On total coloring and equitable total coloring of cubic graphs with large girth[☆]

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Abstract

A k-total-coloring of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The total chromatic number of G, denoted by $\chi''(G)$, is the least k for which G has a k-total-coloring. It was proved by Rosenfeld and Vijayaditya that the total chromatic number of a cubic graph is either 4 or 5. Cubic graphs with $\chi'' = 4$ are said to be Type 1, and cubic graphs with $\chi'' = 5$ are said to be Type 2. A k-total-coloring is equitable if the cardinalities of any two color classes differ by at most one. Similarly, the least k for which G has an equitable k-total-coloring is the equitable total chromatic number of G, denoted by $\chi''_e(G)$. It was proved by Wang that the equitable total chromatic number of a cubic graph is either 4 or 5.

We investigate two questions about total colorings of large girth cubic graphs.

1. Does there exist a Type 2 cubic graph of girth greater than 4?

2. Does there exist a Type 1 cubic graph of girth greater than 4 with $\chi''_e = 5$?

We contribute to both questions by exhibiting infinite families of cubic graphs that indicate that possibly both questions would have a negative answer. In par-

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ticular, we prove that almost all generalized Petersen graphs are Type 1. Furthermore, we present a sufficient condition for a cubic graph to be Type 2, contributing to the search for an answer to the first question.

Keywords: total coloring, equitable total coloring, cubic graphs, girth

1. Introduction

The Total Coloring Conjecture states that the total chromatic number of any graph is at most $\Delta + 2$, where Δ is the maximum degree of a graph [1]. This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (say Type 1) or 5 (say Type 2) [2, 3] (see also [4] for a recent concise proof), but it is NP-hard to decide whether a cubic graph is Type 1, even restricted to bipartite cubic graphs [5, 6]. Motivated by the question proposed by Cavicchioli et al. [7] about the smallest order of a snark (a class of cubic graphs) with girth greater than 4 having total chromatic number equal to 5, we started to investigate the total coloring of large girth cubic graphs. In this paper, we focus on total coloring and equitable total coloring.

The type of all cubic graphs with order up to 16 is established (see [8, 9]), and it is also known that all snarks of girth greater than 4 with order up to 38 [7, 10] are Type 1. Apart from these results for small graphs, there are only few infinite classes of cubic graphs whose total chromatic number has been determined: ladder graphs are Type 1 except for $G(5, 1)$ [8], all Möbius ladder graphs are Type 2 [8], and all members of Flower, Goldberg, Blanuša, and Loupekine snark families [11, 12] have been shown to be Type 1.

The smallest Type 2 cubic graph is K_4 and the smallest Type 2 bipartite cubic graph is $K_{3,3}$. As remarked before, infinite families of Type 2 cubic graphs are known, and all these graphs contain squares or triangles [8]. Type 2 snarks have very recently been found, but so far every known Type 2 snark contains a square [13]. Moreover, computational results showed that a possible Type 2 cubic graph with girth greater than 4 must have at least 34 vertices [13]. So it could be that there exists no Type 2 cubic graph of girth greater than 4.

Recent results on the fractional total chromatic number (see [14] for an exposition about fractional colorings) support the evidence that the girth of a graph is a relevant parameter in the study of total coloring: in particular, it is proved in [15, 16] that if the girth of a cubic graph is sufficiently large then its fractional total chromatic number is 4 (see Theorem 1 for a precise formulation).

All previous arguments naturally lead us to formulate the following question:

Question 1. *Does there exist a Type 2 cubic graph with girth greater than 4?*

In this paper, we present a sufficient condition for a cubic graph to be Type 2, contributing to the search for an answer to Question 1. In addition, we contribute to Question 1 by proving that all members of two infinite families of Blowup and Semi blowup snarks [17] and "almost all" generalized Petersen graphs are Type 1.

The equitable total coloring requires further that the cardinalities of any two color classes differ by at most 1. Similarly to the total coloring, it was conjectured that the equitable total chromatic number of any graph is at most $\Delta + 2$ [18], and this conjecture was proved for cubic graphs in the same work. So, the equitable total chromatic number of a cubic graph is either 4 or 5. No complexity result is known. Some previously known Type 1 cubic graphs admit an equitable 4-total-coloring, for example all members of Flower snarks [11] and of Blanuša snarks [12].

Although there are known examples of graphs such that the total chromatic number is strictly less than the equitable total chromatic number, i.e., $\chi'' = \Delta+1 < \chi''_e = \Delta + 2$ [19], until now no cubic graph with $\chi'' = 4$ and $\chi''_e = 5$ was known. In this paper we present the first known Type 1 cubic graphs such that the equitable total chromatic number is 5; these graphs contain squares or triangles. In this context, we investigate the following additional question about cubic graphs of large girth.

Question 2. *Does there exist a Type 1 cubic graph with girth greater than 4 and equitable total chromatic number 5?*

We contribute to a possible negative answer to Question 2 by proving that infinite families of generalized Petersen graphs have equitable total chromatic number 4.

This paper is organized as follows: Section 2 is devoted to notation and definitions; Section 3 presents a sufficient condition for a cubic graph to need five colors in any total coloring, and examples that satisfy the condition; in Section 4, we prove that for each positive integer k, there exists at most a finite number of generalized Petersen graphs $G(n, k)$ that are Type 2, and we prove that all members of two infinite families of Semi blowup and Blowup snarks are Type 1; Section 5 introduces the first known Type 1 cubic graphs with equitable total chromatic number 5, which motivated Question 2, and we prove that some infinite families of Type 1 generalized Petersen graphs have equitable total chromatic number 4; Section 6 discusses the computational search of 4-total-colorings and equitable 4-total-colorings of generalized Petersen graphs up to order 70 and 40, respectively, which provide further evidences that a positive answer for both Question 1 and 2 would require a large graph; finally, Section 7 contains our final considerations.

2. Notation and definitions

A *semi-graph* is a 3-tuple $G = (V(G), E(G), S(G))$ where $V(G)$ is the set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semi-edges* having one endpoint in $V(G)$. When there is no chance of ambiguity, we simply write V, E or S.

We write edges having endpoints v and w shortly as vw and semi-edges having endpoint v as $v\cdot$. When vertex v is an endpoint of $e \in E \cup S$ we say that v and e are *incident*. Two elements of $E \cup S$ incident to the same vertex, respectively two vertices incident to the same edge, are called *adjacent*.

A *graph* G is a semi-graph with an empty set of semi-edges. In that case we can write $G = (V, E)$. Given a semi-graph $G = (V, E, S)$, we call the graph (V, E) the *underlying graph of* G.

All definitions given below for semi-graphs, that do not require the existence of semi-edges, are also valid for graphs.

Let $G = (V, E, S)$ be a semi-graph. The *degree* $d(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident to v. We say that G is *d-regular* if the degree of each vertex is equal to d. In this paper we are mainly interested in cubic graphs and semi-graphs, also called respectively *cubic graphs* and *cubic semi-graphs*. Given a graph G of maximum degree 3, the semi-graph obtained from G by adding $3 - d(v)$ semi-edges with endpoint v, for each vertex v of G, is called *the cubic semi-graph generated by* G.

For $k \in \mathbb{N}$, a *proper k-vertex-coloring* of G is a map $C^V\colon V \to \{1, 2, ..., k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices. The *chromatic number* of G, denoted by $\chi(G)$, is the least k for which G has a k-vertex-coloring.

Similarly, a *proper k-edge-coloring* of G is a map $C\colon E \cup S \to \{1, 2, ..., k\}$, such that $C(e) \neq C(f)$ whenever e and f are adjacent elements of $E \cup S$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. By Vizing's theorem [20] we have that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class 1*, otherwise G is said to be *Class 2*.

A *k-total-coloring* of G is a map $C^T\colon V \cup E \cup S \to \{1, 2, ..., k\}$, such that

- **(a)** $C^T|_V$ is a proper vertex-coloring,
- **(b)** $C^T|_{E\cup S}$ is a proper edge-coloring,
- **(c)** $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$, and e is incident to v.

The *total chromatic number* of G, denoted by $\chi''(G)$, is the least k for which G has a k-total-coloring. Clearly $\chi''(G) \geq \Delta(G) + 1$. The Total Coloring Conjecture [1] claims that $\chi''(G) \leq \Delta + 2$ and it has been proved for cubic graphs [2, 3]. A cubic graph is said to be Type 1 or Type 2, according to the fact that its total chromatic number is 4 or 5, respectively.

A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G, denoted by $\chi''_e(G)$. In [18] it was conjectured that $\chi''_e(G) \leq \Delta + 2$ for any graph G, and this conjecture was proved for cubic graphs in the same work.

In [13] it is shown that $(k+1)$-total-colorings of k-regular semi-graphs are characterized by some particular proper $(k+1)$-edge-colorings, called strong $(k+1)$-edge-colorings. We recall the definition in the particular case $k = 3$.

Definition 1. *A proper* 4*-edge-coloring C of a cubic semi-graph $G = (V, E, S)$ is called* **strong 4-edge-coloring** *if for each edge $vw \in E$ we have* $|\{C(e)|e \in E \cup S, e \text{ incident to } v \text{ or } w\}| = 4.$

Equivalently a strong 4-edge-coloring of a cubic semi-graph is a proper 4-edge-coloring such that for each edge vw the color not used for the elements of $E \cup S$ incident to v is used for an element incident to w.

Lemma 1. *(Brinkmann et al. [13]) Let $G = (V, E, S)$ be a cubic semi-graph.*

Each strong 4*-edge-coloring C of G can be extended to a* 4*-total-coloring C^T with $C^T|_{E\cup S} = C$ and, for each* 4*-total-coloring C^T of G, $C^T|_{E\cup S}$ is a strong* 4*-edge-coloring.*

REMARK: Lemma 1 implies that there exists a 4-total-coloring C^T of G if and only if there exists a strong 4-edge-coloring C of G.

By Lemma 1, a cubic semi-graph G is Type 1 if and only if there exists a strong 4-edge-coloring of G. Furthermore, a strong 4-edge-coloring has the property that if we assign to each vertex v the color c which is not used for the three edges incident v, we produce a 4-total-coloring of G. In what follows, we say that c is *the color induced on v* by the strong 4-edge-coloring.

A *triangle* is a graph consisting of a cycle of length 3 (or equivalently a complete graph on three vertices) and a *square* is a graph consisting of a chordless cycle of length 4. We denote by K_4 the complete graph on 4 vertices and by $K_{3,3}$ the unique complete bipartite cubic graph.

Section 4 is largely devoted to results on total colorings of generalized Petersen graphs, a well-known class of cubic graphs introduced by Watkins in [21]. Following Watkins' notation, the generalized Petersen graph $G(n, k)$, $n \geq 3$ and

$1 \leq k \leq n-1$, is the graph with vertex-set $\{u_0, u_1, \ldots u_{n-1}, v_0, v_1, v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leq i \leq n-1\}$, with subscripts taken modulo n. Clearly, the graph $G(n, k)$ and the graph $G(n, n-k)$ are isomorphic, so generalized Petersen graphs are usually defined for $k \leq \lceil \frac{n-1}{2} \rceil$. Here, we consider k into the entire interval $[1, n-1]$ in order to avoid boring specifications along the rest of the paper.

Let $G = (V, E)$ be a graph. The *girth of* G is the minimum length of a cycle contained in G, or if G has no cycle, it is defined to be infinity. A *snark* is a cyclically 4-edge-connected (and so of girth at least 4) Class 2 cubic graph (see [12, 13] for more information about Type of snarks).

Along the paper we will use the following construction: let G be a cubic graph and let R be a graph having exactly three vertices of degree 2 and all the others of degree 3. We construct a new cubic graph, denoted by G_v^R, in the following way:

- Remove a vertex v of G (and all edges that are incident to v),
- Connect vertices of degree 2 of $G \setminus \{v\}$ to distinct vertices of degree 2 of R.

In the present paper R will always be a triangle or the complete bipartite graph $K_{2,3}$. Due to the symmetries of each of these two graphs, the choice of the connections in the previous construction is not relevant, so we simply say that G_v^R *is obtained by replacing the vertex* v *of* G *by a copy of* R. We will denote by G^R the cubic graph obtained by replacing every vertex of G by a copy of R.

3. Type 2 cubic graphs

Very little is known about Type 2 cubic graphs. A trivial way to obtain a Type 2 cubic graph is to start from a Type 2 non cubic graph of maximum degree 3 and complete it in a way to obtain a cubic graph. This of course allows to obtain an infinite number of Type 2 cubic graphs starting from the same graph, but these cannot have interesting structures. Remark however that any Type 2 cubic graph containing a bridge can be obtained in this way. This kind of property is not anymore true for Type 2 cubic graphs containing an edge-cut of cardinality greater than 1. It is however easy to see that if a Type 2 cubic bridgeless graph G has an edge-cut of cardinality 2, then we can construct a Type 2 cubic graph smaller than G: more precisely, deleting the two edges of the cut, we may obtain two cubic graphs by adding one edge to each of the new connected components, and at least one of them is Type 2 (see [12]). But no reverse construction is known.

It has been proved by Chetwynd and Hilton in [8] that all Möbius ladders are Type 2 (a Möbius ladder is a cubic graph consisting of an even cycle whose diagonally opposite vertices are joined by an edge). Moreover, they were able to show that, with the exception of the one of order 6, all Möbius ladders are critical: deleting any edge provides a Type 1 graph. We do not know any other infinite family of critical Type 2 cubic graphs.

Using a computer search, Hamilton and Hilton (see [9]) have listed all critical Type 2 graphs of maximum degree 3 and order at most 16, pursuing the list of Chetwynd [22] for graphs of order at most 10. Among these 165 graphs, only one has no chordless cycle of length 4. It is the cubic graph $K'_{3,3}$ of order 12 obtained from $K_{3,3}$ by replacing all three vertices on one side of the bipartition by a triangle (see Figure 2). Moreover, a computer search done by Brinkmann (see [13]) shows that among square-free graphs up to 32 vertices, only another one is of Type 2: the generalized Petersen graph $G(9,3)$ (see Figure 1). There is no other Type 2 square-free cubic graph known, and both $K'_{3,3}$ and $G(9,3)$ contain triangles. We have no intuition of what could explain why a Type 2 cubic graph could need a small cycle. Nevertheless, the girth seems to play a role in total colorings: Reed [23] conjectured that for every $\epsilon > 0$ and every integer Δ there exists g such that the fractional total chromatic number (see [14] for an exposition about fractional colorings) of any graph with maximum degree Δ and girth at least g is at most $\Delta+1+\epsilon$. This conjecture has been proved by Kaiser, King and Král (see [15]) for $\Delta = 3$ and Δ even, and by Kardos, Král and Sereni (see [16]) for the remaining cases. In particular, the following stronger theorem for the cubic case is proved in [15].

Theorem 1 ([15]). *The fractional total chromatic number of any graph with maximum degree 3 and girth at least* 15840 *is 4.*

The proof of this theorem is far from easy, and uses probabilistic arguments. Notice that, if it would be the case that no Type 2 cubic graph with girth more than 4 exists, then we could replace 15840 by 5 in the statement of Theorem 1.

As it is an NP-complete problem to decide whether a cubic graph is Type 1 (see [5, 6]), a nice characterization of Type 2 graphs is unlikely. To show that a cubic graph is of Type 2, so far we did not know any other way than to try to totally color it with four colors until reaching an impossibility. However, we could find a condition that is sufficient to guarantee that a graph is Type 2. It is based on the following observation. Given a 4-total-coloring of a Type 1 cubic graph G, let M_i be the set of edges of a given color i. By definition of a total coloring, M_i is a matching. Furthermore any vertex of G that is not incident to an edge of M_i must be colored with color i, so such vertices form a stable set S_i. Hence, no edge of G could be added to M_i in order to get a larger matching, that is, M_i is a maximal matching. This gives, besides Lemma 1, another characterization of Type 1 cubic graphs.

Proposition 1. *A proper 4-edge-coloring of a cubic graph is strong if and only if the set of edges of each color is a maximal matching.*

Proof. We have already shown that the condition is necessary. Let us assume that a cubic graph G has a partition of its edges into four maximal matchings M_1, M_2, M_3, M_4. Let S_i be the set of vertices uncovered by M_i, $i \in \{1,2,3,4\}$. Since M_i is a maximal matching, each S_i is a stable set. Since G is cubic, for every vertex v of G there is a unique $i \in \{1,2,3,4\}$ such that no edge incident to v in G is in M_i. So S_1, S_2, S_3, S_4 is a partition of the vertices of G into stable sets (we may have some S_i empty) and the function giving color i to every element of $M_i \cup S_i$ ($i \in \{1,2,3,4\}$) is a 4-total-coloring of G. □

In other words, a cubic graph is Type 1 if and only if its set of edges may be partitioned into four maximal matchings. As a consequence, we have a sufficient condition for a cubic graph to need five colors in any total coloring:

Proposition 2. *If a cubic graph G has no maximal matching of cardinality at most $\frac{|E|}{4}$, then G is Type 2.*

As we will see below, this condition is not necessary to be Type 2, but it can be sometimes used with success. In what follows, we will denote by $MMM(G)$ the minimum cardinality of a maximal matching of a graph G. Yannakakis and Gavril showed in [24] that the problem of computing $MMM(G)$ is NP-hard even for bipartite graphs of maximum degree 3 or planar graphs of maximum degree 3. Notice that Proposition 2 can be generalized as follows: if a k-regular graph G has no maximal matching of cardinality at most $\frac{|E|}{k+1}$, then G has no $(k+1)$-total-coloring ($k \geq 1$).

Now, we show how we can use Proposition 2 to furnish a combinatorial easy proof of the already known fact that K_4, $K_{3,3}$, $G(5,1)$ and $G(9,3)$ are Type 2.

Corollary 1. *The graphs K_4, $K_{3,3}$, $G(5,1)$ and $G(9,3)$ are Type 2.*

Proof. As K_4 is a complete graph, any of its maximal matchings is perfect and the condition of Proposition 2 is satisfied.

In $K_{3,3}$ too, all maximal matchings are perfect matchings, and so $K_{3,3}$ also satisfies the condition of Proposition 2.

Now, suppose that the generalized Petersen graph $G(5,1)$ (see Figure 1) has a maximal matching M of size at most $3 = \lfloor \frac{15}{4} \rfloor$; the stable set S of vertices uncovered by M has cardinality 4. Hence, S contains precisely two vertices of the cycle $u_0u_1u_2u_3u_4$ and two vertices of the cycle $v_0v_1v_2v_3v_4$. Assume, without loss of generality, that u_1 and u_3 are in S, hence u_2, v_1 and v_3 cannot be in S, and vertex v_2 must be covered by edge u_2v_2. Now only one more vertex (either v_4 or v_0) could be in S, a contradiction.

Consider now $G(9,3)$ (see Figure 1). This graph has 18 vertices and 27 edges. Suppose that $G(9,3)$ has a maximal matching M of size at most $6 = \lfloor \frac{27}{4} \rfloor$, and let S be the set of vertices not covered by M, so $|S| \geq 6$. By definition, $G(9,3)$ consists in an outer cycle C on nine vertices and three triangles, each connected to the outer cycle by three edges connecting to vertices mutually at distance 3 on C.

Claim 1: The three neighbors in C of the vertices of a triangle cannot be all in S.

Proof of Claim 1: Assume on the contrary that the vertices of a triangle are adjacent to three vertices of S. Then the three vertices of that triangle are pairwise matched by edges of M. This is of course impossible.

Claim 2: If one of the edges connecting a triangle to C belongs to M, and one of the neighbors in C of a vertex of the same triangle belongs to S, then the triangle has no vertex in S.

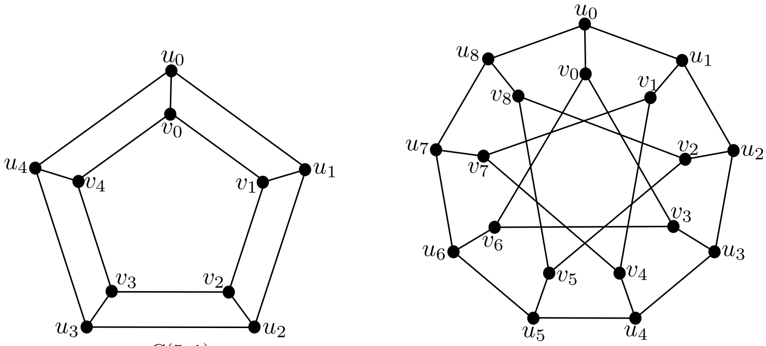
Proof of Claim 2: Denote by t_1, t_2, t_3 the vertices of the triangle, by t_1c the edge in M and t_2c' the edge such that $c' \in C \cap S$. Then, the vertex t_2 cannot be in S, and the only way to have it covered by an edge is if t_2t_3 is in M. So no vertex of the triangle is in S.

As a triangle contains at most one vertex of S, at least three vertices of S belong to C. Then by Claim 1, there should be in C two vertices of S at distance 2. Assume without loss of generality that these are u_1 and u_3. Then u_2v_2 is in M.

Assume that the triangle $v_2v_5v_8$ contains a vertex in S, by symmetry we can assume that $v_5 \in S$. Then u_8 is covered by u_8v_8 and u_0 is covered by u_0v_0. Now, C contains at most one more vertex of S (either u_6 or u_7), and by Claim 2, the triangle $v_3v_6v_0$ contains no vertex of S: a contradiction to the fact that S contains at least 6 vertices.

Assume now that $v_2v_5v_8$ contains no vertex in S. Then C should contain more than three vertices and one of them should be u_5 or u_8, by symmetry we can assume that $u_8 \in S$. Then again u_0v_0 should be in M and by Claim 2 the triangle $v_3v_6v_0$ contains no vertex of S. As C contains at most four vertices of S we obtain again a contradiction to the fact that S contains at least 6 vertices. This completes the proof. □

From [8, 25], we know that among all generalized Petersen graphs $G(n,1)$ with $n \geq 3$, the only Type 2 graph is $G(5,1)$. In the next section, from Theorem 2 and computational results, we will show that among all generalized Petersen graphs $G(n,3)$ with $n \geq 7$, the only one that is Type 2 is the graph $G(9,3)$.



$G(5,1)$ $G(9,3)$

Figure 1: Type 2 generalized Petersen graphs $G(5,1)$ and $G(9,3)$.

The sufficient condition presented in Proposition 2 is not necessary: for instance, we cannot use it to prove that $K'_{3,3}$ is of Type 2, as $MMM(K'_{3,3}) \leq 4$ which is less than $\frac{18}{4}$ (see Figure 2 for a maximal matching of size 4). For completeness, we furnish a combinatorial proof of the fact that $K'_{3,3}$ is Type 2.

Proposition 3. *$K'_{3,3}$ is Type 2.*

Proof. Let us denote by $A, B, C, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ the vertices of $K'_{3,3}$, where a_i, b_i, c_i are the three vertices of the triangles and A is adjacent to all a_i's, B to all b_i's and C to all c_i's ($1 \leq i \leq 3$). There is no maximal matching of size less than 4, since there is obviously no stable set of cardinality 6 in $K'_{3,3}$.

Claim: For any maximal matching M of $K'_{3,3}$ of size 4, the stable set S of four uncovered vertices contains exactly two vertices among A, B, C.

Proof of the claim: The set S contains at least one of A, B, C, since there are four vertices in S and each triangle contains at most one vertex in S. So we can assume without loss of generality that vertex A is in S. Hence, a_1, a_2, a_3 must be covered by three distinct edges of M that are inside the triangles. Hence, since $|M| = 4$ there is precisely one edge of M connecting the triangles to vertices B and C, and precisely one of the vertices B and C remains uncovered by M.

The previous claim proves that any two maximal matchings of size 4 have non disjoint sets of uncovered vertices, and by Proposition 1, a partition of the edges of $K'_{3,3}$ into four maximal matchings cannot contain more than one maximal matching of cardinality 4. Therefore, $K'_{3,3}$ does not admit a partition of its 18 edges into four maximal matchings. $\square$

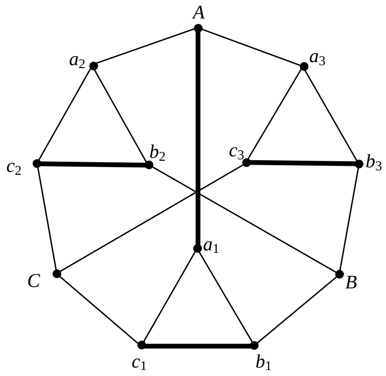


Figure 2: Type 2 graph on 12 vertices $K'_{3,3}$ and a maximal matching of cardinality 4.

4. Type 1 cubic graphs

In the previous section, we have furnished a proof that $G(5,1)$ and $G(9,3)$ are Type 2. As far as we know, these two graphs are the unique examples of Type 2 generalized Petersen graphs. It is proved in [8] that $G(n,1)$, i.e., ladder graphs, are all Type 1 but $G(5,1)$. More generally, we believe that an example of Type 2 cubic graph with large girth (see Question 1) can be hardly found into the class of generalized Petersen graphs.

We give support to this assertion by proving in Section 4.1 that for each positive integer k there exist at most a finite number of Type 2 generalized Petersen graphs $G(n,k)$. Additionally, we prove in Section 4.2 that all members of two infinite families of Semi blowup and Blowup snarks are Type 1.

4.1. Generalized Petersen graphs

In order to prove our theorem, we need to define the following semi-graph, which we denote by $F_{l,k}$, for $l \geq 2k-1$:

- the vertices of $F_{l,k}$ are
  $u_1, u_2, \dots, u_l, v_1, v_2, \dots v_l$,
- the edges of $F_{l,k}$ are
  $u_i u_{i+1}$ for $1 \leq i < l$,
  $u_i v_i$ for $1 \leq i \leq l$,
  $v_i v_{i+k}$ for $1 \leq i \leq l-k$,
- the semi-edges of $F_{l,k}$ are divided in two classes, left semi-edges and right semi-edges. Each class contains $k+1$ semi-edges numbered from 0 to k:

the 0-*th left semi-edge* is $u_1\cdot$,
the i-*th left semi-edge* is $v_i\cdot$, for $1 \leq i \leq k$,
the 0-*th right semi-edge* is $u_l\cdot$,
the $(i-l+k)$-*th right semi-edge* is $v_i\cdot$, for $l-k+1 \leq i \leq l$.

Any semi-graph isomorphic to $F_{l,k}$ will be called a k-*frieze* of length l.

We define the *merge* of a k-frieze F of length l and a k-frieze F' of length l' as the k-frieze FF' of length $l+l'$ obtained by the junction of the i-th right semi-edge of F with the i-th left semi-edge of F' for $0 \leq i \leq k$. The left semi-edges of FF' are those of F and the right ones are those of F'.

We define the *closure* of a k-frieze F as the graph obtained by the junction, for each $0 \leq i \leq k$, of the i-th left semi-edge of F with the i-th right semi-edge of F itself. It is easy to check that the closure of a k-frieze of length $l > 2k$ is the generalized Petersen graph $G(l,k)$.

Given two strong 4-edge-colorings ϕ and ϕ' of k-friezes $F = F_{l,k}$ and $F' = F_{l',k}$ respectively, we say that ϕ is *compatible* with ϕ' if or each i from 0 to k:

- the color given by ϕ to the right i-th semi-edge of F is equal to the color given by ϕ' to the left i-th semi-edge of F',
- the color induced by ϕ on the end-vertex of the right i-th semi-edge of F is distinct from the color induced by ϕ' on the end-vertex of the left i-th semi-edge of F'.

Then obviously ϕ and ϕ' provide a strong 4-edge-coloring of the merge of F and F'.

Definition 2. *Strong* 4*-edge-colorings of a family $\mathcal{F}$ of k-friezes (for some given k) are said to be* ***mutually compatible*** *if for any two (not necessarily distinct) ϕ and ϕ' among these colorings we have ϕ compatible with ϕ'.*

Notice that strong 4-edge-colorings of a family $\mathcal{F}$ of k-friezes are mutually compatible if and only if for each i from 0 to k:

- all i-th semi-edges get the same color,
- the color induced on the end-vertex of any right i-th semi-edge is distinct from the color induced on the end-vertex of any left i-th semi-edge.

Then obviously, any sequence of merges of friezes in $\mathcal{F}$ and also their closure can be provided with a strong 4-edge-coloring.

Theorem 2. *The generalized Petersen graph $G(n,k)$ is Type 1 for any $k \geq 2$, and any n such that $n = 2k\lambda + (2k-1)\mu$ for some non-negative integers λ and μ.*

Proof. The proof proceeds as follows: for every $k \geq 2$ we define two strong 4-edge-colorings, one of the k-frieze $F_{2k,k}$ and one of the k-frieze $F_{2k-1,k}$, that are mutually compatible.

Since we can obtain $G(n,k)$ by the merge of λ k-friezes of length $2k$ and μ k-friezes of length $2k-1$ the assertion will be proved.

We have to distinguish two cases according to the parity of k.

- Case k even (see Figure 3):

  We define a strong 4-edge-coloring Φ_{2k} with colors in the set $\{1,2,3,4\}$ for a k-frieze of length $2k$.

  We set

$$\Phi_{2k}(u_i u_{i+1}) = \begin{cases} 1 & \text{if } i < k, i \text{ even}, \\ 2 & \text{if } i < k, i \text{ odd}, \\ 3 & \text{if } i \geq k, i \text{ even}, \\ 4 & \text{if } i \geq k, i \text{ odd}. \end{cases}$$

$$\Phi_{2k}(u_i v_i) = \begin{cases} 1 & \text{if } k \leq i < 2k, i \text{ even}, \\ 2 & \text{if } k < i < 2k, i \text{ odd}, \\ 3 & \text{if } i < k, i \text{ even, or } i = 2k, \\ 4 & \text{if } i < k, i \text{ odd}. \end{cases}$$

$$\Phi_{2k}(v_i v_{i+k}) = \begin{cases} 3 & \text{if } i \leq k, i \text{ odd}, \\ 4 & \text{if } i \leq k, i \text{ even}. \end{cases}$$

$$\Phi_{2k}(i\text{-th semi-edge}) = \begin{cases} 1 & \text{if } i \text{ odd, or } i = 0, \\ 2 & \text{if } i \neq 0, i \text{ even}. \end{cases}$$

  Now, we define a strong 4-edge-coloring Φ_{2k-1} for a k-frieze of length $2k-1$.

  We set

$$\Phi_{2k-1}(u_i u_{i+1}) = \begin{cases} 1 & \text{if } i < k, i \text{ even}, \\ 2 & \text{if } i < k, i \text{ odd}, \\ 3 & \text{if } i \geq k, i \text{ even}, \\ 4 & \text{if } i \geq k, i \text{ odd}. \end{cases}$$

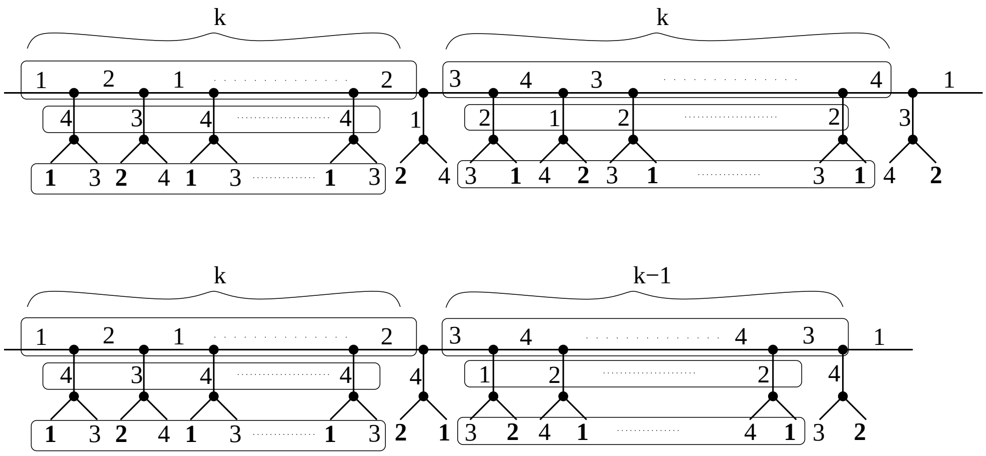


Figure 3: Strong 4-edge-colorings Φ_{2k} and Φ_{2k-1}, for the case k even. The colors of semi-edges are bold.

$$\Phi_{2k-1}(u_iv_i) = \begin{cases} 1 & \text{if } k < i < 2k-1, i \text{ odd}, \\ 2 & \text{if } k < i < 2k, i \text{ even}, \\ 3 & \text{if } i < k, i \text{ even}, \\ 4 & \text{if } i < k, i \text{ odd, or } i = k, 2k-1, \end{cases}$$

$$\Phi_{2k-1}(v_iv_{i+k}) = \begin{cases} 3 & \text{if } i \text{ even}, \\ 4 & \text{if } i \text{ odd}. \end{cases}$$

$$\Phi_{2k}(i\text{-th semi-edge}) = \begin{cases} 1 & \text{if } i \text{ odd, or } i = 0, \\ 2 & \text{if } i \neq 0, i \text{ even}. \end{cases}$$

- Case k odd (see Figure 4):

  We define a strong 4-edge-coloring φ_{2k} with colors in the set $\{1, 2, 3, 4\}$ for a k-frieze of length $2k$. We set

$$\varphi_{2k}(u_iu_{i+1}) = \begin{cases} 1 & \text{if } i < k, i \text{ even, or } i = 2k-1, \\ 2 & \text{if } i < k, i \text{ odd}, \\ 3 & \text{if } i \geq k, i \text{ even}, \\ 4 & \text{if } k \leq i < 2k-1, i \text{ odd}. \end{cases}$$

$$\varphi_{2k}(u_iv_i) = \begin{cases} 1 & \text{if } k < i < 2k-1, i \text{ odd,} \\ 2 & \text{if } k < i < 2k, i \text{ even, or } i = 2k-1, \\ 3 & \text{if } i < k, i \text{ even, or } i = k, \\ 4 & \text{if } i < k, i \text{ odd, or } i = 2k. \end{cases}$$

$$\varphi_{2k}(v_iv_{i+k}) = \begin{cases} 1 & \text{if } i \leq k, i \text{ odd, or } i = k-1, \\ 2 & \text{if } i < k-1, i \text{ even.} \end{cases}$$

$$\varphi_{2k}(i\text{-th semi-edge}) = \begin{cases} 2 & \text{if } i = k, \\ 3 & \text{if } i < k, i \text{ odd, or } i = 0, \\ 4 & \text{if } i \neq 0, i \text{ even.} \end{cases}$$

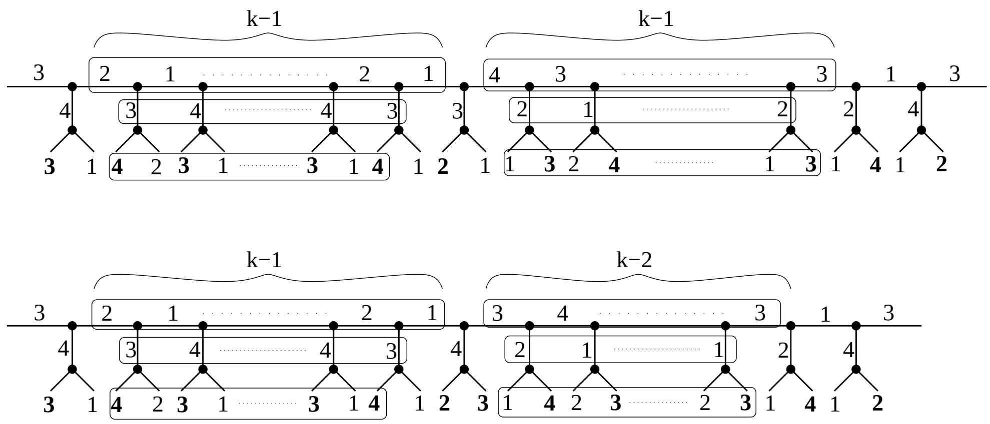


Figure 4: Strong 4-edge-colorings φ_{2k} and φ_{2k-1}, for the case k odd. The colors of semi-edges are bold.

Now, we define a strong 4-edge-coloring φ_{2k-1} for a k-frieze of length $2k-1$. We set

$$\varphi_{2k-1}(u_iu_{i+1}) = \begin{cases} 1 & \text{if } i < k, i \text{ even, or } i = 2k-2, \\ 2 & \text{if } i < k, i \text{ odd,} \\ 3 & \text{if } i \geq k, i \text{ odd,} \\ 4 & \text{if } k < i < 2k-2, i \text{ even.} \end{cases}$$

$$\varphi_{2k-1}(u_iv_i) = \begin{cases} 1 & \text{if } k < i < 2k-1, i \text{ odd,} \\ 2 & \text{if } k < i < 2k-1, i \text{ even,} \\ 3 & \text{if } i \leq k, i \text{ even,} \\ 4 & \text{if } i \leq k, i \text{ odd or } i = 2k-1. \end{cases}$$

$$\varphi_{2k-1}(v_iv_{i+k}) = \begin{cases} 1 & \text{if } i < k, i \text{ odd, or } i = k-1, \\ 2 & \text{if } i < k-1, i \text{ even.} \end{cases}$$

$$\varphi_{2k}(i\text{-th semi-edge}) = \begin{cases} 2 & \text{if } i = k, \\ 3 & \text{if } i < k, i \text{ odd, or } i = 0, \\ 4 & \text{if } i \neq 0, i \text{ even.} \end{cases}$$

We leave to the patient reader to check that Φ_{2k} and Φ_{2k-1}, as well as φ_{2k} and φ_{2k-1}, are mutually compatible. This concludes the proof. $\square$

An application of the method used in the proof of Theorem 2 is shown in Figure 5.

Corollary 2. *For $n \geq (2k-1)(2k-2)$, the generalized Petersen graph $G(n,k)$ is Type 1.*

Proof. It is proved by Sylvester (see [26]) that each integer $n \geq (2k-1)(2k-2)$ can be obtained as the sum of a suitable combination of $2k$ and $2k-1$. Hence the assertion is a direct consequence of Theorem 2. $\square$

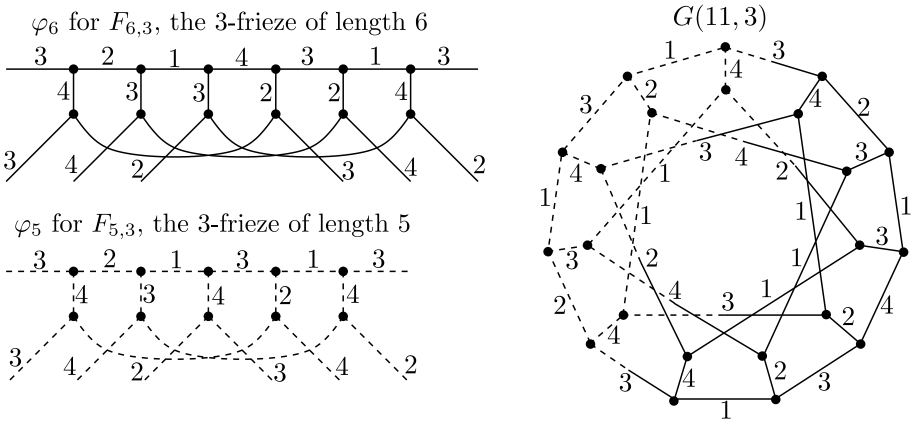


Figure 5: A strong 4-edge-coloring of $G(11,3)$.

Theorem 3. *If $G(n,k)$ is Type 1, then $G(n',k')$ is Type 1, for all $n' \equiv 0 \bmod n$ and all $k' \equiv k \bmod n$.*

Proof. Let $n \geq 3$ and k be two integers such that $G(n,k)$ is Type 1. Then there exists a strong 4-edge-coloring c of the k-frieze $F = F_{n,k}$ satisfying the properties indicated in the proof of Theorem 2 that ensures that any sequence of merges of copies of F, and also their closure, can be provided with a strong 4-edge-coloring. Hence, for any n' that is a p-th multiple of n, we get a strong 4-edge-coloring c' of $G(n',k)$ by merging p copies of F, and this coloring has the property that for each $1 \leq i \leq n$ and $0 \leq p' \leq p$ the edges v_iu_i, v_iv_{i+k} and v_iv_{i-k} have the same colors respectively as $v_{i+p'n}u_{i+p'n}$, $v_{i+p'n}v_{i+p'n+k}$ and $v_{i+p'n}v_{i+p'n-k}$ (any index j bigger than n' is considered as equal to $j - n'$). As a consequence, if $k' - k$ is a q-th multiple of n ($1 \leq q \leq p$), replacing each edge v_iv_{i+k} by an edge v_iv_{i+k+qn} of the same color we obtain a strong 4-edge-coloring of $G(n',k')$. □

By using Theorem 3 we can establish the Type also of some generalized Petersen graphs which do not satisfy the assumption of Theorem 2. For instance, we can construct a 4-total-coloring for $G(12,5)$ starting from a 4-total-coloring of $G(4,1)$.

Furthermore, we have verified by a computer search (see Section 6) that the generalized Petersen graphs of order $2n$ up to 70 are Type 1 with only the two exceptions $G(5,1)$ and $G(9,3)$. For small values of k, the combination of Theorem 2, Theorem 3, and a first computer search leaves uncovered very few cases. For instance, for $k = 4$, the unique open case is the graph $G(41,4)$. For $k = 5$, we cannot establish the type of $G(41,5)$, $G(43,5)$, $G(53,5)$, $G(61,5)$ and $G(71,5)$ from previous arguments. Finally, for $k = 6$, the cases uncovered are eight and the largest one is $G(109,6)$. A further computer search on all these specific cases proves the following theorem.

Theorem 4. *All generalized Petersen graphs $G(n,k)$, $k \leq 6$, are Type 1, but $G(5,1)$ and $G(9,3)$.*

We would like to stress that making use of the well-known isomorphisms between generalized Petersen graphs (see [21]), one can prove that $G(n,k)$ is Type 1 also for some values of n and k not directly covered by Theorem 2; for example $G(127,21)$ is Type 1 since it is isomorphic to $G(127,6)$.

Still considering cubic graphs with girth greater than 4, we describe two families of snarks recently defined by Hägglund [17] and we show that all members of these families are Type 1.

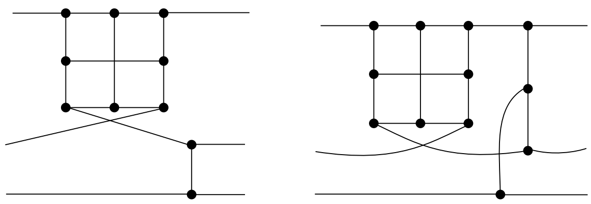
4.2. The n-SemiBlowup and n-Blowup snarks

Hägglund [17] defined two constructions of snarks that have as parameters a graph G and a 2-regular subgraph D of G. Here, we consider the snarks obtained when G is the generalized Petersen graph $G(n, 1)$ and D is the n-cycle of $G(n, 1)$ generated by $u_0, u_1, \ldots, u_{n-1}$. We define them directly.

Each construction uses a cubic semi-graph with six semi-edges: S in the case of SemiBlowup and B in the case of Blowup, as indicated in Figure 6. For any given n we denote by:

- n-SemiBlowup, the graph constituted by a cyclic arrangement of n copies of S such that two consecutive copies are linked as shown in Figure 7 (a).
- n-Blowup, the graph constituted by a cyclic arrangement of n copies of B such that two consecutive copies are linked as shown in Figure 7 (b).

By [17], we know that the n-SemiBlowup and n-Blowup graphs are snarks; and for every $n \geq 5$, they have girth equal to 5.



(a) S (b) B

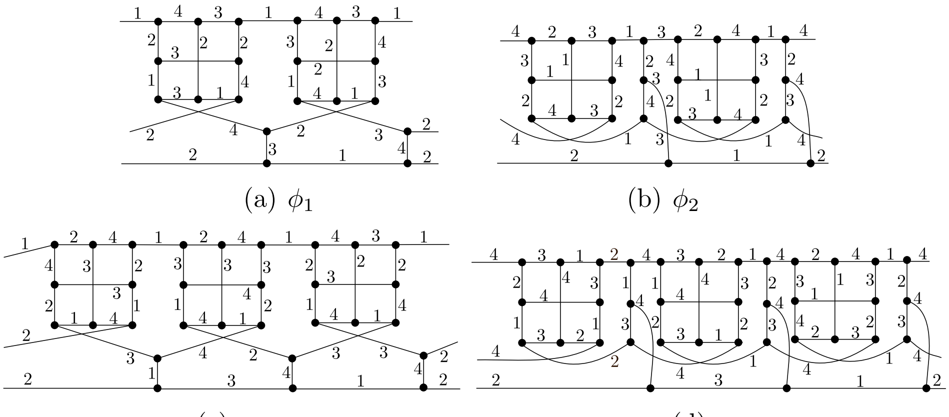
Figure 6: The two cubic semi-graphs S and B used in the construction.

Theorem 5. *For every $n \geq 5$, the n-SemiBlowup and the n-Blowup snarks are Type 1.*

Proof. We define 4-total-colorings of n-SemiBlowup and n-Blowup snarks using partial colorings that can be combined together, similarly as for generalized Petersen graphs.

On Figure 7 (a) and (c), ϕ_1 is a strong 4-edge-coloring of the semi-graph generated by two consecutive copies of S in an n-SemiBlowup and φ_1 is a strong 4-edge-coloring of the semi-graph generated by three consecutive copies of S in an n-SemiBlowup. It can easily be checked that these colorings are "mutually compatible" (in the same sense as defined for k-friezes).

For an even n, by using ϕ_1 on $\frac{n}{2}$ consecutive paired S, we obtain a strong 4-edge-coloring of the n-SemiBlowup snark. For n odd, we need to use once the strong 4-edge-coloring φ_1 and then ϕ_1 on $\frac{n-3}{2}$ consecutive paired S. Therefore, we get in this way a strong 4-edge-coloring of the n-SemiBlowup snark, for n odd.



(a) ϕ_1 (b) ϕ_2

(c) φ_1 (d) φ_2

Figure 7: The strong 4-edge-colorings ϕ_1, ϕ_2, φ_1, and φ_2.

The case of the n-Blowup snark is obtained from above, by replacing ϕ_1 by ϕ_2 and φ_1 by φ_2, see Figure 7 (b) and (d). $\square$

5. Equitable total colorings

In this section, we deal with Question 2 about Type 1 cubic graphs of girth greater than 4 with equitable total chromatic number 5. Although there are known examples of graphs such that the total chromatic number is strictly less than the equitable total chromatic number, i.e., $\chi'' = \Delta + 1 < \chi''_e = \Delta + 2$ [19], until now no cubic graph with $\chi'' = 4$ and $\chi''_e = 5$ was known. This section is divided in two parts. In the first part, we present a construction of Type 1 cubic graphs with equitable total chromatic number 5. All these graphs contain squares or triangles. The second part presents two infinite families of generalized Petersen graphs with equitable total chromatic number 4, which gives support to the assertion of Question 2.

5.1. Equitable total chromatic number 5

Consider the cubic semi-graph K' generated by the complete bipartite graph $K_{2,3}$. The semi-graph K' has interesting coloring properties. First we state a useful lemma, that can be easily checked.

Lemma 2. *In every* 4*-total-coloring of the cubic semi-graph generated by a square, semi-edges incident to adjacent vertices get distinct colors.* $\square$

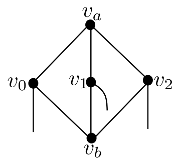


Figure 8: Cubic semi-graph K' generated by $K_{2,3}$.

Lemma 3. *In every 4-total-coloring of K', all semi-edges must have the same color and their endpoints must have different colors.*

Proof. Consider a 4-total-coloring of K'. Let s and e be a semi-edge and an edge of K', respectively. Either s and e are adjacent, so they have different colors, or they are incident to adjacent vertices of a square of K'. Hence by Lemma 2, they have different colors also in this case. Since three colors are necessary to color the edges of K', we get that exactly three colors are used for the edges of K' and the remaining color for the semi-edges. Suppose now that the endpoints of two distinct semi-edges receive the same color. This implies that the edges of the unique square which contains both these two vertices are colored with only two colors, so the two edges not in the square must receive the same color, but they are adjacent and so we have a contradiction. Therefore, the endpoints of the three semi-edges of K' have three different colors. □

By Lemma 3, we are in position to construct infinitely many Type 1 cubic graphs of small girth with equitable total chromatic number 5. Let us denote by K the graph $K_{2,3}$.

Theorem 6. *For every cubic graph H, the cubic graph H^K obtained from H by replacing every vertex by K, is Type 1 and has equitable total chromatic number 5.*

Proof. Let H be a cubic graph with n vertices, H^T be obtained from H by replacing every vertex of H by a triangle and H^K be obtained by replacing every vertex of H by K.

We remark that H^K may also be obtained from H^T by the following transformation: for every triangle $T = v_0v_1v_2$ that replaces a vertex v of H in H^T we delete the three edges of the triangle and we add two vertices v_a and v_b both joined to v_0, v_1 and v_2.

By Brooks' theorem, there exists a 3-vertex-coloring c of H^T with color-set $\{1, 2, 3\}$ and, from c and the construction above, we derive a 4-total-coloring c' of H^K as follows: all vertices that were already in H^T keep their color and all new vertices are colored by 4; all edges that were already in H^T are colored by 4; for every vertex v of H, $c'(v_a v_i) = c(v_{i+1})$ and $c'(v_b v_i) = c(v_{i-1})$ (where indices are taken modulo 3). So H^K is Type 1.

Denote the number of vertices of H^K by $\lambda = 5n$. By Lemma 3, in any 4-total-coloring of H^K, $\frac{3n}{2}$ edges and $2n$ vertices have the same color, say 4. So, there are $\frac{7n}{2}$ elements with color 4. On the other hand, the cardinality of each of the other three color classes is $3n$. Hence, the cardinalities of color class 4 and any other color class differ by $\frac{n}{2}$, and then, as $n \geq 4$, there exists no equitable 4-total-coloring of H^K.

By the main result in [18], H^K has then equitable total chromatic number 5. □

Note that the difference between the cardinalities of two color classes increases according to the number of vertices of H. Unfortunately, the graph H^K contains squares.

Furthermore, it is easy to see that the graph $H^{K,T}$, obtained from any cubic graph H of order at least six by replacing one vertex by a triangle and all other vertices by K, is Type 1, has $\chi''_e = 5$ and contains a triangle. See Figure 9 for two examples of Type 1 cubic graphs with equitable total chromatic number 5.

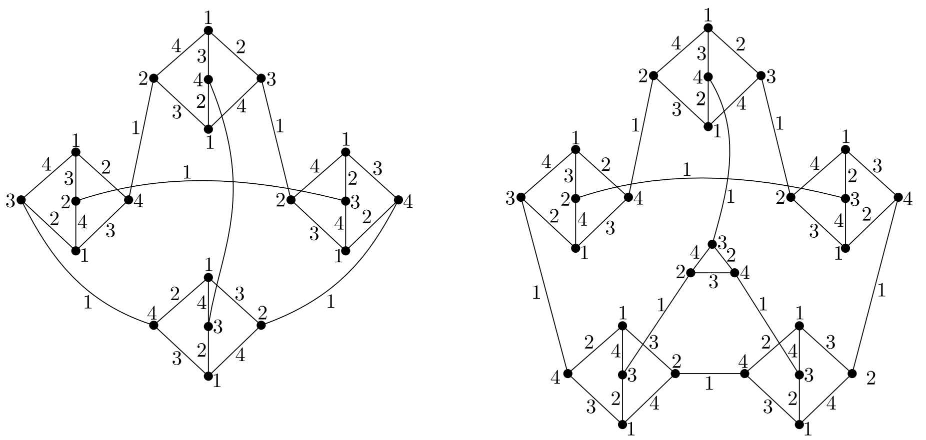


Figure 9: Two Type 1 cubic graphs with equitable total chromatic number 5.

5.2. Equitable total chromatic number 4

For the next results, we use definitions presented in Section 4. Moreover, we will use the following property.

Proposition 4. *Let c be a 4-total-coloring of a cubic graph $G = (V, E)$. The following three statements are equivalent.*

1. *c is equitable,*
2. *$c|_E$ is equitable (the numbers of edges in each color class differ by at most one)*
3. *$c|_V$ is almost equitable (the numbers of vertices in each color class differ by at most two)*

Theorem 7. *For every even k and $n \geq 2k$ such that $n \equiv 0, -1$ or $-2 \bmod 2k$, the generalized Petersen graph $G(n, k)$ has equitable total chromatic number 4.*

Proof. We will show this result using the colorings Φ_{2k} and Φ_{2k-1} defined in the proof of Theorem 2.

Claim 1: The coloring C of the vertices of the frieze $F_{2k,k}$ induced by the strong 4-edge-coloring Φ_{2k} is such that all color classes have the same cardinality k.

Proof of Claim 1: It is straightforward to check that

$$C(u_1), C(u_2), \ldots, C(u_{k-1}), C(u_k), C(u_{k+1}), C(u_{k+2}), \ldots, C(u_{2k-1}), C(u_{2k})$$

$$= 3, 4, \ldots, 3, 4, 1, 2, \ldots, 1, 2,$$

and

$$C(v_1), C(v_2), \ldots, C(v_{k-2}), C(v_{k-1}), C(v_k), C(v_{k+1}), \ldots, C(v_{2k-2}), C(v_{2k-1}), C(v_{2k})$$

$$= 2, 1, \ldots, 1, 2, 3, 4, \ldots, 3, 4, 1$$

(dots mean that the two preceding colors alternate along, an appropriate number of times).

So there are exactly k vertices of each color.

Claim 2: The coloring c of the vertices of the frieze $F_{2k-1,k}$ induced by the strong 4-edge-coloring Φ_{2k-1} is almost equitable: there are $k-2$ vertices colored 4 and all the others color classes have cardinality k.

Proof of Claim 2: It is straightforward to check that

$$c(u_1), c(u_2), \ldots, c(u_{k-1}), c(u_k), c(u_{k+1}), \ldots, c(u_{2k-2}), c(u_{2k-1})$$

$$= 3, 4, \ldots, 3, 1, 2, \ldots, 1, 2,$$

and

$$c(v_1), c(v_2), \ldots, c(v_{k-1}), c(v_k), c(v_{k+1}), \ldots, c(v_{2k-2}), c(v_{2k-1})$$
$$= 2, 1, \ldots, 2, 3, 4, \ldots 3, 1.$$

So there are exactly $k-2$ vertices of color 4 and k vertices of each other color.

Claim 3: The coloring Φ'_{2k-1} of the edges of the frieze $F_{2k-1,k}$ obtained from the strong 4-edge-coloring Φ_{2k-1} by exchanging colors 3 and 4 is strong and the coloring c' of the vertices of $F_{2k-1,k}$ induced by Φ'_{2k-1} is almost equitable: there are $k-2$ vertices colored 3 and all the other color classes have cardinality k.

The proof of this last claim is straightforward.

It is also easy to check that the strong 4-edge-colorings Φ_{2k}, Φ_{2k-1}, Φ'_{2k-1} are mutually compatible. This concludes the proof, since:

- for $n \equiv 0 \bmod 2k$, $G(n,k)$ can be obtained from the closure of the merging of $\frac{n}{2k}$ friezes $F_{2k,k}$,
- for $n \equiv -1 \bmod 2k$, $G(n,k)$ can be obtained from the closure of the merging of $\frac{n-(2k-1)}{2k}$ friezes $F_{2k,k}$ and one frieze $F_{2k-1,k}$,
- for $n \equiv -2 \bmod 2k$, $G(n,k)$ can be obtained from the closure of the merging of $\frac{n-(4k-2)}{2k}$ friezes $F_{2k,k}$ and two friezes $F_{2k-1,k}$.

By Proposition 4, using the strong 4-edge-coloring Φ_{2k} and at most once each of Φ_{2k-1} and Φ'_{2k-1}, we obtain an equitable strong 4-edge-coloring for all cases. □

Theorem 8. *For every $n \geq 4$, the generalized Petersen graph $G(n,2)$ has equitable total chromatic number 4.*

Proof. By Theorem 7 we only have to prove that $G(n,2)$ has equitable total chromatic number 4 whenever $n \equiv 1 \bmod 4$. Let us consider the strong 4-edge-colorings Φ_4 of $F_{4,2}$ and Φ_5 of $F_{5,2}$ shown in Figure 10. These colorings are mutually compatible. The coloring of the 8 vertices of $F_{4,2}$ induced by Φ_4 is such that all color classes have the same cardinality 2 and the coloring of the 10 vertices of $F_{5,2}$ induced by Φ_5 is almost equitable. So using $\frac{n-5}{4}$ times Φ_4 and one time Φ_5, we obtain an equitable strong 4-edge-coloring of $G(n,2)$ when $n \equiv 1 \bmod 4$. □

Theorem 9. *For every $n \geq 3$ and $n \neq 5$, the generalized Petersen graph $G(n,1)$ has equitable total chromatic number* 4.

Proof. We follow the same idea used in the previous proof. Let us consider the strong 4-edge-colorings φ_3 and φ'_3 of $F_{3,1}$, φ_4 of $F_{4,1}$ and φ_9 of $F_{9,1}$ shown in Figure **??**. These colorings are mutually compatible. The coloring of the 8 vertices of $F_{4,1}$ induced by φ_4 is equitable and all the other colorings are almost equitable.

We obtain an equitable strong 4-edge-coloring of $G(n,1)$ by using

- $\frac{n}{4}$ times φ_4, for $n \equiv 0 \bmod 4$;
- $\frac{n-3}{4}$ times φ_4 and one time φ_3, for $n \equiv 3 \bmod 4$;
- $\frac{n-6}{4}$ times φ_4, one time φ_3 and one time φ_3', for $n \geq 6$ and $n \equiv 2 \bmod 4$;
- $\frac{n-9}{4}$ times φ_4 and one time φ_9, for $n \geq 9$ and $n \equiv 1 \bmod 4$.

□

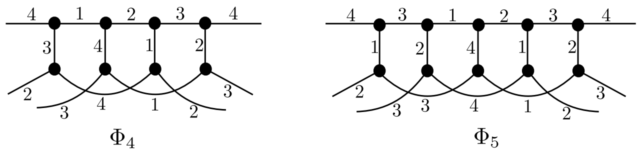


Figure 10: The two strong 4-edge-colorings Φ_4 and Φ_5.

In general for k odd, we know, from the strong 4-edge-coloring φ_{2k} defined in the proof of Theorem 2, that $G(2k, k)$ has equitable total chromatic number 4, but no more.

Furthermore, we have verified by a computer search (see Section 6) that the generalized Petersen graphs of order $2n$ up to 40 have equitable total chromatic number 4 with only the two exceptions $G(5, 1)$ and $G(9, 3)$ that are Type 2.

Blanuša snarks and Flower snarks are other classes of cubic graphs of equitable total chromatic number 4. It is indeed easy to check that the 4-total-colorings of these classes described in [12] and [11] are equitable.

6. Computational Results

The software developed for total coloring the generalized Petersen graphs is a specialized backtrack algorithm that makes use of the special structure of such graphs and try to find a 4-total-coloring. The general idea of the algorithm consists in the following steps.

Let G be a generalized Petersen graph.

(1) Define an order on the vertices of the graph.
(2) Let v be the next uncolored vertex according to the order defined in (1).
  (2.1) If no such v exists, G is already colored with 4 colors. Return such coloring.
  (2.2) For each possible coloring of v and its incident edges compatible with the previously colored elements,

(2.2.1) Color v and its incident edges and recurse in the next uncolored vertex;

(2.2.2) If no coloring was found recursing on the next uncolored vertex, uncolor the colored elements in the previous step.

(2.3) If all the possibilities were tried, return to the previous vertex in the order and try to recolor it. If v is the first vertex, then G is of Type 2.

6.1. Time complexity

Note that, for a vertex v, there can be up to $4! = 24$ possible colorings of v and its incident edges on step (2.2). This could give an overall complexity of $O(24^{|V(G)|})$. However, with a carefully chosen order in step (1), the number of choices can be drastically reduced. The heuristic used when defining such an order was that, when a vertex was chosen to be colored, at least one of its neighbors already had a color assigned. Actually, we used orders in which additionally a vertex had two neighbors occuring before it in the order as frequently as possible. Besides that, as soon as a 4-total-coloring was found, the software returned it, avoiding the worst case complexity. This enabled us to run the software covering besides all generalized Petersen graphs with order $2n$ up to 70, even some instances with more than 200 vertices.

For the equitable total-coloring, a similar approach was used, with an additional procedure for checking, after a 4-total-coloring had been found, whether it is equitable or not. If it was not equitable, the search continued. In this case, greater computational effort was required since, for a given graph, more than one 4-total-coloring could be found in order to find an equitable 4-total-coloring. Except for the two Type 2 graphs $G(5, 1)$ and $G(9, 3)$, all generalized Petersen graphs of order $2n$ up to 40 had their equitable total chromatic number determined to be 4.

The algorithms were implemented using the programming language C, and all the instances were run in a Mac OS X system over a dual core processor.

6.2. Correctness of the program

Although the absence of implementation errors in the program cannot be assured, all the results provided by it can be attested: for the two known cases of Type 2 generalized Petersen graphs, $G(5, 1)$ and $G(9, 3)$, we have explicit proofs that they require $\Delta + 2$ colors in Section 3. For all the other possible graphs given to the software it outputs a proper total coloring with $\Delta + 1$ colors, showing that they are Type 1.

7. Conclusion

In this paper, we investigate two related questions that are motivated by the rich existing literature on total coloring of cubic graphs. Both questions consider total colorability of large girth cubic graphs, with respect to the general and the equitable total coloring. Question 1 searches for a Type 2 large girth cubic graph and Question 2 searches for a large girth cubic graph without an equitable 4-total-coloring.

The constructed examples in Section 5 of the first known Type 1 cubic graphs with equitable total chromatic number 5 suggest a further relation between Questions 1 and 2. Indeed, the used graph $K_{2,3}$ is obtained from the small girth Type 2 $K_{3,3}$ by removing a vertex. Perhaps, a positive answer to Question 1 may provide a suitable large girth Type 2 cubic graph that might be used to achieve a positive answer to Question 2.

We contribute to both questions by exhibiting infinite families of cubic graphs that indicate that possibly both questions would have a negative answer, and by providing computational and theoretical evidence that a positive answer would require a large graph. On the other hand, we present a sufficient condition that contributes to the search for a positive answer to Question 1, and we present the first Type 1 cubic graphs with equitable total chromatic number 5, which motivates the search for a positive answer to Question 2.

Figure 11 presents a diagram summarizing the results on total-coloring and equitable total-coloring of generalized Petersen graphs contained in this work. Note that Theorems 3 and 7 provide infinitely many additional Type 1 and equitable 4-total-colorable generalized Petersen graphs in the white region, respectively.

References

[1] M. Behzad, G. Chartrand, J. K. Cooper Jr, The colour numbers of complete graphs, J. Lond. Math. Soc. 42 (1967) 226–228.

[2] M. Rosenfeld, On the total chromatic number of a graph, Israel J. Math. 9 (1971) 396–402.

[3] N. Vijayaditya, On total chromatic number of a graph, J. London Math. Soc. 2 (1971) 405–408.

[4] Y. Feng, W. Lin, A concise proof for total coloring subcubic graphs, Inform. Process. Lett. 113 (2013) 664–665.

[5] A. Sánchez-Arroyo, Determining the total colouring number is NP-hard, Discrete Math. 78 (1989) 315–319.

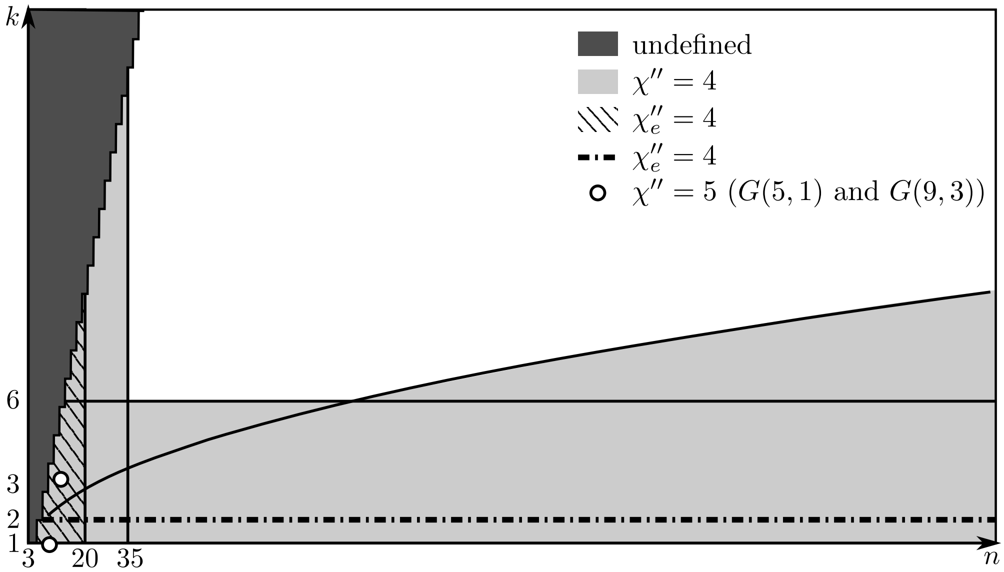


Figure 11: Summary of results about generalized Petersen graphs.

[6] C. McDiarmid, A. Sánchez-Arroyo, Total colouring regular bipartite graphs is NP-hard, Discrete Math. 124 (1994) 155–162.

[7] A. Cavicchioli, T. E. Murgolo, B. Ruini, F. Spaggiari, Special classes of snarks, Acta Appl. Math. 76 (2003) 57–88.

[8] A. G. Chetwynd, A. J. W. Hilton., Some refinements of the total chromatic number conjecture, Congr. Numer. 66 (1988) 195–216.

[9] G. M. Hamilton, A. J. W. Hilton, Graphs of maximum degree 3 and order at most 16 which are critical with respect to the total chromatic number, J. Combin. Math. Combin. Comput. 10 (1991) 129–149.

[10] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, Generation and properties of snarks, J. Comb. Theory B 103 (2013) 468–488.

[11] C. N. Campos, S. Dantas, C. P. Mello, The total-chromatic number of some families of snarks, Discrete Math. 311 (2011) 984–988.

[12] D. Sasaki, S. Dantas, C. M. H. de Figueiredo, M. Preissmann, The hunting of a snark with total chromatic number 5, Discrete Appl. Math. (2013). `http://dx.doi.org/10.1016/j.dam.2013.04.006`.

[13] G. Brinkmann, M. Preissmann, D. Sasaki., Snarks with total chromatic number 5, Submitted (2013).

[14] E. Scheinerman, D. Ullman, Fractional Graph Theory: A rational approach to the theory of graphs, Dover books on Mathematics, 2011.

[15] T. Kaiser, A. King, D. Král, Fractional total colourings of graphs of high girth, J. Comb. Theory B 101 (2011) 383–402.

[16] F. Kardos, D. Král, J. Sereni, The last fraction of a fractional conjecture, SIAM J. Discr. Math. 24 (2010) 699–707.

[17] J. Hägglund, On snarks that are far from being 3-edge-colorable (2012). `arxiv.org/abs/1203.2015`.

[18] W. F. Wang, Equitable total coloring of graphs with maximum degree 3, Graphs Combin. 18 (2002) 677–685.

[19] H. L. Fu, Some results on equalized total coloring, Congr. Numer. 102 (1994) 111–119.

[20] V. G. Vizing, On an estimate of the chromatic class of a p-graph, Metody Diskret. Analiz. 3 (1964) 25–30.

[21] M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory 6 (1969) 152–164.

[22] A. G. Chetwynd, Total colourings of graphs, in: R. Nelson, R. J. Wilson (Eds.), Graph Colourings, 218, Pitman, Essex, UK, 1990.

[23] B. Reed, Fractional total colouring, Presentation at DIMACS Workshop on Graph Coloring and Structure, Princeton, USA, 2009.

[24] M. Yannakakis, M. Gavril, Edge dominating sets in graphs, SIAM J. Appl. Math. 38 (1980) 364–372.

[25] C. N. Campos, C. P. Mello, The total chromatic number of some bipartite graphs, Ars Combin. 88 (2008) 335–347.

[26] J. J. Sylvester, Question 7382, Mathematical Questions from the Education Times 41 (1884) 21.