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COMMENT ON THE PREDICTION OF TWO-LOOP STANDARD CHIRAL PERTURBATION THEORY FOR LOW-ENERGY $\pi\pi$ SCATTERING

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Abstract

Four of the six parameters defining the two-loop $\pi\pi$ scattering amplitude have been determined using Roy dispersion relations. Combining this information with the Standard χPT expressions, we obtain the threshold parameters, low-energy phases and the $O(p^4)$ constants l_1^r, l_2^r are obtained. The result $l_2^r(M_\rho) = (1.6 \pm 0.4 \pm 0.9) \times 10^{-3}$ ($\bar{l}_2 = 4.17 \pm 0.19 \pm 0.43$) reproduces the correct D-waves but it is incompatible with existing Standard χPT analyses of K_{l4} form factors beyond one loop.

Key-Words: chiral symmetries, sum rules, meson-meson interactions, chiral lagrangians

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1. During the last few years there has been a noticeable revival of interest in the high precision analysis of low-energy $\pi\pi$ scattering [1]-[13]. There are at least two reasons for this. First, it has been shown [13, 1, 3] and repeatedly emphasized [14] that the $\pi\pi$ scattering amplitude in the threshold region is particularly sensitive to the strength of quark anti-quark pair condensation in the QCD vacuum: the smaller the condensate, the stronger the isoscalar S-wave $\pi\pi$ interaction. The accurate measurement of S-wave scattering lengths would, indeed, provide the first experimental evidence in favour of, or against, the standardly admitted hypothesis according to which the mechanism of spontaneous chiral symmetry breaking is dominated by the formation of a large $<\bar{q}q>$ condensate. Within QCD, this hypothesis is by no means a logical necessity and its experimental test might well become an important step towards a non-perturbative understanding of the quark-gluon dynamics. The second reason which makes detailed $\pi\pi$ studies topical, is that there are two new high precision experiments currently under preparation: *i*) The phase shift difference $\delta_0^0(E) - \delta_1^1(E)$ at low energies ($E < 400$MeV) will be extracted from a new K_{l4}-decay experiment [15] performed with the KLOE detector at the Frascati ϕ-factory DAΦNE [16]. *ii*) At CERN, the project DIRAC [17] aims at the measurement of the lifetime of $\pi^+\pi^-$ atoms to 10%, implying the determination of the combination of scattering lengths $| a_0^0 - a_0^2 |$ with a 5% accuracy. On the theoretical side an even better precision can be reached by a systematic use of chiral perturbation theory [18, 19] (χPT). The low-energy expansion of the $\pi\pi$ scattering amplitude $A(s|t,u)$ starts at order $O(p^2)$ given by Weinberg more than 30 years ago [20]. Subsequently, the one-loop $O(p^4)$ contribution to $A(s|t,u)$ has been calculated by Gasser and Leutwyler [21, 19]. It is given by four low-energy constants l_1, l_2, l_3, l_4 besides the (charged) pion mass M_π and the decay constant F_π. The present state of the art involves the two-loop $O(p^6)$ order and the present letter concerns this degree of accuracy.

2. The $O(p^6)$ amplitude $A(s|t,u)$ has been first given in Ref. [1] in the form

$$A(s|t,u) = A_{KMSF}(s|t,u;\alpha,\beta;\lambda_1,\lambda_2,\lambda_3,\lambda_4) + O\left[\left(\frac{p}{\Lambda_H}\right)^8, \left(\frac{M_\pi}{\Lambda_H}\right)^8\right]. \qquad (1)$$

The function A_{KMSF}, which depends on the Mandelstam variables s, t, u and on the six parameters α, β, λ_1, $\ldots\lambda_4$, is explicitly displayed in [1]. Here, p denotes the characteristic pion momentum and Λ_H is the mass scale of bound states not protected by the chiral symmetry, $\Lambda_H \sim 4\pi F_\pi \sim 1$ GeV. The result (1) holds independently of the strength of the quark condensate. The latter merely shows up in the size of the constant α: for standard, large values[1] $<\bar{q}q> \simeq -(250\text{ MeV})^3$ one has $\alpha \simeq 1$ and its value increases up to $\alpha \simeq 4$ for $|<\bar{q}q>|$ decreasing down to zero. The parameter β is less sensitive to the value of the condensate, remaining

[1] $<\bar{q}q>$ denotes the single flavour condensate in the $SU(2)\times SU(2)$ chiral limit $m_u = m_d = 0$ at the QCD scale $\nu = 1$ GeV.

always close to unity. It has been shown [3] that the remaining four constants $\lambda_1, \ldots \lambda_4$ can be rather accurately determined from the existing $\pi\pi$ scattering data [22] in the intermediate energy range 0.5 GeV $< E <$ 1.9 GeV using the Roy dispersion relations [23]. The latter explicitly incorporate crossing symmetry and consequently they strongly constrain the $\pi\pi$ amplitude at low energies. Equating the perturbative formula (1) with the Roy dispersive representation in a whole low-energy region of the Mandelstam plane, one infers the values of $\lambda_1, \ldots \lambda_4$, whereas the paramenters α and β remain essentially undetermined. The resulting λ_i's are almost independent of α and β. Here we quote and use the central values corresponding to $\alpha = 1.04$, $\beta = 1.08$ [3],

$$\lambda_1 = (-5.7 \pm 2.2) \times 10^{-3}, \quad \lambda_2 = (9.3 \pm 0.5) \times 10^{-3},$$
$$\lambda_3 = (2.2 \pm 0.6) \times 10^{-4}, \quad \lambda_4 = (-1.5 \pm 0.12) \times 10^{-4}. \tag{2}$$

The quoted errors include experimental uncertainties on $\pi\pi$ phase-shifts and inelasticities in the medium energy region and an estimate of the systematic error arising from neglected higher orders in the low-energy representation (1). The errors due to the uncertainty in the high-energy behaviour of the $\pi\pi$ scattering amplitude are negligible.

3. With the constants λ_i determined, Eq. (1) allows one to convert new high-precision experimental information on low-energy $\pi\pi$ phase shifts and/or threshold parameters into a measurement of α and β and finally, into an experimental determination of the quantity $(m_u + m_d) < \bar{q}q >$ (the detailed relation between α and β and the condensate can be found in Ref. [1]). Conversely, Eq. (1) can be used to predict, for each value of the condensate, all low-energy observables. It is of particular importance to assess with as much accuracy as possible the prediction concerning the standard alternative of a large $< \bar{q}q >$ condensate. The strength of the $< \bar{q}q >$ condensate is conveniently described by the deviation from the Gell-Mann–Oakes–Renner relation, i.e. by the parameter

$$\frac{m}{m_0} = \frac{F_\pi^2 M_\pi^2}{2m \, |< \bar{q}q >|} - 1 \,. \tag{3}$$

Here, $m = \frac{1}{2}(m_u + m_d)$ is the running quark mass and m_0 is a mass scale characteristic of $\bar{q}q$ condensation. The standard alternative of a large condensate corresponds to $m_0 \gtrsim \Lambda_H$. In this special case the ratio (3) can be treated as an expansion parameter, $m/m_0 = O(p^2/\Lambda_H^2)$ and the general low-energy expansion becomes the standard chiral perturbation theory (SχPT) [19]. The complete SχPT two-loop calculation of the $\pi\pi$-scattering amplitude has been recently completed by Bijnens *et al.* [2]. Not surprisingly, this calculation recovers the formula (1) giving, in addition, the expressions of the six parameters $\alpha, \beta, \lambda_1, \ldots, \lambda_4$ in terms of *i*) M_π, F_π, *ii*) four $O(p^4)$ constants $l_1^r(\mu)$, $l_2^r(\mu)$, $l_3^r(\mu)$ and $l_4^r(\mu)$ and finally *iii*) six $O(p^6)$ constants $r_1^r(\mu), \ldots, r_6^r(\mu)$ which appear in the effective lagrangian and are renormalized at a scale μ. These expressions read

$$\alpha = 1+\left(-\frac{1}{2}L+6l_3^r+2l_4^r-\frac{1}{32\pi^2}\right)\frac{M_\pi^2}{F_\pi^2}+\left[-8k_1-\frac{14}{3}k_2-13k_3-\frac{3}{2}k_4\right.$$
$$-24l_3^{r\,2}+20l_3^r l_4^r+5l_4^{r\,2}+\frac{6239}{331776\pi^4}+\frac{1}{\pi^2}\left(-\frac{19}{3456}-\frac{769}{576}L-\frac{1}{6}l_1^r\right.$$
$$\left.\left.+\frac{1}{9}l_2^r-\frac{27}{16}l_3^r-\frac{1}{8}l_4^r\right)+3r_1^r+4r_2^r+4r_3^r-4r_4^r\right]\frac{M_\pi^4}{F_\pi^4} \quad (4)$$

$$\beta = 1+\left(-2L+2l_4^r-\frac{1}{8\pi^2}\right)\frac{M_\pi^2}{F_\pi^2}+\left[\frac{5}{3}k_1-\frac{5}{2}k_3-3k_4-4l_3^r l_4^r+5l_4^{r\,2}\right.$$
$$+\frac{8911}{331776\pi^4}+\frac{1}{\pi^2}\left(-\frac{1}{512}+\frac{727}{864}L-\frac{11}{18}l_1^r-\frac{7}{8}l_2^r-\frac{9}{8}l_3^r-\frac{1}{2}l_4^r\right)$$
$$\left.+r_2^r+4r_3^r-4r_4^r+12r_5^r-4r_6^r\right]\frac{M_\pi^4}{F_\pi^4} \quad (5)$$

$$\lambda_1 = -\frac{1}{3}L+2l_1^r-\frac{1}{36\pi^2}+\left[-\frac{7}{6}k_1-\frac{1}{2}k_2-\frac{1}{3}k_4+8l_1^r l_4^r+\frac{79}{9216\pi^4}\right.$$
$$+\frac{1}{3456\pi^2}\left(-1+2272L-2496l_1^r-2160l_2^r-384l_4^r\right)$$
$$\left.+r_3^r-r_4^r+6r_5^r-2r_6^r\right]\frac{M_\pi^2}{F_\pi^2} \quad (6)$$

$$\lambda_2 = -\frac{1}{3}L+l_2^r-\frac{5}{288\pi^2}+\left[-\frac{1}{3}k_1-\frac{4}{3}k_2-\frac{1}{3}k_4+4l_2^r l_4^r+\frac{1223}{331776\pi^4}\right.$$
$$\left.+\frac{1}{27648\pi^2}\left(17+752L+3840l_1^r+1536l_2^r-1920l_4^r\right)+2r_4^r\right]\frac{M_\pi^2}{F_\pi^2} \quad (7)$$

$$\lambda_3 = -\frac{23}{18}k_1-\frac{37}{36}k_2+\frac{277}{1990656\pi^4}+\frac{1}{41472\pi^2}\left(19+5368L-13056l_1^r\right.$$
$$\left.-9600l_2^r\right)+r_5^r-\frac{1}{3}r_6^r \quad (8)$$

$$\lambda_4 = \frac{5}{36}k_1+\frac{25}{72}k_2+\frac{3311}{3981312\pi^4}+\frac{1}{10368\pi^2}\left(-2-257L+336l_1^r+840l_2^r\right)$$
$$-\frac{4}{3}r_6^r\,, \quad (9)$$

with

$$\mu\frac{dl_i^r}{d\mu}=-\frac{\gamma_i}{16\pi^2}\,, \qquad \gamma_1=\frac{1}{3}\,,\gamma_2=\frac{2}{3}\,,\gamma_3=-\frac{1}{2}\,,\gamma_4=2\,, \quad (10)$$

and

$$L=\frac{1}{16\pi^2}\log\frac{M_\pi^2}{\mu^2}\,; \qquad k_i(\mu)=(4l_i^r(\mu)-\gamma_i L)L\,. \quad (11)$$

(These expressions are obtained from the expansions of the parameters $b_1, \ldots, b_6$ originally given in [2], which are in one-to-one correspondence with α, β, $\lambda_1, \ldots, \lambda_4$. We prefer to work with the latter set of parameters for the reader's convenience: explicit formulae for low-energy observables in terms of α, β, λ_i are given in Ref. [1], whereas similar expressions in terms of the b_i's are at present not available in the literature). A few points are worth recalling. *i)* The parameters α, β, λ_i are μ-independent. This fact, together with Eq.(10) fixes the scale dependence of the low-energy constants $r_i^r(\mu)$. *ii)* Eqs. (4)-(9) fix the expansion of the parameters α, β, λ_i in powers of M_π^2 and $\log M_\pi^2$ (and/or in powers of the quark mass m), since $l_1^r(\mu), \ldots, l_4^r(\mu)$ and the $r_i^r(\mu)$ are quark mass independent. Contributions of successive chiral orders to α, β, λ_i can be identified by counting the powers of M_π^2/F_π^2. Notice that α and β start by an order $O(p^2)$ contribution ($\alpha = 1, \beta = 1$) followed by $O(p^4)$ and $O(p^6)$ corrections. The expansions of λ_1, λ_2 consist of $O(p^4)$ and $O(p^6)$ contributions, whereas λ_3 and λ_4 are entirely of order $O(p^6)$. *iii)* The $O(p^4)$ constants l_3 and l_4 belong to the explicit symmetry breaking sector of the effective lagrangian. They represent the fine tuning of the $< \bar{q}q >$ condensate to its presumed large value: in SχPT, the deviation from the Gell-Mann–Oakes–Renner relation (3) is given by [19]

$$\frac{m}{m_0} = \left[2l_3^r(\mu) + 2l_4^r(\mu) - \frac{3}{2}L\right]\left(\frac{M_\pi}{F_\pi}\right)^2 + \ldots . \tag{12}$$

Similarly, l_4^r controls the deviation of β from 1. On the other hand, the λ_i's are *independent* of l_3^r (and only very weakly dependent on l_4^r) reflecting the fact that they are only marginally sensitive to the size of the $< \bar{q}q >$ condensate. In the sequel, we complete our *definition of the standard χPT by adopting the standardly used central values of l_3^r and l_4^r* [19, 2]:

$$l_3^r(M_\rho) = 0.82 \times 10^{-3}, \qquad l_4^r(M_\rho) = 5.6 \times 10^{-3}. \tag{13}$$

Finally, the constants l_1^r and l_2^r do not describe explicit symmetry breaking effects (they are coefficient of four-derivative terms in $\mathcal{L}^{(4)}$) and they are insensitive to the size of the quark condensate. They control the parameters λ_1 and λ_2.

4. Equations (4)-(9) can be used to predict the parameters α, β, λ_1, $\ldots$, λ_4 and consequently, all low-energy $\pi\pi$ scattering observables, provided the low-energy constants $l_1, \ldots, l_4$ and $r_1, \ldots, r_6$ are determined from the analysis of different processes. This is a path advocated by the authors of Ref. [2]. In the present letter this kind of analysis will be confronted with additional *experimental* information contained in Eq. (2). Bijnens *et al.* [2] have used the values (13) for l_3 and l_4; for l_1^r and l_2^r they have taken the central values obtained from the SχPT analysis of K_{l4} form factors [25]:

$$l_1^r(M_\rho) = -5.40 \times 10^{-3}, \qquad l_2^r(M_\rho) = 5.67 \times 10^{-3}. \tag{14}$$

As for the $O(p^6)$ constants $r_i^r(\mu)$, the authors of [2] take $r_i^r(1\mathrm{GeV}) = 0$ and they check that this approximation confronted with a resonance saturation model produces a negligible error.

With the values (14), and $r_i^r = 0$ at $\mu = 1$ GeV one obtains (in this letter we always use $F_\pi = 93.2$ MeV and $M_\pi = 139.6$ MeV):

$$\begin{aligned} \alpha &= 1.074 \,, & \beta &= 1.105 \,, \\ \lambda_1 &= -8.91 \times 10^{-3} \,, & \lambda_2 &= 14.5 \times 10^{-3} \,, \\ \lambda_3 &= 2.04 \times 10^{-4} \,, & \lambda_4 &= -1.79 \times 10^{-4} \,. \end{aligned} \tag{15}$$

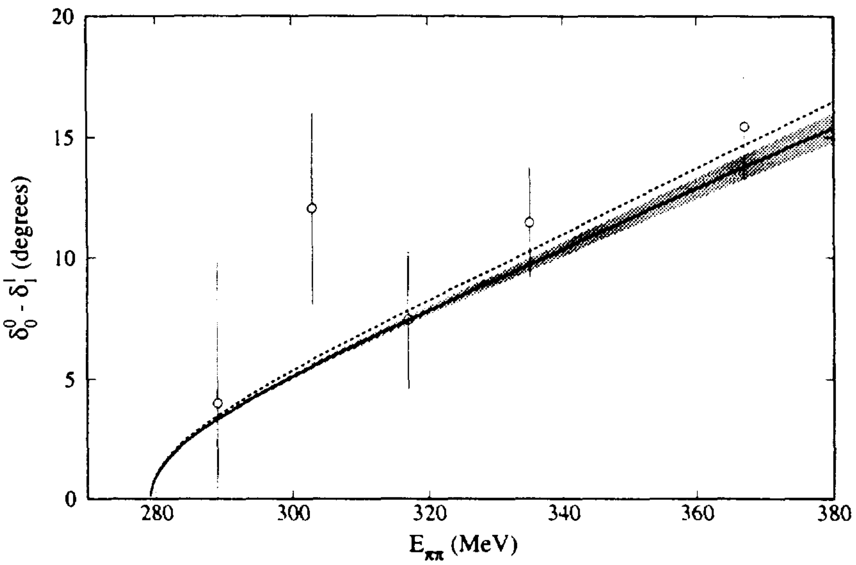

Figure 1: *The phase shift difference $\delta_0^0 - \delta_1^1$ in the energy region of K_{l4} decays. The dashed curve is obtained with the values of Eq. (15) and it coincides with the curve displayed in Fig. 1 of Ref. [2]. The solid line is obtained with the values of Eqs. (2) and (18) while the shaded area results adding the corresponding error bars quadratically. The experimental points are from Ref. [32].*

For these values of the parameters $\alpha, \beta, \lambda_1, \ldots, \lambda_4$, one obtains the S-wave scattering lengths $a_0^0 = 0.218$, $a_0^0 - a_0^2 = 0.259$ corresponding[2] to the predictions given in Ref. [2]. The resulting phase shift difference $\delta_0^0 - \delta_1^1$ (measurable in K_{l4} decays) is shown as a function of the center of mass energy as the dashed line in Fig. 1. It reproduces the curve displayed in Fig. 1 of Ref. [2]. Finally, a few remaining threshold parameters not discussed in Ref. [2] are collected in the first column of Table 1, using the expressions displayed in Appendix D of Ref. [1].

[2]Actually these have to be compared with the numbers given in Eq. (4) of Ref. [2] in parentheses ($r_i^r(1 \text{ GeV}) = 0$). The small difference provides an estimate of $O(p^8)$ effects: it is entirely due to the fact that the amplitude A_{KMSF} of Ref. [1] coincides with the amplitude calculated in Ref. [2] only modulo $O(p^8)$ contributions.

	Bijnens *et al.* [2]	KMSF	Experiment [26]
a_0^0	0.218 (0.2156)	0.209 ± 0.004	0.26 ± 0.05
b_0^0	0.273 (0.271)	0.255 ± 0.010	0.25 ± 0.03
$-10a_0^2$	0.411 (0.4094)	0.44 ± 0.01	0.28 ± 0.12
$-10b_0^2$	0.709 (0.704)	0.80 ± 0.02	0.82 ± 0.08
$a_0^0 - a_0^2$	0.259 (0.2565)	0.254 ± 0.004	0.29 ± 0.05
$10a_1^1$	0.395 (0.3956)	0.373 ± 0.008	0.38 ± 0.02
$10^2 b_1^1$	0.785 (0.784)	0.55 ± 0.07	
$10^2 a_2^0$	0.263 (0.267)	0.16 ± 0.02	0.17 ± 0.03
$10^3 a_2^2$	0.237 (0.2356)	0.09 ± 0.13	0.13 ± 0.30
$10^4 a_3^1$	0.428 (0.478)	0.49 ± 0.07	0.6 ± 0.2

Table 1: *Threshold parameters of $\pi\pi$ scattering (in units of M_{π^+}) in the standard framework using the two-loop expressions of Ref. [1], App. D. The first column results from the values of Eq. (15) (see the text for the numbers in parentheses). The second column is obtained in the same way but taking the values of Eqs. (2) and (18) as input.*

The numbers in parentheses are obtained keeping in higher orders only those components of α, β, λ_1 and λ_2 that actually contribute at most to the order $O(p^6)$. These exactly coincide with the corresponding predictions one would obtain using the amplitude given in [2]. Among the latter it is worth noticing the value predicted for the isoscalar D-wave scattering length $a_2^0 = 26.3 \times 10^{-4}$, which is three standard deviations above the value extracted from the analysis of Roy equations [26]. This disagreement reflects the fact that the value (15) of λ_2 is significantly above the value (2) inferred from experimental phase shifts in Ref. [3]. We would like to stress that both the canonical value $a_2^0 = (17 \pm 3) \times 10^{-4}$ and the determination of the constant $\lambda_2 = (9.3 \pm 0.5) \times 10^{-3}$ are based on the Roy dispersion relations [23] using the experimental $\pi\pi$ data above 500 MeV as input. Furthermore, in both cases, the dominant contribution comes from the P-wave in the $\rho(770)$ region, which explains the relatively small error bars. These facts suggest that the predictions of Ref. [2] based on (15) have to be revised in order to agree with the values (2) of the parameters $\lambda_1, \ldots, \lambda_4$ and with the standard value of a_2^0. We therefore proceed as follows: fixing l_3^r and l_4^r according to Eq. (13), we solve Eqs. (6) and (7) for $l_1^r(M_\rho)$, $l_2^r(M_\rho)$,

$$l_1^r(M_\rho) = (-4.0 \pm 1.0) \times 10^{-3} + \left[-1.1\, r_3^r + 1.0\, r_4^r - 6.3\, r_5^r + 2.1\, r_6^r\right]_{\mu=1\mathrm{GeV}} \quad (16)$$

$$l_2^r(M_\rho) = (1.6 \pm 0.4) \times 10^{-3} + \left[0.1\, r_3^r - 3.5\, r_4^r + 0.5\, r_5^r - 0.2\, r_6^r\right]_{\mu=1\mathrm{GeV}} \quad (17)$$

where the values and errors (2) have been used for λ_1 and λ_2. Eqs. (16) and (17) are then inserted back into the formulae (4) and (5) for α and β. Keeping in mind that α and β are sensitive to l_1 and l_2 only at next-to-next-to-leading level, the unknown constants $r_i^r(1\ \mathrm{GeV})$ are viewed as a source of uncertainty in α and β. Inspired by naïve dimensional analysis [24] we take in the expressions for α and β, $r_i^r(1\ \mathrm{GeV}) = (0 \pm 2) \times 10^{-4}$. Adding the corresponding uncertainties quadratically, we obtain

$$\alpha = 1.07 \pm 0.01 \qquad \beta = 1.105 \pm 0.015. \tag{18}$$

It should be stressed that the error in Eq. (18) does not include the uncertainty in the low-energy constants l_3^r and l_4^r. As in the case of the chiral condensate itself, the constant l_3^r has not yet been determined experimentally and for this reason it is hard to associate an error bar with it. The values (18) have to be viewed as corresponding to the "standard alternative" of a large condensate *defined* by the values (13) of l_3^r and l_4^r. We now use the formulae given in Ref. [1] to generate the predictions for threshold parameters and phase shifts that correspond to α, β (18) and $\lambda_1, \ldots \lambda_4$ (2). Adding the errors quadratically, the resulting threshold parameters are summarized in the second column of Table 1. One observes that the deviations of a_0^0 and $a_0^0 - a_0^2$ from their *central* experimental values are significantly *larger* than predicted in Ref. [2]. Notice that now, the D-wave scattering lengths perfectly agree with their Roy-equation "experimental" values as expected from the manner the values (2) of the constants $\lambda_1, \ldots, \lambda_4$ have been obtained.

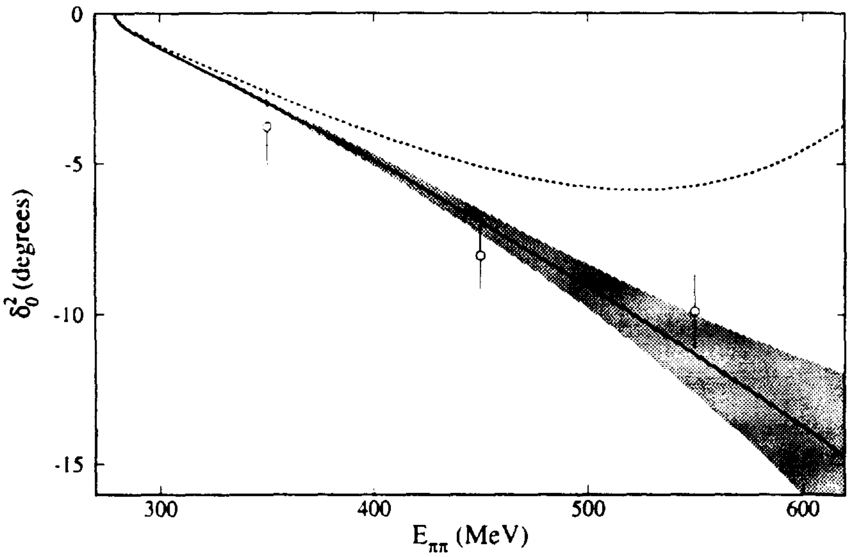

Figure 2: *The isospin 2, S-wave phase shifts at low energies. The different curves are obtained with the values (15) (dashed) and the values (2) and (18) (solid). In the latter case the shaded area shows the corresponding error band. The experimental points are taken from Ref. [22].*

A similar conclusion holds for the phase shift difference $\delta_0^0 - \delta_1^1$, shown as the solid curve in Fig. 1 with the error band indicated by the shaded area: the curve displayed in Ref. [2] is significantly higher, i.e. closer to the experimental central-value points. For illustration, the phase δ_0^2 is also shown in Fig. 2.

5. We finally address the question of interpreting the mismatch described in the previous paragraph. Its origin clearly appears upon comparing eqs (16) and (17) with the values of the constants $l_{1,2}^r(M_\rho)$ extracted in Ref. [25] from the "unitarized" one-loop SχPT K_{l4} form factors (Eq. (5.10) of [25]). Including errors the latter read:

$$l_1^r(M_\rho) = (-5.4 \pm 1.1) \times 10^{-3}, \qquad l_2^r(M_\rho) = (5.7 \pm 1.1) \times 10^{-3}. \tag{19}$$

The question is how close the expressions (16) and (17) can be brought to these values keeping at the same time the $O(p^6)$ constants $r_3^r(1\ \mathrm{GeV}), \ldots, r_6^r(1\ \mathrm{GeV})$ at a reasonable size. If one proceeds as before treating the r_i^r's at 1 GeV as randomly distributed around zero with a standard deviation $\pm 2 \times 10^{-4}$, one gets:

$$\begin{aligned} l_1^r(M_\rho) &= (-4.0 \pm 1.0 \pm 1.8) \times 10^{-3} \ , \\ l_2^r(M_\rho) &= (1.6 \pm 0.4 \pm 0.9) \times 10^{-3} \ , \end{aligned} \tag{20}$$

or

$$\bar{l}_1 = -0.37 \pm 0.95 \pm 1.71 \ , \qquad \bar{l}_2 = 4.17 \pm 0.19 \pm 0.43 \ , \tag{21}$$

where the first error has its origin in λ_1 and λ_2 (Eq. (2)), whereas the second error arises from the presumed uncertainties in the individual r_i's added quadratically. Two cheks of the size of the constants r_i are conceivable. *i)* First, one can make a full use of informations contained in Eq. (2) determining the parameters $l_{1,2}^r(M_\rho)$ and $r_3^r(1\ \mathrm{GeV}), \ldots, r_6^r(1\ \mathrm{GeV})$ by a simultaneous fit to Eqs. (6)-(9) ***and*** to the constraints $r_i^r(1\ \mathrm{GeV}) = 0 \pm 2 \times 10^{-4}$. The resulting $\chi^2/d.o.f.$ is 1.9/2 and one obtains

$$l_1^r(M_\rho) = (-4.0 \pm 0.5) \times 10^{-3}, \qquad l_2^r(M_\rho) = (2.0 \pm 0.3) \times 10^{-3}, \tag{22}$$

compatible with (20), whereas for the r_i's one gets

$$\begin{aligned} & r_3^r(1\ \mathrm{GeV}) = (-0.3 \pm 2.0) \times 10^{-4} \ , \quad r_4^r(1\ \mathrm{GeV}) = (-0.7 \pm 0.9) \times 10^{-4} \ , \\ & r_5^r(1\ \mathrm{GeV}) = (1.5 \pm 0.5) \times 10^{-4} \ , \quad r_6^r(1\ \mathrm{GeV}) = (0.4 \pm 0.9) \times 10^{-4} \ . \end{aligned} \tag{23}$$

This result turns out to be rather stable: if one increases the uncertainties in the r_i's to $\pm 3 \times 10^{-4}$, the new $\chi^2/d.o.f. = 1.24/2$, the values (22) become $(-4.8 \pm 0.5) \times 10^{-3}$ and

$(2.1 \pm 0.3) \times 10^{-3}$ respectively, and the changes in the r_i's also remain rather modest. On the other hand, the errors obtained by this procedure (increase of χ^2 by one unit) and shown in Eqs. (22) and (23) are probably heavily underestimated. *ii*) Next, it is instructive to confront the previous discussion with the estimate of the constants r_i by resonance saturation as quoted recently by Hannah [27]:

$$\begin{aligned} r_1 &= -0.61 \times 10^{-4}, \quad r_2 = 1.3 \times 10^{-4}, \\ r_3 &= -1.70 \times 10^{-4}, \quad r_4 = -1.0 \times 10^{-4}, \\ r_5 &= 1.14 \times 10^{-4}, \quad r_6 = 0.3 \times 10^{-4}. \end{aligned} \tag{24}$$

Estimating low-energy constants by resonance saturation does not, in principle, fix the renormalization scale μ at which the estimate is supposed to hold. Actually, if a constant exhibits a strong scale dependence, its resonance saturation estimate is subject to caution. Interpreting Eqs. (24) as values of $r_i^r(\mu)$ at $\mu = 1$ GeV, one observes a striking coherence with the preceding analysis: (24) is, indeed, consistent not only with dimensional analysis or with the assumption $|r_i^r| < 2 \times 10^{-4}$ but, moreover it agrees with the fit (23). One can even repeat the fit to Eqs. (6)-(9) constraining $r_i^r(1\text{ GeV})$ to the values (24) allowing for a 100% error: the fit is excellent ($\chi^2/d.o.f. = 0.91/2$) and it yields $l_1^r(M_\rho) = (-4.0 \pm 0.5) \times 10^{-3}$, $l_2^r(M_\rho) = (2.1 \pm 0.3) \times 10^{-3}$, again compatible with (20) and (22). On the other hand, one finds that between $\mu = M_\rho$ and $\mu = 1$ GeV, only the constants r_4, r_5 and r_6 show a moderate scale dependence: had we assumed that the values (24) concern the scale $\mu = M_\rho$ (as suggested in Ref. [27]), the comparison with our previous analysis would be less favourable as far as the constant r_3 is concerned, $r_3^r(1\text{ GeV}) = -4.9 \times 10^{-4}$ in this case. Notice however that according to Eq. (17) the correction to the "critical" constant $l_2^r(M_\rho)$ is dominated by r_4^r whose scale dependence is rather weak:

$$r_4^r(1\text{ GeV}) = r_4^r(M_\rho) - 7 \times 10^{-6}. \tag{25}$$

In order that the constant $l_2^r(M_\rho)$ (17) differ from the K_{l4} value (19) by at most two standard deviations, the constant $r_4^r(1\text{ GeV})$ would have to be $r_4^r(1\text{ GeV}) \simeq -5 \times 10^{-4}$. This cannot be excluded but it looks unlikely in the light of the present analysis.

6. The constants l_1 and l_2 (19) have not been obtained from a full two-loop analysis of K_{l4} form factors F and G, which is not yet available. Instead, their determination is based on matching a dispersive representation for the form factor F with the one-loop SχPT expressions, the latter merely serving to fix the subtraction constants. This method of "improving" one-loop χPT calculations has been often used in the past [28] and it suffers from a basic ambiguity: one has to assume that the one-loop and two-loop amplitudes practically coincide in a particular kinematical point M. Even if one admits the very existence of such a matching point M, the results can still depend on its choice. In Ref. [25] the matching point has been chosen at the threshold $s_\pi = 4M_\pi^2$ of the S-wave amplitude $\pi\pi \to K$ + axial current, where s_π stands for the dipion invariant mass squared. We have repeated the analysis of Ref. [25] for other choices of the matching point between $s_\pi = 4M_\pi^2$ and the left-hand-cut branch point $s_\pi = 0$. We reproduce the result (19) and find that it is actually rather insensitive to the matching point except in the vicinity of the singular point $s_\pi = 0$, where the outcome for l_1 (but not l_2) becomes less stable. For instance, with the matching point at $s_\pi = 2M_\pi^2$, we obtain

$$l_1^r(M_\rho) = (-4.8 \pm 2.1) \times 10^{-3}, \qquad l_2^r(M_\rho) = (5.3 \pm 1.0) \times 10^{-3}. \tag{26}$$

Given the present state and quality of K_{l4} experimental data, it seems hard to ascribe the discrepancy described above to the SχPT analysis performed in Ref. [25]. On the other hand, it should be kept in mind that outside the standard framework, i.e. for low values of the condensate $< \bar{q}q >$, the constants l_1 and l_2 extracted from K_{l4} data will be modified already at the one-loop level: since in GχPT the loop contributions are more important, the resulting central values of $|l_1|$ and $|l_2|$ are expected to come out somewhat smaller [29].

7. A few concluding remarks are in order. The past determinations [19, 30, 6, 7] of the constants l_1^r and l_2^r have operated within the $O(p^4)$ order of χPT. They have shown an apparent coherence and compatibility with the K_{l4} analyses of Ref. [25]. This compatibility might be lost at $O(p^6)$ order and we have to understand why. The resonance saturation models are the only ones that determine the constants $l_{1,2}$ directly, integrating out the resonance degrees of freedom from an extended effective lagrangian $\mathcal{L}_{eff}$. However, incorporating resonances into $\mathcal{L}_{eff}$ is not free of ambiguities, especially if one aims at the $O(p^6)$ accuracy. On the other hand, less model-dependent sources of information, such as $\pi\pi$ D-waves [19] and/or sum rules [6, 7] primarily determine the physical parameters λ_1, λ_2. It turns out that this determination is rather stable and barely affected by switching from order $O(p^4)$ to $O(p^6)$. At the $O(p^4)$ level, i.e. neglecting in Eq. (1) the two-loop effects and setting $\lambda_3 = \lambda_4 = 0$, one would get from the a_2^0 and a_2^2 experimental central values $\lambda_1 = -6.4 \times 10^{-3}$ and $\lambda_2 = 10.8 \times 10^{-3}$, to be compared with Eq. (2). In other words, the relationship between D-wave scattering lengths and the parameters λ_1, λ_2 is almost unaffected by $O(p^6)$ effects. The latter however become rather important in the relationship between λ_2 and l_2^r. Rewriting Eq. (7) to make the dependence on $l_2^r(M_\rho)$ appear explicitly, one obtains

$$\lambda_2 = \{l_2^r(M_\rho) + 5.45 \times 10^{-3}\} + \{0.32 \times l_2^r(M_\rho) + 1.7 \times 10^{-3}\} \tag{27}$$

where the first (second) curly brackets collect all $O(p^4)$ ($O(p^6)$) contributions (r_4 has been neglected). The $O(p^6)$ contribution is as large as 30% and it is dominated by double logs, whose importance has been anticipated by Colangelo [4]. It follows that for a given λ_2 (D-waves), the resulting value of $l_2^r(M_\rho)$ can easily differ by a factor ~ 2 depending whether in Eq. (27) one includes the $O(p^6)$ term or not. Whether the consistency with K_{l4} form factors can be understood within the large condensate hypothesis remains to be clarified. It might be, for instance, that at $O(p^6)$ level the K_{l4} form factors also receive an important contribution from double logs, which the unitarization procedure would not take into account [31]. Independently of this issue, the main conclusion of this letter is the following: the predictions of SχPT for a_0^0, $a_0^0 - a_0^2$ and $\delta_0^0 - \delta_1^1$ given in Ref. [2] are systematically overestimated as shown in Fig. 1 and Table 1 of the present paper. A closely related fact is the failure of the values of l_1^r and l_2^r used in Ref. [2] to describe the D-waves in agreement with Roy equations analyses. This agreement is nicely recovered if instead the present determinations of Eq. (20) are used. This shows, once more, that a sensible and sensitive test of QCD in low-energy $\pi\pi$ scattering should be based on a global analysis making use of all theoretical constraints and all pertinent low-energy observables.

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