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ON CONFINEMENT POTENTIALS IN GAUGE THEORY :

THE Z_2 CASE ON A LATTICE

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ABSTRACT : We show that a sufficient decrease of the Wilson loop implies automatically an area decreasse ; i.e. the energy to separate quarks at distance L is either at most Log L or L in the Z_2 case. We believe that it is a general fact on a lattice.

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I. INTRODUCTION

In 1979, R.L. Dobrushin and Pechersky [1] announced the following result : for a wide class of models suppose that the fall-off of some observables decrease at a sufficiently high power of the distance, this implies the exponential decay. More recently in 1980, Simon [2] , Lieb [3] , and Aizenman Simon [4] showed similar results for correlation functions using correlation inequalities. It is clear that such results could be very useful because evidently it is easier to show a decrease at a sufficiently high power than an exponential decay ; this seems to be the case for models in two dimensions exhibiting a SO_3 continuous symmetry [5].

For similar reasons it is tempting to look at the situation of gauge models ; in these models the analogous of long range order are the so-called Wilson loops ; the characteristic functions of the flux of a magnetic field $\vec{B} = \mathrm{curl}\,\vec{A}$ (in the Abelian case) through a surface of size $L \times T$. In this paper we analyse the gauge type model. We show that the following decay of the Wilson loop $\langle W \rangle \leq e^{-\mu T / \ln L}$ for μ sufficiently large implies :

$$\langle W \rangle \leq e^{-\lambda(\mu) T \cdot L} .$$

i.e. the area decay. This implies a very strong restriction on the confinement potential between quark antiquark in the interpretation of Wilson [6] . We think that the situation described for the Z_2, models on a lattice is generic, i.e. the same should occur for Random variables ranging in U_1, SU_2 etc...

II. GAUGE MODELS ON A LATTICE

We consider a lattice Z^d, the random variables are indexed by links between the sites ; usually they range in groups G such as : Z_2, U_1, SU_2, SU_3 in particle physics. These models are associated with a magnetic field $\vec{B} = \mathrm{curl}\,\vec{A}$. We consider for sake of simplicity a three dimensional model (this restriction is irrelevant).

Let $g^1(ijk)$, $g^2(ijk)$, $g^3(ijk)$ the random variables associated to the three links issued from the site (ijk).
A usual gauge model is : $\forall g^\alpha \in G$

$$H = -J\{\sum \mathrm{Tr}\, g^1(i+1jk)\, g^2(ij+1k)\; g^1(ijk)\, g^2(ijk)$$
$$+ \mathrm{Tr}\, g^2(ij+1k)\, g^3(ijk+1)\; g^2(ijk)\, g^3(ijk)$$
$$+ \mathrm{Tr}\, g^3(ijk+1)\, g^1(ijk)\; g^3(ijk)\, g^1(ijk)\}$$

in fact more general models can be investigated. A basic fact of this model is the gauge invariance. Let $\Omega(ijk) \in G$ an arbitrary function of G. The Hamiltonian is invariant under the transformation

$$g^1_{i+1jk} \longrightarrow \Omega_{i+1jk}\, g^1_{i+1jk}\, \Omega^{-1}(ijk).$$
$$g^2_{ij+1k} \longrightarrow \Omega_{ij+1k}\, g^2(ij+1k)\, \Omega^{-1}(ijk)$$
$$g^3_{ijk+1} \longrightarrow \Omega_{ijk+1}\, g^3(ijk+1)\, \Omega^{-1}(ijk)$$

III. THE SIMON INEQUALITIES FOR A GAUGE MODEL

We first recall the Simon inequality derived in [2]. Let H be a ferromagnetic Hamiltonian, A B & C be three sets so that for any $\alpha \in A$, $\beta \in B$, $\gamma \in C$ B separates α and γ. Let $\sigma^D = \prod_{\alpha \in D} \sigma_\alpha$ then :

$$\langle \sigma_A \sigma_C \rangle \leq \sum_{D \subset B} \langle \sigma_A \sigma_D \rangle \langle \sigma_D \sigma_C \rangle$$

where σ_α are the usual spin $\frac{1}{2}$ random variables, we need a slight generalization of these inequalities for gauge model which are derived according to [2].

<u>Definition</u> : The elementary squares of the lattices are called plaquettes.

<u>Definition</u> : A surface S is a numbered set of plaquettes (possibly repeated).

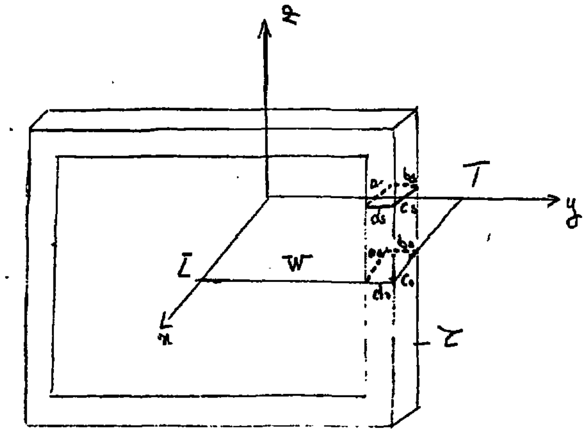
<u>Definition</u> : The boudary operation ∂ is defined as usually in differential geometry.

<u>Definition</u> : A tube T will be the set of transverse plaquettes orthogonal to a close curve C (in fact we will consider square in a plane as curve).

<u>Definition</u> : Let $\{W\}$ the set of surface whose boundary set is ∂W.

<u>Definition</u> : A tube T separates the surface W if it intersects every surface of $\{W\}$ i.e. the cycle ∂W intersects C in the sense of algebraic geometry, see [7], p. 49.

<u>Remark</u> : In four dimensions C is a two dimensional surface.



Picture 1

The tube $\tau^{L,T}$ separates W. $\{T_s\}_{s\in\tau}$ are the plaquettes transverse to the curve C ; a generic plaquette will be denoted by $a_s\, b_s\, c_s\, d_s = T_s$. The bonds linking the plaquette T_s are called the lateral bonds of the tube τ^{LT}; (L,T) are the coordinates of the basis of the tube in xOy. All the following definitions are the same as in [2] but they are adapted to surfaces.

Definition : f is an analytic function of J_p one for each plaquette

$$\partial^S = \frac{\partial^{\Sigma n_p}}{\partial J_p^{\,n_p}} \qquad \text{evaluated at all } J_p = 0$$

T, S be surfaces $T \subset S$; $R = S/T$, we write $S = R \oplus T$

$$\partial^S fg = \sum_{R\oplus T} \partial^T f \, \partial^R g .$$

We derive the following result.

3.1. Theorem Let $H = -\{\Sigma J_p \sigma^1_{i+1jk}\sigma^1_{ijk}\sigma^2_{ij+1k}\sigma^2_{ijk} + \cdots$

$$+ J_p \sigma^2_{ij+1k}\sigma^2_{ijk}\sigma^3_{ijk+1}\sigma^3_{ijk} + J_p \sigma^{1}\sigma^1_{ijk}\sigma^3_{ijk+1}\sigma^3_{ijk}\}$$

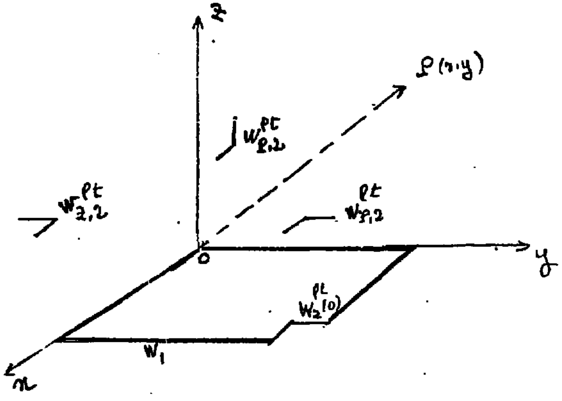
where the Jp's are positive. In the gauge described below :

$$\langle \prod_{\alpha\in W/d_0,c_0} \sigma_\alpha \cdot \sigma_{d_0c_0}\rangle \leq \sum_{s\in\tau_j} \langle \prod_{\alpha\in W/d_0,c_0} \sigma_\alpha \, \sigma_{as}\sigma_{bs}\rangle \langle \sigma_{d_0}\sigma_{c_0}\sigma_{a_s}\sigma_{b_s}\rangle$$

Proof : similar to [2] we replace graphs by surfaces whose boundaries are ∂ W.

To prove the general result we need some comparison between several Wilson loops.

Let W a Wilson loop in the plane xoy ; $W = W_1 \cup W_2$



Picture 2

$W^{pt}_2(s)$ symmetric with respect to P of W

$W^{pt}_{P,2}(s)$ symmetric with respect to P(xy) of W

$W^{pt}_{2,2}(s)$ symmetric with respect to x o y of W

In our case $W_2^{\ell t}$ will be composed of two bonds $d_0 \ell_0$ (Picture 1); we shall omit the coordinates of the basis of the tubes $1, \bar{T}$ when it is not necessary.

By a gauge transformation we can replace by 1 the lateral links of the tube $\mathcal{T}$ starting from W going to W (t) resp. W (t) resp. W (t) (one has to be cautious as it is impossible to replace by 1 all the lateral links of $\mathcal{T}$ by a gauge transformation). We have then the following lemma.

3.2. Lemma

With free and periodic boundary conditions the following inequalities hold

$$\langle \sigma_{W_1 \cup W_2} \rangle \geqslant \langle \sigma_{W_1} \cup W_2(s) \rangle \geqslant \langle \sigma_{W_1} \cup W_{\ell,2}(s) \rangle$$

$$\geqslant \langle \sigma_{W_1} \cup W_2(s) \rangle \geqslant \langle \sigma_{W_1} \cup W_{2,\ell}(s) \rangle$$

Proof : We use as in [8] the Lebowitz inequalities [9] the trick consists in using duplicate variables ; let $P_{\frac{\Delta}{2}}$ the plane parallel to W at $z = \frac{\Delta}{2}$ (if Δ is even we keep the variables contained in the plane $P(\frac{s}{2})$).

$$p_{ijk} = \frac{\sigma_{ijk} - \sigma_{ij\bar{k}}}{2} \qquad q_{ijk} = \frac{\sigma_{ijk} + \sigma_{ij\bar{k}}}{2}$$

where $ij\bar{k}$ denotes the symmetric of ijk with respect to $P_{\frac{\Delta}{2}}$, identified for simplicity with the $x\,0\,y$ plane.
It is easy to show that the Hamiltonian is ferromagnetic in the new variables

$$\sigma_{W_1}\left(\sigma_{W_2} - \sigma_{W_2}(s)\right) \equiv \prod_{W_1}(p+q)\; p_{L,T-1,0}\; p_{L-1,T,0}$$

as Lebowitz inequalities ensure that

$$\langle p_A\, q_B \rangle \geqslant 0$$

We deduce

$$\langle \sigma_{W_1} \sigma_{W_2} \rangle \geq \langle \sigma_{W_1} . \sigma_{W_2}(s) \rangle$$

the other inequalities of the lemma are derived by repeating the duplication first with respect to P(x,y) and then to $z \circ x$. The results follow.

We notice that this kind of results can be generalized to more general gauge models, the only required hypothesis being the validity of the first Lebowitz inequality.

We derive now the final result : let $W_s^{L.T}$ the Wilson loop in τ^{LT} composed with the bonds $\omega_2^{LT}(o)$ and $W_2^{LT}(s)$ (resp. $\omega_2^{LT}(o)$ and $\omega_{\ell 2}^{LT}(s)$) (resp. $W_2^{LT}(o)$, and $W_{2,2}^{LT}(s)$) in the chosen gauge.
Let

$$\text{3.2)} \quad A_{L,T}(\beta) = \beta J \sum_{\ell=1}^{\ell=L} \langle \sigma_{W_s^{\ell T}} \rangle$$

this expression is generally independent of T : $A_L(\beta)$

3.3 Theorem

Suppose $A_\ell(\beta) < 1$; $\forall \ell < L_i$, then we have :

$$\langle \sigma_{W^{LT}} \rangle \leq e^{-\lambda L.T}$$

where

$$\lambda = \underline{\lim} \frac{1}{L} \sum_{\ell=1}^{L} \log A_\ell(\beta)$$

Proof : First we use the inequality (3.1)

$$\langle \sigma_{W^{L.T}} \rangle \leq \sum_{s \in \tau^{LT}} \beta J \langle \sigma_{W_1} \sigma_{c_s} \sigma_{d_s} \rangle \langle \sigma_{W^f(s)} \rangle$$

Now we use the inequality (3.2)

$$\langle \sigma_{W^{LT}} \rangle \leq A_L(\beta) \langle \sigma_{W_1} \sigma_{C_0} \sigma_{d_0} \rangle$$
$$\equiv A_L(\beta) \langle \sigma_{W_1^{L-1,T}} \rangle$$

Using this procedure repeatedly on the slice T , we obtain

$$\langle \sigma_{W^{LT}} \rangle \leq \prod_{\ell=1}^{L} A_\ell(\beta) \langle \sigma_{W^{L,T-1}} \rangle$$

then we derive for the total area :

$$\langle \sigma_{W^{LT}} \rangle \leq \left(\prod_{1}^{L} A_\ell(\beta) \right)^T$$
$$\leq e^{-\lambda L.T}$$

<u>Remark</u>. Condition α is expressed in terms of Wilson loops of length ℓ and of width $T = 1$, i.e. one requires that

$$\langle \sigma_{W_s^\ell} \rangle \leq e^{-\eta \log \ell} \quad \text{for } \eta \text{ large enough.}$$

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CONCLUDING REMARKS

We recall the interpretation of Wilson loops L.T. [6] if $\langle \sigma_{W(L,T)} \rangle \sim e^{-f(L)T}$, T is the time axis and L is the length separating two particles antiparticles or quark antiquark. f(L) is the energy needed to separate quarks ; so if this energy is infinite the quarks are confined. Then in the Z_2 case one can have very few confinement potentials i.e.

$$f(L) \leq \log L \quad \underline{\text{or}} \quad f(L) = \lambda L$$

The situation is very close to the case of spin spin models considered in [2] [3] [4] one expects that such a situation is generic, i.e. that the same is true for the U_1 SU_2 etc. situation. We intend to come back to this case in a further paper. Finally we observe that the Wilson loops are related by duality in 3 dimension to the surface tension and the preceding result gives information on the surface tension.

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