Factorizations, elasticities and zero-sum sequences

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Abstract: The *Fundamental Theorem of Arithmetic* states that every natural number can be written, except for the order of the factors, in a unique way as a product of primes. In other algebraic structures, the factorization may not exist or may not be unique. For instance, if a is an element of a monoid H, then it is possible that $a = u_1 \dots u_k = v_1 \dots v_m$, where u_i, v_j are distinct irreducibles. The *set of lengths* of $a \in H$, $\mathsf{L}(a) = \{k \in N \mid \exists\, u_1, \dots, u_k$ irreducible with $a = u_1 \dots u_k\}$, and the *system of sets of lengths*, $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$, are ways of describing the non-uniqueness of factorizations in H. For $k \in N$, let $\mathcal{U}_k(H) = \{m \in N \mid \exists u_1, \dots, u_k, v_1, \dots, v_m$ irreducibles with $u_1 \dots u_k = v_1 \dots v_m\}$, $\lambda_k(H) = \min \mathcal{U}_k(H)$ and $\rho_k(H) = \sup \mathcal{U}_k(H)$ (*k-th elasticity*). The sets $\mathcal{L}(H)$ and $\mathcal{U}_k(H)$ are, in general, well structured and, under reasonably weak hypotheses, one has $\mathcal{U}_k(H) = [\lambda_k(H), \rho_k(H)]$. Moreover, if H is the set of integers of some algebraic number field K, then $L(a)$ is an interval for almost all $a \in H$. It is known that $\lambda_k(H)$ can be written as a function of $\rho_k(H)$. On the other hand, by a *sequence* over a group we mean a finite sequence of terms of the group whose order is disregarded and repetition is allowed. A sequence S over an abelian group $(G, +, 0)$ is *zero-sum* if the sum of its terms is 0. The *Davenport constant* of a group G, $\mathsf{D}(G)$, denotes the largest possible length among all minimal zero-sum sequences. There is an intrinsic relationship between $\rho_k(H)$ and $\mathsf{D}(G)$, where G is the group of ideal classes of H. This makes $\rho_k(H)$ and $\mathsf{D}(G)$ two of the most important objects for describing the non-uniqueness of factorizations in H. In this talk, we will introduce the relationship between factorization theory, focused on non-uniqueness, and zero-sum problems. We will present the main results and conjectures about $\rho_k(H)$ (therefore about $\mathcal{U}_k(H)$) and $\mathcal{L}(H)$.

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