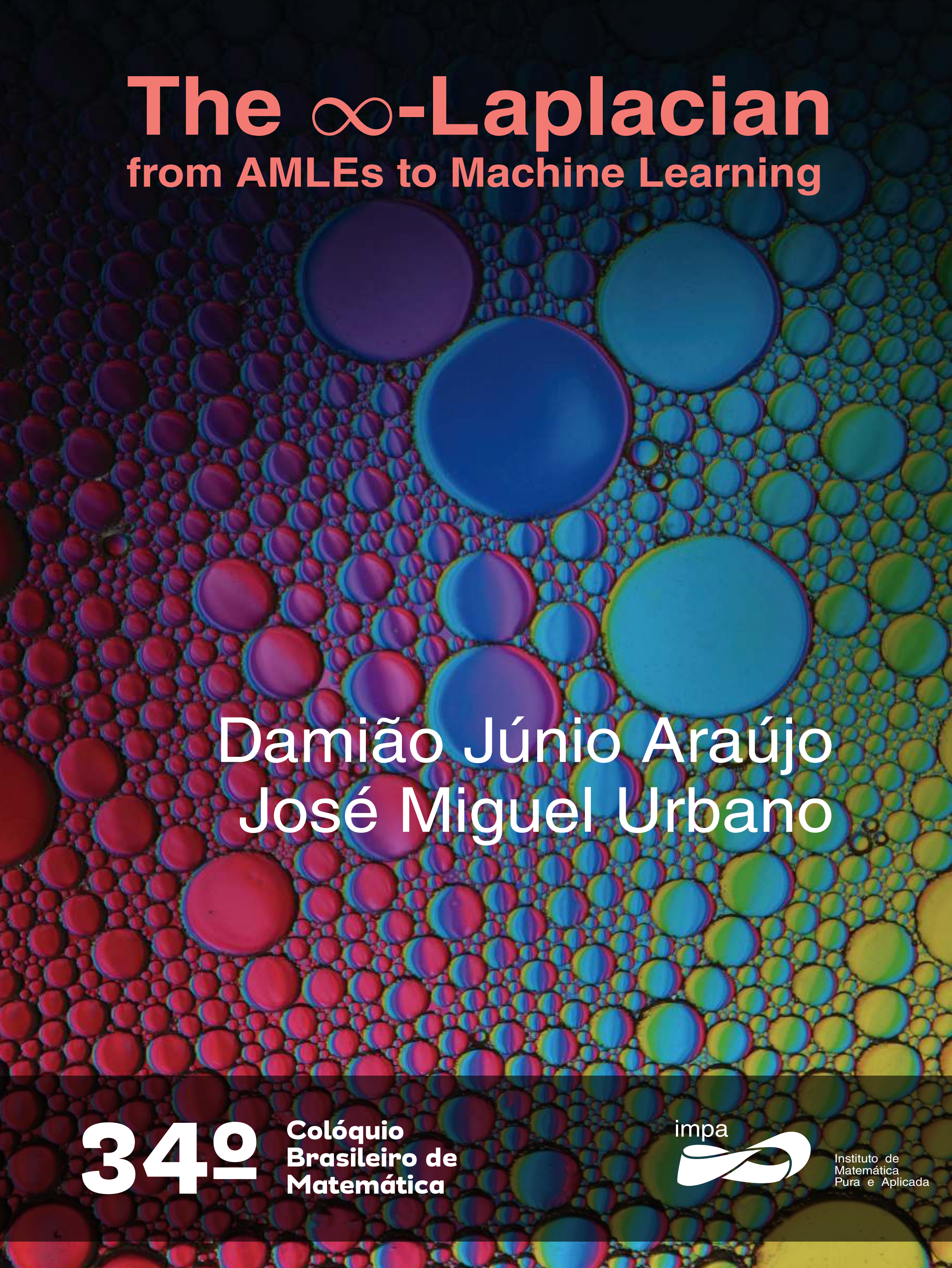
The ∞-Laplacian
from AMLEs to Machine Learning



Damião Júnio Araújo
José Miguel Urbano

34º Colóquio Brasileiro de Matemática

impa
Instituto de Matemática Pura e Aplicada

The ∞-Laplacian

from AMLEs to Machine Learning

The ∞-Laplacian: from AMLEs to Machine Learning
Primeira impressão, setembro de 2023

Publicado no Brasil / Published in Brazil.

ISBN 978-85-244-0529-7 (print)
ISBN 978-85-244-0530-3 (ebook)
MSC (2020) Primary: 35J94, Secondary: 35-02, 35B65, 35J70, 35R35, 26A16

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** IMPA

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

www.impa.br
editora@impa.br

AR663l Araújo, Damião Júnio
The ∞-Laplacian: from AMLEs to Machine Learning / Damião Júnio Araújo e José Miguel Urbano. - 1.ed. -- Rio de Janeiro: IMPA, 2023.
34º Colóquio Brasileiro de Matemática; v. 9, 101p.: il.; 23cm
ISBN 978-85-244-0529-7 (print)
ISBN 978-85-244-0530-3 (ebook)
1. Comparison with Cones. 2. Regularity. 3. Free Boundary Problems. I. Urbano, José Miguel. II. Série. III. Título
UDC: 517.9

Carolina Celano Lima/CRB-7: 2438

To our wives, Oslenne Araújo and Alexandra Urbano

Contents

Preface

These lecture notes serve as an introduction to the analysis of $\infty-$harmonic functions, a subject that has significantly matured in recent years within the field of nonlinear partial differential equations. They are based on the minicourses we taught online during the pandemic at the Universidade Federal da Paraíba (João Pessoa, Brazil) and at the 24th Brazilian Mathematical Colloquium in IMPA (Rio de Janeiro, Brazil) in July 2023. Shorter versions of these lectures were previously delivered by the second author at the Universidade Federal do Ceará (Fortaleza, Brazil) in the (southern hemisphere) Summer of 2013, at Aalto University (Helsinki, Finland) in the (northern hemisphere) Spring of 2013 and at KAUST (Thuwal, Saudi Arabia) early in 2017, and by the first author at the Universidade Federal do Rio de Janeiro (Rio de Janeiro, Brazil) in the (southern hemisphere) Summer of 2020.

The material covered ranges from the Lipschitz extension problem to questions of existence, uniqueness, and regularity for $\infty-$harmonic functions and to free boundary problems ruled by the $\infty-$Laplacian. A rigorous and detailed analysis of the equivalence between being absolutely minimizing Lipschitz, enjoying comparison with cones, and solving the $\infty-$Laplace equation in the viscosity sense is the backbone of the set of lectures. At the heart of the approach adopted lies the notion of comparison with cones, which is pivotal throughout the text. The course includes the proof of the existence of $\infty-$harmonic functions in the case of an unbounded domain, several regularity results (including the Harnack inequality and local Lipschitz continuity), and an easy proof, due to Armstrong and Smart, of the celebrated uniqueness theorem of Jensen. The everywhere differentiability of $\infty-$harmonic functions is treated with detail, and we digress into the optimal regularity issue, an outstanding open problem in the field. The course concludes with an analysis of two free boundary problems involving the $\infty-$Laplacian.

We have written the book with students in mind, aiming to address the difficulties we ourselves encountered when we first approached this subject. While experts may find certain parts trivial, we hope they discover helpful material within these pages. The writing of the first two chapters has been strongly influenced by the survey papers Crandall 2008 and Aronsson, Crandall, and Juutinen 2004, and we claim no originality whatsoever. As the material has evolved from our handwritten notes while studying those sources, it is natural that some portions are reproduced almost verbatim, including some of the proposed exercises. Our contributions lie primarily in the level of detail provided in certain proofs (*that's where the devil is*), the simplification of some arguments, and the overall organization of the text. For instance, we believe the topics covered are well-suited for an advanced graduate course on nonlinear PDEs. We include in the final chapter a list of exercises and propose solutions to most of them. Needless to say, the reader should try hard to solve the exercises before jumping to the solutions.

We are immensely grateful for our interactions with the excellent students who attended the minicourses in Fortaleza, Helsinki, João Pessoa, Rio de Janeiro and KAUST. The suggested solutions to the problems were provided by our PhD students, Aelson Sobral and Ginaldo Sá, whom we warmly thank. Over the years, we have engaged in many exciting discussions on this topic with our colleagues. We want to mention, in particular, Diogo Gomes, Eduardo Teixeira, Juan Manfredi, Juha Kinnunen, Juha Videman, Juhana Siljander, Julio Rossi, Levon Nurbekyan, Mikko Parviainen, Petri Juutinen, and Tuomo Kuusi, whose input directly influenced the writing of some of the proofs. Any remaining typos or inaccuracies are, of course, our sole responsibility.

(The writing of this preface was (slightly) enhanced by the use of Grammarly and ChatGPT.)

João Pessoa and KAUST, August 2023

Damião J. Araújo
Department of Mathematics
Universidade Federal da Paraíba
58059-900, João Pessoa, PB, Brazil
araujo@mat.ufpb.br

José Miguel Urbano
Applied Mathematics and Computational Sciences (AMCS)
Computer, Electrical and Mathematical Sciences and Engineering Division (CEMSE)
King Abdullah University of Science and Technology (KAUST)
Thuwal, 23955-6900, Kingdom of Saudi Arabia

and

University of Coimbra
CMUC, Department of Mathematics
3001-501 Coimbra, Portugal
miguel.urbano@kaust.edu.sa

1 Lipschitz extensions

Extending a given function to a larger domain is a fundamental problem in Analysis, arising in numerous contexts and with significant practical applications. An emblematic example is extending a Lipschitz function from the boundary of an open bounded set in $\mathbb{R}^n$ to its interior without increasing the Lipschitz constant. It is obvious[1] the constant can not be decreased, so keeping it the same is indeed the best we can hope for. As we will see in a while, the problem turns out to be easily solvable, but it is rather ill-posed, not enjoying uniqueness, comparison, stability, or locality. This pushes for the search for a canonical Lipschitz extension satisfying the aforementioned properties, particularly the uniqueness, leading to the notion of Absolutely Minimising Lipschitz Extension or AMLE, in short.

A very contemporary application (see Calder 2019; Calder and Slepčev 2020) emerges in the context of Semi-Supervised Learning (SSL). Labelling vast amounts of data is one of the most prominent challenges in Machine Learning. If performed by a human (say, a medical doctor analysing medical images or a computer analyst classifying websites), this is invariably a costly task, being thus crucial to find reasonable alternatives. Since acquiring unlabelled data has, by comparison, a negligible cost, one of the most effective options is SSL, which uses very little labelled data and explores the topological or geometric structure of the overwhelmingly more abundant unlabelled data in the learning process. The technique is typically implemented using graph-based algorithms, each data point corresponding to a vertex, with edges being assigned weights gauging the similarity of the vertices. Since the problem is highly ill-posed (there are plenty of possible extensions of the labelled data), it is necessary to make a smoothness assumption on the graph,

[1] If not, review the definition of Lipschitz constant.

guaranteeing the learned labels vary smoothly throughout dense regions of the graph. This amounts to minimising a regulariser measuring the smoothness of a labelling, subject to the given label constraints. Classical attempts use Laplacian regularisation leading, in the continuum limit, to the minimisation of the L^2-norm of the gradient of the learned function. It turns out that in the limit of infinite unlabelled data and finite labelled data, the problem degenerates into a constant label that is a sort of average of the given labels with sharp spikes near the labelled data; roughly speaking, the learned function forgets about the labelled data. The underlying reason is that a $W^{1,2}-$function does not necessarily attain boundary data continuously. Minimising, for large p, the L^p-norm of the gradient circumvents this difficulty as placing a heavier penalty on large gradients prevents the formation of spikes. In fact, $W^{1,p}-$functions are known to be Hölder continuous up to the boundary provided p is greater than the dimension of the underlying space, so the learned function transitions more smoothly between labelled and unlabelled data. For data sets of increasing dimension, we may as well consider the limit as $p \to \infty$, thus bringing about Lipschitz learning, consisting of minimising the $L^\infty-$norm of the gradient of the learned function, which turns out to be its Lipschitz constant in the case of a convex domain. The problem is then no other than finding a Lipschitz extension from the labelled to the unlabelled data that conserves the Lipschitz constant.

1.1 The Lipschitz extension problem

We start with the basic definition of a Lipschitz function and by fixing some notation.

Definition 1.1. Let $X \subset \mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is Lipschitz continuous on X, equivalently $f \in Lip(X)$, if there exists a constant $L \in [0, \infty)$ such that

$$|f(x) - f(y)| \leqslant L\,|x - y|, \quad \forall x, y \in X. \tag{1.1}$$

Any $L \in [0, \infty)$ for which (1.1) holds is called a Lipschitz constant for f in X. The least constant $L \in [0, \infty)$ for which (1.1) holds is the Lipschitz constant of f in X and denoted by $\mathrm{Lip}_f(X)$.

If there is no L for which (1.1) holds, we write $\mathrm{Lip}_f(X) = \infty$.

Let $U \subset \mathbb{R}^n$ be open and bounded and denote its boundary with ∂U. We will be concerned with the problem of extending a Lipschitz function defined on ∂U to $\overline{U}$ without increasing its Lipschitz constant. Since decreasing it is out of the question, the best we can hope for is to keep it the same.

The Lipschitz Extension Problem (LEP) . Given $f \in Lip(\partial U)$, find $u \in Lip(\overline{U})$ such that

$$u = f \text{ on } \partial U \qquad \text{and} \qquad \mathrm{Lip}_u(\overline{U}) = \mathrm{Lip}_f(\partial U).$$

In fact, we are both extending the function and minimising the Lipschitz constant.

If $y, z \in \partial U$ and $x \in \overline{U}$, then any solution to **(LEP)** trivially satisfies

$$f(z) - \mathrm{Lip}_f(\partial U)|x - z| \leqslant u(x) \leqslant f(y) + \mathrm{Lip}_f(\partial U)|x - y|$$

since $f(z) = u(z)$ and $f(y) = u(y)$. Let us show that these two bounds belong to $Lip(\overline{U})$.

Let $z \in \partial U$ and put

$$F_z(x) = f(z) - \mathrm{Lip}_f(\partial U)|x - z|, \quad x \in \overline{U}.$$

We then have, for any $x, \widetilde{x} \in \overline{U}$,

$$\begin{aligned} |F_z(x) - F_z(\widetilde{x})| &= \left| f(z) - \mathrm{Lip}_f(\partial U)|x - z| - f(z) + \mathrm{Lip}_f(\partial U)|\widetilde{x} - z| \right| \\ &= \mathrm{Lip}_f(\partial U)\ \left||\widetilde{x} - z| - |x - z|\right| \\ &\leqslant \mathrm{Lip}_f(\partial U)\ |\widetilde{x} - z - x + z| \\ &= \mathrm{Lip}_f(\partial U)\ |x - \widetilde{x}|\,. \end{aligned}$$

This means that $F_z \in Lip(\overline{U})$ and that $\mathrm{Lip}_f(\partial U)$ is a Lipschitz constant for F_z in $\overline{U}$. The result is, in fact, a triviality because F_z is a cone (*cf.* Definition 1.9 and Corollary 1.12). Since $\mathrm{Lip}_f(\partial U)$ is independent of z, it is a common Lipschitz constant for all F_z, $z \in \partial U$.

Given $y \in \partial U$, an entirely analogous reasoning holds for

$$G_y(x) = f(y) + \mathrm{Lip}_f(\partial U)|x - y|, \quad x \in \overline{U}.$$

Definition 1.2. The McShane–Whitney extensions of $f \in Lip(\partial U)$ are the functions defined in $\overline{U}$ by

$$\mathcal{MW}_*(f)(x) := \sup_{z \in \partial U} F_z(x) = \sup_{z \in \partial U} \left\{ f(z) - \mathrm{Lip}_f(\partial U)|x - z| \right\}$$

and

$$\mathcal{MW}^*(f)(x) := \inf_{y \in \partial U} G_y(x) = \inf_{y \in \partial U} \left\{ f(y) + \mathrm{Lip}_f(\partial U)|x - y| \right\}.$$

Proposition 1.3. *The infimum and the supremum of a family of Lipschitz functions $\mathcal{F} = \{f_a : X \to \mathbb{R}\}_{a \in \mathcal{A}}$, with Lipschitz constant L is Lipschitz and has, if it is finite, the same Lipschitz constant .*

Proof. We have

$$f_a(x) \leqslant f_a(y) + L\,|x - y|\,,\ \forall x, y \in X$$

so, taking the supremum, we get

$$\sup_{a \in \mathcal{A}} f_a(x) \leqslant \sup_{a \in \mathcal{A}} f_a(y) + L\,|x - y|\,,\ \forall x, y \in X$$

and thus

$$\left(\sup_{a\in\mathcal{A}} f_a\right)(x) - \left(\sup_{a\in\mathcal{A}} f_a\right)(y) \leqslant L\,|x-y|\,,\ \forall x,y\in X.$$

Interchanging x and y we get the result for the supremum; for the infimum, the proof is analogous. □

Using Proposition 1.3, we conclude that both $\mathcal{MW}_*(f)$ and $\mathcal{MW}^*(f)$ are Lipschitz functions in $\overline{U}$, with Lipschitz constant $\mathrm{Lip}_f(\partial U)$.

We next show that $\mathcal{MW}_*(f) = f$ on ∂U (the same holds, of course, for $\mathcal{MW}^*(f)$). Let $x\in\partial U$. Then

$$\mathcal{MW}_*(f)(x) \geqslant F_x(x) = f(x) - \mathrm{Lip}_f(\partial U)|x-x| = f(x).$$

On the other hand, since $f\in Lip(\partial U)$,

$$f(z) - \mathrm{Lip}_f(\partial U)|x-z| \leqslant f(x),$$

for any $z\in\partial U$, and thus

$$\mathcal{MW}_*(f)(x) = \sup_{z\in\partial U}\left\{f(z) - \mathrm{Lip}_f(\partial U)|x-z|\right\} \leqslant f(x).$$

This implies that

$$\mathrm{Lip}_{\mathcal{MW}_*(f)}(\overline{U}) = \mathrm{Lip}_{\mathcal{MW}^*(f)}(\overline{U}) = \mathrm{Lip}_f(\partial U).$$

We have just proved the following result.

Theorem 1.4. *The McShane–Whitney extensions, $\mathcal{MW}_*(f)$ and $\mathcal{MW}^*(f)$, solve the Lipschitz extension problem for $f\in Lip(\partial U)$ and if u is any other solution to the problem then*

$$\mathcal{MW}_*(f) \leqslant u \leqslant \mathcal{MW}^*(f) \quad \textit{in } \overline{U}.$$

The Lipschitz Extension Problem is then uniquely solvable if

$$\mathcal{MW}_*(f) = \mathcal{MW}^*(f) \quad \text{in } \overline{U},$$

which rarely happens.

Example 1.5. Let $n=1$ and $U=(-1,0)\cup(0,1)$. Consider $f:\partial U\to\mathbb{R}$ defined by $f(-1)=f(0)=0$ and $f(1)=1$. Then $\mathrm{Lip}_f(\partial U)=1$ and a simple computation gives

$$\mathcal{MW}_*(f)(x) = \begin{cases} -x-1 & \text{if } \ -1\leqslant x\leqslant -\frac{1}{2} \\[2ex] \quad x & \text{if } \ -\frac{1}{2}\leqslant x\leqslant 1 \end{cases}$$

and

$$\mathcal{MW}^*(f)(x) = \begin{cases} x+1 & \text{if } \ -1\leqslant x\leqslant -\frac{1}{2} \\[2ex] \ |x| & \text{if } \ -\frac{1}{2}\leqslant x\leqslant 1, \end{cases}$$

which are, of course, different functions.

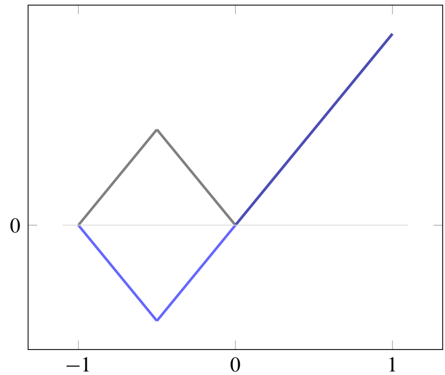


Figure 1.1: Functions $\mathcal{MW}_*(f)(x)$ and $\mathcal{MW}^*(f)(x)$ in Example 1.5.

1.2 Absolutely minimising Lipschitz extensions (AMLEs)

The lack of uniqueness in the Lipschitz extension problem illustrated by the previous example is an issue, but other features are perhaps even more relevant. We will address them again using example 1.5.

Non comparison Take as boundary data $g : \partial U \to \mathbb{R}$ defined by $g(-1) = 0$, $g(0) = \frac{1}{2}$ and $g(1) = 1$. Then $\operatorname{Lip}_g(\partial U) = \frac{1}{2}$ and we easily see that $\mathcal{MW}_*(g) = \mathcal{MW}^*(g)$, so the problem is uniquely solvable. But we have $f \leqslant g$ and, nevertheless, neither

$$\mathcal{MW}^*(f) \leqslant \mathcal{MW}^*(g)$$

nor

$$\mathcal{MW}^*(f) \geqslant \mathcal{MW}^*(g)$$

hold.

Non stability Let $V = \left(-\frac{3}{4}, -\frac{1}{4}\right)$. Then $\mathcal{MW}^*(f)|_{\partial V} \equiv \frac{1}{4}$ and so also

$$\mathcal{MW}^*\left(\mathcal{MW}^*(f)|_{\partial V}\right) \equiv \frac{1}{4} \neq \mathcal{MW}^*(f) \quad \text{in } V.$$

In particular, a repeated application of the McShane-Whitney extension in a subset may decrease the local Lipschitz constant.

Non locality Again let $V = \left(-\frac{3}{4}, -\frac{1}{4}\right)$; then

$$\mathrm{Lip}_{\mathcal{MW}^*(f)}(V) = 1 \neq 0 = \mathrm{Lip}_{\mathcal{MW}^*(f)}(\partial V).$$

The extension defined by

$$u(x) = \begin{cases} 0 & \text{if} \quad -1 \leqslant x \leqslant 0 \\ \\ x & \text{if} \quad 0 \leqslant x \leqslant 1 \end{cases}$$

satisfies this property for any $V \subset\subset U$ (this means $\overline{V}$ is a compact subset of U). In a certain sense, it locally varies as little as possible.

The notion of locality is embedded in the following definition that meets the need to define a sort of *canonical* Lipschitz extension, which we will eventually prove is unique.

Definition 1.6. A function $u \in C(U)$ is absolutely minimising Lipschitz on U, and we write $u \in \mathrm{AML}(U)$, if

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V), \quad \forall\, V \subset\subset U. \tag{1.2}$$

This notion is trivially local in the sense that if $u \in \mathrm{AML}(U)$ and $V \subset U$, then $u \in \mathrm{AML}(V)$. It does not involve boundary conditions, being a property of continuous functions defined on open sets alone.

Still, we can try to recast the Lipschitz extension problem as the following problem: given $f \in Lip(\partial U)$, find $u \in C(\overline{U})$ such that

$$u \in \mathrm{AML}(U) \qquad \text{and} \qquad u = f \;\text{ on } \partial U. \tag{1.3}$$

The two problems do not necessarily have the same solutions (convince yourself through examples), but for a bounded U, it can be shown that a solution to the second problem satisfies the Lipschitz extension problem.

1.3 Strongly absolutely minimising Lipschitz

Given $u \in C(U)$, define the Lipschitz constant of u at the point x, taking values in $[0, \infty]$, as

$$\begin{aligned} T_u(x) \;&:=\; \lim_{r \downarrow 0} \mathrm{Lip}_u\left(B_r(x)\right) \\ &=\; \inf_{0 < r < \mathrm{dist}(x, \partial U)} \mathrm{Lip}_u\left(B_r(x)\right), \quad x \in U. \end{aligned}$$

It is the smallest number satisfying the following property: given $\epsilon > 0$, we can find $r_\epsilon > 0$ such that $T_u(x) + \epsilon$ is a Lipschitz constant for u on $B_{r_\epsilon}(x)$.

Proposition 1.7. *Let $u \in C(U)$. Then*

1. *$T_u(x)$ is upper semi-continuous in U;*
2. *if u is differentiable at $x \in U$, then*
$$T_u(x) \geqslant |Du(x)|;$$
3. *if $x \in U$ and $T_u(x) = 0$ then u is differentiable at x and $Du(x) = 0$.*

Proof.

1. Let $x_0 \in U$ and consider a sequence $x_n \to x_0$, with $x_n \in U$. We want to show that
$$\limsup_{n\to\infty} T_u(x_n) \leqslant T_u(x_0).$$
For n sufficiently large, we have
$$|x_n - x_0| < \frac{1}{2}\operatorname{dist}(x_0, \partial U),$$
and so
$$B_{|x_n-x_0|}(x_n) \subset B_{2|x_n-x_0|}(x_0) \subset U.$$
Then
$$\begin{aligned} T_u(x_n) &= \lim_{r\downarrow 0} \operatorname{Lip}_u(B_r(x_n)) \\ &\leqslant \operatorname{Lip}_u\left(B_{|x_n-x_0|}(x_n)\right) \\ &\leqslant \operatorname{Lip}_u\left(B_{2|x_n-x_0|}(x_0)\right). \end{aligned}$$
Taking the limit when $n \to \infty$, we obtain
$$\begin{aligned} \limsup_{n\to\infty} T_u(x_n) &\leqslant \lim_{n\to\infty} \operatorname{Lip}_u\left(B_{2|x_n-x_0|}(x_0)\right) \\ &= T_u(x_0). \end{aligned}$$

2. If $Du(x) = 0$, the result is trivial so we may suppose $\vec{p} := Du(x) \neq 0$. Let
$$y = x + \lambda\vec{p}, \quad 0 < \lambda \leqslant 1;$$
then
$$\begin{aligned} \operatorname{Lip}_u\left(B_{\lambda|\vec{p}|}(x)\right) &\geqslant \frac{|u(y) - u(x)|}{|y - x|} \\ &= \frac{\left|u(x+\lambda\vec{p}) - u(x)\right|}{\lambda|\vec{p}|} \\ &= \frac{\langle \vec{p}, \lambda\vec{p}\rangle + o(\lambda|\vec{p}|)}{\lambda|\vec{p}|} \\ &= |\vec{p}| + o(1) \quad (\lambda \to 0). \end{aligned}$$

Taking the limit when $\lambda \to 0$, we get

$$T_u(x) \geqslant |\vec{p}| = |Du(x)|.$$

3. If $T_u(x) = 0$, given $\epsilon > 0$, we can find $\delta > 0$ such that

$$0 < r < \delta \Rightarrow \mathrm{Lip}_u\left(B_r(x)\right) < \epsilon.$$

Then

$$|y - x| < \frac{\delta}{2} \Rightarrow \frac{|u(y) - u(x)|}{|y - x|} \leqslant \mathrm{Lip}_u\left(B_{\frac{\delta}{2}}(x)\right) < \epsilon.$$

and u is differentiable at x, with $Du(x) = 0$.

□

Definition 1.8. A function $u \in C(U)$ is strongly absolutely minimising Lipschitz on U, and we write $u \in \mathrm{SAML}(U)$, if

$$\sup_{x\in V} T_u(x) \leqslant \sup_{x\in V} T_v(x), \quad \forall\, V \subset\subset U, \tag{1.4}$$

and all $v \in C(\overline{V})$ such that $v = u$ on ∂V.

We have $\mathrm{AML}(U) = \mathrm{SAML}(U)$ (see Aronsson, Crandall, and Juutinen 2004, Section 4). The reason for using the qualitative strong is that the inclusion $\mathrm{AML}(U) \subset \mathrm{SAML}(U)$ is significantly harder to prove.

1.4 Comparison with cones

We now introduce a more geometric notion, that of comparison with cones. It will be instrumental in most of the analysis hereafter.

Definition 1.9. A cone with vertex $x_0 \in \mathbb{R}^n$ is a function of the form

$$C(x) = a + b|x - x_0|, \quad a, b \in \mathbb{R}.$$

The height of C is a, and its slope is b.

Definition 1.10. For a cone C with vertex at x_0, the half-line

$$\{x_0 + t(x - x_0),\ t \geqslant 0\}$$

is the ray of C through the point x.

Lemma 1.11. *If a set V contains two distinct points on the same ray of a cone C with slope b, then*

$$\mathrm{Lip}_C(V) = |b|.$$

Proof. Let $C(x) = a + b|x - x_0|$. Then, for any $x, y \in \mathbb{R}^n$,

$$\frac{|C(x) - C(y)|}{|x - y|} = |b| \frac{\big||x - x_0| - |y - x_0|\big|}{|x - y|} \leqslant |b|,$$

so $|b|$ is a Lipschitz constant for C in any set.

If w, y are distinct points on the same ray of C, we have, for a certain x^*, $y = x_0 + \alpha(x^* - x_0)$ and $w = x_0 + \beta(x^* - x_0)$, with $\alpha, \beta \geqslant 0$, $\alpha \neq \beta$. Then

$$\begin{aligned}
\frac{|C(y) - C(w)|}{|y - w|} &= \frac{|C(x_0 + \alpha(x^* - x_0)) - C(x_0 + \beta(x^* - x_0))|}{|x_0 + \alpha(x^* - x_0) - x_0 - \beta(x^* - x_0)|} \\
&= \frac{\big|a + b|x_0 + \alpha(x^* - x_0) - x_0| - a - b|x_0 + \beta(x^* - x_0) - x_0|\big|}{|\alpha - \beta||x^* - x_0|} \\
&= |b| \frac{|\alpha - \beta||x^* - x_0|}{|\alpha - \beta||x^* - x_0|} = |b|,
\end{aligned}$$

and if $w, y \in V$ then $\mathrm{Lip}_C(V) = |b|$. □

Corollary 1.12. *Let $V \subset \mathbb{R}^n$ be non-empty and open, and C be a cone with slope b. Then*

$$\mathrm{Lip}_C(V) = |b|.$$

Moreover, if V is bounded and does not contain the vertex of C, then

$$\mathrm{Lip}_C(\partial V) = |b|.$$

Definition 1.13. A function $w \in C(U)$ enjoys *comparison with cones from above* in U if, for every $V \subset\subset U$ and every cone C whose vertex is not in V,

$$w \leqslant C \text{ on } \partial V \quad \Longrightarrow \quad w \leqslant C \text{ in } V.$$

A function w enjoys comparison with cones from below if $-w$ enjoys comparison with cones from above. A function w enjoys comparison with cones if it enjoys comparison with cones from above and from below.

Lemma 1.14. *The following is an equivalent condition to $u \in C(U)$ enjoying comparison with cones from above in U: for every $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$,*

$$u(x) - b|x - z| \leqslant \max_{w \in \partial V} (u(w) - b|w - z|), \quad \forall\, x \in V.$$

Proof. To prove the necessity of the condition, let $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$. We trivially have

$$u(x) - b|x - z| \leqslant \max_{w \in \partial V} \left(u(w) - b|w - z|\right), \quad \forall\, x \in \partial V. \tag{1.5}$$

This can be rewritten as

$$u(x) \leqslant C(x) := \max_{w \in \partial V} \left(u(w) - b|w - z|\right) + b|x - z|, \quad \forall\, x \in \partial V,$$

for the cone C centred at $z \notin V$. Since u enjoys comparison with cones from above in U, (1.5) also holds for any $x \in V$.

Reciprocally, let $V \subset\subset U$ and let

$$C(x) = a + b|x - z|, \quad \text{with } a, b \in \mathbb{R},$$

be any cone with vertex at $z \notin V$ such that $u \leqslant C$ on ∂V. We know that for every $x \in V$,

$$\begin{aligned} & u(x) - b|x - z| \leqslant \max_{w \in \partial V} \left(u(w) - b|w - z|\right) \\ \Rightarrow\ & u(x) - a - b|x - z| \leqslant \max_{w \in \partial V} \left(u(w) - a - b|w - z|\right) \\ \Rightarrow\ & u(x) - C(x) \leqslant \max_{w \in \partial V} \left(u(w) - C(w)\right) \leqslant 0 \end{aligned}$$

since $u \leqslant C$ on ∂V. We conclude that also $u \leqslant C$ in V. □

1.5 Comparison with cones and absolutely minimising Lipschitz

One of the main results in these notes is the following equivalence between being absolutely minimising Lipschitz and enjoying comparison with cones.

Theorem 1.15. *A function $u \in C(U)$ is absolutely minimising Lipschitz in U if, and only if, it enjoys comparison with cones in U.*

Proof. We start with sufficiency. Suppose u enjoys comparison with cones in U and let $V \subset\subset U$. We want to show that

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Since $u \in C(\overline{V})$, we have $\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\overline{V})$ (see Problem 1). Then, as $\partial V \subset \overline{V}$, we trivially have that $\mathrm{Lip}_u(V) \geqslant \mathrm{Lip}_u(\partial V)$, and it remains to prove the other inequality.

First, observe that for any $x \in V$,

$$\mathrm{Lip}_u\left(\partial\left(V \setminus \{x\}\right)\right) = \mathrm{Lip}_u(\partial V \cup \{x\}) = \mathrm{Lip}_u(\partial V). \tag{1.6}$$

To see if this holds, we need only check that, for any $y \in \partial V$,

$$|u(y) - u(x)| \leqslant \mathrm{Lip}_u(\partial V)\,|y - x|,$$

which is equivalent to

$$u(y) - \mathrm{Lip}_u(\partial V)\,|x - y| \leqslant u(x) \leqslant u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|. \tag{1.7}$$

This clearly holds for any $x \in \partial V$, but we want to prove it for $x \in V$. Let's focus on the second inequality in (1.7). The right-hand side can be regarded as the cone

$$C(x) = u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|,$$

centred at $y \in \partial V$. Since $y \notin V$ and u enjoys comparison with cones from above in U, the inequality holds in V because it holds on ∂V. We argue analogously to obtain the first inequality in (1.7), using comparison with cones from below.

Now let $x, y \in V$. Using (1.6) twice, we obtain

$$\mathrm{Lip}_u(\partial V) = \mathrm{Lip}_u\left(\partial\left(V \setminus \{x\}\right)\right) = \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right).$$

Since $x, y \in \partial\left(V \setminus \{x, y\}\right) = \partial V \cup \{x, y\}$, we have

$$|u(x) - u(y)| \leqslant \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right)\,|x - y| = \mathrm{Lip}_u(\partial V)\,|x - y|.$$

Thus

$$\mathrm{Lip}_u(V) \leqslant \mathrm{Lip}_u(\partial V).$$

Now the necessity. Suppose $u \in \mathrm{AML}(U)$. For $V \subset\subset U$, we have

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Due to Lemma 1.14, we want to prove that for every $b \in \mathbb{R}$ and $z \notin V$,

$$u(x) - b|x - z| \leqslant \max_{w \in \partial V}\left(u(w) - b|w - z|\right), \quad \forall\, x \in V.$$

Setting

$$W = \left\{x \in V \;:\; u(x) - b|x - z| > \max_{w \in \partial V}\left(u(w) - b|w - z|\right)\right\},$$

the result will follow by proving that $W = \emptyset$. We will argue by contradiction. Consider the cone

$$C(x) := \max_{w \in \partial V}\left(u(w) - b|w - z|\right) + b|x - z|.$$

Then $W = V \cap (u - C)^{-1}((0, \infty))$ is open and

$$u = C \quad \text{on } \partial W. \tag{1.8}$$

To prove this, note first that, trivially, if $x \in \partial V$, then $(u - C)(x) \leqslant 0$. Now suppose $x \in \partial W$, with $(u - C)(x) > 0$. Then $x \notin \partial V$, and since $\partial W \subset \overline{V}$, $x \in V$, in which case $x \in W$, which is a contradiction since W is open. If $x \in \partial W$, with $(u - C)(x) < 0$, then, since $u - C \in C(U)$, there is a neighbourhood N_x of x such that $u - C < 0$ in N_x. So $N_x \cap W = \emptyset$, again a contradiction.

We then have, since $u \in \mathrm{AML}(U)$,

$$\mathrm{Lip}_u(W) = \mathrm{Lip}_u(\partial W) = \mathrm{Lip}_C(\partial W) = |b|,$$

due to Corollary 1.12, because $z \notin W$, since $z \notin V$ and $W \subset V$.

Take $x_0 \in W$. The ray of C through x_0

$$\{z + t(x_0 - z),\ t \geqslant 0\}$$

contains a segment in W, containing x_0, which meets ∂W at its endpoints. Consider the functions

$$F(t) = C(z + t(x_0 - z)) = a + b|x_0 - z|\,t, \quad t \geqslant 0,$$

with $a = \max_{w \in \partial V}(u(w) - b|w - z|)$, and

$$G(t) = u(z + t(x_0 - z)), \quad t \geqslant 0.$$

They coincide at the segment's endpoints since $u = C$ on ∂W. Now F is affine with slope $|b||x_0 - z|$, while G has $|b||x_0 - z|$ as Lipschitz constant on the segment. In fact,

$$\begin{aligned}
\frac{|G(t_1) - G(t_2)|}{|t_1 - t_2|} &= \frac{|u(z + t_1(x_0 - z)) - u(z + t_2(x_0 - z))|}{|t_1 - t_2|} \\
&\leqslant \frac{|b||(t_1 - t_2)(x_0 - z)|}{|t_1 - t_2|} = |b||x_0 - z|,
\end{aligned}$$

because $\mathrm{Lip}_u(W) = |b|$ and the segment is contained in W. We conclude that F and G are the same function on the segment (*cf.* Problem 6), and since it contains x_0,

$$G(1) = u(x_0) = C(x_0) = F(1).$$

We have reached a contradiction because $x_0 \in W$ and so $u(x_0) > C(x_0)$.

The proof that u satisfies comparison with cones from below in U is analogous and uses a lemma similar to Lemma 1.14. $\square$

2 The ∞–Laplace equation

In this chapter, we explore the connection with ∞–harmonic functions. We introduce the notion of viscosity solution to the ∞–Laplace equation and prove that it is equivalent to enjoying comparison with cones. We then treat questions of existence, uniqueness and regularity.

2.1 The ∞–Laplacian

Definition 2.1. The partial differential operator defined on smooth functions φ by

$$\Delta_\infty \varphi := \sum_{i,j=1}^{n} \varphi_{x_i} \varphi_{x_j} \varphi_{x_i x_j} = \langle D^2\varphi D\varphi, D\varphi \rangle$$

is called the ∞–Laplacian.

This operator is not in divergence form, so we can not (formally) integrate by parts to define a notion of weak solution. The appropriate one to consider is that of viscosity solution.

Definition 2.2. A function $w \in C(U)$ is a viscosity subsolution of $\Delta_\infty u = 0$ (or a viscosity solution of $\Delta_\infty u \geqslant 0$ or ∞–subharmonic) in U if, for every $\hat{x} \in U$ and every $\varphi \in C^2(U)$ such that $w - \varphi$ has a local maximum at $\hat{x}$, we have

$$\Delta_\infty \varphi(\hat{x}) \geqslant 0.$$

A function $w \in C(U)$ is ∞−superharmonic in U if $-w$ is ∞−subharmonic in U. A function $w \in C(U)$ is ∞−harmonic in U if it is at the same time ∞−subharmonic and ∞−superharmonic in U.

Lemma 2.3. *If $u \in C^2(U)$ then u is ∞−harmonic in U if, and only if, $\Delta_\infty u = 0$ in the pointwise sense.*

Proof. Suppose u is ∞−harmonic. Then it is ∞−subharmonic, and we take $\varphi = u$ in the definition. Since every point $x \in U$ will then be a local maximum of $u - \varphi \equiv 0$, $\Delta_\infty u(x) \geqslant 0$, for every $x \in U$. Since also $-u$ is ∞−subharmonic, we get in addition

$$\Delta_\infty(-u)(x) \geqslant 0 \;\Leftrightarrow\; -\Delta_\infty u(x) \geqslant 0 \;\Leftrightarrow\; \Delta_\infty u(x) \leqslant 0, \quad \forall x \in U$$

and so $\Delta_\infty u = 0$ in the pointwise sense.

Reciprocally, suppose $\Delta_\infty u = 0$ in the pointwise sense and take $\hat{x} \in U$ and $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at $\hat{x}$. We want to prove that $\Delta_\infty \varphi(\hat{x}) \geqslant 0$, thus showing that u is ∞−subharmonic (the ∞−superharmonicity is obtained in an analogous way). We have, since $u - \varphi \in C^2(U)$ and $\hat{x} \in U$ is a local maximum,

$$D(u-\varphi)(\hat{x}) = 0 \;\Leftrightarrow\; Du(\hat{x}) = D\varphi(\hat{x})$$

and

$$D^2(u-\varphi)(\hat{x}) \preceq 0 \;\Leftrightarrow\; \langle D^2u(\hat{x})\xi, \xi\rangle \leqslant \langle D^2\varphi(\hat{x})\xi, \xi\rangle, \quad \forall \xi \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
\Delta_\infty \varphi(\hat{x}) &= \langle D^2\varphi(\hat{x}) D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\
&\geqslant \langle D^2u(\hat{x}) D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\
&= \langle D^2u(\hat{x}) Du(\hat{x}), Du(\hat{x})\rangle \\
&= \Delta_\infty u(\hat{x}) \\
&= 0.
\end{aligned}$$

□

We now show that the celebrated flatland example of Aronsson

$$\mathcal{A}(x_1, x_2) = x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}$$

is ∞−subharmonic in $\mathbb{R}^2$. The proof that it is also ∞−superharmonic is analogous.

Take any point $(x_0, y_0) \in \mathbb{R}^2$ and $\varphi \in C^2(\mathbb{R}^2)$ such that $u - \varphi$ has a local maximum at (x_0, y_0). We start by observing that, since $u \in C^1(\mathbb{R}^2)$,

$$D(u-\varphi)(x_0, y_0) = 0$$

and, consequently,

$$\varphi_x(x_0, y_0) = u_x(x_0, y_0) = \frac{4}{3} x_0^{\frac{1}{3}} \tag{2.1}$$

and

$$\varphi_y(x_0, y_0) = u_y(x_0, y_0) = -\frac{4}{3} y_0^{\frac{1}{3}}. \tag{2.2}$$

We first exclude the case $x_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u - \varphi$ has a local maximum at $(0, y_0)$, then

$$\begin{aligned} & (u-\varphi)(x, y_0) \leqslant (u-\varphi)(0, y_0) \\ \Leftrightarrow \quad & x^{\frac{4}{3}} \leqslant \varphi(x, y_0) - \varphi(0, y_0), \end{aligned} \tag{2.3}$$

for every x in a neighbourhood of 0, and this simply can not happen. In fact, letting $F(x) = \varphi(x, y_0) - \varphi(0, y_0)$, we have $F(0) = 0$ and also

$$F'(0) = \varphi_x(0, y_0) = u_x(0, y_0) = 0.$$

Then, by Taylor's theorem,

$$\lim_{x \to 0} \frac{F(x)}{x^2} = \frac{F''(0)}{2} = \frac{\varphi_{xx}(0, y_0)}{2} < +\infty.$$

On the other hand, if (2.3) would hold,

$$\lim_{x \to 0} \frac{F(x)}{x^2} \geqslant \lim_{x \to 0} \frac{x^{\frac{4}{3}}}{x^2} = \lim_{x \to 0} x^{-\frac{2}{3}} = +\infty,$$

a contradiction.

We next consider the case $x_0 \neq 0$ and $y_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u - \varphi$ has a local maximum at $(x_0, 0)$, then

$$\begin{aligned} & (u-\varphi)(x, 0) \leqslant (u-\varphi)(x_0, 0) \\ \Leftrightarrow \quad & x^{\frac{4}{3}} - \varphi(x, 0) \leqslant x_0^{\frac{4}{3}} - \varphi(x_0, 0), \end{aligned} \tag{2.4}$$

for every x in a neighbourhood of x_0. This means that the function

$$G(x) = x^{\frac{4}{3}} - \varphi(x, 0)$$

has a local maximum at the point x_0. Since it is of class C^2 in a neighbourhood of x_0 (small enough that it does not contain 0), we have $G'(x_0) = 0$ and

$$G''(x_0) \leqslant 0 \quad \Leftrightarrow \quad \varphi_{xx}(x_0, 0) \geqslant \frac{4}{9} x_0^{-\frac{2}{3}} \geqslant 0. \tag{2.5}$$

Then, using (2.1), (2.2) and (2.5),

$$\begin{aligned} \Delta_\infty \varphi(x_0, 0) &= \left(\varphi_x^2 \varphi_{xx} + 2\varphi_x \varphi_y \varphi_{xy} + \varphi_y^2 \varphi_{yy}\right)(x_0, 0) \\ &= \varphi_x^2(x_0, 0)\varphi_{xx}(x_0, 0) \geqslant 0 \end{aligned}$$

as required.

Finally, if both $x_0 \neq 0$ and $y_0 \neq 0$, u is C^2 in a neighborhood of (x_0, y_0) and the equation is satisfied in the pointwise sense, the calculation being trivial.

2.2 Comparison with cones and $\infty-$harmonic

A crucial fact for $\infty-$harmonic functions is that they can be characterised through comparison with cones.

Theorem 2.4. *If $u \in C(U)$ is $\infty-$subharmonic, then it enjoys comparison with cones from above.*

Proof. According to Lemma 1.14, we want to prove that, given $V \subset\subset U$, $b \in \mathbb{R}$ and $z \notin V$,

$$u(x) - b|x - z| \leqslant \max_{w \in \partial V} \left(u(w) - b|w - z|\right), \quad \forall\, x \in V. \tag{2.6}$$

Note that if G is smooth, we have (*cf.* Problem 4)

$$\Delta_\infty G(|x|) = G''(|x|)\left[G'(|x|)\right]^2, \quad x \neq 0.$$

Taking $G(t) = bt - \gamma t^2$, we have, for all $x \in V$ (recall that $z \notin V$),

$$\begin{aligned}
\Delta_\infty \left(b|x - z| - \gamma|x - z|^2\right) &= \Delta_\infty G(|x - z|)\\
&= G''(|x - z|)\left[G'(|x - z|)\right]^2\\
&= -2\gamma\left(b - 2\gamma|x - z|\right)^2\\
&< 0
\end{aligned}$$

if $\gamma > 0$ is small enough. In particular, since V is bounded, we must have, if $b > 0$ (the case $b \leqslant 0$ is trivial),

$$\gamma < \frac{b}{2 \sup\limits_{x \in V} |x - z|}.$$

Now, since u is $\infty-$subharmonic in $V \subset\subset U$ (due to the local character of the notion of viscosity sub-solution),

$$u(x) - \left(b|x - z| - \gamma|x - z|^2\right)$$

can not have a local maximum in V. Then

$$u(x) - \left(b|x - z| - \gamma|x - z|^2\right) \leqslant \max_{w \in \partial V}\left(u(w) - \left(b|w - z| - \gamma|w - z|^2\right)\right),$$

for all $x \in V$. Finally, let $\gamma \to 0$ to obtain (2.6) and thus the result. □

Magnificently, the reciprocal also holds.

Theorem 2.5. *If $u \in C(U)$ enjoys comparison with cones from above, then it is $\infty-$subharmonic.*

Proof. We start by observing that, for every $x \in B_r(y) \subset\subset U$,

$$u(x) \leqslant u(y) + \max_{w\in\partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y|. \tag{2.7}$$

The inequality clearly holds for $x \in \partial\left(B_r(y) \setminus \{y\}\right) = \partial B_r(y) \cup \{y\}$ and, since the right-hand side is a cone with vertex at $y \notin B_r(y) \setminus \{y\}$, the open set $B_r(y) \setminus \{y\} \subset\subset U$ and u enjoys comparison with cones from above, it also holds for $x \in B_r(y) \setminus \{y\}$.

Now, we rewrite (2.7) as

$$u(x) - u(y) \leqslant \max_{w\in\partial B_r(y)} \left(u(w) - u(x)\right) \frac{|x - y|}{r - |x - y|}. \tag{2.8}$$

This is just algebra:

$$\begin{aligned}
& u(x) \leqslant u(y) + \max_{w\in\partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y| \\
\Leftrightarrow\ & u(x) \leqslant u(y) + \left(\max_{w\in\partial B_r(y)} u(w) - u(y) \right) \frac{|x - y|}{r} \\
\Leftrightarrow\ & u(x) - \frac{r - |x - y|}{r} u(y) \leqslant \max_{w\in\partial B_r(y)} u(w) \frac{|x - y|}{r} \\
\Leftrightarrow\ & \frac{r}{r - |x - y|} u(x) - u(y) \leqslant \max_{w\in\partial B_r(y)} u(w) \frac{|x - y|}{r - |x - y|} \\
\Leftrightarrow\ & \left(1 + \frac{|x - y|}{r - |x - y|}\right) u(x) - u(y) \leqslant \max_{w\in\partial B_r(y)} u(w) \frac{|x - y|}{r - |x - y|} \\
\Leftrightarrow\ & u(x) - u(y) \leqslant \max_{w\in\partial B_r(y)} \left(u(w) - u(x)\right) \frac{|x - y|}{r - |x - y|}.
\end{aligned}$$

We first prove the result at points of twice differentiability. If u is twice continuously differentiable at x_0, namely if there is a $p \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix X such that, as $x \to x_0$,

$$u(x) = u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o(|x - x_0|^2), \tag{2.9}$$

so that

$$p = Du(x_0) \qquad \text{and} \qquad X = D^2u(x_0),$$

we show that

$$\Delta_\infty u(x_0) = \langle Xp, p \rangle \geqslant 0.$$

We can assume $p \neq 0$ since, otherwise, the result is trivial.

Let $x_0 \in U$ be a point of twice differentiability for u. Denoting

$$\operatorname{dist}(x_0, \partial U) = \inf\{|x_0 - y| \mid y \in \partial U\},$$

choose

$$r < \frac{1}{2}\operatorname{dist}(x_0, \partial U)$$

and λ small enough so that, for $y_0 = x_0 - \lambda Du(x_0)$, $B_r(y_0) \subset\subset U$ and

$$x_0 \in B_r(y_0) \;\Leftrightarrow\; |x_0 - y_0| < r \;\Leftrightarrow\; \lambda < \frac{r}{|Du(x_0)|}.$$

Put $x = y_0$, in (2.9) to obtain, with $p = Du(x_0)$,

$$u(y_0) = u(x_0) + \langle p, -\lambda p\rangle + \frac{1}{2}\langle X(-\lambda p), -\lambda p\rangle + o(|-\lambda p|^2)$$

$$\Leftrightarrow\; u(x_0) - u(y_0) = \lambda|p|^2 - \frac{1}{2}\lambda^2\langle Xp, p\rangle - o(\lambda^2|p|^2),$$

as $\lambda \to 0$.

Then, let $w_{r,\lambda} \in \partial B_r(y_0)$ be such that

$$u(w_{r,\lambda}) = \max_{w \in \partial B_r(y_0)} u(w)$$

and put $x = w_{r,\lambda}$ in (2.9) to obtain

$$u(w_{r,\lambda}) - u(x_0) = \langle p, w_{r,\lambda} - x_0\rangle + \frac{1}{2}\langle X(w_{r,\lambda} - x_0), w_{r,\lambda} - x_0\rangle + o(|w_{r,\lambda} - x_0|^2).$$

Now, choose $x = x_0$ and $y = y_0$ in (2.8) to get, after division by λ,

$$|p|^2 - \frac{1}{2}\lambda\langle Xp, p\rangle - o(\lambda)$$

$$\leqslant \left(\langle p, w_{r,\lambda} - x_0\rangle + \frac{1}{2}\langle X(w_{r,\lambda} - x_0), w_{r,\lambda} - x_0\rangle + o((r + \lambda|p|)^2)\right)\frac{|p|}{r - \lambda|p|}.$$

Note that

$$|w_{r,\lambda} - x_0| = |w_{r,\lambda} - y_0 - \lambda p| \leqslant r + \lambda|p|.$$

We now send $\lambda \downarrow 0$ to get

$$\begin{aligned} |p|^2 &\leqslant \left(\left\langle p, \frac{w_r - x_0}{r}\right\rangle + \frac{1}{2}\left\langle X\left(\frac{w_r - x_0}{r}\right), w_r - x_0\right\rangle\right)|p| + |p|o(r) \\ &\leqslant |p|^2 + \frac{1}{2}\left\langle X\left(\frac{w_r - x_0}{r}\right), w_r - x_0\right\rangle|p| + |p|o(r), \end{aligned} \tag{2.10}$$

where $w_r \in \partial B_r(x_0)$ is any limit point of $w_{r,\lambda}$ and thus

$$\left|\frac{w_r - x_0}{r}\right| = 1.$$

Next, take $r \downarrow 0$ in the first inequality to get, since $|w_r - x_0| = r$,

$$|p| \leqslant \left\langle p, \lim_{r\downarrow 0} \frac{w_r - x_0}{r} \right\rangle \leqslant |p| \cos\alpha,$$

where α is the angle formed by p and $\lim_{r\downarrow 0} \frac{w_r - x_0}{r}$, which is then $\alpha = 0$. It follows that

$$\lim_{r\downarrow 0} \frac{w_r - x_0}{r} = \frac{p}{|p|}, \quad p \neq 0.$$

To conclude this part, pass to the limit as $r \downarrow 0$ in the extremes inequality in (2.10) to obtain, after dividing by r,

$$0 \leqslant \frac{1}{2}\left\langle X\frac{p}{|p|}, \frac{p}{|p|}\right\rangle |p| \ \Leftrightarrow\ 0 \leqslant \langle Xp, p\rangle = \Delta_\infty u(x_0).$$

In the general case, let $\hat{x} \in U$ and $\varphi \in C^2(U)$ be such that $u - \varphi$ has a local maximum at $\hat{x}$. Then, for y, w close to $\hat{x}$,

$$\varphi(\hat{x}) - \varphi(y) \leqslant u(\hat{x}) - u(y)$$

and

$$u(w) - u(\hat{x}) \leqslant \varphi(w) - \varphi(\hat{x}).$$

Then

$$\begin{aligned}
\varphi(\hat{x}) - \varphi(y) &\leqslant u(\hat{x}) - u(y) \\
&\leqslant \max_{w\in\partial B_r(y)} (u(w) - u(\hat{x})) \frac{|\hat{x} - y|}{r - |\hat{x} - y|} \\
&\leqslant \max_{w\in\partial B_r(y)} (\varphi(w) - \varphi(\hat{x})) \frac{|\hat{x} - y|}{r - |\hat{x} - y|}
\end{aligned}$$

and we obtain (2.8) for the twice continuously differentiable function φ. Repeating the reasoning above, we conclude that

$$\Delta_\infty \varphi(\hat{x}) \geqslant 0$$

and the proof is complete. □

Entirely analogous results hold replacing ∞−subharmonic with ∞−superharmonic and comparison with cones from above with comparison with cones from below. We thus obtain the following result.

Theorem 2.6. *A function $u \in C(U)$ is ∞−harmonic if, and only if, it enjoys comparison with cones.*

2.3 Regularity

We now turn to regularity. For an open set U and $x \in U$, we introduce the notation

$$d(x) := \operatorname{dist}(x, \partial U).$$

Our first result is a Harnack inequality.

Lemma 2.7 (Harnack Inequality). *Let $0 \geqslant u \in C(U)$ satisfy*

$$u(x) \leqslant u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y|, \tag{2.11}$$

for $x \in B_r(y) \subset\subset U$.

If $z \in U$ and $R < d(z)/4$, then

$$\sup_{B_R(z)} u \leqslant \frac{1}{3} \inf_{B_R(z)} u. \tag{2.12}$$

Proof. Take arbitrary $x, y \in B_R(z)$. Then (2.11) holds for r sufficiently large. Let $r \uparrow d(y)$ to get, using the fact that $u \leqslant 0$,

$$u(x) \leqslant u(y) \left(1 - \frac{|x - y|}{d(y)} \right). \tag{2.13}$$

We have

$$d(y) \geqslant 3R \qquad \text{and} \qquad |x - y| \leqslant 2R$$

and thus, from (2.13), we obtain

$$u(x) \leqslant u(y) \left(1 - \frac{2R}{3R} \right) = \frac{1}{3} u(y)$$

and the result follows. □

We now sharpen the estimate with a direct proof of the result in Lindqvist and Manfredi 1995, where, alternatively, the proof follows from looking at the ∞–Laplace equation as the limit as $p \to \infty$ of the p–Laplace equation.

Theorem 2.8 (The Harnack inequality of Lindqvist–Manfredi). *Let $0 \geqslant u \in C(U)$ satisfy* (2.11). *If $z \in U$ and $0 < R < d(z)$, then*

$$u(x) \leqslant \exp\left(-\frac{|x - y|}{d(z) - R} \right) u(y), \quad \forall x, y \in B_R(z). \tag{2.14}$$

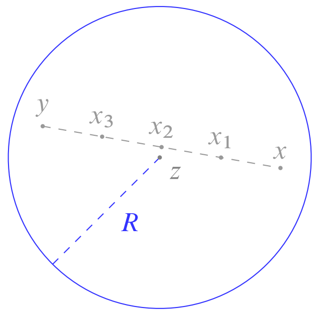


Figure 2.1: Sequence of points x_k for $m = 4$.

Proof. Let $x, y \in B_R(z)$, $m \in \mathbb{N}$ and define (see Figure 2.1)

$$x_k = x + k\,\frac{y - x}{m}, \qquad k = 0, 1, \ldots, m.$$

We have, for every k,

$$|x_{k+1} - x_k| = \frac{|x - y|}{m} < d(x_{k+1}),$$

for m large enough, and

$$d(x_{k+1}) \geqslant d(z) - R.$$

We can then apply (2.13), with $x = x_k$ and $y = x_{k+1}$, to get

$$\begin{aligned} u(x_k) &\leqslant u(x_{k+1})\left(1 - \frac{|x_{k+1} - x_k|}{d(x_{k+1})}\right) \\ &\leqslant u(x_{k+1})\left(1 - \frac{|x - y|}{m(d(z) - R)}\right). \end{aligned}$$

Iterating, we obtain

$$u(x) = u(x_0) \leqslant u(y)\left(1 - \frac{|x - y|}{m(d(z) - R)}\right)^m,$$

and taking $m \to \infty$, we arrive at (2.14). □

This is indeed a sharper Harnack inequality when compared with (2.12). For starters, it is valid for every $R < d(z)$. Moreover, the constant is also better: taking $R = d(z)/4$, we obtain

$$\sup_{B_R(z)} u(x) \leqslant \exp\left(-\frac{d(z)/2}{d(z) - d(z)/4}\right) \inf_{B_R(z)} u(y) = \exp\left(-\frac{2}{3}\right) \inf_{B_R(z)} u(y)$$

and $\exp\left(-\frac{2}{3}\right) \approx 0.5134 > 0.3333 \approx \frac{1}{3}$.

The local Lipschitz regularity for $\infty-$harmonic functions is now a consequence of the Harnack inequality.

Theorem 2.9. *If $u \in C(U)$ is $\infty-$harmonic, then it is locally Lipschitz and hence differentiable almost everywhere.*

Proof. We know u satisfies (2.11) since it enjoys comparison with cones from above. Take $z \in U$, $R < d(z)/4$ and $x, y \in B_R(z)$. Assume first that $u \leqslant 0$. Then (2.13) and the Harnack inequality (2.12) hold, and we get

$$\begin{aligned} u(x) - u(y) &\leqslant -u(y)\frac{|x-y|}{d(y)} \\ &\leqslant -\inf_{B_R(z)} u \, \frac{|x-y|}{3R} \\ &\leqslant -\sup_{B_R(z)} u \, \frac{|x-y|}{R}. \end{aligned}$$

If u is not non-positive, then this holds with u replaced by

$$v = u - \sup_{B_{4R}(z)} u \leqslant 0,$$

since $v = u + const$ still enjoys comparison with cones from above. We thus obtain

$$\begin{aligned} u(x) - u(y) = v(x) - v(y) &\leqslant -\sup_{B_R(z)} v \, \frac{|x-y|}{R} \\ &= \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u\right)\frac{|x-y|}{R} \end{aligned}$$

and, interchanging x and y,

$$|u(x) - u(y)| \leqslant \frac{1}{R}\left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u\right)|x-y|.$$

$\square$

The best regularity result to date is due to Evans and Smart 2011 and asserts that $\infty-$harmonic functions are differentiable *everywhere* (see chapter 3). It remains an outstanding open problem to prove the C^1 or $C^{1,\alpha}$ regularity (see chapter 4), which are known to hold only in two dimensions after the breakthroughs of Savin 2005 and Evans and Savin 2008.

2.4 Existence

It is now time to deal with existence. We will need the following result; a proof is in Aronsson, Crandall, and Juutinen 2004.

Lemma 2.10. *Let $\mathcal{F} \subset C(U)$ be a family of functions that enjoy comparison with cones from above in U. Suppose*

$$h(x) = \sup_{v \in \mathcal{F}} v(x)$$

is finite and locally bounded above in U. Then $h \in C(U)$, and it enjoys comparison with cones from above in U.

The existence result holds for U unbounded if the boundary function f is allowed to grow at most linearly at infinity. Note that it also settles the existence for problem (1.3) since u is $\infty-$harmonic in U if, and only if, $u \in AML(U)$.

Theorem 2.11. *Let $U \subset \mathbb{R}^n$ be open, $0 \in \partial U$ and $f \in C(\partial U)$. Let $A^{\pm}, B^{\pm} \in \mathbb{R}$, $A^+ \geqslant A^-$ and*

$$A^-|x| + B^- \leqslant f(x) \leqslant A^+|x| + B^+, \quad \forall x \in \partial U. \tag{2.15}$$

There exists $u \in C(\overline{U})$ which is $\infty-$harmonic in U and satisfies $u = f$ on ∂U. Moreover,

$$A^-|x| + B^- \leqslant u(x) \leqslant A^+|x| + B^+, \quad \forall x \in \overline{U}. \tag{2.16}$$

The proof is an application of Perron's method. By translation, we can always assume $0 \in \partial U$ so this assumption is not restrictive and is used to simplify the notation.

We start by defining two functions $\underline{h}, \overline{h} : \mathbb{R}^n \to \mathbb{R}$ by

$$\underline{h}(x) = \sup\{\underline{C}(x) \, : \, \underline{C}(x) = a|x - z| + b, a < A^-, z \in \partial U, \underline{C} \leqslant f \text{ on } \partial U\}$$

and

$$\overline{h}(x) = \inf\{\overline{C}(x) \, : \, \overline{C}(x) = a|x - z| + b, a > A^+, z \in \partial U, \overline{C} \geqslant f \text{ on } \partial U\}$$

with the properties stated in the next lemma.

Lemma 2.12. *The functions $\underline{h}$ and $\overline{h}$ are well-defined and continuous. Moreover,*

$$A^-|x| + B^- \leqslant \underline{h}(x) \leqslant \overline{h}(x) \leqslant A^+|x| + B^+, \quad \forall x \in \mathbb{R}^n, \tag{2.17}$$

$$\underline{h} = \overline{h} = f \text{ on } \partial U, \tag{2.18}$$

$\underline{h}$ enjoys comparison with cones from above and $\overline{h}$ enjoys comparison with cones from below.

Proof. We argue for $\overline{h}$; similar arguments hold for $\underline{h}$. First, observe that any cone in the family that is used to define $\overline{h}$ is bounded below by the cone

$$A^+|x-z|+f(z),$$

and then so is $\overline{h}$. The question is to show that the family is non-empty. Since $0 \in \partial U$, we may take

$$\overline{C}(x)=(A^++\epsilon)|x|+B^+, \quad \epsilon>0$$

and so $\overline{h}$ is well defined. This also readily implies that $\overline{h}(x) \leqslant A^+|x|+B^+$.

To show that (2.18) holds for $\overline{h}$, fix $0 \neq z \in \partial U$ and $\epsilon > 0$. By the continuity of f, there exists $\delta > 0$ such that

$$f(x) < f(z)+\epsilon, \quad \forall x \in B_\delta(z) \cap \partial U. \tag{2.19}$$

Then choose $a > \max\{A^+, 0\}$ such that

$$f(z)+\epsilon+a\delta > \max_{\overline{B_\delta(z)}} (A^+|x|+B^+) \tag{2.20}$$

and

$$f(z)+\epsilon+a|z| > B^+. \tag{2.21}$$

Define the cones

$$\overline{C}(x) := a|x-z|+f(z)+\epsilon$$

and

$$C^+(x) := A^+|x|+B^+,$$

and the open set

$$W := \left\{x \in \mathbb{R}^n \setminus \overline{B_\delta}(z) \;:\; \overline{C}(x) < C^+(x)\right\}.$$

Since $a > A^+$, W is bounded:

$$\lim_{|x|\to\infty} \left(a|x-z|-A^+|x|\right) = +\infty.$$

Moreover, by (2.20), $\partial B_\delta(z) \cap \overline{W} = \emptyset$, and then $\overline{C} = C^+$ on ∂W. Since both vertices of the cones, 0 and z, do not belong to W (to see that $0 \notin W$, use (2.21)), we conclude, by Corollary 1.12 and the reasoning at the end of the proof of Theorem 1.15, that $\overline{C} = C^+$ also in W. Thus $W = \emptyset$ and

$$\overline{C}(x) \geqslant C^+(x), \quad \forall x \in \mathbb{R}^n \setminus B_\delta(z).$$

This and (2.19) implies that

$$\overline{C} \geqslant f \ \text{ on } \partial U.$$

Now, $\overline{C}(z) = f(z)+\epsilon$ and so $\overline{h}(z) = f(z)$ and (2.18) holds for $\overline{h}$.

To prove that $\underline{h} \leqslant \overline{h}$, take any two cones

$$\underline{C}(x) = \underline{a}|x - \underline{z}| + \underline{b} \qquad \text{and} \qquad \overline{C}(x) = \overline{a}|x - \overline{z}| + \overline{b}$$

entering in the definition of $\underline{h}$ and $\overline{h}$, respectively. Since

$$\underline{C} \leqslant f \leqslant \overline{C} \ \text{ on } \partial U,$$

$$\underline{z}, \overline{z} \in \partial U$$

and

$$\overline{a} > A^{+} \geqslant A^{-} > \underline{a},$$

the set where $\underline{C} > \overline{C}$ is bounded, contains neither vertex and the two cones agree on its boundary. Arguing as before, we conclude the set is empty, and so $\underline{C} \leqslant \overline{C}$, which implies

$$\underline{h} \leqslant \overline{h} \ \text{ in } R^n$$

and (2.17) is proved. In particular, $\underline{h}$ and $\overline{h}$ are locally bounded, and comparison with cones (respectively, from above and from below) follows from Lemma 2.10 and its variant.

We are left to prove the continuity of $\underline{h}$ and $\overline{h}$. First, observe that $\underline{h}$ is lower semi-continuous (as the supremum of continuous functions) and $\overline{h}$ is upper semi-continuous (as the infimum of continuous functions) in R^n. The continuity in U also follows from Lemma 2.10. Since $\underline{h}$ enjoys comparison with cones from above, the idea is to use the Harnack Inequality (Lemma 2.7, which holds for lower semi-continuous functions) as in the proof of Theorem 2.9.

To prove the continuity of $\underline{h}$ on ∂U, use (2.18) and (2.17) to get

$$f(x) \leqslant \liminf_{y \to x} \underline{h}(y) \leqslant \limsup_{y \to x} \underline{h}(y) \leqslant \limsup_{y \to x} \overline{h}(y) \leqslant f(x), \quad x \in \partial U.$$

The case of $\overline{h}$ is treated analogously. □

We need yet another lemma.

Lemma 2.13. *Suppose $u \in C(U)$ enjoys comparison with cones from above in U but does not enjoy comparison with cones from below in U. Then, there exists a non-empty set $W \subset\subset U$ and a cone $C(x) = a|x - z| + b$, with $z \notin W$, such that $u = C$ on ∂W, $u < C$ on W and the function $\hat{u}$ defined by*

$$\hat{u} = u \ \text{ in } U \setminus W \qquad \text{and} \qquad \hat{u} = C \ \text{ in } W \tag{2.22}$$

enjoys comparison with cones from above in U. Moreover, if u is Lipschitz in U, then so is $\hat{u}$ and

$$\mathrm{Lip}_{\hat{u}}(U) \leqslant \mathrm{Lip}_u(U).$$

Proof. That there exist W and C satisfying the conditions of the lemma follows from the proof of the necessity in Theorem 1.15, more correctly, from its variant corresponding to comparison with cones from below.

Let's show that $\hat{u}$ defined by (2.22) enjoys comparison with cones from above in U. Suppose not; then, again from the proof of Theorem 1.15, there exists a non-empty set $\widetilde{W} \subset\subset U$ and a cone $\widetilde{C}(x) = \widetilde{a}|x - \widetilde{z}| + \widetilde{b}$, with $\widetilde{z} \notin \widetilde{W}$, such that $\hat{u} = \widetilde{C}$ on $\partial\widetilde{W}$ and $\hat{u} > \widetilde{C}$ in $\widetilde{W}$. Since u enjoys comparison with cones from above in U and $u \leqslant \hat{u} = \widetilde{C}$ on $\partial\widetilde{W}$, we have $u \leqslant \widetilde{C}$ also in $\widetilde{W}$. This implies that $\widetilde{W} \subset W$ because

$$u \leqslant \widetilde{C} < \hat{u} \text{ in } \widetilde{W} \qquad \text{and} \qquad u = \hat{u} \text{ in } U \setminus W.$$

Thus, on $\partial\widetilde{W} \subset W \cup \partial W$,

$$\widetilde{C} = \hat{u} = C.$$

Since the vertices of the cones C and $\widetilde{C}$ are outside $\widetilde{W}$, this implies

$$\widetilde{C} \equiv C \equiv \hat{u} \text{ in } \widetilde{W}$$

and so $\widetilde{W} = \emptyset$, a contradiction.

Finally, since

$$\mathrm{Lip}_{\hat{u}}(W) = \mathrm{Lip}_C(W) = \mathrm{Lip}_C(\partial W) = \mathrm{Lip}_u(\partial W) \leqslant \mathrm{Lip}_u(U)$$

(note that the vertex of C is outside W), we conclude that

$$\mathrm{Lip}_{\hat{u}}(U) = \max\{\mathrm{Lip}_{\hat{u}}(W), \mathrm{Lip}_{\hat{u}}(U \setminus W)\} \leqslant \mathrm{Lip}_u(U).$$

□

We are now ready to prove Theorem 2.11.

Proof. Define

$$u(x) := \sup\Big\{v(x) \,:\, \underline{h} \leqslant v \leqslant \overline{h} \text{ and } v \in CCA(U)\Big\}, \quad x \in \overline{U},$$

where, by $v \in CCA(U)$ we mean that v enjoys comparison with cones from above in U.

By Lemma 2.12, the set includes $\underline{h}$ so it is not empty and u is well defined; it follows from Lemma 2.10 that it enjoys comparison with cones from above in U, and, from (2.18), that $u \in C(\overline{U})$ and $u = f$ on ∂U.

If u enjoys comparison with cones, then u is ∞−harmonic, and the proof is complete. Otherwise, u does not enjoy comparison with cones from below and, by Lemma 2.13, there exists a non-empty set $W \subset\subset U$ and a cone C, with vertex outside W, such that

$$u = C \text{ on } \partial W \qquad \text{and} \qquad u < C \text{ in } W,$$

and a continuous function $\hat{u}$, enjoying comparison with cones from above, such that

$$\hat{u} = u \ \text{ in } U \setminus W \qquad \text{and} \qquad \hat{u} = C \ \text{ in } W.$$

It is obvious that $\underline{h} \leqslant u \leqslant \hat{u}$. We also claim that $\hat{u} \leqslant \overline{h}$ in U, which contradicts the definition of u.

Since $\overline{h}$ enjoys comparison with cones from below and

$$\overline{h} \geqslant C = u \ \text{ on } \partial W,$$

we have $\overline{h} \geqslant C = \hat{u}$ also in W. Since in $U \setminus W$, $\hat{u} = u \leqslant \overline{h}$, the proof is complete. □

2.5 Uniqueness

The uniqueness reveals the extent to which the notion of viscosity solution is the appropriate one to deal with the $\infty-$Laplace equation. Given any PDE, we can, of course, define any reasonable notion of solution; what makes the difference is that, for that notion, not only existence but also uniqueness holds.

The uniqueness of $\infty-$harmonic functions remained open for more than two decades before Jensen settled it in Jensen 1993 using the full machinery of viscosity solutions. The proof we will next present is much simpler and exploits the equivalence between being $\infty-$harmonic and enjoying comparison with cones. It is a surprisingly easy and beautiful proof due to Armstrong and Smart Armstrong and Smart 2010.

We start with some notation. Given an open and bounded subset $U \subset \mathbb{R}^n$ and $r > 0$, let

$$U_r := \left\{ x \in U \ : \ \overline{B_r}(x) \subset U \right\}.$$

For $u \in C(U)$ and $x \in U_r$, define

$$u^r(x) := \max_{\overline{B_r}(x)} u \qquad \text{and} \qquad u_r(x) := \min_{\overline{B_r}(x)} u,$$

and let

$$S_r^+ u(x) = \frac{u^r(x) - u(x)}{r} \qquad \text{and} \qquad S_r^- u(x) = \frac{u(x) - u_r(x)}{r}.$$

Note that both $S_r^+ u \geqslant 0$ and $S_r^- u \geqslant 0$.

The first result we prove is a comparison principle at the discrete level for the finite difference equation $S_r^- u = S_r^+ u$.

Lemma 2.14. *Assume $u, v \in C(U) \cap L^\infty(U)$ satisfy*

$$S_r^- u(x) - S_r^+ u(x) \leqslant 0 \leqslant S_r^- v(x) - S_r^+ v(x), \ \forall x \in U_r. \tag{2.23}$$

Then

$$\sup_U (u - v) = \sup_{U \setminus U_r} (u - v).$$

Proof. Suppose the thesis does not hold, *i.e.*,

$$\sup_U (u - v) > \sup_{U \setminus U_r} (u - v).$$

The set

$$E := \left\{ x \in U \ : \ (u - v)(x) = \sup_U (u - v) \right\}$$

is then non-empty, closed, and contained in U_r. Define

$$F := \left\{ x \in E \ : \ u(x) = \max_E u \right\},$$

which is also non-empty and closed, and select a point $x_0 \in \partial F$. Since $u - v$ attains its maximum at x_0 (because $x_0 \in \overline{F} = F \subset E$), we have

$$S_r^- v(x_0) \leqslant S_r^- u(x_0) \ \Leftrightarrow \ u_r(x_0) - v_r(x_0) \leqslant (u - v)(x_0), \tag{2.24}$$

which holds since $\max(f - g) \geqslant \min f - \min g$.[1]

We now consider two cases.

1. $S_r^+ u(x_0) = 0$: from (2.23), we get

$$S_r^- u(x_0) \leqslant 0 \ \Rightarrow \ S_r^- u(x_0) = 0$$

and, from (2.24),

$$S_r^- v(x_0) \leqslant 0 \ \Rightarrow \ S_r^- v(x_0) = 0.$$

Using the other inequality in (2.23),

$$0 \leqslant 0 - S_r^+ v(x_0) \ \Rightarrow \ S_r^+ v(x_0) = 0.$$

So

$$\max_{\overline{B_r}(x_0)} u = u(x_0) = \min_{\overline{B_r}(x_0)} u$$

[1]In fact, $\max(f - g) = \max_x (f(x) - g(x)) \geqslant \max_x (\min f - g(x)) = \min f - \min g$.

and

$$\max_{\overline{B_r}(x_0)} v = v(x_0) = \min_{\overline{B_r}(x_0)} v,$$

and both u and v are constant in $\overline{B_r}(x_0)$. Thus $B_r(x_0) \subset F$; in fact, if $y \in B_r(x_0)$ then, since $x_0 \in E$,

$$(u - v)(y) = (u - v)(x_0) = \sup_U (u - v) \;\Rightarrow\; y \in E;$$

but also

$$u(y) = u(x_0) = \max_E u,$$

since $x_0 \in F$; thus $y \in F$. We conclude that $x_0 \in int(F)$ and so $x_0 \notin \partial F$, a contradiction.

2. $S_r^+ u(x_0) > 0$: select a point $z \in \overline{B_r}(x_0)$ such that

$$r S_r^+ u(x_0) = u(z) - u(x_0).$$

Since $u(z) > u(x_0)$ and $x_0 \in F$, we see that $z \notin E$. From this, we deduce that

$$r S_r^+ v(x_0) \geqslant v(z) - v(x_0) > u(z) - u(x_0) = r S_r^+ u(x_0). \tag{2.25}$$

To justify the strict inequality above, observe that

$$(u - v)(z) \leqslant (u - v)(x_0) = \sup_U (u - v),$$

because $x_0 \in F \subset E$, and equality does not hold since then $z \in E$. Finally, combining (2.24) and (2.25), we get

$$S_r^- v(x_0) - S_r^+ v(x_0) < S_r^- u(x_0) - S_r^+ u(x_0),$$

which contradicts (2.23).

□

The following result establishes a link between the continuous and the discrete levels, showing that solutions of the PDE can be suitably modified to solve the finite difference equation.

Lemma 2.15. *If $u \in C(U)$ is $\infty-$subharmonic in U, then*

$$S_r^- u^r(x) - S_r^+ u^r(x) \leqslant 0, \quad \forall x \in U_{2r},$$

and if $v \in C(U)$ is $\infty-$superharmonic in U, then

$$S_r^- v_r(x) - S_r^+ v_r(x) \geqslant 0, \quad \forall x \in U_{2r}.$$

Proof. We just prove the first statement; the second one follows from the fact that $(-v)^r = -v_r$.

Fix a point $x_0 \in U_{2r}$. Select $y_0 \in \overline{B_r}(x_0)$ and $z_0 \in \overline{B_{2r}}(x_0)$ such that

$$u(y_0) = u^r(x_0) \qquad and \qquad u(z_0) = u^{2r}(x_0).$$

Then,

$$\begin{aligned} r\left[S_r^- u^r(x_0) - S_r^+ u^r(x_0)\right] &= 2u^r(x_0) - (u^r)_r(x_0) - (u^r)^r(x_0) \\ &\leqslant 2u^r(x_0) - u^{2r}(x_0) - u(x_0) \\ &= 2u(y_0) - u(z_0) - u(x_0). \end{aligned}$$

We next justify why the inequality holds.

1. $(u^r)^r(x) = u^{2r}(x)$: we have

$$(u^r)^r(x) = \max_{z \in \overline{B_r}(x)} u^r(z) = \max_{z \in \overline{B_r}(x)} \max_{y \in \overline{B_r}(z)} u(y)$$

and

$$u^{2r}(x) = \max_{y \in \overline{B_{2r}}(x)} u(y).$$

• If $z \in \overline{B_r}(x)$ and $y \in \overline{B_r}(z)$ then $y \in \overline{B_{2r}}(x)$. In fact,

$$|z - x| \leqslant r \ \wedge \ |y - z| \leqslant r$$

$$\Rightarrow |y - x| \leqslant |y - z| + |z - x| \leqslant 2r$$

and thus

$$(u^r)^r(x) \leqslant u^{2r}(x).$$

• If $y \in \overline{B_{2r}}(x)$ then $y \in \overline{B_r}(z)$, for a certain $z \in \overline{B_r}(x)$; just take z to be the middle point of the segment $[x, y]$. So, also

$$u^{2r}(x) \leqslant (u^r)^r(x).$$

2. $(u^r)_r(x) \geqslant u(x)$: we have

$$(u^r)_r(x) = \min_{z \in \overline{B_r}(x)} u^r(z) = \min_{z \in \overline{B_r}(x)} \max_{y \in \overline{B_r}(z)} u(y).$$

Since

$$\max_{y \in \overline{B_r}(z)} u(y) \geqslant u(x), \ \forall z \in \overline{B_r}(x),$$

the result follows.

Now, clearly,

$$u(w) \leqslant u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|w - x_0|, \quad \forall w \in \partial\left(B_{2r}(x_0) \setminus \{x_0\}\right).$$

Since u enjoys comparison with cones from above, because u is ∞−subharmonic, the inequality also holds for every $w \in B_{2r}(x_0) \setminus \{x_0\}$ and so for every $w \in \overline{B_{2r}}(x_0)$.

Putting $w = y_0$ and using the fact that $|y_0 - x_0| \leqslant r$, we get

$$\begin{aligned} u(y_0) &\leqslant u(x_0) + \frac{u(z_0) - u(x_0)}{2r}|y_0 - x_0| \\ &\leqslant u(x_0) + \frac{u(z_0) - u(x_0)}{2} \end{aligned}$$

and then

$$2u(y_0) - u(x_0) - u(z_0) \leqslant 0,$$

and the proof is complete. □

Theorem 2.16 (Jensen's Uniqueness Theorem)**.** *Let $u, v \in C(\overline{U})$ be, respectively, ∞−subharmonic and ∞−superharmonic. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v).$$

Proof. From Lemmas 2.14 and 2.15,

$$\sup_{U_r}(u^r - v_r) = \sup_{U_r \setminus U_{2r}}(u^r - v_r), \quad \forall r > 0.$$

To get the result, let $r \downarrow 0$ and use the local uniform convergence of u^r and v_r to u and v, respectively. □

3 Differentiability everywhere

In this chapter, we discuss ideas from the works Crandall, Evans, and Gariepy 2001 as well as Crandall and Evans 2001. We shall explore the relationship between the comparison with cones property and various findings on linearity for blow-ups and differentiability results.

3.1 Monotonicity properties and consequences

3.1.1 Definitions and main properties

Let $U \subset \mathbb{R}^n$ be an open set and $u : U \to \mathbb{R}$. Similarly to what is done in Section 2.5, we define the quantities

$$L_r^+(u, y) := \max_{z \in S_r(y)} \frac{u(z) - u(y)}{r} \quad \text{and} \quad L_r^-(u, y) := \min_{z \in S_r(y)} \frac{u(z) - u(y)}{r},$$

for $y \in U$ and $r > 0$. Here, we denote

$$S_r(y) = \{x \in \mathbb{R}^n : |x - y| = r\},$$

assuming $r < dist(y, \partial U)$ to guarantee $S_r(y) \subset U$.

Moreover, we define

$$L_0^+(u, y) := \lim_{r \to 0} L_r^+(u, y) \quad \text{and} \quad L_0^-(u, y) := \lim_{r \to 0} L_r^-(u, y), \tag{3.1}$$

and invite the reader to compare these definitions with that of $T_u(y)$, introduced in Section 1.3.

The following result concerns the monotonicity of L_r^+ with respect to r. It follows as a consequence of u enjoying comparison with cones from above.

Proposition 3.1. *(Monotonicity properties) Let u enjoy comparison with cones from above in U. Then, for each $y \in U$ and $r \leqslant \operatorname{dist}(x, \partial U)$,*

$$r \mapsto L_r^+(u, y) \text{ is non-decreasing.} \tag{A}$$

In addition,

$$L_0^+(u, y) \text{ is a non-negative number.} \tag{B}$$

Proof of property (A). For each $y \in S_r(x)$, one holds

$$\frac{u(y) - u(x)}{r} \leqslant \max_{y \in S_r(z)} \frac{u(z) - u(x)}{r} =: L_r^+(u, x).$$

and so,

$$u(y) \leqslant u(x) + L_r^+(u, x)|y - x|$$

for $y \in S_r(x)$. From the fact that u enjoys comparison with cones from above, the inequality also holds for each $y \in B_r(x) \setminus \{x\}$ and so also in $B_r(x)$. In other words, we obtain

$$u(y) \leqslant u(x) + L_r^+(u, x)|y - x|$$

for each $y \in S_\tau(x)$ and $\tau \leqslant r$, which is equivalent to

$$\frac{u(y) - u(x)}{\tau} = \frac{u(y) - u(x)}{|y - x|} \leqslant L_r^+(u, x),$$

for each $y \in S_\tau(x)$. Therefore, $L_s^+(u, x) \leqslant L_r^+(u, x)$ whenever $s \leqslant r$. □

Similarly, we can prove exercise 13.

Proof of property (B). Without loss of generality, we assume $y = 0$. Initially, we prove that

$$-M \leqslant \lim_{r \to 0} L_r^+(u, 0), \quad \text{provided} \quad \lim_{r \to 0} L_r^+(u, 0) < M. \tag{3.2}$$

Then, if $\lim_{r \to 0} L_r^+(u, 0) < 0$, we apply (3.2) for the particular case $M = 0$ and get a contradiction.

In the sequel, we prove that (3.2) holds. From property (A), we find $\overline{r} > 0$ such that, for each $r \leqslant \overline{r}$, we have $L_r^+(u, 0) \leqslant M$. From this,

$$u(y) \leqslant u(0) + M|y| \quad \text{for each } \; y \in S_r(0). \tag{3.3}$$

In what follows, select $x \in B_{\overline{r}}(0)$ such that $0 < |x| \leqslant \overline{r}/3$. First, we observe that

$$u(0) \leqslant u(x) + L^+_{|x|}(u,x)|x| \leqslant u(x) + \max_{z \in S_r(x)} \frac{u(z) - u(x)}{r}|x|,$$

for any $r \geqslant |x|$. In addition, we take $x_r \in S_r(x)$ such that

$$\frac{u(x_r) - u(x)}{r} = L^+_r(u,x).$$

By using (3.3) for $y = x_r$,

$$u(x_r) \leqslant u(0) + M|x_r| \leqslant u(0) + M(|x| + r),$$

Therefore, from the last three estimates, we get

$$u(0) \leqslant u(x) + \frac{u(0) + M(|x| + r) - u(x)}{r}|x|,$$

and so

$$u(0) \leqslant u(x) + \left(\frac{M(|x| + r)}{r} - \frac{u(x) - u(0)}{r} \right) |x|.$$

This gives us

$$-\frac{M(|x| + r)}{r - |x|}|x| \leqslant -\frac{M(|x| + r)}{r}|x| \leqslant (u(x) - u(0)) \left(1 - \frac{|x|}{r} \right).$$

Note that we are free to select any $|x| = \varepsilon$, see Figure 3.1,

$$-M \frac{(\varepsilon + r)}{r - \varepsilon} \leqslant \max_{|x| = \varepsilon} \frac{u(x) - u(0)}{\varepsilon} \left(1 - \frac{\varepsilon}{r} \right).$$

Finally, we let $\varepsilon \to 0$, obtaining

$$-M \leqslant L^+_0(u,0).$$

□

3.1.2 Inferring existence of derivatives

Next, we show some properties assuming pointwise differentiability for u.

Proposition 3.2. *Let $p \in \mathbb{R}^n$ and $a \in \mathbb{R}$. For affine functions $u(x) = a + \langle p, x \rangle$, we have*

$$L^+_s(u,y) = L^+_r(u,y) = -L^-_r(u,y) = |p|,$$

for each $y \in U$ and $0 < r < s < \mathrm{dist}(y, \partial U)$.

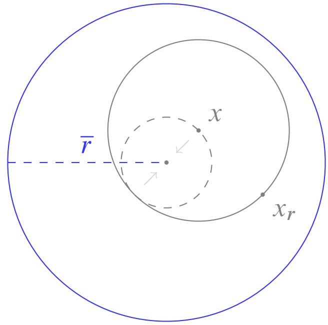


Figure 3.1: The picture shows how $S_{|x|}(0)$ and $S_r(x)$ are located into $B_{\overline{r}}(0)$.

Proof. The second equation follows by observing that if z_r is the maximum point for u in $S_r(y)$, then $-z_r$ is the minimum for u in $S_r(y)$. More generally, since u is affine, we observe that there exists $e \in S_1(0)$, such that

$$0 \leqslant L_r^+(u, y) = \frac{p \cdot (y + re) - p \cdot y}{r} = p \cdot e,$$

which proves the first and the last estimates. □

The following proposition should be compared with Proposition 1.7.

Proposition 3.3. *Assume u is differentiable at $y \in U$. Then*

$$L_0^+(u, y) = |Du(y)|. \tag{3.4}$$

Proof. In case $|Du(y)| \neq 0$, denote

$$\nu = \frac{Du(y)}{|Du(y)|}.$$

From this,

$$|Du(y)| = Du(y) \cdot \nu = \lim_{h \to 0} \frac{u(y + h\nu) - u(y)}{h} \leqslant \lim_{h \to 0} L_h^+(u, y) = L_0^+(u, y). \tag{3.5}$$

In parallel, for each integer $j > 0$, consider $x_j \in S_1(0)$, such that

$$L_{\frac{1}{j}}(u, y) = \max_{x \in S_1(0)} j(u(y + x/j) - u(y)) = j(u(y + x_j/j) - u(y)).$$

In addition, up to a subsequence, we have that $y_j \to y' \in S_1(0)$. Hence,

$$\begin{aligned} j(u(y+\tfrac{1}{j}x_j)-u(y)) &\leqslant j|u(y+\tfrac{1}{j}y_j)-u(y+\tfrac{1}{j}y')|+j|u(y+\tfrac{1}{j}y')-u(y)| \\ &\leqslant Lip_u(U)|y_j-y'|+j|u(y+\tfrac{1}{j}y')-u(y)|, \end{aligned}$$

and so, letting $j \to \infty$, we derive

$$L_0^+(u,y) \leqslant Du(y)\cdot y' \leqslant |Du(y)|\,|y'| = |Du(y)|.$$

Therefore, from the estimate above and (3.5), we conclude (3.4).

Finally, if $|Du(y)| = 0$, for a given $a \in \mathbb{R}^n \setminus \{0\}$ we consider

$$v(x) = u(x) + a\cdot x.$$

We have that $Dv(y) = a$. Therefore, from Proposition 3.2, one holds

$$L_0^+(u,y)+|a| = |L_0^+(u,y)+|a|| = |a|,$$

which implies that $L_0^+(u,y) = 0 = |Du(y)|$. □

We also point out that, from Rademacher's theorem, see for instance Weaver 2018, Theorem 1.41, u is differentiable at almost every point of U, provided u is Lipschitz in U.

Corollary 3.4. *Assume u enjoys comparison with cones from above in U. Then,*

$$\sup_{y\in U} L_0^+(u,y) = \|Du\|_{L^\infty(U)}. \tag{3.6}$$

3.1.3 Some useful consequences

In the sequel, we explore some consequences of the results above. First, we obtain a gradient control at almost every point in U. In addition, essentially from property (A), $\infty-$harmonic functions are locally Lipschitz continuous. Alternatively, from Proposition 3.1, the same holds only requiring comparison with cones from above.

Proposition 3.5. *Assume that u enjoys comparison with cones from above. Then $u \in W^{1,\infty}_{loc}(U)$. Furthermore, there exists a constant C, depending only on n, such that for radii $r \leqslant dist(x,\partial U)$, there holds*

$$|Du(x)| \leqslant L_0^+(u,x) \leqslant \max_{z\in S_r(x)} \frac{u(z)-u(x)}{r} \leqslant 2\frac{\|u\|_{L^\infty(U)}}{dist(x,\partial U)}, \quad a.e.\ x \in U.$$

Proof. Let $x, y \in U$ such that for $r := |x - y|$, we have $S_r(x) \cup S_r(y) \subset U$. From property (A), we immediately obtain

$$u(x) - u(y) \leqslant L_r^+(u, y)|x - y| \quad \text{and} \quad u(y) - u(x) \leqslant L_r^+(u, x)|x - y|.$$

Hence,

$$|u(x) - u(y)| \leqslant \max\{L_r^+(u, y), L_r^+(u, x)\}|x - y|,$$

so u is locally Lipschitz. By Rademacher's theorem, u is differentiable at almost every point of U. Let $x \in U$ be a Lebesgue point for u (which means u is differentiable at x). From the inequality above,

$$|Du(x)| \leqslant \overline{\lim_{r\to 0}} \left(\overline{\lim_{y\to x}} \max\{L_r^+(u, y), L_r^+(u, x)\} \right) = \overline{\lim_{r\to 0}} L_r^+(u, x) = L_0^+(u, x).$$

Here, we only use the upper semi-continuity of $r \mapsto L_0^+(u, r)$. The upper semi-continuity property follows from the continuity of u and the fact that $L_r^+(u, y)$ is evaluated by taking the maximum of u on $S_r(y)$. □

Corollary 3.6. *Assume u enjoys comparison with cones from above in $\mathbb{R}^n$, and $u(x) \leqslant a + \langle p, x\rangle$, for some $a \in \mathbb{R}$ and $p \in \mathbb{R}^n$. Then $|Du| \leqslant |p|$ a.e. in $\mathbb{R}^n$.*

Proof. Assume, without loss of generality that $u(0) = 0$. From Proposition 3.5,

$$|Du(x)| \leqslant \max_{z\in S_r(x)} \frac{u(z)}{r} \leqslant \max_{z\in S_r(x)} \frac{|a| + |p|(|x| + r)}{r}, \quad \text{a.e. in } \mathbb{R}^n.$$

Letting $r \to \infty$, we conclude the proof. □

We conclude the first part of this chapter by exploiting the ideas in the proof of Proposition 3.1, property (B).

Corollary 3.7. *Let u enjoy comparison with cones from above and below. Then, for each $y \in U$, there holds*

$$L_0^+(u, y) = -L_0^-(u, y).$$

Proof. Without loss of generality, assume $y = 0$. From (3.2),

$$-(L_0^+(u, 0) + \varepsilon) \leqslant L_0^-(u, 0), \tag{3.7}$$

for each $\varepsilon > 0$ arbitrary chosen, and so, $-L_0^-(u, 0) \leqslant L_0^+(u, 0)$. Arguing similarly for $-u$, which has the property of comparison with cones from above, and applying exercise 11, we derive $-L_0^+(u, 0) \geqslant L_0^-(u, 0)$. □

3.2 Blow-up analysis

As in the previous section, we assume u enjoys comparison with cones (from above and below). For each $x \in U$, a *blow-up* of u at x is the function

$$u_x(y) := \lim_{k\to\infty} \frac{u(r_k y) - u(x)}{r_k},$$

where $\{r_k\}_k$ is a sequence of positive numbers such that $r_k \to 0$. From property (A), for each integer $k > 0$, the function

$$u_{x,k}(y) := \frac{u(r_k y) - u(x)}{r_k},$$

is locally uniformly Lipschitz, so the limit above exists. We follow ideas in Crandall and Evans 2001 to show that blow-ups of functions enjoying comparison with cones (form above and below) must be linear.

For convenience, we assume $0 \in U$ and proceed with the blow-up analysis at the origin. Next, we prove some properties of $L_r^{\pm}(\cdot, y)$ related to blow-up functions.

Proposition 3.8. *Let u enjoy comparison with cones from above and below. Then, the following property holds:*

$$\max\left\{L_r^+(u_0, y), -L_r^-(u_0, y)\right\} \leqslant L_0^+(u,0), \quad \textit{for each } \; y \in \mathbb{R}^n \textit{ and } r > 0. \tag{C}$$

In addition,

$$L_0(u,0) := L_0^+(u,0) = L_0^+(u_0,0) = -L_0^-(u_0,0). \tag{D}$$

Proof of property (C). Fix any $y \in \mathbb{R}^n$ and let $z_r \in S_r(y)$ be such that

$$L_r^+(u_0, y) = \frac{u_0(z_r) - u_0(y)}{r} = \lim_{j\to\infty} \frac{u(r_j z) - u(r_j y)}{r_j r}. \tag{3.8}$$

From Proposition 3.1, property (A), for $r_j r \leqslant R < dist(r_j y, \partial U)$,

$$\frac{u(r_j z) - u(r_j y)}{r_j r} \leqslant L_{r_j r}^+(u, r_j y) \leqslant L_R^+(u, r_j y).$$

Letting $j \to \infty$ in the estimate above, we obtain

$$L_r^+(u_0, y) = \lim_{j\to\infty} \frac{u(r_j z) - u(r_j y)}{r_j r} \leqslant L_R^+(u,0).$$

Now, letting $R \to 0$, we conclude that

$$L_r^+(u_0, y) \leqslant L_0^+(u,0).$$

□

Proof of property (D). Consider $y = 0$ and $u_0(0) = 0$ in (3.8) to get

$$L_r^+(u_0, 0) = \frac{u_0(z_r)}{r} = \lim_{j\to\infty} \frac{u(r_j z_r) - u(0)}{r_j r}.$$

Hence, from Corollary 3.7 and exercise 17

$$-L_0^-(u_0, 0) = L_0^+(u_0, 0) = \lim_{j\to\infty} L_{r_j r}^+(u, 0) = L_0^+(u, 0).$$

□

Corollary 3.9. *Let u enjoy comparison with cones from above and below. Then,*

$$L_0(u, 0) = \|Du_0\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. From Proposition 3.1, property (A),

$$\|Du_0\|_{L^\infty(\mathbb{R}^n)} \leqslant \sup_{y\in\mathbb{R}^n} L_r^+(u_0, y).$$

Hence, from Proposition 3.8, property (C), we obtain

$$\|Du_0\|_{L^\infty(\mathbb{R}^n)} \leqslant L_0(u_0, 0).$$

On the other hand, from Corollary 3.4, we easily get

$$L_0(u_0, 0) \leqslant \sup_{y\in\mathbb{R}^n} L_0^+(u_0, y) = \|Du_0\|_{L^\infty(\mathbb{R}^n)}.$$

□

The following technical result concerns linearity properties for Lipschitz functions in $\mathbb{R}^n$ under tightness on a given line. This is crucial for obtaining classification results of blow-up functions.

Lemma 3.10. *Assume $v : \overline{B_1} \to \mathbb{R}$ satisfies $Lip_v(\overline{B_1}) = 1$, and there exists $e \in S_1(0)$, such that $v(te) = t$, for each $t \in (-1, 1)$. Then $v(x) = e \cdot x$, for each $x \in B_1$.*

Proof. Let us first denote $(x, y) \in B_1 \subset \mathbb{R}^n$, for $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Up to a rotation, we can assume, with no loss of generality, that $e = (1, 0)$. Given $(x_1, y), (s, 0) \in B_1$, we have

$$\begin{aligned}
|v(x_1, y) - v(s, 0)|^2 &= |v(x_1, y) - v(x_1, 0) + v(x_1, 0) - v(s, 0)|^2 \\
&= 2(v(x_1, y) - v(x_1, 0))(v(x_1, 0) - v(s, 0)) \\
&\quad + |v(x_1, y) - v(x_1, 0)|^2 + |v(x_1, 0) - v(s, 0)|^2 \\
&\geqslant 2(v(x_1, y) - v(x_1, 0))(v(x_1, 0) - v(s, 0)) + |x_1 - s|^2.
\end{aligned}$$

On the other hand, since $Lip_v = 1$,

$$|v(x_1, y) - v(s, 0)|^2 \leqslant |y|^2 + |x_1 - s|^2 \quad \text{for each } \; s \in \mathbb{R},$$

which implies

$$2(v(x_1, y) - v(x_1, 0))(x_1 - s) \leqslant |y|^2 \quad \text{for each } \; s \in \mathbb{R}.$$

Since $x_1 < 1$ and $s \in (-1, 1)$ is arbitrary, we deduce

$$v(x_1, y) = v(x_1, 0) = x_1 v(e) = x_1,$$

which means that v depends only on the first variable. □

Lemma 3.11. *Let* $g : [0, 1] \to \mathbb{R}$ *be such that* $\mathrm{Lip}_g([0, 1]) = |g(1) - g(0)|$. *Then*

$$g(t) = g(1)t + g(0), \quad \forall t \in [0, 1].$$

Proof. To prove this, without loss of generality, we assume $g(0) = 0$. Hence,

$$\frac{|g(t)|}{t} \leqslant \frac{|g(t) - g(0)|}{t} \leqslant |g(1) - g(0)| = |g(1)|.$$

On the other hand,

$$\frac{|g(1)| - |g(t)|}{1 - t} \leqslant \frac{|g(1) - g(t)|}{1 - t} \leqslant |g(1)|.$$

This implies that $|g(t)| - |g(1)| \geqslant -|g(1)|(1 - t)$, and so $|g(t)| \geqslant |g(1)|t$. □

Next, we prove the main result of this section.

Theorem 3.12. *Let u enjoy comparison with cones from above and below in U. Then, u_0 is an affine function.*

Proof. For the sake of simplicity, we assume that $u(0) = 0$. For each radius r, we select points $z_r^+, z_r^- \in S_r(0)$, such that

$$L_r^+(u_0, 0) = \frac{u_0(z_r^+) - u_0(0)}{r} = \frac{u_0(z_r^+)}{r} \quad \text{and} \quad L_r^-(u_0, 0) = \frac{u_0(z_r^-)}{r}. \tag{3.9}$$

By monotonicity, we have that $L_0^+(u_0, 0) \leqslant L_r^+(u_0, 0)$. Hence, from properties (C) and (D) in Proposition 3.8 and exercise 17, we obtain $L_0(u, 0) = L_0(u_0, 0) = L_r^+(u_0, 0)$. Additionally, we have $-L_r^-(u_0, 0) = L_0(u, 0)$. Hence,

$$L_0(u, 0) = \frac{u_0(z_r^+) - u_0(z_r^-)}{2r}. \tag{3.10}$$

Next, consider $g_r : [0,1] \to \mathbb{R}$ defined by

$$g_r(t) = u_0(z_r^- + t(z_r^+ - z_r^-)) - u_0(z_r^-). \tag{3.11}$$

From Proposition 3.8, property (C), since g has Lipschitz constant $\overline{L}$, we obtain

$$\begin{aligned}
\overline{L} := \sup_{t,s\in(0,1)} \frac{|g_r(t)-g_r(s)|}{|t-s|} &\leqslant L^+_{|t-s||z_r^+-z_r^-|}(u_0, z_r^- + s(z_r^+ - z_r^-))|z_r^+ - z_r^-| \\
&\leqslant L^+_{2r}(u_0, z_r^- + s(z_r^+ - z_r^-))|z_r^+ - z_r^-| \\
&\leqslant L_0(u,0)|z_r^+ - z_r^-| \\
&\leqslant L_0(u,0)2r.
\end{aligned}$$

On the other hand, from (3.10), we have

$$\overline{L} \geqslant |g_r(1) - g_r(0)| = |u_0(z_r^+) - u_0(z_r^-)| = 2rL_0(u,0).$$

From the last two estimates, we conclude

$$|z_r^+ - z_r^-|L_0(u,0) = \overline{L} = |g_r(1) - g_r(0)| = 2rL_0(u,0).$$

From this, we deduce some consequences. First, since $|z_r^+ - z_r^-| = 2r$ and $z_r^+, z_r^- \in S_r(0)$, we have that z_r^+ and z_r^- are diametrically opposed, which means

$$z_r^+ = -z_r^-. \tag{3.12}$$

Thus, for each $r > 0$, we have that z_r^+ and z_r^- are uniquely determined. Second, we observe that $|g_r(1) - g_r(0)|$ is the Lipschitz constant of g_r, and so, from Lemma 3.11, we get

$$g_r(t) = 2rL_0(u,0)\,t. \tag{3.13}$$

Consequently, from (3.11), (3.12) and (3.13), we obtain

$$2L_0(u,0)\,t = u_0((2t-1)z_1^+) - u(-z_1^+), \tag{3.14}$$

and so, taking $t = 1$ and $t = 1/2$, we conclude that

$$u(z_1^+) = L_0(u,0).$$

In the sequel, denote $e := z_1^+$. We claim that

$$u_0(te) = tL_0(u,0), \quad \text{for each } 0 \leqslant t \leqslant 1. \tag{3.15}$$

In fact, in the case we find $t_\star \in (0,1)$, such that

$$u_0(t_\star e) < t_\star L_0(u,0),$$

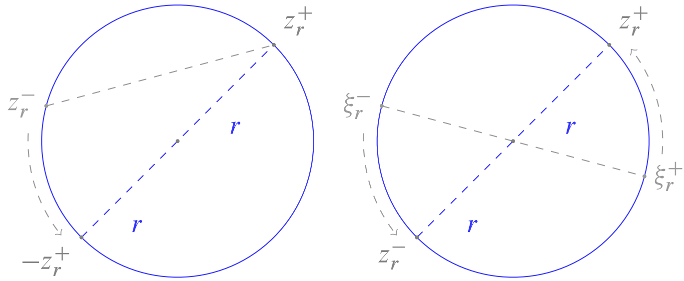


Figure 3.2: Points z_r^+ and z_r^- are diametrically opposed and unique.

then, from Corollary 3.9, we derive

$$u_0(e) - u_0(t_\star e) \leqslant (1 - t_\star)\|Du_0\|_{L^\infty(\mathbb{R}^n)} = (1 - t_\star)L_0(u, 0)$$

and so,

$$u_0(e) < (1 - t_\star)L_0(u, 0) + u_0(t_\star e) \leqslant L_0(u, 0) = u_0(e),$$

which is a contradiction.

From (3.14) and (3.15), we also have

$$u(-z_1^+) = -L_0(u, 0).$$

Hence, arguing as above, we obtain

$$u_0(se) = sL_0(u, 0), \quad \text{for each } \; -1 \leqslant s \leqslant 1.$$

Therefore, we conclude that u_0 is affine on some line crossing the origin. By Lemma 3.10, we obtain that u_0 is affine in B_1.

To conclude the proof, we define

$$\lambda_\star := \max\{\lambda \geqslant 0 \mid u_0(x) = L_0(u, 0)x, \; \text{ for each } |x| \leqslant \lambda\}.$$

Note that $\lambda_\star \geqslant 1$. Now, assume $\lambda_\star < +\infty$, and define

$$u_\star(x) := \frac{u_0((\lambda_\star + 1)x)}{\lambda_\star + 1}.$$

From exercise 15, note that $u_\star$ is a blow-up for u at 0. Hence, from the arguments above

$$u_\star(x) = L_0(u, 0)x, \quad \text{for each } x \in B_1.$$

From this, we easily obtain $u_0(x) = L_0(u, 0)x$, for each $x \in B_{\lambda_\star + 1}$. However, this contradicts the definition above. Therefore, $\lambda_m = \infty$, which is equivalent to obtaining that u_0 is affine in $\mathbb{R}^n$. $\square$

In conclusion, the previous arguments permit us to state the following Liouville-type property.

Lemma 3.13 (Liouville)**.** *Let u enjoy comparison with cones from above and below in $\mathbb{R}^n$ and satisfies properties* (C) *and* (D)*. Then u must be linear.*

3.3 Everywhere differentiability

This section addresses everywhere differentiability of functions that enjoy comparison with cones. It is worth noting that, according to Rademacher's theorem, Lipschitz functions are differentiable almost everywhere. The pioneering work on this topic was done by Crandall and Evans 2001.

3.3.1 Preiss' example

Initially, one could expect that Lipschitz functions, with the property that any blow-up is affine, would be differentiable. However, this is not the case. In ibid., the authors exhibit a counter-example due to D. Priess. Consider

$$P(x) = \begin{cases} x \cdot \sin\left(\log\left(|\log(|x|)|\right)\right) & \text{for} \quad x \in (-1,0) \cap (0,1), \\ 0 & \text{for} \quad x = 0. \end{cases}$$

This function is differentiable everywhere except at the origin and Lipschitz near that point. However, all blow-up limits $u(r_j x)/r_j$ at the origin, as $r_j \to 0$, are affine. Precisely, the blow-ups might exhibit any slope between -1 and 1.

3.3.2 An equivalence for differentiability

Next, we provide the first result towards differentiability, assuming the condition below. Choosing $z_r \in S_r(x_0)$ such that

$$L_r^+(u, x_0) = \max_{z \in S_r(x_0)} \frac{u(z) - u(x_0)}{r} = \frac{u(z_r) - u(x_0)}{r},$$

we shall assume that

$$\lim_{r \to 0} \frac{|z_r - x_0|}{r} \quad \text{exists.} \tag{Lim}$$

We point out that differentiability at x_0 implies that (*Lim*) holds, provided $|Du(x_0)| > 0$. In fact, for $x \in S_r(x_0)$, we consider Taylor's expansion for u at x_0,

$$u(x) = u(x_0) + \langle Du(x_0), x - x_0 \rangle + o(r).$$

From this, we obtain

$$0 \leqslant \frac{u(z_r) - u(x)}{r} = \langle Du(x_0), \frac{z_r - x}{r} \rangle + o(1),$$

for each $x \in S_r(x_0)$. From exercise 18, we conclude the validity of (*Lim*).

Next, we show the equivalence between differentiability and condition (*Lim*). Hereafter, we say u has differentiability condition at a point x_0 if, and only if, condition (*Lim*) holds.

Theorem 3.14. *Assume u satisfies* (*Lim*) *and enjoys comparison with cones in a given domain. Then u is differentiable everywhere.*

Proof. Assume that $x_0 = 0$ and $v(0) = 0$. We shall split the argument into two parts. First, we consider the case $L_0(u, 0) = 0$. Then

$$\frac{|u(x) - u(0)|}{|x|} \leqslant \max\left(L_r^+(u, 0), -L_r^-(u, 0)\right),$$

for $0 < |x| < r$. From this, we conclude that $Du(0) = 0$. More generally, we leave it as an exercise to show that $L_0(u, 0) = 0$ if, and only if, $Du(0) = 0$. Next, we treat the case $L_0(u, 0) > 0$. For a sequence $r_j \to 0$, we consider

$$v(x) = \lim_{j\to\infty} \frac{u(r_j x) - u(0)}{r_j}. \tag{3.16}$$

From Theorem 3.12, there exists $p \in \mathbb{R}^n$, such that

$$v(x) = \langle p, x \rangle.$$

From properties (C) and (D), we have that $|p| = |Dv(0)| = L_0(u, 0)$. To guarantee that u is differentiable at 0, we shall guarantee that p is unique for any blow-up of u at 0. From (*Lim*), we denote

$$\lim_{r\to 0} \frac{z_r}{r} =: \xi \in S_1(0).$$

Then, for each $x \in S_1(0)$, we have

$$v(x) = \lim_{j\to\infty} \frac{u(r_j x) - u(0)}{r_j} \leqslant \lim_{j\to\infty} \frac{u\left(r_j \frac{z_{r_j}}{r_j}\right) - u(0)}{r_j} = v(\xi).$$

In view of this, $\langle p, x \rangle \leqslant \langle p, \xi \rangle$, for each $x \in S_1(0)$, which implies $\xi = p/|p|$. $\square$

3.3.3 Differentiability via uniqueness of blow-ups

Next, we treat differentiability for $\infty-$harmonic functions - remember the equivalence with comparison with cones, see Theorem 2.6. For this, we shall prove that any limit

$$\lim_{r\to 0}\frac{u(ry-x)-u(x)}{r}=\langle a,y\rangle$$

is obtained uniquely for $a=Du(x)$ and $L_0(u,x)=|Du(x)|$. We proceed from ideas in Evans and Smart 2011, omitting some technical details, which can be found precisely in that reference.

Theorem 3.15. *Let u be a viscosity solution of*

$$-\Delta_\infty u=0 \quad in \quad B_1(0).$$

Then u is differentiable at each point in $B_1(0)$.

A flatness property for an auxiliary equation

To prove this, we will consider the following perturbed Dirichlet problem involving the $\infty-$Laplacian,

$$\begin{cases} L_\varepsilon[v]:=-\Delta_\infty v-\varepsilon\Delta v=0 & \text{in} \quad B_1(0),\\ \qquad\qquad\qquad\qquad v=u & \text{on} \quad \partial B_1(0).\end{cases} \tag{3.17}$$

For each $\varepsilon>0$, there exists a unique smooth solution u_ε to (3.17). From ibid., Theorem 2.1, for some universal $C>0$, depending only on dimension and $\|u\|_\infty$, we have

$$\sup_{B_1(0)}|u_\varepsilon|+\sup_{B_{1/2}(0)}|Du_\varepsilon|\leqslant C_1. \tag{3.18}$$

The idea is to differentiate the equation in (3.17), observing that

$$v_\varepsilon:=(|Du_\varepsilon|^2+u^2)/2$$

satisfies $L_\varepsilon[v_\varepsilon]\gtrsim 0$. From this and Arzelà–Ascoli, we conclude that u_ε converge in B_1 uniform in ε. By uniqueness, see Theorem 2.6, we have

$$\lim_{\varepsilon\to 0}u_\varepsilon=u.$$

The core of the argument is the following lemma, which proves that solutions u_ε are universally close to an affine function provided the n-flatness assumption (3.19) holds. The proof can be found in ibid., Theorem 2.2.

Lemma 3.16. *For $\lambda > 0$ small enough, assume*

$$\sup_{B_1(0)} |u_\varepsilon - x_n| \leqslant \lambda. \tag{3.19}$$

Then

$$\sup_{B_{1/4}(0)} |Du_\varepsilon|^2 - D_n u_\varepsilon \leqslant C_2 \left(\lambda^{1/2} + \left(\frac{\varepsilon}{\lambda}\right)^{1/2}\right), \tag{3.20}$$

for C_2 depending only on dimension and C_1, but not depending on ε.

We also need the following technical lemma.

Lemma 3.17. *Consider $b \in S_1(0)$ and let $w : B_\rho(0) \to \mathbb{R}$ be a smooth function satisfying*

$$\max_{y \in B_\rho(0)} \frac{|w(y) - b \cdot y|}{\rho} \leqslant \eta,$$

for some $\eta > 0$. Then, there exists $x \in B_\rho(0)$, such that

$$|Dw(x) - b| \leqslant 4\eta.$$

Proof. We first argue the case $\rho = 1$. Defining $h(y) := b \cdot y - 2\eta|y|^2$, we note that $(w-h)(0) \leqslant \eta$. By the triangle inequality, we have $(w-h)(y) = w(y) - b \cdot y + 2\eta \geqslant \eta$, for each $y \in S_1(0)$. Therefore, $w - h$ attains a minimum at some interior point $x \in B_1(0)$, which implies

$$|Dw(x) - b| = |Dh(x) - b| = 4\eta|x|.$$

To conclude the proof, we consider $w_\rho : B_1(0) \to \mathbb{R}$, given by $w_\rho(y) = w(\rho y)/\rho$, and apply the previous analysis. □

Proof of Theorem 3.15:

We shall proceed with the arguments at the origin. According to Theorem 3.12, consider sequences of positive numbers $\{r_j\}_{j\geqslant 1}$ and $\{s_j\}_{j\geqslant 1}$, converging to zero, such that

$$\left|\frac{u(r_j y)}{r_j} - \langle a, y\rangle\right| + \left|\frac{u(s_j y)}{s_j} - \langle b, y\rangle\right| \longrightarrow 0, \quad \text{as} \quad j \to 0, \tag{3.21}$$

for $a, b \in \mathbb{R}^n$. From exercise 17, we observe that $|a| = |b| = L_0(u, 0)$. As commented in the proof of Theorem 3.14, if $|a| = 0$, we already have $a = b = Du(0) = 0$. We now assume $a, b \in \mathbb{R}^n \setminus \{0\}$, such that $a \neq b$. Without loss of generality, let us consider in addition

$$a = e_n = (0, \cdots, 0, 1), \quad |b| = 1 \quad \text{with} \quad b \neq e_n.$$

Denote $b = (b_1, \cdots, b_n)$ and

$$\theta_b := 1 - |b_n|. \tag{3.22}$$

Note that since $|b| = 1$, we easily have $\theta_b > 0$.

Next, for C_2 as in Lemma 3.16, we choose

$$\lambda = \left(\frac{\theta_b}{8C_2}\right)^2.$$

From (3.21), we take a radius $r_\lambda = r$, such that

$$\max_{B_r(0)} \frac{|u(y) - y_n|}{r} \leqslant \frac{\lambda}{2}.$$

Since u_ε converges uniformly to u, we consider $\varepsilon_1 > 0$ small enough, such that

$$\max_{B_r(0)} \frac{|u_\varepsilon(y) - y_n|}{r} \leqslant \lambda, \tag{3.23}$$

for each $0 < \varepsilon \leqslant \varepsilon_1$. In addition, using (3.21) once more, we find a small ρ, such that

$$\max_{B_\rho(0)} \frac{|u(y) - b \cdot x|}{\rho} \leqslant \frac{\theta_b}{96}.$$

Hence, for some ε_2, we get

$$\max_{B_\rho(0)} \frac{|u_\varepsilon(y) - b \cdot x|}{\rho} \leqslant \frac{\theta_b}{48},$$

for any $0 < \varepsilon \leqslant \varepsilon_2 \leqslant \varepsilon_1$.

From now on, we consider $\varepsilon = \min\{\varepsilon_2, \lambda^2\}$. We use the last estimate and apply Lemma 3.17, thus obtaining

$$|Du_\varepsilon(x) - b| \leqslant \frac{\theta_b}{12},$$

for some point $x \in B_\rho(0)$. In particular,

$$|D_n u_\varepsilon(x) - b_n| \leqslant \frac{\theta_b}{12}. \tag{3.24}$$

By the reverse triangle inequality, we also have

$$1 - \frac{\theta_b}{12} \leqslant |Du_\varepsilon(x_0)|. \tag{3.25}$$

Now, we are ready to apply Lemma 3.16. From the previous choices and (3.23),

$$|Du_\varepsilon(x)|^2 \leqslant D_n u_\varepsilon(x) + 2C_2\lambda^{1/2} = D_n u_\varepsilon(x) + \frac{\theta_b}{4}.$$

However, from (3.24) and (3.25), we derive

$$\left(1-\frac{\theta_b}{12}\right)^2 \leqslant b_n + \frac{\theta_b}{12} + \frac{\theta_b}{4}.$$

Finally, taking into account (3.22), we get

$$\begin{aligned}
\theta_b = 1-|b_n| &\leqslant 1+\frac{\theta_b}{3}-\left(1-\frac{\theta_b}{12}\right)^2 \\
&\leqslant \frac{\theta_b}{3}+\frac{\theta_b}{6}-\frac{\theta_b^2}{144} \\
&\leqslant \frac{(48+24+1)\theta_b}{144} \\
&= \frac{73}{144}\theta_b,
\end{aligned} \tag{3.26}$$

which is a contradiction since we assumed $\theta_b > 0$.

4 Beyond differentiability

This chapter delves into the issue of the optimal regularity of viscosity solutions to the $\infty-$Laplace equation, both in the homogeneous and the non-homogeneous scenarios.

We start by pointing out that in general, a differentiable function might not have continuous derivatives. For example, the function

$$g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable, with derivative

$$g'(x) = \begin{cases} -\cos\left(\dfrac{1}{x}\right) + 2x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

but $g'(x)$ is not continuous at zero since $\cos(1/x)$ oscillates as $x \to 0$. Differentiable functions whose derivatives are continuous are called continuously differentiable. The class of such functions is denoted by C^1.

4.1 $\infty-$harmonic functions

One of the main open problems in the modern theory of PDEs is whether $\infty-$harmonic functions or, equivalently, functions that enjoy comparison with cones, see Theorem 2.6, are continuously differentiable. This conjecture has been answered positively by Savin 2005 in the plane. Evans and Savin 2008 sharpened the result to $C^{1,\alpha}$ for some universal and small $\alpha > 0$, but still in dimension two. More precisely, for a universal $C > 0$, one can show that

$$|Du(x) - Du(y)| \leqslant C|x-y|^{\alpha}, \tag{4.1}$$

for u $\infty-$harmonic in the plane. Nevertheless, no continuity feature of Du can be inferred from their reasoning. The famous example of the $\infty-$harmonic function

$$\mathcal{A}(x_1, x_2) := x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

due to Aronsson, in the late 1960s, sets the ideal optimal regularity theory for such a problem. That is, no universal regularity theory for $\infty-$harmonic functions can go beyond $C^{1,\frac{1}{3}}$.

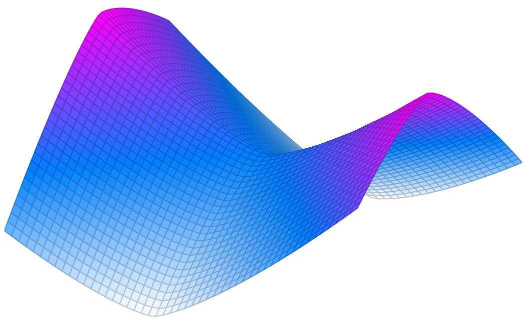


Figure 4.1: Aronson's function in the plane.

Up to our knowledge, it remains entirely open for debate whether $\infty-$harmonic functions satisfy universal $C^{1,\frac{1}{3}}$ regularity estimates.

4.2 The inhomogeneous case

A very natural generalization of the theory of the $\infty-$Laplacian arises in the inhomogeneous case

$$\Delta_{\infty} u = f(x). \tag{4.2}$$

Lu and Wang 2008 show existence, uniqueness, and stability results for viscosity solutions of the Dirichlet problem, assuming that f does not change sign. Lindgren 2014, assuming

f is bounded and continuous, obtained that any blow-up is linear, which is the inhomogeneous counterpart of Theorem 3.12. The author follows the same strategy of Crandall and Evans 2001, obtaining an almost monotonicity property for the following function

$$r \mapsto L_r^+(u, x_0) + r,$$

see Lindgren 2014, Corollary 1. As previously observed, linear blow-ups are not necessarily unique. Given this, assuming f of class C^1, the author proves the uniqueness of blow-ups, which is equivalent to showing that solutions are differentiable.

A possible way to deal with $C^{1,\alpha}$ regularity of solutions to (4.2) would be by exploring the scaling properties of the equation. For instance, if one writes the $\infty-$Laplacian as

$$\Delta_\infty u = (Du)^t \cdot D^2u \cdot Du,$$

it becomes tempting to compare the degeneracy feature of (4.2) with

$$|Du|^2 \cdot \Delta u \sim f \in L^\infty. \tag{4.3}$$

This equation has the same scaling invariance as (4.2). This fact might be closely related to obtaining estimate (4.1) for $\alpha = 1/3$. Assuming u is a viscosity solution of (4.2) in $B_1(0)$, we note that, for positive parameters κ and θ, the rescaled function

$$u_\kappa(x,t) = \frac{u(\kappa x)}{\kappa^\theta} \quad \text{in} \quad B_{1/\kappa}$$

solves, in the viscosity sense (see Definition 2.2),

$$\Delta_\infty u_\kappa = \kappa^{2-\theta+2(1-\theta)} f(\kappa x) = f(\kappa x) =: f_\kappa(x),$$

under the choice $\theta = 4/3$. It is worth noting that f_κ is bounded, as much as f. We arrive at the same conclusion by considering equation (4.3) instead of (4.2).

In relation to equation (4.3), we observe that the uniform ellipticity property of the Laplacian Δ degenerates along the set of critical points

$$\mathcal{C}(u) = \{x \,:\, Du(x) = 0\}.$$

Furthermore, it is reasonable to extend the study of equations of the form (4.2) to the more general context of

$$|Du|^\gamma F(D^2u) = f \in L^\infty, \tag{4.4}$$

for a parameter $\gamma > 0$ and F a concave uniformly elliptic fully-nonlinear operator (see Caffarelli and Cabré 1995 for the definition). Recently, Imbert and Silvestre 2013 proved that solutions to (4.4) are in fact $C^{1,\alpha}$, for α universally small. The optimal $C^{1,\alpha}$ regularity result for the degenerate equation (4.4) was obtained by Araújo, Ricarte, and Teixeira 2015. Precisely, the authors show that estimate (4.1) holds, for

$$\alpha = \frac{1}{1+\gamma}. \tag{4.5}$$

Applying the result above, we consider $\infty-$harmonic functions with separable variables. Notice that Aronsson's example $\mathcal{A}$ is a function of this class.

Theorem 4.1. *Let $u : B_1 \to \mathbb{R}$ be $\infty-$harmonic and assume u is a function of separable variables,*

$$u(x) = \sigma_1(x_1) + \sigma_2(x_2) + \cdots + \sigma_n(x_n)$$

for σ_i continuous in B_1. Then $u \in C^{1,\frac{1}{3}}\left(B_{1/2}\right)$.

Proof. A formal direct computation gives

$$\Delta_\infty u = \left|\sigma_1'(x_1)\right|^2 \sigma''(x_1) + \left|\sigma_2'(x_2)\right|^2 \sigma''(x_2) + \cdots + \left|\sigma_d'(x_d)\right|^2 \sigma''(x_d). \tag{4.6}$$

It is a matter of routine to justify the above computation using the viscosity solutions machinery. We notice, however, that the i-th term in (4.6) depends only upon the variable x_i. Thus, since they sum up to zero, each of them must be constant,

$$\left|\sigma_i'(x_i)\right|^2 \sigma''(x_i) = \tau_i, \quad \sum_{i=1}^{d} \tau_i = 0.$$

Taking $\gamma = 2$ in (4.5), we obtain the $C^{1,\frac{1}{3}}$-regularity of each σ_i. □

In a number of geometrical problems, it is often the case that solutions behave asymptotically radially near singular points. It is therefore interesting to analyze the regularity theory for solutions that are smooth up to a possible radial singularity. More precisely, a function u is called *smooth up to a possible radial singularity at a point x_0* if we can write,

$$u(x) = \varphi(x) + \psi(|x - x_0|) \qquad \text{near } x_0,$$

with $\varphi \in C^2$ and $\psi(x) = O(|x - x_0|^2)$. In the sequel, we shall prove that functions smooth up to a possible radial singularity and whose $\infty-$Laplacian is bounded in the viscosity sense are of class $C^{1,\frac{1}{3}}$. This regularity is optimal as

$$\Delta_\infty |x|^{\frac{4}{3}} = \text{cte}.$$

Theorem 4.2. *Let u satisfy* (4.2) *in $B_1(0)$, in the viscosity sense, and assume u is smooth up to a possible radial singularity. Then u is of the class $C^{1,\frac{1}{3}}$ in $B_{1/2}(0)$.*

Proof. Without loss of generality, we can assume $x_0 = 0$. If $u = \varphi(x) + \psi(|x|)$ is smooth up to a radial singularity near the origin, then formally, a direct computation yields

$$Du(x) = D\varphi(x) + \psi'(|x|)\frac{x}{|x|}$$

and

$$D^2u(x) = D^2\varphi(x) + \frac{1}{|x|^2}\psi''(|x|)x \otimes x + \psi'(|x|)\left[\frac{1}{|x|}\mathrm{Id} - \frac{1}{|x|^3}x \otimes x\right].$$

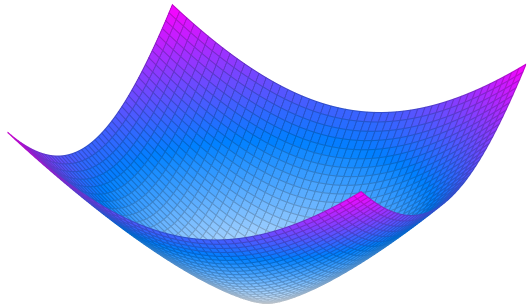


Figure 4.2: The radial function $|x|^{\frac{4}{3}}$.

Owing to the estimates

$$|x|^{-2}|\varphi| + |x|^{-1}|D\varphi| + \left|D^2\varphi\right| \leqslant C_1, \quad |\psi| + |\psi'| \leqslant C_2,$$

and

$$|\Delta_\infty u| \leqslant C_3,$$

we end up with

$$\left(\mathrm{O}(r) + \left|\psi'\right|^2\right) \cdot \left|\psi''\right| \leqslant C_4. \tag{4.7}$$

Since ψ is radial, see exercise 4, we derive

$$\begin{aligned} \Delta_\infty \psi(|x|) &= \sum_{i,j=1}^{n} D_{ij}\psi(|x|)D_i\psi(|x|)D_j\psi(|x|) \\ &= \psi''(|x|)\psi'(|x|)^2 \textstyle\sum_{i,j=1}^{n} \dfrac{x_i x_j}{|x|^4} \\ &\quad + \psi'(|x|)^3 \textstyle\sum_{i,j=1}^{n} \left[\dfrac{x_i x_j \delta_{ij}}{|x|^3} - \dfrac{x_i^2 x_j^2}{|x|^5}\right] \\ &= \psi''(|x|)\psi'(|x|)^2. \end{aligned} \tag{4.8}$$

From this and (4.7), we reach

$$|\Delta_\infty \psi| = |\psi''(\psi')^2| \leqslant C_5,$$

which gives the desired regularity for ψ, applying (4.5) for $\gamma = 2$. □

5 Free boundary problems ruled by the ∞–Laplacian

Free boundary problems (FBPs) belong to a class of mathematical problems characterized by the presence of partial differential equations that are satisfied in regions or domains that depend on the solution itself. In these problems, the boundaries between different regions or domains are not predefined but are determined as part of the solution process. Applications of free boundary problems include the study of fluid flows with evolving interfaces, phase transitions, optimal control problems, financial modelling, shape optimization, and biological processes such as tumour growth or population dynamics. For classical references on the topic of free boundary problems, we recommend the works Petrosyan, Shahgholian, and Uraltseva 2012, Caffarelli and Salsa 2005, Rodrigues 1987, and Friedman 1982.

In this chapter, we shall focus on the study of free-boundary problems with reaction-diffusion equations governed by the ∞–Laplacian

$$\Delta_\infty u = f(x,u) \quad \text{in} \quad B_1 \cap \{u > 0\}, \tag{5.1}$$

$$u \geqslant 0 \quad \text{in} \quad B_1. \tag{5.2}$$

Hereafter, we denote $\{u > 0\} := \{x \in B_1 \, : \, u(x) > 0\}$. Our analysis consists of obtaining improved $C^{1,\alpha}$ regularity at the free boundary

$$\partial\{u > 0\},$$

for an optimal exponent $\alpha > 0$. We also highlight that the results below have no dimensional restriction, so according to the results mentioned in Section 4.1, they are stronger than those obtained locally in higher dimensions.

5.1 Obstacle problems for the $\infty-$Laplacian

In this section, we turn our analysis towards optimal regularity estimates at free boundary points for the following obstacle-type problem:

$$\min\{\Delta_\infty u - f(x), u\} = 0 \quad \text{in } B_1. \tag{5.3}$$

This problem is equivalent to the zero-obstacle problem (5.1), in which solutions are understood in the viscosity sense. The case of the Laplacian is pretty well understood and appears as the Euler–Lagrange equation of the functional

$$\mathcal{J}(u) = \int \frac{1}{2}|Du|^2 + fu\,dx.$$

We refer to Petrosyan, Shahgholian, and Uraltseva 2012 for a complete account of the study of obstacle-type problems. Rossi, Teixeira, and Urbano 2015 study problem (5.3), obtaining existence, uniqueness, and regularity at free boundary points. Later, Araújo, Leitão, and Teixeira 2016 study problem (5.1) with γ-strong absorption $f(x,u) = (u_+)^\gamma$, obtaining, in particular, that non-negative solutions are surprisingly smoother along the boundary of the non-coincidence set.

Our goal here is to present the results in Rossi, Teixeira, and Urbano 2015, showing in particular that solutions to (5.3) grow precisely as

$$[\text{dist}(x, \partial\{u > 0\})]^{4/3}$$

away from the free boundary. This highlights the $C^{1,\frac{1}{3}}$ regularity at free boundary points, in the sense of estimate (4.1). In fact, given $x \in B_1$, consider $x' \in \partial\{u > 0\}$ such that $|x - x'| = \text{dist}(x, \partial\{u > 0\})$. From Corollary 5.4 below, we conclude that

$$|Du(x) - Du(x')| \leqslant C|x - x'|^{1/3}.$$

We shall work under the assumption that $f(x)$ is continuous and bounded away from zero and infinity, *i.e.*, for some $M > 1$ there holds

$$1/M \leqslant f(x) \leqslant M \quad \text{for each } x \in B_1. \tag{5.4}$$

Such a condition is natural in the context of obstacle-type problems and allows us to prove existence and uniqueness for problem (5.3) using a Perron-type method.

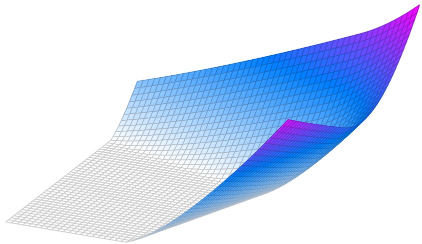


Figure 5.1: Solution of a free boundary problem with zero-obstacle.

Theorem 5.1. *Given a function $g \in C(\partial B_1)$, with $g > 0$, and f satisfying* (5.4)*, there exists a unique function $u \in C(\overline{B_1})$, satisfying*

$$\begin{cases} \min\{\Delta_\infty u - f(x), u\} = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases} \tag{5.5}$$

in the viscosity sense. Assuming further that f is uniformly Lipschitz continuous in B_1, then u is locally Lipschitz continuous in B_1.

Proof. The existence, uniqueness, and continuity up to the boundary follow as in Lu and Wang 2008 (see Rossi, Teixeira, and Urbano 2015 for further details). The solution is given by

$$u(x) := \inf_{v \in \mathcal{A}^+_{f,g}} v(x), \quad \text{for } x \in \overline{B_1}, \tag{5.6}$$

where

$$\mathcal{A}^+_{f,g} := \left\{ v \in C(\overline{B_1}) \;\middle|\; v \geqslant 0,\ \Delta_\infty v \leqslant f(x) \text{ in } B_1, \text{ and } v \geqslant g \text{ on } \partial B_1 \right\}. \tag{5.7}$$

Let us concentrate on the local Lipschitz continuity of u. Locally in $\{u > 0\}$, u satisfies $\Delta_\infty u \in L^\infty$ in the viscosity sense, thus u is locally Lipschitz continuous in the non-coincidence set (see, for instance, Lindgren 2014, Corollary 2). Hence, the result needs only to be proven near the free boundary. By continuity of u and the fact that $g > 0$ on ∂B_1, there exists a small number $\tau_0 > 0$ such that $u > 0$ in $B_1 \setminus B_{1-\tau_0}$. Arguing as before, we find a constant $C > 0$, depending on M and τ_0, such that

$$|Du(x)| < C, \quad \forall x \in B_{1-\frac{\tau_0}{5}} \setminus B_{1-\frac{\tau_0}{10}}. \tag{5.8}$$

For any vector ν, with $|\nu| < \frac{\tau_0}{100}$, define σ_ν by

$$\sigma_\nu^3 := \inf_{B_{1-\frac{\tau_0}{100}}} \frac{f(x)}{f(x+\nu)}.$$

Since f is strictly positive and Lipschitz continuous, it follows that

$$|1-\sigma_\nu| + |1-\sigma_\nu^3| \leqslant K_0|\nu|.$$

In the sequel, let us label $r_0 := 1 - \frac{3}{20}\tau_0$ and define $u_\nu\colon B_{r_0} \to \mathbb{R}$ by

$$u_\nu(x) := \sigma_\nu u(x+\nu) + \left(C + K_0 \sup_{B_1} u\right)|\nu|.$$

We now apply the analysis from the beginning of this proof to the domain B_{r_0}. One simply verifies that u_ν belongs to the set

$$\widetilde{\mathcal{A}}^+_{f,g} := \left\{ v \in C(\overline{B_{r_0}}) \;\middle|\; v \geqslant 0,\ \Delta_\infty v \leqslant f(x) \text{ in } B_{r_0}, \text{ and } v \geqslant u \text{ on } \partial B_{r_0} \right\}.$$

By uniqueness, $u|_{B_{r_0}}$ is the infimum among all functions in $\widetilde{\mathcal{A}}^+_{f,g}$. Thus, we can write, for any $x \in B_{r_0}$,

$$u_\nu(x) \geqslant u(x),$$

which immediately yields

$$u(x+\nu) - u(x) \geqslant -\left(C + 2K_0 \sup_{B_1} u\right)|\nu|$$

and the local Lipschitz estimate for u follows. □

We remark that assuming only the boundedness of $f(x)$, the local Lipschitz continuity of the solution to the infinity obstacle problem is a consequence of the following lemma.

Lemma 5.2. *Let* (5.4) *be in force and let u be the viscosity solution to the obstacle problem* (5.5). *Then*

$$|\Delta_\infty u| \leqslant M,$$

in the viscosity sense.

Proof. The idea of the proof is to perform a singular approximation of the obstacle problem. Let ζ be a non-negative real C^1 function satisfying $\operatorname{supp}\zeta = [0,1]$ and $\int \zeta(t)dt = 1$. For each $\epsilon > 0$, consider the penalized boundary value problem

$$\begin{cases} \Delta_\infty u_\epsilon = f(x)\displaystyle\int_0^{u_\epsilon/\epsilon} \zeta(t)\,dt & \text{in} \quad B_1 \\[2ex] \qquad\qquad u_\epsilon = g & \text{on} \quad \partial B_1. \end{cases} \tag{5.9}$$

Notice that the reaction term

$$\beta(x, u_\epsilon) := f(x)\int_0^{u_\epsilon/\epsilon} \zeta(t)\,dt \tag{5.10}$$

is monotone non-decreasing with respect to u_ϵ. Hence, as before, using a Perron-type method (see Bhattacharya and Mohammed 2011, 2012), the Dirichlet problem (5.9) is uniquely solvable. Clearly,

$$|\beta(x, u_\epsilon)| \leqslant M.$$

Thus, it follows from Lipschitz estimates (*cf.*, for instance, Lindgren 2014, Corollary 2) and uniform continuity up to the boundary that the family $\{u_\epsilon\}_{\epsilon>0}$ is equicontinuous in B_1. By the Arzelà–Ascoli Theorem, u_ϵ converges uniformly, up to a subsequence, to a function v. The limiting function v is non-negative, agrees with g on the boundary and satisfies $|\Delta_\infty v| \leqslant M$, in the viscosity sense. In particular, v is locally Lipschitz continuous in B_1. Now, given a point $z \in \{v > 0\} \cap B_1$, by the triangular inequality, one easily checks that

$$B := B_{\frac{v(z)}{2L}}(z) \subset \left\{ v > \frac{v(z)}{2} > 0 \right\},$$

where L is the Lipschitz norm of v on $B_{1-|z|}$. In particular,

$$\Delta_\infty u_\epsilon = f(x) \text{ in } B,$$

for all $\epsilon < \frac{v(z)}{2}$. By stability, we deduce that $\Delta_\infty v = f(x)$ in B as well. Since $z \in \{v > 0\}$ was taken arbitrary, it follows that v satisfies $\Delta_\infty v = f(x)$ in $\{v > 0\}$. We have verified that v solves the same boundary value problem as u. Thus, by uniqueness, $u = v$, and the lemma is proven. □

We are now ready for our main result, which gives the optimal $C^{1,\frac{1}{3}}$−regularity estimate for solutions of the infinity obstacle problem along the free boundary.

Theorem 5.3 (Optimal $C^{1,\frac{1}{3}}$−regularity at the free boundary)**.** *Let u be a solution to* (5.3) *and $x_0 \in \partial\{u > 0\}$ be a generic free boundary point. Then*

$$\sup_{y \in B_r(x_0)} u(y) \leqslant C\, r^{4/3}, \tag{5.11}$$

for a constant C that depends only upon the data of the problem.

Proof. For simplicity, and without loss of generality, assume $x_0 = 0$. By combining discrete iterative techniques and a continuous reasoning (see, for instance, Caffarelli, Karp, and Shahgholian 2000), it is well established that proving estimate (5.11) is equivalent to verifying the existence of a constant $C > 0$, such that

$$\mathfrak{s}_{j+1} \leqslant \max\left\{ C\, 2^{-4/3(j+1)},\ 2^{-4/3}\mathfrak{s}_j \right\}, \quad \forall\, j \in \mathbb{N}, \tag{5.12}$$

where

$$\mathfrak{s}_j = \sup_{B_{2^{-j}}} u.$$

Let us suppose, for the sake of contradiction, that (5.12) fails to hold, *i.e.*, that for each $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$ such that

$$\mathfrak{s}_{j_k+1} > \max\left\{k\, 2^{-4/3(j_k+1)},\ 2^{-4/3}\mathfrak{s}_{j_k}\right\}. \tag{5.13}$$

Now, for each k, define the rescaled function $v_k : B_1 \to \mathbb{R}$ by

$$v_k(x) := \frac{u(2^{-j_k}x)}{\mathfrak{s}_{j_k+1}}.$$

One easily verifies that

$$0 \leqslant v_k(x) \leqslant \sqrt[3]{16}, \quad \forall x \in B_1; \tag{5.14}$$

$$v_k(0) = 0; \tag{5.15}$$

$$\sup_{B_{\frac{1}{2}}} v_k = 1. \tag{5.16}$$

Moreover, we formally have

$$\begin{aligned}
\Delta_\infty v_k(x) &= \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Du(2^{-j_k}x) \cdot \left(\frac{2^{-2j_k}}{\mathfrak{s}_{j_k+1}} D^2u(2^{-j_k}x)\right) \cdot \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Du(2^{-j_k}x) \\
&= \frac{2^{-4j_k}}{\mathfrak{s}^3_{j_k+1}} \Delta_\infty u(2^{-j_k}x) =: f_k.
\end{aligned}$$

It is a matter of routine to rigorously justify the above calculations using the language of viscosity solutions (see, *e.g.*, Teixeira 2006, section 2). We estimate

$$|f_k| \leqslant \frac{2^{-4j_k}}{2^{-4(j_k+1)}\, k^3} M = \frac{16M}{k^3} \leqslant 16M, \tag{5.17}$$

using Lemma 5.2 and (5.13).

Combining the uniform bounds (5.14), (5.17), and local Lipschitz regularity results for the inhomogeneous $\infty-$Laplace equation (*cf.*, for example, Lindgren 2014, Corollary 2), we obtain both the equiboundedness and the equicontinuity of the sequence $(v_k)_k$. By the Arzelà–Ascoli Theorem, and passing to a subsequence if needed, we conclude that v_k converges locally uniformly to an $\infty-$harmonic function v_∞ in B_1 (observe that $f_k \to 0$) such that

$$0 \leqslant v_\infty \leqslant \sqrt[3]{16} \quad \text{and} \quad v_\infty(0) = 0.$$

We now use the Harnack inequality of Lindqvist–Manfredi (Theorem 2.8) to obtain the bound

$$v_\infty(x) \leqslant e^{2|x|}\, v_\infty(0) = 0, \quad \forall\, x \in B_{1/2}.$$

It follows that $v_\infty \equiv 0$ in $B_{1/2}$, which contradicts (5.16). The theorem is proven. □

As a first consequence, we improve the local Lipschitz regularity estimate provided by Theorem 5.1, where f needs only to satisfy (5.4). Indeed we obtain a finer gradient control near the free boundary.

Corollary 5.4. *Let u be a solution to* (5.3) *in B_1. Then u is locally Lipschitz continuous, and for any point*

$$z \in \{u > 0\} \cap B_1,$$

there holds

$$|Du(z)| \leqslant C \operatorname{dist}(z, \partial\{u > 0\})^{1/3}.$$

Proof. Fix $z \in \{u > 0\} \cap B_{1/2}$ and label $d := \operatorname{dist}(z, \partial\{u > 0\})$. Let $\zeta \in \partial\{u > 0\}$ be a free boundary point satisfying

$$|\zeta - z| = d.$$

From the $C^{1,\frac{1}{3}}$-smoothness of u at ζ, we know

$$\sup_{B_d(z)} u \leqslant \sup_{B_{2d}(\zeta)} u \leqslant C d^{4/3}. \tag{5.18}$$

We now define the auxiliary function $v\colon B_1 \to \mathbb{R}_+$, by

$$v(x) := \frac{u(z + dx)}{d^{4/3}}.$$

As argued before, v satisfies

$$\Delta_\infty v = f(z + dx), \qquad \text{in } B_1. \tag{5.19}$$

From (5.18) we can estimate

$$\sup_{B_1} v \leqslant C. \tag{5.20}$$

Finally, applying the gradient estimate for bounded solutions to (5.19), we conclude

$$|Dv(0)| = d^{-1/3}|Du(z)| \leqslant C_2,$$

and the corollary is proven. □

Our next theorem establishes a $C^{1,\frac{1}{3}}$–estimate from below, which implies that u leaves the zero-obstacle trapped by the graphs of two functions of the order $\operatorname{dist}^{4/3}(x, \partial\{u > 0\})$.

Theorem 5.5 (Optimal non-degeneracy estimates)**.** *Let u be a viscosity solution to* (5.3) *and $y_0 \in \overline{\{u > 0\}}$ be a generic point in the closure of the non-coincidence set. Then*

$$\sup_{B_r(y_0)} u \geqslant c\, r^{4/3},$$

for a constant $c > 0$ that depends only upon M.

Proof. By continuity arguments, it is enough to prove the result for points in the non-coincidence set. For simplicity and without loss of generality, take $y_0 = 0$. Define the barrier

$$\mathcal{B}_\infty(x) := \frac{3}{4}\sqrt[3]{\frac{3}{M}}\,|x|^{4/3},$$

which satisfies, by direct computation,

$$\Delta_\infty \mathcal{B}_\infty = \frac{1}{M}.$$

Hence,

$$\Delta_\infty u = f(x) \geqslant \frac{1}{M} = \Delta_\infty \mathcal{B}_\infty, \quad \text{in } \{u > 0\},$$

in the viscosity sense. On the other hand,

$$u \equiv 0 < \mathcal{B}_\infty \quad \text{on } \partial\{u > 0\} \cap B_r.$$

Therefore, for some point $y^\star \in \partial B_r \cap \{u > 0\}$, there must hold

$$u(y^\star) > \mathcal{B}_\infty(y^\star); \tag{5.21}$$

otherwise, by Jensen's comparison principle for $\infty-$harmonic functions (Jensen 1993), we would have, in particular,

$$0 < u(0) \leqslant \mathcal{B}_\infty(0) = 0.$$

Estimate (5.21) implies the thesis of the theorem. □

As usual, as soon as we establish the precise, sharp asymptotic behaviour for a given free boundary problem, obtaining certain soft geometric properties of the phases becomes possible. We conclude this section by proving that the region where the solution is above the obstacle has uniform positive density along the free boundary, which is then inhibited from developing cusps pointing inwards to the coincidence set. We use $|E|$ for the n-dimensional Lebesgue measure of the set E.

Corollary 5.6. *Let u be a solution to* (5.3) *and $x_0 \in \partial\{u > 0\}$ be a free boundary point. Then*

$$|B_\rho(x_0) \cap \{u > 0\}| \geqslant \delta_\star \rho^n, \tag{5.22}$$

for a constant $\delta_\star > 0$ that depends only upon the data of the problem.

Proof. It follows from Theorem 5.5 that there exists a point

$$z \in \partial B_\rho(x_0) \cap \{u > 0\}$$

such that $u(z) \geqslant c\,\rho^{4/3}$. By $C^{1,\frac{1}{3}}$–bounds along the free boundary, Theorem 5.3, it follows that

$$B_{\lambda\rho}(z) \subset \{u > 0\},$$

where the constant

$$\lambda := \sqrt[4]{\left(\frac{c}{2C}\right)^3}$$

depends only on the data of the problem. In fact, if this were not true, there would exist a free boundary point $y \in B_{\lambda\rho}(z)$. From (5.11), we would reach

$$c\,\rho^{4/3} \leqslant u(z) \leqslant \sup_{B_{\lambda\rho}(y)} u \leqslant C\,(\lambda\rho)^{4/3} = \frac{1}{2}c\,\rho^{4/3},$$

which is a contradiction. Thus,

$$B_\rho(x_0) \cap B_{\lambda\rho}(z) \subset B_\rho(x_0) \cap \{u > 0\}$$

and, finally,

$$|B_\rho(x_0) \cap \{u > 0\}| \geqslant |B_\rho(x_0) \cap B_{\lambda\rho}(z)| \geqslant \delta_\star \rho^n,$$

and the corollary is proven. □

We conclude by remarking that the thesis of Corollary 5.6 implies that the free boundary $\partial\{u > 0\}$ is porous, with porosity constant $\tau > 0$ that depends only on the data of the problem. In particular, the Hausdorff dimension of the free boundary is strictly less than n, and hence it has Lebesgue measure zero.

5.2 ∞−Laplace equations with singular absorptions

In this section, we include some comments on the geometric and analytic properties of non-negative viscosity solutions of the singular free boundary problem

$$\Delta_\infty u = u^{-\gamma} \quad \text{in } B_1 \cap \{u > 0\}, \tag{5.23}$$

for a parameter $0 \leqslant \gamma < 1$. Singular equations as in (5.23) appear in several contexts in the engineering sciences, for example, as simplified stationary models for fluids passing through a porous medium. The Laplacian case, $\Delta u = u^{-\gamma}$, is fairly well understood and appears as the Euler–Lagrange equation of the non-differentiable functional

$$\mathcal{J}_\gamma(u) = \int \frac{1}{2}|Du|^2 + u^{1-\gamma}\,dx.$$

Regularity results for minimizers of $\mathcal{J}_\gamma$ have been studied in Alt and Phillips 1986; Giaquinta and Giusti 1983 (see also Araújo and Teixeira 2013, for a non-variational approach). The pde satisfied in (5.23) can be considered the intermediate case between the infinity obstacle problem, the case $\gamma = 0$ (see Rossi, Teixeira, and Urbano 2015), and the infinity cavitation problem, the case $\gamma = 1$ (see Araújo, Teixeira, and Urbano 2021; Crasta and Fragalá 2020; Ricarte, Silva, and Teymurazyan 2017).

The study of this type of free boundary problem presents significant difficulties since the source term blows up along the *a priori* unknown set $\partial\{u > 0\}$. To circumvent these issues, viscosity solutions to the penalized problem

$$\begin{cases} \Delta_\infty u = \beta_\varepsilon(u)\, u^{-\gamma} & \text{in} \quad B_1 \\ \quad u = \varphi & \text{on} \quad \partial B_1 \end{cases} \tag{P_ε}$$

are considered, here the term $\beta_\varepsilon(s)$ is a suitable approximation of $\chi_{\{s>0\}}$ as in (5.10). Araújo and Sá 2022 provide existence of Perron's solutions for each parameter ε, and ε-uniform oscillation estimates for viscosity solutions of (P_ε), denoted by u_ε. As a consequence, $C^{1,\alpha}$ estimates at free boundary points are derived for limiting solutions of (5.23).

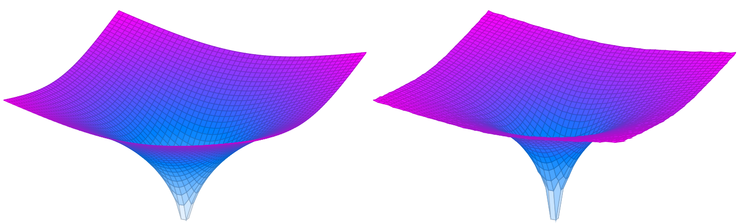


Figure 5.2: Gradient of solutions for (5.23), cases $\gamma = 0$ and $\gamma = 0.996$, respectively. It is observed that the singularity parameter γ plays a crucial role in determining the smoothness of solutions. As the value of γ increases, solutions tend to exhibit less smoothness (up to Lipschitz regularity). This phenomenon highlights the influence of the singularity parameter on the smoothness properties of the solutions.

Theorem 5.7 (Optimal regularity at free boundary points)**.** *Let u be a limit solution of problem* (5.23)*. There exist positive constants C and r_0, depending only on γ, $\|u\|_{L^\infty(\Omega)}$, and dimension, such that, for points*

$$x \in \partial\{u > 0\} \cap \Omega',$$

there holds

$$c\, r^{\frac{4}{3+\gamma}} \leqslant \sup_{B_r(x)} u \leqslant C\, r^{\frac{4}{3+\gamma}}, \tag{5.24}$$

for any $0 < r \leqslant r_0$. Furthermore,

$$\partial\{u > 0\} \subset \{|Du| = 0\},$$

which implies that u is $C^{1,\frac{1-\gamma}{3+\gamma}}$ along $\partial\{u > 0\}$.

We remind that solutions for (5.23) are limits of minimal Perron's solutions of (P_ε). Theorem 5.7 consists in obtaining asymptotic growth estimates derived from Ishii–Lions techniques. From this, the upper estimate in (5.24) follows from a discrete iterative argument, together with the use of Hopf's Lemma (see Araújo and Sá 2022, Theorem 6). The lower estimate is obtained by constructing entire radial supersolutions for (P_ε), whose geometric properties are analyzed in contrast with the minimality of Perron's solutions.

Corollary 5.8. *Let u be a limit solution of problem* (5.23)*. Then for any point*

$$z \in \{u > 0\} \cap B_1,$$

there holds

$$|Du(z)| \leqslant C \operatorname{dist}(z, \partial\{u > 0\})^{\frac{1+\gamma}{3-\gamma}}.$$

Note that the estimate above is reduced to Corollary 5.4 for the choice $\gamma = 0$. In addition, we observe that estimate (5.22) for solutions of (5.23) also holds by following ideas in the proof of Corollary 5.6.

6 Problems with solutions

Problems

1. Let $u \in C(\overline{U})$. Show that

$$\mathrm{Lip}_u(U) = \mathrm{Lip}_u(\overline{U}).$$

2. Let $n = 1$ and $U = (-2, -1) \cup (1, 2)$. Consider $f : \partial U \to \mathbb{R}$ defined by $f(-2) = 0$ and $f(-1) = f(1) = f(2) = 1$.

   (a) Determine $\mathrm{Lip}_f(\partial U)$.

   (b) Compute the MacShane–Whitney extensions of f to U.

   (c) Choose the extension of f which is in $\mathrm{AML}(U)$.

3. Consider the modulus function $u(x) = |x|$ in $\mathbb{R}^n$.

(a) Prove u is ∞–subharmonic.

(b) Give a short justification for the fact that it is not ∞–harmonic.

(c) Use the definition to show the previous fact.

4. Show that, if $u : \mathbb{R} \to \mathbb{R}$ is smooth, then

$$\Delta_\infty u(|x|) = \left[u'(|x|)\right]^2 u''(|x|), \quad x \neq 0.$$

5. Consider the function $v : \mathbb{R} \to \mathbb{R}$ given by

$$v(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Show that

$$T_v(0) > |v'(0)|.$$

6. Let F, G be functions defined in a segment $I = [r, s]$ such that

   $i)$ F is affine, that is, $F(t) = a + bt$, with $b \geqslant 0$;
   $ii)$ $F = G$ at the ends of I;
   $iii)$ $\mathrm{Lip}_G(I) \leqslant b$.

   Show that $F = G$.

7. Show that if $u \in CCA(U)$ then the map

$$r \to u^r(x) = \max_{\overline{B_r}(x)} u$$

is convex on $[0, d(x))$, for every $x \in U$.

8. Show that, as $r \to 0$, u^r converges locally uniformly to u in U.

9. Let $n = 2$, $u(x) = |x|$ and $v(x) = x_1$.

(a) Construct a bounded set $U \subset \mathbb{R}^2 \setminus \{0\}$ such that $v < u$ on ∂U except at two points and $u = v$ on the line segment joining these two points.

(b) Conclude there is no strong comparison principle for ∞−harmonic functions.

10. (Liouville's Theorem) Prove that if u is ∞−harmonic in $\mathbb{R}^n$ and u is bounded below, then u is constant.

11. Show that
$$-L_r^{\pm}(u, y) = L_r^{\mp}(-u, y). \tag{6.1}$$

12. Prove that
$$L_0^-(u, x) = \lim_{r \to 0} L_r^-(u, x)$$
exists and $L_0^-(u, x) \leqslant 0$.

13. Let u be a function enjoying comparison with cones from below. Then, for each $y \in U$, the function $r \mapsto -L_r^-(u, y)$ is non-decreasing. Moreover, prove that $-L_0^-(u, y)$ is a non-negative number.

14. Let u enjoy comparison with cones from above and below in B_1. Show that each blow-up u_x of u at $x \in B_1$ enjoys comparison with cones from above and below in $\mathbb{R}^n$. In addition, prove that
$$Lip_{u_x}(\mathbb{R}^n) \leqslant Lip_u(U).$$

15. Show that, for each $\lambda > 0$, the rescaled blow-up
$$x \mapsto u_0(\lambda x)\lambda^{-1}$$
is also a blow-up for u. Moreover,
$$L_0(u_0(\lambda x)\lambda^{-1}, 0) = L_0(u_0(x), 0).$$

16. Analogously to Proposition 3.8, prove that
$$-L_r^-(u_0, y) \leqslant L_0(u, 0).$$

17. Show that
$$L_0^+(u,0) = L_0^+(u_0,0).$$
Also, conclude that $L_0^+(u,0)$ is invariant with respect to blow-up functions of u at the origin.

18. Let $p \in \mathbb{R}^n \setminus \{0\}$. Assume that for each $r > 0$, there exists $\xi_r \in S_r(0)$ such that $\langle p, x\rangle \leqslant \langle p, \xi_r\rangle + o(r)$, for each $x \in S_r(0)$. Then $\xi_r/r \to p/|p|$, as $r \to 0$.

Solutions

We propose here possible solutions to the exercises, strongly encouraging the reader to try to solve the problems prior to coming to this section.

1. Since $U \subseteq \overline{U}$, if follows that
$$\mathrm{Lip}_u(U) \leqslant \mathrm{Lip}_u(\overline{U}).$$
Now, let $x, y \in \overline{U}$. Then there exists sequences $(x_m)_{m\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ such that
$$x_m \to x \quad \text{and} \quad y_k \to y \quad \text{as } m,k \to \infty. \tag{6.2}$$
Since $x_m, y_k \in U$, we have
$$|u(x_m) - u(y_k)| \leqslant \mathrm{Lip}_u(U)|x_m - y_k|.$$
By continuity, (6.2) and passing to the limit as $m \to \infty$, we obtain
$$|u(x) - u(y_k)| \leqslant \mathrm{Lip}_u(U)|x - y_k|.$$
Now we pass to the limit as $k \to \infty$ to obtain
$$|u(x) - u(y)| \leqslant \mathrm{Lip}_u(U)|x - y|.$$
This implies that
$$\mathrm{Lip}_u(\overline{U}) \leqslant \mathrm{Lip}_u(U),$$
from which the desired equality follows.

2. (a) First note that

$$\partial U = \{-2, -1, 1, 2\}.$$

In order to compute $\text{Lip}_f(\partial U)$, we first calculate all possible Newton quotients of f over ∂U:

- $\left|\frac{f(2)-f(1)}{2-1}\right| = 0;$
- $\left|\frac{f(2)-f(-1)}{2-(-1)}\right| = 0;$
- $\left|\frac{f(2)-f(-2)}{2-(-2)}\right| = \frac{1}{4};$
- $\left|\frac{f(1)-f(-1)}{2-(-1)}\right| = 0;$
- $\left|\frac{f(1)-f(-2)}{1-(-2)}\right| = \frac{1}{3};$
- $\left|\frac{f(-1)-f(-2)}{(-1)-(-2)}\right| = 1.$

Thus,

$$\text{Lip}_f(\partial U) = \max\left\{0, \frac{1}{4}, \frac{1}{3}, 1\right\} = 1.$$

(b) We start with the lower extension. Since

$$\mathcal{MW}_*(f)(x) = \sup_{z\in\partial U}\{f(z) - \text{Lip}_f(\partial U)|x-z|\}$$
$$= \sup\{1-|x-2|, 1-|x-1|, 1-|x+1|, -|x+2|\}$$

we have

$$\mathcal{MW}_*(f)(x) = \begin{cases} 1-|x+1|, & \text{if} \quad x\in[-2,-1] \\ 1-|x-1|, & \text{if} \quad x\in\left[1,\frac{3}{2}\right] \\ 1-|x-2|, & \text{if} \quad x\in\left[\frac{3}{2},2\right] \end{cases}$$

$$= \begin{cases} 2+x, & \text{if} \quad x\in[-2,-1] \\ 2-x, & \text{if} \quad x\in\left[1,\frac{3}{2}\right] \\ -1+x, & \text{if} \quad x\in\left[\frac{3}{2},2\right]. \end{cases}$$

Similarly,

$$\mathcal{MW}^*(f)(x) = \inf_{z\in\partial U}\{f(z) + \text{Lip}_f(\partial U)|x-z|\}$$

$$= \inf\{1+|x-2|, 1+|x-1|, 1+|x+1|, |x+2|\}$$

and so

$$\mathcal{MW}^*(f)(x) = \begin{cases} |x+2|, & \text{if} \quad x \in [-2,-1] \\ 1+|x-1|, & \text{if} \quad x \in \left[1, \frac{3}{2}\right] \\ 1+|x-2|, & \text{if} \quad x \in \left[\frac{3}{2}, 2\right] \end{cases}$$

$$= \begin{cases} 2+x, & \text{if} \quad x \in [-2,-1] \\ x, & \text{if} \quad x \in \left[1, \frac{3}{2}\right] \\ 3-x, & \text{if} \quad x \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

(c) The extension which is in AML(U) is

$$u(x) = \begin{cases} 2+x, & \text{if} \quad x \in [-2,-1] \\ 1, & \text{if} \quad x \in [1,2]. \end{cases}$$

3. (a) Let $\widehat{x} \in \mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^n)$ be such that $(u-\varphi)$ has a local maximum at $\widehat{x}$. Initially, let us consider $\hat{x} \neq 0$. In this case, the function u is of class C^2 in a neighbourhood of $\widehat{x}$, and we can compute its derivatives:

$$\begin{aligned} u_{x_i}(\widehat{x}) &= \frac{\widehat{x}_i}{|\widehat{x}|} \\ u_{x_i x_j}(\widehat{x}) &= \frac{\delta_{ij}|\widehat{x}| - \widehat{x}_i \frac{\widehat{x}_j}{|\widehat{x}|}}{|\widehat{x}|^2}. \end{aligned}$$

Thus,

$$\begin{aligned} Dw(\widehat{x}) &= \frac{\widehat{x}}{|\widehat{x}|} \\ D^2 w(\widehat{x}) &= \frac{1}{|\widehat{x}|}\left(I_n - \frac{\widehat{x}\otimes\widehat{x}}{|\widehat{x}|^2}\right). \end{aligned} \tag{6.3}$$

Since $(u-\varphi)$ has a local maximum at $\widehat{x}$, we have

$$D(u-\varphi)(\widehat{x}) = 0 \quad \Leftrightarrow \quad Du(\widehat{x}) = D\varphi(\widehat{x}) \tag{6.4}$$

and

$$D^2(u-\varphi)(\widehat{x}) \preceq 0 \quad \Leftrightarrow \quad D^2u(\widehat{x}) \preceq D^2\varphi(\widehat{x}). \tag{6.5}$$

Then,

$$
\begin{aligned}
\Delta_\infty \varphi(\widehat{x}) &= \langle D^2\varphi(\widehat{x}) D\varphi(\widehat{x}), D\varphi(\widehat{x})\rangle \\
&\overset{(6.5)}{\geqslant} \langle D^2 u(\widehat{x}) D\varphi(\widehat{x}), D\varphi(\widehat{x})\rangle \\
&\overset{(6.4)}{=} \langle D^2 u(\widehat{x}) Du(\widehat{x}), Du(\widehat{x})\rangle \\
&\overset{(6.3)}{=} \left\langle \frac{1}{|\widehat{x}|}\left(I_n - \frac{\widehat{x}\otimes\widehat{x}}{|\widehat{x}|^2}\right)\frac{\widehat{x}}{|\widehat{x}|}, \frac{\widehat{x}}{|\widehat{x}|}\right\rangle \\
&= \frac{1}{|\widehat{x}|^3}\left(|\widehat{x}|^2 - \frac{1}{|\widehat{x}|^2}\langle(\widehat{x}\otimes\widehat{x})\widehat{x}, \widehat{x}\rangle\right) \\
&= \frac{1}{|\widehat{x}|^3}\left(|\widehat{x}|^2 - \frac{1}{|\widehat{x}|^2}\langle\widehat{x}, \widehat{x}\rangle^2\right) \\
&= \frac{1}{|\widehat{x}|^3}\left(\langle\widehat{x}, \widehat{x}\rangle - \frac{1}{\langle\widehat{x}, \widehat{x}\rangle}\langle\widehat{x}, \widehat{x}\rangle^2\right) \\
&= 0.
\end{aligned}
$$

Now, let's look at the case of the origin. Since u is not differentiable at 0, the computations above do not apply. But as $(u-\varphi)$ has a local maximum at 0, we have

$$(u-\varphi)(0) \geqslant (u-\varphi)(x) = |x| - \varphi(x),$$

for all x in a neighborhood of 0. Thus,

$$\varphi(x) - \varphi(0) \geqslant |x| > 0,$$

that is,

$$\varphi(x) > \varphi(0),$$

for all x in a neighborhood of 0, and φ has a local minimum at 0. Hence,

$$D\varphi(0) = 0 \quad \text{e} \quad D^2\varphi(0) \succeq 0.$$

Therefore,

$$\Delta_\infty \varphi(0) = \langle D^2\varphi(0) D\varphi(0), D\varphi(0)\rangle = 0.$$

This shows that u is a $\infty-$subharmonic function in the viscosity sense.

(b) It clearly does not enjoy comparison with cones from below. Given a set containing the origin, it is possible to find a cone with vertex outside that set such that the cone is below the function on the boundary of the set but not in the interior.

(c) Let's now show that u fails to be $\infty-$superharmonic. The problem is at the origin. Consider $\varphi \in C^2(\mathbb{R}^2)$ given by

$$\varphi(x) = \frac{1}{5}|x|^2 - \langle b, x\rangle,$$

where $|b| = \frac{2}{5}$. Notice that $(u-\varphi)$ has a local minimum at $\widehat{x} = 0$ because

$$\begin{aligned}
(u-\varphi)(x) &= |x| - \frac{1}{5}|x|^2 + \langle b, x\rangle \\
&\geqslant |x| - \frac{1}{5}|x|^2 - |b||x| \\
&\geqslant |x| - \frac{1}{5}|x| - |b||x| \\
&= \left(1 - \frac{1}{5} - \frac{2}{5}\right)|x| \\
&= \frac{2}{5}|x| > 0,
\end{aligned}$$

for $|x| < 1$. Since $(u-\varphi)(0) = 0$, we have

$$(u-\varphi)(0) \leqslant (u-\varphi)(x), \quad x \in B_1.$$

Then, we conclude that

$$(u-\varphi)(0) = \min_{B_1}(u-\varphi).$$

If u were $\infty-$superharmonic, we should have

$$\Delta_\infty \varphi(0) \leqslant 0$$

but in fact

$$\begin{aligned}
\Delta_\infty \varphi(0) &= \langle D^2\varphi(0) D\varphi(0), D\varphi(0)\rangle \\
&= \left\langle \frac{2}{5} I_n(-b), (-b) \right\rangle \\
&= \frac{2}{5}|b|^2 = \frac{8}{125}.
\end{aligned}$$

4. By the chain rule, for $x \neq 0$, we have

$$\begin{aligned}
\partial_i u(|x|) &= u'(|x|)\frac{x_i}{|x|}; \\
\partial_{ij} u(|x|) &= \left(u''(|x|) - \frac{u'(|x|)}{|x|}\right)\frac{x_i x_j}{|x|^2} + \delta_{ij}\frac{u'(|x|)}{|x|}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\Delta_\infty u(|x|) &= \sum_{i,j=1}^{n} \partial_i u(|x|)\partial_j u(|x|)\partial_{ij}u(|x|) \\
&= \sum_{i,j=1}^{n} \left(u'(|x|)\frac{x_i}{|x|}\right)\left(u'(|x|)\frac{x_j}{|x|}\right) \cdot \\
&\quad \cdot \left(\left(u''(|x|) - \frac{u'(|x|)}{|x|}\right)\frac{x_i x_j}{|x|^2} + \delta_{ij}\frac{u'(|x|)}{|x|}\right) \\
&= [u'(|x|)]^2\left(u''(|x|) - \frac{u'(|x|)}{|x|}\right)\sum_{i,j=1}^{n}\frac{x_i^2 x_j^2}{|x|^4} \\
&\quad +[u'(|x|)]^3\sum_{i,j=1}^{n}\frac{x_i x_j}{|x|^3}\delta_{ij} \\
&= [u'(|x|)]^2\left(u''(|x|) - \frac{u'(|x|)}{|x|}\right) + \frac{[u'(|x|)]^3}{|x|} \\
&= [u'(|x|)]^2 u''(|x|).
\end{aligned}
$$

5. At $x = 0$, we have

$$v'(0) = \lim_{h\to 0}\frac{v(0+h)-v(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0.$$

Therefore

$$v'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Observe that v' is not continuous at $x = 0$ because

$$\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$$

does not exist. We want to show that

$$T_v(0) > 0$$

and we offer both a proof by contradiction and a more direct proof.

Suppose that $T_v(0) = 0$. Let r_n be an arbitrary sequence of real numbers such that $r_n \to 0$. By Proposition 1.7, we have

$$-T_v(r_n) \leqslant v'(r_n) \leqslant T_v(r_n)$$

and, by the upper semicontinuous of T_v, also

$$\limsup_{n\to\infty} v'(r_n) \leqslant \limsup_{n\to\infty} T_v(r_n) \leqslant T_v(0) = 0.$$

Moreover,

$$0 \leqslant -\limsup_{n\to\infty} T_v(r_n) = \liminf_{n\to\infty}(-T_v(r_n)) \leqslant \liminf_{n\to\infty} v'(r_n).$$

Thus $\lim\limits_{x\to 0} v'(x)$ exists (and is equal to 0), which is a contradiction.

Alternatively, let $x_n \to 0$, with $x_n \neq 0$. We have

$$\begin{aligned} T_v(0) &\geqslant \limsup_{n\to\infty} T_v(x_n) \\ &\geqslant \limsup_{n\to\infty} \left|v'(x_n)\right| \\ &= \limsup_{n\to\infty} \left|2x_n \sin\left(\frac{1}{x_n}\right) - \cos\left(\frac{1}{x_n}\right)\right| \\ &= 1. \end{aligned}$$

6. Note that, for $t_1, t_2 \in I$, we have

$$\begin{aligned} |G(t_1) - G(t_2)| &\leqslant \operatorname{Lip}_G(I)|t_1 - t_2| \\ &\leqslant b|t_1 - t_2| \\ &= |bt_1 - bt_2| \\ &= |a + bt_1 - (a + bt_2)| \\ &= |F(t_1) - F(t_2)|. \end{aligned}$$

So, given $t \in I = [r, s]$, we have

$$G(t) - G(r) \leqslant |G(t) - G(r)| \leqslant |F(t) - F(r)| = F(t) - F(r),$$

since F is non-decreasing because $b \geqslant 0$. Since $G(r) = F(r)$, we obtain

$$G(t) \leqslant F(t), \quad \forall t \in I.$$

Similarly,

$$G(s) - G(t) \leqslant |G(s) - G(t)| \leqslant |F(s) - F(t)| = F(s) - F(t).$$

As $G(s) = F(s)$, it follows that

$$-G(t) \leqslant -F(t) \quad \Longrightarrow \quad F(t) \leqslant G(t) \quad \forall t \in I.$$

Thus, $G = F$ in I.

7. Denote
$$g(r) = u^r(x) = \max_{w \in \overline{B_r}(x)} u(w).$$
We will show that the function g is convex. We claim that, for $0 < s < r < d(x)$,
$$u(y) \leqslant g(s) + \frac{g(r) - g(s)}{r - s}(|y - x| - s), \tag{6.6}$$
for $s \leqslant |y-x| \leqslant r$. Indeed, observe that the right side of (6.6) is a cone with vertex in x that does not belong to the ring $s \leqslant |y - x| \leqslant r$. Also, on the boundary of this ring, (6.6) holds because trivially
$$\begin{aligned} |y - x| = s \quad &\Longrightarrow \quad u(y) \leqslant g(s), \\ |y - x| = r \quad &\Longrightarrow \quad u(y) \leqslant g(r). \end{aligned}$$
As $u \in CCA(U)$, it follows that (6.6) holds also in $s < |y - x| < r$.

Now, let
$$\tau = \lambda s + (1 - \lambda)r, \quad \text{with} \quad \lambda \in [0, 1].$$
As $u(y) \leqslant g(s)$, for $|y - x| \leqslant s$, then
$$u(y) \leqslant g(s) + \frac{g(r) - g(s)}{r - s}(\tau - s), \tag{6.7}$$
for $0 \leqslant |y - x| \leqslant \tau$. Maximizing the left-hand side of (6.7) over $|y - x| \leqslant \tau$, we have
$$\begin{aligned} &\max_{y \in \overline{B_\tau}(x)} u(y) \leqslant g(s) + \frac{g(r) - g(s)}{r - s}(\lambda s + (1 - \lambda)r - s) \Rightarrow \\ \Rightarrow \quad &g(\tau) \leqslant g(s) + \frac{g(r) - g(s)}{r - s}(1 - \lambda)(r - s) \\ \Rightarrow \quad &g(\tau) \leqslant \lambda g(s) + (1 - \lambda)g(r). \end{aligned}$$
Therefore, the function $g(r)$ is convex.

8. Remember that, given $u \in C(U)$ and $x \in U_r$, we define
$$u^r(x) := \max_{\overline{B_r}(x)} u.$$
Note that if $r < s$, then
$$\overline{B_r}(x) \subset \overline{B_s}(x)$$
and thus
$$u^r(x) \leqslant u^s(x).$$

As u is continuous at x, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u(z) - u(x)| \leqslant \frac{\varepsilon}{2}, \quad \forall z \in B_\delta(x).$$

We can now take the supremum over $B_r(x)$, for $r \leqslant \delta$, to obtain

$$|u^r(x) - u(x)| \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

In other words,

$$\lim_{r \to 0} u^r(x) = u(x).$$

We will now show that the convergence is locally uniform. Take $z \in U$ and $\delta > 0$ such that $\overline{B_\delta}(z) \subset U$. Let $\varepsilon > 0$. Since u is uniformly continuous in $\overline{B_\delta}(z)$, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$|x - y| < \eta \implies |u(x) - u(y)| < \varepsilon, \quad \forall x, y \in \overline{B_\delta}(z).$$

Note that if $x \in \overline{B_{\frac{\delta}{2}}}(z)$ and $r \leqslant \frac{\delta}{2}$, then

$$\overline{B_r}(x) \subset \overline{B_\delta}(z)$$

because, for $y \in \overline{B_r}(x)$,

$$|y - z| \leqslant |y - x| + |x - z| \leqslant r + \frac{\delta}{2} \leqslant \delta.$$

Therefore, if $r < \min\left(\eta, \frac{\delta}{2}\right)$, then

$$\begin{aligned} |u^r(x) - u(x)| &= \left| \max_{\overline{B_r}(x)} u - u(x) \right| \\ &= |u(x_r) - u(x)| \quad \left(x_r \in \overline{B_r}(x)\right) \\ &< \varepsilon, \end{aligned}$$

for all $x \in \overline{B_{\frac{\delta}{2}}}(z)$.

9. *The reader is given here the opportunity to work out a problem without guidance.*

10. *The first correct solution e-mailed to the authors will be included, with an acknowledgement, in a future edition.*

11. *The reader is given here the opportunity to work out a problem without guidance.*

12. From the previous exercises,

$$\begin{aligned} L_0^-(u,x) &= \lim_{r\to 0}(-L_r^-(-u,x)) \\ &= -\lim_{r\to 0} L_r^+(-u,x) \\ &= -L_0^+(-u,x), \end{aligned}$$

which implies,

$$-L_0^-(u,x) = L_0^+(-u,x) \geqslant 0.$$

13. For each $r > 0$, note that

$$\begin{aligned} L_r^-(u,x) &= \min_{y\in S_r(x)} \frac{u(y)-u(x)}{r} \\ &= (-1) \max_{y\in S_r(x)} \frac{(-u)(y)-(-u)(x)}{r} \\ &= -L_r^+(-u,x). \end{aligned}$$

Also, we have that the function

$$r \mapsto L_r^+(u,x)$$

is non-decreasing. Hence, we get that if $s \leqslant r$, then

$$L_s^+(-u,x) \leqslant L_r^+(-u,x).$$

Hence,

$$\begin{aligned} L_s^-(u,x) &= (-1)L_s^+(-u,x) \\ &\geqslant (-1)L_r^+(-u,x) \\ &= L_r^-(u,x). \end{aligned}$$

14. *The reader is given here the opportunity to work out a problem without guidance.*

15. *The reader is given here the opportunity to work out a problem without guidance.*

16. *The reader is given here the opportunity to work out a problem without guidance.*

17. *The reader is given here the opportunity to work out a problem without guidance.*

18. We consider $x = r\frac{p}{|p|}$ and obtain

$$\langle p, r\frac{p}{|p|}\rangle \leqslant \langle p, \xi_r\rangle + o(r),$$

which implies

$$1 \leqslant \langle \frac{p}{|p|}, \frac{\xi_r}{r}\rangle \leqslant 1 + o(r).$$

Hence,

$$\lim_{r\to 0} \frac{\xi_r}{r} = \frac{p}{|p|}.$$

Bibliography

H. W. Alt and D. Phillips (1986). "A free boundary problem for semilinear elliptic equations." *J. Reine Angew. Math.* 368, pp. 63–107. MR: 850615 (cit. on p. 62).

D. J. Araújo, R. Leitão, and E. V. Teixeira (2016). "Infinity Laplacian equation with strong absorptions." *J. Funct. Anal.* 270.6, pp. 2249–2267. MR: 3460240 (cit. on p. 55).

D. J. Araújo, G. Ricarte, and E. V. Teixeira (2015). "Geometric gradient estimates for solutions to degenerate elliptic equations." *Calc. Var. Partial Differential Equations* 53.3-4, pp. 605–625. MR: 3347473 (cit. on p. 51).

D. J. Araújo and G. S. Sá (2022). "Infinity Laplacian equations with singular absorptions." *Calc. Var. Partial Differential Equations* 61.4, Paper No. 132, 16. MR: 4417397 (cit. on pp. 63, 64).

D. J. Araújo and E. V. Teixeira (2013). "Geometric approach to nonvariational singular elliptic equations." *Arch. Ration. Mech. Anal.* 209.3, pp. 1019–1054. MR: 3067831 (cit. on p. 62).

D. J. Araújo, E. V. Teixeira, and J. M. Urbano (2021). "On a two-phase free boundary problem ruled by the infinity Laplacian." *Israel J. Math.* 245.2, pp. 773–785. MR: 4358263 (cit. on p. 63).

S. N. Armstrong and C. K. Smart (2010). "An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions." *Calc. Var. Partial Differential Equations* 37.3-4, pp. 381–384. MR: 2592977 (cit. on p. 27).

G. Aronsson, M. G. Crandall, and P. Juutinen (2004). "A tour of the theory of absolutely minimizing functions." *Bull. Amer. Math. Soc. (N.S.)* 41.4, pp. 439–505. MR: 2083637 (cit. on pp. v, 8, 23).

T. Bhattacharya and A. Mohammed (2011). "On solutions to Dirichlet problems involving the infinity-Laplacian." *Adv. Calc. Var.* 4.4, pp. 445–487. MR: 2844513 (cit. on p. 58).

— (2012). "Inhomogeneous Dirichlet problems involving the infinity-Laplacian." *Adv. Differential Equations* 17.3-4, pp. 225–266. MR: 2919102 (cit. on p. 58).

L. Caffarelli and S. Salsa (2005). *A geometric approach to free boundary problems*. Vol. 68. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. x+270. MR: 2145284 (cit. on p. 54).

L. A. Caffarelli and X. Cabré (1995). *Fully nonlinear elliptic equations*. Vol. 43. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, pp. vi+104. MR: 1351007 (cit. on p. 51).

L. A. Caffarelli, L. Karp, and H. Shahgholian (2000). "Regularity of a free boundary with application to the Pompeiu problem." *Ann. of Math. (2)* 151.1, pp. 269–292. MR: 1745013 (cit. on p. 58).

J. Calder (2019). "Consistency of Lipschitz learning with infinite unlabeled data and finite labeled data." *SIAM J. Math. Data Sci.* 1.4, pp. 780–812. MR: 4039189 (cit. on p. 1).

J. Calder and D. Slepčev (2020). "Properly-weighted graph Laplacian for semi-supervised learning." *Appl. Math. Optim.* 82.3, pp. 1111–1159. MR: 4167693 (cit. on p. 1).

M. G. Crandall, L. C. Evans, and R. F. Gariepy (2001). "Optimal Lipschitz extensions and the infinity Laplacian." *Calc. Var. Partial Differential Equations* 13.2, pp. 123–139. MR: 1861094 (cit. on p. 32).

M. G. Crandall (2008). "A visit with the ∞-Laplace equation." In: *Calculus of variations and nonlinear partial differential equations*. Vol. 1927. Lecture Notes in Math. Springer, Berlin, pp. 75–122. MR: 2408259 (cit. on p. v).

M. G. Crandall and L. C. Evans (2001). "A remark on infinity harmonic functions." In: *Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000)*. Vol. 6. Electron. J. Differ. Equ. Conf. Southwest Texas State Univ., San Marcos, TX, pp. 123–129. MR: 1804769 (cit. on pp. 32, 38, 43, 51).

G. Crasta and I. Fragalá (2020). "Bernoulli free boundary problem for the infinity Laplacian." *SIAM J. Math. Anal.* 52.1, pp. 821–844. MR: 4065639 (cit. on p. 63).

L. C. Evans and O. Savin (2008). "$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions." *Calc. Var. Partial Differential Equations* 32.3, pp. 325–347. MR: 2393071 (cit. on pp. 22, 50).

L. C. Evans and C. K. Smart (2011). "Everywhere differentiability of infinity harmonic functions." *Calc. Var. Partial Differential Equations* 42.1-2, pp. 289–299. MR: 2819637 (cit. on pp. 22, 45).

A. Friedman (1982). *Variational principles and free-boundary problems*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, pp. ix+710. MR: 679313 (cit. on p. 54).

M. Giaquinta and E. Giusti (1983). "Differentiability of minima of nondifferentiable functionals." *Invent. Math.* 72.2, pp. 285–298. MR: 700772 (cit. on p. 62).

C. Imbert and L. Silvestre (2013). "$C^{1,\alpha}$ regularity of solutions of some degenerate fully non-linear elliptic equations." *Adv. Math.* 233, pp. 196–206. MR: 2995669 (cit. on p. 51).

R. Jensen (1993). "Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient." *Arch. Rational Mech. Anal.* 123.1, pp. 51–74. MR: 1218686 (cit. on pp. 27, 61).

P. Juutinen (1998). "Minimization problems for Lipschitz functions via viscosity solutions." *Ann. Acad. Sci. Fenn. Math. Diss.* 115. Dissertation, University of Jyväskulä, Jyväskulä, 1998, p. 53. MR: 1632063.

E. Lindgren (2014). "On the regularity of solutions of the inhomogeneous infinity Laplace equation." *Proc. Amer. Math. Soc.* 142.1, pp. 277–288. MR: 3119202 (cit. on pp. 50, 51, 56, 58, 59).

P. Lindqvist (2006). *Notes on the p-Laplace equation*. Vol. 102. Report. University of Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä, Jyväskylä, pp. ii+80. MR: 2242021.

— (2016). *Notes on the infinity Laplace equation*. SpringerBriefs in Mathematics. BCAM Basque Center for Applied Mathematics, Bilbao; Springer, [Cham], pp. ix+68. MR: 3467690.

— (2019). *Notes on the stationary p-Laplace equation*. SpringerBriefs in Mathematics. Springer, Cham, pp. xi+104. MR: 3931688.

P. Lindqvist and J. J. Manfredi (1995). "The Harnack inequality for ∞-harmonic functions." *Electron. J. Differential Equations* 4, p. 5. MR: 1322829 (cit. on p. 20).

G. Lu and P. Wang (2008). "Inhomogeneous infinity Laplace equation." *Adv. Math.* 217.4, pp. 1838–1868. MR: 2382742 (cit. on pp. 50, 56).

A. Petrosyan, H. Shahgholian, and N. Uraltseva (2012). *Regularity of free boundaries in obstacle-type problems*. Vol. 136. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. x+221. MR: 2962060 (cit. on pp. 54, 55).

G. C. Ricarte, J. V. Silva, and R. Teymurazyan (2017). "Cavity type problems ruled by infinity Laplacian operator." *J. Differential Equations* 262.3, pp. 2135–2157. MR: 3582224 (cit. on p. 63).

J.-F. Rodrigues (1987). *Obstacle problems in mathematical physics*. Vol. 134. North-Holland Mathematics Studies. Notas de Matemática [Mathematical Notes], 114. North-Holland Publishing Co., Amsterdam, pp. xvi+352. MR: 880369 (cit. on p. 54).

J. D. Rossi, E. V. Teixeira, and J. Urbano (2015). "Optimal regularity at the free boundary for the infinity obstacle problem." *Interfaces Free Bound.* 17.3, pp. 381–398. MR: 3421912 (cit. on pp. 55, 56, 63).

O. Savin (2005). "C^1 regularity for infinity harmonic functions in two dimensions." *Arch. Ration. Mech. Anal.* 176.3, pp. 351–361. MR: 2185662 (cit. on pp. 22, 50).

O. Savin, C. Wang, and Y. Yu (2008). "Asymptotic behavior of infinity harmonic functions near an isolated singularity." *Int. Math. Res. Not. IMRN* 6, Art. ID rnm163, 23. MR: 2427455.

E. V. Teixeira (2006). "Optimal regularity of viscosity solutions of fully nonlinear singular equations and their limiting free boundary problems." In: vol. 30. XIV School on Differential Geometry (Portuguese), pp. 217–237. MR: 2373512 (cit. on p. 59).

N. Weaver (2018). *Lipschitz algebras*. Second edition of [MR1832645]. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, pp. xiv+458. MR: 3792558 (cit. on p. 36).

Index of Notation

Index

Títulos Publicados — 34º Colóquio Brasileiro de Matemática

Uma introdução à convexidade em grafos – *Júlio Araújo, Mitre Dourado, Fábio Protti e Rudini Sampaio*

Uma introdução aos sistemas dinâmicos via exemplos – *Lucas Backes, Alexandre Tavares Baraviera e Flávia Malta Branco*

Introdução aos espaços de Banach – *Aldo Bazán, Alex Farah Pereira e Cecília de Souza Fernandez*

Contando retas em superfícies no espaço projetivo – *Jacqueline Rojas, Sally Andria e Wállace Mangueira*

Paths and connectivity in temporal graphs – *Andrea Marino e Ana Silva*

Geometry of Painlevé equations – *Frank Loray*

Implementação computacional da tomografia por impedância elétrica – *Fábio Margotti, Eduardo Hafemann e Lucas Marcilio Santana*

Regularidade elíptica e problemas de fronteiras livres – *João Vitor da Silva e Gleydson Ricarte*

The ∞-Laplacian: from AMLEs to Machine Learning – *Damião Araújo e José Miguel Urbano*

Homotopical dynamics for gradient-like flows – *Guido G. E. Ledesma, Dahisy V. S. Lima, Margarida Mello, Ketty A. de Rezende e Mariana R. da Silveira*

impa

Instituto de
Matemática
Pura e Aplicada

Damião Júnio Araújo

Damião Júnio, aka DJ, is a specialist in free boundary problems and regularity theory for nonlinear partial differential equations. Born in Juazeiro do Norte (Ceará, Brazil) in 1983, Damião obtained his undergraduate degree from URCA (Ceará) in 2005, followed by a master's degree from UFCG (Paraíba) in 2008. He completed his PhD at UFC (Ceará) in 2012 and conducted postdoctoral research at the University of Florida in 2015. Currently, he holds a professor position at UFPB (Paraíba) and is a Junior Associate at the International Centre for Theoretical Physics in Trieste. Besides his academic pursuits, DJ enjoys playing soccer and savouring good IPA beer.

José Miguel Urbano

Born in 1970, he studied in Coimbra, Paris, Lisbon and Chicago. He works on regularity theory for nonlinear partial differential equations and free boundary problems, having authored a book, published over 60 scientific papers and mentored several PhD students and postdocs. He serves as Editor-in-Chief of Portugaliae Mathematica and is a Corresponding Member of the Lisbon Academy of Sciences. Apart from teaching and writing math, he likes reading and watching TV series: Better Call Saul is his all-time favourite, with Mad Men a close second and Succession completing the podium. A father of four, his idea of bliss is to go with his children to Estádio do Dragão on match day and watch FC Porto win.

The ∞-Laplacian

impa
Instituto de
Matemática
Pura e Aplicada

ISBN 978-85-244-0530-3
9 788524 405303