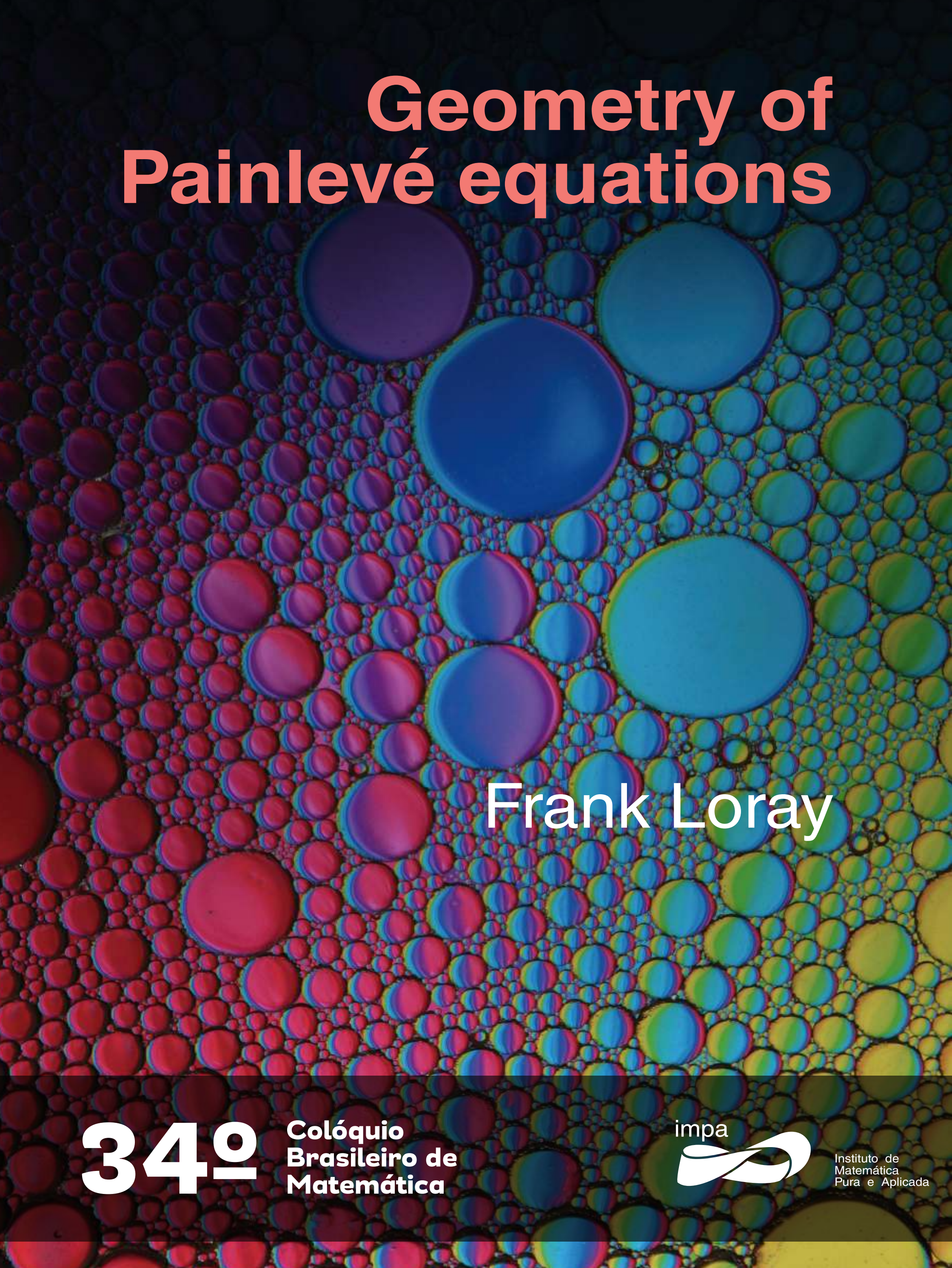
Geometry of Painlevé equations



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Preface

The goal of the course is to provide a description of the foliation associated to the Painlevé VI equation:

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)$$
$$+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\kappa_\infty^2}{2} - \frac{\kappa_0^2}{2}\frac{t}{q^2} + \frac{\kappa_1^2}{2}\frac{t-1}{(q-1)^2} + \frac{1-\kappa_t^2}{2}\frac{t(t-1)}{(q-t)^2}\right).$$

Painlevé equations form 6 families of differential equations, the first 5 arising from degenerescence of the 6th one above, with parameters $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in \mathbb{C}^4$. They were discovered by Painlevé and his students as non linear second order ODE having the Painlevé property: solutions are well-behaved with respect to analytic continuation. These differential equations, or their solutions, are now used in many areas of mathematics and physics. Our goal is to focus on the 6th family and explain how this family arise as isomonodromy equation, and then provide a description of its phase portrait in $\mathbb{C}^3 \ni (t, q, p)$, its semi-compactification, its monodromy, and how this has been used to classify algebraic solutions.

Introduction

Paul Painlevé and his students classified those differential equations $y'' = f(x, y, y')$ which are rational with respect to y and y' and analytic with respect to x, satisfying the following properties:

- Painlevé property: solutions behave well with respect to analytic continuation,
- Irreducibility: integration of the differential equation cannot be reduced by successive integration of linear differential equations, non linear first order differential equations and/or algebraic extension field, starting from the field of rational functions $\mathbb{C}(x)$.

The first property allows us to define the monodromy representation of the differential equation. The second property tells us that solutions define new transcendental functions.

The 19th century was the century of special functions: trigonometric, Bessel, elliptic or hypergeometric functions were discovered and show powerful tools in solving many differential equations that arise from sciences. All of these share the same properties: they can be defined by differential equations satisfying the Painlevé Property. Fuchs and Poincaré independently proved (see Pan and Sebastiani (2004)) that first order differential equations $P(x, y, y') = 0$ having the Painlevé Property either have only algebraic solutions, or reduce by meromorphic change of coordinates/variable to one of the following:

- Riccati differential equations $y' = a(x)y^2 + b(x)y + c(x)$ with a, b, c meromorphic;
- Weierstrass differential equation $(y')^2 = y^3 + ay + b$ with $a, b \in \mathbb{C}$.

One can check that solutions admit meromorphic continuation (i.e. analytic continuation with poles) outside the polar set of a, b, c for the first one, and everywhere for the second one. The dream of Painlevé was to find new transcendental functions for the future, by looking for a generalisation of Poincaré–Fuchs classification for higher order differential equations, namely those having the form $y'' = f(x, y, y')$. Constraints given by the Painlevé Property allow him to reduce to a list of 50 families of differential equations. Then, Irreducibility constraint allows to eliminate most of them, reducing to a list of 6 families of differential equations, namely Painlevé equations:

$$
\begin{array}{lll}
P_{\mathrm{I}} & : & \ddot{q} = 6q^2 + t \\
P_{\mathrm{II}}(\alpha) & : & \ddot{q} = 2q^3 + tq + \alpha \\
P_{\mathrm{III}}(\alpha,\beta,\gamma,\delta) & : & \ddot{q} = \frac{(\dot{q})^2}{q} - \frac{\dot{q}}{t} + \frac{\alpha q^2+\beta}{t} + \gamma q^3 + \frac{\delta}{q} \\
P_{\mathrm{IV}}(\alpha,\beta) & : & \ddot{q} = \frac{(\dot{q})^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2(t^2-\alpha)q + \frac{\beta}{q} \\
P_V(\alpha,\beta,\gamma,\delta) & : & \ddot{q} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(\dot{q})^2 - \frac{\dot{q}}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1} \\
P_{\mathrm{VI}}(\alpha,\beta,\gamma,\delta) & : & \ddot{q} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)(\dot{q})^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\dot{q} \\
 & & \qquad + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right)
\end{array}
$$

In fact, Painlevé missed some of them, due to computation mistakes, and it was his student Gambier who found the complete list. The particular case $P_{VI}(0,0,0,\frac{1}{2})$ was already known by Mazzocco (2001a) (see Picard (1889)). The reduction to this list is detailled by Bureau (1964) in 1964. More conceptual approach of this classification can be found in the work Sakai (2001), and Saito and Takebe (2002) and Saito, Takebe, and Terajima (2002) at the beginning of the current century. However, the Painlevé Property is translated into another geometric property without proof of the equivalence between the two. Nevertheless, these texts provide a very nice set-up where Painlevé equations can be retrieved in a natural way.

It is much later that the Painlevé Property and Irreducibility Property were rigorously proved. Painlevé Property has been proved in Okamoto (1979) for P_{VI}, following an idea of Reeb (1974). This consists in considering first the Painlevé foliation defined by $v = \partial_t + q'\partial_q + f(t,q,q')\partial_{q'}$ in variables $(t,q,q') \in \mathbb{C}^3$, and then compactify, reduce the singular points by blowing-up, and finally prove that non vertical leaves are covering of the punctured plane with respect to the time projection $(t,q,q') \to t$.

Irreducibility Property has been independently proved by Nishioka (1988) and Umemura (1988) for the first Painlevé equation. Umemura developed some non linear Differential Galois Theory to study this problem. A simpler approach using Galois Groupoid for foliations, defined by Malgrange (2001, 2002), was given in Casale (2008).

Finally, last but not least, Painlevé equations were rediscovered as isomonodromy equations by Fuchs (1907) for the Painlevé VI equation, and later by Jimbo and Miwa (1981, 1981/82) and Jimbo, Miwa, and Ueno (1981) for the other families. Bernard Malgrange reobtained the Painlevé Property in Malgrange (1983a,b) using isomonodromy nature of these differential equation. It is not know if there exist differential equations with Painlevé Property that do not arise from an isomonodromic problem.

The Painlevé Property is exactly what is needed to define the monodromy of the differential equation (not only a groupoid or pseudo-group, like it so happens usually for foliations). Then, the monodromy can be explicitely described and studied to derive chaotic properties of the solutions. This has been used in Cantat and Loray (2009) to prove the irreducibility of the Painlevé VI equation, based on Casale–Malgrange's approach. But the main application is surely the classification of algebraic solutions, that correspond to finite orbits under the action of the Painlevé monodromy on the set of initial conditions. This classification was initiated in Dubrovin and Mazzocco (2000), developped by Philip Boalch in a series of papers Boalch (2005, 2006a,b, 2007a,b, 2010), leading to a list which has been shown to be complete by Lisovyy and Tykhyy (2014), thereby closing the classification problem.

In Chapter 1, we provide some basic definitions and properties of linear systems with rational coefficients on the Riemann sphere, and their monodromy representation. The notion of Fuchsian systems is introduced (systems with simple poles) and it is shown, as an example, how to compute the monodromy representation of hypergeometric systems: rank 2 Fuchsian systems with 3 poles on the Riemann sphere. When we increase rank or number of poles, the computation of monodromy representation becomes transcendental, no more explicit. One can only compute local eigenvalues of the system, and derive by an exponential the eigenvalues of local monodromy around poles. Once we fix these local datas, we have moduli spaces of systems up to gauge transformations, and representations up to conjugacy, that turn to have the same dimension. The monodromy map is then almost an isomorphism. In order to really define an isomorphism (a bijection), one has to consider logarithmic connections instead of systems, that is to say a holomorphic vector bundle E equipped with a linear meromorphic connection ∇ having only simple poles.

The notion of holomorphic and meromorphic connection on curves is developped in Chapter 2, and we establish (in the rank two case) the Riemann–Hilbert correspondance, which provides a bijection between moduli spaces of connections and moduli spaces of representations. This is done in the holomorphic and logarithmic setting. Without details we explain how the two moduli spaces can be constructed in an algebraic way. The Riemann–Hilbert correspondance provides a transcendental analytic isomorphism between these quasi-projective moduli spaces. Going back to the Riemann sphere with 4 poles, one can then explicitely describe the two moduli spaces.

In Chapter 3, we define isomonodromic deformations and explain how this notion leads to the Painlevé VI equation. By considering the Riemann–Hilbert correspondance in family, we see that we can uniquely deform the poles (and systems) without deforming the monodromy representation. Schlesinger showed that such a deformation generally comes from slicing a flat connection on the total deformation space. The notion of flat logarithmic connection, and its relation ship with isomonodromic deformations, is developped in Chapter 3. We end the chapter by explaining how to derive Painlevé VI solutions from coefficients of isomonodromic deformation of a rank 2 system with 4 poles on the Riemann sphere.

In Chapter 4, we use this correspondence between Painlevé VI solutions and deformations of systems to explain why Painlevé VI equation satisfies the Painlevé property: this is geometrically traduced into a non linear analytic local system on the deformation of moduli spaces. This allows us to define the (non linear !) monodromy representation of the Painlevé VI equation which is a well-known dynamical system since works of William Goldman: it is the action of the Mapping Class Group on the character variety (moduli space of linear representations). We end the book by explaining how this dynamical system has been used to prove transcendence of first integrals and solutions of Painlevé VI equation, as well as, to classify all special Riccati and algebraic solutions.

We have omitted numerous aspects of Painlevé equations in this text. The goal was to offer an initiation through some geometrical aspect of it. We hope that this text will inspire the reader to learn more about one of the most beautiful subject in the theory of differential equations.

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1 Linear systems on the Riemann sphere

In this chapter, we define the monodromy representations of a linear differential equation. Let $\mathbb{C}$ be the line over the complex numbers, and denote by x the variable. We denote by $\mathcal{O}$ the sheaf of homolorphic functions: for any open set $U \subset \mathbb{C}$, $\mathcal{O}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on U. Denote by $\mathcal{O}^*$ the sheaf of non vanishing functions.

1.1 Linear systems of differential equations with rational coefficients

1.1.1 Linear systems with rational coefficients

In this text, we consider linear systems of differential equations given in matrix form by

$$\frac{dY}{dx} + A(x)Y = 0 \tag{1.1}$$

where $A(x) = \big(a_{ij}(x)\big)_{ij}$ is a matrix of rational functions $a_{ij} \in \mathbb{C}(x)$, of rank r say, and the unknown variable $Y(x)$ is a column vector of local functions $y_i(x)$. Equivalently, one can write

$$\begin{cases} \frac{dy_1}{dx} + a_{11}y_1 + \cdots + a_{1r}y_r &= 0 \\ \vdots & \\ \frac{dy_r}{dx} + a_{r1}y_1 + \cdots + a_{rr}y_r &= 0 \end{cases} \tag{1.2}$$

A *local solution* of (1.1) is the data of an open set $U \subset \mathbb{C}$ and a holomorphic vector $Y(x)$ (or holomorphic functions $y_i(x)$) defined on U satisfying the differential equations (1.1) (resp. (1.2)). The coefficients a_{ij} may have poles, and the union of these poles for all $1 \leqslant i, j \leqslant r$ form a finite set $S = \{x_1, \dots, x_n\} \subset \mathbb{C}$. The differential system is holomorphic on $\mathbb{C} \setminus S$.

1.1.2 Local trivialization

An immediate corollary of Cauchy Theorem yields:

Lemma 1.1.1. *For any point $x_0 \in \mathbb{C} \setminus S$, and any sufficiently small connected open neighborhood $U \ni x_0$, solutions of (1.1) form a vector space of dimension r. In other words, there is a $r \times r$ holomorphic matrix $B(x)$ satisfying*

$$\frac{dB}{dx} + A(x)B(x) = 0 \quad \textit{and} \quad \det(B) \neq 0 \tag{1.3}$$

whose columns $B = (Y^1(x), \dots, Y^r(x))$ form a basis of local solutions.

Proof. On a given open set, one easily checks that solutions form a vector space. By Cauchy–Lipschitz Theorem, any initial condition Y^0 gives rise to a unique local holomorphic solution $Y^0(x)$ such that $Y^0(x_0) = Y^0$. Given a basis $(Y^1, \dots, Y^r)$ for $\mathbb{C}^r$, one easily derives local solutions $B := (Y^1, \dots, Y^r)$ on some common neighborhood U of x_0. One can notice that, maybe shrinking U, $\det(B(x))$ does not vanish on U. □

An equivalent way to state Lemma 1.1.1 is as follows. The system (1.1) defines a holomorphic vector field

$$v = \partial_x - (a_{11}y_1 + \dots + a_{1r}y_r)\partial_{y_1} - \dots - (a_{r1}y_1 + \dots + a_{rr}y_r)\partial_{y_r}$$

which admits a global (w.r.t. coordinate y_i's) flow-box or trivialisation:

$$\Phi_* v = \partial_x \quad \text{where} \quad \Phi(x, Y) = (x, F(x)Y) := (x, B^{-1}(x)Y)$$

is an automorphism of $U \times \mathbb{C}^r$. It satisfies $F^{-1}dF = A(x)dx$, i.e.

$$d(FY) = 0 \quad \Leftrightarrow \quad dY + \underbrace{F^{-1}dF}_{A(x)dx} Y = 0.$$

This local trivialization matrix is unique up to composition $F(x) \rightsquigarrow MF(x)$ by a constant matrix $M \in \mathrm{GL}_r(\mathbb{C})$.

1.1.3 Gauge transformations

A change of local trivialization $Y = M(x)\widetilde{Y}$ (also called *gauge transformation*) of the system (1.1), with $M \in \mathrm{GL}_r(\mathbb{C}(x))$, i.e. with rational coefficients, yields in the new variable $\widetilde{Y}$

$$\frac{d(M\widetilde{Y})}{dx} + A(M\widetilde{Y}) = 0$$

$$\Leftrightarrow M\frac{d\widetilde{Y}}{dx} + \frac{dM}{dx}\widetilde{Y} + AM\widetilde{Y} = 0$$

$$\Leftrightarrow \frac{d\widetilde{Y}}{dx} + \underbrace{\left(M^{-1}AM + M^{-1}\frac{dM}{dx}\right)}_{\widetilde{A}(x)}\widetilde{Y} = 0$$

which gives the gauge relation

$$\widetilde{A} = M^{-1}AM + M^{-1}\frac{dM}{dx}. \tag{1.4}$$

The new system is also with rational coefficients. We will also use gauge transformations with local holomorphic or meromorphic coefficients. In the case M is constant, the gauge transformation is just the conjugacy $M^{-1}A(x)M$.

1.1.4 Trace of a system and reduction to sl_r-systems

Lemma 1.1.2. *Under notations of Lemma 1.1.1, $\delta(x) := \det(B(x))$ is solution of the differential equation*

$$\frac{d\delta}{dx} + \mathrm{tr}(A(x)) \cdot \delta = 0.$$

In particular, if $\mathrm{tr}(A(x)) \equiv 0$, then we can choose $B(x)$, or $F(x)$, taking values into $\mathrm{SL}_r(\mathbb{C})$. We then say that $A(x)$ defines a sl_r-system.

Proof. Let C_B denotes the *comatrix* of B: the i^{th} column of C_B is given by $Z^i = (-1)^{i+1}Y^1 \wedge \cdots \wedge \widehat{Y^i} \wedge \cdots \wedge Y^r$, where hat notation means that Y^i is omitted. We have: ${}^tC_B \cdot B = B \cdot {}^tC_B = \delta \cdot I$. On the other hand, we have:

$$
\begin{aligned}
\frac{d}{dx}\delta &= \frac{d}{dx}\det(B) = \frac{d}{dx}(Y^1 \wedge \cdots \wedge Y^r) \\
&= \frac{dY^1}{dx} \wedge Y^2 \wedge \cdots \wedge Y^r + Y^1 \wedge \frac{dY^2}{dx} \wedge \cdots \wedge Y^r + \cdots + Y^1 \wedge Y^2 \wedge \cdots \wedge \frac{dY^r}{dx} \\
&= \frac{dY^1}{dx} \wedge Z^1 + Y^2 \wedge Z^2 + \cdots + Y^r \wedge Z^r \\
&= \mathrm{tr}\left(C_B^t \cdot \frac{dB}{dx}\right) = -\mathrm{tr}\left(C_B^t \cdot AB\right) = -\mathrm{tr}\left(\underbrace{BC_B^t}_{\delta\cdot I} \cdot A\right) = -\delta \cdot \mathrm{tr}(A).
\end{aligned}
$$

If $\mathrm{tr}(A(x)) \equiv 0$, then $\delta := \det(B(x))$ satisfies $d\delta \equiv 0$, and is therefore constant. If we choose $B(x)$ so that $B(x_0) = I$ corresponds to the canonical basis, then $\det(B(x)) \equiv 1$ and monodromy of $F(x) = B^{-1}(x)$ takes values in $\mathrm{SL}_r(\mathbb{C})$. □

Remark 1.1.3. *More generally, if the matrix $A(x)$ of system (1.1) takes values in a Lie sub-algebra $\mathfrak{g} \subset \mathrm{gl}_r(\mathbb{C})$ (the Lie algebra of $r \times r$ matrices), then the monodromy will take value in the corresponding Lie group $G \subset \mathrm{GL}_r(\mathbb{C})$, i.e. the minimal algebraic group that contains the flows of $\mathfrak{g}$, provided that we conveniently choose the local trivialization $F(x)$: for instance, setting $F(x_0) = I$. Indeed, integrating the system is equivalent to integrating a non autonomous vector field in the Lie algebra. But the converse is not true. For instance, given any rational function $f \in \mathbb{C}(x)$, the rank 1 system $\frac{dy}{dx} - \frac{df}{fdx}y = 0$ has trivial monodromy although taking value in $\mathrm{gl}_1(\mathbb{C})$.*

One can easily modify the trace of a system as follows. Let $Y(x)$ be a solution of a system (1.1) and $f(x)$ be solution of a single linear differential equation (rank 1 system)

$$\frac{df}{dx} + \lambda(x) f = 0$$

then $\widetilde{Y}(x) = f(x)Y(x)$ satisfies

$$\frac{d}{dx}(fY) = \frac{df}{dx} \cdot Y + f \cdot \frac{dY}{dx} = -(\lambda f)Y - f(AY) = -\lambda\widetilde{Y} - A\widetilde{Y}$$

and is therefore solution of the new system

$$\frac{d}{dx}\widetilde{Y} + \underbrace{(\lambda I + A)}_{\widetilde{A}}\widetilde{Y} = 0.$$

For instance, if we set $\lambda = -\frac{\mathrm{tr}(A)}{r}$, then $\mathrm{tr}(\widetilde{A}) = 0$ and $\widetilde{A}$ is the sl_2-system associated to A. It is quite easy to understand solutions of a rank 1 system (see Section 1.5.1) and it is therefore equivalent to understand solutions (and later monodromy) of system A and $\widetilde{A}$. We will often work with sl_r-systems, better than gl_r-systems.

1.1.5 Compactification on the Riemann sphere

It is natural to consider the system (1.1) on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the added point $x = \infty$ plays no role. For instance, changing $x = \frac{1}{z}$ yields

$$\frac{dY}{dz} - \underbrace{z^2 A\left(\frac{1}{z}\right)}_{\widetilde{A}(z)} Y = 0 \tag{1.5}$$

which is also a linear system with rational coefficients. We notice that $x = \infty$ might be singular (a pole of the new matrix $\widetilde{A}$) or not.

1.1.6 Fuchsian system

The system (1.1) has a pole of order k at x_i (finite or infinite) if k is the maximal order of pole for the coefficients of the matrix ($A(x)$ or $\widetilde{A}(z)$) at x_i. We will say that the system (1.1) is *Fuchsian* if it has only simple poles on the Riemann sphere $\widehat{\mathbb{C}}$. A straightforward computation shows that this is the case if, and only if,

$$A(x) = \frac{A_1}{x - x_1} + \cdots + \frac{A_n}{x - x_n}, \quad A_i \in \mathrm{GL}_r(\mathbb{C}); \tag{1.6}$$

moreover, at $x = \infty$, after setting $x = \frac{1}{z}$, the system is given by

$$\frac{dY}{dz} + \left(\frac{A_\infty}{z} + \textit{holomorphic}\right) Y = 0 \quad \text{where} \quad A_\infty = -\sum_{i=1}^{n} A_i. \tag{1.7}$$

1.2 Monodromy representation

1.2.1 Analytic continuation

The local trivialization $F(x)$ constructed in Section 1.1 can be analytically continued along any path avoiding the polar set S. Precisely,

Lemma 1.2.1. *For any local trivialization $F(x)$ of system (1.1) defined on a neighborhood of x_0, and any path $\gamma : [0,1] \to \mathbb{C} \setminus S$ starting at $\gamma(0) = x_0$, then F can be analytically continued along γ and provides a new trivialization F^γ at $x_1 = \gamma(1)$. Moreover, the resulting local trivialization F^γ at x_1 only depends on the homotopy type of γ with fixed enpoints.*

Proof. We can cover the path by open sets U_i equipped with basis of solutions F_i in such a way that i is varying from 0 to n along γ. Therefore, we get

$$\begin{array}{ccccccccc} F = F_0 & = & M^{01}F_1 & = & M^{01}M^{12}F_2 & = & \cdots & = & M^\gamma F_n =: F^\gamma \\ \text{on } U_0 & \uparrow & \text{on } U_1 & \uparrow & \text{on } U_2 & & & & \text{on } U_n \\ & U_0 \cap U_1 & & U_1 \cap U_2 & & & & & \end{array}$$

where M^γ is the composition of transition matrices $M^{i,i+1}$ along γ. Clearly, after deforming a little bit γ with fixed extremities, the sequence of discs is still doing the job. One easily checks for large deformations that we just need to convince that we can add or delete intermediate open sets along γ without modifying the resulting F^γ. □

Corollary 1.2.2. *Over any simply connected open set $U \subset \mathbb{C} \setminus S$, there exists a trivialization $F(x)$ of system (1.1).*

1.2.2 Monodromy representation

When $x_1 = x_0$, i.e. γ is a loop, then F^γ is a new trivialization of (1.1) at x_0 and must write $F^\gamma = M^\gamma F$ for some constant matrix $M^\gamma \in \mathrm{GL}_r(\mathbb{C})$. This gives rise to the *monodromy representation*:

Proposition 1.2.3. *Given a local trivialization F of the system (1.1) at $x_0 \in \mathbb{C} \setminus S$, we have a morphism of groups*

$$\rho_F : \pi_1(\mathbb{C} \setminus S, x_0) \to \mathrm{GL}_r(\mathbb{C}) \ ; \ [\gamma] \mapsto M^\gamma$$

where $M^\gamma F$ is the analytic continuation of F along γ. Moreover, changing for another local trivialization $F' = MF$ has the effect to conjugate:

$$\rho_{F'} = M \rho_F M^{-1}.$$

Proof. The first part is again a direct consequence of Lemma 1.2.1. For the last assertion, just note that

$$(F')^\gamma = (MF)^\gamma = (M^\gamma)(F^\gamma) = M(M^\gamma F) = (MM^\gamma M^{-1})F'$$

(since M is constant, we have $M^\gamma = M$). □

1.3 Fuchs local study

1.3.1 Normalization of the residual matrix

Here, we concentrate on the study of solutions and monodromy near a simple pole of the system (1.1). Assume for simplicity that $x = 0$ is a simple pole of the matrix:

$$\frac{dY}{dx} + \underbrace{\left(\frac{A_{-1}}{x} + \mathit{holomorphic}\right)}_{A(x)} Y = 0, \quad \text{where } A_{-1} \in \mathrm{GL}_r(\mathbb{C}). \tag{1.8}$$

We would like to simplify system (1.8) by local change of coordinates $M(x) \in \mathrm{GL}_r(\mathcal{O})$. The first remark is that $M^{-1}\frac{dM}{dx}$ is always holomorphic, and we promptly deduce:

$$\frac{d\widetilde{Y}}{dx} + \underbrace{\left(\frac{M_0^{-1}A_{-1}M_0}{x} + \mathit{holomorphic}\right)}_{\widetilde{A}(x)}\widetilde{Y} = 0, \quad \text{where } M_0 = M(0).$$

We can therefore assume that A_{-1} is already in Jordan normal form, e.g. in rank 2 case:

$$A_{-1} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} \tag{1.9}$$

and eigenvalues and Jordan types are uniquely defined up to gauge transformation. Eigenvalues of the residue matrix are called *residual eigenvalues*. .

1.3.2 Fuchs' relation

It promptly follows from (1.7) that

$$\operatorname{tr}(A_1) + \cdots + \operatorname{tr}(A_n) + \operatorname{tr}(A_\infty) = 0. \tag{1.10}$$

In particular, if we denote by θ_i^+, θ_i^-, say, the residual eigenvalues at each pole x_i, then we have that the residue of $\operatorname{tr}(A_i)$ is $\theta_i^+ + \theta_i^-$, and therefore

$$(\theta_1^+ + \theta_1^-) + \cdots + (\theta_n^+ + \theta_n^-) + (\theta_\infty^+ + \theta_\infty^-) = 0 \tag{1.11}$$

(known as Fuchs' relation).

1.3.3 Poincaré–Dulac normal forms

For simplicity, we state the complete classification in the rank 2 case:

Theorem 1.3.1 (Poincaré–Dulac in the rank 2 case)**.** *Up to a local holomorphic gauge transformation, we can assume that the matrix of system (1.8) is transformed into*

$$\widetilde{A}(x)dx = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}\frac{dx}{x}, \quad \text{or} \quad \begin{pmatrix} \theta & x^n \\ 0 & \theta + n \end{pmatrix}\frac{dx}{x} \quad \text{for some } n \in \mathbb{Z}_{\geqslant 0}. \tag{1.12}$$

Moreover, this normal form is unique up to permutation $\theta_1 \leftrightarrow \theta_2$.

Remark 1.3.2. *We note that, in the second normal form, we can assume $n = 0$ after meromorphic gauge transformation (1.4) with matrix $M = \operatorname{diag}(1, x^{-n})$. More generally, using meromorphic gauge $M = \operatorname{diag}(x^{n_1}, x^{n_2})$, we can shift eigenvalues of normal forms by arbitrary integer (with only restriction $n_1 \leqslant n_2 + n$ in the second case).*

Proof. Assume that A_{-1} is in Jordan normal form (1.9). The idea is to delete (or simplify at most as possible) holomorphic part $A(x)$ by a formal gauge transformation, and then prove that it is convergent. The formal gauge transformation $Y = \widehat{M}(x)\widetilde{Y}$ is obtained by infinite composition of simple gauge transformations of the form $Y_k = (I + M_k x^k)Y_{k+1}$ with $M_k \in \mathrm{gl}_2(\mathbb{C})$ so that

$$\widehat{M}(x) = (I + M_1 x)(I + M_2 x^2)\cdots(I + M_k x^k)\cdots$$

which is convergent in $\mathrm{GL}_2(\mathbb{C}[[x]])$. Let us see how it works.

We note that

$$(I + M_k x^k)^{-1} = I - M_k x^k + o(x^k)$$

so that, for $M = I + M_k x^k$, we get

$$M^{-1}AM + M^{-1}\frac{dM}{dx} = A + (\{A_{-1}, M_k\} + kM_k)\, x^{k-1} + o(x^{k-1})$$

where $\{A_{-1}, M_k\} = A_{-1}M_k - M_k A_{-1}$ is the Lie bracket. For the respective Jordan normal forms (1.9) for A_{-1}, we find:

$$M_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \{A_{-1}, M_k\} + kM_k = \begin{pmatrix} ka & (\theta_1 - \theta_2 + k)b \\ (\theta_2 - \theta_1 + k)c & kd \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} ka + c & kb + d - a \\ kc & kd - c \end{pmatrix}.$$

In the first case, where $A_{-1} = \mathrm{diag}(\theta_1, \theta_2)$, we get two cases:

- if $\theta_1 - \theta_2 \notin \mathbb{Z}$, then we can successively delete the positive coefficients of $A(x) = A_{-1}\frac{dx}{x} + \sum_{k\geqslant 0} A_k x^k dx$ by solving $A_{k-1} + \{A_{-1}, M_k\} + kM_k = 0$ at each step, and this yields the formal normalization;
- if $\theta_2 - \theta_1 = n \in \mathbb{Z}_{\geqslant 0}$, then at the n^{th} step, i.e. for $k = n$, the $(1,2)$-coefficient of $\{A_{-1}, M_n\} + nM_n$ vanishes and we cannot kill the corresponding coefficient of A_{n-1}: if non zero, we get the second normal form of the statement (after normalizing the non linear coefficient by a constant diagonal gauge transformation). If the $(1,2)$-coefficient of A_{n-1} was zero, then we end with the diagonal normal form.

When A_{-1} is not diagonal, then one can check that equation $A_{k-1} + \{A_{-1}, M_k\} + kM_k = 0$ has always a solution and we can reduce A to its residual part.

In order to prove the convergence, assume for simplicity that $\theta_1 - \theta_2 \notin \mathbb{Z}^*$ so that there is an

$$\widehat{M}(x) = I + M_1 x + M_2 x^2 + \cdots$$

such that $\widehat{M}^{-1}A\widehat{M} + M^{-1}\frac{d\widehat{M}}{dx} = \frac{A_{-1}}{x}$ (with A_{-1} like in (1.9)), i.e.

$$\{A_{-1}, \widehat{M}\} + \frac{d\widehat{M}}{dx} + \underbrace{A_0 + A_1 x + A_2 x^2 + \cdots}_{A_{\geqslant 0}(x)} = 0$$

The coefficients of x^{k-1} writes

$$\{A_{-1}, M_k\} + kM_k + (A_{-1}M_{k-1} + \cdots + A_{k-2}M_1) = 0.$$

Let $\|M\|$ denote the sup norm on $\mathrm{gl}_2(\mathbb{C})$ (the supremum of modulus of coefficients). For the convergence of M, we need to prove that $\frac{\|M_k\|}{\|M_{k-1}\|}$ is uniformly bounded. We first note that

$$\{A_{-1}, M\} + kM + A = 0 \quad \Rightarrow \quad \|M\| \leqslant c\|A\|, \quad \forall M, A \in \mathrm{gl}_2(\mathbb{C})$$

where $c \geqslant \sup_{k\in\mathbb{Z}} \frac{1}{|\theta_1-\theta_2+k|}$ and $c \geqslant 1$. On the other hand, if the local coordinate is such that $A_{\geqslant 0}(x)$ converges in norm on the disc of radius 1, then we have $\sum_{k\geqslant 0} \|A_k\| \leqslant a$ for some constant $a \in \mathbb{C}$. Then we deduce

$$\|M_k\| \leqslant c \cdot a \cdot \sum_{l=1,\ldots,k-1} \|M_l\|$$

which yields by induction that $\|M_k\| \leqslant (c')^{k-1}\|M_1\|$, with $c' = \sum\{1, c \cdot a\}$, whence the convergence.

Finally, when $\theta_2 = \theta_1 + n$ with $n \in \mathbb{Z}_{>0}$ say, we can either modify a bit the previous proof, of proceed as follows. We first normalize A up to order n by truncating the formal gauge normalization $\widehat{M}$ by taking its jet $J^n\{\widehat{M}\}$ of order n. Then we may apply a meromorphic gauge transformation $M = \mathrm{diag}(1, x^{-n})$ to shift eigenvalues and get θ_1 with multiplicity 2. Then the previous proof applies providing some convergent normalizing gauge transformation $M(x)$ (tangent to I up to order n). The one easily check that the composition $J^n\{\widehat{M}\} \cdot \mathrm{diag}(1, x^{-n}) \cdot M \cdot \mathrm{diag}(1, x^n)$ is holomorphic and normalizing A. □

Remark 1.3.3. *In higher rang, we can found a normal form $A(x)$ with monomial coefficients in a very similar way. When $\theta_j = \theta_i + n$ with $n \in \mathbb{Z}_{\geqslant 0}$, then the monomial x^n in coefficient (i, j) of $A(x)$ cannot be killed. A standart assumption due to Deligne (1970) is $0 \leqslant \Re(\theta_i) < 1$ (real part) which implies that there is no resonance $\theta_j - \theta_i \in \mathbb{Z}^*$ and the holomorphic part can be deleted to reduce to the residual matrix in Jordan Normal Form.*

1.3.4 Local monodromy

One can now compute the local monodromy around a pole by integrating the normal form. Precisely, let $\mathbb{D} \subset \mathbb{C}$ be a disc centered at the singular point $x = 0$, small enough so that normalization of Theorem 1.3.1 holds on $\mathbb{D}$. Denote by $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ the punctured disc, and fix some base point $x_0 \in \mathbb{D}^*$. Then $\pi_1(\mathbb{D}^*, x_0)$ is generated by the loop

$$\gamma : [0, 1] \to \mathbb{D}^* \; ; \; t \mapsto e^{2i\pi t} x_0.$$

The monodromy representation $\rho \in \mathrm{Hom}(\pi_1(\mathbb{D}^*, x_0), \mathrm{GL}_2(\mathbb{C}))$ of the system restricted to $\mathbb{D}$, called the *local monodromy* of the global system at $x = 0$, is determined by the monodromy M^γ along γ of a local trivialization $F(x)$ at x_0.

Proposition 1.3.4. *A (multiform) local trivialization for the normal forms (1.12) is respectively given by*

$$F(x) = \begin{pmatrix} x^{\theta_1} & 0 \\ 0 & x^{\theta_2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x^{\theta} & x^{\theta+n}\log(x) \\ 0 & x^{\theta+n} \end{pmatrix}$$

and the monodromy $\Phi^{\gamma} = M^{\gamma}\Phi(x)$ is respectively given by

$$M^{\gamma} = \begin{pmatrix} e^{2i\pi\theta_1} & 0 \\ 0 & e^{2i\pi\theta_2} \end{pmatrix} \quad \text{or} \quad e^{2i\pi\theta}\begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}.$$

Proof. In the first case, after integration, a basis of solutions is given by $B = \begin{pmatrix} x^{-\theta_1} & 0 \\ 0 & x^{-\theta_2} \end{pmatrix}$ and the trivialization is given by $F = B^{-1}$. The second case is similar but needs integration by parts. □

1.4 Link with scalar and Riccati equations

In this section, we mention the relationship with linear scalar equations of higher order, and, in the rank $r = 2$ case, with the Riccati equation. Details can be found in any classical book of ordinary differential equations.

1.4.1 Linear scalar equation of order r

Consider a linear differential equation

$$u^{(r)} + f_{r-1}(x)u^{(r-1)} + \cdots + f_1(x)u' + f_0(x)u = 0 \tag{1.13}$$

with rational coefficients $f_i \in \mathbb{C}(x)$, where $u^{(i)}$ is the i^{th} derivative w.r.t. x. One can associate a system of first order linear equations by setting

$$y_1 = u, \quad y_2 = u', \quad \ldots \quad y_r = u^{(r-1)} \tag{1.14}$$

and get a linear system with rational coefficients in (transposed) companion form:

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{r-1} \\ y_r \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -f_0 & -f_1 & -f_2 & \ldots & -f_r \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{r-1} \\ y_r \end{pmatrix} \tag{1.15}$$

Conversely, given a linear system (1.1), we can recursively define

$$Y_{i+1} = dY_i + A(x)Y_i$$

starting from the constant vector $Y_1 = e_1$, say. If we are lucky, the matrix $M(x) = (Y_1, \ldots, Y_r)$ has non identically-zero determinant and $M \in \mathrm{GL}(\mathbb{C}(x))$ provides a gauge change transforming system (1.1) into companion form (or its dual, we have to transpose the matrix), and deduce a order r scalar equation. We note that the starting vector Y_1 could be an arbitrary (even not constant) vector, and we just require that $\det(M(x)) \not\equiv 0$.

We note that these two operations increase the polar set. The first one increase the order of poles, while the second one is adding poles at those points where $\det(M(x))$ is vanishing. Therefore, we cannot deduce a one-to-one correspondance. However, we can list some remarkable facts without proof:

- If we transform a system-to-scalar, and then scalar-to-system, then the inital and last systems are related by a birational gauge transformation $M(x)$.
- Local solutions of a scalar equation at a point where all f_i's are holomorphic form a vector space, and we can similarly define the monodromy representation; then monodromy of scalar equations and systems corresponding by the two operations above have same monodromy representation (up to conjugacy).
- In some special and important cases, it is possible to construct an almost one-to-one correspondance between families of systems and families of scalar equations (see for instance Dubrovin and Mazzocco (2007)).
- Litterature on differential equations around 1900 was dealing mainly with scalar equation (hypergeometric, Lamé, Heun).
- Some properties of the singular set are easier to characterize on scalar equation (see next section).

Let us detail, in the rank 2 case, the second transformation from system to scalar. Assume we are given a matrix system

$$\frac{d}{dx}Y + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y = 0$$

and assume that $\gamma \not\equiv 0$. Then the matrix $M := \begin{pmatrix} 1 & \alpha \\ 0 & \gamma \end{pmatrix}$ formed by the constant vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and its image by the differential operator $\frac{d}{dx} + A$, has non zero determinant. We can therefore apply gauge transformation and get

$$M^{-1}AM + M^{-1}dM = \begin{pmatrix} 0 & \widetilde{\beta} \\ 1 & \widetilde{\delta} \end{pmatrix} \quad \text{where} \quad \begin{cases} \widetilde{\beta} = \underbrace{\beta\gamma - \alpha\delta}_{-\det(A)} + d\alpha - \alpha\frac{d\gamma}{\gamma} \\ \widetilde{\delta} = \underbrace{\alpha + \delta}_{\mathrm{tr}(A)} + \frac{d\gamma}{\gamma} \end{cases} \tag{1.16}$$

and we derive the scalar equation

$$u'' - \left(\mathrm{tr}(A) + \frac{d\gamma}{\gamma}\right)u' + \left(\det(A) + \alpha\frac{d\gamma}{\gamma} - d\alpha\right)u = 0 \tag{1.17}$$

1.4.2 Fuchs' criterium for regular-singular points

Rational gauge transformations modify poles and their order. It is therefore natural to ask whether if it is possible to transform a given system into a Fuchsian system, i.e. having only simple poles. In general, it is not possible: for each pole, one can define a minimal pole order up to local meromorphic gauge transformation. This is closely related to Poincaré rank of the system at the singular point. Lazarus Fuchs gave a criterium for the possible reduction to simple pole.

Theorem 1.4.1 (Fuchs)**.** *Let $\frac{dY}{dx} + A(x)Y = 0$ be a rational (or meromorphic) system at the neighborhood of $x = 0$, and let (1.13) be a scalar equation corresponding from operations of Section 1.4.1. Then are equivalent:*

- *the system is locally equivalent to a system with a simple pole at $x = 0$ by a meromorphic gauge transformation;*
- *the coefficients of the scalar equation satisfy: f_i has a pole of order $\leqslant r - i$;*
- *solutions[1] of the scalar equation have at most polynomial growth at $x = 0$;*
- *solutions of the system have at most polynomial growth at $x = 0$.*

Since Deligne (1970), such singular points of linear differential equations are called *regular-singular*. Be careful that a system having only regular-singular points needs not be globally equivalent to a Fuchsian system: the reduction to simple pole by local meromorphic transformations might not be simultaneously performed at all poles by a single rational gauge transformation. This is one of the issue in the Riemann–Hilbert problem (see Bolibrukh (1990)).

1.4.3 Riccati equations

By linearity, a system $\frac{dY}{dx} + A(x)Y = 0$ induces a differential equation on projective coordinate $(y_1 : \cdots : y_r) \in \mathbb{P}^{r-1}$. Let us detail the rank $r = 2$ case.

Define $(y_1 : y_2) = (1 : y) \in \mathbb{P}^1$, i.e. y is an affine coordinate (we identify $\mathbb{P}^1 = \widehat{\mathbb{C}}$). Starting from the system

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A(x)} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

[1] local solutions are multiform, and therefore by solutions we mean here their determinations on sectors $\{x\ ;\ \theta_1 < \arg(x) < \theta_2,\ 0 < |x| < r\}$

we deduce the Riccati equation for $y = \frac{y_2}{y_1}$:

$$\frac{dy}{dx} - a_{12}y^2 + (a_{22} - a_{11})y + a_{21} = 0.$$

We observe that the trace of the system disappears through the projectivization. In fact, a Riccati equation

$$\frac{dy}{dx} + a(x)y^2 + b(x)y + c(x) = 0. \tag{1.18}$$

can be viewed as a PGL_2 differential equation, by considering the corresponding vector field

$$\partial_x - (a(x)y^2 + b(x)y + c(x))\partial_y$$

whose right-hand-side is a vector field on $\widehat{\mathbb{C}}$ with parameter x. The Lie algebra is isomorphic to $\mathrm{sl}_2(\mathbb{C})$, and the Riccati equation lifts uniquely as a SL_2 system, namely:

$$\frac{dY}{dx} + \begin{pmatrix} -\frac{b}{2} & -a \\ c & \frac{b}{2} \end{pmatrix} Y = 0$$

i.e. the Riccati equation is closer to a trace-free system. Precisely, there is a unique system with trace $\mathrm{tr}(A(x)) \equiv 0$ inducing the Riccati equation by projectivization.

If we start from the 2^{nd} order linear differential equation

$$u'' + f_1(x)u' + f_0(x)u = 0$$

then setting $y = \frac{u'}{u}$, we get the Riccati equation

$$\frac{dy}{dx} + y^2 + f_1(x)y + f_0(x) = 0.$$

Gauge transformations $Y = M(x)\widetilde{Y}$ on the system induce Moebius transformations (with parameter x) $y = \overline{M}(x)\tilde{y}$ on the Riccati equation. For instance, setting $y = \frac{1}{\tilde{y}}$ in equation (1.18) yields

$$\frac{dy}{dx} - c(x)y^2 - b(x)y - a(x) = 0.$$

If $a(x) \not\equiv 0$, then the change $y = \frac{\tilde{y}}{a(x)}$ gives the new Riccati equation

$$\frac{d\tilde{y}}{dx} + \tilde{y}^2 + \left(b - \frac{a'}{a}\right)\tilde{y} + ac = 0$$

which is equivalent to the scalar equation

$$u'' + \underbrace{\left(b - \frac{a'}{a}\right)}_{f_1}u' + \underbrace{ac}_{f_0}u = 0.$$

Finally, it is interesting to note that the change of coordinate $y = \widetilde{y} + g(x)$ transforms the Riccati / scalar equation

$$u'' + f_1 u' + f_0 u = 0$$

into

$$u'' + \underbrace{(f_1 + 2g)}_{\widetilde{f}_1} u' + \underbrace{(g' + g^2 + f_1 g + f_0)}_{\widetilde{f}_0} u = 0 \tag{1.19}$$

We will use this change to normalize the scalar equation in the next section.

1.4.4 Normalizations of scalar equations

Moebius transformations on the Riccati equation allow to normalize the corresponding scalar equation in (1.19). The most natural normalization consists in choosing $g = -\frac{f_1(x)}{2}$ in (1.19) so that the scalar equation is in Sturm–Liouville normal form

$$u'' + \phi(x)u = 0. \tag{1.20}$$

On the other hand, in the regular-singular case, one might want to have only simple poles. This is indeed feasible in the scalar equation. To see this, one observes that at a given pole, say $x = 0$, the change of coordinate $y = \widetilde{y} + \frac{c}{x}$ transforms

$$u'' + \left(\frac{\nu_1}{x} + \text{holomorphic}\right) u' + \left(\frac{\nu_0}{x^2} + \text{simple pole}\right) u = 0 \tag{1.21}$$

into

$$u'' + \left(\frac{\nu_1 + 2c}{x} + \text{holomorphic}\right) u' + \left(\frac{c^2 + (\nu_1 - 1)c + \nu_0}{x^2} + \text{simple pole}\right) u = 0;$$

for a convenient value of c, we can delete the pole of order 2. We can easily do this globally on $\mathbb{C}$ at all regular-singular poles. One exazmple is the well-known Gauss hypergeometric equation (1.27) which is normalized with simple poles.

In order to deduce a Fuchsian system from a scalar equation with regular-singular points, it is convenient to replace variables (1.14) by

$$y_1 = u, \quad y_2 = P(x)u' \tag{1.22}$$

where $P(x)$ is a reduced equation of poles, so that the resulting system

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{P} \\ P f_0 & f_1 - \frac{P'}{P} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{1.23}$$

has only simple poles.

1.4.5 Local exponents

A differential system is locally characterized at the neighborhood of a simple pole by its eigenvalues, at least when they do not differ by integers. When we projectivize, only the difference of eigenvalues make sense. In the rank 2 case, if we compare scalar equation (1.21) with the corresponding system (1.23)

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \left(\frac{\begin{pmatrix} 0 & -1 \\ \nu_0 & (\nu_1 - 1) \end{pmatrix}}{x} + \text{holomorphic} \right) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

we have the following relationship between difference of the two eigenvalues θ_1, θ_2 of the residual matrix, and the corresponding terms ν_0, ν_1 of the scalar equation (1.21):

$$(\theta_1 - \theta_2)^2 = (\nu_1 - 1)^2 - 4\nu_0$$

This difference $\theta := \theta_1 - \theta_2$, defined up to a sign, is called *exponent* of the differential equation.

1.5 Some examples

1.5.1 Rank one case

In case $r = 1$, then $A = a(x)$ is a rational function and integration yields

$$\int a(x)dx = \theta_1 \log(x - x_1) + \cdots + \theta_n \log(x - x_n) + f(x)$$

for a rational function $f \in \mathbb{C}(x)$. Then solutions take the form

$$y(x) = c(x - x_1)^{-\theta_1} \cdots (x - x_n)^{-\theta_n} e^{-f(x)}, \quad c \in \mathbb{C}^*$$

so that the monodromy representations is given by

$$\begin{array}{rccc} \rho : & \pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}) & \to & \mathrm{GL}_1(\mathbb{C}) \simeq \mathbb{C}^* \\ & \gamma_i & \mapsto & e^{2\sqrt{-1}\pi\theta_i} \end{array}$$

for the standard presentation

$$\pi_1(\widehat{\mathbb{C}} \setminus \{x_1, \ldots, x_n\}) = \langle \gamma_1, \ldots, \gamma_n \, ; \, \gamma_1 \cdots \gamma_n = 1 \rangle$$

with γ_i turning once around x_i and homotopic to 1 in $\mathbb{C} \setminus \{x_j, \ j \neq i\}$.

1.5.2 Two poles on $\hat{\mathbb{C}}$: Euler systems

In case $n = 1$, we get a pole at $x = 0$, say, and the system writes

$$\frac{dY}{dx} + \frac{A_0}{x} = 0$$

and the other pole at $x = \infty$ has residual matrix $-A_0$. The monodromy of the generating loop turning once around 0 in trigonometric sense is given by

$$M_0 = \exp(2i\pi A_0).$$

1.5.3 Three poles on $\hat{\mathbb{C}}$: Hypergeometric systems

Let us consider a rank 2 system with 3 poles on $\widehat{\mathbb{C}}$, say at $x = 0, 1, \infty$. Let us denote by θ_i^+, θ_i^- the two eigenvalues at each pole $x_i = 0, 1, \infty$. Assuming that eigenvectors of θ_0^+ and θ_1^- form a basis, we can express the system in this basis and get (after renormalizing the length)

$$\frac{dY}{dx} + A(x)Y = 0 \quad \text{with} \quad A(x) = \frac{\begin{pmatrix} \theta_0^+ & 1 \\ 0 & \theta_0^- \end{pmatrix}}{x} + \frac{\begin{pmatrix} \theta_1^+ & 0 \\ c & \theta_1^- \end{pmatrix}}{x-1} \tag{1.24}$$

and the constant c is determined by eigenvalues:

$$A_\infty = -\begin{pmatrix} \theta_0^+ + \theta_1^+ & 1 \\ c & \theta_0^- + \theta_1^- \end{pmatrix}, \quad \underbrace{\theta_\infty^+ \theta_\infty^-}_{\det(A_\infty)} = (\theta_0^+ + \theta_1^+)(\theta_0^- + \theta_1^-) - c. \tag{1.25}$$

Note that we also have Fuchs relation (1.11)

$$(\theta_0^+ + \theta_0^-) + (\theta_1^+ + \theta_1^-) + (\theta_\infty^+ + \theta_\infty^-) = 0$$

Following Section 1.1.4, we can further normalize eigenvalues by adding some scalar term $\left(\frac{c_0}{x} + \frac{c_1}{x-1}\right) I$ to the system, which translates eigenvalues $\theta_i^\pm \rightsquigarrow \theta_i^\pm + c_i$, $i = 0, 1$. This preserves the difference $\theta_i = \theta_i^+ - \theta_i^-$. By this way, we can reduce to a sl_2-system $(\theta_i^+, \theta_i^-) = \left(\frac{\theta_i}{2}, -\frac{\theta_i}{2}\right)$. On the other hand, it is interesting to note that our system is related to the famous Gauss' hypergeometric equation after alternate normalization $(\theta_i^+, \theta_i^-) = (0, -\theta_i)$:

$$A(x) = \frac{\begin{pmatrix} 0 & 1 \\ 0 & -\theta_0 \end{pmatrix}}{x} + \frac{\begin{pmatrix} 0 & 0 \\ c & -\theta_1 \end{pmatrix}}{x-1} = \begin{pmatrix} 0 & \frac{1}{x} \\ \frac{c}{x-1} & -\frac{\theta_0}{x} - \frac{\theta_1}{x-1} \end{pmatrix}.$$

Here we have by Fuchs relation and (1.25) that $\theta_\infty^\pm = \frac{\theta_0+\theta_1\pm\theta_\infty}{2}$ and

$$c = \frac{\theta_\infty^2 - (\theta_0+\theta_1)^2}{4}. \tag{1.26}$$

Then, from cyclic vector e_1, we deduce the scalar equation (see (1.17))

$$x(x-1)u'' + ((\alpha+\beta+1)x-\gamma)u' + \alpha\beta u = 0 \tag{1.27}$$

with parameters

$$\alpha = \frac{\theta_0+\theta_1+\theta_\infty}{2}, \quad \beta = \frac{\theta_0+\theta_1-\theta_\infty}{2}, \quad \text{and} \quad \gamma = -\theta_0. \tag{1.28}$$

The monodromy representation of the sl_2-system

$$\frac{dY}{dx} + A(x)Y = 0 \quad \text{with} \quad A(x) = \frac{\begin{pmatrix} \frac{\theta_0}{2} & 1 \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}}{x} + \frac{\begin{pmatrix} \frac{\theta_1}{2} & 0 \\ c & -\frac{\theta_1}{2} \end{pmatrix}}{x-1} \tag{1.29}$$

can be also determined from eigenvalues differences $(\theta_0, \theta_1, \theta_\infty)$ in general. We know that such a representation is determined by matrices

$$M_0,\ M_1,\ M_\infty \in \mathrm{SL}_2(\mathbb{C}), \quad \text{with} \quad M_0 M_1 M_\infty = I$$

such that M_i has eigenvalues $\lambda_i^{\pm 1} = e^{\pm\sqrt{-1}\pi\theta_i}$, and therefore

$$\mathrm{tr}(M_i) = 2\cos(\pi\theta_i).$$

Assuming that eigenvectors of λ_0 and λ_1^{-1} form a basis, the monodromy writes in this basis:

$$M_0 = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \lambda_1 & 0 \\ \mu & \lambda_1^{-1} \end{pmatrix} \tag{1.30}$$

and μ is determined by

$$\underbrace{\lambda_\infty + \lambda_\infty^{-1}}_{2\cos(\pi\theta_\infty)} = \mathrm{tr}(M_\infty) = \mu + \underbrace{\lambda_0\lambda_1 + (\lambda_0\lambda_1)^{-1}}_{2\cos(\pi(\theta_0+\theta_1))}. \tag{1.31}$$

Recall that the representation defined by (M_0, M_1) (or equivalently the group $\langle M_0, M_1\rangle$) is said reducible if there is a common eigendirection for the two matrices.

Proposition 1.5.1. *Let $M_0, M_1 \in \mathrm{SL}_2(\mathbb{C})$ and denote*

$$t_0 = \mathrm{tr}(M_0), \quad t_1 = \mathrm{tr}(M_1) \quad \textit{and} \quad t_\infty = \mathrm{tr}(M_\infty).$$

The monodromy group $\langle M_0, M_1\rangle$ is reducible if, and only if,

$$t_0^2 + t_1^2 + t_\infty^2 - t_0 t_1 t_\infty - 4 = 0. \tag{1.32}$$

Moreover, in the irreducible case, i.e. if triple (t_0, t_1, t_∞) does not satisfy equation (1.32), then all pairs (M_0, M_1) with triple (t_0, t_1, t_∞) are SL_2-conjugated to the pair (1.30) with (1.31).

In other words, the set of SL_2-conjugacy classes of irreducible representations $(M_0, M_1) \in \mathrm{SL}_2(\mathbb{C})^2$ is in one-to-one correspondance with the set

$$\left\{(t_0, t_1, t_\infty) \in \mathbb{C}^3 \ ; \ t_0^2 + t_1^2 + t_\infty^2 - t_0 t_1 t_\infty - 4 \neq 0.\right\}$$

On the other hand, in the reducible case, the pair (M_0, M_1) writes in a convenient basis

$$M_0 = \begin{pmatrix} \lambda_0 & \mu_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \lambda_1 & \mu_1 \\ 0 & \lambda_1^{-1} \end{pmatrix} \tag{1.33}$$

maybe permuting $\lambda_i \leftrightarrow \lambda_i^{-1}$. Then, one easily check that the pairs defined by $(\mu_0, \mu_1) = (0,0)$ or $(0,1)$ are not conjugated (only the first one shares two common eigendirections). However, they define the same triple of traces.

Proof. Assume that the monodromy group is reducible. Then there is a common eigenvector for M_0 and M_1, that we can assume to be e_1 after base change, i.e.

$$M_0 = \begin{pmatrix} \lambda_0 & \mu_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \lambda_1 & \mu_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}$$

maybe permuting $\lambda_i \leftrightarrow \lambda_i^{-1}$. We then have

$$M_\infty = \begin{pmatrix} (\lambda_0\lambda_1)^{-1} & \star \\ 0 & \lambda_0\lambda_1 \end{pmatrix} \quad \text{and} \quad \lambda_0\lambda_1 = \lambda_\infty^{\pm 1}.$$

This is equivalent to say that

$$\underbrace{\lambda_\infty + \frac{1}{\lambda_\infty} - \left(\lambda_0\lambda_1 + \frac{1}{\lambda_0\lambda_1}\right)}_{\mu} = 0$$

Other possibility, after permuting $\lambda_i \leftrightarrow \lambda_i^{-1}$ yield the other equality

$$\lambda_\infty + \frac{1}{\lambda_\infty} - \left(\frac{\lambda_0}{\lambda_1} + \frac{\lambda_1}{\lambda_0}\right) = 0$$

After multiplication, we get (1.32) of the statement.

Conversely, assume first $\lambda_0\lambda_1 \neq \lambda_\infty^{\pm 1}$. Then we can put M_0, M_1 into the normal form (1.30). Eigenvectors are

$$\begin{array}{cccc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ \frac{1}{\lambda_0} - \lambda_0 \end{pmatrix} & \begin{pmatrix} \lambda_1 - \frac{1}{\lambda_1} \\ \mu \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda_0 & \lambda_0^{-1} & \lambda_1 & \lambda_1^{-1} \end{array}$$

Therefore, M_0 and M_1 share a common eigenvector if, and only if, we have

$$\mu = 0, \quad \text{or} \quad \mu = \left(\lambda_0 - \frac{1}{\lambda_0}\right)\left(\lambda_1 - \frac{1}{\lambda_1}\right).$$

But from (1.31), we deduce that $\mu \neq 0$ and the last equality writes

$$\lambda_\infty + \frac{1}{\lambda_\infty} - \left(\frac{\lambda_0}{\lambda_1} + \frac{\lambda_1}{\lambda_0}\right) = 0.$$

So we can conclude the proof of the first part of the statement. The second part follows from the above discussion and normalization. □

We have a family of systems parametrized by $(\theta_0, \theta_1, \theta_\infty) \in \mathbb{C}^3$ through formulae (1.29) with (1.26). This parametrizes a dense open set of SL_2-systems with simple poles $0, 1, \infty$ over the Riemann sphere $\widehat{\mathbb{C}}$. Mind that changing signs $(\pm\theta_0, \pm\theta_1, \pm\theta_\infty)$ provide systems that are SL_2-conjugated in general, for instance when $\langle A_0, A_1 \rangle$ is not reducible: these systems just correspond to different normalizations.

We conclude that have a monodromy map from our family of system to the corresponding monodromy which is given by

$$(\theta_0, \theta_1, \theta_\infty) \quad \mapsto \quad (t_0, t_1, t_\infty) = (2\cos(\pi\theta_0), 2\cos(\pi\theta_1), 2\cos(\pi\theta_\infty)).$$

In particular, the system (1.30) cannot be deformed without deforming its monodromy representation. On the other hand, the global monodromy is determined by conjugacy classes of local monodromy around poles, i.e. by local eigenvalues. We say that it is a *rigid system* (see Katz (1996)).

1.5.4 Schwarz and Klein

There is a natural group action of the space of our family of hypergeometric systems. As explained before, we can change signs $(\pm\theta_0, \pm\theta_1, \pm\theta_\infty)$ and deduce an action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We can also permute the poles by a Moebius transformation, and deduce an action of the symmetric group S_3 by permutations on the triple. And last but not least, we have the following transformation. The gauge change on system (1.29) given by $M(x) =$

$\begin{pmatrix} 0 & 1 \\ \frac{cx}{x-1} & 0 \end{pmatrix}$ yields

$$M^{-1}AM + M^{-1}dM = \frac{\begin{pmatrix} -\frac{\theta_0}{2} & 1 \\ 0 & +\frac{\theta_0}{2} \end{pmatrix}}{x} + \frac{\begin{pmatrix} -\frac{\theta_1}{2} & 0 \\ c & +\frac{\theta_1}{2} \end{pmatrix}}{x-1} + \begin{pmatrix} \frac{1}{x} - \frac{1}{x-1} & 0 \\ 0 & 0 \end{pmatrix}$$

which, after sl_2-reduction, gives system (1.29) with parameter $(1-\theta_0, 1-\theta_1, \theta_\infty)$. Combining with previous transformations, we get an infinite group of transformations Γ acting on parameters. A normal subgroup is given by translations by integers:

$$(\theta_0, \theta_1, \theta_\infty) \mapsto (\theta_0 + n_0, \theta_1 + n_1, \theta_\infty + n_\infty), \quad n_i \in \mathbb{Z}, \quad n_0 + n_1 + n_\infty \in 2\mathbb{Z}.$$

It is important to note that the monodromy group $\langle M_0, M_1 \rangle$ is left unchanged by the action of the group Γ on corresponding systems.

The famous classification of Schwarz can be stated as follows.

Theorem 1.5.2 (Schwarz). *Are equivalent:*

- *the linear system (1.24) has only algebraic solutions;*
- *the monodromy of system (1.24) takes values in a finite group;*
- *exponents $(\theta_0, \theta_1, \theta_\infty)$ take value in the finite list below, up to action of Γ:*
  - D_n: $\left(\frac{1}{2}, \frac{1}{2}, \frac{k}{n}\right)$ *(dihedral);*
  - A_4: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$, *or* $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ *(tetrahedral);*
  - S_4: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$, *or* $\left(\frac{2}{3}, \frac{1}{4}, \frac{1}{4}\right)$ *(octahedral);*
  - A_5: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$, $\left(\frac{2}{5}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, \frac{1}{5}, \frac{1}{5}\right)$, $\left(\frac{1}{2}, \frac{2}{5}, \frac{1}{5}\right)$, $\left(\frac{3}{5}, \frac{1}{3}, \frac{1}{5}\right)$, $\left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right)$, $\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{5}\right)$, $\left(\frac{4}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5}\right)$, *or* $\left(\frac{3}{5}, \frac{2}{5}, \frac{1}{3}\right)$ *(icosahedral).*

The proof contains the following ideas. First of all, if all solutions are algebraic, then they have finitely many determinations, and monodromy action has only finite orbits. This implies that the group $\langle M_0, M_1 \rangle$ is finite. Secondly, for all finite groups in $\mathrm{SL}_2(\mathbb{C})$, one can classify all generating triples of matrices M_0, M_1, M_∞ and deduce their traces (t_0, t_1, t_∞): the list of the statement is a choice of preimages $(\theta_0, \theta_1, \theta_\infty)$ for each of them. Conversely, one has to verify that this list gives rise to only finite groups. Then, solutions are finitely sheeted; Fuchs property for regular singular points implies analytic extension at each point $x = 0, 1, \infty$; global analytic solutions are algebraic.

In fact, there is an alternate statement due to Klein:

Theorem 1.5.3 (Klein). *If the* sl_2*-Fuchsian system* (S) *has finite monodromy group, then there is a rational map* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that* (S) *is rational-gauge equivalent to* f^*S_0 *where* (S_0) *is one of the following systems:*

- C_n: $\left(\frac{k}{n}, \frac{k}{n}\right)$ *(diagonal);*
- D_n: $\left(\frac{1}{2}, \frac{1}{2}, \frac{k}{n}\right)$ *(dihedral);*
- A_4: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$ *(tetrahedral);*
- S_4: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$ *(octahedral);*
- A_5: $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$ *(icosahedral).*

The first case is not hypergeometric, but a system with only two poles.

Sketch of proof. Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be the finite group that contains monodromy of (S). Then, up to conjugacy, G is one of the groups of the statement. Using a cyclic vector, the system (S) can be transformed into a scalar equation $E: \ u'' + \phi(x)u = 0$ whose solutions have finite monodromy. Let $\varphi = u_1/u_2$ be the ratio of two local independent solutions. Then φ is a local diffeomorphism whose global monodromy (under analytic continuation outside poles) takes values into G. Let $\pi : \mathbb{C} \to \mathbb{C}$ be the quotient map by the action of G by Moebius transformations. Then, the map $\pi \circ \varphi : \mathbb{C} \setminus \{x_1, \ldots, x_n\} \to \mathbb{C}$ is well-defined and one can check that it extends holomorphically at poles, providing a holomorphic map $f : \mathbb{C} \to \mathbb{C}$. On the other hand, if we start with the Fuchsian scalar equation (E_0) corresponding to G in the statement, then the corresponding local map φ_0 is, up to composition by a Moebius transformation, just a local determination of π^{-1}. It follows that

$$\varphi = \pi^{-1} \circ \underbrace{\pi \circ \varphi}_{f} = \varphi_0 \circ f.$$

On the other hand, we can retrieve the differential equation from φ (quotient of independent solutions): indeed, we have that $\frac{\phi}{2}$ is the Schwarzian derivative of φ (see de Saint-Gervais (2016, Chapt. IV)). Then it promptly follows that $(E) = f^*(E_0)$. $\square$

1.5.5 More poles

Consider a general Fuchsian system with $n+1$ poles

$$\frac{dY}{dx} + \left(\frac{A_1}{x - x_1} + \cdots + \frac{A_n}{x - x_n}\right) Y = 0, \quad A_\infty = -\sum_{i=1}^{n} A_i, \quad A_i \in \mathrm{SL}_2(\mathbb{C}). \quad (1.34)$$

or equivalently a n^{uple}

$$(A_1, \ldots, A_n) \in (\mathrm{SL}_2(\mathbb{C}))^n$$

For generic systems, one can normalize A_1, A_2 like in (1.29), and we need $3n - 3$ coefficients to describe the family of normalized systems: 3 for (A_1, A_2) and $3(n-2)$ for the remaining $n-2$ matrices. Denote by $\theta_1, \dots, \theta_n, \theta_\infty$ the difference of eigenvalues at each pole:

$$(\theta_i)^2 = -4\det(A_i).$$

Clearly, if we prescribe all θ_i's, there remain $2n-4$ free parameters: the θ_i's can determine the system only in the hypergeometric case. For $n+1 > 3$ poles, the moduli space of systems with fixed poles $x_1, \dots, x_n, \infty$ and eigenvalues $\pm\frac{\theta_i}{2}$ up to conjugacy has dimension $2n-4$.

The monodromy representation of such system is given by

$$(M_1, \dots, M_n), \quad t_i = \mathrm{tr}(M_i) = 2\cos(\pi\theta_i)$$

and in a very similar way, we can count parameters and check that the space of representations with fixed traces $(t_1, \dots, t_n, t_\infty)$ up to conjugacy has also dimension $2n-4$. The next chapter will show that the monodromy map between spaces of systems and representations (with fixed local spectral data and position of poles) is essentially one-to-one. After this, we will show that it is possible to deform poles and systems without changing the monodromy: these are isomonodromic deformations of linear systems, giving rise to Painlevé transcendents.

1.5.6 Moduli of sl_2-systems with 4 poles

Consider a sl_2-Fuchsian system on the Riemann sphere with polar divisor $D = 0 + 1 + t + \infty$

$$\frac{dY}{dx} + \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}\right) Y \tag{1.35}$$

where A_i's are constant matrices, $i = 0, 1, t, \infty$, satisfying

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \in \mathrm{sl}_2(\mathbb{C}) \quad \text{and} \quad A_0 + A_1 + A_t + A_\infty = 0. \tag{1.36}$$

The group $\mathrm{SL}_2(\mathbb{C})$ acts on the Y-variable and thus on residues A_i's by simultaneous conjugacy. Indeed, change of variable $Y = MY'$ induces change of residues $A_i' = M^{-1}A_iM$. The spectrum of each matrix A_i is preserved by this action. Let us fix the spectral data

$$-\det(A_i) = a_i^2 + b_ic_i = \theta_i^2 \tag{1.37}$$

for some $\boldsymbol{\theta} = (\pm\theta_0, \pm\theta_1, \pm\theta_t, \pm\theta_\infty)$. We want to describe the quotient space of those systems by this action:

$$\mathrm{Sys}^{\boldsymbol{\theta}}(X, D) = \left\{(A_0, A_1, A_t, A_\infty) \in (\mathrm{sl}_2(\mathbb{C}))^4 \; ; \; \begin{matrix} \sum_i A_i = 0, \\ \det(A_i) = -\theta_i^2 \end{matrix}\right\}_{/\mathrm{SL}_2(\mathbb{C})}. \tag{1.38}$$

A straightforward computation shows that $\mathrm{Sys}^{\boldsymbol{\theta}}(X,D)$ is expected to be a surface (depending on 4 parameters θ_i's). However, it is non Hausdorff (as a topological space) in general. For instance, the orbit of the triangular system

$$A_i = \begin{pmatrix} \theta_i & b_i \\ 0 & -\theta_i \end{pmatrix} \tag{1.39}$$

under diagonal conjugacy is not closed; its closure contains the diagonal system (with all b_i's vanishing) so that they define infinitesimally closed points in the quotient.

Introduce the following functions on $\mathrm{Sys}^{\boldsymbol{\theta}}(X,D)$:

$$\begin{gathered} x := \det(A_0 + A_1),\ y := \det(A_1 + A_t),\ z := \det(A_0 + A_t) \\ \text{and} \quad w = \mathrm{tr}\,(A_0[A_1, A_\infty]) \end{gathered} \tag{1.40}$$

where $[A,B] = AB - BA$ is the Lie bracket. These functions are clearly invariant under $\mathrm{SL}_2(\mathbb{C})$-action.

Proposition 1.5.4. *Assume $\theta_4 \neq 0$. Then, the map*

$$(x,y,z,w) \ :\ \mathrm{Sys}^{\boldsymbol{\theta}}(X,D) \longrightarrow \mathbb{C}^4 \tag{1.41}$$

sends the moduli space onto the affine cubic surface $S^{\boldsymbol{\theta}}$ defined by

$$\begin{gathered} x + y + z = \theta_0^2 + \theta_1^2 + \theta_t^2 + \theta_\infty^2 \quad \textit{and} \\ \tfrac{w^2}{4} + xyz + (\theta_0^2 - \theta_t^2)(\theta_1^2 - \theta_\infty^2)y + (\theta_1^2 - \theta_t^2)(\theta_0^2 - \theta_\infty^2)z \\ = (\theta_0^2 + \theta_1^2 - \theta_t^2 - \theta_\infty^2)(\theta_0^2\theta_1^2 - \theta_t^2\theta_\infty^2) \end{gathered} \tag{1.42}$$

The map (1.41) above is one-to-one over the smooth part of $S^{\boldsymbol{\theta}}$. Singularities arise when

- *$\theta_i = 0$ and $A_i = 0$ for $i = 1,2,3$;*
- *$\pm\theta_0 \pm \theta_1 \pm \theta_t \pm \theta_\infty = 0$ and all A_i's are simultaneously triangular up to conjugacy.*

In particular, apart from special values of $\boldsymbol{\theta}$ listed just before, the quotient $\mathrm{Sys}^{\boldsymbol{\theta}}(X,D)$ *is Hausdorff and is isomorphic to the smooth cubic surface $S^{\boldsymbol{\theta}}$.*

Proof. Since $\theta_\infty \neq 0$, we can assume $A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$. This normalization is well-defined up to diagonal conjugacy so that the monomials a_i and $b_i c_j$ are invariant. Using $A_0 + A_1 + A_t + A_\infty = 0$, we can now express A_t in function of A_0 and A_1:

$$\begin{cases} a_t &= -a_0 - a_1 - \theta_\infty \\ b_t &= -b_0 - b_1 \\ c_t &= -c_0 - c_1 \end{cases}$$

Spectral data (1.37) gives the following conditions

$$\begin{gathered} a_0^2 + b_0c_0 = \theta_0^2,\quad a_1^2 + b_1c_1 = \theta_1^2 \quad \text{and} \\ 2a_0a_1 + 2\theta_\infty(a_0 + a_1) + b_0c_1 + b_1c_0 + \theta_0^2 + \theta_1^2 - \theta_t^2 + \theta_\infty^2 = 0. \end{gathered} \tag{1.43}$$

On the other hand, we get

$$\begin{aligned} x &= 2\theta_\infty a_t + \theta_\infty^2 + \theta_t^2 \\ y &= 2\theta_\infty a_0 + \theta_\infty^2 + \theta_0^2 \quad \text{and} \quad w = 4\theta_\infty(b_0c_1 - b_1c_0) \\ z &= 2\theta_\infty a_1 + \theta_\infty^2 + \theta_1^2 \end{aligned} \tag{1.44}$$

Once we know x, y, z, w, we get the invariants a_0, a_1 and $b_0c_1 - b_1c_0$. From relations (1.43), we promptly deduce b_0c_0, b_1c_1 and $b_0c_1 + b_1c_0$, and therefore b_0c_1 and b_1c_0. The cubic equation (1.42) just says that these invariants satisfy the obvious relation $(b_0c_0)(b_1c_1) = (b_0c_1)(b_1c_0)$. We can thus recover (b_0, b_1, c_0, c_1) uniquely up to the diagonal action, except when either $b_0 = b_1 = 0$, or $c_0 = c_1 = 0$; in these latter cases, there are several possible solutions (A_0, A_1, A_t) up to diagonal action, but all of them are triangular. $\square$

2 The Riemann–Hilbert correspondance

In order to establish a one-to-one correspondance between families of linear differential equations and linear representations of fundamental groups, one has to enlarge the set of objects, namely replace linear systems by vector bundles with linear connections.

2.1 The language of connections

Let X be a Riemann surface, not necessarily compact. In other words, X is a connected smooth complex manifold of dimension one; basic examples are $\widehat{\mathbb{C}}$ and its connected open sets. Details can be found in Forster (1991).

2.1.1 Vector bundles

A rank r vector bundle over X is a complex manifold E of dimension $r+1$ equipped with a projection $p : E \to X$ such that fibers are r-dimensional vector spaces and the structure of vector spaces vary holomorphically on X. This latter property can be explicited as follows. We have an open covering $X = \cup_i U_i$ by open sets (for analytic topology) and local trivializations:

$$\begin{tikzcd} E|_{U_i} := p^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "p"'] & & U_i \times \mathbb{C}^r \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

where ϕ_i's are biholomorphisms (and pr_1 projection to the first factor), satisfying compatibility condition of each non empty intersection $U_{ij} = U_i \cap U_j$:

$$\phi_i|_{U_{ij}} = \phi_{ij} \circ \phi_j|_{U_{ij}} \quad \text{where} \quad \phi_{ij} = \mathrm{id}_{U_{ij}} \times M_{ij}, \quad M_{ij} \in \mathrm{GL}_r(\mathcal{O}(U_{ij})).$$

Therefore, each fiber $p^{-1}(x)$ has a well-defined structure of r-dimensional vector space (not depending on the local chart ϕ_i we are looking at) and surrounding fibers look like a trivial bundle $U_i \times \mathbb{C}^r$. Of course, the simplest example is the trivial bundle on X defined by $E = X \times \mathbb{C}^r$.

2.1.2 Sections

A local (holomorphic) section of a vector bundle $p : E \to X$ over an open set $U \subset X$ is just a holomorphic map $s : U \to E|_U$ such that $p|_U \circ s : U \to U$ is the identity. The set of holomorphic sections form a vector space $H^0(U, E)$ whose dimension can be infinite. However, when $U = X$ is compact, then $H^0(X, E)$ has finite dimension.

It is convenient (see section 2.1.9) to consider the sheaf $\mathcal{E}$ of holomorphic sections of E which, to an open set U, associates

$$\mathcal{E}(U) \text{ (or } \Gamma(U, \mathcal{E})) := H^0(X, E)$$

See Forster (1991, sections 6) for more details.

2.1.3 Morphisms

A morphism from a vector bundle $p : E \to X$ to another one $p' : E' \to X$ is given by a holomorphic map $\Psi : E \to E'$ which sends fibers to fibers, and restricts as a linear transformation on each fiber. In charts, this means that we have commutative diagrams:

$$\begin{tikzcd} E|_{U_i} \arrow[r, "\Psi|_{U_i}"] \arrow[d, "\phi_i"] & E'|_{U_i} \arrow[d, "\phi_i"] \\ U_i \times \mathbb{C}^r \arrow[r, "\mathrm{id}_{U_i} \times M"] & U_i \times \mathbb{C}^{r'} \end{tikzcd}$$

where $M \in \mathrm{M}_{r,r'}(\mathcal{O}(U_i))$. We note that, maybe rescalling open coverings, we can assume that the trivialization open sets U_i are the same for E and E'. We say that Ψ is an *isomorphism* if it admits an inverse, which is equivalent to say that $r = r'$ and $M \in \mathrm{GL}_r(\mathcal{O}(U_i))$.

A trivialization of $E|_U$ over an open set $U \subset X$ is equivalent to the data of linearly independent holomorphic sections $s_1, \ldots, s_r : U \to E|_U$ on U, i.e. independent at any point $x \in U$. In that case, we can define an isomorphism $\mathcal{O}_U^{\oplus r} \to E|_U$ sending e_i to s_i. Automorphisms of $p : E \to X$ are isomorphisms $\Psi : E \to E$ and form a group.

In sheaf theoretic language, a morphism $\Psi : E \to E'$ induces a morphism $\Psi : \mathcal{E} \to \mathcal{E}'$ between sheaves of sections, namely the collection

$$\Psi|_U : \mathcal{E}(U) \to \mathcal{E}'(U) \; ; \; s \mapsto \Psi \circ s$$

of homomorphisms between vector spaces of sections over all open sets $U \subset X$.

2.1.4 Cohomology viewpoint

In sheaf theoretic language (see Forster (ibid., sections 6, 12, 29)), the collection $\{\phi_{ij}\}$ defines a cocycle of the Čech cohomology group

$$(M_{ij})_{ij} \in H^1(X, \mathrm{GL}_r(\mathcal{O}))$$

and any two cocycles define isomorphic vector bundles if, and only if, they define the same element in $H^1(X, \mathrm{GL}_r(\mathcal{O}))$, i.e. equivalent cocycles. Precisely, the collection (M_{ij}) satisfies the cocycle condition

$$M_{ij} M_{jk} M_{ki} \quad \text{on any triple intersection} \quad U_i \cap U_j \cap U_k \tag{2.1}$$

and any two collections (M_{ij}) and (M'_{ij}) (with same open covering) define isomorphic vector bundles if, and only if, there is a collection of local bundle isomorphism $M_i \in \mathrm{GL}_r(\mathcal{O}(U_i))$ such that

$$M'_{ij} M_j = M_i M_{ij} \quad \text{on any intersection} \quad U_i \cap U_j . \tag{2.2}$$

Then $H^1(X, \mathrm{GL}_r(\mathcal{O}))$ is the set of equivalence classes of cocycles (M_{ij}) (i.e. satisfying (2.1)) up to equivalence relation defined by (2.2), for a sufficiently fine covering (U_i). Therefore, the moduli space of rank r vector bundles on X up to isomorphisms identifies with $H^1(X, \mathrm{GL}_r(\mathcal{O}))$. When X is not compact, then any vector bundle is trivial on X, i.e. $H^1(X, \mathrm{GL}_r(\mathcal{O})) = 0$ (see Forster (ibid., section 30)). However, in the compact case, the study of the structure of this set is tricky in general.

2.1.5 Sum, tensor product, determinant

The sum $E \oplus E'$ of two vector bundles $p : E \to X$ and $p' : E' \to X$ with respective cocycles (M_{ij}) and (M'_{ij}) is the vector bundle of rank $r + r'$ defined by the cocycle

$$E \oplus E' : \begin{pmatrix} M_{ij} & 0 \\ 0 & M'_{ij} \end{pmatrix}.$$

The tensor product $E \otimes E'$ is the vector bundle of rank $r \cdot r'$ given by the cocycle

$$E \otimes E' \;:\; M_{ij} \otimes M'_{ij}.$$

In the particular case $r' = 1$ (a line bundle), then this is just scalar multiplication by $m'_{ij} \in \mathcal{O}^*(U_{ij})$. The determinant of a vector bundle $p : E \to X$ is the line bundle defined by

$$\det(E) \;:\; \det(M_{ij}).$$

We denote by $\mathcal{E} \oplus \mathcal{E}'$, $\mathcal{E} \otimes \mathcal{E}'$ and $\det(\mathcal{E})$ the corresponding sheaves of sections.

2.1.6 Line bundles

Line bundles are particular cases of interest. Since $\mathrm{GL}_1(\mathcal{O}) = \mathcal{O}^*$ is a sheaf of abelian groups, then $H^1(X, \mathcal{O}^*)$ is a group. The multiplicative group law on $H^1(X, \mathcal{O}^*)$ corresponds to the tensor product of line bundles. The trivial line bundle is the neutral element, and the inverse line bundle, given by (m_{ij}^{-1}). This group also identifies with the Picard group of linear equivalence classes of divisors: $\mathcal{O}(D_1) \simeq \mathcal{O}(D_2)$ iff $D_1 \sim D_2$ iff $D_1 - D_2 = \mathrm{div}(f)$ for some meromorphic function $f \in \mathcal{M}(X)$. The degree of a line bundle is the degree of the divisor. When X is a compact curve, then it is well known that

$$H^1(X, \mathcal{O}^*) \simeq \mathrm{Jac}(X) \times \mathbb{Z}$$

where $\mathrm{Jac}(X)$ is a g-dimensional complex torus, where g is the genus of X and $\mathbb{Z}$ stands for the degree of the line bundle. In particular, we have $H^1(\widehat{\mathbb{C}}, \mathcal{O}^*) = \mathbb{Z}$ and let us describe these line bundles.

2.1.7 Over the Riemann sphere

We can cover $\widehat{\mathbb{C}}$ by two charts $U_0 = \mathbb{C}$ and $U_\infty = \widehat{\mathbb{C}} \setminus \{0\} \simeq \mathbb{C}$ with transition chart given by $z = 1/x$:

$$\begin{tikzcd}
U_0 \arrow[r, hook] \arrow[d, "x"] & U_{0\infty} \arrow[r, "m_{\infty 0}"] \arrow[d, "x"] & U_{0\infty} \arrow[d, "z"] & U_\infty \arrow[l, hook'] \arrow[d, "z"] \\
\mathbb{C} \arrow[r, hook] & \mathbb{C}^* \arrow[r, "1/x"] & \mathbb{C}^* & \mathbb{C} \arrow[l, hook']
\end{tikzcd}$$

Then, we define, for $k \in \mathbb{Z}$, the line bundle $\mathcal{O}(k)$ by the cocycle $(\phi_{0\infty})$ where

$$\{m_{0\infty} = x^k, m_{\infty 0} = z^k\}$$

The group law is $\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) = \mathcal{O}(k_1 + k_2)$. Global holomorphic sections of $\mathcal{O}(k)$ are given in charts by polynomials of degree k:

$$y_0 = P(x), \quad y_\infty = z^k P\left(\frac{1}{z}\right), \quad \deg(P) \leqslant k$$

(with convention that $P \equiv 0$ if $\deg(P) < 0$). Therefore, $\mathcal{O}(k)$ is characterized by the dimensions of $H^0(\widehat{\mathbb{C}}, \mathcal{O}(k))$ and $H^0(\widehat{\mathbb{C}}, \mathcal{O}(-k))$, and we deduce that $\mathcal{O}(k_1) \simeq \mathcal{O}(k_2)$ iff $k_1 = k_2$.

Theorem 2.1.1 (Birkhoff–Grothendieck). *On $\widehat{\mathbb{C}}$, any vector bundle $p : E \to X$ splits as a product of line bundles and writes*

$$E = \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r) \quad \textit{for unique} \quad k_1 \leqslant \cdots \leqslant k_r.$$

We have $\det(E) = \mathcal{O}(k_1 + \cdots + k_r)$.

This theorem has been proved completely by Birkhoff. Later Grothendieck gave a shorter proof that algebraic vector bundles on $\widehat{\mathbb{C}}$ are splitting. Be careful that the decomposition is not unique: for instance, the trivial bundle decomposes as product of trivial line bundles in many different ways.

2.1.8 Higher genus

Vector bundles on elliptic curves (i.e. genus $g = 1$) have been classified by Atiyah (1957); some of them are not splitting. Narasimhan and Ramanan described the moduli space of SL_2-vector bundles in Narasimhan and Ramanan (1969): most of vector bundles are not splitting. For general curve with $g > 1$, rank r and degree d, there is a moduli space of semi-stable vector bundles using Mumford's Geometric Invariant Theory, which is an irreducible projective variety of dimension $(g-1)r^2 + 1$.

2.1.9 Connections

Let $p : E \to X$ be a vector bundle and $\mathcal{E}$ the sheaf of holomorphic sections. A linear holomorphic connection on E is a morphism (i.e. $\mathbb{C}$-linear morphism) of sheaves

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X,$$

satisfying the Leibniz rule:

$$\nabla|_U(f \cdot s) = df \otimes s + f \cdot \nabla_U(s), \tag{2.3}$$

for all $U \subset X$ open, $f \in \mathcal{O}(U)$ function, and $s \in \mathcal{E}(U)$ section. In order to avoid multiplication of notations, we will often write

$$\nabla : E \to E \otimes \Omega^1_X,$$

and therefore make confusion between E and its sheaf of sections $\mathcal{E}$. But mind that ∇ does not define neither a morphism of bundles, nor a map between total spaces of bundles.

In each trivialization chart $U_i \times \mathbb{C}^r$, a section is given by a linear combination

$$s = y_1 \cdot e_1 + \cdots + y_r \cdot e_r \quad \leftrightarrow \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

where $y_k \in \mathcal{O}(U_i)$. Through this trivialization, the connection writes:

$$Y \mapsto dY + \mathbf{A}_i \cdot Y, \quad \mathbf{A}_i \in \Gamma(U_i, \mathrm{gl}_r(\mathcal{O}) \otimes \Omega^1)$$

i.e. $\mathbf{A}_i$ is a $r \times r$ matrix whose coefficients $\alpha_{k,l} \in \Omega^1(U_i)$ are given by

$$\nabla(e_l) = \alpha_{1l} e_1 + \cdots + \alpha_{rl} e_r.$$

Compatibility condition between open sets is given by

$$\mathbf{A}_j = M_{ij}^{-1} \mathbf{A}_i M_{ij} + M_{ij}^{-1} dM_{ij}. \tag{2.4}$$

2.1.10 Horizontal sections

Horizontal sections with respect to ∇ (or ∇-horizontal sections) are those sections $s \in \Gamma(U, E)$ satisfying $\nabla(s) = 0$; through a local trivialization chart $U \times \mathbb{C}^r$, they are defined by solutions $Y(x)$ of the linear system of differential equations $dY + \mathbf{A}Y = 0$ which, through a local coordinate x on X, writes $\frac{dY}{dx} + A(x)Y = 0$ where $\mathbf{A} = Adx$. We denote by $\mathcal{E}^\nabla \subset \mathcal{E}$ the subsheaf whose sections are ∇-horizontal sections of E.

Example 2.1.2. *The trivial vector bundle $\mathcal{O}^{\oplus r} = X \times \mathbb{C}^r$ admits the trivial connection $d : Y \mapsto dY$. Horizontal sections are constant sections. Conversely, a connection ∇ on $X \times \mathbb{C}^r$ whose horizontal sections are constant is the trivial connection $\nabla = d$.*

2.1.11 Morphisms

A morphism of connections $\Psi : (E, \nabla) \to (E', \nabla')$ is a morphism of vector bundle sending (local) ∇-horizontal sections to ∇'-horizontal sections. We say that it is an *isomorphism* (resp. *automorphism*) if it is as a morphism of vector bundles.

As an immediate consequence of Lemma 1.1.1, a holomorphic connection (E, ∇) on X is locally trivial: for each point $x \in X$, there is a neighborhood $U \ni x$ and an isomorphism $\Phi : (E, \nabla)|_U \to (\mathcal{O}_U^{\oplus r}, d)$ with the trivial connection on U.

2.1.12 Monodromy

Then we can define the *monodromy representation* of (E, ∇). Start with a local trivialization $\Phi : (E, \nabla)|_U \to (\mathcal{O}_U^{\oplus r}, d)$ at a base point $x_0 \in X$. Given a path $\gamma : [0, 1] \to X$ strating at $x_0 = \gamma(0)$, we can define the analytic continuation Φ^γ of Φ along γ in a very similar way than in Section 1.2.1. When γ is a loop, then we have $\Phi^\gamma = M^\gamma \Phi$ for a matrix $M^\gamma \in \mathrm{GL}_r(\mathbb{C})$. We define the monodromy representation

$$\rho_\Phi : \pi_1(X, x_0) \to \mathrm{GL}_r(\mathbb{C}) \ ; \ [\gamma] \mapsto M^\gamma$$

The map Φ provides, by restriction, an isomorphism $\Phi_{x_0} : E|_{x_0} \overset{\sim}{\to} \mathbb{C}^r$. This allows to view the monodromy representation as a homomorphism

$$\pi_1(X, x_0) \to \mathrm{GL}(E|_{x_0})$$

defined by $(\Phi|_{x_0})^{-1} \circ M^\gamma \circ \Phi|_{x_0}$, which is in some sense a more intrinsic representation, not depending on the choice of Φ.

2.1.13 Reducible connections

A subbundle $F \subset E$ is called ∇-*invariant* if it is locally generated by ∇-horizontal sections. In that case, ∇ restricts to F as a connection $\nabla|_F : F \to F \otimes \Omega^1$. We then say that the connection is *reducible*. This implies that the monodromy group $G = \text{image}(\rho_\Phi)$ is reducible, since it must preserve the subspace $F|_{x_0}$ (viewed as acting on the fiber). There is a converse.

Proposition 2.1.3. *Let $G \subset \mathrm{GL}_r(\mathbb{C})$ be the image of the monodromy representation. There is a one-to-one correspondance between:*

1. *fixed points of G $\leftrightarrow$ global ∇-horizontal sections of E;*
2. *invariant subspaces of G $\leftrightarrow$ ∇-invariant subbundles $F \subset E$;*
3. *linear transformations $\psi \in \mathrm{GL}_r(\mathbb{C})$ commuting with G $\leftrightarrow$ automorphisms $\Psi : (E, \nabla) \to (E, \nabla)$ of the connection.*

In particular, ∇ is reducible if and only if its holonomy group G is reducible.

Proof. Let $s : X \to E$ be a global section; then, clearly, $s|_{x_0} \subset E|_{x_0}$ is fixed by the monodromy action, i.e. $\Phi \circ s \in \mathbb{C}^r$ is fixed by G. Conversely, if $v \in \mathbb{C}^r$ is fixed by G, then the corresponding section $s = \Phi^{-1}(v)$ can be analytically continued along paths as any local ∇-horizontal sections, and satisfies $s^\gamma = s$ whatever is $\gamma \in \pi_1(X, x_0)$. It follows that s extends as a global ∇-horizontal section on X.

Similarly, ∇-invariant subbundles are generated by ∇-horizontal sections and we have a similar correspondance (these sections might have monodromy though).

Finally, if ψ commutes with G, then the local conjugacy Ψ defined as $\Psi := \Phi^{-1} \circ \psi \circ \Phi$ has no monodromy:

$$\Psi^\gamma = (\Phi^\gamma)^{-1} \circ \psi \circ \Phi^\gamma = \Phi^{-1} \circ \underbrace{(M^\gamma)^{-1} \circ \psi \circ M^\gamma}_{\psi} \circ \Phi = \Psi.$$

It is therefore globally defined. The converse is left as an exercise. $\square$

Corollary 2.1.4. *On a simply connected manifold X, all connections are trivial (i.e. isomorphic to the trivial connection).*

Proof. All local ∇-horizontal sections extend as global sections; a maximal family of independent sections $s_1, \ldots, s_r$ provides a global trivialization Φ by sending s_k on e_k. □

In particular, all connections on $\widehat{\mathbb{C}}$ are trivial.

2.1.14 Riemann–Hilbert correspondance

More generally, the monodromy representation determines the isomorphism class of the connection.

Theorem 2.1.5 (Riemann–Hilbert Correspondance: the holomorphic case). *Let (E, ∇) and (E', ∇') be two connections of rank r on X, and Φ, Φ' be respective local trivializations at $x_0 \in X$, and ρ, ρ' their monodromy representations.*

1. *Then*

$$(E, \nabla) \simeq (E', \nabla') \quad \Longleftrightarrow \quad \rho' = M\rho M^{-1} \text{ for some } M \in \mathrm{GL}_r(\mathbb{C}).$$

2. *Given a representation $\rho \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_r(\mathbb{C}))$, then there exists a connection (E, ∇) such that $\rho_{\nabla,\Phi} = \rho$ for some local ∇-trivialization Φ.*

Proof. 1. If we have an isomorphism $\Psi : (E, \nabla) \xrightarrow{\sim} (E', \nabla')$, then for any local trivialization Φ' for ∇', we get a local trivialization $\Phi := \Phi' \circ \Psi$ for ∇. After analytic continuation along a loop γ, we get $M^\gamma \Phi = (M')^\gamma \Phi' \circ \Psi$. Since Ψ is globally defined, we immediately deduce that $M^\gamma = (M')^\gamma$, i.e. that monodromy representations coincide.

Conversely, assume that monodromy representations of $(E, \nabla), (E', \nabla')$ are conjugated as in the statement. One can first assume that $M = I$ (the identity) after changing Φ to $M\Phi$. Then $\Psi := (\Phi')^{-1} \circ \Phi$ is a local isomorphism conjugating (E, ∇) to (E', ∇'). We can make analytic continuation of Φ, Φ' and therefore Ψ along paths. Since monodromy representations coincide for Φ and Φ', we deduce that Ψ is uniform under analytic continuation and defines a global isomorphism $\Psi : (E, \nabla) \xrightarrow{\sim} (E', \nabla')$.

2. We first note that, given a connection (E, ∇) on X, then we can pull-back $\pi^*(E, \nabla)$ on the universal cover $\pi : \widetilde{X} \to X$ and any local trivialization Φ therefore extends by analytic continuation as a global trivialization $\widetilde{\Phi} : \pi^*(E, \nabla) \xrightarrow{\sim} (\mathcal{O}^{\oplus r}, d)$. Moreover, the natural action by covering transformations on the total space $\pi^* E$ preserves (horizontal sections of) ∇; it is conjugated by $\widetilde{\Phi}$ to the twisted action $(\tilde{x}, Y) \mapsto (\gamma \cdot \tilde{x}, M^\gamma \cdot Y)$. Now, let $\rho \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_r(\mathbb{C}))$. Then we have an action

$$\pi_1(X, x_0) \times (\widetilde{X} \times \mathbb{C}^r) \to (\widetilde{X} \times \mathbb{C}^r)\ ;\ (\gamma, (\tilde{x}, Y)) \mapsto (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot Y)$$

which is discrete and free. The quotient $E := (\widetilde{X} \times \mathbb{C}^r)/_{\pi_1(X,x_0)}$ inherits the structure of a holomorphic vector bundle over $X = \widetilde{X}/_{\pi_1(X,x_0)}$ via the projection $(\tilde{x}, Y) \mapsto \tilde{x}$.

Indeed, the action on Y is linear. On the other hand, the trivial connection d on $\widetilde{X}\times\mathbb{C}^r$ is invariant under the action, inducing a connection ∇ on the quotient. By construction, the monodromy representation of ∇ is ρ. $\square$

2.1.15 Trace and sl_r-connections

The *trace* of the connection (E,∇) is the rank 1 connection defined in charts by

$$d+\mathrm{tr}(\mathbf{A}_i)$$

on the line bundle $\det(E)$. When $(\det(E),\mathrm{tr}(\nabla))$ is the trivial connection, then we say that (E,∇) is a sl_r*-connection*; in that case, the monodromy takes values in $\mathrm{SL}_r(\mathbb{C})$. Conversely, if the monodromy is in $\mathrm{SL}_r(\mathbb{C})$, then obviously $(\det(E),\mathrm{tr}(\nabla))$ is the trivial connection, and (E,∇) is a sl_r-connection.

2.1.16 Vector bundles that admit connection

We say that E (resp. (E,∇)) is *decomposable* if there exists a non trivial decomposition $E=F_1\oplus F_2$ (resp. with ∇-invariant subbundles F_i), i.e. where $F_i\neq\{0\},E$. In that case, the connection also splits as $(F_1,\nabla_1)\oplus(F_2,\nabla_2)$. André Weil (1938) proved that a vector bundle E admits a holomorphic connection on a compact Riemann surface X if, and only if, for a maximal decomposition $E=F_1\oplus\cdots\oplus F_l$, we have $\deg(F_i)=0$ for all $i=1,\ldots,l$.

Example 2.1.6. *Holomorphic connections on the trivial bundle* $E=\mathcal{O}^{\oplus r}$ *are given by* $\nabla=d+\mathbf{A}$ *where* $\mathbf{A}$ *is a* $r\times r$ *matrix of holomorphic* 1*-forms on* X*. We note that Weil criterium is satisfied as* $\mathcal{O}^{\oplus r}$ *is a (non unique) maximal decomposition and* $\deg(\mathcal{O})=0$.

Example 2.1.7. *The simplest non trivial example of the Riemann–Hilbert correspondance is given by rank 1 connections on an elliptic curve. Denote* $X=\mathbb{C}/\Gamma$ *where* $\Gamma=\mathbb{Z}+\tau\mathbb{Z}$*, and denote by* α *and* β *the loops given by* $[0,1]\to\mathbb{C}\ ;\ t\mapsto t$ *and* $t\mapsto\tau t$ *respectively. One easily see that the moduli space of representations is*

$$\mathrm{Hom}(\pi_1(X,x_0),\mathrm{GL}_1(\mathbb{C}))=\mathrm{Hom}(\pi_1(X,x_0),\mathbb{C}^*)\simeq\mathbb{C}^*\times\mathbb{C}^*$$

where the latter isomorphism is given by $\rho\mapsto(a,b)=(\rho(\alpha),\rho(\beta))$*. Let us denote by* $(L_{a,b},\nabla_{a,b})$ *the (unique) connection on* X *whose monodromy* ρ *is defined by* $\rho(\alpha)=a$ *and* $\rho(\beta)=b$*. Then we have*

$$L_{a,b}\simeq L_{a',b'}\quad\Longleftrightarrow\quad(a',b')=(e^ca,e^{c\tau}b)\ \ \textit{for some } c\in\mathbb{C}.$$

Indeed, a connection on the trivial bundle $\mathcal{O}$ *writes* $d+c\,dx$ *where* dx *is the holomorphic* 1*-form on* X *given from the universal cover; its monodromy representation is given by* $(e^c,e^{c\tau})$*. The tensor product* $(L_{a,b},\nabla_{a,b})\otimes(\mathcal{O},d+c\,dx)$ *yields a new connection on* *the same line bundle* $L_{a,b} \otimes \mathcal{O} = L_{a,b}$ *with monodromy* $(e^c a, e^{c\tau} b)$. *Conversely, any two connections* ∇ *and* ∇' *on the same line bundle* L *differ by a holomorphic* 1*-form,* $(L,\nabla) \otimes (L,\nabla')^{\otimes(-1)} = (\mathcal{O}, d+\omega)$, *and we can write* $\omega = c\,dx$. *We conclude that we get an equivalence relation on the space of monodromies, namely defining isomorphic line bundles; the cosets for this equivalence relation are given by the integral curves of the vector field* $a\frac{\partial}{\partial a} + \tau b \frac{\partial}{\partial b}$.

One can rephrase as follows. Consider the universal cover

$$\pi : \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}^* \;;\; (u,v) = (e^{2i\pi u}, e^{2i\pi v}).$$

The covering group is generated by

$$(u,v) \mapsto (u+1, v) \quad \textit{and} \quad (u,v) \mapsto (u, v+1).$$

The equivalence relation above (of equivalent line bundles) lifts as the equivalence relation given by integral curves of $\frac{\partial}{\partial u} + \tau \frac{\partial}{\partial v}$, *having first integral* $(u,v) \mapsto v - \tau u$. *In new coordinates*

$$(x,y) = (v - \tau u, -u) \quad \Longleftrightarrow \quad (u,v) = (y - \tau x, -x),$$

the covering group is now generated by

$$(x,y) \mapsto (x+1, y) \quad \textit{and} \quad (x,y) \mapsto (x+\tau, y+1).$$

Let S *denote the quotient: it is analytically equivalent to* $\mathbb{C}^* \times \mathbb{C}^*$ *but can be endowed with an alternate algebraic structure. The projection* $(x,y) \mapsto x$ *induces a map* $S \to X$ *which cannot be algebraic as a map* $\mathbb{C}^* \times \mathbb{C}^* \to X$ *since* X *has genus* 1. *The surface* S *can be interpreted as the moduli space of rank* 1 *holomorphic connections* (L,∇) *on* X, *and the map* $S \to X \simeq \mathrm{Jac(X)}$ *as the forgetful map* $(L,\nabla) \mapsto L$. *Then,* S *has a structure of an affine* $\mathbb{A}^1$*-bundle over* X *and its natural compactification is a ruled surface over* X. *The map* π *above induces an analytic isomorphism* $\pi : S \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}^*$. *This is the Riemann–Hilbert correspondance.*

2.2 Logarithmic connections

A meromorphic connection on E is given in trivialization charts $U_i \times \mathbb{C}^r$ by $\nabla = d + \mathbf{A}_i$ where $\mathbf{A}_i \in \mathrm{gl}_r(\Omega^1 \otimes \mathcal{M})$, i.e. coefficients are meromorphic 1-forms, satisfying compatibility condition (2.4). It is easy to check that poles and their multiplicity do not depend on the trivialization chart, and are therefore well defined by (E,∇). It is natural to associate the divisor of poles, which is $D = n_1[x_1] + \cdots + n_k[x_k]$ where $x_i \in X$ are poles, and $n_i \in \mathbb{Z}_{>0}$ the maximal multiplicity occurring in coefficients of the matrix A. In other words, the connection defines a $\mathbb{C}$-linear morphism of sheaves $\nabla : E \to E \otimes \Omega^1(D)$ satisfying the Leibniz rule. In this text, we will only consider connections with simple poles, i.e. $n_i = 1$, called *logarithmic* connections. On the trivial bundle, logarithmic connections correspond to a Fuchsian system. Let us see examples over non trivial bundles.

Example 2.2.1. *Let $E = \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ be any vector bundle over $\widehat{\mathbb{C}}$, $k_1 \leqslant k_2$, and set $k = k_2 - k_1 \geqslant 0$. We have two trivializations Y_0 near $x = 0$ and Y_∞ near $x = \infty$ related by*

$$Y_\infty = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} Y_0, \quad z = \frac{1}{x}.$$

A logarithmic connection ∇ on E with poles (at most) $x_1, \ldots, x_n, \infty$ writes

$$d + \underbrace{\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}}_{\text{polynomial}} \frac{dx}{P(x)} \quad \text{with} \quad \begin{cases} \deg(A), \deg(D) \leqslant n-1 \\ \deg(B) \leqslant n-k-1 \\ \deg(C) \leqslant n+k-1 \end{cases} \tag{2.5}$$

where $P(x) = (x - x_1) \cdots (x - x_n)$. We note that for $k \geqslant n$, the connection is reducible since $B \equiv 0$ and the factor $\mathcal{O}(k_2) \subset E$ is then ∇-invariant. Automorphisms of E are given in the main chart by the action of $\mathrm{GL}_2(\mathbb{C})$ *if $k = 0$, and by polynomial matrices*

$$Y \mapsto \begin{pmatrix} \lambda_1 & 0 \\ Q(x) & \lambda_2 \end{pmatrix} Y \quad \text{where} \quad \deg(Q) \leqslant k, \quad \lambda_i \in \mathbb{C}^* \quad \text{if } k > 0. \tag{2.6}$$

We deduce that, if we fix eigenvalues at each pole, then the moduli space of connections on E up to automorphisms has dimension:

- $2n - 4$ *if $k = 0, 1$,*
- $2n - k - 3$ *if $0 < k < n$,*
- $n - 3$ *if $k \geqslant n$.*

We conclude that, generically, $k = 0$ or 1.

2.2.1 Residue, eigenvalues and Fuchs' relation

Let (E, ∇) be a logarithmic connection on the curve X and x be a local coordinate at a pole of ∇. Recall that, in a local trivialization, the connection writes

$$d + \left(\frac{A_0}{x} + \text{holomorphic} \right) dx$$

and gauge transformations $M(x)$ act by conjugacy by $M(0)$ on the residual matrix A_0. Therefore, we have well-defined residual endomorphism $\mathrm{Res}_{x=0}\nabla \in \mathrm{End}(E|_{x=0})$ given in local trivialization by A_0, whose eigenvalues do not depend on trivialization. We call them *residual eigenvalues* of the connection ∇. In particular, $\mathrm{tr}(A_0)$ is the residual eigenvalue of the connection $\mathrm{tr}(\nabla)$ on $\det(E)$.

Proposition 2.2.2 (Fuchs' Relation)**.** *Let (E, ∇) be a logarithmic connection on a compact Riemann surface X, and $x_1, \dots, x_n \in X$ the poles. Then we have:*

$$\sum_{i=1}^{n} \operatorname{tr}(\operatorname{Res}_{x=x_i} \nabla) + \deg(E) = 0 \tag{2.7}$$

Proof. This is a result on rank 1 connections, namely for $(L, \zeta) = (\det(E), \operatorname{tr}(\nabla))$. Indeed, (2.7) is, by definition, equivalent to

$$\sum_{i=1}^{n} \operatorname{Res}_{x=x_i} \zeta + \deg(L) = 0.$$

If $L = \mathcal{O}(D)$ for some divisor $D = \sum_j n_j [p_j]$ on X, then this means that L admits a section s with divisor $\operatorname{div}(s) = D$. We can define a birational trivialization $\phi : L \dashrightarrow \mathcal{O}$ by sending s to the section 1. We can assume that poles of s do not intersect poles of $\operatorname{tr}(\nabla)$. The connection ζ is transformed into $d + \omega$ with ω a meromorphic 1-form on X. The 1-form ω inherits poles and residues of ζ, and has additional poles along p_j's, the support of D, with eigenvalues given by multiplicity n_j of D. Therefore, residue formula gives:

$$\sum_{x \in X} \operatorname{Res}_x \omega = \sum_i \operatorname{Res}_{x_i} \zeta + \underbrace{\sum_j \operatorname{Res}_{p_j} n_j}_{\deg(L)} = 0.$$

□

2.3 Moduli spaces in the logarithmic setting

Here we restrict to the rank 2 case for simplicity. We fix a complete irreducible smooth curve X over $\mathbb{C}$ (i.e. compact Riemann surface), a polar divisor $D = x_1 + \cdots + x_n$, with $x_1, \dots, x_n \in X$ pairwise distinct; we denote by $X^* = X \setminus \{x_1, \dots, x_n\}$. We also fix eigenvalues $\theta_i^+, \theta_i^- \in \mathbb{C}$ at each pole x_i, and denote by $\boldsymbol{\theta}$ the collection of these numbers, and $\theta_i := \theta_i^+ - \theta_i^-$. We assume Fuchs' Relation:

$$d := \sum_{i=1}^{n} \theta_i^+ + \theta_i^- \in \mathbb{Z}. \tag{2.8}$$

Then we consider the following set of connections

$$\operatorname{Con}(X, D, \boldsymbol{\theta}) = \left\{ (E, \nabla)\,; \quad \begin{array}{c} E \to X \text{ a rank 2 vector bundle} \\ \nabla : E \to \Omega^1(D) \text{ a logarithmic connection} \\ \operatorname{Res}_{x_i} \nabla \text{ has eigenvalues } \theta_i^{\pm} \end{array} \right\}.$$

We say that any two such connections (E,∇), (E',∇') are *isomorphic*, and denote

$$(E,\nabla)\sim(E',\nabla')$$

if there is an isomorphism $\Phi : E \xrightarrow{\sim} E'$ sending ∇-horizontal sections to ∇'-horizontal sections over open sets $U\subset X^*$. The moduli (or quotient) space of those connections up to isomorphism is denoted

$$\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)=\mathrm{Con}(X,D,\boldsymbol{\theta})/\sim$$

We can consider this space just as a set, without more structure. But it is also possible to put some analytic, even algebraic structure on it, i.e. consider it as a "stack". Beware that it is not a manifold: it ca n be singular, even non Hausdorff in general. We discuss in Section 2.7 how to give more structure to this set.

In a very similar way, we can consider the moduli space of monodromy representations in which the monodromy of connections from $\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)$ are defined. For this, let us recall that $\pi_1(X^*,x_0)$ is isomorphic to:

$$\pi_{g,n}:=\langle\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g,\gamma_1,\ldots,\gamma_n\ ;\ [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\gamma_1\cdots\gamma_n=1\rangle \tag{2.9}$$

where the single relation involves commutators $[\alpha,\beta]=\alpha\beta\alpha^{-1}\beta^{-1}$. Then, we define the following set of representations

$$\mathrm{Rep}_{g,n}^{\boldsymbol{\theta}}=\left\{\rho\in\mathrm{Hom}(\pi_{g,n},\mathrm{GL}_2(\mathbb{C}))\ ;\ \rho(\gamma_i)\text{ has eigenvalues }\lambda_i^{\pm}\right\},\quad \lambda_i^{\pm}=e^{2\pi\sqrt{-1}\theta_i^{\pm}}.$$

This can be more concretely described as

$$\mathrm{Rep}_{g,n}^{\boldsymbol{\theta}}=\left\{\begin{array}{c}(M^{\alpha_1},\ldots,M^{\alpha_g},M^{\beta_1},\ldots,M^{\beta_g},M^{\gamma_1},\ldots,M^{\gamma_g})\in\mathrm{GL}_2(\mathbb{C})^{2g+n}\ ;\\ [M^{\alpha_1},M^{\beta_1}]\ldots[M^{\alpha_g},M^{\beta_g}]M^{\gamma_1}\cdots M^{\gamma_n}=I,\\ M^{\gamma_i}\text{ has eigenvalues }\lambda_i^{\pm}\end{array}\right\}. \tag{2.10}$$

Then, the moduli space of representations up to isomorphism is denoted

$$\mathfrak{Rep}_{g,n}^{\boldsymbol{\theta}}=\mathrm{Rep}_{g,n}^{\boldsymbol{\theta}}/\mathrm{PGL}_2(\mathbb{C}).$$

We can also define $\mathrm{Rep}_{g,n}$ and $\mathfrak{Rep}_{g,n}$ in a similar way without requiring nothing about local eigenvalues. After fixing an isomorphism $\pi_1(X^*,x_0)\simeq\pi_{g,n}$, we get a well defined *monodromy map*

$$\mathrm{Mon}:\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)\to\mathfrak{Rep}_{g,n}^{\boldsymbol{\theta}}. \tag{2.11}$$

which, to a connection (E,∇), associate the monodromy of a local trivialization Φ at x_0 under analytic continuation: $\Phi^\gamma=M^\gamma\circ\Phi$. As we shall see later, this map is almost one-to-one.

2.4 The local Riemann–Hilbert correspondance

Here we restrict to a neighborhood U of a pole. In a local coordinate x at the pole, recall (see Theorem 1.3.1) that the connection ∇ writes $d + A$ and can be normalized to

$$A(x) = \begin{pmatrix} \theta^+ & 0 \\ 0 & \theta^- \end{pmatrix} \frac{dx}{x}, \quad \text{resp.} \quad \begin{pmatrix} \theta & x^n \\ 0 & \theta + n \end{pmatrix} \frac{dx}{x} \quad \text{for some } n \in \mathbb{Z}_{\geqslant 0}.$$

in a convenient local trivialization of the bundle. Then, the connection itself can be trivialized by

$$\Phi(x) = \begin{pmatrix} x^{\theta^+} & 0 \\ 0 & x^{\theta^-} \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} x^{\theta} & x^{\theta+n}\log(x) \\ 0 & x^{\theta+n} \end{pmatrix}$$

which makes sense locally on a punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\} \subset U$. The local monodromy, namely the monodromy of Φ along a simple loop γ turning positively around 0, is therefore given by

$$M^\gamma = \begin{pmatrix} e^{2i\pi\theta^+} & 0 \\ 0 & e^{2i\pi\theta^-} \end{pmatrix}, \quad \text{resp.} \quad e^{2i\pi\theta} \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}.$$

An easy consequence of formulae gives:

Proposition 2.4.1. *Fix residual eigenvalues θ^+, θ^-. Then, the monodromy map induces a one-to-one correspondance*

$$\mathrm{Mon} \;:\; \left\{ \begin{matrix} (E,\nabla) \textit{ logarithmic on } (\mathbb{D},0) \\ \mathrm{Res}_0\nabla \textit{ has eigenvalues } \theta^\pm \end{matrix} \right\}_{/\sim} \xrightarrow{\sim} \left\{ \begin{matrix} M \in \mathrm{GL}_2(\mathbb{C}) \textit{ with} \\ \textit{eigenvalues } e^{2i\pi\theta^\pm} \end{matrix} \right\}_{/\mathrm{PGL}_2(\mathbb{C})}$$

However, there is a subtlety with the injectivity compared with the holomorphic Riemann–Hilbert correspondance. Indeed, given two logarithmic connections (E, ∇), (E', ∇') on $(\mathbb{D}, 0)$, with local trivializations Φ, Φ' near some base point $x_0 \in \mathbb{D}^*$ with same monodromy M, then $\Psi := (\Phi')^{-1} \circ \Phi$ can be analytically continued as a conjugacy

$$\Psi := (\Phi')^{-1} \circ \Phi : (E, \nabla)|_{\mathbb{D}^*} \to (E', \nabla')|_{\mathbb{D}^*}.$$

But, even when ∇, ∇' have same residual eigenvalues θ_i^+, θ_i^-, the conjugacy Ψ needs not extend holomorphically at 0. For instance, in the following case:

$$\nabla = \nabla' = d + \begin{pmatrix} \theta & 0 \\ 0 & \theta + n \end{pmatrix} \frac{dx}{x} \quad \text{and} \quad \Psi = \begin{pmatrix} 0 & x^{-n} \\ x^n & 0 \end{pmatrix}$$

the conjugacy Φ is holomorphic on $\mathbb{D}^*$ but extends meromorphically at 0. We need the following more precise lemma:

Lemma 2.4.2. *Let* $(E, \nabla), (E', \nabla')$ *be two logarithmic connections on* $(\mathbb{D}, 0)$*, and let*

$$\Psi : (E, \nabla)|_{\mathbb{D}^*} \to (E', \nabla')|_{\mathbb{D}^*}$$

be a conjugacy on the punctured disc. Then Ψ *extends as a meromorphic gauge transformation on* $\mathbb{D}$.

Moreover, if the two connections have the same residual eigenvalues θ^+, θ^-*, then* Ψ *extends as a holomorphic gauge transformation on* $\mathbb{D}$ *under the following condition:*

> *when* $\nabla, \nabla' \sim d + \begin{pmatrix} \theta & 0 \\ 0 & \theta + n \end{pmatrix} \frac{dx}{x}$ *with* $n \in \mathbb{Z}_{>0}$*, then* Ψ *extends holomorphically at* 0 *if, and only if, it conjugates the two invariant subbundles of* ∇, ∇' *that restrict as the* θ*-eigendirections at* 0.

Proof. We reduce the two connections to their normal form. We first prove the second statement. The two normal forms must be the same for ∇ and ∇', and Ψ is just a holomorphic symmetry of a given model over $\mathbb{D}^*$. By Proposition 2.1.3, such a symmetry takes the form $\Psi = \Phi^{-1} C \Phi$, where Φ gives a local trivialization, and C commutes with the monodromy.

For the first normal form, we obtain

$$\Psi = \begin{pmatrix} a & bx^{\theta} \\ cx^{-\theta} & d \end{pmatrix} \quad \text{where} \quad C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \quad \theta = \theta^- - \theta^+.$$

- If $\theta \notin \mathbb{Z}$, then the only possibility for Ψ being holomorphic is $b = c = 0$ and Ψ is any constant diagonal matrix. On the other hand, the monodromy M^γ is diagonal with distinct eigenvalues in this case: its centralizeur consists of all diagonal matrices C. Therefore, each matrix C commuting with monodromy gives rise to a global holomorphic symmetry over $\mathbb{D}$.

- If $\theta = n \in \mathbb{Z}_{\geqslant 0}$, then holomorphic symmetries on $\mathbb{D}^*$ take the form

  $$\Psi = \begin{pmatrix} a & bx^{n} \\ cx^{-n} & d \end{pmatrix}.$$

  Then we see that they extend meromorphically at 0, and the extension is holomorphic at 0 if, and only if, either $n = 0$, or $n > 0$ and Ψ is upper triangular in that case.

For the second normal form, a similar computation shows that symmetries on $\mathbb{D}^*$ write

$$\Psi = \begin{pmatrix} a & bx^{n} \\ 0 & a \end{pmatrix}, \quad a \in \mathbb{C}^*, \; b \in \mathbb{C}$$

and extend holomorphically on $\mathbb{D}$.

Finally, for the first assertion, we observe that existence of Ψ implies conjugacy of the monodromies and therefore that eigenvalues of ∇ and ∇' differ by integers (their exponential are the same, namely eigenvalues of the monodromy). Then, maybe composing Ψ by a meromorphic gauge transformation, we can assume that ∇ and ∇' have the same eigenvalues (see Remark 1.3.2). Then we can apply the previous discussion to Ψ and conclude. □

2.5 The global Riemann–Hilbert correspondance

In order to have a true bijection, we need to impose some restriction. The famous one due to Deligne (1970) is to ask $0 \leqslant \Re(\lambda_i^{\pm}) < 1$ (real part of eigenvalues). We can impose a weaker one that $\theta_i^+ - \theta_i^- \notin \mathbb{Z}^*$ which is implied by Deligne's condition. The goal is to avoid those singularities of Lemma 2.4.2 where symmetries of punctured neighborhood do not extend.

Theorem 2.5.1 (Riemann–Hilbert Correspondance: the logarithmic case)**.** *Assume $\theta_i^+ - \theta_i^- \notin \mathbb{Z}^*$ for $i = 1, \ldots, n$. Then the monodromy map (2.11)*

$$\mathrm{Mon} : \mathfrak{Con}^{\boldsymbol{\theta}}(X, D) \to \mathfrak{Rep}_{g,n}^{\boldsymbol{\theta}}.$$

is a bijection.

Proof. For the injectivity, we first use the holomorphic RH correspondance on $X^* = X \setminus |D|$ (on the complement of the support of D): if $(E, \nabla), (E', \nabla')$ admits local trivializations Φ, Φ' near x_0 with same monodromy, then we get a conjugacy

$$\Psi := (\Phi')^{-1} \circ \Phi : (E, \nabla)|_{X^*} \to (E', \nabla')|_{X^*}$$

outside of the polar locus. Then, since $\theta \notin \mathbb{Z}^*$, the conjugacy Ψ extends holomorphically at all punctures by Lemma 2.4.2.

For the surjectivity, we can first realize a given representation by a holomorphic connection (E^*, ∇^*) on X^*. Then take some small discs $\mathbb{D}_i$ at x_i and denote $\mathbb{D}_i^* = \mathbb{D}_i \setminus \{x_i\}$. The monodromy of $\nabla^*|_{\mathbb{D}_i^*}$ can be realized by the monodromy of a logarithmic connection (E_i, ∇_i) on $\mathbb{D}_i$ having a simple pole at x_i with eigenvalues $\{\theta_i^+, \theta_i^-\}$ (Proposition 2.4.1). By the holomorphic Riemann–Hilbert correspondance, there is an isomorphism $\Psi_i : (E^*, \nabla^*)|_{\mathbb{D}_i^*} \stackrel{\sim}{\to} (E_i, \nabla_i)|_{\mathbb{D}_i^*}$ and we can patch (E^*, ∇^*) with (E_i, ∇_i) over $\mathbb{D}_i^*$ by Ψ_i to construct a logarithmic extension of (E^*, ∇^*) at x_i with desired eigenvalues. □

There is a weaker version of the Riemann–Hilbert correspondance which is very simple, though.

Corollary 2.5.2. *Any two logarithmic connections $(E, \nabla), (E', \nabla')$ on (X, D) are meromorphically equivalent, i.e. by a bimeromorphic bundle isomorphism $\Psi : E \dashrightarrow E'$, if, and only if, they have the same monodromy representation.*

Proof. If $\Psi : E \dashrightarrow E'$ is a bimeromorphic bundle isomorphism conjugating the two connections, then it induces a biholomorphic isomorphism over a Zariski open subset $U \subset X^*$. Therefore, we deduce that monodromy of ∇ and ∇' coincide along loops in U. Since the inclusion $\iota : U \hookrightarrow X^*$ induces a surjective morphism $\iota_* : \pi_1(U, x_0) \to \pi_1(X^*, x_0)$, we deduce that monodromies coincide along loops in X^*. Conversely, if monodromy representations of ∇ and ∇' coincide, then we can apply the proof of Theorem 2.5.1 with the first assertion of Lemma 2.4.2 to conclude. □

2.6 Character varieties

The moduli space of representations $\mathfrak{Rep}^{\boldsymbol{\theta}}_{g,n}$ is not Hausdorff in general. For instance, in the hypergeometric case (1.33), upper-triangular representation

$$M_0 = \begin{pmatrix} \lambda_0 & \mu_0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \lambda_1 & \mu_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}$$

with $(\mu_0, \mu_1) = (0, 1)$ can be conjugated to $(\mu_0, \mu_1) = (0, \varepsilon)$ after conjugacy by $M = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$. We promptly see that this conjugacy class is infinitesimally close to the diagonal class $(\mu_0, \mu_1) = (0, 0)$ by making $\varepsilon \to 0$. These two equivalence classes of representations will give non separated points in the quotient space. But there is a Hausdorff quotient $\mathfrak{Rep}^{\boldsymbol{\theta}}_{g,n} \to \mathbf{Rep}^{\boldsymbol{\theta}}_{g,n}$ which admits the structure of an affine variety.

Let us consider the Painlevé case $(g, n) = (0, 4)$ as an example. Set $\{\theta_i^{\pm}\} = \{\pm\frac{\theta}{2}\}$. We are considering the set of $\mathrm{SL}_2(\mathbb{C})$-representations

$$\mathrm{Rep}^{\boldsymbol{\theta}}_{g,n} = \{M_0, M_1, M_t, M_\infty \in \mathrm{SL}_2(\mathbb{C}), \quad M_0 M_1 M_t M_\infty = I, \quad \mathrm{tr}(M_i) = 2\cos(\pi\theta_i)\}$$

We have an action of $\mathrm{SL}_2(\mathbb{C})$ by conjugacy, and we consider the quotient moduli space:

$$\mathrm{Rep}^{\boldsymbol{\theta}}_{0,4} \to \mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}$$

In order to define natural functions on $\mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}$, we are considering the following $\mathrm{SL}_2(\mathbb{C})$-invariant functions

$$x = \mathrm{tr}(M_0 M_1), \quad y = \mathrm{tr}(M_1 M_t) \quad \text{and} \quad z = \mathrm{tr}(M_0 M_t).$$

Then, we have the following relation (see Benedetto and Goldman (1999), Cantat and Loray (2009), Fricke and Klein (1897), and Iwasaki (2003))

$$x^2 + y^2 + z^2 + xyz = c_x x + c_y y + c_z z + c \tag{2.12}$$

where

$$\begin{gathered} c_x = a_0 a_1 + a_t a_\infty, \quad c_y = a_1 a_t + a_0 a_\infty, \quad c_z = a_0 a_t + a_1 a_\infty \\ \text{and} \quad c = 4 - a_0 a_1 a_t a_\infty - a_0^2 - a_1^2 - a_t^2 - a_\infty^2. \\ a_i = \mathrm{tr}(M_i) = 2\cos(\pi\theta_i). \end{gathered}$$

It can be shown that $\mathbb{C}[x,y,z]$ is the ring of $\mathrm{SL}_2(\mathbb{C})$-invariant polynomial functions on $\mathrm{Rep}_{g,n}^{\boldsymbol{\theta}}$ (see Horowitz (1975)). Then the map

$$\mathfrak{Rep}_{0,4}^{\boldsymbol{\theta}} \overset{(x,y,z)}{\longrightarrow} \mathbf{Rep}_{0,4}^{\boldsymbol{\theta}} = \{(x,y,z)\,;\; x^2+y^2+z^2+xyz = c_x x + c_y y + c_z z + c\}$$

provides a Hausdorff quotient which is one-to-one on a Zariski open set. Precisely, for generic values of $\boldsymbol{\theta}$, the cubic surface $\mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$ is smooth and the above map provides a one-to-one correspondance with the set $\mathfrak{Rep}_{0,4}^{\boldsymbol{\theta}}$ of conjugacy classes of representations. We note that, for any holomorphic (resp. rational) family of representations, defined by a family of matrices

$$s \mapsto (M_0(s), M_1(s), M_t(s), M_\infty(s))$$

with coefficients depending holomorphically (resp. rationally) on s, then traces (x,y,z) are holomorphic (resp. rational) and this makes the complex (resp. algebraic) structure of $\mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$ natural, compatible with families of representations.

For special values of $\boldsymbol{\theta}$, singularities arise at the locus of

- reducible representations,
- those representations with a matrix $M_i = \pm I$.

Then, each smooth point corresponds to a single conjugacy class; on the other hand, singular points might correspond to several conjugacy classes. For instance, in the reducible case, the set of reducible conjugacy classes:

$$\begin{pmatrix} \lambda_0 & \mu_0 \\ 0 & \lambda_0^{-1} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \lambda_1 & \mu_1 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad M_t = \begin{pmatrix} \lambda_t & \mu_t \\ 0 & \lambda_t^{-1} \end{pmatrix}$$

can be parametrized by $\widehat{\mathbb{C}}$ in general. Indeed, if $\lambda_0 \neq \pm 1$, then one can diagonalize M_0 and then normalize $\mu_1 = 1$ by a diagonal matrix; this normalization is unique and we get a one-parameter family of non equivalent reducible representations parametrized by

$$\mathbb{C} \ni s \mapsto (\mu_0, \mu_1, \mu_t) = (0, 1, s).$$

We complete by adding the representation $(\mu_0, \mu_1, \mu_t) = (0,0,1)$ which corresponds to $s = \infty$. These representations all go to the same point as traces only depend on μ_i's.

Moduli spaces of representations admit an affine Hausdorff quotient in general. Moreover, they carry a natural holomorphic symplectic structure, as shown in Goldman (1984) and Iwasaki (2003), which in this special case is given by the following holomorphic 2-form

$$\omega = \frac{dx \wedge dy}{2z + xy - c_z} = \frac{dy \wedge dz}{2x + yz - c_x} = \frac{dz \wedge dx}{2y + xz - c_y}. \tag{2.13}$$

Note that these different expressions coincide in restriction to the surface.

2.7 Moduli spaces of logarithmic connections

There is a way to make an open dense set of the moduli space of connections into an algebraic manifold by Mumford's Geometric Invariant Theory (see Inaba (2013), Inaba, Iwasaki, and Saito (2006a), and Nitsure (1993)).

For holomorphic connections (see Hitchin (1987) and Simpson (1994a)), the open set is given by irreducible connections, and for them, the quotient $\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)^{\mathrm{irr}}$ is an irreducible quasi-projective manifold (in particular smooth) of dimension $6g-6$. Moreover, there is a Hausdorff quotient $\mathfrak{Con}^{\boldsymbol{\theta}}(X,D) \to \mathbf{Con}^{\boldsymbol{\theta}}(X,D)$ which admits the structure of irreducible quasi-projective variety of the same dimension, that contains the previous one as a smooth and dense open set. Mind that this latter one has singular points representing families of reducible connections. Let us explain how it works in the logarithmic case.

2.7.1 Parabolic structures and stability condition

It is convenient to consider an additional structure called *parabolic structure*. For each pole x_i of the connection, we have a linear subspace $F_i^+ \subset E|_{x_i}$ which is the eigenspace associated to eigenvalue θ_i^+ for $\mathrm{Res}_{x_i}\nabla$. This is a line except when $\theta_i^+ = \theta_i^-$ and $\mathrm{Res}_{x_i}\nabla = \theta_i^+ \cdot I$. A parabolic structure is a choice at each pole of a one dimensional linear space $l_i \subset F_i^+$, and we denote by $\mathbf{l} = (l_1,\ldots,l_n)$ the collection of these lines. We call $(E,\nabla,\mathbf{l})$ a *parabolic* connection.

Let us fix weights $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n) \in [0,1]^n$. We will say that a connection (E,∇) is $\boldsymbol{\mu}$*-stable* (resp. $\boldsymbol{\mu}$*-semi-stable*) if we have

$$\deg(E) - 2\deg(L) + \sum_{i=1}^{n} \epsilon_i \mu_i > 0 \quad (\text{resp.} \geqslant 0), \quad \text{where} \quad \begin{cases} \epsilon_i = -1 \text{ if } L|_{x_i} = l_i \\ \epsilon_i = 1 \text{ if } L|_{x_i} \neq l_i \end{cases} \tag{2.14}$$

for any ∇-invariant line bundle $L \subset E$. Then, denote by

$$\mathrm{Con}^{\boldsymbol{\theta}}(X,D)^{\mathrm{irr}} \quad \subset \quad \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) \quad \subset \quad \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-ss}(X,D) \quad \subset \quad \mathrm{Con}^{\boldsymbol{\theta}}(X,D)$$

the subsets of irreducible, $\boldsymbol{\mu}$-stable and $\boldsymbol{\mu}$-semistable parabolic connections. We note that irreducible admit no ∇-invariant line bundle by definition, and are automatically $\boldsymbol{\mu}$-stable.

Theorem 2.7.1 (Inaba (2013), Inaba, Iwasaki, and Saito (2006a), and Nitsure (1993))**.** *The moduli space of* $\boldsymbol{\mu}$*-stable parabolic connections*

$$\mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) = \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D)/\sim$$

is an irreducible quasi-projective complex manifold of dimension $2(4g-3+n)$.

On the larger set of semi-stable bundles, there are already non Hausdorff phenomena. Then we have to consider a weaker equivalence $\approx$ which will identify some equivalence cosets that are infinitesimally close to each other, and the moduli space of $\boldsymbol{\mu}$-semistable parabolic connections

$$\mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-ss}(X,D) = \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-ss}(X,D)/\approx$$

is an irreducible quasi-projective variety that might contain singular points. There are also unstable connections, i.e. that are not semi-stable, that we have to take away in this story.

Remark 2.7.2. *If $\boldsymbol{\theta}$ satisfies*

$$\theta_i^+ \neq \theta_i^- \quad \textit{for} \quad i = 1,\ldots,n, \tag{2.15}$$

then we deduce that the parabolic structure $\mathbf{l}$ is determined by the connection (E,∇), and the moduli spaces above are rather moduli spaces of logarithmic connections (without parabolic structure).

2.7.2 Generic eigenvalues

Given a logarithmic connection (E,∇) as before on (X,D), if it is reducible, then there is a ∇-invariant line bundle $L \subset E$ and Fuchs' Relation for the restricted connection gives

$$\sum_{i=1}^{n} \theta_i^{\epsilon_i} + \deg(L) = 0, \quad \epsilon_i \in \{+,-\}.$$

Therefore, if we assume the following generic condition

$$\sum_{i=1}^{n} \theta_i^{\epsilon_i} \notin \mathbb{Z}, \quad \text{for all choices} \quad \epsilon_1,\ldots,\epsilon_n \in \{+,-\} \tag{2.16}$$

then, all connections are irreducible and we have equalities

$$\mathrm{Con}^{\boldsymbol{\theta}}(X,D)^{\mathrm{irr}} = \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) = \mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-ss}(X,D) = \mathrm{Con}^{\boldsymbol{\theta}}(X,D).$$

In that case, Theorem 2.7.1 can be rephrased:

Corollary 2.7.3. *If $\boldsymbol{\theta}$ satisfies (2.8) with genericity conditions (2.15) and (2.16), then the naive moduli space of logarithmic connections*

$$\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)$$

coincides with $\mathrm{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D)$, and is an irreducible quasiprojective manifold of dimension $2(4g-3+n)$.

This corollary makes sense if there is at least one pole, otherwise condition (2.16) cannot be fulfilled, and moreover $n \geqslant 3$ when $g = 0$.

2.7.3 Elementary transformations

We start recalling what is an *elementary transformation* of a rank 2 vector bundle $E \to X$. Given a point $x_0 \in X$ and a parabolic structure at x_0, i.e. a linear subspace $l \in \mathbb{P}(E|_{x_0})$ in the fibre over x_0, one usually defines two birational bundle transformations

$$\mathrm{elm}^+_{x_0,l} : E \dashrightarrow E^+ \quad \text{and} \quad \mathrm{elm}^-_{x_0,l} : E \dashrightarrow E^-,$$

that are uniquely defined up to post-composition by a bundle isomorphism. In restriction to the punctured curve $X^* = X \setminus \{x_0\}$, both $\mathrm{elm}^{\pm}_{x_0,l}$ induce isomorphisms. At the neighborhood of x_0, they can be described as follows. Choose a local coordinate $x : U \to \mathbb{C}$ at x_0 together with a trivialization of $Y : E|_U \to \mathbb{C}^2$ for which the linear subspace l is spanned by $Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This, in particular, induces a trivialization of $E|_{X^*}$ on $U^* = U \setminus \{p\}$. Elementary transformations $\mathrm{elm}^{\pm}_{x_0,l}$ can be defined by the following commutative diagram

$$\begin{tikzcd} E \arrow[d, dashed, "\mathrm{elm}^{\pm}_{x_0,l}"] & E|_{X^*} \arrow[d, equal, "\mathrm{id}"] & E|_{U^*} \arrow[l, hook'] \arrow[r, "Y", "\sim"'] \arrow[d, equal, "\mathrm{id}"] & U^* \times \mathbb{C}^2 \arrow[r, hook, "\mathrm{id}"] \arrow[d, equal, "\mathrm{id}"] & U \times \mathbb{C}^2 \arrow[d, "\phi^{\pm}"] \\ E^{\pm} & E|_{X^*} & E|_{U^*} \arrow[l, hook'] \arrow[r, "Y", "\sim"'] & U^* \times \mathbb{C}^2 \arrow[r, hook, "\phi^{\pm}"] & U \times \mathbb{C}^2 \end{tikzcd}$$

where

$$\phi^+(Y) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} Y \quad \text{and} \quad \phi^-(Y) = \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix} Y.$$

All three bundles E and $E^{\pm}$ are constructed by gluing the local trivial bundle $U \times \mathbb{C}^2$ to the same restricted bundle $E|_{X^*}$ through different bundle isomorphisms (either the identity, or $\phi^{\pm}$) over the punctured neighborhood U^*. Isomorphisms $E|_{X^*} \to E^{\pm}|_{X^*}$ given by this construction extend as birational bundle transformations. We have

$$\det(E^{\pm}) = \det(E) \otimes \mathcal{O}(\pm[x_0]).$$

On the other hand, $\mathrm{elm}^{\pm}_{x_0,l}$ induce the same birational $\mathbb{P}^1$-bundle transformation

$$\mathrm{elm}_{x_0,l} : P = \mathbb{P}(E) \dashrightarrow P'$$

since ϕ^+ and ϕ^- coincide both in $\mathrm{PGL}(2, \mathcal{O}(U^*))$ and $\mathrm{PGL}(2, \mathcal{M}(U))$.

One still has to verify that our construction only depends on the parabolic structure (x_0, l), not on the choice of the local trivialization Y. For another choice

$$\widetilde{Y} = M \cdot Y, \quad M \in \mathrm{GL}(2, \mathcal{O}(U)),$$

one has to check that $\phi^+(\widetilde{Y}) = \phi^+(M \cdot Y) = \widetilde{M} \cdot \phi^+(Y)$ for some $\widetilde{M} \in \mathrm{GL}(2, \mathcal{O}(U))$. Indeed, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\widetilde{M} = \begin{pmatrix} a & bx \\ c/x & d \end{pmatrix}$; since l has to be spanned by $\widetilde{Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have $c(0) = 0$ and $\widetilde{M}$ is holomorphic with $\det(\widetilde{M}) = \det(M) \neq 0$.

A similar computation shows that the line $l^{\pm} \subset E|_{x_0}^{\pm}$, spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the construction above, does not depend on our choices. In other word, given a bundle E equipped with a parabolic structure over x_0, $l \subset E|_{x_0}$, elementary transformations define a birational transformation

$$\mathrm{elm}_{x_0}^{\pm} : (E, l) \dashrightarrow (E^{\pm}, l^{\pm})$$

between parabolic bundles which is well defined up to left-and-right composition by parabolic bundle isomorphisms. It also follows from computations above that

$$\mathrm{elm}_{x_0}^{\pm} \circ \mathrm{elm}_{x_0}^{\mp} : (E, l) \to (E', l')$$

are parabolic bundle isomorphisms. In this sense, $\mathrm{elm}_{x_0}^{+}$ and $\mathrm{elm}_{x_0}^{-}$ are inverse to each other. We can also consider a general rank 2 parabolic bundle $(E, \mathbf{l})$ over (X, D) where $D \subset X$ is a finite subset, and $\mathbf{l} : D \to \mathbb{P}(E|_D)$ a section of the projective bundle induced over D. The elementary transformations $\mathrm{elm}_{x_0}^{\pm} : (E, \mathbf{l}) \dashrightarrow (E^{\pm}, \mathbf{l}^{\pm})$ are defined between parabolic bundles over (X, D) like above when $x_0 \in D$ (note that $\mathrm{elm}_{x_0, \mathbf{l}(x_0)}^{\pm}$ induces an isomorphism of parabolic bundles over (X^*, D^*)) and as the identity when $x_0 \notin D$. Finally, if $x_1, x_2 \in D$ are two distinct points, elementary transformations $\mathrm{elm}_{x_1}^{\pm}$ and $\mathrm{elm}_{x_2}^{\pm}$ commute (up to parabolic bundle isomorphisms) so that one can define $\mathrm{elm}_{D'}^{\pm}$ for any subset $D' \subset D$.

Now, we would like to describe how elementary transformations act on parabolic connections $(E, \nabla, \mathbf{l})$: $(E, \mathbf{l})$ is a parabolic bundle over (X, D) like above and ∇ a meromorphic connection on E. Let $x_0 \in D$ and denote by $\nabla^{\pm}$ the push-forward of ∇ by the elementary transformation $\mathrm{elm}_{x_0}^{\pm} : (E, \mathbf{l}) \dashrightarrow (E^{\pm}, \mathbf{l}^{\pm})$: $\nabla^{\pm}$ is a meromorphic connection on $E^{\pm}$. Under notations above, if ∇ is defined in coordinates (x, Y) by

$$Y \mapsto dY + \mathbf{A}Y, \quad \mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then $\nabla^{\pm}$ is defined in the $E^{\pm}$ local trivialization $Y^{\pm} = \phi^{\pm}(Y)$ by $Y^{\pm} \mapsto dY^{\pm} + \mathbf{A}^{\pm}Y^{\pm}$ where

$$\mathbf{A}^{+} = \begin{pmatrix} \alpha - \frac{dx}{x} & x\beta \\ \frac{\gamma}{x} & \delta \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{-} = \begin{pmatrix} \alpha & x\beta \\ \frac{\gamma}{x} & \delta + \frac{dx}{x} \end{pmatrix}.$$

If x_0 is not a pole of ∇, then $\nabla^{\pm}$ has a logarithmic pole at x_0. If x_0 is a simple pole of ∇, and $\mathbf{l}(x_0)$ is an eigendirection for ∇, then β vanishes at x_0 and $\nabla^{\pm}$ is also logarithmic. One can choose the coordinate Y such that ∇ is normalized as $Y \mapsto dY + AY\frac{dx}{x}$ with

$$A = \begin{pmatrix} \theta^{+} & 0 \\ 0 & \theta^{-} \end{pmatrix}, \quad \begin{pmatrix} \theta & x^n \\ 0 & \theta + n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \theta + n & 0 \\ x^n & \theta \end{pmatrix}, \quad l = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with the restriction $n > 0$ in the last case. Then $\nabla^\pm$ is given in coordinate $Y^\pm = \phi^\pm(Y)$ by $Y^\pm \mapsto dY^\pm + A^\pm Y^\pm \frac{dx}{x}$ with respectively

$$A^+ = \begin{pmatrix} \theta^+ - 1 & 0 \\ 0 & \theta^- \end{pmatrix}, \quad \begin{pmatrix} \theta - 1 & x^{n+1} \\ 0 & \theta + n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \theta + n - 1 & 0 \\ x^{n-1} & \theta \end{pmatrix}, \quad l^+ = \mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$A^- = \begin{pmatrix} \theta^+ & 0 \\ 0 & \theta^- + 1 \end{pmatrix}, \quad \begin{pmatrix} \theta & x^{n+1} \\ 0 & \theta + n + 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \theta + n & 0 \\ x^{n-1} & \theta + 1 \end{pmatrix}, \quad l^- = \mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We now understand in detail how are related local models (1.12) with same monodromy under bimeromorphic gauge transformations, as was already noticed in Remark 1.3.2. In any case, if $\{\theta^+, \theta^-\}$ denote the eigenvalues at x_0, and if l is the eigenline associate to θ^+, then ∇^+ (resp. ∇^-) has eigenvalues $\{\theta^+ - 1, \theta^-\}$ (resp. $\{\theta^+, \theta^- + 1\}$); $\nabla^\pm$ are of diagonal type if, and only if, ∇ is; the new parabolic structure of $l^\pm$ over x_0 corresponds to the eigenvalue θ^- (resp. $\theta^- + 1$).

Let $L \subset E$ be a ∇-invariant line bundle, and denote $L^\pm := \mathrm{elm}^\pm_{x_i} L \subset E^\pm$. One easily check from previous formulae that

- if $L|_{x_i} = l_i$, then $L^\pm|_{x_i} \neq l_i^\pm$, $L^+ \simeq L \otimes \mathcal{O}([x_i])$ and $L^- \simeq L$
- if $L|_{x_i} \neq l_i$, then $L^\pm|_{x_i} = l_i^\pm$, $L^+ \simeq L$ and $L^- \simeq L \otimes \mathcal{O}(-[x_i])$.

The trace of the connection is changed by

$$\mathrm{tr}(\nabla^\pm) = \mathrm{tr}(\nabla) \otimes \zeta^\pm$$

where ζ is the unique logarithmic connection on $\mathcal{O}_X(\pm[p])$ having a single pole at p with residue ± 1 and trivial monodromy. Indeed, the monodromy does not change by a birational bundle transformation.

2.7.4 Isomorphisms between moduli spaces

There are several natural transformations that can be applied in family, giving rise to isomorphisms between moduli spaces of parabolic connections.

Firstly, there is an action of $(\mathbb{Z}_{/2})^n$ on parameters $\boldsymbol{\theta}$ given by changing signs

$$\boldsymbol{\theta} = (\theta_1^\pm, \ldots, \theta_n^\pm) \mapsto \boldsymbol{\theta}' = (\theta_1^{\epsilon_1}, \ldots, \theta_n^{\epsilon_n}), \quad \boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \{+, -\}^n;$$

Obviously $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D) = \mathfrak{Con}^{\boldsymbol{\theta}'}(X, D)$ since the eigenvalues at each pole are $\{\theta_i^+, \theta^-\}$. For generic $\boldsymbol{\theta}$ (see Corollary 2.7.3), we have natural isomorphism

$$\mathfrak{Con}^{\boldsymbol{\theta}}(X, D) \simeq \mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X, D)$$

and we can deduce an isomorphism

$$\boldsymbol{\epsilon} \;:\; \mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) \xrightarrow{\sim} \mathbf{Con}^{\boldsymbol{\theta}'}_{\boldsymbol{\mu}-stab}(X,D)\;;\; (E,\nabla,\mathbf{l}) \mapsto (E,\nabla,\mathbf{l}');$$

mind that the parabolic structure changes with $\boldsymbol{\theta}$: if $\theta_i^+ \leftrightarrow \theta_i^-$, then l_i' is the other eigendirection, attached to θ_i^-. For special values of $\boldsymbol{\theta}$, these isomorphisms might be birational, only defined on a Zariski open set. Indeed, $\boldsymbol{\mu}$-stability changes with $\mathbf{l} \mapsto \mathbf{l}'$.

Secondly, there is an action of rank 1 logarithmic connections on (X,D) by tensor product. Given $L \to X$ a line bundle, and $\zeta : L \to L \otimes \Omega^1(D)$ a connection with simple poles along D, then denote by $(\vartheta_1,\ldots,\vartheta_n) \in \mathbb{C}^n$ the corresponding residual eigenvalues: $\mathrm{Res}_{x_i}\zeta = \vartheta_i$. Then we have the following isomorphism

$$(L,\zeta)\otimes \;:\; \mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) \xrightarrow{\sim} \mathbf{Con}^{\boldsymbol{\theta}'}_{\boldsymbol{\mu}-stab}(X,D)\;;\; (E,\nabla,\mathbf{l}) \mapsto (L\otimes E,\zeta\otimes\nabla,\mathbf{l}')$$

where l_i' is the direction induced by $l_i \subset E|_{x_i}$ on $L \otimes E|_{x_i}$, and $\boldsymbol{\theta}'$ is the translate

$$\boldsymbol{\theta} = (\theta_1^{\pm},\ldots,\theta_n^{\pm}) \mapsto \boldsymbol{\theta}' = (\theta_1^{\pm}+\vartheta_1,\ldots,\theta_n^{\pm}+\vartheta_n).$$

In trivialization charts, this twist operation can be described as follows

$$(L,\zeta) \;:\; d+\omega_i \quad \text{with cocycle} \quad (m_{ij})$$

$$(E,\nabla) \;:\; d+\mathbf{A}_i \quad \text{with cocycle} \quad (M_{ij})$$

$$\rightsquigarrow (L,\zeta)\otimes(E,\nabla) \;:\; d+\mathbf{A}_i+\omega_i I \quad \text{with cocycle} \quad (m_{ij}M_{ij})$$

Rank 1 logarithmic connections on (X,D) form a group for the tensor product. We note that determinant changes as follows

$$\det(L\otimes E) = L^{\otimes 2}\otimes\det(E) \quad \text{and} \quad \mathrm{tr}(\zeta\otimes\nabla) = 2\mathrm{tr}(\zeta)+\mathrm{tr}(\nabla).$$

Using this operation, we can deduce that any moduli space $\mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D)$ with even degree $\sum_{i=1}^n \theta_i^+ + \theta_i^- \in 2\mathbb{Z}$ is isomorphic to a moduli space with holomorphic trace, i.e. with $\boldsymbol{\theta}' = \left(\pm\frac{\theta_i}{2}\right)$ where $\theta_i = \theta_i^+ - \theta_i^-$.

Thirdly, elementary transformations (see Section 2.7.3) also provide isomorphisms:

$$\mathrm{elm}^+_{x_i} \;:\; \mathbf{Con}^{\boldsymbol{\theta}}_{\boldsymbol{\mu}-stab}(X,D) \xrightarrow{\sim} \mathbf{Con}^{\boldsymbol{\theta}'}_{\boldsymbol{\mu}'-stab}(X,D)$$

where $(\boldsymbol{\theta},\boldsymbol{\mu}) \mapsto (\boldsymbol{\theta}',\boldsymbol{\mu}')$ is given by

$$\theta_i^+ \mapsto \theta_i^-, \quad \theta_i^- \mapsto \theta_i^+ - 1, \quad \mu_i \mapsto 1-\mu_i$$

and all remaining parameters/weights left unchanged. Here, the change of weights is exactly what is needed to preserve stability: this follows from discussion in Section 2.7.3.

When $\boldsymbol{\theta}$ is generic, all above transformations preserve the complex algebraic structure, and are therefore biregular isomorphisms between moduli spaces. The combination of them allow to modify

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1^+ & \cdots & \theta_n^+ \\ \theta_1^- & \cdots & \theta_n^- \end{pmatrix} \quad \mapsto \quad \boldsymbol{\theta}' = \begin{pmatrix} \theta_1^+ + \nu_1^+ + \vartheta_1 & \cdots & \theta_n^+ + \nu_n^+ + \vartheta_n \\ \theta_1^- + \nu_1^- + \vartheta_1 & \cdots & \theta_n^- + \nu_n^- + \vartheta_n \end{pmatrix}$$

where $\nu_i^\pm \in \mathbb{Z}$ and $\vartheta_i \in \mathbb{C}$, $\sum_{i=1}^n \vartheta \in \mathbb{Z}$.

We deduce that any moduli space of connections on the Riemann sphere $\widehat{\mathbb{C}}$ with generic $\boldsymbol{\theta}$ is isomorphic to a moduli space of sl_2-connections, i.e. with $\boldsymbol{\theta}' = \left(\pm\frac{\theta_1}{2}, \ldots, \pm\frac{\theta_n}{2}\right)$.

2.7.5 The Riemann sphere with 4 poles

We saw in Section 1.5.6 how to explicitely describe the moduli space of sl_2-systems with 4 simple poles on $\widehat{\mathbb{C}}$. This corresponds to logarithmic connections on the trivial bundle. However, there are many non trivial bundles on $\widehat{\mathbb{C}}$ (see 2.1.1).

Proposition 2.7.4. *Let $X = \widehat{\mathbb{C}}$. For any choice of $\boldsymbol{\theta}$, there are finitely many possible vector bundles $E = \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ occurring in the moduli space $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D)$.*

For $n = 4$, and generic $\boldsymbol{\theta}$ with $\sum_i \theta_i^+ + \theta_i^- + d = 0$, the possible vector bundles are

- *$\mathcal{O}(k) \oplus \mathcal{O}(k)$ or $\mathcal{O}(k-1) \oplus \mathcal{O}(k+1)$ if $d = 2k$ (even),*
- *$\mathcal{O}(k) \oplus \mathcal{O}(k+1)$ if $d = 2k+1$ (odd).*

Proof. It is a straightforward application of formulae in Example 2.2.1 (beware that the number of poles is $n+1$ in that example). We obviously have $k_1 + k_2 = d$; so we just have to upper-bound k_2, or $k_2 - k_1$. If (E, ∇) is irreducible (which is the case for generic $\boldsymbol{\theta}$), then $B(x)$ cannot be zero. This promptly implies by (2.5) that $k_2 - k_1 \leqslant n - 2$; for $n = 4$, we get $k_2 - k_1 \leqslant 2$, whence the list of possible bundles. On the other hand, for non generic $\boldsymbol{\theta}$, we might have reducible connections, i.e. with $B(x) \equiv 0$. This means that $\mathcal{O}(k_2)$ is invariant and we deduce a relation of the form $\theta_1^{\epsilon_1} + \cdots + \theta_n^{\epsilon_n} + k_2 = 0$, $\epsilon_i \in \{+,-\}$. There are finitely many ϵ_i, and therefore finitely many possible k_2. $\square$

On the Riemann sphere $\widehat{\mathbb{C}}$, a connection (E, ∇) is sl_2, i.e. with $\det(E) = \mathcal{O}$ and $\mathrm{tr}(\nabla) = 0$, if and only if $\theta_i^+ + \theta_i^- = 0$ for $i = 1, \ldots, n$. Indeed, it is exactly the condition for which $(\det(E), \mathrm{tr}(\nabla))$ is holomorphic, and since $\widehat{\mathbb{C}}$ is simply connected, its monodromy is trivial, and the rank 1 connection is trivial too (by the Riemann–Hilbert correspondance). In particular, $E = \mathcal{O}(-k) \oplus \mathcal{O}(k)$.

We deduce from Proposition 2.7.4 that the moduli space of logarithmic sl_2-connections with 4 poles on the Riemann sphere $\widehat{\mathbb{C}}$ splits into those defined on the trivial bundle (Fuchsian systems, see Section 1.5.6) and those defined on $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. These later ones can be uniquely normalized to

$$\nabla = d + \begin{pmatrix} 0 & \frac{dx}{x(x-1)(x-t)} \\ \omega & 0 \end{pmatrix}$$

where

$$\omega = \left(\frac{t\theta_0^2}{4x} + \frac{(1-t)\theta_1^2}{4(x-1)} + \frac{t(t-1)\theta_t^2}{4(x-t)} + c + (\theta_\infty^2 - 1)x \right) dx$$

Indeed, we can use (2.5) with $(n,k) = (3,2)$, and action of (2.6) to normalize $B = 1$ and $A = 0$; then, $\mathrm{tr}(\nabla) = 0$ gives $D = 0$, and $C(x)$ is almost determined by the fact that the residual eigenvalues are $\pm\frac{\theta_i}{2}$. The free parameter $c \in \mathbb{C}$ stands for the parametrization of the moduli space of connections on $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$.

If we denote by $\mathfrak{Con}^{\boldsymbol{\theta}}_{(k_1,k_2)}(X,D)$ the moduli space of connections on the bundle $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$, then we deduce:

Corollary 2.7.5. *Let $(\theta_0, \theta_1, \theta_t, \theta_\infty) \in \mathbb{C}^4$, and set* $\boldsymbol{\theta} = \begin{pmatrix} \frac{\theta_0}{2} & \frac{\theta_1}{2} & \frac{\theta_t}{2} & \frac{\theta_\infty}{2} \\ -\frac{\theta_0}{2} & -\frac{\theta_1}{2} & -\frac{\theta_t}{2} & -\frac{\theta_\infty}{2} \end{pmatrix}$. *Assume genericity conditions*

$$\begin{cases} \epsilon_0\theta_0 + \epsilon_1\theta_1 + \epsilon_t\theta_t + \epsilon_\infty\theta_\infty \notin 2\mathbb{Z}, & \forall \epsilon_i \in \{+,-\}, \\ \theta_i \notin \mathbb{Z}, & \forall\, i = 0,1,t,\infty. \end{cases} \tag{2.17}$$

Then, the moduli space of connections $\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)$ is an irreducible and smooth quasi-projective surface, which splits into

$$\mathfrak{Con}^{\boldsymbol{\theta}}(X,D) = \mathfrak{Con}^{\boldsymbol{\theta}}_{(0,0)}(X,D) \sqcup \mathfrak{Con}^{\boldsymbol{\theta}}_{(-1,1)}(X,D).$$

The moduli space of representations is an irreducible and smooth affine surface $\mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4} \subset \mathbb{C}^3$, and we get a biholomorphic map

$$\mathrm{RH} : \mathfrak{Con}^{\boldsymbol{\theta}}(X,D) \xrightarrow{\sim} \mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}.$$

For larger n, and generic sl_2-parameters $(\theta_1, \ldots, \theta_n)$, we get a stratification

$$\mathfrak{Con}^{\boldsymbol{\theta}}(X,D) = \mathfrak{Con}^{\boldsymbol{\theta}}_{(0,0)}(X,D) \sqcup \mathfrak{Con}^{\boldsymbol{\theta}}_{(-1,1)}(X,D) \sqcup \cdots \sqcup \mathfrak{Con}^{\boldsymbol{\theta}}_{(-k,k)}(X,D)$$

with $2k \leqslant n-2$, with decreasing dimension. In the case $n = 4$ however, there is a way to explicitely describe $\mathfrak{Con}^{\boldsymbol{\theta}}(X,D)$ as an irreducible variety. The idea is to apply one elementary transformation, so that the degree is $\deg(E) = 1$ and there is only one possible vector bundle (see Proposition 2.7.4).

2.7.6 Explicit description as 8-blow-up of a line bundle

A ruled surface $S \to X$ on a curve X is a locally trivial bundle with fiber $\mathbb{P}^1$. Examples are given by the total space of the projectivization $\mathbb{P}(E)$ where $E \to X$ is a rank 2 vector bundle. In fact, all ruled surfaces are of this type (see Maruyama (1970)). Precisely, E and E' will define the same ruled surface if, and only if, $E' = L \otimes E$ for a line bundle L. It follows from the Birkhoff–Grothendieck Theorem 2.1.1 that, on the Riemann sphere $\widehat{\mathbb{C}}$, ruled surfaces are so-called Hirzebruch surfaces:

$$\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \to \widehat{\mathbb{C}}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{2.18}$$

The subbundle $\mathcal{O}(k)$ defines a section $h : \widehat{\mathbb{C}} \to \Sigma_k$, with image $H = h(\widehat{\mathbb{C}}) \subset \Sigma_k$, and $\Sigma_k \setminus H \to \widehat{\mathbb{C}}$ can be viewed as the total space of a line bundle $\mathbf{L}_k$ whose zero-section is given by the subbundle $\mathcal{O}$. In coordinates, this writes (with notations of Section 2.1.7)

$$U_0 \times \mathbb{P}^1 \ni (x, y_0) \mapsto (z, y_\infty) = \left(\frac{1}{x}, \frac{y_0}{x^k}\right) \in U_\infty \times \mathbb{P}^1.$$

Then, H is defined by $y_0 = \infty$ or $y_\infty = \infty$, and we have $\mathbf{L}_k = \Sigma_k \setminus H$.

Let $\boldsymbol{\theta} = (\pm\frac{\theta_0}{2}, \pm\frac{\theta_1}{2}, \pm\frac{\theta_t}{2}, \pm\frac{\theta_\infty}{2})$ be generic as in (2.17). Consider the following 8 points on the surface Σ_2:

$$\underbrace{\begin{matrix} p_0^+ = (0, t\kappa_0) & p_1^+ = (1, (1-t)\kappa_1) & p_t^+ = (t, t(t-1)\kappa_t) \\ p_0^- = (0,0) & p_1^- = (1,0) & p_t^- = (t,0) \end{matrix}}_{(x,y_0)} \quad \underbrace{\begin{matrix} p_\infty^+ = (0, -\rho) \\ p_\infty^- = (0, -\rho - \kappa_\infty) \end{matrix}}_{(z,y_\infty)} \tag{2.19}$$

where κ_i's and ρ are defined by

$$(\theta_0, \theta_1, \theta_t, \theta_\infty) = (\kappa_0, \kappa_1, \kappa_t, 1 - \kappa_\infty) \quad \text{and} \quad \kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho = 1. \tag{2.20}$$

Now, define the blow-up $\widehat{\Sigma}_2 \to \Sigma_2$ at the 8 points $p_i^\pm$ and define

- $E_i^\pm \subset \widehat{\Sigma}_2$ the exceptional divisors over $p_i^\pm$;
- $F_i \subset \widehat{\Sigma}_2$ the strict transform of the fiber of Σ_2 at x_i;
- $H \subset \widehat{\Sigma}_2$ the section at infinity;
- $Z = H \cup F_0 \cup F_1 \cup F_t \cup F_\infty$.

Theorem 2.7.6. *Let $\boldsymbol{\theta} = (\pm\frac{\theta_0}{2}, \pm\frac{\theta_1}{2}, \pm\frac{\theta_t}{2}, \pm\frac{\theta_\infty}{2})$ be generic as in (2.17). Then, the moduli space of connections $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D)$ is smooth, isomorphic to $\widehat{\Sigma}_2 \setminus Z$.*

Sketch of proof. We start by twisting (E,∇) by

$$(L,\zeta) = (\mathcal{O}, d + \frac{\theta_0}{2}\frac{dx}{x} + \frac{\theta_1}{2}\frac{dx}{x-1} + \frac{\theta_t}{2}\frac{dx}{x-t}),$$

so that we get new eigenvalues

$$\boldsymbol{\theta}' = \begin{pmatrix} \theta_0 & \theta_1 & \theta_t & \rho \\ 0 & 0 & 0 & \rho-\theta_\infty \end{pmatrix}$$

Then, we mimic the cyclic vector in formula (1.16). Assuming first $E = \mathcal{O}\oplus\mathcal{O}$, we diagonalize $A_\infty = \mathrm{diag}(\rho+\theta_\infty,\rho)$. Then we see that

$$\nabla = d + \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \frac{dx}{x(x-1)(x-t)}$$

with B, C polynomials of degree $\leqslant 1$: the expected degree is 2 but one root is at $x=\infty$. We apply the gauge transformation

$$M^{-1}(x) = \begin{pmatrix} 1 & 0 \\ A(x) & B(x) \end{pmatrix}$$

and get the following normalized connection

$$\nabla'' = d + \begin{pmatrix} 0 & 1 \\ BC-AD & A+D \end{pmatrix} \frac{dx}{x(x-1)(x-t)} + \begin{pmatrix} 0 & 0 \\ A\frac{dB}{B} - dA & -\frac{dB}{B} \end{pmatrix}.$$

This transformation can be thought as an elementary transformation

$$\mathrm{elm}_q^+ \circ \mathrm{elm}_\infty^- : E = \mathcal{O}\oplus\mathcal{O} \to \mathcal{O}(-1)\oplus\mathcal{O}(1)$$

where parabolic at $x=q$ is spanned by e_2. The new eigenvalues are

$$\boldsymbol{\theta}'' = \begin{pmatrix} \theta_0 & \theta_1 & \theta_t & 0 & \rho-\theta_\infty+1 \\ 0 & 0 & 0 & -1 & \rho \end{pmatrix} = \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_t & 0 & \rho+\kappa_\infty \\ 0 & 0 & 0 & -1 & \rho \end{pmatrix}$$

connection writes:

$$\nabla'' = d + \begin{pmatrix} 0 & \frac{dx}{x(x-1)(x-t)} \\ \gamma & \delta \end{pmatrix} \quad \text{with} \quad \begin{cases} \gamma = \left(\frac{p}{x-q} + c_0 + c_1 x\right) dx \\ \delta = \left(\frac{\kappa_0}{x} + \frac{\kappa_1}{x-1} + \frac{\kappa_t}{x-t} - \frac{dx}{x-q}\right) dx \end{cases}$$

for some constants $p, c_0, c_1 \in \mathbb{C}$. The connection at $x=\infty$ writes

$$\nabla = d - \begin{pmatrix} -1 & 1 \\ c_1 & \kappa_0+\kappa_1+\kappa_t \end{pmatrix} \frac{dz}{z} + \text{holomorphic}$$

so that we deduce, by looking at the determinant, that

$$c_1 + \kappa_0 + \kappa_1 + \kappa_t + \rho(\rho + \kappa_\infty) = 0.$$

On the other hand, at $x = q$, the matrix connection expands as

$$A(x) = \frac{\begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix}}{x - q} + \begin{pmatrix} 0 & \frac{1}{q(q-1)(q-t)} \\ c_0 + c_1 q & \frac{\kappa_0}{q} + \frac{\kappa_1}{q-1} + \frac{\kappa_t}{q-t} \end{pmatrix} + O(x - q)$$

and this singular point is apparent (i.e. of diagonal type, without monodromy) if, and only if, the eigenvector $\begin{pmatrix} 1 \\ p \end{pmatrix}$ of the residual matrix corresponding to 0-eigenvalue is also eigenvector of the constant matrix-term, yielding

$$c_0 = \frac{p^2}{q(q-1)(q-t)} - p\left(\frac{\kappa_0}{q} + \frac{\kappa_1}{q-1} + \frac{\kappa_t}{q-t}\right) - q\rho(\rho + \kappa_\infty).$$

It follows that, for $q \in \mathbb{C} \setminus \{0, 1, t\}$, we can uniquely determine ∇'' from $p \in \mathbb{C}$, or equivalently from the parabolic structure $\begin{pmatrix} 1 \\ p \end{pmatrix}$ associated to the 0-eigenvalue at $x = q$. One can then go back to a connection (E, ∇) on $E = \mathcal{O} \oplus \mathcal{O}$ by applying elementary transformation $\mathrm{elm}^-_{q,\infty}$ and then twist by $(L, \zeta)^{\otimes(-1)}$.

When $q \to 0, 1, t$, then the normal form$(\mathcal{O} \oplus \mathcal{O}(2), \nabla'')$ has a limit if, and only if, the parabolic $l_q = \mathbb{C}\begin{pmatrix} 1 \\ p \end{pmatrix} \in \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))|_{x=q}$ tends to one of the eigendirections $p_i^\pm \in \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))|_{x=i}$ given by 2.19 over poles $i = 0, 1, t$. Moreover, the resulting limit depends on the slope of the limit: the set of limit connections at $p_i^\pm$ is parametrized by the exceptional divisor $E_i^\pm$ after blowing-up this point in the surface $\Sigma_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$. Then the new eigenvalues are, for instance when $l_q \to p_0^+$ (resp. p_0^-)

$$\boldsymbol{\theta}'' = \begin{pmatrix} \kappa_0 - 1 & \kappa_1 & \kappa_t & \rho \\ 0 & 0 & 0 & \rho + \kappa_\infty \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} \kappa_0 & \kappa_1 & \kappa_t & \rho \\ -1 & 0 & 0 & \rho + \kappa_\infty \end{pmatrix}$$

and then $\mathrm{elm}^-_{0,\infty}$ provides a connection on the trivial bundle with eigenvalues $\boldsymbol{\theta}$ (after twisting by $(L, \zeta)^{\otimes(-1)}$) for the limiting parabolic structure

$$l_q = \mathbb{C}\begin{pmatrix} 1 \\ t\kappa_0 \end{pmatrix} \quad \text{resp.} \quad \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

When $q \to \infty$, there are two cases depending whether l_q coincides or not with l_∞. If they coincide, then we have to apply two successive elementary transformations to retrieve a connection on $\mathcal{O} \oplus \mathcal{O}$ with expected eignevalues. On the other hand, if they do not coincide, then the two elementary transformations annihilate and provide just a twist by $(\mathcal{O}(-1), d_{\mathrm{can}})$ where d_{can} is trivial in the affine chart U_0. Then, we get a connection on $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ with the expected eigenvalues after twisting by $(L, \zeta)^{\otimes(-1)}$. We conclude that the exceptional divisor E_∞^- stands for those connection on the non trivial bundle. $\square$

The proof implies:

Corollary 2.7.7. *The locus of those connections (E, ∇) in $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D)$ with non trivial bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ coincides with the curve E_∞^-.*

3 Isomonodromic deformations and Painlevé equations

In the Riemann–Hilbert correspondance, we have two moduli spaces, $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D)$ depending of the complex structure of the punctured curve (X, D), and $\mathfrak{Rep}^{\boldsymbol{\theta}}_{g,n}$ depending only on the topology of the curve. We can therefore consider deformation of the curve (X, D) and the Riemann–Hilbert correspondance with parameters. In fact, it is possible, when deforming the curve $t \mapsto (X_t, D_t)$, to deform a given connection $(E_0 \to X_0, \nabla_0)$ as $t \mapsto (E_t \to X_t, \nabla_t)$ in such a way that its monodromy representation is constant (up to conjugacy). This provides a local analytic trivialization of the family $t \mapsto \mathfrak{Con}^{\boldsymbol{\theta}}(X_t, D_t)$: fibers of the Riemann–Hilbert correspondance with parameter

$$\sqcup_t \mathfrak{Con}^{\boldsymbol{\theta}}(X_t, D_t) \to \mathfrak{Rep}^{\boldsymbol{\theta}}_{g,n}$$

are isomonodromic deformations. It turns out that we can explicitely compute the non linear differential equations defining the isomonodromic foliation (whose leaves are isomonodromic deformations). The way of doing this goes back to Schlesinger who noticed that isomonodromic deformations $(E_t \to X_t, \nabla_t)$ are induced by a flat logarithmic connection ∇ on the total space $\mathcal{X} := \sqcup_t X_t$. Then, integrability of ∇ provides isomonodromic differential equations. The particular case of $(g, n) = (0, 4)$ gives rise to the Painlevé VI differential equation.

3.1 Flat holomorphic connections

Let X be a complex manifold of dimension m. Let $E \to X$ be a (locally trivial holomorphic) vector bundle of rank r. A holomorphic connection on E is a $\mathbb{C}$-linear morphism of sheaves $\nabla : E \to E \otimes \Omega^1_X$ satisfying the Leibniz rule (2.3) as in the case $\dim(X) = 1$. Locally, through trivializations of $Y : E|_U \to \mathbb{C}^r$, the connection writes:

$$\nabla : Y \mapsto dY + \mathbf{A}Y, \quad \mathbf{A} \in H^0(U, \mathrm{gl}_r(\Omega^1)).$$

The main difference with dimension 1 case is that there needs not exist local horizontal sections. In local coordinates $x = (x_1, \dots, x_m)$, we can split $\mathbf{A} = A_1(x)dx_1 + \cdots + A_m(x)dx_m$ so that the differential equation for horizontal sections $dY + \mathbf{A}Y = 0$ is more a PDE, that can be viewed as a combination of several systems:

$$\frac{dY}{dx_1} + A_1(x)Y = 0, \quad \cdots \quad , \frac{dY}{dx_m} + A_m(x)Y = 0.$$

These equalities define a m-dimensional distribution on the tangent space TE transversal to the (fibers of the) projection $p : E \to X$, i.e. a m-dimensional linear subspace of T_pE at each point $p \in E$. From this point of view, it is a connection in the sense of Ehresmann (this is much more general, without requiring linear condition), and the existence of local invariant manifolds is given by Frobenius Integrability Theorem.

Lemma 3.1.1. *Are equivalent:*

- $\nabla \cdot \nabla \equiv 0$*;*
- $d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \equiv 0$ *in each trivialization chart* $U \times \mathbb{C}^r$*;*
- (E, ∇) *admits a local trivialization at the neighborhood of any point* $x \in X$.

We say that ∇ *is* flat *(or* integrable*) in that case.*

When X has dimension one, all these properties are automatic, because 2-forms are trivial on X.

Proof. If we iterate twice ∇, we get a $\mathbb{C}$-linear map $\nabla \cdot \nabla : E \to E \otimes \Omega^2_X$. In local trivialization, it is given by

$$Y \overset{\nabla}{\to} dY + \mathbf{A}Y \overset{\nabla}{\to} \underbrace{d(dY)}_{0} + (d\mathbf{A})Y - \mathbf{A} \wedge dY + \mathbf{A} \wedge (dY + \mathbf{A}Y) = (d\mathbf{A})Y + \mathbf{A} \wedge \mathbf{A}Y.$$

This gives the equivalence between the two first assertions.

For the third assertion, first observe that the existence of a local trivialization

$$\Phi : (E, \nabla)|_U \overset{\sim}{\longrightarrow} (\mathcal{O}^{\oplus r}, d) \quad \Leftrightarrow \quad \mathbf{A} = \Phi^{-1} d\Phi$$

implies that

$$d\mathbf{A} = \underbrace{-\Phi^{-1}\, d\Phi\, \Phi^{-1}}_{d(\Phi^{-1})} \wedge d\Phi = -(\Phi^{-1}\, d\Phi) \wedge (\Phi^{-1}\, d\Phi) = -\mathbf{A} \wedge \mathbf{A}$$

giving one implication. For the converse, observe that $\frac{dY}{dx_m} + A_m(x)Y$ can be viewed as a family of systems of a single variable x_m with parameter $(x_1, \ldots, x_{m-1})$. Then, we can solve in family and provide a trivialization Φ with respect to x_m-variable: after applying Φ, we may assume $A_m \equiv 0$. Then, integrability condition gives

$$d\mathbf{A}+\mathbf{A}\wedge\mathbf{A} = -\sum_{i=1}^{m-1} \frac{dA_i}{dx_m} dx_i \wedge dx_m + \sum_{1\leqslant i<j\leqslant m-1} \left(\frac{dA_i}{dx_j} - \frac{dA_j}{dx_i} + [A_i, A_j] \right) dx_i \wedge dx_j$$

where $[A_i, A_j] = A_i A_j - A_j A_i$ is the Lie bracket. We promptly deduce by examining terms in $dx_i \wedge dx_m$ that $\frac{dA_i}{dx_m} = 0$ for $i = 1, \ldots, m-1$. So the variable x_m has disappeared from the story and we can reiterate with trivialization along x_{m-1} variable, and so on, until the connection only depends on x_1 variable. □

Remark 3.1.2. *An alternate proof is given by considering ∇ as a distribution, and flatness as Frobenius integrability condition for foliations. Let us explain in the rank $r = 2$ case. Locally on X, the system rewrites*

$$dY + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} Y = 0 \quad \leftrightarrow \quad \begin{cases} \omega_1 = dy_1 + \alpha y_1 + \beta y_2 = 0 \\ \omega_2 = dy_2 + \gamma y_1 + \delta y_2 = 0 \end{cases}$$

The kernel of $\omega_1 \wedge \omega_2$ defines a corank 2 subbundle $\mathcal{D} \subset TE$ (i.e. a distribution) which is transversal to fibers. By Frobenius Theorem (see Camacho and Lins Neto (1985, Apendix)), this defines a foliation if, and only if

$$\omega_1 \wedge \omega_2 \wedge d\omega_i \equiv 0 \quad \textit{for } i = 1, 2.$$

One easily checks, examining coefficients of $y_i dy_1 \wedge y_2$ $i = 1, 2$, that this condition is equivalent to

$$dA + A \wedge A = \begin{pmatrix} d\alpha & d\beta \\ d\gamma & d\delta \end{pmatrix} + \begin{pmatrix} \beta \wedge \gamma & \alpha \wedge \beta + \beta \wedge \delta \\ \gamma \wedge \alpha + \delta \wedge \gamma & \alpha \wedge \beta \end{pmatrix} = 0.$$

Under this vanishing condition, the distribution $\mathcal{D}$ defines a codimension 2 foliation whose leaves form ∇-horizontal sections, by construction. The existence of a basis of sections is enough to prove local trivialization.

If (E, ∇) is a flat connection, then local trivializations admit analytic continuation, and we can define the monodromy representation. Then, the holomorphic Riemann–Hilbert correspondance still holds, replacing connections by flat connections:

Theorem 3.1.3 (Riemann–Hilbert Correspondance: the holomorphic case)**.** *Let X be a connected complex manifold. Then, the monodromy map induces a one-to-one correspondance*

$$\mathrm{Mon}\ :\ \left\{\begin{matrix}\textit{Flat rank } r \textit{ connections}\\ (E,\nabla)\ \textit{ over }\ X\end{matrix}\right\}_{/\sim}\ \overset{\sim}{\to}\ \left\{\begin{matrix}\textit{Group homomorphisms}\\ \pi_1(X,x_0)\to \mathrm{GL}_r(\mathbb{C})\end{matrix}\right\}_{/\mathrm{GL}_r(\mathbb{C})}$$

3.2 Flat logarithmic connections

A meromorphic connection on E is given in trivialization charts by $\nabla = d + \mathbf{A}$ where $\mathbf{A} \in \mathrm{gl}_r(\Omega^1 \otimes \mathcal{M})$, i.e. coefficients are meromorphic 1-forms. It is easy to check that poles and their multiplicity do not depend on the trivialization chart, and are therefore globally defined on X. It is natural to associate the divisor of poles, which is $D = k_1 D_1 + \cdots + k_n D_n$ where $D_i \subset X$ are irreducible components of the hypersurface of poles of A, and k_i the maximal multiplicity occurring in coefficients of the matrix A. In other words, the connection defines a $\mathbb{C}$-linear map $\nabla : E \to E \otimes \Omega^1(D)$ satisfying the Leibniz rule. In this text, we will only consider connections with simple poles, i.e. $k_i = 1$, and moreover with D_i's smooth and two-by-two disjoints. A meromorphic connection is flat, if it is flat in restriction to the open set where it is holomorphic (complement of poles); equivalently $d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \equiv 0$.

Proposition 3.2.1 (and definition)**.** *Let $(x_1, \ldots, x_n)$ be local coordinates on X such that the connection $\nabla = d + \mathbf{A}$ (on the trivial rank r bundle) has a pole of order* 1 *along $D : \{x_1 = 0\}$, and is flat; denote $x' = (x_2, \ldots, x_m)$. We say that the pole D is* logarithmic*, if both $\mathbf{A}$ and $d\mathbf{A}$ have simple pole along D. This property does not depend on the trivialization, and implies that, after local holomorphic gauge transformation, we have that A only depends on x_1.*

Proof. Since $\mathbf{A}$ has simple pole along $x_1 = 0$, we can write:

$$\mathbf{A} = \frac{A_1(x')dx_1 + \cdots + A_m(x')dx_m}{x_1} + B_1(x)dx_1 + \cdots + B_m(x)dx_m$$

with A_k, B_k holomorphic. Then, $d\mathbf{A}$ has simple pole if, and only if, $A_2 = \cdots = A_m = 0$. This implies that the partial connection

$$\nabla' = d + B_2(x_1, x')dx_2 + \cdots + B_m(x_1, x')dx_m$$

viewed as a family of connections along fibers of $x_1 : U \to \mathbb{C}$, is holomorphic. It is also flat as restriction of a flat connection. We can integrate in family and assume $\nabla' = d$. Therefore, $\mathbf{A} = A(x)\frac{dx_1}{x_1}$ and integrability condition shows that $dA \wedge \frac{dx_1}{x_1} = 0$, i.e. $A(x) = A(x_1)$. $\square$

In the rank 2 case, a logarithmic connection can be locally reduced to models (1.12) in x_1-variable. We can then talk without ambiguity about the type of logarithmic singularity along an irreducible component D_i of the polar divisor, as well as about its eigenvalues. Here follows a particular case of Deligne's version of the Logarithmic Riemann–Hilbert Correspondance (see Briançon (2004) and Deligne (1970)).

Just before this, fix a point $x_0 \in X$, and for each component D_i, a small loop δ_i based at x_i turning around D_i and a path σ_i from x_0 to x_i; then denote by $\gamma_i = \sigma_i^{-1}\delta_i\sigma_i$.

Theorem 3.2.2. *Let X be a connected complex manifold. Let $D = D_1 + \cdots + D_n$ be a smooth divisor with irreducible components D_i. Let $\boldsymbol{\theta} = \left(\theta_1^\pm, \ldots, \theta_n^\pm\right) \in \mathbb{C}^{2n}$ such that $\theta_i^+ - \theta_i^- \notin \mathbb{Z}^*$. Then the monodromy map induces a one-to-one correspondance*

$$\left\{\begin{array}{c}\textit{Flat logarithmic connections}\\ (E,\nabla)\ \textit{ over }\ (X,D)\ \textit{of rank}\ 2\\ \mathrm{Res}_{D_i}\nabla\ \textit{has eigenvalues}\ \theta_i^\pm\end{array}\right\}_{/\sim} \xrightarrow[\text{Mon}]{\sim} \left\{\begin{array}{c}\textit{Group homomorphisms}\\ \rho : \pi_1(X\setminus D, x_0) \to \mathrm{GL}_2(\mathbb{C})\\ \rho(\gamma_i)\ \textit{has eigenvalues}\ e^{2\pi\sqrt{-1}\theta_i^\pm}\end{array}\right\}_{/\mathrm{GL}_2(\mathbb{C})}$$

where γ_i are the loops previously defined.

Proof. The proof is essentially the same as in the one-dimensional case. Denote $X^* = X \setminus D$ and cover each D_i by polydiscs $(U_{i,k})_k$ with coordinates like in Proposition 3.2.1, and denote $U_{i,k}^* = U_{i,k} \setminus D_i|_{U_{i,k}}$.

Injectivity. If (E,∇), (E',∇') have same monodromy representation, then there is an isomorphism $\Psi^* : (E,\nabla)|_{X^*} \to (E',\nabla')|_{X^*}$. On each $U_{i,k}$, the restrictions $(E,\nabla)|_{U_{i,k}}$, $(E',\nabla')|_{U_{i,k}}$ are like 1-variable connections, and $\Psi^*|_{U_{i,k}}$ like a 1-variable isomorphism between them, and therefore extends holomorphically (and uniquely) along $D_i|_{U_{i,k}}$ by Lemma 2.4.2. By this way, Ψ extends holomorphically all along D.

Surjectivity. Any representation can be realized as the monodromy of a holomorphic connection (E^*,∇^*) on X^*. We have to construct a logarithmic extension along D with prescribed exponents. On each open set $U_{i,k}$, one can do this by applying Proposition 3.2.1 with the 1-dimensional Logarithmic Riemann–Hilbert correspondance (Theorem 2.5.1): from the 1-dimensional model (E_i,∇_i) defined by the monodromy, we deduce a model $(E_{i,k},\nabla_{i,k})$ on the polydisc by taking the product with a polydisc of dimension $m-1$. Since $(E^*,\nabla^*)|_{U_{ik}^*}$ and $(E_{i,k},\nabla_{i,k})|_{U_{ik}^*}$ have the same monodromy, then we get an isomorphism $\Psi_{i,k}$ over $U_{i,k}^*$ between the restrictions of (E^*,∇^*) and $(E_{i,k},\nabla_{i,k})$. Over intersections $U_{i,k}\cap U_{i,l}$, we can define isomorphisms $\Psi_{i,k,l}$ from $(E_{i,k},\nabla_{i,k})$ to $(E_{i,l},\nabla_{i,l})$ by extending along D_i the symmetry $\Psi_{i,l}\circ\Psi_{i,k}^{-1}$ which is defined on $U_{i,k}^*\cap U_{i,l}^*$. All these maps provide a way to patch all local connections $(E_{i,k},\nabla_{i,k})$ to (E^*,∇^*) into a global logarithmic connection over (X,D). □

3.3 Deformation of curves and flat extension of logarithmic connections

Let (X, D) be a punctured curve. We would like to consider its deformation, and especially its maximal non trivial deformation.

A deformation of the curve (X_0, D_0) of genus g with n punctures $D = x_1 + \cdots + x_n$ is

- a parameter space $T \ni t_0$, i.e. a connected complex manifold;
- a holomorphic fiber-bundle $\pi : \mathcal{X} \to T$ (i.e. a proper submersion) whose fiber is a (complete smooth) curve of genus g;
- disjoint holomorphic sections $\mathcal{D}_1, \ldots, \mathcal{D}_n : T \to \mathcal{X}$;
- an isomorphism $\varphi : (X_{t_0}, D_{t_0}) \xrightarrow{\sim} (X_0, D_0)$.

where we denote

$$t \mapsto (X_t, D_t) := (\pi^{-1}(t), \mathcal{D}_1(t) + \cdots + \mathcal{D}_n(t)).$$

For any $(g, n) \neq (0,0), (0,1), (0,2), (1,0)$, there is a moduli space $M_{g,n}$ of compact smooth irreducible curves of genus g with n distinct punctures up to isomorphisms preserving the numbering of points $D = x_1 + \cdots + x_n$. This is an irreducible quasi-projective variety of complex dimension $3g - 3 + n$ which can be singular (due to automorphisms of (X, D)), with non trivial topology. On the other hand, there is a *Teichmüller space* $\mathrm{Teich}_{g,n}$ which takes into account a marking of the surface, i.e. a presentation of the fundamental group $\pi_1(X \setminus D, x_0) \simeq \pi_{g,n}$ as in (2.9). Equivalently, we add a homeomorphism $\phi : (X, D) \to (\Sigma_g, p_1 + \cdots + p_n)$ to a fixed topological model, equipped with a system of generators for the fundamental group. This is an analytic (not algebraic) connected manifold of complex dimension $3g - 3 + n$ that can be embedded as a topological ball into $\mathbb{C}^{3g-3+n}$. In fact, the *forgetful map*

$$\mathrm{Teich}_{g,n} \to M_{g,n} \; ; \; (X, D, \phi) \mapsto (X, D)$$

which is the *orbifold* universal cover, is an infinite ramified covering. The Galois group is given by the action of the Mapping Class Group $\mathrm{Mod}_{g,n}$ on $\pi_{g,n}$.

There is a universal curve on $\mathrm{Teich}_{g,n}$, i.e. the data of

- a deformation $(\mathcal{X}, \mathcal{D}) \to \mathrm{Teich}_{g,n}$;
- a topological marking $\Phi : (\mathcal{X}, \mathcal{D}) \to (\Sigma_g, p_1 + \cdots + p_n)$;
- the restriction $(X, D, \phi) = (\mathcal{X}, \mathcal{D}, \Phi)|_t$ is a representative of the isomorphism class $t \in \mathrm{Teich}_{g,n}$.

Let us consider the case $g = 0$ as an example.

We can fix $x = 0, 1, \infty$ and consider variations of remaining points $x_1, \ldots, x_{n-3}$. The moduli space is smooth in this case, given by

$$M_{0,n} = \left\{(x_1, \ldots, x_{n-3}) \in \mathbb{C}^{n-3}\ ;\ x_i \neq x_j,\ x_i \neq 0, 1\right\}$$

and a universal curve is given by

- $\mathcal{X} = M_{0,n} \times \mathbb{P}^1 \ni (x_1, \ldots, x_{n-3}, z)$;
- $\mathcal{D}_i\ :\ \{z = x_i\}$ for $i = 1, \ldots, n-3$,
- $\{z = 0, 1, \infty\}$ for the remaining sections $i = n-2, n-1, n$.

Here, the bundle $(\mathcal{X}, \mathcal{D})$ is even not topologically trivial over the open set $M_{0,n}$. It becomes topologically trivial when lifted on $\mathrm{Teich}_{0,n}$ and we can define a map $\Phi : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to (\mathbb{S}^2, p_1 + \cdots + p_n)$ there. Equivalently, we can provide a retract by deformation $\Phi_0 : \Phi : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}) \to (X_0, D_0)$, to a fiber at $t_0 \in \mathrm{Teich}_{0,n}$.

In Malgrange (1983a), B. Malgrange constructs a maximal isomonodromic deformation $t \mapsto (X_t, D_t, E_t, \nabla_t)$ of a given logarithmic connection (E_0, ∇_0) on a curve (X_0, D_0).

Corollary 3.3.1 (Malgrange)**.** *Let (E_0, ∇_0) be a logarithmic connection over the punctured curve (X_0, D_0) of genus g with n points. Let $(\mathcal{X}, \mathcal{D}, \Phi)$ be the universal curve over* $\mathrm{Teich}_{g,n}$ *with $\Phi : (\mathcal{X}, \mathcal{D}) \to (X_0, D_0)$ a retraction. Then, (E_0, ∇_0) is the restriction of a unique logarithmic connection $(\mathcal{E}, \nabla)$ on $(\mathcal{X}, \mathcal{D})$.*

Idea of proof. Since $\Phi : (\mathcal{X}, \mathcal{D}) \to (X_0, D_0)$ is a retraction, we deduce that $\Phi^* : \pi_1(\mathcal{X} \setminus \mathcal{D}, x_0) \to (X_0 \setminus D_0, x_0)$ is an isomorphism. In particular, there is a one-to-one correspondance with the spaces of representations in $\mathrm{GL}_r(\mathbb{C})$. In the rank 2 case, and assuming that eigenvalues of (X_0, D_0) satisfy $\theta_i^+ - \theta_i^- \notin \mathbb{Z}^*$, then we can use Theorem 3.2.2 to deduce a one-to-one correspondance between flat logarithmic connections on $(\mathcal{X}, \mathcal{D})$, and their restriction to X_0. □

3.4 Schlesinger Theorem

A deformation of a logarithmic connection (E_0, ∇_0) over (X_0, D_0) is the data of

- a deformation $\pi : (\mathcal{X}, \mathcal{D}) \to T \ni t_0$ of the punctured curve (X_0, D_0);
- a vector bundle $\mathcal{E} \to \mathcal{X}$;
- a partial connection $\nabla_{\mathcal{X}/T} : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathcal{X}/T}(\mathcal{D})$;
- an isomorphism $\phi : (E_{t_0}, \nabla_{t_0}) \xrightarrow{\sim} (E_0, \nabla_0)$ over (X_0, D_0)

where we denote

$$t \mapsto (X_t, D_t, E_t, \nabla_t) \quad \text{where} \quad (E_t, \nabla_t) := (\mathcal{E}|_{X_t}, \nabla_{\mathcal{X}/T}|_{X_t}).$$

Locally, in coordinates, the projection is defined by

$$x = (x_1, x_2, \ldots, x_m) \mapsto t = (x_2, \ldots, x_m)$$

and the connection takes the form

$$\nabla_{\mathcal{X}/T} = d + \mathbf{A} \quad \text{where} \quad \mathbf{A} = A(x)dx_1, \quad A \in \mathrm{gl}_r(\mathcal{O}(\mathcal{D})).$$

The deformation $t \mapsto (X_t, D_t)$ is locally topologically trivial, i.e. over an open neighborhood $x_0 \in V \subset T$, we have a topological trivialization

$$\begin{tikzcd}
(\mathcal{X}, \mathcal{D})|_V := \pi^{-1}(V) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & V \times (X_0, D_0) \arrow[dl, "\mathrm{pr}_1"] \\
& V &
\end{tikzcd}$$

Then we can define the monodromy representation in family: we deduce, for instance in the rank 2 case, a map:

$$\mathrm{Mon} \; : \; T \to \mathfrak{Rep}_{g,n}$$

where $\mathfrak{Rep}_{g,n}$ is the moduli space of representations without fixing local eigenvalues. We say that the deformation is *isomonodromic* if, and only if, this map is constant.

A flat logarithmic connection $(\mathcal{E} \to \mathcal{X}, \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathcal{X}}(\mathcal{D}))$ induces a partial connection $\nabla_{\mathcal{X}/T}$, and therefore a deformation $t \mapsto (X_t, D_t, E_t, \nabla_t)$. But the monodromy of $(X_t, D_t, E_t, \nabla_t)$ is that of the flat connection $(\mathcal{E}, \nabla)$ on $(\mathcal{X}, \mathcal{D})$ because the topology is the same after restricting to the transversal $X_t \subset \mathcal{X}$ to the divisor $\mathcal{D}$. So the monodromy must be constant. Therefore, the flat logarithmic connection induces an isomonodromic deformation. The first result of Schlesinger is that the converse is often true. We state it in the rank 2 case:

Theorem 3.4.1 (Schlesinger). *Let (E_0, ∇_0) be a logarithmic rank 2 connection on (X_0, D_0) with eigenvalues $\boldsymbol{\theta}$ satisfying $\theta_i^+ - \theta_i^- \notin \mathbb{Z}^*$. Let $t \mapsto (X_t, D_t, E_t, \nabla_t)$ be an isomonodromic deformation induced by a partial connection $(\mathcal{E}, \nabla)$ on the total space $(\mathcal{X}, \mathcal{D}) \to T$ of the deformation. Then, the partial connection ∇ is induced by a flat logarithmic connection ∇' on $(\mathcal{X}, \mathcal{D})$.*

Proof. Let us start considering the partial connection $(\mathcal{E}, \nabla)$ at a the neighborhood U of a generic point of X_0 (i.e. outside of the polar set) in local coordinates:

$$\nabla' = d + A(x)dx_1.$$

There exists a local trivialization $\Phi(x)$ (in family) given by solving

$$\Phi^{-1}\frac{d\Phi}{dx_1} = A(x).$$

If we choose Φ such that $\Phi(0,t) = I$, then $\Phi(x)$ will depend holomorphically on all variables $x = (x_1, t)$ on U. The corresponding monodromy representation will also depend holomorphically on t. By assumption and Lemma 3.4.2 below, there exists a holomorphic family of matrices $t \mapsto M(t) \in \mathrm{GL}_2(\mathbb{C})$ such that the monodromy of $M \circ \Phi$ along paths in fibers of t does not depend on t. Then, replacing Φ by $M \circ \Phi$, we can assume that Φ has constant monodromy. We note that eigenvalues of the partial connection restricted to curves (X_t, D_t) must have constant eigenvalues, since their exponential are constant (they are eigenvalues of the local monodromy), i.e. given by $\boldsymbol{\theta}$ as in the statement.

On the other hand, the connection (E_0, ∇_0) extends as a flat logarithmic connection $(\mathcal{E}', \nabla')$ on $(\mathcal{X}, \mathcal{D})$, and admits a local trivialization Φ' with the same monodromy representation as Φ. Then $\Psi := (\Phi')^{-1} \circ \Phi$ extends by analytic continuation along curves (X_t, D_t) as a holomorphic isomorphism of vector bundles $\Psi : \mathcal{E} \to \mathcal{E}'$ conjugating ∇ to ∇' along curves (X_t, D_t); by assumption on $\boldsymbol{\theta}$, the conjugacy Ψ extends holomorphically on $\mathcal{D}$. Then $\Psi^*\nabla$ provides the flat extension of ∇. $\square$

Lemma 3.4.2. *Let $t \mapsto (M_1(t), \ldots, M_{2g-2+n}(t))$ be a holomorphic family of representations in* $\mathrm{Rep}_{g,n}$ *(see (2.10)) and assume they are two-by-two conjugated. Then there is holomorphic family of matrices $t \mapsto (M(t))$ such that $M^{-1}M_iM$ is not depending on t anymore.*

Proof. Since $M_i(t)$ has constant eigenvalues, we can choose eigenvectors depending holomorphically on t. We start with the case where matrices $M_i(t)$ are not commuting. Then we can find $M_i(t), M_j(t)$ with distinct eigenvectors $v_i(t) \neq v_j(t)$. In the basis (v_i, v_j) (depending holomorphically on t), the matrices write

$$M_i = \begin{pmatrix} \lambda_i^+ & c_i(t) \\ 0 & \lambda_i^- \end{pmatrix} \quad \text{and} \quad M_j = \begin{pmatrix} \lambda_j^+ & 0 \\ c_j(t) & \lambda_j^- \end{pmatrix}.$$

Then, one of the non-diagonal coefficients is not zero (otherwise commutative), say c_i, and we can set $c_i = 1$ after conjugating by $\mathrm{diag}(c_i, 1)$ (holomorphic in t). This normalization is unique, and any conjugacy between two such normalized pairs of matrices should be the identity. In particular, under this normalization, the matrix $M(t)$ of the statement should be the identity, and therefore holomorphic. In the commutative case, we can proceed similarly with adapted normal form (diagonal or Jordan block). $\square$

3.5 Schlesinger equations

We now restrict to the rank 2 case with 4 poles for simplicity. Consider a Fuchsian system

$$\frac{dY}{dx} + \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}\right)Y = 0, \quad A_\infty = -A_0 - A_1 - A_t.$$

with eigenvalues $\boldsymbol{\theta} = \left(\pm\frac{\theta_0}{2}, \pm\frac{\theta_1}{2}, \pm\frac{\theta_t}{2}, \pm\frac{\theta_\infty}{2}\right)$ satisfying $\theta_i \notin \mathbb{Z}^*$.

Theorem 3.5.1 (Schlesinger equations)**.** *A small holomorphic deformation of the previous system:*

$$t \mapsto A(x,t) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}$$

is isomonodromic if, and only if, up to holomorphic gauge transformation $t \mapsto M(t)$, we have:

$$\frac{\partial A_0}{\partial t} = \frac{[A_0, A_t]}{t}, \quad \frac{\partial A_1}{\partial t} = \frac{[A_1, A_t]}{t-1} \quad \textit{and} \quad \frac{\partial A_\infty}{\partial t} = 0. \tag{3.1}$$

Proof. From Theorem 3.4.1, the assumption can be translated into the existence of a flat logarithmic connection inducing the partial connection $d + A(x,t)dx$. Precisely, there exists a flat logarithmic extension:

$$\nabla = d + A(x,t)dx + B(x,t)dt$$

where B, having only simple poles at $x = 0, 1, t, \infty$, must be of the form:

$$B(x,t) = \frac{B_0(t)}{x} + \frac{B_1(t)}{x-1} + \frac{B_t(t)}{x-t} + C(t) + B_\infty(t)x.$$

Now, the logarithmic condition, requiring that $\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x}$ has only simple poles, implies

$$B_0 = B_1 = B_\infty = 0 \quad \text{and} \quad B_t = -A_t.$$

Finally, integration of $d + C(t)dt$ gives a trivialization $\phi(t,Y) = (t, M(t)Y)$ which, applied globally to the flat connection allows us to make $C = 0$. We arrive at the normal form

$$\nabla = d + A_0(t)\frac{dx}{x} + A_1(t)\frac{dx}{x-1} + A_t(t)\frac{d(x-t)}{x-t}.$$

Then, the flatness of ∇ is equivalent to (3.1). Indeed, flatness writes

$$d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = \left(-\frac{\partial A_0}{\partial t} + \frac{[A_0, A_t]}{t}\right)\frac{dx \wedge dt}{x} + \left(-\frac{\partial A_1}{\partial t} + \frac{[A_1, A_t]}{t-1}\right)\frac{dx \wedge dt}{x-1}$$

$$+ \left(-\frac{\partial A_t}{\partial t} - \frac{[A_0, A_t]}{t} - \frac{[A_1, A_t]}{t-1}\right)\frac{dx \wedge dt}{x-t} = 0.$$

The vanishing of the two first coefficients gives the two first conditions of (3.1), and the vanishing of the sum of the three coefficients

$$-\frac{\partial A_0}{\partial t} - \frac{\partial A_1}{\partial t} - \frac{\partial A_t}{\partial t} = 0$$

gives the third condition of (3.1) after recalling $A_\infty = -A_0 - A_1 - A_t$. □

Remark 3.5.2. *There is a small gap in the presentation of Schlesinger Theorem. First of all, the vector bundle E_t deforms along isomonodromic deformations. However, upper semi-continuity of $t \mapsto \dim H^0(X_t, E_t)$ shows that $t \mapsto k_t$ is upper semi-continuous in Birkhoff decomposition $E_t = \mathcal{O}(-k_t) \oplus \mathcal{O}(k_t)$: the set of parameters for which $k = 0$ (minimum) is open. Therefore, if we start with a system, i.e. a connection on the trivial bundle $E_0 = \mathcal{O} \oplus \mathcal{O}$, then small deformations E_t will also be the trivial bundle. On the other hand, this also implies that $\mathcal{E}$ is trivial on a neighborhood of X_0. Indeed, one can apply Fischer and Grauert (1965) to trivialize $\mathbb{P}(\mathcal{E})$, the deformation of compact surfaces $\mathbb{P}(E_t)$, and then it remains to locally trivialize $\det(\mathcal{E})$ which is easy.*

3.6 The Painlevé VI equation

Let us explain how to derive Painlevé VI equation from the Schlesinger PDE under generic assumptions. In what follows, we denote $P(x) = x(x-1)(x-t)$ and

$$\frac{dY}{dx} + \left(\underbrace{\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}}_{A(x)} \right) Y = 0, \quad A_\infty = -A_0 - A_1 - A_t, \tag{3.2}$$

with

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \quad \text{and} \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}, \; i = 0, 1, t, \infty.$$

We have

$$-\det(A_i) = a_i^2 + b_i c_i = \frac{\theta_i^2}{4}, \quad i = 0, 1, t, \infty. \tag{3.3}$$

Since we can apply a constant gauge transformation to the Schlesinger system without changing equations, we will assume for simplicity $\theta_\infty \neq 0$ and normalize

$$A_\infty = \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{pmatrix}, \quad \theta_\infty \neq 0. \tag{3.4}$$

We have the following equations

$$\begin{cases} a_0 + a_1 + a_t & = & \frac{\theta_\infty}{2} \\ b_0 + b_1 + b_t & = & 0 \\ c_0 + c_1 + c_t & = & 0 \end{cases} \tag{3.5}$$

Our normalization of the system is not unique, and we can still conjugate by a constant diagonal matrix. Invariant functions under this action are given by a_i's and $b_i c_j$'s. They generate the ring of polynomial functions invariant under the action: they define polynomial functions on the moduli space.

We introduce rational coordinates on the moduli space of systems, namely:

$$q=\frac{tb_0}{tb_0+(t-1)b_1}\quad\text{and}\quad p=\frac{-a_0+\frac{\theta_0}{2}}{q}+\frac{-a_1+\frac{\theta_1}{2}}{q-1}+\frac{-a_t+\frac{\theta_t}{2}}{q-t}. \tag{3.6}$$

In what follows, we will work on the open set where q and p are well defined. We note that q and p are invariant under diagonal conjugacy: $x=q$ is the zero of the $(1,2)$ coefficient $b(x)$ of the system, i.e. the point where the $\frac{\theta_\infty}{2}$-eigenvector of A_∞ is also an eigenvector of $A(x)$ with eigenvalue

$$p-\frac{\theta_0}{2q}-\frac{\theta_1}{2(q-1)}-\frac{\theta_t}{2(q-t)}.$$

We claim that we can reconstruct the system from generic $(q,p)\in\mathbb{C}^2$ uniquely up to conjugacy. In order to do this, we first notice that equations (3.5) allow us to express a_t,b_t,c_t in term of a_0,a_1,b_0,b_1,c_1,c_0. Moreover, equations (3.3) for $i=0,1$ allow us to express c_1,c_0 in terms of a_0,a_1,b_0,b_1, and we thus get:

$$\begin{cases} c_0 &= \frac{\frac{\theta_0^2}{4}-a_0^2}{b_0}\\ c_1 &= \frac{\frac{\theta_1^2}{4}-a_1^2}{b_1}\\ a_t &= -a_0-a_1+\frac{\theta_\infty}{2}\\ b_t &= -b_0-b_1\\ c_t &= -\frac{\frac{\theta_0^2}{4}-a_0^2}{b_0}-\frac{\frac{\theta_1^2}{4}-a_1^2}{b_1}\end{cases} \tag{3.7}$$

It is therefore enough to know

$$a_0,\ a_1 \text{ and } \frac{b_1}{b_0}$$

to reconstruct the system up to conjugacy. But there is an extra relation on these three variables which is (3.3) for $i=t$, inducing after substituting (3.7) the following

$$\left(a_0+a_1-\frac{\theta_\infty}{2}\right)^2+(b_0+b_1)\left(\frac{\frac{\theta_0^2}{4}-a_0^2}{b_0}+\frac{\frac{\theta_1^2}{4}-a_1^2}{b_1}\right)=\frac{\theta_t^2}{4}.$$

By definition of q (see (3.6), we have

$$\frac{b_1}{b_0}=-\frac{t}{q}\frac{q-1}{t-1} \tag{3.8}$$

and we can rewrite the previous equality as

$$\frac{q(q-1)}{t(t-1)}\left\{\left(\frac{t}{q}a_0+\frac{t-1}{q-1}a_1\right)^2-\left(\frac{t\theta_0}{2q}\right)^2-\left(\frac{(t-1)\theta_1}{2(q-1)}\right)^2\right\} \tag{3.9}$$

$$-\theta_\infty(a_0+a_1)+\frac{\theta_0^2+\theta_1^2-\theta_t^2+\theta_\infty^2}{4}=0$$

On the other hand, substituting (3.7) in the definition (3.6) of p, we have

$$p=\frac{t}{q(q-t)}a_0+\frac{t-1}{(q-1)(q-t)}a_1+\frac{\theta_0}{2q}+\frac{\theta_1}{2(q-1)}+\frac{\theta_t-\theta_\infty}{2(q-t)} \tag{3.10}$$

from which we deduce

$$\frac{t}{q}a_0+\frac{t-1}{q-1}a_1=(q-t)\left(p-\frac{\theta_0}{2q}-\frac{\theta_1}{2(q-1)}-\frac{\theta_t-\theta_\infty}{2(q-t)}\right) \tag{3.11}$$

This later equation allows us to eliminate quadratic terms in a_0, a_1 in (3.9), so that the combination of (3.9) and (3.11) provide a system of linear equations for a_0, a_1 which allow to express these coordinates in term of q and p. So far, we have constructed a birational map $(p,q) : \mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1,\{0,1,t,\infty\}) \to \mathbb{C}^2$ and we are now going to express the differential constraints (3.1) in terms of variables (p,q), and then finally only on q.

For simplicity, we denote a_i' the derivative $\frac{\partial a_i}{\partial t}$ and so on. Equations (3.1) write

$$\begin{cases} a_0' &= \frac{c_0b_1-b_0c_1}{t} \\ b_0' &= 2\frac{b_0a_1-a_0b_1}{t}-\frac{b_0\theta_\infty}{t} \\ c_0' &= 2\frac{a_0c_1-c_0a_1}{t}+\frac{c_0\theta_\infty}{t} \end{cases} \quad\text{and}\quad \begin{cases} a_1' &= -\frac{c_0b_1-b_0c_1}{t-1} \\ b_1' &= -2\frac{b_0a_1-a_0b_1}{t-1}-\frac{b_1\theta_\infty}{t-1} \\ c_1' &= -2\frac{a_0c_1-c_0a_1}{t-1}+\frac{c_1\theta_\infty}{t-1} \end{cases} \tag{3.12}$$

A combination of above equations yields simpler ones

$$\begin{cases} ta_0'+(t-1)a_1' &= 0 \\ tb_0'+(t-1)b_1' &= -(b_0+b_1)\theta_\infty \\ tc_0'+(t-1)c_1' &= (c_0+c_1)\theta_\infty \end{cases} \tag{3.13}$$

Now, derivating (3.8), one has

$$q'=q(q-1)\left(\frac{b_1'}{b_1}-\frac{b_0'}{b_0}+\frac{1}{t(t-1)}\right). \tag{3.14}$$

Using equations (3.12) for b_0' and b_1', we get

$$\frac{b_1'}{b_1}-\frac{b_0'}{b_0}=2\frac{a_0}{t}\left(\frac{b_1}{b_0}\right)+\left(2\frac{a_0}{t-1}-2\frac{a_1}{t}-\frac{\theta_\infty}{t(t-1)}\right)-2\frac{a_1}{t-1}\left(\frac{b_0}{b_1}\right). \tag{3.15}$$

Substituting (3.15) in (3.14) yields

$$q'=2\frac{q(q-1)}{t(t-1)}\left(a_0\frac{t}{q}+a_1\frac{t-1}{q-1}+\frac{1-\theta_\infty}{2}\right). \tag{3.16}$$

Substituting (3.11), we get

$$q' = 2\frac{q(q-1)(q-t)}{t(t-1)}\left(p - \frac{\theta_0}{2q} - \frac{\theta_1}{2(q-1)} - \frac{\theta_t - 1}{2(q-t)}\right). \tag{3.17}$$

In a very similar way (a bit more complicated though), we can differentiate equation (3.10) with respect to t, and then substitute (3.12) and (3.7) and get

$$p' = -\frac{(3q^2 - 2(t+1)q + t)p^2 - ((\theta_0 + \theta_1 + \theta_t - 1)(2q-1) - \theta_0 t - \theta_1(t-1))p + \rho(\rho + 1 - \theta_\infty)}{t(t-1)} \tag{3.18}$$

where ρ is defined by (2.20)

$$(\theta_0, \theta_1, \theta_t, \theta_\infty) = (\kappa_0, \kappa_1, \kappa_t, 1 - \kappa_\infty) \quad \text{and} \quad \kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 2\rho = 1.$$

We can resume the discussion as follows

Theorem 3.6.1 (Fuchs–Malmquist)**.** *Consider a system (3.2) with generic $\boldsymbol{\theta}$ defined by parameters $(p,q) \in \mathbb{C}^2$. Then, a deformation $t \mapsto (p(t), q(t))$ of system (3.2) is isomonodromic if, and only if, eigenvalues θ_i are fixed, and parameters $(p(t), q(t))$ defined by (3.6) satisfy the non autonomous Hamiltonian system*

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} \quad \textit{and} \quad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q} \tag{3.19}$$

where H is defined by

$$H = \tfrac{q(q-1)(q-t)}{t(t-1)}\left(p^2 - (\tfrac{\kappa_0}{q} + \tfrac{\kappa_1}{q-1} + \tfrac{\kappa_t - 1}{q-t})p + \tfrac{\rho(\kappa_\infty + \rho)}{q(q-1)}\right). \tag{3.20}$$

The differential system (3.20) can be expressed as a vector field

$$v = \partial_t + \frac{\partial H}{\partial p}\partial_q - \frac{\partial H}{\partial q}\partial_p$$

and is also defined by the kernel of the closed rational 2-form

$$\omega = dp \wedge dq + dt \wedge dH.$$

Finally, we deduce

Theorem 3.6.2. *Consider a system (3.2) with generic $\boldsymbol{\theta}$ normalized by (3.4), and define parameter $q := \frac{tb_0}{tb_0 + (t-1)b_1} \in \mathbb{C} \setminus \{0, 1, t\}$. A small deformation $A_i = A_i(t)$ of the normalized system is isomonodromic if, and only if, θ_i's are constant and $q(t)$ satisfies*

$$\frac{dq}{dt} = -2a_0\frac{q-1}{t-1} - 2a_1\frac{q}{t} + (1 - \theta_\infty)\frac{q(q-1)}{t(t-1)} \tag{3.21}$$

and the Painlevé VI equation

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)$$
$$+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\kappa_\infty^2}{2} - \frac{\kappa_0^2}{2}\frac{t}{q^2} + \frac{\kappa_1^2}{2}\frac{t-1}{(q-1)^2} + \frac{1-\kappa_t^2}{2}\frac{t(t-1)}{(q-t)^2}\right). \quad (3.22)$$

with parameter $\boldsymbol{\kappa} = (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (\theta_0, \theta_1, \theta_t, 1-\theta_\infty)$.

Proof. Rewriting (3.17) as

$$p = \frac{t(t-1)}{2q(q-1)(q-t)}q' + \frac{\theta_0}{2q} + \frac{\theta_1}{2(q-1)} + \frac{\theta_t - 1}{2(q-t)} \quad (3.23)$$

we can substitute in (3.18), which gives a second order differential equation in q, namely the Painlevé VI equation. Conversely, given a solution $q(t)$, we can define $p(t)$ as above and check that $(p(t), q(t))$ satisfies Hamiltonian system (3.20). □

3.7 Symmetries of the Painlevé VI equation

A symmetry of the Painlevé VI equation is a birational transformation of $(\boldsymbol{\kappa}, t, q, p)$ that is preserving Painlevé VI solutions. There are countably many, and they have been described in Okamoto (1987), named canonical tranformations. Most of them arise from isomorphisms described in Section 2.7.4, that we can perform in family along deformation t of the polar locus.

First of all, we can permute the role of the 4 poles of system (3.2). For instance, changing $x \mapsto 1-x$ in the system yields a new system with parameters

$$\sigma_{(01)} \;:\; (\theta_0, \theta_1, \theta_t, \theta_\infty, t, q, p) \mapsto (\theta_1, \theta_0, \theta_t, \theta_\infty, 1-t, 1-q, -p)$$

This induces a transformation

$$\sigma_{(01)} \;:\; (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto (\kappa_1, \kappa_0, \kappa_t, \kappa_\infty, 1-t, 1-q, -p)$$

on the Painlevé VI parameters, which preserves (globally) the family of solutions $q(t)$. Similarly, $x \mapsto \frac{x}{t}$ induces

$$\sigma_{(1t)} \;:\; (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_0, \kappa_t, \kappa_1, \kappa_\infty, \frac{1}{t}, \frac{q}{t}, tp\right)$$

and $x \mapsto \frac{1}{x}$

$$\sigma_{(0\infty)} \;:\; (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_\infty, \kappa_1, \kappa_t, \kappa_0, \frac{1}{t}, \frac{1}{q}, -q(pq+\rho)\right).$$

These 3 transformations generate the symmetric group S_4. If we do not want to change the value of t, only the 4 Klein group is acting. For instance, changing $x \to \frac{t}{x}$ in the system yields the transformation

$$\sigma_{(0\infty)(1t)} \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_\infty, \kappa_t, \kappa_1, \kappa_0, t, \frac{t}{q}, -\frac{q(pq+\rho)}{t}\right).$$

We can also change sign of exponents κ_i's, which does not change q, but changes the definition of p:

$$\epsilon_0 \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(-\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p - \frac{\kappa_0}{q}\right)$$

$$\epsilon_1 \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_0, -\kappa_1, \kappa_t, \kappa_\infty, t, q, p - \frac{\kappa_1}{q-1}\right)$$

$$\epsilon_t \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_0, \kappa_1, -\kappa_t, \kappa_\infty, t, q, p - \frac{\kappa_t}{q-t}\right)$$

$$\epsilon_\infty \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto (\kappa_0, \kappa_1, \kappa_t, -\kappa_\infty, t, q, p)$$

Last but not least, the following symmetry has no direct interpretation as a natural transformation of the system and we call it *Okamoto symmetry*:

$$s \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto \left(\kappa_0 + \rho, \kappa_1 + \rho, \kappa_t + \rho, \kappa_\infty + \rho, t, q + \frac{\rho}{p}, p\right)$$

where $\rho = \frac{1-\kappa_0-\kappa_1-\kappa_t-\kappa_\infty}{2}$ as usual. The 8 involutions

$$\sigma_{(0\infty)(1t)}, \ \sigma_{(1\infty)(0t)}, \ \sigma_{(t\infty)(01)}, \ \epsilon_0, \ \epsilon_1, \ \epsilon_t, \ \epsilon_\infty, \ s$$

generate the group W of *Bäcklund transformations* found by Okamoto (1987). It is identified to some affine Weyl group related to the root system D_4 in Boalch (2006a) and Noumi and Yamada (2002). In fact, the induced action on parameter space $\boldsymbol{\kappa}$ is faithful and W identifies to a group of affine transformations of $\mathbb{C}^4 \ni \boldsymbol{\kappa}$ generated by the above 8 involutions.

The group W is infinite since it contains translations

$$(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \mapsto (\kappa_0 + n_0, \kappa_1 + n_1, \kappa_t + n_t, \kappa_\infty + n_\infty), \quad n_i \in \mathbb{Z}, \quad \sum_i n_i \in 2\mathbb{Z}.$$

The corresponding transformations on moduli spaces coincide with symmetries arising from elementary transformations (see Section 2.7.3). For instance, an elementary transformation at $x = 0$ and $x = \infty$ induces a transformation

$$\mathrm{elem}_{0,\infty} \ : \ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto (1-\kappa_0, \kappa_1, \kappa_t, 1-\kappa_\infty, t, \widetilde{q}, \widetilde{p})$$

where

$$\widetilde{q} = t\,\frac{p^2 - \left(\frac{\kappa_1}{q-1} + \frac{\kappa_t}{q-t}\right)p + \frac{\rho(\rho+\kappa_0+\kappa_\infty-1)}{(q-1)(q-t)}}{qp^2 + \left(2\rho - \frac{\kappa_1}{q-1} - \frac{t\kappa_t}{q-t}\right)p + \frac{\rho(\rho q - t(\rho+\kappa_t)-\rho-\kappa_1)}{(q-1)(q-t)}}$$

and

$$\widetilde{p} = -(pq+\rho)\frac{qp^2 + \left(2\rho - \frac{\kappa_1}{q-1} - \frac{t\kappa_t}{q-t}\right)p + \frac{\rho(\rho q - t(\rho+\kappa_t)-\rho-\kappa_1)}{(q-1)(q-t)}}{t\left(p + \frac{\rho}{q-1}\right)\left(p + \frac{\rho}{q-t}\right)}$$

One easily checks that this transformation coincides with the involution

$$\sigma_{(0\infty)(1t)} \circ \epsilon_0 \circ s \circ \epsilon_1 \circ \epsilon_t \circ s$$

of W. We promptly deduce the translation

$$\underbrace{\sigma_{(0\infty)(1t)} \circ \epsilon_0 \circ s \circ \epsilon_1 \circ \epsilon_t \circ s}_{\mathrm{elem}_{0,\infty}} \circ \epsilon_0 \circ \epsilon_\infty :$$

$$(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty, t, q, p) \mapsto (\kappa_0 + 1, \kappa_1, \kappa_t, \kappa_\infty + 1, t, \bar{q}, \bar{p})$$

where we omit the explicit expression of $\bar{q}$ and $\bar{p}$.

4 Monodromy of Painlevé VI equation

4.1 The Painlevé Property

Given a local meromorphic function f on a disc $D \subset \mathbb{C}$, and given a path $\gamma : [0,1] \to \mathbb{C}$ starting at $\gamma(0) = x_0 \in D$, we say that f *admits an analytic continuation with poles along* γ (or *meromorphic continuation*) if there is a sequence of discs $D = D_0, D_1, \ldots, D_n$ the path γ successively passes through, and meromorphic functions $f_i : D_i \to \widehat{\mathbb{C}}$ such that $f_i|_{D_i \cap D_j} \equiv f_j|_{D_i \cap D_j}$. Then, the germ of meromorphic function f^γ defined by f_n at $x_1 := \gamma(1) \in D_n$ is called *analytic continuation* (with poles) of f along γ.

A differential equation $P(x, y, y', \ldots, y^{(n)}) = 0$ with P polynomial is said to have the Painlevé Property if there exists a finite set $S \subset \mathbb{C}$ such that for any local meromorphic solution $f(x)$ at the neighborhood of some point $x_0 \in \mathbb{C} \setminus S$, and for any path $\gamma : [0,1] \to \mathbb{C} \setminus S$ starting at $\gamma(0) = x_0$, then f admits an analytic continuation with poles along γ. Likely as in the holomorphic case, f^γ only depends of the homotopy class of γ with fixed ends in $\mathbb{C} \setminus S$.

A first consequence of the Painlevé Property is that we have a one to one correspondance between local meromorphic solutions at x_0 and local meromorphic solutions at x_1 for any two points $x_0, x_1 \in \mathbb{C} \setminus S$. Indeed, we can define an isomorphism

$$\begin{array}{ccc}\left\{\begin{array}{c} f : (\mathbb{C}, x_0) \to \widehat{\mathbb{C}} \quad \text{local} \\ \text{meromorphic solution at } x_0 \end{array}\right\} & \to & \left\{\begin{array}{c} g : (\mathbb{C}, x_1) \to \widehat{\mathbb{C}} \quad \text{local} \\ \text{meromorphic solution at } x_1 \end{array}\right\} \\ f & \mapsto & f^\gamma \end{array}$$

for any path $\gamma : [0,1] \to \mathbb{C} \setminus S$ from x_0 to x_1.

A second consequence is that, for any $x_0 \in D \subset \mathbb{C} \setminus S$, any local meromorphic solution at (a neighborhood of) x_0 extends uniquely as a meromorphic solution on the whole of D. After lifting the differential equation on the universal cover $\widetilde{\mathbb{C} \setminus S} \to \mathbb{C} \setminus S$, all local meromorphic solutions actually come from global meromorphic solutions. For instance, Painlevé I equation has Painlevé Property for $S = \emptyset$ so that all local solutions are actually meromorphic over $\mathbb{C}$.

Example 4.1.1. *The functions $f_c(x) = \sqrt{x-c}$ are solutions of the differential equation $f' = \frac{1}{2f}$ which does not satisfy Painlevé Property. Indeed, each f_c define local meromorphic solutions, but there is no disc $D \subset \mathbb{C}$ on which all these functions are meromorphic: we are missing those f_c with $c \in D$. The obstruction is given by branching $x = c$ around which the function f_c is multiform. We call this* movable singular points*, because their position c depends on the solution f_c. This illustrates the strength of Painlevé Property.*

A third consequence of the Painlevé Property is that we can define the monodromy representation:

$$\begin{array}{ccc} \pi_1(\mathbb{C} \setminus S, x_0) & \to & \mathrm{Perm}\left\{ \begin{array}{c} f : D \to \widehat{\mathbb{C}} \\ \text{meromorphic solution} \end{array} \right\} \\ \gamma & \mapsto & [f \mapsto f^\gamma] \end{array}$$

which, to a loop γ based at x_0, one associates the permutation of meromorphic solutions on D by analytic continuation along γ, where D is any disc such that $x_0 \in D \subset \mathbb{C} \setminus S$ (remind that any local meromorphic solution at x_0 extends on D). Here, Perm denotes the symmetric or permutation group of the set of solutions. The set of meromorphic solutions on D (or local at x_0) is called *space of initial conditions*, and can be endowed with a complex structure.

Kazuo Okamoto (1979) proved the Painlevé Property for Painlevé equations $y'' = f(x, y, y')$ by considering the foliation defined by trajectories $x \mapsto (x, y(x), z(x))$ of the vector field $v = \partial_x + z\partial_y + f(x, y, z)\partial_z$: these are of the form $x \mapsto (x, y(x), y'(x))$ and correspond to solutions $y(x)$ of the second order differential equation $y'' = f(x, y, y')$. These trajectories define a foliation by curves $\mathcal{F}^o$ on $Y^o = X^* \times \mathbb{C}^2$ which is transversal to the projection

$$\pi : Y^o \to X^* \ ; \ (x, y, z) \mapsto x$$

where $X^* = \mathbb{C} \setminus S$ with

- $S = \emptyset$ for P_I, P_{II},
- $S = \{0\}$ for P_{III}, P_{IV}, P_V,
- $S = \{0, 1\}$ for P_{VI}.

But solutions have poles, meaning that leaves (or trajectories) escape at infinity of the fiber $\pi^{-1}(x_0)$ at some (in fact any) point $x_0 \in X^*$. We have to compactify fibers in order to better understand.

Theorem 4.1.2 (Okamoto (1979)). *For each choice of parameters $\boldsymbol{\theta}$, there exist quasiprojective manifolds $Y^o \subset Y^* \subset \overline{Y}^*$ such that:*

1. *π extends as a submersion $\pi : \overline{Y}^* \to X^*$ with (compact) projective fibers,*
2. *coordinates (p, q) on Y^o extend meromorphically on $\overline{Y}^*$,*
3. *the foliation $\mathcal{F}^o$ extends as a singular holomorphic foliation $\overline{\mathcal{F}}$ on $\overline{Y}^*$,*
4. *the (singular) divisor $\Sigma := \overline{Y}^* \setminus Y^*$ is $\overline{\mathcal{F}}$-invariant, and the restriction $\overline{\mathcal{F}}|_\Sigma$ is vertical, i.e. tangent to* $\ker(\pi^* dx)$,
5. *the restriction $\mathcal{F} := \overline{\mathcal{F}}|_{Y^*}$ is everywhere transversal to π,*
6. *the pair $(\pi : Y^* \to X^*, \mathcal{F})$ is locally analytically trivial over X^* in the following sense: for each $x_0 \in X^*$, there is an open neighborhood $U \ni x_0$ and an analytic diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \pi^{-1}(x_0)$ such that*
   - *$\pi = \pi_0 \circ \Phi$ where $\pi_0 : U \times \pi^{-1}(x_0) \to U$ is the first projection,*
   - *$\mathcal{F}|_{\pi^{-1}(U)}$ is just the pull-back $\Phi^*\mathcal{F}_0$ of the horizontal foliation $\mathcal{F}_0$, whose leaves are fibers of the second projection $U \times \pi^{-1}(x_0) \to \pi^{-1}(x_0)$.*

The meaning of the last item is that the foliated bundle $(Y^*, \mathcal{F})$ can be thought as a non linear analogue of a flat (i.e. locally trivial) connection. This local triviality is exactly what is needed to derive a (non linear) monodromy representation. But let us first derive the Painlevé Property.

Corollary 4.1.3. *For each choice of parameters $\boldsymbol{\theta}$, the Painlevé VI equation (3.22) satisfies the Painlevé Property.*

Proof. Let $q(t)$ be a local meromorphic solution of (3.22). If $q(t)$ is constant, then there is nothing to prove: it admits analytic continuation everywhere. Assume now that $q(t)$ is not constant. Then, $p(t)$ is also meromorphic by (3.23). The curve $t \mapsto (t, p(t), q(t))$ is an integral curve of v, and therefore coincides, outside of the poles of p and q, to a piece of a leaf of $\mathcal{F}$. Let L denotes the complete leaf in Y^*. By item 6 of Theorem 4.1.2, the restriction of $\pi|_L : L \to X^*$ is a covering. On the other hand, item 2 implies that p and q are meromorphic on $L \subset Y^*$. Therefore, any path $\gamma : [0,1] \to X^*$ starting at x_0 can be lifted on L: this gives the analytic continuation of $\pi|_L^{-1}$ along γ. We deduce the analytic continuation $q \circ \pi|_L^{-1}$ of $q(t)$. □

We now turn to ideas of proof of Theorem 4.1.2 following Okamoto.

Idea of proof. We start considering some minimal fiber-wise compactification of $\pi : Y^o \to X^*$: Okamoto choose a fiber-wise compactification $\overline{Y}_0^*$ by Hirzebruch surfaces Σ_2 (see 2.7.6). Then, the vector field v is rational, defining a singular foliation $\overline{\mathcal{F}}_0$ on $\overline{Y}_0^*$.

Then, there are singular points where infinitely many leaves intersect a the same point. This occur along curves of singular points that are sections of the bundle $\pi : \overline{Y}_0^* \to X^*$ We have to blow-up these points in order to separate the leaves. Then Okamoto proceeds to 8 successive blow-ups after which we are in the following situation. The new manifold is $\pi : \overline{Y}^* \to X^*$, the foliation satisfies items 1-5. The divisor $Z = \overline{Y}^* \setminus Y^*$ splits as a simple normal crossing divisor:

$$Z = H \cup F_0 \cup F_1 \cup F_t \cup F_\infty \tag{4.1}$$

where each irreducible component H and F_i intersect each fiber of $\pi : \overline{Y}^* \to X^*$ as a rational curve. They intersect along 4 sections $\sigma_i := H \cap F_i, i = 0, 1, t, \infty$ which are the singular points of the foliation $\overline{\mathcal{F}}$. In fact, the restriction to a fiber of the blow-up process is exactly that one described in Section 2.7.6; we will see why, later in the text.

Now, assume by contradiction that there is a path $\gamma : [0, 1] \to X^*$ starting at x_0 which can be lifted as $\tilde{\gamma}$ in a leaf L of $\mathcal{F}$ via π along $[0, 1)$, but not at $t = 1$. Consider the limit set Ω (or accumulation set) of $\lim_{t\to 1} \tilde{\gamma}(t)$ inside the fiber $\pi^{-1}(x_1)$, $x_1 = \gamma(1)$. If Ω contains a regular point p of $\overline{\mathcal{F}}$, then $\tilde{\gamma}$ must be contained in the (regular) leaf passing through p, which in turn, coincides with L; but there is a limit in this case, what we excluded. Consequently, Ω is contained in the singular set of $\overline{\mathcal{F}}$, i.e. in the intersection point $\sigma_i \cap \pi^{-1}(x_1)$. But a careful study of singular points shows that this cannot occur: non vertical leaves cannot accumulate on this type of foliation singularities. We conclude that the foliation $\mathcal{F}$ has the lifting-path-property with respect to the projection $\pi : Y^* \to X^*$. □

4.1.1 Isomonodromic approach

Isomonodromic approach provides a more conceptual proof of the Painlevé Property, see Jimbo and Miwa (1981, 1981/82), Jimbo, Miwa, and Ueno (1981), and Malgrange (1983a,b). We start proving the following.

Theorem 4.1.4. *Fix generic $\boldsymbol{\theta}$. Then, there is a one-to-one correspondance between local meromorphic solutions $q(t)$ of Painlevé VI equation at $t_0 \in \mathbb{C} \setminus \{0, 1\}$ and local isomonodromic deformations $t \mapsto (E_t, \nabla_t)$ of connections (E_{t_0}, ∇_{t_0}) with poles $D_0 = \{0, 1, t_0, \infty\}$ and eigenvalues $\boldsymbol{\theta}$, i.e. points in $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D_0)$.*

We start by proving two lemmae.

Lemma 4.1.5. *Constant Painlevé solutions are of the form $q(t) = i$ for $i = 0, 1$, and occur exactly when $\theta_i = 0$ respectively. Painlevé solutions $q(t) = t$ only occur when $\theta_t = 0$. In that case, $p(t)$ satisfies a Riccati differential equation.*

When $\theta_\infty = 0$, there is also a stationary leaf that corresponds to $q(t) \equiv \infty$.

Proof. Straightforward computation allows to conclude in the constant case $q = 0, 1$. Now, substituting $q = t$ in (3.20) yields

$$\theta_t = 0 \quad \text{and} \quad \frac{dp}{dt} + p^2 + \left(\frac{1-\kappa_0+\kappa_t}{t} + \frac{1-\kappa_1+\kappa_t}{t-1}\right)p + \frac{\rho(\rho+\kappa_\infty)}{t(t-1)} = 0.$$

□

Lemma 4.1.6. *Let (E_0, ∇_0) belongs to $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D_0)$, and let $t \mapsto (E_t, \nabla_t)$ be an isomonodromic deformation with poles $D = \{0, 1, t, \infty\}$. Then, there is a punctured disc $\mathbb{D}^* = \mathbb{D}\setminus\{t_0\}$ around t_0 such that $E_t = \mathcal{O}\oplus\mathcal{O}$ for $t \in \mathbb{D}^*$, provided that (E_0, ∇_0) is irreducible.*

This is proved in Bolibrukh (1990).

Idea of proof. We have $E_t = \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Following the proof of Corollary 2.7.7, the locus of $\mathfrak{Con}^{\boldsymbol{\theta}}(X, D)$ is E_∞^-. First of all, this means that the set of parameters t for which E_t is the trivial bundle is open. Therefore, if $E_0 = \mathcal{O} \oplus \mathcal{O}$, then there is a disc $\mathbb{D} \ni t$ on which $E_t = \mathcal{O} \oplus \mathcal{O}$. Now, it remains to prove that there are no (non trivial) isomonodromic deformations on $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Here, we need that $\mathcal{O}(1)$ is not ∇_0 invariant, otherwise isomonodromic deformation and Fuch's relation impose the existence of a ∇-invariant line bundle of degree 1 along the deformation, i.e. the the bundle is $E_t \equiv \mathcal{O}(-1) \oplus \mathcal{O}(1)$; whence irreducibility assumption. One way to prove is to apply elementary transformations, for instance $\mathrm{elm}_{0,\infty}$, and check that the resulting connection is on $\mathcal{O} \oplus \mathcal{O}$ with $q = 0$. We can do this in family, and isomonodromy is preserved (monodromy is preserved by elementary transformations). Then Lemma 4.1.5 leads to a contradiction. Another way to prove, is to restrict the flat connection (inducing the isomonodromic deformation) to the section given by $\mathcal{O}(1)$ in $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$ and see that this gives a foliation trivializing the deformation of the poles, again contradiction (see Heu (2009)). We conclude that the locus of the non trivial bundle is a strict analytically closed subset in the parameter space. Therefore, even if $E_0 = \mathcal{O}(-1) \oplus \mathcal{O}(1)$, E_t must be trivial nearby, when t lying on a punctured disc. □

Proof of Theorem 4.1.4. A local meromorphic solution $q(t)$ gives rise to an isomonodromic deformation of system (3.2) by means of Theorem 3.6.2 for t belonging to a punctured disc (as $t = t_0$ might be a pole for q or for coefficients of the system). This can be viewed as an isomonodromic deformation of a connection $t \mapsto (E_t, \nabla_t)$ with $E_t = \mathcal{O} \oplus \mathcal{O}$. Then Corollary 3.3.1 allows to extend $t \mapsto (E_t, \nabla_t)$ at $t = t_0$ as a connection on a possibly non trivial bundle E_{t_0}. Conversely, given an isomonodromic deformation of a connection $t \mapsto (E_t, \nabla_t)$ near $t = t_0$, we can assume by Lemma 4.1.6 that $E_t = \mathcal{O} \oplus \mathcal{O}$ for $t \neq t_0$ (but t close enough to t_0) and normalize this deformation of system, and derive a solution $q(t)$ of Painlevé VI equation like in section 3.6. □

Now, if we construct the family of moduli spaces $\pi : \mathfrak{Con}^{\boldsymbol{\theta}} \to X^*$ whose fiber at $t \in X^*$ is $\mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1, D_t)$ with $D_t = \{0, 1, t, \infty\}$, then isomonodromic deformations provide a foliation $\mathcal{F}$ on the total space $\mathfrak{Con}^{\boldsymbol{\theta}}$ that satisfies items 5 and 6 of Theorem 4.1.2.

These are the main points of Painlevé Property, since items 1–4 are dealing with fiber-compactification. It turns out that the fiber-bundle $\pi : \mathfrak{Con}^{\boldsymbol{\theta}} \to X^*$ is algebraic (see Inaba, Iwasaki, and Saito (2006a,b)) and q defines a rational function on it. Then, through a t-parametrization of a leaf of the foliation $\mathcal{F}$, the function q restricts as a meromorphic function. Then, Painlevé Property promptly follows from item 6. This approach can be immediately generalized to all isomonodromy equation, which in turn satisfy Painlevé Property. Up to now, we do not know an equation satisfying Painlevé property that cannot be derived from an isomonodromic problem. Going back to Painlevé VI case, remind that $\mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1, D_t)$ as constructed in section 2.7.6 exactly coincides with Okamoto space $Y^* \to X^*$. One finds in Inaba, Iwasaki, and Saito (2006c) a modular interpretation of the fiber-wise compactification $Y^* \subset \overline{Y}^*$ which is given in terms of λ-connections and ϕ-connections.

4.2 Non linear monodromy of the Painlevé VI foliation

Item 6 of Theorem 4.1.2 allow us to define a monodromy representation for the pair $(\pi : Y^* \to X^*, \mathcal{F})$. Precisely, given $t_0 \in X^*$ and $\gamma : [0,1] \to X^*$ a loop based at t_0 in X^*, one can lift in $\mathcal{F}$ to deduce an automorphism $\Phi^\gamma \in \mathrm{Aut}(Y_0)$, where $Y_0 = \pi^{-1}(t_0)$ and Φ^γ is an analytic automorphism, i.e. a biholomorphism of Y_0. This can be thought as the monodromy of a non linear local system. Precisely, given a starting point $y_0 \in Y_0$, one can lift γ into the leaf of $\mathcal{F}$ passing through y_0, by lifting path property: the lifted path $\tilde{\gamma}$ starts at $\tilde{\gamma}(0) = y_0$ and ends at $\tilde{\gamma}(1) = y_1$. One derives a (Poincaré, or) *return map* $y_0 \mapsto y_1$ which to any initial condition associates the corresponding ending point. By classical result on the dependance of solutions of ODE on initial condition, the return map is holomorphic; by construction, it can be reversed just by changing γ by γ^{-1}. In fact, let us choose Φ^γ to be the inverse map $y_1 \mapsto y_0$, so that we get a (direct) group morphism

$$\pi_1(X^*, t_0) \to \mathrm{Aut}(Y_0)\ ;\ \gamma \mapsto \Phi^\gamma. \tag{4.2}$$

The analytic automorphisms Φ^γ are transcendental and this is not the right way to understand the global dynamics of the Painlevé foliation $\mathcal{F}$. It is therefore convenient to switch to the space of representations, the monodromy side, where monodromy maps become polynomial, and explicit.

Recall first that we have a natural identification $Y^* \simeq \mathfrak{Con}^{\boldsymbol{\theta}}$, and in particular for the t_0-fiber, $Y_0 \simeq \mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1, D_0)$. On the other hand, the Riemann–Hilbert correspondance (see Theorem 2.5.1)

$$\mathrm{Mon}_0 : \mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1, D_0) \to \mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}, \quad D_0 = \{0, 1, t_0, \infty\},$$

is an analytic diffeomorphism. The right-hand-side does not depend on t_0, but only on an identification of the fundamental group of $\widehat{\mathbb{C}} \setminus D_0$ with $\pi_{0,4}$. Of course, this can be done in family for t close to t_0 and we can therefore define a local map

$$\mathrm{Mon}_U : \mathfrak{Con}^{\boldsymbol{\theta}}|_U \to \mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}$$

where $\mathfrak{Con}^{\boldsymbol{\theta}}|_U$ denotes the restriction of the fiber bundle to a small open neighborhood $t_0 \in U \subset X^*$: $\sqcup_{t \in U} \mathfrak{Con}^{\boldsymbol{\theta}}(\mathbb{P}^1, D_t)$, $D_t = \{0, 1, t_0, \infty\}$. This map is clearly holomorphic, and its fibers coincide with (connected components of) leaves of $\mathcal{F}$ over U: they are isomonodromy leaves. However, the map Mon cannot be extended as a global holomorphic map on $\mathfrak{Con}^{\boldsymbol{\theta}}$. Indeed, the identification $\pi_1(\widehat{\mathbb{C}} \setminus D_0) \simeq \pi_{0,4}$ cannot be globalized over X^*. The reason is that, when t deforms in the large, one has to deform the generators of $\pi_1(\widehat{\mathbb{C}} \setminus D_0)$. Deforming along loops in X^* yields a non trivial transformation on the generators of the fundamental group: thinking of X^* as a moduli space of punctured curves, we get an action of the Mapping Class Group on the space of representations. This natural geometric action is precisely the obstruction to globalize Mon on X^*, and is the origin of non trivial monodromy Φ^γ. Precisely, the local map Mon $:=$ Mon$_U$ admits analytic continuation along any path γ by using the isomonodromic foliation: we can lift γ into isomonodromy leaves, and Mon must be constant along the leaves.

Let us fix $t = t_0 \in X^*$ as above and $x_0 \in X^* \setminus \{t_0\} = \widehat{\mathbb{C}} \setminus D_0$. Then fix loops $\gamma_0, \gamma_1, \gamma_t, \gamma_\infty$ such that γ_i turns around $i = 0, 1, t, \infty$ and γ_i is homotopic to the constant loop in $\widehat{\mathbb{C}} \setminus D_0 \cup \{i\}$. Assume moreover $\gamma_0 \cdot \gamma_1 \cdot \gamma_t \cdot \gamma_\infty = 1$. When we deform t turning around 1, then we have to deform γ_i into:

- $\widetilde{\gamma}_0 = \gamma_0$,
- $\widetilde{\gamma}_1 = (\gamma_1\gamma_t)\gamma_1(\gamma_1\gamma_t)^{-1}$,
- $\widetilde{\gamma}_t = (\gamma_1\gamma_t)\gamma_t(\gamma_1\gamma_t)^{-1}$,
- $\widetilde{\gamma}_\infty = \gamma_\infty$.

Then, starting from a representation

$$\begin{cases} \gamma_0 \mapsto M_0, \\ \gamma_1 \mapsto M_1, \\ \gamma_t \mapsto M_t, \\ \gamma_\infty \mapsto M_\infty \end{cases}$$

we get, after deformation, the new representation

$$\begin{cases} \gamma_0 \mapsto M_0, \\ \gamma_1 \mapsto (M_1M_t)^{-1}M_1(M_1M_t), \\ \gamma_t \mapsto (M_1M_t)^{-1}M_t(M_1M_t), \\ \gamma_\infty \mapsto M_\infty. \end{cases}$$

One easily checks that this provides a non trivial automorphism of $\mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4}$. In order to get things more explicit, we can compose with the Hausdorff quotient $\mathfrak{Rep}^{\boldsymbol{\theta}}_{0,4} \to \mathbf{Rep}^{\boldsymbol{\theta}}_{0,4}$ given in section 2.6. Then, the above transformation writes

$$\Psi_1 \ : \begin{cases} x \mapsto -x - yz + c_x, \\ y \mapsto y, \\ z \mapsto -z + xy + y^2z - c_xy + c_z. \end{cases}$$

where $c_x = a_0a_1 + a_ta_\infty$, $c_y = a_1a_t + a_0a_\infty$, $c_z = a_0a_t + a_1a_\infty$. This is done by computing the action of previous transformations of M_i's on traces of words. In a very similar way, turning t around ∞ induces the transformation

$$\Psi_\infty \ : \ \begin{cases} x \mapsto x, \\ y \mapsto -y + xz + x^2y - c_zx + c_y, \\ z \mapsto -z - xy + c_z. \end{cases}$$

Therefore, setting

$$\Psi_0 \ : \ \begin{cases} x \mapsto -x - yz + c_x, \\ y \mapsto -y + xz + yz^2 - c_xz + c_y, \\ z \mapsto z. \end{cases}$$

we get generators $\Psi_0 \circ \Psi_1 \circ \Psi_\infty = \mathrm{Id}_{\mathbf{Rep}^{\boldsymbol{\theta}}_{0,4}}$ of the monodromy representation of Painlevé VI foliation computed on the character variety

$$\mathbf{Rep}^{\boldsymbol{\theta}}_{0,4} \ : \ \{(x,y,z) \in \mathbb{C}^3 \ ; \ x^2 + y^2 + z^2 + xyz = c_xx + c_yy + c_zz + c\}.$$

4.3 Irreducibility

The dynamics of transformation group generated by Ψ_0 and Ψ_1 on $\mathbf{Rep}^{\boldsymbol{\theta}}_{0,4}$ has been studied at several places Benedetto and Goldman (1999), Cantat (2009), Dubrovin and Mazzocco (2000), Iwasaki and Uehara (2007), and Previte and Xia (2005). On the other hand, Casale (2008) reproved the irreducibility of Painlevé I equation in a dynamical way. One can associate to a foliation the Galois groupoid, following Malgrange (2001, 2002), as a kind of Zariski closure of the holonomy groupoid: we lift the groupoid of tangent-to-$\mathcal{F}$ transformations to jet spaces, then take Zariski closure in fibers, and then take projective limit over jet spaces order. The resulting groupoid is the smallest groupoid of transformations that contains the tangent pseudo-group and which can be defined by polynomial differential equations on the total space. For instance, the Painlevé I foliation has no dynamics from the topological point of view (all solutions are meromorphic on $\mathbb{C}$), but the transcendental oscillation feature near infinity provides an obstruction to trivialize the peudo-group algebraically. At least, there is an algebraic symplectic 2-form transversal to the foliation which is invariant under holonomy dynamics, namely $dp \wedge dq + dH \wedge dt$ (the defining closed 2-form for $\mathcal{F}$), or by ω (2.13) in $\mathbf{Rep}^{\boldsymbol{\theta}}_{0,4}$. Therefore, the Zariski closure of transversal dynamics must preserves this symplectic structure, as it is the case for all Painlevé equations. Now, if the closure coincides with the full symplectic groupoid, then Casale (2008) proved that the corresponding Painlevé equation is irreducible, i.e. cannot admit two independant first integrals that can be obtained from $\mathbb{C}(t,q,p)$ by using successively integration of closed 1-forms, solving linear differential equations, and finite field extensions. Then, Guy Casale used Lie's classification of pseudo-groups in dimension 2 to deduce that it suffices to exclude the existence of invariant curves, webs and affine structure to deduce irreducibility of the Painlevé equation; and he proved irreducibility of Painlevé I equation by this way. In Cantat and Loray (2009), this approach is used to prove irreducibility of all Painlevé VI equations except those related to Picard case $\kappa_0 = \kappa_1 = \kappa_t = \kappa_\infty = 0$ via the symmetry group W:

$$\boldsymbol{\kappa} = (n_0, n_1, n_t, n_\infty) \quad \text{or} \quad \left(\frac{1}{2} + n_0, \frac{1}{2} + n_1, \frac{1}{2} + n_t, \frac{1}{2} + n_\infty\right),$$

$$\text{where} \quad (n_0, n_1, n_t, n_\infty) \in \mathbb{Z}^4 \quad \text{with} \quad \sum_i n_i \in 2\mathbb{Z}.$$

In fact, it is proved, using dynamics of Ψ_0 and Ψ_1 on $\mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$, that there never exist invariant curves or invariant webs for the dynamics; and there exist invariant affine structure only for Picard parameters as above. It is proved in Casale (2007) that, for these special parameters, the Painlevé VI equation is indeed reducible.

So far, we are dealing with transcendance of first integrals. There is however another notion of irreducibility which is closer to the original idea of Painlevé, namely the transcendence of solutions, which is Nishioka–Umemura irreducibility. Let us detail a bit more. A local holomorphic function $f(x)$ is said reducible if there is a tower of extension of differential fields

$$\mathbb{Q}(x) = K_0 \subset K_1 \subset \cdots \subset K_m$$

such that $f \in K_m$ and each K_{l+1} is obtained from K_l by one of the following elementary extensions:

- $K_l \subset K_{l+1}$ is an algebraic extension,
- $K_l \subset K_{l+1}$ is a Picard–Vessiot extension, i.e. $K_{l+1} = K_l(m_{ij})$ where $M = (m_{ij})$ is a fundamental matrix (basis of solutions) of a linear system of differential equations $M' = A(x)M$, with $A \in \mathrm{GL}_r(K_l)$,
- $K_l \subset K_{l+1}$ is an Abelian extension, i.e. $K_{l+1} = K_l(f_1, \ldots, f_n)$ where f_i's form a basis of the field of meromorphic functions on an abelian variety defined on K_l,
- $K_l \subset K_{l+1}$ is an order one extension, i.e. $K_{l+1} = K_l(f)$ with $P(f, f') = 0$ for some polynomial $P \in K_l(X, Y)$.

The results of Nishioka (1988) and Umemura (1988) prove that the general solution of the Painlevé I equation is irreducible. It is proved in Watanabe (1998) that Painlevé VI equation is irreducible whatever are the parameters $\boldsymbol{\theta}$, therefore even in the Picard case: solutions are transcendental enough, but first integrals are special. It is proved in Casale (2009) that irreducibility of first integrals, as considered first, imply irreducibility of the general solution. However, we have to inform that there exist special reducible solutions, even algebraic solutions, as we will see in the next section.

Let us just explain how dynamics on the character variety can be used here, following Cantat and Loray (2009). The cubic affine surface $\mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$ is quadratic when we omit one variable. For instance, fixing $z = z_0$, we obtain a conic, say C_{z_0}. The automorphism Ψ_0

fixes the z-variable and induces an automorphism of each conic C_z. Except for possibly finitely many, C_z has smooth compactification $\simeq \widehat{\mathbb{C}}$ and the restriction $\Psi_0|_{C_z}$ is a Moebius transformation with fixed points at infinity. The type of the Moebius transformation is given by the trace after lifting to $\mathrm{SL}_2(\mathbb{C})$, which can be computed:

$$\mathrm{trace}(\Psi_0|_{C_z}) = z^2 - 2.$$

Therefore, the Moebius type of Ψ_0 along conic fibers is varrying. We promptly deduce:

- if $z = \pm 2$, then $\Psi_0|_{C_z}$ is parabolic or the identity;
- if $z \in (-2, 2)$ (real interval), then $\Psi_0|_{C_z}$ is elliptic; it is moreover periodic if, and only if, $z = 2\cos(\pi\theta)$ with $\theta \in \mathbb{Q} \setminus \mathbb{Z}$;
- if $z \notin [-2, 2]$, then $\Psi_0|_{C_z}$ is hyperbolic or loxodromic, and has unbounded orbits.

Now, it is easy to prove the non existence of invariant algebraic curves or webs. Indeed, such a curve Γ must intersect all but finitely many fibers C_z. On the other hand, the iterate $\Psi_0^{\circ N}$ for $N \in \mathbb{Z}_{>0}$ will be the identity in restriction to those $z_{k,n} = 2\cos(k\pi/N)$, $k \in \{1, \ldots, N-1\}$, and not the identity in restriction to other conic fibers. If Γ is not contained in a z-fiber, we must find infinitely many points where Γ intersects transversely a curve of fixed points $C_{z_{k,n}}$; but Γ cannot be locally invariant by $\Psi_0^{\circ N}$ since nearby dynamics are unbounded. In fact, even the tangent line of Γ is not invariant under action of the differential of $\Psi_0^{\circ N}$. Therefore Γ must be contained in a z-fiber; on the other hand, a similar argument with Ψ_1 proves that Γ is also contained in a y-fiber, contradiction. The same type of arguments can be used to exclude invariant foliations or webs.

Finally, let us notice that Picard parameters correspond to the Cayley cubic

$$\mathcal{C} := \mathbf{Rep}_{0,4}^{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \; : \; \{x^2 + y^2 + z^2 + xyz = 4\}$$

Then, after lifting on the infinite covering

$$\mathbb{C}^2 \mapsto \mathcal{C} \; ; \; (u, v) \mapsto (-2\cos(\pi u), -2\cos(\pi v), -2\cos(\pi uv))$$

the dynamics become affine transformations

$$Y = \begin{pmatrix} u \\ v \end{pmatrix} \mapsto AY + T, \quad A \in \mathrm{SL}_2(Z), \quad T \in \mathbb{C}^2, \quad A \equiv I \bmod 2$$

where Ψ_1 and Ψ_∞ respectively lift as $\widetilde{\Psi}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\widetilde{\Psi}_\infty = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$; translations $Y \mapsto Y + T$ and $\pm I$ arise from deck transformations. We note that Painlevé equation with parameters $\boldsymbol{\theta} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ correspond to deformations of systems with dihedral monodromy:

$$M_0 = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & \frac{\mu}{\lambda} \\ -\frac{\lambda}{\mu} & 0 \end{pmatrix}, \quad \text{and} \quad M_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

or with parabolic generators. The two options correspond via Okamoto symmetry s.

4.4 Special solutions, algebraic solutions

In the work Watanabe (1998), it is proved that reducible solutions of Painlevé VI equation (in the sense of Nishioka–Umemura) are either Riccati or algebraic solutions. Riccati solutions correspond to isomonodromic deformations of

- either reducible systems, yielding a relation $\sum \epsilon_i \theta_i \in 2\mathbb{Z}$, $\epsilon_i = \pm 1$,
- or systems with one apparent singular point (i.e. with $\pm I$ local monodromy), yielding $\theta_i \in \mathbb{Z}$.

They arise as one-parameter family and correspond to (-2) rational curves in the moduli space $\mathfrak{Con}(X, D, \boldsymbol{\theta})$, and to singular points of the character variety (see Saito and Terajima (2001)). For each relation on $\boldsymbol{\theta}$ as above, there is a one parameter family of special solutions parametrized by a rational curve. The Riccati equation corresponds to an hypergeometric equation, i.e. with only 3 poles.

Algebraic solutions have been studied and classified in many papers, among which Boalch (2005, 2006a,b, 2007a,b, 2010), Dubrovin and Mazzocco (2000), and Lisovyy and Tykhyy (2014). Of course, the Okamoto group W acts on the set of algebraic solutions, and one has to classify up to W-action. The classification up to symmetries is as follows.

- There are algebraic solutions occuring as special Riccati solutions; they can be classified by Schwarz' list.
- There is a 2-parameter family of degree 2 solutions, namely $q(t) = \sqrt{t}$, for parameters of type $(\kappa_0, \kappa_1, \kappa_t, \kappa_\infty)$.
- There is a 1-parameter family of degree 3 solutions, namely $q(t)$ implicitely defined by $q^3 - 3tq + 2t^2 = 0$, for parameters of type $(2\kappa, \frac{1}{3}, \kappa, \kappa)$.
- There is a 1-parameter family of degree 4 solutions, namely $q(t)$ implicitely defined by $3q^4 - 4(1+t)q^3 + 6tq^2 - t^2 = 0$, for parameters of type $(\kappa, \kappa, \kappa, 3\kappa)$.
- There is a discrete countable family of solutions for Picard parameter $\boldsymbol{\kappa} = (0, 0, 0, 0)$ that correspond to torsion points on the elliptic curve.
- There are 45 exceptional solutions, each of them arising for a single parameter $\boldsymbol{\kappa}$, whose degree varries between 5 and 72, and genus in $0, 1, 2, 3, 7$.

Algebraic solutions occuring in Riccati families can be classified through the Schwarz' list of algebraic solutions to hypergeometric equation. In Dubrovin and Mazzocco (2000), Dubrovin and Mazzocco classified algebraic solutions with parameter $\boldsymbol{\kappa} = (0, 0, 0, \theta)$ with $\theta \notin \mathbb{Z}$. Picard solutions have been found by Picard (1889), and also appear in Hitchin (1995) and Mazzocco (2001a). It turns out that finite order points along the Legendre family of elliptic curves $y^2 = x(x-1)(x-t)$ deform to define algebraic functions $t \mapsto (x(t), y(t))$; for all of them, $q(t) := x(t)$ is a solution of Picard–Painlevé parameters.

The three continuous families of degree $2, 3, 4$ were classified in Cantat and Loray (2009). Most of the list has been completed by Boalch in Boalch (2005, 2006a,b, 2007a,b, 2010). Most of them come from representations into finite subgroups of $\mathrm{SL}_2(\mathbb{C})$: indeed, as generators have to be sent into a finite group, there are finitely many possibilities. Among exceptional solutions, 1 tetrahedral, 7 octahedral, and 33 icosahedral solutions. One extra exceptional solution does not arise from finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, but from finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ in Boalch (2005) through the Fourier transform.

Another way to construct algebraic solutions, is to construct algebraic isomonodromic deformations. A nice idea is to construct such as pull-back of a fixed differential equation by an algebraic family of ramified covers. This is used to construct algebraic solutions in Doran (2001), and then in Andreev and Kitaev (2002) and Kitaev (2005). They are already in the list above. In fact, every algebraic solution is W-conjugated to a solution obtained by pull-back. For instance, those corresponding to finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ correspond to families of pull-back of a fixed hypergeometric equation in Klein's list, due to Theorem 1.5.3. Doran also found the special exceptional solution not related finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

The classification of algebraic solutions is as follows. One first classify all finite orbits on $\mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$ up to the non linear Painlevé monodromy $\langle \Psi_0, \Psi_1 \rangle$. Those of length $\leqslant 4$ can be done by hands. From length 5 and more, we can check that the point $(X, Y, Z) \in \mathbf{Rep}_{0,4}^{\boldsymbol{\theta}}$ must have coefficients taking value in the countable set $2\cos(\pi\mathbb{Q})$, other wise the orbit under one of the Ψ_i's, $i = 0, 1, \infty$, must be infinite. On the other hand, the image of (X, Y, Z) by one of the Ψ_i's must also be a point with coefficients in $2\cos(\pi\mathbb{Q})$. In fact, replacing W by a larger group W', that contains W with finite index, one breaks these constraints into simpler ones, namely some diophantine equations that can be handled. Of course, finite orbits under W' are related to finite orbits under W as $[W' : W] < \infty$. Then, we have to check that we recover the list above. This is the strategy succesfully followed in Lisovyy and Tykhyy (2014). Then we have to check that the corresponding finite branching solution is indeed infinite. This had been done previously by Boalch, who computed the asymptotics of the solution $q(t)$ when $t \to 0, 1, \infty$ in terms of the local monodromy of the solution, and then found bounds for the bidegree of the solution. Once you know the bidegree of the defining polynomial $P(t, q) = 0$, then is is straightforward (I mean theoretically, i.e. neglecting complexity) to find the polynomial which gives the solution by linear algebra. By this way, Boalch constructed explicitely all exceptional algebraic solutions.

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