

**Differential Equations of
Classical Geometry,
a Qualitative Theory**

Publicações Matemáticas

**Differential Equations of
Classical Geometry,
a Qualitative Theory**

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27^o Colóquio Brasileiro de Matemática

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Preface

These Lecture Notes are addressed to the reader with some familiarity with the Foundations of Ordinary Differential Equations and of Classical Differential Geometry.

The subject developed here centers around the local and global geometry on a surface: Fundamental Forms, principal curvatures, Gauss and Codazzi equations, Gauss–Bonnet Theorem. To represent the level required, references such as [16], [40], [44], [94], [100], [126], [164] and [166] can be mentioned.

The authors believe that from the Introduction provided here the reader will be encouraged to approach and study the papers quoted here, where more complete treatments and details are presented, and become interested in some of the many lines for advanced study and research outlined here, open in the field of interaction between Geometry and Differential Equations (O.D.E's).

This work attempts to illustrate the penetration that ideas such as genericity and structural stability of O.D.E's have in the development of the qualitative theory of differential equations of classical geometry.

Here an effort has been made to present most of the developments addressed to improve the local and global understanding of the structure of principal curvature lines, asymptotic lines and geodesics on

surfaces. The emphasis has been put on those developments derived from the assimilation of ideas coming from the *QTDE* and Dynamical Systems into the classical knowledge on the subject, as presented in prestigious treatises such as Darboux [37], Eisenhart [44], Struik [166], Hopf [85], Spivak [164].

The starting point for the results presented here, concerning principal lines, can be found in the papers of Gutierrez and Sotomayor [71, 72] and in the book of the same authors [75].

The authors acknowledge the influence they received from the well established theories of Structural Stability and Bifurcations which unfolded from the inspiring classical works of Andronov, Pontrjagin, Leontovich [1] and Peixoto [130, 131]. Also the results on bifurcations of principal configurations outlined in [75] and further elaborated along this work are motivated in the work of Sotomayor [158].

The vitality of the *QTDE* and Dynamical Systems, with their remarkable present day achievements, may lead to the belief that the possibilities for directions of future research on the differential equations of lines of curvature and other equations of Classical Geometry are too wide and undefined and that the source of problems in the subject consists mainly in establishing an analogy with one in the above mentioned fields.

While this may partially true in the present work, History shows us that the consideration of problems derived from purely geometrical sources and from other fields such as Control Theory, Elasticity, Image Recognition and Geometric Optics, have also a crucial role to play in determining the directions for relevant research in our subject. In fact, at the very beginning, the works of Monge and Dupin and, in relatively recent times, also the famous Carathéodory Con-

jecture [51], [23], [76], [77, 78, 79], [90], [111], [126, 127], [156, 157], [155], [182], represent geometric sources of research directions leading to clarify the structure of lines of curvature and their umbilic singularities.

Some of the problems and exercises proposed at the end of each chapter are not of routine sort. They are formulated in order to guide the reader into the classical literature on curves and surfaces and also into subjects of current research.

Many students and colleagues contributed with helpful comments and suggestions. In particular Pedro S. Salomão and Luis F. Mello pointed out important improvements to the text. Warm thanks to all them are recorded here.

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Ronaldo Garcia

Jorge Sotomayor

Goiânia and São Paulo, May 2009.

*This work is dedicated to the memory
of our collaborator and friend
Carlos Gutiérrez (1944-2008)*

and to our beloved families

*Cida
Breno (in memoriam)
Flávio*

*Marilda
Mariana
Leonardo*

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Chapter 1

Differential Equations of Classical Geometry

1.1 Introduction

In this chapter the basic notions of differential geometry of curves and surfaces in \mathbb{R}^3 will be reviewed. The differential equations of *geodesics*, *principal curvature lines* and *asymptotic lines* will be obtained.

The references for this chapter are [4], [16], [37], [40], [44], [75], [164] and [166].

The *principal curvature lines* are the curves along which the surface bends extremely. It can be said that the theory of curvature lines was founded by G. Monge (1796), who determined explicitly all the principal curvature lines of the ellipsoid with three different axes. This is probably the first example found in the literature of foliations with singularities.

The *geodesics*, also a classical notion, are obtained as a critical points (local minimizers of the length) applying the Calculus of Variations. They can also be regarded, infinitesimally, as the curves of zero geodesic curvature.

The *asymptotic lines* are characterized geometrically as the curves along which the osculating plane of the curve coincides with the tangent plane of the surface.

1.2 The First Fundamental Form

Let $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^3$ be a C^r , $r \geq 3$, immersion of a smooth surface \mathbb{M} into \mathbb{R}^3 .

The space \mathbb{R}^3 is oriented by a once for all fixed orientation and is endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$.

The induced metric on $T_p\mathbb{M}$ is defined by

$$\langle u, v \rangle_p := \langle D\alpha(p)u, D\alpha(p)v \rangle, \quad \text{where } u, v \in T_p\mathbb{M}.$$

In a local chart $(u, v) : \mathbb{M} \rightarrow \mathbb{R}^2$, consider a parametric curve $c(t) = (u(t), v(t))$. Then it follows that $x(t) = (\alpha \circ c)(t)$ is a curve and $x' = \alpha_u u' + \alpha_v v'$ is a tangent vector of $T_p\mathbb{M}$, where $p = c(0)$.

Therefore, $\langle x', x' \rangle = \langle \alpha_u, \alpha_u \rangle (u')^2 + 2 \langle \alpha_u, \alpha_v \rangle u'v' + \langle \alpha_v, \alpha_v \rangle (v')^2$ and the expression

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{1.1}$$

where $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$ and $\langle \alpha_v, \alpha_v \rangle$, is called the *first fundamental form* of α and is of class C^{r-1} . This form is positive definite, i.e., $E > 0$, $G > 0$ and $EG - F^2 > 0$. The distance, in the

induced metric, between two points $p = c(t_0)$ and $q = c(t_1)$ on the curve c , supposed rectifiable, is defined by:

$$L_c(p, q) = \int_{t_0}^{t_1} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2} dt.$$

The angle between two directions, defined in a local chart by, $dx = \alpha_u du + \alpha_v dv$ and $dy = \alpha_u \delta u + \alpha_v \delta v$ is defined by: $\cos \theta = \frac{\langle dx, dy \rangle}{|dx||dy|}$.

Therefore the *angle* between the parametric curves $u = \text{cte}$ and $v = \text{cte}$ is given by: $\cos \theta = \frac{F}{\sqrt{EG}}$ and $\sin \theta = \pm \frac{\sqrt{EG - F^2}}{\sqrt{EG}}$.

From the metric ds^2 , induced by α it is possible to define a distance d in \mathbb{M} such that (\mathbb{M}, d) become a metric space.

Given two points p and q the distance between them is defined by: $d(p, q) = \inf\{L_\gamma(p, q) : \gamma \text{ is a rectifiable curve connecting } p \text{ to } q\}$.

It can be shown, when (\mathbb{M}, d) is a complete metric space, that given any two points p and q of \mathbb{M} such that $d(p, q) > 0$ there is a curve $\gamma : [a, b] \rightarrow \mathbb{M}$ of class C^1 by parts, such that $\gamma(a) = p$, $\gamma(b) = q$ and $L_\gamma(p, q) = d(p, q)$.

In this way the surface \mathbb{M} has a strictly complete intrinsic distance and (\mathbb{M}, d) is called a *length space*. See [35, Chapter 2]. A useful property of length spaces is that they have the middle point property. Given $p, q \in \mathbb{M}$ such that $d(p, q) = 2\delta > 0$ there is at least a point $r \in \mathbb{M}$ such that $d(p, r) = d(r, q) = \delta$.

In particular $\bar{B}(p, \delta) \cap \bar{B}(q, \delta) \neq \emptyset$. Here $B(p, \delta)$ is the open ball centered at p and radius δ .

1.3 The Second Fundamental Form

In this section assume that \mathbb{M} is oriented by means of collection of *positive charts*, denominated *an atlas*, whose domains constitute a covering. Let N_α be the vector field orthonormal to α defining the positive orientation of \mathbb{M} . This means that if (u, v) is a positive chart then $\{\alpha_u, \alpha_v, N\}$ is a positive frame in \mathbb{R}^3 . This means that N_α is the unit vector $(\alpha_u \wedge \alpha_v)/|\alpha_u \wedge \alpha_v|$, called the *positive normal* of α .

The second fundamental form is introduced in order to define the concept of curvature of a surface. Let $x(s) = \alpha(u(s), v(s))$ be the space curve obtained mapping the curve on \mathbb{M} whose image by a positive chart (u, v) has the parametric equations $u = u(s)$, $v = v(s)$. Suppose that $|x'| = 1$, i.e. x is parametrized by arc length. The curvature vector $k(s) = \frac{dT}{ds}$ where $T(s) = \frac{dx}{ds}$ has the orthogonal decomposition $k = k_n N + k_g N \wedge T$; k_n is called the *normal curvature* and k_g is called the *geodesic curvature*.

From $\langle T, N \rangle = 0$ it follows that $\langle \frac{dT}{ds}, N \rangle = -\langle T, \frac{dN}{ds} \rangle$ and therefore $k_n = -\frac{\langle d\alpha, dN \rangle}{\langle d\alpha, d\alpha \rangle}$.

$$\text{So it is obtained,} \quad k_n = \frac{e du^2 + 2f dudv + g dv^2}{E du^2 + 2F dudv + G dv^2}.$$

Here, $e = -\langle \alpha_u, N_u \rangle$, $2f = -(\langle \alpha_u, N_v \rangle + \langle \alpha_v, N_u \rangle)$ and $g = -\langle \alpha_v, N_v \rangle$.

Also, as $\langle \alpha_u, N \rangle = \langle \alpha_v, N \rangle = 0$ it follows that $e = \langle \alpha_{uu}, N \rangle$, $f = \langle \alpha_{uv}, N \rangle$ and $g = \langle \alpha_{vv}, N \rangle$. Using the expression of $N = \frac{\alpha_u \wedge \alpha_v}{|\alpha_u \wedge \alpha_v|}$ it is obtained that

$$e = \frac{[\alpha_u, \alpha_v, \alpha_{uu}]}{\sqrt{EG - F^2}}, \quad f = \frac{[\alpha_u, \alpha_v, \alpha_{uv}]}{\sqrt{EG - F^2}}, \quad g = \frac{[\alpha_u, \alpha_v, \alpha_{vv}]}{\sqrt{EG - F^2}},$$

where $[\cdot, \cdot, \cdot]$ means the mixed product of three vectors in \mathbb{R}^3 . The quadratic form $II_\alpha = edu^2 + 2fdudv + gdv^2$

is called the *second fundamental form* of α and is of class C^{r-2} .

1.4 Fundamental Equations

The two fundamental forms, which define the length and curvature of curves on surfaces are related to the Fundamental Equations of Surface Theory.

These equations are obtained writing the vectors α_{uu} , α_{uv} , α_{vv} , N_u and N_v in terms of the frame $\{\alpha_u, \alpha_v, N\}$. Here N is the unit normal defined by $N = (\alpha_u \wedge \alpha_v)/|\alpha_u \wedge \alpha_v|$.

Direct calculation gives

$$\begin{aligned}
 \alpha_{uu} &= \Gamma_{11}^1 \alpha_u + \Gamma_{11}^2 \alpha_v + eN \\
 \alpha_{uv} &= \Gamma_{12}^1 \alpha_u + \Gamma_{12}^2 \alpha_v + fN \\
 \alpha_{vv} &= \Gamma_{22}^1 \alpha_u + \Gamma_{22}^2 \alpha_v + gN \\
 (EG - F^2)N_u &= (fF - eG)\alpha_u + (eF - fE)\alpha_v \\
 (EG - F^2)N_v &= (gF - fG)\alpha_u + (fF - gE)\alpha_v
 \end{aligned} \tag{1.2}$$

where it is assumed that α is of class at least C^3 , $\alpha_{uv} = \alpha_{vu}$, and the Christoffel symbols are given by:

$$\begin{aligned}
 \Gamma_{11}^1 &= \frac{E_u G - 2F_u F + E_v F}{2(EG - F^2)} & \Gamma_{11}^2 &= \frac{2F_u E - E_u E - E_u F}{2(EG - F^2)} \\
 \Gamma_{12}^1 &= \frac{E_v G - G_u F}{2(EG - F^2)} & \Gamma_{12}^2 &= \frac{G_u E - E_v F}{2(EG - F^2)} \\
 \Gamma_{22}^1 &= \frac{2F_v G - G_v G - G_v F}{2(EG - F^2)} & \Gamma_{22}^2 &= \frac{G_v E - 2F_v F + G_u F}{2(EG - F^2)}
 \end{aligned} \tag{1.3}$$

The equations (1.2) are the *fundamental equations* of the surface. They express the derivatives of higher order of α in relation to the

frame $\{\alpha_u, \alpha_v, N\}$. In a certain sense this is the correspondent of Frenet equations for space curves.

Now consider the compatibility equations, $(\alpha_{uu})_v = (\alpha_{uv})_u$ and $(\alpha_{uv})_v = (\alpha_{vv})_u$. They are equivalent to six scalar equations, which in fact are redundant and reduce to three essential ones. See [165].

The first compatibility equation, known as Gauss Weingarten equation, is given by:

$$-E \frac{eg - f^2}{EG - F^2} = \frac{\partial \Gamma_{12}^2}{\partial u} - \frac{\partial \Gamma_{11}^2}{\partial v} + \Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2. \quad (1.4)$$

This equation expresses the Gaussian curvature $\mathcal{K} = \frac{eg - f^2}{EG - F^2}$ in terms of the first fundamental form. This means that the Gaussian curvature is an intrinsic entity. The other two compatibility equations are:

$$\begin{aligned} \frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} &= e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2, \\ \frac{\partial f}{\partial v} - \frac{\partial g}{\partial u} &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2. \end{aligned} \quad (1.5)$$

They are called *Codazzi Equations*.

A geometric and dynamical interpretation of Codazzi equations was established in [162].

Remark 1.4.1. *The method of moving frames developed by E. Cartan is also useful to establish the compatibility equations of Surface Theory. See [149].*

1.5 The Fundamental Theorem of Surface Theory

In section 1.4 it was introduced the fundamental forms $I = Edu^2 + 2Fdudv + Gdv^2$, $II = edu^2 + 2fdudv + gdv^2$ and the com-

patibility equations (1.4) and (1.5) between these two forms.

The main results of this section are the following theorems, known as Bonnet's Theorem.

Theorem 1.5.1. *Let $\Omega \subset \mathbb{R}^2$ be a connected and simply connected open set. Consider two forms $I = Edu^2 + 2Fdudv + Gdv^2$, $II = edu^2 + 2fdudv + gdv^2$, I being positive definite. Suppose that the functions E , F and G are of class C^2 and e , f and g are of class C^1 satisfying the compatibility equations given by equations (1.4) and (1.5). Then there exists an immersion $\alpha : \Omega \rightarrow \mathbb{R}^3$ of class C^3 having I as first fundamental form and II as the second fundamental form, i.e, $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$, $G = \langle \alpha_v, \alpha_v \rangle$, $e = \langle \alpha_{uu}, N_\alpha \rangle$, $f = \langle \alpha_{uv}, N_\alpha \rangle$, $g = \langle \alpha_{vv}, N_\alpha \rangle$, $N_\alpha = (\alpha_u \wedge \alpha_v) / |\alpha_u \wedge \alpha_v|$.*

Proof. See [40], [94] and [165]. □

Theorem 1.5.2. *Let $\Omega \subset \mathbb{R}^2$ be a connected and simply connected open set. Let $\alpha : \Omega \rightarrow \mathbb{R}^3$ and $\bar{\alpha} : \Omega \rightarrow \mathbb{R}^3$ two immersions of class C^3 with associated fundamental forms $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$, $II_\alpha = edu^2 + 2fdudv + gdv^2$, $I_{\bar{\alpha}} = \bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2$, $II_{\bar{\alpha}} = \bar{e}du^2 + 2\bar{f}dudv + \bar{g}dv^2$. Suppose that $I_\alpha = I_{\bar{\alpha}}$ and $II_\alpha = II_{\bar{\alpha}}$. Then there exists a vector $v \in \mathbb{R}^3$ and an orthogonal matrix $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\bar{\alpha} = M\alpha + v$.*

Proof. See [40], [94] and [165]. □

The continuity of the immersion α with respect to the forms $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$ and $II_\alpha = edu^2 + 2fdudv + gdv^2$, for some natural topologies in the appropriate space of functions, was established in [30]. This Fundamental Theorem, existence and

uniqueness, in the case where I is only of class C^1 and II is of class C^0 was established in [82] and [83].

1.6 Differential Equations of Curvature Lines

The normal curvature in the direction $[du : dv]$ also denoted by $\lambda = dv/du$ is given by:

$$k_n = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}.$$

The extremal values of k_n are characterized by $\frac{dk_n}{d\lambda} = 0$.

Direct calculations, differentiating k_n with respect to $\lambda = dv/du$ and equating to 0, gives

$$(Fg - Gf)\lambda^2 + (Eg - Ge)\lambda + (Ef - Fe) = 0$$

Or equivalently,

$$(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0. \quad (1.6)$$

Also, the equation above can be interpreted as the annihilation of Jacobian of the map $(du, dv) \rightarrow (II(du, dv), I(du, dv))$.

$$\frac{\partial(II, I)}{\partial(du, dv)} = 4(edu + fdv)(Fdu + Gdv) - 4(fdu + gdv)(Edu + Fdv) = 0.$$

This equation defines two directions $\frac{dv}{du}$, at which k_n attains the extremal values, *minimal* and *maximal*.

They are called *principal directions* and the correspondent curvatures are called *principal curvatures*.

The normal curvature in the principal directions will be denoted by k_1 (minimal curvature) and k_2 (maximal curvature).

The Gaussian curvature is defined by $\mathcal{K} = k_1 k_2$; the (arithmetic) Mean curvature is defined by $\mathcal{H} = (k_1 + k_2)/2$.

Proposition 1.6.1. *The expressions for \mathcal{K} and \mathcal{H} in terms of the coefficients of I_α and II_α are as follows:*

$$\mathcal{K} = \frac{eg - f^2}{EG - F^2}, \quad \mathcal{H} = \frac{eG + gE - 2fF}{2(EG - F^2)}.$$

Proof. Follows from equations (1.2). □

A point p is called, respectively, elliptic, parabolic or hyperbolic when $\mathcal{K}(p) > 0$, $\mathcal{K}(p) = 0$ or $\mathcal{K}(p) < 0$.

See Fig. 1.1 for an illustration of the contact of a surface with the tangent plane at an elliptic, parabolic and hyperbolic point. In the case of parabolic point, in general, the curve of intersection between the tangent plane and the surface has a singularity of cuspidal type (t^2, t^3) at p . The principal directions are well defined outside the

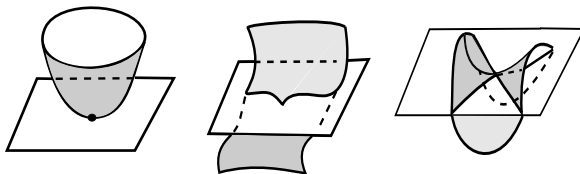


Figure 1.1: A surface and the tangent planes at elliptic, parabolic and hyperbolic points

points where the two fundamental forms are proportional, i. e. $k_1 = k_2$. These points are called *umbilic points*.

The set of umbilic points of α will be denoted by \mathcal{U}_α .

Another geometric interpretation of the differential equation of curvature lines is obtained from Rodrigues' equation $dN + kdp = 0$, which defines the principal curvatures and the principal directions as eigenvalues and eigenvectors of the selfadjoint operator $-dN$. This operator in matrix form, with a change in sign, is given by equations (1.2). See also [166] and [164].

Outside the umbilic set the principal directions are *orthogonal* relative to the metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and define two lines fields, called *principal line fields* which will be denoted by $\mathcal{L}_{1,\alpha}$ and $\mathcal{L}_{2,\alpha}$.

In fact this is a consequence of the selfadjointness of dN . Also taking $\lambda_i, i = 1, 2$, the roots of equation (1.6) in $\lambda = dv/du$, it follows that

$$\begin{aligned} & G\lambda_1\lambda_2 + F(\lambda_1 + \lambda_2) + E \\ &= \frac{-1}{gF - Gf} [G(eF - fE) - F(eG - gE) - E(gF - Gf)] = 0. \end{aligned}$$

The integral curves of these line fields are called *principal curvature lines* or simply *principal lines*. The *principal foliations* defined by the ensemble of the principal lines will be denoted by $\mathcal{P}_{1,\alpha}$ and $\mathcal{P}_{2,\alpha}$. The triple $\mathbb{P}_\alpha = \{\mathcal{P}_{1,\alpha}, \mathcal{P}_{2,\alpha}, \mathcal{U}_\alpha\}$ is called the *principal configuration* of the immersion α .

The implicit differential equation

$$H(u, v, [du : dv]) = (Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0$$

defines a surface, or better a *variety* $H^{-1}(0)$ in the projective bundle $\mathbb{P}\mathbb{M}$. This *variety* in general is regular (i.e. a smooth surface) but it

can present singularities.

The line field in the chart (u, v, p) , $p = \frac{dv}{du}$, defined by,

$$X_H = H_p \frac{\partial}{\partial u} + p H_p \frac{\partial}{\partial v} - (H_u + p H_v) \frac{\partial}{\partial p} \quad (1.7)$$

is called a *Lie-Cartan* line field. In the chart (u, v, q) , $q = \frac{du}{dv}$, the line field is defined by, $X_H = q H_q \frac{\partial}{\partial u} + H_q \frac{\partial}{\partial v} - (H_v + q H_u) \frac{\partial}{\partial q}$.

The projections of the integral curves of X_H by $\pi(u, v, [du : dv]) = (u, v)$ are the principal curvature lines.

The projection π is a double covering outside the set $\pi^{-1}(\mathcal{U}_\alpha)$ and $\pi^{-1}(p_0) = \mathbb{P}_1(\mathbb{R})$ at an umbilic point p_0 .

In chapter 3 will be studied the stability properties of principal configurations \mathbb{P}_α under small perturbations of the immersions α .

1.7 Differential Equations of Asymptotic Lines

The directions where $k_n = 0$ are called *asymptotic directions* and therefore are defined by $II = edu^2 + 2fdudv + gdv^2 = 0$.

The line fields of asymptotic directions will be denoted by $\mathcal{A}_{1,\alpha}$ and $\mathcal{A}_{2,\alpha}$. They are called *asymptotic line fields*.

It can be proved that the ordered pair $\{\mathcal{A}_{1,\alpha}, \mathcal{A}_{2,\alpha}\}$ is well defined in the hyperbolic region of the surface, where these directions are real.

At the parabolic points defined by $\mathcal{K} = 0$ the two asymptotic directions coincide and are well defined when one principal curvature

is not zero. When both principal curvatures are zero (umbilic point) all directions are asymptotic directions.

The triple $\mathbb{A}_\alpha = \{\mathcal{A}_{1,\alpha}, \mathcal{A}_{2,\alpha}, \mathcal{P}_\alpha\}$ is called the *asymptotic configuration* of the immersion α .

Here also the method of Lie-Cartan can be used to consider the implicit differential equation of asymptotic lines as an implicit surface in the projective bundle $\mathbb{P}\mathbb{M}$.

The asymptotic lines are the projections of the integral curves of Lie-Cartan line field. See [4].

Denote by $\mathcal{I}^{r,s}(\mathbb{M}, \mathbb{R}^3)$ the space of immersions of class C^r of \mathbb{M} to \mathbb{R}^3 , endowed with the C^s topology.

Proposition 1.7.1. *Let α be in $\mathcal{I}^{r,r}(\mathbb{M}, \mathbb{R}^3)$. Suppose that $\mathbb{H}_\alpha = \{p : \mathcal{K}(p) \leq 0\}$ is a surface with regular boundary $\partial\mathbb{H}_\alpha = \mathcal{P}_\alpha$. Then the implicit surface of the asymptotic directions $L(u, v, [du : dv]) = edu^2 + 2fdudv + gdv^2 = 0$ is a regular surface in the projective bundle $\mathbb{P}\mathbb{M}$ and the Lie-Cartan line field $X_L = L_p \frac{\partial}{\partial u} + pL_p \frac{\partial}{\partial v} - (L_u + pL_v) \frac{\partial}{\partial p}$*

is globally defined in $L^{-1}(0)$ and it is singular at the points where the asymptotic directions are tangent to the parabolic set \mathcal{P}_α .

Proof. See Proposition 6.2.1, page 130 of Chapter 6. □

The *asymptotic foliations* of α are the integral foliations $\mathcal{A}_{1,\alpha}$ of $\ell_{1,\alpha}$ and $\mathcal{A}_{2,\alpha}$ of $\ell_{2,\alpha}$; they fill out the hyperbolic region \mathbb{H}_α . When non-empty, the region \mathbb{H}_α is bounded by the set \mathcal{P}_α of *parabolic* points of α , on which \mathcal{K}_α vanishes. On \mathcal{P}_α , which generically, i.e. for most α 's, is a regular curve, [12], [18], [45], [93], the pair of asymptotic directions degenerate into a single one or into the whole tangent plane.

This last case happens at flat umbilic points, where $k_1 = k_2 = 0$, which generically are disjoint from the parabolic curve.

In order to characterize the class of immersions which are asymptotic structurally stable, it is useful to consider the following geometric approach. See chapter 6, page 141.

On the projective bundle $\mathbb{P}\mathbb{M} = \{\mathbb{T}\mathbb{M} \setminus \{0\} / \{v = rw, r \neq 0\}$ of \mathbb{M} , consider the submanifold \mathbf{A}_α defined by all the asymptotic directions. That is by the zeros of the second fundamental form of α . The first condition to be imposed on α is precisely that 0 is a regular value of the projectivization of II_α , that is $DK \neq 0$ at parabolic points.

The restriction of the projection Π of $\mathbb{P}\mathbb{M}$ to \mathbf{A}_α covers $\text{Clos}\mathbb{H}_\alpha$. Over \mathbb{H}_α it is a double regular covering. Over \mathbb{P}_α it has a Whitney fold [18], [167], [179]. Therefore the Euler -Poincaré characteristics are related by $\chi(\mathbf{A}_\alpha) = 2\chi(\mathbb{H}_\alpha)$.

Lifting to this manifold the line fields $\mathcal{A}_{1,\alpha}$ and $\mathcal{A}_{2,\alpha}$ defines a single line field $\tilde{\mathcal{L}}_\alpha$ on $\Pi^{-1}(\mathbb{H}_\alpha)$ which, under the conditions of regularity, uniquely extends to a smooth line field $\tilde{\mathcal{L}}_\alpha$ defined on the whole \mathbf{A}_α . Its singularities, when present, are contained in $\mathcal{P}_\alpha = \Pi^{-1}(\mathbb{P}_\alpha)$. These singularities are localized exactly where the asymptotic direction is collinear with the tangent vector to the parabolic line. It can be shown that these singularities are the cuspidal points of the Gauss map $N_\alpha : \mathbb{M} \rightarrow \mathbb{S}^2$, [18], [167].

The surface \mathbf{A}_α is compact, oriented and in a local chart (u, v) is defined by $H(u, v, p) = e + 2fp + gp^2 = 0$, $p = \frac{dv}{du}$ and the line field $\tilde{\mathcal{L}}_\alpha$ defined in \mathbf{A}_α is locally given by the vector field Lie-Cartan line field given by $X = (H_p, pH_p, -(H_u + pH_v))$.

The leaves of integral foliation of this line field, denoted by $\tilde{\mathcal{A}}_\alpha$,

contains the pullback of the leaves of the pair of asymptotic foliations $\mathcal{A}_{\alpha,i}$, $i = 1, 2$.

The projection of the leaves of $\tilde{\mathcal{A}}_\alpha$ into $\text{Clos}(\mathbb{H}_\alpha)$ which intercept the set \mathcal{P}_α are called the *folded asymptotic lines* of α .

A *folded closed asymptotic line* is the projection of a closed integral curve of the single line field $\tilde{\mathcal{L}}_\alpha$ defined on \mathcal{H}_α , which intersects \mathcal{P}_α . A closed asymptotic line contained in \mathbb{H}_α is called *regular*.

1.8 Differential Equations of Geodesics

For a curve c parametrized by arc length s , write its unit tangent vector as $T = u'\alpha_u + v'\alpha_v$. Differentiation of T gives:

$$\begin{aligned} T' &= (u')^2\alpha_{uu} + 2u'v'\alpha_{uv} + (v')^2\alpha_{vv} + u''\alpha_u + v''\alpha_v \\ &= [u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1]\alpha_u \\ &\quad + [v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2]\alpha_v \\ &\quad + (e(u')^2 + 2fu'v' + g(v')^2)N \end{aligned}$$

Therefore the geodesics are characterized by the following system:

$$\begin{aligned} \frac{d^2u}{ds^2} + \Gamma_{11}^1\left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^1\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^1\left(\frac{dv}{ds}\right)^2 &= 0 \\ \frac{d^2v}{ds^2} + \Gamma_{11}^2\left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^2\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^2\left(\frac{dv}{ds}\right)^2 &= 0 \end{aligned} \quad (1.8)$$

Eliminating $ds^2 = Edu^2 + 2Fdu dv + Gdv^2$ from the system above it follows that:

$$\frac{d^2v}{du^2} = \Gamma_{22}^1\left(\frac{dv}{du}\right)^3 + (2\Gamma_{12}^1 - \Gamma_{22}^1)\left(\frac{dv}{du}\right)^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)\frac{dv}{du} - \Gamma_{11}^2 \quad (1.9)$$

Or equivalently,

$$\frac{d^2u}{dv^2} = \Gamma_{11}^2 \left(\frac{du}{dv}\right)^3 - (\Gamma_{11}^1 - 2\Gamma_{12}^2) \left(\frac{du}{dv}\right)^2 - (2\Gamma_{12}^1 - \Gamma_{22}^2) \frac{du}{dv} - \Gamma_{22}^1 \quad (1.10)$$

Remark 1.8.1. *Following [166], an expression for the geodesic curvature $k_g = \langle (N \wedge T), T' \rangle = [T, T', N]$ is given by:*

$$k_g = [\Gamma_{11}^2 (u')^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1) (u')^2 v' + (\Gamma_{22}^2 - 2\Gamma_{12}^1) u' (v')^2 - \Gamma_{22}^1 (v')^3 + u' v'' - u'' v'] \sqrt{EG - F^2}, \text{ where } u' = \frac{du}{ds} \text{ and } v' = \frac{dv}{ds}.$$

For the parametric curves it follows that:

$$(k_g)_{v=v_0} = \Gamma_{11}^2 \frac{\sqrt{EG - F^2}}{E\sqrt{E}}, \quad (k_g)_{u=u_0} = -\Gamma_{22}^1 \frac{\sqrt{EG - F^2}}{G\sqrt{G}}.$$

In particular when the parametric curves are orthogonal ($F = 0$) it holds that:

$$(k_g)_{v=v_0} = -\frac{E_v}{2E\sqrt{G}} = -\frac{d}{ds_2} \ln \sqrt{E}, \quad (1.11)$$

$$(k_g)_{u=u_0} = \frac{G_u}{2G\sqrt{E}} = \frac{d}{ds_1} \ln \sqrt{G}$$

The equation above shows that the geodesic curvature depends only of the first fundamental form and therefore is an intrinsic entity.

The geodesics are the curves of zero geodesic curvature on the surface.

Also the geodesics are defined as the curves of shortest distance between two nearby points. See [40] and [166].

This key connection with the Calculus of Variations leads to fruitful applications to Geodesy and Cartography.

Proposition 1.8.1. *Let \mathbb{M} be a regular surface of class C^r , $r \geq 3$. Then for every $p \in \mathbb{M}$ and $v \in T_p\mathbb{M}$, $v \neq 0$, there exist $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{M}$ such that $\gamma(0) = \gamma(0, p, v) = p$ and $\gamma'(0) = v$.*

The map $\exp : U \subset T\mathbb{M} \rightarrow \mathbb{M} \times \mathbb{M}$ defined by $\exp(p, v) = (p, \exp_p(v)) = \gamma(|v|, p, \frac{v}{|v|})$, $v \neq 0$, and $\exp(p, 0) = (p, p)$ is a local diffeomorphism of class C^{r-2} on an open set U and $D\exp(p, 0) = Id$. Also, $\{\gamma((-\epsilon, \epsilon)), v \in T_p\mathbb{M}, |v| = 1\} = B(p, \epsilon) = \{q \in \mathbb{M} : d(p, q) < \epsilon\}$.

Proof. This follows from the theorem of existence, uniqueness and smooth dependence on initial conditions for ordinary differential equations applied to the C^{r-2} homogenous differential equation (1.8). \square

Proposition 1.8.2. *Let \mathbb{M} be a regular surface of class C^r , $r \geq 3$. Consider the length space (\mathbb{M}, d) . Then for any $p \in \mathbb{M}$ there exists a number $\epsilon > 0$ such that for any points $r, s \in B(p, \epsilon)$, there is a unique geodesic γ passing through r and s , contained in $B(p, \epsilon)$, such that $d(r, s) = L_\gamma(r, s)$. That is, γ is the shortest path in $B(p, \epsilon)$ and so a geodesic line minimizes distances locally. See Fig. 1.2.*

Proof. See [40] and [165]. \square

In Chapter 7 will be discussed some classical results about geodesics such as the Hopf-Rinow theorem. The derivative of the return map near a closed geodesic will be obtained in an elementary way. Also the qualitative theory of geodesics in the ellipsoid with three different axes and in surfaces of revolution will be described.

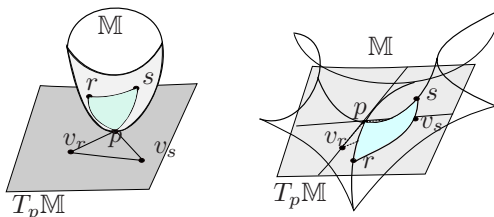


Figure 1.2: Geodesic triangle, positive and negative curvature

1.9 Exercises and Problems

1.9.1. Let p be a non umbilic point of a surface \mathbb{M} . Show that a neighborhood of p can be parametrized by a chart whose coordinate curves are principal lines. Write the Codazzi equations in such chart, called a *principal chart*, which is characterized by $f = F = 0$.

1.9.2. Let p be a hyperbolic point of \mathbb{M} , i.e. $\mathcal{K}(p) < 0$. Show that a neighborhood of p can be parametrized by a chart whose coordinate curves are asymptotic lines. Write the Codazzi equations in such a chart, called an *asymptotic chart* which is characterized by $e = g = 0$.

1.9.3. Suppose that \mathbb{M} is a smooth surface in the euclidian space \mathbb{R}^3 . Let $p \in \mathbb{M}$ such that $\mathcal{K} \neq 0$ and $N : \mathbb{M} \rightarrow \mathbb{S}^2$ the normal Gauss map.

Take an orthonormal frame such that $N(p) = (0, 0, -1)$.

Let also $\pi : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$ be the stereographic projection.

Consider the map $\beta : U \subset \mathbb{R}^2 \rightarrow \mathbb{M}$ defined by $\beta = (\pi \circ N)^{-1}$.

Introduce the support function $f(u, v) = (1 + u^2 + v^2)D(u, v)$ where D is the distance (with sign) from $T_{\beta(u,v)}\mathbb{M}$ to the origin $0 \in \mathbb{R}^3$. It follows

that $\beta(u, v) = (x(u, v), y(u, v), z(u, v))$ is given by:

$$\begin{aligned} x(u, v) &= \frac{1}{2}f_u - u \frac{uf_u + vf_v - f}{u^2 + v^2 + 1}, \\ y(u, v) &= \frac{1}{2}f_v - v \frac{uf_u + vf_v - f}{u^2 + v^2 + 1}, \quad z(u, v) = \frac{uf_u + vf_v - f}{u^2 + v^2 + 1} \end{aligned} \quad (1.12)$$

This parametrization is said to define *Bonnet coordinates* on a surface with non zero Gaussian curvature.

i) Show that in the Bonnet coordinates above the differential equation of curvature lines is given by:

$$f_{uv}(du^2 - dv^2) + (f_{vv} - f_{uu})dudv = 0 \quad (1.13)$$

ii) Show that the differential equation of asymptotic lines is given by:

$$[f_{uu} - 2\frac{uf_u + vf_v - f}{1 + u^2 + v^2}]du^2 + 2f_{uv}dudv + [f_{vv} - 2\frac{uf_u + vf_v - f}{1 + u^2 + v^2}]dv^2 = 0$$

iii) Let $z = u + iv$ and $\frac{\partial}{\partial z} = \frac{\partial}{\partial u} - i\frac{\partial}{\partial v}$, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial u} + i\frac{\partial}{\partial v}$ be the complex differentiation operators. Show that equation (1.13) is equivalent to $Im(f_{\bar{z}\bar{z}}d\bar{z}^2) = Im(f_{zz}dz^2) = 0$. See also [23].

In [111] the same form of the equation in (1.13) is obtained for the function f and coordinates (u, v) of Ribaucour, instead of those of Bonnet, without assuming that $\mathcal{K} \neq 0$. On the other hand, for the study of lines of curvature, whose properties are invariant under inversion, it is always possible to assume that $\mathcal{K} \neq 0$.

1.9.4. Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function of class C^r , $r \geq 2$. Consider the implicit surface $F^{-1}(0)$.

i) Show that the differential equation of principal lines is given by:

$$\begin{vmatrix} dx & dy & dz \\ F_x & F_y & F_z \\ dF_x & dF_y & dF_z \end{vmatrix} = 0, \quad F_x dx + F_y dy + F_z dz = 0.$$

ii) Consider the matrix equation

$$\begin{vmatrix} F'' - \lambda I & F' \\ (F')^t & 0 \end{vmatrix} = a + b\lambda + c\lambda^2, \quad \lambda \in \mathbb{R}$$

where F' is the column gradient vector of F and F'' is the Hessian of F .

Show that $\mathcal{K} = \frac{a/c}{F_x^2 + F_y^2 + F_z^2}$ and $\mathcal{H} = -\frac{b/c}{2(F_x^2 + F_y^2 + F_z^2)^{3/2}}$.

1.9.5. Show that the differential equation of geodesics on an implicit surface $F(x, y, z) = 0$ is given by:

$$\begin{vmatrix} F_x & F_y & F_z \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix} = 0, \quad F_x dx + F_y dy + F_z dz = 0.$$

1.9.6. Let (u, v) be a local positive principal chart on a surface S . Express the geodesic curvatures (see equation (1.11), page 29) of the coordinate lines in function of the principal curvatures k_1 and k_2 and their derivatives, i.e., show that

$$k_g|_{v=v_0}(u, v_0) = \frac{-(k_2)_u}{k_2 - k_1}(u, v_0), \quad k_g|_{u=u_0}(u_0, v) = \frac{-(k_1)_v}{k_2 - k_1}(u_0, v).$$

1.9.7. Given a parametric smooth surface $\alpha : U \rightarrow \mathbb{R}^3$ define the square distance function $D : U \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by the equation $D(u, v, p) = |p - \alpha(u, v)|^2$. Geometrically, from the singularities of D it can be measured the contact between the surface α and spheres in \mathbb{R}^3 .

In terms of critical points of D as a family of functions of (u, v) , depending on the parameter p , define the focal set of α and classify its generic singularities. Regard the focal set as the set of points p where the normal rays issuing from $\alpha(u, v)$ converge. Find that this set is given by the union of two sheets $p = N_\alpha/k_i, i = 1, 2$.

For this project see the steps taken in [137]. This subject is still source of current research.

1.9.8. Show that an oriented connected surface having both principal curvatures constant, or, equivalently, those such that the Mean and Gaussian curvatures are constant, is an open set of the plane, of a sphere, or of a circular right cylinder. See [148].

1.9.9. Let S be a surface of revolution parametrized by

$$\alpha(s, v) = (r(s) \cos v, r(s) \sin v, z(s)).$$

Consider a geodesic line $\gamma(t)$ of S making an angle $\alpha(t)$ with the meridians and let $r(t)$ the radius of the correspondent parallel. Write the differential equation of the geodesic lines and show that $r(t) \sin \alpha(t) = \text{cte}$.

1.9.10. Let c be a closed principal line of a surface S . Show that a tubular neighborhood of c can be parametrized such that the coordinates curves orthogonal to c are principal lines of S .

1.9.11. A non zero symmetric bilinear form V_2 and its associated quadratic form $B_2u = V_2(u, u)$ in \mathbb{R}^2 , with the canonical coordinates (x, y) are given by:

$$V_2(u, v) = V_2uv = u^t B_2 v = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

The bilinear form V_2 and the quadratic form B_2 are called *hyperbolic*, *parabolic* or *elliptic* according as the determinant of its matrix B_2 above is negative, zero or positive.

Two vectors $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are called *conjugate* when $V_2uv = 0$, i.e. $ax_1x_2 + b(x_1y_2 + x_2y_1) + cy_1y_2 = 0$.

A non-zero quadratic form is said to be *right-angled* if it is hyperbolic and its roots are mutually orthogonal with respect to the canonical inner product of \mathbb{R}^2 .

- i) Show that V_2 is parabolic, if and only if there is a non zero vector $u \in \mathbb{R}^2$ such that the linear form $V_2u = 0$. In this case V_2 is a perfect square.
- ii) Let V_2 be a parabolic form in \mathbb{R}^2 with $V_2u = 0$ and let A_1 be a linear form such that $A_1u = 0$. Show that there is a vector $v \in \mathbb{R}^2$ such that $A_1 = B_2v$.
- iii) Show that $B_2(x, y) = ax^2 + 2bxy + cy^2$ is right-angled if and only if $a + c = 0$.

1.9.12. Consider two bilinear symmetric forms V_2 and W_2 in \mathbb{R}^2 . A vector $u \neq 0$ is called a *Jacobian* of the pair if the two linear forms V_2u and W_2u are equivalent, i.e., the determinant of the matrix map $u \rightarrow (V_2(u, u), W_2(u, u))$ is zero.

- i) Show that u is a Jacobian of the quadratic forms $V_2(x, y) = ax^2 + 2bxy + cy^2$ and $W_2(x, y) = Ax^2 + 2Bxy + Cy^2$ if and only if it is a root of the quadratic form

$$\text{Jac}(V_2, W_2) = (aB - Ab)x^2 + (aC - Ac)xy + (bC - Bc)y^2$$

- ii) Show that when W_2 is positive definite (elliptic form) the quadratic form $\text{Jac}(V_2, W_2)$ is hyperbolic and its roots u_1 and u_2 are right-angled with respect to W_2 , i.e., $W_2u_1u_2 = 0$.

1.9.13. Given a three linear symmetric form V_3 its associated cubic form in \mathbb{R}^2 , with the canonical coordinates (x, y) , is given by $V_3(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3 = [a, b, c, d]_3(x, y)$. The determinant of

$$H_V = \begin{pmatrix} ax + by & bx + cy \\ bx + cy & cx + dy \end{pmatrix}$$

is called the Hessian of V and is given by

$$(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2.$$

A non zero vector $u = (x, y)$ is called a *Hessian vector* of V_3 if the bilinear form $V_2 = V_3u$ is parabolic.

- i) Show that the condition for $u = (x, y)$ to be a Hessian vector of V_3 is independent of coordinates of \mathbb{R}^2 .
- ii) Classify the cubic forms V_3 according to the nature of its roots.
- iii) Show that the Hessian of a three linear form V_3 is parabolic if and only if there is a non-zero vector $u \in \mathbb{R}^2$ such that $V_3u^2 = 0$.
- iv) Let $V_3(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3 = [a, b, c, d]_3(x, y)$. Show that there is a non-zero vector $u^t = (x, y)$ such that the matrix H_V is a real multiple of the identity matrix, if and only if the Hessian lines of the cubic, the root lines of its Hessian, are mutually orthogonal, or the Hessian vanishes.

1.9.14. Show that the eigenspaces of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad (a - c)^2 + b^2 \neq 0,$$

are the root lines of the Jacobian of the quadratic forms $V_2 = ax^2 + 2bxy + cy^2$ and $W_2 = x^2 + y^2$. Verify that these root lines are mutually orthogonal. For more on quadratic and cubic forms in \mathbb{R}^2 see [137, Chapter 7].

1.9.15. Let Σ be a quadratic surface in \mathbb{R}^3 and γ be a principal curvature line of Σ . Show that there exist an infinity of quadrics containing γ .

1.9.16. Consider a surface and parametric curves (u, v) such that $I = du^2 + dv^2 + 2 \cos \omega(u, v) du dv$. Parametric curves with this property are called *Tchebychef nets*.

- i) Show that the following partial differential equation holds.

$$\frac{\partial^2 \omega}{\partial u \partial v} + \mathcal{K} \sin \omega = 0.$$

- ii) Show that $\mathcal{K}_T = \omega_2 + \omega_4 - \omega_1 - \omega_3 = 2\pi - \sum_{i=1}^4 \alpha_i$, where \mathcal{K}_T is defined by the integral of the Gaussian curvature with respect to the area element in the parallelogram shown in Fig. 1.3. The integral \mathcal{K}_T is called the total curvature.
- iii) Interpret geometrically and physically the meaning of Tchebychef nets. See [165].

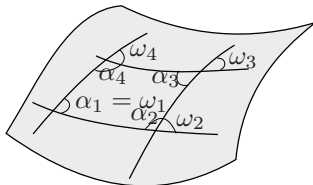


Figure 1.3: Angles of a Tchebychef net

1.9.17. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$f(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2, \quad R > r,$$

$$h(x, y, z) = [y^2 + x(x - a)]^2 + z^2 - r^2.$$

- i) Show that 0 is a regular value of f and h , $R > r > 0$ and $16r^2 - a^4 \neq 0$, respectively. Plot $f^{-1}(0)$ and $h^{-1}(0)$.
- ii) Show that $f^{-1}(0)$ is diffeomorphic to $h^{-1}(0)$ when $16r^2 - a^4 < 0$ and that $h^{-1}(0)$ is diffeomorphic to the unitary sphere \mathbb{S}^2 when $16r^2 - a^4 > 0$. What happens when $16r^2 - a^4 = 0$?
- iii) Obtain regular compact surfaces of genus $g > 0$ as level sets of polynomial functions $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $H : \mathbb{R}^4 \rightarrow \mathbb{R}^2$.

1.9.18. Let $P(x, y)$ be a polynomial of degree $n \geq 2$ and consider the polynomial surface \mathbb{M} defined by the graph $z = P(x, y)$.

- i) Show that the parabolic set of \mathbb{M} , when it is not empty or all \mathbb{M} , is an algebraic curve of degree $l \leq 2(n-2)$.
- ii) **Open problem:** How many (compact) parabolic curves can belong to the graph of a polynomial of a given degree n ? Try the cases of degree 3 and 4. See [8], [6] and [123].
- iii) Show that the umbilic set of \mathbb{M} when it is a discrete set has at most $(3n-4)^2$ points.
- iv) Show that the Gaussian curvature \mathcal{K} of \mathbb{M} is integrable and that $\int_{\mathbb{M}} |\mathcal{K}| dA \leq 2\pi(n-1)^2$. Here dA is the area element of \mathbb{M} . See [124].

1.9.19. Show that the differential equation of asymptotic lines of an implicit surface $f(x, y, z) = 0$ is given by:

$$d(f_x)dx + d(f_y)dy + d(f_z)dz = 0, \quad f_x dx + f_y dy + f_z dz = 0 \quad (1.14)$$

Equivalently show that $v \in T_p f^{-1}(0)$ is an asymptotic direction if $f'(p)v = 0$ and $f''(p)(v, v) = 0$.

1.9.20. Let \mathbb{M} be a surface of class C^3 in \mathbb{R}^3 with first fundamental form $I = Edu^2 + 2Fdudv + Gdv^2$ in a local chart (u, v) .

i) Show that the Gaussian curvature \mathcal{K} of \mathbb{M} is given by:

$$W^4 \mathcal{K} = \begin{vmatrix} F_{uv} - \frac{1}{2}(G_{uu} + E_{vv}) & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}$$

where $W = \sqrt{EG - F^2}$.

- ii) Show that the Gaussian curvature does not depend on coordinates.
- iii) Analyze the items i) and ii) when \mathbb{M} is only of class C^2 .

1.9.21. Let \mathbb{M} be a compact and oriented surface in \mathbb{R}^3 . Suppose that \mathbb{M} is oriented such that the normal vector N is the inner normal vector. Consider

the one parameter family of parallel surfaces defined by $\mathbb{M}_t = \mathbb{M} + tN$ with $|t|$ small.

- i) Show that $A(t) = A(0) - 2t[\int_{\mathbb{M}} \mathcal{H}dA] + t^2[\int_{\mathbb{M}} \mathcal{K}dA]$. Here $A(t)$ is the area of the surface \mathbb{M}_t .
- ii) Show that $V(t) = V(0) - t^2[\int_{\mathbb{M}} \mathcal{H}dA] + \frac{t^3}{3}[\int_{\mathbb{M}} \mathcal{K}dA]$. Here $V(t)$ is the volume of the region bounded by the surface \mathbb{M}_t .
- iii) Show that if \mathbb{M} is convex then $\int_{\mathbb{M}} \mathcal{H}dA > 0$.

1.9.22. Consider the surface defined by the graph $(x, y, h(x, y))$,

$$h(x, y) = \frac{1}{2}ax^2 - by^2 + \frac{1}{6}[Ax^3 + 3Bx^2y + 3B_1xy^2 + Cy^3] + \dots, \quad ab > 0.$$

- i) Compute the curvature of each branch of the planar curve $h(x, y) = 0$ at $p = 0$ (intersection of the surface and the tangent plane at 0).
- ii) Compute the curvature and torsion of each asymptotic line passing through 0.
- iii) Compare the curvatures evaluated in i) and ii) and interpret geometrically the above results.

1.9.23. i) Show that a regular curve $c(s)$ on a surface $\mathbb{M} \subset \mathbb{R}^3$ is a principal curvature line of \mathbb{M} if and only if the ruled surface $\beta(s, v) = c(s) + vN(s)$ is developable (this means that its Gaussian curvature vanishes).

ii) Show that the principal curvatures of β are given by $k_1^\beta(s, v) = k_g(s)/(1 - k_1(s)v)$ and $k_2^\beta = 0$, where $k_1(s)$ and $k_g(s)$ are the principal and geodesic curvature of c as a curve of \mathbb{M} , respectively.

iii) Characterize the singularities and the umbilic set of β . Describe the principal configuration of β .

1.9.24. Let γ be a smooth closed curve in \mathbb{R}^3 of length L parametrized by arc length s . Let $\{t, n, b\}$ the Frenet orthonormal frame associated to γ .

Let $\gamma_\epsilon(s) = \gamma(s, \epsilon) = \gamma(s) + \epsilon(a_1(s)n(s) + a_2(s)b(s))$ a smooth variation of γ with $a_i(s) = a_i(s + L)$.

- i) Compute the total torsion of γ and γ_ϵ and write the first and second derivatives of the total torsion of γ_ϵ with respect to ϵ at $\epsilon = 0$.
- ii) Give an example of a closed curve γ such that the total torsion $\int_\gamma \tau = 2m\pi, m \in \mathbb{Z} \setminus \{0\}$.
- iii) Let $r \in \mathbb{R}$ given. Give an example of a closed curve γ of length L such that $\int_0^L \tau(s)ds = r$. See [29] and [75].

1.9.25. Consider the metric $ds^2 = h(v)(du^2 + dv^2)$, where h is a positive function of class C^1 in neighborhood of 0.

- i) Write the differential equation of the geodesic lines for this metric.
- ii) Give examples of functions h such that the curves $v = 0$ and $v = u^3$ are geodesics through $(0, 0)$.

1.9.26. Consider the parametric surfaces $\alpha(u, v) = (u \cos v, u \sin v, v)$ and $\beta(u, v) = (u \sin v, u \cos v, \log u)$.

- i) Compute the Gaussian curvature of α and β .
- ii) Show that the two surfaces are not isometric.

1.9.27. For a surface $\mathbb{M} \subset \mathbb{R}^3$ of class C^3 define the *third fundamental form* by $III(p)(u, v) = \langle DN(p)u, DN(p)v \rangle, u, v \in T_p\mathbb{M}$. Here $N : \mathbb{M} \rightarrow \mathbb{S}^2$ is the Gauss map of \mathbb{M} .

- i) Show that $III - 2\mathcal{H}II + \mathcal{K}I = 0$.
- ii) Let α be a local immersion $\alpha : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the Gauss map given by $N_\alpha = (\alpha_u \wedge \alpha_v) / |\alpha_u \wedge \alpha_v|$. Suppose that N_α is an immersion. Show that $III_\alpha = I_{N_\alpha} = -II_{N_\alpha}$.

1.9.28. A closed surface \mathbb{M} in \mathbb{R}^3 is called *rigid* if any other surface \mathbb{M}' in \mathbb{R}^3 that is isometric to \mathbb{M} is congruent to it. That is, if there is a

diffeomorphism between \mathbb{M} and \mathbb{M}' preserving the first fundamental forms then it is a restriction of a rigid motion of \mathbb{R}^3 composed (possibly) with a reflection.

- i) Show that any smooth compact convex surface in \mathbb{R}^3 is rigid.
- ii) Show that the torus of revolution is rigid. See [85] and [122].

1.9.29. Let $h : U \subset \mathbb{C} \rightarrow \mathbb{C}$ an analytic complex function defined in the open set U . Write $h(z) = U(z) + iV(z) = W(z)e^{i\beta(z)}$. The graph surface $(z, W(z))$, $z = u + iv = (u, v)$ is called a *modular surface*.

- i) Visualize and analyze the geometry of modular surfaces for $h(z) = z^2$ and $h(z) = \sin \pi z$.
- ii) Show that near a pole of h the modular surface has negative Gaussian curvature. See [99].

1.9.30. Let α and β be two regular curves of \mathbb{R}^3 . Define $\Gamma(u, v) = (\alpha(u) + \beta(v))/2$. The surface Γ is called a *translation surface*. It is regular when $\alpha' \wedge \beta' \neq 0$. See [42], [99] and [135].

- i) Visualize and analyze the geometry of the translation surface Γ defined by $\alpha(u) = (\cos u, \sin u, 0)$ and $\beta(v) = (a + \cos v, 0, \sin v)$, $a \in \mathbb{R}$. Find the singular points of Γ .
- ii) Show that the minimal surface defined by $h(x, y, z) = e^z \cos x - \cos y = 0$ is a translation surface.
- iii) Show that any minimal surface of \mathbb{R}^3 is a translation surface.

Chapter 2

Classical Results on Principal Curvature Lines

2.1 Introduction

In this chapter will be considered triply orthogonal systems of surfaces in \mathbb{R}^3 , envelopes of families of surfaces and also some examples of principal configuration and umbilic points on algebraic surfaces.

The basic references for this chapter are [37], [44], [85], [94], [100], [164], [166] and [174], where additional developments can also be found.

For beautiful illustrations of sculptures and clay models of surfaces see [46] and [84].

2.2 Triply orthogonal systems

Theorem 2.2.1 (Joachimsthal Theorem). *Let two surfaces \mathbb{M}_1 and \mathbb{M}_2 intersecting on a curve γ along which their normals N_1 and N_2 make constant angle, i.e. $\langle N_1, N_2 \rangle|_\gamma$ is a constant, and such that $DN_1\gamma' \wedge \gamma' = 0$. Then it also holds that $DN_2\gamma' \wedge \gamma' = 0$. In other words, if two surfaces intersect with constant angle along a curve which is the union of principal lines and umbilic points of the first one, then this is also the case for the second one.*

Conversely, if $DN_1\gamma' \wedge \gamma' = 0$ and $DN_2\gamma' \wedge \gamma' = 0$ along a curve γ of intersection of \mathbb{M}_1 and \mathbb{M}_2 , then the angle between the surfaces, i.e., $\langle N_1, N_2 \rangle|_\gamma$ is constant.

Proof. By hypothesis the mixed product $[N_1, N_2, \gamma']$ can be assumed to be non zero. Otherwise the conclusion is obvious.

So, differentiating the equation $\langle N_1, N_2 \rangle = c$ it follows that

$$[N_1, DN_2(\gamma)\gamma', \gamma'] = -[DN_1(\gamma)\gamma', N_2, \gamma'] = 0.$$

This implies that $DN_2(\gamma)\gamma' = \lambda\gamma'$. By Rodrigues formula, $dN + kdp = 0$, it follows that γ is the union of curvature lines and umbilic points of \mathbb{M}_2 . The converse is direct. This ends the proof. \square

From this follows directly that the principal configurations for surfaces of revolution are given by the umbilic points which are located at the poles and also along some parallels; the principal foliations are given by arcs of non umbilical meridians and non umbilical circular parallels.

For a non spherical ellipsoid of revolution, however, the unique umbilic points occur at their poles, as follows from a direct calculation.

A more precise formulation of Joachimsthal Theorem is the following.

Theorem 2.2.2. *Consider a principal line γ of an immersion $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ of class C^r . Suppose that the principal curvatures of α are such that $k_1 = k_1^\alpha < k_2^\alpha = k_2$ holds along γ . Let $\beta : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r such that $\langle N_\alpha, N_\beta \rangle = 0$ along γ . Then γ is the union of principal lines of β ; also one principal curvature of β , restricted to γ , is the geodesic curvature $\pm k_g$ of γ considered as a curve on α . Furthermore, any other immersion ξ making a constant angle θ with α along γ , i. e. $\langle N_\alpha, N_\xi \rangle = \cos \theta$, has this curve as the union of principal lines and has one principal curvature equal to $k_1 \cos \theta + k_g \sin \theta$.*

Proof. The Darboux frame $\{t, N_\alpha \wedge t, N_\alpha\}$ associated to γ as a curve of α is defined by the equations:

$$t' = k_g N_\alpha \wedge t + k_1 N_\alpha, \quad (N_\alpha \wedge t)' = -k_g t + 0 \cdot N_\alpha$$

$$N'_\alpha = -k_1 t + 0 \cdot N_\alpha \wedge t, \quad \tau_g = \langle (N_\alpha \wedge t)', N_\alpha \rangle = 0.$$

Along γ the normal vector to β is $N_\beta = \pm N_\alpha \wedge t$. Therefore by Rodrigues equation and the equation $(N_\alpha \wedge t)' + k_g t = N'_\beta + k_g t = 0$ it follows that γ is a principal line of β and one principal curvature is k_g , when the unit normal to β is chosen as $N_\alpha \wedge t$. The other principal curvature of β is the curvature of the plane curve defined by the intersection of β with the plane generated by $[N_\alpha, N_\alpha \wedge t]$.

The case of constant angle is immediate from the above considerations observing that the normal vector N_ξ of the immersion ξ is $N_\xi = \cos \theta N_\alpha + \sin \theta N_\beta$. \square

Definition 2.2.3. *An orientation preserving diffeomorphism*

$H : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ *on an open set where* U , *such that*

$$\langle H_u, H_v \rangle = \langle H_u, H_w \rangle = \langle H_v, H_w \rangle = 0, \quad H_u = \frac{\partial H}{\partial u},$$

is called a triply orthogonal system of coordinates.

The simplest examples of triply orthogonal systems are the cartesian, cylindrical and spherical coordinates in 3-space.

In cylindrical coordinates, two families are given by planes and the other consists of circular cylinders.

In spherical coordinates, the first family is given by concentric spheres with center at 0, the second by planes containing one coordinate axis and the third one by cones with vertex at 0.

For each fixed coordinate, for example when w is fixed, the map $(u, v) \rightarrow H^w(u, v) = H(u, v, w)$ is the parametrization of a surface.

Lemma 2.2.1. *Let* $p = |H_u|$, $q = |H_v|$ *and* $r = |H_w|$ *The following relations hold,*

$$\begin{aligned} H_u \wedge H_w &= \frac{qr}{p} H_u, \quad H_w \wedge H_u = \frac{pr}{q} H_v, \quad H_u \wedge H_v = \frac{pq}{r} H_w, \\ \langle H_u, H_{vw} \rangle &= 0, \quad \langle H_v, H_{uw} \rangle = 0, \quad \langle H_w, H_{uv} \rangle = 0, \\ \langle H_u, H_v \wedge H_w \rangle &= pqr. \end{aligned} \tag{2.1}$$

Proof. Consider the unit fields $N_1 = H_u/|H_u|$, $N_2 = H_v/|H_v|$ and $N_3 = H_w/|H_w|$.

By the hypothesis it follows that $N_1 \wedge N_2 = N_3$, $N_2 \wedge N_3 = N_1$, $N_3 \wedge N_1 = N_2$ and $\langle N_1, N_2 \wedge N_3 \rangle = 1$. Then $H_u \wedge H_v = pN_1 \wedge qN_2 = pqN_3 = \frac{pq}{r} H_w$. The same for the other relations.

Differentiating the equation $\langle H_u, H_v \rangle = 0$ with respect to w it follows that $\langle H_{uw}, H_v \rangle + \langle H_u, H_{vw} \rangle = 0$. Also it holds that $\langle H_{uv}, H_w \rangle + \langle H_u, H_{vw} \rangle = 0$ and $\langle H_{uv}, H_w \rangle + \langle H_v, H_{uw} \rangle = 0$.

So, $\langle H_{uw}, H_v \rangle = -\langle H_u, H_{vw} \rangle = -(-\langle H_{uv}, H_w \rangle) = -\langle H_v, H_{uw} \rangle$. This amounts to $2\langle H_{uw}, H_v \rangle = 0$. The same holds for the other relations. This ends the proof. \square

Proposition 2.2.1. *Consider the parametrized surfaces $S_1 : (u, v) \rightarrow H^w(u, v) = H(u, v, w)$, $S_2 : (w, u) \rightarrow H^v(w, u) = H(u, v, w)$ and $S_3 : (v, w) \rightarrow H^u(v, w) = H(u, v, w)$. Then the coefficients of, I_i , the first, E_i, F_i, G_i , and of, II_i , the second, e_i, f_i, g_i , fundamental forms of the surfaces S_i , $i = 1, 2, 3$ are given by:*

$$\begin{aligned} I_1 &= p^2 du^2 + q^2 dv^2 & II_1 &= -\frac{pp_w}{r} du^2 - \frac{qq_w}{r} dv^2 \\ I_2 &= r^2 dw^2 + p^2 du^2 & II_2 &= -\frac{rr_w}{q} dw^2 - \frac{pp_v}{q} du^2 \\ I_3 &= q^2 dv^2 + r^2 dw^2 & II_3 &= -\frac{qq_u}{p} dv^2 - \frac{rr_u}{p} dw^2 \end{aligned} \quad (2.2)$$

Proof. By lemma 2.2.1 it follows that $F_i = f_i = 0$ for the three parametrized surfaces.

The positive unit normal field to the surface S_1 is given by N_3 , that for S_2 is N_2 and the one for S_3 is N_1 .

Consider the surface $S_2 : (w, u) \rightarrow H^v(w, u)$. Then it follows that

$$\begin{aligned} e_2 &= \langle N_2, H_{ww} \rangle = \left\langle \frac{H_v}{|H_v|}, H_{ww} \right\rangle = -\langle H_{vw}, H_w \rangle / |H_v| = -\frac{|H_w|_v^2}{2q} = -\frac{rr_w}{q}, \\ g_2 &= \langle N_2, H_{uu} \rangle = \left\langle \frac{H_v}{|H_v|}, H_{uu} \right\rangle = -\langle H_{uv}, H_u \rangle / |H_v| = -\frac{|H_u|_v^2}{2q} = -\frac{pp_v}{q}. \end{aligned}$$

The same for the other two surfaces. \square

Remark 2.2.4. *Proposition 2.2.1 implies mean that for the parametrized surfaces S_i of a coordinate system, the coordinates define curvature lines. This is a form of Dupin theorem, revisited below (2.2.6) for coordinate surfaces in an orthogonal system.*

Remark 2.2.5. *The local inverse G of the diffeomorphism $H = (h_1, h_2, h_3)$ is also an orthogonal coordinate system in the space, i.e., for $(G_i = \frac{\partial G}{\partial c_i})$,*

$$\langle G_i, G_j \rangle = 0, \text{ for } i \neq j \quad (2.3)$$

as follows from a direct calculation that gives $G_i = \nabla h_i / |\nabla h_i|^2$.

Theorem 2.2.6 (Dupin). *The intersection of the level surface foliations $\mathbb{M}_2(c_2) = \{h_2 = c_2\}$ and $\mathbb{M}_3(c_3) = \{h_3 = c_3\}$ with a surface $\mathbb{M}_1(a_1) = \{h_1 = a_1\}$ produce on it a net of curves, along each of which, say γ , holds that $DN_1\gamma' \wedge \gamma' = 0$; that is, γ is the union of principal curves and umbilic points of \mathbb{M}_1 .*

Proof. Direct consequence of proposition 2.2.1 and remark 2.2.5. \square

Remark 2.2.7. *Notice that this establishes that the coordinate surfaces of any orthogonal coordinate system G in the space meet along common principal curves. A fact that is actually equivalent to the formulation in remark 2.2.4 of this result due to Dupin.*

Remark 2.2.8. *By taking the ruled surfaces generated by the normal lines to a surface \mathbb{M} along principal lines, two families N_{1v} , based on minimal principal lines (say, $v = \text{constant}$), and N_{2u} , based on maximal principal ones (say $u = \text{constant}$), are produced. These families of surfaces, together with the family \mathbb{M}_r given by parallel translation, are triply orthogonal. This shows that at non umbilic points, any surface can be embedded in a family of surfaces which is part of a triply orthogonal system.*

To prove that inversions $I(p) = p/|p|^2$ preserve lines of curvature, use the fact that these maps are conformal (i.e. preserve angles). Apply Dupin's Theorem to the image of the triply orthogonal system of surfaces just defined.

Remark 2.2.9. A family of surfaces given by $h(u, v, w) = c$, may be part of a triply orthogonal system if the function h satisfies the differential equation

$$\operatorname{div} \left(\frac{\partial}{\partial n} \operatorname{rot}(n) \right) = 0, \quad n = \frac{\nabla h}{|\nabla h|}. \quad \text{See [176].}$$

Theorem 2.2.10 (Darboux). Suppose that two families of orthogonal surfaces intercept along lines of curvature. Then there exists a third family of surfaces orthogonal to the first two families.

Proof. Consider two distributions i.e. fields Δ_1 and Δ_2 of tangent planes to the two families of orthogonal surfaces. Define the distribution Δ_3 orthogonal to both Δ_1 and Δ_2 . Take unit vector fields X, Y such that $X \in \Delta_1 \cap \Delta_3$ and $Y \in \Delta_2 \cap \Delta_3$.

By hypothesis, as the intersection between the two families are curvature lines, it follows that $\nabla_X Y = fX$ and $\nabla_Y X = gY$. Here ∇ is the covariant derivative which is defined by the tangential component of the directional derivative. So, the Lie bracket $[X, Y] = \nabla_X Y - \nabla_Y X = fX - gY \in \Delta_3$. Therefore Δ_3 is integrable and by Frobenius theorem, see [25] and [164], the third family of surfaces exists. \square

Ellipsoid with three different axes

In this subsection will be described the principal configuration on the ellipsoid, working directly on the differential equation of principal

curvature lines.

This is the first non trivial example of a principal configuration established by Monge [115], who directly integrated the differential equations for the principal lines.

Consider the ellipsoid $\mathbb{E} = f^{-1}(0)$, where $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, $a > b > c > 0$.

The differential equation of principal lines in implicit form is given by

$$[d(\nabla f), dp, \nabla f] = 0. \quad (2.4)$$

In the chart (x, y) this differential equation is expressed by:

$$\begin{aligned} & -a^2c^2(b^2 - c^2)xydy^2 + (a^2 - c^2)b^2c^2xydx^2 + \\ & [b^2c^2(c^2 - a^2)x^2 + c^2a^2(b^2 - a^2)y^2 + a^2b^2c^2(a^2 - b^2)]dxdy = 0. \end{aligned} \quad (2.5)$$

Rescaling the coordinates by $x = Au, y = Bv$, $A > 0, B > 0$, with

$$A^8 = \frac{a^2(b^2 - c^2)}{4b^2c^4(a^2 - c^2)^3}, \quad B = \frac{1}{2A^3c^2b^2(a^2 - c^2)}$$

the differential equation of principal curvature lines of the ellipsoid is given by:

$$u v d v^2 + (u^2 - v^2 - \lambda^2) d u d v - u v d u^2 = 0, \quad \lambda = a^2 b^2 c^2 (a^2 - b^2) A B.$$

The coordinates axes u and v and the family of ellipses and hyperbolas

$$\begin{aligned} u(t) &= R \cos t, & v(t) &= r \sin t, & R^2 &= r^2 + \lambda^2 \\ u(t) &= R \cosh t, & v(t) &= r \sinh t, & R^2 + r^2 &= \lambda^2 \end{aligned}$$

are the solutions of the differential equation above.

The points $(\lambda, 0)$ and $(-\lambda, 0)$ are singularities and the interval $(-\lambda, \lambda) \times \{0\}$ can be considered as a degenerated ellipse. The intervals $(-\infty, -\lambda) \times \{0\}$ and $(\lambda, \infty) \times \{0\}$ also can be considered as degenerated hyperbolas. The integral curves of the differential equation above are illustrated in Fig. 2.1; they define a confocal family of quadrics with foci at $(-\lambda, 0)$ and $(\lambda, 0)$.

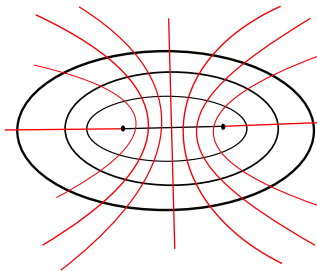


Figure 2.1: Confocal and orthogonal family of quadrics

Below the principal configuration on the ellipsoid \mathbb{E} will be obtained from Dupin's Theorem 2.2.6. To this end notice that it is part of the triply orthogonal family of quadrics: $\mathbf{E}(\lambda), \mathbf{H}_1(\lambda), \mathbf{H}_2(\lambda)$, defined by

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1, \quad a > b > c > 0.$$

See Fig. 2.2 for an illustration. Here, λ ranges on $(-\infty, c^2)$, for $E(\lambda)$, on (c^2, b^2) for $H_1(\lambda)$ and on (b^2, a^2) , for $H_2(\lambda)$.

In fact, for each triple (c_1, c_2, c_3) in $(-\infty, c^2) \times (c^2, b^2) \times (b^2, a^2)$ there is a unique point $p = (x, y, z) = G(c_1, c_2, c_3)$ in the positive or-

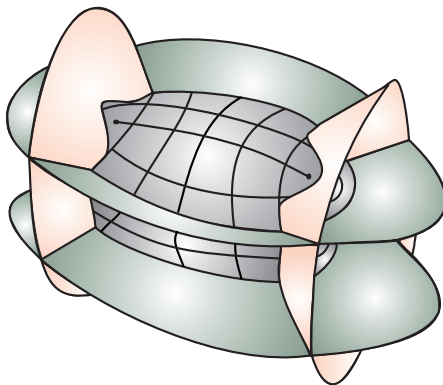


Figure 2.2: Triple orthogonal system of quadratic surfaces

that which is in the intersection of the surfaces $E(c_1)$, $H_1(c_2)$, $H_2(c_3)$. The result follows observing that G is an orthogonal coordinate system (called ellipsoidal) in the positive orthant, which means that the quadrics are defined by the levels of the coordinate functions of $G^{-1} = (h_1, h_2, h_3)$.

The explicit parametrization of the ellipsoid

$$\mathbb{E}_{a,b,c} = \left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

is given by:

$$\alpha(u, v) = \left(\pm \sqrt{\frac{M(u, v, a)}{W(a, b, c)}}, \pm \sqrt{\frac{M(u, v, b)}{W(b, a, c)}}, \pm \sqrt{\frac{M(u, v, c)}{W(c, a, b)}} \right) \quad (2.6)$$

where,

$$M(u, v, w) = w^2(-u + w^2)(-v + w^2), \quad W(a, b, c) = (a^2 - b^2)(a^2 - c^2), \\ u \in (c^2, b^2) \text{ and } v \in (b^2, a^2).$$

The first fundamental form of α is given by:

$$I = ds^2 = Edu^2 + Gdv^2 = \frac{1}{4} \frac{(u-v)u}{h(u)} du^2 + \frac{1}{4} \frac{(v-u)v}{h(v)} dv^2$$

The second fundamental form of α is given by:

$$II = edu^2 + gdv^2 = \frac{abc(u-v)}{4\sqrt{uv}h(u)} du^2 + \frac{abc(v-u)}{4\sqrt{uv}h(v)} dv^2,$$

where $h(x) = (x+a^2)(x+b^2)(x+c^2)$. The four umbilic points (Darbouxian of type D_1 , see proposition 2.4.1) are: $(\pm x_0, 0, \pm z_0) = (\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c\sqrt{\frac{c^2-b^2}{c^2-a^2}})$.

In Fig. 2.3 it is shown in first octant of ellipsoid the principal lines $u = u_0$ and $v = v_0$. The complete configuration is obtained by symmetry in relation to the coordinates planes.

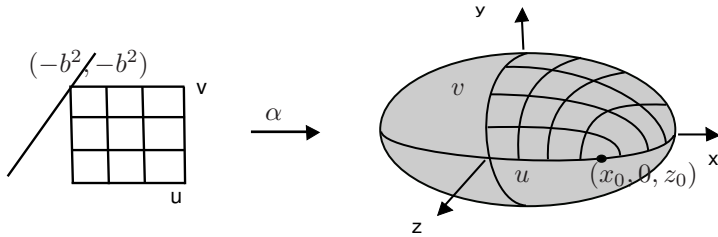


Figure 2.3: Curvature lines of the ellipsoid in the first orthant

A global view of principal lines is given in the Fig. 2.4. All principal lines are closed, except for four open arcs, the connections between Darbouxian umbilics, called umbilic separatrices. See [115], [161] and [163].

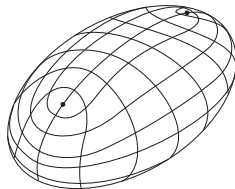


Figure 2.4: Curvature lines of the ellipsoid with three different axes

Remark 2.2.11. In the parametrization of α in equation (2.6) is also usual to take $M(u, v, w) = w^2(w^2 - u)(w^2 - v)$, $W(a, b, c) = (a^2 - b^2)(a^2 - c^2)$ with $u \in (b^2, a^2)$ and $v \in (c^2, b^2)$. See Chapter 7.

2.3 Envelopes of Regular Surfaces

An one parameter family of regular surfaces in \mathbb{R}^3 can be defined by $F(p, \lambda) = 0$ where $F : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is such that for each λ , $\nabla F_\lambda \neq 0$, where $F_\lambda(\cdot) = F(\cdot, \lambda)$.

The variation of this family with respect to the parameter can be defined as $F_\lambda = \frac{\partial F}{\partial \lambda}$. The set defined by

$$C = \{(p, \lambda) \mid F(p, \lambda) = \frac{\partial F}{\partial \lambda} = 0\}$$

is called the *characteristic set* of the family.

The projection $\pi_1(C) = \mathbb{E}$ is called the *envelope* of the family. Here $\pi_1 : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$, $\pi_1(p, \lambda) = p$.

Example 2.3.1. Consider the family the one parameter of spheres defined by

$$F(x, y, z, \lambda) = (x - \lambda)^2 + y^2 + z^2 - r^2 = 0$$

Therefore, $F_\lambda = -2(x - \lambda) = 0$ and so the characteristic set is the hyperplane $\{x = \lambda\}$ and the envelope is the cylinder $y^2 + z^2 = r^2$.

Example 2.3.2. Consider the one parameter family of spheres defined by

$$F(x, y, z, \lambda) = (x - \lambda)^2 + y^2 + z^2 - r^2 + \lambda$$

Therefore, $F_\lambda = -2(x - \lambda) + 1 = 0$ and so the characteristic set is the hyperplane $\{x = 1/2 - \lambda\}$ and the envelope is the paraboloid $x = -(y^2 + z^2) + r^2 + 1/4$.

Example 2.3.3. Consider an one parameter family of spheres of constant radius r with centers ranging along a curve $c(s)$. So the family can be represented by $F(p, s) = \|p - c(s)\|^2 - r^2 = 0$. Therefore it follows that $F_s = -2\langle p - c(s), c'(s) \rangle$. The envelope of this family is called a *canal surface*.

When $c(s) = (R \cos s, R \sin s, 0)$ the envelope is a torus of revolution that can be parametrized by

$$\alpha(s, \theta) = c(s) + r \cos \theta (\cos s, \sin s, 0) + r \sin \theta (0, 0, 1).$$

Intuitively the envelope \mathbb{E} is tangent to the family of surface defined $F_\lambda(p) = 0$. More precisely the following holds

Proposition 2.3.1. *Suppose that \mathbb{E} is a regular surface and $p \in \mathbb{E} \cap F_\lambda^{-1}(0)$. Then the tangent plane $T_p \mathbb{E}$ coincides with the tangent plane of the surface $F(p, \lambda) = 0$.*

Proof. We leave this to the reader. □

Proposition 2.3.2 (Vessiot). *Consider the one parameter family of spheres with center $c(s)$ and variable radius $r(s) > 0$. Suppose that*

the envelope of this family is a regular surface. Then the envelope can be parametrized by

$$\alpha(s, \varphi) = c(s) + r \cos \theta(s)T(s) + r(s) \sin \theta(s)[\cos \varphi N + \sin \varphi B],$$

where $\cos \theta(s) = -r'(s)$ and $\{T, N, B\}$ is the Frenet frame of c . Moreover one family of lines of curvature are circles of radius $r(s) \sin \theta(s)$ and the other is defined by the Riccati differential equation

$$\frac{d\varphi}{ds} = -\tau(s) - k(s) \cotg \theta(s) \sin \varphi.$$

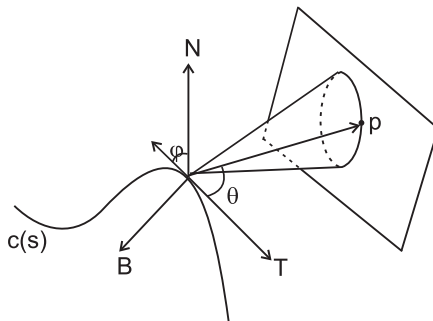


Figure 2.5: Canal surface with variable radius

Proof. The family of spheres is defined by

$$F(s, p) = |p - c(s)|^2 - r(s)^2 = 0.$$

Therefore, $F_s = \langle c'(s), p - c(s) \rangle - 2rr'$.

So the system of equations $F(s, p) = F_s(s, p) = 0$, for each s fixed, defines the intersection of a sphere and a cone with vertex at $c(s)$.

Writing $\langle c', p - c(s) \rangle = |c'| |p - c| \cos \theta(s) = |p - c(s)| \cos \theta(s)$ it follows from $F = F_s = 0$ that $\cos \theta(s) = -r'$.

Therefore by Joachimsthal theorem, 2.2.1, it follows that the circles defined by the one parameter family $C_s = \{p : F(s, p) = F_s(s, p) = 0\}$ is a curvature line of the envelope \mathbb{E} . In order to obtain the other family we observe that in the chart (s, φ) the family of circles is defined by $\dot{s} = 0$. So the orthogonal family is defined by

$$\dot{s} = G(s, v), \quad \dot{\varphi} = -F(s, v), \quad F = \langle \alpha_s, \alpha_v \rangle \quad \text{and} \quad G = \langle \alpha_v, \alpha_v \rangle.$$

Direct calculation gives

$$F(s, v) = \tau r^2 \sin^2 \theta - k(s) r' r^2 \sin \theta \sin \varphi$$

$$G(s, v) = r^2 \sin^2 \theta = r^2(1 - r'^2).$$

Performing the calculations obtain that

$$\frac{d\varphi}{ds} = -\tau(s) - k(s) \cotg \theta(s) \sin \varphi.$$

Finally writing $v = \tan(\varphi/2)$ it is obtained

$$\dot{v} = -\frac{1}{2}\tau(1 + v^2) - k \cotg \theta(s)v,$$

a familiar Riccati equation. □

Remark 2.3.4. *The proof of the proposition above was adapted from [174]. For recent developments about principal curvature lines on canal surfaces and hypersurfaces see [144] and [146].*

Corollary 2.3.5. *Consider the one parameter family of spheres with $c(s)$ a planar simple curve and $r(s) > 0$ variable with $|r'(s)| < 1$.*

Then the envelope, canal surface \mathbb{E} , has two hyperbolic closed principal lines parametrized by $\varphi = 0$ and $\varphi = \pi$ provided

$$\int_0^L \frac{r'}{\sqrt{1-r'^2}} k(s) ds \neq 0.$$

Proof. Direct from the proposition 2.3.2.

□

2.4 Examples of Umbilic Points on Algebraic Surfaces

In this section two examples of principal configurations on algebraic surfaces will be considered.

Proposition 2.4.1. *The ellipsoid $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$, $a > b > c > 0$, has four umbilics given by $(\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm a\sqrt{\frac{b^2-c^2}{a^2-c^2}})$. Moreover these umbilic points are all Darbouxian D_1 . Outside these umbilic points, the principal configuration consists on closed principal lines and four umbilic separatrices, connecting the umbilic points.*

Proof. The differential equation of curvature lines of an implicit surface defined by $f = 0$ is given by $[\nabla f, d(\nabla f), dp] = 0$, where $dp = (dx, dy, dz)$ and $\langle \nabla f, dp \rangle = 0$. Direct calculation shows that the four umbilic points are as stated.

Let $(x_0, 0, z_0)$ be an umbilic point and consider the following

change of coordinates:

$$(x, y, z) = (x_0, 0, z_0) + uE_1 + vE_2 + wE_3, \quad \text{where}$$

$$E_1 = \left(\frac{z_0}{c^2}, 0, -\frac{x_0}{a^2}\right) / \sqrt{\frac{z_0^2}{c^4} + \frac{x_0^2}{a^4}}, \quad E_2 = (0, 1, 0),$$

$$E_3 = \left(\frac{x_0}{a^2}, 0, \frac{z_0}{c^2}\right) / \sqrt{\frac{z_0^2}{c^4} + \frac{x_0^2}{a^4}}.$$

In the coordinates (u, v, w) the ellipsoid has the following parametrization $(u, v, W(u, v))$ where,

$$W(u, v) = -\frac{ac}{2b^3}(u^2 + v^2) - \frac{ac}{2b^6}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}(u^2 + v^2)u + 0(4).$$

This follows from long, but straightforward, calculations. Therefore by the theorem of classification of umbilic points, see chapter 3, page 74, the result follows. \square

Proposition 2.4.2. *The convex surface $f(x, y, z) = x^4 + y^4 + z^4 = 1$, which has the symmetry of the cube, has 14 umbilic points, 8 of the Darbouxian type D_3 and 6 of index 1 (center). The principal configuration is as shown in the Fig. 2.6.*

Proof. Direct calculation gives that the umbilic points are given by:

$$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (\pm a, \pm a, \pm a), \quad a = (1/3)^{1/4}.$$

Notice that the planes of symmetry of the cube intercept the surface orthogonally, and these intersections consist on umbilic points and principal lines. By symmetry arguments it follows that $(1, 0, 0)$ and all the other umbilic points contained in the coordinate planes are of

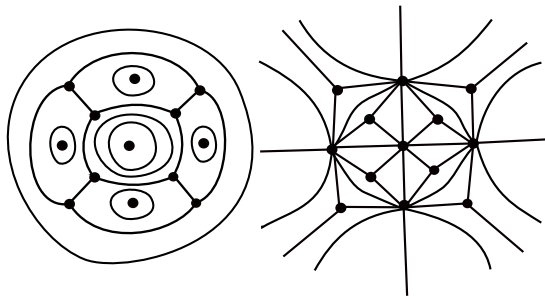


Figure 2.6: Lines of curvature on a convex surface with symmetry of the cube

center type (index 1), surrounded by closed principal lines, for one principal foliation, and it is of nodal type (index 1) for the other principal foliation.

Now near the umbilic point (a, a, a) consider the change of coordinates

$$(x, y, z) = (a, a, a) + \frac{u}{\sqrt{6}}(1, 1, -2) + \frac{v}{\sqrt{2}}(1, -1, 0) - \frac{w}{\sqrt{3}}(1, 1, 1).$$

So the surface $f = 1$ has the following parametrization $(u, v, h(u, v))$, where

$$h(u, v) = \frac{3\sqrt{3}}{4a}(u^2 + v^2) - \frac{\sqrt{2}}{a^2}\left(\frac{1}{6}u^3 - \frac{1}{2}uv^2\right) + O(4).$$

So by the classification of umbilic points, see chapter 3, page 74, it follows that this point is of type D_3 . By the symmetry all umbilic points $(\pm a, \pm a, \pm a)$ are of type D_3 . The principal configuration is as shown in the Fig. 2.6. In this planar representation one umbilic is located at infinity. \square

Remark 2.4.1. *A study of umbilics and the behavior at singular points and at infinity of principal configurations on algebraic surfaces has been carried out by R. Garcia and J. Sotomayor, see [54], [55] and [61].*

2.5 Exercises and Problems

2.5.1. Consider the embedded tube defined by

$$\alpha(s, v) = c(s) + r \cos vn(s) + r \sin vb(s), \quad r > 0, \text{ small.}$$

Here c is a closed curve with curvature $k > 0$ and torsion τ .

- i) Characterize the curves c such that the tube defined above has no umbilic points.
- ii) Give an example of a curve c such that all leaves of one principal foliation of α are dense.

2.5.2. Let $\alpha : \mathbb{M}^{n-1} \rightarrow \mathbb{R}^n$ be an immersion of smooth and oriented manifold \mathbb{M}^{n-1} .

- i) Define lines of curvature for immersed hypersurfaces in \mathbb{R}^n , $n \geq 4$ and write the differential equation in the case $n = 4$.
- ii) Show that the family of quadrics defined by

$$E(\lambda) = \frac{x_1^2}{a_1^2 - \lambda} + \cdots + \frac{x_n^2}{a_n^2 - \lambda} - 1, \quad 0 < a_1 < \cdots < a_n,$$

is an n -orthogonal system of hypersurfaces in \mathbb{R}^n .

- iii) In the higher dimensional situation formulate and provide proofs of suitable extensions of the theorems 2.2.1, 2.2.6 and 2.2.10.

2.5.3. Let $J = \{1, \frac{1}{2}, 0, -\frac{n}{2}, n \in \mathbb{N}\}$. Give examples of umbilic points of topological index $j \in J$. See [119], [120], [156] and [182].

2.5.4. Show that the system of surfaces

$$x^2 + y^2 + z^2 = ux, \quad x^2 + y^2 + z^2 = vy, \quad x^2 + y^2 + z^2 = wz,$$

is triply orthogonal.

2.5.5. Let S be a regular, compact and oriented surface of \mathbb{R}^3 with unit normal N . Show that the parallel surface $S_r = S + rN$ is regular for small $r > 0$ and that there is a diffeomorphism $\varphi : S \rightarrow S_r$ preserving the principal curvature lines.

2.5.6. Let c be a closed principal curvature line of an oriented surface \mathbf{S} and let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a conformal diffeomorphism, that is one which preserves angles. Let k_1 and k_2 the principal curvatures of \mathbf{S} and κ_1 and κ_2 the principal curvatures of $\phi(\mathbf{S}) = \Sigma$.

Define the integrals $r = \int_c \frac{dk_2}{k_2 - k_1}$, $\varrho = \int_\gamma \frac{d\kappa_2}{\kappa_2 - \kappa_1}$, $\gamma = \phi(c)$. Show that γ is a closed principal curvature line of Σ and that $r = \pm\varrho$. The sign of $r = \pm\varrho$ depends on the orientation of the surfaces \mathbf{S} and Σ .

2.5.7. A surface is called a *Weingarten surface* when there is a functional relation between the principal curvatures k_1 and k_2 such as $F(k_1, k_2) = 0$.

i) Give various examples of Weingarten surfaces.

ii) Show that there are Weingarten surfaces, oriented and compact, of any given genus g . See [85].

2.5.8. Show that a curve $c(s)$ is a principal curvature line of an oriented surface \mathbb{M} if and only if the surface $\beta(s, v) = c(s) + vN(s)$ is developable (ruled surface with zero Gauss curvature). Here N is the unit normal vector to S along c . Conclude that c is the union of principal curvature lines of the ruled surface β .

2.5.9. Show that the set of closed principal curvature lines on an oriented developable surface is an open cylinder or an annulus.

2.5.10. Show that a family of quadrics

$$\frac{x^2}{a(u)} + \frac{y^2}{b(u)} + \frac{z^2}{c(u)} = 1,$$

where a , b and c are smooth functions, belongs to a triply orthogonal system of surfaces if and only if the following differential equation holds $a(b-c)a' + b(c-a)b' + c(a-b)c' = 0$. (*)

Find special solutions of the differential equation above.

Suggestion: Show that the solutions of the system $aa' = ah + g$, $bb' = bh + g$, $cc' = ch + g$, where $h = h(u)$ and $g = g(u)$ are arbitrary smooth functions, are solutions of (*). See [48].

2.5.11. Show that the system given by

$$x^2 + y^2 + z^2 = ux, \quad z = vy, \quad (x^2 + y^2 + z^2)^2 = w(y^2 + z^2)$$

defines a triply orthogonal system of surfaces. Visualize the shapes of these surfaces.

2.5.12. Show that a triply orthogonal system is given by:

- the hyperbolic paraboloids $yz = ux$,
- the closed sheets of the surface $(y^2 - z^2)^2 - 2a(2x^2 + y^2 + z^2) + a^2 = 0$,
- the open sheets of the same surface.

2.5.13. Consider the surface S parametrized by $(u, v, h(u, v))$ where,

$$\begin{aligned} h(u, v) = & \frac{1}{2}(au^2 + bv^2) + \frac{1}{6}(Au^3 + 3Bu^2v + 3Cuv^2 + Dv^3) \\ & + \frac{1}{24}(\alpha u^4 + 4\beta u^3v + 6\gamma u^2v^2 + 4\varepsilon uv^3 + \delta v^4) + \dots \end{aligned} \quad (**)$$

Let $c = c(s)$ be a principal curvature line of S passing through 0 and tangent to the u axis. Let k and τ be, respectively, the curvature and the

torsion of c at 0. Show that $k^2\tau = \frac{(3a-b)AB - 3aBC}{(a-b)^2} - \frac{\alpha\beta}{a-b}$. Find the analogous relation for the other principal curvature line and also determine the geodesic curvatures of both principal lines at 0.

2.5.14. Let S be the surface parametrized by equation (***) above. Write the Taylor series of the principal curvatures $k_1 = k_1(u, v)$ and $k_2 = k_2(u, v)$ at 0 up to order two and analyze the level sets of both functions near 0, imposing generic conditions on the coefficients $(a, b, \dots, \varepsilon, \delta)$.

2.5.15. Consider the ellipsoid $E(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let $[dp, d(\nabla E), \nabla E] = 0$, $p = (x, y, z)$, the differential equation of principal curvature lines.

i) Consider the differential equation above complexified and analyze the singular foliations in the complex quadric $E_{\mathbb{C}}$ obtained by complexification of the real ellipsoid.

ii) Investigate the singularities and the complex separatrices of the associated Lie-Cartan vector field defined in a complex analytic surface. For an introduction to complex differential equations, see [26], [89] and [105]. This open problem is based in a question raised by E. Ghys.

2.5.16. Let γ be a closed line of curvature of an oriented surface $\mathbb{M} \subset \mathbb{R}^3$ and let τ the torsion of γ as a curve of \mathbb{R}^3 . Assume that γ is parametrized by arc length.

i) Show that the total torsion $\int_{\gamma} \tau = 2m\pi$, $m \in \mathbb{Z}$.

ii) Show that if \mathbb{M} is convex then $\int_{\gamma} \tau = 0$. See exercise 1.9.24, page 39.

2.5.17. Consider the canal surface given in Proposition 2.3.2.

i) Compute the principal curvatures of the canal surface.

ii) Study the umbilic set of a canal surface.

iii) Give explicit examples of canal surfaces having conical singularities.

2.5.18. Give various examples of surfaces in \mathbb{R}^3 such that:

- i) One family of principal lines consists on plane or spherical curves.
- ii) Both principal foliations consist on plane curves.
- iii) One family of principal lines is formed by plane curves and the other by non planar spherical curves.

2.5.19. Show that the quadratic ellipsoid with three different axes is foliated by parallel circles with two singularities localized at umbilic points.

2.5.20. Let $\gamma_n : \mathbb{R} \rightarrow \mathbb{R}^3$, $n \in \mathbb{N}$, the curve parametrized by

$$\gamma_n(t) = \left(\left(\frac{1}{\sin \frac{2\pi}{n}} + \cos t \right) \cos \frac{t}{n} - \frac{1}{\sin \frac{2\pi}{n}}, \sin t, \left(\frac{1}{\sin \frac{2\pi}{n}} + \cos t \right) \sin \frac{t}{n} \right).$$

- i) Show that $\gamma_n(t + 2\pi n) = \gamma_n(t)$.
- ii) Show that $\gamma_n \rightarrow \gamma$ as $n \rightarrow \infty$, where γ is the circular helix parametrized by $\gamma(t) = (\cos t, \sin t, \frac{t}{2\pi})$.
- iii) Show that the torsion of γ_n is negative for large values of n .
- iv) Compute the total torsion $T_n = \int_{\gamma_n} \tau_n$ of γ_n and analyze the arithmetic properties of the sequence of values $(T_n)_{n \in \mathbb{N}}$. See [16].
- v) The curve γ_n can be a principal line of a surface? See exercise 2.5.16, page 63.

2.5.21. Let γ be a regular arc of \mathbb{R}^3 parametrized by arc length s .

Define the osculating sphere of γ as follows. Take four distinct points $\{p_0 = \gamma(s), p_1 = \gamma(s + s_1), p_2 = \gamma(s + s_2), p_3 = \gamma(s + s_3)\}$ and consider the unique sphere $\Sigma_i = \Sigma(s_1, s_2, s_3)$ passing through these points. The limit (when defined) $\lim_{|s_i| \rightarrow 0} \Sigma(s_1, s_2, s_3) = \Sigma_{\gamma(s)}$ is the osculating sphere of γ at the point $\gamma(s)$.

- i) Show that the curve $c(s) = \gamma(s) + r(s)n(s) + \frac{r'(s)}{\tau(s)}b(s)$, $r(s) = \frac{1}{k(s)}$ and $\{t, n, b\}$ the Frenet frame of γ is the center of the osculating sphere.

ii) Show that the osculating circle \mathcal{O}_s of γ is contained in the osculating sphere $\Sigma_{\gamma(s)}$. The osculating circle is contained in the osculating plane and has radius equal to $r(s)$.

iii) Investigate the geometry (geodesics, principal lines, asymptotic lines and singularities) of the osculating tube

$$\alpha(s, \theta) = \gamma(s) + r(s) \cos \theta n(s) + r(s) \sin \theta b(s).$$

2.5.22. Let $h(u, v) = \frac{1}{2}c \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(u - cv)\right)$.

i) Compute the first and second fundamental form of the graph surface $(u, v, h(u, v))$.

ii) Show that h is a solution of the KdV equation $\theta_v + 6\theta\theta_u + \theta_{uuu} = 0$.

iii) Show that the umbilic set of graph surface is parametrized by $u = cv \pm \frac{2}{\sqrt{c}} \operatorname{arcosh}\left(\frac{\sqrt{6}}{2}\right)$, $z = z(c)$. Describe the principal configuration of the graph of h .

iv) Analyze the structure of the umbilic and parabolic set of the surface defined by the graph of $h(u, v) + \epsilon(cu + v)^3$.

2.5.23. Let $\eta_i = -b_i u + 4b_i^3 v$ and $A = [(b_1 - b_2)/(b_1 + b_2)]^2$. Define

$$h(u, v) = \ln(1 + e^{2\eta_1} + e^{2\eta_2} + Ae^{2\eta_1 + 2\eta_2}) \quad \text{and} \quad H(u, v) = 2h_{uu}.$$

i) Visualize the graph of H . Use the software of A. Montesinos, [116].

ii) Show that H is a solution (2-soliton) of the KdV equation $\theta_v + 6\theta\theta_u + \theta_{uuu} = 0$. See [121].

iii) Show that the graph H has umbilic points and that the Gaussian curvature change signs and $\lim_{|p| \rightarrow \infty} \mathcal{K}(p) = 0$.

2.5.24. Show that the torus of revolution is foliated by four families of circles: parallels, meridians and Villarceau circles.

2.5.25. Show that any Möbius developable surface has at least one umbilic point.

2.5.26. Show that the theory of principal curvature lines on surfaces in \mathbb{R}^3 and that of the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ are equivalent. More precisely, consider the stereographic projection $\Pi : \mathbb{S}^3 \setminus \{N\} \rightarrow \mathbb{R}^3$. Write the expression of Π in coordinates and show that Π is a conformal map. Conclude that Π is a conjugation between the principal configuration of the surface $S \subset \mathbb{R}^3$ and that of $\bar{S} = \Pi^{-1}(S) \subset \mathbb{S}^3$, i. e., Π is an equivalence between principal configurations. See [101] for a geometric proof.

2.5.27. Let \mathbb{M} be a compact surface of class C^r of non negative Gaussian curvature and of *constant width* L , i.e., the orthogonal projection of \mathbb{M} onto every line of \mathbb{R}^3 is an interval of constant length L and so the distance between parallel tangent planes is constant. The pair of points p and q such $N(p) = -N(q)$ are called opposite. See [109].

i) Show that the principal directions at opposite points are parallel.

ii) Give various examples of curves and surfaces of constant width.

iii) Show that $c(t) = (r \cos t + 2 \cos 2t - \cos 4t, r \sin t - 2 \sin 2t - \sin 4t)$, $r > 8$, $0 \leq t \leq 2\pi$, is a convex curve of constant width $2r$ and the revolution of this curve around the x axis is a convex surface (algebraic) of constant width.

iv) A surface \mathbb{M} is said to have *constant brightness* b if and only if the orthogonal projection of \mathbb{M} onto every plane is a region of area b . Show that a surface of constant width and brightness is a sphere. See [87].

2.5.28. Let $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial of degree n . Develop the theory of Darboux integrability, well established for polynomial differential equations on the plane, to the principal configuration of the algebraic surface $h^{-1}(0)$. See [36], [103] and [89].

Chapter 3

Principal curvature configuration stability

3.1 Introduction

In this chapter we formulate and discuss a principal curvature configuration stability result for principal configurations of curvature lines due to C. Gutierrez and J. Sotomayor, [71], [72] and [75].

For a sketch of the history of the theory of qualitative theory of principal lines see [161, 163]. A recent survey on this subject can be found in [63].

3.2 Lines of curvature near Darbouxian umbilics

In this section will be reviewed the behavior of curvature lines near Darbouxian umbilics.

Preliminaries concerning umbilic points

Denote by $\mathbb{P}\mathbb{M}^2$ the projective tangent bundle over \mathbb{M}^2 , with projection Π . For any chart (u, v) on an open set U of \mathbb{M}^2 there are defined two charts $(u, v; p = dv/du)$ and $(u, v; q = du/dv)$ which cover $\Pi^{-1}(U)$.

The differential equation (1.6) of principal lines, being quadratic, is well defined in the projective bundle. Thus, for every α in $\mathcal{I}^r = I^{r,r}(\mathbb{M}, \mathbb{R}^3)$,

$$\mathbb{L}_\alpha = \{\tau_{g,\alpha} = 0, \}$$

defines a variety on $\mathbb{P}\mathbb{M}^2$, which is regular and of class C^{r-2} over $\mathbb{M}^2 \setminus \mathcal{U}_\alpha$. It doubly covers $\mathbb{M}^2 \setminus \mathcal{U}_\alpha$ and contains a projective line $\Pi^{-1}(p)$ over each point $p \in \mathcal{U}_\alpha$.

Recall that the geodesic torsion is given by:

$$\tau_g = \frac{(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2}{(EG - F^2)^{\frac{3}{2}}(Edu^2 + 2Fdudv + Gdv^2)}.$$

Definition 3.2.1. *A point $p \in \mathcal{U}_\alpha$ is Darbouxian if the following two conditions hold:*

T : The variety \mathbb{L}_α is regular also over $\Pi^{-1}(p)$. In other words, the derivative of $\tau_{g,\alpha}$ does not vanish on the points of projective line $\Pi^{-1}(p)$. This means that the derivative in directions transversal to $\Pi^{-1}(p)$ must not vanish.

D : The principal line fields $\mathcal{L}_{i,\alpha}$, $i = 1, 2$ lift to a single line field \mathcal{L}_α of class C^{r-3} , tangent to \mathbb{L}_α , which extends to a unique

one along $\Pi^{-1}(p)$, and there it has only hyperbolic singularities, which must be either

D_1 : a unique saddle

D_2 : a unique node between two saddles, or

D_3 : three saddles.

For calculations will be helpful to express the Darbouxian conditions in a Monge local chart $(u, v): (\mathbb{M}^2, p) \rightarrow (\mathbb{R}^2, 0)$ on \mathbb{M}^2 , $p \in \mathcal{U}_\alpha$, as follows.

Take an isometry Γ of \mathbb{R}^3 with $\Gamma(\alpha(p)) = 0$ such that $\Gamma(\alpha(u, v)) = (u, v, h(u, v))$, with

$$\begin{aligned} h(u, v) = & \frac{k}{2}(u^2 + v^2) + (a/6)u^3 + (b/2)uv^2 + (b'/2)u^2v \\ & + (c/6)v^3 + (A/24)u^4 + (B/6)u^3v + (C/4)u^2v^2 \\ & + (D/6)uv^3 + (E/24)v^4 + O((u^2 + v^2)^{5/2}). \end{aligned} \quad (3.1)$$

To obtain simpler expressions assume that the coefficient b' vanishes.

This is achieved by means of a suitable rotation in the (u, v) -plane.

In the affine chart $(u, v; p = dv/du)$ on $\mathbb{P}(\mathbb{M}^2)$ around $\Pi^{-1}(p)$, the variety \mathbb{L}_α is given by the following equation.

$$\mathcal{T}(u, v, p) = L(u, v)p^2 + M(u, v)p + N(u, v) = 0, \quad p = dv/du. \quad (3.2)$$

The functions L , M and N are obtained from equation (1.6) and

(3.1) as follows:

$$\begin{aligned} L &= h_u h_v h_{vv} - (1 + h_v^2) h_{uv} \\ M &= (1 + h_u^2) h_{vv} - (1 + h_v^2) h_{uu} \\ N &= (1 + h_u^2) h_{uv} - h_u h_v h_{uu}. \end{aligned}$$

Calculation, taking into account the coefficients in equation (3.1) with $b' = 0$, gives:

$$\begin{aligned} L(u, v) &= -bv - (B/2)u^2 - (C - k^3)uv - (D/2)v^2 + M_1^3(u, v) \\ M(u, v) &= (b - a)u + cv + [(C - A)/2 + k^3]u^2 + (D - B)uv \\ &\quad + [(E - C)/2 - k^3]v^2 + M_2^3(u, v) \\ N(u, v) &= bv + (B/2)u^2 + (C - k^3)uv + (D/2)v^2 + M_3^3(u, v), \end{aligned} \tag{3.3}$$

with $M_i^3(u, v) = O((u^2 + v^2)^{3/2})$, $i = 1, 2, 3$.

These expressions are obtained from the calculation of the coefficients of the first and second fundamental forms in the chart (u, v) .

See also [37, 71, 75]. With longer calculations, Darboux [37] provided the full expressions for any value of b' .

Remark 3.2.2. *The regularity condition T in definition 3.2.1 is equivalent to impose that $b(b - a) \neq 0$. In fact, this inequality also implies regularity at $p = \infty$. This can be seen in the chart $(u, v; q = du/dv)$, at $q = 0$.*

Also this condition is equivalent to the transversality of the curves $M = 0$, $N = 0$

The line field \mathcal{L}_α is expressed in the chart $(u, v; p)$ as the one generated by the vector field $X = X_\alpha$, called the *Lie-Cartan* vector field of equation (1.6), which is tangent to \mathbb{L}_α and is given by:

$$\begin{aligned} \dot{u} &= \partial T / \partial p \\ \dot{v} &= p \partial T / \partial p \\ \dot{p} &= -(\partial T / \partial u + p \partial T / \partial v) \end{aligned} \tag{3.4}$$

Similar expressions hold for the chart $(u, v; q = du/dv)$ and the pertinent vector field $Y = Y_\alpha$.

The function T is a first integral of $X = X_\alpha$. The projections of the integral curves of X_α by $\Pi(u, v, p) = (u, v)$ are the lines of curvature. The singularities of X_α are given by $(0, 0, p_i)$ where p_i is a root of the equation $p(bp^2 - cp + a - 2b) = 0$.

Assume that $b \neq 0$, which occurs under the regularity condition T , then the singularities of X_α on the surface \mathbb{L}_α are located on the p -axis at the points with coordinates p_0, p_1, p_2

$$\begin{aligned} p_0 &= 0, \quad p_1 = c/2b - \sqrt{(c/2b)^2 - (a/b) + 2}, \\ p_2 &= c/2b + \sqrt{(c/2b)^2 - (a/b) + 2} \end{aligned} \tag{3.5}$$

Remark 3.2.3. *Assume the notation established in equation (3.1). Suppose that the transversality condition $T : b(b - a) \neq 0$ of definition 3.2.1 and remark 3.2.2 holds. Let $\Delta = -[(c/2b)^2 - (a/b) + 2]$. Calculation of the hyperbolicity conditions at the singularities (3.5) of the vector field (3.4) –see [71]– leads to following equivalences:*

$$D_1) \equiv \Delta > 0, \quad D_2) \equiv \Delta < 0 \text{ and } 1 < \frac{a}{b} \neq 2, \quad D_3) \equiv \frac{a}{b} < 1.$$

See Figs. 3.2 and 3.1 for an illustration of the three possible types of Darbouxian umbilics. The distinction between them is expressed in terms of the coefficients of the 3-jet of equation (3.1), as well as in the lifting of singularities to the surface \mathbb{L}_α . See remarks 3.2.2 and 3.2.3.

The subscript $i = 1, 2, 3$ of D_i denotes the number of *umbilic separatrices* of p . These are principal lines which tend to the umbilic point p and separate regions of different patterns of approach to it. For Darbouxian points, the umbilic separatrices are the projection into \mathbb{M}^2 of the saddle separatrices transversal to the projective line over the umbilic point.

It can be proved that the only umbilic points for which $\alpha \in \mathcal{I}^r$ is locally C^s -structurally stable, $r > s \geq 3$, are the Darbouxian ones. See [20, 75].

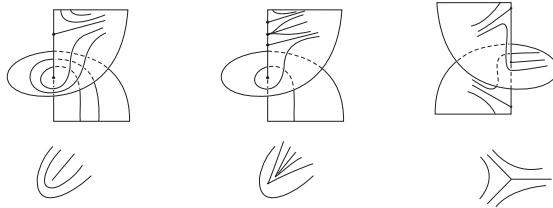


Figure 3.1: Darbouxian Umbilic Points, corresponding \mathbb{L}_α surface and lifted line fields.

The implicit surface $\mathcal{T}(u, v, p) = 0$ is regular in a neighborhood of the projective line if and only if $b(b - a) \neq 0$. Near the singular point $p_0 = (0, 0, 0)$ of X_α it follows that $\mathcal{T}_v(p_0) = b \neq 0$ and therefore, by the Implicit Function Theorem, there exists a function $v = v(u, p)$ such that $\mathcal{T}(u, v(u, p), p) = 0$. Differentiation gives the following

Taylor expansion

$$v(u, p) = -\frac{B}{2b}u^2 + \frac{a-b}{b}up + O(3).$$

For future reference we record the expression the vector field X_α in the chart (u, p) .

$$\begin{aligned} \dot{u} &= \mathcal{T}_p(u, v(u, p), p) \\ &= (b-a)u + \frac{1}{2} \frac{[b(C-A+2k^3) - cB]}{b} u^2 + \frac{c(a-b)}{b} up + O(3) \\ \dot{p} &= -(\mathcal{T}_u + p\mathcal{T}_v)(u, v(u, p), p) = \\ &= -Bu + (a-2b)p - cp^2 + \frac{1}{2} \frac{[B(C-k^3) - a_{41}b]}{b} u^2 \\ &\quad + \frac{[b(A-C-2k^3) + a(k^3-C)]}{b} up + O(3), \end{aligned} \tag{3.6}$$

where a_{41} is $\frac{\partial^5 h}{\partial u^4 \partial v}$, evaluated at $(0, 0)$. However, a_{41} will have no influence in the qualitative analysis that follows.

Theorem 3.2.4. *[Gutierrez, Sotomayor, 1982] Let p an umbilic point of an immersion α of class C^r , $r \geq 4$, given in a Monge chart (u, v) by:*

$$\alpha(u, v) = (u, v, \frac{k}{2}(u^2 + v^2) + \frac{a}{6}u^3 + \frac{b}{2}u^2v + \frac{c}{6}v^3 + o(4))$$

Suppose the following conditions hold:

T) $b(b-a) \neq 0$

D₁) $(\frac{c}{2b})^2 - \frac{a}{b} + 2 < 0$

D₂) $(\frac{c}{2b})^2 + 2 > \frac{a}{b} > 1, \quad a \neq 2b$

\mathbf{D}_3) $\frac{a}{b} < 1$.

Then the behaviors of principal curvature lines near the umbilic point p , in the cases \mathbf{D}_1 , \mathbf{D}_2 and \mathbf{D}_3 , are as in Fig. 3.2. These umbilic points are called *Darbouxian Umbilics*.

An immersion $\alpha \in \mathcal{I}^r$, $r \geq 4$, is C^3 -principally structurally stable at a point $p \in \mathcal{U}_\alpha$ if and only if p is a Darbouxian umbilic point.

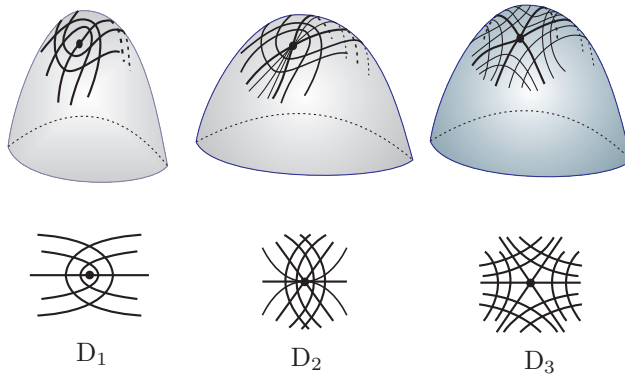


Figure 3.2: Lines of Curvature near Darbouxian Umbilic Points

Remark 3.2.5. *The structure of the curvature lines near umbilic points of analytic surfaces was established by G. Darboux, [37]. He used the methods for singularities of ordinary differential equations developed by H. Poincaré. For C^r , $r \geq 4$, surfaces this analysis was carried out by Gutierrez and Sotomayor [71] and also by Bruce and Fidal [20].*

3.3 Hyperbolic Principal Cycles

A closed line of principal curvature is called a *principal cycle*.

A principal cycle is called *hyperbolic* if the first derivative of the Poincaré return \mathcal{P} – also called *holonomy* – map associated to it is different from one.

Lemma 3.3.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , and $c : [0, l] \rightarrow \mathbb{M}^2$ be a minimal principal cycle parametrized by arc length u and length l . Then the expression:*

$$\alpha(u, v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + \left[\frac{k_2}{2} v^2 + \frac{1}{6} A(u) v^3 + v^3 B(s, v) \right] N(c(u)) \tag{3.7}$$

where $B(u, 0) = 0$ and k_2 is the principal curvature of α , defines a local chart (u, v) of class C^{r-6} in a neighborhood of c . Moreover $A(u) = (k_2)_v(u, 0)$. See Fig.3.3 for an illustration.

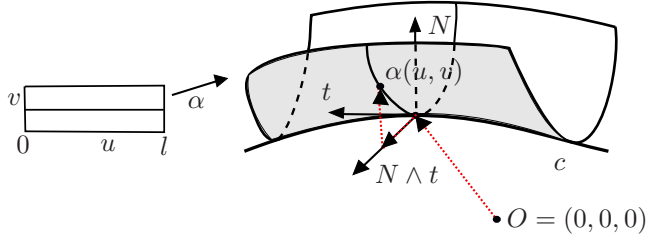


Figure 3.3: Parametrized immersed surface $\alpha(u, v)$ near a principal cycle c .

Proof. The curve c is of class C^{r-1} and the map $\alpha(u, v, w) = (\alpha \circ c)(u) + v(N \wedge T)(u) + wN(us)$ is of class C^{r-2} and is a local diffeomorphism in a neighborhood of the axis u . In fact $[\alpha_u, \alpha_v, \alpha_w](su, 0, 0) = 1$. Therefore there is a function $W(u, v)$ of class C^{r-2} such that $\alpha(u, v, W(u, v))$ is a parametrization of a tubular neighborhood of

$\alpha \circ c$. Now for each u , $W(u, v)$ is just a parametrization of the curve of intersection between $\alpha(\mathbb{M})$ and the normal plane generated by $\{(N \wedge T)(u), N(u)\}$. This curve of intersection is tangent to $(N \wedge T)(u)$ at $v = 0$ and notice that $k_n(N \wedge T)(u) = k_2(u)$. Therefore,

$$\begin{aligned} \alpha(u, v, W(u, v)) &= (\alpha \circ c)(u) + v(N \wedge T)(u) \\ &\quad + \left[\frac{1}{2}k_2(u)v^2 + \frac{1}{6}A(u)6v^3 + v^3B(u, v) \right] N(u), \end{aligned}$$

where A is of class C^{r-5} and $B(u, 0) = 0$.

In the chart (u, v) constructed above it is obtained:

$$E(u, v) = 1 - 2k_g(u)v + h.o.t., \quad F(u, v) = 0 + 0.v + h.o.t.,$$

$$G(u, v) = 1 + 0.v + h.o.t.$$

$$e(u, v) = k_1(u) - 2k_g\mathcal{H}v + h.o.t., \quad f(u, v) = k_2'(u)v + h.o.t.$$

$$g(u, v) = k_2(u) + A(u)v + h.o.t.$$

where in the expressions above, $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$, $G = \langle \alpha_v, \alpha_v \rangle$, $e = \langle \alpha_u \wedge \alpha_v / |\alpha_u \wedge \alpha_v|, \alpha_{uu} \rangle$, $f = \langle \alpha_u \wedge \alpha_v / |\alpha_u \wedge \alpha_v|, \alpha_{uv} \rangle$, and $g = \langle \alpha_u \wedge \alpha_v / |\alpha_u \wedge \alpha_v|, \alpha_{vv} \rangle$.

The functions \mathcal{H} and \mathcal{K} are given by

$$\mathcal{H} = \frac{k_1 + k_2}{2} + \frac{1}{2}[(k_1 - k_2)k_g + A(u)]v + h.o.t.$$

$$\mathcal{K} = k_1k_2 + \frac{1}{2}[(k_1k_2 - k_2^2)k_g - k_1A(u)]v + h.o.t.$$

Therefore the principal curvatures $k_1 = \mathcal{H} - \sqrt{\mathcal{H}^2 - \mathcal{K}}$ and $k_2 = \mathcal{H} + \sqrt{\mathcal{H}^2 - \mathcal{K}}$ are given by

$$k_1(u, v) = k_1 + k_g(k_1 - k_2)v + h.o.t., \quad k_2(u, v) = k_2(u) + A(u)v + h.o.t.$$

This ends the proof. \square

Remark 3.3.1. *The following relations holds*

$$k_g = -\frac{(k_1)_v}{k_2 - k_1}, \quad k_g^\perp = -\frac{k_2'}{k_2 - k_1}, \quad (k_2)_v(u, 0) = A(u),$$

where k_g^\perp is the geodesic curvature of the maximal principal line passing through the point $c(u)$ positively oriented.

Proposition 3.3.1. *Let $c : [0, l] \rightarrow \mathbb{M}^2$ be a minimal principal cycle parametrized by arc length u and of length l . Then the derivative of the Poincaré map \mathcal{P} , associated to it is given by:*

$$\ln \mathcal{P}'(0) = \int_0^l \frac{-k_2'}{k_2 - k_1} du = -\frac{1}{2} \int_c \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} \quad (3.8)$$

where $\mathcal{H} = \frac{k_1 + k_2}{2}$ and $\mathcal{K} = k_1 k_2$ are respectively the Mean Curvature and the Gaussian Curvature.

Proof. The Darboux equations for the positive frame $\{t, N \wedge t, N\}$ are:

$$\begin{aligned} t'(u) &= k_g(u)(N \wedge t)(u) + k_1 N(u) \\ (N \wedge t)'(u) &= -k_g(u)t(u), \quad N'(u) = -k_1(u)t(u) \end{aligned} \quad (3.9)$$

Direct calculations gives that:

$$\begin{aligned} e(u, 0) &= k_1, \quad f(u, 0) = 0, \quad g(u, 0) = k_2, \\ f_v(u, 0) &= k_2', \quad F_v(u, 0) = 0, \quad G(u, 0) = E(u, 0) = 1. \end{aligned} \quad (3.10)$$

The differential equation of the curvature lines in the neighborhood of the line $\{v = 0\}$ is given by:

$$Ef - Fe + (Eg - Ge)\frac{dv}{du} + (Fg - Gf)\left(\frac{dv}{du}\right)^2 = 0 \quad (3.11)$$

Denote by $v(u, r)$ the solution of the (3.11) with initial condition $v(0, r) = r$. Therefore the return map \mathcal{P} is clearly given by $\mathcal{P}(r) = v(l, r)$.

Differentiating equation (3.11) with respect to r , and evaluating at $v = 0$, it results that:

$$[Eg - Ge](u, 0)v_{ur}(u, 0) + [Ef - Fe]_v(u, 0)v_r(u, 0) = 0 \quad (3.12)$$

Therefore, using the expressions for $[Ef - Fe]_v(u, 0)$ calculated in equation (3.10), integration of equation (3.12) along an arc $[u_0, u_1]$ of a minimal curvature line it follows that:

$$\frac{dv}{dr}\Big|_{r=0} = \exp\left[\int_{u_0}^{u_1} \frac{-k'_2}{k_2(u) - k_1(u)} du\right]. \quad (3.13)$$

Therefore,

$$\begin{aligned} \ln \mathcal{P}'(0) &= \int_0^l \frac{-k'_2}{k_2 - k_1} du = \int_0^l \left[-\frac{k'_2 - k'_1}{k_2 - k_1} du - \frac{k'_1}{k_2 - k_1} du \right] \\ &= \int_0^l \left[-[\ln(k_2 - k_1)]' du - \frac{k'_1}{k_2 - k_1} du \right] = \int_0^l \frac{-k'_1}{k_2 - k_1} du. \end{aligned}$$

$$\text{So,} \quad 2 \ln \mathcal{P}'(0) = \int_0^l -\frac{k'_1 + k'_2}{k_2 - k_1} du = \int_0^l -\frac{\mathcal{H}'}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} du.$$

This ends the proof. \square

Remark 3.3.2. *At this point we show how to extend the expression for the derivative of the hyperbolicity of principal curvature cycle established for class C^6 to class C^3 .*

The expression (3.13) is the derivative of the transition map for a principal curvature foliation (which at this point is only of class C^1), along an

arc of a principal curvature line. In fact, this follows by approximating the C^3 immersion by one of class C^6 . The corresponding transition map (now of class C^4) whose derivative is given by expression (3.8) converges to the original one (in class C^1) whose expression must be given by the same integral, since the functions involved there are the uniform limits of the corresponding ones for the approximating immersion.

Proposition 3.3.2. *Let $c : [0, l] \rightarrow \mathbb{M}^2$ be a principal cycle parametrized by arc length u and length l . Suppose that $dk_1|_c \neq 0$. Consider the deformation*

$$\alpha_\epsilon(u, v) = \alpha(u, v) + \epsilon \frac{k'_1}{2} v^2 \delta(v) N(c(u)) \quad (3.14)$$

where δ is a smooth function with small support and $\delta|_{V_0} = 1$. Then for all $\epsilon \neq 0$ small c is a hyperbolic principal cycle of α_ϵ .

Proof. Direct calculation shows that c is a principal cycle and that

$$\mathcal{P}'_\epsilon(0) = \exp \int_0^l -\frac{k'_1}{k_2 + \epsilon - k_1} du, \quad \frac{d\mathcal{P}'_\epsilon(0)}{d\epsilon} \Big|_{\epsilon=0} = \exp \int_0^l \frac{(k'_1)^2}{(k_2 - k_1)^2} du \neq 0.$$

This ends the proof. □

Proposition 3.3.3. *Let c be a hyperbolic principal cycle of length l . Then there exists a principal chart (u, v) , l -periodic in u such that differential equation of curvature lines in a neighborhood of c is given by $du(dv - \lambda du) = 0$, $\lambda = \exp\left(\int_c -\frac{dk_2}{k_2 - k_1}\right)$.*

Proof. See [52] and [53]. □

An immersion $\alpha \in \mathcal{I}^{r,s}$ is C^s -Principally Structurally Stable at a principal cycle c if for every neighborhood V_c of c in \mathbb{M} there must

be a neighborhood \mathcal{V}_α of α in $\mathcal{I}^{k,s}$ such that for every map $\beta \in \mathcal{V}_\alpha$ there must be a principal cycle c_β in V_c and a local homeomorphism h_β on the domain such that $h_\beta : W_c \rightarrow W_{c_\beta}$ between neighborhoods of c and c_β , which maps c to c_β and maps $\mathcal{P}_{1,\alpha}|W_c$ and $\mathcal{P}_{2,\alpha}|W_c$ respectively onto $\mathcal{P}_{1,\beta}|W_{c_\beta}$ and $\mathcal{P}_{2,\beta}|W_{c_\beta}$.

From the discussion above we have the following.

Proposition 3.3.4 (Gutierrez, Sotomayor, 1982). *An immersion $\alpha \in \mathcal{I}^r$, $r \geq 4$, is C^3 -principally structurally stable at a principal cycle c provided one of the following equivalent conditions, H_1 or H_2 , is satisfied:*

$$H_1) \quad \int_c \frac{dk_1}{k_2 - k_1} = \int_c \frac{dk_2}{k_2 - k_1} \neq 0$$

$H_2)$ *The cycle is a hyperbolic principal cycle of the principal foliation which it belongs. That is, the Poincaré return map \mathcal{P} associated to a transversal section to c at a point q is such that $\mathcal{P}'(q) \neq 1$.*

Remark 3.3.3. *The expressions for the higher derivatives of the Poincaré map \mathcal{P} near principal cycles have been established in [73] and [52].*

3.4 A Theorem on Principal Structural Stability

Next we will define the set $\mathcal{S}^r(\mathbb{M}) \subset \mathcal{I}^r$ such that:

- i) All the umbilic points, \mathcal{U}_α , of α are Darbouxian,
- ii) All principal cycles of α are hyperbolic,
- iii) The limit set of every principal line of α is the union of umbilic points and principal cycles,

- iv) There is no umbilic or singular separatrix of α which is separatrix of two umbilic or twice a separatrix of the same umbilic or singular point (i.e. homoclinic umbilic loops are not allowed).

An immersion $\alpha \in \mathcal{I}^{r,s}$ is said to be C^s -*Principally Structurally Stable* if there is a neighborhood \mathcal{V}_α of α in \mathcal{M} such that for every immersion $\beta \in \mathcal{V}_\alpha$ there exist a homeomorphism h_β on the domain such that $h_\beta(\mathcal{U}_\alpha) = \mathcal{U}_\beta$ and h_β maps lines of $\mathcal{P}_{1,\alpha}$, (resp. $\mathcal{P}_{2,\alpha}$) on those of $\mathcal{P}_{1,\beta}$ (resp. $\mathcal{P}_{2,\beta}$).

Theorem 3.4.1 (Gutierrez, Sotomayor, 1982). *Let $r \geq 4$ and \mathbb{M} be a compact oriented two manifold. Then*

- a) *The set $\mathcal{S}^r(\mathbb{M})$ is open in $\mathcal{I}^{r,3}$ and every $\alpha \in \mathcal{S}^r(\mathbb{M})$ is C^3 -principally structurally stable.*
 b) *The set $\mathcal{S}^r(\mathbb{M})$ is dense in $\mathcal{I}^{r,2}$.*

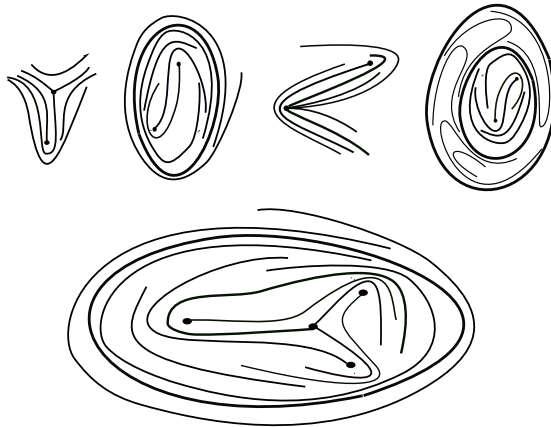


Figure 3.4: Pictorial types of principal configurations

A self sufficient presentation of this theorem was given in [75].

An open problem concerning the theorem 3.4.1 above is to prove (or disprove) that the set $\mathcal{S}^r(\mathbb{M})$ is dense in \mathcal{I}^r .³ The main point here is also related to the Closing-Lemma for Principal Curvature Lines. This is a problem that goes back to the works of Peixoto [130] and Pugh [139].

3.5 Remarks

The origin of the study of qualitative properties of principal lines on surfaces goes back to Monge, [113], [114], [166, page 95], who developed the theory of principal curvature lines motivated by the so called **Transport Problem**, see [175].

In the beginning of last century A. Gullstrand [70], ophthalmologist, developed and applied the theory of principal lines, focal sets and geometric optics to the study of the aberrations of human vision. By this achievements in this field he was awarded with the Nobel Prize (1911).

Also the fundamentals of *principal curvature lines* appears in the study of deformations of shells [134], computational and industrial geometry [168] and geometric theory of conservation laws [154].

3.6 Exercises and Problems

3.6.1. Consider the singular cubic surface defined by

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 + rxyz = 0, \quad (a - b)r \neq 0.$$

- i) Perform an analysis of the qualitative behavior of the principal foliations near the singular point $(0, 0, 0)$. See [54, 55].
- ii) Perform an analysis of the principal foliations near the ends of $f^{-1}(0)$.

3.6.2. Give an explicit example of an algebraic surface having a hyperbolic principal cycle for each principal foliation. See [55].

3.6.3. Consider the cubic surface

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 + rxyz - 1 = 0, \quad (a-1)(b-1)(a-b)r \neq 0.$$

- i) For $r \neq 0$ small study the umbilic points of $S = f^{-1}(0)$.
- ii) Perform simulations of the possible global behaviors of principal foliations of S for small r .
- iii) With the basis of ii) formulate a conjecture about the possibility of dense principal lines on algebraic surfaces of spherical type. See [72, 75] for smooth such surfaces.

3.6.4. Consider the algebraic surface

$$f(x, y, z) = z^2 - [(x - 2a)^2 + y^2 - a^2][(x + 2a)^2 + y^2 - a^2][r^2 - x^2 - y^2] = 0,$$

where $r > 4a$.

- i) Determine the umbilic set of $S = f^{-1}(0)$.
- ii) Determine all planar principal lines of S .
- iii) Using the symmetry of S obtain the principal configuration of S .
- iv) Visualize the shape of S .

3.6.5. Consider the space of quadrics \mathcal{Q} in \mathbb{R}^3 with the topology of coefficients. Define the concept of structural principal stability in this space.

- i) Determine the dimension of \mathcal{Q} .
- ii) Characterize the quadrics which are principally stable.
- iii) Show that the set of quadrics structurally stable \mathcal{S}_0 is open and dense in \mathcal{Q} .
- iv) Characterize the connected components of \mathcal{S}_0 .

3.6.6. In the space of quadrics \mathcal{Q} define the concept of first order structural principal stability. See [160, 158] for the case of vector fields on surfaces.

- i) Characterize the quadrics which are first order principally stable.
- ii) Characterize the connect components of \mathcal{S}_0 .
- iii) Characterize the set $\mathcal{Q} \setminus (\mathcal{S}_0 \cup \mathcal{S}_1)$. Here \mathcal{S}_1 is the set of quadrics which are first order principally stable.

3.6.7. Consider the implicit differential equation

$$(g - \mathcal{H}G)dv^2 + 2(f - \mathcal{H}F)dudv + (e - \mathcal{H}E)du^2 = 0.$$

Here $\mathcal{H} = (k_1 + k_2)/2$ is the arithmetic mean curvature. The integral curves of the equation above are called *arithmetic curvature lines*.

- i) Study the arithmetic curvature lines on the quadrics of \mathbb{R}^3 .
- ii) Determine the patterns of the arithmetic curvature lines near umbilic points and near closed arithmetic curvature lines. See [58, 59].

3.6.8. Given a biregular closed curve $c : [0, l] \rightarrow \mathbb{R}^3$ parametrized by arc length s . Let k and τ the curvature and the torsion of c .

- i) Show that there exists a surface containing c as a principal cycle if and only if $\int_0^l \tau(s)ds = 2k\pi$ for some $k \in \mathbb{Z}$ and c is non circular.
- ii) Show that $\int_0^l \tau(s)ds = 0$ for any biregular spherical curve.

3.6.9. Show that the conditions T and D_i that characterize Darbouxian umbilics are independent of coordinates. More precisely, consider two local charts $(u, v, h(u, v))$ and $(u_1, v_1, h_1(u_1, v_1))$ such that

$$h(u, v) = \frac{k}{2}(u^2 + v^2) + \frac{1}{6}(au^3 + 3buv^2 + 3b'u^2v + cv^3) + h.o.t$$

$$h_1(u_1, v_1) = \frac{k}{2}(u_1^2 + v_1^2) + \frac{1}{6}(a_1u_1^3 + 3b_1u_1v_1^2 + 3b'_1u_1^2v_1 + c_1v_1^3) + h.o.t$$

with $b' = b'_1 = 0$ and $(u_1, v_1) = (\cos \theta u + \sin \theta v, -\sin \theta u + \cos \theta v)$.

- i) Explicit the relations between the coefficients $(a_1, b_1, c_1, a, b, c, \theta)$.
- ii) Show that condition T is independent of coordinates, i.e., $b(b-a) \neq 0$ if, and only if, $b_1(b_1 - a_1) \neq 0$.
- iii) Analogously, show that conditions $D_i, (i=1,2,3)$ are independent of coordinates.
- iv) Show that in the case D_1 the angle θ is 0 or π .

3.6.10. Let $z = h(x, y)$ be a graph of a special Weingarten surface of class C^∞ such 0 is an isolated umbilic point.

- i) Show that if 0 is a Darbouxian umbilic then it is of type D_3 .
- ii) Show that if the special Weingarten surface is given by $z = h_4(x, y) + \dots$, where h_4 is a homogenous polynomial of degree 4 and 0 is an isolated umbilic then each principal foliation near 0 is equivalent to a topological saddle of index -1 .
- iii) Show that the index of an isolated umbilic point of a special Weingarten surface is negative.
- iv) Show that a special Weingarten surface of genus 0 is the sphere \mathbb{S}^2 .

Remark: A smooth surface is called a special Weingarten when there is a functional relation $F(\mathcal{H}, \mathcal{K}) = 0$ such that F is of class C^2 on the region $\mathcal{H}^2 \geq \mathcal{K}$ and when $\mathcal{H}^2 = \mathcal{K}$ it holds that $\frac{1}{2}F_{\mathcal{H}} + \mathcal{H}F_{\mathcal{K}} \neq 0$. See [85].

Chapter 4

Bifurcations of Umbilic Points and Principal Curvature Lines

4.1 Introduction

The local study of principal configurations around an umbilic point received considerable attention in the classical works of Monge [114], Cayley [27], Darboux [37] and Gullstrand [70], among others.

The study of the global features of principal configurations \mathcal{P}_α which remain topologically undisturbed under small perturbations of the immersion α –*principal structural stability*– was initiated by Gutierrez and Sotomayor in [71, 72, 75].

Two generic patterns of bifurcations of umbilic points appear in codimension one. The first one occurs due to the failure of the Darbouxian condition D , while T is preserved, leading to the pattern

called D_2^1 . See chapter 3

The second one occurs when condition T is violated, leading to the pattern denominated $D_{2,3}^1$.

This chapter is based in [142] and focuses only on the simplest bifurcations of umbilic points, referred to also as *codimension one* umbilics, since they appear in generic one-parameter families of immersed surfaces. Codimension two umbilics have been studied in [62].

4.2 Umbilic Points of Codimension One

The D_2^1 Umbilic Bifurcation Pattern

Here will be studied the qualitative changes - bifurcations - of the principal configurations around non Darbouxian umbilic points at which the regularity (or transversality) condition $T : b(a - b) \neq 0$, which implies their isolatedness, is preserved and only the condition D is violated in the mildest possible way.

Definition 4.2.1. *A point $p \in \mathcal{U}_\alpha$ is said to be of type D_2^1 if the following holds:*

T : The variety \mathbb{L}_α is regular along the projective line $\Pi^{-1}(p)$. In other words, the derivative of $\tau_{g,\alpha}$ does not vanish on the points of projective line $\Pi^{-1}(p)$. This means that the derivative in directions transversal to $\Pi^{-1}(p)$ do not vanish.

D_2^1 : The principal line fields $\mathcal{L}_{i,\alpha}$, $i = 1, 2$ lift to a single line field \mathcal{L}_α of class C^{r-3} , tangent to \mathbb{L}_α , which extends to a unique one along $\Pi^{-1}(p)$, and there it has a hyperbolic saddle singu-

larity and a saddle-node whose central line is located along the projective line over p .

In coordinates (u, v) , as in the notation above, this means that $T : b(a - b) > 0$ and either

$$1) a/b = (c/2b)^2 + 2, \quad \text{or} \quad 2) a/b = 2.$$

We point out that due to the particular representation of the 3-jets taken here, with $b' = 0$, the space a, b, c in the case 2) is not transversal, but tangent, to the manifold of jets of type D_2^1 umbilics.

These separatrices bound the parabolic sector of lines of curvature approaching the point; they also constitute the boundary of the hyperbolic sector of the umbilic point.

The bifurcation illustrated in Fig. 4.1 shows that the non-isolated separatrix disappears when the point D_2^1 changes to D_1 and that it turns into an isolated D_2 separatrix when it changes into D_2 . It can be said that D_2^1 represents the simplest transition between D_1 and D_2 Darbouxian umbilic points, which occurs through the annihilation of an umbilic separatrix – the non-isolated one –.

The coefficients of the differential equation of principal curvature lines (1.6) are given by:

$$\begin{aligned} L(u, v) &= -bv - (B/2)u^2 - (C - k^3)uv - (D/2)v^2 + M_1^3(u, v) \\ M(u, v) &= (b - a)u + cv + [(C - A)/2 + k^3]u^2 + (D - B)uv \\ &\quad + [(E - C)/2 - k^3]v^2 + M_2^3(u, v) \\ N(u, v) &= bv + (B/2)u^2 + (C - k^3)uv + (D/2)v^2 + M_3^3(u, v), \end{aligned} \tag{4.1}$$

with $M_i^3(u, v) = O((u^2 + v^2)^{\frac{3}{2}})$.

Condition D_2^1 is equivalent to the existence of a non zero double root for $bp^2 - cp + a - 2b = 0$, which amounts to $b \neq 0$ and $p_1 = p_2 \neq p_0$.

Assuming $b(b - a) \neq 0$, the curves $L = 0$ and $M = 0$ meet transversally at $(0, 0)$ if and only if $b \neq a$. It was shown in [71] that D_1 is satisfied if and only if the roots of $bp^2 - cp + a - 2b = 0$ are non vanishing and purely imaginary.

Also, D_2 is satisfied if and only if $bt^2 - ct + a - 2b = 0$ has two distinct non zero real roots, p_1, p_2 which verify $p_1 p_2 > -1$.

This means that the rays tangent to the separatrices are pairwise distinct and contained in an open right angular sector.

The local configuration of D_2^1 is established now.

Proposition 4.2.1. *Suppose that $\alpha \in \mathcal{I}^r$, $r \geq 5$, satisfies condition D_2^1 at an umbilic point p . Then the local principal configuration of α around p is that of Fig. 4.1 center.*

Proof. Consider the Lie-Cartan lifting X_α as in equation (3.4), which is of class C^{r-3} . If $a = 2b \neq 0$ and $c \neq 0$, it follows that $p_0 = (0, 0, 0)$ is an isolated singular point of quadratic saddle-node type with its center separatrix contained in the projective line –the p axis–. In fact, the eigenvalues of $DX_\alpha(0)$ are $\lambda_1 = -b \neq 0$ and $\lambda_2 = 0$ and the p axis is invariant; there X_α , according to equation (3.6) is given by $\dot{p} = -cp^2 + o(2)$.

The other singular point of X_α is given by $p_1 = (0, 0, \frac{c}{b})$. It follows that

$$DX_\alpha(0, 0, p_1) = \begin{bmatrix} -b & -c & 0 \\ -c & -\frac{c^2}{b} & 0 \\ A_1 & A_2 & \frac{c^2}{b} \end{bmatrix}$$

where,

$$A_1 = \frac{b^2c(A - k^3 - 2C) + bc^2(2B - D) + c^3(C - k^3) - b^3D}{b^3}$$

$$A_2 = \frac{b^2c(B - 2D) + bc^2(2C + k^3 - E) + b^3(k^3 - C) + Dc^3}{b^3}$$

The non zero eigenvalues of $DX(0, 0, p_1)$ are $\lambda_1 = \frac{c^2}{b}$, $\lambda_2 = -\frac{c^2+b^2}{b}$. In fact, p_1 is a hyperbolic saddle point of X_α having eigenvalues given by λ_1 and λ_2 .

Similar analysis can be done when $(\frac{c}{2b})^2 - \frac{a}{b} + 1 = 0$. In this case X_α and $p_1 = (0, 0, \frac{c}{b})$ is a quadratic saddle node, with a local center manifold contained in the projective line. The point $p_0 = (0, 0, 0)$ is a hyperbolic saddle of X_α . This case is equivalent to the previous one, after a rotation in (u, v) that sends de saddle-node to $p = 0$.

□

Remark 4.2.2. *The following structure on the structure of principal curvature lines has been achieved.*

The D_2^1 umbilic point has two separatrices.

The isolated one is characterized by the fact that no other principal line which approaches the umbilic point is tangent to it.

The other separatrix, called non-isolated, has the property that every principal line distinct from the isolated one, that approaches the point does so tangent to it.

Proposition 4.2.2. *Suppose that $\alpha \in \mathcal{I}^r, r \geq 5$, satisfies condition D_2^1 at an umbilic point p . Then there is a function \mathcal{B} of class C^{r-3} on a neighborhood \mathcal{V} of α and a neighborhood V of p such that every $\beta \in \mathcal{V}$ has a unique umbilic point p_β in V .*

- i) $d\mathcal{B}(\alpha) \neq 0$,
- ii) $\mathcal{B}(\beta) > 0$ if and only if p_β is Darbouxian of type D_1 ,
- iii) $\mathcal{B}(\beta) < 0$ if and only if p_β is Darbouxian of type D_2 ,
- iv) $\mathcal{B}(\beta) = 0$ if and only if p_β is of type D_2^1 .

The principal configurations of β around p is that of Fig. 4.1, left, right and center, respectively.

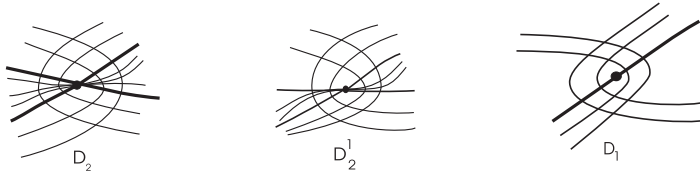


Figure 4.1: Umbilic Point D_2^1 and bifurcation

Proof. Since p is a transversal umbilic point of α , the existence of the neighborhoods \mathcal{V} and V of p_β follow from the Implicit Function Theorem. So we assume that after an isometry Γ_β of \mathbb{R}^3 , with $\Gamma_\beta \beta(0) = 0$, in the neighborhood V are defined coordinates (u, v) , also depending on β , on which it is represented as:

$$h_\beta(u, v) = \frac{k_\beta}{2}(u^2 + v^2) + \frac{a_\beta}{6}u^3 + \frac{b_\beta}{2}uv^2 + \frac{c_\beta}{6}v^3 + O(\beta; (u^2 + v^2)^4).$$

Define the function

$$\mathcal{B}(\beta) = \left[\frac{c_\beta}{2b_\beta} \right]^2 - \frac{a_\beta}{b_\beta} + 2,$$

whose zeros define locally the manifold of immersions with a D_2^1 point.

The derivative of this function in the direction of the coordinate a is clearly non-zero. \square

The $D_{2,3}^1$ Umbilic Bifurcation Pattern

The second case of non-Darbouxian umbilic point studied here, called $D_{2,3}^1$, happens when the regularity condition T is violated.

Definition 4.2.3. *An umbilic point is said of type $D_{2,3}^1$ if the transversality condition T fails at two points over the umbilic point, at which \mathbb{L}_α is non-degenerate of Morse type.*

Proposition 4.2.3. *Suppose that $\alpha \in \mathcal{I}^r$, $r \geq 5$, and p be an umbilic point. Assume the notation in equation (3.1) with $b' = 0$, $b = a \neq 0$ and $\chi = b(C - A + 2k^3) - cB \neq 0$.*

Then p is of type $D_{2,3}^1$ and the local principal configuration of α around p is that of Fig. 4.2, bottom.

Proof. Consider the Lie-Cartan lifting X_α as in equation (3.4), which is of class C^{r-3} . Imposing $a = b \neq 0$, by equations (3.5) and (3.6), the singular points of X_α are p_0 , p_1 and p_2 , roots of the equation $p(bp^2 - cp - b) = 0$.

In fact, if $a = b \neq 0$, it follows that p_0 is a quadratic saddle node with center manifold transversal to the projective line.

From equation (3.6), the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -b$ and all the center manifolds W^c are tangent to the line $p = -\frac{B}{b}u$. By invariant manifold theory it follows that $X|_{W^c}$ is locally topologically equivalent to

$$\dot{u} = \frac{1}{2} \frac{[b(C - A + 2k^3) - cB]}{b} u^2 + o(2) := -\frac{\chi}{2b} u^2 + o(2). \quad (4.2)$$

It follows that

$$DX_\alpha(0, 0, p_i) = \begin{bmatrix} 0 & -2bp_i + c & 0 \\ 0 & -p_i(2bp_i - c) & 0 \\ B_1 & B_2 & 3bp_i^2 - 2cp_i - b \end{bmatrix}$$

where,

$$B_1 = (C - k^3)p_i^3 + (2B - D)p_i^2 + (A - 2C - k^3)p_i - B$$

$$B_2 = Dp_i^3 + (2C + k^3 - E)p_i^2 + (B - 2D)p_i + k^3 - C.$$

The nonzero eigenvalues of $DX_\alpha(0, 0, p_i)$ are:

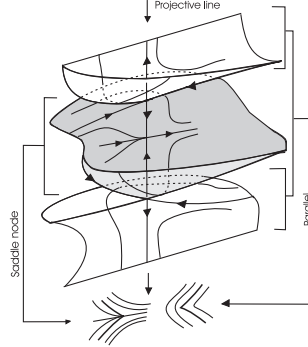
$$\lambda_1 = -2bp_i^2 + cp_i = -b(p_i^2 + 1) \text{ and } \lambda_2 = 3bp_i^2 - 2cp_i - b = b(p_i^2 + 1).$$

By invariant manifold theory, at $(0, 0, p_i)$ has two hyperbolic sectors for X_α , restricted to the conic variety. The phase portrait of X_α near these singularities are as shown in Fig. 4.2.

The two *critical* points p_1 and p_2 are of conic type on the variety \mathbb{L}_α over the umbilic point.

These points are non-degenerate or of *Morse* type, according to the analysis below. At the points $(0, 0, p_i)$ the variety $\mathcal{T}(u, v, p) = 0$ is not regular. In fact:

$\nabla\mathcal{T}(0, 0, p) = [(b - a)p, -bp^2 + cp + b, 0]$. Therefore, for $a = b \neq 0$, at the two roots of the equation $-bp^2 + cp + b = 0$ it follows that $\nabla\mathcal{T}(0, 0, p_i) = (0, 0, 0)$, $i = 1, 2$.

Figure 4.2: Lie-Cartan suspension $D_{2,3}^1$

The Hessian of \mathcal{T} at $\mathbf{p}_i = (0, 0, p_i)$ is

$$Hess(\mathcal{T})(\mathbf{p}_i) = \begin{bmatrix} \frac{p_i(-cB+b(C+2k^3-A))}{b} & \frac{p_i(c(k^3-C)+b(D-B))}{b} & 0 \\ \frac{p_i(c(k^3-C)+b(D-B))}{b} & -\frac{p_i(cD+b(C-E+2k^3))}{b} & c-2bp_i \\ 0 & c-2bp_i & 0 \end{bmatrix}.$$

Direct calculation, using the notation defined in equation (4.2), gives

$$\det(Hess(\mathcal{T})(0, 0, p_i)) = \frac{p_i(-2bp_i + c)^2\chi}{b} = \frac{p_i}{b}(4b^2 + c^2)\chi \neq 0.$$

Therefore, $(0, 0, p_i)$ is a non degenerate critical point of \mathcal{T} of Morse type and index 1 or 2 – a *cone*–, since $\mathcal{T}^{-1}(0)$ contains the projective line. \square

Remark 4.2.4. *Our analysis has shown the equivalence between the conditions a) and b) that follow:*

a) *The non-vanishing on the Hessian of \mathcal{T} on the critical points p_1 and p_2 over the umbilic.*

b) *The presence of a saddle-node at p_0 on the regular portion of the variety \mathbb{L}_α , with central separatrix transversal to the projective line over the umbilic.*

Further direct calculation with equation (4.1) gives that these two conditions are equivalent to

c) *The quadratic contact at the umbilic between the curves $M = 0$ and $N = 0$.*

In fact, from equation (4.1) it follows that $M(u, v(u)) = 0$ for $v = -(B/2b)u^2 + o(2)$ of class C^{r-2} . Therefore $n(u) = N(u, v(u))$ is of class C^{r-2} and $n(u) = -(\chi/2b)u^2 + o(2)$.

Notice also that, unlike the other umbilic points discussed here, the two principal foliations around $D_{2,3}^1$ are topologically distinct.

One of them, located on the *parallel sheet*, has two umbilic separatrices and two hyperbolic sectors

The other, located on the *saddle-node sheet*, has three umbilic separatrices, one parabolic and two hyperbolic sectors.

The separatrix which is the common boundary of the hyperbolic sectors will be called *hyperbolic separatrix*. See Figs. 3.1, 4.1 and Fig. 4.2 for illustrations.

The bifurcation analysis describes the elimination of two umbilic points D_2 and D_3 which, under a deformation of the immersion, collapse into a single umbilic point $D_{2,3}^1$, and then, after a further suitable arbitrarily small perturbation, the umbilic point is annihilated.

Proposition 4.2.4. *Suppose that $\alpha \in \mathcal{I}^r$, $r \geq 5$, satisfies condition $D_{2,3}^1$ at an umbilic point p . Then there is a function \mathcal{B} of class C^{r-3} on a neighborhood \mathcal{V} of α and a neighborhood V of p such that*

i) $d\mathcal{B}(\alpha) \neq 0$,

ii) $\mathcal{B}(\beta) > 0$ if and only if β has no umbilic points in V ,

iii) $\mathcal{B}(\beta) < 0$ if and only if β has two Darbouxian umbilic points of types D_2 and D_3 ,

iv) $\mathcal{B}(\beta) = 0$ if and only if β has only one umbilic point in V , which is of type $D_{2,3}^1$.

The principal configurations of β around p are illustrated in Fig. 4.3, right, left and center, respectively.

Proof. Similar to that given in [158, page 15], for the saddle-node of vector fields, using the equivalence c) of remark 4.2.4. We define \mathcal{B} as follows. An immersion β in a neighborhood \mathcal{V} of α and a neighborhood V of p can be written in a Monge chart as a graph of a function $h_\beta(u, v)$. The umbilic points of β are defined by the equation

$$\begin{aligned} M_\beta &= (1 + ((h_\beta)_u)^2)(h_\beta)_{vv} - (1 + ((h_\beta)_v)^2)(h_\beta)_{uu} = 0 \\ N_\beta &= (1 + ((h_\beta)_u)^2)(h_\beta)_{uv} - (h_\beta)_u(h_\beta)_v(h_\beta)_{uu} = 0. \end{aligned} \tag{4.3}$$

For β in a neighborhood of α it follows that $M_\beta(u, v_\beta(u)) = 0$.

Define $\mathcal{B}(\beta) = n_\beta(u_\beta)$, where u_β is the only critical point of $n_\beta(u) = N_\beta(u, v_\beta(u))$.

Taking $h_\beta(u, v) = h(u, v) + \lambda uv$, where h is as in equation (3.1) it follows by direct calculation that $\frac{d\mathcal{B}(\beta)}{d\lambda}|_{\lambda=0} \neq 0$. \square

The bifurcation of the point $D_{2,3}^1$ can be regarded as the simplest transition between umbilics D_2 and D_3 and non umbilic points. See the illustration in Fig. 4.3, where the maximal and minimal foliations have been drawn separately.

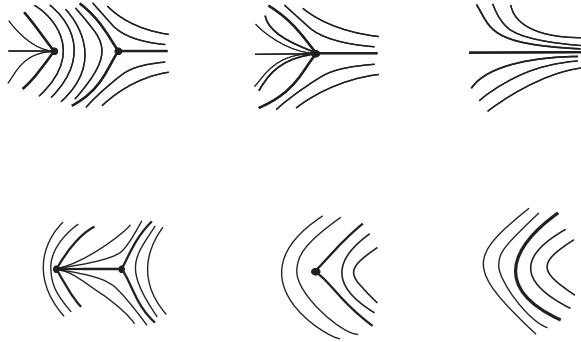


Figure 4.3: Umbilic Point $D_{2,3}^1$ and bifurcation.

4.3 Remarks

For immersions $\alpha : \mathbb{M}^3 \rightarrow \mathbb{R}^4$ there are three principal foliations $\mathcal{F}_i(\alpha)$ which are mutually orthogonal. Here two kind of singularities of the principal line fields $\mathcal{L}_i(\alpha)$ ($i = 1, 2, 3$) can appear. Define the sets, $\mathcal{U}(\alpha) = \{p \in \mathbb{M}^3 : k_1(p) = k_2(p) = k_3(p)\}$, $\mathcal{P}_{12}(\alpha) = \{p \in \mathbb{M}^3 : k_1(p) = k_2(p) \neq k_3(p)\}$, $\mathcal{P}_{23}(\alpha) = \{p \in \mathbb{M}^3 : k_1(p) \neq k_2(p) = k_3(p)\}$ and $\mathcal{P}(\alpha) = \mathcal{P}_{12}(\alpha) \cup \mathcal{P}_{23}(\alpha)$.

The sets $\mathcal{U}(\alpha)$, $\mathcal{P}(\alpha)$ are called, respectively, *umbilic set* and *partially umbilic set* of the immersion α .

Generically, for an open and dense set of immersions $\mathcal{U}(\alpha) = \emptyset$ and $\mathcal{P}(\alpha)$ is either, a submanifold of codimension two or the empty set.

A connected component of $\mathcal{S}(\alpha)$ is called a *partially umbilic curve*.

The study of the principal foliations near $\mathcal{S}(\alpha)$ were carried out in [49, 50], where the local model of the asymptotic behavior of lines of principal curvature was analyzed in the generic case.

Fig. 4.4 shows the qualitative behavior of principal foliations near typical singularities, the partially umbilic lines.

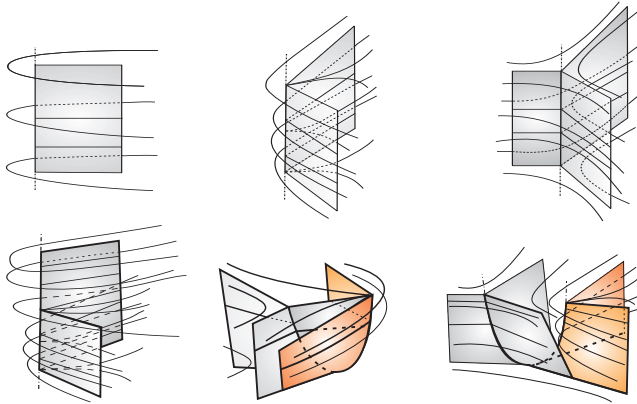


Figure 4.4: Behavior of a principal foliation in the neighborhood of a partially umbilic line and near transition points

4.4 Exercises and Problems

4.4.1. Provide suitable deformations of the surface $f(x, y, z) = x^4 + y^4 + z^4 - 1 = 0$ such that the surface obtained has exactly 20 Darbouxian umbilics, 8 of type D_3 and 12 of type D_1 .

4.4.2. Consider the cubic surface defined by

$$f(x, y, z) = x^2 + y^2 + z^2 + rxyz - 1 = 0, \quad r \neq 0.$$

i) Write the equation for the umbilic points of $f^{-1}(0)$.

ii) Determine the Darbouxian umbilics of $f^{-1}(0)$.

4.4.3. Study the principal curvature lines near non hyperbolic principal cycles and give an example of a surface having a semihyperbolic principal cycle. See [73] and [52].

4.4.4. Give examples of smooth surfaces having separatrix connections (homoclinic and heteroclinic) between Darbouxian umbilic points. For example in the ellipsoid we have connections between separatrices of Darbouxian umbilics of type D_1 . See [74].

4.4.5. Give examples of smooth surfaces with the umbilic set containing regular curves. Analyze the behavior of principal lines near these curves of umbilic points. This study goes back to Caratheodory; see [51] and [60].

4.4.6. Give an example of a surface, homeomorphic to \mathbb{S}^2 , having exactly one singular point and no umbilic points.

Suggestion: Consider the surface $z(x, y) = 2 + \frac{xy}{\sqrt{x^2+y^2}}$ and its inversion with respect to a sphere. See [13].

4.4.7. Give an example of a canal surface, homeomorphic to the torus, having no umbilic points, one principal foliation having all principal lines closed and other principal foliation having all principal lines dense.

4.4.8. Consider the surfaces

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad g(x, y, z) = \frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} - 1 = 0.$$

Suppose that $f^{-1}(0) \cap g^{-1}(0)$ is the union of principal curvature lines on both surfaces.

i) Analyze the behavior of the principal foliations of the surface defined by $h_\epsilon(x, y, z) = f(x, y, z)g(x, y, z) - \epsilon = 0$ for $\epsilon \neq 0$ small.

ii) Visualize the shape of $h_\epsilon^{-1}(0)$. Make simulations using the free software developed by A. Montesinos (Univ. of Valencia, Spain), [116].

4.4.9. Consider the surface $\alpha_\lambda(u, v) = (u, v, \sin^2(u^2 + \lambda v^2))$, $\lambda \in \mathbb{R}$.

- i) Compute the umbilic points of α_λ and analyze the local behavior of the principal configuration for various values of λ .
- ii) Analyze the global behavior of the principal configuration of α_λ for various values of λ .

4.4.10. Let $\alpha(u, v) = (u, v, \frac{k}{2}(u^2 + v^2) + \frac{1}{6}(cv^3 + Bu^3v))$, $cB \neq 0$.

- i) Show that 0 is an isolated umbilic point of α and compute all umbilic points of α .
- ii) Describe the principal configuration of α near 0. See [62].
- iii) Provide suitable deformations of α to obtain umbilic points of types D_1 , D_2 , D_3 , $D_{2,3}^1$ and D_2^1 .

Chapter 5

Lines of Principal Curvature around Whitney Umbrellas

5.1 Introduction

In this chapter are studied the configurations of lines of curvature near a Stable Singularity for maps of surfaces into the space (Whitney Umbrella). The pattern of such configurations is established and characterized in terms of the 3-jet of the map.

The bending or curvature pattern of a smooth mapping $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^3$, where \mathbb{M}^2 is a compact oriented two dimensional manifold, will be represented here by the singular points, \mathcal{S}_α , at which the mapping has rank less than 2 and the bending can be regarded to be infinite; the umbilic points \mathcal{U}_α at which the bending is finite but equal in all directions and by the family of lines of principal curvature $\mathcal{P}_{1,\alpha}$ and

$\mathcal{P}_{2,\alpha}$ defined on $\mathbb{M} \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$ which represent the directions along which the bending, or more precisely, the normal curvature, is extremal: minimal along $\mathcal{P}_{1,\alpha}$ and maximal along $\mathcal{P}_{2,\alpha}$.

These four objects will be assembled into the *principal configuration* of the mapping denoted by $\mathbb{P}_\alpha = (\mathcal{S}_\alpha, \mathcal{U}_\alpha, \mathcal{P}_{1,\alpha}, \mathcal{P}_{2,\alpha})$. The points of \mathcal{S}_α and \mathcal{U}_α are regarded as the singularities of the foliations $\mathcal{P}_{1,\alpha}$ and $\mathcal{P}_{2,\alpha}$.

5.2 Preliminaries

Call by $\mathcal{I}^r = \mathcal{I}^r(\mathbb{M}^2, \mathbb{R}^3)$ the space of C^r mappings of M into \mathbb{R}^3 . When endowed with the C^s topology $s \leq r$, this space will be denoted by $\mathcal{I}^{r,s}$.

Denote by \mathcal{S}_α the set of singular points of α ; that is, where $D\alpha_p$ has rank ≤ 1 . Call \mathcal{U}_α the set of umbilic points of α , i.e., where the *second fundamental form* $II_\alpha(p) = - \langle DN_\alpha(p), D\alpha(p) \rangle$ is proportional to the *first fundamental form* $I_\alpha(p) = - \langle D\alpha(p), D\alpha(p) \rangle$. Here \langle, \rangle is the Euclidean metric on \mathbb{R}^3 and $N_\alpha : \mathbb{M} \setminus \mathcal{S}_\alpha \rightarrow \mathbb{S}^2$ is the normal map of α defined by: $N_\alpha(p) = \alpha_u \wedge \alpha_v / \|\alpha_u \wedge \alpha_v\|$, where $(u, v) : \mathbb{M} \rightarrow \mathbb{R}^2$ is a positive local chart of \mathbb{M} around p , \wedge denotes the exterior product of vectors in \mathbb{R}^3 , determined by a once for all fixed orientation of \mathbb{R}^3 , $\alpha_u = \partial\alpha/\partial u$, $\alpha_v = \partial\alpha/\partial v$ and $\|\cdot\| = \langle, \rangle^{1/2}$ is the Euclidean norm of \mathbb{R}^3 .

Finally, $\mathcal{P}_{1,\alpha}$, (respectively $\mathcal{P}_{2,\alpha}$) denotes the foliation on $\mathbb{M} \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$ by the family of curves of minimal (respectively maximal principal curvature of α). This means that at each point $p \in \mathbb{M} \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$ any vector $v \neq 0$ which spans the line $\mathcal{L}_{1,\alpha}$ (respectively $\mathcal{L}_{2,\alpha}$) tan-

gent to $\mathcal{P}_{1,\alpha}$, (respectively $\mathcal{P}_{2,\alpha}$) provides the minimum $k_{1,\alpha}$ (re-
spec. maximum $k_{2,\alpha}$,) of the normal curvature k_n at p, $k_n(p, u) =$
 $II_a(p)(u, u)/I_a(p)(u, u)$, among all possible directions $u \in T_p\mathbb{M} \setminus \{0\}$.

The function $k_{1,\alpha}$ (respec. $k_{2,\alpha}$) on $\mathbb{M} \setminus \mathcal{S}_\alpha$ is called the minimal
(respectively maximal) principal curvature of α . It is of class C^{r-2}
on $\mathbb{M} \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$.

In a local chart (u, v) , [164], [166] the principal line fields $\mathcal{L}_{1,\alpha}$
and $\mathcal{L}_{2,\alpha}$ for maps are expressed implicitly, by the following quadratic
differential equation:

$$(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2 = 0$$

where $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$ and $II_\alpha = edu^2 + 2fdudv + gdv^2$
are respectively the first and the second fundamental forms of α .

A smooth map $\alpha : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ sending the origin to the
origin is said to be *regular* if $j^1\alpha$ has rank 2; otherwise it is called
singular.

The mapping α is said to have a Whitney Umbrella at 0 provided
it has rank 1 and its first jet extension $j^1\alpha$ is transversal to the
codimension 2 submanifold $S^1(2, 3)$ of 1-jets of rank 1 in $J^1(2, 3)$.
See exercise 5.7.13. In coordinates this means that there exist a local
chart (u, v) such that $\alpha_u(0) \neq 0$, $\alpha_v(0) = 0$ and $[\alpha_u, \alpha_{uv}, \alpha_{vv}] \neq 0$.
Here $[\cdot, \cdot, \cdot]$ means the determinant of the three vectors.

The structure of a smooth map near such point is illustrated in
Fig. 5.1

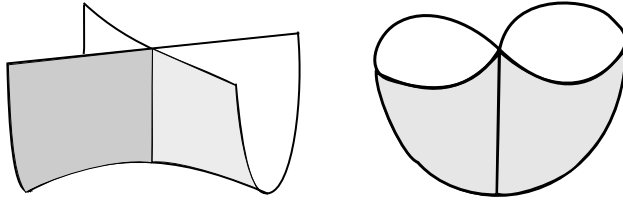


Figure 5.1: Whitney Umbrella Singularity (Hyperbolic and Elliptic)

5.3 Curvature Lines near Whitney Umbrella Singularities

In this section the behavior of the principal curvature lines near a Whitney umbrella singular point will be obtained. Denote by $J^r(2, 3)$ the space of r -jets of smooth mappings of \mathbb{R}^2 to \mathbb{R}^3 , sending the origin to the origin. On this space consider the action of the group \mathcal{G}^r generated by r - jets of smooth diffeomorphisms in the domain and positive isometries and homoteties in the target.

Proposition 5.3.1. *Let $\alpha : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a C^r , $r \geq 5$, map with a Whitney umbrella at 0. Then by the action of the groups \mathcal{G}^r generated by r - jets of smooth diffeomorphisms in the domain and rotations and homoteties of \mathbb{R}^3 , the map α can be written in the*

following form:

$$\begin{aligned}
 x(u, v) &= u \\
 y(u, v) &= uv + \frac{a}{6}v^3 \\
 &\quad + \frac{1}{24}[a_{40}u^4 + 4a_{31}u^3v + 6a_{22}u^2v^2 + 4a_{13}uv^3 + a_{04}v^4] + o(5) \\
 z(u, v) &= \frac{b}{2}u^2 + cuv + v^2 + \frac{A}{6}u^3 + \frac{B}{2}u^2v + \frac{C}{2}uv^2 + \frac{D}{6}v^3 \\
 &\quad + \frac{1}{24}[b_{40}u^4 + 4b_{31}u^3v + 6b_{22}u^2v^2 + 4b_{13}uv^3 + b_{04}v^4] + o(5)
 \end{aligned} \tag{5.1}$$

where $o(5)$ means terms of order greater than or equal to five.

Proof. By the rank 1 condition imposed at 0, we can find a rotation $R : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ and a diffeomorphism $h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $\alpha_1(u, v) = (R \circ \alpha \circ h)(u, v) = (u, s(u, v), t(u, v))$ with $Ds(0) = Dt(0) = 0$. Now using the condition $[\alpha_u, \alpha_{uv}, \alpha_{vv}](0) \neq 0$ we can eliminate the term v^2 of s . More precisely there exists a rotation R_x of \mathbb{R}^3 fixing the x axis such that the following holds.

$$\begin{aligned}
 \alpha_2(u, v) &= (R_x \circ \alpha_1)(u, v) \\
 &= (u, a_1uv + a_2u^2 + s_3(u, v), b_1u^2 + b_2uv + b_3v^2 + t_3(u, v)),
 \end{aligned}$$

where s_3 and t_3 are of order three or more.

Now define an affine change of coordinates in the domain of the form $H(u, v) = (u, a_2u + a_1v)$ to obtain

$$\begin{aligned}
\alpha_3(u, v) &= (\alpha_2 \circ H)(u, v) = \\
&= (u, uv + c_1u^3 + c_2u^2v + c_3uv^2 + c_4v^3 + S_4(u, v), \\
&d_1u^2 + d_2uv + d_3v^2 + T_3(u, v))
\end{aligned}$$

where $[\alpha_{3u}, \alpha_{3uv}, \alpha_{3vv}] = 2d_3 \neq 0$ and S_4 and T_3 are terms of order 4 and 3 respectively.

By a local diffeomorphism $(u, v) \rightarrow (u, v + l_1u^2 + l_2uv + l_3v^2)$ we can reduce α_3 to:

$$\alpha_4(u, v) = (u, uv + Av^3 + \bar{S}_4(u, v), d_1u^2 + d_2uv + d_3v^2 + \bar{T}_3(u, v)).$$

Finally, rescaling the target by the homotety $r(x, y, z) = \epsilon(x, y, z)$ and the domain by the affine map $p(u, v) = (\delta u, v)$, with $\epsilon\delta = 1$ we obtain:

$$\begin{aligned}
\alpha_5(u, v) &= (r \circ \alpha_4 \circ p)(u, v) \\
&= (u, uv + \frac{a}{6}v^3 + y_4(u, v), \frac{b}{2}u^2 + cuv + v^2 + z_3(u, v)).
\end{aligned}$$

This ends the proof. □

Remark 5.3.1. *In J. West [177] a more elaborate and precise normal form is obtained concerning the structure of the fourth order terms. In fact in equation (5.1) it is possible to obtain $a_{40} = a_{31} = a_{22} = a_{13} = 0$.*

Remark 5.3.2. *The change of coordinates in proposition 5.3.1 above does not modify the geometry of the principal configuration of the map α at the singular point.*

An elementary, interesting, application of this proposition follows. See also exercise 5.7.14.

Proposition 5.3.2. *Let $\alpha : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a C^r , $r \geq 5$, map with a Whitney umbrella at 0 given by equation (5.1). Then the curve of double points of α in the domain is given by*

$$u = -\frac{a}{6}v^2 + \frac{1}{36}a(ac - D)v^3 + o(4).$$

Proof. Let $\alpha(u, v) = \alpha(U, V)$. Then $u = U$. Using the equation (5.1) with

$$y(u, v) = uv + \frac{a}{6}v^3 + \frac{1}{24}[a_{40}u^4 + 6a_{31}u^3v + 4a_{22}u^2v^2 + 6a_{13}uv^3 + a_{04}v^4] + o(5)$$

it follows that

$$\begin{aligned} &4a_{31}u^3 + 6a_{22}(v + V)u^2 + 4a_{13}(v^2 + vV + V^2)u + 24u \\ &+ a_{04}(v^3 + v^2V + vV^2 + V^3) + 4a(v^2 + vV + V^2) + o(4) = 0. \end{aligned}$$

By the Implicit Function Theorem it follows that

$$u = -\frac{a}{6}(v^2 + vV + V^2) - \frac{a_{04}}{24}(v^3 + v^2V + vV^2 + V^3) + o(4). \quad (5.2)$$

From $z(u, v) = z(U, V) = z(u, V)$ of equation (5.1) and equation above it follows that

$$-6(v + V) + (ac - D)(v^2 + vV + V^2) + o(3) = 0.$$

Therefore by Implicit Function Theorem it follows that

$$v = -V + \frac{1}{6}(ac - D)V^2 + o(3), \quad \text{or} \quad V = -v + \frac{1}{6}(ac - D)v^2 + o(3).$$

Substituting in equation (5.2) the result follows. \square

An interesting calculation of the curvature and torsion of the image of the double curve is proposed in exercise 5.7.14.

The differential equation of the lines of curvature of the map α around a Whitney Umbrella Singular point $(0, 0)$, as in proposition 5.3.1, is given by:

$$\begin{aligned}
H(u, v, du, dv) &= \\
&= [2bcu^3 + 12cuv^2 + (4 + 4b + 4c^2)u^2v + 8v^3 + l_4(u, v)]dv^2 \\
&+ [2u + Cu^2 + (D - ac)uv - av^2 + (b_{13} - ca_{13} - \frac{1}{2}aB)u^2v \\
&+ (2b^2 - b - 2bc^2 + \frac{1}{3}a_{31} - \frac{1}{2}ca_{22} + \frac{1}{2}b_{22})u^3 \\
&+ \frac{1}{2}(4 - 8b + 4c^2 - aC + b_{04} - ca_{04} - 2a_{13})uv^2 - \frac{2a_{04}}{3}v^3 + m_4(u, v)]dudv \\
&+ [-2v + \frac{B}{2}u^2 + \frac{1}{2}(ac - D)v^2 + (\frac{1}{3}a_{31} - \frac{1}{3}cb_{31} - b^2c)u^3 \\
&- (b + 4b^2 + bc^2 + \frac{1}{2}a_{22} - b_{31} - \frac{1}{2}cb_{22})u^2v + \frac{1}{2}(aB - 12bc - 4b_{22})uv^2 \\
&+ (\frac{1}{2}aC - 2c^2 - 2 - b_{13} - \frac{1}{6}a_{04} + \frac{1}{6}cb_{04})v^3 + n_4(u, v)]du^2 = 0
\end{aligned} \tag{5.3}$$

Proposition 5.3.3. *The implicit surface $H^{-1}(0)$ given by equation (5.3) defines in a punctured neighborhood of $(0, 0)$ the principal line fields $\mathcal{L}_{1,\alpha}$ and $\mathcal{L}_{2,\alpha}$. These line fields extend uniquely to the projective line $\mathbb{P} = (0, 0, [du : dv])$ so that:*

- a) *The extension is regular of class C^{r-2} in $\mathbb{P} \setminus (0, 0, [0 : 1])$ and it is singular at the point $(0, 0, [0 : 1])$.*

b) The singular point $(0, 0, [0 : 1])$ of $H^{-1}(0)$ is of corank 1 and codimension 1. At this point H is locally C^{r-3} equivalent to $uq + 4v^3 = 0$, where $q = du/dv$. Therefore the topological structure of $H^{-1}(0)$ near this singular point is as shown in Fig. 5.2.

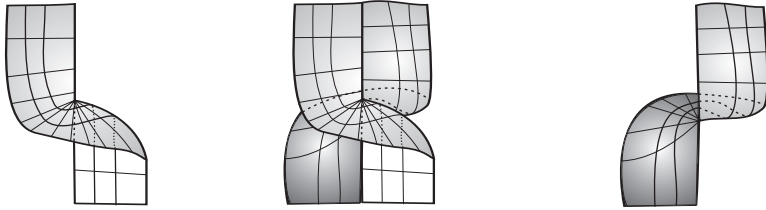


Figure 5.2: Surface $H^{-1}(0)$ near the singular point of corank 1

Proof. In the chart (u, v, p) , $\frac{dv}{du}$ it follows that $\frac{\partial H}{\partial v}(0, 0, p) = -2$. Therefore by the Implicit Function Theorem the surface is regular.

In the chart (u, v, q) , $q = du/dv$, it follows that $\text{grad } H(0, 0, 0) = (0, 0, 0)$.

Direct calculations shows that $\text{rank}(\text{Hess}H) = 2$, $\ker(\text{Hess}H)(0) = (0, 1, 0)$ and $j^3 H|_{\{\ker(\text{Hess}H)(0)\}} \neq 0$. So, by the Parametrized Morse Lemma, [39], H is C^{r-3} equivalent to $uq + 4v^3$. \square

Proposition 5.3.4. *Let $\alpha : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a mapping with a Whitney umbrella as presented in expression (5.1).*

Then outside any sector given by $Q = \{(u, v) : (ev)^2 - u^2 \geq 0\}$ the principal foliations $\mathcal{P}_{1,\alpha}$ and $\mathcal{P}_{2,\alpha}$ are as shown in Fig. 5.3.

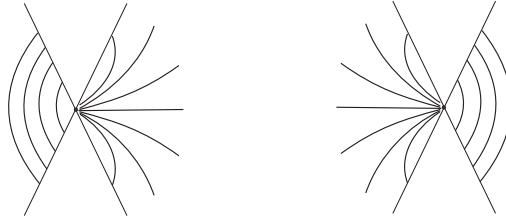


Figure 5.3: Principal foliations: a) $\mathcal{P}_{1,\alpha}$ b) $\mathcal{P}_{2,\alpha}$

Proof. Consider the implicit differential equation of lines of curvature

$$\begin{aligned} H(u, v, p) = & [2bcu^3 + 12cuv^2 + (4 + 4b + 4c^2)u^2v + 8v^3 + l_4(u, v)]p^2 \\ & + [2u + Cu^2 + (D - ac)uv - av^2 + m_3(u, v)]p \\ & + [-2v + \frac{B}{2}u^2 + \frac{1}{2}(ac - D)v^2 + n_3(u, v)] = 0 \end{aligned}$$

in the projective chart (u, v, p) , where $p = dv/du$.

By the Implicit Function Theorem the surface is regular at all points $(0, 0, p)$.

The Lie-Cartan vector field $Z = (H_p, pH_p, -(H_u + pH_v))$, see equation (1.7), is given by:

$$\begin{aligned}
u' &= [2bcu^3 + 12cuv^2 + (4 + 4b + 4c^2)u^2v + 8v^3 + l_3(u, v)]2p \\
&\quad + [2u + Cu^2 + (D - ac)uv - av^2 + m_3(u, v)] \\
v' &= p\{[2bcu^3 + 12cuv^2 + (4 + 4b + 4c^2)u^2v + 8v^3 + l_3(u, v)]2p \\
&\quad + [2u + Cu^2 + (D - ac)uv - av^2 + m_3(u, v)]\} \\
p' &= -\{[Bu + o_1(2)] + p[2Cu + o_2(2)] \\
&\quad + p^2[(D - ac)u + o_3(2)] + p^3[o_4(2)]\}
\end{aligned}$$

In this case the projective line $(0, 0, p)$ is normally hyperbolic for the vector field Z .

Next consider the blowing-up $u = s$, $v = st$. So, $du = ds$ and $dv = sdt + tds$. Therefore the differential equation (5.3) in the variables (s, t) is given by:

$$[(2bc + o(1))s^3]dt^2 + [2 + s(C + o(1))]dsdt + \left[\frac{B}{2} + o(1)\right]ds^2 = 0 \quad (5.4)$$

where $o(1)$ means functions of order 1.

In a neighborhood of $s = 0$ the differential equation (5.3) defines two regular foliations, one having the axis t as an integral curve and other foliation is transversal to this axis. So, outside any angular sector $Q = \{(u, v) : (\epsilon v)^2 - u^2 \geq 0\}$ containing the v axis the principal foliations are as shown in Fig. (5.3). This picture assumes that the domain and target in expression (5.1) have the positive orientation. An inversion in either of these orientations will produce an exchange in the principal foliations in the picture. \square

The analysis of the differential equation (5.3) in a sector containing the v axis is carried out in what follows.

Proposition 5.3.5. *Let $\alpha : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a mapping with a Whitney umbrella as presented in expression (5.1). Consider the blowing-up $\varphi(t, r) = (r^2 \sin t, r \cos t)$. Then in the coordinates (t, r) the differential equation (5.3) is given by:*

$$[L_0(t) + o(r)]dr^2 + r[M_0(t) + o(r)]drdt + r^2[N_0(t) + o(r)]dt^2 = 0 \quad (5.5)$$

where

$$L_0(t) = -2 \cos t[-4 \cos^4 t - 2 \cos^2 t + a \cos^2 t \sin t + 2]$$

$$M_0(t) = a(2 \cos^2 t - 3 \cos^4 t) - 2 \cos^2 t \sin t - 16 \cos^4 t \sin t - 4 \sin t$$

$$N_0(t) = \cos t[-8 \cos^4 t + \cos^2 t - 2 + a \cos^2 t \sin t].$$

The pull-back, $\varphi^*(\mathcal{P}_{1,\alpha})$, of the principal foliation $\mathcal{P}_{1,\alpha}$ has three hyperbolic singularities in the interval $[0, \pi]$, two saddles and one node. The same conclusion holds for $\varphi^*(\mathcal{P}_{2,\alpha})$ in the interval $[\pi, 2\pi]$.

Proof. Performing the blowing-up $u = r^2 \sin t$, $v = r \cos t$, in the equation (5.3), it is obtained:

$$[L_0 + rL_1 + o(r^2)]dr^2 + r[M_0 + rM_1 + o(r^2)]drdt + r^2[N_0 + rN_1 + o(r^2)]dt^2 = 0$$

where, after dividing by r^3 ,

$$\begin{aligned}
L_0 &= -2 \cos t[-4 \cos^4 t - 2 \cos^2 t + a \cos^2 t \sin t + 2] \\
L_1 &= 12 c \cos^4 t \sin t \\
M_0 &= 2 a \cos^2 t - 2 \cos^2 t \sin t - 16 \cos^4 t \sin t \\
&\quad - 3 a \cos^4 t - 4 \sin t \\
M_1 &= -\cos t[-24 c \cos^4 t - D \cos^2 t \sin t \\
&\quad + 24 c \cos^2 t + a c \cos^2 t \sin t + 2 D \sin t - 2 a c \sin t] \\
N_0 &= \cos t[-8 \cos^4 t + 8 \cos^2 t + a \cos^2 t \sin t - 2] \\
N_1 &= -\frac{1}{2} \cos^2 t[24 c \cos^2 t \sin t + a c \cos^2 t \\
&\quad - D \cos^2 t - 2 a c - 24 c \sin t + 2 D]
\end{aligned} \tag{5.6}$$

The following vector fields, $X_\epsilon = T(t, r) \frac{\partial}{\partial t} + R_\epsilon(t, r) \frac{\partial}{\partial r}$, $\epsilon = \pm 1$, are adapted to the equation (5.5), where:

$$\begin{aligned}
T(t, r) &= 2[L_0 + rL_1 + o(r^2)] \\
R(t, r) &= -(M_0 + rM_1)r + \\
&\quad + \epsilon r \sqrt{(M_0^2 - 4L_0N_0) + r(2M_0M_1 - 4L_0N_1 - 4L_1N_0) + o(r^2)}
\end{aligned}$$

Outside their singularities, the vector fields X_ϵ , $\epsilon = \pm 1$, span the line fields $\varphi_*(\mathcal{L}_{1,\alpha})$ and $\varphi_*(\mathcal{L}_{2,\alpha})$.

The singular points of the equation above are given by $\{r = 0\}$ and the roots of

$$-2 \cos t[-4 \cos^4 t - 2 \cos^2 t + a \cos^2 t \sin t + 2] = 0$$

Let $x = \sin t$, and define $l_0(x) = -4x^4 + 10x^2 - 4 + ax(1 - x^2)$ so the equation above writes as:

$$\pm \sqrt{1 - x^2}[-4x^4 + 10x^2 - 4 + ax(1 - x^2)] = 0 \quad (5.7)$$

The polynomial l_0 has the following factorization:

$$-[2x^2 + (\frac{a + \sqrt{a^2 + 32}}{4})x - 2][2x^2 + (\frac{a - \sqrt{a^2 + 32}}{4})x - 2]$$

For every a , the equation (5.7) above has the roots $x = \pm 1$ and one root

$x_1 = \frac{1}{16}[-a - \sqrt{a^2 + 32} + \sqrt{2a^2 + 2a\sqrt{a^2 + 32} + 288}]$ in the interval $(0, 1)$ and the other

$x_2 = \frac{1}{16}[-a + \sqrt{a^2 + 32} - \sqrt{2a^2 - 2a\sqrt{a^2 + 32} + 288}]$ in the interval $(-1, 0)$.

Therefore the differential equation (5.5) has six singularities in the interval $[0, 2\pi]$. So each principal line field, $\varphi^*(\mathcal{L}_{1,\alpha})$ and $\varphi^*(\mathcal{L}_{2,\alpha})$, has three singular points.

For $\epsilon = -1$ the singularities of X_ϵ are $\theta_1 \in (0, \frac{\pi}{2})$, $\theta_2 \in (\frac{\pi}{2}, \pi)$ and $\theta_3 = \frac{\pi}{2}$, where $\sin(\theta_1) = \sin(\theta_2) = t_1$.

For $\epsilon = 1$ the singularities are $\theta_4 \in (-\frac{\pi}{2}, 0)$, $\theta_5 \in (-\pi, -\frac{\pi}{2})$ and $\theta_6 = -\frac{\pi}{2}$, where $\sin(\theta_4) = \sin(\theta_5) = t_2$.

At $x = \pm 1$, that is, for $t = \pm\pi/2$, we have $M_0(\pm\pi/2) = \mp 4$ and $L'_0(\pm\pi/2) = \pm 4$. Therefore the two singular points $(\pm\frac{\pi}{2}, 0)$ are nodes.

The singular points θ_1 , and θ_2 are hyperbolic saddles of the line field X_{-1} and the phase portrait is as shown in figure 5.4 below. Analogous description holds for the line field X_1 .

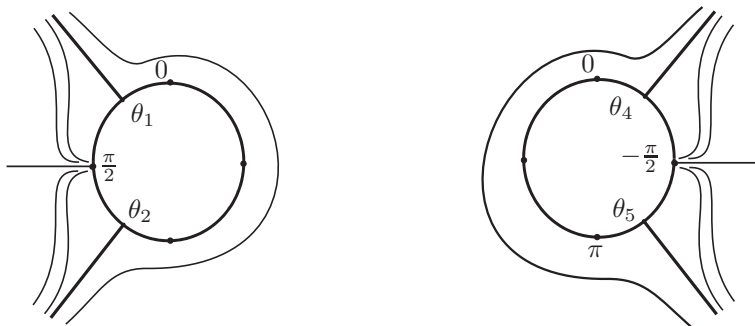


Figure 5.4: Phase portraits of X_{-1} and X_1

In fact we have,

$$DX(t, 0) = \begin{pmatrix} 2L'_0(t) & * \\ 0 & -2M_0(t) \end{pmatrix}, \quad \det DX(t, 0) = -4L'_0(t)M_0(t) \quad (5.8)$$

where, M_0 is given in equation 5.6 and

$$L'_0(t) = 4 \sin t + 6a \sin^2 t \cos^2 t - 2a \cos^4 t - 40 \cos^4 t \sin t - 12 \cos^2 t \sin t.$$

Performing the change of variables $x = \sin t$ it follows that

$$L'_0(x) = -48x + 92x^3 - 40x^5 + a(10x^2 - 8x^4 - 2)$$

$$l_0(x) = -4x^4 + 10x^2 - 4 + ax(1 - x^2)$$

$$M_0(x) = -22x + 34x^3 - 16x^5 + a(-3x^4 + 4x^2 - 1)$$

The resultant of the two polynomials l_0 and M_0 is given by

$$\text{resultant}(l_0(x), M_0(x), x) = 256(a^2 + 81)(a^2 + 36)^2.$$

This calculation is confirmed by Computer Algebra. Therefore L_0 and M_0 have no common roots.

In Fig. 5.5 below are shown the zero level sets of M_0 and L_0 in the plane (t, a) . It follows that the functions M_0 and L'_0 have the same negative sign at the singular points θ_1 and θ_2 . While for the singular points θ_4 and θ_5 the signs of M_0 and L'_0 are positive. Therefore, from equation (5.8) it follows that the points are hyperbolic saddles. \square

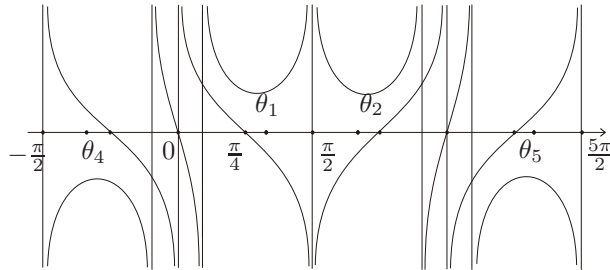


Figure 5.5: Level sets of M_0 and L_0

Remark 5.3.3. *A more direct proof of the proposition above, using a directional blowing-up, was given in [141].*

Theorem 5.3.4. *Let p be a Whitney Umbrella Singularity of a map $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^3$ of class C^r , $r \geq 4$. Then the principal configuration near p has the following structure: Each principal foliation $\mathcal{P}_{i,\alpha}$ of α has a parabolic and a hyperbolic sector at p and the separatrices of these sectors are tangent to the kernel of $D\alpha(p)$. The Fig. 5.6*

illustrates the behavior of curvature lines in the domain of a map α with a Whitney Umbrella Singularity.

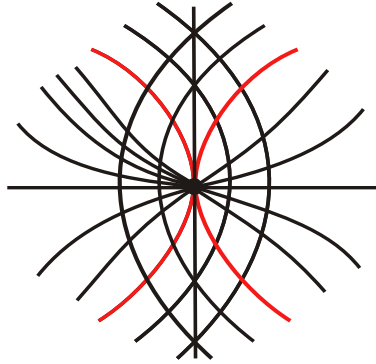


Figure 5.6: Lines of curvature near a Whitney Umbrella Singularity

Proposition 5.3.6. *Consider the differential equation (5.3). Then there are two separatrices, one for each principal foliation, given by $(u_l(v), v)$ and $(u_r(v), v)$, where:*

$$\begin{aligned} u_l(v) &= -(a + \sqrt{a^2 + 32}) \left[\frac{1}{4}v^2 + \frac{2}{3} \frac{a_04 - 9c}{a + 7\sqrt{a^2 + 32}} v^3 + o(4) \right] \\ u_r(v) &= (-a + \sqrt{a^2 + 32}) \left[\frac{1}{4}v^2 + \frac{2}{3} \frac{a_04 - 9c}{a - 7\sqrt{a^2 + 32}} v^3 + o(4) \right] \end{aligned} \quad (5.9)$$

Proof. Direct calculations with equation (5.3). □

5.4 Principal Stability at Whitney umbrellas

The concept of C^s –*Principal Structural Stability* at a point $p \in \mathbb{M}$ can be formulated as follows: For every neighborhood V_p of p in \mathbb{M} there must be a neighborhood \mathcal{V}_α of α in $\mathcal{I}^{r,s}$ such that for every map $\beta \in \mathcal{V}_\alpha$ there must be a point q_β in V_p and a local homeomorphism h_β on the domain such that $h_\beta : W_p \rightarrow W_{q_\beta}$ between neighborhoods of p and q_β , which maps p to q_β and maps $\mathcal{P}_{1,\alpha}|_{W_p}$ and $\mathcal{P}_{2,\alpha}|_{W_p}$ respectively onto $\mathcal{P}_{1,\beta}|_{W_{q_\beta}}$ and $\mathcal{P}_{2,\beta}|_{W_{q_\beta}}$ and such that $\alpha = \beta \circ h_\beta$.

A mapping α is said to be *Principal Structurally Stable* at a singular point p if for any mapping β which is C^3 close to α there is a singular point p_β of β and a homeomorphism of a neighborhood of p into one of p_β sending the foliations $\mathcal{P}_{1,\alpha}$ and $\mathcal{P}_{2,\alpha}$ onto $\mathcal{P}_{1,\beta}$ and $\mathcal{P}_{2,\beta}$.

The theorem below shows that stability of principal configuration at a singular point, in the sense described above, is equivalent to the Whitney Umbrella Singularity condition. A well known result in Singularity Theory establishes that this is the condition which characterizes the stability of singularities of maps of two dimensional into three dimensional manifolds,[66].

Theorem 5.4.1. *A mapping of class C^r , $r \geq 4$, is C^3 – Principally Structurally Stable at a singular point if and only if it has at this point a Whitney Umbrella.*

Proof. Let p a Whitney Umbrella singular point of α . By the intrinsic transversality characterization of Whitney umbrellas, any map β , C^2 -close to α has a unique Whitney umbrella singular point p_β near p .

The C^s -principal structural stability of α at p follows by using the method of canonical regions of differential equations, see [71] and [75]. \square

5.5 Poincaré-Hopf Theorem and Whitney Umbrellas

In this section the following theorem will be proved.

Theorem 5.5.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be a mapping of class C^r , $r \geq 4$, of a compact and oriented two dimensional manifold \mathbb{M} into \mathbb{R}^3 . Suppose that all the umbilic points of α are Darbouxian and the all the singular points of α are Whitney Umbrellas. Then the following expression for the Euler-Poincaré Characteristic of \mathbb{M} holds:*

$$\chi(\mathbb{M}) = \frac{1}{2}[\#(D_1) + \#(D_2) + \#(W) - \#(D_3)]$$

where $\#(D_i)$ $i = 1, 2, 3$ is the number of Darbouxian umbilic points of type D_i and $\#(W)$ is the number of Whitney umbrellas.

Proof. Recall that the index of a line field at a singularity is the total number of turns it accomplishes after running once along the boundary of a disk, positively oriented, containing the singularity in its interior.

For line fields defined by principal directions around a Darbouxian umbilic D_3 the index is $-\frac{1}{2}$ while for the points D_1 and D_2 the index is $\frac{1}{2}$. By Poincaré-Hopf Theorem, see [164] and [85], $\chi(\mathbb{M})$ is equal to the sum of the indices of the singularities of the principal line field

$\mathcal{L}_{1,\alpha}$. The theorem follows by noticing that at a Whitney umbrella the index is $\frac{1}{2}$. \square

5.6 Remarks about curvature lines and singularities

In [54] is established the stable patterns of lines of curvature near a conic singularity of an implicit surface. This situation corresponds to mappings of zero rank, which are degenerate of codimension 6 in the space of mappings. In the space of surfaces (i.e. varieties) defined implicitly the conic singularities (null gradient and hessian with index 2 or 1) have codimension one.

In [55] the stable patterns of lines of curvature at generic ends of algebraic surfaces was determined by the same authors. This amounts to the consideration of singular points at the origin, obtained after the inversion, $I(p) = p/|p|^2$, of the ends of the algebraic surface. The possible generic bifurcations of these patterns has been studied in [61].

The study of principal curvature lines near more degenerate singular points of implicit surfaces in \mathbb{R}^3 or of maps $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is of special interest in Bifurcation Theory.

It follows from the work of Whitney, [178], that umbrella points are isolated singularities and in fact have the following normal form under diffeomorphic changes of coordinates in the source and target (\mathcal{A} -equivalence):

$$x = u, \quad y = uv, \quad z = v^2.$$

5.7 Exercises and Problems

5.7.1. Study the behavior of the principal curvature lines near the singular point $(0, 0, 0)$ of the implicit cubic surface $f^{-1}(0)$, where $f(x, y, z) = z(x^2 + y^2) - x^2 + y^2$. Is the z -axis a principal line?

5.7.2. Consider the local map $\alpha(u, v) = (u, u^2, v^2)$.

- i) Write the differential equation of principal lines of α .
- ii) Study the behavior of principal curvature lines near the singular point of α . **Suggestion:** Explore the symmetry of α with respect to the u -axis.

5.7.3. Consider the ruled surface $\alpha(u, v) = c(u) + vt(u)$, where c is a regular curve and $t(u) = c'(u)$ is the unit tangent to c .

- i) Write the differential equation of principal lines of α .
- ii) Determine the singularities and the umbilic points of α .
- iii) Analyze the behavior of principal curvature lines near the singularities and umbilic points of the surface.
- iv) Give various examples illustrating the generic cases and the bifurcations.

5.7.4. Consider the parametric (singular) surfaces defined by:

$$\alpha(u, v) = (u, v^2, v(u^2 + v^2 - 1)), \quad \beta(u, v) = (u, v^2, v(v^2 - u^2)).$$

- i) Plot the intersection of $\alpha(\mathbb{R}^2)$ and $\beta(\mathbb{R}^2)$ with a sphere of radius $r > 0$ centered at the origin and analyze the geometry of these curves of intersection.
- ii) Determine the singularities of α and β and its double points.

5.7.5. Consider the parametric (singular) surface defined by:

$$\alpha(u, v) = (u, v^3 - v, uv + v^5 - v^3).$$

i) Plot the intersection of $\alpha(\mathbb{R}^2)$ with a sphere of radius $r > 0$ centered at the origin. Determine the double and triple points of this curve of intersection.

ii) Determine the singularities of α and its double and triple points.

5.7.6. Consider the parametric (singular) surface defined by:

$$\alpha_c(u, v) = (u, uv + v^3, uv^2 + cv^4).$$

i) Determine the singularities of α and its double points.

ii) Plot the intersection of $\alpha(\mathbb{R}^2)$ with a sphere of radius $r > 0$ centered at the origin and analyze the geometry of this curve of intersection, in particular for $c = 0.4$, $c = 0.9$ and $c = 1.1$.

5.7.7. Let $f : \mathbb{M} \rightarrow \mathbb{R}^3$ be a smooth immersion of a compact and oriented surface with normal crossings except at a finite set of singularities of cross-cap type. Let $C(f)$ the number of Whitney umbrellas, also denominated *cross-caps* and $T(f)$ the number of triple points.

i) Show that $\chi(f(\mathbb{M})) = \chi(\mathbb{M}) + C(f) + T(f)/2$. Here $\chi(X)$ means the Euler-Poincaré characteristic of a triangulated topological space X . **Suggestion:** Try a relation of the type

$$a_1\chi(f(\mathbb{M})) + a_2\chi(\mathbb{M}) + a_3C(f) + a_4T(f) = 0$$

and work out four or more examples to obtain a linear system in the variables a_i and solve it. Afterwards, obtain the correct formula and prove it. See [91].

5.7.8. Consider the map $\alpha(u, v) = (u, v^3 - v, uv + v^5 - v^3)$.

i) Show that α is a parametrization of a singular surface with two Whitney umbrellas (cross caps) and one triple point. See [108].

ii) Find the implicit equation of α , i.e., obtain an algebraic function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $f(\alpha(u, v)) = 0$.

iii) Calculate the Gaussian curvature and the parabolic points of α .

5.7.9. Consider the one parameter family of maps $\alpha_a(u, v) = (u, v^2, v(a + v^2 - u^2))$.

- i) Determine the singularities of α_a .
- ii) Determine the stable singularities of α_a and analyze the principal configuration of α_a for a small.
- iii) Plot the image of α_a for various values of a .

5.7.10. Let the following functions

Type	Function f
$A_k (k \geq 1)$	$x^{k+1} + y^2 \pm z^2$
$D_k (k \geq 4)$	$xy^{k-1} \pm x^2 \pm z^2$
E_6	$x^3 \pm y^4 \pm z^2$
E_7	$x^3 \pm xy^3 \pm z^2$
E_8	$x^3 \pm y^5 \pm z^2$

- i) Describe the curvature lines of each implicit surface $f^{-1}(0)$ in the list above.
- ii) Plot the intersection of $f^{-1}(0)$ with a sphere of radius $r > 0$ centered at the origin and analyze the geometry of this curve of intersection.

5.7.11. Consider the one parameter families of local maps $\alpha_{\pm} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^3$ defined by $\alpha_{\pm}(u, v, \lambda) = (u, v^2, u^2v \pm v^3 - \lambda v)$.

- i) Show that for $\lambda > 0$ the map α_+ has two Whitney umbrellas which are located in the u -axis.
- ii) Show that for $\lambda < 0$ the map α_+ is an immersion and has two Darbouxian umbilic points of type D_1 located in the v -axis.
- iii) Show that for $\lambda > 0$ the map α_- has two Whitney umbrellas which are located in the u -axis and two Darbouxian umbilic points of type D_3 located in the v -axis.
- iv) Show that for $\lambda < 0$ the map α_- is an immersion and has no umbilic points.

v) Show that in the (u, v, λ) -space the set of double points of α_+ is an elliptic paraboloid. Similarly for α_- show that the locus of double points is hyperbolic paraboloid. Due to this fact these cases are called respectively the *elliptic* and *hyperbolic* Whitney umbrella bifurcations.

vi) Determine the principal configuration of the maps α_+ and α_- for $\lambda = 0$ and $\lambda \neq 0$, near the point $(u = 0, v = 0)$.

5.7.12. Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be a smooth immersion with principal curvatures $k_1 \leq k_2$ and unit normal N_α .

Consider the maps $\alpha_1 = \alpha + (1/k_1)N_\alpha$, $\alpha_2 = \alpha + (1/k_2)N_\alpha$ and $\alpha_3 = \alpha + \frac{1}{2}(1/k_1 + 1/k_2)N_\alpha$. i) Determine the singularities of α_1 , α_2 and α_3 .

ii) Determine the principal configuration of α_1 and α_2 near its singularities and umbilic points.

iii) Let γ be a hyperbolic principal cycle of α . Describe the principal configuration of α_i in a neighborhood of $\gamma_i = \alpha_1(\gamma)$.

5.7.13. Prove that in the 6-dimensional space of 3×2 matrices, identified with $J^1(2, 3)$ those of rank 1, identified with $S^1(2, 3)$, is regular submanifold of codimension 2. To this end, show that the map that maps $(p, \theta, r) \in \mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}_+^1$ to the matrix whose columns are $r \cos(\theta)p$ and $r \sin(\theta)p$ is a regular θ periodic parametrization of $S^1(2, 3)$.

5.7.14. Use proposition 5.3.2 to compute the curvature and torsion of the image of the double curve of a Whitney umbrella at the target.

5.7.15. Let α be any smooth map of $\mathbb{R}^2 \rightarrow \mathbb{R}^3$. Prove that for almost all A in the space of linear mappings of $\mathbb{R}^2 \rightarrow \mathbb{R}^3$, identified with $J^1(2, 3)$, the map $\alpha_A = \alpha + A$ is transversal to $S^1(2, 3)$ and its image is disjoint of the null matrix $S^0(2, 3)$. Use Sard's Theorem. On this basis it can be said that any smooth map has *generically* only Whitney umbrellas at non-regular points.

Chapter 6

Structural Stability of Asymptotic Lines

6.1 Introduction

In this chapter will be considered the simplest qualitative properties of the nets defined by the asymptotic lines of a surface immersed in a Euclidean space.

Asymptotic lines are defined only in the hyperbolic part of the surface, where there are two real asymptotic directions. In the elliptic region the asymptotic directions are complex. At a parabolic point, if not planar we have only one asymptotic direction with multiplicity two. By planar is meant that there $\mathcal{K} = 0$, $\mathcal{H} = 0$.

Conditions for local structural stability of asymptotic lines around parabolic points and closed asymptotic lines are established. This chapter is based mainly in [53] and [140], where the global theory of structural stability of the nets of asymptotic lines was developed.

The study of asymptotic lines goes back to Gaspard Monge (1746-1818), Charles Dupin (1784 - 1873), E. Beltrami (1835-1900), Sophus Lie (1842-1899), G. Darboux (1842-1917) among others. Classical references for this subject are [37], [44], [48], [115], [164], [166] and [167].

6.2 Asymptotic Foliations near Parabolic points

In this section will be established the behavior of the asymptotic nets near parabolic points, to this end conditions expressed in terms of the geometric invariants of the immersion α will be imposed.

Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a regular arc of parabolic points, parametrized positively by arc length u ; that is $[c'(u), \nabla \mathcal{K}(c(u)), N(c(u))] > 0$. Suppose, without loss of generality, that $k_2 \mid c = 0$ and $k_1 \mid c < 0$, where k_1 and k_2 are the principal curvatures of the immersion α . Let $\varphi(u)$ be the oriented angle between $c'(u) = t(u)$ and the principal direction $\mathcal{L}_{2,\alpha}$, corresponding to k_2 , at the point $c(u)$. We assume that $\mathcal{L}_{1,\alpha}$ is positively oriented toward the exterior of hyperbolic region \mathbb{H}_α and that $\mathcal{L}_{2,\alpha}$ is oriented so that $\{\mathcal{L}_{1,\alpha}, \mathcal{L}_{2,\alpha}\}$ defines the positive orientation on the surface. See Fig. 6.1. Denote by $k_g(u)$ the geodesic curvature of c at the point $c(u)$.

The following lemma will be useful in what follows.

Lemma 6.2.1. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a regular arc of parabolic points, which is locally defined by the equation $\mathcal{K} = 0$, parametrized*

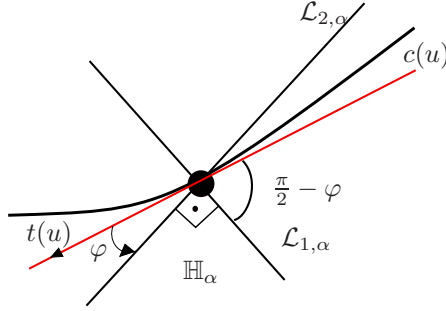


Figure 6.1: Parabolic curve c and angle φ with the principal direction $\mathcal{L}_{2,\alpha}$

by arc length u . Assume here that $dK|_c \neq 0$.

$$\alpha(u, v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + [k_n^\perp(u) \frac{v^2}{2} + v^2 A(u, v)]N(c(u)). \tag{6.1}$$

where, $A(u, 0) = 0$ and $k_n^\perp(u) = k_n(c(u))$, $(N \wedge t)(u)$, defines a local chart of class C^∞ around c .

Proof. The map $\alpha(u, v, w) = (\alpha \circ c)(u) + v(N \wedge t)(u) + wN(u)$ is a local diffeomorphism. Therefore, solving the implicit equation

$$\langle \alpha(u, v, w(u, v)) - (\alpha \circ c)(u), N(u) \rangle = w$$

and using the Hadamard lemma it follows the result asserted. Similar construction was done in Lemma 3.3.1, page 75, in connection with principal configurations. \square

Computation of the Second Fundamental Form

In what follows will be calculated the coefficients and the derivatives of the second fundamental form of α in the chart introduced in 6.2.1.

Calculations in the chart (u, v)

The Darboux equations for the positive frame $\{t, N \wedge t, N\}$ are:

$$\begin{cases} t'(u) = k_g(u)(N \wedge t)(u) + k_n(u)N(u) \\ (N \wedge t)'(u) = -k_g(u)t(u) + \tau_g(u)N(u) \\ (N)'(u) = -\tau_g(u)(N \wedge t)(u) - k_n(u)t(u) \end{cases} \quad (6.2)$$

Using Euler's formula, [166], [40], it follows that, $k_n^\perp = k_1 \cos^2 \varphi$, $k_n = k_1 \sin^2 \varphi$, $k_n^\perp + k_n = 2\mathcal{H}$ and $\tau_g = k_1 \sin \varphi \cos \varphi$.

For the sake of simplicity in the expressions that follow, write $A = A(u, v)$, $N = (N \circ c)(u)$, $k_n^\perp = k_n^\perp(u)$, $k_g = k_g(u)$.

Moreover the following notation will be used:

$$\begin{aligned} E &= \langle \alpha_u, \alpha_u \rangle, & F &= \langle \alpha_u, \alpha_v \rangle, & G &= \langle \alpha_v, \alpha_v \rangle \\ e &= \langle \alpha_u \wedge \alpha_v, \alpha_{uu} \rangle, & f &= \langle \alpha_u \wedge \alpha_v, \alpha_{uv} \rangle, & g &= \langle \alpha_u \wedge \alpha_v, \alpha_{vv} \rangle \end{aligned}$$

Here E, F, G and $e / |\alpha_u \wedge \alpha_v|$, $f / |\alpha_u \wedge \alpha_v|$ and $g / |\alpha_u \wedge \alpha_v|$ are respectively the coefficients of the first and second fundamental forms of α , expressed in the chart (u, v) .

Differentiating equation (6.1) and using equation (6.2), obtain:

$$\begin{aligned}
\alpha_u &= [1 - k_g v - k_n(k_n^\perp \frac{v^2}{2} + v^2 A)]t - \tau_g(k_n^\perp \frac{v^2}{2} + v^2 A)N \wedge t \\
&\quad + [\tau_g v + (k_n^\perp)' \frac{v^2}{2} + v^2 A_u]N, \quad \alpha_v = N \wedge t + (k_n^\perp v + 2vA + v^2 A_v)N \\
\alpha_u \wedge \alpha_v &= -[(1 - k_g v - k_n(k_n^\perp \frac{v^2}{2} + v^2 A))(k_n^\perp v + 2vA + v^2 A_v)]N \wedge t \\
&\quad - [\tau_g v + (k_n^\perp)' v^2 + v^2 A_u + \tau_g(k_n^\perp \frac{v^2}{2} + v^2 A)(k_n^\perp v + 2vA + v^2 A_v)]t \\
&\quad + [1 - k_g v - k_n(k_n^\perp \frac{v^2}{2} + v^2 A)]N
\end{aligned} \tag{6.3}$$

Further differentiation and scalar multiplication give the functions $e(u, v)$, $f(u, v)$ and $g(u, v)$, whose essential properties follow.

$$\begin{aligned}
e(u, 0) &= k_n(u) = k_1 \sin^2 \varphi, \quad f(u, 0) = \tau_g(u) = k_1 \sin \varphi \cos \varphi, \\
g(u, 0) &= k_n^\perp(u) = k_1 \cos^2 \varphi, \\
e_v(u, 0) &= -k_g(2k_n + k_n^\perp) + \tau_g', \\
f_v(u, 0) &= (k_n^\perp)', \quad g_v(u, 0) = -k_g k_n^\perp + 6A_v, \\
E_v(u, 0) &= -2k_g, \quad F_v(u, 0) = 0, \quad G_v(u, 0) = 0.
\end{aligned} \tag{6.4}$$

From the relation, $2\mathcal{H}(EG - F^2)^{\frac{3}{2}} = eG - 2fF + gE$ and equation (6.4), it follows that

$$6A_v(u, 0) = 2\mathcal{H}_v - 6k_g \mathcal{H} + k_g(2k_n + 4k_n^\perp) - \tau_g'. \tag{6.5}$$

Also, from $\mathcal{K}(EG - F^2)^2 = eg - f$, and equations (6.4) and (6.5),

it is obtained that,

$$\mathcal{K}_v(u, 0) = k_g[k_1^2 \cos 2\varphi - 2\tau_g^2] + k_1\tau_g' - 2\tau_g(k_n^\perp)' - 2\mathcal{H}_v k_n \neq 0,$$

which expresses the condition of regularity of the parabolic set in Section 6.2.

The main result of this section is formulated by the following proposition.

Proposition 6.2.1. *Let c be a curve of parabolic points as above. Then the following holds:*

- i) *If $\varphi(u) \neq 0$ the asymptotic net, near $c(u)$, is as shown in Fig. 6.2 (cuspidal type).*
- ii) *If $\varphi(u_0) = 0$ and $\varphi'(u_0) \neq 0$ there are three cases:*

$$a) \frac{k_g(u_0)}{\varphi'(u_0)} < 1, \quad b) 1 < \frac{k_g(u_0)}{\varphi'(u_0)} < 9 \quad \text{and} \quad c) 9 < \frac{k_g(u_0)}{\varphi'(u_0)}.$$

In cases a), b) and c) above the asymptotic net is as shown in the figure 6.2 and correspond respectively to the folded saddle, focus and node types parabolic points.

Proof. i) The cuspidal case: transversal crossing

Suppose that the principal foliation $\mathcal{P}_2(\alpha)$ is transversal to the parabolic line at the point u_0 , this means that $\varphi(u_0) \neq 0$.

From equation (6.4) and using Hadamard lemma, write:

$$e(u, v) = k_n(u) + v[-k_g(2k_n + k_n^\perp) + \tau_g'] + vA_1(u, v),$$

$$f(u, v) = \tau_g(u) + v(k_n^\perp)' + v^2A_2(u, v),$$

$$g(u, v) = k_n^\perp(u) + v[2\mathcal{H}_v - 6k_g\mathcal{H} + k_g(2k_n + 3k_n^\perp) - \tau_g'] + v^2A_3(u, v),$$

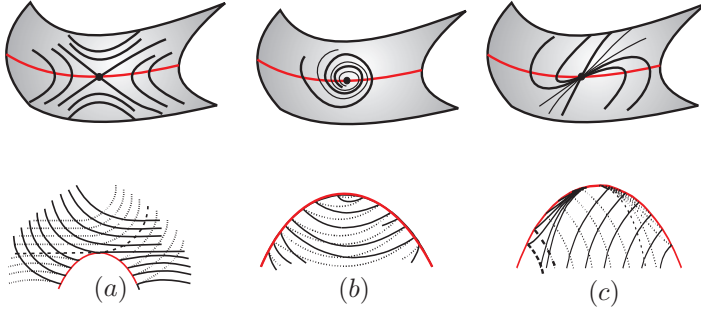


Figure 6.2: Asymptotic lines near parabolic points (folded saddle (a), focus (b) and node (c), separating arcs of parabolic points of cuspidal type).

with, $k_n(u) = k_1 \sin^2 \varphi$, $k_n^\perp(u) = k_1 \cos^2 \varphi$, $\tau_g(u) = k_1 \sin \varphi \cos \varphi$.

The differential equation of the asymptotic lines are given by:

$$e du^2 + 2f dudv + g dv^2 = 0.$$

Then, $du/dv = \frac{-f \pm [f^2 - eg]^{\frac{1}{2}}}{e}$.

Let $v = w^2$. So, it follows that,

$$\left\{ \begin{aligned} \frac{du}{dw} &= -2w \frac{f}{e} \pm \frac{2w^2 W(u, w^2)}{e}, & u(u_0, 0) &= u_0 \end{aligned} \right.$$

where $W(u, 0) = [\mathcal{K}_v(u, 0)]^{\frac{1}{2}} > 0$ by transversality conditions.

Solving the Cauchy problem above it results that:

$$u(u_0, v) = u_0 - \cot g \varphi(u_0) v \pm W(u_0, 0) v^{\frac{3}{2}} + \dots$$

Therefore near a cuspidal parabolic point the net of asymptotic lines is as shown in Fig.6.2, arcs of parabolic points.

Remark 6.2.1. *It follows from [4] that there exist a system of coordinates (U, V) near a cuspidal parabolic point such that the differential equation of the asymptotic lines is given by $(dV/dU)^2 = U$.*

ii) The singular case: point of quadratic tangency.

Now suppose that $\tau_g(u_0) = 0$, $u_0 = 0$. This means that the parabolic line is tangent to the principal foliation $\mathcal{P}_{2,\alpha}$ at u_0 . In fact, at a parabolic point the principal direction corresponding to the zero principal curvature is an asymptotic direction. Suppose also that at the point of tangency u_0 the contact above is quadratic, which is expressed by the conditions $\tau_g(0) = 0$ and $\tau'_g(0) \neq 0$.

Consider the implicit differential equation,

$$F(u, v, p) = e + 2fp + gp^2 = 0, \quad p = \frac{dv}{du}$$

and the line field given locally by the vector field:

$$X = (F_p, pF_p, -(F_u + pF_v)).$$

The projections of the integral curves of X by $\Pi(u, v, p) = (u, v)$ are the asymptotic lines of α .

The singularities of X in $F^{-1}(0)$ are given by: $(u_0, 0, 0)$, where $\tau_g(u_0) = 0$. Suppose $u_0 = 0$.

It results that the Jacobian matrix of $DX(0)$ is given by:

$$DX(0) = \begin{pmatrix} 2f_u & 2f_v & 2g \\ 0 & 0 & 0 \\ -e_{uu} & -e_{uv} & -(2f_u + e_v) \end{pmatrix} \quad (6.6)$$

Using equations (6.4) and (6.6) it results that the eigenvalues of

$DX(0)$ are given by:

$$\lambda_1, \quad \lambda_2 = \left(\frac{k_n^\perp \varphi'}{2}\right) \{(a-1) \pm [(a-1)(a-9)]^{\frac{1}{2}}\}, \quad a = \frac{k_g}{\varphi'}.$$

The eigenspace associated to λ_i is given by:

$$E_i = (1, 0, r_i) = (1, 0, \frac{\varphi'}{4} [(a-5) \pm \sqrt{(a-1)(a-9)}]).$$

The tangent line to the suspension of the parabolic curve at the point $(u_0, 0, 0)$ on the surface $F^{-1}(0)$ is generated by $\Delta = (1, 0, -\varphi')$.

In fact, Δ is a non zero multiple of

$$(\nabla F \wedge \nabla F_p)(u_0, 0, 0) = 2(\varphi' - k_g)k_1^2(1, 0, -\varphi') = 2\varphi'k_1^2(1-a)(1, 0, -\varphi').$$

Therefore E_i is transversal to the singular set $\Pi^{-1}(\mathbb{P}_\alpha) \cap \{F = 0\}$.

In the case of the saddle point ($\lambda_1 \lambda_2 < 0$, which amounts to $a < 1$), the eigenspaces E_i have inclinations of opposite sign with respect to Δ , that is, $(\varphi' + r_1)(\varphi' + r_2) = \frac{1}{2}(\varphi')^2(a-1)$. The vector Δ is interior to the acute angle formed by E_1 and E_2 . This implies that the net of asymptotic lines near a folded saddle parabolic point is as shown in Fig. 6.2 a).

In the case of a focus singularity ($\lambda_1 = \bar{\lambda}_2$, $Re(\lambda_1) \neq 0$, which amounts $1 < a < 9$) the net of asymptotic lines is as shown in Fig. 6.2 b).

In the case of a nodal singularity ($\lambda_1 \lambda_2 > 0$, which amounts $9 < a$) the two eigenspaces have inclinations of the same sign with respect to Δ . Here Δ is interior to the obtuse angle formed by E_1 and E_2 . Also E_2 bisects the angle formed by Δ and E_1 (the tangent space to

the strong separatrix). Therefore near a folded node parabolic point the net of asymptotic lines is as shown in Fig. 6.2 c). \square

An immersion α in \mathcal{I} , is said to be C^r -local asymptotic structurally stable, at p if it has a neighborhood \mathcal{N} in the space \mathcal{I}^r such that for each β in \mathcal{N} there is a smooth diffeomorphism k_β of $\{\mathbb{M}, \text{NE}_\alpha\}$ to $\{\mathbb{M}, \text{NE}\}$ such that $\beta \circ k_\beta$ is local asymptotic topologically equivalent to α at $k_\beta(p)$.

Theorem 6.2.2. *For an open and dense set \mathcal{W} of immersions in \mathcal{I}^r , $r \geq 5$, the asymptotic nets near a parabolic point as described in proposition 6.2.1 are locally asymptotic stable.*

Proof. Follows from proposition 6.2.1 and by the main results of Bleeker and Wilson [18] and Feldman [45]. The construction of the topological equivalence, using the method of canonical regions, can be carried out in the same way as in [140]. \square

6.3 Stability of Closed Asymptotic Lines

In this section will be established an integral expression for the derivative of the first return map of a regular closed asymptotic line in terms of curvature functions of the immersion α .

Also, will be shown how to deform the immersion in order to hyperbolize a regular or a folded closed asymptotic line.

Regular closed asymptotic lines

Recall that a *regular closed asymptotic line* is a closed asymptotic line which is disjoint from the parabolic points.

Lemma 6.3.1. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a closed asymptotic line parametrized by arc length u and length L . Then the expression:*

$$\alpha(u, v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + [\mathcal{H}(u)v^2 + A(u, v)v^2]N(c(u)) \quad (6.7)$$

where $A(u, 0) = 0$ and \mathcal{H} is the Mean Curvature of α , defines a local chart of class C^∞ around c .

Proof. Similar to lemma 6.2.1, where the coefficient of v^2 is given by $k_n^\perp/2$.

Using that $k_n(u) = k_n(c(u), t(u)) = 0$ for an asymptotic line and applying Euler's formula follows that, $k_n^\perp + k_n = 2\mathcal{H}$. \square

Proposition 6.3.1. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a regular closed asymptotic line of length L , parametrized by arc length u . Then the derivative of the Poincaré map Π , associated to it is given by:*

$$\Pi'(0) = \exp \left[\int_0^L \frac{\tau'_g - 2k_g(u)\mathcal{H}(u)}{2\tau_g(u)} du \right]$$

where k_g is the geodesic curvature of c and $\tau_g = (-\mathcal{K})^{\frac{1}{2}}$ is the geodesic torsion of c .

Proof. The Darboux equations for the positive frame $\{t, N \wedge t, N\}$ are:

$$\begin{aligned} t'(u) &= k_g(u)(N \wedge t)(u) \\ (N \wedge t)'(u) &= -k_g(u)t(u) + \tau_g(u)N(u) \\ N'(u) &= -\tau_g(u)(N \wedge t)(u) \end{aligned} \quad (6.8)$$

The same procedure of calculation used in the lemma 6.2.1 gives that:

$$\begin{aligned} e(u, 0) &= 0, & e_v(u, 0) &= \tau'_g - 2\mathcal{H}(u)k_g(u) \\ f(u, 0) &= \tau_g(u) & g(u, 0) &= 2\mathcal{H}(u) \end{aligned} \quad (6.9)$$

The differential equation of the asymptotic lines in the neighborhood of the line $\{v = 0\}$ is given by:

$$e + 2f \frac{dv}{du} + g \left(\frac{dv}{du} \right)^2 = 0 \quad (6.10)$$

Denote by $v(u, r)$ the solution of the (6.10) with initial condition $v(0, r) = r$. Therefore the return map Π is clearly given by $\Pi(r) = v(L, r)$.

Differentiating equation (6.10) with respect to r , it results that:

$$g_r v_r (dv/du)^2 + (2g v_{ur} + 2f_v v_r)(dv/du) + e_v v_r = 0.$$

Evaluating at $v = 0$, it follows that:

$$2f(u, 0)v_{ur}(u, 0) + e_v(u, 0)v_r(u, 0) = 0 \quad (6.11)$$

Therefore, using the expressions for f and e_v found in equation (6.9), integration of equation (6.11) it is obtained:

$$\ln \Pi'(0) = \int_0^L \frac{-\tau'_g + 2\mathcal{H}k_g}{2\tau_g} du$$

This ends the proof. □

Remark 6.3.1. *From [164, vol. III, page 282] it follows that*

$$\omega = \frac{\tau'_g - 2k_g(u)\mathcal{H}(u)}{2\tau_g}$$

is a 1-form evaluated along an asymptotic line.

Proposition 6.3.2. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a regular closed asymptotic line of length L for the immersion α , parametrized by arc length u .*

Consider the following one parameter deformation of α :

$$\alpha_\epsilon(u, v) = \alpha(u, v) + \epsilon w(u) \delta(v) v^2 N(u)$$

where $\delta \mid c = 1$ and has small support. Then c is a regular closed asymptotic line of α_ϵ for all ϵ small and the derivative of the Poincaré map Π_{α_ϵ} , associated to it is given by:

$$\Pi'_{\alpha_\epsilon} = \exp \left[\int_0^L \frac{k_g(u) [\mathcal{H}(u) + \epsilon w(u)]}{\tau_g(u)} du \right].$$

Moreover, taking $w(u) = k_g(u)$ holds that:

$$\frac{d}{d\epsilon} (\Pi'_{\alpha_\epsilon}(0)) \mid_{\epsilon=0} = \int_0^L \frac{k_g(u)^2}{\tau_g(u)} du \neq 0$$

In particular c is a hyperbolic closed asymptotic line for $\alpha_\epsilon, \epsilon \neq 0$.

Proof. Performing the calculation as in Lemma 6.3.1 it follows that:

$$\begin{aligned} e(\epsilon, u, 0) &= 0, \quad f(\epsilon, u, 0) = \tau_g(u), \quad g(\epsilon, u, 0) = 2(\mathcal{H}(u) + \epsilon w(u)), \\ e_v(\epsilon, u, 0) &= -2[\mathcal{H}(u) + \epsilon w(u)]k_g(u) + \tau'_g(u). \end{aligned}$$

Therefore $\{v = 0\}$ is a closed asymptotic line for α_ϵ . Applying Proposition 6.3.1 to α_ϵ and differentiating under the integration sign gives the result stated. \square

Remark 6.3.2. *In order to see that $k_g \mid c$ is not identically zero for a closed asymptotic line, we observe that for an asymptotic line c , the geodesic curvature coincides with the ordinary curvature, considered as a curve in \mathbb{E}^3 . Therefore, if $k_g \mid c = 0$, it follows that c must be a straight line.*

Folded closed asymptotic lines

Here will be established an integral expression for the derivative of the first return map of a folded closed asymptotic line in terms of the curvature functions of the immersion α .

A *folded closed asymptotic line* is a closed asymptotic curve $c : [0, L] \rightarrow \mathbb{M}$ regular by parts, that is, there exist a finite sequence of numbers $a_i, 0 = a_0 < a_1 < \dots < a_l = L$, such that $c_i = c \mid (a_i, a_{i+1}) : (a_i, a_{i+1}) \rightarrow \text{Int}\mathbb{H}$ is an asymptotic line of α and $p_i = c(a_i) \in \mathbb{P}_\alpha$ for $i = 1, \dots, l-1$. In other words, a *folded closed asymptotic line* is the projection of a closed integral curve of the single line field \mathcal{L}_α defined on \mathbb{A} , which intersects \mathcal{P}_α .

Let c be a folded closed asymptotic line. Near each point p_i , consider two transversal sections to c , $\Sigma_{1,i}$ and $\Sigma_{2,i}$, and the Poincaré map $\sigma_i : \Sigma_{1,i} \rightarrow \Sigma_{2,i}$. Denote by $u_i^j = c_i(a_i, a_{i+1}) \cap \Sigma_{j,i}, j = 1, 2$. Denote by $\pi_{i+1,i} : \Sigma_{2,i} \rightarrow \Sigma_{1,i+1}$ the Poincaré map associated to c_i . It follows that the Poincaré map associated to c , $\Pi : \Sigma_{1,1} \rightarrow \Sigma_{1,1}$ is given by: $\Pi = \pi_{l-1,1} \circ \sigma_{l-1} \dots \circ \pi_{i+1,i} \circ \dots \circ \pi_{2,1} \circ \sigma_1$.

Proposition 6.3.3. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a folded closed asymptotic line of length L parametrized by arc length u . Assume the notation above. Then the derivative of the Poincaré map $\pi_{i+1,i}$ associated to*

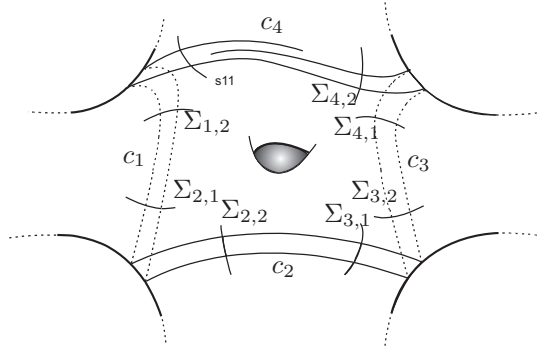


Figure 6.3: Folded closed asymptotic lines

c_i is given by:

$$\pi_{i+1,i}(0) = \exp \left[\int_{u_i^2}^{u_{i+1}^1} \frac{-\tau'_g(u) + 2k_g(u)\mathcal{H}(u)}{2\tau_g(u)} du \right]$$

where k_g is the geodesic curvature of c_i and $\tau_g = (-\mathcal{K})^{\frac{1}{2}}$ is the geodesic torsion of c_i . Moreover the functions σ_i are differentiable.

Proof. Near the point p_i take a system of coordinates (U, V) such that the asymptotic lines are given by the differential equation $(dU/dV)^2 = U$. See [4], [9] and Remark 6.2.1.

In this system of coordinates $\sigma_i : \{V = \epsilon\} \rightarrow \{V = \epsilon\}$ is clearly a translation $\sigma_i(u, \epsilon) = (u + c, \epsilon)$. Therefore σ_i is differentiable.

The expression for the derivative of $\pi_{i+1,i}$ can be obtained in the same way as in the Proposition 6.3.1. \square

Proposition 6.3.4. *Let $c : [0, L] \rightarrow \mathbb{M}^2$ be a folded closed asymptotic line of length L , parametrized by arc length u . Assume the notation*

above and consider the following one parameter deformation of immersions

$$\alpha_\epsilon(u, v) = \alpha(u, v) + \epsilon w(u)\delta(v)vN(u)$$

where $\delta \mid c = 1$ and has small support and $\text{supp}(\delta) \cap c(a_i, a_{i+1}) \neq \emptyset$.

Then c is a folded asymptotic line of α_ϵ for all ϵ small and the derivative of the Poincaré map $\pi_{\alpha_\epsilon, i+1, i}$ associated to it is given by:

$$\Pi'_{\alpha_\epsilon, i+1, i} = \exp \left[\int_{u_i^1}^{u_{i+1}^1} \frac{-\tau'_g(u) + 2k_g(u)[\mathcal{H}(u) + \epsilon w(u)]}{2\tau_g(u)} du \right].$$

Moreover, taking $w(u) = k_g(u)$ holds that:

$$\frac{d}{d\epsilon} (\pi_{\alpha_\epsilon, i+1, i}(0)) \mid_{\epsilon=0} = \int_{u_i^2}^{u_{i+1}^1} \frac{k_g(u)^2}{\tau_g(u)} du \neq 0$$

In particular c is a hyperbolic closed asymptotic folded line for α_ϵ , $\epsilon \neq 0$.

Proof. Similar to the proof of proposition 6.3.2. Here one must take an arc which is not a straight line. \square

From the considerations above we have the following.

Theorem 6.3.3. *Given a regular or folded asymptotic closed line c of the immersion α , then there exist a smooth one parameter family of immersions α_t such that for $t > 0$ small, c is a hyperbolic asymptotic line of α_t .*

6.4 Asymptotic Structural Stability

The following conditions (inspired in [71, 75]) are essential for the definition of the class of immersions which are asymptotic structurally

stable.

Here will be used the notation introduced in section 1.7 of chapter 1.

a) **Condition on parabolic points:** Denote by Σ_a the class of immersions α for which the singularities of the line field $\tilde{\mathcal{L}}_\alpha$, which occur when $\tilde{\mathcal{L}}_\alpha$ is tangent to \mathcal{P}_α , are hyperbolic (non-vanishing real part of eigenvalues). See Proposition 6.2.1 page 130.

b) **Condition on hyperbolic closed asymptotic lines:** Denote by Σ_b the class of immersions for which all the regular and folded asymptotic closed lines, i.e. the closed integral curves of $\tilde{\mathcal{L}}_\alpha$ are hyperbolic (i.e. the derivative of the return map is different from one). See Proposition 6.3.1, page 135 and Proposition 6.3.3, page 138.

c) **Condition on separatrix connections:** Denote by Σ_c the class of immersions such that there are no connection between separatrices of singular points of the foliation $\tilde{\mathcal{A}}_\alpha$ and consequently of the asymptotic foliations $\mathcal{A}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$. See page 28.

d) **Condition on limit sets:** Denote by Σ_d the class of immersions such that for every leaf of $\tilde{\mathcal{A}}_\alpha$ the limit set is a singular point or a closed asymptotic line.

Define the set $\Sigma_{asy}^s = \Sigma_a \cap \Sigma_b \cap \Sigma_c \cap \Sigma_d$.

An immersion $\alpha \in \mathcal{I}^r$ is said to be *C^s -asymptotically structurally stable* if there is a neighborhood \mathcal{V}_α of α in \mathcal{M} such that for every immersion $\beta \in \mathcal{V}_\alpha$ there exist a homeomorphism h_β on the domain such that $h_\beta(\mathcal{P}_\alpha) = \mathcal{P}_\beta$ and h_β maps lines of $\mathcal{A}_{1,\alpha}$, (resp. $\mathcal{A}_{2,\alpha}$) on those of $\mathcal{A}_{1,\beta}$ (resp. $\mathcal{A}_{2,\beta}$). In this notion of stability the homeomorphism of the topological equivalence must preserve the parabolic set,

i.e. the boundary of the hyperbolic region.

This concept of stability has its roots in the theory of stability of differential equations on surfaces, formulated by Pontrjagin, A. Andronov and M. Peixoto, see [1], [130] and [128]. Also A. Davydov considered a similar notion of stability in the theory of control systems, [38]. In [140] it was proved the following theorem.

Theorem 6.4.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 5$, of a compact and oriented surface \mathbb{M} of class C^r . Then:*

i) The set Σ_{asy}^s is open in $\mathcal{I}^{r,s}(\mathbb{M}, \mathbb{R}^3)$, $s \geq 5$.

ii) If $\alpha \in \Sigma_{asy}^s$ then α is C^s - asymptotic structurally stable.

6.5 Examples of Closed Asymptotic Lines

In this section will be given geometric constructions of regular surfaces having closed asymptotic lines.

A hyperbolic closed asymptotic line

In this subsection will be given an example of a surface having a hyperbolic asymptotic line contained in interior of the region of negative Gaussian curvature.

Proposition 6.5.1. *Let $c : [0, L] \rightarrow \mathbb{R}^3$ be a closed biregular curve, parametrized by arc length, such that the curvature $k(s)$ and the torsion $\tau(s)$ of c are different from zero for all $s \in [0, L]$. Consider the surface $\alpha(s, v) = c(s) + vN(s) + \tau(s)\frac{v^2}{2}B(s)$.*

Here $\{T, N, B\}$ is the Frenet orthonormal frame associated to c .

Then c is a regular hyperbolic closed asymptotic line.

Proof. Direct calculation gives that

$$e(s, 0) = 0, \quad f(s, 0) = \tau(s), \quad e_v(s, 0) = \tau'(s) - k(s)\tau(s), \quad g(s, 0) = \tau(s).$$

The Poincaré map given by $\pi(v_0) = v(L, v_0)$, where $v(u, v_0)$ is the solution of the differential equation

$$eds^2 + 2fdsdv + gdv^2 = 0$$

with $v(0, v_0) = v_0$, has the first derivative at 0 given by:

$$\pi'(0) = \exp \int_0^L -\frac{e_v}{2f}(s, 0)ds = \exp \int_0^L \frac{k(s)}{2}ds \neq 1.$$

This ends the proof. □

Remark 6.5.1. *Curves with the above properties are, for example, the toroidal helices, [33]. For appropriate parameters $(m, n) \in \mathbb{N} \times \mathbb{N}$, the closed curve $c_{m,n}$ defined by*

$$c_{m,n}(t) = ((R + r \cos nt) \cos mt, (R + r \cos nt) \sin mt, r \sin nt)$$

has non zero torsion.

A semi hyperbolic closed asymptotic line

In this section will be considered a ruled surface having a non hyperbolic asymptotic line contained in interior of the region of negative Gaussian curvature. Under an integral condition will be proved that the second derivative of the return is different from zero. That is, the closed asymptotic line is semi hyperbolic.

Proposition 6.5.2. *Let $c : [0, L] \rightarrow \mathbb{R}^3$ be a closed biregular curve, parametrized by arc length, such that the curvature $k(s)$ and the torsion $\tau(s)$ of c are different from zero for all $s \in [0, L]$. Consider the ruled surface*

$$\alpha(s, v) = c(s) + vN(s).$$

Then c is a regular semi hyperbolic closed asymptotic line provided

$$\int_c \tau^{-1/2} dk \neq 0.$$

Proof. Direct calculation gives that

$$\begin{aligned} \alpha_s \wedge \alpha_v &= -\tau v T + (1 - vk) B \\ \alpha_{ss} &= -k' v T + [k - (k^2 + \tau^2)v] N + \tau' B \\ \alpha_{sv} &= -k T + \tau B, \quad \alpha_{vv} = 0 \end{aligned}$$

Therefore $e = [\alpha_{ss}, \alpha_s, \alpha_v]$, $f = [\alpha_{sv}, \alpha_s, \alpha_v]$ and $g = [\alpha_{vv}, \alpha_s, \alpha_v]$ are given by.

$$e(s, v) = \tau'(s)v + \left(\frac{k}{\tau}\right)'(s)\tau^2 v^2, \quad f(s, v) = \tau(s), \quad g(s, v) = 0.$$

Therefore one family of asymptotic lines is given by the straight lines $s = cte$ and c is an asymptotic line of the other foliation. The Poincaré map associated to c is defined by $\pi(v_0) = v(L, v_0)$, where $v = v(s, v_0)$ is the solution of the differential equation

$$\left\{ \begin{aligned} \frac{dv}{ds} &= -\frac{\tau'(s)v + \left(\frac{k}{\tau}\right)'(s)\tau^2 v^2}{2\tau}, \quad v(0, v_0) = v_0. \end{aligned} \right.$$

Direct integration of the first variation equation

$$\frac{d}{ds} \left(\frac{dv}{dv_0} \right) = -\frac{\tau'}{2\tau} \frac{dv}{dv_0}$$

evaluated at $v = 0$ gives, $\frac{dv}{dv_0}(s) = \frac{\sqrt{\tau(0)}}{\sqrt{\tau(s)}} = \frac{\sqrt{\tau_0}}{\sqrt{\tau}}$.

Therefore, $\pi'(0) = \frac{\partial v}{\partial v_0}(0, L) = 1$.

Also, the integration of the second variation equation,

$$\frac{d}{ds} \left(\frac{d^2v}{dv_0^2} \right) = -\frac{\tau'}{2\tau} \frac{d^2v}{dv_0^2} - \left(\frac{dv}{dv_0} \right)^2 \left(\frac{k}{\tau} \right)' \tau$$

leads to

$$\begin{aligned} \frac{d^2v}{dv_0^2} &= -\int_0^L \frac{\sqrt{\tau_0}}{\sqrt{\tau}} \left(\frac{k}{\tau} \right)' \tau(s) ds = -\sqrt{\tau_0} \int_0^L \sqrt{\tau} \left(\frac{k}{\tau} \right)' ds \\ &= -\sqrt{\tau_0} \int_c k d(\tau^{-1/2}) = \sqrt{\tau_0} \int_c \tau^{-1/2} dk \end{aligned}$$

This ends the proof. \square

6.6 On a class of dense asymptotic lines

The goal of this section is to present examples of folded recurrent asymptotic lines.

Proposition 6.6.1. *Let \mathbb{T}^2 be the torus of revolution, obtained by the rotation of the circle $(x - R)^2 + z^2 = r^2$, $r < R$, around the z axis. Then the qualitative behavior of the asymptotic lines is as*

shown in Fig. 6.4. Moreover the return map $\Pi : \mathbb{S}(R) \rightarrow \mathbb{S}(R)$, where $\mathbb{S}(R) = \{(x, y, z) : x^2 + y^2 = R^2, z = -r\}$, is a rotation by an angle equal to $4RT(r/R)$, where

$$T\left(\frac{r}{R}\right) = \sum_{n=0}^{\infty} \frac{2a_n}{n!} \left(\frac{r}{R}\right)^n, \quad a_n = \frac{1 \times 3 \times \dots \times (2n-1)}{2^n} \frac{\Gamma(\frac{1}{2})\Gamma(2n + \frac{1}{4})}{\Gamma(2n + \frac{3}{4})}.$$

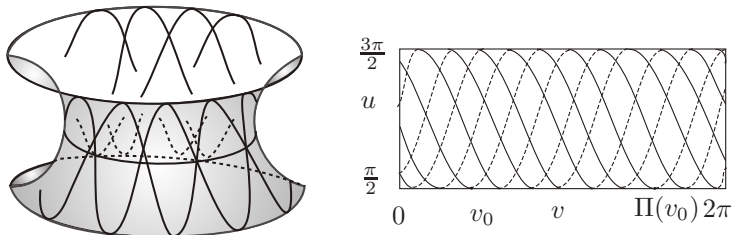


Figure 6.4: Asymptotic lines on the torus

Proof. Consider the following parametrization of the torus of revolution: $(u, v) \rightarrow (\cos v(R + r \cos u), \sin v(R + r \cos u), r \sin u)$. The second fundamental form is given by

$$e(u, v) = R^2, \quad f(u, v) = 0, \quad g(u, v) = R(R + r \cos u) \cos u.$$

Therefore the differential equation of the asymptotic lines is:

$$F(u, v, du/dv) = R(du/dv)^2 + \cos u(R + r \cos u) = 0.$$

Writing $q = du/dv$, consider the vector field X defined by the differential equation

$$(u', v', p') = (qF_q, F_q, -(qF_u + F_v)).$$

After multiplying X by $1/q$ it results that:

$$(u', v', q') = (2Rq, 2R, R \sin u + r \sin 2u).$$

Consider also the projected vector field, $Y(u, p) = (2Rp, R \sin u + r \sin 2u)$. Notice that the orbit of Y through $(\frac{\pi}{2}, 0)$ reaches $(\frac{3\pi}{2}, 0)$.

In fact, from the first integral of Y ,

$$G(u, p) = Rp^2 + R \cos u + \frac{r}{2} \cos 2u,$$

it follows that $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ are in the same connected component of $G^{-1}(\frac{-r}{2})$.

The time spent by an orbit that starts at $(\frac{\pi}{2}, 0)$ to reach the point $(\frac{3\pi}{2}, 0)$ can be calculated as follows:

From $G(u, p) = -\frac{r}{2}$ it results that:

$$q = \left\{ \frac{[-r(1 + \cos 2u) - 2R \cos u]}{2R} \right\}^{\frac{1}{2}}.$$

As $\frac{du}{dt} = 2Rq$, it follows that:

$$T = R^{\frac{1}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{du}{[-\cos u(r \cos u + R)]^{\frac{1}{2}}} = 2 \int_0^{\frac{\pi}{2}} \frac{du}{[\sin u(1 - \frac{r}{R} \sin u)]^{\frac{1}{2}}}.$$

It follows from [67, pages 369 and 950], that the analytic function $T(\frac{r}{R})$ has the following expansion in series

$$T\left(\frac{r}{R}\right) = \sum_{n=0}^{\infty} \frac{2a_n}{n!} \left(\frac{r}{R}\right)^n, \quad a_n = \frac{1 \times 3 \times \dots \times (2n-1)}{2^n} \frac{\Gamma(\frac{1}{2})\Gamma(2n + \frac{1}{4})}{\Gamma(2n + \frac{3}{4})}.$$

Therefore, from $dv/dt = 2R$, it follows that an arc of the asymptotic line that starts at the point $(\frac{\pi}{2}, v_0)$ ends at the point $(\frac{3\pi}{2}, v_1)$, where v_1 is given by $v_1 = 2RT + v_0$.

So the return map $\Pi : \{u = -\frac{\pi}{2}\} \rightarrow \{u = -\frac{\pi}{2}\}$ is given by

$$\Pi(v_0) = v_0 + 4RT\left(\frac{r}{R}\right).$$

As T is clearly non constant, it is possible to select r and R such that the rotation number of Π is irrational. For properties of rotation number see [112] and [128]. \square

6.7 Further developments on asymptotic lines

The geometric approach presented in this Chapter was taken from [56] and [140].

The local analysis of asymptotic lines near parabolic points was also studied in [167], [12] and [93].

The dynamical aspects of asymptotic lines is a source of many difficult problems.

One is the “Closing Lemma”, i.e. the *elimination of recurrent asymptotic lines disjoint from the parabolic set and of folded recurrent asymptotic lines*.

The first kind of recurrences was studied by R. Garcia and J. Sotomayor [64] in embedded torus of \mathbb{S}^3 with suitable deformations of the Clifford torus. Examples of the second kind of recurrence in the torus of revolution have been given in [140].

Another problem is about *the existence of isolated regular closed asymptotic lines in tubes and also a continuum of closed asymptotic lines*. This kind of question is important in the open problem of rigidity of compact surfaces of genus different from zero and also in the study of complete surfaces with negative Gaussian curvature.

Another kind of question is about the structure of the parabolic set of a surface which is the graph $(x, y, p(x, y))$ of a polynomial $p \in \mathbb{R}[x, y]$. See

[6, 8] and [123].

A concrete question is the following: Given a function \mathcal{K} provide local conditions to have \mathcal{K} as the Gaussian curvature of a surface. This question was considered by V. Arnold,[7].

6.8 Exercises and Problems

6.8.1. Consider the embedded tube defined by

$$\alpha(s, v) = c(s) + r \cos vn(s) + r \sin vb(s), \quad r > 0.$$

Here c is a closed Frenet curve with $k > 0$ and torsion τ .

i) Show that the hyperbolic region of α is diffeomorphic to a cylinder and the parabolic set is union of two regular curves.

ii) Characterize the parabolic points (regular, folded saddle, etc.) in terms of (k, τ) and their derivatives.

iii) Examine the possible global behavior of asymptotic foliations in the tube.

iv) Let c be a connected component of $f^{-1}(0) \cap g^{-1}(0)$, where $f(x, y, z) = x^2 + y^2 + z^2 - 1$ and $g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$. Classify the parabolic points of the tube with center c .

6.8.2. Consider the surface S defined by the graph of the polynomial $p(x, y) = \frac{k}{2}(x^2 + y^2) + x^3 - 3xy^2$.

i) Determine the parabolic set of S .

ii) Classify the parabolic points of S according to the local behavior of the asymptotic foliations.

iii) Examine the global behavior of asymptotic foliations on S , including the behavior near the infinity.

6.8.3. Give examples of surfaces (try algebraic) such that:

- i) The parabolic set is the union of two regular curves and the hyperbolic region is diffeomorphic to a cylinder.
- ii) The parabolic set is the union of three regular curves and the hyperbolic region is diffeomorphic to an oriented boundary surface S with Euler characteristic equal to $\psi(S) = -1$.
- iii) The parabolic set is a regular curve and the hyperbolic region is diffeomorphic to a disk.

6.8.4. Show that the quartic algebraic surface defined by

$$p(x, y, z) = 3z^4 + 2(1 + 4xy)z^2 - 2(x^2 + y^2)^2 + 8xy - 1 = 0,$$

determines a smooth negatively curved surface $S \subset \mathbb{R}^3$ homeomorphic to the doubly punctured torus, which has Euler characteristic equal to -2 .

- i) Perform a qualitative analysis of the asymptotic foliations near the ends of $p^{-1}(0)$.
- ii) Visualize the shape of $p^{-1}(0)$. See [31].

6.8.5. Analyze the behavior of asymptotic lines near the Whitney singularities of an immersion of a surface in \mathbb{R}^3 . See [171].

6.8.6. Let $N : S \rightarrow \mathbb{S}^2$ be the normal map associated to a surface $S \subset \mathbb{R}^3$. Classify the parabolic points of S in terms of the singularities (folds and cusps) of N . See [167] and [12].

6.8.7. Let $\alpha : \mathbb{M} \rightarrow \mathbb{S}^3$ be a smooth immersion of a surface \mathbb{M} . Define the first and second fundamental forms of α with respect to the metric of \mathbb{S}^3 induced by the canonical metric of \mathbb{R}^4 and to the normal $N_\alpha = (\alpha_u \wedge \alpha_v \wedge \alpha) / |\alpha_u \wedge \alpha_v \wedge \alpha|$.

A curve $c : I \rightarrow \mathbb{M}$ is called an asymptotic line if $II_\alpha(c(s))(c'(s), c'(s)) = 0$.

- i) Let $\alpha : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ defined by $\alpha(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v)$. The surface $\alpha(\mathbb{S}^1 \times \mathbb{S}^1)$ is the Clifford torus..

- ii) Write the differential equation of asymptotic lines of α .
- iii) Show that the asymptotic lines of the Clifford torus are defined globally and are circles. Compute explicit parametrizations of these circles.

6.8.8. Let $\alpha : S \rightarrow \mathbb{R}^4$ be a smooth immersion of a compact two dimensional surface S and suppose that there exists a unit normal vector field N_α along α .

Define the second fundamental form of α relative to N_α by the equation

$$\begin{aligned} II_\alpha(u, v)(du, dv) &= \langle D^2\alpha(u, v)(du, dv)^2, N_\alpha \rangle \\ &= edu^2 + 2fdudv + gdv^2. \end{aligned}$$

Study the asymptotic lines of α relative to N_α .

6.8.9. Give an example of a connected ruled surface in \mathbb{R}^3 having two hyperbolic asymptotic lines.

6.8.10. Consider the parametric surface defined by

$$\begin{aligned} x(u, v) &= \frac{1}{2A}[(a+u)^3 + (a+v)^3], \quad y(u, v) = \frac{1}{2B}[(b+u)^3 + (b+v)^3] \\ z(u, v) &= \frac{1}{2C}[(a+c)^3 + (a+c)^3] \end{aligned}$$

Obtain the differential equation of asymptotic lines and show that the solutions are given by $u \pm v = c$.

6.8.11. Consider the parametric surface defined by

$$\begin{aligned} x(u, v) &= A(u-a)^m(v-a)^m, \quad y(u, v) = B(u-b)^m(v-b)^m \\ z(u, v) &= C(u-c)^m(v-c)^m, \quad m \in \mathbb{N}. \end{aligned}$$

Obtain the differential equation of the asymptotic lines and find the solutions.

6.8.12. Show that there is no triple system of surfaces cutting mutually in the asymptotic lines of these surfaces. See [43].

6.8.13. Consider the tube defined by

$$\alpha(s, v) = c(s) + r \cos vt(s) + r \sin vn(s), \quad r > 0.$$

Here c is a Frenet curve with $k > 0$ and torsion τ and $\{t, n, b\}$ is the Frenet frame.

i) Characterize the curves c such that the tube defined above is a regular surface.

ii) Analyze the geometry of the tube, classifying the elliptic, parabolic and hyperbolic points and also the singular points.

iii) Analyze the principal and asymptotic lines of the tube.

6.8.14. Develop a study of extrinsic geometry of surfaces of codimension two in \mathbb{R}^4 . In particular analyze the asymptotic lines, mean directionally curved lines and axial curvature lines. See [24], [57], [110], [145] and [143].

6.8.15. Consider the surface S parametrized by $(u, v, h(u, v))$ where,

$$\begin{aligned} h(u, v) = & \frac{1}{2}(au^2 + bv^2) + \frac{1}{6}(Au^3 + 3Bu^2v + 3Cuv^2 + Dv^3) \\ & + \frac{1}{24}(\alpha u^4 + 4\beta u^3v + 6\gamma u^2v^2 + 4\varepsilon uv^3 + \delta v^4) + \dots \end{aligned} \quad (**)$$

Let $c = c(s)$ be an asymptotic line of S passing through 0 and tangent to the u axis. Let k and τ be, respectively, the curvature and the torsion of c at 0. Find the values of k and τ .

Determine the value of the curvature of the branch of the plane curve $h(u, v) = 0$ at 0 which is tangent to the u axis.

6.8.16. Consider the surface defined parametrically by

$$\alpha(u, v) = (2e^v(\sin u - u \cos u), 2e^v(\cos u + u \sin u), u^2 + 3v)$$

- i) Calculate the second fundamental form of α .
- ii) Analyze the asymptotic configuration of α .

6.8.17. Consider a differential equation on the plane \mathbb{R}^2 defined by $dy/dx = f(x, y)$ and suppose that

$$\frac{\partial^2 \theta}{\partial x^2} + 2f \frac{\partial^2 \theta}{\partial x \partial y} + f^2 \frac{\partial^2 \theta}{\partial y^2} = 0, \quad \theta = \arctan f.$$

- i) Show that there is a graph surface $(x, y, h(x, y))$ such that one family of asymptotic lines is defined by the differential equation $y' = dy/dx = f(x, y)$.
- ii) Analyze various explicit examples of differential equations $y' = f$ verifying the partial differential equation above.

6.8.18. Consider a compact oriented surface \mathbb{M} of \mathbb{R}^3 . Let $\mathbb{M}_+ = \{p \in \mathbb{M} : \mathcal{K}(p) > 0\}$, $\mathbb{M}_- = \{p \in \mathbb{M} : \mathcal{K}(p) < 0\}$ and suppose that

A) $\int_{\mathbb{M}_+} \mathcal{K} dS = 4\pi$,

B) for every point $p \in \partial\mathbb{M}_-$ we have that $d\mathcal{K}(p) \neq 0$.

- i) Show that the set $\{p \in \mathbb{M} : \mathcal{K}(p) = 0\}$ is a finite union of regular curves γ_i (parabolic curves).
- ii) Let $N : \mathbb{M} \rightarrow \mathbb{S}^2$ the normal Gauss map. Show that Condition A implies that $N|_{\mathbb{M}_+}$ is a double covering map.
- iii) Show that conditions A e B imply that the parabolic curves γ_i are convex planar curves.
- iv) Show that each connected component of \mathbb{M}_- is diffeomorphic to a cylinder.
- v) (**Open problem:**) Is there a closed asymptotic line contained in the interior of \mathbb{M}_- ? See [122] and [98].

6.8.19. Consider the real ellipsoid $E(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$.

- i) Write the differential equation of asymptotic lines of the ellipsoid.
- ii) Consider the differential equation above complexified and analyze the singular foliations in the complex quadric E_C obtained by complexification of the real ellipsoid.

6.8.20. Let $P(x, y)$ be a polynomial of degree $n \geq 2$ and consider the polynomial surface \mathbb{M} defined by the graph $z = P(x, y)$. Let $H(p) = p_{xx}p_{yy} - p_{xy}^2$ the Hessian of p .

- i) Show that $H(p) = \left(\frac{p_{xx} + p_{yy}}{2}\right)^2 - \left(\frac{p_{xx} - p_{yy}}{2}\right)^2 - p_{xy}^2$.
- ii) Show that the Hessian curve $\{(x, y) : H(p) = 0\}$ is an algebraic curve of degree $l \leq 2(n - 2)$.
- iii) Show that a compact hessian curve has at most $(2n - 5)(n - 3) + 1$ ovals and a non-compact hessian curve has at most $(2n - 5)(n - 3)$ ovals and $2(n - 2)$ unbounded components. See Harnack's Theorem in [14].
- iv) Let $p(x, y) = 12x^2 + 2xy - 2y^2 + 10y^3 + 3xy^2 - 10x^2y - 13x^3 - 11y^4 + 6xy^3 + 9x^2y^2 - 2x^3y - x^4$. Show that the Hessian curve of p is compact, regular and has 4 ovals. See [123].
- v) Analyze items i), ii) and iii) above for an implicit polynomial surface defined by $f(x, y, z) = 0$.

6.8.21. Let $\alpha(u, v) = (u, v, h(u, v))$ be a local parametrization of a surface \mathbb{M} and let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an invertible linear map and consider the surface $\beta(u, v) = (A \circ \alpha)(u, v)$.

- i) Show that the asymptotic lines of β and α are the same. More precisely, show that the differential equation of asymptotic lines of β is given by:

$$\det A[h_{uu}du^2 + 2h_{uv}dudv + h_{vv}dv^2] = 0.$$

ii) Generalize the item i) when $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a projective transformation and show that the asymptotic lines of β and α are the same.

6.8.22. Give an explicit example of a closed Frenet curve c with positive curvature $k > 0$ and torsion $\tau > 0$ such that $\int_c dk/\sqrt{\tau} \neq 0$. **Suggestion:** Try to find torodail helices.

6.8.23. Consider the Möbius band defined by

$$\alpha(u, v) = \left((R - v \sin \frac{u}{2}) \sin u, (R - v \sin \frac{u}{2}) \cos u, v \cos \frac{u}{2} \right),$$

where $u \in [0, 2\pi]$ and $v \in [-r, r]$, $r > 0$ small and $R > 2r0$.

i) Compute the Gauss curvature of α .

ii) Describe the asymptotic configuration of α .

iii) Consider the family of spheres $\Sigma_{(u,v)}(\epsilon)$ of radius $\epsilon > 0$ with center $\alpha(u, v)$. Show that the envelope of $\Sigma_{(u,v)}(\epsilon)$ is a regular and oriented surface \mathbb{M} of class C^1 diffeomorphic to the torus.

iv) Analyze the asymptotic and principal configurations of \mathbb{M} . As the surface is only C^1 take in account the new singularities of asymptotic and principal lines.

6.8.24. Let h be a local equivalence of both principal and asymptotic configurations of a surface immersed in \mathbb{R}^3 , i.e., h is a local homeomorphism which is a equivalence of the 4-web defined by the principal and asymptotic foliations.

i) Investigate the properties of h .

ii) Is h be an isometry of the ambient space?

iii) Investigate the punctual holonomy associated to the 4-web in minimal surfaces of \mathbb{R}^3 or \mathbb{S}^3 . This open problem is based on a question raised by R. Roussarie.

Chapter 7

Geodesics on Surfaces of \mathbb{R}^3

7.1 Introduction

Geometric and dynamical aspects of geodesics on surfaces is a classical subject in Differential Geometry. See for example, [3], [5], [16], [37], [92], [95], [15], [126], [136], [169], [170].

In this chapter classical results are reviewed and the derivative of the Poincaré map associated to a closed geodesic line will be obtained in an elementary way. Also will be discussed the geodesics on the ellipsoid, surfaces of revolution, convex surfaces and quadrics in \mathbb{R}^n .

7.2 General Results

Given a regular surface \mathbb{M} of class C^r the tangent bundle of \mathbb{M} , $T\mathbb{M} = \{(p, v) : p \in \mathbb{M} \text{ and } v \in T_p\mathbb{M}\}$ is a differentiable manifold of

dimension 4 and of class C^{r-1} . The map $\pi : R\mathbb{M} \rightarrow \mathbb{M}$, $\pi(p, v) = p$ is a fibration of class C^{r-1} .

Proposition 7.2.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r . Then there exists a canonical immersion of the tangent bundle $T\mathbb{M}$ in \mathbb{R}^6 .*

Proof. Let $x^{-1} : U \subset \mathbb{M}$ be a local parametrization of \mathbb{M} . Then $\alpha \circ x^{-1}$ is a local parametrization of $\alpha(\mathbb{M})$. By simplicity $\alpha \circ x^{-1}$ will be denoted by α .

Now define $\beta : U \times \mathbb{R}^2 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3$ by $\beta(u, v, x, y) = (\alpha(u, v), x\alpha_u + y\alpha_v)$.

Let $\gamma(t) = \beta((u(t), v(t), x(t), y(t)))$. Then $\gamma' = D\beta(u', v', x', y')$. It follows that

$$\gamma' = (u'\alpha_u + v'\alpha_v, x'\alpha_u + xu'\alpha_{uu} + (xv' + yu')\alpha_{uv} + y'\alpha_v + yv'\alpha_{vv}).$$

As $\{\alpha_u, \alpha_v\}$ is linearly independent, $\gamma' = 0$ implies that $u' = v' = 0$ and $x' = y' = 0$. This shows that $D\beta$ is injective. This ends the proof. \square

Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 3$. The geodesic lines of α are defined by the condition that $k_g = 0$ in the Darboux frame, see equation (6.2), page 128, of a regular immersed curve γ .

As shown in Chapter 1 the geodesics are defined by a second order differential equation which is homogenous in the derivatives. See equations (1.8), (1.9) and (1.10) in Section 1.8.

This property implies that the geodesics are independent of parametrization and so they are geometric entities.

The lifting of a geodesic γ to the tangent bundle $T\mathbb{M}$ is given by $\tilde{\gamma}(s) = (\gamma(s), \gamma'(s))$.

There exists a Hamiltonian vector field X_H defined in $T\mathbb{M}$ such that $\tilde{\gamma}$ is an integral curve of X_H . For more details see [5], [41], [11], [150]. In a local chart (u, v, x, y) the vector field X_H is defined by:

$$\begin{aligned} u' &= x, & v' &= y, \\ x' &= -[x^2\Gamma_{11}^1 + 2xy\Gamma_{12}^1 + y^2\Gamma_{22}^1], \\ y' &= -[x^2\Gamma_{11}^2 + 2xy\Gamma_{12}^2 + y^2\Gamma_{22}^2], \end{aligned} \tag{7.1}$$

where Γ_{ij}^k are given by equation (1.3), page 19.

The flow of X_H is called the *geodesic flow*. It is called *complete* when the domain of every integral curve is \mathbb{R} .

An important aspect of X_H is that it is non singular and its integral curves are transversal (orthogonal with respect to natural metrics) to the fibers $\pi^{-1}(p)$, $p \in \mathbb{M}$ of the fibration π .

Remark 7.2.1. *A natural metric in the tangent bundle is the one induced from \mathbb{R}^6 by the immersion β given in the proof of Proposition 7.2.1. Other natural intrinsic metrics in $T\mathbb{M}$ are the metrics of Sasaki and Cheeger - Gromoll. See [28], [68] and [95].*

In what follows (\mathbb{M}, g) will be \mathbb{M} endowed with the induced Riemannian metric $g_p = I_p = Edu^2 + 2Fdudv + Gdv^2$ associated to an immersion $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$.

A useful concept in the study of geodesics, introduced in Chapter 1, is of the exponential map defined by $\exp : U \subset T\mathbb{M} \rightarrow \mathbb{M} \times \mathbb{M}$ defined by:

$\exp(p, v) = (p, \exp_p(v))$, where $\exp_p(v) = \gamma(1, p, v) = \gamma(|v|, p, \frac{v}{|v|})$, where γ is the geodesic through p with $\gamma'(0) = v$ and U is an open set.

Theorem 7.2.2 (Hopf-Rinow). *Let (\mathbb{M}, g) be a Riemannian manifold, complete as a metric (length) space (\mathbb{M}, d) . Then given any two points $p, q \in \mathbb{M}$ there is a geodesic $\gamma : \mathbb{R} \rightarrow \mathbb{M}$ such that $\gamma(0, p) = p$ and $\gamma(l_{p,q}, p) = q$, where $l_{p,q} = d(p, q)$. Moreover, when α is an embedding the minimizing geodesic γ is a simple curve in $\alpha(\mathbb{M})$.*

Proof. There are several presentations of the proof in this theorem. For instance, see [40] and [165].

The basic fact is that (\mathbb{M}, d) where d is the distance induced by the metric g is a metric space and the exponential map is globally defined in $T\mathbb{M}$. In fact, (M, d) is a length space. Recall that a metric space is called length space when it has the middle point property. See [35].

Here will be shown that a minimizing geodesic is a simple curve.

By contradiction, suppose that γ has a transversal intersection point r given by $\gamma(s_1) = \gamma(s_2) = r$ with $0 < s_1 < s_2 < l_{p,q}$ and $\gamma'(s_1) \neq \gamma'(s_2)$. See Fig. 7.1.

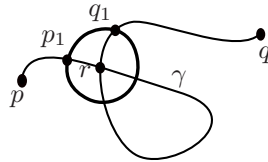


Figure 7.1: Geodesic between the points p and q with self intersection at the point r .

Let $p_1 = \gamma(\bar{s})$, $\bar{s} < s_1$, and $q_1 = \gamma(\tilde{s})$, $\tilde{s} > s_2$ be the points near r in the boundary of a convex ball $B(r, \epsilon)$. The radius ϵ can be defined such that the arc $\gamma|_{[s_1, s_2]}$ is not contained in $B_\epsilon(r)$. Let $\gamma_1 = \gamma|_{[0, \bar{s}]}$ and $\gamma_2 = \gamma|_{[\tilde{s}, l_{p,q}]}$. It follows that $d(p_1, q_1) < d(p_1, r) + d(r, q_1)$. Let

γ_{p_1, q_1} be the unique geodesic connecting p_1 to q_1 . By construction $d(p_1, q_1) < l_{\gamma(\bar{s}), \gamma(\bar{s})}$.

Now consider the geometric curve (connected sum) $\tilde{\gamma} = \gamma_1 * \gamma_{p_1, q_1} * \gamma_2$, that connect the points p and q , formed by juxtaposition of three arcs of geodesics.

It is clear that $l(\tilde{\gamma}) < l_{p, q}$ and so a minimizing geodesic can not have self intersection. \square

7.3 Closed Geodesics on Immersed Surfaces of \mathbb{R}^3

Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 3$.

Recall that a geodesic of \mathbb{M} is a regular curve $\gamma : \mathbb{R} \rightarrow \mathbb{M}$ such that $k_g = 0$, or equivalently, when $\tilde{\gamma} : \mathbb{R} \rightarrow T\mathbb{M}$, $\tilde{\gamma} = (\gamma, \gamma')$ is an integral curve of the differential equation (1.9) given in page 28.

A geodesic γ is *closed* when $\tilde{\gamma}$ is a periodic orbit of the geodesic flow.

Lemma 7.3.1. *Let $\gamma : I \rightarrow \mathbb{M}^2$ be a geodesic line parametrized by arc length s and of length l . Suppose that γ is disjoint from the umbilic set of α . Then the Darboux frame is given by:*

$$\begin{aligned} T' &= k_n N, & (N \wedge T)' &= \tau_g N \\ N' &= -k_n T - \tau_g N \wedge T \end{aligned} \tag{7.2}$$

Moreover, $\tau_g = (k_2 - k_1) \sin \theta \cos \theta$ and $k_n = k_1 \cos^2 \theta + k_2 \sin^2 \theta$, where k_1 and k_2 are the principal curvatures and θ is the angle between $\gamma'(u) = T(u)$ and the principal direction corresponding to the principal curvature k_1 .

Proof. From the Euler equation $k_n = k_1 \cos^2 \theta + k_2 \sin^2 \theta$. Also the geodesic torsion is given by $\tau_g = (k_2 - k_1) \sin \theta \cos \theta$. \square

Lemma 7.3.2. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 6$, and γ be a closed geodesic curve of α , parametrized by arc length u and of length l . Then the expression,*

$$\alpha(u, v) = \alpha \circ \gamma(u) + v(N \wedge T)(u) + \left[\frac{1}{2} k_n^\perp(u) v^2 + \frac{1}{6} A(u, v) v^3 \right] N(u),$$

defines a local chart (u, v) of class C^{r-5} in a neighborhood of γ .

Proof. The curve γ is of class C^{r-1} and the map $\alpha(u, v, w) = \gamma(u) + v(N \wedge T)(u) + wN(u)$ is of class C^{r-2} and it is a local diffeomorphism in a neighborhood of the u axis. In fact $[\alpha_u, \alpha_v, \alpha_w](u, 0, 0) = 1$. Therefore there is a function $W(u, v)$ of class C^{r-2} such that $\alpha(u, v, W(u, v))$ is a parametrization of a tubular neighborhood of $\alpha \circ \gamma$.

Now for each u , $\alpha(u, v, W(u, v))$ is just a parametrization of the curve of intersection of $\alpha(\mathbb{M})$ and the normal plane generated by $\{(N \wedge T)(u), N(u)\}$. This curve of intersection is tangent to $(N \wedge T)(u)$ at $v = 0$ and notice that $k_n^\perp = k_n(N \wedge T)(u)$. Therefore,

$$\alpha(u, v, W(u, v)) = \gamma(u) + v(N \wedge T)(u) + \left[\frac{k_n^\perp}{2} v^2 + A(u, v) \frac{v^3}{6} \right] N(u),$$

where A is of class C^{r-5} . This ends the proof. \square

Remark 7.3.1. *The coordinates (u, v) in the Lemma 7.3.2 are not the usual normal coordinates along γ . The curves $u = u_0$ are not, in general, geodesics, but $k_g^\perp(u, v) = 0$ at $v = 0$.*

In the chart (u, v) constructed above, holds:

$$\begin{aligned}
 E(u, v) &= 1 + (\tau_g^2 - k_n k_n^\perp) v^2 + h.o.t = 1 - \mathcal{K} v^2 + O(v^3) \\
 F(u, v) &= \frac{1}{2} k_n^\perp \tau v^2 + O(v^3), \quad G(u, v) = 1 + (k_n^\perp)^2 v^2 + O(v^3) \\
 e(u, v) &= k_n + O(v), \quad f(u, v) = \tau_g(u) + O(v) \\
 g(u, v) &= k_n^\perp + O(v)
 \end{aligned} \tag{7.3}$$

where in the expressions above, $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$, $G = \langle \alpha_v, \alpha_v \rangle$, $e = \langle \alpha_u \wedge \alpha_v, \alpha_{uu} \rangle$, $f = \langle \alpha_u \wedge \alpha_v, \alpha_{uv} \rangle$ and $g = \langle \alpha_u \wedge \alpha_v, \alpha_{vv} \rangle$.

Therefore from equation (1.3), page 19, the Christoffel symbols are given by:

$$\begin{aligned}
 \Gamma_{11}^1 &= O(v^2), \quad \Gamma_{12}^1 = v(\tau_g^2 - k_n k_n^\perp) + O(v^2), \quad \Gamma_{22}^1 = k_n^\perp \tau_g v + O(v^2) \\
 \Gamma_{12}^2 &= O(v^2), \quad \Gamma_{11}^2 = v(k_n k_n^\perp - \tau_g^2) + O(v^2), \quad \Gamma_{22}^2 = (k_n^\perp)^2 v + O(v^2).
 \end{aligned} \tag{7.4}$$

From equation (1.9), page 28, the differential equations for geodesic lines is given in the coordinates above by:

$$\begin{aligned}
 \frac{dv}{du} &= w \\
 \frac{dw}{du} &= [k_n \tau_g w^3 - k_n^2 w^2 + (\tau_g^2 - k_n k_n^\perp)] v + O(v^2)
 \end{aligned} \tag{7.5}$$

On the unit tangent bundle $T_1\mathbb{M}$ the geodesics are the integral curves of a vector field and so the Poincaré transition map $\pi : \Sigma_0 \rightarrow \Sigma_1$ is well defined between two transversal sections Σ_0 and Σ_1 of a regular orbit γ .

In the local chart (u, v, w) , $w = dv/du$, the differential equation (7.5) have the line $\{v = 0, w = 0\}$ as a regular solution. The transition map between the sections $\Sigma_0 = \{u = 0\}$ and $\Sigma_1 = \{u = u_1\}$ is

defined by

$$\pi(v_0, w_0) = (v(t(v_0, w_0), v_0, w_0), w(t(v_0, w_0), v_0, w_0)).$$

Here $t(v_0, w_0)$ is the first time of the intersection of the orbit through $(0, v_0, w_0)$ with the section Σ_1 .

When the differential equation (7.5) is periodic in u , i.e., the geodesic parametrized by $\{v = 0, w = 0\}$ is closed the transition map between the sections $\Sigma_0 = \{u = 0\}$ and $\Sigma_1 = \{u = l\}$ is the well known *Poincaré return map*. See [128] and [159].

Proposition 7.3.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 4$, and γ be a closed geodesic curve of α , parametrized by arc length and of length l . Then the derivative of the Poincaré return map $\pi(v_0, w_0) = (v(l, v_0, w_0), w(l, v_0, w_0))$ at $(v_0, w_0) = (0, 0)$ associated to equation (7.5) satisfies the following linear system:*

$$\frac{\partial}{\partial u} \begin{pmatrix} \frac{\partial v}{\partial v_0} & \frac{\partial v}{\partial w_0} \\ \frac{\partial w}{\partial v_0} & \frac{\partial w}{\partial w_0} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{K}(u) & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial v_0} & \frac{\partial v}{\partial w_0} \\ \frac{\partial w}{\partial v_0} & \frac{\partial w}{\partial w_0} \end{pmatrix} \quad (7.6)$$

Moreover, $\det(\pi'(0)) = 1$.

Proof. From equation (7.3) it follows that $\mathcal{K} = k_n k_n^\perp - \tau_g^2$. Therefore differentiating equation (7.5) the result follows. The assertion that $\det(\pi'(0)) = 1$ follows from Liouville's formula for systems of linear ODE's. See [159]. \square

Write the linear differential equation (7.6) as $X' = A(u)X$, $A(u + l) = A(u)$, $X(0) = I$.

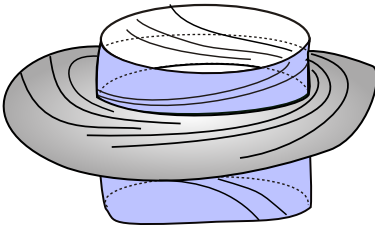
The fundamental matrix is given by $X(l)$. The geodesic γ is called *hyperbolic* when the eigenvalues (Floquet multipliers) λ_1, λ_2 of $X(l)$

are real and satisfy $0 < |\lambda_1| < 1 < |\lambda_2|$. The geodesic γ is called *elliptic* when the eigenvalues $\lambda_1, \lambda_2 = \bar{\lambda}_1$ of $X(l)$ are complex and satisfy $|\lambda_1| = |\lambda_2| = 1$.

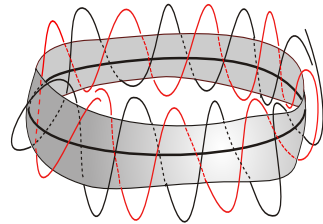
The geodesic γ is called *parabolic* when λ_1, λ_2 are real and $|\lambda_1| = |\lambda_2| = 1$.

A closed geodesic γ is called *stable* (Liapunov stable) when there is a C^0 -tubular neighborhood $\mathcal{U}_\gamma \subset T\mathbb{M}$ such that all integral curves $\varphi(t)$ of the geodesic flow X with $\varphi(0) \in \mathcal{U}_\gamma$ is contained in \mathcal{U}_γ for every $t \in \mathbb{R}$.

Proposition 7.3.2. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$ be an immersion of class C^r , $r \geq 4$, and γ be a closed geodesic curve of α , parametrized by arc length and of length l . Suppose that $\mathcal{K}|_\gamma = k_0$. Then $\det \pi'(0) = 1$ and the eigenvalues of $\pi'(0)$ are given by $\lambda_1 = \exp(l\sqrt{-k_0})$ and $\lambda_2 = \exp(-k\sqrt{-k_0})$. Therefore, if $k_0 < 0$ the geodesic γ is hyperbolic. If $k_0 > 0$ and $l\sqrt{k_0} \neq n\pi, n \in \mathbb{N}$, the geodesic γ is elliptic and stable.*



Hyperbolic geodesic



Elliptic geodesic

Figure 7.2: Behavior of geodesics near a closed geodesic line

Proof. The result follows from direct integration of a linear system

with constant coefficients (7.6) and classification of them. See [128] and [159] for a detailed exposition on linear systems. \square

Let $\gamma : \mathbb{R} \rightarrow \mathbb{M}$ be a geodesic parametrized by arc length s with $\gamma(0) = p$. A *Jacobi equation* along γ is the second order differential equation $(\dagger) \quad y'' + \mathcal{K}(\gamma(s))y = 0$. By the theorem of existence and uniqueness a solution of the Jacobi equation is determined by the initial conditions: $y(0) = 0, y'(0) = v$. The point $q = \gamma(s_1)$ is called a *conjugate point of p along γ* if there is a non zero solution of (\dagger) such that $y(0) = 0$ and $y(s_1) = 0$. A related concept is that of *Jacobi fields*. The conjugate points are characterized by the singularities of the exponential map. See [41] and [94].

Proposition 7.3.3. *Let γ be a closed geodesic curve and suppose that $\mathcal{K}|_\gamma < 0$. Then γ is hyperbolic and there are no conjugate points along γ .*

Proof. Let γ of length l parametrized by arc length s . The periodic linear differential equation (7.6) is equivalent to a second order linear equation $(*) \quad y'' + \mathcal{K}(s)y = 0$.

Let y_1 and y_2 two linearly independent solutions of $(*)$ with initial conditions $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.

As $\mathcal{K}(s) < 0$ it follows that y_1 and y_2 are strictly convex functions and positive in the interval $(0, \infty)$. Also y_i' are increasing functions. So there are no conjugate points along γ . In particular $y_1(l) > 1$ and $y_2'(l) \geq 1$. Define $\sigma = y_1(l) + y_2'(l)$. The eigenvalues of the return map of Poincaré are the eigenvalues of the Floquet matrix

$$X(l) = \begin{pmatrix} y_1(l) & y_2(l) \\ y_1'(l) & y_2'(l) \end{pmatrix}.$$

By Liouville formula it follows that $\det(X(1)) = 1$ and so the eigenvalues are defined by the equation $\lambda^2 - \sigma\lambda + 1 = 0$. As $\sigma > 2$ it follows that the eigenvalues are positive and different from one. So it follows that γ is hyperbolic. \square

Remark 7.3.2. *In $T\mathbb{M}$ near a hyperbolic geodesic $\tilde{\gamma}$ of the geodesic flow there are two invariant surfaces $W^s(\tilde{\gamma})$ (stable manifold) and $W^u(\tilde{\gamma})$ (unstable manifold) of class C^{r-2} such that $W^s(\tilde{\gamma})$ and $W^u(\tilde{\gamma})$ are transversal along $\tilde{\gamma}$. See Fig. 7.7, page 180. This proposition has a natural generalization to Riemannian manifolds with hypothesis on the sectional curvatures. See [95, page 276].*

Remark 7.3.3. *For an analysis of the geodesic flow near closed geodesics on Riemannian manifolds see [95] and [96]. The hyperbolic systems are part of a rich theory of Dynamical Systems and Ergodic Theory, including the so called Anosov and Morse-Smale systems. See [3], [22], [107], [128], [129], [130], [147], [151] and [153]. For the study of structural stability and ergodic stability of the geodesic flow and time-one maps on Riemannian manifolds of negative curvature see [3], [104] and [180].*

Proposition 7.3.4. *Let γ be a closed geodesic curve parametrized by arc length s and of length l . Suppose that $\mathcal{K}|_\gamma \geq 0$ and $\int_0^l \mathcal{K}(s)ds < 4/l$. Then γ is stable.*

Proof. Let $y \neq 0$ be a solution of the second order differential equation $y'' + \mathcal{K}(s)y(s) = 0$. If γ is not stable then, by Floquet theorem, [81], it follows that $y(s) = Y(s)e^{\lambda s}$, with Y l -periodic and $0 \neq \lambda \in \mathbb{R}$. Therefore $y(s+l) = \lambda_1 y(s)$, $\lambda_1 \neq 1$ for every $s \in \mathbb{R}$.

If $y(s) \neq 0$ it follows that $\int_0^l \frac{y''}{y} ds + \int_0^l \mathcal{K}(s) ds = 0$. Integration

by parts gives, $\int_0^l \frac{y''}{y} ds = \frac{y'}{y} \Big|_0^l + \int_0^l \left(\frac{y'}{y}\right)^2 ds = 0 + \int_0^l \left(\frac{y'}{y}\right)^2 ds \geq 0$.

So in this case is established a contradiction.

Next suppose that y has two adjacent zeros $y(a) = y(b) = 0$ with $0 \leq b - a \leq l$ and $y|(a, b) > 0$.

Let $y_{max} = \max\{y(s), s \in [a, b]\} > 0$ and write $y_{max} = y(a+l_1) = y(b-l_2)$, $l_1+l_2 = b-a$. Therefore for the convergent integral $\int_a^b \left|\frac{y''}{y}\right| ds$ it follows that

$$\int_a^b \left|\frac{y''}{y}\right| ds > \int_a^b \left|\frac{y''}{y_{max}}\right| ds = \frac{y'(a) - y'(b)}{y_{max}}.$$

Here it was used that $y''|(a, b) < 0$ and $y'|(a, b)$ is decreasing. It follows by Rolle's Theorem that $y'(s_3) = y_{max}/l_1 < y'(a)$ and $y'(s_4) = -y_{max}/l_2 \leq -y'(b)$, with $s_3 \in (a, a+l_1)$ and $s_4 \in (b-l_2, b)$. So it follows that

$$\begin{aligned} \int_a^b \left|\frac{y''}{y}\right| ds &= \int_a^{a+l_1} \left|\frac{y''}{y}\right| ds + \int_{b-l_2}^b \left|\frac{y''}{y}\right| ds \\ &> \frac{1}{l_1} + \frac{1}{l_2} = \frac{l_1+l_2}{l_1 l_2} \geq \frac{4}{l_1+l_2} = \frac{4}{b-a} > \frac{4}{l}. \end{aligned}$$

Again a contradiction is obtained since $\int_0^l \mathcal{K}(s) ds < 4/l$. \square

Remark 7.3.4. *This kind of result is part of a general theory of Hill's equations $x'' + a(s)x = 0$. See [19], [81] and [106].*

Proposition 7.3.5. *Let γ be a closed geodesic curve and suppose that $1/4 < \mathcal{K}|\gamma < 1$ or, more general, suppose that $\max(\mathcal{K}|\gamma) \leq 4\min(\mathcal{K}|\gamma)$. Then γ is stable.*

Remark 7.3.5. *This result also follows from the analysis of the Hill's equation $y'' + \mathcal{K}(s)y = 0$. The proof, on the higher dimensional case substituting \mathcal{K} by sectional curvature, depends on the notion of index of a geodesic. See [95] and [172].*

7.4 Geodesics on the Ellipsoid

In this section an elementary approach, based on the analysis of implicit differential equations, will be followed to describe the geodesics of the ellipsoid of \mathbb{R}^3 with three different axes.

For a more detailed exposition on this subject see for example [95, pages 303-322]. See also [11], [17], [44], [84] and [117].

Proposition 7.4.1. *Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ parametrized by ellipsoidal coordinates (u, v) . Then the function*

$$\begin{aligned} \mathcal{J}(u, v, u', v') &= v \cos^2 \Omega + u \sin^2 \Omega = \frac{vEu'^2 + uGv'^2}{Eu'^2 + Gv'^2}, \\ \tan \Omega &= \frac{v'}{u'} \sqrt{\frac{G(u, v)}{E(u, v)}} \end{aligned} \tag{7.7}$$

is a first integral of equation the differential equation of geodesics of the ellipsoid.

Remark 7.4.1. *A function $\mathcal{J} : T\mathbb{M} \rightarrow \mathbb{R}$ is a first integral for the geodesic flow of a Riemannian metric g if it is constant on any orbit of the geodesic flow. This means that if $\gamma : \mathbb{R} \rightarrow \mathbb{M}$ is a geodesic then $\mathcal{J}(\gamma(t), \gamma'(t)) = cte$.*

Proof. The differential equation of geodesics when $F = 0$ is given by:

$$2Eu'' = -E_u u'^2 - 2E_v u'v' + G_u v'^2, \quad 2Gv'' = E_v u'^2 - 2G_u u'v' - G_v v'^2.$$

Therefore it follows that

$$(Eu'^2)' = u'v'(G_u v' - E_v u'), \quad (Gv'^2)' = u'v'(-G_u v' + E_v u').$$

$$\text{So,} \quad \mathcal{J}' = \frac{(G + (v - u)G_u)u'v'^2 + (E + (u - v)E_v)u'^2v'}{Eu'^2 + Gv'^2}.$$

In the ellipsoidal coordinates (u, v) , see equation (2.6) and remark 2.2.11 in page 51, with $u \in (b^2, a^2)$ $v \in (c^2, b^2)$ the first fundamental form is given by

$$ds^2 = Edu^2 + Gdv^2 = \frac{u - v}{4} \left(-\frac{udu^2}{h(u)} + \frac{v dv^2}{h(v)} \right),$$

where $h(x) = (x - a^2)(x - b^2)(x - c^2)$. So it follows that $\mathcal{J}' = 0$ and \mathcal{J} is a first integral. \square

Remark 7.4.2. *This kind of first integral for geodesics is valid in any Liouville surface, that is, surfaces with metrics of the form $ds^2 = (A(u) + B(v))(du^2 + dv^2)$.*

Proposition 7.4.2. *The geodesic lines on the ellipsoid with $a > b > c > 0$, in the ellipsoidal coordinates (u, v) where $c^2 < v < b^2 < u < a^2$, are the real integral curves of the implicit differential equation:*

$$\mathcal{G}(u, v, u', v', \lambda) = \frac{h(v)du^2 - h(u)dv^2}{uh(v)du^2 - vh(u)dv^2} = \frac{\lambda^2}{uv} \quad (7.8)$$

$$h(x) = (x - a^2)(x - b^2)(x - c^2), \quad c^2 < \lambda^2 < a^2.$$

The normal curvature in the directions \mathcal{D} defined by equation (7.8) is

$$k_n((u, v), \mathcal{D}) = \frac{abc\lambda}{uv\sqrt{uv}} = \frac{\lambda k_1}{v} = \frac{\lambda k_2}{u} = \frac{\lambda^2 \mathcal{K}^{3/4}}{\sqrt{abc}}.$$

Here $k_1 = e/E$, $k_2 = g/G$ are the principal curvatures with $k_1 \leq k_2$ and $\mathcal{K} = k_1 k_2$ is the Gaussian curvature. The differential equation (7.8) is equivalent to

$$k_n(u, v, [du : dv]) = \frac{e(u, v)du^2 + g(u, v)dv^2}{E(u, v)du^2 + G(u, v)dv^2} = \frac{\lambda^2 \mathcal{K}^{3/4}}{\sqrt{abc}}. \quad (7.9)$$

Proof. The first part follows from algebraic simplification of the equation $J(u, v, dv/du) = \lambda^2$ using the values of E and G in the ellipsoidal coordinates.

For the second part, recall that in the ellipsoidal coordinates (u, v) , see equation (2.6) and remark 2.2.11 in page 51, the second fundamental form is given by

$$II = edu^2 + gdv^2 = \frac{(v-u)abc}{4h(u)\sqrt{uv}}du^2 + \frac{(u-v)abc}{4h(v)\sqrt{uv}}dv^2,$$

where $h(x) = (x - a^2)(x - b^2)(x - c^2)$. Here the orientation of the ellipsoid is such that $k_1 > 0$ and $k_2 > 0$.

So the result follows evaluating k_n in the directions \mathcal{D} which are defined by equation (7.8). \square

Remark 7.4.3. *It is worth to noticing that the geodesics are the only regular solutions of the implicit differential equation (7.8). The singular solutions (envelopes) of equation (7.8) are, in general, not geodesics.*

Proposition 7.4.3. *Consider the binary differential equation (7.8). Then the following holds.*

i) For $\lambda^2 \in (c^2, b^2)$ the real solutions are defined in the region $R = \{(u, v) : b^2 \leq u \leq a^2, c^2 \leq v \leq \lambda^2\}$ and the behavior is as in Fig. 7.3 left.

ii) For $\lambda^2 \in (b^2, a^2)$ the real solutions are defined in the region $R = \{(u, v) : \lambda^2 \leq u \leq a^2, c^2 \leq v \leq b^2\}$ and the behavior is as in Fig. 7.3 right.

iii) For $\lambda = b$ the real solutions are defined in the region $R = \{(u, v) : b^2 \leq u \leq a^2, c^2 \leq v \leq b^2\}$ and the behavior is as in Fig. 7.3 center.

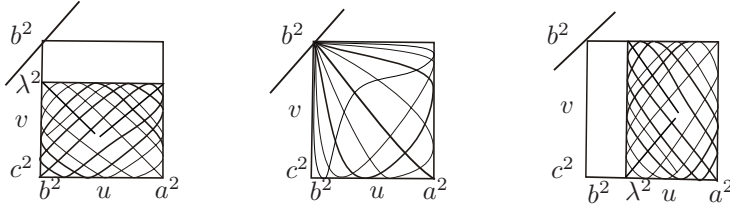


Figure 7.3: Regular and singular solutions of equation (7.8): i, left; ii, right; iii, center.

Proof. Let $\lambda^2 \in (b^2, a^2)$. The implicit differential equation given by equation (7.8) has the lines $u = a^2$, $u = \lambda^2$, $v = c^2$ and $v = b^2$ as singular solution of envelope type. At the points of intersection (λ^2, c^2) , (λ^2, b^2) , (a^2, c^2) , (a^2, b^2) there is an unique separatrix solution in the domain $[\lambda^2, a^2] \times [c^2, b^2]$.

In fact, consider the fold maps $u = a^2 - U^2$ and $v = V^2 - c^2$. Then the equation (7.8) in the new variables $(U, V) \in [0, a^2 - \lambda^2] \times [0, b^2 - c^2]$ is given by:

$$H = (a^2 - U^2)(a^2 - c^2 - V^2)(b^2 - c^2 - V^2)(\lambda^2 - c^2 - V^2)dU^2 - (V^2 + c^2)(a^2 - c^2 - U^2)(a^2 - b^2 - U^2)(a^2 - \lambda^2 - U^2)dV^2 = 0.$$

It follows that $H(0, [dU : dV]) = a^2(a^2 - c^2)(b^2 - c^2)(\lambda^2 - c^2)dU^2 -$

$c^2(a^2 - c^2)(a^2 - b^2)(a^2 - \lambda^2)dV^2$. So this equation defines two non zero real directions which are transversal to the lines $U = 0$ and $V = 0$. In the variables (u, v) it follows that the lines $u = a$ and $v = c$ are singular solutions obtained by the singularities of the fold maps. See Fig. 7.3 right.

Now let $\lambda = b > 0$ and consider the fold maps $u = U^2 + b^2$ and $v = b^2 - V^2$. The differential equation (7.8) is simplified to

$$\begin{aligned} H(U, V, [dU : dV]) &= (b^2 + U^2)V^2(V^2 + a^2 - b^2)(b^2 - c^2 - V^2)dU^2 \\ &\quad - (b^2 - V^2)U^2(a^2 - b^2 - U^2)(b^2 - c^2 + U^2)dV^2 = 0. \end{aligned}$$

This binary equation in the region $[0, a^2 - b^2] \times [0, b^2 - c^2]$ is equivalent to the following ordinary equations:

$$\begin{cases} u' = sU\sqrt{(b^2 - V^2)(a^2 - b^2 - U^2)(b^2 - c^2 + U^2)} \\ v' = V\sqrt{(b^2 + U^2)(V^2 + a^2 - b^2)(b^2 - c^2 - V^2)}, \quad s = \pm 1. \end{cases}$$

For the vector fields above the origin is a hyperbolic node in the case $s = 1$ and a hyperbolic saddle when $s = -1$. In both cases the lines $U = 0$ and $V = 0$ are solutions. In the variables (u, v) it follows that the lines $u = b$ and $v = b$ are singular solutions. See Fig. 7.3 center.

Consider also the fold maps $u = U^2 + b^2$ and $v = c^2 + V^2$. The differential equation (7.8) is simplified to

$$\begin{aligned} H(U, V, [dU : dV]) &= (b^2 + U^2)V^2(a^2 - c^2 - V^2)(b^2 - c^2 - V^2)^2dU^2 \\ &\quad - (V^2 + c^2)U^2(a^2 - b^2 - U^2)(b^2 - c^2 + U^2)dV^2 = 0. \end{aligned}$$

This binary equation in the region $[0, a^2 - b^2] \times [0, b^2 - c^2]$ is equivalent

to the following ordinary equations:

$$\begin{cases} u' = sU\sqrt{(c^2 + V^2)(a^2 - b^2 - U^2)(b^2 - c^2 + U^2)} \\ v' = (b^2 - c^2 - V^2)\sqrt{(b^2 + U^2)(a^2 - c^2 - V^2)}, \quad s = \pm 1. \end{cases}$$

The singular point $(0, 0)$ is regular for both ordinary equations and the line $U = 0$ is a common solution. The analysis of the other cases can be carried out without any more novelty. This ends the proof. \square

For $\lambda^2 \in (b^2, a^2)$, consider the following convergent hyper elliptic integrals:

$$L_1 = \int_{\lambda^2}^{a^2} \sqrt{-\frac{u}{p(u)}} du, \quad L_2 = \int_{c^2}^{b^2} \sqrt{-\frac{v}{p(v)}} dv,$$

$$p(x) = (x - a^2)(x - b^2)(x - c^2)(x - \lambda^2).$$

Also for $\lambda^2 \in (c^2, b^2)$, consider the following convergent hyper elliptic integrals:

$$L_3 = \int_{b^2}^{a^2} \sqrt{-\frac{u}{p(u)}} du, \quad L_4 = \int_{\lambda^2}^{b^2} \sqrt{-\frac{v}{p(v)}} dv.$$

Proposition 7.4.4. *Let $d\sigma_1 = \sqrt{-\frac{u}{p(u)}} du$ and $d\sigma_2 = \sqrt{-\frac{v}{p(v)}} dv$.*

Suppose that $\lambda \neq b$. Then the differential equation (7.8) is equivalent to the product of two linear differential equations $d\sigma_1 + d\sigma_2 = 0$ and $d\sigma_1 - d\sigma_2 = 0$ with

- i) $(\sigma_1, \sigma_2) \in [0, L_1] \times [0, L_2]$ when $(u, v) \in [\lambda^2, a^2] \times [c^2, b^2]$.*
- ii) $(\sigma_1, \sigma_2) \in [0, L_3] \times [0, L_4]$ when $(u, v) \in [b^2, a^2] \times [c^2, \lambda^2]$.*



Figure 7.4: Solutions of the linear differential equations

Proof. The differential equation (7.8) can be written as

$$\frac{v}{(v - \lambda^2)h(v)} dv^2 - \frac{u}{(u - \lambda^2)h(u)} du^2 = 0.$$

So, defining $d\sigma_1 = \sqrt{-\frac{u}{(u-\lambda^2)h(u)}} du$ and $d\sigma_2 = \sqrt{-\frac{v}{(v-\lambda^2)h(v)}} dv$ the result follows taking in care the convergence of the hyperelliptic integrals defining L_i , ($i=1, \dots, 4$). \square

Remark 7.4.4. In Section 8.7 of Chapter 8 a similar approach will be developed to analyze the mean curvature lines on the ellipsoid. A general class of binary differential equations was studied in [59].

Theorem 7.4.5. Consider the ellipsoid $\mathbb{E}_{a,b,c}$ given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a > b > c > 0$, parametrized by ellipsoidal coordinates (u, v) . Then the global behavior of the geodesics is as shown in Figs. 7.5 and 7.6.

i) For $\lambda^2 \in (c^2, b^2)$ the geodesics oscillates as shown in Fig. 7.5 left. The envelopes are curvature lines defined by the intersection of the hyperboloid $\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} + \frac{z^2}{c^2 - \lambda^2} = 1$ of one sheet with the ellipsoid. When L_4/L_3 is irrational all geodesics in the level λ^2 are recurrent, otherwise all are closed. The ellipse contained in the coordinate plane $x = 0$ is a stable geodesic.

ii) For $\lambda^2 \in (b^2, a^2)$ the geodesics oscillates as shown in Fig. 7.5 right. The envelopes are curvature lines defined by the intersection of the hyperboloid $\frac{x^2}{a^2-\lambda^2} + \frac{y^2}{b^2-\lambda^2} + \frac{z^2}{c^2-\lambda^2} = 1$ of two sheets with the ellipsoid. When L_2/L_1 is irrational all geodesics in the level λ^2 are recurrent, otherwise all are closed. The ellipse contained in the coordinate plane $z = 0$ is a stable geodesic.

iii) For $\lambda = b$ the geodesics through an umbilic p_0 pass also through the opposite umbilic $-p_0$. The behavior is as in Fig. 7.6. The ellipse E_y , contained in the coordinate plane $y = 0$, passing through the four Darbouxian umbilics is hyperbolic (saddle). All the other geodesics in this level set accumulate on E_y . The opposite umbilics are conjugate to each other and to no other point.

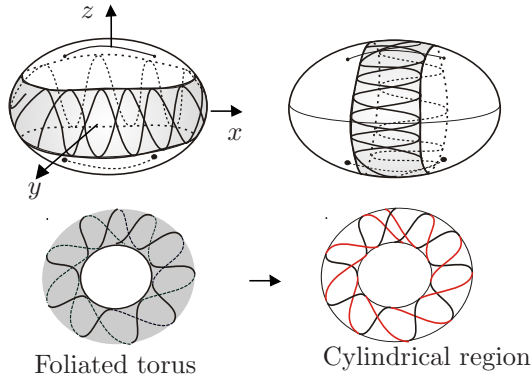


Figure 7.5: Geodesics on the ellipsoid $\mathbb{E}_{a,b,c}$ with three distinct axes at level $\lambda \neq b$ and

Proof. In the ellipsoidal coordinates (u, v) the octants of the ellipsoid $\mathbb{E}_{a,b,c}$ are parametrized by 8 local maps. The intersections of $\mathbb{E}_{a,b,c}$

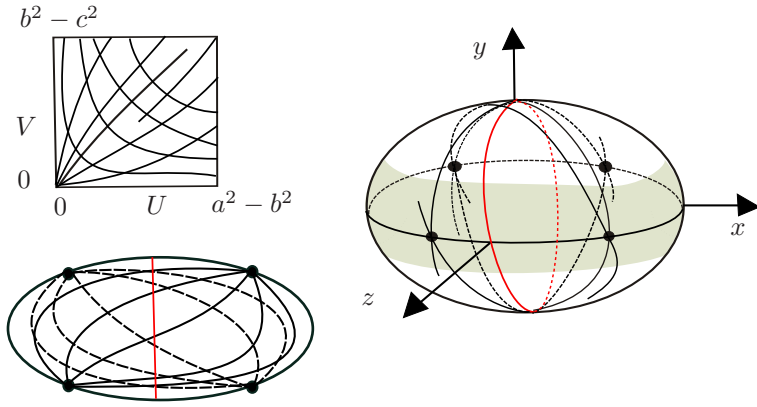


Figure 7.6: Geodesics on the ellipsoid $\mathbb{E}_{a,b,c}$ with three distinct axes at level $\lambda = b$

with the coordinates planes are not covered.

In order to collect these maps in only one consider the change of coordinates $u = b^2 \cos^2 U + a^2 \sin^2 V$ and $v = c^2 \cos^2 U + b^2 \sin^2 V$, $(U, V) \in [0, \pi] \times [0, \pi]$.

It follows that the map defined by $\beta(U, V) = (x(U, V), y(U, V), z(U, V))$ with

$$(x(u, v)^2, y(u, v)^2, z(u, v)^2) = \left(\frac{M(u, v, a)}{W(a, b, c)}, \frac{M(u, v, b)}{W(b, a, c)}, \frac{M(u, v, c)}{W(c, a, b)} \right) \quad (7.10)$$

where, $M(u, v, w) = w^2(u - w^2)(v - w^2)$, $W(a, b, c) = (a^2 - b^2)(a^2 - c^2)$, $u \in (b^2, a^2)$ and $v \in (c^2, b^2)$, is a covering of the ellipsoid by the torus $\mathbb{S}^1 \times \mathbb{S}^1$, with four branch points over the 4 umbilics.

Therefore, it follows from Proposition 7.4.4 that for $\lambda^2 \in (b^2, a^2)$ the geodesics at level λ^2 are integral curves of a linearized flow of the torus $([0, L_1] \times [0, L_2])$ with rotation number equal to L_2/L_1 . So all

the geodesics are recurrent if the rotation number is irrational. See [92] and [112] for more properties of rotation numbers.

By construction, the integral curves of the implicit equation (7.8) are projected in the ellipsoid as shown in Fig. 7.5. In this case the geodesics oscillate between two closed principal lines parametrized by $u = \lambda^2$ which are symmetric in relation to the coordinate plane $z = 0$. Therefore the ellipse $E_z = \mathbb{E}_{a,b,c} \cap \{z = 0\}$ is a stable geodesic. This ends the proof of item ii). The proof of item i) is similar.

For $\lambda = b$, it follows that the differential equation (7.8) is equivalent to the following ordinary equations:

$$\begin{cases} u' = sU\sqrt{(b^2 - V^2)(a^2 - b^2 - U^2)(b^2 - c^2 + U^2)} \\ v' = V\sqrt{(b^2 + U^2)(V^2 + a^2 - b^2)(b^2 - c^2 - V^2)}, \quad s = \pm 1. \end{cases}$$

with (U, V) in the region $[0, a^2 - b^2] \times [0, b^2 - c^2]$. The phase portrait of these equations are as shown Fig. 7.6, left.

From this it follows that any geodesic through an umbilic point p_0 pass also through $-p_0$. By the local structure of saddle near an umbilic point of the phase portrait, see Fig. 7.6, it follows that the ellipse $E_y = \mathbb{E}_{a,b,c} \cap \{y = 0\}$ is not stable, none of the geodesics is contained in a tubular neighborhood of E_y . As E_y is not stable it follows that it is hyperbolic, see [95]. See also Fig. 7.5. The property of accumulation follows from the nodal structure of the phase portrait, see Fig. 7.6, near an umbilic point. A geodesic γ through an umbilic point p_0 , making an angle $\theta_0 \in (0, \pi)$ with the ellipse E_y at p_0 pass through $-p_0$ and return to p_0 making an angle $\theta_1 \neq \theta_0$ with the ellipse E_y . See Fig. 7.6. So it is well defined a return map $\Pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. In this coordinate $\Pi(\theta_0) = \theta_1$. The diffeomorphism Π

has two hyperbolic fixed points 0 and π . This ends the proof. \square

Theorem 7.4.6. *Let $\mathbb{E} = \mathbb{E}_{a,b,c}$ be the ellipsoid with $a \geq b \geq c > 0$. Define $Per(\mathbb{E}) = \{(p, v) \in T_1\mathbb{E} : \text{geodesic } \tilde{\gamma} \text{ through } (p, v) \text{ is closed}\}$. Then $Clos(Per(\mathbb{E})) = T_1\mathbb{M}$.*

For the proof see [95, page 315].

7.5 Geodesics on Surfaces of Revolution

Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$, $\gamma(u) = (r(u), 0, z(u))$, $r > 0$, be a regular curve parametrized by arc length and consider the surface of revolution

$$\alpha(u, v) = (r(u) \cos v, r(u) \sin v, z(u)).$$

The fundamental forms are:

$$I_\alpha = ds^2 = du^2 + r(u)^2 dv^2, \quad II_\alpha = -k(u)du^2 - r(u)z'(u)dv^2,$$

where $k(u)$ is the curvature of the plane curve γ .

The Gaussian curvature of α is $\mathcal{K}(u, v) = \frac{k(u)z'(u)}{r(u)}$.

The Christoffel symbols are given by:

$$\begin{aligned} \Gamma_{11}^1 &= 0, & \Gamma_{11}^2 &= 0, & \Gamma_{12}^1 &= 0, & \Gamma_{22}^2 &= 0 \\ \Gamma_{12}^2 &= \frac{G_u}{2G} = \frac{r'}{r}, & \Gamma_{22}^1 &= -G_u = -2r'r. \end{aligned} \quad (7.11)$$

Therefore the differential equation of the geodesic lines is given by:

$$\begin{aligned} \frac{d^2 u}{ds^2} - G_u \left(\frac{dv}{ds}\right)^2 &= 0, & \frac{d^2 v}{ds^2} + \frac{G_u}{G} \frac{du}{ds} \frac{dv}{ds} &= 0 \\ ds^2 &= du^2 + r(u)^2 dv^2. \end{aligned} \quad (7.12)$$

From the second equation above it follows that:

$$\frac{\frac{d^2v}{ds^2}}{\frac{dv}{ds}} ds = -\frac{G_u}{G} \frac{du}{ds} ds$$

Therefore integrating the equation above it results that, $G \frac{dv}{ds} = c$.

Writing the unit tangent vector to the geodesic $c(s) = (u(s), v(s))$ in the form $c'(s) = (\cos \beta(s), \sin \beta(s))$, where β is the angle of the geodesic γ with the meridians it follows that:

$$G \frac{dv}{ds} = r(u(s))^2 \sin \beta(s) = c.$$

This relation is called *Clairaut formula* and it provides a first integral of the geodesic flow.

Substituting the Clairaut's formula in the equation for ds^2 it results,

$$c^2 du^2 + G(c^2 - G)dv^2 = 0 \tag{7.13}$$

The solutions of the binary equation above are the geodesics contained in the level set of the first integral $G \frac{dv}{ds} = c$.

Proposition 7.5.1. *Consider the differential equation (7.13). Then the following holds:*

i) If in a connected region $R = \{(u, v) : r(u) \geq c\}$ the function r has only one non - degenerate critical point (maximum) u_1 then the Gaussian curvature is positive in R and then the behavior of the solutions of the differential equation (7.13) is as shown in the Fig. 7.7, left. That is, when $2\sqrt{k(u_1)z'(u_1)r(u_1)} \neq n$, $n \in \mathbb{N}$, the geodesic

$u = u_1$ is elliptic and the geodesics oscillate between two principal lines defined by $u = r^{-1}(c)$. See 7.7, right.

ii) If in a connected region $R = \{(u, v) : r(u) \geq c\}$ the function r has only one non degenerate critical point (minimum) then the Gaussian curvature is negative and the behavior of the solutions of the differential equation (7.13) is as shown in the Fig. 7.7, right. That is, the geodesic $u = u_0$ is hyperbolic and the geodesics oscillates to infinity.

iii) If in a connected region $R = \{(u, v) : r(u) \geq c\}$ the function r has only non degenerate critical points (at least two points) then there are homoclinic or heteroclinic points for the return map. That is, there is a hyperbolic geodesic $u = u_0$ with $W^s(u_0) \cap W^u(u_0) \neq \emptyset$ (homoclinic) or there are two hyperbolic geodesics $u = u_0$ and $u = u_1$ such that $W^s(u_0) \cap W^u(u_1) \neq \emptyset$ and unstable $W^u(u_0) \cap W^s(u_1) \neq \emptyset$ (heteroclinic).

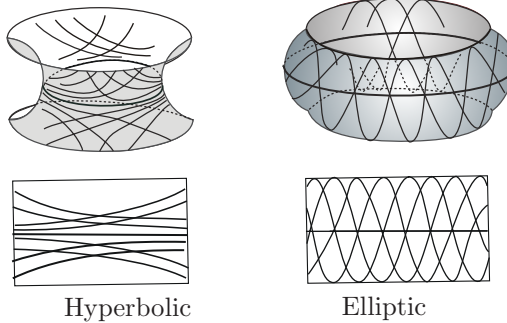


Figure 7.7: Geodesics on the surfaces of revolution

Proof. Consider the implicit surface $F(v, u, p) = c^2 p^2 + r(u)^2 (c^2 - r(u)^2) = 0$, $p = \frac{du}{dv}$.

When $r(u) = c$ and $u = u_0$ is a minimum, resp. maximum, of r the Gauss curvature is negative, resp. positive and $u = u_0$ is geodesic. So the description of the dynamic near $u = u_0$ item cases i) and ii) follows directly from Proposition 7.3.2. Here we observe that in order to obtain elliptical type, when the curvature is positive, is necessary the condition of non resonance given in Proposition 7.3.2.

Also the parallels u_1 such that $G(u_1) = c^2$ are singular solutions (envelope) of equation (7.13).

The existence of homoclinic or heteroclinic orbits is guaranteed from the existence of a first integral for the geodesic flow (Clairaut formula) and the local behavior of hyperbolic and elliptic (stable) closed geodesic lines. \square

7.6 Inverse Problems and Geodesics

An inverse problem for geodesics is the following one.

Let γ be a closed curve in \mathbb{R}^3 parametrized by arc length s . Is there an immersed surface containing γ and having it as a geodesic curve? Is it possible to impose geometric restrictions on the surface or on the geodesic flow near γ ?

Proposition 7.6.1. *Let γ be a closed curve in \mathbb{R}^3 parametrized by arc length s , with positive curvature and associated Frenet frame $\{T, N, B\}$. Consider $\alpha_\delta : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ defined by the equation*

$$\alpha_\delta(s, v) = \gamma(s) + \delta r(1 - \cos v)N(s) + r \sin v B(s), \quad \delta = \pm 1.$$

Then for small radius $r > 0$, α_δ is an immersion and γ is a closed geodesic of α_δ and $\mathcal{K}|\gamma > 0$ for $\delta = 1$ and $\mathcal{K}|\gamma < 0$ for $\delta = -1$. For \bar{s}

fixed the parametrized circle $\gamma_{\bar{s}}(v) = \alpha(s, v)$ is a geodesic of α_δ if and only if $\tau(\bar{s}) = 0$.

Proof. Let $\delta = 1$ and write $\alpha_1 = \alpha$. Direct calculations show that

$$\alpha_s(s, v) = [1 - k(s)r(1 - \cos v)]T(s) - r\tau(s)\sin vN(s) + (1 - \cos v)\tau(s)B(s)$$

$$\alpha_v(s, v) = r\sin vN(s) + r\cos vB(s)$$

$$\begin{aligned} N(s, v) = \alpha_s \wedge \alpha_v &= -[r^2\tau(s)\sin v]T(s) - r\cos v[1 - k(s)(1 - \cos v)]N(s) \\ &\quad + r\sin v[1 - k(s)(1 - \cos v)]B(s) \end{aligned}$$

The coefficients of the first fundamental form are given by:

$$E(s, v) = r^2k^2\cos^2 v + 2r(k - r(k^2 + \tau^2))\cos v + (1 - rk)^2 + 2r^2\tau^2$$

$$F(s, v) = -r^2\tau(1 - \cos v), \quad G(s, v) = r^2$$

Also it follows that, $E(s, 0) = 1$, $F(s, 0) = 0$, $G(s, 0) = r^2$, $e(s, 0) = -\frac{k(s)}{r}$, $f(s, 0) = \tau(s)$ and $g(s, 0) = -1$. Therefore for $r > 0$ small the map α is an embedding.

The Gauss curvature of α restricted to the curve γ is given by $\mathcal{K}(s, 0) = \frac{k(s)}{r} - \tau(s)^2$ and it is positive for small $r > 0$.

The curve γ is a geodesic since $T'(s) = k(s)N(s) + 0(N \wedge T)$.

For $\delta = -1$ it follows that $\mathcal{K}|_\gamma = -k(s)/r + \tau(s)^2$ is negative for small $r > 0$.

Also, as $\langle \alpha_{vv}, \alpha_s \rangle = -r^2\tau(s)\sin v$, it follows that $\gamma_{\bar{s}}$ is a geodesic if and only if $\tau(\bar{s}) = 0$. \square

7.7 Remarks on Geodesics on Compact and Convex Surfaces

This section is devoted to give some comments about geodesics on convex surfaces.

Lemma 7.7.1. *Consider a convex surface $\mathbb{M} \subset \mathbb{R}^3$ enclosing a ball of radius $r_0 > 0$ centered at the origin with unit normal of Gauss N oriented outward. Then for every $p \in \mathbb{M}$ it follows that $\langle N(p), p \rangle \geq r_0$ and the equality holds if and only if p is contained in a sphere of radius r_0 centered at the origin.*

Proof. Let $p \in \mathbb{M}$ and consider the tangent plane $T_p\mathbb{M}$. The distance of $T_p\mathbb{M}$ to the origin is given by $\langle N(p), p \rangle$. As \mathbb{M} is convex and enclose a ball of radius r_0 it follows that $\langle N(p), p \rangle \geq r_0$. \square

Theorem 7.7.1. *Consider a convex surface $\mathbb{M} \subset \mathbb{R}^3$ enclosing a ball of radius $r_0 > 0$ and let $\gamma \subset \mathbb{M}$ be a closed non-trivial geodesic. Then $L(\gamma) \geq 2\pi r_0$ and the equality holds if and only if γ is a great circle of a sphere of radius r_0 .*

Proof. Consider γ parametrized by arc length s and let $k(s) = k_n(s)$ its curvature considered as a space curve. Then $t' = -k(s)N(s)$, where N is the unit normal oriented externally.

We have that $\frac{d}{ds}\langle \gamma, \gamma' \rangle = \langle \gamma', \gamma' \rangle + \langle \gamma'', \gamma \rangle = \langle \gamma', \gamma' \rangle - k(s)\langle N(s), \gamma \rangle$. Therefore,

$$\begin{aligned} L(\gamma) &= \int_0^{L(\gamma)} |\gamma'|^2 ds = \int_0^{L(\gamma)} \left[\frac{d}{ds}\langle \gamma, \gamma' \rangle - \langle \gamma'', \gamma \rangle \right] ds \\ &= \int_0^{L(\gamma)} k(s)\langle N(s), \gamma(s) \rangle ds \geq r_0 \int_0^{L(\gamma)} k(s) ds \geq 2\pi r_0. \end{aligned}$$

If the equality holds then \mathbb{M} is tangent to a sphere of radius r_0 along γ and so γ is a great circle of radius r_0 . \square

Let γ be a closed geodesic, without double points, in a compact convex surface \mathbb{M} of \mathbb{R}^3 .

Then $\mathbb{M} \setminus \gamma$ is the union of two open connected regions \mathbb{M}_i and, in view of Gauss Bonnet theorem, it follows that $\int_{\mathbb{M}_i} \mathcal{K}dS = 2\pi$ for both regions.

Consider the space $\mathcal{C}_{2\pi}$ of regular simple closed curves γ on \mathbb{M} such that $\int_{\mathbb{M}_i} \mathcal{K}dS = 2\pi$ for each connected component \mathbb{M}_i , $i = 1, 2$, of $\mathbb{M} \setminus \gamma$. Let $l(\gamma)$ the length of γ .

Theorem 7.7.2 (H. Poincaré, C. Croke). *A compact and convex surface has at least one closed simple geodesic line.*

Proof. (Idea of proof.) Let γ be a closed simple curve of \mathbb{M} . Suppose γ be parametrized by arc length s and of length $l > 0$.

Consider the two integrals, length of γ and the total curvature of the region \mathbb{M}_i .

$$l(\gamma) = \int_{\gamma} ds, \quad K(\gamma) = \int_{\mathbb{M}_i} \mathcal{K}dS.$$

Consider a small deformation γ_ϵ of γ such that $\gamma_\epsilon = \alpha(s, \epsilon v(s))$

$$\text{where } \alpha(s, v) = \gamma(s) + v(N \wedge t)(s) + [k_n^\perp(s) \frac{v^2}{2} + A(s) \frac{v^3}{6} + h.o.t.]N(s).$$

$$\text{Then } l'(0) = \frac{dl}{d\epsilon}(0) = - \int_0^l k_g(s)v(s)ds, \quad K'(0) = \int_0^l \mathcal{K}(s)v(s)ds.$$

Imposing the condition that $\gamma_\epsilon \in \mathcal{C}_{2\pi}$ it follows that $K(\gamma) = 2\pi$ and $K'(0) = 0$. Also supposing $l(\gamma)$ be a minimum it follows that $l'(0) = 0$. From $l'(0) = 0$ and $K'(0) = 0$ it follows that $k_g(s) = c\mathcal{K}(s)$, $c \in \mathbb{R}$.

By the Gauss-Bonnet theorem we have that

$$\int_0^l k_g(s)ds = 2\pi - \int_{\mathbb{M}_i} \mathcal{K}dS = 2\pi - 2\pi = 0.$$

Since \mathbb{M} is convex, it follows that $\mathcal{K}(s) \geq 0$ and so it follows that $c = 0$ and

$k_g(s) = 0$, that is γ is a geodesic. This approach suggested by Poincaré (1905) was established by C. Croke (1982). See [136] and [34]. \square

Remark 7.7.3. *It is not known if there exists a smooth Riemannian surface (\mathbb{S}^2, g) such that all closed geodesics are hyperbolic. See [32].*

7.8 Remarks on Geodesics in Quadrics

Let \mathbb{M} be the regular surface defined by the quadratic equation $\mathbb{M} = \{p \in \mathbb{R}^n : \langle Ap, p \rangle = 1\}$ with A being a definite positive matrix. The tangent bundle of \mathbb{M} is $T\mathbb{M} = \{(p, v) : p \in \mathbb{M} \text{ and } \langle Ap, v \rangle = 0\}$.

Define $\mathcal{H} : \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\mathcal{H}(p, v) = \frac{1}{2}|v|^2 + \frac{\langle Av, v \rangle}{|Ap|^2} \left(\frac{1}{2} \langle Ap, p \rangle - 1 \right) - \frac{\langle Av, v \rangle^2}{|Ap|^2}.$$

Let $X_{\mathcal{H}}$ the Hamiltonian vector field associated to \mathcal{H} .

Restricted to the tangent bundle $T\mathbb{M}$ it follows that $X_{\mathcal{H}}$ is defined by

$$\dot{x} = v, \quad \dot{v} = -\frac{\langle Av, v \rangle}{|Ap|^2} Ap.$$

The integral curves of $X_{\mathcal{H}}$ with initial conditions in $T\mathbb{M}$ define the geodesic flow of $T\mathbb{M}$. When $A = Id$ we have $\mathbb{M} = \mathbb{S}^{n-1}$ and the geodesic flow is defined by the second order differential equation $p'' = -|p'|^2 p$. Direct analysis shows that all solutions of the above equation on the sphere \mathbb{S}^{n-1} are closed.

Also we mention that after a reparametrization, see [97], the geodesics of \mathbb{M} are given by the Neumann system on the unit sphere

$$q'' = -A^{-1}q + \mu q, q \in \mathbb{R}^n, |q| = 1.$$

The dynamics of the geodesic flow in convex hypersurfaces close to the unit sphere is extremely rich and it is an important area of research. See [118].

7.9 Exercises and Problems

7.9.1. Consider the quadrics $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} - 1 = 0$, $0 < c < b < a$.

i) Show that the hyperboloid of one sheet has one hyperbolic closed planar geodesic.

iii) Show that there are closed geodesics of arbitrary large length in the ellipsoid ($0 < c \leq b < a$). What is the situation in the other quadrics?

iv) Analyze the limit situation when $c \rightarrow 0$.

7.9.2. Show that in the torus of revolution there are closed geodesics of arbitrary length. Discuss the type (elliptic, hyperbolic) of each closed planar geodesic of the torus.

7.9.3. Consider the paraboloid $(x, y, x^2/a^2 + y^2/b^2)$. Analyze the geodesics when $0 \neq a \neq b$ and show that when $a = b \neq 0$ all geodesics passing through 0 goes to infinity.

7.9.4. Determine the geodesic curves of the Clifford torus parametrized by $\alpha(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v)$ in \mathbb{S}^3 with the induced metric $ds^2 = \langle \alpha_u, \alpha_u \rangle du^2 + 2 \langle \alpha_u, \alpha_v \rangle dudv + \langle \alpha_v, \alpha_v \rangle dv^2$.

7.9.5. An oriented line ℓ in \mathbb{R}^3 is defined by $\ell(u, v) = \{v + tu, |u| = 1, t \in \mathbb{R}\}$. Here u is the direction of ℓ and is also the orientation of ℓ . Let \mathcal{L}_3 the space of oriented lines of \mathbb{R}^3 and $T\mathbb{S}^2 = \{(u, v) : |u| = 1, \langle u, v \rangle = 0\}$ the tangent bundle of \mathbb{S}^2 .

i) Show that the map $L : T\mathbb{S}^2 \rightarrow \mathcal{L}_3$ defined by $L(u, v) = \ell(u, v) = [v - \langle v, u \rangle u] + tu$ is a bijection. Here $v - \langle v, u \rangle u$ is the point on the line which is closest to the origin.

ii) Show that $T_{(u,v)}T\mathbb{S}^2 = \{(x, y) : \langle x, u \rangle = 0, \langle x, v \rangle + \langle y, u \rangle = 0\}$.

For a study of the geometric properties of \mathcal{L}_3 see [69] and [150].

7.9.6. Let \mathbb{S}^2 be the unit sphere and $T\mathbb{S}^2$ the associated tangent bundle.

i) Define the Sasaki metric $\langle \cdot, \cdot \rangle_S$ (see [41, page 79]) in $T\mathbb{S}^2$ and determine the geodesics of $T\mathbb{S}^2$.

ii) Show that $T_1\mathbb{S}^2$, the unit tangent sphere bundle, is diffeomorphic to the projective space $\mathbb{P}_1\mathbb{R}^3$ and that the lifting of the geodesic flow of \mathbb{S}^2 to \mathbb{S}^3 (universal covering of $\mathbb{P}_1\mathbb{R}^3$) has all integral curves closed and of the same period.

iii) Show that the orbits of the lifting above define a Seifert fibration of the sphere \mathbb{S}^3 . See [173].

7.9.7. Study the integrability of the geodesic flow on the ellipsoid of \mathbb{R}^n , $n \geq 4$. See [5] and [169].

7.9.8. Give an example of a ruled surface in \mathbb{R}^3 having an isolated closed geodesic line and analyze the associated Poincaré map.

7.9.9. Give examples of developable oriented surfaces in \mathbb{R}^3 having closed geodesic lines and analyze the associated Poincaré map.

7.9.10. Give examples of analytic, non oriented (Möbius band) developable surfaces in \mathbb{R}^3 having closed geodesic lines.

7.9.11. Let \mathbb{M} be a smooth surface of \mathbb{R}^3 . Define a natural form of volume in $T\mathbb{M}$ and show that the geodesic flow on $T\mathbb{M}$ preserves such form. See [41].

7.9.12. Consider the semi-plane $\mathbb{R}_+^2 = \{(u, v) : v > 0\}$ with the metrics $ds^2 = du^2 + \frac{1}{v^2}dv^2$ and $d\sigma^2 = \frac{1}{v^2}(du^2 + dv^2)$. Determine the geodesics of \mathbb{R}_+^2 with respect to each of these metrics.

7.9.13. Give an example of a compact and convex surface enclosed by a ball of radius $r_0 > 0$ and having closed geodesic of arbitrary large length.

7.9.14. Let \mathbb{M} be a compact surface of genus $g \geq 2$. Show “heuristically” that \mathbb{M} has $3g - 3$ simple closed geodesics which defines a partition of \mathbb{M} in $g - 1$ connected regions. Analyze the topology of these regions.

7.9.15. Consider a smooth surface parametrized by a graph:

$$\alpha(u, v) = (u, v, \frac{1}{2}(k_1u^2 + k_2v^2) + \frac{a}{6}u^3 + \frac{b}{2}u^2v + \frac{b_1}{2}uv^2 + \frac{c}{6}v^3 + o(4)).$$

Consider the tangent plan at 0 with orthogonal coordinates (x, y) .

i) Compute the second jet of the exponential map in the charts (x, y) and (u, v) .

ii) Interpret geometrically the second jet of the exponential map.

7.9.16. Let (\mathbb{M}, g) be a complete Riemannian manifold of dimension n and negative curvature (all sectional curvatures bounded above by $\chi < 0$).

i) Show that if $(p_1, p_2) \in \mathbb{M} \times \mathbb{M}$ is a local maximum for the distance function on \mathbb{M} , $d : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}$, then the points p_1 and p_2 are connected by at least $2n + 1$ distinct geodesic segments (i.e. length minimizing).

ii) Show that on the ellipsoid (positive curvature) in \mathbb{R}^3 with three distinct axes the results of item i) does not holds. In fact, show that the points at maximal distance are connected by two geodesic segments only. See [86].

7.9.17. Show that a compact surface can be triangulated choosing the vertices to be an ϵ -dense set, where every point has a geodesically convex neighborhood of radius $r_g > \epsilon$, and choosing the edges of the triangulation to be geodesic segments. For compact surfaces embedded in \mathbb{R}^3 show that the triangulation as above can be defined such that the edges are principal and mean arithmetic curvature lines. See page 84. What can be said about triangulation of immersed or singular surfaces?

7.9.18. Let $\mathbb{M} \subset \mathbb{R}^3$ be a surface of class C^2 with first fundamental form $I = Edu^2 + 2Fdudv + Gdv^2$. Show that every geodesic is uniquely deter-

mined (locally) by its initial conditions. **Suggestion:** The Gauss Bonnet Theorem holds on surfaces of class C^2 . See [80].

7.9.19. Let \mathbb{M} be a compact, smooth manifold of dimension m , and let $\mathcal{R} = \mathcal{R}^r$ be the space of C^r Riemann structures on \mathbb{M} , endowed with the natural C^r topology, $2 \leq r \leq \infty$. Fix $p \in \mathbb{M}$. The k^{th} focal component with respect to $g \in \mathcal{R}$ at p is

$$\sigma_k = \{v \in T_p\mathbb{M} : \exists \text{ exactly } k \text{ vectors } v = v_1, \dots, v_k \in T_p\mathbb{M} \text{ with} \\ |v_1| = \dots = |v_k| \text{ and } \exp(v_1) = \dots = \exp(v_k)\}.$$

The focal decomposition is defined by $T_p\mathbb{M} = \bigcup_k \sigma_k$ where $1 \leq k \leq \infty$. See [132] and [88].

- i) Analyze the sets σ_k when \mathbb{M} is the unit sphere of \mathbb{R}^3 .
- ii) Analyze the sets σ_k when \mathbb{M} is a surface of revolution in \mathbb{R}^3 .

7.9.20. Establish a connection between the planar “billiard problem” and the geodesics on convex surfaces. For example, analyze the geodesics in the C^1 surface obtained from the envelope of a family of spheres of radius ϵ centered in the face, edges and vertexes of an equilateral triangle of length $1 \gg \epsilon$. See [5], [126] and references therein.

7.9.21. Let $\mathbb{M} \subset \mathbb{R}^3$ be a complete surface of class C^r , $r \geq 2$. Let $p \in \mathbb{M}$ and let $\gamma : [0, \infty) \rightarrow \mathbb{M}$ be a geodesic parametrized by arc length s such that $\gamma(0) = p$. Denote by G_p the set of all normalized geodesics through p as above. Let $A_\gamma = \{s \in \mathbb{R} : d(\gamma(0), \gamma(s)) = s\}$ and $s_\gamma = \sup(A_\gamma)$.

This means that γ is the curve of shortest distance between p and any point on γ between p and $q = \gamma(s_\gamma)$, but it is not the curve of shortest distance between p and any point on γ after q . Let $C(p) = \bigcup_{\gamma \in G_p} \{\gamma(s_\gamma)\}$ the *cut locus* or *caustic* of p . See [95, pages 126- 136].

- i)** Show that $A_\gamma = [0, s_\gamma]$ or $[0, \infty)$. Show that in the arc $\gamma([0, s_\gamma])$ there are no conjugate point of p along γ .
- ii)** Let $q \in C(p)$. Show that or q is a conjugate point of p along a geodesic $\gamma \in G_p$ or there exists two distinct geodesics in G_p , with the same length, passing through p and q .
- iii)** Let $p \in \mathbb{M}$ and suppose \mathbb{M} be compact and analytic. Show that generically $C(p)$ is a curve with finite singularities of cuspidal type. What is the structure of the set $\exp_p^{-1}(C(p))$?
- iv)** Let p_0 be an umbilic point of an ellipsoid of three distinct axis. Show that $C(p_0) = \{-p_0\}$.
- v)** Let p_0 be a non umbilic point of an ellipsoid. Show that $C(p_0)$ is a curve with singularities. What is possible to say about the number of singularities of $C(p_0)$? See [152].
- vi)** Consider the paraboloid of revolution $z = a(x^2 + y^2)$, $a \neq 0$ and $p_0 = (0, 0, 0)$. Show that $C(p_0) = \emptyset$.

7.9.22. Carry out a study of various geometrical variational problems, see [138]. For example, consider the following functional

$$F(c) = \int_c k(s)^2 ds, \quad s \text{ is the arc length}$$

in the space of regular curves with positive curvature k

$$\mathcal{C} = \{c : [a, b] \rightarrow \mathbb{R}^3, c(a) = c_a, c'(a) = c_a^1, c(b) = c_b, c'(b) = c_b^1\},$$

with end points fixed. Determine the critical points of F .

7.9.23. Consider an ellipsoid with three different axes $a \gtrsim b \gtrsim c \cong 1$. Let L be a given positive real number. Show that there are no closed geodesic of length smaller than L other than multiples of the three principal ellipses. See [117]. Analyze the case where $a \gg b \gtrsim c > 0$ and show that there are non simple closed geodesics of moderated length.

Chapter 8

Lines of Axial curvature on surfaces immersed in \mathbb{R}^4

8.1 Introduction

The curvature theory of surfaces immersed in \mathbb{R}^3 is among the deepest and most beautiful achievements of Classical Differential Geometry. One of its best understood and accomplished chapters involve the principal curvatures and their elementary symmetric functions: the Mean and Gaussian Curvatures[166]. Intimately associated to the principal curvatures are the principal direction fields, their integral foliations and umbilic singularities. However, the initial steps towards the understanding of the global behavior of these geometric objects have been given only recently. The surveys [76] and [63] discuss these initial steps and provide a list of pertinent references, from

the very classical works, in the tradition of Euler, Monge and Darboux [166] to more recent ones, motivated by the notions of structural stability and genericity, originating from Differential Equations and Dynamical Systems, [75], and Global Analysis, [156]. This chapter is concerned with the extension of these ideas to surfaces immersed in the Euclidean space \mathbb{R}^4 .

Landmarks of the Curvature Theory of surfaces in \mathbb{R}^4 are the works of Wong [181] and Little [102], where a review of properties of the Second Fundamental Form, the Ellipse of Curvature (defined as the image of this form on the unit tangent circle) and related geometric and singularity theoretic notions are presented. These authors give a list of pertinent references to original sources previous to 1969, to which one must add that of Forsyth [48]. Further geometric properties of surfaces in \mathbb{R}^4 have been pursued by Asperti[10] and Fomenko [47], among others.

The global generic structure of the *axial principal* and *mean curvature* lines, along which the second fundamental form points in the direction of the extremes of large and the small axes of the Ellipse of Curvature, is the object of study of this chapter.

The structure around the generic axiumbilic points (for which the ellipse degenerates to a circle) will be established for surfaces immersed with class C^r , $r \geq 4$, in \mathbb{R}^4 . See Fig. 8.2 Section 8.4, for an illustration of the three generic types: E_3 , E_4 , E_5 . An independent previous study of these patterns in the smooth category were can be seen in [24].

The axiumbilic points studied in this chapter must be regarded as the analogues of the Darbouxian umbilics: D_1 , D_2 , D_3 , [37], [71, 75]. In both cases, the subindexes refer to the number of *separatrices*

approaching the singularity.

For an immersion α of a surface \mathbb{M} into \mathbb{R}^4 , the axiumbilic singularities \mathbb{U}_α and the lines of axial curvature are assembled into two *axial configurations*: the *principal axial configuration*: $\mathbb{P}_\alpha = \{\mathbb{U}_\alpha, \mathbb{X}_\alpha\}$ and the *mean axial configuration*: $\mathbb{Q}_\alpha = \{\mathbb{U}_\alpha, \mathbb{Y}_\alpha\}$.

Here, $\mathbb{P}_\alpha = \{\mathbb{U}_\alpha, \mathbb{X}_\alpha\}$ is defined by the axiumbilics \mathbb{U}_α and the field of orthogonal tangent lines \mathbb{X}_α , on $\mathbb{M} \setminus \mathbb{U}_\alpha$, on which the immersion is curved along the extremes of the large axis of the curvature ellipse. The reason for the name given to this object is that for surfaces in \mathbb{R}^3 , \mathbb{P}_α reduces to the classical *principal configuration* defined by the two principal curvature direction fields $\{\mathbb{X}_{\alpha 1}, \mathbb{X}_{\alpha 2}\}$, [71, 75]. Also, in $\mathbb{Q}_\alpha = \{\mathbb{U}_\alpha, \mathbb{Y}_\alpha\}$, \mathbb{Y}_α is the field of orthogonal tangent lines \mathbb{Y}_α on $\mathbb{M} \setminus \mathbb{U}_\alpha$, on which the immersion is curved along the extremes of the small axis of the curvature ellipse.

For surfaces in \mathbb{R}^3 the curvature ellipse reduces to a segment and the crossing \mathbb{Y}_α splits into the two mean curvature line fields $\{\mathbb{Y}_{\alpha 1}, \mathbb{Y}_{\alpha 2}\}$. In this case \mathbb{Q}_α reduces to the *mean configuration* defined by *umbilic points* and line fields along which the normal curvature is equal to the Mean Curvature \mathcal{H} and they make an angle of $\pi/4$ with the principal line fields.

These line fields agree with the *asymptotic line fields* for minimal surfaces.

In this chapter, the notion of principal structural stability of immersions of surfaces into \mathbb{R}^3 is extended to the axial configurations in the case of \mathbb{R}^4 .

Call $\mathcal{J}^r = \mathcal{J}^r(\mathbb{M}^2, \mathbb{R}^4)$ the space of C^r immersions of \mathbb{M}^2 into \mathbb{R}^4 .

An immersion $\alpha \in \mathcal{J}^r$ is said to be *Principal Axial Stable* if it has a C^r , neighborhood $\mathcal{V}(\alpha)$, such that for any $\beta \in \mathcal{V}(\alpha)$ there exist a

homeomorphism $h : \mathbb{M}^2 \rightarrow \mathbb{M}^2$ mapping \mathbb{U}_α onto \mathbb{U}_β and mapping the integral net of \mathbb{X}_α onto that of \mathbb{X}_β . Analogous definition is given for *Mean Axial Stability*.

Sufficient conditions are provided to extend to the present setting the Theorem on Structural Stability for Principal Configurations due to Gutierrez and Sotomayor [75]. The local stability around axiumbilics for smooth immersions, has been carried out in [24].

Two local cases are essential for this extension: the axiumbilic points with their separatrix structure and the axial cycles. Both are treated in detail here.

This chapter is organized as follows:

Section 8.2 is devoted to the analysis of the differential equation of lines of axial curvature, in an arbitrary local chart. It is shown that, for surfaces in \mathbb{R}^3 , this equation factors into the product of the equations of principal and mean curvature lines.

In Section 8.3 the equation of lines of axial curvature is written in a Monge chart. The umbilic condition is explicitly stated in terms of the coefficients of second order jet of the two functions which represent the immersion in a Monge chart.

In Section 8.4 the condition of stability at axiumbilic points is expressed in invariant form involving the third order jets. The local axial configurations at stable axiumbilics is established for C^4 immersions.

In Section 8.5 the derivative of first return map along an axial cycle is established. It consists of an integral expression involving the geometric functions (curvatures, normal and geodesic torsions) along the axial cycle. This expression extends that of \mathbb{R}^3 case given in [71, 75].

In Section 8.6 the results presented in Sections 8.4 and 8.5 are put together to provide sufficient conditions for Axial Stability.

In Section 8.7 the axial configurations of the ellipsoid and of the torus of revolution are discussed.

8.2 Differential equation for lines of axial curvature

Let $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^4$ be a C^r , $r \geq 4$, immersion of an oriented smooth surface \mathbb{M} into \mathbb{R}^4 . This last space is oriented by a once for all fixed orientation and endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$. Let N_1 and N_2 be a frame of vector fields orthonormal to α . Assume that (u, v) is a positive chart and that $\{\alpha_u, \alpha_v, N_1, N_2\}$ is a positive frame.

In a chart (u, v) , the first fundamental form of α is given by:

$$I_\alpha = \langle D\alpha, D\alpha \rangle = Edu^2 + 2Fdu dv + Gdv^2, \quad \text{with}$$

$$E = \langle \alpha_u, \alpha_u \rangle, \quad F = \langle \alpha_u, \alpha_v \rangle \quad \text{and} \quad G = \langle \alpha_v, \alpha_v \rangle.$$

The second fundamental form is given by:

$$II_\alpha = II_{1,\alpha}N_1 + II_{2,\alpha}N_2, \quad \text{where}$$

$$II_{1,\alpha} = \langle N_1, D^2\alpha \rangle = e_1du^2 + 2f_1du dv + g_1dv^2 \quad \text{and}$$

$$II_{2,\alpha} = \langle N_2, D^2\alpha \rangle = e_2du^2 + 2f_2du dv + g_2dv^2.$$

The normal curvature vector at a point p in a tangent direction

$v \in T_p\mathbb{M}$ is given by:

$$k_n = k_n(p, v) = \frac{II_\alpha(v, v)}{I_\alpha(v, v)}.$$

Denote by $T\mathbb{M}$ the tangent bundle of \mathbb{M} and by $N\mathbb{M}$ the normal bundle of α . The image of the unitary circle of $T_p\mathbb{M}$ by $k_n(p) : T_p\mathbb{M} \rightarrow N_p\mathbb{M}$, being a quadratic map is either an ellipse, a point or a segment. In any case, to unify the notation, will be referred to as the *ellipse of curvature* of α and will be denoted by \mathbb{E}_α . See Fig. 8.1.

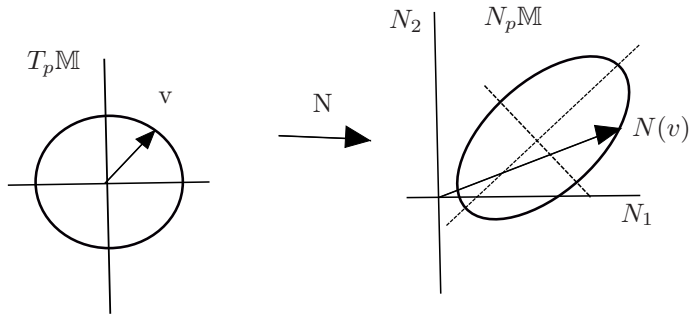


Figure 8.1: Ellipse of curvature \mathbb{E}_α

The *mean curvature vector* \mathcal{H} is defined by:

$$\mathcal{H} = h_1 N_1 + h_2 N_2 = \frac{Eg_1 + e_1 G - 2f_1 F}{2(EG - F^2)} N_1 + \frac{Eg_2 + e_2 G - 2f_2 F}{2(EG - F^2)} N_2.$$

Therefore, the ellipse of curvature \mathbb{E}_α is given by the image of:

$$k_n = (k_n - \mathcal{H}) + \mathcal{H}.$$

The tangent directions for which the normal curvature are the axes, or vertices, of the ellipse of curvature \mathbb{E}_α are characterized by the

following quartic form given by the Jacobian of the pair of forms below, the first being quartic and the second quadratic:

$$Jac(\|k_n - \mathcal{H}\|^2, I_\alpha) = 0,$$

where

$$\begin{aligned} \|k_n - \mathcal{H}\|^2 = & \left[\frac{e_1 du^2 + 2f_1 dudv + g_1 dv^2}{Edu^2 + 2Fdudv + Gdv^2} - \frac{(Eg_1 + e_1G - 2f_1F)}{2(EG - F^2)} \right]^2 \\ & + \left[\frac{e_2 du^2 + 2f_2 dudv + g_2 dv^2}{Edu^2 + 2Fdudv + Gdv^2} - \frac{(Eg_2 + e_2G - 2f_2F)}{2(EG - F^2)} \right]^2. \end{aligned}$$

Expanding the equation above, it follows that the differential equation for the corresponding tangent directions, which define the *axial curvature lines*, is given by:

$$a_4 dv^4 + a_3 dv^3 du + a_2 dv^2 du^2 + a_1 dv du^3 + a_0 du^4 = 0, \quad (8.1)$$

where,

$$\begin{aligned} a_4 = & -4F(EG - 2F^2)(g_1^2 + g_2^2) + 4G(EG - 4F^2)(f_1g_1 + f_2g_2) \\ & + 8FG^2(f_1^2 + f_2^2) + 4FG^2(e_1g_1 + e_2g_2) - 4G^3(e_1f_1 + e_2f_2), \end{aligned}$$

$$\begin{aligned} a_3 = & -4E(EG - 4F^2)(g_1^2 + g_2^2) - 32EFG(f_1g_1 + f_2g_2) \\ & + 16EG^2(f_1^2 + f_2^2) - 4G^3(e_1^2 + e_2^2) + 8EG^2(e_1g_1 + e_2g_2), \end{aligned}$$

$$\begin{aligned} a_2 = & -12FG^2(e_1^2 + e_2^2) + 12E^2F(g_1^2 + g_2^2) \\ & + 24EG^2(e_1f_1 + e_2f_2) - 24E^2G(f_1g_1 + f_2g_2), \end{aligned}$$

$$\begin{aligned}
a_1 &= 4E^3(g_1^2 + g_2^2) + 4G(EG - 4F^2)(e_1^2 + e_2^2) + 32EFG(e_1f_1 + e_2f_2) \\
&\quad - 16E^2G(f_1^2 + f_2^2) - 8E^2G(e_1g_1 + e_2g_2), \\
a_0 &= 4F(EG - 2F^2)(e_1^2 + e_2^2) - 4E(EG - 4F^2)(e_1f_1 + e_2f_2) \\
&\quad - 8E^2F(f_1^2 + f_2^2) - 4E^2F(e_1g_1 + e_2g_2) + 4E^3(f_1g_1 + f_2g_2).
\end{aligned}$$

Lemma 8.2.1. *The following relations hold:*

$$\begin{aligned}
Ea_2 &= -6Ga_0 + 3Fa_1, \quad E^2a_3 = (4F^2 - EG)a_1 - 8FGa_0, \\
E^3a_4 &= G(EG - 4F^2)a_0 + F(2F^2 - EG)a_1.
\end{aligned} \tag{8.2}$$

Proof. Combining and simplifying the expressions above for a_i , $i = 0, \dots, 4$, it can be verified that the following relations hold:

$$Ea_2 = -6Ga_0 + 3Fa_1, \quad 3Ea_3 = -3Ga_1 + 4Fa_2, \quad 6Ea_4 = -Ga_2 + 3Fa_3.$$

Further substitution leads to the result. \square

We have established the following proposition.

Proposition 8.2.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^4$ be a C^r immersion of a smooth and oriented surface. Denote the first fundamental form by $I_\alpha = Edu^2 + 2Fdudv + Gdv^2$ and the second fundamental form by*

$$II_\alpha = (e_1du^2 + 2f_1dudv + g_1dv^2)N_1 + (e_2du^2 + 2f_2dudv + g_2dv^2)N_2,$$

where $\{N_1, N_2\}$ is an orthonormal frame.

i) *The differential equation of axial curvature lines is given by:*

$$\begin{aligned}
A &= [a_0G(EG - 4F^2) + a_1F(2F^2 - EG)]dv^4 \\
&+ [-8a_0EFG + a_1E(4F^2 - EG)]dv^3du \\
&+ [-6a_0GE^2 + 3a_1FE^2]dv^2du^2 \\
&+ a_1E^3dvdu^3 + a_0E^3du^4 = 0,
\end{aligned} \tag{8.3}$$

where,

$$\begin{aligned}
a_1 &= 4G(EG - 4F^2)(e_1^2 + e_2^2) + 32EFG(e_1f_1 + e_2f_2) \\
&+ 4E^3(g_1^2 + g_2^2) - 8E^2G(e_1g_1 + e_2g_2) - 16E^2G(f_1^2 + f_2^2); \\
a_0 &= 4F(EG - 2F^2)(e_1^2 + e_2^2) - 4E(EG - 4F^2)(e_1f_1 + e_2f_2) \\
&- 8E^2F(f_1^2 + f_2^2) - 4E^2F(e_1g_1 + e_2g_2) + 4E^3(f_1g_1 + f_2g_2).
\end{aligned}$$

ii) The axiumbilic points of α are given by: $a_0 = a_1 = 0$.

Remark 8.2.1. The last expression for the differential equation shows that when the (u, v) coordinates are isothermic, i. e. $E = G$ and $F = 0$, it holds that:

$$\begin{aligned}
a_1 &= -a_3 = E^3[e_1^2 + e_2^2 + g_1^2 + g_2^2 - 4(f_1^2 + f_2^2) - 2(e_1g_1 + e_2g_2)], \\
a_0 &= a_4 = -a_2/6 = 4E^3[f_1g_1 + f_2g_2 - (e_1f_1 + e_2f_2)],
\end{aligned}$$

and the differential equation reduces to:

$$a_0(u, v)(dv^4 - 6du^2dv^2 + du^4) + a_1(u, v)(du^2 - dv^2)dudv = 0.$$

Also this equation can be written as $\text{Im} \left[\left(\frac{a_1}{4} + ia_0 \right) dz^4 \right] = 0$.

Proposition 8.2.2. Suppose that the surface \mathbb{M} is contained into \mathbb{R}^3 with $e_2 = f_2 = g_2 = 0$. Then the differential equation (8.3) is the

product of the differential equation of its principal curvature lines and the differential equation of its mean curvature lines, i.e., the quartic differential equation (8.3) is given by

$$Jac(II_\alpha, I_\alpha) \cdot Jac(Jac(II_\alpha, I_\alpha), I_\alpha) = 0. \quad (8.4)$$

Proof. From the differential equation (8.3) it is obtained:

$$a_4 = -4F(EG - 2F^2)g_1^2 + 4G(EG - 4F^2)f_1g_1 + 8FG^2f_1^2 + 4FG^2e_1g_1 - 4G^3e_1f_1;$$

$$a_3 = -4E(EG - 4F^2)g_1^2 - 4G^3e_1^2 - 32EFGf_1g_1 + 16EG^2f_1^2 + 8EG^2e_1g_1;$$

$$a_2 = -12FG^2e_1^2 + 12E^2Fg_1^2 + 24EG^2e_1f_1 - 24E^2Gf_1g_1;$$

$$a_1 = 4E^3g_1^2 + 4G(EG - 4F^2)e_1^2 + 32EFGe_1f_1 - 16E^2Gf_1^2 - 8E^2Ge_1g_1;$$

$$a_0 = 4F(EG - 2F^2)e_1^2 - 4E(EG - 4F^2)e_1f_1 - 8E^2Ff_1^2 - 4E^2Fe_1g_1 + 4E^3f_1g_1. \text{ Therefore the quartic differential equation}$$

$A = a_4dv^4 + a_3dv^3du + a_2du^2dv^2 + a_1dvdv^3 + a_0du^4 = 0$ can be factored as the product of two quadratic forms \mathcal{A}_1 and \mathcal{A}_2 , where:

$$\mathcal{A}_1 = (Ef_1 - Fe_1)du^2 + (Eg_1 - e_1G)dudv + (Fg_1 - f_1G)dv^2$$

and

$$\begin{aligned} \mathcal{A}_2 = & [e_1(EG - 2F^2) + 2EFf_1 - E^2g_1]du^2 \\ & + (4f_1EG - 2EFg_1 - 2FGe_1)dudv \\ & + (g_1(EG - 2F^2) + 2f_1FG - e_1G^2)dv^2. \end{aligned}$$

The second quadratic differential equation above is given by the equivalent quadratic equation:

$$\frac{e_1du^2 + 2f_1dudv + g_1dv^2}{Edu^2 + 2Fdudv + Gdv^2} - \frac{1}{2} \frac{Eg_1 + e_1G - 2Ff_1}{EG - F^2} = 0.$$

The equation above is also equivalent to $Jac(Jac(II_\alpha, I_\alpha), I_\alpha) = 0$. \square

Remark 8.2.2. *The geodesic torsion at a point p in a tangent direction $[du : dv]$ of a curve in $\mathbb{M} \subset \mathbb{R}^3$ is given by:*

$$\tau_g = \frac{(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2}{(EG - F^2)^{\frac{3}{2}}(Edu^2 + 2Fdudv + Gdv^2)}.$$

The extremal values of τ_g are given $\pm\sqrt{\mathcal{H}^2 - \mathcal{K}}$ and they are attained in the mean normal curvature directions. So, the mean curvature lines are also curves of maximal (minimal) geodesic torsion. In a minimal surface they coincide with the asymptotic lines, [166].

8.3 Differential equations of lines of axial curvature in Monge Charts

Take a surface in Monge form: $z = R(x, y)$, $w = S(x, y)$.

The tangent plane at the point over (x, y) is generated by $\{t_1, t_2\}$, where $t_1 = (1, 0, R_x, S_x)$ and $t_2 = (0, 1, R_y, S_y)$.

The normal plane at the point $(x, y, R(x, y), S(x, y))$ is generated by $\{\tilde{N}_1, \tilde{N}_2\}$, where $\tilde{N}_1 = (-R_x, -R_y, 1, 0)$ and $\tilde{N}_2 = t_1 \wedge t_2 \wedge \tilde{N}_1$. Here \wedge means the exterior product of three vectors in \mathbb{R}^4 and is defined by the equation

$$\det(t_1, t_2, \tilde{N}_1, v) = \langle \tilde{N}_2, v \rangle, \quad v \in \mathbb{R}^4.$$

Calculation leads to

$$\tilde{N}_2 = (-S_x(1 + R_y^2) + R_x R_y S_y, -S_y(1 + R_x^2) + R_x R_y S_x, -S_x R_x - R_y S_y, 1 + R_x^2 + R_y^2).$$

Consider also the vectors, $t_{11} = (0, 0, R_{xx}, S_{xx})$, $t_{12} = (0, 0, R_{xy}, S_{xy})$ and $t_{22} = (0, 0, R_{yy}, S_{yy})$.

Therefore the coefficients of the first fundamental form are given by:

$$E = \langle t_1, t_1 \rangle, \quad F = \langle t_1, t_2 \rangle \quad \text{and} \quad G = \langle t_2, t_2 \rangle.$$

Also, those of the second fundamental form are:

$$e_1 = \langle t_{11}, N_1 \rangle, \quad f_1 = \langle t_{12}, N_1 \rangle, \quad g_1 = \langle t_{22}, N_1 \rangle,$$

$$e_2 = \langle t_{11}, N_2 \rangle, \quad f_2 = \langle t_{12}, N_2 \rangle, \quad g_2 = \langle t_{22}, N_2 \rangle,$$

where $N_1 = \tilde{N}_1/|\tilde{N}_1|$ and $N_2 = \tilde{N}_2/|\tilde{N}_2|$

Write

$$R(x, y) = \frac{r_{20}}{2}x^2 + r_{11}xy + \frac{r_{02}}{2}y^2 + \frac{a}{6}x^3 + \frac{d}{2}x^2y + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + h.o.t.$$

$$S(x, y) = \frac{s_{20}}{2}x^2 + s_{11}xy + \frac{s_{02}}{2}y^2 + \frac{A}{6}x^3 + \frac{D}{2}x^2y + \frac{B}{2}xy^2 + \frac{C}{6}y^3 + h.o.t.$$

(8.5)

Direct calculation shows that:

$$e_1 = r_{20} + ax + dy + O(2) \quad e_2 = s_{20} + Ax + Dy + O(2)$$

$$f_1 = r_{11} + dx + by + O(2) \quad f_2 = s_{11} + Dx + By + O(2)$$

$$g_1 = r_{02} + bx + cy + O(2) \quad g_2 = s_{02} + Bx + Cy + O(2)$$

$$E = 1 + O(2) \quad F = O(2) \quad G = 1 + O(2)$$

(8.6)

From the differential equation (8.3) the condition for $(0, 0)$ to be an axiumbilic point is that:

$$r_{11}(r_{20} - r_{02}) + s_{11}(s_{20} - s_{02}) = 0,$$

$$4(r_{11}^2 + s_{11}^2) - (r_{20} - r_{02})^2 - (s_{20} - s_{02})^2 = 0.$$

(8.7)

If 0 is an axiumbilic point, then the differential equation of *lines of axial curvature* is given by:

$$(\bar{a}_0 + A_0)(dy^4 - 6dy^2dx^2 + dx^4) + (\bar{a}_1 + A_1)dxdy(dx^2 - dy^2) = 0, \quad (8.8)$$

where $\bar{a}_0(x, y) = \bar{c}x + \bar{d}y$, $\bar{a}_1(x, y) = \bar{a}x + \bar{b}y$ and

$$\begin{aligned} \bar{c} &= 4d(r_{02} - r_{20}) + 4(b - a)r_{11} + 4(B + D - A)s_{11}; \\ \bar{d} &= 4b(r_{02} - r_{20}) + 4(c - d)r_{11} + 4(B + D)(s_{02} - s_{20}) + 4Cr_{11}; \\ \bar{a} &= 4(b - a)(r_{02} - r_{20}) - 32dr_{11} + 8(B + D - A)(s_{02} - s_{20}); \\ \bar{b} &= 8(c - d)(r_{02} - r_{20}) - 32br_{11} + 8C(s_{02} - s_{20}) - 32(B + D)s_{11}. \end{aligned} \quad (8.9)$$

In the differential equation (8.8) the functions A_0 and A_1 are of type $O(2)$, that is:

$$\frac{\partial A_0}{\partial x}(0, 0) = \frac{\partial A_1}{\partial x}(0, 0) = \frac{\partial A_0}{\partial y}(0, 0) = \frac{\partial A_1}{\partial y}(0, 0) = 0.$$

Define: $\tilde{a} = \bar{a}\bar{d} - \bar{b}\bar{c} \neq 0$, $\tilde{b} = \bar{a} + \bar{d}$, $\tilde{c} = \bar{b} + \bar{c}$, extensive calculation, confirmed by Computer Algebra, shows that these expressions are invariant under positive rotations on the tangent and normal frames. By appropriate choice on the rotation in the plane $\{x, y\}$ and a homotety in \mathbb{R}^4 , it is possible to make $\bar{c} = 0$ and, when $\tilde{a} \neq 0$, also $\bar{d} = 1$.

So, dropping the bar in the coefficients, the differential equation (8.8) reduces to:

$$[y + O(2)](dy^4 - 6dy^2dx^2 + dx^4) + [ax + by + O(2)]dxdy(dx^2 - dy^2) = 0,$$

provided the invariant \tilde{a} doesn't vanish.

This last condition amounts to the *transversality* of the curves $a_0 = 0$, $a_1 = 0$. Axiumbilics satisfying it will be referred to as

transversal axiumbilic points. Therefore, the proposition below is established.

Proposition 8.3.1. *Let p be a transversal axiumbilic point ($\tilde{a} \neq 0$), then there exist a Monge chart (x, y) and a homotety in \mathbb{R}^4 such that the differential equation of the lines of axial curvature is given by:*

$$[y+O(2)](dy^4-6dy^2dx^2+dx^4)+[ax+by+O(2)]dxdy(dx^2-dy^2) = 0, \quad (8.10)$$

where a and b are expressed in terms of the third jet of the surface at p and $O(2)$ means terms of higher order.

Remark 8.3.1. *Taking into account the invariants \tilde{a} , \tilde{b} and \tilde{c} the coefficients a and b of (8.9) can be evaluated directly in any Monge chart. Notice also that in (8.9) a and b are not the coefficients of $j^3R(0)$ given in (8.5).*

8.4 Axial configurations near axiumbilic points on surfaces of \mathbb{R}^4

In this section will be established the qualitative behavior of the axial configurations \mathbb{P}_α and \mathbb{Q}_α in a neighborhood of an axiumbilic point in terms of the parameters (a, b) in equation (8.10).

Let

$$\Delta(a, b) = (a + 1)^2[I^3 - 27J^2] \quad \text{where,} \quad (8.11)$$

$$I = 2a\left(\frac{a}{24} + 1\right) + 4 + \frac{b^2}{4}, \quad J = -\frac{2a}{3}\left[\left(\frac{a}{6} + 1\right)\left(1 - \frac{a}{24}\right) + \frac{b^2}{16}\right].$$

Theorem 8.4.1. *Let $\alpha : \mathbb{M} \rightarrow \mathbb{R}^4$ be an immersion of class C^r , $r \geq 4$. Consider a transversal axiumbilic point, for which $a \neq 0$. Then the following holds:*

- i) If $\Delta(a, b) < 0$, then the axial configurations \mathbb{P}_α and \mathbb{Q}_α are of type E_3 , with three axiumbilic separatrices, as shown in Fig. 8.2, top.
- ii) If $\Delta(a, b) > 0$ and $a < 0$, then the axial configurations \mathbb{P}_α and \mathbb{Q}_α are of type E_4 , with four axiumbilic separatrices and one parabolic sector, as shown in Fig. 8.2, center.
- iii) If $\Delta(a, b) > 0$ and $a > 0$, then the axial configurations \mathbb{P}_α and \mathbb{Q}_α are of type E_5 , with five axiumbilic separatrices, as shown in Fig. 8.2, bottom.

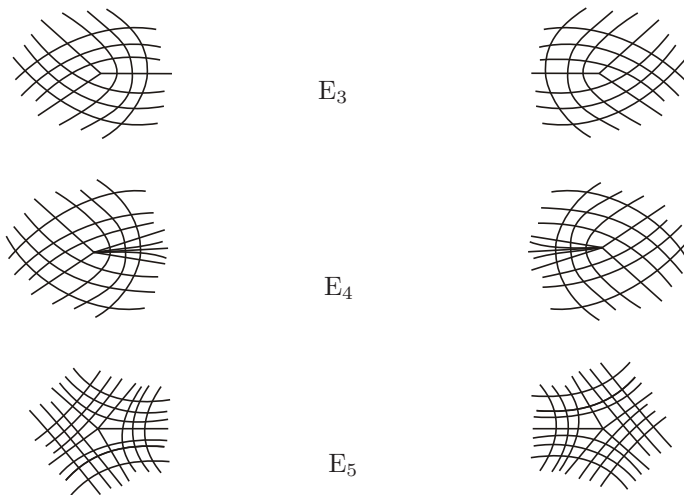


Figure 8.2: Axial Configurations near axiumbilic points E_3 , E_4 and E_5

Proof. Under the conditions above, $a \neq 0$, the implicit surface

$$\mathcal{G}(x, y, p) = y(p^4 - 6p^2 + 1) + (ax + by)p(1 - p^2) + H(x, y, p), \quad p = \frac{dy}{dx},$$

is regular of class C^{r-2} in a neighborhood of the axis p , which represent the projective line. In fact, $\partial\mathcal{G}/\partial x(0, 0, p) = ap(1 - p^2)$, $\partial\mathcal{G}/\partial y(0, 0, p) = (p^4 - 6p^2 + 1) + bp(1 - p^2)$ and the equations $\partial\mathcal{G}/\partial x(0, 0, p) = \partial\mathcal{G}/\partial y(0, 0, p) = 0$ have no common solution in p . Therefore, by the Implicit Function Theorem $\mathcal{G}^{-1}(0)$ is a regular surface around the projective line.

Now consider the Lie-Cartan line field defined, in the coordinates (x, y, p) , by the vector field X , which is of class C^{r-3} :

$$X = \mathcal{G}_p \frac{\partial}{\partial x} + p\mathcal{G}_p \frac{\partial}{\partial y} - (\mathcal{G}_x + p\mathcal{G}_y) \frac{\partial}{\partial p}.$$

The singular points of X located on the projective line are given by:

$$P(p) = pR(p) = p[(p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)] = 0.$$

Consider the polynomial $R(p) = (p^4 - 6p^2 + 1) + (1 - p^2)(a + bp)$ and compute its discriminant Δ_R in terms of the resultant $\text{Res.}(R, R')$:

$$\Delta_R = \text{Res.}(R, R') = (a - 1)^2[16a^5 + (4b^2 + 272)a^4 + (2304 + 16b^2)a^3 - 8(b^2 + 16)(b^2 - 80)a^2 + 96(b^2 + 16)^2a + 4(b^2 + 16)^3].$$

Notice that $\Delta_R = \Delta(a, b)$ is given by equation (8.11). It holds that $R(\pm 1) = -4$ and $\lim_{p \rightarrow \pm\infty} R(p) = \infty$. Therefore R has always two real roots, one bigger than $+1$ and other smaller than -1 .

Therefore, for any pair (a, b) with $\Delta(a, b) \neq 0$, the polynomial R has either four real simple roots in the case $\Delta_R > 0$ or two real simple roots in the case $\Delta_R < 0$.

This follows from the more general fact about quartic polynomials, [21, Ch. VI, page 145], formulated here as a Lemma.

Lemma 8.4.1. *Let R be a quartic real polynomial and Δ_R its discriminant. Then the equation $\Delta_R = 0$ gives the coefficients for which multiple roots arise. Furthermore,*

- i) If $\Delta_R < 0$, R has two real simple and two complex conjugate roots;*
- ii) If $\Delta_R > 0$, R has either four real simple roots or all the roots have non-zero imaginary parts.*

The linear part of X at a singular point $(0, 0, p)$ is given by:

$$DX(0, 0, p) = \begin{pmatrix} a(1 - 3p^2) & 4p^3 + b(1 - 3p^2) - 12p & 0 \\ a(1 - 3p^2)p & p[4p^3 + b(1 - 3p^2) - 12p] & 0 \\ 0 & 0 & -P'(p) \end{pmatrix}$$

The eigenvalues of $DX(0, 0, p)$ are given by:

$$\lambda_1(p) = a(1 - 3p^2) + p[4p^3 + b(1 - 3p^2) - 12p], \quad \lambda_2(p) = -P'(p), \quad \lambda_3 = 0.$$

Now observe that the value of $\lambda_1(p)$ at a root p of R is given by: $-(p^2 + 1)^3 / (p^2 - 1)$.

In fact, writing $R = 0$ as $a = \frac{(6p^2 - p^4 - 1) + bp(p^2 - 1)}{1 - p^2}$ and substituting into λ_1 , the result follows from algebraic simplification.

Therefore, at each singular point of X ,

$$\det D(X|_{\{\mathcal{G}=0\}})(0, 0, p) = \lambda_2(p)\lambda_1(p) = P'(p)(p^2 + 1)^3 / (p^2 - 1), \quad p \neq 0,$$

$$\det(DX|_{\{\mathcal{G}=0\}})(0, 0, 0) = -a(a + 1)$$

The possibilities for the real roots of P and their implications on the types of singularities of X are listed below:

- i) Three real roots $p_1 < -1 < p_0 = 0 < 1 < p_2$, which occurs when $\Delta(a, b) < 0$, implying that all singular points of X are hyperbolic saddles.
- ii) Five real roots $p_1 < -1 < p_2 < p_0 = 0 < p_3 < 1 < p_4$, which occurs when $\Delta(a, b) > 0$, implying for X that all singular points are hyperbolic saddles, in the case $a > 0$, and that there are four hyperbolic saddles and one hyperbolic node if $a < 0$.
- iii) Five real roots $p_1 < -1 < p_2 < p_3 < p_0 = 0 < 1 < p_4$, which occurs when $\Delta(a, b) > 0$ and $a < 0$, implying for X that only one of p_2 and p_3 is a hyperbolic node and all the other singular points are hyperbolic saddles.
- iv) Five real roots $p_1 < -1 < p_0 = 0 < p_2 < p_3 < 1 < p_4$, which occurs when $\Delta(a, b) > 0$ and $a < 0$, implying for X that only one of p_2 and p_3 is a hyperbolic node and all the other singular points are hyperbolic saddles.

The classification above follows from the saddle and node analysis of the hyperbolic singularities contained in the projective line of the vector field X in terms of the sign of $\det(DX|_{\{\mathcal{G}=0\}}(0, 0, p_i))$. The axiumbilic separatrices are the projections of the saddle separatrices of the vector field $X|_{\{\mathcal{G}=0\}}$.

Direct calculation shows that in the chart $(u, v, du/dv = q)$ the Lie-Cartan line field is regular in a neighborhood of $(0, 0, 0)$. The behavior of $X|_{\{\mathcal{G}=0\}}$ near the projective line is as shown in the Fig. 8.3. This ends the proof.

□

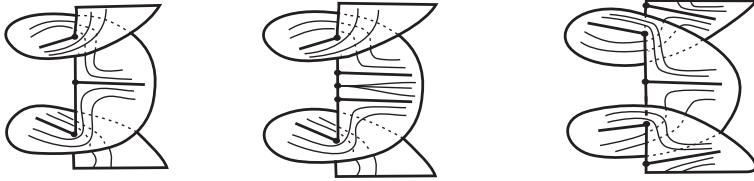


Figure 8.3: Behavior of $X|_{\{g=0\}}$ near the projective line

Remark 8.4.2. *The theorem above was proved for smooth surfaces, using isothermic coordinates, in [24]. The use of Monge coordinates makes it more elementary and allows to check its hypothesis by simpler calculations.*

Theorem 8.4.3. *Let p be an axiumbilic point of an immersion $\alpha \in \mathcal{J}^r$, $r \geq 4$. Then α is locally principal and mean axial stable at p if only if p is one of the types E_3 , E_4 or E_5 shown in Fig 8.2.*

Proof. The construction of the homeomorphism is done using the method of canonical regions, [71], also outlined in global setting in Section 8.6.

The conditions which characterize the axiumbilic points of types E_i , $i = 1, 2, 3$ and their axial configuration (Theorem 8.4.1), are semi-algebraic in the space of coefficients of the third jet of α at p . These conditions amount to the hyperbolicity of singularities of vector fields on surfaces. It is easy to see that the violation of any of them implies in a qualitative change by a small perturbation in the coefficients.

For instance, if $a = 0$, in general, the axiumbilic splits into two (E_5 and E_4) or disappears. Also, if $\Delta(a, b) = 0$ and $a \neq 0$ the type changes into E_3 or E_4 . This ends the proof. \square

8.5 Axial cycles on surfaces immersed in \mathbb{R}^4

In terms of geometric invariants, here is established the formula of the first derivative of the return map of a periodic line of axial curvature, called *axial cycle*. Recall that the return map associated to a cycle is a local diffeomorphism with a fixed point, defined on a cross section normal to the cycle by following the integral curves through this section until they meet again the section. This map is called holonomy in Foliation Theory and Poincaré Map in Dynamical Systems. See [25] and [128].

An axial cycle is called *hyperbolic* if the first derivative of the return map at the fixed point is different from one. The characterization of hyperbolicity of axial cycles is given in Proposition 8.5.1 of this section.

Lemma 8.5.1. *Let $c : I \rightarrow \mathbb{M}^2$ be an axial cycle parametrized by arc length s . Then an orthonormal positive Darboux frame $\{T_1, T_2, N_1, N_2\}$, where $T_1(s) = c'(s)$, $\{N_1, N_2\}$ is an orthonormal frame of the normal bundle associated to ellipse of curvature and $N_2 = T_1 \wedge T_2 \wedge N_1$, verifies the following equations:*

$$\begin{aligned}
 T_1' &= k_g T_2 + (h_1 + k_1) N_1 + h_2 N_2 \\
 T_2' &= -k_g T_1 + \tau_{g,1} N_1 + \tau_{g,2} N_2 \\
 N_1' &= -(h_1 + k_1) T_1 - \tau_{g,1} T_2 + \tau_n N_2 \\
 N_2' &= -h_2 T_1 - \tau_{g,2} T_2 - \tau_n N_1
 \end{aligned} \tag{8.12}$$

Here, $\mathcal{H} = h_1 N_1 + h_2 N_2$ is the mean curvature vector, $\tau_g =$

$\tau_{g,1}N_1 + \tau_{g,2}N_2$ is the geodesic torsion vector, $k_n - \mathcal{H} = k_1N_1$, $k_1 \geq 0$, is the principal axis of the ellipse of curvature and $\tau_n = \langle N'_1, N_2 \rangle$ is the normal torsion of the frame $\{N_1, N_2\}$.

Proof. The result follows by direct differentiation of the equations $\langle T_i, N_j \rangle = 0$, $\langle T_i, T_j \rangle = \delta_{ij}$ and $\langle N_i, N_j \rangle = \delta_{ij}$, $i, j = 1, 2$. \square

Remark 8.5.1. *An axial cycle c of a principal or mean axial configuration is not necessarily a simple regular curve; it can be immersed with transversal crossings. Recall that each axial configuration is a net consisting of orthogonal curves and umbilic points.*

The next lemma shows that $\tau_{g,1}(s) = 0$ for a line of axial curvature.

Lemma 8.5.2. *Let c be an axial cycle, of the axial configuration \mathbb{P}_α , parametrized by arc length s and of length L , and the orthonormal positive Darboux frame $\{T_1, T_2, N_1, N_2\}$ along c . Then the expression,*

$$\begin{aligned} \alpha(s, v) = & c(s) + vT_2(s) + \left[\frac{(h_1 - k_1)}{2}v^2 + a(s, v)\frac{v^3}{6} \right]N_1(s) \\ & + \left[\frac{h_2}{2}v^2 + b(s, v)\frac{v^3}{6} \right]N_2(s), \end{aligned} \quad (8.13)$$

where $a(s, 0) = A(s)$, $b(s, 0) = B(s)$ and $\tau_{g,1}(s) = 0$, defines a local chart, L periodic in s , of class C^{r-5} in a neighborhood of c .

Moreover, $[k_1^2 - \tau_{g,2}^2]$ is the difference between the squares of the two axes of the ellipse of curvature \mathbb{E}_α .

Proof. The assertion about the chart (s, v) follows from the Inverse Function Theorem applied to the map

$$\alpha(s, v, w_1, w_2) = c(s) + vT_2(s) + w_1N_1(s) + w_2N_2(s),$$

which is of class C^{r-1} and defines a tubular neighborhood of c .

At the point $c(s)$, the intersection of the surface with the hyperplane generated by $\{T_2, N_1, N_2\}$ is a curve Γ_s tangent to T_2 of class C^{r-1} .

Then the curve Γ_s can be parametrized by $v \mapsto W_1(s, v)N_1(s) + W_2(s, v)N_2(s)$ and its curvature is $k_n(c(s), T_2(s))$. Now, as the function $k_n(p) : T_p\mathbb{M} \rightarrow N_p\mathbb{M}$, defined by

$$k_n(p, u) = \frac{II_\alpha(p)(u, u)}{I_\alpha(p)(u, u)},$$

is a twice covering map it follows that $k_n(c(s), T_2) = (h_1 - k_1)N_1 + h_2N_2$. So, applying Hadamard's Lemma to the functions W_1 and W_2 the local chart is obtained as stated.

Next will be proved that $\tau_{g,1}(s) = 0$ for an axial line of curvature.

Differentiation of the equation (8.13), evaluated at $(s, 0)$ gives that:

$$\begin{aligned} E(s, 0) &= 1 & F(s, 0) &= 0 & G(s, 0) &= 1 \\ e_1(s, 0) &= h_1 + k_1 & f_1(s, 0) &= \tau_{g,1} & g_1(s, 0) &= h_1 - k_1 \\ e_2(s, 0) &= h_2 & f_2(s, 0) &= \tau_{g,2} & g_2(s, 0) &= h_2 \end{aligned} \quad (8.14)$$

Now, by the differential equation (8.3) of lines of axial curvature, the curve $\{v = 0\}$ is a line of axial curvature if $a_0(s, 0) = 0$ and $a_1(s, 0) \neq 0$.

Substituting equation (8.14) in a_0 given by equation (8.3) it follows that $a_0(s, 0) = -8k_1\tau_{g,1}$. As k_1 is the principal axis of the ellipse of curvature \mathbb{E}_α it follows that $\tau_{g,1} = 0$.

Also, using equation (8.14) and evaluating a_1 , it results that $a_1(s, 0) = 16[k_1^2 - \tau_{g,2}^2]$.

In a direction $T = \cos \theta T_1 + \sin \theta T_2$,

$$\begin{aligned} k_n(c(s), T) = & [e_1 \cos^2 \theta + 2f_1 \cos \theta \sin \theta + g_1 \sin^2 \theta] N_1 \\ & + [e_2 \cos^2 \theta + 2f_2 \cos \theta \sin \theta + g_2 \sin^2 \theta] N_2. \end{aligned}$$

So, it follows that:

$$\begin{aligned} k_n(c(s), \theta) = k_n(c(s), T) = & [h_1 + \tau_{g,1} \sin 2\theta + k_1 \cos 2\theta] N_1 \\ & + [h_2 + \tau_{g,2} \sin 2\theta] N_2 \end{aligned}$$

The normal curvature in the direction of the other axis of the ellipse of curvature is $k_n(\pi/4) = (h_1 + \tau_{g,1})N_1 + (h_2 + \tau_{g,2})N_2$. Therefore $\tau_{g,2}$ is the other axis of the curvature ellipse \mathbb{E}_α .

This allows the interpretation of the coefficient $a_1(s, 0)/16 = [k_1^2 - \tau_{g,2}^2]$ of the differential equation of lines of axial curvature, equation (8.3) as the difference between the square of the two axis of the ellipse of curvature. \square

Remark 8.5.2. *The local chart (s, v) defined by equation (8.13) is similar to that of [71] for principal cycles on surfaces immersed in \mathbb{R}^3 .*

Lemma 8.5.3. *Let c be an axial line of curvature and consider the chart (s, v) given in equation (8.13). Then the orthonormal frame $\{N_1, N_2\}$ of the normal bundle satisfies the following equations:*

$$\begin{aligned} \frac{\partial N_1}{\partial v}(s, 0) = & -(h_1 - k_1)T_2(s) + q_2N_2(s) \\ \frac{\partial N_2}{\partial v}(s, 0) = & -\tau_{g,2}T_1(s) - h_2T_2(s) - q_2N_1(s) \end{aligned} \tag{8.15}$$

where $q_2(s) = \langle \frac{\partial N_1}{\partial v}(s, 0), N_2(s) \rangle$ is the normal torsion of the axial line of curvature orthogonal to c at the point $c(s)$.

Proof. In a chart (u, v) the following relations hold:

$$\begin{aligned} N_{1,v} &= \frac{g_1F - f_1G}{EG - F^2} \alpha_u + \frac{f_1F - g_1E}{EG - F^2} \alpha_v + q_2N_2 \\ N_{2,v} &= \frac{g_2F - f_2G}{EG - F^2} \alpha_u + \frac{f_2F - g_2E}{EG - F^2} \alpha_v - q_2N_2 \end{aligned}$$

Therefore, using equation (8.14) and making the substitutions the result follows. \square

Lemma 8.5.4. *Let c be an axial cycle, then the functions A and B introduced in the chart (s, v) are given by:*

$$\begin{aligned} A &= 2\frac{\partial h_1}{\partial v} - 2k_g k_1 + \tau_{g,2} \tau_n \\ B &= 2\frac{\partial h_2}{\partial v} - \tau'_{g,2} + 2q_2 h_1 \end{aligned} \tag{8.16}$$

Proof. Direct calculation, using that $\tau_{g,1} = 0$ and equations (8.13), (8.14), and (8.15), shows that the following relations hold:

$$E_v(s, 0) = -2k_g \quad F_v(s, 0) = 0 \quad G_v(s, 0) = 0 \tag{8.17}$$

$$\begin{aligned}
e_{1,v}(s, 0) &= -2k_g h_1 - \tau_{g,2} \tau_n + h_2 q_2 \\
e_{2,v}(s, 0) &= -2k_g h_2 + \tau'_{g,2} - (h_1 + q_1) q_2 \\
f_{1,v}(s, 0) &= (h_1 - k_1)' - \tau_n h_2 + q_2 \tau_{g,2} \\
f_{2,v}(s, 0) &= (h_1 - k_1) \tau_n + h'_2 + k_g \tau_{g,2} \\
g_{1,v}(s, 0) &= A + h_2 q_2, \quad g_{2,v}(s, 0) = B - q_2 (h_1 - k_1)
\end{aligned} \tag{8.18}$$

In the coordinates (s, v) the mean curvature vector \mathcal{H} is given by:

$$\begin{aligned}
\mathcal{H}(s, v) &= h_1(s, v) N_1(s, v) + h_2(s, v) N_2(s, v), \\
h_i &= \frac{E g_i + G e_i - 2F f_i}{2(EG - F^2)}.
\end{aligned} \tag{8.19}$$

Therefore, differentiating h_1 and h_2 , using (8.17) and (8.18), the result follows. \square

Proposition 8.5.1. *Let c be an axial cycle of the axial configuration \mathbb{P}_α , parametrized by arc length s and of length L . Then the derivative of the first return map, denoted by π , is given by:*

$$\ln \pi'(0) = -\frac{1}{2} \int_0^L \left[\frac{-k_1 h'_1 + \tau_{g,2} \frac{\partial h_2}{\partial v}}{k_1^2 - \tau_{g,2}^2} + \frac{h_2 k_1 \tau_n + q_2 h_1 \tau_{g,2}}{k_1^2 - \tau_{g,2}^2} \right] ds \tag{8.20}$$

Proof. As $\{v = 0\}$ is an axial cycle it follows that $f_1(s, 0) = \tau_{g,1} = 0$. Direct calculation gives that the derivative of the Poincaré map satisfies the following linear differential equation:

$$\frac{d}{ds}\left(\frac{dv}{dv_0}\right) = -\frac{1}{a_1} \frac{\partial a_0}{\partial v} \cdot \frac{dv}{dv_0} \quad (8.21)$$

From the expression of a_0 , (8.3), using (8.13) and (8.15), it follows that:

$$\begin{aligned} \frac{\partial a_0}{\partial v}(s, 0) &= 4f_1 E_v(3g_1 - 2e_1) + 4f_2 E_v(3g_2 - 2e_2) + 4(g_1 - e_1) \frac{\partial f_1}{\partial v} \\ &\quad + 4f_1 \left(\frac{\partial g_1}{\partial v} - \frac{\partial e_1}{\partial v} \right) + 4(g_2 - e_2) \frac{\partial f_2}{\partial v} + 4f_2 \left(\frac{\partial g_2}{\partial v} - \frac{\partial e_2}{\partial v} \right) \end{aligned}$$

Now, using (8.16), (8.17) and (8.18), it follows that:

$$\frac{\partial a_0}{\partial v} = 8\tau_{g,2} \left[\frac{\partial h_2}{\partial v} + q_2 h_1 - \tau'_{g,2} \right] + 8k_1 [\tau_n h_2 + k'_1 - h'_1]. \quad (8.22)$$

Therefore using that $a_1(s, 0) = 16[k_1^2 - \tau_{g,2}^2]$ it follows that,

$$\frac{\frac{\partial a_0}{\partial v}}{a_1} = \frac{-k_1 h'_1 + \tau_{g,2} \frac{\partial h_2}{\partial v}}{2(k_1^2 - \tau_{g,2}^2)} + \frac{k_1 \tau_n h_2 + \tau_{g,2} q_2 h_1}{2(k_1^2 - \tau_{g,2}^2)} + \frac{1}{4} d[\ln(k_1^2 - \tau_{g,2}^2)]$$

Using the expression above and performing the integration of the linear differential equation (8.21) ends the proof. \square

Remark 8.5.3. *In the Proposition 8.5.1 it was assumed that c is an axial cycle of the axial configuration \mathbb{P}_α . The corresponding formula for the first derivative of the first return map is the same when c is a cycle of the mean configuration \mathbb{Q}_α .*

Corollary 8.5.4. *Let c be a cycle of \mathbb{P}_α such that $\alpha(\mathbb{M}) \subset \mathbb{R}^3$ parametrized by arc length s and of total length L . Then the derivative of the first return map is given by:*

$$\ln \pi'(0) = \frac{1}{2} \int_0^L \frac{h_1'}{k_1} ds = \frac{1}{2} \int_0^L \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} \quad (8.23)$$

Here $h_1 = \mathcal{H}$ is the mean curvature and \mathcal{K} is the Gaussian curvature.

Proof. As $\alpha(\mathbb{M}) \subset \mathbb{R}^3$ and c is a principal curvature line it follows that $\tau_{g,1} = \tau_{g,2} = \tau_n = q_2 = 0$. By the equation (8.20) of the derivative of the return map the result follows observing that k_1 is one half of the difference between the principal curvatures and so $k_1 = \sqrt{\mathcal{H}^2 - \mathcal{K}}$. \square

Corollary 8.5.5. *Let c be an axial cycle of the axial configuration \mathbb{Q}_α such that $\alpha(\mathbb{M}) \subset \mathbb{R}^3$ parametrized by arc length s and of total length L . Then the derivative of the first return map is given by:*

$$\ln \pi'(0) = \frac{1}{2} \int_0^L \frac{(h_2)_v}{\tau_{g,2}} ds = \frac{1}{2} \int_0^L \frac{\mathcal{H}_v}{\sqrt{\mathcal{H}^2 - \mathcal{K}}} ds. \quad (8.24)$$

Here $h_2 = \mathcal{H}$ is the mean curvature and \mathcal{K} is the Gaussian curvature.

Proof. As $\alpha(\mathbb{M}) \subset \mathbb{R}^3$ and c is a mean curvature line it follows that $k_1 = \tau_{g,1} = \tau_n = q_2 = 0$ and $\tau_{g,2}$ is one half of the difference between the principal curvatures. From the equation (8.20) for derivative of the first return map the result follows. \square

Remark 8.5.6. *The formula obtained in the Corollary 8.5.4 (resp. Corollary 8.5.5) agrees with that of [71, 75], for principal curvature lines, (resp. with that of [58], for mean curvature lines). See equation (3.8) in page 77.*

Notice also that the sign in the integral expressions (8.23) and (8.24) depend on the orientation on \mathbb{M} and on the cycle.

Theorem 8.5.7. *Let c be an axial cycle of an immersion α . Then c is locally axial stable if and only if it is hyperbolic.*

Proof. It follows from standard perturbation analysis, [75] and [128], and is similar to that of vector fields on surfaces. \square

8.6 Axial Structural Stability

Let \mathbb{M}^2 be a compact, smooth and oriented surface. Denote by \mathcal{J}^r be the space of C^r immersions of \mathbb{M} into the Euclidean space \mathbb{R}^4 , endowed with the C^r topology. Consider the subsets \mathcal{P}^r (resp. \mathcal{Q}^r) of immersions α defined by the following conditions:

- a) all axiumbilic points are of types: E_3 , E_4 or E_5 , Section 8.4;
- b) all principal (resp. mean) axial cycles are hyperbolic, Section 8.5;
- c) the limit set of every axial line of curvature is contained in the set of axiumbilic points and principal (resp. mean) axial cycles of α ;
- d) all axiumbilic separatrices are associated to a single axiumbilic point; this means that there are no connections or self connections of axiumbilic separatrices, Section 8.4.

Theorem 8.6.1. *Let $k \geq 5$. The following holds:*

- i) *The subsets \mathcal{P}^r and \mathcal{Q}^r are open in \mathcal{J}^r ;*
- ii) *Every $\alpha \in \mathcal{P}^r$ is Principal Axial Stable;*

iii) Every $\alpha \in \mathcal{Q}^r$ is Mean Axial Stable.

Proof. Outline of Proof.

On the Tangent Projective Bundle $\mathbb{PM} = \{\mathbb{TM} \setminus \{0\}\} / \{v = rw, r \neq 0\}$ of \mathbb{M} , consider the submanifold $\{\mathcal{G}_\alpha = 0\}$ defined by the axial directions. In coordinates $(u, v; p = dv/du)$, this surface is defined by the implicit differential equation (8.3). The submanifold $\{\mathcal{G}_\alpha = 0\}$ is regular, under the axiumbilic hypothesis. The restriction of the projection $\pi : \mathbb{PM} \rightarrow \mathbb{M}$ to $\{\mathcal{G}_\alpha = 0\}$ is a four-fold covering outside the preimage of the axiumbilic set \mathbb{U}_α . The surface $\{\mathcal{G}_\alpha = 0\}$ is the union of two regular surfaces \mathbf{P}_α and \mathbf{Q}_α with common boundary along $\pi^{-1}(\mathbb{U}_\alpha)$. These surfaces consist on the liftings to \mathbb{PM} of the crossings \mathbb{X}_α and \mathbb{Y}_α . The restriction of π to \mathbf{P}_α (resp. \mathbf{Q}_α) will be denoted by π_{P_α} (resp. π_{Q_α}). Therefore π_{P_α} (resp. π_{Q_α}) is a double regular covering outside $\pi^{-1}(\mathbb{U}_\alpha)$.

On \mathbb{PM} is defined the involution $I(u, v, [du : dv]) = (u, v, [dv : -du])$ which amounts to a rotation of lines by an angle $\pi/2$. The surfaces \mathbf{P}_α and \mathbf{Q}_α are invariant under I . The restriction of I to them will be denoted by I_{P_α} and I_{Q_α} .

On $\{\mathcal{G}_\alpha = 0\}$ the lifting of the crossings \mathbb{X}_α and \mathbb{Y}_α define a single line field \mathcal{L}_α on $\{\mathcal{G}_\alpha = 0\} \setminus \pi^{-1}(\mathbb{U}_\alpha)$. In a local chart (u, v, p) this line field is defined by the Lie-Cartan vector field $X_\alpha = (\mathcal{G}_\alpha)_p \frac{\partial}{\partial u} + p(\mathcal{G}_\alpha)_p \frac{\partial}{\partial v} - ((\mathcal{G}_\alpha)_u + p(\mathcal{G}_\alpha)_v) \frac{\partial}{\partial p}$ and has a unique regular extension to $\pi^{-1}(\mathbb{U}_\alpha)$.

Considering the induced line field $(I_{P_\alpha})_* X_\alpha$ (respec. $(I_{Q_\alpha})_* X_\alpha$), it is obtained a transversal pair $\{X_\alpha, (I_{P_\alpha})_* X_\alpha\}$ on $\mathbf{P}_\alpha \setminus \pi^{-1}(\mathbb{U}_\alpha)$ (respectively $\{X_\alpha, (I_{Q_\alpha})_* X_\alpha\}$ on $\mathbf{Q}_\alpha \setminus \pi^{-1}(\mathbb{U}_\alpha)$). This procedure defines

a net outside $\pi^{-1}(\mathbb{U}_\alpha)$ on \mathbf{P}_α (resp. \mathbf{Q}_α). This net is invariant under I and by π projects to the integral nets of \mathbb{P}_α (respectively \mathbb{Q}_α).

At this point one should acknowledge the similarity of the structure on \mathbb{P}_α with the situation of two principal line fields and their canonical regions, dealt with in [71, 75]. In fact, here the construction and continuation to a small neighborhood $\mathcal{V}(\alpha)$ of α of canonical regions follow also from the openness and unique continuation, for β near α , of singularities (and their separatrices and parabolic sectors) and of cycles (and their local invariant manifolds) due to the hyperbolicity of these elements in the fields of the pair $\{X_\alpha, (I_{P_\alpha})_*X_\alpha\}$. This leads to the openness of \mathcal{P}^r and gives uniquely a correspondence between axiumbilics, separatrices, cycles and their intersections for $\{X_\alpha, (I_{P_\alpha})_*X_\alpha\}$ and $\{X_\beta, (I_{P_\beta})_*X_\beta\}$. The extension of this correspondence so as to define a topological equivalence $H : \mathbf{P}_\alpha \rightarrow \mathbf{P}_\beta$ between $\{X_\alpha, (I_{P_\alpha})_*X_\alpha\}$ and $\{X_\beta, (I_{P_\beta})_*X_\beta\}$ which commutes with the involution I and gives, by projection, a topological equivalence $h : \mathbb{M} \rightarrow \mathbb{M}$ between $\mathbb{P}_\alpha = \{\mathbb{X}_\alpha, \mathbb{U}_\alpha\}$ and $\mathbb{P}_\beta = \{\mathbb{X}_\beta, \mathbb{U}_\alpha\}$ is carried out as in the case of nets of asymptotic lines on surfaces immersed in \mathbb{R}^3 , [140].

Similar considerations hold for the Mean Axial Stability. □

Remark 8.6.2. *It can be proved, by transversality methods, that conditions a), b) and d) are dense in the C^r topology. However, the density of condition c) presents new difficulties (yet unsolved) related to the "Closing Lemma" for nets. In [75] and [58], this has been proved for the C^2 topology, for Principal and Mean Curvature nets on surfaces in \mathbb{R}^3 . In these cases the nets actually split into pairs of singular foliations. This simplification does not hold in general for the nets considered in this chapter.*

8.7 Examples of Axial Configurations

In this section is made a preliminary comparative study of Principal and Mean curvature configurations for surfaces in \mathbb{R}^3 . Also, two examples about the global behavior of mean curvature lines will be analyzed.

Analogy and discrepancy with two classical results

Proposition 8.7.1. *Let $I : \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ be the inversion with respect to the unitary sphere \mathbb{S}^2 , $I(x, y, z) = \frac{1}{x^2+y^2+z^2}(x, y, z) = \frac{1}{r^2}(x, y, z)$. Then the umbilics, mean curvature and the principal curvature lines of a surface \mathbb{M} are mapped in the same by the inversion, exchanging minimal and maximal in both cases.*

Proof. As the inversion is a conformal map and the principal directions are preserved, it follows that the mean curvature directions are also preserved. □

Proposition 8.7.2. *Let α be an immersion of a surface \mathbb{M} into \mathbb{R}^3 and consider the displacement $\alpha_\epsilon = \alpha + \epsilon N_\alpha$, where N_α is the normal map of α . Then the principal lines are preserved along α_ϵ while the mean curvature lines are not. In fact, the mean curvature lines are locally rotated by a nonzero angle.*

Proof. Let (u, v) be a principal chart on the surface \mathbb{M} .

The coefficients of the fundamental forms of α and of α_ϵ in the

principal chart (u, v) are related by:

$$\begin{aligned}\bar{E} &= (1 - \epsilon k_1)^2 E, & \bar{e} &= (1 - \epsilon k_1) e \\ \bar{F} &= F = 0, & \bar{f} &= f = 0 \\ \bar{G} &= (1 - \epsilon k_2)^2 G, & \bar{g} &= (1 - \epsilon k_2) g\end{aligned}$$

So, the differential equation of mean curvature lines for the immersion α_ϵ is given by:

$$(1 - \epsilon k_1)^2 E du^2 - (1 - \epsilon k_2)^2 G dv^2 = 0. \text{ Therefore, } \frac{dv}{du} = \pm \sqrt{\frac{E}{G} \frac{1 - \epsilon k_1}{1 - \epsilon k_2}}.$$

It follows that $\frac{d}{d\epsilon} \left(\frac{dv}{du} \right) |_{\{\epsilon=0\}} = \pm \sqrt{\frac{E}{G}} (k_2 - k_1) \neq 0$. \square

Mean curvature lines on the Torus of Revolution

Proposition 8.7.3. *Consider a torus of revolution $T(r, R)$. Define the function*

$$\varrho = \varrho\left(\frac{R}{r}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{ds}{\frac{R}{r} + \cos(s)}.$$

Then the mean curvature lines on $T(r, R)$ are all closed or all recurrent according to $d \in \mathbb{Q}$ or $d \in \mathbb{R} \setminus \mathbb{Q}$ and both cases occur.

Proof. The torus of revolution $T(r, R)$ is parametrized by

$$\alpha(s, \theta) = ((R + r \cos(s)) \cos(\theta), (R + r \cos(s)) \sin(\theta), r \sin(s))$$

Direct calculation shows that $E = r^2$, $F = 0$, $G = [R + r \cos(s)]^2$ and $f = 0$. Clearly (s, θ) is a principal chart.

The differential equation of the mean curvature lines, in the principal chart (s, θ) , is given by $E(s, \theta) ds^2 - G(s, \theta) d\theta^2 = 0$. This is equivalent to $r^2 ds^2 = [R + r \cos(s)]^2 d\theta^2$.

Solving the equation above it is follows that,

$\theta(2\pi) = \theta_0 \pm 2\pi d(\frac{R}{r}) = \theta_0 \pm \int_0^{2\pi} \frac{ds}{\frac{R}{r} + \cos(s)}$. So the two Poincaré maps, $\pi_{\pm} : \{s = 0\} \rightarrow \{s = 2\pi\}$, defined by $\pi_{\pm}(\theta_0) = \theta_0 \pm 2\pi d(\frac{R}{r})$ have rotation number equal to $\pm d$. The function $d(\frac{R}{r})$ is strictly decreasing and its image is the interval $(0, \infty)$, both the rational and irrational cases occur. This ends the proof. \square

Remark 8.7.1. *All the principal curvature lines are closed in the torus of revolution $T(r, R)$*

Proposition 8.7.4. *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. Then $\mathbb{E}_{a,b,c}$ have four umbilic points located in the plane of symmetry orthogonal to middle axis; they are of the type D_1 for the principal curvature lines. The mean curvature configuration is topologically equivalent to the principal configuration near the umbilic points which are D_1 .*

Proof. Without lost of generality suppose that $\mathbb{E}_{a,b,c}$ is defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, with $c = 1$, and write $A = \frac{1}{a^2}$ and $B = \frac{1}{b^2}$. Consider the parametrization of the ellipsoid $\alpha(x, y) = (x, y, h(x, y)) = (x, y, \sqrt{1 - Ax^2 - By^2})$. Calculation shows that:

$$\begin{aligned} E &= 1 + \left(\frac{Ax}{h}\right)^2 & F &= \frac{ABxy}{h^2} & G &= 1 + \left(\frac{By}{h}\right)^2 \\ e &= -\frac{Ah^2 - A^2x^2}{h^3} & f &= -\frac{ABxy}{h} & g &= -\frac{Bh^2 - B^2y^2}{h^3} \end{aligned}$$

So, $L = Fg - Gf$, $M = Eg - Ge$ and $N = Ef - Fe$ are given by:

$$\begin{aligned} L(x, y) &= \frac{AB(1-B)xy}{h^3}, & N(x, y) &= \frac{AB(A-1)xy}{h^3} \\ M(x, y) &= \frac{(A-B) + AB(1-A)x^2 + AB(B-1)y^2}{h^3} \end{aligned}$$

As $A < B < 1$ it follows that the four umbilic points are:

$$(\pm x_0, 0, \pm z_0) = \left(\pm \sqrt{\frac{B-A}{AB(1-A)}}, 0, \pm \sqrt{\frac{A(1-B)}{B(1-A)}} \right).$$

In a neighborhood of the umbilic point $(x_0, 0, z_0)$ it follows that the first order jet of the differential equation $Ldy^2 + Mdx dy + Ndx^2 = 0$ of principal curvature lines is given by:

$$[AB(1-B)\frac{x_0}{z_0^3}y]dy^2 + [2AB(1-A)\frac{x_0}{z_0^3}(x-x_0)]dx dy + [AB(A-1)\frac{x_0}{z_0^3}y]dx^2 = 0.$$

Performing a change of coordinates $x = x_0 + \bar{x}$, $y = \sqrt{\frac{1-A}{1-B}}\bar{y}$ the following equation is obtained: $\bar{y}(d\bar{y})^2 + 2\bar{x}d\bar{x}d\bar{y} - \bar{y}(d\bar{x})^2 = 0$.

Therefore this umbilic point is topologically a Darbouxian of type D_1 (see Theorem 3.2.4, page 73) and the same holds for the all the other umbilic points. In a neighborhood of the umbilic point $(x_0, 0, z_0)$ the first order jet of the differential equation of the mean curvature lines is given by:

$$[(1-A)(x-x_0)]dy^2 + [2(A-1)y]dx dy + \left[-\frac{(A-1)^2}{1-B}(x-x_0)\right]dx^2 = 0.$$

In the differential equation above perform the change of coordinates

$$x = x_0 + \bar{x}, \quad y = \sqrt{\frac{1-A}{1-B}}\bar{y} \quad \text{to obtain: } \bar{x}(d\bar{y})^2 - 2\bar{y}d\bar{x}d\bar{y} - \bar{x}(d\bar{x})^2 = 0.$$

The implicit equation above has an unique real separatrix which is $\bar{x} = 0$ and the behavior of the integral curves near 0 is the same of an umbilic point of type D_1 .

Therefore it follows that the mean curvature configuration near an umbilic point of the ellipsoid with three distinct axes is topologically equivalent to the configuration of principal lines near a Darbouxian umbilic point D_1 . \square

Proposition 8.7.5. *Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$. On the ellipse $\Sigma \subset \mathbb{E}_{a,b,c}$, containing the four umbilic points, $p_i, i = 1, \dots, 4$, counterclockwise oriented, denote by s_1 (resp. s_2) the elliptic distance between the adjacent umbilic points p_1 and p_4 (resp. p_1 and p_2). Define $d = \frac{s_2}{s_1}$.*

Then if $d \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $d \in \mathbb{Q}$) all the mean curvature lines are recurrent (resp. all, with the exception of the mean curvature umbilic separatrices, are closed). See Fig. 8.4

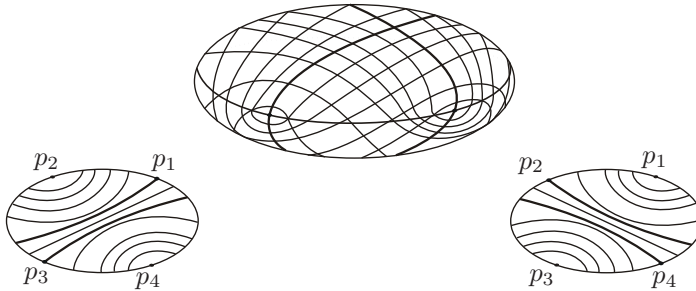


Figure 8.4: Mean curvature lines on the ellipsoid $\mathbb{E}_{a,b,c}$

Proof. The ellipsoid $\mathbb{E}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics,

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \text{ with } a > b > c > 0, \text{ see also [164] and [166].}$$

The following parametrization $\alpha(u, v) = (x(u, v), y(u, v), z(u, v))$ of $\mathbb{E}_{a,b,c}$, where

$$x(u, v) = \pm \sqrt{\frac{a^2(u+a^2)(v+a^2)}{(b^2-a^2)(c^2-a^2)}}, \quad y(u, v) = \pm \sqrt{\frac{b^2(u+b^2)(v+b^2)}{(b^2-a^2)(b^2-c^2)}},$$

$$z(u, v) = \pm \sqrt{\frac{c^2(u+c^2)(v+c^2)}{(c^2-a^2)(c^2-b^2)}},$$

defines the ellipsoidal coordinates (u, v) on $\mathbb{E}_{a,b,c}$, with $u \in (-b^2, -c^2)$ and $v \in (-a^2, -b^2)$.

The first fundamental form of $\mathbb{E}_{a,b,c}$ is given by:

$$ds^2 = \frac{1}{4} \frac{(u-v)u}{(u+a^2)(u+b^2)(u+c^2)} du^2 + \frac{1}{4} \frac{(v-u)v}{(v+a^2)(v+b^2)(v+c^2)} dv^2$$

The four umbilic points are given by:

$$(\pm x_0, 0, \pm z_0) = \left(\pm a \sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c \sqrt{\frac{c^2-b^2}{c^2-a^2}} \right).$$

On the ellipse $\Sigma = \{(x, 0, z) | \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1\}$ the distance between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_4 = (x_0, 0, -z_0)$ is given by $s_1 = \int_{-b^2}^{-c^2} \frac{\sqrt{u}}{(u+a^2)(u+c^2)} du$ and that between the umbilic points $p_1 = (x_0, 0, z_0)$ and $p_2 = (-x_0, 0, z_0)$ is given by $s_2 = \int_{-a^2}^{-b^2} \frac{\sqrt{v}}{(v+a^2)(v+c^2)} dv$.

It is obvious that the ellipse Σ is the union of four umbilic points and four principal umbilical separatrices for the principal foliations. So $\Sigma \setminus \{p_1, p_2, p_3, p_4\}$ is a transversal section of both mean curvature foliations. The differential equation of the mean curvature lines in the principal chart (u, v) is given by $Edu^2 - Gdv^2 = 0$, which is equivalent to $(\sqrt{E}du)^2 = (\sqrt{G}dv)^2$, which amounts to $ds_1 = \pm ds_2$. Therefore near the umbilic point p_1 the mean curvature lines with a mean curvature umbilic separatrix contained in the region $\{y > 0\}$ define a

the return map $\sigma_+ : \Sigma \rightarrow \Sigma$ which is an isometry, reverting the orientation, with $\sigma_+(p_1) = p_1$. This follows because in the principal chart (u, v) this return map is defined by $\sigma_+ : \{u = -b^2\} \rightarrow \{v = -b^2\}$ which satisfies the differential equation $\frac{ds_2}{ds_1} = -1$. By analytic continuation it results that σ_+ is a isometry reverting orientation with two fixed points $\{p_1, p_3\}$. The geometric reflection σ_- , defined in the region $y < 0$ have the two umbilic $\{p_2, p_4\}$ as fixed points. So the Poincaré return map $\pi_1 : \Sigma \rightarrow \Sigma$ (composition of two isometries σ_+ and σ_-) is a rotation with rotation number given by s_2/s_1 .

Analogously for the other mean curvature configuration, with the Poincaré return map given by $\pi_2 = \tau_+ \circ \tau_-$ where τ_+ and τ_- are two isometries having respectively $\{p_2, p_4\}$ and $\{p_1, p_3\}$ as fixed points.

□

8.8 Exercises and Problems

8.8.1. Let $\alpha(u, v) = h(u)(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v)$, where h is a smooth function.

- i) Compute the first and second fundamental form of α .
- ii) Find h such that the mean curvature vector is zero.

8.8.2. Consider an immersed surface $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^4$ and the associated mean curvature vector \mathcal{H} . Let $II_{\mathcal{H}} = e_{\mathcal{H}}du^2 + 2f_{\mathcal{H}}dudv + g_{\mathcal{H}}dv^2$, where (u, v) is a local chart and $e_{\mathcal{H}} = \langle \alpha_{uu}, \mathcal{H} \rangle$, $f_{\mathcal{H}} = \langle \alpha_{uv}, \mathcal{H} \rangle$ and $g_{\mathcal{H}} = \langle \alpha_{vv}, \mathcal{H} \rangle$.

- a) Define the configuration of α relative to \mathcal{H} , i.e., relative to the pair of quadratic forms $I_{\alpha} = Edu^2 + 2Fdudv + Gdv^2$ and $II_{\mathcal{H}}$ by the equation $\text{Jac}(II_{\mathcal{H}}, I_{\alpha}) = 0$.
- b) Analyze the types of singularities of this configuration and show that in the generic case they are topologically equivalent to the Darbouxian

umbilic points of surfaces in \mathbb{R}^3 or \mathbb{S}^3 .

c) Analyze the closed leaves and obtain an integral formula to express the hyperbolicity of the Poincaré return map. See [65] and [145].

8.8.3. Consider an immersed surface $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^4$ and the associated mean curvature vector \mathcal{H} .

a) Consider the implicit differential equation $DN(p)v = \lambda(p)\mathcal{H}(p)$. See Fig. 8.5. Geometrically the directions v_1 and v_2 in the tangent plane $T_p\mathbb{M}$ are the preimages of the vectors $\lambda_1\mathcal{H}$ and $\lambda_2\mathcal{H}$ (intersection of the ellipse of curvature with the straight line passing through $0 \in N_p\mathbb{M}$ with vector direction \mathcal{H}). Show that this differential equation is a binary differential equation.

b) Define the configuration of α in relation to the differential equation obtained in a).

c) Analyze the singularities of this configuration.

d) Analyze the closed leaves and gives a characterization of the hyperbolic ones. See [110].

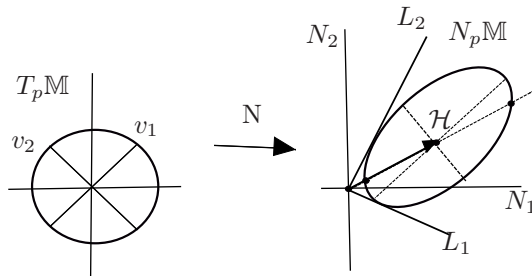


Figure 8.5: Ellipse of curvature \mathbb{E}_α and the mean curvature vector \mathcal{H}

8.8.4. Let $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^4$ be an immersion of class C^r . Consider the implicit differential equation $DN(p)v_i = \lambda L_i$. See Fig. 8.5. Geometrically

the directions v_1 and v_2 in the tangent plane $T_p\mathbb{M}$ are the preimages of the directions L_1 and L_2 (intersection of the ellipse of curvature with the straight line passing through $0 \in N_p\mathbb{M}$ which are tangent to the ellipse). The directions v_1 and v_2 are called asymptotic directions.

- a) Show that $DN(p)v_i = \lambda L_i$ defined above is a binary differential equation.
- b) Write the differential equation above when the surface is parametrized by a graph $(u, v) \rightarrow (u, v, f(u, v), g(u, v))$.
- c) Define the configuration of α in relation to the equation obtained in a).
- d) Analyze the singularities of this configuration.
- e) Analyze the closed leaves and give a characterization of the hyperbolic ones. For items a) and b) see [143]. The analysis of the other items do not appear in the literature.

8.8.5. Consider the quadrics ($a > b > c > 0$) $\mathbb{Q}(a, b, c) = \{(x, y, z, w) : \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, w = 0\}$ as surfaces of \mathbb{R}^4 .

- a) Write the differential equation of the mean (mean axial) curvature lines of $\mathbb{Q}(a, b, c)$.
- b) Show that the singularities of mean (mean axial) configuration of the quadrics are topologically equivalent to the Darbouxian umbilic points D_1 .
- c) Give explicit examples of ellipsoids such that all mean curvature lines are dense.
- d) Give examples of ellipsoids such that all mean curvature lines are periodic with the exception of the umbilic separatrices.
- e) Draw pictures illustrating the mean and the principal axial configurations of the quadrics $\mathbb{Q}(a, b, c)$. See [58] and Section 8.7.

8.8.6. Consider a surface $\alpha(u, v) = (u, v, h(u, v))$ in \mathbb{R}^3 having a Darbouxian umbilic D_i , ($i=1,2,3$), at 0. Let $\alpha_\epsilon = (u, v, h(u, v), \epsilon r(u, v))$.

- a) Give examples of deformations α_ϵ such that α_ϵ have two axiumbilic points E_{i+2} , ($i=1,2,3$), in a neighborhood of 0.
- b) Draw pictures illustrating the mean and the principal axial configurations of α_ϵ .

8.8.7. Let $\alpha : \mathbb{M}^2 \rightarrow \mathbb{R}^n$, $n \geq 5$ be an immersion.

- a) Develop the theory of the ellipse of curvature for immersions α of codimension greater than or equal to 3 and analyze all the possibilities for this ellipse (point, segment, circle, etc.).
- b) Develop the theory of mean and principal axial configurations of immersions of codimension greater or equal to 3.
- c) Analyze the behavior of the mean and principal axial configurations near the axiumbilic points.

8.8.8. Consider the rotation $R : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by

$$R_v = \begin{pmatrix} \cos v & \sin v & 0 & 0 \\ -\sin v & \cos v & 0 & 0 \\ 0 & 0 & \cos v & \sin v \\ 0 & 0 & -\sin v & \cos v \end{pmatrix} \quad (8.25)$$

and the regular curve $\gamma(s) = (x(s), y(s), z(s), w(s))$ parametrized by arc length s .

Let $\alpha(s, v) = R_v \gamma(s)$, $s \in \text{Dom}(\gamma)$, $v \in [0, 2\pi]$.

- a) Compute the first and second fundamental forms of α .
- b) Calculate the mean curvature vector \mathcal{H} of α .
- c) Analyze the global behavior of mean and principal axial configurations of α in special cases, for example, when γ is a circle or a straight line.

8.8.9. Consider the maps $\alpha, \beta : \mathbb{S}^2 \rightarrow \mathbb{R}^4$ defined by

$$\alpha(u, v) = \left(\frac{a}{2} \sin^2 \cos 2u, \frac{a}{2} \sin^2 u \sin 2u, a \sin u \cos v, a \sin u \sin v \right)$$

$$\beta(u, v) = \left(\frac{4a}{3} \cos^3 \frac{u}{2}, \frac{4a}{3} \sin^3 \frac{u}{2}, a \sin u \cos v, a \sin u \sin v \right)$$

where, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$. The unitary sphere \mathbb{S}^2 is parametrized by $x = a \sin u \cos v$, $y = a \sin u \sin v$, $z = a \cos u$ and its first fundamental form is given by $ds^2 = a^2 du^2 + a^2 \sin^2 u dv^2$.

- Compute the first fundamental form of α and β .
- Show that $\alpha(\mathbb{S}^2)$ and $\beta(\mathbb{S}^2)$ are not contained in any hyperplane of \mathbb{R}^4 .
- Show that α and β are isometric immersions of $\mathbb{S}^2 \subset \mathbb{R}^3$ into \mathbb{R}^4 .
- Calculate the mean curvature vector \mathcal{H} of α and β .
- Show that there is no homeomorphism $F: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $F(\alpha(\mathbb{S}^2)) = \beta(\mathbb{S}^2)$. See [125].

8.8.10. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^4$ be a closed regular curve parametrized by arc length s . Analyze the geometric conditions for:

- γ to be a mean axial cycle.
- γ to be a principal axial cycle. See exercise 2.5.16, page 63.

8.8.11. Consider the ruled surface $\alpha(s, v) = c(s) + ve_2(s)$.

Here $\{e_1, e_2, e_3, e_4\}$ is the Frenet frame of a closed curve c supposed parametrized by arc length s , i. e.

$$c' = e_1, \quad e_1' = k_1 e_2, \quad e_2' = -k_1 e_1 + k_2 e_3,$$

$$e_3' = -k_2 e_2 + k_3 e_4, \quad e_4' = -k_3 e_3.$$

- Compute the first and second fundamental forms of α .

Suggestion: Take $N_1 = (\alpha_s \wedge e_2 \wedge e_4) / |(\alpha_s \wedge e_2 \wedge e_4)|$ and $N_2 = e_4$.

- Show that c is an axial curvature line (mean) of α when $k_2 \neq 0$.

iii) Analyze the axial configuration of α near the curve c .

8.8.12. Consider the space \mathbb{R}^4 with canonical base $\{e_1, e_2, e_3, e_4\}$. Let Γ be a regular, smooth, nontrivial, knot of $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. This means that Γ does not bound a topological disk in \mathbb{R}^3 . For each point $p \in \Gamma$ consider the segments $l_+(p) = tp + (1-t)e_4$ and $l_-(p) = tp - (1-t)e_4$ with $0 \leq t \leq 1$. Define $\mathbf{S} = (\bigcup_{p \in \Gamma} l_+(p)) \cup (\bigcup_{p \in \Gamma} l_-(p)) = \mathbf{S}_+ \cup \mathbf{S}_-$.

i) Show that \mathbf{S}_\pm is homeomorphic to a closed disk and so \mathbf{S} is homeomorphic to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

ii) Show that \mathbf{S} is knotted in \mathbb{R}^4 , i.e., show that the first fundamental group of $\mathbb{R}^4 \setminus \mathbf{S}$ is isomorph to the fundamental group of $\mathbb{R}^3 \setminus \Gamma$ and so is different of \mathbb{Z} .

iii) Show that $\alpha(s, v) = v\Gamma(s) + (1-v)e_4$, $0 < v \leq 1$, is a regular parametrization of $\mathbf{S}_+ \setminus \{e_4\}$. Analyze the axial configuration of α .

8.8.13. Let $\gamma : \mathbb{S}^1(r) \rightarrow \mathbb{S}^3$ be a smooth knot parametrized by arc length s . Define the conical surface $\alpha(s, v) = v\gamma(s)$, $v \in \mathbb{R}$.

i) Compute the first and second fundamental forms of α .

ii) Characterize the axial umbilic set of α .

iii) Analyze the axial configuration of α near 0.

iv) Analyze the axial configuration of α near $v = \pm\infty$.

8.8.14. Let $\alpha : \mathbb{S}^2 \rightarrow \mathbb{R}^4$ defined by

$$\alpha(u, v, w) = (u^2 - v^2, uv, uw, vw), \quad u^2 + v^2 + w^2 = 1.$$

i) Show that α is an immersion and compute the first and second fundamental forms of α .

ii) Characterize the axial umbilic set of α .

iii) Show that $\alpha(\mathbb{S}^2)$ is homeomorphic to the projective plane $\mathbb{P}_2\mathbb{R}$.

8.8.15. Show that there are compact oriented surfaces in \mathbb{R}^4 having no regular normal vector field globally defined. See [2, pages 124-131].

8.8.16. Write the fundamental equations of compatibility of Gauss-Codazzi-Ricci for surfaces in \mathbb{R}^4 and enunciate the Bonnet's theorem.

8.8.17. Let $\gamma : [0, l] \rightarrow \mathbb{R}^4$ be a curve parametrized by arc length s with associated Frenet frame $e'_1 = k_1 e_2$, $e'_2 = -k_1 e_1 + k_2 e_3$, $e'_3 = -k_2 e_2 + k_3 e_4$, $e'_4 = -k_3 e_3$. The curvatures of γ are $k_1 > 0$, $k_2 > 0$ and k_3 . Consider the tubes defined by

$$\alpha(s, v) = \gamma(s) + r \cos v e_2 + r \sin v e_3, \quad \beta(s, v) = \gamma(s) + r \cos v e_3 + r \sin v e_4.$$

i) Compute the first and second fundamental forms of α and β .

ii) Analyze the axial configurations of α and β .

8.8.18. Consider the sphere \mathbb{S}^3 as the unit quaternions $\{q \in \mathbb{H} : q\bar{q} = 1\}$. Recall that $\mathbb{H} = \{q = a + bi + cj + dk, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, (a, b, c, d) \in \mathbb{R}^4\}$ is a non commutative ring and $\bar{q} = a - bi - cj - dk$. Let $\gamma : [0, l] \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arc length s and define the map $\alpha(s, \theta) = e^{i\theta} \gamma(s)$.

i) Show that α is an immersion (Hopf cylinder) and compute the first and second fundamental forms of α .

ii) Analyze the axial configuration of α when γ is closed.

iii) Consider the stereographic projection $\Pi : \mathbb{S}^3 \setminus \{p_0\} \rightarrow \mathbb{R}^3$ and define $\beta = \Pi \circ \alpha$. Give various examples of closed curves γ and visualize the shape of the image of β . See [133].

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Glossary

$J^r(2, 3)$	The space of r-jets of smooth mappings of \mathbb{R}^2 to \mathbb{R}^3 , sending the origin to the origin , 105
$(\mathbb{M}, d), (\mathbb{M}, g)$	Riemannian manifold \mathbb{M} with distance d induced by the metric g , 19, 161, 190
E, F, G	Coefficients of the first fundamental form, 18
I_α, II_α	First and Second fundamental forms of the immersion α , 20
$T_p\mathbb{M}$	The tangent plan of \mathbb{M} at point p , 18
Γ_{ij}^k	Christoffel symbols with $i, j, k = 1, 2, 21$
Σ_{asy}^s	Class of immersions C^s - <i>asymptotic structurally stable</i> , 143
E_α	Ellipse of curvature, 198
$\mathbb{P}_\alpha = \{\mathbb{U}_\alpha, \mathbb{X}_\alpha\}$	Principal axial configuration, 195
\mathbb{X}_α and \mathbb{Y}_α	Fields of orthogonal tangent lines on which the immersion is curved along the extremes of the large and small axes of the curvature ellipse, 195
$\mathbb{A}_\alpha = \{\mathcal{A}_{1,\alpha}, \mathcal{A}_{2,\alpha}, \mathcal{P}_\alpha\}$	Asymptotic configuration of an immersion α , 27
PM	The projective tangent bundle of \mathbb{M} , 28, 70
$\mathbb{P}_\alpha = \{\mathcal{P}_{1,\alpha}, \mathcal{P}_{2,\alpha}, \mathcal{U}_\alpha\}$	Principal configuration of an immersion α , 26
$\mathbb{P}_{i,\alpha}$	Principal foliation of an immersion α , 26
\mathbb{R}, \mathbb{R}^n	The set of real numbers and Euclidean spaces, 17
\mathcal{H}	Mean curvature of of \mathbb{M} , 25
\mathcal{K}	Gaussian curvature of of \mathbb{M} , 22, 25, 219
τ_g	Geodesic torsion of a curve, 70, 203, 213

$\tilde{\mathcal{A}}_\alpha$	Pullback of the leaves of the pair of asymptotic foliations $\mathcal{A}_{\alpha,i}$, $i = 1, 2$ to the surface \mathbf{A}_α , 30
$\{T, N, B\}$	$\{T, N, B\}$ and $\{t, n, b\}$ are the Frenet frames of a curve, 57, 66, 183
$\{t, N_\alpha \wedge t, N_\alpha\}$	Darboux frame of a curve, 46
e, f, g	Coefficients of the second fundamental form of the immersion α , 20
k_1	Principal minimal curvature of \mathbb{M} , 24
k_2	Principal maximal curvature of \mathbb{M} , 24
k_g	Geodesic curvature of a curve, 20
k_n	Normal curvature of a curve, 20
\mathbb{P}_α	$\mathbb{P}_\alpha = (\mathcal{S}_\alpha, \mathcal{U}_\alpha, \mathcal{P}_{1,\alpha}, \mathcal{P}_{2,\alpha})$ is the principal configuration of the mapping α with singular set \mathcal{S}_α , umbilic set \mathcal{U}_α and the family of lines of principal curvature $\mathcal{P}_{1,\alpha}$ and $\mathcal{P}_{2,\alpha}$, 104
$\mathbb{Q}_\alpha = \{\mathbb{U}_\alpha, \mathbb{Y}_\alpha\}$	Mean axial configuration, 195
$\mathcal{A}_{i,\alpha}$	Asymptotic foliation of the immersion α , 28
$\mathcal{I}^{r,s}$	The space of immersions α of class C^r with the C^s -topology of Whitney, 83
$\mathcal{I}^{r,s}(\mathbb{M}, \mathbb{R}^3)$	The space of immersions of class C^r of \mathbb{M} to \mathbb{R}^3 , endowed with the C^s topology, 28
\mathcal{J}^r	$\mathcal{J}^r = \mathcal{J}^r(\mathbb{M}^2, \mathbb{R}^4)$ is the space of C^r immersions of \mathbb{M}^2 into \mathbb{R}^4 , 195
$\mathcal{L}_{i,\alpha}$	Principal line field of an immersion α , 26
\mathcal{P}_α	parabolic set of the immersion $\alpha : \mathbb{M} \rightarrow \mathbb{R}^3$, 28
$\mathcal{S}^r(\mathbb{M})$	Class of immersions C^r - principally structurally stable, 82
\mathcal{U}_α	Umbilic points of an immersion α , 25, 70
\mathcal{S}_α	Set of singular points of a map α , 104
D_i	Darbouxian umbilic of type D_i , ($i=1,2,3$), 59, 60, 71, 74–76, 226