The Einstein Constraint Equations

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Preface

This book was written as lecture notes for a mini-course on the Einstein constraint equations (ECE) delivered in the 33rd Brazilian Colloquium of Mathematics. It is directed to a wide audience of students and researchers interested in the overlap of Riemannian geometry, geometric analysis and physics. The focus of these notes is to provide a quite thorough description of the so-called conformal method, which translates the geometric ECE into an elliptic system of partial differential equations (PDEs) in a nearly self contained presentation, ranging from classical results to recent progress. This is a subject which intersects several traditional problems in geometric analysis, such as scalar curvature prescription and the Yamabe problem, and which has its roots in the evolution problem of initial data in general relativity (GR). As such, it has become a whole area of research within mathematical GR and its intersection with classic problems in geometric analysis has produced plenty of feedback between these areas. We shall assume the reader is familiarised with classical topics and language in both differential geometry and Riemannian geometry as well as with standard functional analysis, which is used within PDE theory. We do not assume the reader to be necessarily acquainted with elliptic equations and, with this in mind, we have built an appendix compiling the necessary tools which are used in the core of the book. Also, some of the most recurrent functional analytic tools are also compiled within the first appendix of the book, with emphasis on Sobolev space theory, which provides the reader with all the necessary tools to follow the main chapters without many outside references.

The organisation of the book is intended to deliver a clear exposition highlighting the relevance of the analysis of the ECE, their many subtleties and an up-to-date

presentation of the results available in this area. In doing so, we have been inspired by recent literature in the subject, most notably the monograph of Choquet-Bruhat (2009) and several recent papers such as Holst, Nagy, and Tsogtgerel (2009) and Maxwell (2005a,b, 2009). We have gone through the classical constant mean curvature (CMC) classifications on closed manifolds originated in Isenberg (1995), but putting them in light of these recent advances, and thus presented them in low regularity and also contemplating non-vacuum situations. Along these lines, we have complemented several of these recent references. Furthermore, we have made emphasis in the analysis on asymptotically Euclidean (AE) manifolds, incorporating boundary value problems, and, as a novelty in a book on the subject, we have introduced recent advances on far-from-CMC existence of solutions.

Chapter 1 is meant to be an introduction to general relativity with the objective of setting up the problem, reviewing the context in which the ECE arise, producing some intuitions and motivating the analysis of boundary problems associated to black hole solutions as well as highly coupled systems exemplified by charged fluids. Also, in this chapter we set most of our notational conventions. The topics here included are standard for any specialist in GR, but are intended to serve as a good introduction for the unfamiliarised reader, from whom we do not assume any sophisticated knowledge of physics.

Chapter 2 starts by presenting the conformal method and translating the ECE into a geometric elliptic system. In doing so, we contemplate very general situations which incorporate the conformal formulation of the Gauss–Codazzi constraints coupled with a further electromagnetic constraint. Then, we start our analysis with the CMC case admitting sources which allow the system to be fully decoupled and thus the core of the analysis is devoted to the associated Lichnerowicz equation. During this chapter we will give a near state-of-the-art presentation of this problem following Maxwell (2005a), and therefore establishing an L^p-low-regularity complete CMC classification on closed manifolds which incorporates several physical sources. In the process of doing so, we shall review results concerning the Yamabe classification in this low regularity setting.

In Chapter 3, we move to the analysis of the Lichnerowicz equation on AE manifolds and introduce boundary value problems which model black hole initial data within the conformal method. We deliver a quite self-contained presentation of the necessary elliptic theory on AE manifolds, which appeals to analysis on weighted Sobolev spaces. We introduce the basic machinery associated to these problems merely assuming basic acquaintance of the reader with the corresponding theory on compact manifolds. We shall present a wide variety of results associated to classical papers such as Bartnik (1986), Cantor (1981), Choquet-Bruhat

and Christodoulou (1981), Lockhart (1981), McOwen (1979), and Nirenberg and Walker (1973). After doing this, the main results related to the ECE will be an exposition of Maxwell (2005b).

Chapter 4 is devoted to a presentation of far-from-CMC results. These are quite recent advances in the analysis of the ECE which rely on the application of some fixed-point-theorem ideas and make use of the full machinery developed in previous chapters. We shall first review some near CMC results, attainable through implicit function techniques, and then provide a presentation of the far-from-CMC results established in Maxwell (2009), which followed the pioneering work of Holst, Nagy, and Tsogtgerel (2009). These results concern the coupled ECE in vacuum on closed manifolds. Finally, we will move towards the analysis of the ECE for a charged perfect fluid on AE manifolds with black hole boundary data and present the far-from-CMC results of Avalos and Lira (2019).

Although during the main core of the text the reader is assumed to be familiarised with elliptic theory on closed manifold, in order to provide a self-contained presentation, we have provided most of the necessary tools within two appendixes, where the reader can consult all the results which are used in the main chapters. The first of these appendixes is concerned with some functional analytic tools while the second one with elliptic theory. Since these are extensive areas on their own right, our presentation has been more expository in nature, attempting to provide the reader with full proofs whenever possible, and, when the details exceed the scope of these notes, provide full references as well as the basic intuitions on the ideas behind the actual proofs.

We expect these notes to help researchers within theoretical physics and pure and applied mathematics to become familiarised with some of the many interesting problems in the analysis of the ECE. Some related topics had to be left outside due to time constraints for our course, but a thorough list of references has been provided which the interested reader can use to substantially expand the scope of this book.

Contents

1 Introduction to general relativity

The objective of these notes is to analyse the so-called Einstein constraint equations (ECE). Naturally, these equation arise in the context of general relativity (GR), more specifically within the initial value formulation of this theory. In particular, solution to the ECE provide us with suitable initial data which we can then evolve into solutions of the space-time Einstein equations. Being GR the best known description of gravitational phenomena up to this date, this alone provides enough motivation for the analysis of the ECE. Nevertheless, from a purely mathematical standpoint, they relate with classical problems in Riemannian geometry, such as scalar curvature prescription problems and related geometric partial differential equation (PDE) problems, which further motivates their analysis.

The aim of this first introductory chapter is to provide a review of the setting where the ECE appear naturally, which is the initial value formulation of GR. In this way we can most effectively motivate their relevance, present model situations of interest and provide intuitions about what is expected to occur in their analysis. Since this is a topic which gathers researchers and students ranging from theoretical physics to geometric analysis, we intend to review several notions which are well-known to experts in each of these areas and should be within reach without too much effort for those who are not. In doing so, we will assume acquaintance with differential geometry as well as Riemannian and semi-Riemannian geome-

try.[1] As a remark regarding notational conventions, let us highlight that, besides standard notations within geometry, we will use whenever it may be more convenient Einstein's index and summation conventions for coordinate expressions, without further comments.

   With the above in mind, the organisation of this chapter will be as follows. First, we will review some definitions and results related specifically to Lorentzian geometry. Our main motivations here will be to introduce enough language from causality theory so that, later on, we can introduce notions such as black hole solutions as well as those of Cauchy hypersurfaces and global hyperbolicity. Then, we will present the skeleton of the theory of special relativity. There, the aim is to introduce notions that will be of relevance in subsequent analysis, such us the basic fields which we shall couple to gravity and for which we shall analyse the existence of appropriate initial data. After this, we will promote this discussion to the context of GR, introducing the Einstein equations and presenting these relevant systems in this general context. Also, we will try to develop some intuitions by presenting a few classical well-known exact solutions. In particular, we intend to provide some rudimentary intuitions concerning black hole solutions by describing the Schwarzschild solution. The objective at this point will be to provide us with the right notions to motivate our discussion on black hole initial data. But, before doing this, we will describe the initial value formulation of general relativity. This, in particular, is a topic which deserves a complete book on its own due to its many subtleties (which the interested reader can actually find, for instance, in Ringström (2009)),[2] and therefore we will merely review those results which are of most relevance to us.

1.1   Some elements of Lorentzian geometry

Let us now introduce some notions related to Lorentzian geometry, most of which can be found in standard references, such as Choquet-Bruhat (2009), Hawking and Ellis (1973), and O'Neill (1983) as well as references therein. Let us first state that,

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[1]If needed, the interested reader can consult differential geometric topics in classic textbooks such as J. M. Lee (2013) and Spivak (1999e), Riemannian geometry topics in do Carmo (1992) and Spivak (1999a,b,c,d) and textbooks adapted to semi-Riemannian geometry such as Bishop and Goldberg (1980) and O'Neill (1983).

[2]We further recommend references such as Choquet-Bruhat (2009) for a self-contained presentation of the general problem, as well as Christodoulou and Klainerman (1993) and Klainerman and Nicolò (2003) for issues related to the stability of Minkowski and Dafermos and Rodnianski (2013) for topics related with black hole evolution and stability.

during all this text, manifolds will be assumed to be Hausdorff and second countable and, whenever specifying the dimensionality of a manifold M is relevant, we write M^d for a d-dimensional manifold.

Definition 1.1.1. *A semi-Riemannian manifold (V, g) will be called Lorentzian if the metric g has constant index equal to* 1.

Let us recall that the index of a symmetric bilinear form on a vector space is defined to be the dimension of the largest subspace where its restriction is negative definite. Therefore, using a local orthonormal frame $\{\theta\}_{\alpha=0}^n$, we can write g as

$$g = -\theta^0 \otimes \theta^0 + \sum_{i=1}^n \theta^i \otimes \theta^i.$$

As above, we will typically reserve the 0-th direction to be the one over which g is negative definite. In particular, the above shows that one can split tangent vectors $v \in T_p V$ into three cases, which determine their *causal character*.

Definition 1.1.2. *Let (V, g) be a Lorentzian manifold and let $p \in V$. We will say that a vector $v \in T_p V$, $v \neq 0$, is time-like if $g_p(v, v) < 0$; light-like (or null) if $g_p(v, v) = 0$ and space-like if $g_p(v, v) > 0$. Along these lines, we define the light-cone (or null-cone) at p as the subset of $T_p V$ formed by all the null-vectors.*

Whenever we consider a smooth curve $\gamma : I \subset \mathbb{R} \mapsto V$, if its causal character is constant, that is, if γ' is everywhere time-like, null or space-like, then we will say that γ is time-like, null or space-like respectively. Clearly, an arbitrary curve will not fall into any of these categories since its causal character may change, but, in particular, geodesics have a fixed causal character.[3] In order to clarify some of this terminology, let us anticipate that, in the context of relativity theory, massive particles trace time-like paths in space-time while massless particles (such as photons) trace light-like paths. On the other hand, since no signal can travel faster than light, space-like paths do not represent the dynamics of any kind of particles. In particular, points which are space-like related do not have the possibility of influencing each other. We will therefore say that a curve is *causal* if it is either time-like or light-like.

Let us now highlight the special role played the the following Lorentzian manifold.

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[3]During these notes, we will always work with Riemannian (metric compatible and torsion-free) connections, and therefore parallel transport is an isometry.

Definition 1.1.3. *The manifold $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric η given by*

$$\eta = -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i,$$

where $\{x^\alpha\}_{\alpha=0}^{n}$ stand for (global) canonical coordinates for $\mathbb{R}^n$, is referred to as the Minkowski space-time, and we will denote it by $\mathbb{M}^{n+1}$.

Therefore, just as Euclidean space is the local model of a Riemannian manifold, in a Lorentzian manifold (V^{n+1}, g) we have $(T_p V, g_p) \cong \mathbb{M}^{n+1}$. In particular, the Minkowski space-time is the arena where special relativity takes place.

We will now endow our Lorentzian manifolds with further structure than the minimal one imposed above. In particular, **we will always consider *time-orientable* Lorentzian manifolds, which we shall also refer to as *space-times*.**

Definition 1.1.4. *(O'Neill 1983, Page 145) Let (V, g) be a Lorentzian manifold. At each point $p \in V$, in $T_p V$ we have two null-cones. A choice of one of these null-cones is a time-orientation for $T_p V$. A smooth function τ on V which assigns to each $p \in V$ a null-cone in $T_p V$ is said to be a time-orientation for V. We say (V, g) is time-orientable if it admits such a time-orientation function.*

It is straightforward to see that a Lorentzian manifold is time-orientable if and only if it admits a (global) time-like vector field (see, for instance, O'Neill (ibid., Lemma 32, Chapter 5).) Although in time-orientable Lorentzian manifolds there is a consistent way to distinguish past from future, these are still quite general structures which may inherit some exotic (maybe *undesirable*) properties. For instance, any compact Lorentzian manifold admits a closed time-like curve (see, for instance, O'Neill (ibid., Lemma 10, Chapter 14)). Since, within physics, causal paths represent the history of actual particles, this property is typically deemed as *pathological* allowing for potential travels to the past, and therefore excluded. Such exclusion is made by appealing to topological properties which guarantee a *good* causal structure on our space-time. Let us therefore introduce the relevant concepts.

Let (V, g) be a (time-orientable) Lorentzian manifold and $p, q \in V$.[4] Then, we will write:

1. $p \ll q$ if there is a future-pointing time-like curve in V from p to q;

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[4]From now on, the time-orientability hypothesis will be implicitly assumed.

2. $p < q$ if there is a future-pointing causal curve in V from p to q;

3. $p \leqslant q$ if either $p < q$ or $p = q$;

4. Given a subset $A \subset V$, we define the *chronological* future $\mathcal{I}^+(A)$ and past $\mathcal{I}^-(A)$ of A by

$$\mathcal{I}^+(A) \doteq \{q \in V \; : \; \exists \, p \in A \text{ with } p \ll q\},$$
$$\mathcal{I}^-(A) \doteq \{q \in V \; : \; \exists \, p \in A \text{ with } q \ll p\},$$

and the *causal* future $\mathcal{J}^+(A)$ and past $\mathcal{J}^-(A)$ of A by

$$\mathcal{J}^+(A) \doteq \{q \in V \; : \; \exists \, p \in A \text{ with } p \leqslant q\},$$
$$\mathcal{J}^-(A) \doteq \{q \in V \; : \; \exists \, p \in A \text{ with } q \leqslant p\}.$$

There are several immediate consequences of these definitions, such as the fact the $\ll$ is always an open relation, implying that $\mathcal{I}^+(A)$ is always open, and also some subtleties, such as the fact that $\mathcal{J}^+(A)$ is not always closed (for a simple counter example, see O'Neill (ibid., Example 4, Chapter 14)). Nevertheless, since we shall only *use* this language to introduce relevant concepts and results, we will not be concerned with such subtleties and refer the interested reader to standard references, such as O'Neill (ibid.) or Hawking and Ellis (1973). Let us now introduce the following causality condition, which is related to our previous discussion.

Definition 1.1.5. *Let (V, g) be a Lorentzian manifold. We will say that* the strong causality condition *holds at $p \in V$ if for any given neighbourhood $\mathcal{U}$ of p there is a neighbourhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve with endpoints in $\mathcal{V}$ is entirely contained in $\mathcal{U}$.*

The above causality condition is basically tailored to exclude the possibility of *almost closed* causal-curves, since it implies that causal curves which leave a fixed neighbourhood of $p \in V$ cannot return to arbitrarily close to p. Again, deleting appropriate subsets of simple Lorentz manifolds can be shown to create a Lorentzian manifold without closed causal curves but with causal curves which are almost closed, and we intend to avoid this. In fact, it can be seen that if the strong causality condition holds in a compact subset K of a space-time (V, g), then *future-inextendible* causal curves in K eventually leave K and never return to it (O'Neill 1983, Lemma 13, Chapter 14).
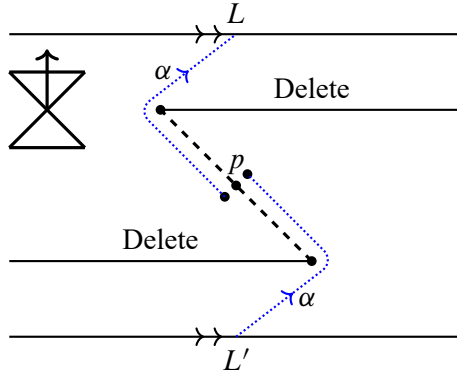
Figure 1.1: $S^1 \times \mathbb{R}$ obtained from identification of sides L and L' and equipped with the metric $-d\theta^2 + dx$ and with the highlighted regions deleted. It possesses *almost* closed time-like curves although no closed ones.

Given two points $p, q \in V$ and $p < q$, we use the notation $\mathcal{J}(p,q) \doteq \mathcal{J}^+(p) \cap \mathcal{J}^-(q)$, which is the smallest set containing all future-pointing causal curves from p to q. Then, we have the following important definition.

Definition 1.1.6. *We say that a Lorentzian manifold (V, g) is **globally hyperbolic** if:*

1. *The strong causality condition holds in V;*

2. *If $p, q \in V$ and $p < q$, then $\mathcal{J}(p,q)$ is compact.*

In particular, in globally hyperbolic space-times, the relation $\leqslant$ is closed (O'Neill 1983, Lemma 22, Chapter 14). Furthermore, globally hyperbolic space-times have a particularly nice topological structure, which makes them natural in the context of evolution problems. To make this precise, let us introduce one further definition.

Definition 1.1.7. *A **Cauchy hypersurface** in a Lorentzian manifold (V, g) is a subset M that is met exactly once by every inextendible time-like curve in V.*

The following result links the two notions of global hyperbolicity and Cauchy surfaces:

Theorem 1.1.1. *(Bernal and Sánchez 2003) Any globally hyperbolic space-time* (V, g) *admits a smooth space-like Cauchy hypersurface* M.[5] *Furthermore, V is diffeomorphic to* $\mathbb{R} \times M$.

The above theorem stands as an improvement to the smooth category of the corresponding topological result, which is a classical celebrated result by Geroch (1970). In this last result, the author obtained a *topological* Cauchy surface and an *homemorphism* with $\mathbb{R} \times M$. There is a rich history concerning the evolution of these kinds of results which can be consulted in Bernal and Sánchez (2003). Furthermore, the above result can be strengthened, establishing that (V, g) is *isometric* to $(\mathbb{R} \times M, -N^2 d\mathcal{T}^2 + \bar{g})$, with $\mathcal{T} : \mathbb{R} \times M \mapsto \mathbb{R}$ the natural projection, $N : \mathbb{R} \times M \mapsto (0, \infty)$ a smooth function, $\bar{g}$ a symmetric $(0, 2)$-tensor field which, for each $\mathcal{T} = cte$, restricts to a Riemannian metric on $\{\mathcal{T}\} \times M \cong M$, and where $\nabla \mathcal{T}$ is time-like and past-pointing, *i.e*, $\mathcal{T}$ is a *time-function* (Bernal and Sánchez 2005). A further generalisation of these ideas can be obtained for *globally hyperbolic manifolds with (appropriate) boundary*. For such results, we refer the reader to Hau, Dorado, and Sánchez (2021).

There are a couple of interesting consequences of the above theorem. First, notice that any non-trivial topology in a globally-hyperbolic space-time must be contained within its Cauchy surface. Second, and more directly related with our discussions to come, a Cauchy hypersurface in a globally hyperbolic space-time is a suitable subset where we can pose initial conditions for evolution problems. In fact, our task will be to start with a Cauchy surface M and initial data on it, and then show that we can evolve such initial data to create space-time solutions to the Einstein equations. Although general existence results only provide us with a slab $[0, T] \times M$ on which the space-time solution is guaranteed to exist, whenever solutions are guaranteed to exist for all times, we recover a globally-hyperbolic space-time by evolution.

We shall return and appeal to the above causality-theory ideas in Section 1.5 when we discuss general black hole space-times and singularity theorems.

1.2 Special Relativity

We shall now introduce some elements from the theory of special relativity which will be useful in upcoming sections. Along the lines of the previous section, we

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[5]By space-like hypersurface, we mean that the induced metric h by g on M is a properly Riemannian metric.

will not enter into details and instead refer the interested reader to standard references in the subject.

Newtonian space-time

Let us start by briefly describing the Newtonian picture of physics and its space-time formulation. In this setting, one starts assuming that the notions of space and time and fixed and, in particular, do not play any dynamical role. Physical particles interact and evolve within 3-dimensional Euclidean space $\mathbb{E}^3 = (\mathbb{R}^3, \cdot)$, which represents the physical space and time is universal, in particular there is a universal agreement on which events are simultaneous. In this context, one distinguishes the set of *inertial reference frames* (special coordinate systems) being those in uniform rectilinear motion, all of which move with constant velocity with respect to each other. On these frames, Newton's laws of mechanics are valid and the principle of Galilean relativity holds. That is, the physical laws of mechanics are the same in all inertial frames. Then, the coordinate transformations that relate different inertial frames define the Galilean group, whose action preserves the laws of mechanics.

Already in this context we can introduce the notion of space-time, which simply refers to the collection of all physical *events*. Galilean space-time is therefore given by the manifold $\mathbb{R} \times \mathbb{R}^3$, where the first factor refers to time and second one to space and where events are labelled by their space and time coordinates. Since particles are described by curves $\alpha : \mathbb{R} \mapsto \mathbb{E}^3$, within space-time the same particles are described by *worldlines*, which are curves of the form $\gamma : \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}^3$, given by $\gamma(t) = (t, \alpha(t))$. Also, Galilean transformations act on space-time relating the coordinate systems adopted by different (Galilean) inertial frames.

In pre-relativistic physics, the above description of mechanics was supplemented by Maxwell's description of electromagnetic phenomena. This already presents a tension in the physical description, since electromagnetic phenomena do not respect the same kind of Galilean invariance alluded to above. In particular, such tension led physicists of the time to believe that there was a preferred reference frame (the *aether* frame) with respect to which Maxwell's equations were written in their usual form and which provided a medium for electromagnetic waves to propagate. Nevertheless, this hypothesis became increasingly difficult to hold in light of experimental results failing to detect such aether frame and needing of certain additional ad hoc hypotheses to account for their negative results. These discussions seem to have been at the core of Einstein's reasoning towards relativity theory.[6]

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[6]For some historical discussions and description of experiments shifting the physical paradigm
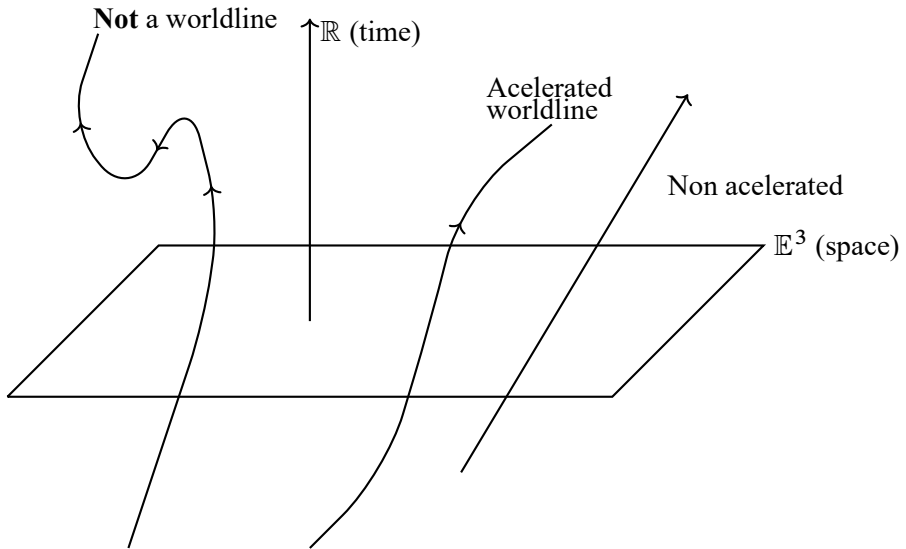
Figure 1.2: Newtonian space-time

Special Relativity and the Minkowski space-time

In the context described above and in order to reconcile the tensions alluded to, Einstein proposed the following two principles, which are now known as the postulates of special relativity:

1. All the laws of nature have the same form in every inertial frame;

2. The speed of light is equal to the same universal constant in every inertial frame, independent of the motion of the source.

The first of the above two principles is an extension of the Galilean principle of relativity to include electromagnetic phenomena. When the above principles are put together, they can be used to lead us to the transformation rules relating different inertial systems, which are no longer the Galilean transformations. In turn, they are now the Lorentz transformations. In order to introduce them, if we assume that we have two inertial Cartesian systems $S = (t, x^i)$ and $S' = (t', x'^i)$, whose origins coincide initially and we assume that the direction of relative motion of

of the time, we refer the reader to references such Møller (1952, Chapter 1) and Jackson (1999, Chapter 12).

S' with respect to S coincides with a particular coordinate axis, say x^1, then the relation between these inertial systems is given by
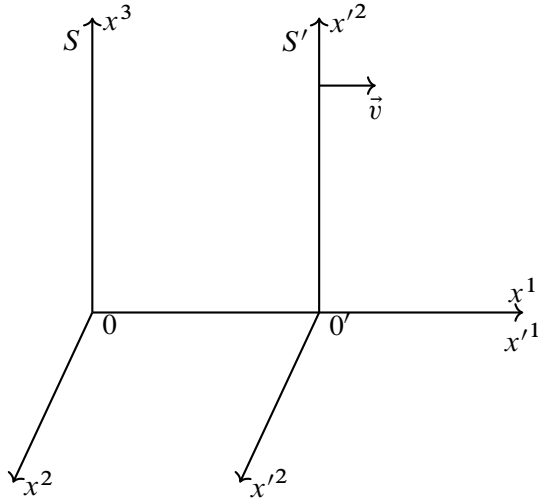
Figure 1.3: Inertial systems in relative motion

$$t' = \frac{t - \frac{v}{c^2}x^1}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{1}{2}}},$$

$$x'^1 = \frac{x^1 - vt}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{1}{2}}},$$

$$x'^2 = x^2,$$

$$x'^3 = x^3,$$

(1.1)

where $v = |\vec{v}|$ stands for the magnitude of the relative speed between the two systems and c for the speed of light. The above relations readily extend to a general situation by composing with rotations (on $\mathbb{R}^3$) and space-time translations. Many well-known consequences of special relativity now follow by direct application of physical invariance under Lorentz transformations. Effects such as those of time-dilation and Lorentz-contraction are two such examples. Furthermore, the above

relations between inertial frames impose a paradigm-shift concerning the notion of simultaneity, since, clearly from the first of the above relations, observers in relative motion do not agree on this concept although their perspectives are all equally valid. In this context, in order to avoid the recurrent appearances of the factor c in every expression, the speed of light is set equal to $c = 1$ and physical units are redefined accordingly. From now on, we will follow this convention. Detailed discussions on all these physical effects can be found in the previously cited references, both on special and general relativity.

One further important consequence of the above principles of relativity and the corresponding Lorentz group $\mathbb{L}$, is that these transformations are precisely the linear isometries associated to the Minkowski metric. That is,

$$\mathbb{L}(\mathbb{R}^4) = \{A \in \mathbb{GL}(\mathbb{R}^4) \ : \ \eta(Ax, Ay) = \eta(x, y) \text{ for all } x, y \in \mathbb{R}^4\}.$$

Allowing for space-time translations, we arrive at the Poincaré group representing the full isometry group of Minkowski's space $\mathbb{M}^4$ (see, for instance, O'Neill (1983, Proposition 10, Chapter 9)). All this motivates us to introduce the Minkowski space-time $\mathbb{M}^4 = (\mathbb{R}^4, \eta)$ as the space-time model of special-relativity. Several further modifications must be imposed to the Newtonian paradigm to make the physical description compatible with the principles of relativity, in particular with the new needed invariance of physical laws for inertial systems under the action of Lorentz transformations. Let us attempt to describe the main setting, which will become useful latter on.

First of all, in Minkowski's space-time, light-rays clearly represent null-curves. On the other hand, massive particles are represented by time-like worldlines. This last fact is based on the empirical evidence that no massive particle has ever been detected to travel at the speed of light or faster. Although whether this is ultimately possible is up to Nature to decide, there are also strong a priori arguments against this possibility. For instance, a particle (massive or not) that appears to be travelling faster-than-light in one inertial frame will appear to be travelling backwards in time in some other inertial frame, as can be seen by appealing to the above Lorentz transformations. Furthermore, a massive particle which starts with velocity lower than that of light, cannot be accelerated up to the speed of light as a consequence of relativistic effects (although it can be brought as close as we want to it). Be it as it may, this universal speed limit is a tenant of contemporary physics which has passed every test to this day. With this in mind, we can introduce the notion of *proper time* associated to a massive particle with worldline γ, simply as its arch-length between two events along its history. That is, given a time-like curve $\gamma : I \mapsto \mathbb{M}^4$ and two events $p = \gamma(s_1)$ and $q = \gamma(s_2)$, $p \ll q$, we define the

elapsed proper-time $\Delta\tau$ as measured by γ as

$$\tau_2 - \tau_1 \doteq \int_{s_1}^{s_2} \sqrt{-\eta(\gamma', \gamma')} ds, \tag{1.2}$$

which is independent of the parametrisation used for γ. Basically, proper-time is the Lorentzian analogue of arch-length for time-like curves. We know from standard arguments that any such curve can be reparametrised by proper-time and that this reparametrisation is precisely the one which normalises its velocity. That is, given a time-like curve γ, if we reparametrise by proper-time, then $\eta(\gamma'(\tau), \gamma'(\tau)) = -1$. This normalisation is standard for time-like particles and therefore, when we consider massive particles, we will assume it. In fact, such preferred parametrisation also has a clear physical interpretation, since it represents the elapsed time as *experienced* by a γ itself.[7] This kind of language spreads within relativity theory. For instance, the mass of a particle as measured by an observer for which the particle is at rest, is referred to as *proper mass*. Similarly, the charge of a particle measured under these conditions is referred to as *proper charge*, and so on.

Let us highlight that, although in this context things such as simultaneity become relative to a reference frame, *causality relations are universal*. That is, using the language of the previous section, the causal relations between events represented by $\ll$ and $\leqslant$ depend only on the Lorentzian structure of Minkowski's space-time, and are therefore invariant by Lorentz transformations. Therefore, we see how the geometric structure of Minkowski space now plays a fundamental role in determining physical relations.

The above geometric description of a worldline of a massive particle allows us to replace Newton's second law in this context quite naturally. This is necessary since Newton's second law, which is invariant under Galilean transformations between inertial frames, is not invariant under the full group of Lorentzian transformations. In this context, given a point-like particle with worldline γ_τ and of proper mass $m_0 > 0$, the well-known Newtonian second law is replaced by

$$\frac{D}{d\tau}(m_0\gamma'_\tau) = f, \tag{1.3}$$

where the left-hand side stands for the covariant derivative of $m_0\gamma'$ along γ and the right-hand side for a space-time force field, typically referred to as a *4-force*.

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[7]Recall that, in contrast to Newtonian physics, time intervals between fixed events are relative to the observer in relativity, as seen by using Lorentz transformations.

Let us notice that, in absence of forces (and with $m_0 = cte$), the above reduces to the geodesic equation for γ. Furthermore, the vector field $p = m_0 \gamma'_\tau$ plays also a special role. Above, it is actually playing an analogue role to that of the Newtonian linear momentum, and that is why p is referred to as the 4-momentum of such particle. Let us fix an inertial coordinate system (t, x^i) and consider the dynamics of γ with respect to it.[8] Let us first notice that we can always parametrise γ by the coordinate time t, since

$$\frac{dt}{d\tau} = -\eta(\partial_t, \gamma'_\tau) > 0 \text{ along } \gamma_\tau.$$

This kind of reparametrisation is typically useful to make contact between the relativistic description in Minkowski space-time and the Newtonian perspective which can help us develop intuitions of new concepts in their low-velocity limit. Then, notice that

$$\gamma'_\tau = \frac{dt(\tau)}{d\tau} \frac{\partial}{\partial t}\Big|_{\gamma_\tau} + \frac{dx^i(\tau)}{d\tau} \frac{\partial}{\partial x^i}\Big|_{\gamma_\tau} = \frac{dt}{d\tau}\left(\frac{\partial}{\partial t}\Big|_{\gamma_\tau} + \frac{dx^i(t)}{dt} \frac{\partial}{\partial x^i}\Big|_{\gamma_\tau}\right),$$

where

$$\frac{dt}{d\tau} = \left(1 - \left|\frac{d\vec{x}}{dt}\right|^2\right)^{-\frac{1}{2}},$$

and $\vec{v} = \frac{d\vec{x}}{dt}$ is the Newtonian velocity of the alleged particle as seen in the (t, x^i) reference frame. That is,

$$\gamma'_\tau = \left(1 - \left|\frac{d\vec{x}}{dt}\right|^2\right)^{-\frac{1}{2}}\left(\frac{\partial}{\partial t}\Big|_{\gamma_\tau} + \frac{dx^i(t)}{dt} \frac{\partial}{\partial x^i}\Big|_{\gamma_\tau}\right). \tag{1.4}$$

Therefore, for a point-like particle of proper mass m_0, the 4-momentum can be written as

$$p = \left(\frac{m_0}{\left(1 - \left|\frac{d\vec{x}}{dt}\right|^2\right)^{\frac{1}{2}}}, \frac{m_0\vec{v}}{\left(1 - \left|\frac{d\vec{x}}{dt}\right|^2\right)^{\frac{1}{2}}}\right).$$

---

[8]To avoid confusion, we adopt the convention that when we refer to a time-like curve representing a physical particle, we assume its parametrisation is chosen so that it is future pointing.

The space-part of the above vector field looks as a suggestive modification of the Newtonian momentum $\vec{p} = m_0 \vec{v}$. In fact, recalling that we set $c = 1$ and thus for Newtonian particles we have $v \ll 1$, we see that

$$\frac{m_0}{\left(1 - v^2\right)^{\frac{1}{2}}} = m_0 \left(1 + \frac{1}{2}v^2 + o(v^4)\right) = m_0 + \frac{1}{2}m_0 v^2 + o(v^4),$$

$$\frac{m_0 \vec{v}}{\left(1 - v^2\right)^{\frac{1}{2}}} = \vec{p} \left(1 + \frac{1}{2}v^2 + o(v^4)\right) = \vec{p} + o(v^3).$$

The usual interpretation of the above relations is that the 4-momentum p represents the *energy-momentum* vector field associated with the point-like mass m_0. Its time-component converges to its Newtonian kinetic energy with an added energy contribution due its mass as $v \to 0$, while its space-component approaches its Newtonian momentum in this limit. Therefore, the energy-momentum vector field p associated with m_0 is actually understood as the suitable relativistic generalisation of the associated Newtonian concepts of energy and momentum, and recovers these last concepts in the low velocity limit. Thus, in (1.3), the time-component of the 4-force f can be understood as the relativistic generalisation of the work done on m_0 while the space-part can be understood as the relativistic generalisation of the usual Newtonian force acting on it. Typically, understanding the situation in the Newtonian limit $v \to 0$ and appealing to a Lorentz-covariant generalisation work as the guiding principles to obtain the suitable relativistic 4-force f.

Let us finally notice that, although now we see that the energy and momentum associated to a particle of mass m_0 depend on our reference frame, and therefore their values in different inertial systems are linked by Lorentz transformations, the proper mass m_0 is a universal invariant quantity, which, in any inertial frame, reads

$$m_0^2 = -\eta(p, p) = E^2 - |\vec{p}|^2. \tag{1.5}$$

Furthermore, the concept of energy-momentum vector can be generalised to massless particles, for which the above relation also holds, establishing that for such particles (for instance photons) we have $E^2 = |\vec{p}|^2$.

Energy-Momentum tensors and continuous matter

Above, among other things, we described the basic elements entering in the dynamics of point-like particles in the relativistic context, which represents a useful idealisation in many situations. Nevertheless, when we deal with systems of many

particles, we can typically neglect details of the specific individual particles and consider certain coarse-grained properties which dictate the overall dynamics of the systems. In these situations, we can idealise such systems as continuous matter distributions, typically modelled as a fluid, whose dynamics is controlled by the corresponding dynamics of certain hydrodynamical parameters, such as its energy density, pressure density and velocity field. These fluid parameters evolve obeying conservation laws relating their rates of change in a given region with their flux in and out of it. All these conservation relations of energy and momentum are best captured by introducing an energy-momentum tensor field $T \in \Gamma(T_2^0 \mathbb{M}^4)$ associated with the fluid. This is a symmetric tensor field, which, in a given inertial coordinate system $(x^0 = t, x^i)$, relates to the *energy density* ϵ and *momentum density* J of the fluid via

$$
\begin{aligned}
\epsilon &= T(\partial_t, \partial_t) = T_{00}, \\
J_i &= -T(\partial_t, \partial_{x^i}) = -T_{0i}
\end{aligned}
\tag{1.6}
$$

and the conservation laws are expressed via

$$
\mathrm{div}_\eta T = 0 \quad \left(\eta^{\mu\nu} \partial_\mu T_{\nu\sigma} = 0 \right).
\tag{1.7}
$$

In an inertial coordinate system, the time component of the above equation put together with Stokes theorem implies an energy-conservation law, while the space-components a momentum conservation law.

In this context, let us introduce one further useful notation. We can fix a reference frame (maybe not inertial) by considering the flow of a time-like vector field v with flow-lines γ_s. We could think about such fame as attached to an idealised fluid with these flow-lines. In case such fluid is inertial, *i.e*, it moves with constant velocity with respect to our inertial frame, then both frames are related via Lorentz transformations and the coordinates of the energy-momentum tensor field T in both frames are also related via these coordinate changes. In case the frame given by v is not inertial, we can nevertheless present the physical description as experienced by such observers by simply applying the appropriate coordinate change between these two frames. Therefore, in this more general setting, we define the energy and momentum densities of the fluid with energy momentum tensor T as seen by γ_s' as

$$
\begin{aligned}
\epsilon &= T(\gamma_s', \gamma_s'), \\
J(X) &= -T(\gamma_s', X), \quad \text{for all } X \perp_\eta \gamma_s',
\end{aligned}
\tag{1.8}
$$

which reduces to (1.6) when we consider $\gamma_s' = \partial_t$.

The above treatment via continuous fluids and hydrodynamic equations models situations ranging from the classical dynamics of fluids, to stellar physics and up to the overall dynamics of the universe, where its matter content is modelled in this way. Below, we will provide a few examples of these situation, limiting to those which we shall encounter in our future analysis. The interested reader may consult the physical details as well as more exhaustive discussion in references such as Weinberg (1972).[9]

Perfect fluids

Perfect fluids are among the simplest examples we can present within the above discussion. These are fluids characterised by their 4-velocity field u, its energy density μ and pressure density p and are defined by the condition that an observer moving along with the fluid should see it as *isotropic* (see, for instance, Weinberg (ibid., Section 10, Chapter 2)). In an arbitrary inertial frame, the corresponding energy-momentum tensor field is deduced to have the form:

$$T = (\mu + p)u^\flat \otimes u^\flat + p\eta, \tag{1.9}$$

where $u^\flat$ denotes the 1-form metrically equivalent to u. The equations of motion for such a fluid, known as the Euler equations, are obtained through (1.7), and typically have to be supplemented by a suitable *state equation*, which provides a relation between the state variables p and μ. Such an equation of state depends on the characteristics of the kinds of matter of which the fluid is made, and is typically derived via methods of statistical mechanics.

The above procedure is particularly simple when $p = 0$. Such a *pressureless* perfect fluid is known as as *dust*. In this case, the equations of motion read

$$\begin{aligned} 0 &= \eta^{\alpha\beta} \nabla_\alpha(\mu u_\beta u_\sigma) \\ &= \mathrm{div}_\eta(\mu u)\, u_\sigma + \mu u^\alpha \nabla_\alpha u_\sigma, \end{aligned} \tag{1.10}$$

The above equation is simplified by projecting it parallel and orthogonal to u. Recalling the normalisation convention $\eta(u, u) = -1$ for massive particles, the parallel component gives

$$0 = -\mathrm{div}_\eta(\mu u) + \mu\eta(u, \nabla_u u),$$

---

[9]To obtain a quite direct acquaintance with the topic of hydrodynamics, the interested reader can find a mathematically oriented brief presentation in Abraham, Marsden, and Ratiu (1988, Chapter 9) in the Newtonian context and, in the other end, a detailed presentation in the general relativistic context in Choquet-Bruhat (2009, Chapter IX).

which can be further simplified, since the normalisation condition on u implies that $\eta(u, \nabla_u u) = 0$. Thus, we obtain

$$\text{div}_\eta(\mu u) = 0 \tag{1.11}$$

Using this information in (1.10), we find ($\mu \not\equiv 0$)

$$\nabla_u u = 0. \tag{1.12}$$

That is, the flow lines of a dust fluid are given by geodesics. Furthermore, (1.11) is simply a continuity equations, representing the conservation of matter.

Maxwell's equations

The idea of this section is to set up the notations for the Maxwell equations consistently while presenting their formulation in the context of special relativity. The usual Maxwell equations, written in some inertial coordinate system (t, x) on $\mathbb{R} \times \mathbb{R}^3$ are given by[10]

$$\partial_t E - \text{Curl } B = -j, \quad \partial_t B + \text{Curl } E = 0$$
$$\text{div } E = \rho, \quad \text{div } B = 0, \tag{1.13}$$

where ρ and j represent, respectively, the total charge and current densities produced by sources. These two quantities are not independent, since putting together the two equations in left column provides us with the continuity equation

$$\partial_t \rho + \text{div } j = 0, \tag{1.14}$$

which simply expresses the conservation of charge. In modern form, the above equations are recast as equations on tensor fields defined on Minkowski's spacetime. This is part of an interesting analysis *revealing* the Maxwell equations as *Lorentz invariant*. In fact, after applying the corresponding Lorentz transformations to (1.13) relating two inertial frames, say (t, x) and (t', x'), we discover that, if these equations are to hold in every inertial frame, then certain transformation rules must be inherited by the physical fields E, B, ρ and j.[11] In particular, these transformation rules suggest that ρ and j can be put together to form the vector

---

[10]In writing the Maxwell equations, we are adopting suitable conventions on the definitions of the fields and systems of units so as to avoid introducing universal physical constants.

[11]For a review on this topic, we refer the reader to classic text books, such as Jackson (1999) and references therein. Also, for an interesting and relevant discussion on this topic, see Redžić (2016).

field on space-time defined by $\mathcal{J} \doteq (\rho, j)$, which reproduces the charge and current densities on any inertial frame transforming via Lorentz transformations. In fact, that this should be the case is strongly suggested by the continuity equation (1.14),[12] which now reads as the space-time equation

$$\mathrm{div}_\eta \mathcal{J} = \nabla_\mu \mathcal{J}^\mu = 0. \tag{1.15}$$

On the other hand, the suggested transformation rules for E and B are not understood in so simple terms, but they can be elegantly shown to the consequence of these fields being special decompositions of an *electromagnetic* 2-form. Thus, let us introduce the following definition.

Definition 1.2.1. *Let* $\mathbb{M} = (\mathbb{R} \times \mathbb{R}^3, \eta)$ *be the Minkowski space-time. Consider an inertial coordinate system* (x^0, x^i) *and define the Faraday electromagnetic 2-form* $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ *by*

$$F_{\alpha\beta} \doteq \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

where E and B stand for the electric and magnetic fields associated to the Maxwell equations on $\mathbb{M}$.

According to the above definition, for a given space-time family of observers with flow lines γ_s, the space-time tensor field F is resolved as

$$E_\alpha \doteq F_{\alpha\beta} \gamma'^\beta_s,$$
$$F_{ij} \doteq F(e_i, e_j),$$

where E is the electric 1-form as measured by such observers, F_{ij} the magnetic part of the electromagnetic 2-form and $\{\gamma'_s, e_i\}_{i=1}^3$ denotes a frame along γ_s. In fact, with this terminology, we can show that following holds.

Proposition 1.2.1. *The Maxwell equations (1.13) written on a fixed inertial system* (x^0, x^i) *are equivalent to the exterior system*

$$\delta_\eta F = \mathcal{J}^\flat, \quad dF = 0, \tag{1.16}$$

where $\delta_\eta F$ *denotes the 1-form defined by* $\delta_\eta F_\mu \doteq -\eta^{\alpha\beta} \partial_\alpha F_{\beta\mu}$.

---

[12]For a discussion related to this topic, we refer the reader, for instance, to Appendix 2 in Møller (1952).

Proof. First, in our inertial coordinate system we can compute

$$-\delta_\eta F_\beta = -\partial_0 F_{0\beta} + \partial_i F_{i\beta},$$

which splits as

$$-\delta_\eta F(\partial_0) = \partial_i F_{i0} = \partial_i E_i,$$
$$-\delta_\eta F(\partial_j) = -\partial_0 F_{0j} + \partial_i F_{ij} = \partial_0 E_j + \epsilon_{ijk}\partial_i B_k = \partial_0 E_j - \epsilon_{jik}\partial_i B_k.$$

where ϵ_{ijk} denote the completely antisymmetric Levi–Civita symbols, allowing us to write $F_{ij} = \epsilon_{ijk} B_k$ in our inertial system. These relations imply

$$-\delta_\eta F(\partial_0) = \operatorname{div} E = \rho,$$
$$-\delta_\eta F(\partial_k) = \partial_t E_k - \operatorname{Curl} B_k = -j_k,$$

where ρ and j stand by the electric charge and current densities as measured by this particular inertial system. Therefore, we find that

$$\delta_\eta F = \mathcal{J}^\flat, \tag{1.17}$$

where $\mathcal{J}^\flat$ is the 1-form metrically isomorphic to $\mathcal{J}$, which, in an inertial inertial frame has components $\mathcal{J}^\flat = -\rho dx^0 + j_k dx^k$.

For the second half of the Maxwell equations, compute

$$dF = \frac{1}{2}\partial_\gamma F_{\alpha\beta} dx^\gamma \wedge dx^\alpha \wedge dx^\beta,$$

$$= \frac{1}{2}\left(\partial_0 F_{ij} dx^0 \wedge dx^i \wedge dx^j + 2\partial_i F_{0j} dx^i \wedge dx^0 \wedge dx^j + \partial_i F_{jl} dx^i \wedge dx^j \wedge dx^l\right),$$

$$= \frac{1}{2}\left(\epsilon_{kij}\partial_0 B_k + 2\partial_i E_j\right) dx^0 \wedge dx^i \wedge dx^j + \frac{1}{2}\epsilon_{jlk}\partial_i B_k dx^i \wedge dx^j \wedge dx^l$$

$$= \frac{1}{2}\left(2\partial_0 B_{\hat{k}} + 2\left(\partial_i E_j - \partial_j E_i\right)\right) dx^0 \wedge dx^{(i} \wedge dx^{j)} + \frac{1}{2}\epsilon_{jlk}\partial_i B_k dx^i \wedge dx^j \wedge dx^l,$$

where the convention $a_{ij} dx^{(i} \wedge dx^{j)}$ implies the summation is to be done only for $i < j$ and the index $\hat{k}$ stands for the only space index different to both i and j. Also, from the antisymmetry properties, it follows that $\epsilon_{jlk} dx^i \wedge dx^j \wedge dx^l = 2\epsilon_{jlk} dx^i \wedge dx^{(j} \wedge dx^{l)}$ and $\epsilon_{jlk}\partial_i B_k dx^i \wedge dx^{(j} \wedge dx^{l)}$ is non-zero only for $k = i$. Thus,

$$dF = \left(\partial_t B_{\hat{k}} + \operatorname{Curl} E_{\hat{k}}\right) dx^0 \wedge dx^{(i} \wedge dx^{j)} + \sum_{i=1}^{3}\epsilon_{ijl}\partial_i B_i dx^i \wedge dx^{(j} \wedge dx^{l)},$$

$$= \left(\partial_t B_{\hat{k}} + \operatorname{Curl} E_{\hat{k}}\right) dx^0 \wedge dx^{(i} \wedge dx^{j)} + \operatorname{div} B\, dx^1 \wedge dx^2 \wedge dx^3.$$

Therefore, we get that[13]

$$*_\eta dF = \epsilon_k \left(\partial_t B_k + \text{Curl } E_k \right) dx^k - \text{div} B \, dt,$$

where $\epsilon_k = \pm 1$. Therefore, the second half of the Maxwell equations hold iff $*dF = 0 \Leftrightarrow dF = 0$, which establishes the final claim. □

From the above discussion, the conclusion is that on the 4-dimensional flat space-time of special relativity, the Maxwell equations (1.13) can be rewritten as tensor equations for the electromagnetic 2-form F, given by (1.16). Furthermore, the dynamical equation $\delta_\eta F = \mathcal{J}^\beta$ contains the charge conservation statement (1.15), since $\delta_\eta \mathcal{J}^\flat = \delta_\eta^2 F = 0$, which is a restatement of the same fact.

In order to complete our description of electromagnetic phenomena in the relativistic context, we introduce the electromagnetic energy-momentum tensor field:

$$T_{\alpha\beta}^{EM} = F_\alpha{}^\lambda F_{\beta\lambda} - \frac{1}{4} \eta_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}. \tag{1.18}$$

In particular, the energy density as observed by the inertial system (x^0, x^i) of the electromagnetic field is computed as

$$\mathcal{E} \doteq T^{EM}(\partial_0, \partial_0) = F_0{}^i F_{0i} + \frac{1}{4} F^{\lambda\mu} F_{\lambda\mu} = |E|^2 + \frac{1}{4} \left(-2F_0{}^i F_{0i} + F^{ij} F_{ij} \right),$$

$$= \frac{1}{2} \left(|E|^2 + \frac{1}{2} |\tilde{F}|^2 \right),$$

where $\tilde{F}$ denotes the magnetic part of the electromagnetic 2-form, where $|\tilde{F}|^2 = 2|B|^2$. Similarly, the momentum density J is given by

$$J_i \doteq -T^{EM}(\partial_0, \partial_i) = -F_0{}^j F_{ij} = F_{ij} E^j,$$

$$= \epsilon_{ijk} E^j B^k = (E \times B)_i.$$

That is, we arrive at the usual expression for the *pointing vector* $S = E \times B$ as the electromagnetic momentum density.

Finally, let us notice how the Maxwell-equations (1.16) relate to the energy-momentum conservation equations associated to the above energy-momentum tensor field.

---

[13] The operator $*_\eta$ denotes the Hodge star operator, associated to the volume form $dV_\eta = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$.

Proposition 1.2.2. *Consider the 2-form F on Mikowski's space-time satisfying Maxwell's equations, then*

$$\text{div}_\eta T^{EM} = \mathcal{J} \lrcorner F, \tag{1.19}$$

where $\mathcal{J} \lrcorner F \doteq F(\mathcal{J}, \cdot)$ is known as the Lorentz force.

Proof. Direct computation shows that

$$
\begin{aligned}
\eta^{\sigma\alpha} \nabla_\sigma T^{EM}_{\alpha\beta} &= -\mathcal{J}^\lambda F_{\beta\lambda} + F^{\sigma\lambda} \nabla_\sigma F_{\beta\lambda} - \frac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu}, \\
&= -\mathcal{J}^\lambda F_{\beta\lambda} + \frac{1}{2} F^{\sigma\lambda} \left(\nabla_\sigma F_{\beta\lambda} - \nabla_\lambda F_{\beta\sigma} \right) - \frac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu}, \\
&= -\mathcal{J}^\lambda F_{\beta\lambda} + \frac{1}{2} F^{\sigma\lambda} \left(-\nabla_\beta F_{\lambda\sigma} - \nabla_\lambda F_{\sigma\beta} - \nabla_\lambda F_{\beta\sigma} \right) - \frac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu}, \\
&= -\mathcal{J}^\lambda F_{\beta\lambda} - \frac{1}{2} F^{\sigma\lambda} \nabla_\beta F_{\lambda\sigma} - \frac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu}, \\
&= -\mathcal{J}^\lambda F_{\beta\lambda} + \frac{1}{2} F^{\sigma\lambda} \nabla_\beta F_{\sigma\lambda} - \frac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu}, \\
&= F_{\lambda\beta} \mathcal{J}^\lambda,
\end{aligned}
$$

where in the first line we used $\delta_\eta F = \mathcal{J}^\flat$, in the second one the antisymmetry of F, in the third one the fact that $dF = 0$ is equivalent to the local expression written in arbitrary coordinates $\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0$, and finally we appealed again to the antisymmetry of F. $\qquad\square$

The above proposition shows that the changes in the energy and momentum of an electromagnetic field are due to the work done on a system of charges and currents $\mathcal{J}$. In the following section, we will come back to the description of electromagnetic field already within the context of general relativity and push this description a little bit further. In particular, let us only comment that the current density $\mathcal{J}$ is itself produced by some sort of charged matter. Such matter will be described by its own energy momentum distributions, represented by some energy-momentum tensor field T^{matter} and the full energy-momentum tensor of the complete system will consist of the sum $T^{\text{total}} = T^{EM} + T^{\text{matter}}$. This last tensor must obey (1.7), so that the total energy-momentum contributions are balanced.

1.3 General Relativity - The Einstein equations

Similarly to the starting point of the previous section, the starting point of GR was to resolve the existing tensions between the principles of special relativity with

Newtonian theory of gravitation. In contrast to the case of electromagnetism previously described, this turned out to be radically more subtle, and again produced another paradigm shift within physics. There is a long and rich history describing the state of affairs concerning the status of Newtonian gravity at the time when Einstein came along. Besides subtle discrepancies with Mercury's perihelia, this theory had been extremely successful in describing solar system physics, and we refer the interested reader to references such as Poisson and Will (2014) and Weinberg (1972) for discussions concerning this history. For us, it is enough motivation to realize that the Newtonian theory of gravity is not compatible with the kind of Lorentz invariance described in the previous section.

As the guiding principle of GR, Einstein put forward the *principle of equivalence*. There are actually at least three versions of such principle. The weakest one, going back to Galileo and known as the *weak equivalence principle*, is a recognition of the experimentally verified fact that the inertial mass (the one responsible for its resistance to change its inertial state) and the gravitational mass (the one appearing in the Newtonian universal law of gravitation, and therefore responsible for its gravitational interaction) are the same for any body.[14] This principle has as a consequence the well-known universality of free-fall, which states that *freely falling test bodies*[15] fall at the same rate in an homogeneous external gravitational field. In fact, under such conditions, one can go from an inertial system which sees a system of (possibly interacting) particles falling in a uniform gravitational field, to a non-inertial *freely-falling* coordinate system, which falls along with such particles. In both frames observers will agree on the laws of mechanics, although they will disagree on the existence of a gravitational field. That is, in a uniform (and static) gravitational field, the equivalence of inertial and gravitational masses allows us to cancel the effects of gravity by moving to an accelerated frame.

In reality, no truly homogeneous gravitational field exists and, therefore, in the above discussion, some *locality* hypothesis has to be added. That is, we must consider that the above cancellation of gravity by acceleration is valid (to a sufficiently high degree of approximation) only *locally*, within a space-time region where the inhomogeneities and time-variation of the gravitational field can be neglected, which leads us to actual equivalence principle used in GR. This states that given a space-time point, there is a sufficiently small neighbourhood of it where we can cancel out the effects of gravitation by moving to a *locally inertial coordi-*

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[14]See Poisson and Will (2014) for a detailed account of the precision to which this has been verified.

[15]These are bodies free of any other interaction than gravity and whose own gravitational field can be neglected in such an experiment.

nate system, where the laws of nature are described by those of special relativity. In this context there is still some discussion on whether such principle should refer to laws of nature for test particles (with negligible gravitational self-interaction) or whether it applies to all phenomena. The stronger version is known as the *strong equivalence principle*. The subtle distinction between all these versions of the equivalence principle relies on the degree to which each of them has been experimentally verified, and for these discussion, we refer the interested reader to Weinberg (1972, Chapter 3) and Poisson and Will (2014, Chapters 1 and 13).

Accepting the above principle of equivalence leads us to the conclusion that, *locally*, physics is sufficiently well approximated by special relativity and therefore space-time is nearly Minkowskian. The picture that is then adopted is that space-time is actually modelled by a Lorentzian manifold $(V, \bar{g})$, and then the existence normal coordinates provides us with the *locally inertial* coordinate systems, where, up to second order in a neighbourhood of an arbitrary point, physics looks Minkowkian. Then, higher-order effects due to gravitation are codified in the curvature of space-time. Since, after all, the choice of a particular coordinate system is for our benefit but does not affect the actual physical happenings, the guiding principle is now to appeal to special relativity locally, and then find coordinate-free laws which generalise for any frame of reference, which is sometimes referred to as *the principle of general covariance*. For instance, freely-falling particles at any particular point will obey (1.3) with $f = 0$, and then their generalisation is taken to be the geodesic equation for the space-time metric $\bar{g}$. Although powerful, this principle does not always lead to a unique possible generalisation, as is illustrated in Wald (1984, Chapter 4, Section 3) and in such cases further considerations must be taken into account.

From the above discussion, we see that in our new picture space-time is modelled as a (a-priori arbitrary) Lorentzian manifold $(V, \bar{g})$ and that gravitational effects are encoded in the choice of Lorentzian metric $\bar{g}$. Therefore, Newton's universal law of gravitation has to be upgraded to some equation on $\bar{g}$. The equivalence principle provides a strong guide towards the correct equations. In particular, an appeal to such a principle put together with a comparison in the low-velocity weak-field limit with Newtonian gravity (which we know to be an extremely good approximation in this limit) and an appeal to certain conservation principles guide

us towards the Einstein equations:[16]

$$\text{Ric}_{\bar{g}} - \frac{1}{2}R_{\bar{g}}\bar{g} + \Lambda\bar{g} = T(\bar{g}, \bar{\psi}), \tag{1.20}$$

where in the left-hand side $\text{Ric}_{\bar{g}}$ and $R_{\bar{g}}$ denote the Ricci tensor and scalar curvature associated to $\bar{g}$, while Λ denotes a constant referred to as *the cosmological constant*. On the right-hand side T denotes the energy-momentum tensor field associated to the matter fields which are sourcing the gravitational field, which (as seen in previous sections) will typically depend on the space-time metric $\bar{g}$ and some collection of *physical fields*, here collectively denoted by $\bar{\psi}$. Let us be clear concerning our notations and explicitly write down our curvature conventions:

$$\begin{aligned}
R_{\bar{g}}(X,Y)Z &\doteq \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_{[X,Y]}Z, \quad \forall\, X,Y,Z \in \Gamma(TV) \\
R^{\alpha}_{\mu\nu\beta}(\bar{g}) &\doteq dx^{\alpha}(R_{\bar{g}}(\partial_{\beta},\partial_{\nu})\partial_{\mu}), \\
\text{Ric}_{\mu\nu}(\bar{g}) &\doteq R^{\alpha}_{\mu\nu\alpha}(\bar{g}), \\
R_{\bar{g}} &\doteq \bar{g}^{\mu\nu}\text{Ric}_{\mu\nu}(\bar{g}),
\end{aligned} \tag{1.21}$$

where $\bar{\nabla}$ denotes the Riemannian connection associated with $\bar{g}$ and $\{x^{\alpha}\}_{\alpha=0}^{4}$ is an arbitrary coordinate system on V. Let us also point out that the left-hand side of (1.20) contains the *Einstein tensor*

$$G_{\bar{g}} \doteq \text{Ric}_{\bar{g}} - \frac{1}{2}R_{\bar{g}}\bar{g} \tag{1.22}$$

which obeys the local conservation law

$$\text{div}_{\bar{g}} G_{\bar{g}} = 0. \tag{1.23}$$

This directly implies that the right-hand side of (1.20) obeys the same kind of

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[16]For this kind of derivation of the Einstein equations see Weinberg (1972, Chapter 7) or Wald (1984, Chapter 4) for somewhat different approach making use of the geodesic deviation equations. Along these lines, we would like to further point the interested reader to the insightful notes of Geroch (2013). Furthermore, let us highlight that the Einstein equations can be obtained as the Euler-Lagrange equations of a Lagrangian involving $R_{\bar{g}}$ (see Choquet-Bruhat (2009, Chapter 3, Section 7) or Wald (1984, Appendix E)). Clearly such a procedure, although elegant, requires impositions on boundary and asymptotic conditions.

conservation law, given by[17]

$$\operatorname{div}_{\bar{g}} T(\bar{g}, \bar{\psi}) = 0. \tag{1.24}$$

The above equations are necessary conditions for (1.20) and therefore have to be coupled to the system. In particular, they will imply conservation laws for the matter fields, which complement the Einstein equations. Notice that the system (1.20)-(1.24) has to be solved simultaneously, and thus we have a strong (non-linear) coupling between matter fields sourcing the gravitational field (described by $\bar{g}$) and the space-time geometry dictating how matter should move. That is, as was famously put by John A. Wheeler, "*space-time tells matter how to move and matter tells space-time how to curve*".

Let us highlight that the above discussion, both on general and special relativity has only been limited to 4-dimensional space-times because this is actually the (main) object of interest in physics, but the mathematical tools and model work fine for any number of space-dimensions with almost no modifications. Since higher-dimensional space-times are objects of interest in contemporary theoretical physics, let us from now on work on space-times $(V^{n+1}, \bar{g})$, with $n \geqslant 3$ being the number of space-dimensions. We can condense the above presentation as follows.

Definition 1.3.1. *An $(n + 1)$-dimensional space-time is defined to be an $(n + 1)$-dimensional time-oriented Lorentzian manifold $(V^{n+1}, \bar{g})$ satisfying the Einstein equations (1.20).*

Typically in physics there are underlying hypotheses concerning what constitutes a *physically reasonable* solution, and this reduces some of the above freedom. For instance, along the lines of Section 1.1, reasonable causality conditions maybe imposed a priori on space-time demanding V to be globally hyperbolic and therefore $V^{n+1} \cong \mathbb{R} \times M^n$. This will be the situation that we will have in mind in the future.[18] Furthermore, as we have commented when describing the electromagnetic interaction in the context of special relativity, in case we have further

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[17]We have already claimed that, according to the equivalence principle, free-falling test particles follow geodesics of the space-time metric $\bar{g}$. Noticing that such a test particle arises as an idealisation of some matter distribution whose motion is already dictated by (1.24), we should be able to prove that in some idealised limit these last equations predict the geodesic equation for the test particle. This intuitive statement is not actually trivial, and two nice versions of it have been established in Ehlers and Geroch (2004) and Geroch and Weatherall (2018).

[18]We should caution the reader that there are interesting solutions of the Einstein equations which are not globally hyperbolic, such as the Anti de Sitter space-time, which plays a distinguished role in many discussions in contemporary physics.

fundamental fields (such as the electromagnetic one) coupled with gravity, such fields will carry over their own field equations (for instance, Maxwell's equations (1.16)) which must be further coupled to (1.20)-(1.24). Below, we shall exemplify this for a few cases of interest.

1.3.1   Field Sources

Let us now present a few examples of energy-momentum tensor fields which model interesting situations and for which we shall construct initial data sets in upcoming chapters.

Scalar fields

Scalar fields are used both within particle physics (for instance the Higgs field) and cosmology (for instance the *inflaton* field of inflationary cosmology). In our case of interest, let us consider a real-valued scalar field $\bar{\phi}$ on a space-time $(V^{n+1}, \bar{g})$. Such a field is described by an energy-momentum tensor field of the form

$$T(\bar{g}, \bar{\phi}) \doteq d\bar{\phi} \otimes d\bar{\phi} - \frac{1}{2}\bar{g}\langle d\bar{\phi}, d\bar{\phi}\rangle_{\bar{g}} - U(\bar{\phi})\bar{g}, \qquad (1.25)$$

where $U : I \mapsto \mathbb{R}$ is a real valued function referred to as the potential of the field $\bar{\phi}$. An elementary computation shows that

$$\operatorname{div}_{\bar{g}} T(\bar{g}, \bar{\phi}) = \left(\Box_{\bar{g}}\bar{\phi} - U'(\bar{\phi})\right) d\bar{\phi},$$

where $\Box_{\bar{g}} \doteq \bar{g}^{\alpha\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}$ denotes the wave operator in the metric $\bar{g}$. Therefore, the full system of equations for such a scalar field coupled with gravity is given by

$$\begin{aligned}\operatorname{Ric}_{\bar{g}} - \frac{1}{2}R_{\bar{g}}\bar{g} + \Lambda\bar{g} &= d\bar{\phi} \otimes d\bar{\phi} - \frac{1}{2}\bar{g}\langle d\bar{\phi}, d\bar{\phi}\rangle_{\bar{g}} - U(\bar{\phi})\bar{g}, \\ \Box_{\bar{g}}\bar{\phi} - U'(\bar{\phi}) &= 0.\end{aligned} \qquad (1.26)$$

Fluid sources

We have already introduced these kind of sources in the case of special relativity. Here, we shall focus on the case of perfect fluids, described by their energy density, pressure density and velocity field, given by $\bar{\psi} = (\bar{\mu}, \bar{p}, \bar{u})$ and whose energy momentum tensor field on the space-time $(V^{n+1}, \bar{g})$ is given by

$$T = (\bar{\mu} + \bar{p})\bar{u}^{\flat} \otimes \bar{u}^{\flat} + \bar{p}\bar{g}. \qquad (1.27)$$

Along the same lines discussed in Section 1.2, the equations of motion for the fluid are given by the conservation law (1.24), which explicitly read as

$$\bar{\nabla}_{\bar{u}}(\bar{\mu} + \bar{p})\bar{u}^{\flat} + (\bar{\mu} + \bar{p})(\text{div}_{\bar{g}}\bar{u}\bar{u}^{\flat} + \bar{\nabla}_{\bar{u}}\bar{u}^{\flat}) + d\bar{p} = 0.$$

We can simplify the above equation by splitting it into its parallel and orthogonal components to $\bar{u}$. The parallel one gives

$$\bar{u}(\bar{\mu}) + (\bar{\mu} + \bar{p})\text{div}_{\bar{g}}\bar{u} = 0.$$

Feeding this back into the original equation, it reduces it to

$$\begin{aligned}
(\mu + p)\bar{\nabla}_{\bar{u}}u + \bar{u}(p)u + \bar{\nabla}\bar{p} &= 0, \\
\bar{u}(\bar{\mu}) + (\bar{\mu} + \bar{p})\text{div}_{\bar{g}}\bar{u} &= 0,
\end{aligned} \tag{1.28}$$

where we can check that the left-hand side of the first equation is orthogonal to $\bar{u}$.[19] Now these equations must be coupled with (1.20) and must typically be supplemented by a state equation. We refer the reader to Choquet-Bruhat (2009, Chapter IX) for such details.

Similarly to the case analysed in Section 1.2, the case of a dust fluid ($\bar{p} = 0$) is particularly simple. Along the same lines described there, in such a case we find that the equation for the fluid reduce to

$$\text{div}_{\bar{g}}(\bar{\mu}\bar{u}) = 0, \quad \bar{\nabla}_{\bar{u}}\bar{u} = 0,$$

which we must couple with the Einstein equations (1.20) to obtain

$$\begin{aligned}
\text{Ric}_{\bar{g}} - \frac{1}{2}R_{\bar{g}}\bar{g} + \Lambda\bar{g} &= \mu\bar{u}^{\flat} \otimes \bar{u}^{\flat}, \\
\text{div}_{\bar{g}}(\bar{\mu}\bar{u}) &= 0, \\
\bar{\nabla}_{\bar{u}}\bar{u} &= 0.
\end{aligned} \tag{1.29}$$

Electromagnetic fields

We have already introduced the basic elements concerning the description of the electromagnetic field in Section 2.1. In particular, equations (1.16) are already written in a coordinate independent fashion and are regarded as the correct equations describing the electromagnetic interaction, in the absence of gravitation, via

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[19]Recall that we use the convention $\bar{g}(\bar{u}, \bar{u}) = -1$.

the 2-form F. Therefore, via the equivalence principle, on a general 4-dimensional physical space-time V these equations still represent the appropriate electromagnetic field equations, obviously coupled to the Einstein equations. In this case, any time-like curve γ resolves the the electromagnetic 2-form into its electric and magnetic parts via

$$E = F(\cdot, \gamma') \ , \quad F_{ij} = F(e_i, e_j). \tag{1.30}$$

Finally, we can extend these notions to general dimensions by considering that, on a space-time $(V, \bar{g})$ of arbitrary (space) dimensions, the electromagnetic field is represented by a space-time 2-form, say F, which is decomposed into its electric part and magnetic parts by space-time observers with flow lines γ_s according to (1.30), and which satisfies the field equations

$$\delta_{\bar{g}} F = \mathcal{J}^\flat \ , \quad dF = 0 \tag{1.31}$$

coupled to the Einstein equations through the generic energy-momentum tensor of an electromagnetic field, given by

$$T_{\alpha\beta}^{EM} = F_\alpha{}^\lambda F_{\beta\lambda} - \frac{1}{4} \bar{g}_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}, \tag{1.32}$$

where indices are, as usual, raised and lowered with the space-time metric $\bar{g}$. Let us highlight that the electromagnetic current $\mathcal{J}$ must be generated by charged particles, which are themselves described by some energy-momentum tensor field. As an illustrative example, let us consider the simplest case of *charged dust*. This model is defined by a dust fluid described by an energy-momentum tensor of the form

$$T^{fluid} = \bar{\mu} \bar{u}^\flat \otimes \bar{u}^\flat, \tag{1.33}$$

where $\bar{\mu}$ represents the proper mass density of the fluid and $\bar{u}$ stands for the time-like vector field whose integral curves are the flow lines of the fluid. We assume that this fluid contains charged particles, and the *proper charge density is* given by a function $\bar{q}$ and therefore, the associated electromagnetic current is given by

$$\mathcal{J} = \bar{q}\bar{u}. \tag{1.34}$$

This gives us all the ingredients to write down a closed system of equations, given by

$$\begin{aligned} G_{\bar{g}} + \Lambda \bar{g} &= T^{fluid} + T^{EM}, \\ \delta_{\bar{g}} F &= \bar{q}\bar{u}^\flat, \\ dF &= 0. \end{aligned} \tag{1.35}$$

Furthermore, the conservation laws associated to the dynamics of the fluid are given by

$$\operatorname{div}_{\bar{g}} T^{fluid} + \operatorname{div}_{\bar{g}} T^{EM} = 0.$$

In particular, we known from Section 1.2 that $\operatorname{div}_{\bar{g}} T^{EM}_{\beta} = F_{\beta}{}^{\lambda} \mathcal{J}_{\lambda}$, implying that

$$0 = \bar{\nabla}^{\alpha}(\bar{\mu} \bar{u}_{\alpha} \bar{u}_{\beta}) + F_{\beta}{}^{\lambda} \mathcal{J}_{\lambda} = \bar{\nabla}^{\alpha}(\bar{\mu} \bar{u}_{\alpha}) \bar{u}_{\beta} + \bar{\mu} \bar{u}_{\alpha} \bar{\nabla}^{\alpha} \bar{u}_{\beta} + F_{\beta}{}^{\lambda} \mathcal{J}_{\lambda}.$$

The parallel component to $\bar{u}$ gives us that

$$0 = -\bar{\nabla}^{\alpha}(\bar{\mu} \bar{u}_{\alpha}) + \bar{\mu} \bar{u}^{\alpha} \bar{g}(\bar{\nabla}_{\alpha} \bar{u}, \bar{u}) + F_{\beta}{}^{\lambda} \mathcal{J}_{\lambda} u^{\beta},$$

where the condition $\bar{g}(\bar{u}, \bar{u}) = -1$ implies $\bar{g}(\bar{\nabla}_{\alpha} \bar{u}, \bar{u}) = 0$. Also, $F_{\beta}{}^{\lambda} \mathcal{J}_{\lambda} \bar{u}^{\beta} = \bar{q} F_{\beta}{}^{\lambda} \bar{u}_{\lambda} u^{\beta} = \bar{q} F(\bar{u}, \bar{u}) = 0$. Thus,

$$\operatorname{div}_{\bar{g}}(\bar{\mu} \bar{u}) = 0. \tag{1.36}$$

Therefore, the system of equations for the fluid is

$$\begin{aligned} \operatorname{div}_{\bar{g}}(\bar{\mu} \bar{u}) &= 0, \\ \bar{\mu} \bar{\nabla}_{\bar{u}} \bar{u} - \bar{q} \bar{u} \lrcorner F &= 0. \end{aligned} \tag{1.37}$$

where the first equation represents the local conservation of mass and last equation one stands for the Lorentz force-law in this generalised context. We finally see that the the full system of space-time equations is given by

$$\begin{aligned} G_{\bar{g}} + \Lambda \bar{g} &= T^{fluid}(\bar{g}, \bar{\mu}, \bar{u}) + T^{EM}(\bar{g}, F), \\ dF &= 0, \\ \operatorname{div}_{\bar{g}} F \doteq -\delta_{\bar{g}} F &= -\bar{q} \bar{u}^{b}, \\ \operatorname{div}_{\bar{g}}(\bar{\mu} \bar{u}) &= 0, \\ \bar{\mu} \bar{\nabla}_{\bar{u}} \bar{u} - \bar{q} \bar{u} \lrcorner F &= 0. \end{aligned} \tag{1.38}$$

The case where $\bar{q} \equiv 0$ reduces to a dust fluid with no charge and, if furthermore $\bar{\mu} \equiv 0$, we fall into the so-called *Einstein–Maxwell* system, also referred to as *electro-vacuum*.

1.3.2 The Schwarzschild solution

In this subsection, we will review some useful properties and constructions related to the so-called Schwarzschild solution, which represents the appropriate geometry describing the exterior of an isolated spherically symmetric massive body, such as an idealised star. In this case, the exterior is taken to be vacuum, that is, $\mathrm{Ric}_{\bar{g}} = 0$, and the resulting solution has the form

$$\bar{g}_{Sc} = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_{\mathbb{S}^2}. \qquad (1.39)$$

In the above form, the solution is defined in an *exterior region*, given by $r > 2m$, and an *interior region*, given by $0 < r < 2m$, where the parameter m is called *the mass* of the associated spherically symmetric body generating our gravitational field. We will limit our discussion to the case $m \geqslant 0$.

Let us highlight that the restriction of the above solution to $r > 2m$ does not represent a substantial initial drawback, since the above solution was intended to model *the exterior region* of an idealised star. In particular, the so-called Schwarzschild radius $r_{Sc} \doteq 2m$, in appropriate units, produces a value which would be deep inside the interior of any star. A model taking into consideration the interior of the star must be a non-vacuum solution, which, in idealised situations, would have compactly supported sources.[20] Such an *interior solution* would have to be glued to (1.39) to provide a complete description of a model situation. Nevertheless, the above exterior solution is good enough to test, for instance, solar system gravitational phenomena. In fact, it provided the tools to produce the first predictions of general relativity, such the advances in the perihelion of Mercury and the deflection of light by the sun.[21]

On the other hand, the existence of sufficiently dense objects living inside its Schwarzschild radius is by now very well-known: such objects represent black holes. In order to understand this terminology, let us point out that, in the interior region $0 < r < 2m$, the dynamics of particles and light-rays is really peculiar. In particular, all future directed causal curves end within a finite proper time and no causal signal can escape this region (see O'Neill 1983, Proposition 30 in Chapter 13). This last property is what gives the name of black hole region to such interior

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[20]For discussion of such idealised models, we refer the interested reader to Wald (see 1984, Chapter 6) and Weinberg (see 1972, Chapter 10).

[21]See O'Neill (1983, Chapter 13) for a particularly nice geometric treatment of the problem and Poisson and Will (2014) for an quite exhaustive treatment of the physics involved.

solution. Black holes represent extremely interesting objects within physics, being a probe for the most extreme gravitational phenomena we are aware of, and, also, they have been the subject of extensive mathematical research. Being the Schwarzschild black hole, modelled by (1.39), the simplest example of such situation, how to appropriately join the interior and exterior regions to construct single *vacuum* solution becomes an interesting question. Since the Schwarzschild black hole solution serves as a building block in many problems within mathematical general relativity, we will make a brief review of this construction.

Let us start by denoting the exterior solution associated to (1.39) by $(N, \bar{g}_{Sc})$ and the interior black hole solution by $(B, \bar{g}_{Sc})$. Both these solutions can be described as warped products $P_i \times_r \mathbb{S}^2$, $i = 1, 2$, where P_i stands for the restriction of the (t, r) half-plane $\mathbb{R} \times \mathbb{R}_+$ to the domains $r > 2m$ and $r < 2m$ respectively. These planes are furnished with the metric $b = -h(r)dt^2 + h^{-1}(r)dr^2$, where $h(r) \doteq 1 - \frac{2m}{r}$. Now, the appropriate way to join these two solutions is through the so-called Kruskal space-time. We will follow closely O'Neill (ibid.) in this topic and refer the interested reader to Wald (1984) for several intuitions behind these constructions.

Let us start by defining the function $f : \mathbb{R}_+ \mapsto (-\frac{2m}{e}, \infty)$ by

$$f(r) = (r - 2m)e^{\frac{r}{2m}-1}. \tag{1.40}$$

Since $f' > 0$, f defines an diffeomorphism. Let Q be the region in the (u, v)-plane given by $uv > -\frac{2m}{e}$, then $r(u, v) \doteq f^{-1}(uv)$ defines a smooth positive function of Q, implicitly defined by $f(r) = uv$. Let us notice that the level sets of the function r are given by the hyperbolas $uv = $ cte, except for $r = 2m$, which corresponds to the coordinate axes. Furthermore, the function r approaches the value $r = 0$ as we move towards the boundary hyperbola $uv = -\frac{2m}{e}$, which is not part of Q. In this setting, by deleting the coordinate axes we will divide Q into four open quadrants $Q_1, \cdots, Q_4$ as depicted in Figure 1.4. We now define the *Kruskal plane of mass* $m > 0$ as the region Q endowed with the metric tensor[22]

$$g_K = \frac{8m^2}{r}e^{1-\frac{2m}{r}} (du \otimes dv + dv \otimes du) \tag{1.41}$$

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[22]Notice that the case $m = 0$ is trivial, since in this case we reduce to Minkowski space-time.
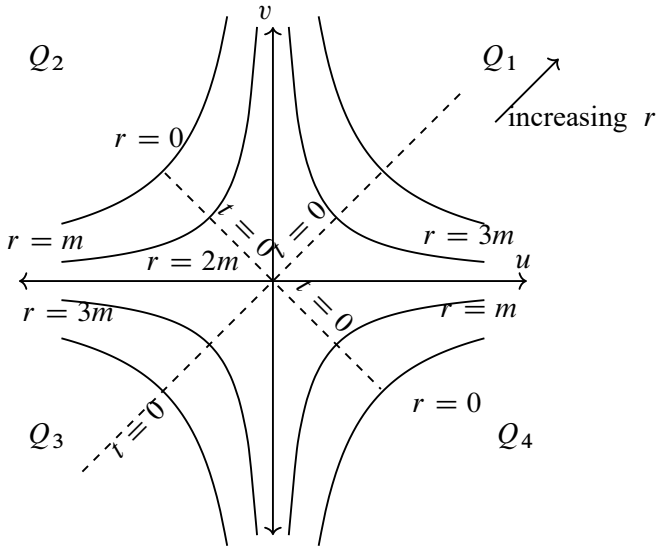
Figure 1.4: Kruskal's Plane

Let us point out a couple of direct consequences of the above definitions. First, the null geodesics of the Kruskal plane are given parametrizations of the coordinate lines $u = cte$ and $v = cte$. Furthermore, the mapping $(u, v) \overset{\phi}{\mapsto} (-u, -v)$ is an isometry of Q, since it preserves r, and, actually, restricts to an isometry between Q_1 and Q_3 as well as between Q_2 and Q_4. Finally, let us define the function $t \doteq 2m \ln |\frac{v}{u}|$ outside the coordinate axes. Let us notice that the level sets of this function are given by rays from the origin in Q (see Figure 1.4). It is now a straightforward procedure to prove that the mapping $\psi : Q_1 \cup Q_2 \mapsto P_1 \cup P_2$, given by $(u, v) \mapsto (t(u, v), r(u, v))$ is an isometry which maps Q_i onto P_i, $i = 1, 2$, and restricts to an isometry there (see Proposition 24 Chapter 13 O'Neill (1983)). Therefore, having found that $Q_i \cong P_i$, $i = 1, 2$, where that Q_1 and Q_2 fit nicely together in Q, we have provided isometric embeddings of P_1 and P_2 into a single manifold.

Let us now define the Kruskal space-time as the warped product $\mathcal{K} \doteq Q \times_r \mathbb{S}^2$, each factor with its natural metric. Then, in the above figures, we can visualise $\mathcal{K}$ by replacing each point by a 2-sphere of radius $r(u, v)$. In this context we denote the corresponding open quadrants by K_i, $i = 1, \cdots, 4$, and we can now extend the isometries ϕ and ψ to $\phi \times id$ and $\psi \times id$ in an obvious way and therefore get

isometries

$$K_3 \cong K_1 \cong N \quad \text{and} \quad K_4 \cong K_2 \cong B. \tag{1.42}$$

Therefore, we have found isometric embeddings of the interior and exterior solutions B and N into a single Ricci-flat (vacuum) space-time $\mathcal{K}$. In order to produce some more intuitions about the special behaviour of this solution in its black hole regions, let us notice that we can give a consistent time orientation to $\mathcal{K}$, since $\partial_v - \partial_u$ is a globally defined non-vanishing time-like vector field. We chose the orientation that makes ∂_t future pointing in the region K_1. This, in particular, implies that ∂_v and $-\partial_u$ are future pointing null vector fields (see Figure 1.5).
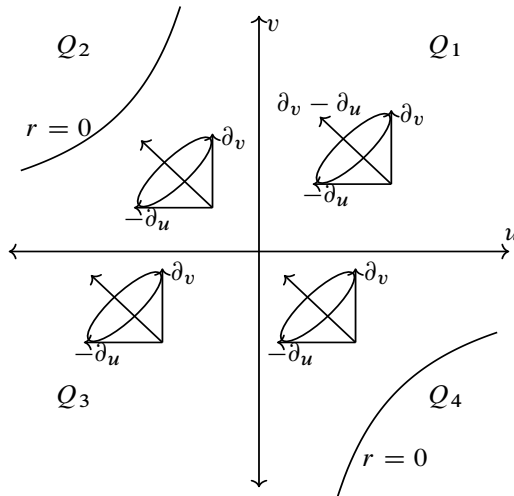
Figure 1.5: Kruskal's space-time orientation

In the above figure, it is clear that the future of any particle beyond this horizon inevitably ends at the central singularity, while only light-like particles can hover over the horizon without falling in. We can make use of the isometry ψ to map $Q_1 \cup Q_2 \mapsto P_1 \cup P_2$ and see how, in our chosen orientation, light-cones are actually tilting as we approach the horizon $r = 2m$ (see Figure 1.6).
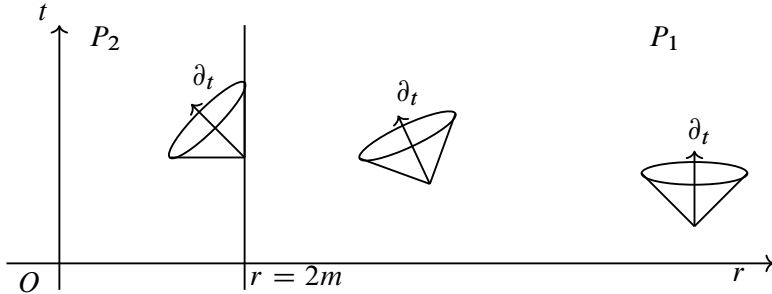
Figure 1.6: Tilting of future cones

In order to finish our discussion on the Schwarzschild solution, let us make one further observation. Above, we have embedded the usual Schwarzschild exterior solution into the Kruskal space-time, which not only contains the additional interior black hole solution, but also an additional copy of each of these parts in the quadrants K_3 and K_4 respectively. Let us fix our attention to the space-like $t = 0$ hypersurfaces highlighted in Figure 1.4. We can see that they belong to a single well-behaved hypersurface which contains a copy of the exterior Schwarzschild solution on each side. We will now rewrite the space-time metric adapted to this hypersurface. The aim behind this exercise is that we will obtain a *complete* Riemannian metric g_{Sc} on this $t = 0$ slice, which (together with some extrinsic information) provides us with *initial data* which describes the full Schwarzschild black hole. This turns out to be the most useful analytic picture and is part of the standard analytic tool kit of general relativity. Thus, let us start by considering the exterior solution at the $t = 0$ slice. The induced Riemannian metric on this slice is given by

$$g_{Sc} = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 g_{\mathbb{S}^2}. \tag{1.43}$$

Spherical symmetry implies that this metric is actually conformally-flat. Actually, we can compute such conformal factor explicitly. If we write $g_{Sc} = u^4(|x|)\delta = u^4(\rho)\left(d\rho^2 + \rho^2 g_{\mathbb{S}^2}\right)$, appealing to a coordinate change of the form $\rho = \rho(r)$, we straightforwardly find that $u^2\rho = r$ and

$$\left(u^2(\rho)\frac{d\rho}{dr}\right)^2 = \frac{1}{1 - \frac{2m}{r}}$$

are the necessary conditions. Imposing $\frac{d\rho}{dr} > 0$, we obtain an ordinary differential equation of the form

$$\frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{r\sqrt{1 - \frac{2m}{r}}}. \tag{1.44}$$

We can apply the change of variable given by $r = m(1 + \cosh\omega)$, with $\omega > 0$, which implies $\ln\rho = \omega + c$. That is $\rho = Ce^{\omega}$, for some constant $C > 0$. From this, we get

$$\frac{r}{\rho} = m\left(\frac{1}{\rho} + \frac{1}{2C} + \frac{C}{2\rho^2}\right) = m\left(\frac{\sqrt{\frac{C}{2}}}{\rho} + \frac{1}{\sqrt{2C}}\right)^2. \tag{1.45}$$

Finally, imposing that $\frac{r}{\rho} \to 1$ as $\rho \to \infty$, we find $m = 2C$, which implies that

$$u^2(\rho) = \frac{r}{\rho} = \left(1 + \frac{m}{2\rho}\right)^2. \tag{1.46}$$

Therefore, we see that

$$g_{Sc} = \left(1 + \frac{m}{2|x|}\right)^4 \delta \tag{1.47}$$

where the above change of variable corresponds to mapping the exterior region $r > 2m$ to $|x| > \frac{m}{2}$. Nevertheless, clearly, (1.47) is well-defined for all $x \neq 0$. In fact, an inversion of coordinates $z = \left(\frac{m}{2}\right)^2 \frac{x}{|x|^2}$, maps the punctured ball $0 < |x| < \frac{m}{2}$, to its exterior $\mathbb{R}^3 \backslash \overline{B}_{\frac{m}{2}}(0)$ while it preserves the sphere $\mathbb{S}^2_{\frac{m}{2}}(0)$. Furthermore, we find that $|z| = \left(\frac{m}{2}\right)^2 \frac{1}{|x|}$ and thus $x = \left(\frac{2}{m}\right)^2 |x|^2 z = \left(\frac{2}{m}\right)^2 \left(\frac{m}{2}\right)^4 \frac{z}{|z|^2} = \left(\frac{m}{2}\right)^2 \frac{z}{|z|^2}$. Therefore,

$$\partial_{z^i} = \left(\frac{m}{2|z|}\right)^2 \left(\delta_i^k - 2|z|^{-2}z^i z^k\right)\partial_{x^k},$$

implying

$$g_{Sc}(\partial_{z^i}, \partial_{z^j}) = \left(1 + \frac{m}{2|x|}\right)^4 \left(\frac{m}{2|z|}\right)^4 \delta_{ij} = \left(1 + \frac{2|z|}{m}\right)^4 \left(\frac{m}{2|z|}\right)^4 \delta_{ij},$$

$$= \left(1 + \frac{m}{2|z|}\right)^4 \delta_{ij},$$

which proves that the punctured ball is isometric the exterior solution. That is, the Riemannian manifold $\left(\mathbb{R}^3\setminus\{0\}, g_{Sc} \doteq \left(1 + \frac{m}{2|x|}\right)^4 \delta\right)$, is a complete manifold which contains the two copies of the $t = 0$ *initial data* for the exterior Schwarzschild solution of the quadrants K_1 and K_3 smoothly glued along their boundaries, and is thus isometric to the $t = 0$ slice of the Kruskal space-time. For completeness, let us highlight that, in these coordinates, the space-time Schwarzschild solution reads

$$\bar{g}_{Sc} = -\frac{\left(1 - \frac{m}{2|x|}\right)^2}{\left(1 + \frac{m}{2|x|}\right)^2}dt^2 + \left(1 + \frac{m}{2|x|}\right)^4 \delta \tag{1.48}$$

where the appeal to the time-coordinate t clearly excludes the coordinate axes separating the quadrants K_i in the Kruskal space-time.

Finally, let us point out that there are analogous higher-dimensional generalisations of all of the above constructions, which can be obtained along the same lines. Just for the record and future reference, let us point out the the n-dimensional complete $t = 0$ slice of an $(n + 1)$-dimensional Schwarzschild space-time is given by the Riemannian manifold $\left(\mathbb{R}^n\setminus\{0\}, g_{Sc} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta\right)$.

1.3.3   Some cosmological solutions

Let us now present another set of physically relevant solutions which can be worked out explicitly. These solutions concern *cosmological* situations, which is a setting in which we analyse the dynamics of the universe as a whole. In order to be able to do this several idealisations have to be made. Along these lines, if we are concerned only with analysing the overall dynamics of the universe, we can average its properties over large scales and produce a very course-grained description of it. In such a situation a point in space-time is meant to represent large regions in the universe such as a whole galaxy of even clusters of them. In particular, going to sufficiently large distances, there seems to be compelling experimental evidence in favour of the fact that the universe (in such scales) is highly symmetric. More explicitly, in cosmological scales the universe is approximately *homogeneous* and *isotropic*. As we will see below, the presence of these symmetries allows us to reduce our problem to a very compact system of ordinary differential equations which can be dealt with explicitly in some situations, and, more generally, can be used to describe a general picture concerning the overall evolution of the universe.

Let us start considering a 4-dimensional space-time $(V, \bar{g})$ and comment on how the above hypotheses translate into the mathematical model. First of all, the assumption of isotropy actually singles out a distinguished global time-like vector field ∂_t whose simultaneity spaces M_t globally split ($V \cong I \times M, \bar{g} = -dt^2 + \bar{g}_t$), where $t \in I \subset \mathbb{R}$ and whose orthogonal space-like hypersufaces $(M_t, \bar{g}_t)$ are isotropic in the usual sense. That is, they have no preferred direction. This last conditions implies that, for every $p \in M$, the sectional curvatures of all the planes in $T_p M_t$ must be equal. Thus, the sectional curvature K_t of $(M_t, \bar{g}_t)$ at p depends only on p, *i.e*, $K_t = K_t(p)$. Then, the contracted Bianchi identities imply that a Riemannian manifold which is isotropic at every point must have constant sectional curvature, that is $K_t = cte$ (see, for instance, Choquet-Bruhat (2009, Chapter V, Theorem 3.4)). To simplify our discussion, let us then assume that space is simply connected. In such a case, for each time t we have $(M_t, \bar{g}_t) \cong \mathbb{E}^3$ if $K = 0$, and if $K \not\equiv 0$ we can consider the conformal scaling $\bar{g} = |K|^{-1}\gamma_\epsilon$, which implies that γ_ϵ has constant sectional curvature equal to $\epsilon = \text{sign}(K)$, and therefore

$$(M_t, \gamma_1) \cong \mathbb{S}^3,$$
$$(M_t, \gamma_{-1}) \cong \mathbb{H}^3,$$

where $\mathbb{S}^3$ stands for the round unit 3-sphere and $\mathbb{H}^3$ for the standard hyperbolic 3-space of constant curvature -1. Therefore, to contemplate the three cases at once, the space-time metric for can be written as

$$\bar{g} = -dt^2 + a^2(t)\gamma_\epsilon, \tag{1.49}$$

where now $\gamma_0 = \delta$ the standard flat Euclidean metric, and in the cases $\epsilon = \pm 1$ we have $a^2(t) = |K_t|^{-1}$. The warping factor $a(t)$ is referred to as the *scale factor* and becomes the only geometric degree of freedom in the problem. To determine it and have our cosmological description complete, we have to assume something for the matter content of the universe. In this setting of homogeneous and isotropic cosmologies, it is typical to model the matter content as a perfect fluid with flow lines $\bar{u} = \partial_t$ and therefore homogeneous and isotropic, implying that the energy and pressure densities are functions only of time. Now, plugging all this into the Einstein-perfect-fluid equations gives a set of ordinary differential equations which

dictate the dynamics of the system (see Choquet-Bruhat (2009, Chapter V)):

$$-3\frac{\ddot{a}}{a} = \frac{1}{2}(\bar{\mu} + 3\bar{p}) - \Lambda,$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\epsilon}{a^2} = \frac{1}{2}(\bar{\mu} - \bar{p}) + \Lambda,$$

$$(\mu + p)\bar{\nabla}_{\bar{u}}u + \bar{u}(p)u + \bar{\nabla}\bar{p} = 0,$$

$$\bar{u}(\bar{\mu}) + (\bar{\mu} + \bar{p})\text{div}_{\bar{g}}\bar{u} = 0,$$

where a dot over a quantity denotes a derivative with respect to time. These equations can be further simplified by using the first one to eliminate the second order term in the second one, which gives

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{\epsilon}{a^2} = \bar{\mu} + \Lambda.$$

The fluid equations can also be simplified under our hypotheses. Actually, the first one is a tautology, since our construction implies that the flow-lines ∂_t are geodesics, so the equation actually read as $\bar{\nabla}\bar{p} = -\dot{p}\partial_t$, which is the definition of the gradient since $\bar{p} = \bar{p}(t)$. Also the second fluid equation can be simplified to give

$$\dot{\bar{\mu}} + 3(\bar{\mu} + \bar{p})\frac{\dot{a}}{a} = 0.$$

Therefore, the full system of Einstein equations gets reduced to the so called *Friedman–Lemaître equations*:

$$-3\frac{\ddot{a}}{a} = \frac{1}{2}(\mu + 3p) - \Lambda,$$

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{\epsilon}{a^2} = \bar{\mu} + \Lambda, \tag{1.50}$$

$$\dot{\bar{\mu}} + 3(\bar{\mu} + \bar{p})\frac{\dot{a}}{a} = 0.$$

The above system can be integrated explicitly for some simple state equations relating $\bar{\mu}$ and $\bar{p}$, such as for dust ($\bar{p} = 0$), radiation ($\bar{\mu} - 3\bar{p} = 0$) and also the Λ-vacuum cases which lead to deSitter and anti-deSitter solutions. Some of these cases model specific stages in the history of the universe. More importantly, under

reasonable assumptions of $\bar{\mu}$ and $\bar{p}$, equations (1.50) are good enough to give us an overall picture of the dynamics of the system. To a first approximation, this gives a good cosmological qualitative description. We refer the interested reader to Wald (1984, Chapter 5) for discussions of this kind.[23]

1.4 The initial value formulation

Let us now enter into the core of this chapter and present the main ideas concerning the initial value formulation of the Einstein equations. As was stated in the beginning of the chapter, this is a subtle topic which involves ongoing research in geometric analysis and PDE theory. Any self-contained presentation needs to appeal to a decent amount of hyperbolic PDE theory, in particular of non-linear wave equations. The interested reader can find such presentations in references such as Choquet-Bruhat (2009) and Ringström (2009) and the many references therein. Our presentation will be merely expository, appealing to the main ideas and skipping completely the hyperbolic PDE issues.

Let us start by putting forward a couple of strong motivations for the analysis to come. First, it is within the standard paradigm of physics that physical theories should be useful to make predictions concerning the future evolution of a system. This is done typically by evolving initial data sets, and works in models ranging from classical Newtonian mechanics to relativistic electrodynamics and even the Schrödinger equation of quantum mechanics. Clearly, this is quite useful for the physicist, who can then model a specific situation at a particular time via suitable initial conditions and find out how physics plays out by evolving such system, and permeates deeply into the issue of predictability of a physical theory. Furthermore, let us notice that the complicated and non-linear nature of the Einstein equations does not allow us to solve them explicitly unless appealing to idealised highly symmetrical situations, of the kind we have reviewed in previous sections. Moreover, one would like to have information concerning generic properties of solutions, their stability against perturbations, global properties of generic solutions and also to have some systematic way of producing more general solutions. Some experience in PDE theory can anticipate that some of these questions could be settled by providing a suitable PDE treatment of the Einstein equations. Let us use this as motivation for the following analysis.

---

[23]Let us also point out to the reader the discussion presented in the lecture notes of Blau (2020), which can be quite useful for an understanding of the physical consequences of these cosmological models.

Let us start by considering globally hyperbolic vacuum $(n + 1)$-dimensional space-times $(V^{n+1} = \mathbb{R} \times M^n, \bar{g})$ so that the Einstein equations get reduced to

$$\mathrm{Ric}_{\bar{g}} = 0. \tag{1.51}$$

The objective is to be able to give initial data on M and guarantee that we can evolve it into such a solution. But, as we will see below, there are some immediate subtleties in this procedure. First, notice that in this analysis we will have to make a clear space-time splitting and therefore, let us introduce a time parameter t along the $\mathbb{R}$ factor, and the global future pointing time-like vector-field ∂_t tangent to the time-curves $t \mapsto (t, x) \in V$. Then, let us denote the tangential component of ∂_t to M_t by X, which is a time-dependent vector field tangent to M known as the *shift vector* and the normal component to M_t will be denoted by a function $N > 0$ referred to as *the lapse function*. These objects allow us to build adapted local frames $\{e_\alpha\}_{\alpha=0}^n$ of the form

$$\begin{aligned} e_0 &= \partial_t - X \perp M_t, \\ e_i &= \partial_{x^i}, \end{aligned} \tag{1.52}$$

for any coordinate system $\{x^i\}_{i=1}^n$ on M, and their dual co-frames $\{\theta^\alpha\}_{\alpha=0}^n$ then read as

$$\begin{aligned} \theta^0 &= dt, \\ \theta^i &= dx^i + X^i dt, \end{aligned}$$

as can be readily checked. Using such frames, the space-time metric can be locally put in the form

$$\bar{g} = -N^2 dt \otimes dt + \bar{g}_t, \tag{1.53}$$

where the induced metric $\bar{g}_t$ on M_t has the local form $\bar{g}_t = \bar{g}_{ij} \theta^i \otimes \theta^j$.

$$V = M \times \mathbb{R}$$
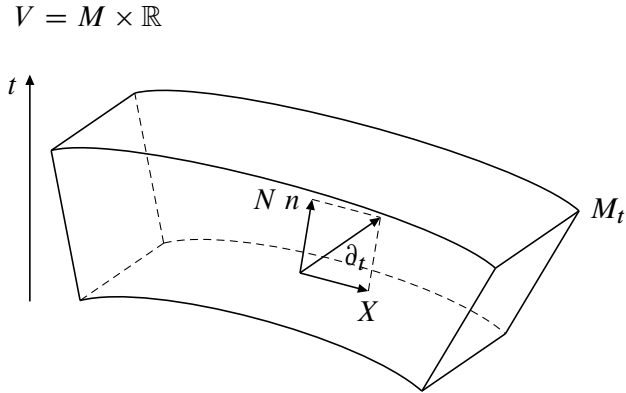
Figure 1.7: Lapse–Shift space-time splitting

Notice that the future pointing unit normal to each M_t can then be written as

$$n = \frac{1}{N} (\partial_t - X) \tag{1.54}$$

In the above space-time splitting, the choice of our family of time-like curves defined by the vector field ∂_t is uniquely determined by the choice of lapse and shift, since $\partial_t = Nn + X$. So, each choice of $N > 0$ and X satisfying $-N^2 + |X|^2_{\bar{g}_t} < 0$ determines a unique such family of space-time *observers* and vice-versa. So, as could be suspected from the beginning, our choice of space-time splitting according to a preferred ∂_t should work as a *gauge* choice, not playing a major role at the end of our analysis.

Now, notice that as PDE operator on the space-time metric $\bar{g}$, the Ricci tensor is a second order operator. In fact, in an arbitrary coordinate system reads as

$$\text{Ric}_{\mu\nu}(\bar{g}) = -\frac{1}{2} \bar{g}^{\alpha\beta} \partial_{\alpha\beta} \bar{g}_{\mu\nu} + f_{\mu\nu}(\bar{g}, \partial\bar{g}) + \frac{1}{2} \left(\bar{g}_{\mu\lambda} \partial_\nu F^\lambda + \bar{g}_{\nu\lambda} \partial_\mu F^\lambda \right), \tag{1.55}$$

where $f_{\mu\nu}(\bar{g}, \partial\bar{g})$ are smooth functions of their arguments, in particular quadratic on $\partial\bar{g}$, and the functions F^λ are given by

$$F^\lambda \doteq \bar{g}^{\alpha\beta} \bar{\Gamma}^\lambda_{\alpha\beta}.$$

As we will see, it is precisely the last two terms involving derivatives of F that pose some extra difficulties in this problem. For now, let us notice that, if we

are to have a well-posed evolution problem associated to the vacuum field equations, then we will have to prescribe initial data for both $\bar{g}_{\mu\nu}$ and $\partial_t \bar{g}_{\mu\nu}$ at $t = 0$. The geometric picture here is to split the initial data geometrically into the induced metric on $t = 0$, given by $\bar{g}_t|_{t=0}$, the initial data for lapse and shift $N|_{t=0}, X|_{t=0}$ (completing the initial data $\bar{g}|_{t=0}$) and then the first order initial data, $\partial_t \bar{g}_t|_{t=0}, \partial_t N|_{t=0}, \partial_t X|_{t=0}$. It is well-known from standard submanifold theory that in such a situation $\partial_t \bar{g}_t|_{t=0}$ is related with the extrinsic curvature $K \in \Gamma(T_2^0 M)$ of $M \cong \{t = 0\} \times M$ as an embedded hypersurface in $(V, \bar{g})$. Explicitly, we have

$$K = -\frac{1}{2N} \left(\partial_t \bar{g}_t - \mathscr{L}_X \bar{g}_t \right) |_{t=0}, \tag{1.56}$$

where $\mathscr{L}_X \bar{g}_t$ stands for the Lie derivative of $\bar{g}_t$ with respect to X, and our conventions for the extrinsic curvature are

$$K(X, Y) \doteq \bar{g}(\mathbb{II}(X, Y), n), \text{ for all } X, Y \in \Gamma(TM)$$
$$\mathbb{II}(X, Y) \doteq \left(\bar{\nabla}_{\bar{X}} \bar{Y} \right)^{\perp} \tag{1.57}$$

and where $\bar{X}, \bar{Y}$ denote arbitrary extensions of X, Y to V and $\mathbb{II} : \Gamma(TM) \times \Gamma(TM) \mapsto \Gamma(TM^{\perp})$ denotes the second fundamental form of $M \hookrightarrow (V, \bar{g})$. Therefore, we see that the geometric problem becomes more transparent. We attempt to prescribe a Riemannian manifold (M^n, g) equipped with a symmetric $(0, 2)$-tensor field K and initial data for the lapse-shift $(N, X, \partial_t N, \partial_t X)|_{t=0}$ (which determine the family of observers along whose integral curves we intend to evolve the initial data), and then find an isometric embedding $\iota : (M, g) \mapsto (V = I \times M, \bar{g})$ with $I \subset \mathbb{R}^n$ such that:

1. $\bar{g}$ solves the space-time Einstein equations. In the vacuum case given by $\text{Ric}_{\bar{g}} = 0$;

2. K stands as the extrinsic curvature of $M \hookrightarrow (V, \bar{g})$.

$$V = M \times \mathbb{R}$$
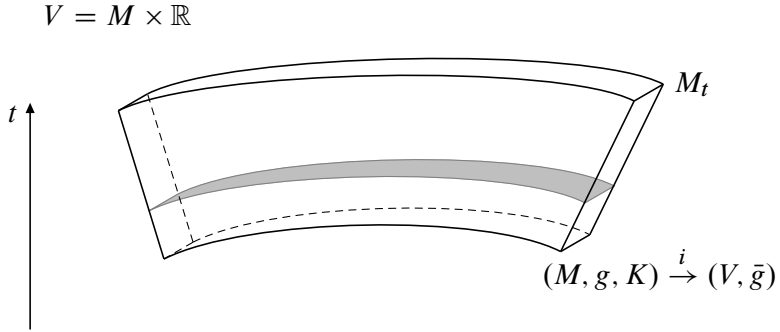
Figure 1.8: Geometric picture associated to the vacuum Cauchy problem

By leaving the data $(N, X, \partial_t N, \partial_t X)|_{t=0}$ outside of the above requirements, we intend to exploit the freedom in choosing the flow lines along which we shall evolve. This demands having enough freedom so as to guarantee that at the end of the problem $\partial_t|_{t=0} = (Nn + X)|_{t=0}$ is time-like. In fact, we will see that $(N, X)|_{t=0}$ are completely free for us to prescribe, but $(\partial_t N, \partial_t X)|_{t=0}$ will be fixed in terms of (g, K, N, X) conveniently.

Now that we have stated clearly what is our geometric problem, we immediately have to realise that, in contrast to classical situations in physics, the initial data for the evolution problem in GR is not free! This follows from the well-known Gauss–Codazzi equations for hypersurfaces, which for a space-like hypersurface (M, g, K) isometrically immersed in a Lorentzian manifold $(V, \bar{g})$ read as:

$$\bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) \qquad \text{(Gauss' Eq.)}$$
$$- (K(X, Z)K(Y, W) - K(Y, Z)K(X, W)),$$
$$\bar{g}(\bar{R}(X, Y)Z, n) = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) \quad \text{(Codazzi's Eq.)},$$
$$(1.58)$$

where $X, Y, Z \in \Gamma(TM)$; n stands for the future-pointing unit normal vector field to M and the quantities without a bar on top are constructed with the intrinsic induced Riemannian metric g on M. That is, for instance, ∇ refers to the Riemannian connection on M associated to g. The above equations are a priori necessary conditions that (g, K) must satisfy. In fact, they imply the following constraint equations:

Proposition 1.4.1. *Let (M, g, K) be a space-like hypersurface isometrically immersed in a Lorentzian manifold $(V, \bar{g})$ satisfying the Einstein equations $G_{\bar{g}} +$*

$\Lambda \bar{g} = T$ *for some energy-momentum tensor T. Then, g and K satisfy the follow-*
*ing **constraint** equations on M:*

$$
\begin{aligned}
R_g - |K|_{\bar{g}}^2 + (\operatorname{tr}_g K)^2 - 2\Lambda &= 2\epsilon, \\
\operatorname{div}_g K - d(\operatorname{tr}_g K) &= J,
\end{aligned}
\tag{1.59}
$$

where $\epsilon \doteq T(n,n)$ and $J \doteq -T(n,\cdot) \in \Gamma(TM)$ denote the energy and momen-
tum densities induced on M.

Proof. Given any local orthonormal frame $\{n, e_i\}_{i=1}^n$, from the Gauss equation
we can compute that

$$
\begin{aligned}
\sum_{i,j=1}^n \bar{g}(\bar{R}(e_i, e_j)e_j, e_i) &= \sum_{i,j=1}^n g(R(e_i, e_j)e_j, e_i) \\
&\quad - \sum_{i,j=1}^n \left((K(e_i, e_j)K(e_j, e_i) - K(e_j, e_j)K(e_i, e_i)) \right), \\
&= R_g - \sum_{i,j=1}^n \left(K(e_i, e_j)K(e_j, e_i) - K(e_j, e_j)K(e_i, e_i) \right), \\
&= R_g - |K|_g^2 + (\operatorname{tr}_g K)^2.
\end{aligned}
$$

Furthermore, since

$$
\begin{aligned}
\operatorname{Ric}_{\bar{g}}(e_i, e_j) &= \sum_{\alpha=0}^n \bar{g}(e_\alpha, e_\alpha)\bar{g}(\bar{R}(e_\alpha, e_i)e_j, e_\alpha), \\
&= -\bar{g}(\bar{R}(n, e_i)e_j, n) + \sum_{k=1}^n \bar{g}(\bar{R}(e_k, e_i)e_j, e_k),
\end{aligned}
$$

we get that

$$R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 = \operatorname{Ric}_{\bar{g}}(n,n) + \sum_{i=1}^{n} \operatorname{Ric}_{\bar{g}}(e_i, e_i)$$

$$= 2\operatorname{Ric}_{\bar{g}}(n,n) + \left(-\operatorname{Ric}_{\bar{g}}(n,n) + \sum_{i=1}^{n} \operatorname{Ric}_{\bar{g}}(e_i, e_i)\right),$$

$$= 2\operatorname{Ric}_{\bar{g}}(n,n) + R_{\bar{g}} = 2\left(\operatorname{Ric}_{\bar{g}} - \frac{1}{2}\bar{g}R_{\bar{g}}\right)(n,n),$$

$$= 2(T - \Lambda\bar{g})(n,n) = 2T(n,n) + 2\Lambda.$$

Thus, from the definition $T(n,n) \doteq \epsilon$, we get

$$R_g - |K|_g^2 + (\operatorname{tr}_g K)^2 = 2(\epsilon + \Lambda). \tag{1.60}$$

Now, consider the Codazzi equation, so that

$$\operatorname{Ric}_{\bar{g}}(n, e_i) = \sum_{\alpha=0}^{n} \bar{g}(e_\alpha, e_\alpha)\bar{g}(\bar{R}(e_\alpha, n)e_i, e_\alpha) = \sum_{j=1}^{n} \bar{g}(\bar{R}(e_j, n)e_i, e_j),$$

$$= \sum_{j=1}^{n} \bar{g}(\bar{R}(e_i, e_j)e_j, n),$$

$$= \sum_{j=1}^{n} (\nabla_{e_i} K)(e_j, e_j) - \sum_{j=1}^{n} (\nabla_{e_j} K)(e_i, e_j) = \operatorname{tr}_g(\nabla_{e_i} K) - \operatorname{div}_g K(e_i),$$

$$= \nabla_{e_i} \operatorname{tr}_g K - \operatorname{div}_g K(e_i).$$

Thus, since $\operatorname{Ric}_{\bar{g}}(n, e_i) = T(n, e_i)$, we get that

$$d\left(\operatorname{tr}_g K\right)(e_i) - \left(\operatorname{div}_g K\right)(e_i) = T(n, e_i).$$

Now, from the definition of the physical momentum density is $J = -T(n, \cdot)$ we arrive at the momentum constraint:

$$\operatorname{div}_g K - d\left(\operatorname{tr}_g K\right) = J. \tag{1.61}$$

$\square$

The above proposition establishes (1.59) as necessary conditions to be satisfied by any initial data set for which we may attempt to find a well-posed evolution problem. It is a remarkable fact that in all of the situations of interest for us, these are also sufficient conditions. This last statement goes back to the pioneering work of Y. Choquet-Bruhat (see Choquet-Bruhat (1962) and Y. Fourès-Bruhat (1952)). We shall now briefly describe the main steps in this construction, where we shall follow the exposition of Choquet-Bruhat (2009, Chapter VI). Thus, let us equip M with a some fixed smooth and complete Riemannian metric e,[24] then trivially embed M into $V = \mathbb{R} \times M$ and fix a background Riemannian metric $\hat{e} = dt^2 + e$ on V. From now on, quantities constructed from $\hat{e}$ will be denoted with a hat on top. For instance, its Riemannian covariant derivative will be denoted by $\hat{D}$. Then, similarly to (1.55), we can write the Ricci tensor as

$$\mathrm{Ric}_{\mu\nu}(\bar{g}) = -\frac{1}{2}\bar{g}^{\alpha\beta}\hat{D}_{\alpha}\hat{D}_{\beta}\bar{g}_{\mu\nu} + \hat{f}_{\mu\nu}(\bar{g},\hat{D}\bar{g}) + \frac{1}{2}\left(\bar{g}_{\mu\lambda}\hat{D}_{\nu}\hat{F}^{\lambda} + \bar{g}_{\nu\lambda}\hat{D}_{\mu}\hat{F}^{\lambda}\right),$$
(1.62)

where now $\hat{F}$ denotes the *vector field* defined via

$$\hat{F}^{\lambda} \doteq \bar{g}^{\gamma\sigma}\left(\Gamma^{\lambda}_{\gamma\sigma}(\bar{g}) - \hat{\Gamma}^{\lambda}_{\gamma\sigma}\right),$$
(1.63)

and $\hat{f}(\bar{g}, \hat{D}\bar{g})$ denotes a tensor field, depending smoothly on its arguments, which is in particular a quadratic function on $\hat{D}\bar{g}$. Then, let us consider the *reduced Ricci tensor*, given by

$$\mathrm{Ric}^{(\hat{e})}_{\mu\nu}(\bar{g}) \doteq -\frac{1}{2}\bar{g}^{\alpha\beta}\hat{D}_{\alpha}\hat{D}_{\beta}\bar{g}_{\mu\nu} + \hat{f}_{\mu\nu}(\bar{g},\hat{D}\bar{g}).$$
(1.64)

The idea is first to consider the reduced Einstein equations given by

$$\mathrm{Ric}^{(\hat{e})}_{\bar{g}} = 0.$$
(1.65)

The advantage now is that this is a set of quasi-linear wave equations where some *standard* PDE theory theorems guarantee that, for appropriate initial data on $\bar{g}$, the system possesses one and only one solution. By appropriate initial data we mean (g, K, N) in some appropriate $\overset{\circ}{H}{}^{s}_{loc}$-Sobolev space and $(K, \partial_t N|_{t=0}, \partial_t X|_{t=0})$ in

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[24]The *initial-boundary value problem* for manifolds with boundary is more subtle than what we will describe. We refer the interested reader to references such as Friedrich (2009), Friedrich and Nagy (1999), Kreiss et al. (2009), and Reula and Sarbach (2011) for further discussion on this topic.

the corresponding $\overset{\circ}{H}{}^{s-1}_{loc}$, with $s > \frac{n}{2} + 1$ (see, for instance, Choquet-Bruhat (ibid., Theorem 7.4, Chapter VI) for detailed statements). The solution to this problem provides us with a Lorentzian metric $\bar{g}$ on $[0, T) \times M$ for some $T > 0$. We now intend to show that if our initial data set (M, g, K) solves the vacuum constraint equations (1.59) (with $\epsilon = \Lambda = 0$ and $J = 0$), then an appropriate choice of the gauge data $\partial_t N|_{t=0}, \partial_t X|_{t=0}$ guarantees that $\hat{F} = 0$, which implies that $\mathrm{Ric}_{\bar{g}} = 0$ and $(V, \bar{g})$ is therefore our desired Cauchy development of (M, g, K). For this, let us first notice that (1.62) implies that our solution $\bar{g}$ to (1.65) satisfies

$$G_{\mu\nu}(\bar{g}) = \frac{1}{2}\left(\bar{g}_{\mu\lambda}\hat{D}_\nu\hat{F}^\lambda + \bar{g}_{\nu\lambda}\hat{D}_\mu\hat{F}^\lambda - \hat{D}_\lambda\hat{F}^\lambda\bar{g}_{\mu\nu}\right),$$

Therefore, the contracted Bianchi identities imply that $\hat{F}$ *must* satisfy the equation

$$0 = \bar{g}^{\alpha\mu}\bar{\nabla}_\alpha G_{\mu\nu}(\bar{g}) = \bar{g}^{\alpha\mu}\hat{D}_\alpha G_{\mu\nu}(\bar{g}) - \bar{g}^{\alpha\mu}S^\sigma_{\alpha\mu}G_{\sigma\nu}(\bar{g}) - \bar{g}^{\alpha\mu}S^\sigma_{\alpha\nu}G_{\mu\sigma}(\bar{g}),$$

where $S^\sigma_{\alpha\mu} \doteq \Gamma^\sigma_{\alpha\mu}(\bar{g}) - \hat{\Gamma}^\sigma_{\alpha\mu}$. Then, with some computational effort, the above can be rewritten as

$$\bar{g}^{\alpha\mu}\hat{D}_\alpha\hat{D}_\mu\hat{F}_\nu + \mathcal{B}^{\alpha\mu}_\nu(\bar{g})\hat{D}_\alpha\hat{F}_\mu + \mathcal{C}^\mu_\nu(\bar{g})\hat{F}_\mu = 0.$$

This last equations reads as a linear wave equation on $\hat{F}$, where the explicit expressions for the coefficients and the regularity properties of $\bar{g}$ guarantee that the solution to such an equation is unique in appropriate functional spaces. Therefore, if in particular $\hat{F}|_{t=0}, \partial_t\hat{F}|_{t=0} = 0$, then we have $\hat{F} \equiv 0$. These conditions can be further simplified by a straightforward computation which show that if

A) The initial data for the solution $\bar{g}$ to (1.65) solves the vacuum constraints associated to (1.59);

B) $\hat{F}|_{t=0} = 0$,

then $\partial_t\hat{F}|_{t=0} = 0$ (ibid., Lemma 8.2, Chapter VI). Since we know that the constraints already are a necessary hypotheses we must assume on our initial data, it is only the second condition that is posing an obstruction. But now, let us consider an adapted frame $\{e_\alpha\}^n_{\alpha=0}$ of the form of (1.52) and assume that we have constructed $\bar{g}$ out of initial data (M, g, K) satisfying the constraints and with $N|_{t=0} = 1$ and $X|_{t=0} = 0$. Then, a straightforward computation gives us

$$F^0|_{t=0} = -(\partial_t N|_{t=0} + g^{ij}K_{ij}),$$
$$F_i|_{t=0} = -\partial_t X_i|_{t=0} + g_{ij}g^{kl}(\Gamma^j_{kl}(g) - \Gamma^j_{kl}(e)).$$

Therefore, we can fix the initial conditions $\partial_t N|_{t=0}$, $\partial_t X_i|_{t=0}$ on M so as to satisfy $\hat{F}|_{t=0} = 0$. Then, from the above discussion, we see that the corresponding solution $\bar{g}$ to (1.65) with initial data satisfying:

1. (M, g, K) solve the vacuum constraint equations;

2. $N|_{t=0} = 1$ and $X|_{t=0} = 0$;

3. $\partial_t N|_{t=0}$, $\partial_t X_i|_{t=0}$ are picked so as to satisfy $\hat{F}|_{t=0} = 0$,

solves the full vacuum Einstein equations on V and is therefore an appropriate (short-time) Cauchy development of (M, g, K) (see Choquet-Bruhat (2009, Theorem 8.3, Chapter VI) for a precise statement involving the precise regularity properties).

   The above presentation leaves the following question open: Does our choice of special observers picked by conditions 1)-2) above on N, X at $t = 0$ play some fundamental role? As the geometric picture suggests, the answer to this question is no. In particular, if we have two Cauchy developments $(V_i, \bar{g}_i)$, $i = 1, 2$, of the same geometric data (M, g, K), and therefore implying that their initial data can differ only via the initial data of N, X which selects the space-time observers, then these developments are isometric (see Choquet-Bruhat (ibid., Theorem 8.4, Chapter VI) and also Ringström (2009, Theorem 14.3)). This is sometimes referred to as *geometric uniqueness*. Furthermore, a celebrated result by Choquet-Bruhat and Geroch (1969) states that there is unique (up to isometries) *maximal* globally hyperbolic development of any such vacuum initial data set.[25] Let us also highlight that, as might be expected, the solutions to these problems have the right causality behaviour. That is, they exhibit the finite-speed propagation associated to solutions of wave equations inherited via hyperbolic theory applied to (1.65). In particular, the limit speed of propagation is given by that of that of the null curves of $\bar{g}$ (see Choquet-Bruhat (2009, Theorems 8.8 and 8.9, Chapter VI)).

   Finally, let us notice that the above discussion can be readily extended along the same lines to non-vacuum situations. The case for scalar fields can be consulted explicitly in Ringström (2009) and fluid sources are analysed in Choquet-Bruhat (2009, Chapter IX), including cases such as perfect fluids and charged fluids. But, let us highlight that these last cases which involve an electromagnetic field actually present one further subtlety, which is that the Maxwell equations of electromagnetism also impose constraints on the admissible initial data for the electromagnetic 2-form F. Below, we will derive such enlarged system of constraints

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[25] See also Chruściel (2013) for a version of this result under weaker regularity conditions.

constraints, which in Chapter 4 will work as a model for the analysis of a highly coupled system of constraints for *realistic* initial data. As we will see in Chapter 2, the Gauss–Codazzi constraints (1.59) admit a nice PDE formulation which decouples them in a variety of interesting situations. In contrast, the constraints associated to a charged fluid will not decouple and thus present a more delicate well-motivated problem, which is analysed in Section 4.3 of Chapter 4.

Electromagnetic sources

Let us consider the constraint equations associated to a charged fluid, for instance, such as that considered in equations (1.38). Notice that the initial data for such a system would consist not only on the initial data for $\bar{g}$, but also on the initial data for F, $\bar{u}$ and $\bar{\mu}$. The initial data for $\bar{u}$ and $\bar{\mu}$ is not subject to any constraints, but the initial data for F is. This is clear since the space-time 2-form F induces a 2-form on M, say $\tilde{F}$, given by the restriction of F to tangent vectors to M. Then, the equation $dF = 0$ also implies $d\tilde{F} = 0$ on M. That is, $\tilde{F}$ on M has to be closed. This is an additional constraint which must be coupled to the above Gauss–Codazzi constraints.

Furthermore, the evolution equation $\delta_{\bar{g}} F = \mathcal{J}^\flat$, when projected orthogonally to M also gives us a constraint on the initial data. To see this, consider a space-time orthonormal frame adapted to M. That is, a frame $\{n, e_i\}_i^n$, where n is future pointing unit normal and e_i are tangent to M. Then, we get

$$-\delta_{\bar{g}} F(n) = \operatorname{div}_{\bar{g}} F(n) = \sum_{\alpha=0}^{n} \bar{g}(e_\alpha, e_\alpha) \bar{\nabla}_{e_\alpha} F(e_\alpha, n)$$

$$= \sum_{\alpha=0}^{n} \bar{g}(e_\alpha, e_\alpha) \left(e_\alpha \left(F(e_\alpha, n) \right) - F(\bar{\nabla}_{e_\alpha} e_\alpha, n) - F(e_\alpha, \bar{\nabla}_{e_\alpha} n) \right).$$

Now, since $\bar{g}(n, n) = -1$, it follows that $\bar{g}(\bar{\nabla}_{e_i} n, n) = 0$. Therefore $\bar{\nabla}_{e_i} n$ is tangent to M. Also, since $\bar{g}(\bar{\nabla}_{e_j} n, e_i) = -\bar{g}(n, \bar{\nabla}_{e_j} e_i) = -K_{ij}$, it holds that $\bar{\nabla}_{e_i} n = -\sum_{i=1}^{n} K_{ij} e_j$. Using this in the above expression we get that

$$-\delta_{\bar{g}} F(n) = -n \left(F(n, n) \right) + \sum_{i=1}^{n} \left(e_i (F(e_i, n)) - F(\bar{\nabla}_{e_i} e_i, n) - F(e_i, \bar{\nabla}_{e_i} n) \right),$$

$$= \sum_{i=1}^{n} \left(e_i (F(e_i, n)) - F((\bar{\nabla}_{e_i} e_i)^\top, n) + \sum_{j=1}^{n} K_{ij} F(e_i, e_j) \right),$$

where in the second identity we used that $\bar{\nabla}_{e_i} e_i = \left(\bar{\nabla}_{e_i} e_i\right)^\top - K(e_i, e_i)n$ and, since $F(n, n) = 0$, it follows that $F(\bar{\nabla}_{e_i} e_i, n) = F(\left(\bar{\nabla}_{e_i} e_i\right)^\top, n)$. On the other hand, we know that $\left(\bar{\nabla}_{e_i} e_i\right)^\top = \nabla_{e_i} e_i$, where we are denoting by ∇ the Riemannian connection associated to each t-dependent Riemannian metric $\bar{g}_t$ on M_t, which establishes

$$-\delta_{\bar{g}}\, F(n) = \sum_{i=1}^{n} \left(e_i(F(e_i, n)) - F(\nabla_{e_i} e_i, n)\right) + \langle K, F\rangle_{\bar{g}_t},$$

$$= \sum_{i=1}^{n} \left(e_i(F(e_i, n)) - F(\nabla_{e_i} e_i, n)\right),$$

where the last step is a consequence of the symmetry of K and antisymmetry of F.

Recalling the definition of the electric 1-form field, we conclude that the t-dependent 1-form on M given by

$$\bar{E}(X) \doteq F(X, n) = N^{-1} F(X, e_0), \quad \forall \ X \in \Gamma(TM). \tag{1.66}$$

represents the electric field as measured by observers whose flow lines are the integral curves of the future oriented unit normal field n to each M_t. This, implies that

$$-\delta_{\bar{g}}\, F(n) = \sum_{i=1}^{n} \left(e_i(\bar{E}(e_i)) - \bar{E}(\nabla_{e_i} e_i)\right) = \operatorname{div}_{\bar{g}_t} \bar{E}.$$

Thus, defining $E \doteq \bar{E}|_{t=0}$, we get the constraint

$$\operatorname{div}_g E = -\mathcal{J}^\flat(n)|_{t=0} \doteq \rho, \tag{1.67}$$

where ρ denotes the charge density at $t = 0$ as measured by observes following the integral curves of n. Since the above equation depends only on initial data, then it represents a constraint.

We have therefore found the two electromagnetic constraints, given by

$$\operatorname{div}_g E = \rho, \ , \ d\tilde{F} = 0, \tag{1.68}$$

which must be coupled to the Gauss–Codazzi constraints. That is, for a charged fluid, the corresponding constraint equations on M for the full system of space-time field equations is given by

$$
\begin{aligned}
R_g - |K|_g^2 + \left(\operatorname{tr}_g K\right)^2 - 2\Lambda &= 2\epsilon, \\
\operatorname{div}_g K - d(\operatorname{tr}_g K) &= J, \\
\operatorname{div}_g E &= \rho, \\
d\widetilde{F} &= 0,
\end{aligned}
\tag{1.69}
$$

where ϵ, J and ρ are fields dependent on the specific model for the charged fluid.

1.5   Black hole solutions

In the next chapters we will treat in some detail the problem of constructing initial data for black hole solutions to the Einstein equations in a wide variety of situations. We have already encountered the most basic and illustrative example of a black hole solution in Section 1.3.2 and, there, the relevance of the analysis of such objects within general relativity was pointed out. In this section we would like to introduce some notions concerning general black hole solutions, which generalise the discussion of Section 1.3.2. This subject has received plenty of attention both within mathematics and physics, which, unfortunately, we will not be able to cover and give its deserved detailed treatment. We refer the interested reader to references such us Choquet-Bruhat (2009), Hawking and Ellis (1973), and Wald (1984) for further details.

In what follows, we will be particularly interested in describing properties within initial data sets which signal the presence of black holes in their associated evolutions. This is highly motivated by (at least) two facts. On the one hand, it is a fact that these objects are out there in reality and therefore their understanding as well as modelling becomes essential for physics. On the other hand, many deeply interesting mathematical problems are related to these solution. In particular, it has been part of the folklore in general relativity that *generic* solutions of the Einstein equations possess black hole regions. We will actually use this last point as the starting point for our discussion and motivation to introduce models of black hole initial data sets.

We should start our discussion by defining more precisely what is meant by a black hole region within a solution of the Einstein equations. A large part of the necessary intuitions for such concept can be extracted from the analysis presented

for the Schwarzschild solution. There, we noticed that the black hole region B was somehow characterised by its impossibility of sending signals to its exterior. Nevertheless, this property alone is not what captures the essence of what is going on in the region B. Notice that the interior of any light cone in Minkowski space also shares this property, although nothing special is going on there, and therefore we must be more careful. In particular, what is special about the region B is that this is a *bounded region* which can never send signals to its exterior. This is actually what captures the essence behind what a black hole is.

Usually, the formal definition of the *the black hole region* B of a space time $(V, \bar{g})$ is given in terms of certain properties of *a conformal completion* of such space-times. More concretely, the idea is to consider those null geodesics whose canonical parametrisations can be extended indefinitely (geodesics that *escape to infinity*), and add to V *idealised endpoints* to such geodesics. These endpoints represent a kind of boundary for V within a conformally related space-time, which is commonly referred to as the *null-infinity*, denoted by $\mathcal{S}$. In such situations, if we consider the *future null-infinity* $\mathcal{S}^+$, we could then define the black hole region of V as the complement of the past of $\mathcal{S}^+$, which, appealing to some causality theory is denoted as $\mathcal{J}^-(\mathcal{S}^+)$, where $\mathcal{J}^-(A)$ denotes the causal past of a set A.[26] Then, *the event horizon* of B is taken to be the boundary of B. This procedure is depicted for the Schwarzschild space-time in Figure 1.9 below.
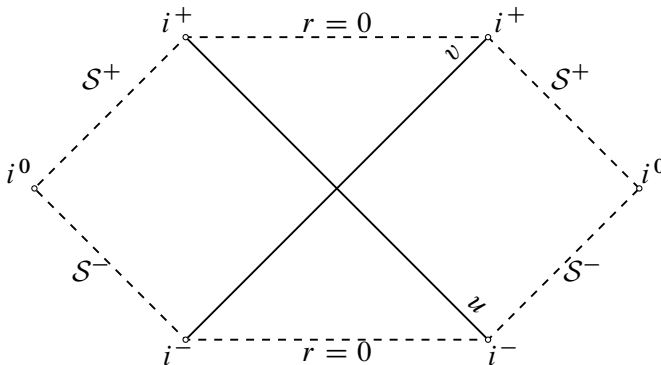

Figure 1.9: Schwarzschild's conformal completion

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[26]In this context, the causal past $\mathcal{J}^-(\mathcal{S}^+)$ is taken with respect of the *unphysical* conformally related space-time.

The above procedure for defining black holes within a space time works well in a range of interesting situations, typically restricted to *asymptotically Minkowskian* space-times which are *asymptotically simple*, this last concept being related to the existence of the necessary conformal compactification used above. The existence, properties and usefulness of such compactifications are not at all trivial, and things such are the appropriate regularity of the conformal factor at infinity are key. Nevertheless, assuming their existence together with certain properties gives a powerful machinery which allows one to prove very nice theorems (see, for instance, Chruściel, Delay, et al. 2001; Hawking and Ellis 1973; Wald 1984). We refer the interested reader to Wald (1984, Chapter 12) and Hawking and Ellis (1973, Chapter 9) for more details concerning all these concepts.

Besides the above comments, one of the major drawbacks of the characterisation of black holes via conformal completions, is that, in order to find out whether (or where) a space time has a black hole region, we need to know the complete space-time beforehand. This is particularly inconvenient if we want to model black hole space-times evolving appropriate initial data, which is the paramount procedure in physics. Therefore, in what follows, we will look for some characterisation related to initial data sets that signals the existence of a black hole region in the evolving space-time. In this direction, we will appeal to two very important problems in general relativity: the existence of singularities and their relation to black hole formation. The precise definition of singular space-time is given by geodesic incompleteness. In particular, through the foundational work of Roger Penrose (Penrose 1965) and subsequent collaborations with Steven Hawking (Hawking and Penrose 1970), it was shown that under a wide variety of compelling hypotheses, a space-time must be singular. In particular, Penrose's singularity theorem states the following (Penrose 1965):

Theorem 1.5.1 (Penrose). *Let $(V, \bar{g})$ be a space-time satisfying the following conditions: (1) $\mathrm{Ric}_{\bar{g}}(v, v) \geqslant 0$ for all null tangent vectors v; (2) V has a non-compact Cauchy surface M; (3) There is a closed trapped surface in V. Then, $(V, \bar{g})$ cannot be null geodesically complete.*

In the above theorem we have introduced the concept of a *trapped surface*, which we will define and characterise precisely below. For now, let us say that a trapped surface is an $(n-1)$-dimensional closed space-like submanifold of V along which all future pointing null orthogonal geodesics are *converging*. That is, this is a submanifold along which future pointing light rays are *focusing*. Let us also point out that condition (1) on the Ricci tensor is known as the *null energy condition* (NEC) and belongs to a family of energy conditions used in general

relativity. Notice that through the Einstein equations, the NEC gets translated into a condition on the energy-momentum tensor, and thus it becomes a hypothesis on the matter fields present in a physical model. In particular, the NEC is a very compelling condition to be imposed, at least on classical fields.

The above singularity theorem has played a foundational role in general relativity, being the first of a series of results which showed that the formation of singularities is a typical behaviour for general solutions in general relativity and not an artefact of idealised highly symmetrical situations, as was once believed. In particular, several generalisations and complementary theorems have been proven contemplating, for instance, compact Cauchy surfaces (Geroch 1966; Hawking 1966, 1967; Hawking and Penrose 1970), averaged energy conditions (Borde 1987; Chicone and Ehrlich 1980) and low regularity space-times (Graf 2020; Graf et al. 2017).

Let us now notice that the premises of Theorem 1.5.1 can be translated into hypotheses of an initial data set. As was highlighted above, condition (1) can be cast as a condition of the energy-momentum sources (ϵ, J) of an initial data set. For instance, initial data for sources which are meant to satisfy the dominant or weak energy condition will satisfy the null energy condition. Also, in the context of the evolution problem for initial data sets, condition (2) has a trivial interpretation, while condition (3) can be imposed on an initial data set (M, g, K, ϵ, J). Therefore, we can try to generate initial data sets which will evolve into singular space-times. In practice, this would imply having a good characterisation of what a trapped surface looks like within an initial data set. Such a characterisation is actually the main objective in this section, and will be done in detail below. But, before doing this, let us explain why all this discussion is relevant in the context of black hole initial data sets.

The singularity theorems alluded to above prove the existence of a *pathological* behaviour of probably most physically reasonable solutions to the Einstein equations. Physically, they imply that certain particles would either cease to exist or come out of *nowhere* at a singularity. Such a pathological behaviour would also have implications for the breakdown of predictability from initial data. Whether these are actually problematic points depends on the nature of the singularities. Although the singularity theorems themselves do not tell us much about what is actually going wrong near the singularity, if, for instance, it happens to be true that singularities are hidden within black holes, it is by now widely accepted that quantum effect should kick in at some point in such extreme gravitational situations and thus, the *generic* existence singularities would be one further signal about the break down of general relativity as an accurate description of gravitational phenomena in these extreme situations, rather an actual pathology existing in Nature.

These undesirable physical problems led Roger Penrose to pose two conjectures about the nature of singularities, which are known as the *Cosmic Censorship Conjectures* (Penrose 1969).

Intuitively, the *weak cosmic censorship conjecture* states that, outside certain special cases, *generic* singularities should be *hidden* within black holes. This would imply that observers outside the black holes do not see any pathological behaviour in space-time. The precise formulations available of this conjecture are posed on asymptotically Euclidean initial data sets, and, in particular, the most common of them make use of conformal completions. The idea is that adding some completeness criteria to the future conformal null infinity $\mathcal{S}^+$, can be used to guarantee that light rays which can escape to infinity are actually complete to the past, i.e, they cannot appear at a singularity out of nowhere, avoiding what are called *naked singularities*, which are singularities visible from infinity.

Clearly, there are many subtleties related to the above conjecture. For instance, there are known counterexamples (see Christodoulou 1994),[27] although they have been shown to be *unstable* (see Christodoulou 1999b), which proves that a kind of *genericity* hypothesis is needed for the conjecture to be sensible. In this context, a property is typically said to be non-generic if it is not stable under small perturbations, which seems to be a good physical criterion. Another subtlety related to the above conjecture is, once more, the appeal to a conformal completion, which brings about all the same concerns we commented before. Thus, another formulation of the same conjecture which retains the same physical intuitions but does not appeal to conformal completions has been proposed in Christodoulou (1999a). In particular, the author has been able to show what is, to the best of our knowledge, the only concrete proof of weak cosmic censorship, although limited to spherically symmetric solutions (see Christodoulou 1999b). We refer the reader to Wald (1999) for a very nice review of this topic, which presents a positive case for the validity (of a suitable version) of this conjecture.

If we accept weak cosmic censorship (at least as a good working hypothesis), then, from the singularity theorems, we can conclude that the presence of a trapped surface in a non-compact initial data set is a signal of the existence of a black hole region. In fact, from the very definition of a trapped surface, we may venture that this is the case in general. Furthermore, as we will now see, trapped surfaces (and analogous geometric objects) can be very nicely characterised within initial data sets in a manner which is amenable to be treated be standard analytic tools, and therefore we will appeal to them to distinguish black hole initial data sets.

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[27]See also the related work of Christodoulou (1987, 1991, 1993).

Trapped Surfaces

Let us now characterise the trapped surface condition in concrete terms. Along this process, we will fix some notation that will be used in subsequent chapters.

Let (M, g, K, ϵ, J) be an initial data set whose evolution results in a time-orientable space-time $(V, \bar{g})$. Let us assume that M is a manifold with boundary $\partial M = \cup_{i=1}^{m} \Sigma_i$, consisting of m compact connected components $\{\Sigma_i\}_{i=1}^{m}$. Let us now concentrate on a particular component Σ and introduce some geometric concepts associated to it.

Since $(V, \bar{g})$ is time orientable, we have a time-like future pointing vector field T defined along Σ. Since M is space-like, then $T - T^{\top} \perp M$ and future pointing. Normalising this vector field we construct a unit normal future pointing vector field to M, which will be denoted by n, and can be restricted to Σ. Furthermore, since Σ is a boundary component of M, we necessarily have a space-like outward pointing unit normal vector field ν. That is, the normal bundle of Σ is orientable. Let us then define the future pointing null vector fields $N_{\pm} \in \Gamma(T\Sigma^{\perp})$, given by

$$N_{\pm} \doteq n \mp \nu,$$

and therefore $\{N_+, N_-\}$ produce a basis of $T_p\Sigma^{\perp}$ at each $p \in \Sigma$.
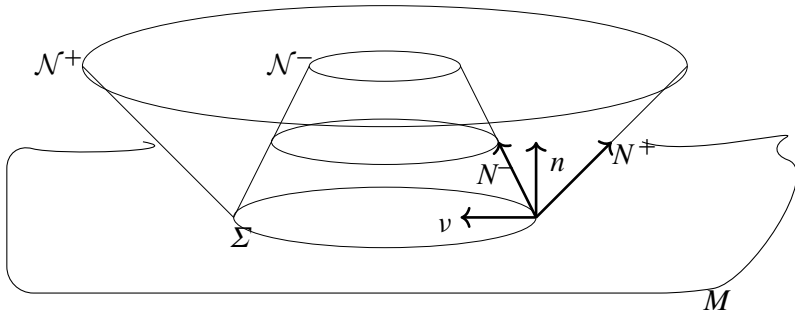
Let us now consider the null space-time geodesics starting on Σ with initial conditions given by $N_{\pm}$ and denote by

$$\Phi_{\pm} : \Sigma \times (-\epsilon, \epsilon) \mapsto V,$$
$$(x, s) \mapsto \exp_x(sN_{\pm})$$

the associated geodesic flows. At least near Σ, this defines two n-manifolds $\mathcal{N}_\pm$ embedded in V. We can extend $N_\pm$ to null vector fields tangent to the corresponding geodesics ruling $\mathcal{N}_\pm$ via the fields $\frac{d}{ds}\exp_x(sN_\pm(x))$, and then we can extend these fields to a neighbourhood in V of $\mathcal{N}_\pm$. To simplify notation, we will still denote all these extensions as $N_\pm$.

Definition 1.5.1. *In the above setting, we define the null extrinsic curvatures $\chi_\pm \in \Gamma(T_2^0\Sigma)$ by*

$$\chi_\pm(X,Y) \doteq \bar{g}(\bar{\nabla}_{\bar{X}}\bar{Y}, N_\pm), \text{ for all } X, Y \in \Gamma(T\Sigma), \tag{1.70}$$

where $\bar{X}, \bar{Y} \in \Gamma(TV)$ denote arbitrary extensions of X, Y to V in a neighbourhood of Σ. Furthermore, let us denote by h the Riemannian metric induced by $\bar{g}$ on Σ and define the expansion scalars (null mean curvatures) by

$$\theta_\pm \doteq -\mathrm{tr}_h \chi_\pm. \tag{1.71}$$

The above definitions are standard in the analysis of the extrinsic geometry of submanifolds, and it is an easy exercise to check that the definition of χ does not depend on how we extend the fields X and Y. Thus, in the future, we will not differentiate notationally the fields on Σ (such as X and Y) from their extensions to V.

Proposition 1.5.1. *Consider the above setting and notations. Then, the expansion scalars satisfy the following identity*

$$\theta_\pm = \mathrm{div}_{\bar{g}} N_\pm, \tag{1.72}$$

and they can be computed in terms of the initial data (M, g, K) as

$$\theta_\pm = K(\nu, \nu) - \mathrm{tr}_g K \pm \mathrm{tr}_h k, \tag{1.73}$$

where we have defined the extrinsic curvature of Σ as a hypersurface of M as $k(X,Y) = g(\nabla_X Y, \nu)$, for any vector fields X, Y tangent to Σ.[28]

Proof. Let us first extend ν to a vector field in a neighbourhood of a point $p \in \Sigma$ such that $\bar{g}(\nu, \nu) = 1$ and $\bar{g}(\nu, n) = 0$. Then, consider an orthonormal frame

---

[28]To facilitate comparison with other references, let us highlight that, under our conventions, $\mathrm{tr}_h k = -\mathrm{div}_g \nu$.

around $p \in \Sigma \hookrightarrow V$ of the form $\{n, \nu, E_i\}_{i=1}^n$, with $\{E_i\}_{i=1}^{n-1}$ tangent to Σ. Since $N_\pm$ belong to the normal bundle of Σ, it holds that

$$
\begin{aligned}
\theta_\pm &= -\sum_{i=1}^{n-1} \bar{g}(\bar{\nabla}_{E_i} E_i, N_\pm) = \sum_{i=1}^{n-1} \bar{g}(E_i, \bar{\nabla}_{E_i} N_\pm), \\
&= \operatorname{div}_{\bar{g}} N_\pm - \bar{g}(\nu, \bar{\nabla}_\nu N_\pm) + \bar{g}(n, \bar{\nabla}_n N_\pm), \\
&= \operatorname{div}_{\bar{g}} N_\pm - \bar{g}(\nu, \bar{\nabla}_\nu N_\pm) + \bar{g}(n, \bar{\nabla}_n n) \mp \bar{g}(n, \bar{\nabla}_n \nu)
\end{aligned}
$$

But, since $\bar{g}(n, n) = -1$, we know that $\bar{g}(n, \bar{\nabla}_n n) = 0$ and thus

$$
\theta_\pm = \operatorname{div}_{\bar{g}} N_\pm - \bar{g}(\nu, \bar{\nabla}_\nu N_\pm) \pm \bar{g}(\nu, \bar{\nabla}_n n).
$$

Also, since $\bar{g}(\nu, \nu) = 1$, it follows that $\bar{g}(\bar{\nabla}_n \nu, \nu) = 0$, and thus $\bar{g}(\nu, \bar{\nabla}_n n) = \bar{g}(\nu, \bar{\nabla}_n N_\pm)$. Therefore, we find that

$$
\theta_\pm = \operatorname{div}_{\bar{g}} N_\pm \pm \bar{g}(\nu, \bar{\nabla}_{n \mp \nu} N_\pm).
$$

But since $\bar{\nabla}_{n \mp \nu} N_\pm = \bar{\nabla}_{N_\pm} N_\pm = 0$ by construction of $N_\pm$, the first claim follows. In order to establish the second claim, let us use the notation $E_n \doteq \nu$ and notice that

$$
\begin{aligned}
-\theta_\pm &= \sum_{i=1}^{n-1} \bar{g}(\bar{\nabla}_{E_i} E_i, N_\pm) = \sum_{i=1}^{n} \bar{g}(\bar{\nabla}_{E_i} E_i, n \mp \nu) - \bar{g}(\bar{\nabla}_\nu \nu, n \mp \nu), \\
&= \sum_{i=1}^{n} \bar{g}(\bar{\nabla}_{E_i} E_i, n) \mp \sum_{i=1}^{n-1} \bar{g}(\bar{\nabla}_{E_i} E_i, \nu) - \bar{g}(\bar{\nabla}_\nu \nu, n), \\
&= \operatorname{tr}_g K \mp \operatorname{tr}_h k - K(\nu, \nu),
\end{aligned}
$$

and we have used that $\bar{g}(\bar{\nabla}_{\bar{E}_i} \bar{E}_j, \nu) = \bar{g}(\nabla_{E_i} E_j + \mathbb{II}(E_i, E_j), \nu) = g(\nabla_{E_i} E_j, \nu) = k(E_i, E_j)$, where $\mathbb{II}$ is the second fundamental form of M as a hypersurface of V. $\qquad \square$

In this context we can introduce the following definitions.

Definition 1.5.2. *In the above setting, we will say that Σ is a trapped surface if $\theta_\pm < 0$; is marginally trapped if $\theta_\pm \leqslant 0$; is outer marginally trapped if $\theta_+ \leqslant 0$ and is an apparent horizon if $\theta_+ = 0$.*

From (1.72) we see that all of the above conditions have the interpretation we were looking for. That is, they represent conditions which show that light rays (even those emitted pointing away from Σ) cannot scape towards infinity. For weakly censored space-times, due to singularity theorems, this translates into the existence of black hole regions. Furthermore, the characterisation (1.73) is precisely of the kind we were looking for since it is expressed solely in terms in the initial data set.

After having established the above criteria for black hole initial data, it is instructive to go back to the Schwarzschild case and compare. From (1.47), we know that the initial data for the two ended Schwarzschild space-time is given by $(\mathbb{R}^3\backslash\{0\}, g_{Sc} = \left(1 + \frac{m}{2|x|}\right)^4 \delta, K \equiv 0)$. Let us analyse the location of what we know to be the event horizon, that is the sphere connecting the two ends given by $r = \frac{m}{2}$. It is straightforward to compute that this surface is totally geodesic in $\mathbb{R}^3\backslash\{0\}$. That is, $k \equiv 0$, and therefore we see that this surface satisfies $\theta_+ = 0$ and represents a apparent horizon.

Actually, trapped surfaces tend to appear *inside* black hole regions, and, although in the kind of models we will analyse the hope is that putting trapped surfaces *sufficiently far apart* from each other inside initial data sets will produce a corresponding black hole associated to each of them, this is far from obvious as can be seen from the discussion presented in Chruściel and Mazzeo (2003). Nevertheless, this is a canonical way to address this problem in a systematic way which is amenable to a very nice analytic treatment (see Maxwell 2005b).

2

An overview of classical results

During this chapter we will start our analysis of the constraint equations for the general relativistic initial data sets presented in Section 1.4. The first objective will be to cast the ECE as system of geometric elliptic PDEs. With this aim in mind, the first thing to realise (for instance looking at (1.59)) is that the ECE seen as equations for (g, K) on M^n are a highly under-determined system. Thus, in particular, we have some freedom to look for a useful decomposition of g and K into *prescribed data* and *unknowns* which may turn it into a determined (elliptic) system. The ideal objective would be that such splitting is *natural* both from a geometric and a physical stand point. The best known method to achieve these goals is the so called *conformal method*, which goes back to ideas of Bruhat (1944) and was developed by Choquet-Bruhat (1962), Y. C. Fourès-Bruhat (1957), Ó Murchadha and York Jr. (1974), and York Jr. (1973). This method splits g into a prescribed conformal class and an unknown conformal factor, while it splits K into a prescribed trace part (mean curvature) and unknown traceless part, which itself undergoes a further slitting allowing to write the momentum constraint as an elliptic equation on some vector field X.[1]

---

[1] Let us further refer the reader to some more recent developments and other splitting proposals related to the conformal method, such as Pfeiffer and York Jr. (2003) and York Jr. (1999), and, in particular to the discussion presented in Maxwell (2014). Furthermore, let us also highlight the

We will begin this chapter by describing the above conformal method in detail. As we will see, under special geometric conditions which involve a constant mean curvature (CMC) hypothesis, the conformal method decouples the Gauss–Codazzi constraints of (1.59). It is in such situations that this method is most effective. The analysis of the resulting equations relies on the application of analytic tools such as those described in Appendix B. Therefore, in this chapter, we will start by analysing the ECE on closed manifolds where the tools of that appendix readily apply. In particular, we will present results which include the CMC vacuum classification of Isenberg (1995) as well as the more recent remarkable developments of Maxwell (2005a). Our discussion also contemplates CMC results appearing in Choquet-Bruhat (2004). In subsequent chapters we shall extend this analysis to special non-compact manifolds called asymptotically Euclidean manifolds, which in particular model isolated gravitational systems. In Chapter 3 we will extend the analysis for CMC initial data to such non-compact case and, furthermore, include boundary conditions modelling black hole initial data. In Chapter 4 we will analyse the coupled system allowing for freely prescribed mean curvature initial data in this last non-compact setting.

2.1   The conformal method

The idea of this section is to rewrite the constraint equations we encountered in Section 1.4 in a manner which is amenable to PDE analysis. In doing so, we will follow closely the presentation given in Choquet-Bruhat (2009, Chapter 7). Let us start by recalling the Gauss–Codazzi constraints (with $\Lambda = 0$), given on a Riemannian manifold (M^n, g) by[2]

$$
\begin{aligned}
R_g - |K|_g^2 + \left(\mathrm{tr}_g K\right)^2 &= 2\epsilon, \\
\mathrm{div}_g K - d\left(\mathrm{tr}_g K\right) &= J,
\end{aligned}
\tag{2.1}
$$

where $\epsilon \doteq T(n, n)$ and $J = -T(n, \cdot)$ stand for the energy and momentum densities on the initial data set, with n being the future pointing unit normal vector field. In what follows, we will adopt the notation $\tau \doteq \mathrm{tr}_g K$ for the *mean curvature*. For a fixed matter model, the above equations could be thought of as equation for the geometric data g and K, with ϵ and J prescribed, and this works well in a series

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modifications proposed in Maxwell (2021), aiming to deal with specific issues of the conformal method in the non-CMC setting.

[2]The necessary modification to contemplate $\Lambda \neq 0$ are straightforward in most cases.

of situations we shall encounter below. As we commented in the introduction, in such context equations (2.1) stands as a highly under-determined system posed for (g, K), and we attempt to exploit this freedom to split (g, K) in some clever way into prescribed data and unknowns for the system following the *the conformal method*, which translates (2.1) into a determined elliptic PDE systems. Let just highlight that, in this context, the energy constraint in (2.1) has the form of a generalised scalar curvature prescription problem, and therefore it is not surprising that conformal deformations work quite nicely. Let us start by recalling the following computational result.

Proposition 2.1.1. *Let (M^n, g) be a Riemannian manifold with $n \geq 3$. Suppose that $g = \varphi^{\frac{4}{n-2}} \gamma$ for some other Riemannian metric γ on M. Then, the following transformation rule for the scalar curvature holds*

$$R_g = \varphi^{-\frac{n+2}{n-2}} \left(R_\gamma \varphi - \frac{4(n-1)}{n-2} \Delta_\gamma \varphi \right), \tag{2.2}$$

where Δ_γ stands for the negative Laplace operator.

In the above context we will denote by ∇ the Riemannian connection associated to g and by D the corresponding connection associated to γ, so that, for instance, $\Delta_\gamma \varphi = \gamma^{ij} D_i D_j \varphi$. Also, the second order linear operator appearing in the right hand side of (2.2), given by $L_g \doteq \Delta_\gamma - c_n R_\gamma$ will be referred to as *the conformal Laplacian*. This language is standard, since L_g is a very well-known operator playing a key role in conformal scalar curvature deformation problems. Equation (2.2) transforms the energy constraint in (2.1) into

$$\Delta_\gamma \varphi - c_n R_\gamma \varphi + c_n \left(|K|_g^2 - \tau^2 + 2\epsilon \right) \varphi^{\frac{n+2}{n-2}} = 0, \tag{2.3}$$

where $c_n = \frac{1}{4} \frac{n-2}{n-1}$.

We now need to stipulate a splitting for the extrinsic curvature. To begin with, we will split it into its trace and traceless parts. In doing so, the aim is to leave the trace as a parameter which is free for us to specify. As is typical in geometric PDE problems, this will lead to natural geometric conditions which greatly simplify a difficult non-linear problem. Now, the trace part $\tau = \text{tr}_g K$ of K will naturally inherit some scaling under conformal deformations. Nevertheless, we will need to impose some good ad hoc scaling for the traceless part under conformal transformations. In doing so, we will follow the so-called *York splitting*. Explicitly, let us split the extrinsic curvature as follows:

$$K = \varphi^{-2} \widetilde{K} + \frac{\tau}{n} g,$$

where $\widetilde{K}$ is a γ-traceless (and thus g-traceless) $(0, 2)$-tensor field, where we take the convention that $\widetilde{K}$ *moves its indices* with the conformal metric γ, while the physical extrinsic curvature K moves its indices with the physical metric g. That is,

$$
\begin{aligned}
K_{ij} &= \varphi^{-2} \widetilde{K}_{ij} + \frac{\tau}{n} g_{ij}, \\
K^{ij} &= \varphi^{-2\frac{n+2}{n-2}} \widetilde{K}^{ij} + \frac{\tau}{n} g^{ij}.
\end{aligned}
\tag{2.4}
$$

This in particular implies that

$$
|K|_g^2 = \varphi^{-\frac{4n}{n-2}} |\widetilde{K}|_\gamma^2 + \frac{1}{n} \tau^2,
$$

and therefore we can make a further decomposition to the conformally formulated energy constraint (2.3) and rewrite it as

$$
\Delta_\gamma \varphi - c_n R_\gamma \varphi + c_n |\widetilde{K}|_\gamma^2 \varphi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \varphi^{\frac{n+2}{n-2}} = 0. \tag{2.5}
$$

We will refer to the above equation as *the Lichnerowicz equation* in general. This equation will take different forms depending of our physical model, which determines the form of ϵ and J, as well as the remaining geometric data, related to the extrinsic curvature.

Let us now concentrate on how to obtain a similar form for the momentum constraint in (2.1). In particular, the aim is to rewrite (2.1) as a *determined* elliptic PDE system, and therefore, we shall attempt to rewrite the momentum constraint as a PDE on *some vector field* linked to K (actually $\widetilde{K}$) in a natural way. The first step in this direction is the following computational result.

Proposition 2.1.2. *Let us consider the Riemannian manifold (M, g), with $g = \varphi^{\frac{4}{n-4}} \gamma$ for some other Riemannian metric γ on M. Let $K \in \Gamma(T_2^0 M)$ be symmetric and let us split it as in (2.4). Then the g and γ divergences of K are related via the following expression*

$$
\operatorname{div}_g K = \varphi^{-\frac{2n}{n-2}} \operatorname{div}_\gamma \widetilde{K} + \frac{1}{n} d\tau. \tag{2.6}
$$

Proof. First, using (2.4), we can compute

$$
\begin{aligned}
\nabla_i K^{ij} &= \nabla_i \left(\varphi^{-2\frac{n+2}{n-2}} \widetilde{K}^{ij} \right) + \frac{1}{n} g^{ij} \nabla_i \tau, \\
&= \varphi^{-2\frac{n+2}{n-2}} \nabla_i \widetilde{K}^{ij} - 2\frac{n+2}{n-2} \varphi^{-2\frac{n+2}{n-2}-1} \widetilde{K}^{ij} \nabla_i \varphi + \frac{1}{n} g^{ij} \nabla_i \tau.
\end{aligned}
$$

Also, the covariant derivatives of $\widetilde{K}$ in the metrics g and γ are related via

$$\nabla_i \widetilde{K}^{ij} = D_i \widetilde{K}^{ij} + S^i_{il} \widetilde{K}^{lj} + S^j_{il} \widetilde{K}^{il}$$

where

$$S^k_{il} = \frac{2}{n-2} \varphi^{-1} \gamma^{ka} \left(D_i \varphi \gamma_{la} + D_l \varphi \gamma_{ia} - D_a \varphi \gamma_{il} \right).$$

Thus, it follows that

$$\nabla_i \widetilde{K}^{ij} = D_i \widetilde{K}^{ij} + \frac{2(n+2)}{n-2} \varphi^{-1} \widetilde{K}^{ij} D_i \varphi.$$

Putting together the above, we find

$$\nabla_i K^{ij} = \varphi^{-2\frac{n+2}{n-2}} D_i \widetilde{K}^{ij} + \frac{1}{n} g^{ij} \nabla_i \tau.$$

Then, the claim follows lowering the free index and remembering $g = \varphi^{\frac{4}{n-2}} \gamma$, where $\widetilde{K}$ moves indices with γ. $\square$

The above proposition explains the choice of scaling imposed for the traceless part of K under conformal transformations. That is, just as much as we the choice $g = \varphi^{\frac{4}{n-2}} \gamma$ is good to get rid of the first order derivatives of the conformal factor in (2.2), the choice given in (2.4) avoids the first order contributions in (2.6). Also, appealing to the above proposition, we can rewrite the momentum constraint in (2.1) as

$$\operatorname{div}_\gamma \widetilde{K} - \left(\frac{n-1}{n} d\tau + J \right) \varphi^{\frac{2n}{n-2}} = 0. \tag{2.7}$$

We have therefore, thus far, rewritten the constraints (2.1) as the system

$$\Delta_\gamma \varphi - c_n R_\gamma \varphi + c_n |\widetilde{K}|^2_\gamma \varphi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \varphi^{\frac{n+2}{n-2}} = 0,$$
$$\operatorname{div}_\gamma \widetilde{K} - \left(\frac{n-1}{n} d\tau + J \right) \varphi^{\frac{2n}{n-2}} = 0, \tag{2.8}$$

where, above, the input geometric data would be the metric γ, which fixes the conformal class of physical metric g, the mean curvature τ as well as the physical

information ϵ and J. Then, the equations (2.8) are posed for the conformal factor u and the traceless tensor $\tilde{K}$. Let us point out that in the case of vacuum ($\epsilon, J = 0$) *maximal* ($\tau = 0$) initial data, the system (2.8) decouples. In such a case, we must first find a traceless tensor which is γ-divergence free (such tensors are called TT-tensors),[3] which works as an input in the resulting equation for the conformal factor. In such a case, all of the analysis falls on the associated Lichnerowicz equation. On the other hand, for non-vacuum and/or non-maximal solutions, in general, we have coupled system.

In order to get the final decomposition for K associated with the conformal method we need to impose one further splitting for $\tilde{K}$. We will introduce such final decomposition assuming that M is closed, since in that case it follows naturally.

Let (M^n, γ) be a Riemannian manifold as above, with $n \geqslant 3$, and assume that $\gamma \in W^{2,p}$, with $p > \frac{n}{2}$. Then, define the conformal Killing Laplacian (CKL) operator

$$\begin{aligned}
\Delta_{\gamma,\mathrm{conf}} : W^{2,p}(TM) &\mapsto L^p(T^*M), \\
X &\mapsto \mathrm{div}_\gamma \left(\mathscr{L}_{\gamma,\mathrm{conf}} X \right),
\end{aligned} \tag{2.9}$$

where $\mathscr{L}_{\gamma,\mathrm{conf}} X \doteq \mathscr{L}_X \gamma - \frac{2}{n} \gamma \, \mathrm{div}_\gamma X$ stands for the conformal Lie derivative, whose kernel is given by conformal Killing fields (CKF) of the metric γ. Also, let us recall from Appendix B that (2.9) is an elliptic operator. In this context, the following theorem follows the lines of Berger and Ebin (1969) and it was introduced in the context of general relativity by York Jr. (1974).

Theorem 2.1.1. *Let (M^n, γ) be a smooth closed Riemannian manifold, $n \geqslant 3$. Then, for any $1 < p < \infty$, the following splitting holds*

$$W^{1,p}(\overset{\circ}{S_2}M) = \mathrm{Ker}\,(L_1) \oplus \mathrm{Im}\,(L_2), \tag{2.10}$$

where $L_1 : W^{1,p}(\overset{\circ}{S_2}M) \mapsto L^p(T^*M)$ is given by $L_1 W \doteq \mathrm{div}_\gamma W$, while $L_2 : W^{2,p}(TM) \mapsto W^{1,p}(\overset{\circ}{S_2}M)$ is given by $L_2 X = \mathscr{L}_{\gamma,conf} X$, and we have denoted by $\overset{\circ}{S_2}M$ the vector bundle whose fibres consist of traceless symmetric $(0,2)$ tensor fields on M.

Proof. Let us first notice that $\Delta_{\gamma,\mathrm{conf}} = L_1 \circ L_2$ and thus, appealing to Theorem A.1.1, we need to show that $\mathrm{Ker}(\Delta_{\gamma,\mathrm{conf}}) = \mathrm{Ker}\,(L_2)$ and $\mathrm{Im}(\Delta_{\gamma,\mathrm{conf}}) = $

---

[3]TT stands for traceless and transverse tensors.

$\mathrm{Im}(L_1)$. In order to deal with the first identity, it is clear that $\mathrm{Ker}(\Delta_{\gamma,\mathrm{conf}}) \supset \mathrm{Ker}(L_2)$ and therefore we just need to show the other opposite inclusion. Nevertheless, the Kernel of $\Delta_{\gamma,\mathrm{conf}} : W^{2,p} \mapsto L^p$ is actually smooth by Theorem B.4. Thus, through Theorem B.8, $X \in \mathrm{Ker}\,\Delta_{\gamma,\mathrm{conf}}$ iff X is a conformal Killing field of γ, which is equivalent to $X \in \mathrm{Ker}(L_2)$. Similarly, the inclusion $\mathrm{Im}(\Delta_{g,\mathrm{conf}}) \subset \mathrm{Im}(L_1)$ is trivial, so we need only consider the opposite one. In this case, from Theorem A.1.3, $X \in \mathrm{Im}(\Delta_{\gamma,\mathrm{conf}})$ iff $X \in \mathrm{Ker}^{\perp}(\Delta^*_{\gamma,\mathrm{conf}})$, where $\Delta^*_{\gamma,\mathrm{conf}} : L^{p'} \mapsto W^{-2,p'}$. In particular, any $X \in \mathrm{Ker}^{\perp}(\Delta^*_{\gamma,\mathrm{conf}})$ is an $L^{p'}$-weak solution to $\Delta_{\gamma,\mathrm{conf}}X = 0$ and is therefore smooth due to elliptic regularity results. In particular, this again implies through Theorem B.8 that any such X is a smooth CKF of γ. Let us then consider $Y \in \mathrm{Im}(L_1)$. That is

$$Y = \mathrm{div}_{\gamma} U,$$

for some $U \in W^{2,p}(\overset{\circ}{S_2}M)$. Then, for any $X \in \mathrm{Ker}(\Delta^*_{g,\mathrm{conf}})$, it follows that

$$\begin{aligned}
\int_M \langle X, Y \rangle_{\gamma}\, dV_{\gamma} &= -\int_M \langle DX, U \rangle_{\gamma}\, dV_{\gamma}, \\
&= -\frac{1}{2}\int_M \langle \mathscr{L}_X \gamma, U \rangle_{\gamma}\, dV_{\gamma}, \\
&= -\frac{1}{2}\int_M \langle \mathscr{L}_{\gamma,\mathrm{conf}} X, U \rangle_{\gamma}\, dV_{\gamma}, \\
&= 0,
\end{aligned}$$

where the first identity comes from integration by parts, the second one from the symmetry of U, the third one from its traceless property and the final one from X being a CFK. Therefore $\mathrm{Im}(L_1) \subset \mathrm{Ker}^{\perp}(\Delta^*_{g,\mathrm{conf}}) = \mathrm{Im}(\Delta_{g,\mathrm{conf}})$, which finishes the proof. $\qquad\square$

Therefore, at least for smooth data γ on closed manifolds, we can always split the traceless-part of our extrinsic data via

$$\widetilde{K} = \mathscr{L}_{\gamma,\mathrm{conf}} X + U, \tag{2.11}$$

where X is a vector field and U is the TT-part associated to it by the above theorem. We will keep this decomposition for γ in any regularity and even for M non-compact as an ad hoc one, although analogue versions can be established in some of these related situations (see Cantor 1981). Using (2.11) as the imposed

decomposition for $\widetilde{K}$ in (2.8), we finally arrive at the standard form of the conformally formulated Gauss–Codazzi constraints, explicitly given by

$$\Delta_\gamma \varphi - c_n R_\gamma \varphi + c_n |\widetilde{K}|_\gamma^2 \varphi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \varphi^{\frac{n+2}{n-2}} = 0,$$

$$\Delta_{\gamma,\mathrm{conf}} X - \left(\frac{n-1}{n} d\tau + J \right) \varphi^{\frac{2n}{n-2}} = 0.$$

(2.12)

Above, fixing a given physical model determining the form of the sources ϵ and J, the above equations form an elliptic system posed for (φ, X) with geometric data $\mathcal{I} \doteq (\gamma, \tau, U)$, where γ is a fixed $W^{2,p}$-Riemannian metric, τ a fixed $W^{1,p}$-function standing for the mean curvature of the initial set and U is a fixed $W^{1,p}$ γ-TT tensor.

Let us point out that the physical sources will typically inherit some natural scaling under conformal transformation of the initial data (g, K). This is easy to realise in the models introduced in Section 1.3. The dependence of the corresponding energy-momentum tensor on the space-time metric $\bar{g}$ will naturally induce some scaling for (ϵ, J). Below, we will make this precise for some cases of interest, but, before, let us highlight the following interesting case.

Definition 2.1.1. *We will say that the physical sources (ϵ, J) in an initial data set (g, K, ϵ, J) are York-scaled if, under the conformal decomposition of (g, K) described above, their scaling on the initial data set induces a change in the momentum density of the form $J = \varphi^{-\frac{2n}{n-2}} \widetilde{J}$, where $\widetilde{J}$ is a 1-form constructed with the conformal data (γ, τ, U) plus additional prescribed data.*

The feature that makes York-scaled sources special is that, under an additional CMC-condition, they transform the conformally formulated momentum constraint into

$$\Delta_{\gamma,\mathrm{conf}} X = \widetilde{J},$$

(2.13)

which is completely decoupled from the associated Lichnerowicz equation. Therefore, in some sense, this generalises the CMC vacuum case mentioned above. In this case, appealing to the analysis in Appendix B, we can deal with this linear PDE, solve for X, which completes all the information in $\widetilde{K}$, and then, once more, the core of the analysis falls on the corresponding Lichnerowicz equation.

2.1.1   Some Model Sources

Let us now analyse the explicit form of the induced energy-momentum densities in a few cases which will appear in the following sections and chapters, together with their conformal scaling in the above decomposition. During such decompositions, we will adopt the convention of putting *tildes* on top of freely specified quantities on (M, γ), which, after some scaling, correspond to their physical counterparts on the initial data set (M, g).

Remark 2.1.1. *Since notation will get heavier as we move along, we will avoid putting bars on top of space-time quantities when there is no danger of confusion between them and their evaluations on M at $t = 0$. Also, we will use the notations related to the space-time splitting introduced in Section 1.4, but, in order to avoid confusion between the shift vector and the vector field X introduced in the conformal splitting of the TT part of K, we will denote the shift vector by β.*

Scalar fields

Recall from Chapter 1 that the energy-momentum tensor of a (real) scalar field ϕ on a space-time $(M \times \mathbb{R}, \bar{g})$ with self-interacting potential $V : \mathbb{R} \mapsto \mathbb{R}$, is given by

$$T = d\phi \otimes d\phi - \frac{1}{2}|d\phi|^2_{\bar{g}}\bar{g} - V(\phi)\bar{g}. \tag{2.14}$$

From this, we immediately get $T(n, X) = d\phi(n)d\phi(X)$ for all $X \in \Gamma(TM)$, which implies $J = -d\phi(n)d\phi$. In this context, we denote by $\pi \doteq d\phi(n)$, so that

$$J = -\pi d\phi. \tag{2.15}$$

Similarly,

$$\epsilon = \frac{1}{2}\left(\pi^2 + |\nabla\phi|^2_g\right) + V(\phi). \tag{2.16}$$

Notice that in order to analyse the scaling properties of (2.15)-(2.16), we need to analyse the scaling of the normal vector n to M, which basically depends on the scaling of the lapse function (see Section 1.4). In fact, since $n = N^{-1}(\partial_t - \beta)$, with $\beta = (\partial_t)^\top$, we see that β is independent of the conformal class of $\bar{g}$ since ∂_t is. On the other hand, our conformal transformations on the geometric initial data (g, K) do not, a priori, imply any specific transformation rule for the lapse function.

In fact, recalling that the choice of initial data for N is a *gauge choice*, we have some freedom to impose a convenient transformation rule. Although imposing N^2 to scale in the same way as g might seem a first sensible possibility, the choice

$$N \doteq \varphi^{\frac{2n}{n-2}} \tilde{N}, \tag{2.17}$$

turns out to be more convenient in several situations, where $\tilde{N}$ is the freely specified function on the initial data set. This choice implies $n = \varphi^{-\frac{2n}{n-2}} \tilde{n}$, with $\tilde{n} \doteq \tilde{N}^{-1}(\partial_t - \beta)$ and therefore

$$\epsilon_\phi = \frac{1}{2} \left(\varphi^{-\frac{4n}{n-2}} \tilde{\pi}^2 + \varphi^{-\frac{4}{n-2}} |\nabla \phi|^2_\gamma \right) + V(\phi),$$
$$J_\phi = -\varphi^{-\frac{2n}{n-2}} \tilde{\pi} d\phi, \tag{2.18}$$

where $\tilde{\pi} \doteq d\phi(\tilde{n})$. Denoting $\tilde{J} \doteq \tilde{\pi} d\phi$, we see that (with the choice (2.17)) the initial data of a self-interacting real scalar field are York-scaled.

Fluid sources

Let us consider a perfect fluid, of the kind introduced in Chapter 1, which is described by an energy-momentum tensor field of the form

$$T = (\mu + p)u^\flat \otimes u^\flat + p\bar{g}, \tag{2.19}$$

where μ, p are scalar functions on space-time denoting the mass and pressure densities of the fluid, while u denotes the fluid's velocity field. It is straightforward to compute

$$J_i = -(\mu + p)\bar{g}(u, n)u_i = Nu^0(\mu + p)u_i,$$
$$\epsilon = (\mu + p)(\bar{g}(u, n))^2 - p = (\mu + p)(Nu^0)^2 - p. \tag{2.20}$$

In order to analyse the scaling properties of these fields, let us first notice that the fluid's velocity field is subject to the normalisation condition $\bar{g}(u, u) = -1$, which makes it metric-dependent. In an adapted frame, we have that

$$-(\bar{N}u^0)^2 + \bar{g}_{ij}u^i u^j = -1. \tag{2.21}$$

We have to give initial data for both u^0 and u^i, where their combination must satisfy the above relation at $t = 0$. In particular, considering the conformal scaling for g, we can put

$$\tilde{u}^i = \varphi^{\frac{2}{n-2}} u^i, \tag{2.22}$$

which implies that $g_{ij}u^i u^j = \gamma_{ij}\tilde{u}^i \tilde{u}^j$. Picking the scaling for u^0 satisfying

$$(Nu^0)^2 = 1 + |\tilde{u}|^2_\gamma, \tag{2.23}$$

we will get a pair consisting of a scalar and vector field on M satisfying (2.21). In particular, for the choice of densitised lapse (2.17), we get

$$u^0 = \varphi^{-\frac{2n}{n-2}}\tilde{u}^0, \tag{2.24}$$

so that the pair $(\tilde{u}^0, \tilde{u})$ is of unit speed with respect to $\tilde{g} = -\tilde{N}^2\theta^0 \otimes \theta^0 + \gamma$. Using these conventions, we find

$$J_i = \varphi^{\frac{2}{n-2}}\tilde{N}\tilde{u}^0(\mu + p)\tilde{u}_i = \varphi^{\frac{2}{n-2}}\left(1 + |\tilde{u}|^2_\gamma\right)^{\frac{1}{2}}(\mu + p)\tilde{u}_i,$$
$$\epsilon = (\mu + p)(\tilde{N}\tilde{u}^0)^2 - p = (\mu + p)\left(1 + |\tilde{u}|^2_\gamma\right) - p. \tag{2.25}$$

As they stand, the above sources are not York-scaled. Clearly, we can impose any ad hoc convenient scaling we want for μ and p, for instance to force J to be York-scaled and induce some scaling in ϵ. Nevertheless, if such a scaling is not well-motivated, maybe via some microscopic theory that gives us μ and p as the correct averaged macroscopic data, it might be desirable to have the freedom to control the final densities. Notice that if we impose some scaling for μ and p and write these quantities in terms of some prescribed densities $\tilde{\mu}$ and $\tilde{p}$, unless we have some a priori strong control on φ, we lose any kind of control on the behaviour of μ and p, which might be an undesirable feature in physical modelling. Thus, if we attempt to gain such a freedom when modelling the sources, the momentum constraint will not decouple from the hamiltonian constraint, *even under CMC assumptions*. We will follow this last choice.

Electromagnetic sources

Let us now analyse the electromagnetic case, described by the energy momentum tensor

$$T_{\alpha\beta} = F_\alpha{}^\mu F_{\mu\beta} - \frac{1}{4}\bar{g}_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}.$$

Straightforward computations of the kind done in Section 1.2, give us

$$\epsilon = \frac{1}{2}\left(|E|^2_g + \frac{1}{2}|\tilde{F}|^2_g\right),$$
$$J_k = -\tilde{F}_{ik}E^i. \tag{2.26}$$

   We can analyse how these sources scale under conformal transformations. First, since $\tilde{F}$ is metric independent, it does not scale with conformal transformations. On the other hand, from the definition of the electric field, if the initial data N for the lapse function $\bar{N}$ scales, then E will also scale. For the choice of the densitized (2.17), we fix the electric field scaling is as follows

$$E^i = \varphi^{-\frac{2n}{n-2}} \tilde{E}^i. \tag{2.27}$$

Above, we regard $\tilde{E}$ as the variable we have to solve for in the electric constraint in (1.69). Taking into consideration (2.27), we see that the conformally formulated energy momentum sources of the electromagnetic field scale as

$$\epsilon = \frac{1}{2}\left(|\tilde{E}|^2_\gamma \varphi^{-4\frac{n-1}{n-2}} + \frac{1}{2}|\tilde{F}|^2_\gamma \varphi^{-\frac{8}{n-2}}\right),$$
$$J_k = -\tilde{F}_{ik}\tilde{E}^i \varphi^{-\frac{2n}{n-2}}, \tag{2.28}$$

which shows that the momentum J is York-scaled. Also, all this implies that

$$\mathrm{div}_g\, E = \varphi^{-\frac{2n}{n-2}}\mathrm{div}_g\, \tilde{E} - \frac{2n}{n-2}\varphi^{-\frac{2n}{n-2}-1}D_i\varphi\tilde{E}^i,$$
$$= \varphi^{-\frac{2n}{n-2}}\left(\mathrm{div}_\gamma\, \tilde{E} + \frac{2n}{n-2}\varphi^{-1}D_i\phi\tilde{E}^i\right) - \frac{2n}{n-2}\phi^{-\frac{2n}{n-2}-1}D_i\varphi\tilde{E}^i,$$
$$= \varphi^{-\frac{2n}{n-2}}\mathrm{div}_\gamma\, \tilde{E}.$$

This shows that in the electro-vacuum case, the constraint

$$\mathrm{div}_g\, E = 0$$

is conformally invariant. Thus, in such a case, in the conformally formulated version of the system (1.69), the electric constraint reads $\mathrm{div}_\gamma\, \tilde{E} = 0$. Therefore, in this situation and for this particular scaling for the lapse function, the electromagnetic constraints decouple from the Gauss–Codazzi constraints. Then, we can first choose $\tilde{E}$ being γ-divergence-free and $\tilde{F}$ being closed, which can be solved independently, and then put this information in the Gauss–Codazzi constraints, where now $\tilde{E}$ and $\tilde{F}$ become data. On the other hand, if we have electromagnetic sources, then we need to analyse how they scale and solve for the electric constraint coupled to the conformally formulated Gauss–Codazzi constraints.

Charged fluids

In view of the discussion presented above, let us formulate the conformal problem associated to the full constraint system (1.69) for a charged fluid. In this case, we assume the energy momentum tensor is given by

$$T = (\mu + p)u^\flat \otimes u^\flat + p\bar{g} + T^{EM}, \qquad (2.29)$$

which implies

$$\epsilon = (\mu + p)\left(1 + |\tilde{u}|_\gamma^2\right) - p + \frac{1}{2}\left(|\tilde{E}|_\gamma^2 \varphi^{-4\frac{n-1}{n-2}} + \frac{1}{2}|\tilde{F}|_\gamma^2 \varphi^{-\frac{8}{n-2}}\right),$$

$$J_i = \varphi^{\frac{2}{n-2}}\left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}}(\mu + p)\tilde{u}_i - \tilde{F}_{ik}\tilde{E}^i \varphi^{-\frac{2n}{n-2}}. \qquad (2.30)$$

Also, in this context, we need to evaluate how the current $\mathcal{J}$ scales. In fact, from (1.69), we need to consider the scaling of $\rho = -\mathcal{J}^\flat(n)$. Typically, one writes the current density associated to the flow of charged particles with velocity field u as

$$\mathcal{J} = qu + j, \qquad (2.31)$$

where qu is referred to as the *convective current*, with q the *proper charge density* of the fluid, while $j \perp u$ as the *conductive current*, which, for instance, for conductive fluids with linear response can be written as $j = \sigma E_u$, where σ is the *conductivity* of the fluid and E_u denotes the electric field as observed by u. In what follows, we shall assume that $j = 0$ and therefore the fluid has zero conductivity and we may write $\mathcal{J} = qu$, implying[4]

$$\rho = -\mathcal{J}^\flat(n) = -\bar{g}(\mathcal{J}, n) = qNu^0 = q\left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}}. \qquad (2.32)$$

Putting (2.32) together with the scaling $\mathrm{div}_g\, E = \varphi^{-\frac{2n}{n-2}}\mathrm{div}_\gamma\, \tilde{E}$, we see that the electric constraint reads as follows.

$$\mathrm{div}_\gamma\, \tilde{E} = q(1 + |\tilde{u}|_\gamma^2)^{\frac{1}{2}}\phi^{\frac{2n}{n-2}}, \qquad (2.33)$$

where we have chosen not to scale the charge density q under the same arguments as with μ and p.

---

[4]In a more general case we should add a contribution from the conductive current, which would further couple the electric field. For instance, if $j = \sigma E_u$, then $-\mathcal{J}^\flat(n) = \sigma F(n, u) = -\sigma u^i E_i$. Along these lines, we could also notice from Section 1.4 that our choice of space-time splitting is arbitrary, and if we chose to evolve the initial data along u, that is $n = u$, then in all the above expressions we would find $u^0 = 1$, $u^i = 0$ which could be used to simplify the analysis of a conductive fluid.

Full system of constraints

Above, we presented the analysis related to some of the most common *classical* physical sources of energy-momentum densities. Scalar fields find applications in different scenarios in astrophysics and cosmology, for instance within inflationary cosmology, and are typically a test case for more elaborated constructions. Fluid sources are the fundamental tool to model continuous matter distributions, ranging from star-modelling to cosmological models for the Universe. In particular, perfect fluids find several applications, for instance, within homogeneous and isotropic cosmological models. Finally, electromagnetism represents the second (besides gravitation) *fundamental* classical field in physics. Therefore, putting of all of above together we can present a quite general situation given by a charged perfect fluid which couples with a self-interacting scalar field, besides the electromagnetic one. For such a system, we have found that

$$
\epsilon = \frac{1}{2} \left(\varphi^{-\frac{4n}{n-2}} \tilde{\pi}^2 + \varphi^{-\frac{4}{n-2}} |\nabla \phi|_\gamma^2 \right) + \frac{1}{2} \left(|\tilde{E}|_\gamma^2 \varphi^{-4\frac{n-1}{n-2}} + \frac{1}{2} |\tilde{F}|_\gamma^2 \varphi^{-\frac{8}{n-2}} \right)
$$
$$
+ V(\phi) + (\mu + p)\left(1 + |\tilde{u}|_\gamma^2\right) - p,
$$
$$
J_i = -\varphi^{-\frac{2n}{n-2}} \tilde{\pi} d\phi - \tilde{F}_{ik} \tilde{E}^i \varphi^{-\frac{2n}{n-2}} + \varphi^{\frac{2}{n-2}} \left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}} (\mu + p)\tilde{u}_i,
$$
$$
\tag{2.34}
$$

and therefore, we can write down the associated conformally formulated system (1.69) as

$$
\begin{aligned}
&\Delta_\gamma \varphi - f(\varphi, X, \tilde{E}, \tilde{F}) = 0, \\
&\Delta_{\gamma,\text{conf}} X - \mathcal{F}(\varphi, X, \tilde{E}, \tilde{F}) = 0, \\
&\text{div}_\gamma \tilde{E} - q(1 + |\tilde{u}|_\gamma^2)^{\frac{1}{2}} \varphi^{\frac{2n}{n-2}} = 0, \\
&d\tilde{F} = 0,
\end{aligned}
\tag{2.35}
$$

where, denoting by $\psi \doteq (\varphi, X, \tilde{E}, \tilde{F})$, we have defined

$$
f(\psi) \doteq r\varphi - c_n \left(|\tilde{K}|_\gamma^2 + \tilde{\pi}^2 \right) \varphi^{-\frac{3n-2}{n-2}} + a_\tau \varphi^{\frac{n+2}{n-2}} - c_n |\tilde{E}|_\gamma^2 \varphi^{-3} - \frac{c_n}{2} |\tilde{F}|_\gamma^2 \varphi^{\frac{n-6}{n-2}},
$$
$$
\mathcal{F}(\psi) \doteq \omega_\phi - \tilde{E}\lrcorner \tilde{F} + \omega_\tau \varphi^{\frac{2n}{n-2}} + \omega_\mu \varphi^{2\frac{n+1}{n-2}},
$$

with

$$r \doteq c_n \left(R_\gamma - |\nabla \phi|_\gamma^2 \right), \quad a_\tau \doteq \frac{n-2}{4n}\tau^2 - 2c_n\epsilon_0, \quad \omega_\mu \doteq \left(1 + |\widetilde{u}|_\gamma^2 \right)^{\frac{1}{2}} (\mu + p)\widetilde{u}^\flat,$$

$$\epsilon_0 \doteq V(\phi) + (\mu + p)\left(1 + |\widetilde{u}|_\gamma^2 \right) - p, \quad \omega_\phi \doteq -\widetilde{\pi}d\phi, \quad \omega_\tau \doteq \frac{n-1}{n}d\tau,$$

Notice that the only equation that always decouples in the above system is the magnetic constraint $d\widetilde{F} = 0$, which demands us to chose a closed 2-form on M. After that, $\widetilde{F}$ becomes a datum in the remaining system, and therefore we will disregard this last equation.

On the other hand, the remaining system, consisting on the first three equations in (2.35) will in general be completely coupled, *even under a CMC assumption*, since the presence of non-York scaled momenta such as ω_μ couples the momentum constraint with the conformal factor explicitly, but also the York-scaled momentum $\widetilde{F}_{ij}\widetilde{E}^i$ couples the momentum constraint with the electric one, which is explicitly coupled with the conformal factor. Therefore, unless we decide to neglect all fluid contribution, that is $\mu, p, q = 0$, the system will be fully coupled. In the case we *switch-off* the fluid's contributions and adopt the CMC hypothesis, we obtain the decoupled system given by

$$\Delta_\gamma\varphi - r\varphi + a_{TT}\varphi^{-\frac{3n-2}{n-2}} - a_\tau\varphi^{\frac{n+2}{n-2}} + a_E\varphi^{-3} + a_{\widetilde{F}}\varphi^{\frac{n-6}{n-2}} = 0,$$

$$\Delta_{\gamma,\mathrm{conf}}X = \omega_\phi - \widetilde{E}\lrcorner\widetilde{F},$$

$$\mathrm{div}_\gamma\widetilde{E} = 0,$$

$$d\widetilde{F} = 0$$

$$(2.36)$$

where we have introduced the additional notations

$$a_{TT} \doteq c_n \left(|\widetilde{K}|_\gamma^2 + \widetilde{\pi}^2 \right), \quad a_E \doteq c_n|\widetilde{E}|_\gamma^2, \quad a_{\widetilde{F}} \doteq \frac{c_n}{2}|\widetilde{F}|_\gamma^2.$$

System (2.36) is completely decoupled. We must first choose a closed 2-form $\widetilde{F}$ and a divergence-free vector field $\widetilde{E}$ on (M, γ) to satisfy the source-free electromagnetic constraints. Then, the right-hand side in the momentum constraint becomes a source, fixed by the chosen free data. The well-posedness of the system relies on this last equation being solvable. Assuming it is, we find X and then the TT-part of the extrinsic curvature becomes fixed by the free data, which implies that all of the coefficients appearing in the associated Lichnerowicz equation are fixed by the free data, and the remaining work must be devoted to the analysis of this semi-linear PDE.

2.1.2 Conformal covariance

It may seem intuitive that there should be some nice relation between two different conformal initial data sets built from conformally related metrics γ and $\gamma' = \theta^{\frac{4}{n-2}}\gamma$. This can be explicitly seen to be the case as follows. First, notice that associated to conformal data $\vartheta \doteq (\gamma, \tau, U, \tilde{N}, \phi, \tilde{\pi}, \mu, p, \tilde{u}^0, \tilde{u}, \tilde{F})$, we have the solution $(\varphi, X, \tilde{E})$, from which we construct the physical initial data

$$g = \varphi^{\frac{4}{n-2}}\gamma, \quad K = \varphi^{-2}\left(\mathscr{L}_{\gamma,\text{conf}}X + U\right) + \frac{\tau}{n}g, \quad E = \varphi^{-\frac{2n}{n-2}}\tilde{E},$$
$$N = \varphi^{\frac{2n}{n-2}}\tilde{N}, \quad \pi = \varphi^{-\frac{2n}{n-2}}\tilde{\pi}, \quad u = \varphi^{-\frac{2n}{n-2}}\tilde{u}^0 e_0 + \varphi^{-\frac{2}{n-2}}\tilde{u}, \tag{2.37}$$

and we have adopted the notation $\tilde{K} \doteq \mathscr{L}_{\gamma,\text{conf}}X + U$. Such a solution to the constraint equations solves the system

$$L_\gamma\varphi = -c_n|d\phi|_\gamma^2\varphi - c_n\left(|\tilde{K}|_\gamma^2 + \tilde{\pi}^2\right)\varphi^{-\frac{3n-2}{n-2}} + a_\tau\varphi^{\frac{n+2}{n-2}} - c_n|\tilde{E}|_\gamma^2\varphi^{-3}$$
$$\qquad - \frac{c_n}{2}|\tilde{F}|_\gamma^2\varphi^{\frac{n-6}{n-2}},$$
$$\text{div}_\gamma\tilde{K} = -\tilde{\pi}d\phi - \tilde{E}\lrcorner\tilde{F} + \omega_\tau\varphi^{\frac{2n}{n-2}} + \left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}}(\mu + p)\tilde{u}^\flat\varphi^{2\frac{n+1}{n-2}},$$
$$\text{div}_\gamma\tilde{E} = q(1 + |\tilde{u}|_\gamma^2)^{\frac{1}{2}}\varphi^{\frac{2n}{n-2}},$$
$$d\tilde{F} = 0, \tag{2.38}$$

and associated to each solution of such a system with $\varphi > 0$, we have the physical data (2.37) solving the physical constraint equations (1.69) with sources given by a charged perfect fluid interacting with a scalar field. Now, let us consider the conformally related data

$$\gamma' \doteq \theta^{\frac{4}{n-2}}\gamma, \quad U' \doteq U, \quad \tau' \doteq \tau, \quad \tilde{N}' \doteq \theta^{\frac{2n}{n-2}}\tilde{N}, \quad \phi' \doteq \phi, \quad \tilde{\pi}' \doteq \theta^{-\frac{2n}{n-2}}\tilde{\pi},$$
$$\mu' \doteq \mu, \quad p' \doteq p, \quad \tilde{u}'^0 \doteq \theta^{-\frac{2n}{n-2}}\tilde{u}^0, \quad \tilde{u}'^i \doteq \theta^{-\frac{2}{n-2}}\tilde{u}^i. \tag{2.39}$$

Below, we will show that $\psi \doteq (\varphi, \tilde{K}, \tilde{E}, \tilde{F})$ is a solution of (2.38) associated to free data given by ϑ iff $\psi' \doteq (\varphi', \tilde{K}', \tilde{E}', \tilde{F}')$ is a solution of the same system associated to free data ϑ' related to ϑ via (2.39), where

$$\varphi' \doteq \theta^{-1}\varphi, \quad \tilde{K}' \doteq \theta^{-2}\tilde{K}, \quad \tilde{E}' \doteq \theta^{-\frac{2n}{n-2}}\tilde{E}, \quad \tilde{F}' \doteq \tilde{F}. \tag{2.40}$$

In either case, the physical solution is constructed from (ψ, ϑ) (or (ψ', ϑ')) via the relations (2.37) using primed or unprimed variables consistently. That is,

$$g' = \varphi'^{\frac{4}{n-2}} \gamma' = \varphi^{\frac{4}{n-2}} \gamma = g, \quad K' = \varphi'^{-2} \tilde{K}' + \frac{\tau'}{n} g' = \varphi^{-2} \tilde{K} + \frac{\tau}{n} g = K,$$

$$N' = \varphi'^{\frac{2n}{n-2}} \tilde{N}' = \varphi^{\frac{2n}{n-2}} \tilde{N} = N, \quad \pi' = \varphi'^{-\frac{2n}{n-2}} \tilde{\pi}' = \varphi^{-\frac{2n}{n-2}} \tilde{\pi} = \pi,$$

$$u' = \varphi'^{-\frac{2n}{n-2}} \tilde{u}'^0 e_0 + \varphi'^{-\frac{2}{n-2}} \tilde{u}' = \varphi^{-\frac{2n}{n-2}} \tilde{u}^0 e_0 + \varphi^{-\frac{2}{n-2}} \tilde{u} = u,$$

$$E' = \varphi'^{-\frac{2n}{n-2}} \tilde{E}' = \varphi^{-\frac{2n}{n-2}} \tilde{E} = E, \quad \tilde{F}' = \tilde{F}.$$

$$(2.41)$$

Therefore, the *physical* solution is the same in both cases. That is, we find an action of the conformal group on the conformal data (ψ, ϑ), which makes it a kind of *gauge* group.

With the above in mind, let us first present the following computational result.

Proposition 2.1.3. *Let us consider a Riemannian manifold (M^n, γ), $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and a conformally related Riemannian metric $\gamma' = \theta^{\frac{4}{n-2}} \gamma$ with $\theta \in W^{2,p}$. The conformal Laplacian operators L_γ and $L_{\gamma'}$ associated to γ and γ' respectively satisfy the following conformal covariance property:*

$$L_\gamma \varphi = \theta^{\frac{n+2}{n-2}} L_{\gamma'} \varphi' \ \forall \ \varphi \in W^{2,p}. \tag{2.42}$$

where $\varphi' = \theta^{-1} \varphi \in W^{2,p}$.

Proof. First, notice that L_γ and $L_{\gamma'}$ are well-defined mappings from $W^{2,p} \mapsto L^p$ (see Appendix B). Furthermore, since $W^{2,p}$ is an algebra under point wise multiplication under our hypotheses, we have $\varphi' \in W^{2,p}$. Now, the proof follows by straightforward local computations. First, notice that the following formulae hold:

$$\nabla'_i \varphi' = \theta^{-1} \nabla'_i \varphi - \theta^{-2} \varphi \nabla'_i \theta,$$

$$\Delta_{\gamma'} \varphi' = \theta^{-1} \left(\Delta_{\gamma'} \varphi - \theta^{-1} \left(\varphi \Delta_{\gamma'} \theta + 2\langle \nabla \theta, \nabla \varphi \rangle_{\gamma'} \right) + 2\theta^{-2} \varphi |\nabla \theta|^2_{\gamma'} \right),$$

$$\Gamma^k_{ij}(\gamma') = \Gamma^k_{ij}(\gamma) + \frac{2}{n-2} \theta^{-1} \left(\partial_i \theta \delta^k_j + \partial_j \theta \delta^k_i - \gamma^{kl} \partial_l \theta \gamma_{ij} \right).$$

Therefore

$$\Delta_{\gamma'} \varphi = \theta^{-\frac{4}{n-2}} \Delta_\gamma \varphi + 2\theta^{-1} \langle \nabla \theta, \nabla \varphi \rangle_{\gamma'},$$

$$\Delta_{\gamma'} \theta = \theta^{-\frac{4}{n-2}} \Delta_\gamma \theta + 2\theta^{-1} |\nabla \theta|^2_{\gamma'},$$

and we recall from Proposition 2.1.1 that

$$R_{\gamma'} = \theta^{-\frac{n+2}{n-2}} \left(\theta R_\gamma - \frac{4(n-1)}{n-2} \Delta_\gamma \theta \right).$$

Putting all the above together, we find

$$L_{\gamma'} \varphi' = \theta^{-\frac{4}{n-2}-1} \Delta_\gamma \varphi - \theta^{-\frac{4}{n-2}-2} \varphi \Delta_\gamma \theta - \frac{n-2}{4(n-1)} \theta^{-\frac{n+2}{n-2}} R_\gamma \varphi + \theta^{-\frac{n+2}{n-2}-1} \varphi \Delta_\gamma \theta,$$

$$= \theta^{-\frac{n+2}{n-2}} \left(\Delta_\gamma \varphi - \frac{n-2}{4(n-1)} R_\gamma \varphi \right),$$

$$= \theta^{-\frac{n+2}{n-2}} L_\gamma \varphi,$$

which establishes the claim. $\qquad\square$

We can now use Propositions 2.1.2 and 2.1.3 to establish the following.

Lemma 2.1.1. *The tuple $(\varphi, \tilde{K}, \tilde{E}, \tilde{F})$ is a solution to (2.38) associated to the freely prescribed conformal data ϑ iff $(\varphi', \tilde{K}', \tilde{E}', \tilde{F}')$ is a solution to (2.38) associated to the freely prescribed data ϑ', where ϑ and ϑ' are related via (2.39), while $(\varphi', \tilde{K}', \tilde{E}', \tilde{F}')$ are related via (2.40).*

Proof. First, notice that the relations (2.39) imply

$$|\tilde{K}|^2_\gamma = \theta^{\frac{4n}{n-2}} |\tilde{K}'|^2_{\gamma'}, \ |\tilde{E}|^2_\gamma = \theta^{4\frac{n-1}{n-2}} |\tilde{E}'|^2_{\gamma'}, \ |\tilde{u}|^2_\gamma = |\tilde{u}'|^2_{\gamma'}$$

$$\tilde{u}_j = \theta^{-\frac{2}{n-2}} \tilde{u}'_j, \ \omega_\mu = \theta^{-\frac{2}{n-2}} \omega'_\mu.$$

Using the above relations together with (2.39), from Proposition 2.1.3, the following holds:

$$\theta^{\frac{n+2}{n-2}} L_{\gamma'} \varphi' = - c_n |\nabla \phi|^2_\gamma \varphi - c_n \left(|\tilde{K}|^2_\gamma + \tilde{\pi}^2 \right) \varphi^{-\frac{3n-2}{n-2}} + a_\tau \varphi^{\frac{n+2}{n-2}} - c_n |\tilde{E}|^2_\gamma \varphi^{-3}$$

$$- \frac{c_n}{2} |\tilde{F}|^2_\gamma \varphi^{\frac{n-6}{n-2}},$$

$$= - c_n \theta^{\frac{4}{n-2}} |d\phi|^2_{\gamma'} \varphi - c_n \theta^{\frac{4n}{n-2}} \left(|\tilde{K}'|^2_{\gamma'} + \tilde{\pi}'^2 \right) \varphi^{-\frac{3n-2}{n-2}} + a_\tau \varphi^{\frac{n+2}{n-2}}$$

$$- c_n \theta^{\frac{4n-4}{n-2}} |\tilde{E}'|^2_{\gamma'} \varphi^{-3} - \frac{c_n}{2} \theta^{\frac{8}{n-2}} |\tilde{F}'|^2_{\gamma'} \varphi^{\frac{n-6}{n-2}},$$

$$= \theta^{\frac{n+2}{n-2}} \left(- c_n |d\phi|^2_{\gamma'} \varphi' - c_n \left(|\tilde{K}'|^2_{\gamma'} + \tilde{\pi}'^2 \right) \varphi'^{-\frac{3n-2}{n-2}} + a_\tau \varphi'^{\frac{n+2}{n-2}} \right.$$

$$\left. - c_n |\tilde{E}'|^2_{\gamma'} \varphi'^{-3} - \frac{c_n}{2} |\tilde{F}'|^2_{\gamma'} \varphi'^{\frac{n-6}{n-2}} \right),$$

Similarly, from Proposition 2.1.2, we find

$$\theta^{\frac{2n}{n-2}} \operatorname{div}_{\gamma'} \widetilde{K}' = -\widetilde{\pi} \, d\phi - \widetilde{E} \lrcorner \widetilde{F} + \omega_\tau \varphi^{\frac{2n}{n-2}} + \omega_\mu \varphi^{2\frac{n+1}{n-2}},$$
$$= \theta^{\frac{2n}{n-2}} \left(-\widetilde{\pi}' \, d\phi - \widetilde{E}' \lrcorner \widetilde{F} + \omega_\tau \varphi'^{\frac{2n}{n-2}} + \omega'_\mu \varphi'^{2\frac{n+1}{n-2}} \right),$$

and we have already seen that the relation $\operatorname{div}_{\gamma'} \widetilde{E}' = \theta^{-\frac{2n}{n-2}} \operatorname{div}_\gamma \widetilde{E}$ holds, which implies

$$\theta^{\frac{2n}{n-2}} \operatorname{div}_{\gamma'} \widetilde{E}' = q(1 + |\widetilde{u}|^2_\gamma)^{\frac{1}{2}} \varphi^{\frac{2n}{n-2}} = \theta^{\frac{2n}{n-2}} q(1 + |\widetilde{u}|^2_\gamma)^{\frac{1}{2}} \varphi'^{\frac{2n}{n-2}}$$

Putting together the above relations proves the claim.   □

The above lemma becomes very useful, for instance, when we can find some *preferred* element in a conformal class which simplifies the problem (2.38). As we will see, this tends to be the case when we split the space of Riemannian metrics on M into its disjoint *Yamabe classes*. In such a case, our conformal class will belong to exactly one Yamabe class, and that allows us to select a conformal representative in $[\gamma]$ with fixed sign on the scalar curvature, which can be used to control the behaviour of L_γ as well as the existence of simple barriers for the Lichnerowicz equation, as we shall see in the next section. In such a case, we first fix a useful conformal representative to solve our problem, knowing that the final physical initial data will remain unaltered by these gauge choices.

2.2   CMC-solutions on closed manifolds

Let us now start with the analysis of the decoupled system (2.36), which corresponds to the choices $\tau = cte$ and $q, \mu, p \equiv 0$. Through this section we will consider M to be a closed n-dimensional manifold, with $n \geq 3$. Since the electromagnetic constraints are completely decoupled, we assume that we have fixed a priori a closed 2-form $\widetilde{F}$ and a γ-divergence-free vector field $\widetilde{E}$ together with the remaining free data $(\gamma, U, \tau, \phi, \widetilde{\pi})$. Let us first concentrate on the decoupled momentum constraint given by

$$\Delta_{\gamma,\text{conf}} X = -\widetilde{\pi} \, d\phi + \widetilde{E} \lrcorner \widetilde{F} \doteq \widetilde{J}. \tag{2.43}$$

Let us organise our functional hypotheses corresponding to the above equation in the following proposition.

Proposition 2.2.1. *Let (M^n, γ), $n \geqslant 3$, be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and assume that $\phi \in W^{2,p}$, $\widetilde{\pi}, \widetilde{E}, \widetilde{F} \in W^{1,p}$, with $p > \frac{n}{2}$. Then, $\widetilde{J}$ in equation (2.43) is in L^p.*

The proof of the above proposition is a straightforward application of the Sobolev multiplication properties.[5] With the aid of the above proposition and appealing to Theorem B.8, we see that (2.43) is solvable iff $\widetilde{J}$ is $L^2(M, dV_\gamma)$ orthogonal to the set of CKF of γ. Therefore, we obtain the following result.

Lemma 2.2.1. *Let (M^n, γ), $n \geqslant 3$, be a closed Riemannian manifold with $\gamma \in W^{2,p}$ with $p > \frac{n}{2}$ and assume that $\phi \in W^{2,p}$, $\widetilde{\pi}, \widetilde{E}, \widetilde{F} \in W^{1,p}$. Then, (2.43) is solvable for $X \in W^{2,p}$ iff $\widetilde{J}$ is L^2-orthogonal to the space of CKF of γ. In such a case, the solution is unique up to the addition a CKF.*

From the above lemma, we find a *minor* obstruction in order to be able to find solutions to the ECE in this setting, which is given by the necessity to choose $\phi, \widetilde{\pi}, \widetilde{E}$ and $\widetilde{F}$ so that $\widetilde{J} \perp \mathrm{Ker}(\Delta_{\gamma,\mathrm{conf}})$. This is not a dramatic constraint on the admissible free data, since it is also a condition that can be satisfied *generically*. This follows since, as might be expected, metrics with no non-trivial CKF are generic. This means that, within suitable topologies for the space of Riemannian metrics on M, metrics with no CKF can be shown to be dense. We refer the reader to Beig, Chruściel, and R. Schoen (2005) for detailed statements and also to Liimatainen and Salo (2012) for a proof in the case of closed manifolds and smooth metrics. Therefore, unless we are faced with a rather *exotic* situation, we will not face any obstructions of the sort described in the above lemma. In fact, we do not loose much generality in assuming that γ possesses no CKF from the start, from which we find unique solutions to (2.43) for any $\widetilde{J} \in L^p$. This kind of hypothesis will be exploited, for instance, in upcoming chapters.

Finally, let us also highlight that there are tangible geometric conditions which guarantee the non-existence of CKF on a closed Riemannian manifold. One such condition is that Riemannian metrics with negative definite Ricci tensor do not have any CKF (see Choquet-Bruhat 2009, page 203). A beautiful related result is that any manifold M^n, $n \geqslant 3$, carries a metric with negative definite Ricci tensor (see Lohkamp 1994, Theorem A).

In what follows we will focus on the associated Lichnerowicz equation, where the core of the analysis for this CMC problem relies.

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[5]See Appendix A.2.

The Lichnerowicz equation

We now need to consider the equation

$$\Delta_\gamma \varphi - a_r \varphi + a_{TT} \varphi^{-\frac{3n-2}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} + a_E \varphi^{-3} + a_{\widetilde{F}} \varphi^{\frac{n-6}{n-2}} = 0, \qquad (2.44)$$

where we recall the notations

$$a_r \doteq r = c_n \left(R_\gamma - |\nabla\phi|_\gamma^2 \right), \;\; a_\tau = \frac{n-2}{4n}\tau^2 - 2c_n V(\phi), \;\; a_{TT} \doteq c_n \left(|\tilde{K}|_\gamma^2 + \tilde{\pi}^2 \right),$$

$$a_E \doteq c_n |\tilde{E}|_\gamma^2, \;\; a_{\widetilde{F}} \doteq \frac{c_n}{2}|\tilde{F}|_\gamma^2.$$

Let us summarise our functional choices in the following proposition.

Proposition 2.2.2. *Let (M^n, γ), $n \geq 3$, be a Riemannian manifold with $\gamma \in W^{2,p}$ and assume that $\phi, X \in W^{2,p}$, $\tau, \tilde{\pi}, \tilde{E}, \tilde{F}, U \in W^{1,p}$ and $V : W^{2,p} \mapsto L^p$, with $p > \frac{n}{2}$. Then, all the coefficients a_I in (2.44) are in L^p.*

Proof. First, notice that $\gamma \in W^{2,p}$ implies $R_\gamma \in L^p$ through multiplication properties. Also, $\phi, X \in W^{2,p}$ imply $\nabla\phi, \mathscr{L}_{\gamma,\text{conf}} X \in W^{1,p}$ and the continuous multiplication property $W^{1,p} \otimes W^{1,p} \hookrightarrow L^p$, $p > \frac{n}{2}$, shows that the quadratic terms are in L^p. Finally, $V(\phi) \in L^p$ by definition of V as a map $W^{2,p} \mapsto L^p$. □

Remark 2.2.1. *Notice that the* choice $X \in W^{2,p}$ *is the natural choice from the other functional hypotheses put together with Lemma 2.2.1.*

With the above proposition in mind, we now aim to analyse a generic scalar equation on a closed Riemannian manifold (M^n, γ), $\gamma \in W^{2,p}$, of the form

$$\Delta_\gamma \varphi = \sum_I a_I \varphi^I, \qquad (2.45)$$

where the exponents I determine the type of non-linearities present in a specific problem, and we assume $a_I \in L^p$. In the following section, we will present the technical tools used to prove existence results to equations of this form.

2.2.1 The monotone iteration scheme

During this section we will describe an iterative method used to prove existence results associated to equations of the form of (2.45), which is based on the existence of *barrier* functions. Methods of this type are well-known within elliptic

PDE theory and encounter several applications in geometric analysis. In particular, they were introduced by Isenberg (1995) to analyse the Lichnerowicz equation associated to vacuum CMC initial data. Subsequently this method has gone through several improvements and generalisations, due to Choquet-Bruhat (2004) and Maxwell (2005a,b, 2006). Before establishing the main result of this section, given by Theorem 2.2.1 below, let us introduce the following forms of the maximum principle adapted to our regularity setting. In particular, in the next lemmas, we follow ideas of Maxwell (2005a,b, 2006).

Lemma 2.2.2 (Weak Maximum Principle). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Let us also consider a function $V \in L^p$ and assume that $V \geqslant 0$ a.e, $V \not\equiv 0$. Then, given $\varphi \in W^{2,p}$ the following implication holds*

$$\Delta_\gamma \varphi - V\varphi \geqslant 0 \implies \varphi \leqslant 0. \tag{2.46}$$

Proof. Let us consider the function $\varphi^+ \doteq \max\{\varphi, 0\}$ and then the equation

$$\varphi^+ \Delta_\gamma \varphi - V\varphi^+ \varphi \geqslant 0 \text{ a.e} \tag{2.47}$$

The idea is to integrate the above equation over M and then justify an integration by parts procedure. Let us concentrate in this last step. First, let us consider the cases where $\frac{n}{2} < p < n$ since they contain the general statement. Then, the first claim is that the embedding $W^{2,p} \hookrightarrow W^{1,2}$ holds. This is obvious for $n > 3$ and in the case $n = 3$ it follows from Theorem A.2.5 as long as $2 \leqslant \frac{np}{n-p}$, which is equivalent to $p \geqslant \frac{2n}{n+2}$. Since $\frac{n}{2} > \frac{2n}{n+2}$ for $n \geqslant 3$, then this embedding holds under our hypotheses. Now, notice that $\varphi \in W^{1,2}$ implies $\varphi^+ \in W^{1,2}$ (Kesavan 1989, see, for instance, Theorem 2.2.5). We furthermore claim that $W^{1,2} \hookrightarrow L^{p'}$. Once more appealing Theorem A.2.5, we know that this holds as long as $p' \leqslant \frac{2n}{n-2}$, which is equivalent to

$$\frac{1}{p'} = 1 - \frac{1}{p} \geqslant \frac{n-2}{2n} = \frac{1}{2} - \frac{1}{n} \iff \frac{1}{2} \geqslant \frac{1}{p} - \frac{1}{n} = \frac{n-p}{np} \iff 2 \leqslant \frac{np}{n-p},$$

which we already knows that hols under our present hypotheses, and therefore this second claim also follows. In particular, all this implies that $\varphi^+ \Delta_\gamma \varphi \in L^1(M)$.

Now, consider sequences $\{\varphi_k^+\}_{k=0}^\infty, \{\varphi_k\}_{k=0}^\infty \subset C^\infty(M)$ such that

$$\varphi_k^+ \xrightarrow{W^{1,2}} \varphi^+, \quad \varphi_k \xrightarrow{W^{2,p}} \varphi, \tag{2.48}$$

and then compute

$$\left| \int_M \varphi^+ \Delta_\gamma \varphi \, dV_\gamma - \int_M \varphi_k^+ \Delta_\gamma \varphi_k \, dV_\gamma \right| \leqslant \int_M |\varphi^+ - \varphi_k^+||\Delta_\gamma \varphi| dV_\gamma$$
$$+ \int_M |\varphi_k^+||\Delta_\gamma(\varphi - \varphi_k)| dV_\gamma,$$
$$\leqslant \|\varphi^+ - \varphi_k^+\|_{L^{p'}} \|\Delta_\gamma \varphi\|_{L^p}$$
$$+ \|\varphi_k^+\|_{L^{p'}} \|\Delta_\gamma(\varphi - \varphi_k)\|_{L^p}.$$

The right-hand side of the above expression goes to zero due to (2.48) put together with the embedding $W^{1,2} \hookrightarrow L^{p'}$ and the continuity of $\Delta_\gamma : W^{2,p} \mapsto L^p$ (see Proposition B.2). Therefore, we find that

$$\int_M \varphi^+ \Delta_\gamma \varphi \, dV_\gamma = \lim_{k \to \infty} \int_M \varphi_k^+ \Delta_\gamma \varphi_k \, dV_\gamma,$$
$$= -\lim_{k \to \infty} \int_M \langle \nabla \varphi_k^+, \nabla \varphi_k \rangle_\gamma \, dV_\gamma, \tag{2.49}$$

where in the last step, the integration by parts now follows from arguments of Theorem B.7. In order to establish that the right-hand side of the above expression converges to the corresponding limit, now notice that $W^{2,p} \hookrightarrow W^{1,2}$ implies that $\nabla \varphi_k \xrightarrow{L^2} \nabla \varphi$ and (2.48) implies $\nabla \varphi_k^+ \xrightarrow{L^2} \nabla \varphi^+$. Therefore, proceeding as above, we find

$$\left| \int_M \langle \nabla \varphi^+, \nabla \varphi \rangle_\gamma \, dV_\gamma - \int_M \langle \nabla \varphi_k^+, \nabla \varphi_k \rangle_\gamma \, dV_\gamma \right| \leqslant \int_M |\langle \nabla(\varphi^+ - \varphi_k^+), \nabla \varphi \rangle_\gamma| dV_\gamma$$
$$+ \int_M |\langle \nabla \varphi_k^+, \nabla(\varphi - \varphi_k) \rangle_\gamma| dV_\gamma,$$
$$\leqslant \|\nabla(\varphi^+ - \varphi_k^+)\|_{L^2} \|\nabla \varphi\|_{L^2}$$
$$+ \|\nabla \varphi_k^+\|_{L^2} \|\nabla(\varphi - \varphi_k)\|_{L^2},$$

where the right-hand side again goes to zero by previous arguments. Putting all this together, we justified the integration by parts of (2.47):

$$\int_M \left(\varphi^+ \Delta_\gamma \varphi - V\varphi^+ \varphi \right) dV_\gamma = -\int_M \left(\langle \nabla \varphi^+, \nabla \varphi \rangle_\gamma + V\varphi^+ \varphi \right) dV_\gamma \geqslant 0.$$

Since $\varphi = \varphi^+$ on $\mathrm{supp}(\varphi^+)$, we find

$$- \int_M \left(|\nabla \varphi^+|_\gamma^2 + V \left(\varphi^+ \right)^2 \right) dV_\gamma \geqslant 0.$$

The above implies, in particular, that φ^+ is constant. If $\varphi^+ = c \neq 0$, then by continuity of φ, we find $\varphi = c > 0$ over all of M. But then $-Vc \geqslant 0$ a.e, which implies $V = 0$ a.e, which contradicts our hypotheses. Therefore, it follows that $\varphi^+ \equiv 0$ and the claim holds. $\square$

The above maximum principle is robust enough to allow us to establish the monotone iteration scheme which is used in the analysis of semi-linear equations of the form of (2.45). Nevertheless, for geometric problems, we sometimes need a stronger version which excludes the possibility of φ vanishing. This is the content of the following lemma.

Lemma 2.2.3 (Strong Maximum Principle). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Let us also consider a function $V \in L^p$ and assume that $V \geqslant 0$ a.e, $V \not\equiv 0$. Then, given $\varphi \in W^{2,p}$ satisfying the inequality*

$$\Delta_\gamma \varphi - V\varphi \geqslant 0, \tag{2.50}$$

if $\varphi(x) = 0$ for some $x \in M$, then $\varphi \equiv 0$.

Proof. The proof done via a connectivity argument. That is, if we show that the subset of M were $\varphi = 0$ is open, then the result follows since M is assumed to be connected. In order to establish this last claim, we follow arguments of Maxwell (see 2006, Lemma 5.3), which appeals to the so-called *weak Harnack inequality* presented in Trudinger (1973, Theorem 5.2). The argument goes as follows. Let us consider a coordinate ball $\mathcal{B}$ in M, where we can write

$$\Delta_\gamma \varphi - V\varphi = \partial_i \left(g^{ij} \partial_j \varphi \right) - \left(\partial_i g^{ik} + g^{ij} \Gamma_{ij}^k \right) \partial_k \varphi - V\varphi. \tag{2.51}$$

In Trudinger (ibid., Section 5), the author studies elliptic operators of the form

$$Lu = -\partial_i \left(a^{ij} \partial_i u + a^i u \right) + b^i \partial_i u + au \tag{2.52}$$

under very general hypotheses on the coefficients a^{ij}, a^i, b^i and a on domains $\Omega \subset \mathbb{R}^n$. In particular, for Ω bounded, the choices a^{ij} continuous, $a^i, b^i \in L^t$ and $a \in L^{\frac{t}{2}}$, with $t > n$, satisfy the hypotheses of Trudinger (ibid., Theorem 5.2).

Under these conditions, if $u \in W^{1,2}(\Omega)$ satisfies $Lu \geqslant 0$ (weakly) and $u \geqslant 0$ on a ball $B_{5R}(x_0) \subset \Omega$, then, for a sufficiently large $s > 0$ it follows that

$$\|u\|_{L^s(B_{2R}(x_0))} \leqslant C R^{\frac{n}{s}} \inf_{B_R(x_0)} u, \tag{2.53}$$

where the constant C depends on the operator, the dimension and s, but not on u. We intend to apply the above inequality to φ. First, notice that since $p > \frac{n}{2}$ by hypothesis, the Sobolev embeddings imply $W^{2,p}(\mathcal{B}) \hookrightarrow W^{1,2}(\mathcal{B})$. Also, by hypothesis $g^{ij} \in W^{2,p}$ and is therefore continuous. Furthermore, comparing the operators in (2.51) and (2.52), we see that $a^{ij} = g^{ij}$ satisfies the imposed conditions. Also, $a^i \equiv 0$ in our case, while $b^i = \partial_k g^{ki} + g^{kl}\Gamma^i_{kl} \in W^{1,p}$. Again, using Sobolev embeddings, if $p \geqslant n$, we directly have $W^{1,p} \hookrightarrow L^t$, with $t > n$. In case $\frac{n}{2} < p < n$, then this last embedding holds if $n < t \leqslant \frac{np}{n-p}$. Such number t exists iff $p > \frac{n}{2}$, which proves that b^i also satisfies the required conditions. Finally, $a = V \in L^p$, with $p > \frac{n}{2}$ already satisfies the requirements. Finally, $\varphi \leqslant 0$ due to Lemma 2.2.2 and therefore $\varphi' \doteq -\varphi \geqslant 0$ satisfies

$$-\partial_i \left(g^{ij}\partial_j\varphi' \right) + \left(\partial_i g^{ik} + g^{ij}\Gamma^k_{ij} \right) \partial_k\varphi' + V\varphi' \geqslant 0$$

together will all the requirements to apply (2.53). In particular, if there some $x_0 \in M$ where $\varphi(x_0) = 0$, picking $\mathcal{B}$ containing x_0 and applying (2.53) in a ball $B_R(x_0) \subset \mathcal{B}$, we find $\|\varphi\|_{L^s(B_{2R})(x_0)} = 0$, and therefore $\varphi|_{B_{2R}(x_0)} \equiv 0$, proving that the set $\varphi^{-1}(0)$ is open and establishing the claim. $\qquad\square$

Remark 2.2.2. *We would like to highlight that similar (more general) versions of the above two maximum principles have been established in the literature. We refer the reader to Holst, Nagy, and Tsogtgerel (2009), Holst and Tsogtgerel (2013), and Maxwell (2006) for further references.*

Let us now introduce the following concepts concerning barriers of an equation of the form (2.45). First, let us define

$$f : M \times \mathcal{I} \mapsto \mathbb{R},$$
$$(x, y) \mapsto f(x, y) \doteq \sum_I a_I(x)y^I, \tag{2.54}$$

where $\mathcal{I} \subset \mathbb{R}$ stands for an interval, and, as in (2.45), the coefficients $a_I \in L^p$. Furthermore, we assume that $\partial_y f(x, y)$ exists and is continuous on $\mathcal{I}$. Notice that this is an imposition on $\mathcal{I}$ more than on f, since, due to the form of f, this is satisfied by any interval $\mathcal{I} = [l, m] \subset \mathbb{R}^+$ with $l > 0$.

Definition 2.2.1. *Let (M^n, γ) be a $W^{2,p}$-Riemannian manifold, with $p > \frac{n}{2}$. We say that $\varphi_- \in W^{2,p}$ is a **subsolution** of the equation $\Delta_\gamma \varphi = f(x, \varphi)$ if*

$$\Delta_\gamma \varphi_- \geqslant f(x, \varphi_-). \tag{2.55}$$

*Analogously, we say that $\varphi_+ \in W^{2,p}$ is a **supersolution** of the same equation if*

$$\Delta_\gamma \varphi_+ \leqslant f(x, \varphi_+). \tag{2.56}$$

We can now establish the following theorem.

Theorem 2.2.1. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Consider the equation $\Delta_\gamma \varphi = f(x, \varphi)$, with f given as in (2.57). If this equation admits a pair of $W^{2,p}$ sub and supersolutions $0 < l \leqslant \varphi_- \leqslant \varphi_+ \leqslant m$ with $[l, m] \subset \mathcal{I}$, then there is a solution $\varphi \in W^{2,p}$ satisfying $\varphi_- \leqslant \varphi \leqslant \varphi_+$.*

Proof. Let us construct the solution by iterations of solutions to linear problems. That is, we will start considering the sequence $\{\varphi_k\}_{k=0}^\infty \subset W^{2,p}$ generated via

$$\Delta_\gamma \varphi_{k+1} - a\varphi_{k+1} = f(x, \varphi_k) - a\varphi_k, \tag{2.57}$$

where $a \in L^p$ is a function satisfying $a \geqslant 0$ a.e, $a \not\equiv 0$, to be fixed below, and, in order to start the iteration we fix $\varphi_0 \doteq \varphi_-$. Since the right hand side of the above equation remains in L^p at any step due to Sobolev multiplication properties, the non-negativity of the coefficient $a \in L^p$ guarantees that our sequence is well-defined through Theorem B.7, since $\Delta_\gamma - a : W^{2,p} \mapsto L^p$ is an isomorphism. Let us now fix a choice for $a \in L^p$. This will be done imposing the function

$$f_a : M \times [l, m] \mapsto \mathbb{R},$$
$$(x, y) \mapsto f_a(x, y) \doteq f(x, y) - ay$$

to be a decreasing function on $y \in [l, m]$. We achieve this by choosing a satisfying

$$a > \sum_I |I a_I| \sup_{y \in [l,m]} y^{I-1},$$

noticing that $\sup_{y \in [l,m]} y^{I-1}$ is going to be given in terms of a power of either l or m depending on the sign of $I - 1$. Let us now show that our sequence of solutions is trapped between the barriers φ_- and φ_+. With this in mind, consider

$$\Delta_\gamma (\varphi_1 - \varphi_-) - a(\varphi_1 - \varphi_-) = f(x, \varphi_-) - a\varphi_- - \Delta_\gamma \varphi_- + a\varphi_-,$$
$$= -(\Delta_\gamma \varphi_- - f(x, \varphi_-)) \leqslant 0,$$

and therefore Lemma 2.2.2 implies $\varphi_1 \geqslant \varphi_-$. Similarly,

$$
\begin{aligned}
\Delta_\gamma (\varphi_1 - \varphi_+) - a (\varphi_1 - \varphi_+) &= f(x, \varphi_-) - a\varphi_- - \Delta_\gamma\varphi_+ + a\varphi_+, \\
&= -\left(\Delta_\gamma\varphi_+ - f(x,\varphi_+) + f(x,\varphi_+) - f(x,\varphi_-)\right) \\
&\quad + a (\varphi_+ - \varphi_-), \\
&= -\left(\Delta_\gamma\varphi_+ - f(x,\varphi_+)\right) \\
&\quad - (f(x,\varphi_+) - a\varphi_+ - (f(x,\varphi_-) - a\varphi_-)) \geqslant 0,
\end{aligned}
$$

where the last inequality comes from $y \mapsto f_a(x, y)$ being a decreasing function of $y \in [l, m]$. Once more, the weak maximum principle gives us $\varphi_1 \leqslant \varphi_+$.

We now proceed by induction. Let us now assume that $\varphi_- \leqslant \varphi_{k-1} \leqslant \varphi_k \leqslant \varphi_+$ for some $k \geqslant 1$, and notice that the case $k = 1$ has just been established. Then

$$
\begin{aligned}
\Delta_\gamma (\varphi_{k+1} - \varphi_k) - a (\varphi_{k+1} - \varphi_k) &= f(x, \varphi_k) - a\varphi_k - f(x, \varphi_{k-1}) + a\varphi_{k-1}, \\
&= f_a(x, \varphi_k) - f_a(x, \varphi_{k-1}) \leqslant 0,
\end{aligned}
$$

where the last inequality comes from the inductive hypothesis and the decreasing property of $f_a(x, y)$ on y. Therefore, $\varphi_{k+1} \geqslant \varphi_k$. Similarly

$$
\begin{aligned}
\Delta_\gamma (\varphi_{k+1} - \varphi_+) - a (\varphi_{k+1} - \varphi_+) &= f(x, \varphi_k) - a\varphi_k - \Delta_\gamma\varphi_+ + a\varphi_+, \\
&= -\left(\Delta_\gamma\varphi_+ - f(x,\varphi_+) + f(x,\varphi_+) - f(x,\varphi_k)\right) \\
&\quad + a (\varphi_+ - \varphi_k), \\
&= -\left(\Delta_\gamma\varphi_+ - f(x,\varphi_+)\right) \\
&\quad - (f(x,\varphi_+) - a\varphi_+ - (f(x,\varphi_k) - a\varphi_k)) \geqslant 0,
\end{aligned}
$$

from which we find $\varphi_{k+1} \leqslant \varphi_+$ and therefore the inductive proof is finished, establishing that the sequence $\{\varphi_k\}_{k=0}^\infty$ satisfies $\varphi_- = \varphi_0 \leqslant \varphi_1 \leqslant \cdots \leqslant \varphi_k \leqslant \varphi_{k+1} \leqslant \cdots \leqslant \varphi_+$. That is, $\{\varphi_k\} \subset W^{2,p}$ is bounded in C^0. Since $W^{2,p} \hookrightarrow C^0$ is compact for $p > \frac{n}{2}$, there must be a C^0-convergent subsequence, to which we now restrict. Let us show that such subsequence is Cauchy is $W^{2,p}$ by considering the following elliptic estimates

$$
\begin{aligned}
\|\varphi_k - \varphi_l\|_{W^{2,p}} &\leqslant C \|\Delta_\gamma (\varphi_k - \varphi_l) - a (\varphi_k - \varphi_l)\|_{L^p}, \\
&= C \|f(\cdot, \varphi_{k-1}) - f(\cdot, \varphi_{l-1}) - a (\varphi_{k-1} - \varphi_{l-1})\|_{L^p}, \\
&\leqslant C \sum_I \|a_I\|_{L^p} \|\varphi_{k-1}^I - \varphi_{l-1}^I\|_{C^0} + \|a\|_{L^p} \|\varphi_{k-1} - \varphi_{l-1}\|_{C^0}.
\end{aligned}
$$

The C^0-convergence of $\{\varphi_k\}$ implies that the right-hand side of the above expression goes to zero as k, l go to infinity and therefore $\{\varphi_k\}$ is Cauchy in $W^{2,p}$. Thus, there is some $\varphi \in W^{2,p}$, such that

$$\varphi_k \xrightarrow{W^{2,p}} \varphi.$$

Since $\Delta_\gamma - a : W^{2,p} \mapsto L^p$ is continuous, passing to the limit in (2.57) we see that φ solves $\Delta_\gamma \varphi = f(x, \varphi)$. Furthermore, by construction we must have $\varphi_- \leqslant \varphi \leqslant \varphi_+$. $\qquad\square$

The above theorem will be our main tool when proving existence results for the Lichnerowicz equation. Therefore, we see that our task will be reduced to finding suitable barrier functions $\varphi_- \leqslant \varphi_+$ to our associated equation. In doing so, we will see that the behaviour of the linear term a_r in (2.45) plays a particularly special role. Therefore, certain classification results concerning conformal deformations of scalar curvature are specially useful, which motivates the analysis presented in the next section concerning the Yamabe problem. Finally, let us close this section by referring the interested reader to some refined low-regularity versions of Theorem 2.2.1, such as Maxwell (2005a, Proposition 6.2).

2.2.2   The Yamabe classification

Given a smooth closed Riemannian manifold (M^n, γ), $n \geqslant 3$, the Yamabe problem in Riemannian geometry consists in finding a conformal deformation of γ into a metric of constant scalar curvature. From (2.2), we know that this amounts to finding a smooth positive solution to the equation

$$-a_n \Delta_\gamma \varphi + R_\gamma \varphi = \widetilde{R} \varphi^{\frac{n+2}{n-2}}, \tag{2.58}$$

for some constant $\widetilde{R}$, where $a_n \doteq \frac{4(n-1)}{n-2}$. If we can achieve this goal, then the metric $g = \varphi^{\frac{4}{n-2}} \gamma$ has constant scalar curvature $R_g = \widetilde{R}$. Yamabe observed that the above equation is actually the Euler–Lagrange equation of the functional

$$\mathcal{Q}(g) = \frac{\int_M R_g \, dV_g}{\mathrm{Vol}_g(M)^{\frac{2}{p^*}}}, \tag{2.59}$$

where $2^* \doteq \frac{2n}{n-2}$ denotes the *critical* exponent for the Sobolev embedding $W^{1,2} \hookrightarrow L^{p^*}$,[6] and we consider a minimization problem in a conformal class $[\gamma]$. That is,

---

[6]Notice that this is precisely the exponent where the embedding looses its compactness.

we consider the above functional on metrics on the form $g = \varphi^{\frac{4}{n-2}}\gamma$, and write

$$\mathcal{Q}(\varphi) = \frac{E(\varphi)}{\|\varphi\|^2_{L^{2^*}(M,dV_\gamma)}}, \tag{2.60}$$

where

$$\begin{aligned}
E(\varphi) &\doteq \int_M \left(R_\gamma \varphi^2 - a_n \varphi \Delta_\gamma \varphi\right) dV_\gamma, \\
&= \int_M \left(a_n |\nabla \varphi|^2_\gamma + R_\gamma \varphi^2\right) dV_\gamma.
\end{aligned} \tag{2.61}$$

A now straightforward computation shows that a critical point φ of (2.60) is a weak solutions to (2.58), with $\widetilde{R} \doteq \frac{E(\varphi)}{\|\varphi\|^{2^*}_{L^{2^*}}}$. Notice that Hölder's inequality put together with $W^{1,2} \hookrightarrow L^{2^*}$ implies that

$$E(\varphi) \geq -\left|\int_M R_\gamma \varphi^2 dV_\gamma\right| \geq -\|R_\gamma\|_{L^{q'}}\|\varphi^2\|_{L^q} = -\|R_\gamma\|_{L^{q'}}\|\varphi\|^2_{L^{p*}},$$

where $q \doteq \frac{n}{n-2}$ and $q' = \frac{q}{q-1}$. Therefore, $\mathcal{Q}(\varphi)$ admits an infimum, given by

$$\mathcal{Y}([\gamma]) \doteq \inf_{\substack{\varphi \in C^\infty(M) \\ \varphi \not\equiv 0}} \mathcal{Q}_\gamma(\varphi), \tag{2.62}$$

and we will refer to $\mathcal{Y}([\gamma])$ as the *Yamabe invariant*, which is clearly a conformal invariant. The core of the standard analysis at this stage is to use variational techniques to find a positive smooth minimizer for this problem. The fact that the problem involves the critical (non-compact) embedding $W^{1,2} \hookrightarrow L^{p^*}$ presents an additional difficulty, which is part of the rich history around this problem. We refer the reader to J. M. Lee and Parker (1987) and Aubin (1998, Chapter 5) for detailed reviews on this topic, where the final resolution is due to the remarkable work of several authors, most notably Aubin (1976), R. Schoen (1984), Trudinger (1968), and Yamabe (1960). In these works, the problem was solved by Trudinger (1968) in non-positive Yamabe case, then it was shown by Aubin (1976) that the problem was solvable as long as $\mathcal{Y}([\gamma]) < \mathcal{Y}(S^n)$, where S^n here represents the round sphere, obviously with positive Yamabe invariant. This lead to the resolution of the Yamabe problem in dimensions $n \geq 6$ for non-locally conformally flat manifolds. The remaining cases were addressed by R. Schoen (1984), using a strategy which links the Yamabe problems with the positive mass theorem of general relativity (R. Schoen and S. T. Yau 1979; R. M. Schoen and S.-T. Yau 1979, 1988).

The resolution of the Yamabe problem implies that smooth metrics on closed manifolds get classified into three disjoint classes given by the sign of their Yamabe invariant, where $\mathcal{Y}([\gamma]) > 0$ (resp. $= 0, < 0$) iff γ admits a conformal deformation to constant positive (resp. zero and negative) scalar curvature. This kind of control over the sign of the scalar curvature in a conformal class is what we want to exploit when constructing barrier functions for the Lichnerowicz equation (2.44). In what follows, our aim is to provide a similar classification in our low-regularity setting. The subtlety of the problem described above should caution us not to be overambitious, and, actually, we will attempt to simply control the sign of the scalar curvature, rather than guaranteeing the existence of conformal deformations to constant scalar curvature, which, in the low regularity setting seems to still be a partially open problem to the best of our knowledge. The results we will present are based on developments given by Holst, Nagy, and Tsogtgerel (2009) and Maxwell (2005a).

Let us consider a Riemannian manifold (M^n, γ) with $n \geqslant 3$, $\gamma \in W^{2,p}$ and $p > \frac{n}{2}$. Then, let us define

$$
\begin{aligned}
\mathcal{A} : W^{1,2} &\times W^{1,2} \mapsto \mathbb{R} \\
(\varphi_1, \varphi_2) &\mapsto \int_M \left(a_n \langle \nabla \varphi_1, \nabla \varphi_2 \rangle_\gamma + R_\gamma \varphi^2 \right) dV_\gamma
\end{aligned}
\tag{2.63}
$$

Notice that the Sobolev embedding $W^{1,2} \hookrightarrow L^{2q}$ for all $1 \leqslant q \leqslant \frac{n}{n-2}$ implies that if $\varphi_1, \varphi_2 \in W^{1,2}$, then $|\varphi_1|^q, |\varphi_2|^q \in L^2$, implying

$$
\int_M |\varphi_1 \varphi_2|^q dV_\gamma \leqslant \|\varphi_1\|_{L^{2q}} \|\varphi_2\|_{L^{2q}}.
$$

That is, $\varphi_1 \varphi_2 \in L^q$ for all $1 \leqslant q \leqslant \frac{n}{n-2}$. Furthermore, $R_\gamma \in L^{q'}$ if $q' \leqslant p$ and

$$
q' \leqslant p \iff \frac{1}{p} \leqslant \frac{1}{q'} = 1 - \frac{1}{q} \leqslant \frac{2}{n},
$$

that is $q' \leqslant p$ if $p \geqslant \frac{n}{2}$, which satisfies our hypotheses. Therefore

$$
\left| \int_M R_{\gamma'} \varphi_1 \varphi_2 dV_\gamma \right| \leqslant \|R_{\gamma'}\|_{L^{q'}} \|\varphi_1 \varphi_2\|_{L^q} \leqslant C \|R_{\gamma'}\|_{L^{q'}} \|\varphi_1\|_{W^{1,p}} \|\varphi_2\|_{W^{1,p}}.
$$

That is, $\mathcal{A} : W^{1,p} \times W^{1,p} \mapsto \mathbb{R}$ is a continuous bilinear functional. In what follows, we will keep the notation $E(\varphi) \doteq \mathcal{A}(\varphi, \varphi)$ for the associated quadratic

form and define

$$J_{\gamma,q}(\varphi) \doteq \frac{E(\varphi)}{\|\varphi\|_{L^{2q}}^2} = \frac{\int_M \left(a_n |\nabla \varphi|_\gamma^2 + R_\gamma \varphi^2\right)}{\left(\int_M \varphi^{2q}\right)^{\frac{1}{q}}}. \tag{2.64}$$

We know get the following simple result.

Lemma 2.2.4. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geq 3$. Then, the functionals $J_{\gamma,q}$ are all bounded from below for any $1 \leq q \leq \frac{n}{n-2}$.*

Proof. Notice that under our hypotheses $R_\gamma \in L^{q'}$ from the discussion presented above. Therefore, it follows that

$$\mathcal{A}(\varphi, \varphi) \geq -\left|\int_M R_\gamma \varphi^2 dV_\gamma\right| \geq -\|R_{\gamma'}\|_{L^{q'}} \|\varphi^2\|_{L^q} = -\|R_{\gamma'}\|_{L^{q'}} \|\varphi\|_{L^{2q}}^2,$$

which implies

$$J_{\gamma,q}(\varphi) \geq -\|R_{\gamma'}\|_{L^{q'}},$$

and the claim follows. □

Using the above lemma, we can introduce the following notation for the infima of these functionals:

$$\mathcal{Y}_{\gamma,q} \doteq \inf_{\substack{\varphi \in W^{1,p} \\ \varphi \not\equiv 0}} J_{\gamma,q}(\varphi). \tag{2.65}$$

We will refer to the numbers $\mathcal{Y}_{\gamma,q}$ as the q-th Yamabe number. In particular, we write $\lambda_\gamma \doteq \mathcal{Y}_{\gamma,1}$ for the *first eigenvalue* of the conformal Laplacian and note that the Yamabe quotient is given by $Q_\gamma(\varphi) = J_{\gamma,\frac{n}{n-2}}(\varphi)$. Thus, the Yamabe invariant is $\mathcal{Y}([\gamma]) = \mathcal{Y}_{\gamma,\frac{n}{n-2}}$. Let us now present the following useful analytical property associated to $J_{\gamma,1}$.

Lemma 2.2.5. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geq 3$. Then, the map $W^{1,2} \mapsto \mathbb{R}$ given by*

$$u \mapsto \int_M R_\gamma u^2 dV_\gamma \tag{2.66}$$

is weakly continuous.

Proof. Considering $1 \leqslant q < \frac{n}{n-2}$, we know that that $W^{1,2} \hookrightarrow L^{2q}$ is compact. Then, $W^{1,2} \mapsto L^q, u \mapsto u^2$, is weakly continuous. To see this, consider a weakly convergent sequence $\{u_k\} \subset W^{1,2}$ with limit $u \in W^{1,2}$. Since such a sequence must be $W^{1,2}$-bounded, we know that $\{u_k\}$ has an L^{2q}-convergent subsequence with limit $\tilde{u} \in L^{2q}$. But then $u_k \rightharpoonup \tilde{u}$ and since $L^{(2q)'} \hookrightarrow W^{-1,2}$, we must have $\tilde{u} = u$. Then, for all $f \in L^{q'}$ it follows that

$$
\left| \int_M f(u_k^2 - u^2) \right| = \left| \int_M f(u_k(u_k - u) + u(u_k - u)) \right|,
$$

$$
\leqslant \|f\|_{L^{q'}} \left(\|u_k(u_k - u)\|_{L^q} + \|u(u_k - u)\|_{L^q} \right),
$$

$$
\leqslant \|f\|_{L^{q'}} \left(\left\| |u_k|^q |u_k - u|^q \right\|_{L^1}^{\frac{1}{q}} + \left\| |u|^q |u_k - u|^q \right\|_{L^q}^{\frac{1}{q}} \right),
$$

$$
\leqslant \|f\|_{L^{q'}} \left(\|u_k\|_{L^{2q}}^{\frac{1}{q}} \|u_k - u\|_{L^{2q}}^{\frac{1}{q}} + \|u\|_{L^{2q}}^{\frac{1}{q}} \|u_k - u\|_{L^{2q}}^{\frac{1}{q}} \right),
$$

where we have appealed to Hölder's inequality in the second line and Cauchy–Schwartz's inequality in the last one. Then, the strong convergence $u_k \xrightarrow{L^{2q}} u$ proves our initial claim.

Finally, let us fix q such that $R_\gamma \in L^{q'}$. From Hölder's inequality this follows if $q' \leqslant p$, which is equivalent to $\frac{1}{p} \leqslant 1 - \frac{1}{q}$, where

$$
1 - \frac{1}{q} \leqslant 1 - \frac{n-2}{n} = \frac{2}{n}.
$$

That is $q' \leqslant p \iff p \geqslant \frac{n}{2}$, which is satisfied under our hypotheses. Therefore $R_\gamma \in L^{q'}$, for any such choice of q and it follows that

$$
\int_M R_\gamma u_k^2 \to \int_M R_\gamma u^2.
$$

$\square$

The following theorem is key in the low-regularity Yamabe classification.

Theorem 2.2.2. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geqslant 3$. Then, there exists a $W^{2,p}$ function $\varphi > 0$ such that*

$$
-a_n \Delta_\gamma \varphi + R_\gamma \varphi = \lambda_\gamma \varphi. \tag{2.67}
$$

In particular, γ is conformal to a metric with continuous scalar curvature having the same sing as λ_γ.

Proof. Let us first notice that (2.67) is the Euler–Lagrange equation associated to the functional $J_{\gamma,1}$. Consider then a minimizing sequence $\{\varphi_k\} \subset W^{1,2}$ of $J_{\gamma,1}$, which we take to be L^2-normalised (i.e $||\varphi_k||_{L^2} = 1$). Such sequence must be bounded in $W^{1,2}$ and thus, by compactness of the embedding $W^{1,2} \hookrightarrow L^2$, there is an L^2 convergent subsequence to which we now restrict, with limit $\varphi_0 \in L^2$. Furthermore, since $W^{1,2}$ is reflexive, it is weakly sequentially compact, and therefore we can extract a subsequence which converges weakly to some $\varphi_1 \in W^{1,2}$. Since strong convergence implies weak convergence, $\varphi_k \rightharpoonup \varphi_0$, and since the weak limit is unique, we must have $\varphi_0 = \varphi_1 \doteq \varphi \in W^{1,2}$. Also, it follows that $||\varphi||_{L^2} = 1$ and thus $\varphi \neq 0$. Also, using Theorem A.1.4, we see that

$$||\varphi||_{W^{1,2}} \leqslant \liminf_{k\to\infty} ||\varphi_k||_{W^{1,2}}.$$

Since $||\varphi_k||_{L^2} = ||\varphi||_{L^2} = 1$ for all k, the above implies

$$||\nabla\varphi||_{L^2}^2 \leqslant \liminf_{k\to\infty} ||\nabla\varphi_k||_{L^2}^2.$$

Since the map $u \in W^{1,2} \mapsto \int_M R_\gamma u^2$ is weakly continuous, we see that

$$
\begin{aligned}
\lambda_\gamma &= \lim_{k\to\infty} \int_M \left(a_n|\nabla\varphi_k|_\gamma^2 + R_\gamma\varphi_k^2\right) dV_\gamma, \\
&\geqslant \int_M a_n|\nabla\varphi|_\gamma^2 dV_\gamma + \lim_{k\to\infty} \int_M R_\gamma\varphi_k^2 dV_\gamma, \\
&= \int_M \left(a_n|\nabla\varphi|_\gamma^2 + R_\gamma\varphi^2\right) dV_\gamma.
\end{aligned}
$$

This implies that $J_{\gamma,1}(\varphi) \leqslant \lambda_\gamma$, with $\varphi \in W^{1,2}$, therefore $J_{\gamma,1}(\varphi) = \lambda_\gamma$ and hence $\varphi \in W^{1,2}$ is a minimizer. This, in particular, proves that φ is a weak solution of (2.67) and since $J_{\gamma,1}(\varphi) = J_{\gamma,1}(|\varphi|)$ there is no loss in generality assuming that $\varphi > 0$. Then, elliptic regularity gives us $\varphi \in W^{2,p}$, which finishes the first part of the proof.

Finally, to prove the scalar curvature statement, we can consider the Riemannian metric $g = \varphi^{\frac{4}{n-2}}\gamma$, where $\varphi \in W^{2,p}$ is the minimizer just constructed above. Appealing to Proposition 2.1.1, we know that

$$R_g = \varphi^{-\frac{n+2}{n-2}} \left(-a_n\Delta_\gamma\varphi + R_\gamma\varphi\right) = \lambda_\gamma\varphi^{1-\frac{n+2}{n-2}} = \lambda_\gamma\varphi^{-\frac{4}{n-2}}, \qquad (2.68)$$

which, since $\varphi > 0$ is continuous, proves that R_g is continuous and has the same sing as λ_γ. $\qquad\square$

We can now state the main result of this section.

Theorem 2.2.3. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geq 3$. Then, the following statements hold:*

1. $\mathcal{Y}([\gamma]) > 0$ *iff γ is conformal to a metric of continuous positive scalar curvature;*

2. $\mathcal{Y}([\gamma]) = 0$ *iff γ is conformal to a metric of constant zero scalar curvature;*

3. $\mathcal{Y}([\gamma]) < 0$ *iff γ is conformal to a metric of continuous negative scalar curvature,*

where in the three cases above the conformal deformation is of the form $g = \varphi^{\frac{4}{n-2}} \gamma$, with $\varphi \in W^{2,p}$.

Proof. Let us start for establishing all the "backwards" implications. That is, assume that γ is conformal to a metric of continuous scalar curvature with fixed sign, and since $\mathcal{Y}([\gamma])$ is a conformal invariant, we can actually assume that γ is such a metric. Then, if $R_\gamma < 0$, testing $\mathcal{Q}_\gamma$ on constant functions we find $\mathcal{Q}_\gamma(c) < 0$, and therefore $\mathcal{Y}([\gamma]) < 0$. In case $R_\gamma \geq 0$, it follows that $\mathcal{Q}_\gamma(\varphi) \geq 0$ for all $\varphi \in W^{1,2}$. In particular, if $R_\gamma \equiv 0$, again testing $\mathcal{Q}$ on constant functions gives $\mathcal{Q}(c) = 0$, and therefore $\mathcal{Y}([\gamma]) = 0$. On the other hand, if $R_\gamma > 0$, then $E(\varphi)$ represents and equivalent norm to the standard one in $W^{1,2}$, and therefore we have $1 \leq CE(\varphi)$ for all $\varphi \in W^{1,2}$ such that $\|\varphi\|_{L^{2*}} = 1$, with $C > 0$, which shows that $\mathcal{Y}([\gamma])$ must be positive in this case, since we can write

$$\mathcal{Y}([\gamma]) = \inf_{\substack{\varphi \in W^{1,2} \\ \|\varphi\|_{L^{2*}} = 1}} E(\varphi).$$

Let us now prove the "forward" implications. First assume that $\mathcal{Y}([\gamma]) < 0$. This means that there is some $\varphi \in W^{1,2*}$ such that $E(\varphi) < 0$. Since $W^{1,2*} \hookrightarrow W^{1,2}$, we can use the same function φ to test $J_{\gamma,1}(\varphi) < 0$. Therefore, we must have $\mathcal{Y}_{\gamma,1} < 0$ and Theorem 2.2.2 guarantees the existence of the corresponding conformal deformation to a metric with continuous and negative scalar curvature.

Now, let us consider the case $\mathcal{Y}([\gamma]) > 0$. In particular, since $L^{\frac{2n}{n-2}} \hookrightarrow L^{2q}$ for all $1 \leq q \leq \frac{n}{n-2}$, then $\|\varphi\|_{L^{2q}}^2 \leq C \|\varphi\|_{L^{\frac{2n}{n-2}}}^2$

$$Q_\gamma(\varphi) = \frac{E(\varphi)}{\|\varphi\|_{L^{2*}}^2} \leq C \frac{E(\varphi)}{\|\varphi\|_{L^{2q}}^2},$$

which implies

$$\mathcal{Y}([\gamma]) = \inf_{\substack{\varphi \in C^\infty \\ \varphi \neq 0}} \frac{E(\varphi)}{\|\varphi\|_{L^{2*}}^2} \leqslant \inf_{\substack{\varphi \in C^\infty \\ \varphi \neq 0}} \frac{E(\varphi)}{\|\varphi\|_{L^{2q}}^2} = C\mathcal{Y}_{\gamma,q}, \ \ 1 \leqslant q \leqslant \frac{n}{n-2}.$$

In particular, since $\mathcal{Y}([\gamma]) > 0 \Rightarrow \mathcal{Y}_{\gamma,1} > 0$. Once more Theorem 2.2.2 finishes the proof in this case.

Now assume that $\mathcal{Y}([\gamma]) = 0$, which, as a first step implies $\mathcal{Y}_{\gamma,1} \geqslant 0$. Since $\mathcal{Y}([\gamma])$ is a conformal invariant, we do not loose any generality in fixing γ within its conformal class so that R_γ is continuous and has fixed (non-negative) sign, according to Theorem 2.2.2. First, notice that $R_\gamma > 0$ is not possible, since in this case $E(\varphi)$ is an equivalent norm to standard one in $W^{1,2}$ and $\mathcal{Y}([\gamma]) = 0$ implies the existence of a sequence $\{\varphi_k\} \subset W^{1,2}$ of L^{2*} normalized functions satisfying $E(\varphi_k) \to 0$. But this implies that $\varphi_k \to 0$ in $W^{1,2} \hookrightarrow L^{2*}$, and so $\|\varphi_k\|_{L^{2*}} \to 0$, which contradicts the fact that the sequence is L^{2*}-normalized. Thus, $R_\gamma = 0$, which finishes the proof. □

The extended scalar-Yamabe classification

As we have already explained, the Yamabe classification will prove to be a useful tool in controlling the linear part of the Lichnerowicz equation appealing to some a priori conformal deformation. This will become clear in subsequent sections. Nevertheless, let us notice that when dealing with the Lichneroiwicz equation associated to the constraints with a scalar field as a source, the corresponding linear operator appearing in (2.44) has a contribution coming from the scalar field in the linear (scalar-curvature) term. That is, the linear part of (2.44) is given by

$$\widetilde{L}_\gamma \doteq \Delta_\gamma - c_n \left(R_\gamma - |\nabla \phi|_\gamma^2 \right), \tag{2.69}$$

where ϕ is the prescribed scalar field and we denote by $r_\gamma \doteq R_\gamma - |\nabla \phi|_\gamma^2$. In this case, we would like to have a classification such as that given by Theorem 2.2.3 involving r_γ instead of R_γ. Actually, most of the work we did in the previous section can be translated *mutatis mutandis* to achieve this goal. Namely, let us consider the following modifications. Let us fix $\phi \in W^{2,p}$, so that $|\nabla \phi|_\gamma^2 \in L^p$ and define $\widetilde{E}_\gamma : W^{1,2} \mapsto \mathbb{R}$ by

$$\widetilde{E}_\gamma(\varphi) \doteq \int_M \left(a_n |\nabla \varphi|_\gamma^2 + r_\gamma \varphi^2 \right) dV_\gamma$$

which is well defined using the same arguments that were used for $E(\varphi)$ in the previous section. In particular, we can also introduce the quotients

$$\tilde{J}_{\gamma,q}(\varphi) \doteq \frac{\tilde{E}_\gamma(\varphi)}{\|\varphi\|^2_{L^{2q}}}, \quad \text{for } 1 \leqslant q \leqslant \frac{n}{n-2}, \tag{2.70}$$

and, again following the arguments of the previous section, we know that these invariants are bounded from below, which allows us to introduce the corresponding invariants

$$\mathcal{SY}_{\gamma,q} \doteq \inf_{\substack{\varphi \in W^{1,2} \\ \varphi \neq 0}} \tilde{J}_{\gamma,q}(\varphi). \tag{2.71}$$

One substitute we need in order to parallel the discussion of the previous section is the fact that $\mathcal{SY}_{\gamma,\frac{n}{n-2}}$ is a conformally invariant. Let us define $g \doteq \theta^{\frac{4}{n-2}}\gamma$, for $\theta \in W^{2,p}$ with $\theta > 0$ and $\varphi' \doteq \theta\varphi$ with $\varphi \in C^\infty(M)$, and notice that

$$\begin{aligned}
\tilde{E}_g(\varphi) &= \int_M \left(a_n \langle \nabla\varphi, \nabla\varphi \rangle_g + R_g \varphi^2 - |\nabla\phi|^2_g \varphi^2 \right) dV_g, \\
&= \int_M \left(-a_n \varphi L_g \varphi - |\nabla\phi|^2_g \varphi^2 \right) dV_g, \\
&= \int_M \left(-a_n \varphi' L_\gamma \varphi' - |\nabla\phi|^2_\gamma \varphi'^2 \right) dV_\gamma, \\
&= \tilde{E}_\gamma(\varphi').
\end{aligned}$$

Similarly,

$$\int_M \varphi^{\frac{2n}{n-2}} dV_g = \int_M \varphi'^{\frac{2n}{n-2}} dV_\gamma.$$

Therefore

$$\tilde{J}_{g,\frac{n}{n-2}}(\varphi) = \tilde{J}_{\gamma,\frac{n}{n-2}}(\varphi'), \quad \forall\, \varphi \in C^\infty(M), \tag{2.72}$$

which implies the infima of both functionals over functions in $W^{1,2}$ are equal. That is, $\mathcal{SY}_{\gamma,\frac{n}{n-2}}$ is actually a conformal invariant, and we can thus refer to it as $\mathcal{SY}([\gamma])$, which we refer to as the *scalar Yamabe invariant*. Also, the first eigenvalue λ_γ is replaced by $\tilde{\lambda}_\gamma \doteq \mathcal{SY}_{\gamma,1}$. In this context, we consider the equations

$$-a_n \Delta_\gamma \varphi + r_\gamma \varphi = \tilde{\lambda}_\gamma \varphi, \tag{2.73}$$

and look for positive $W^{2,p}$ solutions. We see that the first part of Theorem 2.2.2 establishes the existence of such positive solutions without any further modifications. Then, considering the metric $g = \varphi^{\frac{4}{n-2}}\gamma$ constructed with such positive solution, we have the following conformal covariance property

$$
\begin{aligned}
r_g &= \varphi^{-\frac{n+2}{n-2}} \left(R_\gamma \varphi - a_n \Delta_\gamma \varphi \right) - \varphi^{-\frac{4}{n-2}} |\nabla \phi|_\gamma^2, \\
&= \varphi^{-\frac{n+2}{n-2}} \left(R_\gamma \varphi - a_n \Delta_\gamma \varphi \right) - \varphi^{-\frac{n+2}{n-2}} |\nabla \phi|_\gamma^2 \varphi, \\
&= \varphi^{-\frac{n+2}{n-2}} \left(r_\gamma \varphi - a_n \Delta_\gamma \varphi \right) = \tilde{\lambda}_\gamma \varphi^{-\frac{n+2}{n-2}+1}, \\
&= \tilde{\lambda}_\gamma \varphi^{-\frac{4}{n-2}}.
\end{aligned}
\tag{2.74}
$$

That is, r_g is continuous with the same sign as the first eigenvalue $\tilde{\lambda}_\gamma$. Therefore, we have all the tools to establish the following result, which corresponds to Theorem 2.2.3.

Theorem 2.2.4. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geqslant 3$. Then, the following statements hold:*

1. *$\mathcal{SY}([\gamma]) > 0$ iff γ is conformal to a metric with continuous positive scalar r_γ;*

2. *$\mathcal{SY}([\gamma]) = 0$ iff γ is conformal to a metric of constant zero scalar r_γ;*

3. *$\mathcal{SY}([\gamma]) < 0$ iff γ is conformal to a metric of continuous negative scalar r_γ,*

where in the three cases above the conformal deformation is of the form $g = \varphi^{\frac{4}{n-2}}\gamma$, with $\varphi \in W^{2,p}$.

The following proposition refines the classification in non-negative case.

Proposition 2.2.3. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geqslant 3$. If $r_\gamma \geqslant 0$, then $\mathcal{SY}([\gamma]) = 0$ if and only if $r_\gamma = 0$.*

Proof. First, notice that the "backwards" implication is trivial, since if $r_\gamma \equiv 0$, then $\mathcal{SY}([\gamma]) \geqslant 0$ and then testing against constant functions gives $\tilde{J}_{\gamma, \frac{n}{n-2}}(cte) = 0$, implying $\mathcal{SY}([\gamma]) = 0$. Thus, let us concentrate on the "forward implication". Since the sign of $\mathcal{SY}([\gamma])$ is the same as of $\tilde{\lambda}_\gamma$, we find $\mathcal{SY}([\gamma]) = 0$ iff $\tilde{\lambda}_\gamma = 0$ and we will work with this last condition. Let us consider a minimizing sequence $\{\varphi_k\} \subset W^{1,2}$ for $\tilde{J}_{\gamma,1} \to \tilde{\lambda}_\gamma$, which we can take to be non-negative and normalised via $\|\varphi_k\|_{L^2} = 1$. Since such sequence must be bounded in $W^{1,2}$, we

have a weakly convergent subsequence with limit $\varphi \in W^{1,2}$. We must then have $\|\varphi\|_{L^2} = 1$ and

$$\|\varphi\|_{W^{1,2}} \leqslant \liminf_{k \to \infty} \|\varphi_k\|_{W^{1,2}},$$

which then implies $\|\nabla \varphi\|_{L^2} \leqslant \liminf_{k \to \infty} \|\nabla \varphi_k\|_{L^2}$ and the conditions $r_\gamma \geqslant 0$ and $\widetilde{\lambda}_\gamma = 0$ imply

$$\|\nabla \varphi\|_{L^2} \leqslant \liminf_{k \to \infty} \|\nabla \varphi_k\|_{L^2} = 0.$$

That is, $\nabla \varphi = 0$ a.e and therefore φ is constant. But then, weak continuity of

$$W^{1,2} \mapsto \mathbb{R},$$

$$f \mapsto \int_M r_\gamma f^2 dV_\gamma,$$

implies that $\widetilde{J}_{\gamma,1}(\varphi) = 0$. Which, since φ is constant, gives

$$\int_M r_\gamma dV_\gamma = 0,$$

and therefore $r_\gamma = 0$ a.e. $\square$

2.2.3 Non-existence and uniqueness

Before embarking on the analysis of existence results, let us first consider the cases of non-existence and uniqueness of solutions for (2.44). The following theorem concerns some straightforward non-existence results.

Theorem 2.2.5 (Non-Existence). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$ and $n \geqslant 3$ and let us consider the Lichnerowicz equation (2.44), given by*

$$\Delta_\gamma \varphi - a_r \varphi + a_{TT} \varphi^{-\frac{3n-2}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} + a_E \varphi^{-3} + a_{\widetilde{F}} \varphi^{\frac{n-6}{n-2}} = 0. \quad (2.75)$$

If all the coefficients are in L^1, then, if either of the following situations

1. *$a_r, a_\tau \geqslant 0$ and $a_{TT}, a_E, a_{\widetilde{F}} \leqslant 0$;*

2. *$a_r, a_\tau \leqslant 0$ and $a_{TT}, a_E, a_{\widetilde{F}} \geqslant 0$,*

and not all of these coefficients vanish identically. Then, the above equation admits no positive solutions.

Proof. The proof is quite straightforward. Since the coefficients in (2.75) are integrable and we are assuming $\varphi \in W^{2,p}$, $\varphi > 0$, we can integrate this equation over M. In particular, Stokes' theorem and an approximation argument of the type given in Theorem B.7 prove that $\Delta_\gamma \varphi$ integrates to zero. Then, in case (1) above, we have

$$0 \leqslant \int_M \left(a_r \varphi + a_\tau \varphi^{\frac{n+2}{n-2}} \right) dV_\gamma = \int_M \left(a_{TT} \varphi^{-\frac{3n-2}{n-2}} + a_E \varphi^{-3} + a_{\widetilde{F}} \varphi^{\frac{n-6}{n-2}} \right) dV_\gamma$$
$$\leqslant 0,$$

which implies that all coefficients must vanish. The same arguments applies to case (2) with opposite inequalities.   □

Remark 2.2.3. *Firstly, let us highlight that for the physically motivated equation (2.44), the condition $a_{TT}, a_E, a_{\widetilde{F}} \leqslant 0$ is actually equivalent to $a_{TT}, a_E, a_{\widetilde{F}} \equiv 0$, since these coefficients are non-negative by definition. Secondly, let us highlight that in case we allowed all coefficients to vanish, then φ must equal a positive constant.*

Let us now present the following uniqueness result, which makes use of the geometric origin of Lichnerowicz's equation.

Theorem 2.2.6 (Uniqueness). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$ and assume that the coefficients of equation (2.44) satisfy the hypotheses of Proposition 2.2.2. Suppose, furthermore, that $n \leqslant 6$ and $a_\tau \geqslant 0$, and let φ_1 and φ_2 be two positive $W^{2,p}$-solution of (2.44), then either $\varphi_1 \equiv \varphi_2$ or $a_{TT}, a_\tau, a_E, a_{\widetilde{F}} \equiv 0$, $\mathcal{SY}([\gamma]) = 0$ and $\varphi_1 = c\varphi_2$ for some constant $c > 0$.*

Proof. Under our hypotheses, define $\varphi \doteq \varphi_2 \varphi_1^{-1}$ and let $g_1 \doteq \varphi_1^{\frac{4}{n-2}} \gamma$. Then, from conformal covariance

$$L_{g_1} \varphi = - c_n |d\phi|_{g_1}^2 \varphi - c_n \left(|\widetilde{K}_1|_{g_1}^2 + \widetilde{\pi}_1^2 \right) \varphi^{-\frac{3n-2}{n-2}} + \left(\frac{n-2}{4n} \tau^2 - 2c_n V(\phi) \right) \varphi^{\frac{n+2}{n-2}}$$
$$- c_n |\widetilde{E}_1|_{g_1}^2 \varphi^{-3} - \frac{c_n}{2} |\widetilde{F}|_{g_1} \varphi^{\frac{n-6}{n-2}}$$

where $\tilde{K}_1 = \varphi_1^{-2} \tilde{K}$, $\tilde{\pi}_1 = \varphi_1^{-\frac{2n}{n-2}} \tilde{\pi}$ and $\tilde{E}_1 = \varphi^{-\frac{2n}{n-2}} \tilde{E}$. Furthermore, by construction $\left(g_1, K_1 = \tilde{K}_1 + \frac{\tau}{n} g_1, \phi, \pi_1 = \tilde{\pi}_1, E_1 = \tilde{E}_1, \tilde{F} \right)$ solve the Gauss-constraint. In particular,

$$R_{g_1} = \tilde{\pi}_1^2 + |d\phi|_{g_1}^2 + 2V(\phi) + |\tilde{E}_1|_{g_1}^2 + \frac{1}{2}|\tilde{F}|_{g_1}^2 + |K_1|_{g_1}^2 - \tau^2,$$

$$= \tilde{\pi}_1^2 + |d\phi|_{g_1}^2 + 2V(\phi) + |\tilde{E}_1|_{g_1}^2 + \frac{1}{2}|\tilde{F}|_{g_1}^2 + |\tilde{K}_1|_{g_1}^2 - \frac{n-1}{n}\tau^2$$

which implies

$$\begin{aligned}
\Delta_{g_1}(\varphi - 1) = &- c_n \left(|\tilde{K}_1|_{g_1}^2 + \tilde{\pi}_1^2 \right) \left(\varphi^{-\frac{3n-2}{n-2}} - \varphi \right) \\
&+ \left(\frac{n-2}{4n} \tau^2 - 2 c_n V(\phi) \right) \left(\varphi^{\frac{n+2}{n-2}} - \varphi \right) \qquad (2.76) \\
&- c_n |\tilde{E}_1|_{g_1}^2 \left(\varphi^{-3} - \varphi \right) - \frac{c_n}{2} |\tilde{F}|_{g_1} \left(\varphi^{\frac{n-6}{n-2}} - \varphi \right).
\end{aligned}$$

Let us multiply the above equation by $(\varphi - 1)^+ \doteq \max\{\varphi - 1, 0\} \in W^{1,p}$. Applying the arguments of Lemma 2.2.2, we can integrate the left-hand side by parts with respect to dV_{g_1} to get

$$\begin{aligned}
\int_M (\varphi - 1)^+ \Delta_{g_1}(\varphi - 1) dV_{g_1} &= - \int_M \langle \nabla(\varphi - 1)^+, \nabla(\varphi - 1) \rangle_{g_1} dV_{g_1}, \\
&= - \int_{\varphi > 1} |\nabla(\varphi - 1)|_{g_1}^2 dV_{g_1} \leqslant 0.
\end{aligned}$$

On the other hand, since $\frac{n-2}{4n}\tau^2 - 2c_n V(\phi) \geqslant 0$, all the coefficients in the right-hand side of (2.76) are non-negative. Furthermore, for $\varphi > 1$, $(\varphi^{-\frac{3n-2}{n-2}} - \varphi) < 0$, $(\varphi^{-3} - \varphi) < 0$, while $(\varphi^{\frac{n+2}{n-2}} - \varphi) > 0$ and $(\varphi^{\frac{n-6}{n-2}} - \varphi) < 0$ if $n \leqslant 6$. This means that

$$\int_M (\varphi - 1)^+ \Delta_{g_1}(\varphi - 1) dV_{g_1} = \int_{\varphi > 1} (\varphi - 1)^+ \Delta_{g_1}(\varphi - 1) dV_{g_1} \geqslant 0,$$

and therefore

$$0 = \int_M (\varphi - 1)^+ \Delta_{g_1}(\varphi - 1) dV_{g_1} = \int_{\varphi > 1} |\nabla(\varphi - 1)|_{g_1}^2 dV_{g_1} \qquad (2.77)$$

We can apply a similar argument to $(\varphi - 1)^- \doteq \min\{0, \varphi - 1\} \in W^{1,p}$ and we get the same results, that is

$$0 = \int_M (\varphi - 1)^- \Delta_{g_1} (\varphi - 1) dV_{g_1} = \int_{\varphi < 1} |\nabla (\varphi - 1)|^2_{g_1} dV_{g_1}. \qquad (2.78)$$

Since φ is continuous, it follows that $\varphi - 1 \equiv cte$ and therefore $\varphi_1 = c\varphi_2$. If $c \neq 1$, then

$$0 = \int_M \left\{ -c_n \left(|\tilde{K}_1|^2_{g_1} + \tilde{\pi}_1^2 \right) \left(\varphi^{-\frac{3n-2}{n-2}} - \varphi \right) (\varphi - 1) \right.$$
$$+ \left(\frac{n-2}{4n} \tau^2 - 2c_n V(\phi) \right) \left(\varphi^{\frac{n+2}{n-2}} - \varphi \right) (\varphi - 1) - c_n |\tilde{E}_1|^2_{g_1} \left(\varphi^{-3} - \varphi \right) (\varphi - 1)$$
$$\left. - \frac{c_n}{2} |\tilde{F}|_{g_1} \left(\varphi^{\frac{n-6}{n-2}} - \varphi \right) (\varphi - 1) \right\} dV_{g_1}.$$

Since $\varphi = c \neq 1$, by the same type of arguments used above, all the terms are non-negative a.e, implying that each coefficient must identically vanish. That is $\tilde{K}, \tilde{\pi}, \tilde{E}, \tilde{F} \equiv 0$ and $\frac{n-2}{4n} \tau^2 - 2c_n V(\phi) \equiv 0$. This, in turn, implies that $R_{g_1} - |d\phi|^2_{g_1} \equiv 0$, which gives us $\mathcal{SY}([\gamma]) = 0$ through Theorem 2.2.4. $\qquad \square$

Remark 2.2.4. *Notice that in the above theorem the dimensional restriction $n \leqslant 6$ relates only to the magnetic term $|\tilde{F}|^2_\gamma$.*

2.2.4   Existence results for the Lichnerowicz equation

During this section we will present a sequence of results which completely classify the low regularity CMC vacuum constraint equations and also provide partial classifications for the decoupled system in the presence of different sources. The vacuum classification follows from the classical foundational work of Isenberg (1995) for smooth coefficients and its low regularity version, in the vacuum case, has been obtained by Maxwell (2005a), who we will follow closely. In particular, we will adapt the results presented in Maxwell (ibid.) to account for our selected sources and also present them in the L^p-setting, which does not require much work starting from the low regularity L^2-theory presented there. First, let us rewrite (2.44) as follows:

$$-a_n \Delta_\gamma \varphi + r_\gamma \varphi = a_{TT} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}|^2_\gamma \varphi^{-3} + \frac{1}{2} |\tilde{F}|^2_\gamma \varphi^{\frac{n-6}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} \qquad (2.79)$$

where we have redefined the coefficients $a_{TT} \doteq |\tilde{K}|_\gamma^2 + \tilde{\pi}^2$ and $a_\tau \doteq b_n \tau^2 - 2V(\phi)$, where $b_n \doteq \frac{n-1}{n}$.

During all this section **we will restrict ourselves** to the case $a_\tau \geq 0$, which, in particular, contains the vacuum case. Some results can be translated to settings where this condition is weakened and we refer the interested reader to Choquet-Bruhat, Isenberg, and Pollack (2007). The following lemma deals with the existence of solutions to the above equation when $\mathcal{SY}([\gamma]) \geq 0$.

Lemma 2.2.6 (Existence - $\mathcal{SY} \geq 0, a_\tau \geq 0$). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geq 3$. If $\mathcal{SY}([\gamma]) \geq 0$, then equation (2.79) admits a positive solution $\varphi \in W^{2,p}$ if and only if one of the following conditions hold:*

1. $\mathcal{SY}([\gamma]) = 0$, $a_{TT}, a_\tau, \tilde{E}, \tilde{F} \equiv 0$;

2. $\mathcal{SY}([\gamma]) > 0$, $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$ and $a_\tau \equiv 0$;

3. $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$ and $a_\tau \not\equiv 0$.

Proof. Let us first consider the backwards implications. If 1) holds, then we must solve the equation $-a_n \Delta_\gamma \varphi + r_\gamma \varphi = 0$ for some positive element in $W^{2,p}$. Such solution is guaranteed to exist via Theorem 2.2.4 since $\mathcal{SY}([\gamma]) = 0$ and therefore the claim follows. Let us therefore concentrate in cases 2) and 3). In both of these cases, let us start considering a conformal deformation $\gamma_1 = \varphi_1^{\frac{4}{n-2}} \gamma$ fixing the sign of r_{γ_1} according to the sign of $\mathcal{SY}([\gamma])$. From Lemma 2.1.1, we know that there is a solution of (2.79) iff there is a solution of

$$-a_n \Delta_{\gamma_1} \varphi + r_{\gamma_1} \varphi = a_{TT}^{(1)} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}_1|_{\gamma_1}^2 \varphi^{-3} + \frac{1}{2} |\tilde{F}|_{\gamma_1}^2 \varphi^{\frac{n-6}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}},$$
$$(2.80)$$

where $r_{\gamma_1} = R_{\gamma_1} - |d\phi|_{\gamma_1}^2$, $a_{TT}^{(1)} = \varphi_1^{-\frac{4n}{n-2}} a_{TT}$ and $\tilde{E}_1 = \varphi_1^{-\frac{2n}{n-2}} \tilde{E}$. Let us then analyse existence of solutions to (2.80), where we now control the sign of r_γ, from which we know that $r_\gamma + a_\tau \geq 0$ and $r_\gamma + a_\tau \not\equiv 0$. Therefore, Theorem B.7 guarantees a solution $\varphi_2 \in W^{2,p}$ to

$$-a_n \Delta_{\gamma_1} \varphi_2 + (r_{\gamma_1} + a_\tau) \varphi_2 = a_{TT}^{(1)} + |\tilde{E}_1|_{\gamma_1}^2 + \frac{1}{2} |\tilde{F}|_{\gamma_1}^2 \geq 0. \qquad (2.81)$$

Lemma 2.2.2 implies that $\varphi_2 \geq 0$ and Lemma 2.2.3 applied to $-\varphi_2$ implies that either $\varphi_2 > 0$ or $\varphi_2 \equiv 0$, but the second case contradicts our hypotheses, since

$a_{TT}^{(1)} + |\tilde{E}_1|^2_{\gamma_1} + \frac{1}{2}|\tilde{F}|^2_{\gamma_1} \not\equiv 0$. Therefore $\varphi_2 > 0$ and we can consider one further conformal deformation $\gamma_2 \doteq \varphi_2^{\frac{4}{n-2}} \gamma_1$. Appealing to Equation (2.35) once more, we know that (2.80) admits a positive $W^{2,p}$-solution iff

$$-a_n \Delta_{\gamma_2}\varphi + r_{\gamma_2}\varphi = a_{TT}^{(2)}\varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}_2|^2_{\gamma_2}\varphi^{-3} + \frac{1}{2}|\tilde{F}|^2_{\gamma_2}\varphi^{\frac{n-6}{n-2}} - a_\tau\varphi^{\frac{n+2}{n-2}},$$
$$\tag{2.82}$$

admits a positive $W^{2,p}$-solution, where $r_{\gamma_2} = R_{\gamma_2} - |d\phi|^2_{\gamma_2}$, $a_{TT}^{(2)} = \varphi_2^{-\frac{4n}{n-2}} a_{TT}^{(1)}$ and $\tilde{E}_2^i = \varphi_2^{-\frac{2n}{n-2}} \tilde{E}_1^i$. Furthermore, our choice of φ_2 implies that

$$\begin{aligned}
r_{\gamma_2} &= \varphi_2^{-\frac{n+2}{n-2}} \left(-a_n \Delta_{\gamma_1}\varphi_2 + r_{\gamma_1}\varphi_2\right), \\
&= \varphi_2^{-\frac{n+2}{n-2}} \left(a_{TT}^{(1)} + |\tilde{E}_1|^2_{\gamma_1} + \frac{1}{2}|\tilde{F}|^2_{\gamma_1} - a_\tau\varphi_2\right), \\
&= \varphi_2^{-\frac{n+2}{n-2}} \left(a_{TT}^{(2)}\varphi_2^{\frac{4n}{n-2}} + \varphi_2^{\frac{4n-4}{n-2}}|\tilde{E}_2|^2_{\gamma_2} + \frac{1}{2}\varphi_2^{\frac{8}{n-2}}|\tilde{F}|^2_{\gamma_2} - a_\tau\varphi_2\right), \\
&= a_{TT}^{(2)}\varphi_2^{\frac{3n-2}{n-2}} + \varphi_2^3|\tilde{E}_2|^2_{\gamma_2} + \frac{1}{2}\varphi_2^{-\frac{n-6}{n-2}}|\tilde{F}|^2_{\gamma_2} - a_\tau\varphi_2^{-\frac{4}{n-2}}
\end{aligned}$$

Then, φ_+ is a constant supersolution if

$$\left(a_{TT}^{(2)}\varphi_2^{\frac{3n-2}{n-2}} + \varphi_2^3|\tilde{E}_2|^2_{\gamma_2} + \frac{1}{2}\varphi_2^{-\frac{n-6}{n-2}}|\tilde{F}|^2_{\gamma_2} - a_\tau\varphi_2^{-\frac{4}{n-2}}\right)\varphi_+ \geqslant$$
$$a_{TT}^{(2)}\varphi_+^{-\frac{3n-2}{n-2}} + |\tilde{E}_2|^2_{\gamma_2}\varphi_+^{-3} + \frac{1}{2}|\tilde{F}|^2_{\gamma_2}\varphi_+^{\frac{n-6}{n-2}} - a_\tau\varphi_+^{\frac{n+2}{n-2}}$$

Consider φ_+ to be a constant sufficiently large so as to satisfy

$$\varphi_+^{\frac{3n-2}{n-2}+1} = \varphi_+^{4\frac{n-1}{n-2}} \geqslant \max_M \varphi_2^{-\frac{3n-2}{n-2}}, \quad \varphi_+^4 \geqslant \max_M \varphi_2^{-3},$$
$$\varphi_+^{1-\frac{n-6}{n-2}} = \varphi_+^{\frac{4}{n-2}} \geqslant \max_M \varphi_2^{\frac{n-6}{n-2}}, \quad \varphi_+^{\frac{4}{n-2}} \geqslant \max_M \varphi_2^{-\frac{4}{n-2}}.$$

which is always possible, since φ_2 is continuous and positive, and, since all the powers on φ_+ are positive, the above are all condition which bound φ_+ from below. Thus, for a large enough constant, we find a constant supersolution associated to

(2.82). To find a constant subsolutions, we must reverse inequalities and change $\max_M \to \min_M$. That is, let us chose a constant $\varphi_- > 0$ satisfying

$$\varphi_-^{4\frac{n-1}{n-2}} \leq \min_M \varphi_2^{-\frac{3n-2}{n-2}}, \quad \varphi_-^4 \leq \min_M \varphi_2^2, \quad \varphi_-^{\frac{4}{n-2}} \geq \min_M \varphi_2^{\frac{n-6}{n-2}}, \quad \varphi_-^{\frac{4}{n-2}} \leq \min_M \varphi_2^{-\frac{4}{n-2}},$$

It is also clear that under these conditions $\varphi_- \leq \varphi_+$ and therefore we have provided barriers satisfying the conditions of Theorem 2.2.1 for equation (2.82). Thus, there is a positive solution $\varphi \in W^{2,p}$ to these last equations, which proves via Lemma 2.1.1 that the original equation admits a positive $W^{2,p}$ solution.

Let us now prove the necessity of conditions 1) − 3). We need to show that under the condition $\mathcal{SY}([\gamma]) \geq 0$, then the following claims follows:

- $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \equiv 0 \Longrightarrow \mathcal{SY}([\gamma]) = 0$ and all the coefficients vanish;

- $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$ and $a_\tau \equiv 0 \Longrightarrow \mathcal{SY}([\gamma]) > 0$.

Notice that if the above claims hold, then, the first one implies that if $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \equiv 0$, then we are in case 1). On the other hand, if $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$, then either $a_\tau \equiv 0$ or $a_\tau \not\equiv 0$. In the latter case we are in case 3), while in the former case the second item above implies $\mathcal{SY}([\gamma]) > 0$ and therefore we must fall into case 2) and the necessity of conditions 1) − 3) follows. Let us then prove the above two claims.

Assume that we have a positive solution $\varphi \in W^{2,p}$ to (2.79), which implies that the metric $g = \varphi^{\frac{4}{n-2}} \gamma$ satisfies

$$\begin{aligned} r_g &= \varphi^{-\frac{n+2}{n-2}} \left(r_\gamma \varphi - a_n \Delta_\gamma \varphi \right), \\ &= \varphi^{-\frac{n+2}{n-2}} \left(a_{TT} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}|_\gamma^2 \varphi^{-\frac{3n-2}{n-2}} + \frac{1}{2} |\tilde{F}|_\gamma^2 \varphi^{\frac{n-6}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} \right). \end{aligned}$$

Considering the first item, we see that $r_g = -a_\tau \leq 0$. Therefore, it follows that $\mathcal{SY}([\gamma]) \leq 0$, implying that $\mathcal{SY}([\gamma]) = 0$. Furthermore, if $a_\tau \not\equiv 0$, we must have $\mathcal{SY}([\gamma]) < 0$, simply by testing $\tilde{J}_{\gamma,\frac{n}{n-2}}$ on constant functions. Since this contradicts our hypotheses, we find $a_\tau \equiv 0$ and the first claim follows.

In order to establish the second claim, notice that if $a_\tau \equiv 0$, then

$$r_g = \varphi^{-\frac{n+2}{n-2}} \left(a_{TT} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}|_\gamma^2 \varphi^{-\frac{3n-2}{n-2}} + \frac{1}{2} |\tilde{F}|_\gamma^2 \varphi^{\frac{n-6}{n-2}} \right) \geq 0.$$

Now, Proposition 2.2.3 implies that if $\mathcal{SY}([g]) \geq 0$ and $r_g \geq 0$, then $\mathcal{SY}([g]) = 0$ iff $r_g = 0$ a.e. Since by hypotheses $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$, then $r_g \not\equiv 0$ and we cannot have $\mathcal{SY}([g]) = 0$, which proves that $\mathcal{SY}([g]) > 0$ and finishes the proof. $\qquad\qquad\square$

The Yamabe negative case demands us to sharpen our classification of Theorem 2.2.4 in the negative case. This will be done in Lemma 2.2.7 below, but let us first present the following proposition which explicitly tells us when equation (2.79) admits solutions in this setting. Both results below, just as the lemma above, rely on mild adaptations from the results presented in Maxwell (2005a).

Proposition 2.2.4. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geq 3$. If $\mathcal{SY}([\gamma]) < 0$, then $\varphi \in W^{2,p}$ is a solution of (2.79) if and only if there is a conformal deformation $\gamma_1 \doteq \varphi^{\frac{4}{n-2}} \gamma$ with $\varphi_1 \in W^{2,p}$ such that $R_{\gamma_1} = -a_\tau$.*

Proof. First, notice that if $a_{TT}, \tilde{E}, \tilde{F} \equiv 0$, then the Lichnerowicz equation is precisely given by

$$-a_n \Delta_\gamma \varphi + r_\gamma \varphi = -a_\tau \varphi^{\frac{n+2}{n-2}}.$$

Then, the existence of a positive solution is equivalent to a deformation to $r_{\varphi^{\frac{4}{n-2}}\gamma} = \varphi^{-\frac{n+2}{n-2}} \left(-a_n \Delta_\gamma \varphi + r_\gamma \varphi \right) = -a_\tau$, and therefore this case is trivial. Thus, from now on, let us assume that $a_{TT} + |\tilde{E}|_\gamma^2 + |\tilde{F}|_\gamma^2 \not\equiv 0$. Let us now start assuming that there is a conformal deformation $\gamma_1 = \varphi_1^{\frac{4}{n-2}} \gamma$ to $r_{\gamma_1} = -a_\tau$ and analyse the existence of solutions to

$$-a_n \Delta_{\gamma_1} \varphi + r_{\gamma_1} \varphi = a_{TT}^{(1)} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}_1|_{\gamma_1}^2 \varphi^{-3} + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2 \varphi^{\frac{n-6}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}},$$
(2.83)

where $r_{\gamma_1} = R_{\gamma_1} - |d\phi|_{\gamma_1}^2$, $a_{TT}^{(1)} = \varphi_1^{-\frac{4n}{n-2}} a_{TT}$ and $\tilde{E}_1 = \varphi_1^{-\frac{2n}{n-2}} \tilde{E}$. Since $\mathcal{SY}([\gamma]) < 0$ and $r_{\gamma_1} = -a_\tau \leq 0$, we must have $a_\tau \not\equiv 0$ to avoid $\mathcal{SY}([\gamma]) = 0$. Let us first rewrite (2.83) using our hypothesis

$$-a_n \Delta_{\gamma_1} \varphi - a_\tau \varphi = a_{TT}^{(1)} \varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}_1|_{\gamma_1}^2 \varphi^{-3} + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2 \varphi^{\frac{n-6}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}}.$$
(2.84)

In order to prove the existence of a solution, we will exhibit barriers for this equation. First, consider the unique solution $\varphi_2 \in W^{2,p}$ to the equation

$$-a_n \Delta_{\gamma_1} \varphi_2 + a_\tau \varphi_2 = a_{TT}^{(1)} + |\tilde{E}_1|_{\gamma_1}^2 + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2,$$

which is guaranteed to exist via Theorem B.7 since $a_\tau \geqslant 0$ and $a_\tau \not\equiv 0$. Also, since $a_{TT}^{(1)} + |\tilde{E}_1|_{\gamma_1}^2 + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2 \geqslant 0$ and does not vanish identically, we can use the maximum principles to guarantee that φ_2 is strictly positive. Then, we consider the metric $\gamma_2 = \varphi^{\frac{4}{n-2}}\gamma_1$ from which it follows that

$$
\begin{aligned}
r_{\gamma_2} &= \varphi_2^{-\frac{n+2}{n-2}}\left(r_{\gamma_1}\varphi_2 - \Delta_{\gamma_1}\varphi_2\right), \\
&= \varphi_2^{-\frac{n+2}{n-2}}\left(-2a_\tau\varphi_2 - + a_{TT}^{(1)} + |\tilde{E}_1|_{\gamma_1}^2 + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2\right), \\
&= -2a_\tau\varphi_2^{-\frac{4}{n-2}} + \varphi_2^{\frac{3n-2}{n-2}}a_{TT}^{(2)} + \varphi_2^3|\tilde{E}_2|_{\gamma_2}^2 + \frac{1}{2}\varphi_2^{-\frac{n-6}{n-2}}|\tilde{F}|_{\gamma_2}^2,
\end{aligned}
$$

and we intend to solve the conformally related equation

$$-a_n\Delta_{\gamma_2}\varphi + r_{\gamma_2}\varphi = a_{TT}^{(2)}\varphi^{-\frac{3n-2}{n-2}} + |\tilde{E}_2|_{\gamma_2}^2\varphi^{-3} + \frac{1}{2}|\tilde{F}|_{\gamma_2}^2\varphi^{\frac{n-6}{n-2}} - a_\tau\varphi^{\frac{n+2}{n-2}}, \tag{2.85}$$

looking for constant barriers. For the supersolution we must find

$$\left(-2a_\tau\varphi_2^{-\frac{4}{n-2}} + \varphi_2^{\frac{3n-2}{n-2}}a_{TT}^{(2)} + \varphi_2^3|\tilde{E}_2|_{\gamma_2}^2 + \frac{1}{2}\varphi_2^{-\frac{n-6}{n-2}}|\tilde{F}|_{\gamma_2}^2\right)\varphi_+ \geqslant$$

$$a_{TT}^{(2)}\varphi_+^{-\frac{3n-2}{n-2}} + |\tilde{E}_2|_{\gamma_2}^2\varphi_+^{-3} + \frac{1}{2}|\tilde{F}|_{\gamma_2}^2\varphi_+^{\frac{n-6}{n-2}} - a_\tau\varphi_+^{\frac{n+2}{n-2}}.$$

In order to satisfy these conditions, pick a constant φ_+ satisfying

$$\varphi_+^{4\frac{n-1}{n-2}} \geqslant \max_M \varphi_2^{-\frac{3n-2}{n-2}}, \quad \varphi_+^4 \geqslant \max_M \varphi_2^{-3}, \quad \varphi_+^{\frac{4}{n-2}} \geqslant \max_M \varphi_2^{-\frac{n-6}{n-2}}, \quad \varphi_+^{\frac{4}{n-2}} \geqslant 2\max_M \varphi_2^{-\frac{4}{n-2}},$$

Similarly to the analysis of Lemma 2.2.6, we can find a constant subsolution $\varphi_- > 0$ choosing

$$\varphi_-^{4\frac{n-1}{n-2}} \leqslant \min_M \varphi_2^{-\frac{3n-2}{n-2}}, \quad \varphi_-^4 \leqslant \min_M \varphi_2^{-3}, \quad \varphi_-^{\frac{4}{n-2}} \geqslant \min_M \varphi_2^{-\frac{n-6}{n-2}}, \quad \varphi_-^{\frac{4}{n-2}} \leqslant 2\min_M \varphi_2^{-\frac{4}{n-2}},$$

Therefore, through Theorem 2.2.1 we have a positive solution to (2.85) and therefore, via Lemma 2.1.1 we have a solution of (2.83).

For the converse, let us assume the existence of a positive $W^{2,p}$ solution to (2.79) and solve the prescribed curvature problem $r_g = -a_\tau$, with g conformal to γ. We know that, our hypotheses, we can start deforming γ to $\gamma_1 = \varphi_1^{\frac{4}{n-2}}\gamma$ so that $r_{\gamma_1} < 0$ and $\varphi_1 \in W^{2,p}$. Let us then consider the conformal deformation $\gamma_1 = \varphi_1^{\frac{4}{n-2}}\gamma$ and look for a conformal deformation to $r_g = -a_\tau$ but starting from γ_1. That is, we search for a positive $W^{2,p}$-solution to

$$-\Delta_{\gamma_1}\varphi + r_{\gamma_1}\varphi = -a_\tau \varphi^{\frac{n+2}{n-2}}. \tag{2.86}$$

The existence of a solution to (2.79) implies the existence of a positive solution $\varphi \in W^{2,p}$ to (2.83). Then, set $\varphi_+ \doteq \varphi_1$ and notice that this is a supersolution to (2.86), since

$$-a_n\Delta_{\gamma_1}\varphi_1 + r_{\gamma_1}\varphi_1 = a_{TT}^{(1)}\varphi_1^{-\frac{3n-2}{n-2}} + |\tilde{E}_1|_{\gamma_1}^2\varphi_1^{-3} + \frac{1}{2}|\tilde{F}|_{\gamma_1}^2\varphi_1^{\frac{n-6}{n-2}} - a_\tau\varphi_1^{\frac{n+2}{n-2}},$$

$$\geq -a_\tau\varphi_1^{\frac{n+2}{n-2}}.$$

A subsolution can be constructed considering the family of equations

$$-\Delta_{\gamma_1}\varphi_\epsilon - r_{\gamma_1}\varphi_\epsilon = -r_{\gamma_1} - \epsilon a_\tau, \tag{2.87}$$

with ϵ in a neighbourhood of zero. Since $r_{\gamma_1} < 0$, we know that there is a unique solution to this equation for any such ϵ and, in particular, $\varphi_0 \equiv 1$. Furthermore,

$$\|\varphi_{\epsilon_1} - \varphi_{\epsilon_2}\|_{C^0} \leq C\|\varphi_{\epsilon_1} - \varphi_{\epsilon_2}\|_{W^{2,p}} \leq C'\|a_\tau\|_{L^p}|\epsilon_1 - \epsilon_2|,$$

which implies that for ϵ sufficiently small $\varphi_\epsilon > 0$, and we now restrict to a neighbourhood of $\epsilon = 0$ such that $\varphi_\epsilon > \frac{1}{2}$. Let us now consider a positive constant $\eta > 0$ and the function $\varphi_- \doteq \eta\varphi_\epsilon > 0$. First, pick such η so that $\varphi_- < \varphi_+$ and notice that

$$-\Delta_{\gamma_1}\varphi_- + r_{\gamma_1}\varphi_- = \eta\left(r_{\gamma_1}\varphi_\epsilon - r_{\gamma_1} - \epsilon a_\tau + r_{\gamma_1}\varphi_\epsilon\right),$$

$$= \eta(\underbrace{r_{\gamma_1}}_{<0}\underbrace{(2\varphi_\epsilon - 1)}_{>0} - \epsilon a_\tau) \leq -\eta\epsilon a_\tau.$$

Therefore, if $\eta^{\frac{n+2}{n-2}-1} = \eta^{\frac{4}{n-2}} \leq \epsilon \min_M \varphi_\epsilon^{-\frac{n+2}{n-2}}$, it follows that

$$-\Delta_{\gamma_1}\varphi_- + r_{\gamma_1}\varphi_- \leq -a_\tau(\eta\varphi_\epsilon)^{\frac{n+2}{n-2}} = -a_\tau\varphi_-^{\frac{n+2}{n-2}}, \tag{2.88}$$

and $0 < \varphi_- \leqslant \varphi_+$ is a subsolution. Therefore, via Theorem 2.2.1 we have a solution to (2.86), which finishes the proof. $\qquad\square$

Lemma 2.2.7. *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geqslant 3$. Let $\mathcal{SY}([\gamma]) < 0$ and assume that $a_\tau \in L^p$ satisfies $a_\tau \geqslant c > 0$ a.e for some positive constant c. Then, there is a a positive function $\varphi \in W^{2,p}$ such that $g \doteq \varphi^{\frac{4}{n-2}} \gamma$ has $r_g = -a_\tau$.*

Proof. First, we start considering a deformation $\gamma_1 = \varphi_1^{\frac{4}{n-2}} \gamma$ into a metric with $r_{\gamma_1} < 0$ and continuous, with $\varphi_1 \in W^{2,p}$. Then, there are constants $c_1, c_2 > 0$ such that

$$-c_1 (1 + a_\tau) \leqslant r_{\gamma_1} \leqslant -c_2 (1 + a_\tau).$$

We intend to prove the existence of a positive $W^{2,p}$-solution to

$$-a_n \Delta_{\gamma_1} \varphi + r_{\gamma_1} \varphi = -a_\tau \varphi^{\frac{n+2}{n-2}}. \tag{2.89}$$

With this in mind, consider φ_- a positive constant. Then,

$$-a_n \Delta_{\gamma_1} \varphi_- + r_{\gamma_1} \varphi_- = r_{\gamma_1} \varphi_- \leqslant -c_2 (1 + a_\tau) \varphi_- \leqslant -a_\tau c_2 \varphi_-.$$

Then, choosing $0 < \varphi^{\frac{4}{n-2}} < c_2$, we find $-a_n \Delta_{\gamma_1} \varphi_- + r_{\gamma_1} \varphi_- \leqslant -a_\tau \varphi^{\frac{n+2}{n-2}}$. Similarly, let φ_+ be a a positive constant. Then,

$$-a_n \Delta_{\gamma_1} \varphi_+ + r_{\gamma_1} \varphi_+ = r_{\gamma_1} \varphi_+ \geqslant -c_1 (1 + a_\tau) \varphi_+$$

We intend to fix φ_+ so that

$$a_\tau \varphi_+^{\frac{n+2}{n-2}} \geqslant c_1 (a_\tau + 1) \varphi_+ \quad a.e,$$

which, since $a_\tau \geqslant c > 0$ a.e, is equivalent to

$$\varphi_+^{\frac{4}{n-2}} \geqslant c_1 \left(1 + \frac{1}{a_\tau}\right) \quad a.e.$$

Then, if $\varphi_+^{\frac{4}{n-2}} > c_1 \left(1 + \frac{1}{c}\right) \geqslant c_1 \left(1 + \frac{1}{a_\tau}\right) > 0$, it follows that

$$-a_n \Delta_{\gamma_1} \varphi_+ + r_{\gamma_1} \varphi_+ \geqslant -a_\tau \varphi_+^{\frac{n+2}{n-2}}$$

and therefore such φ_+ gives us a superoslution to (2.89). Therefore, once more via Theorem 2.2.1, we have proven the existence of a solution to (2.89) and the main claim follows. $\qquad\square$

Remark 2.2.5. *Let us highlight that the above Lemma improves the Yamabe classification given in Theorem 2.2.4, since it guarantess that $\mathcal{SY}([\gamma]) < 0$ if and only if there is a conformally related metric $g = \varphi^{\frac{4}{n-2}}\gamma$ with constant negative curvature $r_\gamma = -c$.*

We can now state the main result of this section, which classifies the solutions to the Lichnerowicz equation (2.79).

Theorem 2.2.7 (Existence - $a_\tau \geqslant 0$). *Let (M^n, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > \frac{n}{2}$, and $n \geqslant 3$. Then, equation (2.79) admits a positive solution $\varphi \in W^{2,p}$ if of the following conditions hold:*

1. *$\mathcal{SY}([\gamma]) = 0$, $a_{TT}, a_\tau, \widetilde{E}, \widetilde{F} \equiv 0$;*

2. *$\mathcal{SY}([\gamma]) > 0$, $a_{TT} + |\widetilde{E}|_\gamma^2 + |\widetilde{F}|_\gamma^2 \not\equiv 0$ and $a_\tau \equiv 0$;*

3. *$\mathcal{SY}([\gamma]) \geqslant 0$, $a_{TT} + |\widetilde{E}|_\gamma^2 + |\widetilde{F}|_\gamma^2 \not\equiv 0$ and $a_\tau \not\equiv 0$;*

4. *$\mathcal{SY}([\gamma]) < 0$ and $a_\tau \geqslant c > 0$.*

In case a_τ is constant, then the above conditions are also necessary conditions for existence.[7]

Proof. Notice that the first three cases correspond to Lemma 2.2.6. The last one follows from Lemma 2.2.7 put together with Proposition 2.2.4. Finally, the necessity of these conditions follows from Lemma 2.2.6 for the first three cases and, if a_τ is constant, since $a_\tau \geqslant 0$, then either $a_\tau \equiv 0$ or $a_\tau \equiv c > 0$. But, if $a_\tau \equiv 0$ and φ solves (2.79), then

$$-a_n \Delta_\gamma \varphi + r_\gamma \varphi = a_{TT}\varphi^{-\frac{3n-2}{n-2}} + |\widetilde{E}|_\gamma^2 \varphi^{-\frac{3n-2}{n-2}} + \frac{1}{2}|\widetilde{F}|_\gamma^2 \varphi^{-\frac{n-6}{n-2}} \geqslant 0.$$

But then

$$
\begin{aligned}
r_{\varphi^{\frac{4}{n-2}}\gamma} &= \varphi^{-\frac{n+2}{n-2}}\left(-a_n\Delta_\gamma\varphi + r_\gamma\varphi\right), \\
&= \varphi^{-\frac{n+2}{n-2}}\left(a_{TT}\varphi^{-\frac{3n-2}{n-2}} + |\widetilde{E}|_\gamma^2\varphi^{-\frac{3n-2}{n-2}} + \frac{1}{2}|\widetilde{F}|_\gamma^2\varphi^{-\frac{n-6}{n-2}}\right) \geqslant 0,
\end{aligned}
$$

which would imply that $\mathcal{SY}([\gamma]) \geqslant 0$, contradicting our hypotheses. Therefore, we must have $a_\tau \equiv c > 0$. □

---

[7]Notice that the necessity part of the theorem in cases $(1) - (3)$ holds *without* CMC assumptions due to Lemma 2.2.6.

Let us highlight that if we neglect the physical sources and consider only the vacuum CMC case, then the momentum constraint becomes

$$\Delta_{\gamma,\text{conf}}X = 0, \tag{2.90}$$

which implies that X is a $W^{2,p}$-conformal Killing field and thus $\tilde{K} = U$ becomes a TT-tensor, which is freely prescribed. Then, (2.79) is given by

$$-a_n \Delta_\gamma \varphi + R_\gamma \varphi = |U|^2_\gamma \varphi^{-\frac{3n-2}{n-2}} - b_n \tau^2 \varphi^{\frac{n+2}{n-2}} \tag{2.91}$$

with $b_n = \frac{n-1}{n}$. Then, the above theorem gives us the following classification for the existence of solutions to (2.91):

	$\tau = 0\ U = 0$	$\tau = 0\ U \neq 0$	$\tau \neq 0\ U = 0$	$\tau \neq 0\ U \neq 0$
$\mathcal{Y}_\gamma > 0$	No	Yes	No	Yes
$\mathcal{Y}_\gamma = 0$	Yes	No	No	Yes
$\mathcal{Y}_\gamma < 0$	No	No	Yes	Yes

The content of the above table is well-known from Isenberg (1995) when the coefficients are smooth and from Maxwell (2005a) when the coefficients are of low (even *rough*) regularity.

3

Maximal Solutions on asymptotically Euclidean manifolds

In this Chapter we want to analyse a somehow complementary case to the one considered in the previous one. In particular, we would like to analyse initial data modelling isolated systems. In physics, many times, a system which at large distances interacts sufficiently weakly with the rest of the Universe can be idealised as *isolated*. In such situations, we have a compact core region where fields and matter may be interacting (even very strongly) but, as we move away, fields will fall off as we approach a vacuum condition at *infinity*. In the context of gravitational systems, we consider that the vacuum condition at (space) infinity is given by approaching (in some sense to be made precise below) the Minkowski solution. Thus, being concerned with initial data sets, we will attempt to construct initial data which outside some compact set falls to Minkowski's initial data. This is made precise by first fixing our manifold structure to be a non-compact manifold, which, outside some compact set, consists of a finite number of ends diffeomorphic to the exterior of a compact set in $\mathbb{R}^n$. Then, we shall impose decaying conditions

for our fields.

During this section, we will restrict ourselves to the vacuum CMC case and, in order to make such initial data sets more *realistic*, we will consider boundary value problems with boundary conditions on some (possibly multiply connected) inner compact boundary Σ. Whatever lies inside such a boundary could be thought of as the source of our gravitational field. It is particularly interesting to consider boundary conditions which model black hole initial data, which gives us the opportunity to analyse quite general black hole solutions. These boundary conditions will be extracted from the analysis of Section 1.5. In such a case, the removal of the compact region inside the black hole would not present problems from the point of view of predictability of the exterior region. Finally, let us highlight that the not necessarily vacuum case will be treated in the next chapter, in the context of freely prescribed mean curvature initial data sets.

3.1 AE manifolds - Analytical tools

Let us now introduce some definitions and technical results concerning AE manifolds. First, as has been stated above, we will consider manifolds M^n (possibly with boundary) which consist of a compact core K such that $M \setminus K$ is the disjoint union of a finite number of open sets U_i, such that each U_i is diffeomorphic to the complement of a closed ball in Euclidean space. Such manifolds, in part of the classic literature, are referred to as *Euclidean at infinity* (see Choquet-Bruhat and Christodoulou (1981)). The diffeomorphisms $\Phi_i : U_i \subset M \mapsto \mathbb{R}^n \setminus \overline{B}$ induce charts, which are referred to as *end coordinate systems* and are said to provide a *structure of infinity* (Bartnik 1986).

On these model manifolds we want to control the behaviour of fields near infinity. This can be done in different ways. For instance, some authors opt to fix the end coordinate systems and impose decay rates for fields written in those coordinates.[1] Another common option is to introduce function spaces with weights adapted to our manifold structure which provide good controls of the asymptotic behaviour of the fields (see, for instance, Bartnik (1986), Cantor (1981), Choquet-Bruhat and Christodoulou (1981), and Maxwell (2005b)). For our purposes, the latter option is best, since, as we will see below, we can also tailor such weighted spaces to have good analytic properties useful for our PDE analysis. Such spaces

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[1]This option is quite common within the analysis of conserved quantities, such as the analysis of the positive mass theorem (D. A. Lee 2019; R. Schoen and S. T. Yau 1979; R. M. Schoen and S.-T. Yau 1979).

have been investigated for a long time by different authors, such as Bartnik (1986), Choquet-Bruhat and Christodoulou (1981), Lockhart (1981), McOwen (1979), and Nirenberg and Walker (1973). In what follows, we will adopt the conventions given in Bartnik (1986) for the weight parameters, which has become the most common one in current literature and comment on how translate to the more classical notations of Choquet-Bruhat and Christodoulou (1981). Let us start with the following definition on $\mathbb{R}^n$.

Definition 3.1.1 (Weighted Spaces). *Let $E \to \mathbb{R}^n$ be vector bundle over $\mathbb{R}^n$. The weighted Sobolev space $W_\delta^{k,p}$, with k a non-negative integer, $1 < p < \infty$ and $\delta \in \mathbb{R}$, of sections u of E, is defined as the subset of $W_{loc}^{k,p}$ for which the norm*

$$\|u\|_{W_\delta^{k,p}(\mathbb{R}^n)} \doteq \sum_{|\alpha| \leq k} \|\sigma^{-\delta - \frac{n}{p} + |\alpha|} \partial^\alpha u\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

is finite, where $\sigma(x) \doteq (1 + |x|^2)^{\frac{1}{2}}$ and α denotes an arbitrary multi-index.

Similarly, the weighted C_δ^k-spaces are given by sections $u \in \Gamma(E)$, whose components are k-times continuously differentiable and which satisfy

$$\|u\|_{C_\delta^k} \doteq \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} \sigma^{-\delta + |\alpha|} |\partial^\alpha u(x)| < \infty. \tag{3.2}$$

The two types of spaces introduced above are easily seen to be Banach spaces. Let us in particular notice the duality $\left(L_\delta^p\right)' \cong L_{-\delta-n}^{p'}$, from which we see that these spaces are reflexive. Also, $u \in L_\delta^p \xrightarrow{\Phi} \sigma^{-\delta - \frac{n}{p}} u \in L^p$ provides an isometry between L_δ^p and $\Phi(L_\delta^p)$, which is a closed subspace of L^p, proving that L_δ^p is separable. Then, a standard argument shows that $W_\delta^{k,p}$ are also separable and reflexive (see, for instance, Adams (1975, page 46)). Furthermore, for $k \geq 0$, we define $W_{-\delta-n}^{-k,p}(\mathbb{R}^n) \doteq (\mathring{W}_\delta^{k,p}(\mathbb{R}^n))'$, where $\mathring{W}_\delta^{k,p}(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ in the $W_\delta^{k,p}$-norm. In particular, $\mathring{W}_\delta^{k,p}(\mathbb{R}^n) = W_\delta^{k,p}(\mathbb{R}^n)$, for any p, δ and k.

Remark 3.1.1. *In the definition given above, for a section $u \in \Gamma(E)$, locally, over any trivialization $E \cong U \times \mathbb{R}^N$, we have $u = (u^1, \cdots, u^N)$, with $N = \dim(E_x)$ for any $x \in U$. Induced on $E|_U$ we have the Euclidean norm on the vector valued function u, which we denote simply by $|u|$ and the $L^p(U)$-norm of u is given by*

$$\|u\|_{L^p(U)}^p = \int_U |u|^p dx.$$

The weight parametrization chosen in (3.1), introduced by Bartnik (1986), has the advantage to give us an *heuristic estimate* on the the behaviour of fields at infinity. Morally speaking, $u \in L^p_\delta \cap C^0$ will behave like $|u| = o(|x|^\delta)$. This is made precise via Sobolev embeddings associated to these weighted spaces, presented below.

Remark 3.1.2. *Many classical references, such as* Cantor (1981), Choquet-Bruhat *and* Christodoulou (1981), Lockhart (1981), McOwen (1979), *and* Nirenberg and *Walker* (1973) *adopt a different convention for the weight parameter, which is given by introducing the related norms:*

$$\|u\|_{\widetilde{W}^{k,p}_\rho} \doteq \sum_{|\alpha| \leqslant k} \|\sigma^{\rho+|\alpha|}\partial^\alpha u\|_{L^p}. \tag{3.3}$$

Therefore, these weighted Sobolev spaces relate to the ones introduce in (3.1) by

$$\widetilde{W}^{k,p}_\rho = W^{k,p}_{-(\rho+\frac{n}{p})}. \tag{3.4}$$

There is one further equivalent norm, introduced by Bartnik (1986), which proves to be very useful to translate results valid for compact manifold, and furthermore exploits some natural scaling properties associated these weighted spaces. In order to introduce this new norm, let us first introduce the scaling operator

$$S_R : L^p_{loc} \mapsto L^p_{loc},$$
$$u(x) \mapsto u(Rx)$$

for some given $R > 0$. Also, let us denote by A_r the annuli defined by $B_r \backslash \overline{B_{\frac{r}{2}}}$. Since $\frac{\sigma}{|x|}$ is continuous, positive and bounded on $\mathbb{R}^n \backslash \overline{B_{\frac{1}{2}}}$, then, there are constants $m, M > 0$ such that

$$m \leqslant \frac{\sigma(x)}{|x|} \leqslant M, \text{ for all } x \in \mathbb{R}^n \backslash \overline{B_{\frac{1}{2}}},$$

which implies that $m|x| \leqslant \sigma(x) \leqslant M|x|$ for all $x \in \mathbb{R}^n \backslash \overline{B_{\frac{1}{2}}}$, and therefore there are constants $c_1, c_2 > 0$ (independent of $r > 0$) such that for any fixed $\alpha \in \mathbb{R}$, it holds that

$$c_1 r^\alpha \leqslant \sigma^\alpha(x) \leqslant c_2 r^\alpha, \text{ for all } x \in A_r.$$

This, in particular, implies that for any $u \in W_\delta^{|\beta|,p}$ it holds that

$$r^{-\delta-\frac{n}{p}+|\beta|}\|\partial^\beta u\|_{L^p(A_r)} \lesssim \|\sigma^{-\delta-\frac{n}{p}+|\beta|}\partial^\beta u\|_{L^p(A_r)} \lesssim r^{-\delta-\frac{n}{p}+|\beta|}\|\partial^\beta u\|_{L^p(A_r)}$$

Let us also notice that

$$\int_{A_1} |\partial_x^\beta(S_r u)(x)|^p dx = r^{-n+p|\beta|} \int_{A_r} |\partial_x^\beta u(x)|^p dx,$$

implying that $\|\partial_\beta u\|_{L^p(A_r)} = r^{\frac{n}{p}-|\beta|}\|\partial_\beta(S_r u)\|_{L^p(A_1)}$. Putting together these inequalities, we find

$$c_1(\beta)r^{-\delta}\|\partial^\beta(S_r u)\|_{L^p(A_1)} \leq \|\sigma^{-\delta-\frac{n}{p}+|\beta|}\partial^\beta u\|_{L^p(A_r)} \leq c_2(\beta)r^{-\delta}\|\partial^\beta(S_r u)\|_{L^p(A_1)}.$$

Let us also notice that, for any multi-index β, we also have positive constants $c_3(\delta, p, n, \beta)$ and $c_4(\delta, p, n, \beta)$, so that

$$c_3(\delta, p, n, \beta)\|\partial^\beta u\|_{L^p(B_1)} \leq \|\sigma^{-\delta-\frac{n}{p}+|\beta|}\partial^\beta u\|_{L^p(B_1)}^p \leq c_4(\delta, p, n, \beta)\|\partial^\beta u\|_{L^p(B_1)}.$$

Rewriting

$$\|u\|_{W_\delta^{k,p}(\mathbb{R}^n)}^p = \|u\|_{W_\delta^{k,p}(B_1)}^p + \sum_{j=1}^\infty \|u\|_{W_\delta^{k,p}(A_{2^j})}^p,$$

the above estimates imply that the norm

$$\|u\|_{\tilde{W}_\delta^{k,p}(\mathbb{R}^n)}^p \doteq \|u\|_{W^{k,p}(B_1)}^p + \sum_{j=1}^\infty 2^{-j\delta p}\|S_{2^j}u\|_{W^{k,p}(A_1)}^p, \qquad (3.5)$$

is an equivalent norm on $W_\delta^{k,p}(\mathbb{R}^n)$. Using this alternative norm, the following properties are derived from those in Appendix A.2 quite straightforwardly.

Theorem 3.1.1. *Let $E \to \mathbb{R}^n$ be a vector bundle as in Definition 3.1.1. Then, the following continuous embeddings hold:*

1. *If $1 < p \leq q < \infty$ and $\delta_2 < \delta_1$, then $L_{\delta_2}^q \hookrightarrow L_{\delta_1}^p$;*

2. *If $kp < n$, then $W_\delta^{k,p} \hookrightarrow L_\delta^q$ for all $p \leq q \leq \frac{np}{n-kp}$;*

3. If $kp = n$, then $W^{k,p}_\delta \hookrightarrow L^q_\delta$ for all $q \geqslant p$;

4. If $kp > n$, then $W^{k+l,p}_\delta \hookrightarrow C^l_\delta$ for any $l = 0, 1, 2 \cdots$;

5. For any given $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that, for all $u \in W^{2,p}_\delta$, $1 < p < \infty$ the following inequality holds

$$\|u\|_{W^{1,p}_\delta} \leqslant \epsilon \|u\|_{W^{2,p}_\delta} + C_\epsilon \|u\|_{L^p_\delta}. \tag{3.6}$$

6. If $1 < p \leqslant q < \infty$ and $k_1 + k_2 > \frac{n}{q} + k$ where $k_1, k_2 \geqslant k$ are non-negative integers, then, we have a continuous multiplication property $W^{k_1,p}_{\delta_1} \otimes W^{k_2,q}_{\delta_2} \mapsto W^{k,p}_\delta$ for any $\delta > \delta_1 + \delta_2$. In particular, $W^{k,p}_\delta$ is an algebra under multiplication for $k > \frac{n}{p}$ and $\delta < 0$.

The first five of the above properties can be found, for instance, in Bartnik (1986, Theorem 2.1), while the multiplication property can be found in Cantor (1981, Lemma 5.5) and the corresponding L^2-version can also be found in Choquet-Bruhat and Christodoulou (1981, Lemma 2.5).[2] Furthermore, we have a version of the Rellich–Kondrachov theorem, given by:

Theorem 3.1.2. *Under the same assumption as in Theorem 3.1.1, if $k \geqslant 1$ and $\delta' < \delta$, the $W^{k,p}_{\delta'} \hookrightarrow W^{k-1,p}_\delta$ is compact.*

Proof. Let us consider a normalised sequence $\{u_j\} \subset W^{k,p}_{\delta'}$ and highlight that it is enough for us to prove any such sequence has a convergent subsequence in $W^{k-1,p}_\delta$. Let $B_R(0) \subset \mathbb{R}^n$ be the ball of radius R (to be fixed latter) with center at the origin. Let η be a cut-off function equal to one in B_R and supported in B_{2R}, and write $u_j = \eta u_j + (1 - \eta)u_j$. Then, it follows that $\{\eta u_j\} \subset W^{k,p}(B_{2R})$ is a bounded sequence, and therefore from Theorem A.2.1, the embedding $W^{k,p}(B_{2R}) \hookrightarrow W^{k-1,p}(B_{2R})$ is compact and thus there is a $W^{k-1,p}(B_{2R})$-convergent subsequence, to which we now restrict. Let us now show that the cor-

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[2]Some of these proofs are done for scalar functions in the cited references. The adaptation for the case of vector valued functions follows the same lines as those commented in Appendix A.2.

responding subsequence $\{u_j\} \subset W_\delta^{k-1,p}$ is Cauchy.

$$\|u_j - u_l\|_{W_\delta^{k-1,p}}^p = \sum_{|\alpha| \leqslant k-1} \int_{\mathbb{R}^n} \sigma^{-(\delta + \frac{n}{p})p} |\partial^\alpha (u_j - u_l)|^p dx,$$

$$\leqslant C' \sum_{|\alpha| \leqslant k-1} \int_{B_R} |\partial^\alpha (u_j - u_l)|^p dx$$

$$+ C \sum_{|\alpha| \leqslant k-1} \int_{\mathbb{R}^n \setminus B_R} |x|^{-(\delta - \delta')p} |x|^{-(\delta' + \frac{n}{p})p} |\partial^\alpha (u_j - u_l)|^p dx,$$

$$\leqslant C' \|u_j - u_l\|_{W^{k-1,p}(B_R)}^p$$

$$+ C R^{-(\delta - \delta')p} \|u_j - u_l\|_{W_{\delta'}^{k-1,p}(\mathbb{R}^n \setminus B_R)}^p,$$

where we have used that outside B_R we can estimate $\sigma(x) \lesssim |x|$, while inside of B_R this quantity is bounded. Also, in the last inequality, we used that $\delta - \delta' > 0$. Now, since $\{\eta u_j\}$ is convergent in $W^{k-1,p}(B_{2R})$, there is a limit function $\tilde{u} \in W^{k-1,p}(B_{2R})$, and since $\eta \equiv 1$ on B_R, we know that $\{u_j\} \subset W^{k-1,p}(B_R)$ is Cauchy. Therefore, given $\epsilon > 0$, we can pick j, l sufficiently large so that $C' \|u_j - u_l\|_{W^{k-1,p}(B_R)}^p \leqslant \epsilon$. Also, since $\{u_j\} \subset W_{\delta'}^{k,p}$ is bounded, then

$$\sum_{|\alpha| \leqslant k} \int_{\mathbb{R}^n \setminus B_R} |x|^{-(\delta' + \frac{n}{p})p} |\partial^\alpha (u_j - u_l)|^p dx \leqslant C'',$$

for some fixed constant $C'' > 0$ independent of R. We can then chose R in our argument so that $R^{-(\delta - \delta')} < \frac{\epsilon}{CC''}$. Therefore,

$$\|u_j - u_l\|_{W_\delta^{k-1,p}}^p \leqslant 2\epsilon, \tag{3.7}$$

which proves that $\{u_j\}$ is Cauchy in $W_\delta^{k-1,p}$, and is therefore convergent, establishing the compactness of the embedding $W_{\delta'}^{k,p} \hookrightarrow W_\delta^{k-1,p}$. $\square$

Let us now present the following result result, due to Choquet-Bruhat (2009, Theorem 3.5, Appendix I), which gives us a more subtle interpolation inequality in this context. Such inequality is useful when dealing with low regularity problems.

Theorem 3.1.3. *Let $1 < q < \infty$, k, m be integers $0 \leqslant k < m$, θ be a real number in the interval $\frac{k}{m} \leqslant \theta \leqslant 1$ and*

$$\frac{1}{p} = \frac{k - m\theta}{n} + \frac{1}{q}, \tag{3.8}$$

If $m - k - \frac{n}{q} \notin \mathbb{N}$, then, for any given $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that, for all $u \in W_\delta^{m,q}(\mathbb{R}^n)$,

$$\|\partial^k u\|_{L^p_{\delta-k}(\mathbb{R}^n)} \leqslant \epsilon \|u\|_{W^{m,q}_\delta(\mathbb{R}^n)} + C_\epsilon \|u\|_{L^q_\delta(\mathbb{R}^n)}, \tag{3.9}$$

where $\partial^k u$ denotes any k-th order derivative of u.

Proof. We will prove the claim for the norm (3.5). Thus, let us first fix some multi-index β with $|\beta| = k$, and notice that

$$\|S_{2^j} \partial^\beta u\|^p_{L^p(A_1)} = \int_{A_1} |\partial^\beta u(2^j x)|^p dx = 2^{-jp|\beta|} \|\partial^\beta (S_{2^j} u)\|^p_{L^p(A_1)}.$$

Now, using (A.30) on A_1, we see that

$$\|S_{2^j} \partial^\beta u\|^p_{L^p(A_1)} \leqslant C 2^{-jp|\beta|} \|S_{2^j} u\|^{\theta p}_{W^{m,q}(A_1)} \|S_{2^j} u\|^{(1-\theta)p}_{L^q(A_1)},$$

where the constant C depends on A_1, which is fixed. Also, we know from Theorem A.2.7 that

$$\|\partial^\beta u\|^p_{L^p(B_1)} \leqslant C \|u\|^{p\theta}_{W^{m,q}(B_1)} \|u\|^{p(1-\theta)}_{L^q(B_1)}.$$

Therefore, from (3.5), we find

$$\|\partial^\beta u\|^p_{\dot{L}^p_{\delta-|\beta|}} \doteq \|\partial^\beta u\|^p_{L^p(B_1)} + \sum_{j=1}^\infty 2^{-j(\delta-|\beta|)p} \|S_{2^j} (\partial^\beta u)\|^p_{L^p(A_1)},$$

$$\lesssim \|u\|^{p\theta}_{W^{m,q}(B_1)} \|u\|^{p(1-\theta)}_{L^q(B_1)} + \sum_{j=1}^\infty 2^{-j\delta p} \|S_{2^j} u\|^{\theta p}_{W^{m,q}(A_1)} \|S_{2^j} u\|^{(1-\theta)p}_{L^q(A_1)}.$$

Now, recall that for any given $\epsilon' > 0$ and for any fixed $1 < \lambda, \lambda' < \infty$ satisfying $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, there is a constant $C_{\epsilon'} > 0$ such that for all $a, b > 0$ it holds that

$$ab \leqslant \epsilon' a^\lambda + C_{\epsilon'} b^{\lambda'}. \tag{3.10}$$

Let us then chose $\frac{1}{\lambda} \doteq \theta$, $\frac{1}{\lambda'} \doteq 1-\theta$ and apply this inequality to $a = \|u\|_{W^{m,q}(B_1)}^{p\theta}$ and $b = \|u\|_{L^q(B_1)}^{p(1-\theta)}$ to get

$$\|u\|_{W^{m,q}(B_1)}^{p\theta}\|u\|_{L^q(B_1)}^{p(1-\theta)} \leqslant \epsilon'\|u\|_{W^{m,q}(B_1)}^{p\theta\lambda} + C_{\epsilon'}\|u\|_{L^q(B_1)}^{p(1-\theta)\lambda'}$$
$$= \epsilon'\|u\|_{W^{m,q}(B_1)}^{p} + C_{\epsilon'}\|u\|_{L^q(B_1)}^{p}.$$

Similarly, let us apply (3.10) to $a_j \doteq \|S_{2^j}u\|_{W^{m,q}(A_1)}^{\theta p}$ and $b_j \doteq \|S_{2^j}u\|_{L^q(A_1)}^{(1-\theta)p}$ to get

$$\|S_{2^j}u\|_{W^{m,q}(A_1)}^{\theta p}\|S_{2^j}u\|_{L^q(A_1)}^{(1-\theta)p} \leqslant \epsilon'\|S_{2^j}u\|_{W^{m,q}(A_1)}^{p} + C_{\epsilon'}\|S_{2^j}u\|_{L^q(A_1)}^{p}.$$

Putting all the above together, we find

$$\|\partial^\beta u\|_{\bar{L}^p_{\delta-|\beta|}}^p \lesssim \epsilon'\|u\|_{W^{m,q}(B_1)}^{p} + C_{\epsilon'}\|u\|_{L^q(B_1)}^{p}$$
$$+ \sum_{j=1}^\infty 2^{-j\delta p}\left(\epsilon'\|S_{2^j}u\|_{W^{m,q}(A_1)}^{p} + C_{\epsilon'}\|S_{2^j}u\|_{L^q(A_1)}^{p}\right).$$

Also, notice that $q \leqslant p$ since

$$\frac{1}{p} = \frac{k-\theta m}{n} + \frac{1}{q} = \frac{1}{q} + (\frac{k}{m} - \theta)\frac{m}{n} \leqslant \frac{1}{q}.$$

Then, recall that $\ell^q \hookrightarrow \ell^p$, which implies that

$$\|\partial^\beta u\|_{\bar{L}^p_{\delta-|\beta|}} \lesssim \left(\epsilon'^{\frac{q}{p}}\left(\|u\|_{W^{m,q}(B_1)}^{q} + \sum_{j=1}^\infty 2^{-j\delta q}\|S_{2^j}u\|_{W^{m,q}(A_1)}^{q}\right)\right.$$
$$\left. + C_{\epsilon'}^{\frac{q}{p}}\left(\|u\|_{L^q(B_1)}^{q} + \sum_{j=1}^\infty 2^{-j\delta q}\|S_{2^j}u\|_{L^q(A_1)}^{q}\right)\right)^{\frac{1}{q}}$$

where the right-hand side is finite by hypothesis. Thus, for some fixed constant $C > 0$ independent of u, it holds that

$$\|\partial^\beta u\|_{\bar{L}^p_{\delta-|\beta|}} \leqslant C\left(\epsilon'^{\frac{1}{p}}\|u\|_{W^{m,q}_\delta} + C_{\epsilon'}^{\frac{1}{p}}\|u\|_{L^q_\delta}\right), \tag{3.11}$$

from which we obtain our statement after the choice $\epsilon'^{\frac{1}{p}} = \frac{\epsilon}{C}$.

$\square$

Let us now consider a manifold M^n Euclidean at infinity (recall that this only fixed the topological structure of the ends of M) which may have non-empty boundary within the compact region K. Notice that we have a finite number of end charts, say $\{U_i, \Phi_i\}_{i=1}^{N_0}$, with $\Phi(U_i) \simeq \mathbb{R}^n \backslash \overline{B}$, and a finite number of coordinate charts covering the compact region K, say $\{U_i, \Phi_i\}_{i=N_0+1}^N$. We can consider a partition of unity $\{\eta_i\}_{i=1}^N$ subordinate to the coordinate cover $\{U_i, \Phi_i\}_{i=1}^N$, and let V_i be equal to either $\mathbb{R}^n$ or $\mathbb{R}_+^n$, depending on whether U_i, $i \geq N_0 + 1$, is an interior or boundary chart respectively. Then, given a vector bundle $E \xrightarrow{\pi} M$, we can define $W_\delta^{k,p}(M; E)$ to be the subset of $W_{loc}^{k,p}(M; E)$ such that

$$\|u\|_{W_\delta^{k,p}} \doteq \sum_{i=1}^{N_0} \|\Phi_i^{-1*}(\eta_i u)\|_{W_\delta^{k,p}(\mathbb{R}^n)} + \sum_{i=N_0+1}^N \|\Phi_i^{-1*}(\eta_i u)\|_{W^{k,p}(V_i)} < \infty.$$

$$(3.12)$$

We can now extend the embedding and multiplication properties of Theorems 3.1.1 and 3.1.2 to a general manifold M^n Euclidean at infinity by an appeal to localization of fields using a partition of unity and using Theorems 3.1.1, 3.1.2 and A.2.1. That is, the following holds:

Theorem 3.1.4. *Let M^n be a manifold Euclidean at infinity and $E \to M$ a vector bundle over M. Then, all the properties of Theorem 3.1.1 as well as the compact embedding of Theorem 3.1.2 hold for $W_\delta^{k,p}(E)$ under the same conditions stated in those theorems. Furthermore, for all $k > \frac{1}{p}$, we have a continuous trace map $\tau : W_\delta^{k,p}(M, E) \to W^{k-\frac{1}{p},p}(\Sigma, E)$ and a continuous extension map* $\mathrm{ext} : W^{k-\frac{1}{p},p}(\Sigma, E) \mapsto W_\delta^{k,p}(M, E)$.

The last claim in the above theorem concerning the trace and extension properties of these $W_\delta^{k,p}$-spaces follows directly from the corresponding properties stated in Theorem A.2.1, since the boundary $\Sigma \doteq \partial M$ is assumed to be compact and the trace and extension properties depend only on the field in neighbourhood in Σ, not on what happens at infinity. Let us also clarify that, in order to avoid introducing unnecessarily heavy notation, in the theorem above we have denoted the bundle obtained by the restriction of E to Σ by the same symbol E. Finally, let us highlight that the same type of localization properties prove that $C_0^\infty(M)$ (M with its boundary) is dense in $W_\delta^{k,p}$ for all $k \geq 0$, $1 < p < \infty$ and $\delta \in \mathbb{R}$.

Now that we can measure the behaviour of fields at infinity appealing to Sobolev spaces, let us introduce the key concept of AE manifolds,

Definition 3.1.2 (AE manifolds). *We will say that a Riemanian manifold (M^n, g) is asymptotically Euclidean if:*

1. *M^n is Euclidean at infinity, with end charts $\{\Phi_i\}$;*

2. *$g \in W_{loc}^{k,p}$ for some $k > \frac{n}{p}$;*

3. *$g - \Phi_i^* e \in W_\delta^{k,p}(E_i)$ for some $\delta < 0$ and all end charts Φ_i,*

where, above, "e" denotes the Euclidean metric on $\mathbb{R}^n$.

3.2   Some elliptic theory on AE manifolds

Besides providing us with a precise way to control fields near infinity, the above weighted Sobolev spaces provide us with an appropriate context to prove good mapping properties for elliptic operators on manifolds Euclidean at infinity. In particular, it has been known for a long time that, in general, elliptic operators do not have good mapping properties acting on (unweighed) Sobolev spaces on non-compact manifolds (Cantor 1979; Lockhart 1981). This can be exemplified by considering the Euclidean Laplacian Δ on $\mathbb{R}^n$.

Although Δ has good mapping properties acting on $W_0^{2,p}(\Omega)$ for bounded domains (Fredholm properties), it does not have such nice properties acting over all of $\mathbb{R}^n$ and, in particular, it does not have closed range. This has been explained very nicely in Maxwell (2004) and can be seen as follows. First, notice that any $W^{2,p}(\mathbb{R}^n)$ solution to $\Delta u = 0$, must be zero. This follows by multiplying the equation by u and integrating by parts (combined with a suitable approximation argument to get rid of the boundary term). Therefore, if Δ had closed range $X \subset L^p$, then X would be a Banach space with the L^p norm and we would have an isomorphism

$$\Delta : W^{2,p} \mapsto X,$$

with the inequality

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} \lesssim \|\Delta u\|_{L^p(\mathbb{R}^n)} \quad \forall \, u \in W^{2,p}(\mathbb{R}^n). \tag{3.13}$$

But this can be seen to be impossible from the following argument. Consider the scaling operator $S_r : L^p \mapsto L^p$, $(S_r u)(x) \doteq u(rx)$ for $r > 0$. Then, for any $u \in W^{2,p}$, it follows that

$$\Delta(S_r u)(x) = r^2 \Delta u(rx) = r^2 (S_r \Delta u)(x) \text{ and } \|S_r u\|_{L^p(\mathbb{R}^n)} = r^{-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)}.$$

Therefore, if (3.13) were to hold, we would find that

$$\|u\|_{L^p(\mathbb{R}^n)} = r^{-\frac{n}{p}} \|S_{\frac{1}{r}} u\|_{L^p(\mathbb{R}^n)} \lesssim r^{-\frac{n}{p}} \|S_{\frac{1}{r}} u\|_{W^{2,p}(\mathbb{R}^n)},$$

$$\lesssim r^{-\frac{n}{p}} \|\Delta(S_{\frac{1}{r}} u)\|_{L^p(\mathbb{R}^n)} = r^{-\frac{n}{p}-2} \|S_{\frac{1}{r}}(\Delta u)\|_{L^p(\mathbb{R}^n)} = r^{-2} \|\Delta u\|_{L^p(\mathbb{R}^n)},$$

for all $u \in W^{2,p}$ and any $r > 0$. But this last inequality can be falsified for any $u \not\equiv 0$ by taking r sufficiently large. Therefore, we must conclude that Δ is not Fredholm on $W^{2,p}(\mathbb{R}^n)$. This shows that, in order to get good Fredholm properties, we must appeal to spaces with better scaling properties. The weighted Sobolev spaces introduced above are such spaces.

In this context of AE manifolds and weighted spaces, it is possible to develop linear elliptic PDE analysis with the same level of completeness as was presented in Appendix B for compact manifolds. Since the appropriate theory for boundary value problems is more delicate (the same is true in the compact case), we will develop such analysis explicitly for the kind of second order operators involved in the constraints, subject to natural boundary conditions in this context. In the case where $\partial M = \emptyset$ the full theory can be developed following lines similar to those of Appendix B, first establishing the appropriate estimates for trivial sections on $\mathbb{R}^n$ and operators acting between weighted spaces and then localising the problem via a partition of unity argument to apply such estimates together with estimates for compact manifolds, such as those of Theorem B.4. We will provide the main arguments in this section and also refer the reader to Bartnik (1986), Cantor (1979, 1981), Choquet-Bruhat and Christodoulou (1981), Lockhart (1981), Maxwell (2006), McOwen (1979), and Nirenberg and Walker (1973) and Choquet-Bruhat (2009, Appendix II) for several details.

Let us first start the analysis considering a constant coefficient elliptic operator of order m of the form

$$L_\infty \doteq \sum_{|\alpha|=m} A_\alpha^\infty \partial^\alpha : W_\delta^{m,p}(\mathbb{R}^n) \mapsto L_{\delta-m}^p(\mathbb{R}^n). \tag{3.14}$$

Then, the following result holds:

Lemma 3.2.1. *For any given real number ρ, if $u \in W_{loc}^{m,p}(\mathbb{R}^n) \cap L_\rho^p$, $1 < p < \infty$, and $L_\infty u \in L_{\rho-m}^p$, then $u \in W_\rho^{m,p}$ and*

$$\|u\|_{W_\rho^{m,p}} \lesssim C \left(\|L_\infty u\|_{L_{\rho-m}^p} + \|u\|_{L_{\rho-m}^p} \right) \tag{3.15}$$

Proof. We will again appeal to the scaling properties associated to weighted spaces. First, let $A_r = B_r \setminus \overline{B_{\frac{r}{2}}}$ and notice that using $(S_r u)(x) = u(rx)$ and $\partial_x^\beta(S_r u)(x) = r^{|\beta|}\partial^\beta u(rx)$ for all $|\beta| \leq m$, we find that $r^{|\beta|p-n}\|\partial^\beta u\|_{L^p(A_r)}^p = \|\partial^\beta (S_r u)\|_{L^p(A_1)}^p$. Then, use interior elliptic estimates to write

$$
\int_{A_r} r^{|\beta|p-n}|\partial_x^\beta u(x)|^p dx = \int_{A_1} |\partial_x^\beta (S_r u)(x)|^p dx \leq \|S_r u\|_{W^{m,p}(A_1)}^p,
$$
$$
\leq C \int_{\frac{1}{4}\leq|x|\leq 4} \left(|L_\infty(S_r u)(x)|^p + |(S_r u)(x)|^p\right) dx,
$$
$$
= C \int_{\frac{r}{4}\leq|x|\leq 4r} \left(r^{mp-n}|L_\infty u(x)|^p + r^{-n}|u(x)|^p\right) dx,
$$

where the important point is that $C > 0$ depends on L_∞, p, n but neither on u nor $r > 0$. Let us now multiply the above inequality by $r^{-\rho p}$, to get

$$
\int_{r\leq|x|\leq 2r} \left|r^{-(\rho-|\beta|+\frac{n}{p})}|\partial_x^\beta u(x)|\right|^p dx \leq
$$
$$
\leq C \int_{\frac{r}{4}\leq|x|\leq 4r} \left(\left|r^{-(\rho-m+\frac{n}{p})}|L_\infty u(x)|\right|^p + \left|r^{-\rho-\frac{n}{p}}|u(x)|\right|^p\right) dx,
$$

Notice now that above implies that there is another constant C', which may now also depend on ρ and $|\beta| \leq m$ (but is still independent of r), for which we have

$$
\int_{r\leq|x|\leq 2r} \left||x|^{-(\rho-|\beta|+\frac{n}{p})}|\partial_x^\beta u(x)|\right|^p dx \leq
$$
$$
\leq C' \int_{\frac{r}{4}\leq|x|\leq 4r} \left(\left||x|^{-(\rho-m+\frac{n}{p})}|L_\infty u(x)|\right|^p + \left||x|^{-\rho-\frac{n}{p}}|u(x)|\right|^p\right) dx.
$$

For $r \geq 1$, by modifying C' we can replace in the above inequality $|x|$ by $\sigma(x)$. Then, we can pick $r = 2^j$, $j \in \mathbb{N}$, and sum over j to get

$$
\int_{\mathbb{R}^n\setminus B_1} \left||\sigma(x)|^{-(\rho-|\beta|+\frac{n}{p})}|\partial_x^\beta u(x)|\right|^p dx \leq C' \left(\|L_\infty u\|_{L^p_{\rho-m}(\mathbb{R}^n)}^p + \|u\|_{L^p_\rho(\mathbb{R}^n)}^p\right),
$$

where the right-hand side is finite by hypotheses. Summing over $|\beta| \leq m$, we find

$$
\|u\|_{W^{m,p}_\rho(\mathbb{R}^n\setminus B_1)}^p \leq C'' \left(\|L_\infty u\|_{L^p_{\rho-m}(\mathbb{R}^n)}^p + \|u\|_{L^p_\rho(\mathbb{R}^n)}^p\right), \tag{3.16}
$$

for some other constant $C'' > 0$, which implies the desired statements. $\square$

Let us now consider operators

$$L = \sum_{|\alpha| \leqslant m} A_\alpha \partial^\alpha, \tag{3.17}$$

with coefficients satisfying the following hypotheses

1. $A_\alpha - A_\alpha^\infty \in W^{m,p}_{\delta_m}$, $m > \frac{n}{p}$ and $\delta_m < 0$ for all $|\alpha| = m$;

2. $A_\alpha \in W^{|\alpha|,p}_{\delta_{|\alpha|}}$, $\delta_{|\alpha|} < \delta_m - m + |\alpha|$ for all $|\alpha| < m$.

Lemma 3.2.2. *The operator (3.17) satisfying the above conditions is continuous from $W^{m,p}_\delta \mapsto L^p_{\delta-m}$, for any $p > \frac{n}{m}$ and $\delta \in \mathbb{R}$.*

Proof. Consider $u \in W^{m,p}_\delta$ and write

$$Lu = L_\infty u + \sum_{|\alpha|=m} \left(A_\alpha - A_\alpha^\infty\right) \partial^\alpha u + \sum_{|\alpha|<m} A_\alpha \partial^\alpha u.$$

We already know that $\|L_\infty u\| \lesssim \|u\|_{W^{m,p}_\delta}$. Now, the multiplication property guarantees that $\left(A_\alpha - A_\alpha^\infty\right) \partial^\alpha u \in L^p_{\delta-m}$ if $m > \frac{n}{p}$ and $\delta - m > \delta_m + \delta - m$ and both conditions are satisfied by hypothesis. Then, the continuity of the multiplication property implies

$$\left\| \sum_{|\alpha|=m} \left(A_\alpha - A_\alpha^\infty\right) \partial^\alpha u \right\|_{L^p_{\delta-m}} \lesssim \sum_{|\alpha|=m} \|A_\alpha - A_\alpha^\infty\|_{W^{m,p}_{\delta_m}} \|\partial^\alpha u\|_{L^p_{\delta-m}},$$

$$\lesssim \left(\sum_{|\alpha|=m} \|A_\alpha - A_\alpha^\infty\|_{W^{m,p}_{\delta_m}} \right) \|u\|_{W^{m,p}_\delta}.$$

Similarly, for $|\alpha| < m$, $A_\alpha \partial^\alpha u \in L^p_{\delta-m}$ if $m > \frac{n}{p}$ and $\delta - m > \delta_\alpha + \delta - |\alpha|$. The first of these conditions is obviously satisfied, and for the second notice that it is equivalent to $\delta_\alpha < -m + |\alpha|$, which is also satisfied by hypothesis, since $\delta_m < 0$. The continuity of the multiplication property implies

$$\left\| \sum_{|\alpha|<m} A_\alpha \partial^\alpha u \right\|_{L^p_{\delta-m}} \lesssim \sum_{|\alpha|<m} \|A_\alpha\|_{W^{|\alpha|,p}_{\delta_{|\alpha|}}} \|u\|_{W^{m-|\alpha|,p}_{\delta-|\alpha|}},$$

$$\lesssim \left(\sum_{|\alpha|<m} \|A_\alpha\|_{W^{|\alpha|,p}_{\delta_{|\alpha|}}} \right) \|u\|_{W^{m,p}_\delta}.$$

Putting together all the above estimates, we find $\|Lu\|_{L_{\delta-m}^p} \leqslant C \|u\|_{W_\delta^{m,p}}$, where the constant C depends on the norms of the coefficients. $\qquad\square$

We can now present the following result for an operator of the form of (3.17).

Theorem 3.2.1 (Nirenberg–Walker)**.** *Consider the operator $L_0 = \sum_{|\alpha|=m} A_\alpha^0 \partial^\alpha$ of the form of (3.17) consisting only of terms of order m. If $p > \frac{n}{m}$, then, for all $u \in W_{loc}^{m,p} \cap L_{\rho-m}^p$ satisfying $L_0 u \in L_{\rho-m}^p$, it holds that $u \in W_\rho^{m,p}$ and there is a constant $C > 0$ such that for all such u we have the following elliptic estimate:*

$$\|u\|_{W_\rho^{m,p}} \leqslant C \left(\|L_0 u\|_{L_{\rho-m}^p} + \|u\|_{L_{\rho-m}^p} \right). \tag{3.18}$$

Remark 3.2.1.   *1. Let us first notice that the condition $p > \frac{n}{m}$ is used to guarantee that $W_{\delta_m}^{m,p} \hookrightarrow C_{\delta_m}^0$, with $\delta_m < 0$, which implies $A_\alpha^0 - A_\alpha^\infty \in C_{\delta_m}^0$. This condition allows us to appeal to the "freezing of coefficients" technique, where, within some sufficiently large compact set we can use the techniques of Appendix B, while at infinity the appeal is to Lemma 3.2.1 and notice that $\sup |A_\alpha^0 - A_\alpha^\infty|$ is controlled via the decay of L_0 to L_∞;*

   2. In Nirenberg and Walker (1973, Theorem 3.1), the authors consider operators admitting lower order terms, but ask for all coefficients to be continuous, which is a stronger condition than what we intend to use. Below, we will use Theorem 3.2.1 to obtain results for operators with less regular coefficients;

   3. Also, in Nirenberg and Walker (ibid., Theorem 3.1), the result is stated for sections $u \in W^{m,p}(\mathbb{R}^n) \cap L_\rho^p(\mathbb{R}^n)$. Nevertheless, as has been previously noted by other authors such as Choquet-Bruhat and Christodoulou (1981, see the remark after Theorem 4.1), the proof of Theorem 3.1 in Nirenberg and Walker (1973) works perfectly for $u \in W_{loc}^{m,p}(\mathbb{R}^n) \cap L_\rho^p(\mathbb{R}^n)$.

We can now establish the following result, for more general operators.

Theorem 3.2.2. *Consider the operator L of the form of (3.17). If $m > \frac{n}{p} + 1$, then, there is a constant $C > 0$ such that for all such for all $u \in W_\rho^{m,p}$ we have the following elliptic estimate:*

$$\|u\|_{W_\rho^{m,p}} \leqslant C \left(\|Lu\|_{L_{\rho-m}^p} + \|u\|_{L_\rho^p} \right). \tag{3.19}$$

Proof. Denote by $L_0 = \sum_{\alpha=m} A_\alpha \partial$ the top order part of L. Then, clearly $L_0 u \in W_{\rho-m}^{m,p}$ and we can apply the previous theorem to get

$$
\begin{aligned}
\|u\|_{W_\rho^{m,p}} &\leqslant C\left(\left\|\left(L - \sum_{|\alpha|<m} A_\alpha \partial^\alpha\right)u\right\|_{L_{\rho-m}^p} + \|u\|_{L_\rho^p}\right), \\
&\leqslant C\left(\|Lu\|_{L_{\rho-m}^p} + \sum_{|\alpha|<m}\|A_\alpha \partial^\alpha u\|_{L_{\rho-m}^p} + \|u\|_{L_\rho^p}\right).
\end{aligned}
\tag{3.20}
$$

Now, from the multiplication property, if $|\alpha| + m - |\alpha| - 1 = m - 1 > \frac{n}{p}$ for all $0 \leqslant |\alpha| \leqslant m - 1$ and $\rho - m > \delta_{|\alpha|} + \rho - |\alpha|$, then

$$
\begin{aligned}
\|A_\alpha \partial^\alpha u\|_{L_{\rho-m}^p} &\leqslant C'\|A_\alpha\|_{W_{\delta_{|\alpha|}}^{|\alpha|,p}}\|\partial^\alpha u\|_{W_{\rho-|\alpha|}^{m-1-|\alpha|,p}}, \\
&\leqslant C'\|A_\alpha\|_{W_{\delta_{|\alpha|}}^{|\alpha|,p}}\|\partial^\alpha u\|_{W_{\rho-|\alpha|}^{m-1,p}}.
\end{aligned}
\tag{3.21}
$$

The first of the above conditions is precisely $m > \frac{n}{p} + 1$, satisfied by hypothesis. Similarly, the condition for the weights is equivalent to $\delta_\alpha < |\alpha| - m$, which also follows by the hypothesis 2 on the coefficients of (3.17). Thus, (3.21) holds and implies for some other constant C'

$$
\begin{aligned}
\sum_{|\alpha|<m}\|A_\alpha \partial^\alpha u\|_{L_{\rho-m}^p} &\leqslant C'\left(\sum_{|\alpha|<m}\|A_\alpha\|_{W_{\delta_{|\alpha|}}^{|\alpha|,p}}\right)\|\partial^\alpha u\|_{W_\rho^{m-1,p}}, \\
&\leqslant \epsilon C''\|u\|_{W_\rho^{m,p}} + C_\epsilon C''\|u\|_{L_\rho^p},
\end{aligned}
$$

where we have used interpolation and absorbed the norms of the coefficients within $C'' > 0$. Going back to (3.20), picking $\epsilon > 0$ sufficiently small, and relabelling the constant C, we get the inequality

$$
\|u\|_{W_\rho^{m,p}} \leqslant C\left(\|Lu\|_{L_{\rho-m}^p} + \|u\|_{L_\rho^p}\right) \quad \forall u \in W_\rho^{m,p}(\mathbb{R}^n).
\tag{3.22}
$$

$\square$

Let us highlight that the condition $m > \frac{n}{p} + 1$, which implies $A_\alpha \in C^{1,\lambda}$, for $|\alpha| = m$ and some $\lambda \in (0,1)$, is quite common in classic literature (Bartnik 1986; Cantor 1981; Choquet-Bruhat and Christodoulou 1981). In the above proof, this

condition becomes necessary to get the estimates (3.21) through the multiplication property. One important aspect of (3.21), is that the norms of u in the right-hand side go only up to order $m - 1$. This allows us to use interpolation and chose ϵ small enough so as to absorb the term of order m into the left-hand side. If we were to attempt the same proof but relaxing to $m > \frac{n}{p}$, the multiplication property, in the case of $|\alpha| = 0$, would incorporate a $W_\rho^{m,p}$-norm of u in the right-hand side of (3.21), which we would not be able to handle via interpolation. Nevertheless, there is a way around this, appealing to more subtle estimates. Let us simply exemplify this here for our main case if interest, which are equation of second order. Thus, assume $m = 2$, $p > \frac{n}{2}$ and consider the term $A_0 u$, with $A_0 \in L_{\delta_0}^p$ and $u \in W_\rho^{2,p}$. Let us assume $p < n$, since the other cases have been covered above. Then, let us appeal to the Sobolev embeddings of Theorem 3.1.4 to notice

$$W_\rho^{1+1,p} \hookrightarrow W_\rho^{1,q} \text{ for any } p < q < \frac{np}{n-p}, \tag{3.23}$$

which implies $u \in W_\rho^{1,q}$. Also, the multiplication property says that

$$L_{\delta_0}^p \otimes W_\rho^{1,q} \hookrightarrow L_{\rho-2}^p,$$

as long as $1 > \frac{n}{q}$, $q \geqslant p$ and $\delta_0 + \rho < \rho - 2$. To guarantee the first of these conditions, notice that

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{q} < \frac{1}{p},$$

and the assumption $\frac{n}{2} < p < n$, implies $\frac{1}{n} < \frac{1}{p}$ and $\frac{1}{p} < \frac{2}{n}$, which gives us

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{n} < \frac{1}{p}.$$

Therefore, we can chose q in the range allowed by (3.23) and satisfying

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{q} < \frac{1}{n} < \frac{1}{p},$$

which implies our first condition, equivalent to $q > n$, where the second condition $q > p$ is also satisfied. The last condition, $\delta_0 + \rho < \rho - 2$, holds since $\delta_0 < -2$ by hypotheses 2 following (3.17). Therefore, we find

$$\|A_0 u\|_{L_{\rho-2}^p} \lesssim \|A_0\|_{L_{\delta_0}^p} \|u\|_{W_\rho^{1,q}}.$$

Now, since $m - 1 - \frac{n}{p} = 1 - \frac{n}{p} < 1$, then $m - 1 - \frac{n}{p} \notin \mathbb{N}$ and we can use Theorem 3.1.3 with $k = 1$, $m = 2$ and $\frac{1}{q} = \frac{1-2\theta}{n} + \frac{1}{p}$ as long as we can find θ satisfying

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{q} \doteq \frac{1-2\theta}{n} + \frac{1}{p} < \frac{1}{n}. \tag{3.24}$$

Notice that since θ is restricted to $\frac{1}{2} \leqslant \theta \leqslant 1$, then $-1 \leqslant 1 - 2\theta \leqslant 0$ and therefore

$$0 < \frac{1}{p} - \frac{1}{n} \leqslant \frac{1}{p} + \frac{1-2\theta}{n},$$

which implies that, for any choice of θ in the interval $\frac{1}{2} < \theta \leqslant 1$ the first inequality in (3.24) is satisfied. For the second one, notice that it is equivalent to

$$1 - 2\theta + \frac{n}{p} < 1 \iff \frac{n}{2p} < \theta.$$

Since $\frac{n}{2p} < 1$ by hypotheses, then, there is some $\frac{1}{2} < \theta < 1$ which satisfies (3.24). Therefore, we find

$$\|A_0 u\|_{L^p_{\rho-2}} \lesssim \|A_0\|_{L^p_{\delta_0}} (\epsilon \|u\|_{W^{2,p}_\rho} + C_\epsilon \|u\|_{L^p_\rho}),$$

and then we can proceed as in the above proof. The interested reader may find further details in Choquet-Bruhat (2009, see Lemma 3.2 and Theorem 3.3, Appendix II). The final result, is the following:

Theorem 3.2.3. *Consider a second order elliptic operator L of the form of (3.17) satisfying the hypotheses 1 and 2 following (3.17). Then, there is a constant $C > 0$ such that*

$$\|u\|_{W^{2,p}_\rho} \leqslant C \left(\|Lu\|_{L^p_{\rho-2}} + \|u\|_{L^p_\rho} \right) \ \forall \ u \in W^{2,p}_\rho. \tag{3.25}$$

Remark 3.2.2. *Let us notice that, along the same lines of Remark 3.2.1, once the (3.25) has been established, under the hypotheses of the above theorem, the proof of Nirenberg and Walker (1973, Theorem 3.1) can be used to obtain the following regularity result:*

$$If \ u \in W^{2,p}_{loc} \cap L^p_\rho \ and \ Lu \in L^p_{\rho-2} \implies u \in W^{2,p}_\rho. \tag{3.26}$$

Let us now simply point out that, using what by now are standard localisation arguments, applying the interior estimates of Appendix B to a covering by small balls of the compact region of an AE manifold with no boundary, and combining this with the above theorems, we get the precise analogues of Theorems 3.2.2 to 3.2.3, but for sections on the whole AE-manifold. We leave these small adjustments to the reader, who can consult different versions of these statements in previously cited references.

3.3   Some boundary value problems

The objective of this section is to introduce the corresponding analysis associated to the boundary value problems which play a role within the conformally formulated ECE. The general elliptic theory of boundary value problems is quite subtle and will not be treated here. In particular, there are *admissible* types of boundary conditions which *complement* the equations. Boundary conditions which arise naturally in well-motivated physical problems tend to fall into this class, and this is the case in the context of the conformal problem associated to the ECE. Below, we will first introduce the so-called black hole boundary conditions and present their conformal formulation. This will provide us with natural boundary conditions for the conformal Laplacian and Killing Laplacian involved in the analysis of the ECE. Then, we will show how to modify the above general elliptic theory in this context, and finally present the general properties of these two boundary value problems. Most of this section, as well as the remaining parts of this chapter, is based upon Maxwell (2005b).

3.3.1   Conformally formulated black hole initial data

From Section 1.5 in Chapter 1, if M^n is a manifold with compact boundary Σ, $n \geqslant 3$, we have a characterisation of the appropriate boundary conditions that an initial data set (M^n, g, K) for Einstein equations should satisfy on Σ in order to evolve into a space-time containing black holes. These conditions were given by

$$\theta_\pm = K(\nu, \nu) - \tau \pm \mathrm{tr}_h k \leqslant 0, \text{ on } \Sigma \qquad (3.27)$$

where we recall that $\theta_\pm$ denote the *expansion* coefficients and the condition $\theta_\pm \leqslant 0$ represents a *convergence* condition on future pointing light-rays within the evolving space-time, signalling the presence of a black hole, and we denote by $\tau \doteq \mathrm{tr}_g K$ the mean curvature of the initial data set. Also, we denoted by $h = g|_\Sigma$ and k the

extrinsic curvature of $\Sigma \hookrightarrow (M, g)$, with respect to the outward-pointing g-unit normal ν (see Proposition 1.5.1).

Now, we would like to re-express (3.27) in terms of the conformal data described in Chapter 2. Let us first notice that

$$\theta_- = \theta_+ - 2\mathrm{tr}_h k. \tag{3.28}$$

We will now consider boundary conditions for the conformal problem arising by imposing restrictions on the expansion coefficients $\theta_\pm$ so as to satisfy (3.27). Such types of conditions have been analysed by different authors, among which we would like to highlight Avalos and Lira (2019), Dain (2004), Holst and Tsogtgerel (2013), and Maxwell (2005b). Along these lines, we will consider two different possibilities below. The first one consists in freely prescribing $\theta_- \leqslant 0$, which implies that

$$\mathrm{tr}_h k + \theta_- + \tau - K(\nu, \nu) = 0 \tag{3.29}$$

is our boundary condition. In this case, we can rewrite

$$\begin{aligned}
\theta_+ = \theta_- + 2\mathrm{tr}_h k &= \theta_- - 2(\theta_- + \tau - K(\nu, \nu)), \\
&= -\theta_- - 2\tau + 2K(\nu, \nu).
\end{aligned}$$

Then,

$$\theta_+ \leqslant 0 \iff K(\nu, \nu) \leqslant \frac{1}{2}\theta_- + \tau. \tag{3.30}$$

Therefore, our boundary conditions are

$$\begin{aligned}
\mathrm{tr}_h k + \theta_- + \tau - K(\nu, \nu) &= 0, \\
K(\nu, \nu) &\leqslant \frac{1}{2}\theta_- + \tau.
\end{aligned} \tag{3.31}$$

Using our conventions for the conformal problem (see Section 2.1),[3] we write $g = \phi^{\frac{4}{n-2}}\gamma$ and $K = \phi^{-2}(\mathscr{L}_{\gamma,\mathrm{conf}}X + U) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma$, which implies that

$$K(\nu, \nu) = \phi^{-\frac{4}{n-2}-2}(\mathscr{L}_{\gamma,\mathrm{conf}}X(\hat{\nu}, \hat{\nu}) + U(\hat{\nu}, \hat{\nu})) + \frac{\tau}{n},$$

---

[3] During this chapter, since we will not include scalar field sources, we will denote the conformal factor in the conformal method by ϕ instead of φ. The advantage of this change in notation will become clear as we move forward.

with $v = \phi^{-\frac{2}{n-2}}\hat{v}$, where $\hat{v}$ is the outward point unit normal with respect to γ. Furthermore, since

$$\mathrm{tr}_h k = -\mathrm{div}_g v = -\nabla_i v^i = -\phi^{-\frac{2}{n-2}}\nabla_i \hat{v}^i + \frac{2}{n-2}\phi^{-\frac{2}{n-2}-1}\hat{v}(\phi),$$

$$= -\phi^{-\frac{2}{n-2}}(\hat{\nabla}_i \hat{v}^i + S^i_{ik}\hat{v}^k) + \frac{2}{n-2}\phi^{-\frac{2}{n-2}-1}\hat{v}(\phi),$$

where $S^j_{ik} = \frac{2}{n-2}\phi^{-1}\gamma^{jl}(\gamma_{kl}\partial_i\phi + \gamma_{il}\partial_k\phi - \gamma_{ik}\partial_l\phi)$, implying that $S^i_{ik}\hat{v}^k = \frac{2n}{n-2}\phi^{-1}\hat{v}(\phi)$, which, in turn, implies that

$$\mathrm{tr}_h k = -\phi^{-\frac{2}{n-2}-1}\left(\phi\hat{\nabla}_i\hat{v}^i + 2\frac{n-1}{n-2}\hat{v}(\phi)\right) = -\phi^{-\frac{2}{n-2}-1}\left(2\frac{n-1}{n-2}\hat{v}(\phi) + \phi H\right),$$

where $H = \mathrm{div}_\gamma \hat{v}$ is the mean curvature of (Σ, h) as an embedded hypersurface of (M, γ), taken with respect to $-\hat{v}$. This implies

$$\frac{1}{2}a_n\hat{v}(\phi) + H\phi - (\theta_- + b_n\tau)\phi^{\frac{n}{n-2}} + \widetilde{K}(\hat{v},\hat{v})\phi^{-\frac{n}{n-2}} = 0, \qquad (3.32)$$

recalling that $a_n = 4\frac{n-1}{n-2}$, $b_n = \frac{n-1}{n}$ and $\widetilde{K} = \mathscr{L}_{\gamma,cong}X + U$ is our TT-part of the extrinsic curvature. The above boundary conditions now looks quite nicely like an appropriate (*Robin-type*) complementing condition for the Lichnerowicz equation.

Now, the additional condition $K(v,v) \leqslant \frac{1}{2}\theta_- + \tau$ translates naturally to a boundary condition for the conformally formulated momentum constraint, since, explicitly, it reads

$$\mathscr{L}_{\gamma,\mathrm{conf}}X(\hat{v},\hat{v}) \leqslant -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)\phi^{\frac{2n}{n-2}} - U(\hat{v},\hat{v}). \qquad (3.33)$$

In order to satisfy such an inequality, we use shall impose a link between θ_- and τ on Σ, given by $\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0$. Then, under this assumption we consider the boundary condition

$$\mathscr{L}_{\gamma,\mathrm{conf}}X(\hat{v},\cdot) = -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\hat{v} - U(\hat{v},\cdot),$$
$$v \geqslant \phi|_\Sigma, \qquad\qquad (3.34)$$

where we have introduced the function v on Σ and imposed the *a priori* inequality $v \geqslant \phi|_\Sigma$. Notice that (3.34) implies (3.33) and hence it implies $\theta_+ \leqslant 0$.

Also, let us highlight that (3.34) is a good *complementing* boundary condition for the conformally formulated momentum constraint. We should observe that the condition $v \geqslant \phi|_\Sigma$ being a priori, is a constraint on an admissible solution. Part of the work in the associated PDE analysis is related to proving the resulting solution satisfies this condition and therefore satisfies the (marginally) trapped condition $\theta_+ \leqslant 0$.

We have thus found the following conformal formulation of the black hole boundary conditions (3.31), given by

$$\frac{1}{2}a_n\hat{v}(\phi) + H\phi - (\theta_- + b_n\tau)\,\phi^{\frac{n}{n-2}} + \tilde{K}(\hat{v},\hat{v})\phi^{-\frac{n}{n-2}} = 0,$$

$$\mathscr{L}_{\gamma,\mathrm{conf}}X(\hat{v},\cdot) = -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\hat{v} - U(\hat{v},\cdot),$$

$$v \geqslant \phi|_\Sigma,$$

$$\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0$$

(3.35)

recalling that the third condition is merely used to guarantee that $\theta_+ \leqslant 0$ and the last one is a condition between two free parameters of the problem (which is trivial for maximal initial data), which can always be satisfied a priori.

Remark 3.3.1. *Notice that, in general, (3.35) couples the conformally formulated Gauss–Codazzi constraints (2.12) through the boundary conditions even for vacuum.*

Taking into account the above remark, we can present a simplifying case which is of particular interest. Notice that if we impose decaying conditions on the initial data at infinity, in particular for τ, then the CMC condition implies $\tau \equiv 0$. In this case, we can pose black hole boundary conditions which decouple proceeding as follows.

First, notice that appealing to the above computations, the *apparent horizon condition* $\theta_+ = 0$ can be conformally formulated as

$$\frac{1}{2}a_n\hat{v}(\phi) + \phi H - \tilde{K}(\hat{v},\hat{v})\phi^{-\frac{n}{n-2}} = 0. \tag{3.36}$$

Furthermore, in this case, from (3.28), one finds that

$$\theta_- = -2\mathrm{tr}_h k, \tag{3.37}$$

which, put together with (3.27), implies that

$$-\mathrm{tr}_h k = K(\nu, \nu). \tag{3.38}$$

Therefore, we see that the condition $\theta_- \leqslant 0$ is implied by

$$\mathrm{tr}_h k \geqslant 0 \iff K(\nu, \nu) \leqslant 0 \iff \tilde{K}(\hat{\nu}, \hat{\nu}) \leqslant 0$$

where the last of the above three conditions is independent of the conformal factor. Therefore, in this case, we impose

$$\frac{1}{2} a_n \hat{\nu}(\phi) + \phi H - \tilde{K}(\hat{\nu}, \hat{\nu}) \phi^{-\frac{n}{n-2}} = 0,$$
$$\tilde{K}(\hat{\nu}, \hat{\nu}) \leqslant 0, \tag{3.39}$$

where these boundary conditions correspond to the black hole boundary conditions given by $\theta_+ = 0$ and $\theta_- = 2K(\nu, \nu) \leqslant \theta_+ = 0$.

Notice that the above conditions, having been imposed for the system (2.12) with $\tau \equiv 0$, in the vacuum case, give us the following boundary value problem:

$$\begin{cases} -a_n \Delta_\gamma \phi + R_\gamma \phi - |\tilde{K}|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} &= 0, \\ \frac{1}{2} a_n \hat{\nu}(\phi) + \phi H - \tilde{K}(\hat{\nu}, \hat{\nu}) \phi^{-\frac{n}{n-2}} &= 0, \text{ on } \Sigma, \end{cases} \tag{3.40}$$

$$\begin{cases} \Delta_{\gamma,\mathrm{conf}} X &= 0, \\ \tilde{K}(\hat{\nu}, \hat{\nu}) &\leqslant 0, \text{ on } \Sigma. \end{cases} \tag{3.41}$$

In the above system, the equation (3.41) can be solved independently of (3.40), and then the analysis of the boundary value problem for Lichnerowicz becomes the main object of analysis. The pair of equations (3.40)-(3.41) was analysed in the context of AE manifolds in detail by Maxwell (2005b), and, in this chapter, we will present the results obtained in this reference. This will give us the opportunity to introduce the necessary analytical tools related to the conformal Laplacian, conformal Killing operator and Yamabe problem in the context of AE manifolds with boundary. Then, in Chapter 4, we shall analyse the fully coupled system with boundary conditions (3.35).

3.3.2 The Poisson and Conformal Killing operators

In this section we shall present the analytical tools associated to the boundary value problems described in the previous one. The roadmap for this analysis is basically the same as the one presented in Section 2.2, adapted to AE manifolds with boundary. Thus, one our main goals will be to establish a version of the monotone iteration scheme in this context. Let us recall that, in the compact case, the isomorphism properties associated to the operators $\Delta_\gamma - a$, with $a \geqslant 0$ a.e, were one of the key elements. Thus, in what follows, we shall investigate these properties in the context of AE manifolds. This will allow us to deal with the Lichnerowicz equation. Also, we shall analyse the mapping properties of the conformal Killing Laplacian in order to deal with the momentum constraint. Let us start by presenting the following estimates, which replace Equation (3.25) in the context of AE manifolds with boundary.

Proposition 3.3.1. *Let (M, γ) be a $W^p_{2,\rho}$-AE manifold with $p > \frac{n}{2}$ and $\rho < 0$. Consider $\delta < 0$ and let $a \in L^p_{\delta-2}$. Consider the operators $L \doteq \Delta_\gamma - a :$ $W^p_{2,\delta}(M) \mapsto L^p_{\delta-2}(M)$, $\Delta_{\gamma,conf} : W^p_{2,\delta}(M) \mapsto L^p_{\delta-2}(M)$, as well as the boundary operators*

$$\begin{aligned}
\mathcal{B}_1 &: W^p_{2,\delta}(M) \mapsto W^{1-\frac{1}{p},p}(\Sigma), \quad u \mapsto \nu(u)|_\Sigma, \\
\mathcal{B}_2 &: W^p_{2,\delta}(M) \mapsto W^{1-\frac{1}{p},p}(\Sigma), \quad X \mapsto \mathcal{L}_{\gamma,conf}X(\nu,\cdot)|_\Sigma
\end{aligned} \tag{3.42}$$

Then, there is a constant $C > 0$ such that the following estimates hold

$$\|u\|_{W^{2,p}_\delta} \leqslant C \left(\|Lu\|_{L^p_{\delta-2}(M)} + \|u\|_{L^p_\delta(M)} + \|\mathcal{B}_1 u\|_{W^{1-\frac{1}{p},p}(\Sigma)} \right),$$

$$\|X\|_{W^{2,p}_\delta} \leqslant C \left(\|\Delta_{\gamma,conf}X\|_{L^p_{\delta-2}(M)} + \|X\|_{L^p_\delta(M)} + \|\mathcal{B}_2 X\|_{W^{1-\frac{1}{p},p}(\Sigma)} \right),$$

$$\tag{3.43}$$

for all u and X in $W^{2,p}_\delta(M)$.

The proof of the above proposition can be consulted, for instance, in Maxwell (ibid.). Let us merely notice that the only thing we have missing up to now are the boundary estimates. These are obtained by working in a neighbourhood of a boundary point and appealing to techniques such as those of Appendix B and Section 3.2. In particular, the proof appeals to the *freezing of coefficients* technique,

since the corresponding boundary estimates for the associated constant coefficient operators are valid due to foundational work of Agmon, Douglas, and Nirenberg (1964). It is a this point that having good *complementing* boundary conditions plays a key role. Then, once the freezing of coefficients technique gives us the above estimates near the boundary, a standard localisation procedure combined with the interior estimates of Appendix B and the estimates on $\mathbb{R}^n$ of Section 3.2 (which provide the estimates at infinity) complete the proof.

The Poisson operator

Let us recall from Chapter 2, that one of our main tools in the analysis of the Lichnerowicz equation were the maximum principles of Lemmas 2.2.2 and 2.2.3. Thus, let us start our analysis presenting their corresponding analogues for AE manifolds.

Lemma 3.3.1 (Weak maximum principle). *Consider an AE manifold (M^n, γ), $n \geqslant 3$, and let ψ be a $W_{loc}^{2,p}$-solution to the boundary problem:*

$$\Delta_\gamma \psi - a\psi \leqslant 0, \text{on } M,$$
$$-\nu(\psi) - b\psi \leqslant 0, \text{on } \Sigma, \tag{3.44}$$

where Σ is compact, $\gamma \in W_\delta^{2,p}$, $p > \frac{n}{2}$, $\delta < 0$, $a \in L_{\delta-2}^p(M)$, $a \geqslant 0$ a.e, ν stands for the outward unit normal on Σ and $b \in W^{1-\frac{1}{p},p}(\Sigma)$ satisfies $b \geqslant 0$ a.e. Furthermore, suppose that ψ tends to constants $A_j \geqslant 0$ on each end E_j, then $\psi \geqslant 0$ on M.

Proof. Let $\epsilon \geqslant 0$ be a given (small) number and define $v \doteq (\psi + \epsilon)^-$. Since $\psi \to A_j \geqslant 0$ on each end, then v must have compact support. Now, notice that $W_{loc}^{2,p} \hookrightarrow W_{loc}^{1,2}$. Since $p > \frac{n}{2}$ and $n \geqslant 3$, this last statement is not obvious only for $n = 3$ and $\frac{3}{2} < p < 2$. But in this case we can appeal to the Sobolev embedding $W_{loc}^{2,p} \hookrightarrow W_{loc}^{1,q}$ for $q = \frac{np}{n-p}$, where $p > \frac{3}{2}$ implies that $q > 3$, thus $W_{loc}^{1,q} \hookrightarrow W_{loc}^{1,2}$, which implies that $W_{loc}^{2,p} \hookrightarrow W_{loc}^{1,2}$. Therefore, this first claim follows. Furthermore, since in the support of v we know that $\psi \leqslant 0$, we get that $v\psi \geqslant 0$ a.e. Finally, notice that on the support of v, it holds that $\nabla\psi = \nabla v$, and

then compute the following

$$\|\nabla v\|_{L^2}^2 = \int_M \langle \nabla \psi, \nabla v \rangle_\gamma dV_\gamma = \int_M (-v)\Delta_\gamma \psi dV_\gamma + \int_\Sigma (-v)(-\nu(\psi))d\Sigma,$$
$$\leqslant - \int_M av\psi dV_\gamma - \int_\Sigma bv\psi d\Sigma \leqslant 0,$$

where in the second line we used equations (3.44). From the above, we get that v is constant and has compact support, hence $v \equiv 0$, which implies that $\psi \geqslant -\epsilon$ $\forall$ $\epsilon \geqslant 0$. Thus, $\psi \geqslant 0$. $\square$

As noted in Maxwell 2005b, quite general existence results for solutions to semi-linear equations whose linear part obeys the (weak) maximum principle as stated above can be derived by applying the above version of the weak maximum principle. Nevertheless, in the investigation of the Lichnerowicz equation we will need the following version of the strong maximum principle.

Lemma 3.3.2 (Maxwell (ibid.)). *Suppose that* (M^n, γ) *is a* $W_\rho^{2,p}$*-AE manifold with* $p > \frac{n}{2}$ *and* $\rho < 0$ *and let* $a \in L_{\rho-2}^p$ *and* $b \in W^{1-\frac{1}{p},p}(\Sigma)$ *satisfy* $a, b \geqslant 0$ *a.e. Suppose that* $u \in W_{loc}^{2,p}$ *is non-negative and satisfies*

$$\Delta_\gamma u - au \leqslant 0,$$
$$-\nu(u) - bu \leqslant 0, \ on \ \Sigma, \tag{3.45}$$

then, if $u(x) = 0$ *for some* $x \in M$, *it follows that* $u \equiv 0$.

The proof of the above lemma can be consulted in Maxwell (ibid., Lemma 4), which follows the lines of Lemma 2.2.3, where the objective is to prove that the the subset $u^{-1}(0)$ is open in M. If x is an interior point, then the proof runs as in Lemma 2.2.3 appealing to the weak Harnack inequality of Trudinger (1973). The new part is the analysis of the case when x is a boundary point. This is achieved by considering the corresponding equations in an hemisphere $B_1(0) \cap \mathbb{R}_+^n$ (a co-ordinate neighbourhood of x) and constructing a related equation on the whole ball $B_1(0)$ to which we can apply the Harnack inequality of Trudinger (ibid.). The details can be consulted in the previously cited reference.

Let us now establish the following notation. Consider the linear operator

$$\mathcal{P}_1 : W_\delta^{2,p}(M) \mapsto L_{\delta-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma),$$
$$u \mapsto (\Delta_\gamma u - au, -(\nu(u) + bu)|_\Sigma), \tag{3.46}$$

where γ is $W_\delta^{2,p}$-AE, with $p > \frac{n}{2}$, $\delta < 0$, $a \in L_{\delta-2}^p(M)$, $b \in W^{1-\frac{1}{p},p}(\Sigma)$, and ν is the outward pointing normal to Σ. Then, we get that the following holds.[4]

Theorem 3.3.1. *Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold with p and δ satisfying the above conditions and $n \geq 3$. If $2 - n < \delta < 0$, then the operator $\mathcal{P}_1$ as defined in (3.46) is an isomorphism as long as $a \geq 0$ and $b \geq 0$ a.e.*

Proof. First, notice that Lemma 3.3.1 implies that $u \in \mathrm{Ker}(\mathcal{P}_1) \implies u \geq 0$, but the same argument applies to $-u$ implying $-u \geq 0$. Thus, $u \equiv 0$ and $\mathcal{P}_1$ is injective. In order to establish the surjectivity claim, let us show that the adjoint operator

$$\mathcal{P}_1^* : L_{2-n-\delta}^{p'}(M) \times W^{-1+\frac{1}{p},p'}(\Sigma) \mapsto W_{-\delta-n}^{-2,p'}(M),$$

is injective. Thus, consider $(v, w) \in \mathrm{Ker}(\mathcal{P}_1^*)$, which implies $\langle (v, w), \mathcal{P}_1 u \rangle = 0$ for all $u \in W_\delta^{2,p}(M)$. That is,

$$0 = \langle v, \Delta_\gamma u - au \rangle_{L_{2-n-\delta}^{p'} \times L_{\delta-2}^p} + \langle w, -\nu(u) - bu \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}}.$$

Considering $u \in C_0^\infty(\overset{\circ}{M})$, we find that v is a weak solution to $\Delta_\gamma v - av = 0$ in $\overset{\circ}{M}$, implying through elliptic regularity that $v \in W_{loc}^{2,p'} \cap L_{2-n-\delta}^{p'}$ and $\Delta_\gamma v - av = 0$ then that $v \in W_{2-n-\delta}^{2,p'}$ is a strong solution. Then,

$$\langle v, \Delta_\gamma u - au \rangle_{L_{2-n-\delta}^{p'} \times L_{\delta-2}^p} = \int_M v \left(\Delta_\gamma u - au \right) dV_\gamma,$$

$$= -\int_M \left(\langle \nabla v, \nabla u \rangle_\gamma + auv \right) dV_\gamma + \int_\Sigma v\nu(u) d\Sigma,$$

$$= \int_M \underbrace{\left(\Delta_\gamma v - av \right)}_{=0} u \, dV_\gamma + \int_\Sigma (v\nu(u) - \nu(v)u) d\Sigma,$$

$$= \int_\Sigma (v\nu(u) - \nu(v)u) d\Sigma \quad \forall \, u W_\delta^{2,p},$$

where the above identities hold classically for smooth compactly supported functions, and an approximation argument shows that they also hold in this case. Now,

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[4]To the best of our knowledge, Theorem 3.3.1 is due to Maxwell (2005b, Proposition 1).

since the trace map

$$\tau : W^{2,p}(U) \mapsto W^{2-\frac{1}{p},p}(\Sigma) \times W^{1-\frac{1}{p},p}(\Sigma),$$
$$u \mapsto (\tau(u), \tau(v(u)))$$

is surjective, where U denotes a neighbourhood of Σ, then the above implies that

$$0 = \int_{\Sigma} (v\chi_1 - v(v)\chi_2)d\Sigma + \langle w, -\chi_1 - b\chi_2 \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}}, \quad (3.47)$$

for all $\chi_1 \in W^{1-\frac{1}{p},p}(\Sigma)$ and $\chi_2 \in W^{2-\frac{1}{p},p}(\Sigma)$. Fixing $\chi_2 = 0$, we find

$$\langle w, \chi \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}} = \int_{\Sigma} v\chi d\Sigma, \quad \forall \chi \in W^{1-\frac{1}{p},p}(\Sigma).$$

Feeding this information in the previous expression with $\chi_1 = 0$, we find

$$0 = \int_{\Sigma} v(v)\chi_2 d\Sigma + \langle w, b\chi_2 \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}},$$
$$= \int_{\Sigma} v(v)\chi_2 d\Sigma + \int_{\Sigma} bv\chi_2 d\Sigma,,$$
$$= \int_{\Sigma} (v(v) + bv)\chi_2 d\Sigma, \quad \forall \chi_2 W^{2-\frac{1}{p},p}(\Sigma).$$

The above implies that $(v(v)+bv)|_{\Sigma} = 0$. That is, $v \in \mathrm{Ker}\left(\mathcal{P}_1 : W^{2,p'}_{-n-\delta+2}(M) \mapsto L^{p'}_{-n-\delta}(M) \times W^{1-\frac{1}{p'},p}(\Sigma)\right)$. But, actually, elliptic regularity gives us $v \in W^{2,p}_{loc} \cap L^{p'}_{2-\delta-n}$, which implies that $u = o(1)$ at infinity. Then, just as the beginning of the proof, the fact that $v \in W^{2,p}_{loc}$ solves

$$\Delta_\gamma v - av = 0,$$
$$-(v(v) + bv) = 0 \quad (3.48)$$

and $v \to 0$ at infinity, implies that both $v, -v \geqslant 0$ due to Lemma 3.3.1, and therefore $v \equiv 0$. That is, the condition $2-n < \delta$, implies that $\mathcal{P}_1 : W^{2,p'}_{2-n-\delta}(M) \mapsto L^{p'}_{-n-\delta}(M) \times W^{1-\frac{1}{p'},p}(\Sigma)$ is injective. With this information, we now know that

$$\langle w, -v(u) - bu \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}} = 0, \quad \forall u \in W^{2,p}_{\delta}(M). \quad (3.49)$$

Once more, this implies that

$$0 = \langle w, \chi_1 \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}} + \langle w, b\chi_2 \rangle_{W^{-1+\frac{1}{p},p'} \times W^{1-\frac{1}{p},p}}$$

for all $\chi_1 \in W^{1-\frac{1}{p},p}(\Sigma)$ and $\chi_2 \in W^{2-\frac{1}{p},p}(\Sigma)$. Setting $\chi_2 = 0$, implies that $w = 0$, and proves that $\mathcal{P}_1^*$ is injective, finishing the proof. □

Remark 3.3.2. *Notice that, under the conditions of the above theorem, the elliptic estimates of Proposition 3.3.1 are improved to*

$$\|u\|_{W_\delta^{2,p}} \leqslant C \|\mathcal{P}_1 u\|_{L_{\delta-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma)} \quad \forall u \in W_\delta^{2,p}(M). \tag{3.50}$$

In particular, fixing $a, b = 0$ above, we see that

$$\|u\|_{W_\delta^{2,p}} \leqslant C \left(\|\Delta_\gamma u\|_{L_{\delta-2}^p(M)} + \|v(u)\|_{W^{1-\frac{1}{p},p}(\Sigma)} \right) \quad \forall u \in W_\delta^{2,p}(M). \tag{3.51}$$

Finally, let us highlight the following useful result concerning the kernel of $\mathcal{P}_1$.

Lemma 3.3.3. *Let (M^n, γ) be an AE manifold satisfying the hypotheses of Theorem 3.3.1 and suppose that $u \in W_\rho^{2,p}$, $\rho < 0$, is in the kernel of $\mathcal{P}_1$. Then, $u \in W_{\rho'}^{2,p}$ for all $\rho' \in (2 - n, 0)$.*

Proof. Since $a \in L_{\delta-2}^p(M)$ and $b \in W^{1-\frac{1}{p},p}(\Sigma)$, then $au \in L_{\delta'-2}^p(M)$ for any $\delta' > \delta + \rho$ and $bu \in W^{1-\frac{1}{p},p}(\Sigma)$. Actually, if $2 - n < \rho$, since $\delta < 0$, this implies that there is some $\max\{\rho + \delta, 2 - n\} < \delta' < \rho$ such that $(\Delta_\gamma u, -v(u)) = (au, bu) \in L_{\delta'-2}^p \times W^{1-\frac{1}{p},p}$. But the above theorem guarantees that $(\Delta_\gamma, -v) : W_{\delta'}^{2,p} \mapsto L_{\delta'-2}^p \times W^{1-\frac{1}{p},p}$ is an isomorphism for all $2 - n < \delta' < 0$. That is, we have found $u \in W_{\delta'}^{2,p}(M)$ with $\max\{\rho + \delta, 2 - n\} < \delta' < \rho$ and we can iterate the argument starting with $2 - n < \delta' < \rho < 0$ in order to establish that, given any $2 - n < \rho' < 0$, $u \in W_{\rho'}^{2,p}$. □

The conformal Killing Laplacian

Let us analyse the behaviour of the CKL in the context of AE manifolds. The ultimate objective is to prove an isomorphism result of the type of Theorem 3.3.1, but, first, we need some tool to replace the maximum principle in the proof of injectivity. That is, as a first step we are looking for a proof of the claim that AE metrics have no CKF which decay at infinity. The intuition for this comes from the fact that this is case for the Euclidean metric in $\mathbb{R}^n$ and this has been established under different hypotheses by Christodoulou and O'Murchadha (1981) and Maxwell (2005b). Since the latter proof extends to less regular metrics, we shall follow Maxwell (2005b). Let us first review a few basic facts concerning the CKF of $(\mathbb{R}^n, \cdot)$. First, recall the following general fact.

Proposition 3.3.2. *Let (M^n, g) be a smooth Riemannian manifold. If X is a Killing field, then it obeys the following equation*

$$\nabla_i \nabla_j X_k = R_{iljk} X^l. \tag{3.52}$$

Proof. By definition, we have that $\nabla_i X_j = -\nabla_j X_i$, implying

$$\nabla_k \nabla_i X_j = -\nabla_k \nabla_j X_i = -\left(R_{iljk} X^l + \nabla_j \nabla_k X_i \right)$$

which we rewrite as

$$\nabla_k \nabla_i X_j + \nabla_j \nabla_k X_i = -R_{iljk} X^l. \tag{3.53}$$

Now, notice that the first Bianchi identity implies that

$$\left(R_{lijk} + R_{ljki} + R_{lkij} \right) X^l = 0.$$

Rewriting this explicitly in terms of covariant derivatives of X, and using the fact that X is Killing, we have

$$\nabla_k \nabla_j X_i + \nabla_j \nabla_i X_k + \nabla_i \nabla_k X_j = 0.$$

Putting the above equation together with (3.53), we find (3.52). □

Corollary 3.3.1. *Any Killing field X of $(\Omega, \cdot)$, $\Omega \subset \mathbb{R}^n$ an open set endowed with the Euclidean metric, is of the form*

$$X_i(x) = c_i + \omega_{ij} x^j, \tag{3.54}$$

where $\{x^i\}_{i=1}^n$ are rectangular coordinates, c_i and ω_{ij} are constants, where $\omega_{ij} = -\omega_{ji}$. That is, $\omega \in \mathfrak{o}(n)$ is an antisymmetric matrix.

Proof. From (3.52), we see that in the case of $(\Omega, \cdot)$ we have

$$\partial_{x^k} \partial_{x^j} X_i = 0, \text{ for all } k, j, i = 1, \cdots, n,$$

thus the solution must be of the form $X_i = c_i + \omega_{ij} x^j$ and plugging this in the Killing equation we find that $\omega_{ij} = -\omega_{ji}$. $\qquad\square$

The following is also an easy result related to the conformal field of $\mathbb{R}^n$.

Proposition 3.3.3. *Let X be a conformal Killing field of $(\Omega, \cdot)$, $\Omega \subset \mathbb{R}^n$ an open set endowed with the Euclidean metric, $n \geqslant 3$. Then*

$$\operatorname{div} X(x) = a + b_i x^i, \tag{3.55}$$

where $\{x^i\}_{i=1}^n$ are rectangular coordinates and a and b_i are constants.

Proof. Recall that X is a CKF iff $\partial_i X_j + \partial_j X_i = \frac{2}{n} \operatorname{div} X \, \delta_{ij}$. Thus,

$$\frac{2}{n} \delta_{ij} \Delta(\operatorname{div} X) = \partial_{ik}(\partial_k X_j) + \partial_{jk}(\partial_k X_i) = 2\partial_{ik}\left(-\partial_j X_k + \frac{2}{n}\operatorname{div} X \, \delta_{jk}\right),$$

$$= 2\left(-\partial_{ij}(\operatorname{div} X) + \frac{2}{n}\partial_{ij}(\operatorname{div} X)\right),$$

$$= 2\frac{2-n}{n}\partial_{ij}(\operatorname{div} X),$$

implying $(2-n)\partial_{ij}(\operatorname{div} X) = \delta_{ij}\Delta(\operatorname{div}(X))$. Tracing the equation, we find $\Delta(\operatorname{div}(X)) = 0$, which in turn implies $\partial_{ij}(\operatorname{div}(X)) = 0$ for all $i, j = 1, \cdots, n$, implying (3.55). $\qquad\square$

We can reduce the analysis of CKF to that of Killing fields as follows.

Proposition 3.3.4. *Any conformal Killing field X of $(\Omega, \cdot)$, $\Omega \subset \mathbb{R}^n$ an open set endowed with the Euclidean metric and $n \geqslant 3$, is of the form*

$$X_i(x) = c_i + \omega_{ij} x^j + \frac{1}{n} b \cdot x \, x^i - \frac{1}{2n}|x|^2 b^i + \frac{1}{n} a x^i, \tag{3.56}$$

where $\{x^i\}_{i=1}^n$ are rectangular coordinates, c_i and ω_{ij} are as in Proposition 3.3.4 while a and b are as in (3.55).

Proof. Set $Y_i \doteq X_i - \frac{1}{n}b \cdot x\, x^i + \frac{1}{2n}|x|^2 b^i - \frac{1}{n}a x^i$. We claim that this is an Euclidean Killing field. This follows computationally, since

$$\partial_j Y_i = \partial_j X_i - \frac{1}{n}\left(b_j x^i - x^j b_i + b \cdot x \delta_{ij}\right) - \frac{1}{n}a\delta_{ij},$$

$$\partial_i Y_j = \partial_i X_j - \frac{1}{n}(b_i x^j - x^i b_j + b \cdot x \delta_{ij}) - \frac{1}{n}a\delta_{ij},$$

implying

$$\partial_j Y_i + \partial_i Y_j = \partial_j X_i + \partial_i X_j - \frac{2}{n}\underbrace{(a + b \cdot x)}_{=\mathrm{div}(X)}\delta_{ij} = 0,$$

where the last equality if the definition of X being a CKF. Thus, from Proposition 3.3.4 we find that Y must have the form $Y_i = c_i + \omega_{ij} x^j$, which implies the final result.   □

The above proposition provides a proof for our original claim that in Euclidean space there are no CKF decaying at infinity. Furthermore, it gives us the tool to prove the following proposition (see Maxwell (2005b, Lemma 6) for another proof in the same spirit).

Proposition 3.3.5. *Suppose that X is a non-trivial CKF on $(B_1(0), \cdot\,)$, where $B_1(0)$ denotes the unit ball in $\mathbb{R}^n$, $n \geqslant 3$. If $X(0) = 0$ and $\nabla X(0) = 0$, then $X(x) \neq 0$ if $x \neq 0$.* [5]

Proof. From Proposition 3.3.4, we write

$$X_i(x) = c_i + \omega_{ij} x^j + \frac{1}{n}b \cdot x\, x^i - \frac{1}{2n}|x|^2 b_i + \frac{1}{n}a x^i.$$

Since $X(0) = 0$, then $c_i = 0$ for all $i = 1, \cdots, n$. Also, from

$$0 = \partial_k X_i(0) = \omega_{ik} + \frac{1}{n}a\delta_{ik} + \frac{1}{n}\left(b_k x^i - x^k b_i + b \cdot x\,\delta_{ik}\right)|_{x=0},$$

$$= \omega_{ik} + \frac{1}{n}a\delta_{ik} \text{ for all } i, k = 1, \cdots, n.$$

---

[5] In this statement ∇ denotes the Euclidean Riemannian connexion.

But the above implies that $\omega_{ik} = -\frac{a}{n}\delta_{ik}$, which, due to the symmetries of each side, can be satisfied iff $\omega = 0$ and $a = 0$. Thus,

$$X_i(x) = \frac{1}{n} b \cdot x \, x^i - \frac{1}{2n}|x|^2 b_i.$$

Since $X \not\equiv 0$ by assumption, then $b \neq 0$. Assume that there is some $x_0 \neq 0$ such that $X_i(x_0) = 0$. That is

$$b \cdot x_0 \, x_0 - \frac{1}{2}|x_0|^2 b = 0. \tag{3.57}$$

Taking inner product with x_0, we find

$$b \cdot x_0 \, |x_0|^2 - \frac{1}{2}|x_0|^2 b \cdot x_0 = \frac{1}{2}|x_0|^2 b \cdot x_0 = 0.$$

Since $x_0 \neq 0$ by hypothesis, we find that $b \cdot x_0 = 0$. But then (3.57) implies that $|x_0|^2 b = 0$ from which, for $x_0 \neq 0$, we must have $b = 0$, which implies that $X \equiv 0$ and therefore we have a contradiction. Thus, no such $x_0 \in B_1(0)$ can exist. □

We will combine the above result with the following one (due to Maxwell (2005b, Lemma 7)) to prove the injectivity of conformal Killing Laplacian in this context.

Lemma 3.3.4. *Let (M^n, g) be an AE manifold of class $W_\rho^{2,p}$, $p > \frac{n}{2}$, $\rho < 0$ and $n \geq 3$. Suppose that $X \in W_\delta^{2,p}$ is a conformal Killing field with $\delta < 0$. Then X vanishes identically in a neighbourhood of infinity.*

Proof. Let us fix an end of M and denote it by $E_1 \cong \mathbb{R}^n \setminus \overline{B_1(0)}$. Then, let $\{x^i\}_{i=1}^n$ be rectangular end coordinates and denote by e the euclidean metric on E_1. Consider the sequence of metrics $\{g_k\}_{k=1}^\infty$ on E_1 defined via $g_k(x) \doteq g(2^k x)$ and notice that

$$\|g_k - e\|_{W_\rho^{2,p}(E_1)} \lesssim 2^{k\rho}\|g - e\|_{W_\rho^{2,p}(\mathbb{R}^n \setminus B_{2^k}(0))} \xrightarrow[k \to \infty]{} 0.$$

The above implies that, as maps on $W_\rho^{2,p}(E_1)$,

$$\Delta_{g_k,\mathrm{conf}} \to \Delta_{e,\mathrm{conf}},$$
$$\mathscr{L}_{g_k,\mathrm{conf}} \to \mathscr{L}_{e,\mathrm{conf}}.$$

Now, assume that X is a CKF of g which does not vanish identically outside a compact set. Set E_R to be the exterior of $B_R(0)$ in E_1 and consider $\hat{X}_k(x) \doteq X(2^k x)$, which by hypotheses does not vanish identically. Then, consider $\bar{X}_k \doteq \frac{\hat{X}_k}{\|\hat{X}_k\|_{W_\rho^{2,p}}}$, which is well-defined due to our last assumption. Then, since $W_\rho^{2,p} \hookrightarrow W_{\rho'}^{1,p}$ for any $\rho < \rho'$, we know that $\bar{X}_k \to \bar{X}$ in $W_{\rho'}^{1,p}$. Now, let us apply the elliptic estimates of Proposition 3.3.1 to the conformal Killing Laplacian (and its boundary operator) constructed with the Euclidean metric e, so that

$$
\begin{aligned}
\|\bar{X}_k - \bar{X}_l\|_{W_\rho^{2,p}(E_1)} &\lesssim \|\Delta_{e,\mathrm{conf}}(\bar{X}_k - \bar{X}_l)\|_{W_\rho^{2,p}(E_1)} + \|\bar{X}_k - \bar{X}_l\|_{L_\rho^p(E_1)} \\
&\quad + \|\mathscr{L}_{e,\mathrm{conf}}(\bar{X}_k - \bar{X}_l)\|_{W^{1-\frac{1}{p},p}(\partial B_1)}, \\
&\lesssim \|\Delta_{e,\mathrm{conf}}\bar{X}_k\|_{W_\rho^{2,p}(E_1)} + \|\Delta_{e,\mathrm{conf}}\bar{X}_l\|_{W_\rho^{2,p}(E_1)} \\
&\quad + \|\mathscr{L}_{e,\mathrm{conf}}(\bar{X}_k - \bar{X}_l)\|_{W_\rho^{2,p}(E_1)} + \|\bar{X}_k - \bar{X}_l\|_{L_\rho^p(E_1)}, \\
&\lesssim \|(\Delta_{e,\mathrm{conf}} - \Delta_{g_k,\mathrm{conf}})\bar{X}_k\|_{W_\rho^{2,p}(E_1)} \\
&\quad + \|(\Delta_{e,\mathrm{conf}} - \Delta_{g_l,\mathrm{conf}})\bar{X}_l\|_{W_\rho^{2,p}(E_1)} \\
&\quad + \|(\mathscr{L}_{e,\mathrm{conf}} - \mathscr{L}_{g_k,\mathrm{conf}})\bar{X}_k\|_{W_\rho^{2,p}(E_1)} \\
&\quad + \|(\mathscr{L}_{e,\mathrm{conf}} - \mathscr{L}_{g_l,\mathrm{conf}})\bar{X}_l\|_{W_\rho^{2,p}(E_1)} + \|\bar{X}_k - \bar{X}_l\|_{L_{\rho'}^p}, \\
&\lesssim \|\Delta_{e,\mathrm{conf}} - \Delta_{g_k,\mathrm{conf}}\|_{Op} + \|\Delta_{e,\mathrm{conf}} - \Delta_{g_l,\mathrm{conf}}\|_{Op} \\
&\quad + \|\mathscr{L}_{e,\mathrm{conf}} - \mathscr{L}_{g_k,\mathrm{conf}}\|_{Op} + \|\mathscr{L}_{e,\mathrm{conf}} - \mathscr{L}_{g_l,\mathrm{conf}}\|_{Op} \\
&\quad + \|\bar{X}_k - \bar{X}_l\|_{L_{\rho'}^p},
\end{aligned}
$$

where in the second inequality we used the continuity of the trace map and in the third one we used that $\hat{X}_k$ is a g_k-CKF and thus $\Delta_{g_k,\mathrm{conf}}\hat{X}_k = \mathscr{L}_{g_k,\mathrm{conf}}\hat{X}_k = 0$ and also that $L_\rho^p \hookrightarrow L_{\rho'}^p$ for $\rho < \rho'$. From our construction, the right-hand side of the above expression goes to zero and therefore $\{\bar{X}_k\}$ is Cauchy in $W_\rho^{2,p}(E_1)$ and therefore $\bar{X} \xrightarrow{W_\rho^{2,p}} \bar{X} \in W_\rho^{2,p}$, but then $\mathscr{L}_{g_k,\mathrm{conf}}\bar{X}_k \to \mathscr{L}_{e,\mathrm{conf}}\bar{X}$, which implies that $\bar{X}$ is an Euclidean CKF since $\mathscr{L}_{g_k,\mathrm{conf}}\bar{X}_k = 0$ for all k, and since $\bar{X} \in W_\rho^{2,p}$, with $\rho < 0$, we conclude that $\bar{X} \equiv 0$ since there are no Euclidean CKF decaying at infinity. But this contradicts the fact that $\bar{X}_k \to \bar{X}$ and $\bar{X}_k$ is normalised. Therefore, we have a contradiction and no such CKF X can exist. $\square$

We can now prove the following non-existence result for CKF decaying at

infinity. The proof we shall present is due to David Maxwell, who has established the result under different functional hypotheses in Maxwell (2005b, Theorem 4) and Maxwell (2006, Proposition 4.5).

Theorem 3.3.2. *Let (M^n, g) be an AE manifold of class $W^{2,p}_\rho$, $p > n$, $\rho < 0$ and $n \geq 3$. Then, there are no non-trivial CKF in $W^{2,p}_\delta$ for any $\delta < 0$.*

Proof. We intend to prove that if $X \in W^{2,p}_\delta$ is a CKF of g, then $X^{-1}(0) = M$. First, notice that Lemma 3.3.4 implies that $X^{-1}(0)$ contains an open neighbourhood of infinity in M. Thus, due to the continuity of X, if the claim is not true there must be an interior point $p_0 \in \partial X^{-1}(0) \cap \overset{\circ}{M}$ and p_0 must be contained within a compact set. Let us then fix a coordinate ball $B_1(0)$ centred at p_0, with coordinates $\{x^i\}_{i=1}^n$, $x(p) = 0$, and from now on work on such a neighbourhood. Furthermore, chose coordinates so that $g_{ij}(0) = \delta_{ij}$. Finally, in this construction, notice that X is a CKF of g in $B_1(0)$ and, since $W^{2,p}(B_1) \hookrightarrow C^1(B_1)$ for $p > n$, continuity of X and ∇X shows that $X(0), \nabla X(0) = 0$.

The idea now is to show that the assumption that $\partial X^{-1}(0) \neq \emptyset$ is contradictory. We shall do this by proving that, if this were true, then would be able to construct a CFK for the Euclidean metric which violates Proposition 3.3.5. For this, first notice that if $p_0 \in \partial X^{-1}(0) \cap \overset{\circ}{M}$, then $X \not\equiv 0$ for any neighbourhood of p_0 and also, for each $0 < r_k \leq 1$ there is some $x_k \in B_{\frac{r_k}{2}}(0) \cap X^{-1}(0)$. We therefore consider a sequence $\{r_k\}_{k=1}^\infty$, with $r_k \searrow 0$ and $r_k \leq 1$, and a corresponding selection of points $\{\hat{x}_k\}_{k=1}^\infty \subset B_1(0)$, such that $|\hat{x}_k| = \frac{r_k}{2}$ and $X(\hat{x}_k) = 0$.

Now, construct the sequence of metrics $\{g_k\}_{k=1}^\infty$ on $B_1(0)$ defined by

$$g_k(x) \doteq g(r_k x), \ x \in B_1(0),$$

which is well-defined since $r_k \leq 1$ for all k. It follows from $g \in W^{2,p}(B_1)$ that $g_k \xrightarrow{W^{2,p}(B_1)} e$, where e stands for the Euclidean metric on B_1. Similarly to the above lemma, this implies that

$$\Delta_{g_k,\text{conf}} \to \Delta_{e,\text{conf}},$$
$$\mathcal{L}_{g_k,\text{conf}} \to \mathcal{L}_{e,\text{conf}}$$

as operators on $W^{2,p}(B_1)$. Therefore, to construct an Euclidean CKF in $B_1(0)$ we follow the same ideas as in Lemma 3.3.4 and consider the sequence $\{\hat{X}_k\}_{k=1}^\infty$ defined by

$$\hat{X}_k(x) \doteq X(r_k x), \ x \in B_1(0).$$

Since, by hypothesis, $X \not\equiv 0$ on any ball centred at 0, it holds that $\hat{X}_k \not\equiv 0$ for any k and therefore we can construct the $W^{2,p}$-normalised sequence $\bar{X}_k \doteq \frac{\hat{X}_k}{\|\hat{X}_k\|_{W^{2,p}(B_1)}}$. By compactness of $W^{2,p}(B_1) \hookrightarrow W^{1,p}(B_1)$ we have a subsequece, to which we restrict, satisfying $\bar{X}_k \xrightarrow{W^{1,p}(B_1)} \bar{X}_0 \in W^{1,p}(B_1)$. We can now apply the same reasoning as in Lemma 3.3.4 to show (through elliptic estimates) that $\{\bar{X}_k\}$ is Cauchy $W^{2,p}(B_1)$ and achieve strong $W^{2,p}$-convergence to $\bar{X}_0 \in W^{2,p}(B_1)$. Following also such arguments, we again find that $\bar{X}_0$ is an Euclidean CKF and, also, $p > n$ implies C^1-convergence and therefore

$$\bar{X}_0(0), \nabla\bar{X}_0(0) = 0. \tag{3.58}$$

But now notice that our sequence $\{\hat{x}_k\} \subset B_1$ selected above, provides us with a sequence $\{x_k\} \subset B_1(0)$, satisfying $|x_k| = \frac{1}{2}$ and $\bar{X}_k(x_k) = \hat{X}_k(x_k) = 0$ for all k. Now, being bounded in $\overline{B_1(0)}$, the sequence $\{x_k\}$ has a convergent subsequence (to which we restrict) with limit x satisfying $|x| = \frac{1}{2}$, and, along such sequence, we find $\bar{X}_0(x) = 0$. But, this last result together with (3.58) contradict Proposition 3.3.5, proving the non-existence of $p_0 \in \partial X^{-1}(0) \cap \overset{\circ}{M}$. $\qquad\square$

Remark 3.3.3. *Let us highlight that, in the above proof, the condition $p > n$ was only used within a fixed ball $B_1(0)$ centred at p_0, with p_0 contained in a compact subset of M, where this last statement only requires $p > \frac{n}{2}$ due to Lemma 3.3.4. Thus, the same conclusion follows for any metric $g \in W^{2,q}_{loc}$ which is $W^{2,p}_\rho(M)$-AE with $p > \frac{n}{2}, q > n$ and $\rho < 0$. Thus, for instance, if $g \in C^\infty$ is $W^{2,p}_\rho$-AE with $p > \frac{n}{2}$ and $\rho < 0$, the result follows and we conclude that g possesses no CKF in $W^{2,p}_\delta$ for any $\delta < 0$.*

The above theorem will allow us to guarantee injectivity of the conformal Killing Laplacian under the corresponding regularity hypotheses. Therefore, in order to analyse the corresponding analogue to Equation (3.46), let us consider an AE manifold (M^n, γ) and introduce the notation

$$\begin{aligned} \mathcal{P}_2 : W^{2,p}_\delta(M) &\mapsto L^p_{\delta-2} \times W^{1-\frac{1}{p},p}(\Sigma), \\ X &\mapsto (\Delta_{\gamma,\mathrm{conf}}X, \mathscr{L}_{\gamma,\mathrm{conf}}X(\nu,\cdot)) \end{aligned} \tag{3.59}$$

where, as usual, where γ is $W^{2,p}_\rho$-AE, with $p > \frac{n}{2}$, $\rho < 0$ and ν is the outward pointing normal to Σ. Let us now present the following result.

Theorem 3.3.3 (Maxwell (2005b)). *Let (M^n, γ) be an AE manifold satisfying the above hypotheses and consider the operator (3.59). If $2 - n < \delta < 0$, then $\mathcal{P}_2$: $W_\delta^{2,p} \mapsto L_{\delta-2}^p \times W^{1-\frac{1}{p},p}(\Sigma)$ is a Fredholm operator of index zero. Furthermore, its kernel consists of $W_\delta^{2,p}$-CKF and therefore it is an isomorphims if $p > n$.*

The proof of the above theorem follows the basic steps of those of Theorem 3.3.1 with some subtle modifications. Due to their similarities, and since the interested reader can find it completely spelled out in Maxwell (ibid., Proposition 6), we omit the proof and merely comment on its basic ingredients.

The first step is to establish the claim for smooth metrics, which admit one further reduction. That is, if $g \in C^\infty$ is $W_\delta^{2,p}$-AE and if we denote by $\mathcal{P}$ the action of $\mathcal{P}_2$ on $W_\delta^{2,2}(M)$, that is $\mathcal{P} \doteq \mathcal{P}_2 : W_\delta^{2,2}(M) \mapsto L_{\delta-2}^2(M) \times W^{\frac{1}{2},2}(\Sigma)$, if $\mathcal{P}$ is isomorphism, then (due to elliptic regularity arguments) so is $\mathcal{P}_2$ acting as in (3.59). Then, we proceed to prove the isomorphism claim for $\mathcal{P}$. The injectivity follows by an integration by parts procedure on any element $X \in \text{Ker}(\mathcal{P})$ (justified via approximation arguments). This establishes that any such X is actually a CKF of the smooth metric, and, in such a case, we know that X must vanish due to our previous discussion and references. Then, the surjectivity follows by the injectivity of the adjoint map under the bound $2 - n < \delta$, where the analysis is quite parallel to that of Theorem 3.3.1. Thus, this establishes the isomorphism of $\mathcal{P}$ and therefore also of $\mathcal{P}_2$ for smooth metrics, which shows that $\mathcal{P}_2$ is a Fredholm operator of zero index in this case. Then, we use Fredholm stability properties to derive the corresponding claim under the more general hypotheses of the theorem, which is achieved approximating g by smooth metrics and using Theorem A.1.2. This establishes the Fredholm claims of the theorem. Finally, we can use Theorem 3.3.2 to finish.

Remark 3.3.4. *Let us point out that the same comments made in Remark 3.3.2 hold also for $\mathcal{P}_2$ under the conditions of Theorem 3.3.3 which guarantee that $\mathcal{P}_2$ is an isomorphism.*

The monotone iteration scheme

When investigating solutions for the conformal problem associated to the ECE on AE manifold we attempt to use the above analytic theory. Nevertheless, notice that the conformal factor (for which the Lichnerowicz equation is posed) should not vanish at infinity. Therefore, the idea is to split it as $\phi = \omega + \varphi$, with $\varphi \in W_\delta^{2,p}$, $\delta < 0$, and ω capturing some prescribed behaviour of ϕ at infinity. In order for

ω not to interfere significantly in the analysis, we shall construct it according to the following lemma, which follows the ideas of Dilts, Isenberg, et al. (2014) and Holst and Meier (2014).

Lemma 3.3.5. *Let (M^n, g) be a $W^{2,p}_\rho$-AE manifold, with $\rho < 0$ with ends $\{E_j\}_{j=1}^N$ and let $\{A_j\}_{j=1}^N$ be bounded functions on M with $\Delta_g A_j \in L^p_{\delta-2}$ for some $\delta < 0$. Then, there is a unique solution to the equation*

$$\Delta_g \omega = 0,$$
$$\nu(\omega) = 0 \ \ on \ \Sigma,$$
(3.60)

such that ω tends to A_j in E_j as we go to infinity.[6] Furthermore,

$$\min_{1 \le j \le N} \inf_M A_j \le \omega \le \max_{1 \le j \le N} \sup_M A_j.$$
(3.61)

Proof. Let $\omega_1 = \sum_{j=1}^N \chi_j A_j$, where χ_j is a cutoff function equal to 1 on E_j and supported in a neighbourhood of E_j. Thus, since $\Delta_g A_j \in L^p_{\delta-2}$ for all $j = 1, \cdots, N$, we get that $\Delta_g \omega_1 \in L^p_{\delta-2}$ and also $\nu(\omega_1) \in W^{1-\frac{1}{p},p}(\Sigma)$, which, from Theorem 3.3.1, implies that there is a unique $\omega_2 \in W^{2,p}_\delta$ satisfying $\Delta_g \omega_2 = \Delta_g \omega_1$ with boundary condition $\nu(\omega_2) = \nu(\omega_1)$, implying that $\omega \doteq \omega_2 - \omega_1$ is harmonic, satisfies $\nu(\omega) = 0$ along ∂M and tends asymptotically to A_j on each E_j as we move towards infinity, which guarantees the existence of a solution to (3.60). Now, for the uniqueness claim, consider two such solutions ω_1 and ω_2, then their difference satisfies (3.60) but now $\omega_1 - \omega_2 \to 0$ at infinity. Then, Lemma 3.3.1 implies that both $\omega_1 - \omega_2 \ge 0$ and $-(\omega_1 - \omega_2) \ge 0$ and therefore $\omega_1 = \omega_2$.

In order to establish (3.61), consider $\epsilon < \min_{1 \le j \le N} \inf_M A_j$ and consider $v \doteq (\omega - \epsilon)^-$. Then, by definition of ϵ, since ω is asymptotic to A_j on each E_j, there must be a compact set K such that $v|_{M \setminus K} \equiv 0$. That is, v must have compact support. Thus, we have the following

$$\|\nabla v\|^2_{L^2} = \int_M \langle \nabla \omega, \nabla v \rangle_g \, dV_g = -\int_M v \Delta_g \omega \, dV_g = 0,$$

---

[6]This means that, given $\epsilon > 0$, there exists a compact set $K \subset M$ such that

$$\sup_{E_j \cap (M \setminus K)} |\omega - A_j| < \epsilon.$$

which implies that $v \equiv 0$. Therefore $\omega \geqslant \epsilon \; \forall$ such $\epsilon < \min_{1 \leqslant j \leqslant N} \inf_M A_j$, implying that $\omega \geqslant \min_{1 \leqslant j \leqslant N} \inf_M A_j$. Similarly, if we consider $\epsilon > \max_{1 \leqslant j \leqslant N} \sup_M A_j$ and we define $v \doteq (\epsilon - \omega)^-$, then, again, this implies that v must have compact support. Then, the same kind of integration by parts argument as above implies that $v \equiv 0$, and hence implies that for any such ϵ it holds that $\epsilon > \omega$, implying that $\max_{1 \leqslant j \leqslant N} \sup_M A_j \geqslant \omega$. $\square$

We will now consider the existence of positive solutions to Lichnerowicz-type semilinear equations of the form

$$
\begin{aligned}
\Delta_\gamma \psi &= f_1(\cdot, \psi) = \sum_I a_I \psi^I \text{ on } M, \\
-\nu(\psi) &= f_2(\cdot, \psi) = \sum_J b_J \psi^J \text{ on } \Sigma,
\end{aligned}
\tag{3.62}
$$

where $f_1 : M \times \mathcal{I}_1 \mapsto \mathbb{R}$ and $f_2 : \Sigma \times \mathcal{I}_2 \mapsto \mathbb{R}$ are functions of the form:

$$
\begin{aligned}
f_1(x, y) &\doteq \sum_I a_I(x) y^I, \\
f_2(x, y) &\doteq \sum_J b_J(x) y^J,
\end{aligned}
\tag{3.63}
$$

where the summation is carried out along exponents $I, J \in \mathbb{R}$, we consider $a_I \in L^p_{\delta-2}(M)$ and $b_J \in W^{1-\frac{1}{p},p}(\Sigma)$, with $\delta < 0$, for all I and J, and $\mathcal{I}_1$ and $\mathcal{I}_2$ are intervals in the real line on which the functions $y \mapsto y^I$ and $y \mapsto y^J$ are smooth for all the exponents I, J involved. Let us notice that, since we shall split $\phi = \omega + \varphi$ according to Lemma 3.3.5, then (3.62) is recast as the following boundary problem for $\varphi \in W^{2,p}_\delta$:

$$
\begin{aligned}
\Delta_\gamma \varphi &= f_1(\cdot, \omega + \varphi) \text{ on } M, \\
-\nu(\varphi) &= f_2(\cdot, \omega + \varphi) \text{ on } \Sigma,
\end{aligned}
\tag{3.64}
$$

where we have used that ω is constructed to satisfy (3.60). In this context, we refine the notion of barriers introduced in Definition 2.2.1 as follows.

Definition 3.3.1. *Let* (M^n, γ) *be a* $W^{2,p}_\delta$*-AE manifold, with* $p > \frac{n}{2}$. *We say that* $\phi_- \in W^{2,p}_{loc}$ *is a **subsolution** of the equation (3.62) if*

$$
\begin{aligned}
\Delta_\gamma \phi_- &\geqslant f_1(x, \phi_-), \\
-\nu(\phi_-) &\geqslant f_2(x, \phi_-).
\end{aligned}
\tag{3.65}
$$

*Analogously, we say that $\phi_+ \in W^{2,p}_{loc}$ is a **supersolution** of the same equation if*

$$\Delta_\gamma \phi_+ \leq f_1(x, \phi_+),$$
$$-\nu(\phi_+) \leq f_2(x, \phi_+).$$
$$(3.66)$$

Let us highlight that, in practice, we shall consider barriers which have the form $\phi_\pm = \omega_\pm + \varphi_\pm$, where $\omega_\pm$ are constructed according to Lemma 3.3.5 and prescribe the behaviour of the barriers at infinity, while $\varphi_\pm \in W^{2,p}_\delta$ and satisfy the corresponding inequalities. In this context we have the following analogue of Theorem 2.2.1.[7]

Theorem 3.3.4. *Consider equation (3.62) where γ is $W^{2,p}_\delta$-AE, $p > \frac{n}{2}$ and $\delta < 0$. Suppose that this equation admits a pair of sub and supersolutions $\phi_- \leq \phi_+$ such that*

$$l \leq \phi_- \leq \phi_+ \leq m,$$

where $l \leq m$ are some non-negative real numbers satisfying $[l, m] \subset \mathcal{I}_1 \cap \mathcal{I}_2$ and also $\phi_- - c_- \in W^p_{2,\delta}$ and $\phi_+ - c_+ \in W^p_{2,\delta}$ for some numbers $c_- \leq c_+$. Furthermore, let ω be as in Lemma 3.3.5, where each A_j satisfies $c_- \leq A_j \leq c_+$. Then, equation (3.62) admits a solution ϕ satisfying $\phi - \omega \in W^{2,p}_\delta$ and $\phi_- \leq \phi \leq \phi_+$.

Proof. The first step, just in Theorem 2.2.1, is to introduce shifts on the equation the gain certain monotonicity properties. That is, we consider functions $a \in L^p_{\delta-2}(M)$ and $b \in W^{1-\frac{1}{p},p}(\Sigma)$ such that

$$a(x) \geq \sum_I |I| |a_I(x)| \left| \sup_{y \in [l,m]} y^{I-1} \right|, \quad b(x) \geq \sum_J |J| |b_J(x)| \left| \sup_{y \in [l,m]} y^{J-1} \right|,$$

with these choices of a and b, consider the operator $\mathcal{P}_{a,b}$ given by $\mathcal{P}_1$ in (3.46). Then, rewrite $\phi \doteq \omega + \varphi$, with $\varphi \in W^{2,p}_\delta$ so that ω captures the asymptotics of ϕ and then pose (3.62) for φ. Then, use Theorem 3.3.1 to build the sequence $\{\phi_k = \omega + \varphi_k\}_{k=1}^\infty$ with φ_k defined iteratively via

$$\mathcal{P}_{a,b}\varphi_{k+1} = (f_1(x, \phi_k) - a\phi_k, f_2(x, \phi_k) - b\phi_k) \in L^p_{\delta-2}(M) \times W^{1-\frac{1}{p},p}(\Sigma),$$

---

[7]Different versions of Theorem 3.3.4 with slightly different hypotheses can be found in the literature, some of these with weaker hypotheses than ours. For further references, we refer the reader to Choquet-Bruhat (2009), Holst and Meier (2014), and Maxwell (2005b, 2006).

starting with $\varphi_0 \doteq \phi_- - c_-$. We can in fact use the maximum principle of Lemma 3.3.1 to guarantee that $\phi_- \leqslant \phi_1 \leqslant \cdots \leqslant \phi_k \leqslant \cdots \leqslant \phi_+$. This is seen inductively as follows. First, consider ϕ_1 and notice that

$$\Delta_\gamma(\phi_1 - \phi_-) - a(\phi_1 - \phi_-) = f_1(x, \phi_-) - a\phi_- - \Delta_\gamma\phi_- + a\phi_-,$$
$$-\nu(\phi_1 - \phi_-) - b(\phi_1 - \phi_-) = f_2(x, \phi_-) - b\phi_- + \nu(\phi_-) - b\phi_-.$$

Now, since ϕ_- is a subsolution, from (3.65), we have

$$-\Delta_\gamma\phi_- \leqslant -f_1(x, \phi_-),$$
$$\nu(\phi_-) \leqslant -f_2(x, \phi_-),$$

implying

$$\Delta_\gamma(\phi_1 - \phi_-) - a(\phi_1 - \phi_-) \leqslant f_1(x, \phi_-) - f_1(x, \phi_-) = 0,$$
$$-\nu(\phi_1 - \phi_-) - b(\phi_1 - \phi_-) \leqslant f_2(x, \phi_-) - f_2(x, \phi_-) = 0.$$

Finally, notice that $\phi_1 - \phi_- \to A_j - c_- \geqslant 0$ at infinity in each end. Therefore, the maximum principle of Lemma 3.3.1 implies that $\phi_1 \geqslant \phi_-$. Similarly, let us analyse

$$\Delta_\gamma(\phi_+ - \phi_1) - a(\phi_+ - \phi_1) = \Delta_\gamma\phi_+ - a\phi_+ - (f_1(x, \phi_-) - a\phi_-),$$
$$-\nu(\phi_+ - \phi_1) - b(\phi_+ - \phi_1) = -(\nu(\phi_+) + b(\phi_+)) - (f_2(x, \phi_-) - b\phi_-).$$

Since ϕ_+ is a supersolution, then

$$\Delta_\gamma\phi_+ \leqslant f_1(x, \phi_+),$$
$$-\nu(\phi_+) \leqslant f_2(x, \phi_-),$$

implying

$$\Delta_\gamma(\phi_+ - \phi_1) - a(\phi_+ - \phi_1) \leqslant f_1(x, \phi_+) - a\phi_+ - (f_1(x, \phi_-) - a\phi_-),$$
$$-\nu(\phi_+ - \phi_1) - b(\phi_+ - \phi_1) \leqslant f_2(x, \phi_+) - b(\phi_+) - (f_2(x, \phi_-) - b\phi_-).$$

In this case, since by hypotheses $\phi_- \leqslant \phi_+$ and by construction of a and b the functions $y \to f_1(x, y) - ay$ and $y \to f_2(x, y) - by$ are decreasing functions for all $y \in [l, m]$, then

$$\Delta_\gamma(\phi_+ - \phi_1) - a(\phi_+ - \phi_1) \leqslant 0,$$
$$-\nu(\phi_+ - \phi_1) - b(\phi_+ - \phi_1) \leqslant 0.$$

Again, $\phi_+ - \phi_1 \to c_+ - A_j \geqslant 0$ at infinity and therefore Lemma 3.3.1 implies $\phi_+ \geqslant \phi_1$. Then, assume that for some $k \geqslant 1$ it holds that $\phi_- \leqslant \phi_{k-1} \leqslant \phi_k \leqslant \phi_+$ and notice that

$$\Delta_\gamma(\phi_{k+1} - \phi_k) - a(\phi_{k+1} - \phi_k) = f_1(x, \phi_k) - a\phi_k - (f_1(x, \phi_{k-1}) - a\phi_{k-1}) \leqslant 0,$$
$$-v(\phi_{k+1} - \phi_k) - b(\phi_{k+1} - \phi_k) = f_2(x, \phi_k) - b\phi_k - (f_2(x, \phi_{k-1}) - b\phi_{k-1}) \leqslant 0,$$

where the final inequalities come from the fact that, by our choice of functions a and b, $y \mapsto f_1(x, y) - ay$ and $y \mapsto f_2(x, y) - by$ are decreasing for any $y \in [l, m]$, and by the inductive hypothesis, $l \leqslant \phi_{k-1} \leqslant \phi_k \leqslant m$. Therefore, the Lemma 3.3.1 implies $\phi_{k+1} \geqslant \phi_k$. Similarly

$$\Delta_\gamma(\phi_+ - \phi_{k+1}) - a(\phi_+ - \phi_{k+1}) \leqslant f_1(x, \phi_+) - a\phi_+ - (f_1(x, \phi_k) - a\phi_k) \leqslant 0,$$
$$-v(\phi_+ - \phi_{k+1}) - b(\phi_+ - \phi_{k+1}) \leqslant f_2(x, \phi_+) - b\phi_+ - (f_2(x, \phi_k) - b\phi_k) \leqslant 0,$$

where we have used that since ϕ_+ is a supersolution then (3.66) is satisfied by hypotheses, and also the decreasing properties of the right-hand side put together with the inductive hypothesis $\phi_k \leqslant \phi_+$. Therefore, we find $\phi_{k+1} \leqslant \phi_+$ by Lemma 3.3.1, and the inductive claim is obtained.

Next, the idea is to prove that $\{\varphi_k\}_{k=0}^\infty \subset W_\delta^{2,p} \hookrightarrow C^0(M)$ is bounded. As in Theorem 2.2.1, this is achieved via elliptic estimates. Thus, appealing to Equation (3.43) we have

$$\|\varphi_{k+1}\|_{W_\delta^{2,p}(M)} \leqslant C \Big\{ \sum_I \|a_I\|_{L_{\delta-2}^p(M)} \|\phi_k^I\|_{C^0(M)} + \|a\|_{L_{\delta-2}^p(M)} \|\phi_k\|_{C^0(M)}$$
$$+ \sum_J \|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} + \|b\phi_k\|_{W^{1-\frac{1}{p},p}(\Sigma)} \Big\}.$$

$$(3.67)$$

Notice that the first line in the above inequality if uniformly bounded for all k since the functions $y \mapsto y^I$ are smooth for $y \in [l, m]$ by hypotheses and $l \leqslant \phi_- \leqslant \phi_k \leqslant \phi_+ \leqslant m$ by construction. Bounding the second line is more subtle. In this case, let U be a bounded neighbourhood of Σ (which we fix from now on) and consider extensions $\tilde{b}_J \in W^{1,p}(U)$ of the coefficients b_J, so that $\tilde{b}_J \phi^J$ is a $W^{1,p}(U)$ extension of $b_J \phi^J \in W^{1-\frac{1}{p},p}(\Sigma)$. Then

$$\|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim \|\tilde{b}_J \phi_k^J\|_{W^{1,p}(U)}.$$

We can now appeal to an analysis analogous to that preceding Theorem 3.2.3. That is, let us restrict our attention to the case $\frac{n}{2} < p < n$, since this case implies the

general one. Then, notice that from Lemma A.2.1 we know that $\phi_k^I \in W^{2,p}(U)$. Therefore, from Sobolev embeddings $W^{2,p}(U) \hookrightarrow W^{1,q}(U)$ for any q in the interval

$$\frac{1}{p} - \frac{1}{n} < \frac{1}{q} < \frac{1}{n} < \frac{1}{p}, \tag{3.68}$$

where the last inequality follows from our restriction to $p < n$. Thus, in particular, $q > n > p$ and then Sobolev multiplication implies that $W^{1,p}(U) \otimes W^{1,q}(U) \hookrightarrow W^{1,p}(U)$, and therefore

$$\|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim \|\tilde{b}_J \phi_k^J\|_{W^{1,p}(U)} \lesssim \|\tilde{b}_J\|_{W^{1,p}} \|\phi_k^J\|_{W^{1,q}(U)},$$
$$\lesssim \|\phi_k^J\|_{L^q(U)} + \|J\phi_k^{J-1}\nabla\phi_k\|_{L^q(U)},$$

where in the last inequality the implicit constant depends on the coefficient b_J and on U (which is fixed), but not on k. Then, since we already know that $0 < l \leq \phi_- \leq \phi_k \leq \phi_+ \leq m$, the first term in the last inequality is also uniformly bounded, and we can actually write

$$\|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim 1 + \|\phi_k\|_{W^{1,q}(U)} \lesssim 1 + \|\omega\|_{W^{1,q}(U)} + \|\varphi_k\|_{W^{1,q}(U)}$$

Now, from the discussion preceding Theorem 3.2.3, we know that we can chose $\frac{1}{q} = \frac{1-2\theta}{n} + \frac{1}{p}$, with $\frac{1}{2} < \theta < 1$ so that q satisfies (3.68) and therefore, through the Gagliardo–Nirenberg interpolation, for any $\epsilon > 0$, we find

$$\|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim 1 + \epsilon\|\varphi_k\|_{W^{2,p}(U)} + C_\epsilon\|\varphi_k\|_{L^p(U)},$$

where we can once more bound the last term using the barriers, so that

$$\|b_J \phi_k^J\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim 1 + C_\epsilon + \epsilon\|\varphi_k\|_{W^{2,p}(U)}, \tag{3.69}$$

where the implicit constant in front of the right-hand side depends on the coefficients b_J, U, ω and the barriers, but is independent of k and also of ϵ. We can now use the above estimate in (3.67) to get

$$\|\varphi_{k+1}\|_{W^{2,p}_\delta(M)} \lesssim \epsilon\|\varphi_k\|_{W^{2,p}(U)} + C_\epsilon + 1.$$

Therefore, we can chose $\epsilon > 0$ sufficiently small so that

$$\|\varphi_{k+1}\|_{W^{2,p}_\delta(M)} \lesssim \frac{1}{2}\|\varphi_k\|_{W^{2,p}_\delta(M)} + C, \tag{3.70}$$

for some constant $C > 0$ which depends on the barriers, the coefficients of the equation, the neighbourhood U and our choice of $\epsilon > 0$, but is independent of k. Thus, inductively, the above implies

$$\|\varphi_{k+1}\|_{W^{2,p}_{\delta}(M)} \lesssim \|\varphi_-\|_{W^{2,p}_{\delta}(M)} + C\sum_{i=0}^{k} 2^{-i} \leq \|\varphi_-\|_{W^{2,p}_{\delta}(M)} + 2C,$$

which proves that the sequence $\{\varphi_k\}_{k=1}^{\infty} \subset W^{2,p}_{\delta}$ is uniformly bounded. Therefore, by reflexivity of $W^{2,p}_{\delta}$, we now that it admits a weakly convergent subsequence, implying that there exists some $\varphi \in W^{2,p}_{\delta}$ such that $\varphi_k \to \varphi$ weakly in $W^{2,p}_{\delta}$. Let us now show that φ solves the equation. For this, first notice that we have a compact embeddings $W^{2,p}_{\delta} \hookrightarrow W^{1,p}_{\delta'}$, with $\delta < \delta'$, implying that $\varphi_k \to \varphi$ strongly in $W^{1,p}_{\delta'}$, and also we have C^0-convergence on compacts, due to the compact embeddings for compact manifolds. Therefore,

$$\int_M (f_1(x,\varphi_k) - a\varphi_k)\,\psi\,dV_\gamma \to \int_M (f_1(x,\phi) - a\varphi)\,\psi\,dV_\gamma,$$

$$\int_\Sigma (f_2(x,\varphi_k) - b\varphi_k)\,\psi\,d\Sigma \to \int_\Sigma (f_2(x,\phi) - b\varphi)\,\psi\,d\Sigma,$$

$$\int_M \left(\langle\nabla\varphi_k,\nabla\psi\rangle_\gamma + a\varphi_k\psi\right)dV_\gamma \to \int_M \left(\langle\nabla\varphi,\nabla\psi\rangle_\gamma + a\varphi\psi\right)dV_\gamma,$$

for all $\psi \in C^\infty_0(M)$. Therefore, we find

$$\int_M \left(\langle\nabla\varphi,\nabla\psi\rangle_\gamma + a\varphi\psi\right)dV_\gamma = \lim_{k\to\infty}\left\{-\int_M (\Delta_\gamma\varphi_k - a\varphi_k)\,\psi\,dV_\gamma + \int_\Sigma v(\varphi_k)\psi\,d\Sigma\right\}$$

$$= \lim_{k\to\infty}\left\{-\int_M (f_1(x,\phi_{k-1}) - a\varphi_{k-1})\,\psi\,dV_\gamma\right.$$

$$\left. + \int_\Sigma (-f_2(x,\phi_{k-1}) + b\varphi_{k-1} - b\varphi_k)\,\psi\,d\Sigma\right\}$$

$$= -\int_M (f_1(x,\phi) - a\varphi)\psi\,dV_\gamma - \int_\Sigma f_2(x,\phi)\psi\,d\Sigma.$$

Also, integrating by parts the left-hand side and cancelling out the shift terms, we find

$$\int_M \left(\Delta_\gamma\varphi - f_1(x,\phi)\right)\psi\,dV_\gamma + \int_\Sigma \left(-v(\psi) - f_2(x,\phi)\right)\psi\,d\Sigma = 0,$$

for all $\psi \in C_0^\infty(M)$ and with $\varphi \in W_\delta^{2,p}(M)$, which implies that φ solves the original boundary value problem. $\square$

3.4 Maximal black hole vacuum initial data

The tools developed above will play a important role in the analysis of very general versions of the conformally formulated ECE. The more general case, which allows for far from CMC initial data, further constraint coupled to the Gauss–Codazzi ones as well as coupling through boundary conditions and interactions with matter fields will be dealt with in Chapter 4. For now, let us analyse the decoupled system (3.40)-(3.41), which we rewrite below for the reader's convenience.[8]

$$\begin{cases} -a_n \Delta_\gamma \phi + R_\gamma \phi - |\widetilde{K}|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} = 0, \\ \frac{1}{2} a_n \hat{v}(\phi) + \phi H - \widetilde{K}(\hat{v}, \hat{v}) \phi^{-\frac{n}{n-2}} = 0, \text{ on } \Sigma, \end{cases} \tag{3.71}$$

$$\begin{cases} \Delta_{\gamma,\text{conf}} X = 0, \\ \mathscr{L}_{\gamma,\text{conf}} X(\hat{v}, \cdot) = \alpha, \text{ on } \Sigma. \end{cases} \tag{3.72}$$

where α is a 1-form defined on a neighbourhood of Σ, which we shall chose so that $\alpha(v) \leqslant 0$. Then, we shall consider that $\widetilde{K} \doteq \mathscr{L}_{\gamma,\text{conf}} X$ in the conformal problem. This implies that the boundary conditions satisfy (3.40) and therefore solutions (with $\phi > 0$) to (3.71)-(3.72) which follow these conventions will provide us with initial data

$$\begin{aligned} g &= \phi^{\frac{4}{n-2}} \gamma, \\ K &= \phi^{-2} \mathscr{L}_{\gamma,\text{conf}} X, \end{aligned} \tag{3.73}$$

which solve (recall $\tau = 0$)

$$\begin{aligned} R_g - |K|_g^2 + \tau^2 &= 0, \\ \text{div}_g K - d\tau &= 0, \\ \theta_+ &= 0 \text{ on } \Sigma, \\ \theta_- &\leqslant 0 \text{ on } \Sigma, \end{aligned} \tag{3.74}$$

providing us with appropriate black hole vacuum initial data for the evolution problem. Similarly to the case for closed manifolds, the analysis of the momentum constraint is now quite straightforward.

---

[8] The results we shall present concerning the analysis of (3.71)-(3.72) are due to Maxwell (2005b).

Lemma 3.4.1. *Let (M^n, γ) be a $W^{2,p}_\delta$-AE manifold with compact boundary Σ satisfying $p > \frac{n}{2}$, $n \geqslant 3$ and $2 - n < \delta < 0$. Let us assume that either γ possesses no CKF or that $p > n$. If α in (3.72) is $W^{1,p}_{loc}(M)$, then (3.72) admits a unique solution $X \in W^{2,p}_\delta$.*

Proof. The proof is a direct application of Theorem 3.3.3. Since we are under its hypotheses, it follows that if γ has no CKF or if $p > n$ (which excludes the possibility of CKF in $W^{2,p}_\delta$), then the operator on the left-hand side of (3.72) is an isomorphism acting from $W^{2,p}_\delta \mapsto L^p_{\delta-2}(M) \times W^{1-\frac{1}{p},p}(\Sigma)$. Thus, since the condition $\alpha \in W^{1,p}_{loc}(M)$ implies $\alpha|_\Sigma \in W^{1-\frac{1}{p},p}(\Sigma)$, the claim follows from the isomorphism property. $\square$

3.4.1 The Lichnerowicz equation

Let us now analyse the boundary value problem (3.71). First, let us present the following proposition which shows that (3.71) is of the form of the equations treated in Theorem 3.3.4.

Proposition 3.4.1. *Let (M^n, γ) be a $W^{2,p}_\delta$-AE manifold with $n \geqslant 3$, $p > \frac{n}{2}$ and $\delta < 0$. Let $\widetilde{K} = \mathscr{L}_{\gamma,conf}X$, with $X \in W^{1,p}_\delta$. If $H \in W^{1-\frac{1}{p},p}(\Sigma)$, then the coefficients of (2.75) satisfy the hypotheses of Theorem 2.2.1.*

Proof. Notice that we just need to show that $R_\gamma, |\widetilde{K}|^2_\gamma \in L^p_{\delta-2}(M)$ and $\widetilde{K} \in W^{1-\frac{1}{p},p}(\Sigma)$. The condition on the scalar curvature follows directly from γ being $W^{2,p}_\delta$-AE and the multiplication property. Also, $\widetilde{K} \in W^{1,p}_{\delta-1}$, which implies that $|\widetilde{K}|^2_\gamma \in L^p_{\delta-2}$ from the multiplication property and $\widetilde{K}|_\Sigma \in W^{1-\frac{1}{p},p}(\Sigma)$ from the trace theorem. $\square$

Therefore, just as in Chapter 2, we see that we have reduced our task to constructing barriers for (3.71). In this case, the sign of both the scalar curvature of γ and the mean curvature of $\Sigma \subset (M^n, \gamma)$ play an important role. Therefore, let us introduce the following elements concerning the analysis of the Yamabe problem on AE manifolds with boundary.[9] Let us consider the following Yamabe-type

---

[9]We would like to point the interested reader to Escobar (1992, 1996) for the analysis of the Yamabe problem on compact manifolds wit boundary and to Dilts and Maxwell (2018) for the analysis of the Yamabe problem on AE manifolds.

quotient

$$Q_\gamma(f) \doteq \frac{\int_M \left(a_n |\nabla f|_\gamma^2 + R_\gamma f^2\right) dV_\gamma + \int_\Sigma 2H f^2 d\Sigma_\gamma}{\|f\|^2_{L^{2^*}(M, dV_\gamma)}}, \tag{3.75}$$

for all $f \in C_0^\infty(M)$, and, following Maxwell (2005b), we define

$$\lambda_\gamma \doteq \inf_{\substack{f \in C_0^\infty(M) \\ f \not\equiv 0}} Q_\gamma(f). \tag{3.76}$$

Let us first show that this is a conformal invariant. First, let us define

$$
\begin{aligned}
E_\gamma(f) &\doteq \int_M \left(a_n |\nabla f|_\gamma^2 + R_\gamma f^2\right) dV_\gamma + \int_\Sigma 2H f^2 d\Sigma_\gamma, \\
&= \int_M \left(R_\gamma f - a_n \Delta_\gamma f\right) f dV_\gamma + 2 \int_\Sigma \left(\frac{1}{2} a_n \nu(f) + Hf\right) f d\Sigma_\gamma.
\end{aligned}
$$

Given a conformal metric $\gamma' \doteq u^{\frac{4}{n-2}} \gamma$, from Proposition 2.1.3, we know that

$$f\left(R_\gamma f - a_n \Delta_\gamma f\right) = u^{\frac{n+2}{n-2}+1} f' \left(R_{\gamma'} f' - a_n \Delta_{\gamma'} f'\right), \quad f' \doteq u^{-1} f.$$

This implies that

$$
\begin{aligned}
\int_M \left(R_\gamma f - a_n \Delta_\gamma f\right) f dV_\gamma &= \int_M u^{\frac{2n}{n-2}} \left(R_{\gamma'} f' - a_n \Delta_{\gamma'} f'\right) f' dV_\gamma, \\
&= \int_M \left(R_{\gamma'} f' - a_n \Delta_{\gamma'} f'\right) f' dV_{\gamma'}.
\end{aligned}
$$

Let us now analyse the boundary terms. From the computations of Section 3.3.1, we know that $\nu' = u^{-\frac{2}{n-2}} \nu$ and $H' = u^{-\frac{n}{n-2}} \left(\frac{a_n}{2} \nu(u) + Hu\right)$, where H' denotes the mean curvature of $\Sigma \hookrightarrow (M^n, \gamma')$. Then,

$$
\begin{aligned}
f'\left(\frac{1}{2} a_n \nu'(f') + H' f'\right) &= u^{-\frac{n}{n-2}-1} f \left(\frac{1}{2} a_n \nu(f) + Hf\right), \\
&= u^{-\frac{2(n-1)}{n-2}} f \left(\frac{1}{2} a_n \nu(f) + Hf\right).
\end{aligned}
$$

Then, noticing that $d\Sigma_{\gamma'} = u^{\frac{2(n-1)}{n-2}} d\Sigma_\gamma$, we find

$$\int_\Sigma \left(\frac{1}{2} a_n \nu'(f') + H' f'\right) f' d\Sigma_{\gamma'} = \int_\Sigma \left(\frac{1}{2} a_n \nu(f) + Hf\right) f d\Sigma_\gamma.$$

Finally, we already know that $\|f'\|^2_{L^{2*}(M,dV_{\gamma'})} = \|f\|^2_{L^{2*}(M,dV_\gamma)}$, and therefore we have established that

$$\mathcal{Q}_{\gamma'}(f') = \mathcal{Q}_\gamma(f), \tag{3.77}$$

which proves the infimum among all $C_0^\infty(M)$ is a conformally invariant number.

Let us now present the following characterisation for $\lambda_\gamma > 0$. First, define the 1-parameter family of operators

$$\begin{aligned}
\mathcal{P}_\eta : W_\delta^{2,p}(M) &\mapsto L_{\delta-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma), \\
u &\mapsto (-a_n \Delta_\gamma u + \eta R_\gamma u, \frac{a_n}{2} v(u) + \eta H u),
\end{aligned} \tag{3.78}$$

for each $\eta \in [0,1]$.

Proposition 3.4.2. *(ibid., Proposition 3) Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold, with $p > \frac{n}{2}$, $2 - n < \delta < 0$ and $n \geq 3$. Then, the following conditions are equivalent:*

1. *There is a conformal factor $\phi > 0$ such that $1 - \phi \in W_\delta^{2,p}$ and such that $(M^n, \gamma' = \phi^{\frac{4}{n-2}} \gamma)$ is scalar flat and has minimal surface boundary;*

2. *$\lambda_\gamma > 0$;*

3. *$\mathcal{P}_\eta : W_\delta^{2,p}(M) \mapsto L_{\delta-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma)$ is an isomorphism for each $\eta \in [0,1]$.*

We now have the following straightforward corollary.

Corollary 3.4.1. *(ibid., Corollary 1) Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold, with $p > \frac{n}{2}$, $2 - n < \delta < 0$ and $n \geq 3$. If $\lambda_\gamma > 0$, then there is a conformal factor $\phi > 0$, such that $1 - \phi \in W_\delta^{2,p}$ and such that $\gamma' \doteq \phi^{\frac{4}{n-2}} \gamma$ satisfies $R_{\gamma'} = 0$ and $\Sigma \hookrightarrow (M^n, \gamma')$ has negative mean curvature.*

Proof. Using Proposition 3.4.2, we can start assuming that $R_\gamma = 0$ and $H_\gamma = 0$, since, if this were not the case, we could achieve it via a a conformal transformation which preserves $\lambda([\gamma]) > 0$. Thus, we intend to find a positive function $\phi > 0$,

with $1 - \phi \in W_\delta^{2,p},$ solving

$$-\Delta_\gamma \phi = 0 \qquad \left(R_{\phi^{\frac{4}{n-2}}\gamma} = 0 \right),$$

$$v(\phi) < 0 \qquad \left(H_{\phi^{\frac{4}{n-2}}\gamma} < 0 \right). \tag{3.79}$$

With this in mind, fix $\epsilon > 0$ and consider the equation for $\varphi \in W_\delta^{2,p}$

$$-\Delta_\gamma \varphi = 0,$$
$$v(\varphi) = -\epsilon,$$

which has a unique solution $\varphi_\epsilon \in W_\delta^{2,p}$ due to Theorem 3.3.1. From the embedding $W_\delta^{2,p} \hookrightarrow C_\delta^0$ for $p > \frac{n}{2}$ and elliptic estimates, we know that φ_ϵ depends continuously on ϵ in the C_δ^0 topology. Thus, since $\varphi_0 \equiv 0$, we see that for ϵ sufficiently small $\varphi_\epsilon > -1$, implying that $\phi_\epsilon \doteq 1 + \varphi_\epsilon > 0$ for all such small $\epsilon > 0$ and $(-\Delta_\gamma \phi_\epsilon, v(\phi_\epsilon)) = (0, -\epsilon)$. Thus, ϕ_ϵ solves (3.79) and $\gamma' = \phi_\epsilon^{\frac{4}{n-2}}\gamma$ satisfies our claims. □

We will now present the main result of the section, which proves the existence of solutions to (3.71) when the conformal data is Yamabe positive. This is also a consequence of Maxwell (2005b, Theorem 1)

Theorem 3.4.1. *Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold, with $p > \frac{n}{2}$, $2 - n < \delta < 0$ and $n \geq 3$. Suppose $\lambda_\gamma > 0$, $R_\gamma = 0$, $H_\gamma \leq 0$ and $\tilde{K}$ is a $W_{\delta-1}^{1,p}$ γ-TT tensor. If $H_\gamma \leq \tilde{K}(\hat{v}, \hat{v}) \leq 0$ along Σ, then there is a conformal factor ϕ solving (3.71), with $\phi - 1 \in W_\delta^{2,p}$, and therefore $(g = \phi^{\frac{4}{n-2}}\gamma, K = \phi^{-2}\tilde{K})$ solve (3.74).*

Proof. Appealing to Theorem 3.3.4, we know that what we need to do is to exhibit barriers for (3.71). Since we are assuming $R_\gamma = 0$ the system becomes

$$-a_n \Delta_\gamma \phi = |\tilde{K}|_\gamma^2 \phi^{-\frac{3n-2}{n-2}},$$
$$\frac{1}{2} a_n \hat{v}(\phi) = -H\phi + \tilde{K}(\hat{v}, \hat{v})\phi^{-\frac{n}{n-2}}, \text{ on } \Sigma. \tag{3.80}$$

Notice that $\tilde{K}(\hat{v}, \hat{v}) - H_\gamma \geq 0$ along Σ by assumption. Then $\phi_- \equiv 1$ is a subsolution. Let us now look for a supersolution $\phi_+ \geq 1$. With this in mind, consider

the family of equations

$$-a_n \Delta_\gamma \varphi = |\tilde{K}|_\gamma^2 \geq 0,$$
$$\frac{1}{2} a_n \hat{v}(\varphi) + \eta H \varphi = -\eta H \geq 0,$$

(3.81)

with $\eta \in [0, 1]$, which have unique solutions $\varphi_\eta \in W_\delta^{2,p}$ from Proposition 3.4.2. Then, $\phi_\eta \doteq 1 + \varphi_\eta$ satisfy

$$-a_n \Delta_\gamma \phi_\eta = |\tilde{K}|_\gamma^2,$$
$$\frac{1}{2} a_n \hat{v}(\phi_\eta) + \eta H \phi_\eta = 0.$$

(3.82)

We first want to prove that $\phi_1 > 0$. Notice that since H might be negative, we cannot apply the maximum principle directly. Nevertheless, we know that $\phi_0 \geq 0$ due to the weak maximum principle, and due to the strong maximum principle of Lemma 3.3.2, we know that $\phi_0 > 0$, since if it vanishes at one point it must then vanish identically and would contradict that $\phi_0 \to 1$ at infinity. Thus, the set $I \doteq \{\eta \in [0, 1] : \phi_\eta > 0\} \neq \emptyset$. We intend to show that $I = [0, 1]$ by showing that I is both open and closed. To see that it is open, we use a similar argument to the one in the above corollary. That is, the embedding $W_\delta^{2,p} \hookrightarrow C_\delta^0$ and elliptic estimates prove that the family φ_η is continuous with respect to η in the C_δ^0 topology. Therefore, the condition $\varphi_\eta > -1 \Leftrightarrow \phi_\eta > 0$ is open in η. To see that it is closed, assume that $\eta_0 \in \bar{I} \subset [0, 1]$. Then $\phi_{\eta_0} \geq 0$ and satisfies

$$-a_n \Delta_\gamma \phi_{\eta_0} = |\tilde{K}|_\gamma^2 \geq 0,$$
$$\frac{1}{2} a_n \hat{v}(\phi_{\eta_0}) \geq -\eta_0 H \phi_{\eta_0} \geq 0,$$

and, again, since $\phi_{\eta_0} \to 1$ at infinity, Lemma 3.3.2 implies that $\phi_{\eta_0} > 0$. That is, $\eta_0 \in I$ and therefore I is closed, implying by connectedness that $I = [0, 1]$. That is, we have established that $\phi_1 > 0$. But now this implies that

$$-a_n \Delta_\gamma \varphi_1 = |\tilde{K}|_\gamma^2 \geq 0,$$
$$\frac{1}{2} a_n \hat{v}(\varphi_1) = -H \phi_1 \geq 0,$$

showing that $\varphi_1 \geqslant 0$ through the weak maximum principle of Lemma 3.3.1. Therefore, setting $\phi_+ \doteq \phi_1 = 1 + \varphi_1 \geqslant \phi_- = 1$ and notice that

$$
\begin{aligned}
-a_n \Delta_\gamma \phi_+ &= |\tilde{K}|^2_\gamma \geqslant |\tilde{K}|^2_\gamma \phi_+^{-\frac{3n-2}{n-2}}, \quad (\phi_+ > 0) \\
\frac{a_n}{2} \hat{v}(\phi_+) &= -H_\gamma \phi_+ \geqslant -H_\gamma \phi_+ + \tilde{K}(\hat{v}, \hat{v}) \phi_+^{-\frac{n}{n-2}}, \quad (\tilde{K}(\hat{v}, \hat{v}) \leqslant 0),
\end{aligned}
\tag{3.83}
$$

proving that we have constructed barriers $0 < \phi_- \leqslant \phi_+$ satisfying the hypotheses of Theorem 3.3.4, and thus we have a positive solution ϕ to (3.71), with $\phi_- \leqslant \phi \leqslant \phi_+$ and $\phi - 1 \in W^{2,p}_\delta$. $\qquad\square$

At first sight, the geometric hypotheses in the above theorem may not seem easy to satisfy. Nevertheless, notice that we can always proceed as follows. If (M^n, γ) is Yamabe positive, then (if necessary), first deform to $\gamma' = \theta^{\frac{4}{n-2}} \gamma$ according to Equation (3.79), so that $R_{\gamma'} \equiv 0$ and $H_{\gamma'} < 0$. Then, on (M^n, γ'), solve the momentum constraint (3.72) using Lemma 3.4.1 and fixing $\alpha \doteq h v'^\flat$, where v' is the γ' unit normal to Σ and $h \in W^{1,p}_{loc}(M)$ is any function satisfying $H_{\gamma'} \leqslant h \leqslant 0$ along Σ. This implies that

$$
\tilde{K}_{\gamma'}(v', v') = \mathcal{L}_{\gamma',\mathrm{conf}} X(v', v') = h \geqslant H_{\gamma'}.
\tag{3.84}
$$

Therefore, the data $(M^n, \gamma', \tilde{K}_{\gamma'})$ satisfy all the hypotheses of Theorem 3.4.1 and we can find a solution to our problem. Furthermore, we have a conformal covariance property which allows us to build a related solution starting with γ which gives rise to the same physical data (g, K) as follows.

Lemma 3.4.2. *Let* (M^n, γ) *be a Riemannian manifold,* $\gamma \in W^{2,p}_{loc}$, $n \geqslant 3$, $p > \frac{n}{2}$. *Consider a positive function* $\theta \in W^{2,p}_{loc}$ *and define the Riemannian metric* $\gamma' = \theta^{\frac{4}{n-2}} \gamma$. *Then, given a TT-tensor* $\tilde{K} \in W^{1,p}_{loc}$ *and a positive function* $\phi \in W^{2,p}_{loc}$, *the pair* $(\phi, \tilde{K})$ *is a solution of*

$$
\begin{aligned}
-a_n \Delta_\gamma \phi + R_\gamma \phi - |\tilde{K}|^2_\gamma \phi^{-\frac{3n-2}{n-2}} &= 0, \\
\frac{1}{2} a_n v(\phi) + H_\gamma \phi - \tilde{K}(v, v) \phi^{-\frac{n}{n-2}} &= 0, \text{ on } \Sigma, \\
\tilde{K}(v, v) &\leqslant 0,
\end{aligned}
\tag{3.85}
$$

if and only if $\phi' = \theta^{-1}\phi$ satisfies

$$-a_n \Delta_{\gamma'}\phi' + R_{\gamma'}\phi' - |\tilde{K}'|_{\gamma}^2 \phi'^{-\frac{3n-2}{n-2}} = 0,$$
$$\frac{1}{2}a_n v'(\phi') + H_{\gamma'}\phi' - \tilde{K}'(v',v')\phi'^{-\frac{n}{n-2}} = 0, \text{ on } \Sigma, \qquad (3.86)$$
$$\tilde{K}'(v',v') \leqslant 0,$$

where $\tilde{K}' = \theta^{-2}\tilde{K}$. Furthermore, both solutions give rise to the same maximal vacuum physical initial data

$$g = \phi'^{\frac{4}{n-2}}\gamma' = \phi^{\frac{4}{n-2}}\gamma, \quad K = \phi'^{-2}\tilde{K}' = \phi^{-2}\tilde{K} \qquad (3.87)$$

solving the problem

$$\begin{aligned} R_g - |K|_g^2 &= 0, \\ \text{div}_g K &= 0, \\ \theta_+ &= 0 \text{ on } \Sigma, \\ \theta_- &\leqslant 0 \text{ on } \Sigma, \end{aligned} \qquad (3.88)$$

Proof. From Lemma 2.1.1, we know that

$$-a_n \Delta_{\gamma'}\phi' + R_{\gamma'}\phi' - |\tilde{K}'|_{\gamma}^2 \phi'^{-\frac{3n-2}{n-2}} = 0 \Longleftrightarrow$$
$$-a_n \Delta_{\gamma'}\phi' + R_{\gamma'}\phi' - |\tilde{K}'|_{\gamma}^2 \phi'^{-\frac{3n-2}{n-2}} = 0.$$

Furthermore, since $v' = \theta^{-\frac{2}{n-2}}v$, then, by definition of $\tilde{K}'$,

$$\tilde{K}(v,v) \leqslant 0 \Leftrightarrow \tilde{K}'(v',v') \leqslant 0.$$

Therefore, we must concentrate on the boundary condition. But we already know that

$$\frac{a_n}{2}v'(\phi') + H_{\gamma'}\phi' = \theta^{-\frac{n}{n-2}}\left(\frac{a_n}{2}v(\phi) + H_{\gamma}\phi\right),$$

which we can put together with

$$\tilde{K}'(v',v')\phi'^{-\frac{n}{n-2}} = \theta^{-\frac{4}{n-2}-2}\tilde{K}(v,v)\theta^{\frac{n}{n-2}}\phi^{-\frac{n}{n-2}},$$
$$= \theta^{-\frac{n}{n-2}}\tilde{K}(v,v)\phi^{-\frac{n}{n-2}},$$

to get

$$\frac{a_n}{2} v'(\phi') + H_{\gamma'}\phi' - \tilde{K}'(v',v')\phi'^{-\frac{n}{n-2}} =$$

$$= \theta^{-\frac{n}{n-2}} \left(\frac{a_n}{2} v(\phi) + H_\gamma \phi - \theta^{-\frac{n}{n-2}} \tilde{K}(v,v)\phi^{-\frac{n}{n-2}} \right).$$

implying that the boundary conditions are satisfied in one set of variables iff they are satisfied in the other. Therefore, we have established that $(3.85) \Leftrightarrow (3.86)$, and then (3.87) follows immediately by our definitions of ϕ' and $\tilde{K}'$. Also, since $\mathrm{tr}_g K = \mathrm{tr}_\gamma \tilde{K} = \mathrm{tr}_{\gamma'} \tilde{K}' = 0$ by definition, from Proposition 2.1.2 we know that

$$\mathrm{div}_g K = 0 \Leftrightarrow \mathrm{div}_\gamma \tilde{K} = 0 \Leftrightarrow \mathrm{div}_{\gamma'} \tilde{K}' = 0.$$

Finally, we know from Section 3.3.1 that

$$\begin{cases} R_g &= |K|_g^2, \\ \theta_+ &= 0 \text{ on } \Sigma, \\ \theta_- &\leqslant 0 \text{ on } \Sigma, \end{cases} \Longleftrightarrow \begin{cases} -a_n \Delta_\gamma \phi + R_\gamma \phi - |\tilde{K}|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} &= 0, \\ \frac{1}{2}a_n v(\phi) + H_\gamma \phi - \tilde{K}(v,v)\phi^{-\frac{n}{n-2}} &= 0, \text{ on } \Sigma, \\ \tilde{K}(v,v) &\leqslant 0, \text{ on } \Sigma. \end{cases}$$

$\square$

In Chapter 4 we will return to the analysis of black hole initial data in the presence of matter fields and also without invoking CMC assumptions.

Finally, let us notice that the above results immediately extend to the case $\Sigma = \emptyset$, simply by ignoring the analysis on the boundary while working only on the bulk (See Maxwell 2005b, Theorem 2). In this context, we can present a uniqueness result which is quite close to the one presented in Theorem 2.2.6.

Theorem 3.4.2 (Uniqueness). *Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold without boundary satisfying $p > \frac{n}{2}$, $\delta < 0$ and $n \geqslant 3$. Furthermore, assume that $\tilde{K} \in W_{\delta-1}^{1,p}$ in (3.71). Let ϕ_1 and ϕ_2 be two positive $W_{loc}^{2,p}$-solutions of (3.71), with $\phi_i - 1 \in W_\delta^{2,p}$, $i = 1, 2$, then either $\phi_1 \equiv \phi_2$ or $\tilde{K} \equiv 0$, $\lambda_\gamma > 0$ and $\phi_1 = c\phi_2$ for some constant $c > 0$.*

Proof. Under our hypotheses, define $\phi \doteq \phi_2 \phi_1^{-1}$ and let $g_1 \doteq \phi_1^{\frac{4}{n-2}} \gamma$. Then, from conformal covariance

$$a_n \Delta_{g_1} \phi - R_{g_1} \phi = -|\tilde{K}_1|_{g_1}^2 \phi^{-\frac{3n-2}{n-2}},$$

where $\tilde{K}_1 = \phi_1^{-2}\tilde{K}$ and, by construction, $\left(g_1, K_1 = \tilde{K}_1\right)$ solve the Gauss-constraint. In particular,

$$R_{g_1} = |\tilde{K}_1|^2_{g_1},$$

which implies

$$a_n \Delta_{g_1}(\phi - 1) = -|\tilde{K}_1|^2_{g_1}\left(\phi^{-\frac{3n-2}{n-2}} - \phi\right), \tag{3.89}$$

Notice that $|\tilde{K}_1|^2_{g_1} \in L^p_{\delta-2}$ and therefore, since $\phi^{-\frac{3n-2}{n-2}} - \phi \in W^{2,p}_\delta$, we find $|\tilde{K}_1|^2_{g_1}\left(\phi^{-\frac{3n-2}{n-2}} - \phi\right) \in L^p_{\delta'-2}$ for any $0 > \delta' > 2\delta$. Hence, there is some $\max\{2\delta, 2-n\} < \delta' < \delta$ such that the right-hand side of equation (3.89) lies in $L^p_{\delta'-2}$. Therefore, from Theorem 3.3.1, we have $\phi - 1 \in W^{2,p}_{\delta'}$. Similarly to Lemma 3.3.3, we can iterate the procedure to get $\phi - 1 \in W^{2,p}_{\delta'}$ for any $\delta' > 2-n$.

Let us now multiply the above equation by $(\phi-1)^+ \doteq \max\{\phi-1, 0\} \in W^{1,2}_{\delta'}$. We would like to integrate the left-hand side of the resulting equality by parts with respect to dV_{g_1}. Let us spell out this procedure considering the case $\frac{n}{2} < p < n$ which implies the other ones. In this case, we have

$$W^{1,2}_{\delta'} \hookrightarrow L^{p'}_{2-\delta'-n}, \tag{3.90}$$

as long as $p' \leqslant \frac{2n}{n-2}$ and we can pick $2 - \delta' - n > \delta' > 2 - n$. The first of these conditions is equivalent to

$$1 - \frac{1}{p} \geqslant \frac{1}{2} - \frac{1}{n} \iff \frac{1}{2} + \frac{1}{n} \geqslant \frac{1}{p} \iff \frac{n+2}{2n} \geqslant \frac{1}{p} \iff p \geqslant \frac{2n}{n+2},$$

and since $n \geqslant 3$, then $\frac{n}{2} \geqslant \frac{2n}{n+2}$. Thus, from $p > \frac{n}{2}$, this conditions is fulfilled. Concerning the second one, it can be satisfied as along as we can pick δ' in the range

$$1 - \frac{n}{2} > \delta' > 2 - n,$$

which we can always do. Then, consider sequences $\{\varphi_k^+\}, \{\varphi_k\} \subset C_0^\infty(M)$ converging to $\varphi^+ \doteq (\phi - 1)^+$ and $\varphi \doteq (\phi - 1)$ in $W^{1,2}_{\delta'}$ and $W^{2,p}_{\delta'}$ respectively. This

implies that

$$\left| \int_M \varphi^+ \Delta_{g_1} \varphi \, dV_{g_1} - \int_M \varphi_k^+ \Delta_{g_1} \varphi_k \, dV_{g_1} \right| \leqslant \int_M |(\varphi^+ - \varphi_k^+) \Delta_{g_1} \varphi| \, dV_{g_1}$$
$$+ \int_M |\varphi_k^+ \Delta_{g_1} (\varphi - \varphi_k)| \, dV_{g_1},$$
$$\leqslant \|\varphi^+ - \varphi_k^+\|_{L^{p'}_{2-\delta'-n}} \|\Delta_{g_1} \varphi\|_{L^p_{\delta'-2}}$$
$$+ \|\varphi_k^+\|_{L^{p'}_{2-\delta'-n}} \|\varphi - \varphi_k\|_{L^p_{\delta'-2}},$$

where the right-hand side goes to zero. Therefore,

$$\int_M \varphi^+ \Delta_{g_1} \varphi \, dV_{g_1} = \lim_{k \to \infty} \int_M \varphi_k^+ \Delta_{g_1} \varphi_k \, dV_{g_1},$$
$$= - \lim_{k \to \infty} \int_M \langle \nabla \varphi_k^+, \nabla \varphi_k \rangle_{g_1} \, dV_{g_1} \tag{3.91}$$

To deal with the right-hand side in the last identity, we must apply a similar reasoning to that given above. That is, notice that $W^{1,p}_{\delta'-1} \hookrightarrow L^2_{\delta'-1}$ as long as $2 \leqslant \frac{np}{n-p}$, which is equivalent to $p \geqslant \frac{2n}{n+2}$, where this last condition is now known to be satisfied due to the previous discussion. Therefore $\nabla \varphi^+, \nabla \varphi \in L^2_{\delta'-1}$. Notice that $L^2_{\delta'-1} \hookrightarrow L^2_{1-\delta'-n}$ as long as $1 - \delta' - n > \delta' - 1$, implying that the embedding holds as long as $1 - \frac{n}{2} > \delta' > 2 - n$, which is the case. Therefore, the same argumentation as above shows that the right-hand side in (3.91) also converges to its limit, justifying the integration by parts:

$$\int_{\phi>1} (\phi - 1)^+ \Delta_{g_1} (\phi - 1) \, dV_{g_1} = \int_M (\phi - 1)^+ \Delta_{g_1} (\phi - 1) \, dV_{g_1},$$
$$= - \int_{\phi>1} |\nabla(\phi - 1)|^2_{g_1} \, dV_{g_1} \leqslant 0.$$

On the other hand, if $\phi > 1$, then $\phi^{-\frac{3n-2}{n-2}} - \phi < 0$, implying

$$\int_{\phi>1} (\phi - 1)^+ \Delta_{g_1} (\phi - 1) \, dV_{g_1} = - \int_{\varphi>1} (\phi - 1)^+ |\tilde{K}_1|^2_{g_1} \left(\phi^{-\frac{3n-2}{n-2}} - \phi \right) dV_{g_1} \geqslant 0,$$

and therefore

$$0 = \int_M (\varphi - 1)^+ \Delta_{g_1} (\varphi - 1) \, dV_{g_1} = \int_{\varphi>1} |\nabla(\varphi - 1)|^2_{g_1} \, dV_{g_1}. \tag{3.92}$$

That is, $\phi - 1 = cte$ on the open subset $\phi > 0$.

Applying a similar argument to $(\phi - 1)^- \doteq \min\{0, \phi - 1\} \in W_{\delta'}^{1,2}$ we get the same results, that is

$$0 = \int_M (\phi - 1)^- \Delta_{g_1}(\phi - 1) dV_{g_1} = -\int_{\phi<1} |\nabla(\phi - 1)|_{g_1}^2 dV_{g_1}, \qquad (3.93)$$

implying $\phi - 1 = cte$ on $\phi < 1$. Since ϕ is continuous, it follows that $\phi - 1 \equiv cte$ and therefore $\phi_1 = c\phi_2$. If $c \neq 1$, then

$$0 = \int_M |\tilde{K}_1|_{g_1}^2 \left(\phi^{-\frac{3n-2}{n-2}} - \phi\right) dV_{g_1}$$

Since $\phi = c \neq 1$, then $\tilde{K} \equiv 0$. This, in turn, implies that $R_{g_1} \equiv 0$, which implies $\lambda_\gamma > 0$ through Proposition 3.4.2. $\qquad\square$

Let us end this chapter with a brief note on the status of the analysis of the Lichnerowicz equation on non-compact manifolds. In particular, we would like to point out to the reader that this equation has been analysed on non-compact manifolds with other interesting asymptotic geometries, such as asymptotically cylindrical and/or hyperbolic ends, for instance in Chruściel and Mazzeo (2015). Also, general complete non-compact manifolds were analysed in Albanese and Rigoli (2016) and, more generally, metrically complete manifolds (which may admit even non-compact boundaries) were studied in Albanese and Rigoli (2017). Also, useful results on Yamabe-type equations on non-compact manifolds can be found in Mastrolia, Rigoli, and Setti (2012).

4

Far from CMC solutions

In Chapters 2 and 3 we have seen how to produce solutions to the ECE both on closed manifolds and AE manifolds under a CMC condition, which allows us to decouple their conformal formulation. In this chapter, the ultimate goal is to analyse the existence of solutions to systems of the form of (1.69) through their conformal formulation (2.38).[1] This will demand us to deal with a fully coupled system of semi-linear PDEs. As was noted in Chapter 2, the system (2.38) does not even decouple under CMC assumptions, which makes it substantially more subtle. Furthermore, if we attempt to produce black hole initial data as in Chapter 3, using the results of Section 3.3.1, in particular (3.31), we will add further coupling through the boundary conditions. All this provides us with a strong motivation to attempt to prove robust existence theorems, which can deal with general systems of this kind, which will be the content of Section 4.3. But, before that, we will review some recent remarkable results from Holst, Nagy, and Tsogtgerel (2009) and Maxwell (2009) where the authors were able to establish far-from-CMC results for the Gauss–Codazzi system (2.1).

In order to present the results referred to above, this chapter will be organised as follows. In Section 4.1, we will start by describing implicit function techniques

---

[1]In Section 4.3, we will analyse a slightly less general system, which arises by neglecting the scalar field and the pressure terms arising from the perfect fluid in (2.38).

(due to Choquet-Bruhat (2004)) which allow us to build non-CMC solutions in a neighbourhood of CMC-data (see also Isenberg and Moncrief (1996)). Then, we will actually exhibit the results of Maxwell (2009) to produce vacuum solutions to (2.1) with freely prescribed mean curvature. These result belong to a family of recent results, which started with the work of Holst, Nagy, and Tsogtgerel (2009), and have systematically improved existence results for (2.1) allowing for far-from-CMC constructions under different assumptions on the coefficients and the manifold M (see Dilts, Isenberg, et al. (2014), Holst and Meier (2014), Holst, Meier, and Tsogtgerel (2018), Nguyen (2016), Premoselli (2014), and Vâlcu (2020)).[2] Building on ideas of these authors, we will analyse the conformally formulated system (2.38), and present far-from-CMC results results for black hole initial data on AE manifolds, which are due to Avalos and Lira (2019). Finally, let us just notice that, contrary to Chapters 2 and 3, uniqueness results in this setting are much more subtle and non-uniqueness is known to occur (see, for instance, Isenberg and Ó Murchadha (2004), Maxwell (2011), and Premoselli (2015) as well as references therein).

4.1 Near CMC-solutions

Let us start by recalling the conformal formulation of the constraints (2.1) with sources given by a self-interacting scalar field and a perfect fluid. Such a systems can be derived from (2.35) by setting to zero the electromagnetic contributions.[3] Explicitly, we find

$$
\begin{aligned}
&\Delta_\gamma \varphi - r_\gamma \varphi + a_{TT} \varphi^{-\frac{3n-2}{n-2}} - a_\tau \varphi^{\frac{n+2}{n-2}} = 0, \\
&\Delta_{\gamma,\mathrm{conf}} X - \omega_\tau \varphi^{\frac{2n}{n-2}} - \omega_\mu \varphi^{2\frac{n+1}{n-2}} - \omega_\phi = 0,
\end{aligned}
\tag{4.1}
$$

---

[2]Let us also point out to the reader the non-CMC existence and non-existence results obtained by Dahl, Gicquaud, and Humbert (2012) on closed manifolds as well as the related analysis on asymptotically hyperbolic manifolds of Gicquaud and Sakovich (2012).

[3]Neglecting electromaagnetic contributions is not really a restriction for tools presented in this section, but allows for a cleaner and more straightforward presentation. Similar results can be obtained in those cases, and, since the main objective of this chapter is to obtain far-from-CMC results, we leave such modifications for the reader, who can find some of these extensions in Choquet-Bruhat (2004) as well as Choquet-Bruhat (2009, Chapter VII, Section 8).

with

$$r_\gamma \doteq c_n \left(R_\gamma - |\nabla\phi|_\gamma^2 \right),$$

$$a_\tau \doteq \frac{n-2}{4n}\tau^2 - 2c_n\epsilon_0, \quad \epsilon_0 \doteq V(\phi) + (\mu+p)\left(1 + |\tilde{u}|_\gamma^2\right) - p,$$

$$a_{TT} \doteq c_n \left(|\tilde{K}|_\gamma^2 + \tilde{\pi}^2 \right),$$

$$\omega_\phi \doteq -\tilde{\pi}d\phi,$$

$$\omega_\tau \doteq \frac{n-1}{n}d\tau,$$

$$\omega_\mu \doteq \left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}}(\mu+p)\tilde{u}^\flat,$$

and recalling that $\tilde{K}(X) = \mathscr{L}_{\gamma,\mathrm{conf}}X + U$, where U is a freely prescribed TT-tensor field.

Let us consider the above system on a closed manifold M^n, $n \geq 3$ and rewrite the above system more compactly, slitting the conformal data for the unknowns. That is, consider $\psi \doteq (\gamma, \tau, U, \phi, \tilde{\pi}, \mu, p, \tilde{u}) \in \mathcal{B}_2$, where

$$\mathcal{B}_2 \doteq \mathcal{M}^{2,p} \times W^{1,p} \times W^{1,p} \times W^{2,p} \times W^{1,p} \times W^{1,p} \times W^{1,p} \times W^{2,p}, \quad (4.2)$$

where $\mathcal{M}^{2,p}$ denotes the open cone of $W^{2,p}$-Riemannian metrics on M, where $p > \frac{n}{2}$. Finally, let us also assume that the potential function $V : \mathbb{R} \mapsto \mathbb{R}$ is smooth, so that from our choice of $\phi \in W^{2,p}$ and Lemma A.2.1, we know that $V(\phi) \in W^{2,p}$. Then, denote by $\mathcal{X} \doteq (\varphi, X) \in \mathcal{B}_1 \doteq W^{2,p} \times W^{2,p}$, and also

$$\begin{aligned}\Phi : \mathcal{B}_1 \times \mathcal{B}_2 &\mapsto \mathcal{B}_3, \\ (\mathcal{X}, \psi) &\mapsto (\mathcal{H}(\mathcal{X}, \psi), \mathcal{M}(\mathcal{X}, \psi)),\end{aligned} \quad (4.3)$$

where we havev denoted

$$\begin{aligned}\mathcal{H}(\mathcal{X}, \psi) &\doteq \Delta_\gamma\varphi - r_\gamma\varphi + a_{TT}\varphi^{-\frac{3n-2}{n-2}} - a_\tau\varphi^{\frac{n+2}{n-2}}, \\ \mathcal{M}(\mathcal{X}, \psi) &\doteq \Delta_{\gamma,\mathrm{conf}}X - \omega_\tau\varphi^{\frac{2n}{n-2}} - \omega_\mu\varphi^{2\frac{n+1}{n-2}} - \omega_\phi,\end{aligned} \quad (4.4)$$

and also $\mathcal{B}_3 \doteq L^p \times L^p$. The fact that the map (4.3) is well-defined and acts between the stated functional spaces is a consequence of the Sobolev multiplication properties, which by now are well-known. Then, the system (4.1) can be compactly rewritten as

$$\Phi(\mathcal{X}, \psi) = 0. \quad (4.5)$$

The strategy to produce near-CMC solutions is a classical one, following from implicit function arguments. That is, suppose we start with a known CMC solution $(\mathcal{X}_0, \psi_0)$ to (4.5), and consider the linearisation of Φ with respect to $\mathcal{X}$ at the point $(\mathcal{X}_0, \psi_0)$, which is given by the linear map

$$
\begin{aligned}
D_{\mathcal{X}}\Phi_{(\mathcal{X}_0,\psi_0)} &: \mathcal{B}_1 \mapsto \mathcal{B}_3, \\
\mathcal{Y} &\mapsto \left(D_{\mathcal{X}}\mathcal{H}_{(\mathcal{X}_0,\psi_0)} \cdot \mathcal{Y}, D_{\mathcal{X}}\mathcal{M}_{(\mathcal{X}_0,\psi_0)} \cdot \mathcal{Y}\right),
\end{aligned}
\tag{4.6}
$$

where $\mathcal{Y} = (v, Y) \in \mathcal{B}_1$ and the linear maps $D_{\mathcal{X}}\mathcal{H}_{(\mathcal{X}_0,\psi_0)} : \mathcal{B}_1 \mapsto \mathcal{B}_3$ and $D_{\mathcal{X}}\mathcal{M}_{(\mathcal{X}_0,\psi_0)} : \mathcal{B}_1 \mapsto \mathcal{B}_3$ can be deduced from (4.1) to be

$$
\begin{aligned}
D_{\mathcal{X}}\mathcal{H}_{(\mathcal{X}_0,\psi_0)} \cdot \mathcal{Y} &= D_{\varphi}\mathcal{H}_{(\mathcal{X}_0,\psi_0)} \cdot (v,0) + D_X\mathcal{H}_{(\mathcal{X}_0,\psi_0)} \cdot (0,Y), \\
&= \Delta_{\gamma_0} v - \left(\overset{0}{r}_\gamma + \frac{3n-2}{n-2}\overset{0}{a}_{TT}\varphi_0^{-4\frac{n-1}{n-2}} + \frac{n+2}{n-2}\overset{0}{a}_\tau\varphi_0^{\frac{4}{n-2}}\right) v \\
&\quad + \varphi_0^{-\frac{3n-2}{n-2}} (D_X a_{TT})_{\psi_0} \cdot Y, \\
D_{\mathcal{X}}\mathcal{M}_{(\mathcal{X}_0,\psi_0)} \cdot \mathcal{Y} &= D_{\varphi}\mathcal{M}_{(\mathcal{X}_0,\psi_0)} \cdot (v,0) + D_X\mathcal{M}_{(\mathcal{X}_0,\psi_0)} \cdot (0,Y), \\
&= \Delta_{\gamma_0,\mathrm{conf}} Y - \left(\frac{2n}{n-2}\overset{0}{\omega}_\tau\varphi_0^{\frac{n+2}{n-2}} + 2\frac{n+1}{n-2}\overset{0}{\omega}_\mu\varphi_0^{\frac{n+4}{n-2}}\right) v,
\end{aligned}
\tag{4.7}
$$

where the quantities with a zero on top a computed with the data ψ_0, and one can check that all the operators are well defined and act between the corresponding functional spaces using Sobolev multiplication properties. In this context, if we show that $D_{\mathcal{X}}\Phi_{(\mathcal{X}_0,\psi_0)}$ is an isomorphism, then, the implicit function theorem (see Abraham, Marsden, and Ratiu (1988, Theorem 2.5.7)) guarantees that there are neighbourhoods $\mathcal{U}$ of ψ_0 in $\mathcal{B}_2$ and $\mathcal{V}$ of $\mathcal{X}_0$ in $\mathcal{B}_1$, and a unique C^1-map

$$
h : \mathcal{U} \mapsto \mathcal{V},
\tag{4.8}
$$

such that

$$
\Phi(h(\psi), \psi) = 0, \ \forall \psi \in \mathcal{U}.
\tag{4.9}
$$

Furthermore, if $\varphi_0 > 0$, then since $W^{2,p} \hookrightarrow C^0$ for $p > \frac{n}{2}$, then the solution φ given above is also positive if we take $\mathcal{U}$ small enough. Therefore, our task is to show that this linearisation is actually an isomorphism around appropriate CMC-data. Before presenting the corresponding result, let us highlight the following fact. Let $(M^n, g, K, \phi, \pi, \mu, p, u)$ be a solution to the Einstein - scalar field -

perfect fluid system associated to (2.1), with M^n closed, $g \in W^{2,p}$, $K \in W^{1,p}$, $\phi \in W^{2,p}$, $\pi, \mu, p \in W^{1,p}$ and $u \in W^{2,p}$. If g has no CKF, then, the splitting $K = \overset{\circ}{K} + \frac{\tau}{n}g$, where $\overset{\circ}{K}$ stands for the traceless part of K, admits one further decomposition. Explicitly, let $X \in W^{2,p}$ be the unique solution to (guaranteed to exist and be unique due to the no CKF condition and Theorem B.8)

$$\Delta_{g,\mathrm{conf}}X = \mathrm{div}_g \overset{\circ}{K},$$

and define $U \doteq \overset{\circ}{K} - \mathscr{L}_{g,\mathrm{conf}}X$, which guarantees that $\mathrm{div}_g U = 0$ and therefore U is a g-TT tensor. Then, we can write

$$K = \mathscr{L}_{g,\mathrm{conf}}X + U + \frac{\tau}{n}g, \tag{4.10}$$

with U the TT-part associated to K. Notice then, that the conformal problem (4.1) with conformal data $(g, \tau, U, \phi, \pi, \mu, p, u)$ has the obvious solution ($\varphi = 1, X$), with X given by (4.10). Therefore, given such a CMC solution, we have conformal data $\psi_0 \doteq (g, \tau, U, \phi, \pi, \mu, p, u) \in \mathcal{B}_2$ with the associated solution $\mathcal{X}_0 \doteq (1, X) \in \mathcal{B}_1$, and shall therefore linearise at $(\mathcal{X}_0, \psi_0)$.

Theorem 4.1.1. *Let $(M^n, g, K, \phi, \pi, \mu, p, u)$ be a CMC-solution to the Einstein - scalar field - perfect fluid system associated to (2.1), with M^n closed, $n \geq 3$, $g \in W^{2,p}$, $K \in W^{1,p}$, $\phi \in W^{2,p}$, $\pi \in W^{1,p}$ with smooth potential function $V : \mathbb{R} \mapsto \mathbb{R}$ for the scalar field and trivial fluid data $\mu = 0, p = 0, u = 0$. If* [4]

$$\frac{n-2}{4n}\tau^2 - 2c_n V(\phi) \geq 0,$$

and g possesses no CKFs, then there is a $\mathcal{B}_2$-neighbourhood $\mathcal{U}$ of the conformal data $\psi_0 = (g, \tau, U, \phi, \pi, \mu = 0, p = 0, u = 0)$, where U is the TT-tensor defined by (4.10), such that (4.8)-(4.9) are satisfied and $h(\psi) \in \mathcal{B}_1$ solve the conformal problem (4.1).

Proof. From our previous discussion, we need to show that $D_{\mathcal{X}}\Phi_{(\mathcal{X}_0, \psi_0)}$ is an isomorphism with ψ_0 taken as in the theorem and $\mathcal{X}_0 \doteq (1, X) \in \mathcal{B}_1$, where X is given by (4.10). Using our assumptions on such a background solution $(\mathcal{X}_0, \psi_0)$, this is equivalent to proving that the system

$$\Delta_g v - av + (D_X a_{TT})_{\psi_0} \cdot Y = f,$$
$$\Delta_{g,\mathrm{conf}}Y = F, \tag{4.11}$$

---

[4]Recall from Chapter 2 that $c_n = \frac{1}{4}\frac{n-2}{n-1}$.

has a unique solution $(v, Y) \in W^{2,p} \times W^{2,p}$ for $(f, F) \in L^p \times L^p$, where

$$a \doteq \overset{0}{r}_g + \frac{3n-2}{n-2}\overset{0}{a}_{TT} + \frac{n+2}{n-2}\overset{0}{a}_\tau \in L^p \tag{4.12}$$

Since the second equation above decouples, we know that the no CKF condition guarantees that it is uniquely solvable for each $F \in L^p$. This makes the last term in the left-hand side of the first equation a datum and therefore, the problem is reduced to proving that $\Delta_g - a : W^{2,p} \mapsto L^p$ is an isomorphism. From our hypotheses, notice that if $\overset{0}{r}_g \geq 0$ a.e, then $a \geq 0$ a.e. and then the isomorphism property follows from Theorem B.7. In the general case, notice that (g, K, ϕ, π) by hypothesis solve

$$R_g = |K|_g^2 - \tau^2 + \pi^2 + |\nabla\phi|_g^2 + 2V(\phi),$$
$$= |\overset{\circ}{K}|_g^2 + \frac{1-n}{n}\tau^2 + \pi^2 + |\nabla\phi|_g^2 + 2V(\phi),$$

which is equivalent to

$$r_g = c_n(R_g - |\nabla\phi|_g^2) = c_n\left(|\overset{\circ}{K}|_g^2 + \pi^2\right) - c_n(\frac{n-1}{n}\tau^2 - 2V(\phi)),$$
$$= \overset{0}{a}_{TT} - \overset{0}{a}_\tau.$$

This implies that

$$a = 4\frac{n-1}{n-2}\overset{0}{a}_{TT} + \frac{4}{n-2}\overset{0}{a}_\tau \geq 0 \text{ a.e},$$

and therefore the isomorphism claim follows. $\qquad \square$

With the aid of the above theorem, we can can construct low-regularity near CMC initial data for the Einstein - scalar field - perfect fluid system in a neighbourhood of some prior CMC solution with zero fluid data. Furthermore, the techniques described above can clearly be used to accommodate other kinds of matter fields, and can also be extended outside the case of closed manifolds as long as the corresponding functional analytic and linear PDE properties remain valid, which is the case, for instance, for AE-manifolds appealing the weighted spaces of Chapter 3. We leave such modifications for the reader, who can also consult a similar version of the above theorem in this non-compact context in Choquet-Bruhat (Chapter VII 2009, Theorem 12.2).

4.2   Vacuum solution with freely specified mean curvature

Let us now move to the core is this chapter, which is the analysis of initial data with freely prescribed mean curvature. First, let us highlight the effectiveness of the conformal method described in Chapter 2 to deal with the ECE when these equations decouple, which always involves a CMC condition. On closed manifolds, such effectiveness is reflected by the detailed description of space of solutions provided in Theorem 2.2.7. In this case, we needed to impose strong conditions on the coefficients of (2.35) which introduce coupling between the equations (among them, the mean curvature) but, besides that, the restrictions on the remaining coefficients were mild. In the previous section we have widened this description by allowing for near CMC data, but it has actually been a challenge in the analysis of the ECE to be able to produce initial data with freely prescribed mean curvature, which is something that inevitably faces us with the difficulty of having to deal with the coupled system.

Although outside the CMC context the conformal method has proven to be less effective that in the CMC-case, there have been recent advances in this new direction starting with the work of Holst, Nagy, and Tsogtgerel (2009), where the authors were able to produce low-regularity far-from-CMC initial data. Not surprisingly, this new freedom on the mean curvature comes at the expense of demanding certain smallness conditions on the remaining coefficients of the system. Furthermore, in this original work, the authors also needed to exclude the vacuum case and considered certain phenomenological sources which can accommodate some classical fields. Nevertheless, the vacuum case has been shown not be outside the scope of their techniques, and has been dealt with in Maxwell (2009). In this last case, the analysis trades the CMC-condition for some smallness condition of the TT-part of the extrinsic curvature. Below, we shall present these vacuum results which exemplify the ideas introduced in these two foundational papers. We shall follow Maxwell (ibid.), since the techniques used in this last paper are closer to those we have presented in Chapter 2, allowing for a more direct presentation. Besides, in the following section (through slightly different techniques), we will incorporate matter fields, which can even include further coupling with electromagnetic sources and boundary data. In order to provide a cleaner presentation, we shall restrict to the 3-dimensional case, a restriction which shall be lifted in the

final section. Therefore, we now intend to analyse the system

$$
\begin{aligned}
-8\Delta_\gamma\varphi + R_\gamma\varphi &= -\frac{2}{3}\tau^2\varphi^5 + |\mathscr{L}_{\gamma,\mathrm{conf}}X + U|^2\varphi^{-7}, \\
\Delta_{\gamma,\mathrm{conf}}X &= \frac{2}{3}\varphi^6 d\tau,
\end{aligned}
\tag{4.13}
$$

posed on a 3-dimensional Riemannian manifold (M^3, γ), and following our conventions we will denote by $\tilde{K}(X) = \mathscr{L}_{\gamma,\mathrm{conf}}X + U$.

In order to describe the strategy adopted in Maxwell (ibid.) to deal with the system (4.13), let us first present the following characterisation for the solutions of the Lichnerowicz equation

$$
-8\Delta_\gamma\varphi + R_\gamma\varphi = -\frac{2}{3}\tau^2\varphi^5 + |\tilde{K}|^2\varphi^{-7}
\tag{4.14}
$$

for fixed $(\gamma, \tau, \tilde{K})$.

Theorem 4.2.1. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$. Then, equation (4.14) with $\tau, \tilde{K} \in W^{1,p}$ admits a positive solution $\varphi \in W^{2,p}$ if and only if one of the following conditions hold:*

1. *$\mathcal{Y}([\gamma]) > 0$ and $\tilde{K} \not\equiv 0$;*

2. *$\mathcal{Y}([\gamma]) \geqslant 0$, $\tilde{K} \not\equiv 0$ and $\tau \not\equiv 0$;*

3. *$\mathcal{Y}([\gamma]) < 0$ and there is a conformal deformation of γ to γ' with $R_{\gamma'} = \frac{2}{3}\tau^2$;*

4. *$\mathcal{Y}([\gamma]) = 0$, $\tilde{K}, \tau \equiv 0$.*

Furthermore, in the first three cases the solution is unique, while in the last one any two such solutions solutions φ_1 and φ_2 are of the form $\varphi_2 = c\varphi_1$, with $c > 0$.

Proof. the uniqueness claim is a direct application of Theorem 2.2.6. Concerning existence, cases (1), (2) and (4) are also a direct application of Lemma 2.2.6, and case (3) a direct application of Proposition 2.2.4. □

A few remarks are now in order. First, notice that we have presented the above theorem appealing to more regularity than what is actually necessary. This is because in our analysis of the coupled system (4.13) we will need to have some point

wise control over $\widetilde{K}(X)$. If $p > 3$, then $W^{1,p}$ is an algebra under multiplication, and therefore if $\widetilde{K}(X) \in W^{1,p}$, we can estimate

$$
\begin{aligned}
\left\| |\widetilde{K}(X)|^2_\gamma \right\|_{W^{1,p}} &\lesssim \|\widetilde{K}(X)\|^2_{W^{1,p}} \leqslant 2 \left(\|\mathscr{L}_{\gamma,\mathrm{conf}} X\|^2_{W^{1,p}} + \|U\|^2_{W^{1,p}} \right), \\
&\lesssim \|X\|^2_{W^{2,p}} + \|U\|^2_{W^{1,p}}.
\end{aligned}
$$

Also, we have $W^{1,p} \hookrightarrow C^0$, which allows us to estimate

$$
|\widetilde{K}(X)|^2_\gamma \lesssim \|\widetilde{K}(X)\|^2_{W^{1,p}} \lesssim \|X\|^2_{W^{2,p}} + \|U\|^2_{W^{1,p}},
$$

and therefore we gain point wise control of the coefficient $|\widetilde{K}(X)|^2_\gamma$ in terms of $\|X\|^2_{W^{2,p}}$, which, in turn, whenever X is constructed from a solution of the momentum constraint with fixed source φ, can be estimated appealing to elliptic estimates on the conformal Killing Laplacian.

Also, let us notice that item (4) in the above theorem will not play an important role in subsequent analysis, since this case falls back on the CMC analysis of Chapter 2.

Finally, fixing our conformal data (γ, τ, U) with $\tau \not\equiv 0$, the above theorem provides a clear description of the domain $\mathcal{D}_{\gamma,\tau}$ of the map[5]

$$
\begin{aligned}
\mathcal{L}_1 : \mathcal{D}_{\gamma,\tau} &\subset W^{1,p} \mapsto W^{2,p}, \\
\widetilde{K} &\mapsto \varphi = \mathcal{L}_1(\widetilde{K}),
\end{aligned}
\tag{4.15}
$$

which assigns to $\widetilde{K} \in \mathcal{D}_{\gamma,\tau} \subset W^{1,p}$ the corresponding solution to (4.14) according to Theorem 4.2.1. That is, according to Theorem 4.2.1, such map is well-defined on

$$
\mathcal{D}_{\gamma,\tau} = \begin{cases} W^{1,p} \backslash \{0\}, & \text{if } \mathcal{Y}([\gamma]) \geqslant 0, \\ W^{1,p}, & \text{if } \mathcal{Y}([\gamma]) < 0 \text{ and } \exists\, \gamma' \in [\gamma] \text{ such that } R_{\gamma'} = -\frac{2}{3}\tau^2, \end{cases}
\tag{4.16}
$$

furthermore (from the "only if" part of the theorem) these are maximal domains of definition for $\mathcal{L}_1$ in $W^{1,p}$. This motivates the following definition.

Definition 4.2.1. *Let (M^3, γ) be a closed Riemannian manifold and $\tau, \widetilde{K} \in W^{1,p}$ a function and a symmetric tensor field respectively. We will say that γ and τ are **Lichnerowicz compatible** is they satisfy one the conditions of cases 1-3 in Theorem 4.2.1 and we will say that $\widetilde{K}$ is **admissible** is it satisfies the corresponding condition of this theorem.*

---

[5]Notice that, the uniqueness claim in Theorem 4.2.1 is key to guarantee that the map $\mathcal{L}_1$ is well-defined.

Going back to the strategy we shall adopt to deal with the system (4.13), Suppose we fix our conformal data $(\gamma, \tau, U) \in W^{2,p} \times W^{1,p} \times W^{1,p}$, $p > 3$, assume that γ has no CKF. Then, given any positive $\varphi \in L_+^\infty$, the momentum constraint in (4.13) is uniquely solvable. That is, we get a map

$$\mathcal{L}_2 : L_+^\infty \mapsto W^{2,p}, \tag{4.17}$$
$$\varphi \mapsto X_\varphi$$

which assigns to $\varphi \in L_+^\infty$ the corresponding unique solution to the momentum constraint $X_\varphi = \mathcal{L}_2(\varphi)$, where are have denoted by L_+^∞ the set of positive elements in L^∞. This map is well-defined, since under the above hypothesis the right-hand side of the momentum constraint in (4.13) is in L^p and the no CKFs condition implies that $\Delta_{\gamma,\mathrm{conf}} : W^{2,p} \mapsto L^p$ is an isomorphism through Theorem B.8.

Proposition 4.2.1. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and assume that γ possesses no CKFs. Let $\tau \in W^{1,p}$ be a function and $U \in W^{1,p}$ a TT-tensor. Assume that γ and τ and Lichnerowicz compatible and that U is admissible. Then, given any $\varphi \in L_+^\infty$, the symmetric tensor field $\tilde{K}(X_\varphi) = \mathcal{L}_{\gamma,\mathrm{conf}}(X_\varphi) + U$ is Lichnerowicz admissible.*

Proof. Since γ has no CKF, the map $\mathcal{L}_2$ is defined on all of L_+^∞ and therefore $\tilde{K}(X_\varphi) = \mathcal{L}_{\gamma,\mathrm{conf}}(X_\varphi) + U$ is well-defined for all $\varphi \in L_+^\infty$. Notice that if $\mathcal{Y}([\gamma]) < 0$ with γ and τ Lichnerowicz compatible, then $\tilde{K}(X)$ is Lichnerowicz admissible since in this case $\mathcal{D}_{\gamma,\tau} = W^{1,p}$. On the other hand, if $\mathcal{Y}([\gamma]) \geqslant 0$, since U is admissible by hypothesis, it must be the case that $U \not\equiv 0$. Let us show that $\tilde{K}(X_\varphi)$ is admissible is this case as well. Since U is γ-TT, it follows that

$$\int_M |\tilde{K}(X_\varphi)|_\gamma^2 dV_\gamma = \int_M |\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi|_\gamma^2 dV_\gamma + \int_M |U|_\gamma^2 dV_\gamma$$
$$+ 2\int_M \langle \mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi, U\rangle_\gamma dV_\gamma,$$
$$= \int_M |\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi|_\gamma^2 dV_\gamma + \int_M |U|_\gamma^2 dV_\gamma + 4\int_M \langle DX_\varphi, U\rangle_\gamma dV_\gamma,$$

where, in the second line, we have used the traceless condition of U as well as its symmetry. Integrating by parts the last term, the transverse condition on U shows that this term vanishes. That is, $\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi$ and U are L^2-orthogonal. Therefore, we find

$$\int_M |\tilde{K}(X_\varphi)|_\gamma^2 dV_\gamma \geqslant \int_M |U|_\gamma^2 dV_\gamma > 0, \tag{4.18}$$

since $U \not\equiv 0$. Thus, $\widetilde{K}(X_\varphi)$ is Lichnerowicz admissible. $\square$

From the above proposition, we immediately get the following

Corollary 4.2.1. *Under the assumptions of the above proposition,* $\mathrm{Im}(\mathcal{L}_2) \subset \mathcal{D}_{\gamma,\tau}$.

Therefore, under the assumptions of Proposition 4.2.1, the map

$$\mathcal{N} : L_+^\infty \mapsto W_+^{2,p} \subset L_+^\infty,$$
$$\varphi \mapsto \hat{\varphi} = \mathcal{N}(\varphi) \doteq \mathcal{L}_1(\underbrace{\mathscr{L}_{\gamma,\mathrm{conf}}(\mathcal{L}_2(\varphi)) + U}_{X_\varphi}) \tag{4.19}$$

is well-defined. Furthermore, the solutions of (4.13) are in 1-1 correspondence with fixed point of the above map. Therefore, our task will be to guarantee the existence of such fixed point. For this, we will appeal to the following well-known fixed point theorem by Leray–Schauder (see, for instance, Taylor (2011c, Corollary B.3)).

Theorem 4.2.2. *Let* $\mathcal{U}$ *be a closed, convex set in a Banach space* V *and let* $\mathcal{F} : \mathcal{U} \mapsto \mathcal{U}$ *be a continuous map such that* $\overline{\mathcal{F}(\mathcal{U})}$ *is compact. Then* $\mathcal{F}$ *has a fixed point.*

In order to appeal to the above theorem to extract a fixed point out of $\mathcal{N}$, we need to find an appropriate closed and convex subset $\mathcal{U} \subset L_+^\infty$, invariant under the action of $\mathcal{N}$, and also prove that $\mathcal{N}$ is continuous.[6] In previous chapters, when analysing the decoupled Lichnerowicz equation, we saw that the existence of barriers $0 < \varphi_- \leqslant \varphi_+$ permitted us to trap solutions to the Lichnerowicz equation in $[\varphi_-, \varphi_+] \subset L^\infty$, where by $[\varphi_-, \varphi_+]$, we mean those elements φ of $W_+^{2,p}$ satisfying $\varphi_- \leqslant \varphi \leqslant \varphi_+$. We intend to find a similar subset in this context to be the domain of $\mathcal{N}$. But recall that the invariance property of $\mathcal{L}_1$ on $[\varphi_-, \varphi_+]$ in the context of the decoupled Lichnerowicz equations relied on these endpoints being sub and super-solution, which depend on the coefficients of the equation. But in our new context, the coefficient $\widetilde{K}(X_\varphi)$ would depend on $\varphi \in [\varphi_-, \varphi_+]$. From this discussion, we see that, in order for similar method to succeed in this context, we would need a fixed set of barriers which works for every such $\widetilde{K}(X_\varphi)$ obtained as a solution of the momentum equation with source $\varphi \in [\varphi_-, \varphi_+]$. This motivates the following definition.

---

[6]Notice that, although such a fixed point φ is a priori found in L_+^∞, since $\varphi = \mathcal{N}(\varphi)$ and $\mathcal{N}(L_+^\infty) \subset W_+^{2,p} \subset L_+^\infty$, then actually $\varphi \in W_+^{2,p}$.

Definition 4.2.2. *We say that $\varphi_+ \in W^{2,p}$, $p > 3$, is a **global supersolution** of (4.13) if, whenever $0 < \varphi \leqslant \varphi_+$, $\varphi \in W^{2,p}$, then*

$$-8\Delta_\gamma \varphi_+ + R_\gamma \varphi_+ \geqslant -\frac{2}{3}\varphi_+^5 + |\mathcal{L}_{\gamma,conf}X_\varphi + U|_\gamma^2 \varphi_+^{-7}, \qquad (4.20)$$

*where $X_\varphi = \mathcal{L}_2(\varphi)$ is the solution obtained via (4.17). Similarly, we say that $\varphi_- \in W^{2,p}$ is a **global subsolution** if, whenever $\varphi \geqslant \varphi_-$, $\varphi \in W^{2,p}$, then*

$$-8\Delta_\gamma \varphi_- + R_\gamma \varphi_- \leqslant -\frac{2}{3}\varphi_-^5 + |\mathcal{L}_{\gamma,conf}X_\varphi + U|_\gamma^2 \varphi_-^{-7}. \qquad (4.21)$$

With the above definitions in mind, we shall attempt to construct the invariant subset $\mathcal{U} \subset L^\infty$ of Theorem 4.2.2 for $\mathcal{N}$ to be of the form $[\varphi_-, \varphi_+]$, with $\varphi_\pm$ global barriers for (4.13). Notice that such subset is automatically closed and convex in L^∞, and the remaining properties will be a consequence of the mapping properties of $\mathcal{N}$.

Now that the strategy to deal with (4.13) is clear, we will devote the next two subsections to construct the invariant subset $\mathcal{U}$ for the map $\mathcal{N}$ and then prove that $\mathcal{N}$ is a continuous mapping. With these results at hand, we shall present the existence results for (4.13) of Maxwell (2009).

The invariant subset $\mathcal{U}$

Following Maxwell (ibid.), let us start with the following conformal covariance property for barriers.

Lemma 4.2.1. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and consider the Lichnerowicz equation (4.2.1) with $\tau, \tilde{K} \in W^{1,p}$ and let $\theta \in W_+^{2,p}$. Then, ϕ is a subsolution (respectively supersolution) of (4.2.1) iff $\phi' \doteq \theta^{-1}\phi$ is a subsolution (respectively supersolution) of the conformally related equation*

$$-8\Delta_{\gamma'}\phi' + R_{\gamma'}\phi' = -\frac{2}{3}\tau^2\phi'^5 + |\tilde{K}'|_{\gamma'}^2\phi'^{-7}, \qquad (4.22)$$

where, as usual, $\gamma' = \theta^4\gamma$ and $\tilde{K}' = \theta^{-2}\tilde{K}$.

Proof. For any $\phi \in W_+^{2,p}$ and $\phi' = \theta^{-1}\phi$ we know from Proposition 2.1.3 that

$$-8\Delta_{\gamma'}\phi' + R_{\gamma'}\phi' = \theta^{-5}\left(-8\Delta_\gamma\phi + R_\gamma\phi\right),$$

and it follows directly that

$$-\frac{2}{3}\tau^2\phi'^5 + |\tilde{K}'|^2_{\gamma'}\phi'^{-7} = \theta^{-5}\left(-\frac{2}{3}\tau^2\phi^5 + |\tilde{K}|^2_\gamma\phi^{-7}\right),$$

implying

$$-8\Delta_{\gamma'}\phi' + R_{\gamma'}\phi' + \frac{2}{3}\tau^2\phi'^5 - |\tilde{K}'|^2_{\gamma'}\phi'^{-7} =$$

$$\theta^{-5}\left(-8\Delta_\gamma\phi + R_\gamma\phi + \frac{2}{3}\tau^2\phi^5 - |\tilde{K}|^2_\gamma\phi^{-7}\right). \quad (4.23)$$

Therefore, the claim follows since both sides have the same sign. □

As usual, the above lemma guarantees that we can look for barriers in some preferred element in our conformal class where the task may be simplified.

The follows result guarantees that if we have a pair of sub and super-solutions $\varphi_\pm$, there is no real obstruction in making them compatible. More precisely, the following holds.[7]

Lemma 4.2.2. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and consider the Lichnerowicz equation (4.2.1) with $\tau, \tilde{K} \in W^{1,p}$. If $\varphi_+ \in W^{2,p}_+$ is a supersolution of (4.14), then so is $\varphi^+_\alpha \doteq \alpha\varphi_+$ for all $\alpha \geqslant 1$. Similarly, if $\varphi_- \in W^{2,p}_+$ is a subsolution of (4.14), then so is $\varphi^-_\alpha \doteq \alpha\varphi_-$ for all $0 < \alpha \leqslant 1$.*

Proof. By direct computation we see that for $\alpha \geqslant 1$,

$$-8\Delta_\gamma\varphi^+_\alpha + R_\gamma\varphi^+_\alpha + \frac{2}{3}\tau^2\varphi^{+5}_\alpha - |\tilde{K}|^2_\gamma\varphi^{+-7}_\alpha = \alpha\left(-8\Delta_\gamma\varphi_+ + R_\gamma\varphi_+\right) + \alpha^5\frac{2}{3}\tau^2\varphi^5_+$$

$$-\alpha^{-7}|\tilde{K}|^2_\gamma\varphi^{-7}_+$$

$$\geqslant \alpha\left(-8\Delta_\gamma\varphi_+ + R_\gamma\varphi_+\right) + \alpha\frac{2}{3}\tau^2\varphi^5_+$$

$$-\alpha|\tilde{K}|^2_\gamma\varphi^{-7}_+$$

$$\geqslant \alpha\left(-8\Delta_\gamma\varphi_+ + R_\gamma\varphi_+ + \frac{2}{3}\tau^2\varphi^5_+\right.$$

$$\left. - |\tilde{K}|^2_\gamma\varphi^{-7}_+\right) \geqslant 0.$$

The proof in the subsolution case follows along the same lines. □

---

[7]Although we will not use it, the claims of Lemma 4.2.2 clearly hold under weaker regularity assumptions.

Let us now introduce the following lemma, which guarantees that any super-solution to (4.14) provides an upper bound for the map $\mathcal{L}_1$ of (4.15).

Lemma 4.2.3. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and consider the Lichnerowicz equation (4.14) with τ, $\widetilde{K} \in W^{1,p}$. Suppose that (γ, τ) are compatible, $\widetilde{K}$ is admissible and that $\varphi_+ \in W_+^{2,p}$ is a supersolution to (4.14), then $\mathcal{L}_1(\widetilde{K}) \leqslant \varphi_+$. Similarly, under these conditions, if $\varphi_- \in W_+^{2,p}$ is a subsolution of (4.14), then $\mathcal{L}_1(\widetilde{K}) \geqslant \varphi_-$.*

Proof. Let us start with the supersolution. First, recall from the proofs of Lemma 2.2.6 and Proposition 2.2.4 that if (γ, τ) are compatible and $\widetilde{K}$ is admissible, then the Lichnerowicz equation (4.14) admits a compatible pair of sub and super solutions $\bar{\varphi}_\pm$. Let us now consider the subsolution and, using a constant $\alpha \leqslant 1$ sufficiently small, we can fix $\varphi'_- \doteq \alpha\bar{\varphi}_- \leqslant \varphi_+$. Then, from Theorem 2.2.1, we have a solution $\varphi' \in W_+^{2,p}$ to (4.14) with $\varphi'_- \leqslant \varphi' \leqslant \varphi_+$. But, since from Theorem 4.2.1 the solution associated to our conformal data is unique, it follows that $\varphi' = \mathcal{L}_1(\widetilde{K}) \leqslant \varphi_+$.

Similarly to what we did above, if we know chose $\alpha \geqslant 1$ sufficiently large so that $\varphi'_+ \doteq \alpha\bar{\varphi}_+ \geqslant \varphi_-$, then through the monotone iteration scheme we find a solution $\varphi' \in W_+^{2,p}$ to (4.14) satisfying $\varphi_- \leqslant \varphi' \leqslant \varphi'_+$. Again, since the solution to this problem is unique under our compatibility-admissibility conditions, we must have $\mathcal{L}_1(\widetilde{K}) = \varphi' \geqslant \varphi_-$. $\qquad\square$

Corollary 4.2.2. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and assume that γ possesses no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$. Suppose that (γ, τ) are compatible, U is admissible and that $\varphi_+ \in W_+^{2,p}$ is a **global supersolution** to (4.13). If $\varphi \in L_+^\infty$ satisfies $\varphi \leqslant \varphi_+$, then $\mathcal{N}(\varphi) \leqslant \varphi_+$.*

Proof. First, from our hypotheses, from Corollary 4.2.1, we know that $\mathcal{N}$ is defined over all of L_+^∞. Now, consider $\varphi \in L_+^\infty$ and define $\bar{\varphi} = \mathcal{N}(\varphi)$, so that it solves

$$-8\Delta_\gamma\bar{\varphi} + R_\gamma\bar{\varphi} = -\frac{2}{3}\tau^2\bar{\varphi}^5 + |\mathcal{L}_{\gamma,\text{conf}}X_\varphi + U|_\gamma^2\bar{\varphi}^{-7}, \qquad (4.24)$$

where $X_\varphi = \mathcal{L}_2(\varphi) \in W^{2,p}$. Now, since φ_+ is a global supersolution and $\varphi \leqslant \varphi_+$, by hypothesis, it holds that φ_+ is a supersolution of (4.24). But then Lemma 4.2.3 guarantees that $\bar{\varphi} = \mathcal{N}(\varphi) \leqslant \varphi_+$, which establishes the claim. $\qquad\square$

The objective now is to show that, given a global supersolution φ_+ to (4.13), there is a constant global subsolution $\varphi_- = K_0$, so that $\mathcal{N}$ is invariant on $\mathcal{U} \doteq \{\varphi \in L_+^\infty : K_0 \leqslant \varphi \leqslant \varphi_+\}$, which reduces the task of finding barriers to that of finding just a global supersolution. In order to establish this claim, we need to split the cases $\mathcal{Y}([\gamma]) \geqslant 0$ from $\mathcal{Y}_\gamma < 0$. Let us start with the first of these two cases, which is more delicate. For this, we need the following technical result.

Proposition 4.2.2. *(Maxwell 2009, Porposition 8) Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and let $V \in L^p$ be a function such that $V \geqslant 0$ a.e, $V \not\equiv 0$. Then, the Poisson operator $-\Delta_\gamma + V$ admits a Green function G, which satisfies a lower bound*

$$G(x, y) \geqslant m_G \tag{4.25}$$

for some constant $m_G > 0$.

Remark 4.2.1. *First, let us recall that the Green function associated to a $-\Delta_\gamma + V$ is a function G defined on $M \times M$ satisfying*

$$-\Delta_{\gamma,y} G(x, y) + V(y)G(x, y) = \delta_x, \tag{4.26}$$

where δ_x stands for the δ-distribution with support on $x \in M$. In particular, it satisfies that if φ is a solution of $-\Delta_\gamma \varphi + V\varphi = f \in L^p$, then

$$\varphi(x) = \int_M G(x, y) f(y) d V_\gamma(y). \tag{4.27}$$

We refer the reader to Druet, Hebey, and Robert (2004, Appendix A) or Aubin (1998, Chapter 4) for the general properties and constructions related to such a Green function. Furthermore, under the regularity properties of the above proposition, the Green function is continuous outside the diagonal of $M \times M$.

The proof of the above proposition, relies on properties of the Green function of the Laplacian of γ plus some elliptic properties, all of which can be consulted in the above references. Appealing to the above proposition we can now establish the following result.

Proposition 4.2.3. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and let $V \in L^p$ be a function such that $V \geqslant 0$ a.e, $V \not\equiv 0$. There are positive constants C_1, C_2 such that for every $f \in L^p$, $f \geqslant 0$ a.e, the solution of*

$$-\Delta_\gamma \phi + V\phi = f \tag{4.28}$$

satisfies

$$\max_M \phi \leqslant C_1 \|f\|_{L^p},$$
$$\min_M \phi \geqslant C_2 \|f\|_{L^1}. \tag{4.29}$$

Proof. First, notice that the first estimate in (4.29) follows from the isomorphism property of the Poisson operator presented in Theorem B.7 and the embedding $W^{2,p} \hookrightarrow C^1$. The second one follows from Proposition 4.2.2, since from (4.28) it follows

$$\phi(x) = \int_M G(x,y) f(y) dV_\gamma(y) \geqslant m_G \int_M f(y) dV_\gamma(y) = m_G \|f\|_{L^1},$$

where we have used that f is non-negative. $\qquad\square$

Using the results presented above, we can now prove the following.

Proposition 4.2.4. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, with $\mathcal{Y}([\gamma]) \geqslant 0$, and assume that γ possesses no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$. Suppose that (γ, τ) are compatible, U is admissible and that $\varphi_+ \in W_+^{2,p}$ is a global supersolution to (4.13). Then, there is a constant $K_0 > 0$ such that for all $\varphi \in L_+^\infty$ satisfying $0 < \varphi \leqslant \varphi_+$, it follows that*

$$\mathcal{N}(\varphi) \geqslant K_0. \tag{4.30}$$

Proof. The elegant strategy adopted in Maxwell (2009) is the following one. Fix any $\varphi_0 \in L^\infty$ such that $0 < \varphi_0 \leqslant \varphi_+$, associate the corresponding solution to the momentum constraint $X_{\varphi_0} = \mathcal{L}_2(\varphi_0) \in W^{2,p}$ and then consider the equation

$$-8\Delta_\gamma \varphi + R_\gamma \varphi = -\frac{2}{3}\tau^2 \varphi^5 + |\mathcal{L}_{\gamma,\mathrm{conf}} X_{\varphi_0} + U|_\gamma^2 \varphi^{-7}. \tag{4.31}$$

The objective now is to show that the above equation admits a subsolution φ_- which is bounded from below by a constant K_0 where this constant is independent of the specific choice φ_0. Below, we shall see that such K_0 can be found as long as $\varphi \leqslant \varphi_+$. Once this is established, Lemma 4.2.3 guarantees that $K_0 \leqslant \mathcal{N}(\varphi)$ for any $\varphi \in L_+^\infty$ such that $\varphi \leqslant \varphi_+$, and the claim follows.

To prove the statements in the above paragraph, let us first appeal to the conformal covariance for the barriers established in Lemma 4.2.1. Since $\mathcal{Y}([\gamma]) \geqslant 0$, we know from Theorem 2.2.3 that there is a conformal deformation $\gamma' = \theta^4 \gamma$,

$\theta \in W^{2,p}_{+}$, such that $R_{\gamma'}$ is continuous and has the same sign as $\mathcal{Y}([\gamma])$. Now, appealing to Lemma 4.2.1, let us first find a subsolution of the conformally related equation:

$$-8\Delta_{\gamma'}\varphi' + R_{\gamma'}\varphi' = -\frac{2}{3}\tau^2\varphi'^5 + |\widetilde{K}'|^2_{\gamma'}\varphi'^{-7}, \qquad (4.32)$$

where $\widetilde{K}' = \theta^{-2}(\mathcal{L}_{\gamma,\text{conf}}X_{\varphi_0} + U)$. For this, let $\eta \in W^{2,p}_{+}$ be the unique solution to

$$-8\Delta_{\gamma'}\eta + \left(R_{\gamma'} + \frac{2}{3}\tau^2\right)\eta = |\widetilde{K}'|^2_{\gamma'}. \qquad (4.33)$$

From our hypotheses $R_{\gamma'} + \frac{2}{3}\tau^2 \geqslant 0$ and $R_{\gamma'} + \frac{2}{3}\tau^2 \not\equiv 0$ (recall that $\tau \not\equiv 0$), Theorem B.7 guarantees the existence of such a solution and the maximum principles of Lemmas 2.2.2 and 2.2.3 guarantee that $\eta > 0$. We claim that for $\alpha > 0$ sufficiently small $\alpha\eta$ is a subsolution of (4.32). Using (4.33), we find that

$$-8\Delta_{\gamma'}(\alpha\eta) + R_{\gamma'}(\alpha\eta) + \frac{2}{3}\tau^2(\alpha\eta)^5 - |\widetilde{K}'|^2_{\gamma'}(\alpha\eta)^{-7} = -\frac{2}{3}\tau^2\alpha\eta + \alpha|\widetilde{K}'|^2_{\gamma'}$$
$$+ \frac{2}{3}\tau^2(\alpha\eta)^5 - |\widetilde{K}'|^2_{\gamma'}(\alpha\eta)^{-7},$$
$$= \frac{2}{3}((\alpha\eta)^4 - 1)\tau^2\alpha\eta$$
$$+ |\widetilde{K}'|^2_{\gamma'}(\alpha - \alpha^{-7}\eta^{-7}).$$

We therefore see that if $\alpha \leqslant \min_M(\eta^{-1})$ and also satisfies $\alpha^8 \leqslant \min_M(\eta^{-7})$, then the right-hand side of the above expression in non-positive and $\alpha\eta$ becomes a subsolution. The choice $\alpha \doteq \min\{1, (\max_M \eta)^{-1}\}$ satisfies both of these conditions.

Let us now show that $\alpha\eta = \min\{1, (\max_M \eta)^{-1}\}\eta \geqslant K'_0$ with K'_0 independent of φ_0. This implies showing that η is bounded by above and below by numbers independent of φ_0.[8] For this, applying Proposition 4.2.3, we know that

$$\max_M \eta \leqslant C_1\big\||\widetilde{K}'|^2_{\gamma'}\big\|_{L^p},$$
$$\min_M \eta \geqslant C_2\big\||\widetilde{K}'|^2_{\gamma'}\big\|_{L^1}, \qquad (4.34)$$

---

[8]Notice that the definition of η via (4.33) depends on φ_0 in the right-hand side.

for constants $C_1, C_2 > 0$ independent of φ_0. Let us now work with the first inequality above. First, notice that $|\tilde{K}'|^2_{\gamma'} = \theta^{-12}|K|^2_{\gamma}$, which implies

$$\left\| |\tilde{K}'|^2_{\gamma'} \right\|_{L^p} \leqslant \|\theta^{-12}\|_{C^0} \left\| |\tilde{K}|^2_{\gamma} \right\|_{L^p}.$$

Also, it holds that

$$|\tilde{K}|^2_{\gamma} \leqslant 2\left(|\mathscr{L}_{\gamma,\text{conf}} X_{\varphi_0}|^2_{\gamma} + |U|^2_{\gamma} \right)$$

and therefore

$$\left\| |\tilde{K}'|^2_{\gamma'} \right\|_{L^p} \leqslant 2\|\theta^{-12}\|_{C^0} \left(\left\| |\mathscr{L}_{\gamma,\text{conf}} X_{\varphi_0}|^2_{\gamma} \right\|_{L^p} + \left\| |U|^2_{\gamma} \right\|_{L^p} \right). \tag{4.35}$$

Now, from the Sobolev multiplication property we have $W^{1,p} \otimes W^{1,p} \hookrightarrow L^p$, which implies

$$\left\| |\mathscr{L}_{\gamma,\text{conf}} X_{\varphi_0}|^2_{\gamma} \right\|_{L^p} \lesssim \left\| |DX_{\varphi_0}|^2_{\gamma} \right\|_{L^p} \lesssim \left\| DX_{\varphi_0} \right\|^2_{W^{1,p}} \leqslant \|X_{\varphi_0}\|^2_{W^{2,p}},$$
$$\left\| |U|^2_{\gamma} \right\|_{L^p} \lesssim \|U\|^2_{W^{1,p}}.$$

But now, since $\Delta_{\gamma,\text{conf}} : W^{2,p} \mapsto L^p$ is an isomorphism under our hypotheses, it follows that

$$\|X_{\varphi_0}\|_{W^{2,p}} \leqslant C\|\varphi_0^6 d\tau\|_{L^p} \leqslant C\|d\tau\|_{L^p} \|\varphi_+^6\|_{L^\infty}, \tag{4.36}$$

where we have used that $\varphi_0 \in L^\infty$ satisfies $\varphi_0 \leqslant \varphi_+$. Putting all of the above together, we find that for some other constant $C > 0$ independent of φ_0 it holds

$$\left\| |\tilde{K}'|^2_{\gamma'} \right\|_{L^p} \leqslant C\|\theta^{-12}\|_{C^0} \left(\|d\tau\|_{L^p} \|\varphi_+^6\|_{L^\infty} + \|U\|^2_{W^{1,p}} \right), \tag{4.37}$$

which provides an upper bound for the first inequality in (4.34) which is independent of $\varphi_0 \leqslant \varphi_+$. Let us now work with the lower bound of the second inequality in (4.34). This inequality is easier, since

$$\begin{aligned}
\left\| |\tilde{K}'|^2_{\gamma'} \right\|_{L^1} &= \int_M \theta^{-12}|\tilde{K}|^2_{\gamma} dV_{\gamma'} = \int_M \theta^{-6}|\tilde{K}|^2_{\gamma} dV_{\gamma}, \\
&\geqslant \min_M \theta^{-6} \int_M |\tilde{K}|^2_{\gamma} dV_{\gamma}, \\
&= \min_M \theta^{-6} \left(\int_M |\mathscr{L}_{\gamma,\text{conf}} X_{\varphi_0}|^2_{\gamma} dV_{\gamma} + \int_M |U|^2_{\gamma} dV_{\gamma} \right), \\
&\geqslant \min_M \theta^{-6} \int_M |U|^2_{\gamma} dV_{\gamma} > 0,
\end{aligned}$$

where the last inequality holds since U is admissible and we have used the L^2 orthogonality of $\mathscr{L}_{\gamma,\text{conf}}X_{\varphi_0}$ and the TT-tensor U. Therefore, we have a lower bound for the second inequality in (4.34) which is independent of φ_0. Therefore, we have established that $\varphi'_- \doteq \alpha\eta$ is a subsolution for (4.32) which is bounded from below by a constant K'_0 independent of φ_0. From Equation (4.22), we now that $\varphi_- = \theta\varphi'_-$ is a subsolution of (4.31) bounded from below by $K_0 = (\min_M \theta)K'_0$, and then Lemma 4.2.3 implies that $\mathcal{N}(\varphi) \geqslant K_0$ for all $\varphi \in L^\infty$ satisfying $0 < \varphi \leqslant \varphi_0$. $\qquad\square$

Let us now show the corresponding result to the above proposition in the Yamabe negative case.

Proposition 4.2.5. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, with $\mathcal{Y}([\gamma]) < 0$, and assume that γ possesses no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$. Suppose that (γ, τ) are compatible and U is admissible. Then, there is a constant $K_0 > 0$ such that for all $\varphi \in L^\infty_+$ it follows that*

$$\mathcal{N}(\varphi) \geqslant K_0. \tag{4.38}$$

Proof. Since (γ, τ) are compatible, then there is a conformal deformation $\gamma' = \theta^4\gamma$, $\theta \in W^{2,p}_+$, has scalar curvature $R_{\gamma'} = -\frac{2}{3}\tau^2$, which implies that

$$-8\Delta_\gamma\theta + R_\gamma\theta = -\frac{2}{3}\tau^2\theta^5,$$

Therefore, for any $\varphi \in L^\infty$, it follows that

$$-8\Delta_\gamma\theta + R_\gamma\theta + \frac{2}{3}\tau^2\theta^5 - |\mathscr{L}_{\gamma,\text{conf}}X_\varphi + U|^2_\gamma\theta^{-7} = -|\mathscr{L}_{\gamma,\text{conf}}X_\varphi + U|^2_\gamma\theta^{-7} \leqslant 0,$$

implying that $\varphi_- \doteq \theta$ is a subsolution for (4.14) for any coefficient $\tilde{K} = \mathscr{L}_{\gamma,\text{conf}}X_\varphi + U$ constructed from any $\varphi \in L^\infty_+$, and therefore setting $K_0 \doteq \min_M \theta > 0$ Lemma 4.2.3 implies $\mathcal{N}(\varphi) \geqslant K_0$. $\qquad\square$

Using the results of this subsection, we have established the following.

Corollary 4.2.3. *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, and assume that γ possesses no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$. Suppose that (γ, τ) are compatible, U is admissible and that φ_+ is a global supersolution of (4.13). Then, there is a constant $K_0 > 0$ such that the set $\mathcal{U} \doteq \{\varphi \in L^\infty_+ : K_0 \leqslant \varphi \leqslant \varphi_+\}$ is invariant under $\mathcal{N}$. That is, $\mathcal{N} : \mathcal{U} \mapsto \mathcal{U}$.*

Proof. Corollary 4.2.2 shows that under our hypotheses, for any given $\varphi \in \mathcal{U}$, $\mathcal{N}(\varphi) \leqslant \varphi_+$ regardless of our Yamabe class. Now, if $\mathcal{Y}([\gamma]) \geqslant 0$, then pick K_0 according to Proposition 4.2.4 to obtain $\mathcal{N}(\varphi) \geqslant K_0$, while if $\mathcal{Y}([\gamma]) < 0$ pick such K_0 from Proposition 4.2.5 to obtain the same result. Therefore, we find that in any case $\mathcal{N}(\varphi) \in \mathcal{U}$, which proves our claim. $\qquad\square$

The above corollary establishes the existence of the invariant set needed for the application of Theorem 4.2.2, and in particular this set is already convex and closed by construction.

Mapping properties of the solution map $\mathcal{N}$

With the results of the previous subsection at hand, in order to apply Theorem 4.2.2, we see that we need to prove that $\mathcal{N} : \mathcal{U} \mapsto \mathcal{U}$ is continuous and that $\overline{\mathcal{N}(\mathcal{U})} \subset L^\infty$ is compact, where $\mathcal{U}$ refers to the invariant set constructed in Corollary 4.2.3. Let us start with the continuity claim, which is the one that requires a little bit of more work.

Proposition 4.2.6. *Under the hypotheses of Corollary 4.2.3, the map $\mathcal{N} : \mathcal{U} \mapsto \mathcal{U}$ is continuous.*

Proof. Recall that $\mathcal{N} = \mathcal{L}_1 \circ \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined via (4.15) and (4.17) respectively. In the case of $\mathcal{L}_2$, since γ has no CKF, we know from Theorem B.8 that $\Delta_{\gamma,\mathrm{conf}} : W^{2,p} \mapsto L^p$ is a continuous isomorphism, and therefore it follows that $\mathcal{L}_2 = \Delta_{\gamma,\mathrm{conf}}^{-1}$ is also continuous. The case for the solution map of the Lichnerowicz equation is a little bit more delicate and the idea goes as follows.

We intend to show that $\mathcal{L}_1 : \mathcal{D}_{\gamma,\tau} \mapsto W^{2,p}$ is continuous. Fix any $\tilde{K}_0 \in \mathcal{D}_{\gamma,\tau}$ and let us analyse continuity at this point. Let $\varphi_0 \doteq \mathcal{L}_1(\tilde{K}_0) \in W_+^{2,p}$. Then, consider $\gamma_0' = \varphi_0^4 \gamma$ and the corresponding Lichnerowicz equation

$$-8\Delta_{\gamma_0'}\varphi + R_{\gamma_0'}\varphi = -\frac{2}{3}\tau^2\varphi^5 + |\tilde{K}_0'|_{\gamma_0'}^2 \varphi^{-7}, \tag{4.39}$$

where, as usual, $K_0' = \varphi_0^{-2}\tilde{K}_0$ and, by conformal covariance, we know that this equation is also uniquely solvable and we denote by $\mathcal{L}_1' : \mathcal{D}_{\gamma',\tau} = \mathcal{D}_{\gamma,\tau} \mapsto W_+^{2,p}$ its solution map. Then, we know that the two solution maps $\mathcal{L}_1$ and $\mathcal{L}_1'$ are related via

$$\mathcal{L}_1(\tilde{K}) = \varphi_0 \mathcal{L}_1'(\varphi_0^{-2}\tilde{K}). \tag{4.40}$$

This implies that $\mathcal{L}_1(\tilde{K})$ is continuous at $\tilde{K}_0$ iff $\mathcal{L}'_1$ is continuous at $\tilde{K}'_0 = \varphi_0^{-2}\tilde{K}_0$. Furthermore, let us notice that (4.40) implies that

$$\varphi_0 = \mathcal{L}_1(\tilde{K}_0) = \varphi_0 \mathcal{L}'_1(\varphi_0^{-2}\tilde{K}_0),$$

implying $\mathcal{L}'_1(\varphi_0^{-2}\tilde{K}_0) = 1$, which through (4.39) implies

$$R_{\gamma'_0} = -\frac{2}{3}\tau^2 + |\tilde{K}'_0|^2_{\gamma'_0}. \tag{4.41}$$

Let us now write the Lichnerowicz equation (4.39) in operator form as

$$\Psi(\varphi, \tilde{K}') = 0, \tag{4.42}$$

where $\Psi : W^{2,p}_+ \times W^{1,p} \mapsto L^p$ is defined via

$$\Psi(\varphi, \tilde{K}') = -8\Delta_{\gamma'_0}\varphi + R_{\gamma'_0}\varphi + \frac{2}{3}\tau^2\varphi^5 - |\tilde{K}'|^2_{\gamma'_0}\varphi^{-7}, \tag{4.43}$$

where the solution map $\mathcal{L}'_1 : \mathcal{D}_{\gamma,\tau} \mapsto W^{2,p}_+$ satisfies $\Psi(\mathcal{L}'_1(K'), K') = 0$ for all $K' \in \mathcal{D}_{\gamma,\tau}$. The map (4.43) is easily seen to be a C^1-Frechét map between the above functional spaces, whose partial derivative on its first argument $D_1\Psi_{(\varphi,\tilde{K}')} :$ $W^{2,p}_+ \mapsto L^p$ at a point $(\varphi, \tilde{K}')$ is given by

$$D_1\Psi_{(\varphi,\tilde{K}')} \cdot v = -8\Delta_{\gamma'_0}v + \left(R_{\gamma'_0} + \frac{10}{3}\tau^2\varphi^4 + 7|\tilde{K}'|^2_{\gamma'_0}\varphi^{-8}\right)v$$

Therefore, at the point $(1, \tilde{K}'_0)$, we have

$$\Psi(1, \tilde{K}'_0) = 0,$$
$$D_1\Psi_{(1,\tilde{K}'_0)} \cdot v = -8\Delta_{\gamma'_0}v + \left(R_{\gamma'_0} + \frac{10}{3}\tau^2 + 7|\tilde{K}'_0|^2\right)v. \tag{4.44}$$

Using (4.41), we find that

$$D_1\Psi_{(1,\tilde{K}'_0)} \cdot v = -8\Delta_{\gamma'_0}v + \left(\frac{8}{3}\tau^2 + 8|\tilde{K}'_0|^2_{\gamma'_0}\right)v, \tag{4.45}$$

which is an isomorphism from $W^{2,p} \mapsto L^p$ since $\frac{8}{3}\tau^2 + 8|\tilde{K}'_0|^2_{\gamma'_0} \geq 0$ and $\frac{8}{3}\tau^2 + 8|\tilde{K}'_0|^2_{\gamma'_0} \not\equiv 0$. Therefore, the implicit function theorem (see, for instance, Abraham,

Marsden, and Ratiu (1988, Theorem 2.5.7)) implies that there are neighbourhoods $\mathcal{V} \subset W^{1,p}$ and $\mathcal{W} \subset W^{2,p}$ of $\tilde{K}'_0$ and 1 respectively, and a unique C^1-map $\Phi : \mathcal{V} \mapsto \mathcal{W}$ such that

$$\Psi(\Phi(\tilde{K}'), \tilde{K}') = 0, \quad \forall \tilde{K}' \in \mathcal{V}. \tag{4.46}$$

The uniqueness of the above map implies that $\Phi = \mathcal{L}'_1$, which proves that $\mathcal{L}'_1$ is C^1 in a neighbourhood of $\tilde{K}'_0$, implying then through (4.40) that $\mathcal{L}_1$ is C^1 in a neighbourhood of $\tilde{K}_0$ and finishing the proof. $\qquad\square$

Let us now establish the pre-compactness property of $\mathcal{U}$:

Proposition 4.2.7. *Under the hypotheses of Corollary 4.2.3, the subset $\overline{\mathcal{N}(\mathcal{U})} \subset L^\infty_+$ is compact.*

Proof. Let $\varphi \in \mathcal{U}$ be arbitrary. Then, using elliptic estimates associated to $-8\Delta_\gamma + R_\gamma$, we have that

$$\|\mathcal{N}(\varphi)\|_{W^{2,p}} \leqslant C\left(\left\|-\frac{2}{3}\tau^2\varphi^5 + |\mathcal{L}_{\gamma,\mathrm{conf}}X_\varphi + U|^2_\gamma\varphi^{-7}\right\|_{L^p} + \|\varphi\|_{L^p}\right),$$

$$\leqslant C\left(K_0^{-7}\left\||\mathcal{L}_{\gamma,\mathrm{conf}}X_\varphi + U|^2_\gamma\right\|_{L^p} + \frac{2}{3}\|\tau^2\|_{L^p}\|\varphi^5_+\|_{L^\infty} + \|\varphi_+\|_{L^p}\right),$$

where we have used that $K_0 \leqslant \varphi \leqslant \varphi_+$. From Sobolev multiplication, we know that

$$\left\||\mathcal{L}_{\gamma,\mathrm{conf}}X_\varphi + U|^2_\gamma\right\|_{L^p} \lesssim \|DX_\varphi\|^2_{L^p} + \|U\|^2_{L^p} \leqslant \|X_\varphi\|^2_{W^{2,p}} + \|U\|^2_{L^p},$$

$$\lesssim \|d\tau\|^2_{W^{2,p}}\|\varphi^6_+\|^2_{L^\infty} + \|U\|^2_{L^p},$$

where we have used the estimates $\|X_\varphi\|_{W^{2,p}} \leqslant C\|d\tau\varphi^6\|_{L^p}$ for a solution of the momentum constraint. Putting all of the above together, we see that $\mathcal{N}(U)$ is bounded in $W^{2,p}_+$. Since the inclusion $W^{2,p}_+ \hookrightarrow L^\infty_+$ is compact, it follows that $\overline{\mathcal{N}(U)}$ is compact in L^∞_+. $\qquad\square$

Solutions to the coupled system

We now have all the ingredients to apply Theorem 4.2.2 to the map $\mathcal{N} : \mathcal{U} \mapsto \mathcal{U}$.

Theorem 4.2.3. *(Maxwell 2009) Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$ and assume that γ has no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$ and suppose that one of the following conditions holds:*

1. *$\mathcal{Y}([\gamma]) > 0$ and $U \not\equiv 0$;*

2. *$\mathcal{Y}([\gamma]) \geqslant 0$, $U \not\equiv 0$ and $\tau \not\equiv 0$;*

3. *$\mathcal{Y}([\gamma]) < 0$ and there is a conformal deformation of γ to γ' with $R_{\gamma'} = \frac{2}{3}\tau^2$.*

If (4.13) admits a global supersolution φ_+, then the system admits a solution $(\varphi, X) \in W_+^{2,p} \times W^{2,p}$.

Proof. Under our conditions the data (γ, τ) are compatible and U is admissible. Thus, Corollary 4.2.3 implies that there is a constant $K_0 > 0$ such that the set $\mathcal{U} = \{\varphi \in L_+^\infty : K_0 \leqslant \varphi \leqslant \varphi_+\}$ satisfies $\mathcal{N} : \mathcal{U} \mapsto \mathcal{U}$. Then Proposition 4.2.6 implies that $\mathcal{N}$ is continuous acting on $\mathcal{U}$ (which is closed and convex) while Proposition 4.2.7 implies that $\overline{\mathcal{N}(U)}$ is compact in L^∞. We can then apply Theorem 4.2.2 to guarantee the existence of a fixed point $\varphi \in \mathcal{U}$. Since $\mathcal{N}(\mathcal{U}) \subset W_+^{2,p} \subset L_+^\infty$, then $\varphi = \mathcal{N}(\varphi) \in W^{2,p}$, and (by construction) $(\varphi, X_\varphi = \mathcal{L}_2(\varphi)) \in W^{2,p} \times W^{2,p}$ solve (4.13). $\square$

We have therefore reduced the task of finding solutions to (4.13) to that of proving existence of global supersolutions. The following Proposition, due to Holst, Nagy, and Tsogtgerel (2009), provides such a supersolution in the Yamabe positive case.

Proposition 4.2.8 (Supersolution $\mathcal{Y} > 0$). *Let (M^3, γ) be a closed Riemannian manifold with $\gamma \in W^{2,p}$, $p > 3$, with $\mathcal{Y}([\gamma]) > 0$, and assume that γ possesses no CKFs. Consider the system (4.13) with $\tau, U \in W^{1,p}$. If $\|U\|_{L^\infty}$ is sufficiently small, then there is a global supersolution to (4.13).*

Proof. From Theorem 2.2.3, we know that there is a conformal deformation $\gamma' = \theta^4 \gamma$, $\theta \in W_+^{2,p}$, such that $R_{\gamma'} > 0$ is continuous. Then, θ satisfies

$$-8\Delta_\gamma \theta + R_\gamma \theta = R_{\gamma'} \theta^5$$

The claim is now that $\varphi_+ \doteq \epsilon \theta$ is a global supersolution if ϵ is sufficiently small. To prove this consider $0 < \varphi \leqslant \varphi_+$, $\varphi \in L^\infty$, let $X_\varphi = \mathcal{L}_2(\varphi) \in W^{2,p}$ and

compute

$$-8\Delta_\gamma \varphi_+ + R_\gamma \varphi_+ + \frac{2}{3}\varphi_+^5 - |\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi + U|_\gamma^2 \varphi_+^{-7} = \epsilon R_{\gamma'}\theta^5 + \frac{2}{3}\tau^2\varphi_+^5$$
$$- |\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi + U|_\gamma^2 \varphi_+^{-7},$$
$$\geqslant \epsilon R_{\gamma'}\theta^5 - 2|U|_\gamma^2 \varphi_+^{-7}$$
$$- 2|\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi|_\gamma^2 \varphi_+^{-7}.$$

Let us now estimate the last term above as follows. Since $p > 3$, then $|\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi|_\gamma \lesssim |DX_\varphi| \in W^{1,p} \hookrightarrow C^0$. Therefore

$$|\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi| \lesssim \max_M |DX_\varphi| \lesssim \|DX_\varphi\|_{W^{1,p}} \leqslant \|X_\varphi\|_{W^{2,p}},$$
$$\leqslant C\|d\tau\|_{L^p}\|\varphi_+^6\|_{L^\infty} = \epsilon^6 C\|d\tau\|_{L^p} \max_M \theta^6,$$

for some constant $C > 0$ independent of $\varphi \leqslant \varphi_+$. Then,

$$\epsilon R_{\gamma'}\theta^5 - 2|U|_\gamma^2 \varphi_+^{-7} - 2|\mathcal{L}_{\gamma,\mathrm{conf}} X_\varphi|_\gamma^2 \varphi_+^{-7} \geqslant \epsilon \min(R_{\gamma'})\min(\theta^5)$$
$$- \epsilon^5 C^2\|d\tau\|_{L^p}^2 (\max\theta)^5 \quad (4.47)$$
$$- 2|U|_\gamma^2 \varphi_+^{-7}.$$

Now, since

$$\epsilon \min(R_{\gamma'})\min(\theta^5) - \epsilon^5 C^2\|d\tau\|_{L^p}^2 (\max\theta)^5 = \epsilon C^2\|d\tau\|_{L^p}^2 (\max\theta)^5 \times$$
$$\left(\frac{\min(R_{\gamma'})(\min(\theta))^5}{C^2\|d\tau\|_{L^p}^2 (\max\theta)^5} - \epsilon^4 \right),$$

let us pick ϵ small enough so that the right-hand side of the above expression is positive. Once such an ϵ is fixed, if $|U|_\gamma$ is small enough, then the right-hand side of (4.47) remains non-negative, and therefore φ_+ is global supersolution of our system. □

We therefore get the following existence result as a corollary of Theorem 4.2.3 and Proposition 4.2.4.

Corollary 4.2.4. *Under the conditions of Proposition 4.2.4 the system (4.13) admits a solution* $(\varphi, X) \in W_+^{2,p} \times W^{2,p}$.

We refer the reader to Holst, Nagy, and Tsogtgerel (2009) for a variety of constructions of global barriers that allows one to deal with other Yamabe classes, sometimes invoking a near-CMC assumption. Furthermore, although we have presented results addressing the vacuum case, the above methods translate nicely to different non-vacuum situations, such as scalar fields (Premoselli 2014; Vâlcu 2020) and Yorked-scaled sources which do not add further constraints (Holst, Nagy, and Tsogtgerel 2009). Also, they translate to boundary value problems (Holst, Meier, and Tsogtgerel 2018) as well as AE-manifolds (Dilts, Isenberg, et al. 2014; Holst and Meier 2014).[9] In the next section we will deal with a broad generalisation of many of these problems, including sources which are not York-scaled, black hole boundary conditions for AE-initial data, and further constraints arising from charged fluids.

Finally, let us highlight that above we have produced techniques which allow us to establish existence results, but, contrary to the CMC case, we do not have in general information about uniqueness, which is still an open important problem in mathematical general relativity. Furthermore, as was anticipated at the beginning of the section, in order to gain freedom on the mean curvature, we have imposed strong smallness conditions on the TT-part U. This seems to be a feature of the techniques presented above which will reappear in the next section. Therefore, the analysis of the constraint equations (4.13) is still an open problem for general conformal data.

4.3   Far-from-CMC solutions for charged fluids

The idea of this section is to extend the analysis presented for the coupled system of constraints in the previous one to more general situations, which can incorporate realistic matter fields as well as natural boundary conditions. Along these lines, as we have already described in Section 2.1 of Chapter 2, coupling charged matter will typically couple further constraints to the our systems. In particular, the case of a charged perfect fluid was discussed in detail and its conformal formulation

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[9]The kind of fixed point map may vary among the cited references, but the main ideas and techniques are similar.

was given by (2.35), which we explicitly recall below:

$$a_n \Delta_\gamma \phi = R_\gamma \phi - |\widetilde{K}(X)|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} + a_\tau \phi^{\frac{n+2}{n-2}} - |\widetilde{E}|_\gamma^2 \phi^{-3} - \frac{|\widetilde{F}|_\gamma^2}{2} \phi^{\frac{n-6}{n-2}},$$

$$\Delta_{\gamma,\mathrm{conf}} X = -\widetilde{E} \lrcorner \widetilde{F} + \omega_\tau \phi^{\frac{2n}{n-2}} + \omega_\mu \phi^{2\frac{n+1}{n-2}},$$

$$\mathrm{div}_\gamma \widetilde{E} = \tilde{q} \phi^{\frac{2n}{n-2}},$$

$$(4.48)$$

with

$$a_\tau \doteq b_n \tau^2 - 2\epsilon_0, \quad \epsilon_0 \doteq (\mu + p)\left(1 + |\tilde{u}|_\gamma^2\right) - p,$$

$$\omega_\tau \doteq \frac{n-1}{n} d\tau, \quad \omega_\mu \doteq \left(1 + |\tilde{u}|_\gamma^2\right)^{\frac{1}{2}} (\mu + p)\tilde{u}^\flat, \quad \tilde{q} \doteq q(1 + |\tilde{u}|_\gamma^2)^{\frac{1}{2}}$$

where we recall that μ, p and $\tilde{u}$ denote initial data for the energy density, pressure density and velocity field of the fluid respectively, while $\widetilde{F}$ stands for a closed 2-form which represents the initial data for the magnetic-part of the electromagnetic field. Finally, $\widetilde{E}$ stands for the initial data for the electric part of the electromagnetic field.

Comparing with (2.35), we should notice that we have neglected the contributions arising from the scalar field. We have chosen to do this, since the more general case can be dealt with along the same lines. Actually, the scalar field introduces no further coupling between the equations, which is our main interest during this section. Therefore, we prefer to omit this contribution in favour of a slightly cleaner presentation.

Along the lines of the above paragraph, let us highlight once more that the system (4.48) is by nature more subtle than (4.13) or (2.12), since, in particular, it cannot be decoupled without losing its defining properties. That is, as long as we intend to analyse a charged fluid, we cannot decouple the above three equations, since to achieve this we must set to zero q and ω_μ. The first of these conditions, neglects the charge of the (charged!) fluid, while the second one demands setting either $\tilde{u} = 0$ or $\mu = -p$. The first of these choices is puts the totality of the fluid at rest and is therefore a very special and non-generic situations, while the second one can be satisfied for instance by a "*cosmological constant case*", this is not the case of interest here, since this case is not charged by definition and, furthermore, equations of state for conventional (relativistic) matter or radiation do not satisfy this condition (see, for instance, Poisson and Will (2014)). Therefore, as long as

we want to analyse a realistic charged fluid, we are stuck from the go with the fully coupled system (4.48).

As an additional motivation, let us notice that the initial data for more general charged Yang-Mills fields interacting with gravity obey similar constraints to those described above (see, for instance, Choquet-Bruhat (1992) and Holm (1987)). Being these fields the building blocks of modern fundamental physics, we see that the understanding of systems of the above kind is extremely well-motivated both for its relevance within physics and its mathematical subtlety.

Boundary conditions

Along the lines of our analysis in Chapter 3, we intend to analyse the system (4.48) with black hole boundary data. Let us therefore recall the conformal formulation of the black hole boundary conditions (3.27), deduced in Section 3.3, to which we add a corresponding condition for the electric field as follows:

$$
\frac{1}{2}a_n\hat{v}(\phi) + H\phi - (\theta_- + b_n\tau)\,\phi^{\frac{n}{n-2}} + \widetilde{K}(\hat{v},\hat{v})\phi^{-\frac{n}{n-2}} = 0,
$$

$$
(\mathscr{L}_{\gamma,\mathrm{conf}}X(\hat{v},\cdot) = -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\hat{v} - U(\hat{v},\cdot),
$$

$$
\langle \widetilde{E}, \hat{v}\rangle_\gamma = E_{\hat{v}}, \tag{4.49}
$$

$$
v \geqslant \phi|_\Sigma,
$$

$$
\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0,
$$

where the functions $\theta_\pm$ stand for the *expansion coefficients*; $\hat{v}$ is the outward-pointing γ-unit normal; H stands for the mean curvature of the boundary $\Sigma = \partial M$ with respect to $-\hat{v}$ and we introduced an a priori arbitrary function $E_{\hat{v}}$, which stands for the prescribed value of the normal component of the electric field across Σ.[10] Below, we will see that this leads to a Neumann-type condition for the electric potential. All the definitions related to the dimensional coefficients can be consulted in Section 3.3.

Full constraint system

Our aim is to rewrite our PDE system (4.48)-(4.49) as an elliptic system such that we can construct solutions by iteration. In order to rewrite the electric constraint as

---

[10]From well-known theory of electromagnetism, prescribing the normal components of the electric field across the boundary is a natural condition (Jackson 1999).

a second order equation, we will apply a Helmholtz decomposition of the electric field so as to decompose it as the sum of an exact and co-exact 1-forms. In our main case of interest this is quite straightforward. In fact, let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold with compact boundary Σ, assume $p > \frac{n}{2}$, $n \geqslant 3$ and $\delta < 0$, and, for $\rho \in \mathbb{R}$, define

$$
\begin{aligned}
\nabla_\gamma^N : W_{\rho+1}^{2,p}(M) &\mapsto W_\rho^{1,p}(TM) \times W^{1-\frac{1}{p},p}(\Sigma), \\
\phi &\mapsto (\nabla_\gamma \phi, \nu(\phi)), \\
\mathcal{L} : W_{\rho+1}^{2,p}(M) &\mapsto L_{\rho-1}^p(TM) \times W^{1-\frac{1}{p},p}(\Sigma), \\
\phi &\mapsto (\Delta_\gamma \phi, \nu(\phi)).
\end{aligned}
\tag{4.50}
$$

where ν stands for the outward point γ-unit normal to Σ. Therefore, since $\Delta_\gamma = \mathrm{div}_\gamma \circ \nabla_\gamma : W_{\rho+1}^{2,p} \mapsto L_{\rho-1}^p$, we see that

$$
\mathcal{L} = \mathcal{L}_2 \circ \mathcal{L}_1,
$$

where $\mathcal{L}_1 = \nabla_\gamma^N : W_{\rho+1}^{2,p}(M) \mapsto W_\rho^{1,p}(TM) \times W^{1-\frac{1}{p},p}(\Sigma)$ and $\mathcal{L}_2 = (\mathrm{div}_\gamma, \mathrm{Id}) : W_\rho^{1,p}(TM) \times W^{1-\frac{1}{p},p}(\Sigma) \mapsto L_{\rho-1}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma)$. In this setting, we have the following decomposition.

Theorem 4.3.1 (Helmholtz decomposition). *Let (M^n, γ) be an n-dimensional $W_\delta^{2,p}$-AE, with $n \geqslant 3$, $p > \frac{n}{2}$ and $\delta < 0$. If $2 - n < \rho < 0$, then the following decomposition holds:*

$$
W_{\rho-1}^{1,p}(M; TM) \times W^{1-\frac{1}{p},p}(\Sigma) = \nabla_\gamma^N(W_\rho^{2,p}) \oplus \mathrm{Ker}(\mathcal{L}_2).
\tag{4.51}
$$

Proof. Appealing to Theorem A.1.1, we need to show that $\mathrm{Ker}(\mathcal{L}) = \mathrm{Ker}(\mathcal{L}_1)$ and $\mathrm{Im}(\mathcal{L}) = \mathrm{Im}(\mathcal{L}_2)$. Clearly, the inclusion $\mathrm{Ker}(\mathcal{L}_1) \subset \mathrm{Ker}(\mathcal{L})$ holds. To see the opposite, notice that under our hypotheses $\mathcal{L}$ is an isomorphism due to Theorem 3.3.1, which now clearly implies that $\mathrm{Ker}(\mathcal{L}) = \mathrm{Ker}(\mathcal{L}_1) = \emptyset$. Similarly, the inclusion $\mathrm{Im}(\mathcal{L}) \subset \mathrm{Im}(\mathcal{L}_2)$ is also trivial and the converse also follows from Theorem 3.3.1, since if $(f, h) \in \mathrm{Im}(\mathcal{L}_2) \subset L_{\rho-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma)$, then Theorem 3.3.1 implies that the there is some $u \in W_\rho^{2,p}$ such that $\mathcal{L}(u) = (f, h)$. Therefore the theorem holds. □

Remark 4.3.1. *The above theorem holds under weaker conditions to those stated in it, which in particular allow for the operators involved to have kernel. Nevertheless, in those cases the proof is more involved, although it appeals to strategies*

similar to those used in the proof of Theorem 3.3.1. In the case of AE-manifolds without boundary such a decomposition was addressed in Cantor (1981) *and in the case of manifolds with boundary in* Avalos and Lira (2019). *In these weaker scenarios there are some subtleties (even in the case $\Sigma = \emptyset$) when M has more than one end, which allows for harmonic functions in* $\mathrm{Ker}(\mathcal{L})$ *to be non-constant, which is something that can be observed from Equation* (3.60). *We refer the reader to* Avalos and Lira (ibid.) *for a discussion concerning these problems.*

Let us now apply the above theorem to the electric 1-form $E \in W_{\delta-1}^{1,p}$ in (4.48). Since we treat the case of (M^n, γ) being $W_\delta^{2,p}$-AE satisfying the hypotheses of Theorem 4.3.1, we obtain that

$$E = df + \vartheta \tag{4.52}$$

for some $f \in W_\delta^{2,p}$ and $\vartheta \in W_{\delta-1}^{1,p}$ such that $\mathrm{div}_\gamma \vartheta = 0$. In this scenario, we can rewrite the electric constraint as follows

$$\Delta_\gamma f - \tilde{q}\phi^{\frac{2n}{n-2}} = 0, \tag{4.53}$$

where we must solve for f. Taking this into account, we get that the PDE system (4.48) reads as a semi-linear second order PDE system for (ϕ, f, X), which is explicitly given by

$$a_n \Delta_\gamma \phi = R_\gamma \phi - |\tilde{K}(X)|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} + a_\tau \phi^{\frac{n+2}{n-2}} - |\tilde{E}|_\gamma^2 \phi^{-3} - \frac{|\tilde{F}|_\gamma^2}{2}\phi^{\frac{n-6}{n-2}},$$

$$\Delta_\gamma f = \tilde{q}\phi^{\frac{2n}{n-2}},$$

$$\Delta_{\gamma,\mathrm{conf}} X = -\tilde{E}\lrcorner\tilde{F} + \omega_\tau \varphi^{\frac{2n}{n-2}} + \omega_\mu \varphi^{2\frac{n+1}{n-2}}, \tag{4.54}$$

and is subject to the boundary conditions

$$\frac{1}{2}a_n \hat{v}(\phi) + H\phi - (\theta_- + b_n\tau)\phi^{\frac{n}{n-2}} - \left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\phi^{-\frac{n}{n-2}} = 0,$$

$$\hat{v}(f) = E_{\hat{v}},$$

$$\mathcal{L}_{\gamma,\mathrm{conf}} X(\hat{v}, \cdot) = -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\hat{v} - U(\hat{v}, \cdot), \tag{4.55}$$

where, in order for a solution to the above system to represent initial data with (marginally) trapped boundary conditions, we must impose $\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0$ a priori (which is something we control) and we need to guarantee that the solution satisfies the bound $v \geqslant \phi|_\Sigma$. Since v is a datum which must be fixed a priori, we need to show that this is an attainable *a priori bound* at least for those solutions which are meant to represent black hole initial data.

Along the same lines as in the analysis presented in Chapter 3, regarding the hamiltonian constraint, we are looking for bounded solutions with some prescribed asymptotic behaviour. Taking into account Lemma 3.3.5, we know that we can capture the behaviour at infinity of a solution of the Lichnerowicz equation by considering a harmonic function ω with zero Neumann boundary conditions, which is asymptotic to some positive values $\{A_j\}_{j=1}^N$ on each end $\{E_j\}_{j=1}^N$, and letting $\phi = \omega + \varphi$ with $\varphi \in W_\delta^{2,p}$. With this in mind, define the vector bundle $E \doteq (M \times \mathbb{R}) \oplus (M \times \mathbb{R}) \oplus TM$, and consider $W_\delta^{2,p}$-sections of this vector bundle. Then, we get the following differential operator

$$\mathcal{P} : W_\delta^{2,p}(M;E) \mapsto L_{\delta-2}^p(M;E) \times W^{1-\frac{1}{p},p}(\partial M;E),$$
$$(\varphi, f, X) \mapsto (\Delta_\gamma\varphi, \Delta_\gamma f, \Delta_{\gamma,\mathrm{conf}}X, -\nu(\varphi)|_{\partial M}, -\nu(f)|_{\partial M}, \mathscr{L}_{\gamma,\mathrm{conf}}X(\nu,\cdot)|_{\partial M})$$

Now, denote by $\mathbf{F}$ the map taking $(\phi, f, X) \to \mathbf{F}(\phi, f, X)$, where $\mathbf{F}(\phi, f, X)$ stands for the function appearing in the right hand side of (4.54)-(4.55). In this setting, we rewrite the above system more compactly as

$$\mathcal{P}(\psi) = \mathbf{F}(\psi), \tag{4.56}$$

where $\psi \in W_\delta^{2,p}(M;E)$. At this point the idea is to solve the above problem by solving a sequence of linear problems. In particular, given $\psi_0 \in W_\delta^{2,p}(M;E)$, if we get a unique solution for $\mathcal{P}(\psi) = \mathbf{F}(\psi_0)$, which is given by $\psi_1 = \mathcal{P}^{-1}\mathbf{F}(\psi_0)$, we can begin an iteration scheme, where we could now use ψ_1 as a source and solve the linear problem for this source and begin an iteration procedure. If we find a fixed point $\bar{\psi}$ in this iteration, then such fixed point solves

$$\mathcal{P}(\bar{\psi}) = \mathbf{F}(\bar{\psi}),$$

which is equivalent to solving the original system (4.54)-(4.55). If, furthermore, we get that $\phi > 0$, then such solution actually solves the conformal problem associated to a charged fluid. In order to satisfy this last condition, we will need to produce barriers ϕ_- and ϕ_+, and make sure that the iteration stays within $[\phi_-, \phi_+]_{C^0}$. This idea will be made precise in the upcoming sections.

Shifted system

Before going into the analysis of the constraint system, we should note that the analysis of (4.56) shall be done through the corresponding analysis of the following *shifted system*:

$$a_n \Delta_\gamma \varphi - a\varphi = R_\gamma \phi - |\widetilde{K}(X)|^2_\gamma \phi^{-\frac{3n-2}{n-2}} + a_\tau \phi^{\frac{n+2}{n-2}} - |\widetilde{E}|^2_\gamma \phi^{-3}$$

$$- \frac{|\widetilde{F}|^2_\gamma}{2} \phi^{\frac{n-6}{n-2}} - a\varphi$$

$$\Delta_\gamma f = \widetilde{q} \phi^{\frac{2n}{n-2}},$$

$$\Delta_{\gamma,\mathrm{conf}} X = \frac{n-1}{n} D\tau \phi^{\frac{2n}{n-2}} + \omega_1 \phi^{2\frac{n+1}{n-2}} - \omega_2,$$

(4.57)

with boundary conditions

$$\hat{v}(\varphi) - b\varphi = -a_n H\phi + (d_n \tau + d_n \theta_-)\phi^{\frac{n}{n-2}} + \left(\frac{1}{2}|\theta_-| - r_n \tau\right) v^{\frac{2n}{n-2}} \phi^{-\frac{n}{n-2}} - b\varphi,$$

$$\hat{v}(f) = E_{\hat{v}},$$

$$\mathcal{L}_{\gamma,\mathrm{conf}} X(\hat{v}, \cdot) = -\left(\left(\frac{1}{2}|\theta_-| - c_n \tau\right) v^{\frac{2n}{n-2}} + U(\hat{v}, \hat{v})\right) \hat{v},$$

(4.58)

with $a \in L^p_{\delta-2}(M)$, $b \in W^{1-\frac{1}{p},p}(\Sigma)$ satisfying $a, b \geqslant 0$ a.e, $\phi = \omega + \varphi$ and ω is a harmonic function with zero Neumann boundary conditions which captures the behaviour of ϕ at infinity (see Lemma 3.3.5). Following analogous conventions to those of (4.56), we will denote the linear operator appearing in the left-hand side by

$$\mathcal{P}_{a,b} : W^{2,p}_\delta \mapsto L^p_{\delta-2}(M, E) \times W^{1-\frac{1}{p},p}(\Sigma, E),$$

and $\mathbf{F}_{a,b}(\psi)$ by the right-hand side of (4.57)-(4.58). Furthermore, we will constraint the choices of θ_- and τ so as to satisfy the constraint (4.49), and we need to show that, given some $v \in W^{1-\frac{1}{p},p}(\Sigma)$, the solutions of the above boundary value problem satisfy $(v-\phi)|_\Sigma \geqslant 0$, so as to satisfy the marginally trapped surface condition. Then, we can rewrite the shifted system as

$$\mathcal{P}_{a,b}(\psi) = \mathbf{F}_{a,b}(\psi).$$

(4.59)

Notice that the operator $\mathcal{P}_{a,b}$ as defined above is invertible for $p > n$ and $2 - n < \delta < 0$, so that fixing some $\psi_0 \in W^{2,p}_\delta$, the sequence $\{\psi_k\}^\infty_{k=0} \subset W^{2,p}_\delta$ given inductively by $\psi_{k+1} \doteq \mathcal{P}^{-1}_{a,b}(\mathbf{F}_{a,b}(\psi_k))$ is well-defined. Furthermore, continuity of both $\mathcal{P}_{a,b}$ and $\mathbf{F}_{a,b}$ implies that, if we can extract a $W^{2,p}_\delta$-convergent subsequence with limit ψ, then this limit will solves $\mathcal{P}(\bar{\psi}) = \mathbf{F}(\bar{\psi})$. Now, since $(\Delta_\gamma \varphi, \hat{v}(\varphi)|_\Sigma) = (\Delta_\gamma \phi, \hat{v}(\phi)|_\Sigma)$, we see that such procedure provides us with a solution to the full constraint system with marginally trapped boundary conditions.

4.3.1   Existence results

Motivated by the discussion presented above, we intend to analyse the system (4.59). Let us first rewrite this system in the following form, which suggest possible generalisations that the reader can find available in Avalos and Lira (2019). Fixing the relevant vector bundle $E = (M \times \mathbb{R}) \oplus (M \times \mathbb{R}) \oplus TM$, we will write the corresponding sections as (ϕ, Y), with $Y = (f, X)$, and let us rewrite (4.59)

$$
\begin{aligned}
\Delta_\gamma \phi &= \sum_I a^0_I(Y)\phi^I, \\
L^i(Y^i) &= \sum_J a^i_J(Y)\phi^J, \quad i = 1, 2, \\
-\hat{v}(\phi) &= \sum_K b^0_K(Y)\phi^K, \text{ on } \Sigma \\
B^i(Y^i) &= \sum_L b^i_L(Y)\phi^L, \quad i = 1, 2 \ \Sigma,
\end{aligned}
\tag{4.60}
$$

where (L^i, B^i) represent continuous linear elliptic second order operators with boundary conditions, acting between $W^{2,p}_\delta(M) \mapsto L^p_{\delta-2}(M) \times W^{1-\frac{1}{p},p}(\Sigma)$, which are invertible for appropriate choices of $p > n$ and $\delta < 0$. In order to prove an existence theorem based on the ideas described in the previous section, we will need a couple of properties concerning the coefficients of the above system, and, to establish these properties, we will appeal to the following results.

Proposition 4.3.1. *The embedding $W^{1,p}_\delta(\mathbb{R}^n) \hookrightarrow C^0_{\delta'}(\mathbb{R}^n)$ is compact for any $\delta < \delta'$ and $p > n$.*

Proof. Consider the ball $\overline{B_{2R}} \subset \mathbb{R}^n$ which is a compact manifold with smooth boundary. Then, consider a cut off function χ_R, which is equal to one on $\bar{B}_R$,

equal to zero on $\mathbb{R}^n \setminus \overline{B_{2R}}$ and $0 \leqslant \chi_R \leqslant 1$. Then, let $\{u_k\}_{k=1}^{\infty} \subset W_\delta^{1,p}$ be a bounded sequence, *i.e*, $u_k \leqslant 1$ for all k, and split $u_k = \chi_R u_k + (1 - \chi_R) u_k$. Thus, since $\chi_R u_k$ is a bounded sequence supported in $\overline{B_{2R}}$, then $\{\chi_R u_n\} \subset W^{1,p}(\overline{B_{2R}})$. Now consider

$$
\begin{aligned}
\|u_n - u_m\|_{C_{\delta'}^0} &= \sup_{\mathbb{R}^n} |u_n - u_m|(1 + |x|^2)^{-\frac{\delta'}{2}} \\
&\leqslant \sup_{\overline{B_{2R}}} |\chi_R(u_n - u_m)|(1 + |x|^2)^{-\frac{\delta'}{2}} \\
&\quad + \sup_{\mathbb{R}^n \setminus \overline{B_R}} |(1 - \chi_R)(u_n - u_m)|(1 + |x|^2)^{-\frac{\delta'}{2}}, \\
&\leqslant (1 + 4R^2)^{-\frac{\delta'}{2}} \sup_{\overline{B_{2R}}} |\chi_R u_n - \chi_R u_m| \\
&\quad + \sup_{\mathbb{R}^n \setminus \overline{B_R}} |(1 - \chi_R)(u_n - u_m)|(1 + |x|^2)^{-\frac{\delta}{2} + \frac{\delta - \delta'}{2}}.
\end{aligned}
$$

Now, since $\delta - \delta' < 0$ and $|1 - \chi_R| \leqslant 1$, we get

$$
\begin{aligned}
\|u_n - u_m\|_{C_{\delta'}^0} &\leqslant (1 + 4R^2)^{-\frac{\delta'}{2}} \sup_{\overline{B_{2R}}} |\chi_R u_n - \chi_R u_m| \\
&\quad + (1 + R^2)^{\frac{\delta - \delta'}{2}} \sup_{\mathbb{R}^n \setminus \overline{B_R}} |(1 - \chi_R)(u_n - u_m)|(1 + |x|^2)^{-\frac{\delta}{2}}, \\
&\leqslant (1 + 4R^2)^{-\frac{\delta'}{2}} \sup_{\overline{B_{2R}}} |\chi_R u_n - \chi_R u_m| \\
&\quad + (1 + R^2)^{\frac{\delta - \delta'}{2}} \sup_{\mathbb{R}^n} |u_n - u_m|(1 + |x|^2)^{-\frac{\delta}{2}},
\end{aligned}
$$

Also, under our hypotheses, we have a continuous embedding $W_\delta^{1,p} \hookrightarrow C_\delta^0$, which implies that there is a fixed constant $C > 0$ such that

$$
\begin{aligned}
\|u_n - u_m\|_{C_{\delta'}^0} &\leqslant (1 + 4R^2)^{-\frac{\delta'}{2}} \|\chi_R u_n - \chi_R u_m\|_{C^0(\overline{B_{2R}})} \\
&\quad + C(1 + R^2)^{\frac{\delta - \delta'}{2}} \|u_n - u_m\|_{W_\delta^{1,p}(\mathbb{R}^n)}, \\
&\leqslant (1 + 4R^2)^{-\frac{\delta'}{2}} \|\chi_R u_n - \chi_R u_m\|_{C^0(\overline{B_{2R}})} + 2C(1 + R^2)^{\frac{\delta - \delta'}{2}}.
\end{aligned}
$$

Now, fixing $\epsilon > 0$, since $\delta - \delta' < 0$, there is a radius R_ϵ sufficiently large such that $(1 + R^2)^{\frac{\delta-\delta'}{2}} < \frac{\epsilon}{4C}$. Once we have fixed such an $R_\epsilon > 0$, since for $p > n$ $W^{1,p}(\overline{B_{2R}})$ is compactly embedded in $C^0(\overline{B_{2R}})$, there is a subsequence which is convergent in $C^0(\overline{B_{2R}})$, which we will still denote in the same way. After restricting to a such subsequence, $\{\chi_{R_\epsilon} u_n\} \subset W^{1,p}(\overline{B_{2R_\epsilon}})$ is Cauchy in $C^0(\overline{B_{2R_\epsilon}})$, which implies that there is an $N = N(\epsilon)$, such that $\forall\, n, m \geqslant N$ it holds that $\|\chi_R u_n - \chi_R u_m\|_{C^0(\overline{B_{2R_\epsilon}})} < \frac{\epsilon}{2}(1 + 4R_\epsilon^2)^{\frac{\delta'}{2}}$. Thus, we get

$$\|u_n - u_m\|_{C^0_{\delta'}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving that this subsequence is Cauchy in $C^0_{\delta'}$, and therefore convergent. $\qquad\square$

From this, we get the following corollary.

Corollary 4.3.1. *The embedding $W^{2,p}_\delta(\mathbb{R}^n) \hookrightarrow C^1_{\delta'}(\mathbb{R}^n)$ is compact for any $\delta < \delta'$ and $p > n$.*

Proof. Let $\{u_k\}_{k=1}^\infty \subset W^{2,p}_\delta$ be a bounded sequence and recall

$$\|u_k\|_{C^1_{\delta'}} = \|u_k\|_{C^0_{\delta'}} + \|\partial u_k\|_{C^0_{\delta'-1}}.$$

From the above proposition, we know that u_k admits a $C^0_{\delta'}$-convergent subsequence, to which we restrict. But then, $\{\partial u_k\}_{k=1}^\infty \subset W^{1,p}_{\delta-1}$ satisfies $p > n$ and $\delta - 1 < \delta' - 1$, and therefore using again the above proposition, we can extract one further subsequence so that $\{\partial u_k\}_{k=1}^\infty$ converges in $C^0_{\delta'-1}$. This last subsequence is therefore convergent in $C^1_{\delta'}$, and therefore the claim follows. $\qquad\square$

In order to extend the above compact embeddings to AE-manifolds, we just need to appeal to an argument using cut-offs.

Lemma 4.3.1. *Let (M^n, γ) be an AE-manifold, possibly with smooth compact boundary Σ. Then, $W^{2,p}_\delta(M) \hookrightarrow C^1_{\delta'}(M)$ is compact for $p > n$ and $\delta < \delta' < 0$.*

Proof. Consider a bounded sequence $\{u_k\}_{k=1}^\infty \subset W^{2,p}_\delta$, $p > n$ and $\delta < 0$, take a compact set K with smooth boundary given by spheres sufficiently far away in the ends of M, then pick a cut-off function $0 \leqslant \eta \leqslant 1$ equal to one in the compact core of M (where Σ is contained) and zero outside of K and write

$u_k = \eta u_k + (1 - \eta)u_k$. On K the norms $C^1_{\delta'}(K)$ and $C^1(K)$ are equivalent and the same is true for $W^{1,p}_{\delta'}(K)$ and $W^{1,p}(K)$. Also, we have a compact embedding $W^{2,p}(K) \hookrightarrow C^1(K)$, and therefore $\{\eta u_k\}^\infty_{k=1}$ admits a $C^1_{\delta'}(K)$-convergent subsequence $\{\eta u_{k_j}\}^\infty_{j=1}$, to which we now restrict. But then, due to Corollary 4.3.1, $\{(1 - \eta)u_{k_j}\}^\infty_{j=1} \subset W^{2,p}_\delta(\mathbb{R}^n)$ admits one further $C^1_{\delta'}(\mathbb{R}^n)$-convergent subsequence to which we restrict and denote in the same way. Therefore, for i, j, large enough

$$\|u_{k_i} - u_{k_j}\|_{C^1_{\delta'}(M)} \lesssim \|\eta(u_{k_i} - u_{k_j})\|_{C^1_{\delta'}(K)} + \|(1 - \eta)(u_{k_i} - u_{k_j})\|_{C^1_{\delta'}(\mathbb{R}^n)} \to 0,$$

proving that $\{u_{k_j}\}^\infty_{j=1} \subset W^{2,p}_\delta(M)$ has a $C^1_{\delta'}(M)$ convergent subsequence, and therefore $W^{2,p}_\delta(M) \hookrightarrow C^1_{\delta'}(M)$ is also compact in this case. $\qquad\square$

Finally, using the Sobolev embedding $W^{1,p}_\delta \hookrightarrow C^0_\delta$, valid for $p > n$, we see that for $u \in W^{1,p}_\delta$ it holds that

$$|u| = \sigma^\delta |u| \sigma^{-\delta} \leq \sigma^\delta \|u\|_{C^0_\delta} \lesssim \sigma^\delta \|u\|_{W^{1,p}_\delta}.$$

Notice that if r is a smooth function which in the ends, sufficiently near infinity, agrees with the euclidean radial function $|x|$, then there are constants C_1 and C_2, such that $C_1 r(x) \leq \sigma(x) \leq C_2 r(x)$. Thus, using the above relations, we get the following.

Proposition 4.3.2. *Let (M^n, γ) be a $W^{2,p}_\delta$-AE manifold, $n \geq 3$, $p > n$ and $\delta < 0$. Consider a function r on M which near infinity, in each end, agrees with $|x|$. Then, there is a constant $C > 0$ such that*

$$|u|(x) \leq C r^\delta(x) \|u\|_{W^{1,p}_\delta}, \quad \forall u \in W^{1,p}_\delta. \tag{4.61}$$

We can now extract the following properties concerning the coefficients of the system (4.54)-(4.55)

Lemma 4.3.2. *Let (M^n, γ) be a $W^{2,p}_\delta$-AE manifold with $p > n$, $n \geq 3$, $\delta < 0$ and consider the system (4.54)-(4.55). Suppose that the prescribed data for the problem satisfies the functional hypotheses $\mu, p, \tilde{u}, \tilde{q} \in W^{1,p}_{\delta-2}(M)$, $U, \tau, \vartheta, \tilde{F} \in W^{1,p}_{\delta-1}(M)$, $H, \theta_-, E_{\hat{v}} \in W^{1-\frac{1}{p},p}(\Sigma)$ and $v \in W^{2-\frac{1}{p},p}(\Sigma)$. Let $M_Y = \sum_i M_{Y^i}$,*

where $B_{M_{Y^i}} \subset W_{\delta}^{2,p}(M; E_i)$, $i = 1, 2$, *denotes the closed ball of radius* $M_{Y^i} >$ 0, *then there are functions* $f_I \in L_{\delta-2}^p(M)$, $g_K \in W^{1-\frac{1}{p},p}(\Sigma)$ *and constants* $C_K > 0$, *independent of* $Y = (f, X)$, *such that*

$$|a_I^0(Y)| \leqslant f_I \text{ for any } Y \in B_{M_Y},$$
$$|b_K^0(Y)| \leqslant g_K \text{ for any } Y \in B_{M_Y}, \tag{4.62}$$
$$\|b_K^0(Y)\|_{W^{1-\frac{1}{p},p}} \leqslant C_K \text{ for any } Y \in B_{M_Y},$$

Proof. Notice that if μ, p, $\tilde{u} \in W_{\delta-2}^{1,p}$, then so is ϵ_0. Also, under our hypotheses on δ and p we get that $|\tilde{F}|_\gamma^2 \in W_{\delta-2}^{1,p}$ from the multiplication property and the same holds for $\tau^2 \in W_{\delta-2}^{1,p}$. Clearly, we also have that $R_\gamma \in L_{\delta-2}^p$, and none of these coefficients depends on f or X. Now, let us consider the coefficients $\tilde{K}$ and $|\tilde{E}|_\gamma^2$. Notice that

$$|\tilde{K}|_\gamma^2 = |\mathscr{L}_{\gamma,\mathrm{conf}} X|_\gamma^2 + 2\langle U, \mathscr{L}_{\gamma,\mathrm{conf}} X\rangle_\gamma + |U|_\gamma^2,$$
$$|\tilde{E}|_\gamma^2 = |df|_\gamma^2 + 2\langle \vartheta, df\rangle_\gamma + |\vartheta|_\gamma^2.$$

The multiplication property implies that both these coefficients are in $W_{\delta-2}^{1,p}$. Notice that (4.62) is clear for the coefficients a_τ, $|\tilde{F}|_\gamma^2$ and R_γ which all belong to $L_{\delta-2}^p$ and are independent of Y. Also, according to Proposition 4.3.2, a function $u \in W_{\delta-1}^{1,p}$ satisfies $|u| \lesssim r^{\delta-1}\|u\|_{W_{\delta-1}^{1,p}}$. Then, in the case of $|\tilde{K}|_\gamma^2$, we have the following estimate:

$$|\tilde{K}(Y)|_\gamma^2 = |\mathscr{L}_{\gamma,\mathrm{conf}} X|_\gamma^2 + 2\langle U, \mathscr{L}_{\gamma,\mathrm{conf}} X\rangle_\gamma + |U|_\gamma^2 \lesssim 2|DX|_\gamma^2 + 2|U|_\gamma^2,$$
$$\lesssim r^{2(\delta-1)}(\|DX\|_{W_{\delta-1}^{1,p}}^2 + \|U\|_{W_{\delta-1}^{1,p}}^2),$$
$$\lesssim (M_Y^2 + \|U\|_{W_{\delta-2}^{1,p}}^2)r^{\delta-2}r^\delta \quad \forall \ Y \in B_{M_Y}.$$

Then, since under our hypotheses $r^{\delta-2}r^\delta \in L_{\delta-2}^p$, we see that $|\tilde{K}|_\gamma^2$ satisfies (4.62). A similar result holds for $|\tilde{E}|_\gamma^2$, implying that they both satisfy (4.62). Now, let us examine this property for the boundary coefficients. First, notice that none of these coefficients depend on f or X. Also, from our choices of functional spaces, we know that U and τ have $W^{1-\frac{1}{p},p}$-traces on Σ and θ_-, $H \in W^{1-\frac{1}{p},p}$. All this together implies that $b_J^0(Y) \in W^{1-\frac{1}{p},p}$ for all J, and these coefficients are actually independent of Y. Therefore, (4.62) holds. $\square$

Also, the following property will prove to be very useful in our analysis.

Lemma 4.3.3. *Let us consider the system (4.54)-(4.55) under the same assumptions as in Lemma 4.3.2. Given a bounded $W^{2,p}_\delta$-sequence $\{Y_k\}^\infty_{k=1}$ and $\delta < \delta' < 0$, if $Y_k \xrightarrow{C^1_{\delta'}} Y$, then it holds that*

$$
\begin{aligned}
a^\alpha_I(Y_k) &\xrightarrow[k\to\infty]{L^p_{\delta-2}} a^\alpha_I(Y), \ \alpha = 0, 1, 2 \\
b^\alpha_J(Y_k) &\xrightarrow[k\to\infty]{W^{1-\frac{1}{p},p}} b^\alpha_J(Y), \ \alpha = 0, 1, 2.
\end{aligned}
\tag{4.63}
$$

Proof. Considering a bounded sequence $\{Y_k\}^\infty_{k=1} \subset W^{2,p}_\delta$, such that $Y_k \xrightarrow{C^1_{\delta'}} Y$, we get that

$$
\begin{aligned}
\left| |\tilde{K}(Y)|^2_\gamma - |\tilde{K}(Y_k)|^2_\gamma \right| &\leqslant |\langle \mathscr{L}_{\gamma,\mathrm{conf}}X, \mathscr{L}_{\gamma,\mathrm{conf}}X - \mathscr{L}_{\gamma,\mathrm{conf}}X_k\rangle_\gamma| \\
&\quad + |\langle \mathscr{L}_{\gamma,\mathrm{conf}}X_k, \mathscr{L}_{\gamma,\mathrm{conf}}X - \mathscr{L}_{\gamma,\mathrm{conf}}X_k\rangle_\gamma| \\
&\quad + 2|\langle U, \mathscr{L}_{\gamma,\mathrm{conf}}X - \mathscr{L}_{\gamma,\mathrm{conf}}X_k\rangle_\gamma|, \\
&\lesssim |DX|_\gamma|D(X - X_k)|_\gamma + |DX_k|_\gamma|D(X - X_k)|_\gamma \\
&\quad + |U|_\gamma|D(X - X_k)|_\gamma, \\
&\lesssim r^{\delta-1}r^{\delta'-1}\left(\|DX\|_{W^{1,p}_{\delta-1}} + \|DX_k\|_{W^{1,p}_{\delta-1}} + \|U\|_{W^{1,p}_{\delta-1}} \right)\cdot \\
&\quad \|D(X - X_k)\|_{C^0_{\delta'-1}},
\end{aligned}
$$

thus, since $\delta' < 0$, then $r^{\delta-2}r^{\delta'} \in L^p_{\delta-2}$, which, since $\{X_k\} \subset W^{2,p}_\delta$ is supposed to be bounded, implies that

$$
\begin{aligned}
\left\| |\tilde{K}(Y)|^2_\gamma - |\tilde{K}(Y_k)|^2_\gamma \right\|_{L^p_{\delta-2}} &\lesssim \left(\|DX\|_{W^{1,p}_{\delta-1}} + \|DX_k\|_{W^{1,p}_{\delta-1}} + \|U\|_{W^{1,p}_{\delta-1}} \right)\cdot \\
&\quad \|X - X_k\|_{C^1_{\delta'}} \to 0.
\end{aligned}
$$

The same line of reasoning proves the analogous statement for $|\tilde{E}|^2_\gamma$, and the coefficients which are independent of Y trivially satisfy this property. We also need to analyse the coefficient $\tilde{E}\lrcorner\tilde{F} = \tilde{F}(Df, \cdot) + \tilde{F}(\vartheta, \cdot)$. Clearly under our hypotheses $\tilde{E}\lrcorner\tilde{F} \in L^p_{\delta-2}$ and also

$$
|\tilde{E}\lrcorner\tilde{F}(Y) - \tilde{E}\lrcorner\tilde{F}(Y_k)|_\gamma \lesssim r^{\delta-2}r^{\delta'}\|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|Df - Df_k\|_{C^0_{\delta'-1}}.
$$

Thus, using again the fact that $r^{\delta-2}r^{\delta'} \in L^p_{\delta-2}$, we get that

$$\|\tilde{E} \lrcorner \tilde{F}(Y) - \tilde{E} \lrcorner \tilde{F}(Y_k)\|_{L^p_{\delta-2}} \lesssim \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|Df - Df_k\|_{C^0_{\delta'-1}} \to 0.$$

Finally, concerning the b^α_j-coefficients associated with the boundary conditions, since we have already noticed that none of these coefficients depend on f or X, we conclude that (4.63) holds for them as well, which establishes the lemma. $\quad\square$

Before presenting the main existence theorem, let us explain the relevance of the above Lemmas in our discussion. First, let us situate ourselves in the discussion presented in the previous section concerning the sequence of solutions $\{\psi_k\}^\infty_{k=1} \subset W^{2,p}_\delta$ associated to shifted system (4.59). The objective of Lemma 4.3.2 is to provide monotonicity properties associated to the corresponding Lichnerowicz equation which allow us to trap the sequence of conformal factors $\{\phi_k\}^\infty_{k=1}$ between a priori (suitable) barriers. On the other hand, following the lines of the traditional monotone iteration scheme, the idea will be to first show that the sequence is bounded (appealing to elliptic estimates) and the extract a convergent subsequence. Lemma 4.3.3 is tailored to help us in this task. That is, once we know that our sequence of solutions is actually bounded, we can use Lemma 4.3.1 to obtain a $C^1_{\delta'}$-convergent subsequence. It is then that Lemma 4.3.3 becomes useful when put together with elliptic estimates to recover $W^{2,p}_\delta$-convergence. The one piece that is still missing here is an appropriate way to trap the sequence of conformal factors. Notice that if we were to use barriers such as those of Chapter 3, since the coefficients of the equation change at each step of the iteration procedure, then the barriers would also typically change. A similar issue was encountered in Section 4.2, where we introduced the stronger notion of global barriers. Inspired in this idea, let us introduce the following notion of barriers.

Definition 4.3.1. *Consider the Lichnerowicz equation associated to the conformal problem for the Einstein constraint equations, and let us write it as follows.*

$$\begin{aligned}
\Delta_\gamma \phi &= \sum_I a_I(Y)\phi^I, \\
-\hat{v}(\phi) &= \sum_J b_J(Y)\phi^J \text{ on } \Sigma
\end{aligned} \tag{4.64}$$

where $\gamma \in W^p_{2,\delta}$; $\hat{v}$ is the outward pointing unit normal with respect to γ, $Y = (f, X)$, and "I" and "J" denote the exponents which define the non-linearities

*of the Lichnerowicz equation. We will say that ϕ_- is a **strong global subsolution** if there are positive numbers M_f and M_X such that*

$$\Delta_\gamma \phi_- \geqslant \sum_I a_I(Y)\phi_-^I \quad \forall \ Y \in \times_i B_{M_{Y_i}}$$

$$-\hat{v}(\phi_-) \geqslant \sum_J b_J(Y)\phi_-^J \ on \ \Sigma \ \forall \ Y \in \times_i B_{M_{Y_i}} \tag{4.65}$$

*where B_{M_f}, B_{M_X} denote the closed balls in $W_\delta^{2,p}$ of radius M_f and M_X respectively. A **strong global supersolution** is defined in the same way with the opposite inequality. Also, if the same set of numbers M_f and M_X serve for both the sub and supersolution, and $0 < \phi_- \leqslant \phi_+$ we will say the the barriers are **compatible**.*

Before presenting the following existence theorem, let us fix some notation. In Lemma 3.3.5 we established that given a $W_\delta^{2,p}$-AE manifold we can always find a harmonic function, say ω, which tends to some fixed constants, say $\{A_j\}_{j=1}^N$, on each end of the manifold $\{E_j\}_{j=1}^N$ and satisfies homogeneous Neumann boundary conditions. Furthermore, we can use such harmonic functions in order to capture the behaviour of our barriers at infinity. That is, we will say that $\phi_\pm$ is asymptotic to $\omega_\pm$ if $\phi_\pm - \omega_\pm \in W_\delta^{2,p}$, which implies that $\phi_\pm \xrightarrow{E_j} A_j^\pm$. With this in mind, when a function $\phi \in W_{loc}^{2,p}$ is asymptotic to such a harmonic function ω we write it as $\phi = \omega + \varphi$, with $\varphi \in W_\delta^{2,p}$, where ω is capturing the behaviour at infinity of ϕ.

Theorem 4.3.2. *Let (M^n, γ) be a $W_\delta^{2,p}$-AE manifold, with $p > n$, $n \geqslant 3$ and $2 - n < \delta < 0$, and consider the system (4.54)-(4.55) on M. Assume that the Lichnerowicz equation admits a compatible pair of strong global sub and supersolutions given by ϕ_- and ϕ_+, which are, respectively, asymptotic to harmonic functions $\omega_\pm$ tending to positive constants $\{A_j^\pm\}_{j=1}^N$ on each end $\{E_j\}_{j=1}^N$. Fix a harmonic function ω asymptotic to constants $\{A_j\}_{j=1}^N$ on each end satisfying $0 < A_j^- \leqslant A_j \leqslant A_j^+$, and suppose that the solution map*

$$\mathcal{F}_{a,b} : W_{2,\delta}^p(M; E) \mapsto W_{2,\delta}^p(M; E),$$

$$\psi = (\varphi, Y) \mapsto \mathcal{F}_{a,b}(\psi) \doteq \mathcal{P}_{a,b}^{-1} \circ \boldsymbol{F}_{a,b}(\psi).$$

is actually invariant on the set $B_{M_f} \times B_{M_X} \subset W_\delta^{2,p} \times W_\delta^{2,p}$ given in the definition of the barriers ϕ_-, ϕ_+ for any $\phi_- \leqslant \varphi + \omega \leqslant \phi_+$. Then, the system admits a

solution ($\phi = \omega + \varphi$, f, X), with (φ, f, X) $\in W_\delta^{2,p}(M; E)$, and, furthermore, $\phi > 0$.

Proof. First, consider the shifted system associated with (4.60), where we will pick the shift functions $a \in L_{\delta-2}^p$ and $b \in W^{1-\frac{1}{p},p}$ below. We have our strong global sub and supersolutions fixed together with the balls $B_{M_{Yi}} \subset W_\delta^{2,p}(M, E_i)$. From our hypotheses on the barriers $\phi_\pm$, we know that these are bounded functions, which implies that there are positive numbers, say $l \leq m$, such that $l \leq \phi_- \leq \phi_+ \leq m$. Thus, consider $Y \in \times_i B_{M_{Yi}}$, and define the functions

$$h_Y^a(\phi) \doteq h_Y(\phi) - a(\phi - \omega) = \sum_I a_I^0(Y)\phi^I - a(\phi - \omega),$$

$$g_Y^b(\phi) \doteq g_Y(\phi) - b(\phi - \omega) = \sum_K b_K^0(Y)\phi^K - b(\phi - \omega) \text{ on } \Sigma.$$

We want to pick the functions a and b such that both $h_Y(y)$, $g_Y(y)$ are decreasing functions on $y \in [l, m]$, for all $Y \in \times_i B_{M_{Yi}}$. Appealing to Lemma 4.3.2, notice that

$$\frac{\partial}{\partial y} \sum_I a_I^0(Y)y^I \leq \sum_I |I| |a_I^0(Y)| y^{I-1}$$

$$\leq \sum_I |I| f_I y^{I-1} \leq \sum_I |I| \sup_{l \leq y \leq m} y^{I-1} \, f_I \in L_{\delta-2}^p, \, \forall \, Y \in B_{M_Y}$$

From similar arguments, we get that

$$\frac{\partial}{\partial y} \sum_K b_K^0(Y)y^K \leq \sup_{l \leq y \leq m} \sum_K |K| |b_K^0(Y)| y^{K-1} \leq \sup_{l \leq y \leq m} \sum_K |K| g_K y^{K-1},$$

$$\leq \sum_K |K| \sup_{l \leq y \leq m} y^{K-1} \, g_K \in W^{1-\frac{1}{p},p}, \, \forall \, Y \in B_{M_Y},$$

where we have again used the boundedness property (4.62). Thus, if we pick $a \in L_{\delta-2}^p(M)$ and $b \in W^{1-\frac{1}{p},p}(\Sigma)$ satisfying

$$a > \sum_I |I| \sup_{l \leq y \leq m} y^{I-1} \, f_I \; ; \; b > \sum_K |K| \sup_{l \leq y \leq m} y^{K-1} \, g_K, \qquad (4.66)$$

we get that

$$\frac{\partial}{\partial y} h_Y^a(y) \leqslant 0, \text{ and } \frac{\partial}{\partial y} g_Y^b(y) \leqslant 0, \text{ for all } (y, Y) \in [l, m] \times_i B_{M_{Y_i}},$$

implying that $h_Y^a(\phi)$ and $h_Y^b(\phi)$ are decreasing functions on the interval $[\phi_-, \phi_+]_{C^0}$.

Now, consider $\psi_0 = (\phi_0, Y_0)$ with $\phi_0 = \phi_-$ and $Y_0 \in \times_i B_{M_{Y_i}}$, and consider the sequence $\{\psi_k = (\phi_k = \omega + \varphi_k, Y_k)\}_{k=1}^{\infty}$ defined by an iteration procedure of the form:

$$\mathcal{P}_{a,b} \psi_k = \mathbf{F}_{a,b}(\psi_{k-1}).$$

From the linear properties associated with the operator $\mathcal{P}_{a,b}$ we know that the sequence is well-defined, since for each step $\mathbf{F}_{a,b}(\psi_{k-1}) \in L_{\delta+2}^p$. Furthermore, from our hypotheses, we know that $Y_k \in \times_i B_{M_{Y_i}}$ for all k as long as we guarantee that $\phi_k = \omega + \varphi_k$ stays in the interval $[\phi_-, \phi_+]_{C^0}$. We can prove this last statement inductively along the lines of the proof of Theorem 3.3.4. First consider ϕ_1, which satisfies

$$\Delta_\gamma(\phi_1 - \phi_-) - a((\phi_1 - \omega) - (\phi_- - \omega)) = h_{Y_0}^a(\phi_-) - \Delta_\gamma \phi_- + a(\phi_- - \omega)$$
$$-(\hat{v}(\phi_1 - \phi_-) + b(\phi_1 - \omega - (\phi_- - \omega)) = g_{Y_0}^b(\phi_-) + \hat{v}(\phi_-) + b(\phi_- - \omega).$$

Then, since $h_{Y_0}^a(\phi_0) = h_{Y_0}(\phi_-) - a(\phi_- - \omega)$ and $g_{Y_0}^b(\phi_-) = g_{Y_0}(\phi_-) - b(\phi_- - \omega)$, we find

$$\Delta_\gamma(\phi_1 - \phi_-) - a((\phi_1 - \omega) - (\phi_- - \omega)) = -(\Delta_\gamma \phi_- - h_{Y_0}(\phi_-)) \leqslant 0$$
$$-(\hat{v}(\phi_1 - \phi_-) + b(\phi_1 - \omega - (\phi_- - \omega)) = -(-\hat{v}(\phi_-) - g_{Y_0}(\phi_-)) \leqslant 0,$$

where the final inequalities is a consequence of ϕ_- being a strong global subsolution. Notice that ϕ_- is in $W_{loc}^{2,p}$ and is asymptotic to ω_-, which itself tends to positive constants $\{A_j^-\}_{j=1}^N$ in each end $\{E_j\}_{j=1}^N$ respectively, and, by construction, ϕ_1 is asymptotic to ω which tends to positive constants $\{A_j\}_{j=1}^N$ in each end, satisfying $A_j > A_j^-$. Thus, we get that $\phi_1 - \phi_- \to A_j - A_j^- > 0$ in each end E_j. Then, we can apply the weak maximum principle given in Lemma 3.3.1, and conclude that $\phi_1 \geqslant \phi_-$. Similarly, we have that

$$\Delta_\gamma(\phi_+ - \phi_1) - a((\phi_+ - \omega) - (\phi_1 - \omega)) = \Delta_\gamma \phi_+ - a(\phi_+ - \omega) - h_{Y_0}^a(\phi_-),$$
$$- \hat{v}(\phi_+ - \phi_1) - b((\phi_+ - \omega) - (\phi_1 - \omega)) = -\hat{v}(\phi_+) - b(\phi_+ - \omega) - g_{Y_0}^b(\phi_-).$$

We can rearrange the right-hand side in the above expression as

$$\Delta_\gamma \phi_+ - a(\phi_+ - \omega) - h^a_{Y_0}(\phi_-) = \Delta_\gamma \phi_+ - h_{Y_0}(\phi_+) + h_{Y_0}(\phi_+) - a(\phi_+ - \omega)$$
$$- h^a_{Y_0}(\phi_-),$$
$$= \Delta_\gamma \phi_+ - h_{Y_0}(\phi_+) + h^a_{Y_0}(\phi_+) - h^a_{Y_0}(\phi_-) \leqslant 0,$$

and similarly,

$$-\hat{v}(\phi_+) - b(\phi_+ - \omega) - g^b_{Y_0}(\phi_-) = -\hat{v}(\phi_+) - g_{Y_0}(\phi_+) + g_{Y_0}(\phi_+) - b(\phi_+ - \omega)$$
$$- g^b_{Y_0}(\phi_-),$$
$$= -\hat{v}(\phi_+) - g_{Y_0}(\phi_+) + g^b_{Y_0}(\phi_+) - g^b_{Y_0}(\phi_-) \leqslant 0$$

where the final inequalities follows from ϕ_+ being a strong global supersolution, and also $h^a_Y(\cdot)$ and $g^b_Y(\cdot)$ being a decreasing functions of $\phi \in [\phi_-, \phi_+]_{C^0}$ for any $Y \in \times_i B_{M_{Yi}}$. All this implies

$$\Delta_\gamma(\phi_+ - \phi_1) - a((\phi_+ - \omega) - (\phi_1 - \omega)) \leqslant 0,$$
$$- \hat{v}(\phi_+ - \phi_1) - b((\phi_+ - \omega) - (\phi_1 - \omega)) \leqslant 0$$

Then, noticing again that ϕ_+ is asymptotic to ω_+, which tends to constants $\{A_j^+\}_{j=1}^N$ in each end, satisfying $A_j^+ > A_j$, we get that $\phi_+ - \phi_1 \xrightarrow{E_j} A_j^+ - A_j > 0$, which implies through the maximum principle that $\phi_+ \geqslant \phi_1$.

Now, suppose that $\phi_- \leqslant \phi_k = \omega + \varphi_k \leqslant \phi_+$ and $Y_k \in \times_i B_{M_{Yi}}$, and then consider

$$\Delta_\gamma(\phi_{k+1} - \phi_-) - a((\phi_{k+1} - \omega) - (\phi_- - \omega)) = -\Big(\Delta_\gamma \phi_- - h_{Y_k}(\phi_-)$$
$$+ h^a_{Y_k}(\phi_-) - h^a_{Y_k}(\phi_k)\Big) \leqslant 0$$
$$-\hat{v}(\phi_{k+1} - \phi_-) - b((\phi_{k+1} - \omega) - (\phi_- - \omega)) = -\Big(- \hat{v}(\phi_-) - g_{Y_k}(\phi_-)$$
$$+ g^b_{Y_k}(\phi_-) - g^b_{Y_k}(\phi_k)\Big) \leqslant 0$$

where the last inequality holds since ϕ_- is, by hypothesis, a strong global subsolution. Furthermore, h^a_Y and g^b_Y are decreasing functions of $\phi \in [\phi_-, \phi_+]_{C^0}$ for any $Y \in B_{M_f} \times B_{M_X}$, and because of the inductive hypothesis $\phi_- \leqslant \phi_k \leqslant \phi_+$

and $Y_k \in B_{M_f} \times B_{M_X}$. Thus, since $\phi_{k+1} - \phi_- \xrightarrow{E_j} A_j - A_j^- > 0$, from the maximum principle we get $\phi_{k+1} \geqslant \phi_-$. Similarly,

$$\Delta_\gamma(\phi_+ - \phi_{k+1}) - a((\phi_+ - \omega) - (\phi_{k+1} - \omega)) = \Delta_\gamma \phi_+ - h_{Y_k}(\phi_+)$$
$$+ h_{Y_k}^a(\phi_+) - h_{Y_k}^a(\phi_k) \leqslant 0$$

$$-\hat{v}(\phi_+ - \phi_{k+1}) - b((\phi_+ - \omega) - (\phi_{k+1} - \omega)) = -\hat{v}(\phi_+) - g_{Y_k}(\phi_+)$$
$$+ g_{Y_k}^b(\phi_+) - g_{Y_k}^b(\phi_k) \leqslant 0$$

where the last inequality holds because ϕ_+ is a strong global supersolution; $h_Y^a(\cdot)$ and $g_Y^b(\cdot)$ are decreasing function of $\phi \in [\phi_-, \phi_+]_{C^0}$ for all $Y \in \times_i B_{M_{Y_i}}$ and the inductive hypothesis. Then, the maximum principle implies that $\phi_+ \geqslant \phi_{k+1}$. All this implies that $\phi_- \leqslant \phi_{k+1} \leqslant \phi_+$, which finishes the inductive proof. Hence we have produced the sequence of solutions $\{(\varphi_k, Y_k)\}_{k=0}^\infty \subset W_{2,\delta}^p(M; E)$, where we know that $Y_k \in \times_i B_{M_{Y_i}}$ and that $\phi_k = \varphi_k + \omega \in [\phi_-, \phi_+]_{C^0}$ for all k. Notice that this implies that

$$\|\varphi_k\|_{W_\delta^{2,p}} \lesssim \sum_I \|a_I^0(Y_{k-1})\|_{L_{\delta-2}^p} \|\phi_{k-1}^I\|_{C^0} + \|a\|_{L_{\delta-2}^p} \|\phi_{k-1}\|_{C^0}$$

$$+ \sum_K \|b_K^0(Y_{k-1})\phi_{k-1}^K\|_{W^{1-\frac{1}{p},p}} + \|b\phi_{k-1}\|_{W^{1-\frac{1}{p},p}},$$

$$\lesssim \sum_K \|b_K^0(Y_{k-1})\phi_{k-1}^K\|_{W^{1-\frac{1}{p},p}} + \|b\phi_{k-1}\|_{W^{1-\frac{1}{p},p}}$$

$$+ \sum_I \|f_I\|_{L_{\delta-2}^p} \|\phi_\pm^I\|_{C^0} + \|a\|_{L_{\delta-2}^p} \|\phi_+\|_{C^0},$$

We proceed to estimate the boundary terms in a similar (somewhat simpler) manner to Theorem 3.3.4. Since $b_K^0(Y_{k-1}) \in W^{1-\frac{1}{p},p}(\Sigma)$ for any k and any K, we know that there are (non-unique) extensions $\tilde{b}_K^0(Y_{k-1}) \in W^{1,p}(U)$, where U is some smooth neighbourhood of Σ with compact closure, and we also know that both the extension operator and the trace map are continuous. Therefore, $\|b_K^0(Y_{k-1})\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim \|\tilde{b}_K^0(Y_{k-1})\|_{W^{1,p}(M)}$, and, clearly, $\tilde{b}_K^0(Y_{k-1})\phi^K \in W^{1,p}(U)$ is an extension of $b_K^0(Y_{k-1})\phi^K|_\Sigma$. Therefore, using the fact that $W^{1,p}(U)$ is an algebra under multiplication for $p > n$, we see that

$$\|b_K^0(Y_{k-1})\phi_{k-1}^K\|_{W^{1-\frac{1}{p},p}(\Sigma)} \lesssim \|\tilde{b}_K^0(Y_{k-1})\phi_{k-1}^K\|_{W^{1,p}(U)},$$

$$\lesssim \|\tilde{b}_K^0(Y_{k-1})\|_{W^{1,p}(U)} \|\phi_{k-1}^K\|_{W^{1,p}(U)}.$$

Also,

$$\|\phi_{k-1}^K\|_{W^{1,p}(U)} \lesssim \|\phi_{k-1}^K\|_{L^p(U)} + \|\nabla\phi_{k-1}^K\|_{L^p(U)},$$
$$\lesssim \|\phi_{\pm}^K\|_{C^0} + \|\phi_{\pm}^{K-1}\|_{C^0}\|\nabla\phi_{k-1}\|_{L^p(U)},$$
$$\lesssim 1 + \|\phi_{k-1}\|_{W^{1,p}(U)} \lesssim 1 + \|\omega\|_{W^{1,p}(U)} + \|\varphi_{k-1}\|_{W^{1,p}(U)},$$

where in the second line $\phi_{\pm}$ stands for the subsolution is the exponents are negative are the supersolution if they are non-negative, and the implicit constant depends on the barriers, but not on k. We can now proceed just us in Theorem 3.3.4 and appeal to interpolation inequalities in the above estimates, and then go back to $\|\varphi_k\|_{W_{\delta}^{2,p}}$ to obtain

$$\|\varphi_k\|_{W_{\delta}^{2,p}} \leq \frac{1}{2}\|\varphi_{k-1}\|_{W_{\delta}^{2,p}} + C, \tag{4.67}$$

where C is a fixed constant that only depends on the parameters of the problem. That is, it depends on the barriers, the functions f_I, the shift functions a and b, the constants C_K and the choice of ϵ in the interpolation inequalities, but not on k. Similarly to Theorem 3.3.4, we can iterate the above procedure to get that

$$\|\varphi_k\|_{W_{\delta}^{2,p}} \leq \|\varphi_-\|_{W_{\delta}^{2,p}} + 2C \quad \forall \ k. \tag{4.68}$$

The above estimate implies that there is a constant $M_\varphi > 0$, depending on the barriers $\phi_\pm$, the norms of the functions f_I and the constants C_K, such that $\{\varphi_k\}_{k=0}^{\infty} \subset B_{M_\varphi} \subset W_{\delta}^{2,p}$, where B_{M_φ} stands for the closed ball of radius M_φ. This implies that the sequence $\{(\varphi_k, Y_k)\}_{k=0}^{\infty} \subset B_{M_\varphi} \times B_{M_f} \times B_{M_X}$. Then, since the embedding $W_{\delta}^{2,p} \hookrightarrow C_{\delta'}^1$ is compact for any $\delta < \delta' < 0$ (see Lemma 4.3.1), we get that, up to restricting to a subsequence,

$$(\varphi_k, Y_k) \xrightarrow[k\to\infty]{C_{\delta'}^1} (\varphi, Y).$$

Finally, the aim is to show that $\{(\varphi_k, Y_k)\}_{k=0}^{\infty}$ is Cauchy in $W_{\delta}^{2,p}$. This is done appealing to the estimates associated to the linear parts of the shifted system in very much the same way as in Theorem 2.2.1, but now incorporating the boundary terms. The main difference is now that the coefficients of the system also depend on the previous step of the sequence. This can be circumvented appealing to Lemma 4.3.3, which guarantees the $L_{\delta-2}^p$-convergence of the coefficients $a_I^\alpha(Y_k) \to a_I^\alpha(Y)$.

Since the details of this procedure are quite direct although lengthy, we leave them to the reader, who can also consult them in Avalos and Lira (2019).

☐

From the above theorem we see that we have once more reduced our task to constructing (strong) global barriers for the constraint system, and to proving invariance of the solution map $\mathcal{F}_{a,b}(\varphi, \cdot, \cdot)$ on the balls $B_{M_f} \times B_{M_Y}$ for any $\varphi \in [\varphi_-, \varphi_+]_{C^0}$. Let us do this construction in the Yamabe positive case, which is the one that allows for far-from-CMC results.

A priori estimates on the electromagnetic constraint

Going back to the constraint system, within the kind of iteration scheme described above, we need to get a global fixed estimates for solutions of linear equations of the form

$$\begin{aligned} \Delta_\gamma f &= \tilde{q} \bar{\phi}^{\frac{2n}{n-2}}, \\ -\hat{v}(f) &= E_{\hat{v}} \text{ on } \Sigma, \end{aligned} \tag{4.69}$$

where, in the right-hand side, the functions $\bar{\phi} = \omega + \varphi$, $\varphi \in W_\delta^{2,p}(M)$, $\tilde{q} \in L_{\delta-2}^p(M)$ and $E_{\hat{v}} \in W^{1-\frac{1}{p},p}(\Sigma)$, with $p > n$ and $2 - n < \delta < 0$, are considered as given data. Thus, the right-hand side is in $L_{\delta-2}^p(M) \times W^{1-\frac{1}{p},p}(\Sigma)$. With all this settled, we notice that any $W_\delta^{2,p}$-solution of (4.69) satisfies the the following a priori elliptic estimate.

$$\|f\|_{W_\delta^{2,p}} \leqslant C \left(\|\tilde{q}\|_{L_{\delta-2}^p} \|\bar{\phi}^{\frac{2n}{n-2}}\|_{C^0} + \|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}} \right). \tag{4.70}$$

In particular, notice that in the above estimate we can get rid of the dependence on the specific $\bar{\phi}$ by admitting the existence of global barriers for the hamiltonian constraint, more specifically, by admitting the existence of a global supersolution.

A priori estimates on the momentum constraint

Similarly to what we did above, we want to get uniform estimates on the sequence of solutions generated by the momentum constraint. Notice that the momentum

constraint read as follows

$$\Delta_{\gamma,\text{conf}}X = -\tilde{E}\lrcorner\tilde{F} + \omega_\tau\varphi^{\frac{2n}{n-2}} + \omega_\mu\varphi^{2\frac{n+1}{n-2}},$$

$$\mathscr{L}_{\gamma,\text{conf}}X(\hat{v},\cdot) = -\left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\hat{v} - U(\hat{v},\cdot),$$

(4.71)

where $\omega_\mu = (\mu + p)\left(1 + |\tilde{u}|^2_\gamma\right)^{\frac{1}{2}}\tilde{u}^\flat$. Let us assume that $\tau, U \in W^{1,p}_{\delta-1}$ and that γ being a $W^p_{2,\delta}$-AE metric are given, where, as above, $p > n$ and $2 - n < \delta < 0$, and consider that $\phi = \omega + \varphi$, with $\varphi \in W^{2,p}_\delta$, and finally that $f \in W^{2,p}_\delta$, $\vartheta \in W^{1,p}_{\delta-1}$, $\theta_- \in W^{1-\frac{1}{p},p}$ and $v \in W^{2-\frac{1}{p},p}$. Also, suppose that $\mu, p \in L^p_{\delta-2}$ and $\tilde{F} \in W^{1,p}_{\delta-1}$, which implies that $\tilde{F} \otimes \tilde{E} \in W^{1,p}_{\delta-2}$, since $\tilde{E} = df + \vartheta \in W^{1,p}_{\delta-1}$. Then, since the right-hand side of the above equation is in $L^p_{\delta-2} \times W^{1-\frac{1}{p},p}$ we can associate a unique solution to it, say $X_{\phi,f} \in W^{2,p}_\delta$, and we can then estimate the $W^{2,p}_\delta$-norm of $X_{\phi,f}$ in terms of ϕ, f and the free data. That is,

$$\|X_{\phi,f}\|_{W^{2,p}_\delta} \lesssim \|d\tau\|_{L^p_{\delta-2}}\|\phi^{\frac{2n}{n-2}}\|_{C^0} + \|\omega_\mu\|_{L^p_{\delta-2}}\|\phi^{2\frac{n+1}{n-2}}\|_{C^0} + \|\tilde{E}_f\lrcorner\tilde{F}\|_{L^p_{\delta-2}},$$

$$+ \|\frac{1}{2}|\theta_-| - b_n\tau\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}},$$

where we have used that $W^{2,p}_\delta \hookrightarrow C^0$. Notice that

$$\|\tilde{E}_f\lrcorner\tilde{F}\|_{L^p_{\delta-2}} \lesssim \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|df\|_{W^{1,p}_{\delta-1}} + \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|\vartheta\|_{W^{1,p}_{\delta-1}}$$

which explicitly gives us that

$$\|X_{\phi,f}\|_{W^{2,p}_\delta} \leqslant \kappa\bigg\{\|d\tau\|_{L^p_{\delta-2}}\|\phi^{\frac{2n}{n-2}}\|_{C^0} + \|\omega_\mu\|_{L^p_{\delta-2}}\|\phi^{2\frac{n+1}{n-2}}\|_{C^0} +$$

$$+ \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|df\|_{W^{1,p}_{\delta-1}} + \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\|\vartheta\|_{W^{1,p}_{\delta-1}} +$$

$$+ \|\frac{1}{2}|\theta_-| - b_n\tau\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}}\bigg\}.$$

(4.72)

In case we have a pair of compatible strong global barriers $\phi_- \leqslant \phi_+$, then, there are radii $M_f, M_Y \subset W^{2,p}_\delta$ such that Definition 4.3.1 works for the Lichnerowicz equation for any $f \in B_{M_f}$ and any $X \in B_{M_X}$. In such a case, notice

that for any $f \in B_{M_f}$ and any $\phi_- \leqslant \phi \leqslant \phi_+$, we get that

$$
\|X_{\phi,f}\|_{W_\delta^{2,p}} \lesssim \|d\tau\|_{L_{\delta-2}^p}\|\phi^{\frac{2n}{n-2}}\|_{C^0} + \|\omega_\mu\|_{L_{\delta-2}^p}\|\phi^{2\frac{n+1}{n-2}}\|_{C^0} +
$$
$$
+ \|\widetilde{F}\|_{W_{\delta-1}^{1,p}} M_f + \|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|\vartheta\|_{W_{\delta-1}^{1,p}} +
$$
$$
+ \|\tfrac{1}{2}|\theta_-| - b_n\tau\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}},
$$

Regarding the momentum constraint, we we will need one further estimate. Suppose that $X_{\phi,f}$ is a solution of the momentum constraint for source functions (ϕ, f). Since $p > n$ and $X \in W_\delta^{2,p}$, we can appeal to Proposition 4.3.2 to estimate $|DX_{\phi,f}|_\gamma \lesssim r^{\delta-1}\|X_{\phi,f}\|_{W_\delta^{2,p}}$, which implies that

$$
|\mathcal{L}_{\gamma,\mathrm{conf}}X_{\phi,f}|_\gamma \lesssim r^{\delta-1}\Big\{\|d\tau\|_{L_{\delta-2}^p}\|\phi^{\frac{2n}{n-2}}\|_{C^0} + \|\omega_\mu\|_{L_{\delta-2}^p}\|\phi^{2\frac{n+1}{n-2}}\|_{C^0}
$$
$$
+ \|\widetilde{q}\|_{L_{\delta-2}^p}\|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|\phi^{\frac{2n}{n-2}}\|_{C^0} + \|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}}\|\widetilde{F}\|_{W_{\delta-1}^{1,p}}
$$
$$
+ \|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|\vartheta\|_{W_{\delta-1}^{1,p}} + \|\tfrac{1}{2}|\theta_-| - b_n\tau\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0}
$$
$$
+ \|U\|_{W^{1-\frac{1}{p},p}}\Big\},
$$

$$(4.73)$$

where we have assumed that the function f (appearing in the right-hand side of the momentum constraint through the electric field $\widetilde{E}$) is a solution of the electric constraint for the same fixed source $\phi \in W_\delta^{2,p}$.

Barriers for the Lichnerowicz equation

Let us now consider the Lichnerowicz equation associated to system (4.54)-(4.55), explicitly given by:

$$
a_n\Delta_\gamma\phi - R_\gamma\phi + |\widetilde{K}|_\gamma^2\phi^{-\frac{3n-2}{n-2}} - \big(b_n\tau^2 - 2\epsilon_0\big)\phi^{\frac{n+2}{n-2}} + |\widetilde{E}|_\gamma^2\phi^{-3} + \frac{|\widetilde{F}|_\gamma^2}{2}\phi^{\frac{n-6}{n-2}} = 0,
$$
$$
\tfrac{1}{2}a_n\hat{v}(\phi) + H\phi - \big(\theta_- + b_n\tau\big)\phi^{\frac{n}{n-2}} - \Big(\tfrac{1}{2}|\theta_-| - b_n\tau\Big)v^{\frac{2n}{n-2}}\phi^{-\frac{n}{n-2}} = 0, \text{ on } \Sigma
$$

$$(4.74)$$

where we fix the assumptions that (M^n, γ) is a $W^{2,p}_\delta$-AE manifold with $n \geqslant 3$, $p > n, 2 - n < \delta < 0$ as well as

$$
\begin{gathered}
\tau, U, \vartheta, \widetilde{F} \in W^{1,p}_{\delta-1}, \quad \mu, p \in W^{1,p}_{2(\delta-1)}, \quad \widetilde{u} \in W^{1,p}_{\delta-1}, \\
\theta_- \in W^{1-\frac{1}{p},p}, \quad v \in W^{2-\frac{1}{p},p}.
\end{gathered}
\tag{4.75}
$$

where **we will assume** $\epsilon_0 \geqslant 0$, which is physically reasonable.

According to the analysis made in the previous sections, we need to show the following two properties:

- Equation (4.74) admits a compatible pair of strong global sub and supersolutions $0 < \phi_- < \phi_+$, where $\phi_\pm = \omega_\pm + \varphi_\pm$ with $\varphi_\pm \in W^p_{2,\delta}$ and $\omega_\pm$ are given harmonic functions, satisfying Neumann boundary conditions, asymptotic to positive constants $\{A^\pm_j\}^N_{j=1}$ in each end $\{E_j\}^N_{j=1}$ respectively, and these barriers work for any $f \in B_{M_f}$ and $X \in B_{M_X}$ for suitable $M_f, M_Y > 0$.

- The solution map $\mathcal{F}_{a,b} : [\phi_-, \phi_+]_{C^0} \times B_{M_f} \times B_{M_X} \mapsto W^p_{2,\delta}(M; E)$ associated to the shifted system must be invariant on $B_{M_f} \times B_{M_X}$ for any $\phi \in [\phi_-, \phi_+]_{C^0}$.

If the above two properties hold, then, the iteration scheme should work for the complete system.

As usual, the construction of the barriers will strongly depend on the Yamabe class of γ. Appealing to results of Chapter 3, in particular Proposition 3.4.2, the Yamabe positive case allows for a far-from-CMC construction, which we will describe below. Our construction is taken from Avalos and Lira (2019) and arises as a modification of Holst and Meier 2014 to our present context. Let us highlight that the marginally trapped condition required that the data θ_- and τ satisfy $\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0$ and recall that $\theta_- \leqslant 0$. Then, notice that

$$
|\theta_-| - b_n\tau \geqslant \frac{1}{2}|\theta_-| - b_n\tau \geqslant 0,
\tag{4.76}
$$

under these black hole boundary conditions.

Lemma 4.3.4. *Let (M, γ) be a $W^{2,p}_\delta$-Yamabe positive AE-manifold, with $p > n$ and $2 - n < \delta < 0$. Consider the system (4.54)-(4.55) and let τ be an arbitrary*

datum. If $\left(\frac{1}{2}|\theta_-| - b_n\tau\right) \geqslant 0$ along Σ and $v > 0$, then, under smallness assumptions on the remaining coefficients of the system,[11] the Lichnerowicz equation associated to the coupled Einstein–Maxwell system for a charged perfect fluid admits compatible strong global barriers $0 < \phi_- < \phi_+$, such that $\phi_\pm - \omega_\pm \in W^{2,p}_\delta$ for some harmonic functions $\omega_\pm$ which tend to positive constants $\{A^\pm_j\}^N_{j=1}$ on each end. Furthermore, the solution map associated to the shifted system is invariant on balls $B_{M_f}, B_{M_X} \subset W^p_{2,\delta}$ where the strong global barriers work.

Proof. Since the construction of these barriers is quite long and delicate, we will present the main ideas behind it a refer the reader to Avalos and Lira (2019) for the complete details.[12] The basic idea goes as follows. First, appealing to the Yamabe positive condition and to Proposition 3.4.2 we can start assuming that $R_\gamma \equiv 0$ on M and $H \equiv 0$ along Σ. Then, we will present a 1-parameter family of $W^{2,p}_{loc}$-strong global subsolutions $\phi_- = \alpha(\varphi_- + \omega) > 0$, $\alpha \in (0, \alpha_{max})$, where these subsolutions work for any $f, X \in W^{2,p}_\delta$. That is, there are no restrictions of the radii M_f, M_X imposed by this family. After this, we will present a 1-parameter family of *candidate* $W^{2,p}_{loc}$-strong supersolutions of the form $\phi_+ = \beta(\varphi_+ + \omega) > 0$. To make these candidate barriers compatible with ϕ_-, we fix the choice α to satisfy

$$0 < \alpha < \min\left\{\alpha_{max}, \beta \min_M \frac{\omega + \varphi_+}{\omega + \varphi_-}\right\}, \tag{4.77}$$

which is only fixed after choosing β. Also, we fix the choice of (β-dependent) radii

$$
\begin{aligned}
M_f &= C\left\{\|\tilde{q}\|_{L^p_{\delta-2}}\|\phi_+^{\frac{2n}{n-2}}\|_{C^0} + \|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}}\right\}, \\
M_X &= \kappa\left\{\|d\tau\|_{L^p_{\delta-2}}\|\phi_+^{\frac{2n}{n-2}}\|_{C^0} + \|\omega_\mu\|_{L^p_{\delta-2}}\|\phi_+^{2\frac{n+1}{n-2}}\|_{C^0} + \right. \\
&\quad + \|\tilde{F}\|_{W^{1,p}_{\delta-1}}\left(M_f + \|\vartheta\|_{W^{1,p}_{\delta-1}}\right) + \\
&\quad \left. + \left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}}\right\}
\end{aligned}
\tag{4.78}
$$

---

[11] The smallness conditions can be made to fall with different strengths on each coefficient. For instance, it is enough to assume that $\mu, p, \tilde{q}, \tilde{F}, v$ and U are small enough on their corresponding functional spaces.

[12] We caution the interested reader that in Avalos and Lira (2019) the conventions of the dimensional coefficients, as well as the weight parameters are slightly different.

where C and κ are the constant appearing in the estimates (4.70) and (4.72) respectively, and consider the balls B_{M_f} and B_{M_X} in $W_{2,\delta}^p$. Thus, once β is fixed, the radii (4.78) as well as α are fixed. Notice that with these choices for the balls B_{M_f} and B_{M_X}, given $\bar{f} \in B_{M_f}$, $\bar{X} \in B_{M_X}$ and $\bar{\phi} \leqslant \phi_+$, it follows from the a priori elliptic estimates (4.70)-(4.72) that the solution map $\mathcal{F}_{a,b}$ associated to the shifted system (4.57) taking $(\bar{\phi}, \bar{f}, \bar{X}) \mapsto (\phi, f, X) = \mathcal{F}_{a,b}(\bar{\phi}, \bar{f}, \bar{X})$ satisfies that $f \in B_{M_f}$ and $X \in B_{M_X}$, implying the desired invariance property. Therefore, in this scenario, all that is left to do is to chose β appropriately so that ϕ_+ is a strong global supersolution for these choices of balls B_{M_f} and B_{M_X}, and then follow the choices (4.77) and (4.78) for the subsolution and the radii.

Let us begin by considering the existence of a strong global subsolution. We need to satisfy[13]

$$\mathcal{H}^1_{f,X}(\phi_-) \doteq a_n \Delta_\gamma \phi_- - \left(b_n \tau^2 - 2\epsilon_0\right) \phi_-^{\frac{n+2}{n-2}} + |\tilde{K}(X)|_\gamma^2 \phi_-^{-\frac{3n-2}{n-2}} + \epsilon_2(f)\phi_-^{-3}$$
$$+ \epsilon_3 \phi_-^{\frac{n-6}{n-2}} \geqslant 0,$$

$$\mathcal{H}^2_{f,X}(\phi_-) \doteq -\frac{a_n}{2}\hat{v}(\phi_-) + (b_n\tau + \theta_-)\phi_-^{\frac{n}{n-2}} + \left(\frac{1}{2}|\theta_-| - b_n\tau\right) v^{\frac{2n}{n-2}}\phi_-^{-\frac{n}{n-2}} \geqslant$$
$$\geqslant 0 \text{ on } \Sigma,$$

for all $f \in B_{M_f}$ and $X \in B_{M_X}$, where we have denoted by $\epsilon_2(f) \doteq |\tilde{E}(f)|_\gamma^2$ and by $\epsilon_3 \doteq \frac{|\tilde{F}|_\gamma^2}{2}$. Since $\tau \in W_{\delta-1}^{1,p}$, we get that $\tau^2 \in L_{\delta-2}^p$, also, since $(|\theta_-| - b_n\tau) \geqslant 0$, we can define $\varphi_- \in W_\delta^{2,p}$ as the unique solution to

$$a_n\Delta_\gamma\varphi_- - b_n\tau^2\varphi_- = b_n\omega\tau^2,$$
$$-\frac{a_n}{2}\hat{v}(\varphi_-) - (|\theta_-| - b_n\tau)\varphi_- = \omega(|\theta_-| - b_n\tau), \tag{4.79}$$

and ω is a harmonic function with homogeneous Neumann boundary conditions, asymptotic to positive constants $\{A_j\}_{j=1}^N$ on each end. Then, define $\phi_- \doteq \alpha(\omega + \varphi_-)$, where $\alpha > 0$ is a constant to be fixed, and notice that

$$a_n\Delta_\gamma\phi_- - b_n\tau^2\phi_- = 0,$$
$$-a_n\hat{v}(\phi_-) - (|\theta_-| - b_n\tau)\phi_- = 0. \tag{4.80}$$

---

[13]Recall we started assuming $R_\gamma \equiv 0$ and $H_\gamma \equiv 0$ due to the Yamabe positive condition.

Then, the weak maximum principle given in Lemma 3.3.1, we known that $\phi_- \geqslant 0$, and then the strong maximum principle, given in Lemma 3.3.2, guarantees that $\phi_- > 0$ since $\phi_- \to \alpha A_j > 0$ in each end E_j. Now, consider

$$\mathcal{H}^1_{f,X}(\phi_-) = b_n \tau^2 \left(\phi_- - \phi^{\frac{n+2}{n-2}}_- \right) + \text{ non-negative terms,}$$

$$\mathcal{H}^2_{f,X}(\phi_-) = (|\theta_-| - b_n \tau) \left(\phi_- - \phi^{\frac{n}{n-2}}_- \right) + \text{ non-negative terms}$$

Since ω and φ_- are independent of α, we can pick α sufficiently small so that the highlighted terms in the above expressions are non-negative. Such choice of $\alpha > 0$ guarantees that $\mathcal{H}^{1,2}_{f,X}(\phi_-) \geqslant 0 \ \forall \ f, X \in W^p_{2,\delta}$, proving that ϕ_- is a strong global subsolution.

Now, let us consider the supersolution. In this case, we need to find ϕ_+ satisfying

$$\mathcal{H}^{1,2}_{f,X}(\phi_+) \leqslant 0, \ \ \forall \ f \in B_{M_f}, X \in B_{M_X},$$

for our choice of radii M_f, M_X of (4.78). With this in mind, let $\Lambda \in L^p_{\delta-2}$ be a positive function which agrees with $r^\delta r^{\delta-2}$ in a neighbourhood of infinity, in each end, and $\lambda \in W^{1-\frac{1}{p},p}(\Sigma)$ a positive function on the boundary. Define $\varphi_+ \in W^{2,p}_\delta$ as the unique solution to

$$\begin{aligned} a_n \Delta_\gamma \varphi_+ &= -\Lambda, \\ \frac{1}{2} a_n \hat{v}(\varphi_+) &= \lambda \text{ on } \Sigma, \end{aligned} \tag{4.81}$$

and then define $\phi_+ \doteq \beta(\omega + \varphi_+)$, where β is some positive constant to be determined. Since then it holds that

$$\begin{aligned} a_n \Delta_\gamma \phi_+ &= -\beta\Lambda \leqslant 0, \\ -\frac{a_n}{2} \hat{v}(\phi_+) &= -\beta\lambda \leqslant 0 \text{ on } \Sigma, \end{aligned} \tag{4.82}$$

appealing to the maximum principles given in Lemmas 3.3.1 to 3.3.2, we find $\phi_+ > 0$. Being aware of the existence of strong global subsolutions of the form $\phi_- = \alpha(\omega + \varphi_-)$ whenever α is sufficiently small, we us fix the relation between α and β obeying (4.77) so that $\phi_- < \phi_+$. Now, consider the following

$$\mathcal{H}^1_{f,X}(\phi_+) \leqslant -\beta\Lambda + 2\epsilon_0 \phi^{\frac{n+2}{n-2}}_+ + |\tilde{K}(X)|^2_\gamma \phi^{-\frac{3n-2}{n-2}}_+ + \epsilon_2(f)\phi^{-3}_+ + \epsilon_3 \phi^{\frac{n-6}{n-2}}_+,$$

$$\mathcal{H}^2_{f,X}(\phi_+) = -\beta\lambda - (|\theta_-| - b_n \tau)\phi^{\frac{n}{n-2}}_+ + \left(\frac{1}{2}|\theta_-| - b_n \tau \right) v^{\frac{2n}{n-2}} \phi^{-\frac{n}{n-2}}_+,$$

First, let us apply the estimate

$$|\widetilde{K}(X)|_\gamma^2 \leqslant 2|\mathscr{L}_{\gamma,\mathrm{conf}}X|_\gamma^2 + 2|U|_\gamma^2.$$

Then, since $|\mathscr{L}_{\gamma,\mathrm{conf}}X|_\gamma \lesssim |DX|_\gamma$, we get that for any $X \in B_{M_X}$, it holds that

$$|\mathscr{L}_{\gamma,\mathrm{conf}}X|_\gamma^2 \lesssim r^{2(\delta-1)}\|X\|_{W_\delta^{2,p}}^2 \leqslant r^{2(\delta-1)}M_X^2.$$

Since the objective is to produce a far-from-CMC barrier, the idea is now to separate the $d\tau$-term in M_X from all the remaining ones. We can do this appealing to the elementary estimate $(a+b)^2 \leqslant 2(a^2+b^2)$. Using this in (4.78) to compute M_X^2 and then using $|U|_\gamma^2 \lesssim r^{2(\delta-1)}\|U\|_{W_{\delta-1}^{1,p}}^2$, we find

$$
\begin{aligned}
|\widetilde{K}(X)|_\gamma^2 \lesssim r^{2(\delta-1)}\Big\{ &\kappa^2\Big\{\|d\tau\|_{L_{\delta-2}^p}^2\|\phi_+^{\frac{4n}{n-2}}\|_{C^0} + \Big(\|\omega_\mu\|_{L_{\delta-2}^p}\|\phi_+^{2\frac{n+1}{n-2}}\|_{C^0} \\
&+ \|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\big(M_f + \|\vartheta\|_{W_{\delta-1}^{1,p}}\big) + \Big\|\tfrac{1}{2}|\theta_-| - b_n\tau\Big\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} \\
&+ \|U\|_{W^{1-\frac{1}{p},p}}\Big)^2 + \|U\|_{W_{\delta-1}^{1,p}}^2\Big\},
\end{aligned}
\tag{4.83}
$$

Going back to the estimate on the Lichnerowicz equation, introducing the implicit constant in the above estimate as C_n and also using the explicit expression of M_f given by (4.78), we get that

$$
\begin{aligned}
\mathcal{H}^1(\phi_+) \leqslant &-\beta\Lambda + C_n r^{2(\delta-1)}\|d\tau\|_{L_{\delta-2}^p}^2\|\phi_+^{\frac{4n}{n-2}}\|_{C^0}\phi_+^{-\frac{3n-2}{n-2}} \\
&+ C_n r^{2(\delta-1)}\Big(\|\omega_\mu\|_{L_{\delta-2}^p}\|\phi_+^{2\frac{n+1}{n-2}}\|_{C^0} + C\|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|\widetilde{q}\|_{L_{\delta-2}^p}\|\phi_+^{\frac{2n}{n-2}}\|_{C^0} \\
&+ C\|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}} + \|\widetilde{F}\|_{W_{\delta-1}^{1,p}}\|\vartheta\|_{W_{\delta-1}^{1,p}} \\
&+ \Big\|\tfrac{1}{2}|\theta_-| - b_n\tau\Big\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}}\Big)^2\phi_+^{-\frac{3n-2}{n-2}} \\
&+ C_n r^{2(\delta-1)}\|U\|_{W_{\delta-1}^{1,p}}^2\phi_+^{-\frac{3n-2}{n-2}} + 2\epsilon_0\phi_+^{\frac{n+2}{n-2}} + \epsilon_2(f)\phi_+^{-3} + \epsilon_3\phi_+^{\frac{n-6}{n-2}}
\end{aligned}
$$

Also, recall that for any $f \in B_{M_f}$ it holds that

$$
\begin{aligned}
\epsilon_2(f) = |df + \vartheta|_\gamma^2 &\lesssim r^{2(\delta-1)}\|df + \vartheta\|_{W_{\delta-1}^{1,p}}^2, \\
&\leqslant 2r^{2(\delta-1)}\Big(2C^2\|\widetilde{q}\|_{L_{\delta-2}^p}^2\|\phi_+^{\frac{4n}{n-2}}\|_{C^0} + 2C^2\|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}}^2 + \|\vartheta\|_{W_{\delta-1}^{1,p}}^2\Big).
\end{aligned}
\tag{4.84}
$$

Before introducing this explicit expression in the previous estimate, let us also highlight that the choices $\mu, p \in W^{1,p}_{2(\delta-1)}$, $\tilde{u} \in W^{1,p}_{\delta-1}$ (which imply $\epsilon_0 \in W^{1,p}_{2(\delta-1)}$) and $\tilde{F} \in W^{1,p}_{\delta-1}$ that we have made, imply the estimates

$$
\begin{aligned}
\epsilon_0 &= (\mu + p)\left(1 + |\tilde{u}|^2_\gamma\right)^{\frac{1}{2}} - p \leq C^{(1)}_n r^{2(\delta-1)} \|\epsilon_0\|_{W^{1,p}_{2(\delta-1)}}, \\
\epsilon_3 &= \frac{1}{2}|\tilde{F}|^2_\gamma \leq C^{(2)}_n r^{2(\delta-1)} \|\tilde{F}\|^2_{W^{1,p}_{\delta-1}},
\end{aligned}
\tag{4.85}
$$

and therefore we get

$$
\begin{aligned}
\mathcal{H}^1(\phi_+) &\leq -\beta\Lambda + C_n r^{2(\delta-1)} \|d\tau\|^2_{L^p_{\delta-2}} \|\phi_+^{\frac{4n}{n-2}}\|_{C^0} \phi_+^{-\frac{3n-2}{n-2}} \\
&\quad + r^{2(\delta-1)}\Bigg\{ C_n\Big(\|\omega_\mu\|_{L^p_{\delta-2}} \|\phi_+^{2\frac{n+1}{n-2}}\|_{C^0} + C\|\tilde{F}\|_{W^{1,p}_{\delta-1}} \|\tilde{q}\|_{L^p_{\delta-2}} \|\phi_+^{\frac{2n}{n-2}}\|_{C^0} \\
&\quad + C\|\tilde{F}\|_{W^{1,p}_{\delta-1}} \|E_{\hat{v}}\|_{W^{1-\frac{1}{p},p}} + \|\tilde{F}\|_{W^{1,p}_{\delta-1}} \|\vartheta\|_{W^{1,p}_{\delta-1}} \\
&\quad + \|\tfrac{1}{2}|\theta_-| - b_n\tau\|_{W^{1-\frac{1}{p},p}} \|v^{\frac{2n}{n-2}}\|_{C^0} + \|U\|_{W^{1-\frac{1}{p},p}}\Big)^2 \phi_+^{-\frac{3n-2}{n-2}} \\
&\quad + C_n\|U\|^2_{W^{1,p}_{\delta-1}} \phi_+^{-\frac{3n-2}{n-2}} + 2C^{(1)}_n \|\epsilon_0\|_{W^{1,p}_{2(\delta-1)}} \phi_+^{\frac{n+2}{n-2}} + C^{(2)}_n\|\tilde{F}\|^2_{W^{1,p}_{\delta-1}} \\
&\quad + 2C^{(3)}_n\Big(2C^2\|\tilde{q}\|^2_{L^p_{\delta-2}} \|\phi_+^{\frac{4n}{n-2}}\|_{C^0} + 2C^2\|E_{\hat{v}}\|^2_{W^{1-\frac{1}{p},p}} + \|\vartheta\|^2_{W^{1,p}_{\delta-1}}\Big) \phi_+^{-3}\Bigg\}
\end{aligned}
\tag{4.86}
$$

The first objective is now to show that the first line in the above expression is negative for a suitable choice of $\beta > 0$. Since this is trivial of $d\tau = 0$, we shall assume that $d\tau \not\equiv 0$ in what follows. With this in mind, notice that

$$
\|\phi_+^{\frac{4n}{n-2}}\|_{C^0} \phi_+^{-\frac{3n-2}{n-2}} = \beta^{\frac{n+2}{n-2}} \|(\omega + \varphi_+)^{\frac{4n}{n-2}}\|_{C^0} (\omega + \varphi_+)^{-\frac{3n-2}{n-2}}.
$$

Near infinity we know that $\Lambda = r^{2(\delta-1)}$. Thus, if we pick β sufficiently small, independently of how large $\|d\tau\|_{L^p_{\delta-2}}$ might be, we get that

$$
-\beta\Lambda + C_n r^{2(\delta-1)} \|d\tau\|^2_{L^p_{\delta-2}} \|\phi_+^{\frac{4n}{n-2}}\|_{C^0} \phi_+^{-\frac{3n-2}{n-2}} < 0.
$$

In fact, from $\|d\tau\| \neq 0$, we just need to satisfy

$$
0 < \beta^{\frac{n+2}{n-2}-1} = \beta^{\frac{4}{n-2}} < \inf_M \frac{\Lambda r^{-2(\delta-1)}}{C_n\|d\tau\|^2_{L^p_{\delta-2}} \|(\omega + \varphi_+)^{\frac{4n}{n-2}}\|_{C^0}}(\omega + \varphi_+)^{\frac{3n-2}{n-2}}
$$

Therefore, the first line in the above $\mathcal{H}^1(\phi_+)$-estimate is negative from our choice of β Furthermore, since $\Lambda = r^{2(\delta-1)}$ near infinity, we see that this decay controls all the terms in this estimate. Thus, under smallness assumptions all the remaining coefficients, we can keep the right-hand side non-positive over all of M.

Concerning the boundary inequality, since $-\beta\lambda - (|\theta_-| - a_n\tau)\phi_+^{\frac{n}{n-2}} < 0$ along Σ, under a smallness assumptions on $v \in W^{1-\frac{1}{p},p}$, we get that $\mathcal{H}^2_{f,X}(\phi_+) \leqslant 0$ for any $f \in B_{M_f}$ and any $X \in B_{M_X}$. All this implies that, under the present assumptions and with the choices of balls B_{M_f} and B_{M_X} made above, ϕ_- and ϕ_+ form a compatible pair of strong global barriers for the Hamiltonian constraint. Moreover, due to the arguments put forward at the beginning of this proof, the solution map $\mathcal{F}_{a,b}(\bar{\phi}, \cdot, \cdot)$ is invariant on these balls and therefore the construction is finished.

$\square$

The above lemma provides us with appropriate barriers to apply Theorem 4.3.2 to the system (4.54)-(4.55). Nevertheless, we haven not yet satisfied the black hole boundary conditions, since for this we need to guarantee the a priori estimate $\phi|_\Sigma \leqslant v$ for the solution delivered by Theorem 4.3.2. We can achieve this arguing as follows.

In the above lemma, the definition of the strong global supersolution is independent of the boundary function v. Thus, after defining ϕ_+ just as in (4.81)-(4.82), fix $v \doteq \phi_+|_\Sigma \in W^{1-\frac{1}{p},p}(\Sigma)$. This only demands us to substitute this choice for v in the final inequalities for $\mathcal{H}^{1,2}_{f,X}(\phi_+)$. In the case of (4.86), this only affects the first term in the fourth line, which is of the form

$$\left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}\|v^{\frac{2n}{n-2}}\|_{C^0} = \left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}\|\phi_+^{\frac{2n}{n-2}}\|_{C^0}.$$

In order to control the size of this term, we can now appeal to a smallness condition on $\left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}$. On the other hand, concerning the boundary estimates, this choice implies

$$\mathcal{H}^2_{f,X}(\phi_+) = -\beta\lambda - (|\theta_-| - b_n\tau)\phi_+^{\frac{n}{n-2}} + \left(\frac{1}{2}|\theta_-| - b_n\tau\right)v^{\frac{2n}{n-2}}\phi_+^{-\frac{n}{n-2}},$$

$$= -\beta\lambda - \frac{1}{2}|\theta_-|\phi_+^{\frac{n}{n-2}} < 0,$$

$$(4.87)$$

and therefore the boundary inequality is automatically verified. Thus, we conclude that adding a smallness condition on $\left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}$ makes the a priori choice $v = \phi_+|_\Sigma$ admissible in the construction presented in Lemma 4.3.4. The interesting point to be made at this point, is that Theorem 4.3.2 guarantees that the solution (ϕ, f, X) obtained for system (4.54)-(4.55) from these barrier functions satisfies the a priori bound $\phi \leqslant \phi_+$, implying $\phi|_\Sigma \leqslant v = \phi_+|_\Sigma$, and therefore (ϕ, f, X) satisfy the full (marginally) trapped surface boundary conditions (4.49). That is, we have established the following:

Theorem 4.3.3. *Consider the same assumptions as in Lemma 4.3.4 and the strong global barriers constructed therein. Then, under the additional assumption that $\left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}$ is sufficiently small, the choice of data $v = \phi_+|_\Sigma$ is compatible with the construction, and under these choices the resulting barriers ϕ_- and ϕ_+ provide initial data which satisfy the (marginally) trapped conditions (4.49).*

We can therefore present the main result of this section:

Theorem 4.3.4. *Let (M^n, γ) be a $W^{2,p}_\delta$-Yamabe positive AE manifold with compact boundary Σ, $p > n$, $n \geqslant 3$ and $2 - n < \delta < 0$. Consider the system (4.54)-(4.55) with conformal data $\tau, U, \widetilde{F}, \vartheta \in W^{1,p}_{\delta-1}$, $\mu, p \in W^{1,p}_{2(\delta-1)}, \widetilde{u} \in W^{1,p}_{\delta-1}, \widetilde{q} \in L^p_{\delta-2}$ and $\theta_-, E_{\hat{v}} \in W^{1-\frac{1}{p},p}(\Sigma)$. If $\frac{1}{2}|\theta_-| - b_n\tau \geqslant 0$ and $\theta_- < 0$ along Σ, and $U, \widetilde{F}, E_{\hat{v}}, \vartheta, \mu, p, \widetilde{q}$ and $\left\|\frac{1}{2}|\theta_-| - b_n\tau\right\|_{W^{1-\frac{1}{p},p}}$ are sufficiently small, then, there is a $W^{2,p}_\delta$-solution to the conformal problem (4.54)-(4.55) satisfying marginally trapped boundary conditions.*

Proof. Let us begin by fixing a harmonic function ω_0 tending to positive constants $\{A_j\}_{j=1}^N$ on each end $\{E_j\}_{j=1}^N$ respectively. Then, from Theorem 4.3.3, we know that under our smallness assumptions we can produce a compatible pair of strong global barriers $\phi_\pm$ asymptotic to $\alpha\omega_0$ and $\beta\omega_0$ for sufficiently small constants α and β. Furthermore, we know that the solution map $\mathcal{F}_{a,b}$ associated to the shifted system (4.57) is invariant on the balls B_{M_f} and B_{M_X} in $W^p_{2,\delta}$. Then, the result follows from Theorem 4.3.2. $\qquad\square$

Let us finally highlight that there are corresponding results for the other Yamabe classes which the interested reader may find in Avalos and Lira (2019), although they also demand a near-CMC condition. Nevertheless, let us once more point out that contrary to the the vacuum case (or cases without charge), even the

CMC case associated with (4.54)-(4.55) is fully coupled and therefore such near-CMC results for $\mathcal{Y} \leqslant 0$ should be appreciated in this light. For instance, we cannot in this case consider a close CMC solution obtained by decoupling the equations and then consider this result as a consequence of a perturbation technique of the kind discussed at the beginning of the chapter. We are free to assume the existence of such a CMC solution and apply these techniques, but we cannot obtain this first CMC solution by analysis a decoupled system. As we anticipated in the beginning of the section, we are stuck with the fully coupled systems from the first moment, and hence even near CMC solutions are quite delicate.

Finally, along the lines of the final comments of Section 4.2, notice that we do not have any claims about uniqueness, which is an important open problem, and, also, results for general data (without smallness assumptions) do not seem to be within reach with current techniques. Both these problems relate to the objective of using the conformal method as a way of parametrising the space of solutions of the ECE.

A
Some Analytic Tools

In this appendix we shall collect several analytic results which are useful in the analysis of the elliptic problems treated in the core in this book. First, to provide a presentation as self-contained as possible, we shall collect some well-known functional analytic results for future reference and then we will collect several very important results concerning Sobolev spaces. This will be essential in Appendix B in the analysis of linear elliptic partial differential operators.

A.1   Functional analytic results

Let us start recapitulating a few useful results from functional analysis which shall play a role in our constructions. The proof of the following theorem can be found, for instance, in Cantor (1981, Lemma 2.2).

Theorem A.1.1. *Let $T : X \mapsto Y$ and $S : Y \mapsto Z$ be bounded linear operators between Banach spaces. Then, the following are equivalent:*

$$1) \operatorname{Ker}(S \circ T) = \operatorname{Ker}(T) \text{ and } \operatorname{Im}(S \circ T) = \operatorname{Im}(S).$$
$$2) Y = \operatorname{Im}(T) \oplus \operatorname{Ker}(S).$$

Furthermore, in case one of the above holds, then $\operatorname{Im}(T)$ is closed in Y.

Let us now introduce the notion of a *Fredholm* map. First, let us consider a bounded linear map $T : X \mapsto Y$ acting between Banach spaces X, Y and introduce the following definition. If $\text{Im}(T) \subset Y$ is closed, then we can define the quotient space $\text{Coker}(T) \doteq Y/\text{Im}(T)$ (see Abraham, Marsden, and Ratiu 1988, for instance, Proposition 2.1.13).

Definition A.1.1. *A bounded linear operator $T : X \mapsto Y$ acting between Banach spaces X and Y is said to be a Fredholm map if (1) $\text{Ker}(T)$ is finite dimensional, (2) $\text{Im}(T)$ is closed and (3) $\text{Coker}(T)$ is also finite dimensional. In case only (1) and (2) hold, we say that T is semi-Fredholm. Furthermore, the Fredholm index of a Fredholm map T is defined to be the number*

$$\text{Ind}(T) = \dim(\text{Ker}(T)) - \dim(\text{Coker}(T)). \tag{A.1}$$

Fredholm maps, as well as their index, have particularly nice stability properties, as can be seen from the following result (see Hörmander 2007, Corollary 19.1.6):

Theorem A.1.2. *Let X, Y be Banach spaces and $\mathcal{L}(X, Y)$ denote the set of bounded linear maps from $X \mapsto Y$. Then, the set of Fredholm operators in $\mathcal{L}(X, Y)$ is open. Furthermore, if $T \in \mathcal{L}(X, Y)$ is Fredholm, then $\dim(\text{Ker})(T)$ is upper semi-continuous and $\text{Ind}(T)$ is constant on each component.*

Our interest in Fredholm operators relies on the fact that, very generally, elliptic operators are Fredholm, we will sometimes exploit this property appealing to the above theorem. For a complete discussion on the Fredholm properties of elliptic operators, we refer the reader to Hörmander (ibid.).

Duals, Weak Convergence and Compactness

Let us now collect a few well-known definitions and results concerning duality relations which are useful in the analysis of linear PDEs. For details, we refer the reader, for instance, to Rudin (1991, Chapter 4).

Definition A.1.2. *Let X, Y be Banach spaces with dual spaces X' and Y' respectively. Let $T : X \mapsto Y$ be a bounded linear map, then we define its dual (or adjoint) $T^* : Y' \mapsto X'$ by $\langle T^* y', x \rangle \doteq \langle y', Tx \rangle$ for all $y' \in Y'$ and $x \in X$. That is, $T^* y' = y' \circ T \in X'$.*

Adjoint maps of linear partial differential operators play an important role in the existence of solutions to PDE problems, as we shall see shortly. This can be anticipated by introducing the following notions.

Definition A.1.3. *Let X be a Banach space and let $M \subset X$ and $N \subset X'$ be subspaces. We define their annihilator spaces $M^\perp$ and $N^\perp$ by*

$$M^\perp \doteq \{x' \in X' \ : \ \langle x', x \rangle = 0 \ for \ all \ x \in M\},$$
$$N^\perp \doteq \{x \in X \ : \ \langle x', x \rangle = 0 \ for \ all \ x' \in N\}. \tag{A.2}$$

Using this terminology, the following classical result is now obtained obtained by putting together Theorem 4.7, Theorem 4.12 and Theorem 4.14 of Rudin (1991).

Theorem A.1.3. *Suppose that $T : X \mapsto Y$ is a bounded linear transformation between reflexive Banach spaces and that T has closed range. Then, if $T^* : Y' \mapsto X'$ denotes the adjoint of T, it holds that $\mathrm{Ker}(T)^\perp = T^*(Y')$. Furthermore, the range of T^* is closed and $\mathrm{Ker}(T^*)^\perp = T(X)$.*

In the above theorem, if we think of T as a partial differential operator acting between appropriate functional spaces $T : X \mapsto Y$, then, existence of solutions to equations of the form $Tx = y$, with $y \in Y$ can be analysed by having information on $\mathrm{Ker}(T^*)$, which is something we shall exploit. Notice that, in doing so, we will have to show (a priori) that the range of T is closed, which is something we will prove to hold for *elliptic* partial differential operators (see Appendix B).

Let us also notice the following property related to the above theorem and Fredholm operators. Let $T : X \mapsto Y$ be a Fredholm operator acting between Banach spaces X, Y. Then, due to its *Fredholmness*, we know that $\mathrm{Coker}(T) = Y/\mathrm{Im}(T)$ is finite dimensional and that $\mathrm{Im}(T)$ is closed. This last condition guarantees that the following spaces are isomorphic (see ibid., Theorem 4.9(b))

$$(Y/\mathrm{Im}(T))' \cong \mathrm{Im}(T)^\perp.$$

But also, it follows directly from its definition that $\mathrm{Im}(T)^\perp = \mathrm{Ker}(T^*)$ (see ibid., Theorem 4.12), which implies

$$(Y/\mathrm{Im}(T))' \cong \mathrm{Ker}(T^*).$$

Now, since $\mathrm{Coker}(T) = Y/\mathrm{Im}(T)$ is finite dimensional, then $(Y/\mathrm{Im}(T))' \cong \mathrm{Coker}(T)$, which finally implies

$$\mathrm{Coker}(T) \cong \mathrm{Ker}(T^*). \tag{A.3}$$

Therefore, we find that

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*)). \tag{A.4}$$

The above formula is again useful when dealing with elliptic PDE problems.

Let us now recall some definitions and results concerning weak convergence and compactness.

Definition A.1.4. *A sequence $\{x_i\}_{i=1}^{\infty}$ in a normed linear space $(E, \|\cdot\|)$, is said to* converge weakly *to $x \in E$ if $u(x_i) \to u(x)$ for every $u \in E'$. A subset $B \subset E$ is said to be* weakly sequentially compact, *if every sequence in B contains a subsequence which converges weakly to a point in B.*

Theorem A.1.4. *(Brezis 2011, Chapter 3, Section 3.2) A weakly convergent sequence $\{x_i\}_{i=1}^{\infty}$ in a normed linear space E has a unique limit x, is bounded, and*

$$\|x\| \lesssim \liminf_{i \to \infty} \|x_i\|. \tag{A.5}$$

Theorem A.1.5. *(ibid., Theorem 3.17) A Banach space E is reflexive iff its closed unit ball $\overline{B}_1(0)$ is weakly sequentially compact.*

Distributions

We fix our notation for distributions as follows: We denote by $\mathcal{D}(\Omega)$ the space of test functions on some domain Ω, by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the Schwartz space of functions of rapid decrease and by $\mathcal{D}'$ and $\mathcal{S}'$ the corresponding dual spaces of distributions with domains implicitly understood. Also, we denote by $\mathcal{E}'$ the set of distributions of compact support (see Hörmander 1990, Chapter 2). Let us also introduce the following useful concepts.

Definition A.1.5. *A distribution u in $\mathbb{R}^n \backslash \{0\}$ is called* homogeneous of degree a *if*

$$\langle u, \phi \rangle = t^a \langle u, \phi_t \rangle \quad \forall \ \phi \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\}), \tag{A.6}$$

where $\phi_t(x) \doteq t^n \phi(tx)$ with $t > 0$. If, furthermore, u is a distribution in $\mathbb{R}^n$ and the above property holds for all $\phi \in C_0^{\infty}(\mathbb{R}^n)$, then u is said to be homogeneous of degree a in $\mathbb{R}^n$.

Notice that this is motivated, as usual, by the case where $u \in L_{loc}^1(\mathbb{R}^n \backslash \{0\})$. In this case, u is homogeneous of degree a if $u(tx) = t^a u(x)$, when $x \neq 0$ and

$t > 0$. This implies that

$$t^a \int_{\mathbb{R}^n} u(x)\phi_t(x)dx = t^a \int_{\mathbb{R}^n} u(x)t^n\phi(tx)dx,$$

$$= t^a \int_{\mathbb{R}^n} \underbrace{u\left(\frac{y}{t}\right)}_{=t^{-a}u(y)} \phi(y)dy = \langle u, \phi \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R}^n \backslash \{0\}).$$

Our main interest related to homogeneous distributions is that *symbols* associated to differential operators define homogeneous functions (see Appendix B). Therefore let us recall a few useful results.

Theorem A.1.6. *(Hörmander 1990, Theorem 7.1.16) If $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a, then $\hat{u} \in \mathcal{S}'$ is homogeneous of degree $-a-n$, where $\hat{u}$ denotes the Fourier transform of u.*[1]

Theorem A.1.7. *(ibid., Theorem 7.1.18) If $u \in \mathcal{D}'(\mathbb{R}^n)$ and the restriction to $\mathbb{R}^n \backslash \{0\}$ is homogeneous of degree a, then $u \in \mathcal{S}'$. If, addition $u \in C^\infty(\mathbb{R}^n \backslash \{0\})$, then $\hat{u} \in C^\infty(\mathbb{R}^n \backslash \{0\})$.*

Remark A.1.1. *The above two theorems imply that the Fourier transform is a bijection from the set of distributions in $\mathcal{S}' \cap C^\infty(\mathbb{R} \backslash \{0\})$ which are homogeneous of degree a onto the set of distributions $\mathcal{S}' \cap C^\infty(\mathbb{R} \backslash \{0\})$ which are homogeneous of degree $-a-n$.*

Theorem A.1.8. *(ibid., Theorem 7.9.5) Let $k \in \mathcal{S}'(\mathbb{R}^n)$ and assume that $\hat{k} \in L^1_{loc}$ satisfies*

$$\sum_{|\alpha| \leqslant s} \int_{\frac{R}{2} < |\xi| < 2R} |R^{|\alpha|} D^\alpha \hat{k}(\xi)|^2 \frac{d\xi}{R^n} \leqslant C < \infty \quad R > 0, \tag{A.7}$$

where $s > \frac{n}{2}$ is an integer. Then, for any $1 < p < \infty$, it follows that

$$\|k * u\|_{L^p} \leqslant C_p \|u\|_{L^p} \quad \forall \ u \in L^p \cap \mathcal{E}'. \tag{A.8}$$

In addition,

$$\tau\mu\{x; \ |k * u(x)| > \tau\} \leqslant C \|u\|_{L^2} \quad \forall \ u \in L^2 \cap \mathcal{E}'. \tag{A.9}$$

---

[1]Moving forward, we will keep this notation for the Fourier transform.

A.2 Sobolev spaces

In this section we will compile several useful results concerning the theory of Sobolev spaces on compact manifolds, possibly with boundary. Since this is a very rich subject in analysis, we do not intend to prove the large amount of content that will be introduced, rather give a presentation which allows the unacquainted reader to become a user of this theory. We will provide several references where the interested reader can consult detailed proofs. In particular, let us highlight the classic book by Adams (1975), to which we will frequently refer. This reference presents an exhaustive analysis of Sobolev theory for scalar function on domains of $\mathbb{R}^n$. Several of the necessary generalisations to scalar functions on manifolds (not necessarily compact) can be found in Aubin (see 1998, Chapter 2), in particular for spaces of integer degree of regularity. There, in the case of compact manifolds, the necessary localisation techniques are highlighted explicitly. Along these lines, we also refer the reader to Schwarz (see 1995, Chapter 1) and Holst, Nagy, and Tsogtgerel (see 2009, Appendix A.4) for an account of how to use these localisation techniques to extend the results to vector bundle sections. Finally, we would also like to refer the reader to Palais (1968, Section 9) for a exhaustive treatment of the general problem. In this case, interpolating spaces are defined via complex interpolation, resulting in different spaces with similar properties. Along these lines, we also refer the interested reader to Taylor (2011a, Chapter 4) and (Taylor 2011c, Chapter 13)

Local Theory

Definition A.2.1. *Let $U \subset \mathbb{R}^n$ be an arbitrary domain, k be a non-negative integer and $1 \leqslant p \leqslant \infty$ a real number. We define the Sobolev space $W^{k,p}(U)$ as the space of functions $f \in L^p(U)$ which possess weak derivatives $\{\partial^\alpha f\}_{0 \leqslant |\alpha| \leqslant k}$ of order up to k in $L^p(U)$. That is,*

$$W^{k,p}(U) \doteq \{f \in L^p(U) \; : \; \partial^\alpha f \in L^p(U) \; \forall \, 0 \leqslant |\alpha| \leqslant k\}. \tag{A.10}$$

We equip this vector subspaces of L^p with the norm

$$\|f\|_{W^{k,p}(U)} \doteq \left(\sum_{|\alpha|=0}^{k} \|\partial^\alpha f\|_{L^p(U)} \right)^{\frac{1}{p}} \quad \text{if } 1 \leqslant p < \infty,$$

$$\|f\|_{W^{k,\infty}(U)} \doteq \max_{0 \leqslant |\alpha| \leqslant k} \|\partial^\alpha u\|_{L^\infty(U)}. \tag{A.11}$$

Let us start by collecting a few well-known facts about these spaces. First of all, these are Banach spaces. Furthermore, they are separable for $1 \leqslant p < \infty$ and reflexive for $1 < p < \infty$. In particular, $W^{k,2}(U) \doteq H^2(U)$ is a separable Hilbert space with inner product

$$(u, v)_{H^k(U)} \doteq \sum_{0 \leqslant |\alpha| \leqslant k} (\partial^\alpha u, \partial^\alpha v)_{L^2(U)}.$$

Furthermore, $W^{k,p}(U)$ coincides with the closure of $C^\infty(U)$ in the norm (A.11). Let us then define the space $W_0^{k,p}(U)$ as the closure of $C_0^\infty(U)$ in the norm (A.11). Clearly $W_0^{k,p}(U) \subset W^{k,p}(U)$, but the equality is not true in general. In particular, it is a standard result that $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Let us now introduce some notation concerning the duals of these Sobolev spaces. In particular, for $1 \leqslant p < \infty$ and k a non-negative integer, we *define* $W^{-k,p'}(U) \doteq \left(W_0^{k,p}(U) \right)'$, where $p' = 1 - \dfrac{1}{p}$ denotes the conjugate exponent to p. These spaces have a clear characterization: since the space of *test functions* $\mathcal{D}(U)$ is continuously embedded in $W_0^{k,p}(U)$ and it is also dense, $W^{-k,p'}(U)$ consists of all the distributions $\mathcal{D}'(U)$ which posses continuous extensions to $W_0^{k,p}(U)$. On the other hand, whenever $W_0^{k,p} \neq W^{k,p}$, the latter space does not admit a characterisation of this type. In particular, in such situations $(W^{k,p})'$ contains objects which are not distributions.

Let us comment on one further useful property of the spaces $W^{-k,p'}(U)$. As a consequence of Hölder's inequality, any given $v \in L^{p'}(U)$ defines an object $\phi_v \in W^{-k,p'}(U)$ via

$$\phi_v(u) \doteq (u, v)_{L^2} \ \forall \, u \in W_0^{k,p}(U).$$

It can be checked straightforwardly that the map $\phi : L^{p'}(U) \mapsto W^{-k,p'}(U)$ is an isometry. That is,

$$\|v\|_{L^{p'}(U)} = \sup_{\|u\|_{W_0^{k,p}(U)} \leqslant 1} |\phi_v(u)| = \|\phi_v\|_{W^{-k,p'}(U)}.$$

Furthermore, $V \doteq \{\phi_v : v \in L^{p'}(U)\} \subset W^{-k,p'}(U)$ can easily be seen to be a dense subset. Therefore, for each $L \in W^{-k,p'}(U)$, there is a sequence $\{v_k\} \subset L^{p'}(U)$ such that

$$L(u) = \lim_{k \to \infty} \phi_{v_k}(u) = \lim_{k \to \infty} (u, v_k)_{L^2} \ \forall \, u \in W_0^{k,p}(U).$$

Since $C_0^\infty(U)$ is dense in both $W_0^{k,p}(U)$ and $L^{p'}(U)$, then, actually, we see that the bilinear map

$$C_0^\infty(U) \times C_0^\infty(U) \mapsto \mathbb{R},$$
$$(v, u) \mapsto (u, v)_{L^2}$$

extends by continuity to a dual pairing $W^{-k,p'}(U) \times W_0^{k,p}(U) \mapsto \mathbb{R}$.

The above spaces have several nice properties which make them, for instance, particularly useful in PDE analysis. Among these, we find the celebrated Sobolev embedding theorems which are presented below. Let us point out that their validity depends on some regularity assumptions on the boundary of our domains, such as having the *cone property* or Lipschitz properties. A domain $\Omega \subset \mathbb{R}^n$ is said to have the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. We refer the reader to Adams (1975, page 66) for the precise definitions of all the relevant related regularity assumptions, which take more space to define. We highlight the useful fact that bounded domains with smooth boundary satisfy all of these properties. For the proof of the following theorem, see Adams (ibid., Theorem 5.4)

Theorem A.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with the cone property. Let j and m be non-negative integers and p a real number satisfying $1 \leqslant p < \infty$. Then, the following embeddings hold and are continuous*

1. *If $mp < n$, then $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}$, whenever $p \leqslant q \leqslant \frac{np}{n-mp}$;*

2. *If $p = \frac{n}{m}$, then $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$, for all $p \leqslant q < \infty$;*

3. *If $m > \frac{n}{p}$, then $W^{j+m,p}(\Omega) \hookrightarrow C^j(\Omega)$;*

4. *If Ω has the strong local Lipschitz property and $mp > n > (m-1)p$ then $W^{j+m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega})$, for all $0 < \alpha < m - \frac{n}{p}$;*

5. *If Ω has the strong local Lipschitz property and $n = (m-1)p$ then it follows that $W^{j+m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega})$, for all $0 < \alpha < 1$.*

Furthermore, all the above conclusion are valid on arbitrary domains if we replace W-spaces by W_0-spaces.

Remark A.2.1. *In the above theorem, in the third embedding we understand the space $C^r(U)$ as the space of r-times continuously differentiable functions on U, equipped with the norm*

$$\|u\|_{C^r(U)} \doteq \sum_{0 \leqslant |\beta| \leqslant r} \sup_{x \in U} |\partial^\alpha u|.$$

In the fourth and fifth embedding we denoted by $C^{r,\alpha}(U)$ the space of $C^r(U)$ functions satisfying the Hölder condition of order α. That is, their Hölder norms

$$\|u\|_{C^{r,\alpha}(U)} = \sum_{0 \leqslant |\beta| \leqslant r} \|\partial^\beta u\|_{C^{0,\alpha}(U)},$$

$$\|u\|_{C^{0,\alpha}(U)} = \sup_{x \in U} |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

are finite.

Another crucial result in this direction comes from the fact that, under suitable restricting assumptions, the above embeddings are actually compact. This turns out to be an essential property when analysing PDE operators acting between Sobolev function spaces. For the proof of the following theorem, see Adams (1975, Theorem 6.2)

Theorem A.2.2 (Rellich–Kondrachov). *Let Ω be a domain in $\mathbb{R}^n$ and Ω_0 a bounded subdomain of Ω. Let j, m be integers with $j \geqslant 0$, $m \geqslant 1$ and let $1 \leqslant p < \infty$.*

1. *If Ω has the cone property and $mp \leqslant n$, the embeddings $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_0)$ are compact if either $0 < n - mp < n$ and $1 \leqslant q < \frac{np}{n-mp}$, or if $n = mp$ and $1 \leqslant q < \infty$;*

2. *If Ω has the cone property and $mp > n$, the following embeddings are compact:*

$$W^{j+m,p}(\Omega) \hookrightarrow C^j(\Omega),$$
$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_0), \text{ if } 1 \leqslant q \leqslant \infty$$

3. *If Ω has the strong local Lipschitz property, then the following embeddings are compact:*

$W^{j+m,p}(\Omega) \hookrightarrow C^j(\overline{\Omega}_0), \text{ if } mp > n,$

$W^{j+m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega}_0), \text{ if } mp > n \geqslant (m-1)p \text{ and } 0 < \alpha < m - \frac{n}{p}.$

If Ω is an arbitrary domain in $\mathbb{R}^n$, all the above embeddings are compact if we replace W-spaces by W_0-spaces.

Along the lines of the above comments concerning compactness properties and their use in PDE analysis, let us highlight that, when analysing the range a PDE operator, specially in those cases where the coefficients belong to some Sobolev space, we need to know to what space do products of Sobolev functions belong. The following theorem, which follows from the above Sobolev embedding theorems, gives a useful characterisation for the answer to this question.

Theorem A.2.3. *Consider a bounded domain $U \subset \mathbb{R}^n$ having the cone property. Let k, l and s be non-negative integers and $1 < p \leqslant q \leqslant \infty$, then the following continuous multiplication property holds*

$$W^{k,p}(U) \otimes W^{l,q}(U) \hookrightarrow W^{s,p}(U) \tag{A.12}$$

as long as $k + l > \frac{n}{q} + s$, $k \geqslant s$ and $l \geqslant s$. In particular, $W^{k,p}(U)$ is an algebra under multiplication whenever $k > \frac{n}{p}$.

Proof. We will follow closely Choquet-Bruhat and DeWitt-Morette (2000, Chapter VI). Let us first establish the claim for $s = 0$. That is, given $1 < p \leqslant q \leqslant \infty$ and l, k non-negative integers satisfying $l + k > \frac{n}{q}$ it must hold that

$$\begin{aligned} W^{k,p}(U) \times W^{l,q}(U) &\mapsto L^p(U), \\ (f, g) &\mapsto fg \end{aligned} \tag{A.13}$$

and $||fg||_{L^p} \leqslant C||f||_{W^{k,p}}||g||_{W^{l,q}}$. If either $k > \frac{n}{p}$ or $l > \frac{n}{q}$, then the claim follows trivially from the embedding of the corresponding Sobolev spaces in $C^0(U)$ and the inclusion $L^q \hookrightarrow L^p$ for U bounded. Thus, let us assume that $k \leqslant \frac{n}{p}$ and $l \leqslant \frac{n}{p}$. Then, the condition $k + l > \frac{n}{q}$ guarantees the existence of real numbers $r, t \geqslant 1$ satisfying

$$i)\ k > \frac{n}{p} - \frac{n}{r} \quad ii)\ l > \frac{n}{q} - \frac{n}{t} \quad iii)\ \frac{1}{r} + \frac{1}{t} = \frac{1}{p}.$$

Also, from the Sobolev embeddings and the boundedness of U, we know that

$$W^{k,p}(U) \hookrightarrow L^{\bar{p}}(U), \ \forall\ \bar{p} \leqslant \frac{np}{n - kp},$$

$$W^{l,q}(U) \hookrightarrow L^{\bar{q}}(U), \ \forall\ \bar{q} \leqslant \frac{nq}{n - lq}.$$

Therefore, we have $W^{k,p}(U) \hookrightarrow L^r(U)$ and $W^{l,q}(U) \hookrightarrow L^t(U)$. Thus, $f \in L^r$ and $g \in L^t$, which, together with Hölder's inequality, implies

$$\|fg\|_{L^p(U)} \lesssim \|f\|_{L^r(U)}\|g\|_{L^t(U)} \lesssim C\|f\|_{W^{k,p}(U)}\|g\|_{W^{l,q}(U)}, \quad \text{(A.14)}$$

which establishes our preliminary claim. For the general case, consider first $f, g \in C^\infty(U)$ and notice that $\partial^\alpha f \in L^p(U)$ for all $|\alpha| \leq k$, $\partial^\beta g \in L^q(U)$ for all $|\beta| \leq l$ and

$$\partial^\alpha(fg) = \sum_{|\beta|=0}^{|\alpha|} C_\beta \partial^{|\beta|} f \partial^{|\alpha|-|\beta|} g, \quad \text{(A.15)}$$

where C_β denote constants and $\partial^{|\beta|}$ stands for some derivative of order $|\beta|$ while the summation runs through all multi-indices of each order up to $|\alpha|$, for some $|\alpha| \leq s$. Notice that, for any given term in (A.15), we have

$$\partial^{|\beta|} f \in W^{k-|\beta|,p} \ ; \ \partial^{|\alpha|-|\beta|} g \in W^{l-|\alpha|+|\beta|,q}.$$

Thus, from our preliminary claim, we know that $\partial^{|\beta|} f \partial^{|\alpha|-|\beta|} g \in L^p$ and, as long as $k + l > \frac{n}{p} + |\alpha|$ (which holds by hypotheses for $|\alpha| \leq s \leq k, l$) the continuity estimate (A.14) holds. Therefore, in this case it follows that

$$\|fg\|_{W^{s,p}(U)} \lesssim C\|f\|_{W^{k,p}(U)}\|g\|_{W^{l,q}(U)}. \quad \text{(A.16)}$$

Finally, for $f \in W^{k,p}(U)$ and $g \in W^{l,q}(U)$ arbitrary, we can approximate by smooth functions $\{f_j\}, \{g_j\} \subset C^\infty(U)$ converging to f and g respectively in $W^{k,p}(U)$ and $W^{l,q}(U)$ and use (A.16) to show that $f_j g_j \xrightarrow{W^{s,p}(U)} fg$ where the same estimate holds when we pass to the limit. □

Closely related to the above theorem, we will now present another useful result when analysis the rage of non-linear mappings acting on Sobolev functions.

Lemma A.2.1 (Composition Lemma). *Let U be a bounded domain $\mathbb{R}^n$ having the cone property. Let $F : I \mapsto \mathbb{R}$ be a function of class C^m on some open interval $I \subset \mathbb{R}$ and $f \in W^{m,p}(U)$ with $m > \frac{n}{p}$ and $1 \leq p < \infty$, satisfying $\overline{f(U)} \subset I$. Then, $F \circ f \in W^{m,p}(U)$.*

Proof. First, let us notice that since $m > \frac{n}{p}$, then $f(U)$ is bounded in I and hence $F \in C^m(f(U))$. Therefore, $F \circ f \in L^p(U)$. Similarly, any first weak derivative $\partial^i(F \circ f) = F'(f)\partial^i f \in L^p(U)$, since $F' \in L^\infty(U)$ and $\partial^i f \in L^p(U)$. Let us then consider the case of derivatives of order $2 \leqslant k \leqslant m$ and let us start assuming that $f \in C^\infty(U)$. In this case, given any multi-index γ such that $|\gamma| = k$, any weak derivative $\partial^\gamma(F \circ f)$ is given by a sum of terms of the form $F^{(l)}(f)\partial^{\beta_1} f \cdots \partial^{\beta_j} f$, with $0 \leqslant l, j \leqslant k$ an integer and β_i multi-indices satisfying $|\beta_i| \geqslant 1$ and $\beta_1 + \cdots + \beta_j = \gamma$. For any such term, it holds that

$$|F^{(l)}(f)\partial^{\beta_1} f \cdots \partial^{\beta_j} f| \leqslant C|\partial^{\beta_1} f \cdots \partial^{\beta_j} f|,$$

where the constant C depends on the $C^m(f(U))$-norm of F. Also, if we fix numbers $q_i \doteq \frac{k}{|\beta_i|}$, we see that

$$\sum_{i=1}^{j} \frac{1}{q_i} = 1.$$

Furthermore, if we assume that $1 + \frac{n}{p} > m > \frac{n}{p}$, then $(m - |\beta_i|)p < n$ and

$$q_i\, p \leqslant \frac{np}{n - (m - |\beta_i|)p} \iff k(n - mp + p|\beta_i|) \leqslant n|\beta_i|$$

$$\iff k\underbrace{(n - mp)}_{<0} \leqslant (n - kp)\underbrace{|\beta_i|}_{\leqslant k}$$

$$\iff \frac{k}{|\beta_i|}(n - mp) \leqslant n - kp,$$

Since $\frac{k}{|\beta_i|} \geqslant 1$ and $(n - mp) < 0$, then $\frac{k}{|\beta_i|}(n - mp) \leqslant n - mp$. In turn, $n - mp \leqslant n - kp$ iff $m \geqslant k$, which is satisfied. This implies that $\frac{k}{|\beta_i|}(n - mp) \leqslant n - kp$ and therefore $q_i\, p \leqslant \frac{np}{n - (m - |\beta_i|)p}$. Thus, under this restricting condition, we can appeal to the Sobolev embedding $W^{m - |\beta_i|, p} \hookrightarrow L^{r_i}$, with $r_i \doteq q_i\, p$. In particular,

$$\sum_{i=1}^{j} \frac{1}{r_i} = \frac{1}{p}.$$

From this, we conclude that $\partial^{\beta_i} f \in L^{r_i}$ and we can apply Hölder's generalised

inequality combined with the Sobolev continuous embedding to get

$$\|\partial^{\beta_1} f \cdots \partial^{\beta_j} f\|_{L^p} \leqslant \prod_{i=1}^{j} \|\partial^{\beta_i} f\|_{L^{r_i}} \leqslant \prod_{i=1}^{j} C_i \|\partial^{\beta_i} f\|_{W^{m-|\beta_i|,p}}.$$

The above shows that the multi-linear mapping

$$W^{m-|\beta_1|,p}(U) \times \cdots \times W^{m-|\beta_j|,p}(U) \mapsto L^p(U),$$

$$(\partial^{\beta_1} f, \cdots, \partial^{\beta_j} f) \mapsto \partial^{\beta_1} f \cdots \partial^{\beta_j} f,$$

is continuous for any $f \in W^{m,p}(U)$. Therefore, for a general $f \in W^{m,p}(U)$, we can approximate it by a sequence $\{f_i\} \subset C^\infty(U)$, which since $m > \frac{n}{p}$ we can take so that $f_i(U) \subset I$. Then, the above analysis shows that we can pass to the limit and the weak derivatives up to order m of $F \circ f$ are given by the usual chain rule. Finally, the same estimates proved above hold in the limit, and therefore we see that the claim holds for $1 + \frac{n}{p} > m > \frac{n}{p}$, since we can estimate each L^p-norm of the weak derivatives of $F \circ f$, up to order m, in terms of the $C^m(f(U))$-norm of F together with the $W^{m,p}(U)$-norm of f. The general case follows trivially since, for $m \geqslant \frac{m}{p} + 1$, $W^{m,p}(U) \hookrightarrow W^{t,p}(U)$ for $1 + \frac{n}{p} > t > \frac{n}{p}$. $\qquad\square$

Let us now consider a domain Ω in $\mathbb{R}^n$ with boundary $\Sigma \doteq \partial\Omega$. Notice that if we intend to analyse a boundary value problem on $\overline{\Omega}$, we will have boundary values in some Sobolev space intrinsic to Σ, while we will have PDE operators acting on Sobolev spaces on Ω. In particular, a given solution in Ω should induce the appropriate boundary values on Σ. Therefore, the relation between the Sobolev spaces on Ω to those on Σ is key to be able to present well-posed problems within this functional framework. A sharp characterisation in this direction is provided introducing Sobolev spaces of *fractional order*.

Definition A.2.2. *Let Ω be a domain in $\mathbb{R}^n$ and fix a real number $s \in (0,1)$. For any $1 \leqslant p < \infty$, define the space*

$$W^{s,p}(\Omega) \doteq \left\{ u \in L^p(\Omega) : \frac{u(x) - u(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}, \qquad (A.17)$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} \doteq \left\{ \|u\|_{L^p(\Omega)}^p + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx dy \right\}^{\frac{1}{p}}. \qquad (A.18)$$

For these spaces, we also have a couple of nearly immediate properties. First, these are also Banach spaces which satisfy $W^{s_2,p}(\Omega) \hookrightarrow W^{s_1,p}(\Omega)$ for any $0 < s_1 < s_2 < 1$, and this can be extended to $s_2 = 1$ if Ω has Lipschitz boundary Di Nezza, Palatucci, and Valdinoci (see 2012, Propositions 2.1 and 2.2). Therefore, we can genuinely regard the spaces $W^{s,p}$ defined above as *interpolating* spaces between L^p and $W^{1,p}$. Similarly, we define

Definition A.2.3. *Let Ω be a domain in $\mathbb{R}^n$ and fix a real number $s \geqslant 0$ and, whenever s is not an integer, we write $s = m + \sigma$, with m an integer and $\sigma \in (0,1)$. For any $1 \leqslant p < \infty$, define the space*

$$W^{s,p}(\Omega) \doteq \{ u \in W^{m,p}(\Omega) \ : \ \partial^\alpha u \in W^{\sigma,p} \ \forall \ |\alpha| = m \}, \tag{A.19}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} \doteq \left\{ \|u\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} \|\partial^\alpha u\|_{W^{\sigma,p}(\Omega)}^p \right\}^{\frac{1}{p}}. \tag{A.20}$$

The above define a family of Banach spaces which interpolate in between the usual Sobolev spaces with integer values. In particular, the following corollary holds (see ibid., Corollary 2.3)

Corollary A.2.1. *Let $1 \leqslant p < \infty$ and $0 \leqslant s_1 \leqslant s_2 < \infty$ real numbers. Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary. Then, the following continuous inclusion holds $W^{s_2,p}(\Omega) \hookrightarrow W^{s_1,p}(\Omega)$.*

Furthermore, it still holds that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$ for all real $s \geqslant 0$ (see ibid., Theorem 2.4). We keep the notation $W_0^{s,p}(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in the $W^{s,p}(\Omega)$-norm, and for $s < 0$, we still *define* $W^{-s,p}(\Omega) \doteq \left(W_0^{s,p}(\Omega) \right)'$. Let us also highlight that, for $1 < p < \infty$, $W^{s,p}(\Omega)$ is a reflexive space (Adams (1975, page 205) and Behzadan and Holst (2017, Theorem 7.33 for a direct approach)).

Finally, we can characterise the traces of these spaces on the boundary of Ω. For this, let us first consider Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega \doteq \Sigma$. For a given $m \in \mathbb{N}$, we define the trace map as

$$\gamma : C^\infty(\overline{\Omega}) \mapsto \times_{i=0}^{n-1} C^\infty(\Sigma), \tag{A.21}$$
$$u \mapsto \gamma u = (\gamma_0 u, \cdots, \gamma_{m-1} u),$$

where $\gamma_j u \doteq \partial_\nu^j u|_\Sigma$ denotes the j-th derivative of u in the (outward pointing) normal direction ν. Then, the following theorem holds (see Adams 1975, Theorem 7.53)

Theorem A.2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega \doteq \Sigma$. Also, let $1 < p < \infty$ and $m \in \mathbb{N}$. Then, the trace map (A.21) extends by continuity to a linear homeomorphism of $W^{m,p}(\Omega)/\operatorname{Ker}(\gamma)$ onto*

$$\prod_{j=0}^{m-1} W^{m-j-\frac{1}{p},p}(\Sigma). \tag{A.22}$$

Let us highlight that the above theorem can be extended to domains with much weaker assumptions. For instance, the smoothness of boundary can be relaxed substantially (see ibid., Theorem 7.53). Nevertheless, the above theorem will be enough for our purposes.

Compact Manifolds

Let us now extend the above constructions to spaces on compact manifolds M, possibly with smooth boundary Σ, in which case Σ represents a closed (potentially multiply connected) manifold. We will start by fixing some smooth Riemannian metric g on M. Considering a Riemannian vector bundle $E \to M$ over M, we can define $L^p(E, dV_g)$ spaces using the Riemannian measure dV_g as the set of measurable sections $u \in \Gamma(E)$ on M such that

$$\|u\|_{L^p(E)} = \left\{ \int_M |u|_E^p \, dV_g \right\}^{\frac{1}{p}} < \infty, \tag{A.23}$$

which have all the standard properties associated to L^p-spaces. Let us notice that, if $\{U_i, \varphi_i\}_{i=1}^r$ is an open covering by coordinate systems of M trivialising E, and $\{\eta_i\}_{i=1}^r$ is a partition of unity subordinate to this cover, then, since the components of g and $\langle \cdot, \cdot \rangle_E$ (together with its derivatives to all orders) in any coordinate system are uniformly bounded, it holds that

$$u \in L^p(E, dV_g) \iff \eta_i u^j \in L^p(U_i, dx) \; \forall \, i = 1, \cdots, r \text{ and } j = 1, \cdots, k.$$

Therefore, for compact manifolds, we see that L^p-spaces and independent of our choice of smooth Riemannian metrics g and $\langle \cdot, \cdot \rangle_E$. Thus, let us consider the trivialisations $E|_{U_i} \cong U_i \times \mathbb{R}^k$, and, putting the Euclidean metric on the fibres, we

denote by $L^p(U_i) \doteq L^p(E|_{U_i}, dx)$ the corresponding space of L^p vector-valued functions on U_i. Then, the norm

$$\|u\|_{\widetilde{L^p}(E)} \doteq \sum_{i=1}^{r} \|\eta_i u\|_{L^p(U_i)} \tag{A.24}$$

is an equivalent norm to (A.23) for any such coordinate cover and any such partition of unity. Seeing that we can localise the definition of L^p spaces, let us define Sobolev spaces $W^{s,p}(E)$ as follows. Let $1 \leqslant p < \infty$ and $s \geqslant 0$ real numbers, define the spaces $W^{s,p}(E)$ as

$$W^{s,p}(E) = \{u \in L^p(E) : \eta_i u \in W^{s,p}(U_i, \mathbb{R}^k) \text{ for all } i = 1, \cdots, r\}. \tag{A.25}$$

equipped with the norm

$$\|u\|_{W^{s,p}(E)} \doteq \sum_{i=1}^{r} \|\eta_i u\|_{W^{s,p}(U_i, \mathbb{R}^k)}. \tag{A.26}$$

For integer values of $s = k$, these spaces are seen to be equivalently defined as the space of $L^p(E, dV_g)$ sections having weak covariant derivatives in the metric g up to order k in $L^p(E, dV_g)$. Let us highlight that these equivalences do not hold for general non-compact manifolds.

Let us now point out that, via the above localisation, we can reduce most properties of $W^{s,p}(E)$ to the corresponding properties for functions on domains of $\mathbb{R}^n$ (see Schwarz (1995, Chapter 1) for several of these explicit constructions). In particular, note that if (U_i, φ_i) is an interior chart, then $\eta_i u \in W_0^{s,p}(U_i)$, while if is a boundary chart, then $\eta_i u \circ \varphi^{-1} \in W^{s,p}(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n$ denotes the half-space $x^n \geqslant 0$ where $\varphi_i^{-1}\{x^n = 0\} = \Sigma \cap \overline{U_i}$. In particular, writing as usual $u = \sum_i \eta_i u$ for any $u \in W^{s,p}(E)$, we deduce that $C^\infty(E)$ is dense in $W^{s,p}(E)$ for all $s \geqslant 0$ and $1 \leqslant p < \infty$ and these spaces remain reflexive for $1 < p < \infty$. Furthermore, under these conditions, if M is closed, then the dual paring

$$C^\infty(E) \times C^\infty(E) \mapsto \mathbb{R},$$
$$(u, v) \mapsto \int_M \langle u, v \rangle_E \, dV_g \tag{A.27}$$

extends by continuity to $W^{-s,p}(E) \times W_0^{s,p}(E) \mapsto \mathbb{R}$, and every $f \in W^{-s,p}$ can be computed as a limit $f(v) = \lim_{k \to \infty} (f_k, v)_{L^2(E)}$, with $f_n \in C^\infty(E)$ and $v \in W^{s,p}(E)$.

Finally, the embedding, compactness, multiplication and trace properties for E can be deduced quite straightforwardly from those of each $\eta_i u$. We collect these properties in the following theorem:

Theorem A.2.5. *Let M be a compact manifold with smooth boundary Σ; π : $E \mapsto M$ be a vector bundle over M and let $W^{s,p}(E)$ denote the Sobolev space of section of E where $1 < p < \infty$ and $s \geqslant 0$ is a real number. Then, it follows that*

1. *If $0 < s < \frac{n}{p}$ and $p \leqslant q \leqslant \frac{np}{n-sp}$, then $W^{s+m,p} \hookrightarrow W^{m,q}$. If, furthermore, $q < \frac{np}{n-sp}$, then the embedding is compact;*

2. *If $s = \frac{n}{p}$, then $W^{s,p} = L^q$ for all $p \leqslant q < \infty$;*

3. *If $s > \frac{n}{p} + \alpha$, $\alpha \in (0,1)$ and $j \in \mathbb{N}$, then $W^{s+j,p} \hookrightarrow C^{j,\alpha}$. Furthermore, if s is a non-negative integer satisfying the previous hypotheses, the embedding is actually compact.*

4. *If $j + l > k + \frac{n}{p}$ where $j, l \geqslant k$ are non-negative integers, then, the multiplication mapping $(u,v) \to u \otimes v$, defines a continuous bilinear map between $W^{j,p} \times W^{l,p} \to W^{k,p}$;*

5. *If $s > \frac{1}{p}$ the trace map $u \mapsto u|_\Sigma$ defines a continuous map $\tau : W^{s,p}(M) \mapsto W^{s-\frac{1}{p},p}(\Sigma)$. Also, under the same conditions, we have a continuous extension map from $E : W^{s-\frac{1}{p},p}(\Sigma) \mapsto W^{s,p}(M)$.*

For completeness, let us explicitly highlight a useful corollary of item 1. If, under its hypotheses, we impose $p = q$, then the restrictions for the compact embedding $W^{s+m,p} \hookrightarrow W^{m,p}$ become $0 < s < \frac{n}{p}$. But, in case $s \geqslant \frac{n}{p}$, then $W^{m+s,p}$ is continuously embedded in $W^{m+s',p}$ for some $0 < s' < \frac{n}{p} < s$, and we have the chain of embeddings

$$W^{m+s,p} \hookrightarrow W^{m+s',p} \overset{\text{compact}}{\hookrightarrow} W^{m,p}.$$

Since the composition of a continuous map with a compact one gives a compact map, we find that $W^{m+s,p} \hookrightarrow W^{m,p}$ is compact for all $s > 0$.

Finally, let us finish by presenting the following *interpolation inequality*.

Theorem A.2.6. *Let M be a compact manifold with boundary Σ; $\pi : E \mapsto M$ be a vector bundle over M and let $W^{k,p}(E)$ denote the Sobolev space of section*

of E where $1 < p < \infty$ and $k \geq 0$ is a non-negative integer. For any $\epsilon > 0$ there is a number $C_\epsilon > 0$, such that for all $u \in W^{k,p}$, if and $j < k$, then

$$\|u\|_{W^{j,p}} \leq \epsilon \|u\|_{W^{k,p}} + C_\epsilon \|u\|_{L^p}. \tag{A.28}$$

Proof. Assume the statement does not hold. Then, given $\epsilon > 0$ and any constant $C > 0$, there must be some $u_C \in W^{k,p}$ for which the estimate fails. That is, for such a section u it holds that

$$\|u_C\|_{W^{j,p}} > \epsilon \|u_C\|_{W^{k,p}} + C \|u_C\|_{L^p}.$$

Let us then consider the sequence of $\{u_m\} \subset W^{k,p}$ generated in this way by the increasing selection $C = m \in \mathbb{N}$ at each step. Since clearly we must have $u_m \neq 0$, we can divide by $\|u\|_{W^{k,p}}$ the above inequality and get the analogous version for the normalised sequence. Thus, there is no loss in generality in assuming that if the statement fails, then there is a normalised sequence $\{u_m\} \subset W^{k,p}$ for which the inequality

$$1 \geq \|u_m\|_{W^{j,p}} > \epsilon + m \|u_m\|_{L^p}$$

holds for all $m \in \mathbb{N}$. But since in our setting the embedding $W^{k,p} \hookrightarrow W^{j,p}$ is compact, then, there is some $u \in W^{k,p}$ such that $u_m \xrightarrow{W^{j,p}} u$, which in particular implies that $u_m \xrightarrow{L^p} u$. Then, for the above inequality to hold for all $m \in \mathbb{N}$, we must have $u = 0$, which contradicts the normalisation condition for $\{u_m\}$ and we have found a contradiction, and thus the claim of the theorem follows. $\qquad\square$

The above interpolation inequalities can be deduced from more subtle interpolating properties, such as those given in the following theorem:

Theorem A.2.7 (Gagliardo–Nirenberg). *Let Ω be a compact manifold with smooth boundary, let $1 \leq q, r \leq \infty$, j, m be integers $0 \leq j < m$, θ be a real number in the interval $\frac{j}{m} \leq \theta \leq 1$ and*

$$\frac{1}{p} = \frac{j}{n} + \theta \left(\frac{1}{r} - \frac{m}{n} \right) + (1 - \theta) \frac{1}{q}. \tag{A.29}$$

If $m - j - \frac{n}{r}$ is not a non-negative integer, then there is a constant $C > 0$ such that, for all $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$,

$$\|D^j u\|_{L^p(\Omega)} \leq C \|u\|_{W^{m,r}(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}. \tag{A.30}$$

If $m - j - \frac{n}{r}$ is a non-negative integer, then (A.30) holds for $\theta = \frac{j}{m}$.

The above theorem is originally due to Gagliardo (1958) and Nirenberg (1959) and can also be found in Friedman (1969) and Leoni (2017).

Let us notice that using the above theorem, if we fix $q = r$, then

$$\frac{1}{p} = \frac{j - \theta m}{n} + \frac{1}{q} = (\frac{j}{m} - \theta)\frac{m}{n} + \frac{1}{q} \leqslant \frac{1}{q}.$$

Then, for any such $p \geqslant q$, as long as $m - j - \frac{n}{q}$ and θ satisfies the constraints of the above theorem, we have (A.30). This always holds for $p = q$ and $\theta = \frac{j}{m}$, then

$$\|D^j u\|_{L^q(\Omega)} \leqslant C \|u\|_{W^{m,q}(\Omega)}^{\frac{j}{m}} \|u\|_{L^q(\Omega)}^{1 - \frac{j}{m}}.$$

Let us now recall the following well-known inequality. For any $1 < \lambda, \lambda' < \infty$ satisfying $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, given $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that, for any $a, b > 0$, it holds that

$$ab \leqslant \epsilon a^\lambda + C_\epsilon b^{\lambda'}, \tag{A.31}$$

Let us then chose $\lambda = \frac{m}{j}$, $\frac{1}{\lambda'} = 1 - \frac{j}{m}$, $a = \|u\|_{W^{m,q}(\Omega)}^{\frac{j}{m}}$ and $b = \|u\|_{L^q(\Omega)}^{1 - \frac{j}{m}}$, to arrive, once again, at

$$\|D^j u\|_{L^q(\Omega)} \leqslant C \left(\|u\|_{W^{m,q}(\Omega)} + C_\epsilon \|u\|_{L^q(\Omega)} \right), \tag{A.32}$$

valid for all $j \leqslant m$.

B *Elliptic Operators*

Let us start this section considering linear partial differential operators (PDO) acting between sections of vector bundles, say $E \to M$ and $F \to M$, over a manifold M^n and let ∇ be a connection on E. Then, let us consider PDOs of the form

$$L = \sum_{l=0}^{m} A_l \cdot \nabla^l, \tag{B.1}$$

where $A_l \in \Gamma(\mathrm{Hom}((\otimes TM)^l \otimes E, F))$ and $\mathrm{Hom}(E, F)$ denotes the vector bundle of linear transformations between fibres of E and F. That is, $\mathrm{Hom}_x(E, F) = L(E_x, F_x)$, where E_x and F_x denote the fibres of E and F over $x \in M$. These operators can be written locally, in some coordinate system (U, x), as

$$L = \sum_{|\alpha| \leq m} A_\alpha(x) \partial_x^\alpha, \tag{B.2}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes a multi-index, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}}$, $|\alpha| = \sum_i \alpha_i$. We will refer to the set of such m-th order PDO as $\mathbf{PDO}^m(E, F)$.

In the analysis of PDOs, their highest order part (its *principal part*) plays a special role. Notice that for an m-th order linear PDO such as (B.1), associated to

its principal part, we can define the map

$$\sigma(L) : T^*M \mapsto \operatorname{Hom}(\pi^*E, \pi^*F),$$
$$\xi \mapsto \sigma(L)(\xi) \doteq A_m(\xi) \tag{B.3}$$

where $A_m(\xi)$ denotes the contraction between A_m and $(\otimes\xi)^m$ along the obvious indices, and π^*E, π^*F denote the corresponding pullback bundles. We will refer to this map as the *principal symbol* of the PDO L.

Definition B.1. *We will say that a PDO of the form of (B.1) is elliptic if $\sigma(L)$ defines an isomorphism at each point $x \in M$ for all $\xi \neq 0$.*

Let us illustrate the use of this definition by introducing two elliptic operators, which are actually the ones analysed in detail during the main part of this text. Consider a smooth Riemannian manifold (M, g) and define the Laplacian operator acting on functions as $\Delta_g \doteq \operatorname{tr}_g \nabla^2 = g^{ij}\nabla_i\nabla_j$, where ∇ denotes the Riemannian connection compatible with g. Then, it is a trivial exercise to see that $\sigma(\Delta_g)(\xi)$ equals the operator which multiples a given function by $|\xi|_g^2$, which proves that Δ_g is elliptic.

A slightly less trivial example is given the *conformal Killing Laplacian*. This operator is defined on Riemannian manifold (M^n, g) as the map

$$\Delta_{g,conf} : \Gamma(TM) \mapsto \Gamma(T^*M),$$
$$X \mapsto \operatorname{div}_g\left(\mathscr{L}_{g,conf} X\right) \tag{B.4}$$

where $\mathscr{L}_{g,conf} X \doteq \mathscr{L}_X g - \frac{2}{n}\operatorname{div}_g Xg$ denotes the conformal Lie derivative. It is quite straightforward to see that $\sigma(\Delta_{g,conf})_x(\xi)$ is injective for all $\xi \neq 0$ and every $x \in M$, which establishes the isomorphism property by dimensional analysis. These statements can be consulted explicitly in Choquet-Bruhat (2009, Appendix II).

Although the above two operator serve us as the motivating examples to analyse elliptic operators, we should highlight that elliptic operator form an extremely rich family and have been found to have deep geometric applications. For instance the Hodge Laplacian on differential forms, given by $dd^* + d^*d$, is an elliptic operator, where $d : \Gamma(\Lambda^k(T^*M)) \mapsto \Gamma(\Lambda^{k+1}(T^*M))$ denotes the exterior differential, $d^* : \Gamma(\Lambda^{k+1}(M)) \mapsto \Gamma(\Lambda^k(M))$ its dual operator and $\Lambda^k(T^*M)$ the bundle of differential k-forms over M. The application of elliptic theory to these operators finds geometric applications, for instance, in cohomology theory (see Jost 2005; Schwarz 1995; Taylor 2011a, for instance). Another extremely rich application is found in the analysis of Dirac operators (Taylor 2011b, see).

Formal Adjoints

Let us introduce the notion of formal adjoints for operators in $\mathbf{PDO}^m(E_1, E_2)$. Recall that, whenever we have a continuous map L acting between Banach spaces $B_1 \mapsto B_2$, we have an adjoint map $T^* : B_2' \mapsto B_1'$, given by $\langle T^*u, v \rangle \doteq \langle u, Tv \rangle$, for all $u \in B_2'$ and $v \in B_1$. Typically, we will consider PDO acting between theses type of Banach spaces and define the adjoint map just in this way. Nevertheless, C_0^∞ is typically a dense subspace of useful function spaces for PDE theory, and, therefore, having L defined as a continuous map on the closure of C_0^∞ in some norms, it tends to be the case that we can compute its adjoint on this dense subspace more explicitly, which motivates the following definition.

Definition B.2. *Consider $L \in \mathbf{PDO}(E_1, E_2)$ acting between sections of Riemannian vector bundles $E_1, E_2 \to M$ over some smooth Riemannian manifold (M, g). An operator $P \in \mathbf{PDO}(E_2, E_1)$ is said to be the formal adjoint of L if*

$$\int_M \langle Pu, v \rangle_{E_1} dV_g = \int_M \langle u, Lv \rangle_{E_2} dV_g, \quad \forall\, u \in C_0^\infty(E_2) \text{ and } \forall\, v \in C_0^\infty(E_1). \tag{B.5}$$

The following lemma is an easy, but important consequence of the definition.

Lemma B.1. *An operator $L \in \mathbf{PDO}(E_1, E_2)$ acting between sections of Riemannian vector bundles $E_1, E_2 \to M$ over some smooth Riemannian manifold (M, g) can admit, at most, one formal adjoint.*

Proof. Assume that L admits two formal adjoints $P_1, P_2 \in \mathbf{PDO}(E_2, E_1)$. Then, it must follow that

$$\int_M \langle (P_1 - P_2)u, v \rangle_{E_1} dV_g = 0 \quad \forall\, u \in C_0^\infty(E_2) \text{ and } \forall\, v \in C_0^\infty(E_1).$$

Since $\mathrm{supp}((P_1 - P_2)u) \subset \mathrm{supp}(u)$ (this property of PDO is called *locality*), then $(P_1 - P_2)u \in C_0^\infty(E_1)$. Thus, the above identity being valid for all $v \in C_0^\infty(E_1)$ clearly implies that $(P_1 - P_2)u = 0$ for each $u \in C_0^\infty(E_2)$. Now, if $u \in C^\infty(E_2)$ is not compactly supported, use a partition of unity $\{\eta_\alpha\}_{\alpha \in \mathcal{I}}$ subordinate to some open cover of M to write

$$(P_1 - P_2)u = \sum_\alpha \underbrace{(P_1 - P_2)(\eta_\alpha u)}_{=0\,\forall\,\alpha} = 0,$$

since $\eta_\alpha u \in C_0^\infty(E_2)$ for all $\alpha \in \mathcal{I}$. $\qquad\square$

Another direct consequence of the above definition is related to the adjoint of a composition of PDOs. If $L_1 : C^\infty(E_1) \mapsto C^\infty(E_2)$ and $L_2 : C^\infty(E_2) \mapsto C^\infty(E_3)$ admit formal ajoints $L_1^* : C^\infty(E_2) \mapsto C^\infty(E_1)$ and $L_2^* : C^\infty(E_3) \mapsto C^\infty(E_2)$, then $L_2 \circ L_1 : C^\infty(E_1) \mapsto C^\infty(E_3)$ admits a formal adjoint given by

$$(L_2 \circ L_1)^* = L_1^* \circ L_2^*.$$

Furthermore, we have the following result concerning existence of formal adjoints.

Proposition B.1. *Let $E_1, E_2 \to M$ be two Riemannian vector bundles over a smooth Riemannian manifold (M^n, g). Then, any $L \in \mathbf{PDO}(E_1, E_2)$ admits a formal adjoint L^*.*

Proof. Let us start by localising the problem. Thus, consider a covering $\{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{I}}$ of M by coordinate systems such that, on each of them, we have orthonormal frames $\{e_i\}_{i=1}^{k_1}$ and $\{f_I\}_{I=1}^{k_2}$ for E_1 and E_2 respectively, with $k_i = \dim(E_i)$. Let $\{\eta_\alpha\}_\alpha$ be a partition of unity subordinate to such a covering and consider $u_i \in C_0^\infty(E_i)$, $i = 1, 2$. Then, write $u_i = \sum_\alpha \eta_\alpha u_i$ and compute

$$\int_M \langle Lu_1, u_2 \rangle_{E_2} dV_g = \sum_\alpha \int_{\mathcal{U}_\alpha} \langle L(\eta_\alpha u_1), u_2 \rangle_{E_2} dV_g,$$

where, locally, we have

$$L(\eta_\alpha u_1)^I = \sum_{|\beta| \leq m} \sum_{j=1}^{k_1} A_{j\beta}^I \partial^\beta(\eta_\alpha u_1^j), \quad \eta_\alpha u_1 \in C_0^\infty(\mathcal{U}_\alpha).$$

and $I = 1, \cdots, k_2$. We can then write $\langle A_\beta \partial^\beta(\eta_\alpha u_1), u_2 \rangle_{E_2} = \langle \partial^\beta(\eta_\alpha u_1), A_\beta^* u_2 \rangle_{E_1}$, where A_β^* stands for the adjoint of the linear map A_β. Therefore, it follows that

$$\int_{\mathcal{U}_\alpha} \langle L(\eta_\alpha u_1), u_2 \rangle_{E_2} dV_g = \sum_{|\beta| \leq m} \int_{\mathcal{U}_\alpha} \langle \partial^\beta(\eta_\alpha u_1), A_\beta^* u_2 \rangle_{E_1} dV_g.$$

We can simplify computations by writing u_1 and $A_\beta^* u_2$ in components corresponding to $\{e_j\}_{j=1}^{k_1}$, so that

$$\int_{\mathcal{U}_\alpha} \langle \partial^\beta(\eta_\alpha u_1), A_\beta^* u_2 \rangle_{E_1} dV_g = \sum_{i=1}^{k_1} \int_{\mathcal{U}_\alpha} \partial^\beta(\eta_\alpha u_1)^i \left(A_\beta^* u_2 \right)^i \sqrt{\det(g)} dx,$$

Since $\eta_\alpha u_1 \in C_0^\infty(\mathcal{U}_\alpha)$, classical integration by parts shows that each ∂_{x^i} has a (local) ajoint given by $-\partial_i - \partial_i \ln(\sqrt{\det(g)})$, therefore giving

$$\sum_{|\beta|\leqslant m} \int_{\mathcal{U}_\alpha} \langle \partial^\beta(\eta_\alpha u_1), A_\beta^* u_2 \rangle_{E_1} dV_g = \int_{\mathcal{U}_\alpha} \langle u_1, \eta_\alpha \sum_{i=1}^{k_1} \sum_{|\beta|\leqslant m} (\partial^\beta)^* \left(A_\beta^* u_2\right)^i e_i \rangle_{E_1} dV_g.$$

Let us then define

$$(L_\alpha^* \phi)^i \doteq \eta_\alpha \sum_{|\beta|\leqslant m} (\partial^\beta)^* \left(A_\beta^* \phi\right)^i \quad \forall \phi \in C^\infty(E_2|_{\mathcal{U}_\alpha}).$$

Summing over α, we find

$$\int_M \langle Lu_1, u_2 \rangle_{E_2} dV_g = \int_M \langle u_1, L^* u_2 \rangle_{E_2} dV_g,$$

where $L^* : C^\infty(E_2) \mapsto C^\infty(E_1)$ is defined by

$$L^*(u) = \sum_{\alpha \in \mathcal{I}} L_\alpha^* u, \quad \forall u \in C^\infty(E_2).$$

Notice that the uniqueness of L^* establishes that L^* is consistently defined independently of the choices of our open coordinate cover, partition of unity or orthonormal frames. $\qquad \square$

Remark B.1. *Let us highlight the dependence of the formal adjoint in the above proposition on the Riemannian structures of both E_1 and E_2 as well as (M, g).*

We can extract an important corollary from the above computations. Notice that if $L \in \mathbf{PDO}^m(E_1, E_2)$, then the principal part of L^* is given (up to a sign) in any coordinate system by

$$\sum_{|\beta|=m} A_\beta^* \partial^\beta,$$

from which we find that

$$\sigma(L^*) = \pm(\sigma(L))^*, \tag{B.6}$$

where in the right hand side the adjoints are taken as linear maps between finite dimensional inner product spaces. That is, $(\sigma(L))^*$ is basically the point wise transpose of the matrix representation of $\sigma(L)$ as a linear map from $(E_1)_x \mapsto (E_2)_x$ for each $x \in M$. In particular, this shows that L elliptic implies L^* elliptic. All of the above translates equally well for complex vector bundles over M with hermitian metrics (see Nicolaescu 2020, Chapter 9).

Analysis on closed manifolds

Let us know introduce part of the basic skeleton of the theory of linear elliptic operators restricting ourselves for the most part to closed manifolds. This will allow us to localise the analysis and reduce it to its analysis on $\mathbb{R}^n$ by standard arguments and, at the same time, have at our disposal all the standard functional analytic properties associated to appropriate function spaces, such as Sobolev spaces, which are specially well-suited to serve as appropriate domains for PDE operators to act on.

The objective of this section is to exhibit results such as elliptic estimates, basic regularity theory and Fredholm properties of linear elliptic operators in the above restricted framework. Some extensions of this analysis, involving boundary value problems and non-compact manifolds are explicitly treated in the main part of this text. Finally, let us highlight a few standard extremely detailed references for the interested reader, such as Taylor (2011a, Chapter 5 for a detailed analysis of the corresponding L^2-theory), Gilbarg and Trudinger (2001, for an exhaustive analysis of second order scalar elliptic equations) and Hörmander (2007, for a thorough analysis through pseudo-differential operator techniques).

Taking into consideration the above comments, let us start with the analysis of local theory in $\mathbb{R}^n$ and consider a trivial vector bundle $E \to \mathbb{R}^n$ over $\mathbb{R}^n$ with fibre $\mathbb{R}^k$. Then consider an elliptic operator of the form

$$L = \sum_{|\alpha| \leqslant m} A_\alpha(x)\partial^\alpha : C^\infty(E) \mapsto C^\infty(E) \tag{B.7}$$

where A_α are smooth maps from $\mathbb{R}^n$ to the set of $k \times k$ matrices. For fixed $r \in \mathbb{N}_0$ and $R > 0$ define

$$\|A\|_{k,R} \doteq \sum_{\substack{|\alpha| \leqslant m \\ \beta \leqslant r+m-|\alpha|}} \sup_{x \in B_R(0)} \|\partial^\beta A_\alpha(x)\|$$

Theorem B.1. *Consider an operator such as (B.7) which has the special form* $L_0 = A_m \partial^m$ *and* $A_m : \mathbb{R}^n \mapsto \mathrm{GL}(\mathbb{R}^k, \mathbb{R}^k)$ *is a constant matrix. Fix* $p \in (1, \infty)$ *and* $R > 0$. *Then, there is some constant* $C = C(L, p, n, R) > 0$ *such that for all* $u \in C_0^\infty(E|_{B_R(0)})$ *the following estimate holds*

$$\|u\|_{W^{m,p}} \leqslant C \left(\|Lu\|_{L^p} + \|u\|_{L^p}\right). \tag{B.8}$$

Proof. For this special case, we can start by considering $v = L_0 u$, with $u, v \in C_0^\infty(E|_{B_R(0)})$. Since L_0 has constant coefficients, it holds that

$$\partial^\beta v = L_0(\partial^\beta u)$$

for any multi-index β. In particular, we are interested in the case $|\beta| = m$. Now, let us apply (component wise) the Fourier transform to this identity, that is map

$$v = \left(v^1, \cdots, v^k\right) \mapsto \hat{v} \doteq \left(\hat{v}^1, \cdots, \hat{v}^k\right)$$

so that

$$\sigma\left(A_m\right)(\xi) \cdot \widehat{\partial^\beta u} = \xi^\beta \, \hat{v}.$$

Since L_0 is elliptic, then $\sigma\left(A_m\right)(\xi)$ is an isomorphism for all $\xi \neq 0$ which is homogeneous of degree m in ξ. In particular, its inverse is homogeneous of degree $-m$ and therefore $\hat{m} \doteq \sigma\left(A_m\right)(\xi)^{-1} \xi^\beta$ is homogeneous of degree zero. In particular, the components $\hat{m}_{ij}$ are in $L^1_{loc}(\mathbb{R}^n, d\xi)$ and therefore, through Theorem A.1.7 they define tempered distributions, and therefore there exist $m_{ij} \in \mathcal{S}'$ such that $\hat{m}_{ij} \in \mathcal{S}'$ are their Fourier transforms. Then, since $\hat{v}_j \in \mathcal{S}$, the equation

$$\widehat{\partial^\beta u_i} = \sum_j \hat{m}_{ij} \hat{v}_j \tag{B.9}$$

can be rewritten in convolution terms as $\partial^\beta u_i = \sum_j m_{ij} * v_j$. Also, the distributions m_{ij} satisfy the hypotheses of the distribution k in Theorem A.1.8 and $v \in L^p \cap \mathcal{E}'$ for all $p \in (1, \infty)$. Then, we can estimate

$$\|\partial^\beta u_i\|_{L^p} \leqslant \sum_j \|m_{ij} * v_j\|_{L^p} \leqslant C(m, p)\|v\|_{L^p} = C(m, p)\|L_0 u\|_{L^p} \tag{B.10}$$

for all $|\beta| = m$. With this we get estimates for the top derivative norms in $\|u\|_{W^{m,p}}$ in terms of $\|L_0 u\|_{L^p}$. Now, the intermediary derivatives can be estimated via interpolation, so as to get

$$\begin{aligned}
\|u\|_{W^{m,p}} \leqslant{}& C(A, p)\|L_0 u\|_{L^p} + c(n, m)\epsilon\|u\|_{W^{m,p}} \\
& + c(n, m)C_\epsilon\|u\|_{L^p} + \|u\|_{L^p}.
\end{aligned} \tag{B.11}$$

Finally, let us pick $\epsilon < \frac{1}{c(n,m)}$, and pick $C = \frac{1}{1-\epsilon}\max\{C(A, p), c(n, m)C_\epsilon + 1\}$, we find the desired estimate

$$\|u\|_{W^{m,p}} \leqslant C\left(\|L_0 u\|_{L^p} + \|u\|_{L^p}\right). \tag{B.12}$$

$\square$

Theorem B.2 (Local Elliptic estimates). *Consider the operator (B.7), fix $p \in (1, \infty)$ and $R > 0$. Then, there is some constant $C = C(L, p, n, R) > 0$ such that for all $u \in C_0^\infty(E|_{B_R(0)})$ the following estimate holds*

$$\|u\|_{W^{m,p}} \leq C \left(\|Lu\|_{L^p} + \|u\|_{L^p} \right). \tag{B.13}$$

Proof. The proof runs as follows. Let us first cover $\overline{B_R(0)}$ by a finite number of small balls of the form $B_r(x_i)$, centred at points $\{x_i\}_{i=1}^M \subset \overline{B_R(0)}$ and with radii $r > 0$ small, to be fixed latter. Then, consider a partition of unity subordinate to this cover of the form $\{\eta_i\}_{i=1}^M$ and localise any such u via

$$u = \sum_i \eta_i u, \tag{B.14}$$

where we will denote $u_i \doteq \eta_i u$. Then, defining $v_i \doteq L u_i$ and $L_i \doteq A_m(x_i)\partial^m$ we can rewrite

$$v_i = L_i u_i + (A_m(x) - A_m(x_i)) \partial^m u_i + \sum_{|\alpha| < m} A_\alpha(x) \partial^\alpha u_i. \tag{B.15}$$

Let us then apply (B.8) to u_i and L_i, so as to get[1]

$$\|u_i\|_{W^{m,p}(B_r(x_i))} \leq C \Big(\|v_i\|_{L^p(B_r(x_i))} + \| (A_m(x) - A_m(x_i)) \partial^m u_i \|_{L^p(B_r(x_i))}$$
$$+ \sum_{|\alpha| < m} \|A_\alpha(x)\partial^\alpha u_i\|_{L^p(B_r(x_i))} + \|u_i\|_{L^p(B_r(x_i))} \Big),$$
$$\leq C(\|v_i\|_{L^p(B_r(x_i))} + \| (A_m(x) - A_m(x_i)) \partial^m u_i \|_{L^p(B_r(x_i))}$$
$$+ \|u_i\|_{W^{m-1,p}(B_r(x_i))} + \|u_i\|_{L^p(B_r(x_i))}),$$

where the constant C in the second line depends on the norms of the A_α coefficients on $B_R(0)$. Notice that we can use the mean value inequality to estimate $|A_m(x) - A_m(x_i)| \leq C'r$ on $B_r(x_i)$, where the constant C' can be made independent of r by taking $C' = \sup_{y \in B_R(0)} |DA_m(y)|$, which is bounded by hypothesis. Thus, we find that

$$\|u_i\|_{W^{m,p}(B_r(x_i))} \leq C(\|v_i\|_{L^p(B_r(x_i))} + r\|\partial^m u_i\|_{L^p(B_r(x_i))} + \|u_i\|_{W^{m-1,p}(B_r(x_i))}$$
$$+ \|u_i\|_{L^p(B_r(x_i))}),$$

---

[1] The constants C can change from line to line.

where $C = C(L, p, n, m)$. Notice that the above expression can be simplified since $\|\partial^m u_i\|_{L^p(B_r(x_i))} \lesssim \|u_i\|_{W^{m,p}(B_r(x_i))}$ and then, we can fix r so that $Cr < \frac{1}{2}$ for instance, so that

$$\|u_i\|_{W^{m,p}(B_r(x_i))} \leq C(\|v_i\|_{L^p(B_r(x_i))} + \|u_i\|_{W^{m-1,p}(B_r(x_i))} + \|u_i\|_{L^p(B_r(x_i))}),$$

Let us now estimate the first term in the right-hand side. Notice that $v_i = L(\eta_i u) = \eta_i L u + [L, \eta_i]u$ and that $[L, \eta_i]u$ is an operator of order $m - 1$ whose coefficients are smooth and bounded to all orders. In fact, we can bound the derivatives of all the η_i up to order m by some constant. Then, $\|[L, \eta_i]u\|_{L^p(B_r(0))} \lesssim C\|u\|_{W^{m-1,p}(B_r(x_i))} \leq C\|u\|_{W^{m-1,p}(B_R(0))}$, where C depends on the norm of L but not on r. Also $\|\eta_i L u\|_{L^p(B_r(x_i))} \lesssim \|L u\|_{L^p(B_r(x_i))}$, therefore, we find that

$$\|u_i\|_{W^{m,p}(B_r(x_i))} \leq C(\|L u\|_{L^p(B_R(0))} + \|u\|_{W^{m-1,p}(B_R(0))} + \|u\|_{L^p(B_R(0))}).$$

Summing all the contributions we get

$$\|u\|_{W^{m,p}(B_R(0))} \leq C(\|L u\|_{L^p(B_R(0))} + \|u\|_{W^{m-1,p}(B_R(0))} + \|u\|_{L^p(B_R(0))}), \tag{B.16}$$

We can then use interpolation to get rid of the intermediate spaces, so that

$$\|u\|_{W^{m,p}(B_R(0))} \leq C(\|L u\|_{L^p(B_R(0))} + \|u\|_{L^p(B_R(0))}), \tag{B.17}$$

which is the desired estimate. $\qquad\square$

Let us notice that, given $u \in C_0^\infty(E|_{B_R(0)})$, differentiating the equation $Lu = v$ we get a linear elliptic equation of the form $L(\partial^\beta u) = v^\beta$, where v^β involves derivatives up to order $|\beta|$ of the coefficients A_α. As long as these derivatives remain uniformly bounded, we can apply the above estimates to $\partial^\beta u$ and get $W^{k+m,p}$ estimates in terms of $W^{k,p}$ norm of Lu. Below, we will present another related result which provides local *interior estimates* for an arbitrary $u \in C^\infty(E)$. Its proof relies on an application of the above elliptic estimates to a function of the form ηu, where η is a cut-off function equal to one on $B_r(0)$ for some chosen $0 < r < R$ and compactly supported inside of $B_R(0)$. In this case, a more careful choice of our cut-off function η, put together with interpolation inequalities establishes the following corollary. Such a construction can be consulted explicitly in Gilbarg and Trudinger (2001, Theorem 9.11) for the case of second order operators, and analogous arguments work for the general case.

Corollary B.1. *Consider the operator (B.7), fix $p \in (1, \infty)$ and $0 < r < R$. Then, there is some constant $C = C(L, p, n, R) > 0$ such that for all $u \in C^\infty(E)$ the following estimate holds*

$$\|u\|_{W^{m,p}(B_r(0))} \leqslant C \left(\|Lu\|_{L^p(B_R(0))} + \|u\|_{L^p(B_R(0))} \right). \tag{B.18}$$

Another consequence of Theorem B.2 is that, since $C_0^\infty(E|_{B_R})$ is dense in $W_0^{m,p}(E|_{B_R})$, and, furthermore, operators such as B.7 define continuous maps $L : W_0^{m,p}(E|_{B_R}) \mapsto L^p(E|_{B_R})$, then the following results also follows.

Corollary B.2. *Consider the operator (B.7), fix $p \in (1, \infty)$ and $R > 0$. Then, there is some constant $C = C(L, p, n, R) > 0$ such that for all $u \in W_0^{m,p}(E|_{B_R(0)})$ the following estimate holds*

$$\|u\|_{W^{m,p}} \leqslant C \left(\|Lu\|_{L^p} + \|u\|_{L^p} \right). \tag{B.19}$$

Now, let us notice that the above analysis for trivial bundles over $\mathbb{R}^n$ can be readily extended to closed manifolds by standard localisation arguments. That is, let $E, F \to M$ be vector bundles over a closed manifold M. Let us fix some (finite) covering by small coordinate balls $\{B_r(x_i)\}$ trivialising both E and F. Then, any linear differential operator with smooth coefficients acting between sections of E and F can be locally written as an operator of the form (B.7), but with coefficients in $\text{Hom}(E, F)$. In particular, if this operator is elliptic, then the fibres of both bundles are isomorphic to some fixed $\mathbb{R}^k$ and, over any of these common trivialisations, we can see L as an operator acting from $W_0^{m,p}(E_i') \mapsto L^p(E_i')$, with $E_i' = B_r(x_i) \times \mathbb{R}^k$ and the coefficients are smooth $k \times k$ matrices such as is (B.7), and therefore we reduce, locally, the general case to the analysis of (B.7). Also, given a partition of unity $\{\eta_i\}$ subordinate to our coordinate cover, we can localise any section $u = \sum_i \eta_i u$ and apply the above local results to each $u_i = \eta_i u \in W_0^{m,p}(E_i')$. In particular, we get estimates of the form of (B.19) of reach u_i and adding them up we get the analogous estimate for u, where the elliptic constant is the (finite) sum of all the local ones. That is, the following theorem holds:

Theorem B.3. *Let M be a smooth compact manifold (possibly with boundary), $E, F \to M$ Riemannian vector bundles over M and L an elliptic linear partial differential operator of order m with smooth coefficients acting between section of E and F. Then, the following elliptic estimate holds for all $u \in W_0^{m,p}(E)$*

$$\|u\|_{W^{m,p}(E)} \leqslant C \left(\|Lu\|_{L^p(F)} + \|u\|_{L^p(F)} \right). \tag{B.20}$$

In particular, if M is closed, then the same estimate holds for all $u \in W^{m,p}(E)$

Remark B.2. *In the above theorem, notice that if $\partial M \neq \emptyset$ then $W_0^{m,p} \subsetneq W^{m,p}$, since $W_0^{m,p}$ is the closure of $C_0^\infty(\overset{\circ}{M})$.*

From the above theorem, we can deduce the following regularity result.

Theorem B.4. *Let (M, g) be a smooth compact Riemannian manifold (possibly with boundary), $E, F \to M$ be Riemannian vector bundles over M and let $L : W_0^{m,p}(E) \mapsto L^p(F)$, $1 < p < \infty$, be an elliptic linear partial differential operator of order m with smooth coefficients. Then, for any $u \in W_0^{m,p}(E)$, the following implication holds*

$$Lu \in W_0^{k,p}(F) \Rightarrow u \in W_0^{m+k,p}(E), \ k \geqslant 1, \tag{B.21}$$

accompanied by the corresponding improved interior estimate

$$\|u\|_{W^{m+k,p}(E|_\Omega)} \leqslant C \left(\|Lu\|_{W^{k,p}(F|_V)} + \|u\|_{L^p(E|_V)} \right), \tag{B.22}$$

for any open sets $\Omega \subset\subset V \subset\subset \overset{\circ}{M}$.

Proof. Consider any cut-off function η supported in a small ball B_r trivialising both E and F within a coordinate system with coordinates $\{x^i\}_{i=1}^n$ and write $\varphi \doteq \eta u$, u satisfying the hypotheses of the theorem. Then, it follows that $L\varphi \in W_0^{k,p}(E|_{B_r})$. Let us assume $k = 1$ and establish $\varphi \in W_0^{m+1,p}(E|_{B_r})$. For h a sufficiently small real number and $j = 1, \cdots, n$, consider the operators $D_{j,h}\varphi(x) = \frac{\varphi(x+he_j)-\varphi(x)}{h} = \frac{(\tau_{j,h}\varphi)(x)-\varphi(x)}{h}$, where e_j denotes the j-th canonical basis vector. We clearly have $D_{j,h}\varphi(x) \in W_0^{m,p}(E|_{B_r})$ and we can apply to it interior elliptic estimates to get[2]

$$\begin{aligned}
\|D_{j,h}\varphi\|_{W_0^{m,p}(E|_{B_r})} &\leqslant C \left(\|L\left(D_{j,h}\varphi\right)\|_{L^p(F|_{B_r})} + \|D_{j,h}\varphi\|_{L^p(E|_{B_r})} \right), \\
&\leqslant C(\|D_{j,h}\left(L\varphi\right)\|_{L^p(F|_{B_r})} + \|[D_{j,h}, L]\varphi\|_{L^p(E|_{B_r})} \\
&\quad + \|D_{j,h}\varphi\|_{L^p(E|_{B_r})}).
\end{aligned}$$

Notice that for an operator of the form of (B.7) it holds that $[D_{j,h}, A_\alpha \partial^\alpha]\varphi = (D_{j,h}A_\alpha)\partial^\alpha(\tau_{j,h}\varphi)(x)$ for any $|\alpha| \leqslant m$. Since the coefficients A_α are smooth,

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[2]During this proof, as well as other parts of the text, a constant C appearing in estimates may change from line to line, avoiding the introduction of new constants, $C', C'', \cdots$.

we can estimate $\|D_{j,h} A_\alpha\|_{C^0(B_r)} \leqslant C_\alpha$ by a constant depending of the norm of the derivative of A_α and independent of h. We can therefore estimate

$$\|[D_{j,h}, L]\varphi\|_{L^p(F|_{B_r})} \leqslant C\|\varphi\|_{W^{m,p}(E|_{B_r})},$$

for some constant C depending on L but independent of h. We therefore get

$$
\begin{aligned}
\|D_{j,h}\varphi\|_{W_0^{m,p}(E|_{B_r})} &\leqslant C(\|D_{j,h}(L\varphi)\|_{L^p(F|_{B_r})} + \|\varphi\|_{W^{m,p}(E|_{B_r})} \\
&\quad + \|D_{j,h}\varphi\|_{L^p(E|_{B_r})}),
\end{aligned}
\tag{B.23}
$$

for some other constant $C > 0$ independent of h. This shows that we can take the limit $h \to 0$ and see that, for any $j = 1, \cdots, n$, the weak derivative $\partial_j \varphi$ satisfies a bound of the form

$$\|\partial_j \varphi\|_{W_0^{m,p}(E|_{B_r})} \leqslant C\left(\|L\varphi\|_{W^{1,p}(F|_{B_r})} + \|\varphi\|_{W^{m,p}(E|_{B_r})}\right),$$

and therefore $\varphi \in W_0^{m+1,p}(E|_{B_r})$. Now, a partition-of-unity type argument proves that $u \in W_0^{m+1,p}(E)$. Furthermore, the same line of reasoning that establishes the interior estimates of Corollary B.1, establishes their improved $W^{m+1,p}$-local version. Finally, fixing our choices $\Omega \subset\subset V \subset\subset \overset{\circ}{M}$, and covering Ω by sufficiently small balls where these improved interior estimates work, we get

$$\|u\|_{W^{m+1,p}(E|_\Omega)} \leqslant C\left(\|L\varphi\|_{W^{1,p}(F|_V)} + \|\varphi\|_{L^p(E|_V)}\right) \tag{B.24}$$

This now establishes an inductive proof, which establishes the claim.

$\square$

Remark B.3. *It follows from the proof above that, in case the manifold M in the above theorem is closed, then, the interior estimate (B.22) holds on M itself. Furthermore, let us highlight that if $\partial M \neq \emptyset$, our restriction to $W_0^{m,p}$ can be understood as the analysis for boundary value problems with trivial boundary conditions. The more general case of non-trivial boundary conditions and the corresponding analysis on $W^{m,p}$-spaces with $W^{m-\frac{1}{p},p}$-traces on ∂M relies on the analysis of* boundary estimates. *This will be treated explicitly when necessary.*

Let us now appeal to the above theorem to prove another fundamental result.

Theorem B.5. *Let M be a closed manifold and consider an elliptic operator L satisfying the hypotheses of Theorem B.3 the following claims follow:*

1. Ker(L) *is finite dimensional and* Im(L) *is closed;*

2. *If* Ker(L) $= 0$*, then there is a constant* $C = C(L, n, p)$ *such that the following estimate holds for all* $u \in W^{m,p}(E)$

$$\|u\|_{W^{m,p}(E)} \leqslant C\|Lu\|_{L^p(E)} \tag{B.25}$$

3. *A function* $f \in L^p(E) \in$ Im(L) *if and only if*

$$\int_M \langle f, \phi \rangle_E \, dV_g = 0 \ \forall \ \phi \in \text{Ker}(L^*). \tag{B.26}$$

In particular, L is surjective iff L^* is injective.

Proof. In order to prove the first claim, notice that Ker(L) $\subset W^{m,p}$ is closed and therefore a Banach space in its own right, equipped with the $W^{k,p}$ norm. Let us show that under the hypotheses of item (1) the closed unit ball B in Ker(L) is compact. Thus, let $\{u_k\} \subset$ Ker(L) be a normalised sequence. From the elliptic estimates, it then follows that $\|u_k - u_l\|_{W^{m,p}} \leqslant C\|u_k - u_l\|_{L^p}$. Since the embedding $W^{m,p} \hookrightarrow L^p$ is compact, then we can restrict to a subsequence which converges in L^p, establishing that the sequence is Cauchy in $W^{m,p}$ and therefore there is a limit $u \in W^{m,p}$. This establishes that $B \subset$ Ker(L) is compact. Let us now show that Im(L) is closed. First, let us prove the following claim:

Claim: Let Ker$^{\perp}(L)$ be a complement for Ker(L) in $W^{k,p}$.[3] Then, there is a constant $C > 0$ such that following estimate holds for all $u \in$ Ker$^{\perp}(L)$

$$\|u\|_{W^{m,p}(E)} \leqslant C\|Lu\|_{L^p(E)}. \tag{B.27}$$

Assume the above estimate does not hold. Then, there must be a normalised sequence $\{u_k\} \subset$ Ker$^{\perp}(L)$ such that $\|Lu_k\|_{L^p(E)} \to 0$. Since L^p is complete, $\{Lu_k\}$ admits a Cauchy subsequence and let us restrict form now on to the corresponding subsequence $\{u_k\}$. Since $\{u_k\}$ admits a convergent L^p subsequence, to which we once more restrict, applying the elliptic estimates to it, we find that $\|u_k - u_l\|_{W^{m,p}} \leqslant C(\|Lu_k - Lu_l\|_{L^p} + \|u_k - u_l\|_{L^p})$, where the right hand side goes to zero and thus $\{u_k\}$ is Cauchy in $W^{m,p}$ with limit denoted by $u \in W^{m,p}$. Since Ker$^{\perp}(L)$ is closed, then $u \in W^{m,p} \cap$ Ker$^{\perp}(L)$ and, also,

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[3]Notice that, since Ker(L) is finite dimensional, we know that such complement exists due to an application of the Hahn–Banach theorem (see, for instance, Brezis (2011, Section 2.4, Chapter 2)).

we must have $||u||_{W^{m,p}} = 1$. But, by continuity of L and construction of $\{u_k\}$, it also holds that $Lu = 0$, which contradicts $u \in \mathrm{Ker}^{\perp}(L)$. Thus, the estimate (B.27) must hold.

Having established (B.27), first notice that we need only show that $L(\mathrm{Ker}^{\perp}(L))$ is closed in L^p. Then, given a convergent sequence in $\mathrm{Im}(L)$ is a sequence of the form Lu_k, with limit say $v \in L^p$ and with $u_k \in W^{k,p}$, above estimate implies that the sequence $\{u_k\}$ converges in $W^{m,p}$, say to $u \in W^{m,p}$. But then, by continuity of L, we have $Lu_k \xrightarrow{L^p} Lu = v$, which establishes the claim. Also, notice that (B.25) follows from (B.27) whenever L is injective.

Finally, let us notice that (B.26) follows from Theorem A.1.3, since $L^* : L^{p'} \mapsto W^{-m,p'}$ where the action of $L^{p'} = (L^p)'$ on L^p is given by the dual pairing

$$L^{p'} \times L^p \mapsto \mathbb{R},$$

$$(\phi, f) \mapsto \phi(f) = \int_M \langle f, \phi \rangle_E \, dV_g$$

$\square$

Let us highlight that the above theorem proves that the contemplated class of elliptic operators are semi-Fredholm maps. In fact, we can say more than that. If L is an elliptic operator satisfying the hypotheses of the above theorem, then we know that the *formal adjoint* L^* of L is an elliptic m-th order operator with smooth coefficients, which satisfies all the hypotheses of the above theorem as well. In particular, $L^* : C^{\infty}(E) \mapsto C^{\infty}(E)$ extends by continuity to the actual adjoint operator on $L^{p'} \mapsto W^{-m,p'}$, described the duality $\langle L^*u, v \rangle = \langle u, Lv \rangle$ for all $u \in L^{p'}(E)$ and all $v \in W^{m,p}(E)$. *If we knew that* $\mathrm{Ker}(L^* : L^{p'} \mapsto W^{-m,p'}) = \mathrm{Ker}(L^* : W^{m,p'} \mapsto L^{p'})$, then the above theorem would imply that these spaces are finite dimensional. Then, through Theorem A.1.3, we would conclude that $\mathrm{Coker}\, L$ is finite dimensional and hence L would be a Fredholm map. Up to this point, such a claim is conditional upon a regularity claim concerning the kernel of the elliptic operator L^*. The fact that such regularity claim is valid under very general assumptions, is known as *elliptic regularity* (see Hörmander 2007, Chapter XIX).

Using the above paragraph as a motivation, let us briefly discuss some regularity properties associated to elliptic equations. First, recall the following definitions

concerning different notions of solutions to a PDE of the form

$$Lu = v, \tag{B.28}$$

where L is given as a linear operator of the form of (B.8).

Definition B.3. *Let $u \in \Gamma(E)$ and $v \in \Gamma(F)$ be measurable sections of the vector bundles $E, F \to M$ over a smooth Riemannian manifold (M, g). We say that: 1) u is classical solution of the (B.28) if there is some $\alpha \in (0, 1)$ such that $v \in C^{0,\alpha}_{loc}$, $u \in C^{2,\alpha}_{loc}$ and the equation holds everywhere. 2) u is said to be a strong L^p solution if $v \in L^p_{loc}$ and $u \in W^{m,p}_{loc}$ and (B.28) holds almost everywhere. 3) u is said to be an L^p weak solution if $u, v \in L^p_{loc}$ and*

$$\int_M \langle u, L^* \phi \rangle_E \, dV_g = \int_M \langle v, \phi \rangle_F \, dV v_g \quad \forall \, \phi \in C^\infty_0(F). \tag{B.29}$$

Let us first highlight that Theorem B.4 implies that the regularity of strong solutions to (B.28) is controlled by Lu. Also, clearly, any strong solution is a weak solution and a family of results within *elliptic regularity* prove the converse under different degrees of smoothness on the coefficients and the original weak solution. In particular, if L has smooth coefficients, $Lu = f$ weakly and $f \in C^\infty$, then $u \in C^\infty$. Nevertheless, these results (even for operator with smooth coefficients) rely on more subtle constructions than those presented above. In particular, they rely on the construction of *parametrices* and *fundamental solutions* associated to the elliptic constant coefficient operator obtained by freezing the coefficients of the principal part of L. In this context the notion of fundamental solution is well-known, and a parametrix is basically a distribution which differs from the fundamental solution via smooth functions (Hörmander 1990, Definition 7.1.21). For details on these constructions, we refer the reader to references such as Hörmander (2005, 2007) and Taylor (2011a,c).

The tools presented above are good enough to analyse existence and regularity of solutions to linear elliptic PDEs with smooth coefficients on closed manifolds, which contemplates a wide variety of interesting situations. Nevertheless, we will be interested in more particular situations, such as operator with low regularity coefficients. Below, we will comment on these cases. Let us notice that achieving elliptic estimates, which are the key to establishing Theorem B.5, actually relied in Theorem B.1 plus a standard localisation technique and the idea of *freezing coefficients* with these localisations around a finite fixed set of points. In this process

the extra regularity of the coefficients is only necessary to obtain extra regularity for the solutions. In case we have less regular coefficients, the same method of proof works fine, but paying attention to multiplication properties of the coefficients, which can be based, for instance, on Sobolev multiplication properties. In this scenario, the above theory for linear systems of arbitrary order adapted to *low regularity coefficients* can be found in the L^2-regularity case in Choquet-Bruhat and Christodoulou (see 1981, Section 3) and for even lower regularity (so called *rough* coefficients) in Maxwell (see 2006, Section 3), while the corresponding $W^{m,p}$-regularity theory for rough coefficients can be found, for instance, in Holst, Nagy, and Tsogtgerel (see 2009, Appendix A.5). In the case of second order equations with low regularity coefficients, a very nice presentation is given Choquet-Bruhat (see 2009, Appendix II). More specifically, these extensions to low regularity have the following form.

Consider a linear elliptic operator L on a closed manifold M which can be locally written as an operator of the form (B.8) and whose coefficients satisfy the following regularity requirements

$$A_\alpha \in W^{|\alpha|,p}(\mathbb{R}^n, \mathbb{R}^{k \times k}), \quad \forall \ 0 \leqslant |\alpha| \leqslant m, \tag{B.30}$$

where, as usual, k is the dimension of the fibre of E. A priori, we can think of L as a map from $C^\infty(E) \mapsto \mathcal{D}'(E)$. But actually, the following holds:

Lemma B.2. *Consider a linear elliptic operator $L : C^\infty \mapsto \mathcal{D}'(E)$ of order m whose coefficients satisfy (B.30) on a closed manifold M^n. Then, for $p > \frac{n}{m}$, L extends to a bounded map from $W^{m,p}(E) \mapsto L^p(E)$.*

The proof of the above Lemma is basically an application of the Sobolev multiplication properties. A more general version of the same result, which extends L to less regular spaces, can be found in Holst, Nagy, and Tsogtgerel (see 2009, Lemma 31). Then, as explained above, via localisation and freezing of coefficients arguments, the following result follows, which generalises Theorem B.5.

Theorem B.6. *Let M^n be a closed manifold and L a linear elliptic operator of order m satisfying the regularity conditions (B.30). Then, for $p > \frac{n}{m}$, there is a constant $C > 0$ such that*

$$||u||_{W^{m,p}} \leqslant C \left(||Lu||_{L^p} + ||u||_{L^p} \right) \quad \forall \ u \in W^{m,p}(E). \tag{B.31}$$

Furthermore, $L : W^{m,p}(E) \mapsto L^p(E)$ is semi-Fredholm. Finally, if $\mathrm{Ker}(L) = 0$, then there is a constant $C > 0$ such that

$$||u||_{W^{m,p}} \leqslant C||Lu||_{L^p} \quad \forall \ u \in W^{m,p}(E). \tag{B.32}$$

The above theorem, whose proof can be consulted, for instance, in Holst, Nagy, and Tsogtgerel (see ibid., Lemma 34), is the extension of Theorem B.5 to the low regularity scenario. Basically, since the proof of Theorem B.5 depends on the elliptic estimates plus general functional analytic arguments, once the first part of the above theorem is established, the second part follows along the same lines as in Theorem B.5. Finally, along the lines of the remark following Theorem B.5, notice that we can approximate the coefficients of an operator satisfying the regularity hypotheses in the above theorem by smooth ones, and then consider elliptic operators with smooth coefficients converging to L. It is typically the case that computing the index along this sequence is easier than dealing directly with the low regularity coefficient operator. This will be exemplified when dealing with the Laplace operator of a metric with low regularity.

Let us now apply the above machinery to the two operators given by the Laplace–Beltrami operator associated to closed Riemannian manifold (M, g) and the conformal Killing operator introduced in (B.4).

The Poisson and conformal Killing operators

Let (M^n, g) be a closed Riemannian manifold with $g \in W^{2,p}$ and $p > \frac{n}{2}$, so that g is continuous. Then define

$$\Delta_g : W^{2,p} \mapsto L^p,$$
$$f \mapsto \operatorname{tr}_g \nabla^2 f = g^{ij} \nabla_i \nabla_j f \tag{B.33}$$

where ∇ denotes the associated Riemannian connection, where the range of Δ_g in the above definition follows from the Sobolev multiplication properties. Then, for any fixed function $a \in L^p(M)$, let us define the Poisson operator L_g as

$$L_g : W^{2,p} \mapsto L^p.$$
$$f \mapsto \Delta_g f - af \tag{B.34}$$

Proposition B.2. *In the above context, the Poisson operator L_g defines a continuous map from $W^{2,q} \mapsto L^q$ for all $1 < q \leqslant p$.*

Proof. Let us simply write $\Delta_g u = g^{ij} \partial_{ij} u + g^{ij} \Gamma_{ij}^l \partial_l u$ on a small coordinate ball B, for $u \in W^{2,q}$. We can apply Theorem A.2.3 to guarantee that $W^{2,p} \otimes L^q$ is continuously embedded in L^q for $p > \frac{n}{2}$, implying $\|g^{ij} \partial_{ij} u\|_{L^q(B)} \leqslant C \|g^{-1}\|_{W^{2,p}(B)} \|u\|_{W^{2,q}(B)}$. Also, $g^{ij} \Gamma_{ij}^l \in W^{1,p}$ for similar reasons, with the

estimate $\|g^{ij}\,\Gamma_{ij}^l\|_{W^{1,p}(B)} \leqslant C\|g^{-1}\|_{W^{2,p}(B)}\|\partial g\|_{W^{1,p}(B)}$. Now, the multiplication property of Theorem A.2.3 again guarantees that $W^{1,p} \otimes W^{1,q} \hookrightarrow L^q$ for $p > \frac{n}{2}$ and $1 < q \leqslant p$, implying $\|g^{ij}\,\Gamma_{ij}^l\,\partial_l u\|_{L^q(B)} \leqslant C\|g^{ij}\,\Gamma_{ij}^l\|_{W^{1,p}(B)}\|\partial u\|_{W^{1,q}(B)}$. Therefore, we locally get

$$\|\Delta_g u\|_{L^q(B)} \leqslant C \left(\|g^{-1}\|_{W^{2,p}(B)}(1 + \|g\|_{W^{2,p}B})\right)\|u\|_{W^{2,q}(B)}.$$

Finally, $L^p \otimes W^{2,q} \hookrightarrow L^q$ holds once more via Theorem A.2.3, implying $\|au\|_{L^q} \leqslant C\|a\|_{L^p}\|u\|_{W^{2,q}}$. A partition of unity argument establishes the final claim. $\qquad\square$

Let us now highlight the following *classical property* for smooth metrics. If g and a are smooth, then coefficients of L_g are smooth and classical elliptic regularity guarantees that if $L_g u = 0$ weakly for any $u \in \mathcal{D}'$, then actually u is a strong smooth solution. In particular $\mathrm{Ker}(L_g : W^{2,p} \mapsto L^p) \subset C^\infty(M)$ is independent of $p \in (1, \infty)$. This implies that $\mathrm{Ker}(L_g^* : L^{p'} \mapsto W^{-2,p'}) = \mathrm{Ker}(L_g : W^{2,p'} \mapsto L^{p'}) = \mathrm{Ker}(L_g : W^{2,p} \mapsto L^p)$, and therefore $\mathrm{Index}(L_g) = 0$ in this smooth case. We can exploit this to compute properties of L_g^* in low regularity appealing to index properties of Fredholm operators, as we will exemplify in the following theorem.

Theorem B.7. *Let (M^n, g) be a closed Riemannian manifold with $g \in W^{2,p}$, $p > \frac{n}{2}$. Let $a \in L^p(M)$ be a fixed function and consider the associated Poisson operator $L_g = \Delta_g - a$. If $a \geqslant 0$ a.e and not identically zero, then $L_g : W^{2,p} \mapsto L^p$ is an isomorphism.*

Proof. First, let us prove that the theorem is true if the coefficients are smooth. In that case, we can use the associated Riemannian measure dV_g to induce a the L^2-dual pairing between $W^{2,p}$ and $W^{-2,p'}$ in a standard way. Then, from a standard integration by parts argument we see that L_g is formally self-adjoint. Thus, if $\mathrm{Ker}(L) = 0$ we are done. Therefore, assume that $u \in \mathrm{Ker}(L_g) \subset C^\infty(M)$ so that $\Delta_g u - au = 0$. Multiplying this equation by u and integrating by parts with respect to dV_g, we see that

$$\int_M \left(|\nabla u|_g^2 + au^2\right) dV_g = 0 \ \forall \ u \in \mathrm{Ker}(L_g).$$

Since $a \geqslant 0$ and $a \not\equiv 0$, the above implies $u \equiv 0$ and the conclusion follows.

Let us now address the general case. First, approximate g in $W^{2,p}$ be smooth metrics $\{g_k\}$, a in L^p by smooth functions $\{a_k\}$ satisfying $a_k \geqslant 0$ and put some

fixed background smooth metric γ into the picture, denote by D its Riemannian connection, so that

$$\Delta_g f = g^{ij} \nabla_i D_j f = g^{ij} \left(D_i D_j f - S_{ij}^l D_l f \right) = g^{ij} D_i D_j f - g^{ij} S_{ij}^l D_l f, \tag{B.35}$$

where $S_{ij}^l = \Gamma_{ij}^l(g) - \Gamma_{ij}^l(\gamma) = \frac{g^{la}}{2} \left(D_i g_{aj} + D_j g_{ai} - D_a g_{ij} \right)$. Then,

$$(\Delta_g - \Delta_{g_k})f = (g^{ij} - g_k^{ij})D_i D_j f - (g^{ij} - g_k^{ij})S_{ij}^l(g)D_l f$$
$$- g_k^{ij} \left(S_{ij}^l(g) - S_{ij}^l(g_k) \right) D_l f.$$

Therefore

$$\|(L_g - L_{g_k})f\|_{L^p} \lesssim \left(\|g^{-1} - g_k^{-1}\|_{W^{2,p}} (1 + \|S(g)\|_{W^{1,p}}) \right.$$
$$\left. + \|g^{-1}\|_{W^{2,p}} \|S(g) - S(g_k)\|_{W^{1,p}} + \|a - a_k\|_{L^p} \right)\|f\|_{W^{2,p}}.$$

The above shows that $L_{g_k} \to L_g$ in the operator norm. Thus, from Theorem A.1.2, we see that L_g is also a Fredholm map with the same index as L_{g_k}. Since each L_{g_k} is an isomorphism, we see that L_g has zero index and therefore its injectivity proves the general claim. This follows similarly to the smooth case, but via an initial approximation argument. To start with, we know that

$$\int_M f_k L_{g_k} f_k dV_{g_k} = - \int_M \left(|\nabla f_k|_{g_k}^2 + a_k f_k^2 \right) dV_{g_k}, \quad \forall \ f_k \in C^\infty(M). \tag{B.36}$$

Consider $f \in W^{2,p}(M)$ and a sequence $f_k \xrightarrow{W^{2,p}} f$. We want to show that both sides of the above equation converge to their limits. First assume that M is orientable so that $dV_g = h_g dV_\gamma$ and $dV_{g_k} = h_{g_k} dV_\gamma$, with h_g, h_{g_k} positive continuous functions. Also, notice that since $p > \frac{n}{2}$, the multiplication

$$C^0 \times L^p \mapsto L^p$$
$$(u, v) \mapsto uv$$

is a continuous bilinear map and $W^{2,p} \hookrightarrow C^0$. Since $L^p \hookrightarrow L^1$ because M is compact, then these maps are continuous from $C^0 \times L^p \mapsto L^1$. This proves that $f_k L_{g_k} f_k h_{g_k} \xrightarrow{L^1} f L_g f h_g$, and similarly that $a_k f_k^2 h_{g_k} \xrightarrow{L^1} a f^2 h_g$. Similarly,

we have a continuous multiplication property $W^{1,p} \otimes W^{1,p}$ into $L^p \hookrightarrow L^1$, which can be put together with the continuous multiplication $C^0 \otimes L^p$ into L^1 we just discussed to prove $|\nabla f_k|^2_g h_{g_k} \xrightarrow{L^1} |\nabla f|^2_g h_g$. All this implies that we can take limits in (B.36) and get L^1 convergence on both side to prove the identity

$$\int_M f L_g f \, dV_g = - \int_M \left(|\nabla f|^2_g + af^2 \right) dV_g, \quad \forall \ f \in W^{2,p}(M). \quad \text{(B.37)}$$

Once more, this implies $\text{Ker}(L_g) = 0$ in this more general case.

The remaining case is that of M non-orientable. In this case, the above arguments holds in the orientable double cover. Since each of the integrals in (B.36) and (B.37) can be computed as half of the pulled-back integrals in the orientable double cover manifold, then this case gets reduced to the previous one. $\square$

Let us now comment on one further regularity property associated to L_g. Consider L_g satisfying the hypotheses of the above theorem but acting on $W^{2,q}$ with $1 < q \leqslant p$. We know that L_g is continuous on theses spaces. Also, since $W^{2,p} \hookrightarrow W^{2,q}$, we know that $V_1 \doteq \text{Ker}(L_g : W^{2,p} \mapsto L^p) \subset \text{Ker}(L_g : W^{2,q} \mapsto L^q) \doteq V_2$ and therefore $\dim(V_1) \leqslant \dim(V_2)$. From the above proof, we also know that $L_g : W^{2,p} \mapsto L^p$ has index zero, and thus $\dim(V_1) = \dim\left(\text{Ker}(L_g^* : L^{p'} \mapsto W^{-2,p'}) \right)$, where we will denote $V_3 \doteq \text{Ker}(L_g^* : L^{p'} \mapsto W^{-2,p'})$. But let us also notice that, in the same manner we did in the above proof, if $f \in W^{2,q}$ and $\{g_k\}$ and $\{a_k\}$ are a sequence of smooth metrics and functions converging to g and a in $W^{2,p}$ and L^p respectively, then

$$\begin{aligned}
\|(L_g - L_{g_k})f\|_{L^q} &\leqslant \big(\|g^{-1} - g_k^{-1}\|_{W^{2,p}} (1 + \|S(g)\|_{W^{1,p}}) \\
&\quad + \|g^{-1}\|_{W^{2,p}} \|S(g) - S(g_k)\|_{W^{1,p}} + \|a - a_k\|_{L^p} \big) \|f\|_{W^{2,q}}.
\end{aligned}$$

follows from (B.35) using arguments such as those of Proposition B.2. Again, this shows that

$$L_{g_k} \to L_g \ \text{ for any } \ 1 < q \leqslant p,$$

in the space of bounded linear maps from $W^{2,q}$ to L^q. Then, Theorem A.1.2 guarantees that $L_g : W^{2,q} \mapsto L^q$ is a Fredholm map of index zero. This implies that $\dim(V_2) = \dim\left(\text{Ker}\left(L_g^* : L^{q'} \mapsto W^{-2,q'} \right) \right)$, where we will denote $V_4 \doteq \text{Ker}\left(L_g^* : L^{q'} \mapsto W^{-2,q'} \right)$. Now, recalling that $L_g^* : L^{q'} \mapsto W^{-2,q'}$ is defined

via the identity $\langle L_g^* u, v \rangle = \langle u, L_g v \rangle$ for all $u \in L^{q'}$ and $v \in W^{2,q}$, notice that if $u \in V_4$, then since $W^{2,p} \hookrightarrow W^{2,q}$, we have

$$\langle L_g^* u, v \rangle = 0, \text{ for all } v \in W^{2,p}.$$

That is $V_4 \subset V_3$, which now implies

$$\dim(V_1) \leqslant \dim(V_2) = \dim(V_4) \leqslant \dim(V_3) = \dim(V_1), \qquad \text{(B.38)}$$

and therefore all the inequalities become equalities, and the inclusions $V_1 \subset V_2$ and $V_4 \subset V_3$ imply $V_1 = V_2$ and $V_3 = V_4$. Finally, if $u \in V_1$ and $v \in W^{2,p}$ is arbitrary, then, integration by parts of the type justified in the above theorem, shows that

$$\langle u, L_g v \rangle = \int_M u L_g v \, dV_g = \int_M v L_g u \, dV_g = 0,$$

implying that $u \in V_3$. That is, $V_1 \subset V_3$ which implies $V_1 = V_3$ since the index is zero. Putting everything together, we have found $V_1 = \cdots = V_4$, which implies

$$\mathrm{Ker}(L_g : L^{q'} \mapsto W^{-2,q'}) = \mathrm{Ker}(L_g : W^{2,p} :\mapsto L^p) \text{ for all } 1 < q \leqslant p.$$
$$\text{(B.39)}$$

That is, any weak L^q solution to $L_g u = 0$ is a strong L^p-solution for any $1 < q \leqslant p$.

Along the same lines as the above theorem, let us consider the following result concerning the conformal Killing Laplacian (CKL) introduced in (B.4).

Theorem B.8. *Let (M^n, g) be a closed Riemannian manifold with $g \in W^{2,p}$, $p > \frac{n}{2}$ and $n \geqslant 3$. Let us consider the operator $\Delta_{g,conf} : W^{2,p}(TM) \mapsto L^p(T^*M)$. Then, $\Delta_{g,conf}$ is a Fredholm map where $\mathrm{Ker}(\Delta_{g,conf})$ equals the space of $W^{2,p}$-conformal Killing field of the metric g. It is in particular an isomorphism if g does not posses any conformal Killing field.*

Proof. This proof follows the lines of the proof of the previous theorem. Thus, first consider the smooth case, that is, $g \in C^\infty(M)$. Then, we know that $\Delta_{g,conf}$ is Fredholm from Theorem B.6 and we can proceed as in the above theorem to show that, given a sequence $\{g_k\}$ of smooth metrics such that $g_k \xrightarrow{W^{2,p}} g$, it follows that $\Delta_{g_k,conf} \to \Delta_{g,conf}$ in the operator norm. Therefore, we conclude from

Theorem A.1.2 that $\Delta_{g,conf}$ is a Fredholm map. Now, consider $X \in W^{2,p}$ and take a sequence $\{X_k\} \subset C^\infty$ such that $X_k \xrightarrow{W^{2,p}} X$ and notice that

$$\int_M \langle \Delta_{g_k,conf} X_k, X_k \rangle_{g_k} dV_{g_k} = -\int_M |\mathscr{L}_{g_k,conf} X_k|^2_{g_k} dV_{g_k}. \qquad (B.40)$$

Just as we did above, we want to prove that both sides in the above expression both side converge to the corresponding limits and the same multiplication properties described in the previous theorem establish this claim. So, we get that

$$\int_M \langle \Delta_{g,conf} X, X \rangle_g dV_g = -\int_M |\mathscr{L}_{g,conf} X|^2_g dV_g. \qquad (B.41)$$

Then, the above equation shows that

$$X \in \mathrm{Ker}(\Delta_{g,conf}) \iff \mathscr{L}_{g,conf} X = 0,$$

which finishes the proof. $\square$

Let us just highlight that similar remarks to those following Theorem B.7, concerning the regularity of solutions, apply to the conformal Killing Laplacian as well.

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