Real and Complex Gaussian Multiplicative Chaos

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Contents

Summary

These notes were written in prevision of an introductory course on Gaussian Multiplicative Chaos (GMC) given at the 32nd *Colóquio Brasileiro de Matemática*. Their aim is to provide a very accessible and mostly self-contained introduction to a currently very active domain of research. For five hours of lectures, we add to make a drastic selection on the material be presented. We provide some motivation and a short historical introduction to the subject which is by no mean exhaustive. The reader can refer to Rhodes and Vargas (2014) for a more throughout review of all the development the field has known in the past decade.

We have chosen to focus on the problem of construction of GMC in the real and complex setup rather than on applications. We have chosen to present a lot of proofs in the simpler framework of multiplicative cascades which while easier to handle, displays exactly the same phenomenology as the GMC. The last chapter is dedicated to the proof of convergence of Gaussian Multiplicative Chaos in a very general setup. Up to minor modifications and simplifications the proofs we present are borrowed from the literature. We provide the references for the original source for most recent results.

1 Introduction to Gaussian Multiplicative chaos

1.1 Some recall about Gaussian Random Variables, Vectors, Fields

Before introducing the reader to Gaussian Multiplicative Chaos, our main object of study, let us start with a very short recall about Gaussian processes. The reader can refer e.g. to Zeitouni (2015) for a complete introduction to the subject and to Janson (1997) for complements.

1.1.1 Definitions and basic properties

The Gaussian distribution $\mathcal{N}(m,\sigma^2)$ with mean $m \in \mathbb{R}$ and variance σ^2, $\sigma > 0$ is the probability distribution whose density w.r.t. to Lebesgue measure is given by

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma}}. \tag{1.1.1}$$

By convention, when $\sigma = 0$, $\mathcal{N}(m,0)$ is simply the Dirac mass at m. Given $k \in \mathbb{N}$, a random vector $(X_1,\dots,X_k)$ taking value in $\mathbb{R}^k$ is said to be Gaussian if for all $\lambda_1,\dots,\lambda_k \in$

$\mathbb{R}$, the variable

$$Z := \lambda_1 X_1 + \cdots + \lambda_k X_k, \tag{1.1.2}$$

is a Gaussian random variable. Finally if $\mathcal{X}$ is an arbitrary set, we say that the collection $(X(x))_{x\in\mathcal{X}}$ is a Gaussian field indexed by $\mathcal{X}$ if, for every $k\geqslant 1$ and $x_1, \dots, x_k \in \mathcal{X}$, $(X(x_i))_{i=1}^k$ is a Gaussian vector.

The important thing to know about Gaussian fields is that their distribution is completely determined by their mean and covariance. Let us detail this fact a little bit: We say that a function $\Sigma : \mathcal{X}^2 \to \mathbb{R}$ is positive definite if for any $n\geqslant 1$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \Sigma(x_i, x_j) \geqslant 0, \tag{1.1.3}$$

As the r.h.s. in (1.1.3) corresponds to the variance of the variable Z in (1.1.2) (with $X_i = X(x_i)$) the covariance of a Gaussian field $K(x,y) := \mathbb{E}[X(x)X(y)]$ is necessarily positive definite function. The following reciprocal statement also holds

Proposition 1.1.1 (Existence of Gaussian Fields)**.** *Given $m : \mathcal{X} \to \mathbb{R}$ arbitrary and $\Sigma : \mathcal{X} \to \mathbb{R}^2$ positive definite, there exists a unique distribution $\mathbb{P}$ on $(\mathbb{R}^{\mathcal{X}}, \mathcal{B}(\mathbb{R})^{\otimes \mathcal{X}})$, under which the coordinate projections $(X(x))_{x\in\mathcal{X}}$ constitutes a Gaussian field with average m and covariance K.*

Sketch of proof. The statement is about existence and uniqueness of a distribution. Uniqueness follows from the fact that being Gaussian the information of the mean and covariance completely determines the Fourier transform. Indeed as for any $u \in \mathbb{C}$ if Z is a Gaussian random variable of variance m and variance σ^2 we have

$$\mathbb{E}[e^{uZ}] = e^{um + \frac{u^2}{2\sigma}}. \tag{1.1.4}$$

Hence we have for any $\xi_1, \dots \xi_k \in \mathbb{R}$

$$\mathbb{E}[e^{\sum_{k=1}^n \xi_k X(x_k)}] := e^{\sum_{k=1}^n \xi_k m(x_k) + \frac{1}{2}\sum_{1\leqslant j,k\leqslant n} \xi_j \xi_k K(x_j x_k)}, \tag{1.1.5}$$

which by injectivity of the Laplace transform, fully determines the distribution of all finite dimensional marginals.

Existence can be obtained by describing the density explicitly in the vector case (that is $\mathcal{X}$ finite). The construction can then be extended to the infinite case using Kolmogorov extension Theorem. (an explicit construction based on Hilbert spaces is given in Zeitouni (2015, Lemma 7, Proposition 1)) □

Let us conclude this brief section by recalling some properties of Gaussian variables used throughout these lectures. We include their short proof for the sake of completeness.

Lemma 1.1.2 (Gaussian tail bound)**.** *If X has distribution $\mathcal{N}(0,\sigma^2)$ then we have for every $u > 0$*

$$\mathbb{P}[X > u] \leqslant \frac{\sigma}{\sqrt{2\pi} u} e^{-\frac{u^2}{2\sigma^2}} \tag{1.1.6}$$

Proof. Simply observe that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_u^\infty e^{-\frac{v^2}{2\sigma^2}} \, \mathrm{d}v \leqslant \frac{\sigma}{\sqrt{2\pi} u} \int_u^\infty \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} \, \mathrm{d}v \tag{1.1.7}$$

□

Proposition 1.1.3 (Cameron Martin Formula)**.** *On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $X := (X(x))_{x \in \mathcal{X}}$ be a centered Gaussian field and Z be a centered Gaussian random variable such that (X, Z) is also jointly Gaussian. Let $\widetilde{\mathbb{P}}$ be the probability whose density with respect to $\mathbb{P}$ is given by*

$$\mathrm{d}\widetilde{\mathbb{P}} / \mathrm{d}\mathbb{P} = e^{Z - \frac{1}{2}\mathbb{E}[Z^2]}.$$

Then under $\widetilde{\mathbb{P}}$, X is a Gaussian field, with the same covariance but with mean given by

$$\widetilde{\mathbb{E}}[X(x)] = \mathbb{E}[X(x) Z]. \tag{1.1.8}$$

Sketch of proof. Considering $x_1, \dots, x_k \in \mathcal{X}$, we can compute the Laplace transform of $(X(x_1), \dots, X(x_k))$ under $\widetilde{\mathbb{P}}$, by using (1.1.5) for the Gaussian vector $(Z, X(x_1), \dots, X(x_k))$ under $\widetilde{\mathbb{P}}$. We obtain

$$\begin{aligned} \widetilde{\mathbb{E}}[e^{\sum_{k=1}^n \xi_k X(x_k)}] &= \mathbb{E}[e^{Z - \frac{1}{2}\mathbb{E}[Z^2] + \sum_{k=1}^n \xi_k X(x_k)}] \\ &= e^{\frac{1}{2}\sum_{1 \leqslant j,k \leqslant n} \xi_j \xi_k K(x_j x_k) + \sum_{j=1}^n \mathbb{E}[X(x_j) Z]} \end{aligned} \tag{1.1.9}$$

which by injectivity of the Laplace transform and (1.1.5), implies the desired results.

□

1.1.2 Regularity criterion for fields on $\mathbb{R}^d$

Let us focus now on the case where $\mathcal{X} = D$ is a bounded regular subset of $\mathbb{R}^d$ and the covariance Σ possesses some regularity property. In that case we are interested in making sense of integrals of functionals of the Gaussian field X such as

$$\int_D f(X(x)) \, \mathrm{d}x$$

for continuous functions f. If Proposition 1.1.1 guarantees the existence of Gaussian fields it does not say anything about its regularity and thus X and $f \circ X$. To resolve this, we define the field on a probability space with more structure. Let $\mathcal{C}^0(D)$ denote the set of continuous function on D, considered with the topology of uniform convergence on compact set and let $\mathcal{F}$ be the associated σ-algebra. The following classical result shows that we can always consider that X is continuous provided that Σ is sufficiently regular (the assumptions made below are far from optimal but sufficient for the application we have in mind)

Theorem 1.1.4. *If we assume that $\Sigma(x, y)$ has continuous second derivatives then there exists a probability $\mathbb{P}$ on $(\mathcal{C}_0(D), \mathcal{F})$, under which the field formed by projection coordinates $(X(x))_{x \in D}$ is a centered Gaussian field with covariance Σ.*

Proof. From Kolmogorov-Chentsov Theorem Lalley (2011, Theorem 1), there exists a continuous version of the field provided that there exists $p > 0$ and $\beta > 0$ such that

$$\mathbb{E}[|X(x) - X(y)|^p] \leqslant C|x - y|^{d+\beta}. \tag{1.1.10}$$

The variable $X(x) - X(y)$ is a Gaussian and its variance satisfies

$$\mathbb{E}[(X(x) - X(y))^2] = \Sigma(x, x) + \Sigma(y, y) - 2\Sigma(x, y) \leqslant C|x - y|^2, \tag{1.1.11}$$

where the inequality is the consequence of Σ being twice differentiable and is valid uniformly on D. Hence by scaling, there exists C_p such that we have

$$\mathbb{E}[|X(x) - X(y)|^p] \leqslant C_p|x - y|^p,$$

and hence (1.1.10) is satisfied for $\beta = p - d$ for $p > d$. $\square$

1.2 log-Correlated Gaussian field

In this Section, we make use of the definition introduced in the previous section to give a mathematical definition of log-correlated fields, which are needed in the construction of Gaussian Multiplicative chaos.

1.2.1 Definition

We want now to introduce a notion of Gaussian field which is not covered by the previous definition. Given a bounded smooth domain $D \subset \mathbb{R}^d$, we consider a real valued kernel K defined on $D^2 \setminus \{(x, x) : x \in D\}$ and which satisfies:

$$K(x, y) := \log \frac{1}{|x - y|} + L(x, y), \tag{1.2.1}$$

where $L(x, y)$ is a continuous function on D^2. We let $\mathcal{C}_c^\infty(\mathbb{R}^d)$ denote the set of compactly supported smooth (i.e. infinitely differentiable) functions in $\mathbb{R}^d$, and we assume the following continuous analog of (1.1.3) for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) f(y) K(x, y) \,\mathrm{d}x \,\mathrm{d}y \geqslant 0, \tag{1.2.2}$$

where by convention $K(x, y) = 0$ if either x or y is not in D. We say that the Kernel K is positive definite if (1.2.2) is satisfied. We want to introduce a notion of centered Gaussian field "indexed by D" with covariance K.

For such a definition to make sense, we must abandon the idea of defining X pointwise, and only define the values of X integrated against sufficiently regular measures. We let $\mathcal{M}_K$ denote the set of measure that can be written in the form $\rho_+ - \rho_-$ where ρ_+ and ρ_- are two positive measures which satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |K(x, y)| \rho_\pm(\mathrm{d}x) \rho_\pm(\mathrm{d}y) < \infty. \tag{1.2.3}$$

Now we can construct a centered Gaussian field index by $\mathcal{M}_K$ whose covariance is given by

$$\Sigma_K(\rho_1, \rho_2) := \int K(x, y) \rho_1(\mathrm{d}x) \rho_2(\mathrm{d}y). \tag{1.2.4}$$

Note that as a consequence of (1.1.3) Σ_K is a positive definite quadratic form on $\mathcal{M}_K$. We use the notation $\langle X, \rho \rangle$ for the field, and sometimes write improperly $\int X \, \mathrm{d}\rho$. Note that for any $\alpha \in \mathbb{R}$, $\rho_1, \rho_2 \in \mathcal{M}_K$, we have almost-surely

$$\langle X, \alpha\rho_1 + \beta\rho_2 \rangle = \alpha \langle X, \rho_1 \rangle + \beta \langle X, \rho_2 \rangle. \tag{1.2.5}$$

1.2.2 Examples

We provide the reader a few examples of kernels K which satisfies (1.1.3). For more motivation behind these examples we refer to the review paper Rhodes and Vargas (2014).

Kernels which are invariant by rotation and translation If the distribution of the field is invariant by translation and rotations in $\mathbb{R}^d$ it implies that the covariance function must be of radial form, that is $K(x, y) = \varphi(|x - y|)$ for some function $\varphi : (0, \infty) \to \mathbb{R}$.

When $d = 1$ we can deduce from Pólya's criterion (see Feller (1971, pp. 509)) that any choice of non-negative convex φ makes $\varphi(|x - y|)$ definite positive. Many generalized versions of this criterion in arbitrary dimension have been proved (see e.g. Gneiting (2001)). A particular example valid in all dimension (see Rhodes and Vargas (2014) for a proof) is given by

$$K(x, y) = \log_+ \left(\frac{|x - y|}{T} \right) \tag{1.2.6}$$

for an arbitrary value of $T > 0$. Another family of log-correlated field is given by star-scale invariant Kernels. Given $k : \mathbb{R}_+ \to \mathbb{R}$ a differentiable bounded function which is such that $k(|x - y|)$ is positive definite $k(0) = 1$, and $\int_0^\infty |k(u)| \, \mathrm{d}u < \infty$ set

$$K(x, y) := \int_0^\infty k(e^u |x - y|) \, \mathrm{d}u$$

It is then a simple exercise to check that it satisfies (1.2.1).

The Gaussian Free Fields on $\mathbb{R}^2$ Given an open set $\mathcal{D} \subset \mathbb{R}^2$ with smooth boundary, we consider G to be the Green kernel associated with the Laplace operator on Ω with Dirichlet boundary condition, that is, such that for any continuous f in Ω, $g_0 := \int G(\cdot, x) f(x)\,\mathrm{d}x$. is the unique solution

$$\begin{cases} \Delta g(x) = -2\pi f(x) & \text{on } \mathcal{D}, \\ g(x) = 0 & \text{on } \mathcal{D}^{\complement} \end{cases} \tag{1.2.7}$$

Note that this characterization implies positive definiteness since by integration by part, we have for all continuous function f with compact support in D, using the notation $g = \int G(\cdot, x) f(x)\,\mathrm{d}x$

$$\int_{\mathcal{D}^2} f(x) G(x,y) f(y)\,\mathrm{d}y = -2\pi \int_{\mathcal{D}^2} \Delta g(x).g(x)\,\mathrm{d}x = 2\pi \int |\nabla g(x)|^2\,\mathrm{d}x \geqslant 0. \tag{1.2.8}$$

Let us briefly justify why G satisfies (1.2.1) on any compact subset $D \subset \mathcal{D}$ relying on some property of the Brownian motion. It is known that G can be expressed as

$$G(x,y) := \pi \int_0^\infty P_t^*(x,y)\,\mathrm{d}t \tag{1.2.9}$$

where P_t^* is the heat Kernel corresponding to a two dimensional Brownian motion killed when hitting the boundary of D. We are going to show that the integral from 1 to infinity above yields a continuous function on $\mathcal{D}$ while that from 0 to 1 can be written in the form given in Equation (1.2.1). Note first that we have

$$\pi P_t^*(x,y) := \frac{1}{2t} e^{-\frac{|x-y|^2}{2t}} Q(t,x,y,\mathcal{D}) \tag{1.2.10}$$

where $Q(t,x,y,\mathcal{D}) := \mathrm{P}_{x,y,t}(B_s \in \mathcal{D}, \forall s \in [0,t])$ and $\mathrm{P}_{x,y,t}$ is the distribution of the Brownian bridge of length t from x to y.

As in each interval of time of the form $[n, n+1]$, conditioned on the rest of the trajectory, B_s has a positive probability of exiting $\mathcal{D}$, we can find a constant $C_{\mathcal{D}}$ satisfying

$$Q(t,x,y,\mathcal{D}) \leqslant e^{-C_{\mathcal{D}} t} \quad \forall t \geqslant 1,$$

which is sufficient to ensure that $\int_1^\infty P_t^*(x,y)\,\mathrm{d}t$ is a continuous function on $\mathcal{D}^2$. Now if the segment $[x,y]$ is at a distance ε from the boundary of $\mathcal{D}$, that is

$$\min_{t \in [0,1], z \in \mathcal{D}^{\complement}} |xt + y(1-t) - z| > \varepsilon,$$

then standard Brownian estimates yield

$$Q(t,x,y,D) \geqslant 1 - Ce^{-\frac{\varepsilon^2}{2t}},$$

which is sufficient show that $\int_1^0 \pi P_t^*(x,y)\,\mathrm{d}t$ is of the form (1.2.1).

One can also consider the integral Kernel corresponding to the inverse of the massive Laplace operator $\Delta - m^2$, $m > 0$, with Dirichlet boundary condition given by

$$G_m(x,y) := \pi \int_0^\infty e^{-\frac{m^2 t}{2}} P_t^*(x,y)\,\mathrm{d}t.$$

In this case, the integral also converge when $\mathcal{D}$ is unbounded, in the particular case where $D = \mathbb{R}^2$ we obtain

$$G_m^*(x,y) = \int_0^\infty \frac{1}{2t} e^{-\frac{m^2 t}{2} - \frac{|x|^2}{2t}}\,\mathrm{d}t. \tag{1.2.11}$$

The Gaussian fields with correlation function given by one of the Green functions mentioned above are called Gaussian free fields. They are important object in theoretical physics, and provide important application to the theory of Gaussian multiplicative chaos: The Multiplicative chaos associated with the Gaussian Free Fields are connected with 2D Liouville Quantum Gravity, introduced in study Polyakov (1987), which attracted a lot of attention in the probability community following the pioneering work of Duplantier and Sheffield (2011).

1.3 Gaussian multiplicative chaos

The Gaussian Multiplicative Chaos associated with a log-correlated field X of given covariance kernel K, associated with the parameter $\gamma > 0$ is formally defined as a positive random measure on D, whose density is given by the exponential of the field X

$$M_\gamma(A) := \int_A :e^{\gamma X}:\,\mathrm{d}x. \tag{1.3.1}$$

Above, we used the Wick notation, if Z is a centered Gaussian random variable it is defined by $:e^Z: := e^{Z - \mathbb{E}[Z^2]}$. The issue with Equation (1.3.1) is that X is not defined as a function of $x \in D$ so that some extra work is required in order to give a meaning to the integral. A reasonable way to try to give a meaning to (1.3.1) is to apply a smooth convolution Kernel to X. In the remainder of the notes θ will denote a positive function in $\mathcal{C}^\infty(\mathbb{R}^d)$ supported on the Euclidean ball of radius one centered at the origin $B(0,1)$ and such that $\int_{B(0,1)} \theta(x)\,\mathrm{d}x = 1$. Given $\varepsilon \in (0,1)$ we use the notation $\theta_\varepsilon := \varepsilon^{-d}\theta(\varepsilon^{-1}\cdot)$.

Now we consider X_ε to be the Gaussian field indexed by D given by $\langle X, \theta_\varepsilon\rangle$. Using the formula (1.2.4), we obtain that the covariance of the field X_ε is given by

$$K_\varepsilon(x,y) := \int_{\mathbb{R}^d} \theta_\varepsilon(z_1 - x)\theta_\varepsilon(z_2 - x) K(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \tag{1.3.2}$$

(recall that by convention $K(z_1, z_2) = 0$ outside of D^2). We write $K_\varepsilon(x)$ for the variance of the field $K_\varepsilon(x, x)$. An important observation is that K_ε is a smooth function x and y and hence by Theorem 1.1.4 we can find a continuous (hence measurable) version of X_ε. Given $\gamma > 0$ we define

$$M_\varepsilon^{(\gamma)} = \int e^{\gamma X_\varepsilon - \frac{\gamma^2}{2} K_\varepsilon(x)} \,\mathrm{d}x. \tag{1.3.3}$$

In order to show that (1.3.1) is well defined, one would like to show that $M_\varepsilon^{(\gamma)}$ converges when the radius of convolution ε tend to zero. Also we do not want the object (1.3.1) to depend on the approximation scheme which is used for X, and hence to prove that the limit does not depend on the particular choice of the smoothing kernel θ. This is the content of the following result.

Theorem 1.3.1. *If $\gamma < \sqrt{2d}$, the limit*

$$M_0^{(\gamma)} := \lim_{\varepsilon \to 0} M_\varepsilon^{(\gamma)},$$

exists and the convergence holds in the $\mathbb{L}^1$ sense. Furthermore limiting random variable does not depend on the choice of the approximation Kernel ρ.

Remark 1.3.2. *Our result concerns only the multiplicative chaos integrated on the whole domain D but can also be extended to a convergence result for the measure measure $M_\varepsilon^{(\gamma)}(\mathrm{d}x) := e^{\gamma X_\varepsilon - \frac{\gamma^2}{2} K_\varepsilon(x)} \,\mathrm{d}x$. We have chosen to restrict to the proof convergence of the total mass for the sake of exposition.*

Let us dwell a bit on Theorem 1.3.1 which has a long history (we refer also to the introduction of Berestycki (2017) for a more detailed account). A first version of the result is due to Kahane: In Kahane (1985), the convergence of another approximation sequence of (1.3.1) (which is not obtained by convolution), under the stronger assumption that that the Kernel K is of σ-positive type (that is $K(x, y) = \sum_{n \geqslant 1} K_n(x, y)$ where K_n is a sequence of bounded positive definite Kernel which satisfy $K_n(x, y) \geqslant 0$). The result was then considerably extended in Robert and Vargas (2010) where convergence of $M_\varepsilon^{(\gamma)}$ defined above and uniqueness of the limiting distribution was shown without assuming σ-positivity. Finally in Shamov (2016), uniqueness of the limit in $\mathbb{L}_1$ was established in a very general framework. The proof of the result we present in these notes (see Chapter 4), which is short and self contained, is, up to minor modifications, the one presented Berestycki (2017).

There are various motivations to study the random measure (1.3.1). The original motivation in Kahane and Peyrière (1976) is that they form a natural class of self-similar random measure but further application were found in mathematical finance and in the study of three dimensional turbulence (see the introduction of Robert and Vargas (2010) and references therein). In recent years, the Gaussian Multiplicative Chaos associated the exponential of the $2D$-Gaussian Free Field has been the object of numerous studies due to its relation to the theory of two dimensional quantum gravity (see Garban (2013) for a review).

1.4 Complex setup and Sine-Gordon representation of log-gas

Multiplicative chaos given by Equation (1.3.1) has also been studied in the setup where γ is allowed to take complex values (see Barral, Jin, and Mandelbrot (2010b) and Lacoin, Rhodes, and Vargas (2015) and references therein for a general presentation and application). In this note we present an introduction to the imaginary case where $\gamma \in i\mathbb{R}$. For notational convenience and also because of the Sine-Gordon connection exposed below, we look only at the real part of the GMC, that is, consider $\cos(\beta X)$ rather than $e^{i\beta X}$. For the sake of simplicity, in these notes we are only going to prove results in the hierarchical setup of multiplicative cascades (see Section 1.5.1 below) but we present in this section some motivation to study the problem of log-correlated fields (we refer to Lacoin, Rhodes, and Vargas (2019) and references for a more complete introduction).

Given $\beta > 0$, and X a log-correlated field with covariance Kernel K on D, we want, in analogy with the previous section, to make sense of the following integral

$$M^{(\beta)} := \int_D :\cos(\beta X):\, \mathrm{d}x, \tag{1.4.1}$$

where $:\cos:$ denotes the Wick renormalization of the cosine, which, for a Gaussian random variable Z, is defined as follows

$$:\cos(Z): = e^{\frac{\beta^2}{2} Z} \cos Z.$$

As before the problem comes from the fact that $X(x)$ is not a random variable, and the integral must be computed via approximations.

The study of $M^{(\beta)}$ can be motivated by that of gas of electrically charged particle in D , whose interaction is proportional to the kernel K. We place ourselves in the so-called *grand canonical* setup, where the number of particles in the gas in not fixed. The state space is given by

$$\Omega := \coprod_{n \geqslant 0} D^n \times \{-1, 1\}^n.$$

An element of Ω is a triplet $\omega = (n, \mathrm{x}, \lambda)$ where n denotes the number of particles, $\mathrm{x} = (x_1, \ldots, x_n)$ their position, and $\lambda := (\lambda_1, \ldots, \lambda_n)$ their charge. To each configuration we associate an energy which is given by

$$H(n, \mathrm{x}, \lambda) := \sum_{1 \leqslant i < j \leqslant n} \lambda_i \lambda_j K(x_i, x_j). \tag{1.4.2}$$

In two dimension, this a very natural Hamiltonian to consider since the Coulomb potential is in that case equal to $\log \frac{1}{|x-y|}$. Now given α and $\beta > 0$ two parameters ($\alpha/2$ corresponds to the individual particle activity and β^2 to the inverse temperature), we wish to consider $\mathrm{P}_{\alpha,\beta}$ the measure on Ω defined by

$$\mathrm{P}_{\alpha,\beta}(A) := \frac{1}{\mathcal{Z}_{\alpha,\beta}} \sum_{n\geqslant 0} \frac{(\alpha/2)^2}{n!} \sum_{\lambda\in\{-1,1\}^n} \int_{D^n} e^{-\beta^2 H(n,\mathrm{x},\lambda)} 1_{\{(n,\mathrm{x},\lambda)\in A\}}\, \mathrm{dx}. \tag{1.4.3}$$

where

$$\mathcal{Z}_{\alpha,\beta} := \sum_{n\geqslant 0} \frac{(\alpha/2)^2}{n!} \sum_{\lambda\in\{-1,1\}^n} \int_{D^n} e^{-\beta^2 H(n,\mathrm{x},\lambda)}\, \mathrm{dx},$$

is the partition function. For the formula (1.4.3) to make sense we need that $\mathcal{Z}_{\alpha,\beta} < \infty$.

In order to study this problem, physicists have been studied this partition function under an alternative form called the *Sine-Gordon* representation. Let us consider K_ε given by (1.3.2) and H_ε to be the Hamiltonian (1.4.2) with K replaced by $K(\varepsilon)$

$$H_\varepsilon(n,\mathrm{x},\lambda) := \sum_{1\leqslant i<j\leqslant n} \lambda_i\lambda_j K_\varepsilon(x_i,x_j). \tag{1.4.4}$$

and define subsequently $\mathrm{P}_{\alpha,\beta,\varepsilon}$, $\mathcal{Z}_{\alpha,\beta,\varepsilon}$. If X_ε is a Gaussian field of covariance K_ε then $2H$ is, up to the addition of diagonal term, the variance of a linear combination of the coordinates of X_ε

$$2H_\varepsilon(n,\mathrm{x},n) + \sum_{j=1}^n K_\varepsilon(x_j) = \mathbb{E}\left[\left(\sum_{j=1}^n \lambda_j X_\varepsilon(x_j)\right)^2\right]$$

In particular this yields

$$e^{-\beta^2 H_\varepsilon(n,\mathrm{x},\lambda)} = e^{\frac{\beta^2}{2}\sum_{j=1}^n K_\varepsilon(x_j)}\mathbb{E}\left[e^{i\beta\sum_{j=1}^n \lambda_j X_\varepsilon(x_j)}\right]. \tag{1.4.5}$$

Then summing over λ yields

$$2^{-n}\sum_{\lambda\in\{-1,1\}^n} e^{-\beta^2 H_\varepsilon(n,\mathrm{x},\lambda)} = e^{\frac{\beta^2}{2}\sum_{j=1}^n K_\varepsilon(x_j)}\mathbb{E}\left[\prod_{j=1}^n \cos(\beta X(x_j))\right]. \tag{1.4.6}$$

Integrating and summing, we obtain that

$$\mathcal{Z}_{\alpha,\beta,\varepsilon} = \sum_{\alpha\geqslant 0}\frac{\alpha^n}{n!}\left(\int_D e^{\frac{\beta^2}{2}K_\varepsilon(x)}\cos(\beta X)\,\mathrm{d}x\right)^n = \mathbb{E}\left[e^{\alpha M_\varepsilon^{(\beta)}}\right]. \tag{1.4.7}$$

with $M_\varepsilon^{(\beta)} := \int_D e^{\frac{\beta^2}{2}K_\varepsilon(x)} \cos(\beta X(x))\, \mathrm{d}x$. Note that as $K_\varepsilon(x)$ is uniformly bounded on D, this implies immediately that $\mathcal{Z}_{\alpha,\beta,\varepsilon}$ is.

The study of the convergence or renormalization of $\mathcal{Z}_{\alpha,\beta,\varepsilon}$ (although formulated in other terms) when ε tends to zero dates back to Fröhlich (1976). What the results in Fröhlich (ibid.) establish is that when $\beta < \sqrt{d}$ $M_\varepsilon^{(\beta)}$ converges to a non-degenerate limit variable $M_0^{(\beta)}$, and that

$$\mathcal{Z}_{\alpha,\beta} = \mathbb{E}[e^{\alpha M_0^{(\beta)}}] < \infty. \tag{1.4.8}$$

When $\beta \geqslant \sqrt{d}$ the reader can immediately check that the term corresponding to $n = 2$, $\lambda_1 = -\lambda_2 = 1$ which is equal to

$$\int_{D^2} e^{\beta^2 K(x,y)}\, \mathrm{d}x\, \mathrm{d}y,$$

diverges (the integrand behaves like $|x-y|^{-\beta^2}$, and the question becomes: "Does the probability measure $\mathrm{P}_{\alpha,\beta,\varepsilon}$ associated with the partition function $\mathcal{Z}_{\alpha,\beta,\varepsilon}$ of Equation (1.4.7) converge?". This question is intimately related to the study of the divergence of $\mathcal{Z}_{\alpha,\beta,\varepsilon}$ when ε tends to zero, and has first been addressed in Benfatto, Gallavotti, and Nicolò (1982) and Nicolò, Renn, and Steinmann (1986). Roughly speaking the results of Benfatto, Gallavotti, and Nicolò (1982) and Nicolò, Renn, and Steinmann (1986) say that when $\beta \in (\sqrt{d}, \sqrt{2d}$, only finitely many cumulants of $M_\varepsilon^{(\beta)}$ diverge when ε tends to zero. As a consequence, $\mathcal{Z}_{\alpha,\beta,\varepsilon}$ converges after multiplying by an appropriate counter-term which is the exponential of a polynomial in α. We present here a similar result in the case of complex multiplicative cascade (see Theorem 1.5.5).

Multiplicative cascades (introduced in Kahane and Peyrière (1976) and Mandelbrot (1974a,b)) is a model which is to many respect simpler to study than multiplicative chaos but which displays a similar qualitative behavior.

1.5 Hierarchical setup and multiplicative cascades

While they have been studied first Kahane and Peyrière (1976) and Mandelbrot (1974a,b), Gaussian multiplicative cascade can be considered as a *hierarchical* version of Gaussian multiplicative chaos. *Hierarchical models* are models of statistical mechanics for which the partition function of a system at one scale can be obtained by an easy operation on the partition function of the model on a smaller scale (see Equation (1.5.7)). This feature allows for rigorous renormalization group computation.

Multiplicative cascades can be interpreted as a Multiplicative chaos associated with a field on $[0,1)^d$ $(d \geqslant 1)$ which is log-correlated for the *dyadic distance* instead of the usual Euclidean one. Gaussian Multiplicative Cascades are obtained by considering the exponential of a Gaussian field on whose covariance is given by

$$Q(x,y) = \log_2\left(\frac{1}{d_2(x,y)}\right), \tag{1.5.1}$$

where $\log_2(x) := \log(x)/\log 2$ and $d_2(x,y)$ is the dyadic distance on $[0,1]^d$ defined as follows. For $n \geqslant 1$, and $\mathrm{i} := (i_1, \dots, i_d) \in [\![1, 2^n]\!]^d$ (we use the notation $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$ for $a, b \in \mathbb{Z}$) we define the dyadic box $B_{\mathrm{i}}^{(n)}$ to be

$$B_{\mathrm{i}}^{(n)} := \prod_{j=1}^{d} [2^{-n}(i_j - 1), 2^{-n} i_j) \tag{1.5.2}$$

The dyadic distance between two points is defined as the size of the smallest dyadic box they both fit in

$$\log_2(d_2(x,y)) := -\max\{n \geqslant 0 \ : \ \exists \mathrm{i} \in [\![1, 2^n]\!]^d, x, y \in B_{\mathrm{i}}^{(n)}\}. \tag{1.5.3}$$

The idea is that as $d_2(x,y)$ is very analogous to the Euclidean distance and hence the field will be very similar to a log correlated field.

One can formally construct a Gaussian field with covariance Q in the following manner. We first consider a collection of IID standard Gaussian random variables $Z_{\mathrm{i}}^{(n)}$ $\mathrm{i} \in [\![1, 2^n]\!]^d$, and set

$$X_N(x) := \sum_{n=1}^{N} Z_n(x), \tag{1.5.4}$$

where $Z_n(x)$ is defined as the value of $Z_{\mathrm{i}}^{(n)}$ in the dyadic box to which x belongs

$$Z_n(x) = \sum_{\mathrm{i} \in [\![1, 2^n]\!]} Z_{\mathrm{i}}^{(n)} 1_{\{x \in B_{\mathrm{i}}^{(n)}\}}.$$

Note that we have

$$\mathbb{E}[X_N(x) X_N(y)] := Q(x,y) \wedge N. \tag{1.5.5}$$

Hence $X = \lim_{N \to \infty} X_N$ can informally be considered as a random field of covariance Q (one could also prove for instance that X_N converges as a distribution but this is not needed for our purpose). Thus given $\gamma > 0$, we define the multiplicative chaos associated with X as the limit of the following sequence (provided it exists)

$$M_N^{(\gamma)} := \int_{[0,1]^d} e^{\gamma X_N(x) - \frac{\gamma^2}{2} N} \, \mathrm{d}x. \tag{1.5.6}$$

An important consequence of the hierarchical structure of the model that M_{N+1} can be obtained by making an operation on 2^d independent copies of M_N. For simplicity we consider $d = 1$ (the general case is completely similar) and set

$$Z_n^{(1)}(x) := Z_{n+1}(x/2), \quad Z_n^{(2)}(x) := Z_{n+1}((x+1)/2).$$

We let $X_N^{(i)} := \sum_{n=1}^N Z_n^{(i)}$ for $i \in \{1,2\}$ and let $M_N^{(i)}$ denote the associated chaos. It is not difficult to check that $X_N^{(i)}$ (and thus $M_N^{(i)}$) have the same distribution as M_N, furthermore we have

$$M_{N+1} = \left(\frac{e^{\gamma Z_1^{(1)} - \frac{\gamma^2}{2}} M_N^{(1)} + e^{\gamma Z_2^{(1)} - \frac{\gamma^2}{2}} M_N^{(2)}}{2} \right), \tag{1.5.7}$$

where Z_0 is the common value taken by $Z_0(x)$ on $[0,1]$, and is independent of the $M_N^{(i)}$s. Beside the hierarchical structure, another feature that makes cascades easier to study is the fact that our sequence X_N of approximations of the field has independent increments, which implies that $M_N^{(\gamma)}$ is a martingale.

Proposition 1.5.1. *The sequence $M_N^{(\gamma)}$ is a martingale with respect to $\mathcal{F}_N := \sigma(Z_{\mathrm{i}}^{(n)}, n \leqslant N)$. As a consequence it converges to a limit $M_\infty^{(\gamma)}$.*

Proof.

$$\mathbb{E}[M_{N+1} \mid \mathcal{F}_N] := \int e^{\gamma X_N(x) - \frac{\gamma^2}{2} N} \mathbb{E}\left[e^{\gamma Z_N(x) - \frac{\gamma^2}{2}} \mid \mathcal{F}_N \right] \mathrm{d}x = M_N. \tag{1.5.8}$$

□

Note that the relationship (1.5.7) is also valid replacing N by infinity ($(M_N^{(i)})_{N \geqslant 0}$ are also a martingales). As $M_N^{(1)}$ $M_N^{(2)}$ are independent and distributed as M_N, this implies that $\mathbb{P}[M_\infty = 0] = \mathbb{P}[M_\infty = 0]^2$, and thus that the event has probability either 0 or 1. The following is the equivalent of Theorem 1.3.1 in the cascade setup, but turns out to be much easier to prove. It first appeared in Kahane and Peyrière (1976), but the proof we present is inspired by the presentation found in Buffet, Patrick, and Pulé (1993).

Theorem 1.5.2. *We have $M_\infty > 0$, $\mathbb{P}$-a.s. if and only if $\gamma^2 < \sqrt{2d \log 2}$.*

We have chosen to introduce the cascade in a Gaussian setup for practical reasons but it is worth mentioning that the result above (and the proof we present) are valid in much larger generality.

1.5.1 Complex cascades

Multiplicative Cascades have also been considered in the complex setup with various motivations (see Barral, Jin, and Mandelbrot (2010a) and Derrida, Evans, and Speer (1993)). In analogy with (1.4.1) we set

$$M_N^{(\beta)} := \int_{[0,1]^d} e^{\frac{N\beta^2}{2}} \cos(\beta X_N(x))\, \mathrm{d}x. \tag{1.5.9}$$

Repeating the proof of Proposition 1.5.1 we see that $M_N^{(\beta)}$ is a martingale and we want to establish its convergence. Motivated by Section 1.4 above (in particular (1.4.7)) we are concerned not only about $M_N^{(\beta)}$ but also about the finiteness of its Laplace transform. The analogous result in $\mathbb{R}^d$ implies the finiteness of the log-gas partition function has first been proved in Fröhlich (1976).

Theorem 1.5.3. *For every $\beta < \sqrt{d \log 2}$, the limit $\lim_{N\to\infty} M_N^{(\beta)} = M_\infty^{(\beta)}$ exists in $\mathbb{L}_2$. Furthermore we have for every $\alpha \in \mathbb{R}$*

$$\lim_{N\to\infty} \mathbb{E}\left[e^{\alpha M_N^{(\beta)}}\right] = \mathbb{E}\left[e^{\alpha M_\infty^{(\beta)}}\right] < \infty. \tag{1.5.10}$$

The above result is in a sense optimal in the sense that $M_N^{(\beta)}$ does not converge for $\beta < \sqrt{d \log 2}$. More precisely $M_N^{(\beta)}$, after proper renormalization, converges to a Gaussian in distribution.

Proposition 1.5.4. *For every $\beta \geqslant \sqrt{d \log 2}$ we have*

$$\lim_{N\to\infty} \mathbb{E}[(M_N^{(\beta)})^2] = \infty, \tag{1.5.11}$$

and

$$\frac{M_N^{(\beta)}}{\sqrt{\mathbb{E}[(M_N^{(\beta)})^2]}} \overset{N\to\infty}{\Rightarrow} \mathcal{N}(0,1) \tag{1.5.12}$$

where $\Rightarrow$ stands for convergence in distribution.

This result can be pushed in fact a level further. When $\beta \geqslant \sqrt{d \log 2}$, as a consequence of Proposition 1.5.4, $\mathbb{E}\left[e^{\alpha M_N^{(\beta)}}\right]$ diverges. The next result we present gives an information about this divergence.

To enunciate this result, we must recall the notion of cumulant of a random variable. Note that as $M_N^{(\beta)}$ is bounded, the above Laplace transform is analytic in α. Hence we can write its power expansion and we have

$$\log \mathbb{E}\left[e^{\alpha M_N^{(\beta)}}\right] =: \sum_{i\geqslant 0} \frac{\alpha^i}{i!} \mathcal{C}_i^{(\beta)}(N). \tag{1.5.13}$$

The quantity $\mathcal{C}_i^{(\beta)}(N)$ is called the i-th cumulant of M_N. The first two cumulants are respectively given by the mean and variance, and the i-th cumulant can always be expressed as

$$\mathcal{C}_i^{(\beta)}(N) = \mathbb{E}[(M_N)^i] + R(\mathbb{E}[(M_N)^{i-1}], \dots, \mathbb{E}[M_N])$$

where R is a polynomial in $i-1$ variables.

When $\beta > \sqrt{d \log 2}$, the variance $\mathcal{C}_2^{(\beta)}(N)$ diverges. However, it happens that if β is sufficiently close to $\sqrt{d \log 2}$ all other cumulants remain bounded. More precisely we have a sequence of distinct threshold for β which correspond to the divergence of even order cumulant and $\sqrt{d \log 2}$ is only the first one. Let us set $\beta_n = \sqrt{d \log 2 \left(2 - \frac{1}{n}\right)}$

Theorem 1.5.5. *When $\beta < \sqrt{2d \log 2} = \beta_\infty$ then*

$$\lim_{N\to\infty} \mathcal{C}_{2i+1}^{(\beta)}(N) = \mathcal{C}_{2i+1}^{(\beta)}(\infty) \tag{1.5.14}$$

exists and is finite for every i, and when $\beta < \beta_n$ then

$$\lim_{N\to\infty} \mathcal{C}_{2i}^{(\beta)}(N) =: \mathcal{C}_{2i}^{(\beta)}(\infty) \tag{1.5.15}$$

exists for all $i \geqslant n$. Furthermore $\mathbb{E}\left[e^{\alpha M_N^{(\beta)}}\right]$ converges if one subtracts the term corresponding to diverging cumulants.

$$\lim_{N\to\infty} \mathbb{E}\left[e^{\alpha M_N^{(\beta)}}\right] e^{-\sum_{i=1}^{n-1} \mathcal{C}_{2i}^{(\beta)}(N)} = \bar{\mathcal{Z}}_{\alpha,\beta}. \tag{1.5.16}$$

The proof we present are taken from Lacoin, Rhodes, and Vargas (2019) and adapted to the simpler cascade setup. The Central Limit Theorem (Proposition 1.5.4) is proved using the same tools. An alternative proof can be achieved by computing all the moments of the partition function as done in Lacoin, Rhodes, and Vargas (2015).

1.6 Organization of the notes

The rest of the notes is organized as follows. In Chapter 2 we provide a short proof of Theorem 1.5.2 about convergence of multiplicative cascades. In Chapter 3, we prove all the results presented in Section 1.5.1 concerning renormalization of complex cascades. Finally Chapter 4 is devoted to the proof of Theorem 1.3.1 concerning convergence of Gaussian Multiplicative Chaos.

2 Gaussian Multiplicative cascades

2.1 Directed polymer on tree and multiplicative cascades

For notational simplicity and without any loss of generality we decide to stick to the case $d = 1$. First of all for a better graphical representation of the problem, we find an alternative representation of the sequence $M_N^{(\gamma)}$ (defined in Section 1.5. We) as the partition function of a disordered model indexed by the dyadic tree $\mathbb{T}_2$. The elements of $\mathbb{T}_2$ are word represented as words of finite length using the alphabet $\{1, 2\}$ (the root corresponding to the empty word see Figure 3.1). Let us introduce some notation: For $u \in \mathbb{T}_2$ we let $|u| \geqslant 0$ denote the length (i.e. the number of letters) of u. For $0 \leqslant n \leqslant |u|$ we let u_n be the ancestor of u at generation n which is simply the word that consists in the first n letters of u. Finally for $u, v \in \mathbb{T}_2$, we let $u \wedge v$ denote the most recent common ancestor

$$u \wedge v := u_{n*} \text{ where } n^* := \max\{n \leqslant |u| \wedge |v| \ : \ u_n = v_n\}.$$

Now we consider an IID field of centered Gaussian variables with unit variance $(\omega_v)_{v \in \mathbb{T}_2}$ indexed by the vertices of $\mathbb{T}_2$. Now for $u \in \mathbb{T}_2$ we also define a variable X_u which is obtained by summing ω_v along the path linking u to the root

$$X_u := \sum_{n=1}^{|u|} \omega_{u_n}.$$

Now the reader can check that X is a centered Gaussian field indexed by $\mathbb{T}_2$ whose covariance is given by

$$\mathbb{E}[X_u X_v] := |u \wedge v| \tag{2.1.1}$$

Finally we set

$$M_N^{(\gamma)} := 2^{-N} \sum_{\{u \in \mathbb{T}_2 \,:\, |u|=N\}} e^{\gamma X_u - \frac{\gamma^2 N}{2}} \tag{2.1.2}$$

It is not difficult to check that this definition is equivalent to the one given by Equation (1.5.6) in the case $d = 1$ (For the general case it is sufficient to replace the dyadic tree with a 2^d-adic one). In particular, by Proposition 1.5.1, $(M_N^{(\gamma)})_{N \geqslant 0}$ is a martingale and converges to a limit. The aim of this chapter is to understand for which values of γ the limit is non-degenerate.

Theorem 2.1.1. *When $\gamma \in (0, \sqrt{2 \log 2})$, we have the following convergence result in $\mathbb{L}_1$ and almost surely*

$$\lim_{N \to \infty} M_N^{(\gamma)} = M_\infty^{(\gamma)}. \tag{2.1.3}$$

When $\gamma \geqslant \sqrt{2 \log 2}$ then we almost surely have

$$\lim_{N \to \infty} M_N^{(\gamma)} = 0. \tag{2.1.4}$$

As mentioned in Section 1.5, as $M_N^{(\gamma)}$ is a positive martingale, it is sufficient (for the convergence part) to prove that it is uniformly integrable.

Our proof of Theorem 2.1.1 is organized as follows. In Section 2.2, we show using a very simple computation that the martingale in $\mathbb{L}_2$ (and hence converge) if $\gamma < \sqrt{2}$. In Section 2.3, we prove (2.1.4) for $\gamma > \sqrt{2 \log 2}$ using so called fractional moment techniques, and in Section 2.4 we show how the same technique can be adapted to prove a convergence result when $\gamma < \sqrt{2 \log 2}$). Finally in Section 2.5.1, we prove (2.1.4) for the threshold value $\gamma = \sqrt{2 \log 2}$ using so called spine techniques.

2.2 Convergence in $\mathbb{L}^2$

In order to get some intuition on the problem, we start with a very simple explicit computation, which allows to prove convergence for some values of γ.

Proposition 2.2.1. *When $\gamma \in (0, \sqrt{\log 2})$, the sequence $M_N^{(\gamma)}$ is bounded in $\mathbb{L}_2$, and thus as a martingale also converges in $\mathbb{L}_2$ to $M_\infty^{(\gamma)}$*

Remark 2.2.2. *The important observation here is that second moment estimates do not yield an optimal result: for $\gamma \in [\sqrt{\log 2}, \sqrt{2 \log 2})$ the martingale $M_N^{(\gamma)}$ is not bounded in $\mathbb{L}_2$ but is uniformly integrable.*

Proof. The computation of the second moment yields

$$\begin{aligned}\mathbb{E}[M_N^2] &= 4^{-N} \sum_{\{u,v\in\mathbb{T}_2, |u|=|v|=N\}} \mathbb{E}[e^{\gamma(X_u+X_v)-\gamma^2 N}] \\ &= 4^{-N} \sum_{\{u,v\in\mathbb{T}_2, |u|=|v|=N\}} e^{\frac{\gamma^2}{2}(\mathbb{E}[(X_u+X_v)^2]-N)} = 4^{-N} \sum_{\{u,v\in\mathbb{T}_2, |u|=|v|=N\}} e^{\gamma^2|u\wedge v|}.\end{aligned} \tag{2.2.1}$$

Now to compute the above sum, we can observe that

$$\#\{u,v \ : \ |u|=|v|=N, \ |u\wedge v| = k\} = \begin{cases} 2^{2N-k-1} & \text{if } 0\leqslant k\leqslant N-1, \\ 2^N & \text{if } k=N. \end{cases} \tag{2.2.2}$$

and conclude that

$$\begin{aligned}\mathbb{E}[M_N^2] = 4^{-N}\sum_{k=0}^{N-1} 2^{2N-k-1} e^{\gamma^2 k} + 4^{-N} 2^N e^{\gamma^2 N} \\ = \frac{1-2^{-N}e^{\gamma^2}}{2-e^{\gamma^2}} + 2^{-N}e^{\gamma^2 N} \leqslant \frac{1}{2-e^{\gamma^2}},\end{aligned} \tag{2.2.3}$$

where the last inequality is valid when $\gamma^2 < \log 2$. Thus M_N is a bounded sequence in $\mathbb{L}_2$ for these value of γ. □

2.3 Convergence to zero for $\gamma > \sqrt{2\log 2}$ using fractional moments

Let us now move to the case of large γ. In this section we wish to prove that the limit is degenerate for $\gamma > \sqrt{2\log 2}$. Although convergence to zero also holds in the case $\gamma = \sqrt{2\log 2}$, but the proof of this case is more subtle and is detailed in Section 2.5

Proposition 2.3.1. *When $\gamma \geqslant \sqrt{2\log 2}$ then we almost surely have*

$$\lim_{N\to\infty} M_N^{(\gamma)} = 0. \tag{2.3.1}$$

The proof of this result is a particular case of what physicists call fractional moment methods. It relies on estimating $\mathbb{E}\left[(M_N^{(\gamma)})^\theta\right]$ for some non-integer value of $\theta \in (0,1)$.

In contrast with the computation (2.2.1) made in the case $\theta = 2$, for θ non-integer there is no explicit explicit expression for the moment. However we can obtain upper or lower bound on fractional moments by using convexity inequalities. One very simple inequality (easily proved by induction) will be of used several times and is valid for any finite or countable collection of positive numbers $(a_i)_{i\in\mathcal{I}}$ and $\theta \in (0,1)$

$$\left(\sum_{i\in\mathcal{I}} a_i\right)^\theta \leqslant \sum_{i\in\mathcal{I}} a_i^\theta. \tag{2.3.2}$$

Proof of Proposition 2.3.1. We drop the dependence in γ in the notation for commodity. We are going to show that there exists $\theta < 1$ such that

$$\lim_{N\to\infty} \mathbb{E}\left[M_N^\theta\right] = 0, \tag{2.3.3}$$

which by Fatou's Lemma implies that $\mathbb{E}[(\lim_{N\to\infty} M_N^\theta)^\theta] = 0$. Now using (2.3.2) we have

$$M_N^\theta = 2^{-N\theta}\left(\sum_{|u|=n} e^{\gamma X_u - \frac{\gamma^2}{2}N}\right)^\theta \leqslant 2^{-N\theta}\sum_{|u|=n} e^{\gamma\theta X_u - \frac{\theta\gamma^2}{2}N} \tag{2.3.4}$$

and thus taking expectations and choosing $\theta = \sqrt{2\log 2}/\gamma < 1$ we obtain

$$\mathbb{E}\left[M_N^\theta\right] \leqslant \exp\left(N(1-\theta)\left[\log 2 - \frac{\theta\gamma^2}{2}\right]\right) = e^{-\frac{N}{2}(\gamma - \sqrt{2\log 2})^2}. \tag{2.3.5}$$

□

2.4 Convergence when $\gamma < \sqrt{2\log 2}$

In this section, we prove that the martingale limit is non-degenerate whenever $\gamma < \sqrt{2\log 2}$ which is the optimal result. The proof relies on another application of the fractional moment method introduced in the previous section.

Proposition 2.4.1. *When $\gamma \geqslant \sqrt{2\log 2}$ then the following limit is almost surely positive.*

$$\lim_{N\to\infty} M_N^{(\gamma)} = M_\infty^{(\gamma)}, \tag{2.4.1}$$

Furthermore the convergence also holds in $\mathbb{L}^q$ for $q \in [1, \max(\frac{2\log 2}{\gamma^2}, 2))$.

Proof. As M_N is a martingale, its convergence in $\mathbb{L}^q$ is equivalent to boundedness of the sequence in $\mathbb{L}^q$. We choose $q \in (1,2)$, and observe that $M_N^q = (M_N^2)^{q/2}$. We use the decomposition (1.5.7) of M_{N+1} using the partition functions $M_N^{(1)}$ and $M_N^{(2)}$ corresponding to the the two subtrees rooted at 1 and 2 respectively (recall that ω_1 and ω_2 denote the value $(\omega_u)_{u\in\mathbb{T}_2}$ at the two vertices of the first generation of $\mathbb{T}_2$).

$$M_{N+1} = \frac{e^{-\gamma^2/2}}{2}\left(e^{\gamma\omega_1}M_N^{(1)} + e^{\gamma\omega_2}M_N^{(2)}\right). \tag{2.4.2}$$

Taking the square of this recursive expression we obtain

$$M_{N+1}^2 = \frac{e^{-\gamma^2}}{4}\left(e^{2\gamma\omega_1}(M_N^{(1)})^2 + e^{2\gamma\omega_2}(M_N^{(2)})^2 + 2e^{\gamma(\omega_1+\omega_2)}M_N^{(1)}M_N^{(2)}\right). \tag{2.4.3}$$

Finally, we use (2.3.2) for $\theta = q/2 < 1$ and obtain that

$$M_{N+1}^q \leqslant \left(2e^{\frac{\gamma^2}{2}}\right)^{-q}\left(e^{q\gamma\omega_1}(M_N^{(1)})^q + e^{q\gamma\omega_2}(M_N^{(2)})^q + e^{\frac{q\gamma}{2}(\omega_1+\omega_2)}(2M_N^{(1)}M_N^{(2)})^{q/2}\right). \tag{2.4.4}$$

Hence using the independence of the $M_N^{(j)}$s and taking expectation on both sides we have

$$\mathbb{E}[M_{N+1}^q] \leqslant 2^{1-q}e^{\frac{(q^2-q)\gamma^2}{2}}\mathbb{E}[M_N^q] + 2^{-q/2}e^{\frac{(q^2-2q)\gamma^2}{4}}\mathbb{E}[M_N^{q/2}]^2 \tag{2.4.5}$$

Now as we have $\mathbb{E}[M_N^{q/2}] \leqslant 1$ by Jensen's inequality and as $2^{-q/2}e^{\frac{(q^2-2q)\gamma^2}{4}} \leqslant 1$, we obtain that the sequence $a_N := \mathbb{E}[M_{N+1}^q]$ satisfies $a_0 = 1$ and

$$a_{N+1} \leqslant \alpha_q a_N + 1$$

with $\alpha_q = 2^{1-q}e^{\frac{(q^2-q)\gamma^2}{2}}$. When $q < (2\log 2)/\gamma^2$ we have $\alpha_q < 1$ which allows to conclude that a_N is bounded by $1/(1-\alpha_q)$.

□

2.5 Proof of convergence to zero in the critical case $\gamma = \sqrt{2\log 2}$

The argument of Section 2.3 falls short in the critical case $\gamma = \sqrt{2\log 2}$. In order to settle this case we have to use a more refined argument, which involves Cameron-Martin shifts of the Gaussian field (that is considering $e^{\gamma X_u - \frac{\gamma^2|u|}{2}}$ as a probability density and using Cameron-Martin formula (Proposition 1.1.3)). This approach turns out to be very powerful and can also be used in the non hierarchical setup (see Chapter 4).

Proposition 2.5.1. *When $\gamma = \sqrt{2\log 2}$ we have*

$$\lim_{N\to 0} M_N^{(\gamma)} = 0. \tag{2.5.1}$$

Proof. The strategy to prove the above is to identify a sequence of events $\mathcal{A}_N$ whose probability tend to one while

$$\lim_{N\to\infty} \mathbb{E}[M_N 1_{\mathcal{A}_N}] = 0. \tag{2.5.2}$$

Then we can deduce from it that M_N tends to zero in probability by applying the Markov inequality to $M_N 1_{\mathcal{A}_N}$.

$$\mathbb{P}[M_N > \varepsilon] \geqslant \varepsilon^{-1}\mathbb{E}[M_N 1_{\mathcal{A}_N}] + \mathbb{P}[\mathcal{A}_N^\complement]. \tag{2.5.3}$$

The important thing to notice here is that as $\mathbb{E}[M_N] = 1$, $\widetilde{\mathbb{P}}_N[\mathcal{A}_N] = \mathbb{E}[M_N 1_{\mathcal{A}_N}]$ defines a probability measure. Hence the problem reduces to find an even $\mathcal{A}_N$ which while very typical under $\mathbb{P}$, becomes very atypical under the probability $\widetilde{\mathbb{P}}_N$.

To this purpose, it is important to have a more explicit description of $\widetilde{\mathbb{P}}_N$. Given $u \in \mathbb{T}_2$ we can define $\widetilde{\mathbb{P}}_u$ as the measure whose density with respect to $\mathbb{P}$ is given by $e^{\gamma X_u - \frac{\gamma^2}{2}|u|}$. We have

$$\widetilde{\mathbb{P}}_N := \frac{1}{2^N}\sum_{|u|=N}\widetilde{\mathbb{P}}_u.$$

Now using Proposition 1.1.3 it is worth mentioning that $\widetilde{\mathbb{P}}_u$ corresponds to the measure under which the ω_vs are still Gaussians of variance one, but with average equal to γ if v is an ancestor of u and 0 if not.

Let us set

$$\mathcal{A}_N := \left\{\forall n \in [\![1, N]\!],\ \max_{x\in[0,1]} X_n(x) \leqslant \sqrt{2\log 2}n + \log N\right\}$$

The proof of our proposition then follows from the following estimates

Lemma 2.5.2. *When $\gamma = \sqrt{2\log 2}$ we have*

(A) $\lim_{N\to\infty}\mathbb{P}[\mathcal{A}_N^\complement] = 0$,

(B) $\lim_{N\to\infty}\widetilde{\mathbb{P}}_N[\mathcal{A}_N] = 0$.

□

Proof of Lemma 2.5.2. Let us start with (A). We can use union bound and the Gaussian tail bound (1.1.6) we have

$$\begin{aligned}\mathbb{P}[\mathcal{A}_N^\complement] &= \mathbb{P}\left[\exists n \in [\![1, N]\!], |u| = n, X_u > \sqrt{2\log 2}n + \log N\right]\\ &\leqslant \sum_{n=1}^{N}\sum_{|u|=n}\mathbb{P}[X_u > \sqrt{2\log 2}n + \log N] \leqslant \sum_{n=1}^{N} 2^n e^{-\frac{(\sqrt{2\log 2}n + \log N)^2}{2n}},\end{aligned} \tag{2.5.4}$$

and we conclude by observing that neglecting the last term in the expansion of the square $(\sqrt{2\log 2}n + \log N)^2$ we have

$$e^{-\frac{(\sqrt{2\log 2}n+\log N)^2}{2n}} \leqslant 2^{-n} N^{-\sqrt{2\log 2}}. \tag{2.5.5}$$

As $\sqrt{2\log 2} > 1$, this allows to conclude.

To prove (B) we observe that by symmetry $\widetilde{\mathbb{P}}_u[\mathcal{A}_N]$ does not depend on the choice of u when $|u| = N$. Now u being fixed, let us use the notation $(Y_n)_{n=1}^N := (X_{u_n})_{n=1}^N$ for the value of X along the path of ancestors of u. Now using the fact that under $\widetilde{\mathbb{P}}_u$, the increments of Y_n have mean $\sqrt{2\log 2}$ we obtain that

$$\begin{aligned}\widetilde{\mathbb{P}}_u[\mathcal{A}_N] \leqslant \widetilde{\mathbb{P}}_u[\forall n \in [\![0, N]\!], Y_n \leqslant \sqrt{2\log 2}n + \log N]& \\ = \mathbb{P}[\forall n \in [\![0, N]\!], Y_n \leqslant \log N].&\end{aligned} \tag{2.5.6}$$

The probability of the right hand side is of course independent of u and the convergence of $Y_{\lceil xN \rceil}/\sqrt{N}$ to Brownian motion implies that the right hand side converges to $\mathrm{P}[B_t \leqslant 0 \forall t \in [0, 1]] = 0$ (see below for a self-contained short proof).

□

Random walk estimates

Lemma 2.5.3. *Let $(Y_n)_{n\geqslant 0}$ a random sequence (P denotes its distribution) starting from $Y_0 = 0$ and whose increments are IID with distribution $\mathcal{N}(0, 1)$.*

$$\lim_{N\to\infty} \mathrm{P}[\max_{n\in[\![1,N]\!]} Y_n \leqslant \log N] = 0. \tag{2.5.7}$$

Proof. For $n\geqslant 0$, we set $n_k := 2^{k^2} - 1$, and $Z_k := Y_{n_k} - Y_{n_{k-1}}$. We observe that the event whose probability we want to bound is included in $\mathcal{A}_N \cup \mathcal{B}_N$ where

$$\begin{aligned}\mathcal{A}_N &:= \{\forall k \in [\![(\log N)^{1/4}, (\log N)^{1/3}]\!],\ Z_k \leqslant \sqrt{n_k - n_{k-1}}\}\\ \mathcal{B}_N &:= \{\exists n \in [\![1, N[\![,\ Y_n \leqslant -(\log N)\sqrt{n}\}.\end{aligned} \tag{2.5.8}$$

Indeed if neither $\mathcal{A}_N$ nor $\mathcal{B}_N$ holds, considering $k \in [\![(\log N)^{1/4}, (\log N)^{1/3}]\!]$ such that $Z_k > \sqrt{n_k}$, we have

$$Y_{n_k} = Z_k + Y_{n_{k-1}} \geqslant -(\log N)\sqrt{n_{k-1}} + \sqrt{n_k - n_{k-1}} \leqslant \log N. \tag{2.5.9}$$

As $Z_k/\sqrt{n_k - n_{k-1}}$ are IID standard Gaussians, we have

$$\mathrm{P}[\mathcal{A}_N] = \mathrm{P}[Y_1 \leqslant 1]^{\#[\![(\log N)^{1/4},(\log N)^{1/3}]\!]}, \tag{2.5.10}$$

and tends to zero. As Y_n is a centered Gaussian of variance n, using the Gaussian tail estimate (1.1.6), we obtain that

$$\mathbb{P}[Y_n \leqslant -(\log N)\sqrt{n}] \leqslant e^{-\frac{(\log N)^2}{2}}, \tag{2.5.11}$$

and hence by union bound that $\mathrm{P}[\mathcal{B}_N] \leqslant N e^{-\frac{(\log N)^2}{2}}$.

□

3 Complex cascades

3.1 Continuous tree

For the sake of studying complex multiplicative cascade, we decide to extend $M_N^{(\beta)}$ into a continuous time martingale $M_t^{(\beta)}$.

For this we consider the continuous dyadic tree $\mathcal{T}_2$ which is obtained by adding a segment of length one between vertex of $\mathbb{T}_2$ and its immediate ancestor. Formally $\mathcal{T}_2$ can be obtained by quotienting the space $\{(u,t), u \in \mathbb{T}_2, t \leqslant |u|\}$ by the equivalence

$$(u,t) \sim (v,t) \text{ if } u_{\lceil t \rceil} = v_{\lceil t \rceil}.$$

where as introduced in the previous chapter, u_n denotes the ancestor of u at generation n (the word composed of the fist n letters of u. We write u_t for the equivalence class of (u,t) in $\mathcal{T}_2$. Informally, u_t is the point located at distance t from the root (or empty word) on the path going to u (see Figure 3.1). We extend the notion of length and most recent common ancestor to $\mathcal{T}_2$ by setting $|u_t| = t$ and

$$u_t \wedge v_s = (u \wedge v)_{t \wedge s}$$

Now we wish to consider a field indexed by $\mathcal{T}_2$ with covariance function

$$\mathbb{E}[X(u)X(v)] := u \wedge v, \tag{3.1.1}$$

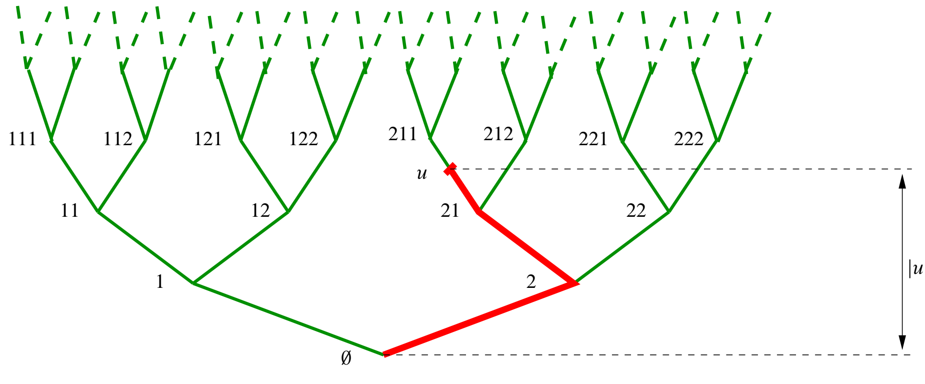

FIGURE 3.1. Graphical representation of the dyadic tree $\mathbb{T}_2$ and its continuous counterpart $\mathcal{T}_2$. The discrete tree $\mathbb{T}_2$ is the graph whose vertices are finite words in the alphabet $\{1, 2\}$. Edges in $\mathbb{T}_2$ link pairs of words for which one can be obtained by adding a letter to the other. The continuous tree $\mathcal{T}_2$ is obtained informally by adding segments of length 1 for each edges in $\mathbb{T}_2$. Here we have schematically represented a point $\boldsymbol{u} \in \mathcal{T}_2$, together with the path which is linking it to the root.

(the same as (2.1.1) but extended to the continuous tree $\mathcal{T}_2$). We consider a continuous version of the field that is continuous with respect to the distance in $\mathcal{T}_2$ which is defined by $d_{\mathcal{T}_2}(u, v) = |u| + |v| - 2|u \wedge v|$. It is possible to construct such a process by simply adding independent Brownian bridges to bridge the increment of $(X(u_n))_{n=0}^{|u|}$ resulting from the field on $\mathbb{T}_2$. Observing that for each $t > 0$ there are $2^{\lceil t \rceil}$ vertices in $\mathcal{T}_2$ satisfying $|u| = t$, we can define

$$M_t^{(\beta)} := 2^{-\lceil t \rceil} e^{\frac{\beta^2 t}{2}} \sum_{\{u \in \mathcal{T}_2 \,:\, |u|=t\}} \cos(\beta X(u)). \tag{3.1.2}$$

We can check that the following holds

Proposition 3.1.1. *The process $M_t^{(\beta)}$ is a continuous martingale, for the filtration given by $\mathcal{F}_t := \sigma(X(u), |u| \leqslant t)$.*

Proof. Continuity of M_t follows from the continuity of $X(u)$. To prove that the process in the martingale, observe that for $s \leqslant t$, and $|u| = t$ we have

$$\begin{aligned} \mathbb{E}\left[\cos(\beta X(u)) \mid \mathcal{F}_s\right] &= \cos(\beta X(u_s))\mathbb{E}[\cos(\beta(X(u) - X(u_s))) \mid \mathcal{F}_s] \\ &\quad - \sin(\beta X(u_s))\mathbb{E}[\sin(\beta(X(u) - X(u_s))) \mid \mathcal{F}_s] \\ &= e^{-\frac{(t-s)\beta^2}{2}} \cos(\beta X(u_s)) \end{aligned} \tag{3.1.3}$$

where above, we have used that $X(u)-X(u_s)$ is a centered Gaussian variable independent from $\mathcal{F}_s$ of variance $t-s$ and that for centered Gaussian variable of variance σ^2 we have

$$\mathbb{E}[\cos(\beta Z)] = \mathbb{E}[e^{i\beta Z}] = e^{-\frac{\beta^2\sigma^2}{2}}. \tag{3.1.4}$$

Hence, keeping in mind that each u with $|u| = s$ has $2^{\lceil t\rceil - \lceil s\rceil}$ offspring at level t, we have

$$\mathbb{E}\left[M_t^{(\beta)} \mid \mathcal{F}_s\right] = 2^{-\lceil t\rceil} e^{\frac{\beta^2 s}{2}} \sum_{\{u\in\mathcal{T}_2\,:\,|u|=t\}} \cos(\beta X(u_s)) = 2^{-\lceil s\rceil} e^{\frac{\beta^2 s}{2}} \sum_{\{u\in\mathcal{T}_2\,:\,|u|=s\}} \cos(\beta X(u)) \tag{3.1.5}$$

□

The aim of this chapter is to understand the asymptotic behavior of $M_t^{(\beta)}$ when t tends to infinity. A first result, that can be obtained by repeating the proof of Proposition 2.2.1, is that the martingale is bounded in $\mathbb{L}_2$ if and only if $\beta < \sqrt{\log 2}$.

Proposition 3.1.2. *When $\beta < \sqrt{\log 2}$, we have*

$$\sup_{t>0} \mathbb{E}[M_t^2] < \infty, \tag{3.1.6}$$

in particular the martingale M_t converges in $\mathbb{L}^2$ to a limit M_∞.

Proof. We have

$$\mathbb{E}[M_t^2] = 2^{-2\lceil t\rceil} e^{\beta^2 t} \sum_{\{|u|=|v|=t\}} \mathbb{E}[\cos(\beta X(u))\cos(\beta X(v))] \tag{3.1.7}$$

and using (3.1.4)

$$\begin{aligned}\mathbb{E}[\cos(\beta X(u))\cos(\beta X(v))]&\\ &= \mathbb{E}\left[\cos(\beta(X(u)-X(v)) + \cos(\beta(X(u)+X(v))\right]\\ &= \frac{e^{\beta^2(t+|u\wedge v|)} + e^{\beta^2(t-|u\wedge v|)}}{2} = e^{\beta^2 t}\cosh(\beta^2|u\wedge v|).\end{aligned} \tag{3.1.8}$$

Hence using the analog of (2.2.2) ofr the continuous tree we have

$$\mathbb{E}[M_t^2] = \sum_{k=0}^{\lceil t\rceil -1} 2^{-(k+1)}\cosh(\beta^2 k) + 2^{-\lceil t\rceil}\cosh(\beta^2 t), \tag{3.1.9}$$

The right hand side converges as t tends to infinity if and only if $\beta^2 < \sqrt{\log 2}$.

□

To complete the proof of Theorem 1.5.3 presented in our introduction, a bound on the second moment is clearly not sufficient. In order to a better control on the fluctuation of M_t we use stochastic calculus to compute its quadratic variation. We are not going to use advanced tools of stochastic calculus, but the reader needs to be familiar with Itô's formula and Girsanov's Theorem (see e.g. Karatzas and Shreve (1991) and Revuz and Yor (1999) for introductions to the subject).

3.2 Computation of the quadratic variation of M_t

We let $\langle M\rangle_t$ denote the quadratic variation of $(M_s)_{s\geqslant 0}$ on the interval $[0,t]$. The aim of this section is to prove bound the quadratic variation in order to prove finiteness of the Laplace transform of M_∞.

Proposition 3.2.1. *For $\beta < \sqrt{\log 2}$ there exists a constant K_β such that almost surely we have*

$$\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t < K_\beta. \tag{3.2.1}$$

As a consequence we have

$$\mathbb{E}[e^{\alpha M_\infty}]\leqslant e^{\alpha+\frac{\alpha^2 K}{2}}, \tag{3.2.2}$$

Before proving the proposition, let us explain why (3.2.2) is a consequence of the bound on the quadratic variation. This relies on the notion of exponential martingale of M_t (see e.g. Revuz and Yor (1999, Proposition (3.4), pp. 148)).

Proposition 3.2.2. *If M is a continuous martingale, $u\in\mathbb{R}$, and $\langle M\rangle_t$ denotes its quadratic variation up to time t, then the process N defined by*

$$N_t = e^{uM_t-\frac{u^2}{2}\langle M\rangle_t}$$

is a martingale for the same filtration.

The above result implies in particular that

$$\mathbb{E}\left[e^{uM_t-\frac{u^2}{2}\langle M\rangle_t}\right] = \mathbb{E}\left[e^{uM_0}\right] = e^u, \tag{3.2.3}$$

and thus trivially implies our result. Now in order to prove Proposition 3.2.1, we must compute the quadratic variation of M_t.

For $t\in(n-1,n]$, we can rewrite M_t as

$$M_t^{(\beta)} = 2^{-n}\sum_{|u|=n} e^{\frac{\beta^2 t}{2}}\cos(\beta X(u_t)). \tag{3.2.4}$$

Hence using the usual rules of Itô calculus to compute the infinitesimal increments, we obtain that for $t \in (n-1, n)$

$$\mathrm{d}M_t^{(\beta)} = -\beta 2^{-n} \sum_{|u|=n} e^{\frac{\beta^2 t}{2}} \sin(\beta X(u_t))\, \mathrm{d}X(u_t), \tag{3.2.5}$$

where $\mathrm{d}X(u_t)$ denotes the increment of the martingale $t \mapsto X(u_t)$. Thus

$$\mathrm{d}\langle M^{(\beta)}\rangle_t = \beta^2 2^{-2n} \sum_{|u|=|v|=n} e^{\beta^2 t} \sin(\beta X(u_t)) \sin(\beta X(v_t))\, \mathrm{d}\langle X(u_t), \rangle X(v_t)\rangle, \tag{3.2.6}$$

The increments of $X(u_t)$ are Brownian and are independent on different branches, hence for u, v with $|u| = |v| = n$ we have by construction

$$\mathrm{d}\langle X(u_t), X(v_t)\rangle = 1_{\{u=v\}}\, \mathrm{d}t.$$

Hence integrating (3.2.6) between 0 and t we obtain that

$$\langle M^{(\beta)}\rangle_t = \beta^2 \int_0^t 2^{-2\lceil s\rceil} e^{\beta^2 s} \sum_{|u|=s} \sin^2(\beta X(u))\, \mathrm{d}s. \tag{3.2.7}$$

With this expression, the proof of the proposition is immediate

Proof of Proposition 3.2.1. When $\beta < \sqrt{\log 2}$, simply using the fact that $\sin^2(x) \leqslant 1$ and that the sum is over $\lceil s\rceil$ terms we obtain that

$$\langle M^{(\beta)}\rangle_t \leqslant \beta^2 \int_0^t 2^{-\lceil s\rceil} e^{\beta^2 s}\, \mathrm{d}s < \infty. \tag{3.2.8}$$

□

3.3 Central limit theorem in beyond the $\mathbb{L}_2$ threshold

When $\beta^2 > \sqrt{\log 2}$, the second moment of $M_t^{(\beta)}$ diverges. With some minor effort, we can obtain the following simple asymptotic of the variance can be deduced from (3.1.9)

$$\sigma_t^2 := \begin{cases} \frac{t}{4} & \text{when } \beta^2 = \log 2, \\ \frac{2^{-\lceil t\rceil}}{2}\left[\frac{e^{\beta^2\lceil t\rceil}}{e^{\beta^2}-2} + e^{\beta^2 t}\right] & \text{when } \beta^2 > \log 2. \end{cases} \tag{3.3.1}$$

We are going to show in this section that in this regime M_t renormalized by its standard deviation (or by σ_t which is asymptotically equivalent to it) converges to a standard centered Gaussian

Theorem 3.3.1. *For all $\beta > \sqrt{\log 2}$ we have*

$$\frac{M_t^{(\beta)}}{\sigma_t} \overset{t\to\infty}{\Longrightarrow} \mathcal{N}(0,1) \tag{3.3.2}$$

where the arrows stands for convergence in law.

Let us briefly explain the strategy we use to prove (3.3.1). We are going to prove the convergence of the Laplace transform of $\frac{M_t^{(\beta)}}{\sigma_t}$ towards that of a Gaussian, in other words that for all $\alpha \in \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{E}\left[e^{\frac{\alpha M_t}{\sigma_t}}\right] = e^{\frac{\alpha^2}{2}}. \tag{3.3.3}$$

This implies in particular that $e^{\frac{\alpha M_t}{\sigma_t}}$ is uniformly integrable, and thus converges for all $\alpha \in \mathbb{C}$. By analytic continuation, (3.3.3) must also be valid for $\alpha \in i\mathbb{R}$ and then Lévy's continuity Theorem implies convergence towards a Gaussian random variable.

To prove (3.3.3), the starting point is to observe that from Proposition 3.2.2 we have

$$\mathbb{E}\left[e^{uM_t - \frac{u^2}{2}\langle M\rangle_t}\right] = \mathbb{E}\left[e^{uM_0}\right] \tag{3.3.4}$$

For $u = \alpha/\sigma_t$, $\mathbb{E}\left[e^{uM_0}\right]$ converges to 1 and proving the result amounts to being able to replace $\langle M\rangle_t$ by σ_t^2 (which corresponds roughly to its expectation). Hence the core of the proof is to prove that $\langle M\rangle_t$ concentrates around its mean (we postpone the proof of this estimate to the end of the section)

Lemma 3.3.2. *When $\beta > \sqrt{2\log 2}$ we have*

$$\lim_{t\to\infty} \sigma_t^{-2}\langle M\rangle_t = 1, \tag{3.3.5}$$

and for sufficiently large values of t we have almost surely

$$\sigma_t^{-2}\langle M\rangle_t \leqslant 3 \tag{3.3.6}$$

Proof of Theorem 3.3.1. We prove (3.3.3) by proving separately an upper and a lower bound (in this order). Using Hölder inequality we have for any $u > 0$, and $p > 1$

$$\begin{aligned} &\mathbb{E}[e^{uM_t}] = \mathbb{E}[e^{uM_t - \frac{pu^2\langle M\rangle_t}{2}} e^{\frac{pu^2\langle M\rangle_t}{2}}] \\ &\leqslant \mathbb{E}\left[e^{puM_t - \frac{p^2u^2\langle M\rangle_t}{2}}\right]^{1/p} \mathbb{E}\left[e^{\frac{p^2u^2\langle M\rangle_t}{2(1-p)}}\right]^{p/(p-1)} = \mathbb{E}\left[e^{\frac{p^2u^2\langle M\rangle_t}{2(1-p)}}\right]^{p/(p-1)} \end{aligned} \tag{3.3.7}$$

Taking $u = \alpha\sigma_t^{-1}$ we can deduce that for any fixed $p > 1$ we have

$$\limsup_{t\to\infty} \mathbb{E}\left[e^{\frac{\alpha M_t}{\sigma_t}}\right] \leqslant \lim_{t\to\infty} \mathbb{E}\left[e^{\frac{p^2\alpha\langle M\rangle_t}{2\sigma_t^2(p-1)}}\right]^{(p-1)/p} = e^{\frac{\alpha^2 p}{2}}. \tag{3.3.8}$$

The last equality is simply obtained by dominated convergence, using Lemma 3.3.2. The inequality is also valid for $p = 1$ by continuity. For the lower bound we have

$$1 = \mathbb{E}[e^{uM_t - \frac{u^2\langle M\rangle_t}{2}}] \leqslant \mathbb{E}[e^{puM_t}]^{1/p}\mathbb{E}\left[e^{-\frac{(p-1)u^2\langle M\rangle_t}{2p}}\right]^{\frac{p}{p-1}}. \tag{3.3.9}$$

Using this for $u = \alpha\sigma_t^{-1}p^{-1}$ we obtain also using dominated convergence that

$$\liminf_{t\to\infty} \mathbb{E}\left[e^{\frac{\alpha M_t}{\sigma_t}}\right] \geqslant \lim_{t\to\infty} \mathbb{E}\left[e^{-\frac{\alpha(p-1)\langle M\rangle_t}{2p^3\sigma_t^{-2}}}\right]^{-\frac{p}{p-1}} = e^{\frac{\alpha^2}{2p^2}}, \tag{3.3.10}$$

which concludes the proof.

□

Proof of Lemma 3.3.2. We split $\langle M^{(\beta)}\rangle_t$ (recall (3.2.7)) into two parts, with the idea in mind to prove that the first one (which is non-random) is asymptotically equivalent to σ_t and that the second part (which is random) in negligible w.r.t. σ_t.

$$\begin{aligned}\langle M^{(\beta)}\rangle_t = \frac{\beta^2}{2}\int_0^t 2^{-\lceil s\rceil}e^{\beta^2 s}\,\mathrm{d}s - \frac{\beta^2}{2}\int_0^t 2^{-2\lceil s\rceil}e^{\beta^2 s}\sum_{|u|=s}\cos(2\beta X(u))\,\mathrm{d}s \\ =: A_t + \zeta_t.\end{aligned} \tag{3.3.11}$$

Note in particular that we have $\langle M^{(\beta)}\rangle_t \leqslant 2A_t$ almost surely. To conclude we need to prove that

$$\lim_{t\to\infty} A_t\sigma_t^{-1} = 1 \quad \text{and} \quad \lim_{t\to\infty} \zeta_t\sigma_t^{-1} = 0, \tag{3.3.12}$$

where the second inequality holds in probability.

To compute A_t let us observe that

$$\int_n^{n+1} \beta^2 e^{\beta^2 s} = e^{\beta^2 n}(e^{\beta^2} - 1) \quad \text{and} \quad \int_{\lceil t\rceil-1}^t \beta^2 e^{\beta^2 s} = e^{\beta^2 t} - e^{\beta^2(\lceil t\rceil-1)}, \tag{3.3.13}$$

When $\beta^2 = \log 2$ this yields

$$A_t := \frac{(\lceil t\rceil - 1)}{4} + \frac{1}{4}(2^{1+t-\lceil t\rceil} - 1),$$

when $\beta^2 > \log 2$ we have

$$A_t = \frac{1}{2}\left((e^{\beta^2}-1)\frac{(e^{\beta^2}/2)^{\lceil t\rceil-1}-1}{e^{\beta^2}-2} + 2^{-\lceil t\rceil}(e^{\beta^2 t}-e^{\beta^2\lceil t\rceil-1)})\right) = \sigma_t^2 - \frac{e^{\beta^2}-1}{2(e^{\beta^2}-2)}, \tag{3.3.14}$$

which proves our statements concerning A_t. To establish the convergence in probability of ζ_t, we compute its second moment. Writing an upper bound (to avoid having to write integer parts, recall that all terms are positively correlated, our constant C may change from line to line and depend on β), we obtain

$$\begin{aligned}\mathbb{E}[\zeta_t^2] \leqslant & C\int_{0\leqslant r\leqslant s\leqslant t}\sum_{|u|=s,|v|=r} 4^{-s+r}e^{\beta^2(s+r)}\mathbb{E}[\cos(2\beta X(u))\cos(2\beta X(v))]\,\mathrm{d}s\,\mathrm{d}r \\ & \leqslant C\int\sum_{|u|=r,|v|=s} 4^{-(s+r)}e^{-\beta^2(s+r)}e^{4\beta^2|u\wedge v|}\,\mathrm{d}s\,\mathrm{d}r \end{aligned} \tag{3.3.15}$$

where the second inequality uses (recall (3.1.8))

$$\mathbb{E}[\cos(2\beta X(u))\cos(2\beta X(v))] = e^{-2\beta^2(s+t)}\cosh(4\beta^2|u\wedge v|).$$

Counting the number of possibility for $|u\wedge v| = k$ like in (2.2.2) we obtain

$$\sum_{|u|=r,|v|=s} e^{4\beta^2|u\wedge v|} \leqslant C\,2^{r+s}\sum_{k=0}^{\lceil s\rceil} 2^{-k}e^{4\beta^2 k} \leqslant C\,2^s e^{4\beta^2 r}, \tag{3.3.16}$$

which finally yields

$$\mathbb{E}[\zeta_t^2] \leqslant \begin{cases} C & \text{if } \beta^2 < \frac{3}{2}\log 2, \\ Ct & \text{if } \beta^2 = \frac{3}{2}\log 2, \\ e^{2\beta^2 t}2^{-3t} & \text{if } \beta^2 > \frac{3}{2}\log 2, \end{cases} \tag{3.3.17}$$

which in every case, is much smaller than $(\sigma_t)^4$, hence ζ_t/σ_t^2 converges to zero in probability. □

3.4 Transitions for cumulants

The last computation highlights that when $\beta^2 \in \left[\log 2, \frac{3}{2}\log 2\right)$, the quadratic variations $\langle M\rangle_t$ diverges but $\langle M\rangle_t = \mathbb{E}[M_t^2]$ remains tight (or bounded in $\mathbb{L}_2$), and this seems to indicate that another transition might occur at $\beta = \sqrt{\frac{3}{2}\log 2}$.

In order to observe these transitions, we need to look at the cumulants of M_t. They are defined (up to a multiplicative factor $i!$) as the coefficient of the power series corresponding to the Laplace transform of M_t

$$\log \mathbb{E}\left[e^{\alpha M_t^{(\beta)}}\right] =: \sum_{i=1}^{n} \frac{1}{i!} \mathcal{C}_i^{(\beta)}(t). \tag{3.4.1}$$

The first two cumulants are given by the mean and variance of M_t respectively and we have

$$\mathcal{C}_i^{(\beta)}(t) = \mathbb{E}[(M_t^{(\beta)})^i] + R_i(t), \tag{3.4.2}$$

where $R_i(t)$ is a polynomial in lower order moments of M_t. We are going to prove that while the variance diverges when $\beta^2 \geqslant \log 2$, all but finitely many cumulants converges when $\beta^2 \in [\log 2, 2\log 2)$.

Theorem 3.4.1. *When $\beta < \sqrt{2\log 2} = \beta_\infty$ then*

$$\lim_{t\to\infty} \mathcal{C}_{2n+1}^{(\beta)}(t) = \mathcal{C}_{2n+1}^{(\beta)}(\infty) \tag{3.4.3}$$

exists and is finite for every $n \geqslant 0$. We also have convergence of the cumulant of order $2n$ when $\beta^2 < \log 2(2 - \frac{1}{n})$.

$$\lim_{N\to\infty} \mathcal{C}_{2n}^{(\beta)}(t) =: \mathcal{C}_{2n}^{(\beta)}(\infty). \tag{3.4.4}$$

To prove the theorem, we first present a formula which allows to compute the cumulant directly (without the need of computing the moments) using Itô calculus (see Proposition 3.4.2). This is done in Section 3.4.1. Then we make explicit computation to obtain a general form of the cumulant in Section 3.4.2.

3.4.1 A general formula for martingale cumulants

We consider $(M_t)_{t\geqslant 0}$ a continuous martingale with respect to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ which starts from $\mathcal{F}_0 = \{0, \Omega\}$ (this last assumption ensures that M_0 is almost surely constant). We define inductively a sequence of processes $A_t^{(i)}$ and $(M_s^{(i,t)})_{s\in[0,t]}$ as follows. First set

$$A_t^{(1)} = M_t \quad \text{and} \quad (M_s^{(1,t)})_{s\in[0,t]} := \mathbb{E}[A_t^{(1)} \,|\mathcal{F}_s] - \mathbb{E}[A_t^{(1)}] = M_s - M_0. \tag{3.4.5}$$

Then for $i \geqslant 2$ we define $A^{(i)}$ in terms of the quadratic variations of previous order martingales provided that they are well defined

$$A_t^{(i)} := \frac{1}{2}\sum_{j=1}^{i-1} \langle M^{(j,t)}, M^{(i-j,t)} \rangle_t \quad \text{and} \quad M_s^{(i,t)} := \mathbb{E}[A_t^{(i)} \,|\mathcal{F}_s] - \mathbb{E}[A_t^{(i)}]. \tag{3.4.6}$$

While the result might hold with greater generality, we assume for simplicity that all the quantities above are well defined and the quadratic variations above are essentially bounded in the sense that for every i and t

$$\|\langle M^{(i,t)}\rangle_t\|_\infty < \infty \quad \text{where} \quad \|Z\|_\infty := \inf\{u\geqslant 0 : \mathbb{P}[|Z| > u] = 0\}. \tag{3.4.7}$$

Proposition 3.4.2. *Under the assumption (3.4.7), the i-th cumulant of M_t is given by*

$$\mathcal{C}_i(M_t) := i!\mathbb{E}[A_t^{(i)}]. \tag{3.4.8}$$

Proof. Given $j\geqslant 1$, let us consider the martingale $(N_s^{(j,\alpha,t)})_{s\in[0,t]}$ defined by

$$N_s^{(i,\alpha,t)} := \sum_{i=1}^{j} \alpha^i M_s^{(i,t)}. \tag{3.4.9}$$

We are going to prove that the log-Laplace transform of M_t can be rewritten in the following form

$$\log \mathbb{E}\left[e^{\alpha M_t}\right] = \sum_{i=1}^{j} \alpha^i \mathbb{E}[A_t^{(i)}] + \log \mathbb{E}\left[e^{N_t^{(j,\alpha,t)} - \frac{1}{2}\langle N^{(j,\alpha,t)}\rangle_t} e^{Q_t^{(j,\alpha)}}\right]. \tag{3.4.10}$$

where

$$Q_t^{(j,\alpha)} := \frac{1}{2} \sum_{i=j+1}^{2j} \alpha^i \sum_{k=i-j}^{j} \langle M^{(k,t)}, M^{(i-k,t)}\rangle_t. \tag{3.4.11}$$

To conclude from (3.4.10), from the characterization (3.4.1) of the cumulants, we only need to prove that the last term is of order $O(\alpha^{j+1})$. Interpreting $e^{N_t^{(j,\alpha,t)} - \frac{1}{2}\langle N^{(j,\alpha,t)}\rangle_t}$ as a probability density (which we can do according to Proposition 3.2.2), we have (using our assumption (3.4.7))

$$\left|\log \mathbb{E}\left[e^{N_t^{(j,\alpha,t)} - \frac{1}{2}\langle N^{(j,\alpha,t)}\rangle_t} e^{Q_t^{(j,\alpha)}}\right]\right| \leqslant \|Q_t^{(j,\alpha)}\|_\infty \leqslant j^2 \alpha^{j+1} \max_{i\leqslant j} \|\langle M^{(i,t)}\rangle_t\|_\infty. \tag{3.4.12}$$

To prove that (3.4.10) holds we observe that for $j\geqslant 1$ using the definition of $N^{(j,\alpha,t)}$ we have

$$\begin{aligned}
\langle N^{(j,\alpha,t)}\rangle_t &= \sum_{i=2}^{2j} \alpha^i \sum_{k=1}^{i-1} 1_{\{\max(k,i-k)\leqslant j\}} \langle M^{(k,t)}, M^{(i-k,t)}\rangle_t \\
&= 2\sum_{i=2}^{j} \alpha^i A_t^{(i)} + \sum_{i=j+1}^{2j} \alpha^i \sum_{k=i-j}^{j} \langle M^{(k,t)}, M^{(i-k,t)}\rangle_t \qquad (3.4.13) \\
&= 2\left[(N_t^{(j,\alpha,t)} - \alpha M_t) + \sum_{i=1}^{j} \alpha^i \mathbb{E}[A_t^{(i)}] + Q_t^{(j,\alpha)}\right].
\end{aligned}$$

Hence we have

$$\mathbb{E}[e^{\alpha M_t}] = \mathbb{E}\left[\exp\left(N_t^{(j,\alpha,t)} - \frac{1}{2}\langle N^{(j,\alpha,t)}\rangle_t + \sum_{i=1}^{j} \alpha^i \mathbb{E}[A_t^{(i)}] + Q_t^{(j,\alpha)}\right)\right], \quad (3.4.14)$$

which is the desired result. □

3.4.2 A recursive approach for the cumulants of complex cascade

Using Proposition 3.4.2 we can now try to compute all the cumulant in a recursive manner. However the recursion (3.4.6) produces a large number of integral terms after only a few steps. Our task is thus to find the right manner to group and estimate these terms. Due to the high level of symmetry of the problem we are able to rewrite the $A_t^{(i)}$ in the following form

$$A_t^{(i)} := \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^t F^{(i,p)}(s,t)\left(2^{-\lceil s \rceil} \sum_{|u|=s} \cos(p\beta X(u))\right) \mathrm{d}s, \quad (3.4.15)$$

where $F^{(i,p)} \equiv 0$ if i and p do not have the same parity. In order to prove convergence of cumulants, we are going to use dominated convergence. Hence we need to prove that $\lim_{t\to\infty} F^{(i,p)}(s,t)$ exists and that the function is dominated by something integrable. This is the purpose of the following result.

Proposition 3.4.3. *For every $i \geqslant 2$, $A_t^{(i)}$ can be written in the form (3.4.15). Furthermore, when $\beta^2 < 2\log 2$ the functions $F^{(i,p)}$ satisfy*

(A) There exists a constant C_i such that for all p we have

$$|F^{(i,p)}(s,t)| \leqslant C_i 2^{-(i-1)s} e^{\frac{i\beta^2 s}{2}}. \quad (3.4.16)$$

(B) There exists a function $\bar{F}^{(i,p)}$ such that for all u we have

$$\lim_{t\to\infty} \bar{F}^{(i,p)}(s,t) = \bar{F}^{(i,p)}(s). \quad (3.4.17)$$

Before proving this result, let us expose how the theorem can be deduced from it.

Proof of Theorem 3.4.1 from Proposition 3.4.3. From (3.4.15) and (3.4.8), setting as a convention $F^{(i,p)}(s,t) = 0$ if $s > t$ we have

$$i!\mathcal{C}_i(t) = \sum_{p=0}^{\lfloor i/2 \rfloor} \int_0^t F^{(i,p)}(s,t)\left(\sum_{|u|=s} \mathbb{E}[\cos(p\beta X(u))]\right) \mathrm{d}s$$

$$= \sum_{p=0}^{i} \int_0^\infty F^{(i,p)}(s,t) 2^{\lceil s \rceil} e^{-\frac{\beta^2 p^2 u}{2}} \mathrm{d}s. \quad (3.4.18)$$

The integrand in (3.4.16) is dominated by $2^{-(i-1)s} e^{\frac{\beta^2(i-p^2)s}{2}}$. When $p \geqslant 1$ and $\beta^2 < 2\log 2$ this is integrable in u. When i is odd, $F^{(i,0)}(s,t) = 0$ and thus the convergence

$$\lim_{t\to\infty} \mathcal{C}_i(t) := \frac{1}{i!} \sum_{p=1}^{\lfloor i/2 \rfloor} \int_0^\infty \bar{F}^{(i,p)}(s) 2^{\lceil s \rceil} e^{-\frac{\beta^2 p^2 s}{2}} \, \mathrm{d}u$$

is is the consequence of the dominated convergence theorem applied to each term of the sum.

When $\beta^2 < \left(2 - \frac{1}{n}\right) \log 2$ also by dominated convergence that the term with $p = 0$ converges when $i \leqslant 2n$

$$\lim_{t\to\infty} \int_0^\infty F^{(i,0)}(s,t) 2^{\lceil s \rceil} \, \mathrm{d}s = \int_0^\infty \bar{F}^{(i,0)}(s) 2^{\lceil s \rceil} \, \mathrm{d}u.$$

This implies the convergence of $\mathcal{C}_{2i}(t)$ for $i \leqslant n$.

□

Proof of Proposition 3.4.3. Let us start by observing that (3.3.11) (recall that $A_t^{(2)} := \langle M \rangle_t$) already gives us the expression for $F^{(2,p)}(s,t)$. We have

$$F^{(2,0)}(s,t) = -F^{(2,2)}(s,t) = \frac{\beta^2}{2} 2^{-\lceil s \rceil} e^{\beta^2 s},$$

and these functions (which to not depend on t but this is specific to the case $i = 2$) satisfy the desired assumption.

Now for higher order we prove the statement by induction. We hence assume that (3.4.15), (3.4.16) and (3.4.17) are valid for $i-1$ and prove the same statement for $i \geqslant 3$. Recall that for $s \leqslant r$ we have (recall (3.1.3))

$$\mathbb{E}\left[2^{-\lceil r \rceil} \sum_{|u|=r} \cos(p\beta X(u)) \mid \mathcal{F}_s \right] = e^{-\frac{\beta^2 p^2 (r-s)}{2}} 2^{-\lceil s \rceil} \sum_{|u|=s} \cos(p\beta X(u)). \quad (3.4.19)$$

Hence we have for $j \in [\![2, i-1]\!]$

$$M_s^{(j,t)} = \int_0^t \sum_{p=1}^{j} F^{(j,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}} \left(e^{\frac{\beta^2 p^2 (s\wedge r)}{2}} 2^{-\lceil s\wedge r \rceil} \sum_{|u|=s\wedge r} \cos(p\beta X(u)) \right) \mathrm{d}r \quad (3.4.20)$$

and thus computing the infinitesimal variation of $\left(e^{\frac{\beta^2 p^2 s}{2}} \sum_{|u|=s} \cos(p\beta X(u)) \right)$ like we did in (3.2.5) we obtain

$$\mathrm{d}M_s^{(j,t)} = -\beta \sum_{p=1}^{j} \left(\int_s^t pF^{(j,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}} \, \mathrm{d}r \right) \left(e^{\frac{\beta^2 p^2 s}{2}} 2^{-\lceil s \rceil} \sum_{|u|=\lceil s \rceil} \sin(p\beta X(u_s)) \, \mathrm{d}X(u_s) \right) \quad (3.4.21)$$

We can then compute the martingale brackets $\langle M^{(j,t)}, M^{(i-j,t)}\rangle_t$. Let us first consider $\langle M^{(1,t)}, M^{(i-1,t)}\rangle_t$ which is special, recalling that when $|u| = |v| = \lceil s\rceil$, $\mathrm{d}\langle X(u_s), X(v_s)\rangle = 1_{\{u=v\}}\,\mathrm{d}s$, we have from (3.2.5) and (3.4.21)

$$\langle M^{(1,t)}, M^{(i-1,t)}\rangle_t := \beta^2 \int_0^t \sum_{p=1}^{i-1} \left(\int_s^t pF^{(i-1,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}}\,\mathrm{d}r \right) \times 2^{-2\lceil s\rceil} e^{\frac{\beta^2(p^2+1)s}{2}} \sum_{|u|=s} \sin(p\beta X(u)) \sin(\beta X(u))\,\mathrm{d}s. \quad (3.4.22)$$

After using the formula $2\sin(p\beta X)\sin(\beta X) = \cos((p-1)\beta X) - \cos((p+1)\beta X)$ the right hand side above assume the desired form and it remains only to check that (3.4.16) and (3.4.17) hold for all terms in the sum. More precisely it is necessary to prove that for $\beta^2 \leqslant 2\log 2$ the generic function G defined by

$$G(s,t) := 2^{-\lceil s\rceil} e^{\frac{\beta^2(p^2+1)s}{2}} \int_s^t F^{(i-1,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}}\,\mathrm{d}r, \quad (3.4.23)$$

converges and satisfies $G(s,t) \leqslant C 2^{-s(i-1)} e^{\frac{\beta^2 s}{2}}$. These two statements are immediately obtained by using the induction hypothesis for $F^{(i-1,p)}(u,t)$ and dominated convergence. Similarly to (3.4.22), we obtain that for $j \geqslant 2$ (we assume also without loss of generality that $j \leqslant i/2$)

$$\begin{aligned} &\langle M^{(j,t)}, M^{(i-j,t)}\rangle_t \\ &= \beta^2 \int_0^t \sum_{p=1}^{j} \sum_{q=1}^{i-j} \left(\int_s^t pF^{(j,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}}\,\mathrm{d}r \right) \left(\int_s^t qF^{(i-j,q)}(r,t) e^{-\frac{\beta^2 q^2 r}{2}}\,\mathrm{d}r \right) \\ &\qquad \times 2^{-2\lceil s\rceil} e^{\frac{\beta^2(p^2+q^2)s}{2}} \sum_{|u|=s} \sin(p\beta X(u)) \sin(\beta q X(u))\,\mathrm{d}s. \end{aligned} \quad (3.4.24)$$

To conclude we must prove the domination and convergence (3.4.16) and (3.4.17) for the function

$$H(s,t) := 2^{-\lceil s\rceil} e^{\frac{\beta^2(p^2+q^2)s}{2}} \left(\int_s^t pF^{(j,p)}(r,t) e^{-\frac{\beta^2 p^2 r}{2}}\,\mathrm{d}r \right) \left(\int_s^t qF^{(i-j,q)}(r,t) e^{-\frac{\beta^2 q^2 r}{2}}\,\mathrm{d}r \right) \quad (3.4.25)$$

This is done in the same manner using the induction hypothesis.

□

4 Convergence of Gaussian Multiplicative Chaos

4.1 Setup and main result

In this chapter, our object of study is the convergence in the limit when ε goes to zero, of the random quantity

$$M_\varepsilon^{(\gamma)} = \int_D e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)]}\,\mathrm{d}x \tag{4.1.1}$$

where $(X_\varepsilon)_{\varepsilon>0}$ is the convolution log correlated field X of with covariance K with a smoothing Kernel $\theta_\varepsilon := \varepsilon^{-d}\theta(\varepsilon\cdot)$ (as defined in Section 1.2). Recall that the fields $(X_\varepsilon)_{\varepsilon>0}$ are all constructed on the same probability space for all values of $\varepsilon \in (0,1)$, and from (1.2.4) the doubly indexed process $(X_\varepsilon(x))_{\varepsilon>0, x\in D}$ is a Gaussian field with covariance given by

$$\mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = K_{\varepsilon,\varepsilon'}(x,y) := \int_{\mathbb{R}^d} \theta_\varepsilon(z_1 - x)\theta_\varepsilon(z_2 - x)K(z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2. \tag{4.1.2}$$

When $\varepsilon = \varepsilon'$ we simply write K_ε and set $K_\varepsilon(x) := K_\varepsilon(x,x)$. The main result we prove in this chapter is the following.

Theorem 4.1.1. *Whenever $\gamma < \sqrt{2d}$ we have*

$$\lim_{\varepsilon \to 0} M_\varepsilon^{(\gamma)} = M_0^{(\gamma)}, \tag{4.1.3}$$

exists and the convergence holds in $\mathbb{L}_1$. Furthermore the limit M_∞ does not depend on the sequence of approximation kernel θ. When $\gamma > \sqrt{2d}$ we have

$$\lim_{\varepsilon \to 0} M_\varepsilon^{(\gamma)} = 0, \tag{4.1.4}$$

in probability

Remark 4.1.2. *Let us mention that, similarly to the case of Multiplicative Cascades, (4.1.4) is also valid in the case $\gamma = \sqrt{2d}$, but we exclude it from the proof for the sake of keeping things simple (we refer to Duplantier, Rhodes, et al. (2014) for more on the marginal case). For the sake of conciseness, we also do not include the proof of the positivity of the limit*

$$\mathbb{P}[M_0^{(\gamma)} > 0] = 1, \text{ when } \gamma < \sqrt{2d}. \tag{4.1.5}$$

Note that convergence in $\mathbb{L}_1$ implies that the limit cannot be almost surely equal to zero since we must have $\mathbb{E}[M_0^{(\gamma)}] = \lim_{\varepsilon\to 0} \mathbb{E}[M_\varepsilon^{(\gamma)}] = |D|$, where here and in the rest of the chapter we use $|\cdot|$ to denote Lebesgue measure of subsets of $\mathbb{R}^d$. Positivity follows from Kolmogorov's $\{0,1\}$-law but requires an additional construction to find a martingale approximation of $M_0^{(\gamma)}$ similar to the one obtained in the cascade case (recall Proposition 1.5.1).

The remainder of the chapter is organized as follows: In Section 4.2, we prove convergence in $\mathbb{L}_2$ in the simpler case $\gamma < \sqrt{d}$ where $M_\varepsilon^{(\gamma)}$. The proof is short and can help to understand how to proceed in the general case $\gamma < \sqrt{2d}$. Then in Section 4.3 we prove convergence to zero in the case $\gamma > \sqrt{2d}$ using size biasing techniques. Finally in Section 4.4, we prove the most delicate part of the result which is convergence in $\mathbb{L}_1$ when $\gamma < \sqrt{2d}$, and give the main guidelines to prove uniqueness of the limit.

4.2 The $\mathbb{L}_2$ case, convergence for $\gamma < \sqrt{d}$

As seen in the case of Cascades (Section 2.2), we are going to prove that when $\gamma < \sqrt{d}$, the second moment of $M_\varepsilon^{(\gamma)}$ remains bounded and this allows for a very simple proof of convergence.

Proposition 4.2.1. *If $\gamma < \sqrt{d}$, there exists $M_0^{(\gamma)}$ such that the following convergence holds in $\mathbb{L}_2$*

$$\lim_{\varepsilon \to 0} M_\varepsilon^{(\gamma)} = M_0^{(\gamma)}.$$

Proof. To prove the convergence, it is sufficient to show that the sequence $(M_\varepsilon)_{\varepsilon>0}$ is Cauchy in $\mathbb{L}_2$, that is

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbb{E}[(M_{\varepsilon_1} - M_{\varepsilon_2})^2] = 0 \tag{4.2.1}$$

As we have

$$\mathbb{E}[(M_{\varepsilon_1} - M_{\varepsilon_2})^2] = \mathbb{E}[M_{\varepsilon_1}^2] + \mathbb{E}[M_{\varepsilon_2}^2] - 2\mathbb{E}[M_{\varepsilon_1} M_{\varepsilon_2}], \tag{4.2.2}$$

it is sufficient to show that $\mathbb{E}[M_\varepsilon M_{\varepsilon'}]$ converges when ε and ε' both converge to zero because it implies that the three terms in the r.h.s. of (4.2.2), $\mathbb{E}[M_{\varepsilon_1}^2]$, $\mathbb{E}[M_{\varepsilon_2}^2]$ and $2\mathbb{E}[M_{\varepsilon_1} M_{\varepsilon_2}]$ converge to the same limit. To compute $\mathbb{E}[M_\varepsilon M_{\varepsilon'}]$, we use the expression for the Laplace transform of a Gaussian variable (see (1.1.4)) and the identity

$$\mathbb{E}[(X_\varepsilon(x) + X_{\varepsilon'}(y))^2] = K_\varepsilon(x) + K_{\varepsilon'}(y) + 2K_{\varepsilon\varepsilon'}(x,y). \tag{4.2.3}$$

We obtain

$$\begin{aligned}\mathbb{E}[M_\varepsilon M_{\varepsilon'}] = \int_{D^2} \mathbb{E}\left[e^{\gamma(X_\varepsilon(x)+X_{\varepsilon'}(y)) - \frac{\gamma^2}{2}(K_{\varepsilon'}(x)+K_{\varepsilon'}(y))} \right] \mathrm{d}x\, \mathrm{d}y \\ = \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\, \mathrm{d}x\, \mathrm{d}y,\end{aligned} \tag{4.2.4}$$

Hence to conclude we only need to prove that

$$\lim_{\varepsilon,\varepsilon'\to 0} \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\, \mathrm{d}x\, \mathrm{d}y = \int_{D^2} e^{\gamma^2 K(x,y)}\, \mathrm{d}x\, \mathrm{d}y. \tag{4.2.5}$$

The statement is in fact valid for every value of γ but the r.h.s. is finite only if $\gamma < \sqrt{d}$. This convergence is a direct consequence of the dominated convergence Theorem and of rather standard estimates for the convoluted Kernel $K_{\varepsilon,\varepsilon'}$ exposed below (Lemma 4.2.2). The following lemma goes slightly beyond what is required for the present proof, but will also be used in the remainder of the chapter.

In what follows we use the following notation for the distance between a point and a set

$$d(x,A) := \inf_{y\in A} |x-y|. \tag{4.2.6}$$

Because of our convention that K is zero outside the domain, we must sometimes make assumption on the distance to the boundary in our estimates for $K_{\varepsilon,\varepsilon'}(x,y)$.

Lemma 4.2.2. *We have for every $x, y \in D$*

$$\lim_{\varepsilon,\varepsilon'\to 0} K_{\varepsilon,\varepsilon'}(x,y) = K(x,y) \tag{4.2.7}$$

(the limit being infinite when $x = y$). Furthermore there exists a constant depending only on the kernel K and on θ which is such that for every $0 < \varepsilon' < \varepsilon < 1$ and every $x, y \in D$ we have

$$K_{\varepsilon,\varepsilon'}(x, y) \leqslant \log\left(\frac{1}{|x-y| \vee \varepsilon}\right) + C_{K,\theta}, \tag{4.2.8}$$

If furthermore we have $d(x, D^{\complement}) > \varepsilon$, $d(y, D^{\complement}) > \varepsilon'$, we also have

$$K_{\varepsilon,\varepsilon'}(x, y) \geqslant \log\left(\frac{1}{|x-y| \vee \varepsilon}\right) - C_{K,\theta} \tag{4.2.9}$$

□

Proof of Lemma 4.2.2. The convergence (4.2.7) holds for $x \neq y$ simply because by continuity, given $\delta > 0$, if ε and ε' are chosen sufficiently small, we have $|K(z_1, z_2) - K(x, y)| \leqslant \delta$ on the support of the integral (which by assumption is contained in $B(x, \varepsilon) \times B(y, \varepsilon')$ where $B(z, r)$ denotes the Euclidean ball of radius r centered at z),

$$\int_{D^2} \theta_\varepsilon(z_1 - x)\theta_{\varepsilon'}(z_2 - y)K(z_1, z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2. \tag{4.2.10}$$

A similar reasoning works on the diagonal.

Now for the second estimate, using the assumption (1.2.1) we can replace K by $\log \frac{1}{|x-y|}$ at the cost of modifying the constant C_K. Now we split the reasoning into two cases according to whether $\varepsilon \leqslant |x - y|/3$, or $\varepsilon \geqslant |x - y|/3$.

In the first case, we have $\frac{|x-y|}{3} \leqslant |z_1 - z_2| \leqslant \frac{5|x-y|}{3}$ on the full support of the integral, and thus we have

$$-\log\frac{5}{3} \leqslant \int_{D^2} \theta_\varepsilon(z_1 - x)\theta_\varepsilon(z_2 - x)\log\frac{1}{|z_1 - z_2|}\,\mathrm{d}z_1\,\mathrm{d}z_2 - \log\frac{1}{|x-y|} \leqslant \log 3. \tag{4.2.11}$$

The lower bound require x, y away from the boundary so that the support of $\theta_\varepsilon(z_1 - x)$ and $\theta_{\varepsilon'}(z_2 - y)$ are included in D.

We can now move to the second case, $\varepsilon \geqslant |x - y|/3$. In that case we are going to prove that the inequalities (4.2.8)-(4.2.9) are satisfied for the quantity

$$\int_D \theta_\varepsilon(z_1 - x)K(z_1, z_2)\,\mathrm{d}z_1.$$

for all $z_2 \in B(y, \varepsilon')$. The result is then obtained by integrating along z_2. Setting $w = \varepsilon^{-1}(z_1 - z_2)$ and observing that $\log\left(\frac{1}{|w|}\right)$ is positive only on a ball of radius one we have

$$\begin{aligned}&\int_D \log\left(\frac{1}{|z_1 - z_2|}\right)\theta_\varepsilon(z_1 - x)\,\mathrm{d}z_1 \\ &\qquad = \log\frac{1}{\varepsilon} + \int_{\varepsilon^{-1}(D - z_2)} \log\left(\frac{1}{|w|}\right)\theta(w - \varepsilon^{-1}(x + z_2))\,\mathrm{d}y\end{aligned} \tag{4.2.12}$$

For the upper bound, it is sufficient to observe that keeping only the positive part of the log we have

$$\int_{\varepsilon^{-1}(D-z_2)} \log\left(\frac{1}{|w|}\right) \theta(w-\varepsilon^{-1}(x+z_2))\, \mathrm{d}w \leqslant \|\theta\|_\infty \int_{B(0,1)} \log\left(\frac{1}{|w|}\right) \mathrm{d}w. \quad (4.2.13)$$

For the lower bound, we just need to observe that on the support of $\theta(\cdot - \varepsilon^{-1}(x+z_2))$, we have $|w| \leqslant \varepsilon^{-1}|x-y| + 2 \leqslant 5$. □

4.3 Convergence to zero in the supercritical case $\gamma > \sqrt{2d}$

In this section we are going to prove the second part of Theorem 4.1.1, that is displayed in Equation (4.1.4). To do so we are going to rely on the notion of size biased measure, which allows to transform the problem of convergence to 0 into one of convergence to $+\infty$.

Given $\varepsilon > 0$, as $\mathbb{E}[M_\varepsilon] = |D|$ (recall that $|\cdot|$ is used to denote the Lebesgue measure) we can define $\widetilde{\mathbb{P}}_\varepsilon$ as the distribution whose Radon Nikodým density with respect to $\mathbb{P}$ is given by $|D|^{-1}M_\varepsilon$,

$$\widetilde{\mathbb{P}}_\varepsilon[A] := |D|^{-1}\mathbb{E}[M_\varepsilon \mathbf{1}_A]. \quad (4.3.1)$$

The measure $\widetilde{\mathbb{P}}_\varepsilon$ is usually referred to as the size biased measure. An interesting property is that the convergence to 0 of M_ε to zero under the original measure is equivalent to the convergence to infinity under the (sequence of) size-biased measure. More precisely we have

Lemma 4.3.1. *If for any $N > 0$*

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{P}}_\varepsilon[M_\varepsilon > N] = 1 \quad (4.3.2)$$

then $\lim_{\varepsilon\to 0} M_\varepsilon = 0$ *in probability.*

Proof. Assuming that (4.3.2) holds, given $a > 0, \delta > 0$ we have

$$\mathbb{P}[M_\varepsilon > a] = \mathbb{P}[M_\varepsilon > 2\delta^{-1}] + \mathbb{P}[M_\varepsilon \in (a, 2\delta^{-1}]] \leqslant \frac{\delta}{2} + a\widetilde{\mathbb{P}}[M_\varepsilon \leqslant 2\delta^{-1}] \leqslant \delta, \quad (4.3.3)$$

where the last inequality is valid when δ is sufficiently small. □

The most convenient way to have an intuition about the measure $\mathbb{P}_\varepsilon$, is write

$$\widetilde{\mathbb{P}}_\varepsilon := \frac{1}{|D|}\int_D \mathbb{P}_{\varepsilon,x}\, \mathrm{d}x$$

where for $x \in D$ the measure $\mathbb{P}_{\varepsilon,x}$ is defined by

$$\frac{\mathrm{d}\mathbb{P}_{\varepsilon,x}}{\mathrm{d}\mathbb{P}} := e^{X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)]}. \quad (4.3.4)$$

By Cameron Martin's formula (Proposition 1.1.3), under $\mathbb{P}_{\varepsilon,x}$, X_ε is a Gaussian field with the same covariance as before but with mean given by

$$\mathbb{E}_{\varepsilon,x}[X_\varepsilon(y)] = \gamma K_\varepsilon(x,y). \tag{4.3.5}$$

We have

$$\mathbb{P}_\varepsilon[M_\varepsilon > N] := \frac{1}{|D|}\int_D \mathbb{E}_{\varepsilon,x}[M_\varepsilon > N]\,\mathrm{d}x. \tag{4.3.6}$$

Hence combining (4.3.1) with the above, the convergence in Equation (4.1.4) is a consequence of the following estimate.

Lemma 4.3.2. *There exists a function* $\delta(\varepsilon, N)$ *which satisfies* $\lim_{\varepsilon\to 0}\delta(\varepsilon,N) = 0$ *for every* $N > 0$ *such that for all* x *which satisfies* $d(x, D^{\complement}) > 2\varepsilon$,

$$\mathbb{E}_{\varepsilon,x}[M_\varepsilon > N] \geqslant 1 - \delta(\varepsilon, N). \tag{4.3.7}$$

Indeed (4.3.7) implies that

$$\int_D \mathbb{E}_{\varepsilon,x}[M_\varepsilon > N]\,\mathrm{d}x \geqslant (1-\delta(\varepsilon,N))|\{x \,:\, d(x, D^{\complement}) > 2\varepsilon\}| \tag{4.3.8}$$

which converges to $|D|$ when ε tends to zero. □

Proof of Lemma 4.3.2. According to (4.3.5), under $\mathbb{P}_{\varepsilon,x}$ the mean of the field X_ε is larger in the neighborhood of x. Hence in order to find a lower bound for M_ε we restrict the integral to $B(x,\varepsilon)$. Note that our assumption on x ensures that for all $y \in B(x,\varepsilon)$, $d(y, D^{\complement}) \geqslant \varepsilon$. We let κ_d denote the volume of the d-dimensional ball of radius one. We have, using Jensen inequality for the uniform probability on $B(x,\varepsilon)$.

$$\begin{aligned} M_\varepsilon \geqslant & \kappa_d \varepsilon^d \frac{1}{\kappa_d \varepsilon^d}\int_{B(x,\varepsilon)} e^{\gamma X_\varepsilon(y) - \frac{\gamma^2}{2}K_\varepsilon(y)}\,\mathrm{d}y \\ & \geqslant \kappa_d \varepsilon^d \exp\left(\frac{1}{\kappa_d\varepsilon^d}\int_{B(x,\varepsilon)} \gamma X_\varepsilon(y) - \frac{\gamma^2}{2}K_\varepsilon(y)\,\mathrm{d}y\right). \end{aligned} \tag{4.3.9}$$

Let us use the shorthand notation $Y(\varepsilon, x) := \frac{1}{\kappa_d \varepsilon^d}\int_{B(x,\varepsilon)}[\gamma X_\varepsilon(y) - \frac{\gamma^2}{2}K_\varepsilon(y)]\,\mathrm{d}y$. By (4.3.5) of the mean of this variable is given by

$$\mathbb{E}_{\varepsilon,x}\left[Y(\varepsilon,x)\right] = \frac{1}{\kappa_d\varepsilon^d}\int_{B(x,\varepsilon)} \gamma^2\left(K_\varepsilon(x,y) - \frac{1}{2}K_\varepsilon(y,y)\right)\mathrm{d}y \geqslant \frac{\gamma^2}{2}\log(\varepsilon^{-1}) - C, \tag{4.3.10}$$

where the last inequality is obtained applying the bounds (4.2.8) and (4.2.9) to replace K_ε by $\log(\varepsilon^{-1})$. The variance is the same that the under the original measure that is (also using (4.2.8))

$$\mathrm{Var}\left[Y(\varepsilon,x)\right] = \frac{\gamma^2}{\kappa_d^2\varepsilon^{2d}}\int_{B(x,\varepsilon)^2} K_\varepsilon(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \leqslant \gamma^2\log(\varepsilon^{-1})+C. \tag{4.3.11}$$

Hence using Chebychev inequality (which is very sub-optimal for Gaussian variables but sufficient for what we want to prove), we have in particular for ε sufficiently small

$$\mathbb{P}_{\varepsilon,x}[Y(\varepsilon,x)\geqslant(\gamma^2/2-\delta)\log(\varepsilon^{-1})]\leqslant\frac{2\gamma^2}{\delta\log(\varepsilon^{-1})}. \tag{4.3.12}$$

Thee above probability tends to 0 when ε goes to zero. When $Y(\varepsilon,x)\geqslant(\gamma^2/2-\delta)\log(\varepsilon^{-1})$, our lower bound (4.3.9) implies that $M_\varepsilon \geqslant \kappa_d\,\varepsilon^{d-\frac{\gamma^2}{2}+\delta}\geqslant N$ for ε sufficiently small.

□

4.4 The $\mathbb{L}_1$ convergence when $\gamma < \sqrt{2d}$

4.4.1 Uniform integrability via restriction

When $\gamma\in(\sqrt{d},\sqrt{2d})$ Equation (4.2.5) (which is also valid in that case by Fatou's Lemma) shows that the second moment of M_ε diverges when ε tends to 0. To prove convergence, we must find another way to prove that M_ε is uniformly integrable. Note also that contrary to the the multiplicative cascade case, there is no martingale structure so that we must go a bit beyond proving only uniform integrability.

For this we decide to excludes large values of X_ε from our integral. We do so by restricting the integral as follows. We let $A_q(x)$ be an event which we allow to depend on x and on an integer variable $q\geqslant 1$ and set

$$\widehat{M}_{\varepsilon,q} := \int_D e^{\gamma X_\varepsilon(x)-\frac{\gamma^2}{2}K_\varepsilon(x)}\mathbf{1}_{A_q(x)}\,\mathrm{d}x. \tag{4.4.1}$$

The following result is sufficient to prove (4.1.3) in Theorem 1.3.1.

Proposition 4.4.1. *There exists an increasing sequence of $A_q(x)$ of events indexed by $q\geqslant 1$ and $x\in D$ which is increasing in q ($A_p(x)\subset A_q(x)$ if $p\leqslant q$) such that*

$$\lim_{q\to\infty}\inf_{\varepsilon\in(0,1)}\mathbb{E}\left[\widehat{M}_{\varepsilon,q}\right] = |D|. \tag{4.4.2}$$

and for all $q\geqslant 1$ we have

$$\lim_{\varepsilon,\varepsilon'\to 0}\mathbb{E}[(\widehat{M}_{\varepsilon,q}-\widehat{M}_{\varepsilon',q})^2]=0. \tag{4.4.3}$$

The fact that the limit does not depend on the choice of the smoothing kernel is somehow a consequence of the proof (we provide more detail in Section 4.5).

Proof of (4.1.3) *from Proposition 4.4.1.* Now Equation (4.4.3) implies that the sequence $\widehat{M}_{\varepsilon,q}$ is Cauchy in $\mathbb{L}_2$ and thus converges in $\mathbb{L}_2$ (and thus also in $\mathbb{L}_1$) to a limit $\widehat{M}_{0,q}$. Now because $A_q(x)$ is monotonous in x, the sequence $(\widehat{M}_{0,q})_{q\geqslant 1}$ is increasing in q and thus converges to a limit M_0. As a consequence of the definition we have

$$M_0 \leqslant \liminf_{\varepsilon\to 0} M_\varepsilon \tag{4.4.4}$$

Using monotone convergence (first equality), convergence in $\mathbb{L}_1$ (second equality) and (4.4.2) we obtain that

$$\mathbb{E}[M_0] = \lim_{q\to\infty} \mathbb{E}[\widehat{M}_{0,q}] = \lim_{q\to\infty} \lim_{\varepsilon\to 0} \mathbb{E}[\widehat{M}_{\varepsilon,q}] = |D|. \tag{4.4.5}$$

As M_ε and M_0 have the same expectation we have

$$\mathbb{E}[|M_\varepsilon - M_0|] = 2\mathbb{E}[(M_0 - M_\varepsilon)_+]$$

and we can conclude by observing that by dominated convergence

$$\lim_{\varepsilon\to 0} \mathbb{E}[(M_0 - M_\varepsilon)_+] = 0. \tag{4.4.6}$$

□

4.4.2 Introduction of thick points and proof of (4.4.2)

We introduce now a family of event $A_q(x)$ which satisfy the desired property. Using the definition of $\mathbb{P}_{x,\varepsilon}$ in (4.3.4) we notice that

$$\mathbb{E}[\widehat{M}_{\varepsilon,q}] = \int_D \mathbb{P}_{x,\varepsilon}[A_q(x)]\, \mathrm{d}x.$$

Hence the property (4.4.2) states that for most $x \in D$, $A_q(x)$ has probability close to one under $\mathbb{P}_{x,\varepsilon}$.

In Section 4.3, we have seen that under $\mathbb{P}_{x,\varepsilon}$, $X_\varepsilon(x)$ has an average given by $\gamma K_\varepsilon(x,x)$ which is approximately $\gamma|\log\varepsilon|$. This indicates that the expectation of M_ε is typically supported by points for which $X_\varepsilon(x)$ is of order $(\gamma + o(1))|\log\varepsilon|$. This points have been referred to as γ-thick points in the literature Hu, Miller, and Peres (2010). For the second moment property (4.4.3) to be satisfied, we want to exclude realizations of X for which $X_\varepsilon(x)$ takes too large values. With those two things in mind, we choose α satisfying

$$\alpha \in (\gamma, 2\gamma) \quad \text{and} \quad \gamma^2 - \frac{(2\gamma-\alpha)^2}{2} < d. \tag{4.4.7}$$

The existence of such an α is guaranteed by the fact that $\gamma^2 < 2d$. Given $k \geqslant 1$, with a small abuse of notation we use X_k for X_ε with $\varepsilon = e^{-k}$, and define

$$A_q(x) := \{\forall k \geqslant 1, \quad X_k(x) \leqslant \alpha k + q\} \cap \{d(x, D^{\complement}) \geqslant 1/q\}. \tag{4.4.8}$$

Note that points near $D^{\complement}$ are excluded in order to avoid boundary effects. The following estimate readily implies that $A_q(x)$ satisfies (4.4.2).

Lemma 4.4.2. *There exists a constant $c_1 > 0$ such that whenever q is sufficiently large, for all x such that $d(x, D^{\complement}) \geqslant 1/q$ and $\varepsilon \in (0,1)$*

$$\widetilde{\mathbb{P}}_{\varepsilon,x}[(A_q(x))^{\complement}] \leqslant 2e^{-c_1 q}.$$

Indeed we have for every $\varepsilon \in (0,1)$

$$\mathbb{E}\left[\widehat{M}_{\varepsilon,q}\right] = \int_D \widetilde{\mathbb{P}}_{\varepsilon,x}[(A_q(x))]\,\mathrm{d}x \geqslant (1 - 2e^{-c_1 q})|\{x \ : \ d(x, D^{\complement}) \geqslant 1/q\}| \tag{4.4.9}$$

and the right hand side tends to $|D|$ when q tends to infinity.

Proof. We have

$$\mathbb{P}_{\varepsilon,x}[(A_q(x))^{\complement}] \leqslant \sum_{k \geqslant 1} \mathbb{P}_{\varepsilon,x}\left[X_k(x) \leqslant \alpha k + q\right] \tag{4.4.10}$$

From Cameron Martin Formula (Proposition 1.1.3), under $\mathbb{P}_{\varepsilon,x}$, $X_k(x)$ is a Gaussian random variable with mean and variance given by

$$\mathbb{P}_{\varepsilon,x}[X_k(x)] = \gamma K_{\varepsilon, e^{-k}}(x,x) \quad \text{ and } \quad \mathrm{Var}_{\mathbb{P}_{\varepsilon,x}}(X_k(x)) = K_{e^{-k}}(x,x). \tag{4.4.11}$$

Using Lemma 4.2.2, we have for some universal constant C

$$\mathbb{E}_{\varepsilon,x}[X_k(x)] \leqslant \gamma(k \wedge \log(\varepsilon^{-1})) + C \quad \text{ and } \quad \mathrm{Var}_{\mathbb{P}_{\varepsilon,x}}(X_k(x)) \leqslant k + C. \tag{4.4.12}$$

Hence considering q sufficiently large, using Gaussian tail estimates (1.1.6), we obtain that for any $\varepsilon \in (0,1)$ we have as long as $q \geqslant 2C$,

$$\begin{aligned}\mathbb{P}_{\varepsilon,x}\left[X_k(x) \leqslant \alpha k + q\right] &\leqslant e^{-\frac{((\alpha-\gamma)k+q-C)^2}{2(k+C)}} \\ &\leqslant e^{-\frac{(\alpha-\gamma)^2 k}{2(k+C)}} e^{-\frac{kq}{2(k+C)}} \leqslant e^{-\frac{(\alpha-\gamma)^2 k}{2(k+C)}} e^{-\frac{q}{2(1+C)}}\end{aligned} \tag{4.4.13}$$

We conclude by observing that the first term (which does not depend on q) is summable that the result is valid e.g. for $c_1 = \frac{1}{3(1+C)}$. ∎

4.4.3 Proof of the convergence of $\widehat{M}_{\varepsilon,q}$

Similarly to the observation made below (4.2.2) for the $\mathbb{L}_2$ case, it is sufficient for us to prove that for any finite q, $\lim_{\varepsilon,\varepsilon'\to 0}\mathbb{E}[\widehat{M}_{\varepsilon,q}\widehat{M}_{\varepsilon',q}]$ exists and is finite. This is the content of the following result.

Lemma 4.4.3. *We have*

$$\lim_{\varepsilon,\varepsilon'}\mathbb{E}[\widehat{M}_{\varepsilon,q}\widehat{M}_{\varepsilon',q}] = \int_D e^{\gamma^2 K(x,y)}\mathbb{P}[\mathcal{A}_q(x,y)]\,\mathrm{d}x\,\mathrm{d}y < \infty \tag{4.4.14}$$

where

$$\begin{aligned}\mathcal{A}_q(x,y) := \{\forall k\geqslant 1,\ X_k(x)\leqslant \alpha k + q - h(x,y,k)\,\}\\ \cap\{\forall k\geqslant 1,\ X_k(y)\leqslant \alpha k + q - h(y,x,k)\}\cap\{d(x,D^{\complement})\vee d(y,D^{\complement}\geqslant 1/q\}.\end{aligned} \tag{4.4.15}$$

with

$$h(k,x,y) = \int_{D^2}[K(x,z)+K(y,z)]\theta_{e^{-k}}(z-x)\,\mathrm{d}z, \tag{4.4.16}$$

Proof. We assume without loss of generality that $\varepsilon' < \varepsilon < 1/q$. We have

$$\begin{aligned}\mathbb{E}[\widehat{M}_{\varepsilon,q}\widehat{M}_{\varepsilon',q}] &= \int_{D^2}\mathbb{E}\left[e^{\gamma[X_\varepsilon(x)+X_{\varepsilon'}(y)]-\frac{\gamma^2}{2}(K_\varepsilon(x)+K_\varepsilon(y))}1_{A_q(x)\cap A_y}\right]\\ &= \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}_{\varepsilon,\varepsilon',x,y}\left[A_q(x)\cap A_q(y)\right].\end{aligned} \tag{4.4.17}$$

where $\mathbb{P}_{\varepsilon,\varepsilon',x,y}$ is the probability whose density w.r.t. to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{P}_{\varepsilon,\varepsilon',x,y}}{\mathrm{d}\mathbb{P}} = e^{\gamma[X_\varepsilon(x)+X_{\varepsilon'}(y)]-\frac{\gamma^2}{2}(K_\varepsilon(x)+K_\varepsilon(y)+2K_{\varepsilon,\varepsilon'})}. \tag{4.4.18}$$

By Cameron-Martin Formula (Proposition 1.1.3), under $\mathbb{P}_{\varepsilon,\varepsilon',x,y}$, X is a Gaussian field with the same covariance but with changed mean. We have in particular

$$\mathbb{E}_{\varepsilon,\varepsilon',x,y}[X_k(z)] = \int_{D^2}K(w_1,w_2)(\theta_\varepsilon(w_1-x)+\theta_{\varepsilon'}(w_1-y))\theta_{e^{-k}}(w_2-z)\,\mathrm{d}w_1\,\mathrm{d}w_2. \tag{4.4.19}$$

We set $h_{\varepsilon,\varepsilon'}(k,x,y) := \mathbb{E}_{\varepsilon,\varepsilon',x,y}[X_k(x)]$ (note that by symmetry $h_{\varepsilon',\varepsilon}(k,y,x) := \mathbb{E}_{\varepsilon,\varepsilon',x,y}[X_k(y)]$. Then subtracting the mean of X to retrieve a centered field we obtain that

$$\mathbb{E}[\widehat{M}_{\varepsilon,q}\widehat{M}_{\varepsilon',q}] = \int_{D^2} e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]\,\mathrm{d}x\,\mathrm{d}y \tag{4.4.20}$$

where

$$\begin{aligned}\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y) := \{\forall k\geqslant 1,\ X_k(x)\leqslant \alpha k + q - h_{\varepsilon,\varepsilon'}(x,y,k)\,\}\\ \cap\{\forall k\geqslant 1,\ X_k(y)\leqslant \alpha k + q - h_{\varepsilon',\varepsilon}(y,x,k)\}\cap\{d(x,D^{\complement})\vee d(y,D^{\complement}\geqslant 1/q\}.\end{aligned} \tag{4.4.21}$$

Hence to conclude we only need to show that $e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]$ is dominated by some integrable function and converges to $e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}[\mathcal{A}_q(x,y)]$.

For this, we mostly require some uniform estimates and convergence results for $h_{\varepsilon,\varepsilon'}(k,x,y)$. All of these are direct consequences of Lemma 4.2.2, details are postponed to the end of the proof.

Lemma 4.4.4. *We have for any $x, y \in D$*

$$\lim_{\varepsilon,\varepsilon'\to 0} h_{\varepsilon',\varepsilon}(x,y,k) = h(x,y,k) \tag{4.4.22}$$

Furthermore,there exists a constant such that for all $x, y, \varepsilon, \varepsilon'$ satisfying $d(x, D^{\complement})\geqslant\varepsilon$ $d(y, D^{\complement}\geqslant\varepsilon'$ we have

$$\left|h_{\varepsilon',\varepsilon}(k,x,y) - \gamma(k\wedge\log\frac{1}{\varepsilon}) - \gamma(k\wedge\log\frac{1}{|x-y|\vee\varepsilon'})\right|\leqslant C. \tag{4.4.23}$$

Using (4.4.23) for $k = k(x,y) := \lceil \log\frac{1}{|x-y|\vee\varepsilon}\rceil$, we obtain (provided that q is sufficiently large)

$$\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]\leqslant\mathbb{P}[X_k(x)\leqslant(\alpha-2\gamma)k+2q]\leqslant Ce^{-\frac{(2\gamma-\alpha)^2k}{2}}, \tag{4.4.24}$$

where the last estimates is a consequence of the fact that $X_k(x)$ has an approximately variance k (cf. Lemma 4.2.2). Using Lemma 4.2.2 again to estimate $K_{\varepsilon,\varepsilon'}(x,y)$ and replacing k by its value we obtain

$$e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]\leqslant C(|x-y|\vee\varepsilon)^{-\left[\gamma^2-\frac{(\alpha-2\gamma)^2}{2}\right]}\leqslant C|x-y|^{-\left[\gamma^2-\frac{(\alpha-2\gamma)^2}{2}\right]}. \tag{4.4.25}$$

The conditions (4.4.7) implies that the r.h.s. is an integrable function. Hence we simply need to prove pointwise convergence of $e^{\gamma^2 K_{\varepsilon,\varepsilon'}(x,y)}\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]$ and apply dominated convergence. Of course given $q\in\mathbb{N}$, we only need to consider $x, y\in D$, $x\neq y$ at a distance at least $1/q$ from the boundary. Given $\delta > 0$ we let k_0 (depending on x and y) be such that

$$\begin{aligned}\sup_{\varepsilon,\varepsilon'\in(0,1)} \mathbb{P}\Big[\forall k > k_0,\ X_k(x)\leqslant\alpha k + q - h_{\varepsilon,\varepsilon'}(x,y,k)&\\ \text{and } X_k(y)\leqslant\alpha k + q - h_{\varepsilon',\varepsilon}(y,x,k)\Big]\leqslant\delta/3,&\end{aligned} \tag{4.4.26}$$

and

$$\mathbb{P}\left[\forall k > k_0,\ X_k(x)\leqslant\alpha k + q - h(x,y,k);\, X_k(y)\leqslant\alpha k + q - h(y,x,k)\right]\leqslant\delta/3, \tag{4.4.27}$$

The existence of such a k_0 is immediate from Gaussian tails estimates (1.1.6) using the fact that $h_{\varepsilon,\varepsilon'}(x,y,k)$ is uniformly bounded in ε,ε'. Now using (4.4.22) we obtain that

$$\begin{aligned}&\lim_{\varepsilon,\varepsilon'\to 0}\mathbb{P}\left[\forall k\leqslant k_0,\ X_k(x)\leqslant\alpha k+q-h_{\varepsilon,\varepsilon'}(k,x,y);X_k(y)\leqslant\alpha k+q-h_{\varepsilon',\varepsilon}(k,y,x)\right]\\&=\mathbb{P}\left[\forall k\leqslant k_0,\ X_k(x)\leqslant\alpha k+q-h(k,x,y);X_k(y)\leqslant\alpha k+q-h(k,y,x)\right].\end{aligned}\tag{4.4.28}$$

Thus combining (4.4.26) and (4.4.28) we obtain that for ε and ε' sufficiently small, we have

$$|\mathbb{P}[\mathcal{A}_q^{\varepsilon,\varepsilon'}(x,y)]-\mathbb{P}[\mathcal{A}_q(x,y)]|\leqslant\delta,\tag{4.4.29}$$

which is the desired convergence result. □

Proof of Lemma 4.4.4. The converge towards $h(k,x,y)$ boils down to proving that given k we have for any x,y at a positive distance from the boundary of D

$$\lim_{\varepsilon\to 0}\int_{D^2}K(w_1,w_2)\theta_\varepsilon(w_1-x)\theta_{e^{-k}}(w_2-y)\,\mathrm{d}w_1\,\mathrm{d}w_2=\int_{D_2}K(x,w_2)\theta_{e^{-k}}(w-y)\,\mathrm{d}w.\tag{4.4.30}$$

Note that from the proof of Lemma 4.2.2 we have

$$\int_D K(w_1,w_2)\theta_\varepsilon(w_1-x)\,\mathrm{d}w_1\leqslant C\log\frac{1}{|w_2-x|\vee\varepsilon},$$

and the l.h.s converges to $K(x,w_2)$, so that (4.4.30) holds by dominated convergence. The estimate (4.4.23) is a direct consequence of (4.2.8)-(4.2.9). □

4.5 Uniqueness of the limit

Little is to be added to the proof to ensure that the limit does not depend of the convolution Kernel. Let us consider $\widetilde{X}_\varepsilon$ the field obtained by using another smoothing kernel $\widetilde{\theta}_\varepsilon$ ($\widetilde{K}_\varepsilon$ denotes the covariance of the field), and set $\widetilde{M}_{\varepsilon,q}$ be defined as

$$\widetilde{M}_{\varepsilon,q}=\int_D e^{\gamma\widetilde{X}_\varepsilon-\widetilde{K}_\varepsilon(x,x)}1_{A_q(x)}\,\mathrm{d}x,\tag{4.5.1}$$

where $A_q(x)$ is defined in (4.4.8). This is important here to stress that A_q is defined in terms of X_k, and not $\widetilde{X}_k$. Now, repeating the proof of the previous section (line to line, all estimates remain valid), one can prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\widetilde{M}_{\varepsilon,q}-\widehat{M}_{\varepsilon,q}|^2\right]=0.\tag{4.5.2}$$

This implies that $\widetilde{M}_{\varepsilon,q}$ also converges to $\widehat{M}_{0,q}$ for every value of q, from which we can deduce, as in Section 4.4.1 that

$$\lim_{\varepsilon\to 0}\int_D e^{\gamma\widetilde{X}_\varepsilon-\widetilde{K}_\varepsilon(x,x)}1_{A_q(x)}\,\mathrm{d}x=M_0,$$

and thus that the limit is the same for kernels θ and $\widetilde{\theta}$.

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