Scale Calculus and M-Polyfolds

An Introduction

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32º Colóquio Brasileiro de Matemática

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ISBN 978-85-244-0486-3
MSC (2010) Primary: 57P99, Secondary: 57P05

Coordenação Geral Nancy Garcia

Produção Books in Bytes **Capa** Sergio Vaz

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

Telefones: (21) 2529-5005
2529-5276
www.impa.br
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Dies ist das Geheimnis der Liebe,
daß sie solche verbinde,
deren jedes für sich sein könnte
und doch nichts ist und sein kann ohne das andere

Friedrich Wilhelm Joseph von Schelling
1775–1854

Preface

This text originates from lecture notes written during the graduate course "MM805 Tópicos de Análise I" held from March through June 2018 at UNICAMP. The manuscript has then been slightly modified in order to serve as accompanying text for an advanced mini-course during the 32nd *Colóquio Brasileiro de Matemática*, CBM-32, IMPA, Rio de Janeiro, in July 2019.

Scope

Our aim is to give an introduction to the new calculus, called *scale calculus*, and the generalized manifolds, called *M-polyfolds*, that were introduced by Hofer, Wysocki, and Zehnder (2007, 2009a,b, 2010) in their construction of a generalized differential geometry in infinite dimensions, called *polyfold theory*. In this respect we recall and survey in the appendix the incarnations of the usual (Fréchet) calculus in various contexts - from topological vector spaces (TVS) to complete normed vector spaces, that is Banach spaces.

Recently the construction of abstract polyfold theory has been concluded and made available in the form of a book by Hofer, Wysocki, and Zehnder (2017). The door is now open, not only to reformulate and reprove past moduli space problems using the new language and tools, but to approach open or new problems.

Content

There are two parts plus an appendix. In part one we introduce scale calculus, starting with the linear theory (scale Banach spaces, scale linear maps, in particular, scale Fredholm operators – these are related to scale shifts), then we define scale continuity and scale differentiability. The latter is compared to usual (Fréchet) differentiability, then the chain rule is established for scale calculus. Part one concludes with boundary and, more surprisingly, corner recognition in scale calculus and with the construction of scale manifolds.

Part two is concerned with the construction of M-polyfolds in analogy to manifolds, just locally modeled not only on Banach space (Banach manifolds), neither only on scale Banach space (scale manifolds), but on a generalization of retracts called scale retracts. This choice of local model spaces is motivated by Cartan's last theorem which we therefore review first. Part two concludes with the construction of the scale version of vector bundles, called strong bundles over M-polyfolds, whose local models are strong trivial-bundle retracts. To accommodate Fredholm sections one introduces a double scale structure from which one then extracts two individual scales.

The appendix recalls and reviews relevant background and results in topology and analysis, particularly standard calculus.

Audience

The intended audience are graduate students. The necessary background is a basic knowledge of functional analysis including the definition of Sobolev spaces such as $W^{k,p}(\mathbb{R}, \mathbb{R}^n)$.

Acknowledgements

It is a pleasure to thank Brazilian tax payers for the excellent research and teaching opportunities at UNICAMP and for generous financial support through Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), process nº 2017/19725-6. I am indebted to the selection committee of CBM-32 for giving me once more the opportunity to teach an advanced mini-course at these bi-annual meetings.

I'd like to thank Daniel Ferreira Machado, Darwin Gregorio Villar Salinas, and José Lucas Pereira Luiz for their interest and a very pleasant atmosphere in the graduate course "MM805 Tópicos de Análise I" held in the first semester of 2018 at UNICAMP.

Last not least, I am very grateful to Kai Cieliebak for handing me out his fine Lecture Notes Cieliebak (2018) while they were still in progress and to Helmut Hofer for a useful comment.

Campinas,
June 2019

Joa Weber

Contents

Introduction

The central problem in areas of global analysis such as Morse, Floer, or Gromov–Witten theory is to study spaces of solutions to nonlinear ordinary or partial differential equations $\mathcal{F}(v) = 0$. The so-called *moduli spaces*

$$\mathfrak{M} := \{\mathcal{F} = 0\}, \qquad \mathfrak{m} := \mathfrak{M}/G$$

consist in case of $\mathfrak{M}$ of *parametrized solutions* $v\colon \Sigma \to S$ taking values in a manifold or – after localization – in a vector space S, often divided out by a group G that acts on $\mathfrak{M}$ by reparameterising the domain manifold Σ. The elements of $\mathfrak{m}$ are then called *unparametrized solutions*. In case of Morse and Floer homology an element τ of the group $G = (\mathbb{R}, +)$ acts on the domain $\Sigma = \mathbb{R}$ by *time-shift*

$$(\tau_* v)(t) := v(t + \tau)$$

for $t \in \mathbb{R}$. The shift map $\Psi : \mathbb{R} \times \mathrm{Map}(\mathbb{R}, S) \to \mathrm{Map}(\mathbb{R}, S)$ is defined by $(\tau, v) \mapsto \tau_* v$. The peculiar different behavior in τ and in v of this simple map, namely linearity, hence smoothness, in v, whereas differentiation with respect to t causes v to loose a derivative, eventually led to the discovery of a new notion of smoothness in infinite dimensions – *scale smoothness* due to Hofer, Wysocki, and Zehnder (2007, 2017). Scale smoothness is connected to interpolation theory Triebel (1978). It was the crucial insight of Hofer, Wysocki, and Zehnder that requiring compactness of the scale embeddings causes that scale smoothness satisfies the chain rule and therefore is suitable to patch together pieces of scale Banach spaces to obtain scale manifolds, or more generally M-polyfolds – new spaces in infinite dimensions.

From holomorphic curves to polyfold theory.

In 1985 Gromov (1985) generalized holomorphic curves from complex analysis to symplectic geometry and thereby discovered that there is a symplectic topology. Right after Gromov's seminal ideas Floer (1986, 1988b, 1989) "morsified" holomorphic curves. He used a perturbed holomorphic curve equation to construct a semi-infinite dimensional Morse homology, called *Floer homology*, which meanwhile has a huge range of applications, from Hamiltonian and contact dynamics through symplectic topology to topological field theories; cf. the survey Abbondandolo and Schlenk (2018). Floer's construction also motivated further developments like the discovery of Fukaya A_∞-categories Fukaya (1993) and Fukaya, Oh, et al. (2009) and Symplectic Field Theory Eliashberg, Givental, and Hofer (2000).

All these applications face difficult transversality and compactness issues largely caused by the fact that one does not find oneself working in a single Banach manifold, but rather in a union of such and one has to deal with each strata individually and even do analysis across neighboring ones. To deal with these problems Fukaya and Ono (1999) discovered the notion of Kuranishi structures based on finite dimensional approximation.
In contrast Hofer, Wysocki, and Zehnder stay in infinite dimension and generalize calculus. Traditionally moduli spaces were studied by cumbersome ad-hoc methods all of whose steps had to be carried out, although rather analogous, for each moduli problem from scratch, usually filling hundreds of pages. Even in one specific setup, the differential operator $\mathcal{F}$ might act on maps γ whose domains and targets vary, in general. Consequently $\mathcal{F}$ cannot be defined on some single Banach manifold $\mathcal{B}$ of maps with values in some single Banach bundle $\mathcal{E}$ over $\mathcal{B}$. Therefore the occurring singular limits, e.g. broken trajectories or bubbling off phenomena, cause difficult compactness/gluing and transversality problems for $\mathcal{F}$ when defined on many individual Banach manifolds $\mathcal{B}_\beta$ that are at most strata of a common ambient space $\mathcal{B}$. While in traditional approaches the ambient spaces $\mathcal{B}$ itself are usually inaccessible to calculus, in a series of papers Hofer, Wysocki, and Zehnder (2007, 2009a,b, 2017) construct ambient spaces in the form of generalized manifolds, called *M-polyfolds*,[1] which are accessible to a customized generalized calculus called *scale* or *sc-calculus*. Now *polyfolds* generalize M-polyfolds like orbifolds generalize manifolds.

Roughly speaking, *polyfold theory* is a mixture of a generalized differential geometry, a generalized non-linear analysis, and some category theory.

Shift map motivates scale calculus.

The discovery of scale calculus was triggered by the properties of the shift map. That map shows up already for one of the simplest non-trivial scenarios, namely, the downward gradient equation $\mathcal{F}(\gamma) := \dot\gamma + (\nabla f)\circ\gamma = 0$ for paths $\gamma\colon \mathbb{R} \to M$ and associated to a given Morse function $f\colon M \to \mathbb{R}$ on a closed Riemannian manifold. Given critical points $x \neq y$ of f, the moduli space $\mathfrak{M}_{xy}$ consists of all solutions $\gamma\colon \mathbb{R} \to M$ of $\mathcal{F}(\gamma) = 0$ which asymptotically connect x to y, i.e. $\lim_{t\to -/+\infty} \gamma(t) = x/y$. Time shift by $\tau \in \mathbb{R}$ produces again a solution

$$(\tau_*\gamma)(t) := \gamma(t+\tau).$$

[1] The "M" is a reminder that M-polyfolds are constructed in analogy to Banach manifolds, just replace the local model Banach space by some (sc-retract of a) scale Banach space. The more general polyfolds are useful in problems having local symmetries.

Having the same image in M one calls γ and $\tau_*\gamma$ equivalent and denotes the space of equivalence classes by $\mathfrak{m}_{xy} := \mathfrak{M}_{xy}/\mathbb{R}$. While the quotient of a manifold by a free and smooth action inherits a manifold structure, unfortunately, the time shift action is not smooth at all.

To illustrate non-smoothness let us simplify the scenario in that we consider the time shift action of $\mathbb{R}$ on the compact[2] domain $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ of $v \in C^k = C^k(\mathbb{S}^1, \mathbb{R})$ where $k \in \mathbb{N}_0$. The derivative of the *shift map*

$$\Psi\colon \mathbb{R} \times C^{k+1} \to C^{k+1}, \quad (\tau, v) \mapsto \tau_* v \tag{0.0.1}$$

taken at $(\tau, v) \in \mathbb{R} \times C^{k+1}$ does not respect the target space C^{k+1}. Indeed

$$d\Psi_{(\tau,v)}(T, V) = (\tau_*\dot{v})\, T + \tau_* V \in C^k, \qquad (T, V) \in \mathbb{R} \times C^{k+1}$$

takes values only in C^k, because $\dot{v} := \frac{d}{dt} v$ does. But then there is no reason to ask the second summand $\tau_* V$ to be better than C^k and for this the assumption $V \in C^k$ suffices. While $\Psi(\tau, v)$ behaves terribly in τ it is extremely tame in v, namely linear.

If one accepts different differentiability classes of domain and target spaces, the shift map has the following still respectable properties for $k \in \mathbb{N}_0$.

(a) The shift map $\Psi\colon \mathbb{R} \times C^k \to C^k$ is continuous.

(b) The shift map as a map $\Psi\colon \mathbb{R} \times C^{k+1} \to C^k$ is pointwise differentiable in the usual sense with (Fréchet) derivative $d\Psi_{(\tau,v)} \in \mathcal{L}(\mathbb{R} \times C^{k+1}, C^k)$.

(c) At $(\tau, v) \in \mathbb{R} \times C^{k+1}$ the derivative $d\Psi_{(\tau,v)}$ extends uniquely (C^{k+1} is dense in C^k) from $\mathbb{R} \times C^{k+1}$ to a continuous linear map $\mathbb{R} \times C^k \to C^k$, denoted by $D\Psi_{(\tau,v)} \in \mathcal{L}(\mathbb{R} \times C^k, C^k)$ and called the *scale derivative*.

(d) The extension $D\Psi\colon \mathbb{R} \times C^{k+1} \to \mathcal{L}_{\mathrm{c}}(\mathbb{R} \times C^k, C^k)$ is continuous in the compact-open topology,[3] equivalently, it is continuous as a map

$$D\Psi\colon (\mathbb{R} \times C^{k+1}) \times (\mathbb{R} \times C^k) \to C^k, \quad (\tau, v, T, V) \mapsto D\Psi_{(\tau,v)}(T, V).$$

Properties (a–d) suggest that instead of considering Ψ as a map between one domain and one target, both of the same regularity (the same level), one should use

[2]Compactness of the domain $\mathbb{S}^1$ is crucial that inclusion $C^{k+1} \hookrightarrow C^k$ is compact.

[3]But it is not continuous in the norm topology on $\mathcal{L}(\mathbb{R} \times C^k, C^k)$; see Remark 1.4.8.

the whole nested sequence (scale) of Banach spaces and consider Ψ as a map $(\mathbb{R} \times C^k)_{k\in\mathbb{N}_0} \to (C^k)_{k\in\mathbb{N}_0}$ between scales.

The proof of (a–d) hinges on (i) compactness of the linear operator $C^{k+1} \hookrightarrow C^k$ given by inclusion and (ii) on density of the intersection $E_\infty := \bigcap_{k=1}^{\infty} C^k$ in each of the Banach spaces (levels) $E_k := C^k$. A nested sequence of Banach spaces $E = (E_k)$ satisfying (i) and (ii) is called a *Banach scale* or an **sc**-*Banach space* and E_k is called *level* $\boldsymbol{k}$ of the scale.

Now one turns properties (a–d) into a definition calling maps between sc-Banach spaces satisfying them *continuously* **sc**-*differentiable* or of *class* $\mathbf{sc^1}$; cf. Remark 1.0.1 and Definition 1.4.6. The new class sc^1 generalizes the usual class C^1 in the following sense: Suppose that $f : E \to F$ is a map between Banach scales whose restriction to any domain level E_m actually takes values in the corresponding level F_m of the target and all the so-called **level maps** $f_m := f|_{E_m} : E_m \to F_m$ are of class C^1. Then f is of class sc^1; see Lemma 1.5.6.

Sc-manifolds are modeled on scale Banach spaces.

In complete analogy to manifolds a scale or sc-manifold is a paracompact Hausdorff space X just locally modeled on a scale Banach space E, as opposed to an ordinary Banach space, and requiring the transition maps to be sc-diffeomorphisms. In finite dimension sc-calculus specializes to standard calculus and sc-manifolds are manifolds.

M-polyfolds are modeled on sc-retracts.

Motivated by Cartan's last theorem Cartan (1986) M-polyfolds are described locally by retracts in scale Banach spaces, replacing the open sets of Banach spaces in the familiar local description of manifolds. As a consequence M-polyfolds may have locally varying dimensions; see Figure 2.1. Enlarging the class of smooth maps one risks loosing vital analysis tools such as the implicit function theorem – which indeed is not available for sc-smooth maps; see Filippenko, Zhou, and Wehrheim (2018). However, for moduli space problems one only needs to work in the subclass of sc-Fredholm maps on which an implicit function theorem is available.

Outlook.

Given abstract polyfold theory Hofer, Wysocki, and Zehnder (2017), it is now up to the scientific community to work out and provide modules, or blackboxes, also called LEGO pieces, that uniformly cover large classes of applications, say in Morse and Floer theory. A shift map LEGO has been provided by Frauenfelder and Weber (2018).

Appendix on topology and analysis.

In the appendix we review the incarnations of the usual (Fréchet) calculus in various contexts - from topological vector spaces (TVS) to Banach spaces. For self-consistency of the text we recall many results of standard calculus in topology and analysis which are used in the main body.

Notes to the Reader.

Each of the two chapters begins with a detailed summary and survey of its contents. Read both of these two chapter summaries first to get an idea of what about is this text.

In the end the present lecture notes only grew to two chapters plus an appendix providing some background of calculus – from topology to functional analysis. In class we also treated, though briefly, scale Fredholm theory and, as an application, the shift map LEGO Frauenfelder and Weber (ibid.) for Morse and Floer path spaces. In a planned extension we shall add these topics in the form of two additional chapters.

Unless mentioned differently, we (closely) follow Hofer, Wysocki, and Zehnder (2017). Two other great sources are Fabert et al. (2016) and Cieliebak (2018).

1 Scale calculus

The ubiquitous “sc”, a priori, abbreviates *scale*, but in the context of scale linear operators and maps it stands for *scale continuous*. The latter is denoted in the context of general, possibly non-linear, maps by sc^0 or by sc^k for k times scale continuously differentiable maps. In a linear context *subspace* means linear subspace.

Section 1.1 “Scale structures” introduces the notion of a Banach scale which is a nested sequence of sets $E = E_0 \supset E_1 \supset \dots$ called levels – each one being actually a Banach space – and subject to two more axioms. A subset $A \subset E$ of the top level generates, we also say induces, naturally a new nested sequence $A^{\cap E}$ by intersecting A with each level E_m. The new levels $A_m := A \cap E_m$ form the nested sequence $A^{\cap E} = (A = A_0 \supset A_1 \supset \dots)$. Of course, not every nested sequence is of the form $A^{\cap E}$.

The three axioms for a *Banach scale* E, also called a scale Banach space or an **sc**-*Banach space*, are the following: Each level E_m is a Banach space under its own norm $|\cdot|_m$, all inclusions $E_m \hookrightarrow E_{m-1}$ are *compact* linear operators, and the intersection $E_\infty := \bigcap_m E_m$ of all levels is *dense* in every level Banach space E_m. The points of E_m are called points of regularity m and those of E_∞ smooth points. A *Banach subscale* of E is a Banach scale B whose levels are Banach subspaces of the corresponding levels of E. Is every Banach subscale B generated by its top level B_0, i.e. is $B = (B_0)^{\cap E}$? You bet. However, not every closed subspace A of a scale Banach space E generates a Banach subscale. In general, there is no reason that $A^{\cap E}$ satisfies the density axiom, consider e.g. cases of trivial intersection $A \cap E_\infty = \{0\}$. Those closed subspaces that do generate a Banach subscale are of crucial significance, they are called **sc**-*subspaces*.

Because Fredholm theory is a fundamental tool in the analysis of solution spaces of differential equations, sc-subspaces K of finite dimension will be key players, as well as sc-subspaces Y of finite codimension. *Finite* dimensional sc-subspaces K of an sc-Banach space E are characterized as follows. For finite dimensional subspaces K of E it holds:

$$K \subset E_\infty \quad \Leftrightarrow \quad K \text{ is an sc-subspace (generates a Banach subscale) of } E.$$

Although simple to prove, this equivalence is far reaching. In particular, since due to finite dimension the generated Banach subscale is constant (all levels $K_m = K$ are necessarily equal).

Section 1.2 "Examples" presents a number of examples of Banach scales, e.g. Sobolev scales and weighted Sobolev scales, that arise frequently in the study of solution spaces of differential equations on manifolds. The desire to simplify and, most importantly, to unify the many cumbersome steps of the classical treatment of analyzing solution spaces actually was the motivation to invent scale calculus; see e.g. the introductions to Hofer, Wysocki, and Zehnder (2005) and Hofer (2006).

Section 1.3 "Scale linear theory" carries over fundamental notions of linear operators to Banach scales. For example a *scale linear* operator is a linear operator $T\colon E \to F$ between sc-Banach spaces which preserves levels, that is $T(E_m) \subset F_m$ $\forall m$. For such T the restriction to level E_m takes values in F_m. The restriction as a map $T_m := T|_{E_m}\colon E_m \to F_m$ is called a *level operator*. Now one can carry over (some) standard notions and properties of linear operators, say continuity, compactness, projections, and so on, by requiring each level operator to have that property. For instance, a s̲cale c̲ontinuous operator, called **sc**-*operator*, is a scale linear operator $T\colon E \to F$ such that all level operators are continuous, that is $T_m \in \mathcal{L}(E_m, F_m)$ $\forall m$.

However, as soon as it comes to sc-Fredholm operators, not level preservation $T(E_m) \subset F_m$, but level – better regularity – *improvement* $S(E_m) \subset F_{m+1}$ $\forall m$ becomes a key property. The latter are called **sc**$^+$-*operators*. They have the property that all their level operators are compact.
Similarly, as mentioned earlier for Fredholm operators in the usual sense, finite dimensional and finite codimensional sc-subspaces will enter the definition of sc-Fredholm operators. Thus one needs the following two notions:

Firstly, the notion of **sc**-*splitting* of $E = F \oplus G$ into an sc-direct sum of sc-subspaces F and G called **sc**-*complements* of one another. Just as for Banach spaces any finite dimensional sc-subspace admits an sc-complement.

Secondly, the notion of **sc**-*quotient* E/A. This allows to establish for finite *co*dimensional sc-subspaces existence of an sc-complement (Proposition 1.3.20) and characterize them as follows (Lemma 1.3.21). For finite codimensional subspaces A of E it holds:

$$A \text{ closed in } E \quad \Leftrightarrow \quad A \text{ is an sc-subspace of } E.$$

It seems that so far the literature missed to spell out these two facts explicitly.

An **sc**-*Fredholm operator* is an sc-operator $T\colon E \to F$ such that there are sc-splittings $E = K \oplus X$ with levels $E_m = K \oplus X_m$ and $F = Y \oplus C$ with levels $F_m = Y_m \oplus C$ where $K = \ker T$ is the kernel and $Y = \operatorname{im} T$ is the image of T and both K and C are of finite dimension. Looks fine already? Well, there is one condition missing yet.[1] The operator T as a map $T\colon X \to Y$ must be an **sc**-*isomorphism* (a bijective sc-operator whose inverse is level preserving). This enforces *level regularity of* T in the sense that $Te \in F_m$ implies $e \in E_m$. And it assures that the levels $Y \cap F_m$ generated by the sc-subspace $Y = \operatorname{im} T$ coincide with the images $T_m(X_m)$ of the level operators.
It is then a consequence that the level operators $T_m\colon E_m \to F_m$ are all Fredholm with the same kernel K and the same Fredholm index. Vice versa, if the level operators of an sc-operator $T\colon E \to F$ are Fredholm and T is level regular in the above sense, then T is sc-Fredholm.
The classical stability result that Fredholm property and index are preserved under addition of a compact linear operator carries over this way: The sc-Fredholm property is preserved under addition of sc^+-operators.

Section 1.4 "Scale differentiability" is where the revolution happens. Free difference quotients! Away with Fréchet mainstream suppression! Hofer, Wysocki, and Zehnder (2017) just did it, at least in infinite dimensions..

Remark 1.0.1. Let $U \subset E$ and $V \subset F$ be open subsets of sc-Banach spaces. An open subset of E_m is given by $U_m := U \cap E_m$. A *scale continuous*[2] map $f: U \to V$ is called *continuously scale differentiable* or *of class* $\mathbf{sc^1}$ if

[1] While T as a map $X \to Y := \operatorname{im} T$ is an isomorphism, this is not yet guaranteed for the level operators as maps $T_m\colon X_m = X \cap E_m \to (\operatorname{im} T) \cap F_m = Y_m$. Their images $T(X_m) \subset Y_m$, a priori, are only subspaces. To get isomorphisms one needs to exclude elements of higher levels $X \setminus X_m$ getting mapped under T to level F_m, in symbols $T(X \setminus X_m) \cap F_m = \emptyset$.

[2] Also called *of class* $\mathbf{sc^0}$ which by definition means level preserving and continuity of all restrictions $f_m := f|_{U_m}\colon U_m \to V_m$, called *level maps*.

- the upmost so-called *diagonal map* (of height one), namely f as a map $f: U_1 \to V_0$ is pointwise differentiable and
- its derivative $df(x) \in \mathcal{L}(E_1, F_0)$ admits a continuous linear extension

$$\begin{tikzcd} E_0 \arrow[r, dashed, "Df(x)"] & F_0 \\ E_1 \arrow[u, hook, "I_1"] \arrow[ur, "{df(x)\in\mathcal{L}(E_1,F_0),\ x\in U_1}"'] & \end{tikzcd}$$

from the dense subset E_1 to E_0 itself, called the **sc**-*derivative of* $\boldsymbol{f}$ *at* $\boldsymbol{x} \in \boldsymbol{U_1}$ and denoted by $Df(x)$. Furthermore, it is required that

- the *tangent map* $Tf: TU \to TV$ defined by

$$Tf(x, \xi) := (f(x), Df(x)\xi)$$

is of class sc^0. Here the *tangent bundle of* $\boldsymbol{U}$ is the open subset[3]

$$TU := U^1 \oplus E^0$$

of the Banach scale $E^1 \oplus E^0$.

The third axiom, the one requiring level preservation and continuity of the level maps associated to the tangent map Tf, has a lot of consequences caused by the shift in the definition of the tangent bundle $TU := U^1 \oplus E^0$. For instance $Df(x)\colon E_0 \to F_0$ restricts at points of better regularity, say $x \in U_{m+1} \subset U_1$, to (continuous) *level operators* $D_\ell f(x) := Df(x)|_{E_\ell}\colon E_\ell \to F_\ell$ for all levels between 0 and down to level m.

In general, the scale derivative only admits level operators $D_\ell f(x)$ for all levels ℓ down to the level right above the x-level!

The sc-derivative $U_{m+1} \ni x \mapsto Df(x)$ viewed (horizontally) between equal levels $E_m \to F_m$ enjoys only *continuity with respect to the compact open topology*[4] whereas viewed as a diagonal map Df is *continuous with respect to the operator norm topology*, i.e. C^0 as a map

$$U_{m+1} \to \mathcal{L}(E_{m+1}, F_m)$$

[3]To get the shifted scale U^k forget the first k levels: Its m^{th} level is $(U^k)_m := U_{m+k}$.
[4]If $x_\nu \to x$ in U_{m+1}, then for each fixed $\zeta \in E_m$ one has $Df(x_\nu)\zeta \to Df(x)\zeta$ in F_m.

where the target carries the operator norm. But for these domains $Df = df$ pointwise, so sc^1 implies that all diagonal maps of height one $f\colon U_{m+1} \to V_m$ are of class C^1 in the usual sense and this brings us to

Section 1.5 "Differentiability – Scale vs Fréchet". Here we will see that higher scale differentiability $f \in \mathrm{sc}^k(U, V)$ implies that all $(\forall m)$ diagonal maps $f\colon U_{m+\ell} \to V_m$ of *all heights up to* k $(\forall \ell \in \{0, \dots, k\})$ are of class C^k in the usual Fréchet sense. Vice versa, for a map $f\colon U \to V$ there is the following *criterion to be of class* $\mathbf{sc}^{\boldsymbol{k}}$: For each $\ell \in \{0, \dots, k\}$ all diagonal maps $f\colon U_{m+\ell} \to V_m$ exist and are of class $C^{\ell+1}$.

Section 1.6 "Chain rule" states and proves this building block of calculus. It allows to construct scale *manifolds* by patching together local pieces of sc-Banach spaces. If $f\colon U \to V$ and $g\colon V \to W$ are both of class sc^1, then the composition $g \circ f$ is, too! The exclamation mark is due to the fact that applying an sc-derivative one looses one level (of regularity), so one might expect to loose two levels when composing two sc^1 maps. One doesn't! And this relies on the compactness requirement of the inclusion $E_k \hookrightarrow E_{k+1}$ operators in a Banach scale.

Section 1.7 "Boundary recognition" introduces the *degeneracy index* $d_C(x)$ of a point x in what is called a partial quadrant C in a Banach scale E. It takes the value 0 on *interior points* x, value 1 on *boundary points* in the usual sense, and points with $d_C(x) \geqslant 2$ are *corner points*. We state without proof invariance of $d_C(x)$ under $\mathbf{sc}^{\mathbf{1}}$*-diffeomorphisms*, that is sc^1-maps with sc^1-inverses. It is remarkable that sc-smooth diffeomorphisms recognize boundary points and corners. In contrast, homeomorphisms also recognize boundaries, but not corners.

Section 1.8 "sc-manifolds" defines an **sc**-*manifold* as a paracompact Hausdorff space X endowed with an equivalence class of sc-smooth atlases. A continuous map $f\colon X \to Y$ between sc-manifolds is called sc-smooth if so are all representatives with respect to sc-charts of X and Y. An sc-chart of X takes values in an sc-Banach space E and so, due to compatibility of sc-charts through sc-diffeomorphisms, the level structure of E is inherited by the sc-manifold X. An important class of sc-manifolds consists of loop spaces $X := W^{1,2}(\mathbb{S}^1, M)$ for finite dimensional manifolds M. These are even *strong* **sc**-*manifolds*, or $\mathbf{ssc}^{\boldsymbol{\infty}}$-*manifolds*, in the sense that already level maps are smooth, as opposed to only the diagonal maps as is required for sc^∞. Given an sc-manifold X, its *tangent bundle* is a map of the form

$$p\colon TX \to X^1$$

that projects on the level shifted sc-manifold after forgetting level 0 of X.

After this survey of Chapter 1 you could, upon first reading, skip the remainder of Chapter 1 and proceed with the introduction to Chapter 2.

1.1 Scale structures

Scales of sets

Definition 1.1.1 (Scales). A *scale of sets* or a *scale structure on a set* $\boldsymbol{A}$ is a nested sequence of subsets

$$A = A_0 \supset A_1 \supset A_2 \supset \dots$$

The subset A_m is called the *level* $\boldsymbol{m}$ *of the scale* and its elements *points of regularity* $\boldsymbol{m}$. The elements of the intersection

$$A_\infty := \bigcap_{m \in \mathbb{N}_0} A_m$$

are called the *smooth points* of the scale. Given a level A_m, the enclosing levels A_0, $\dots, A_{m-1} \supset A_m$ are called *superlevels*, the enclosed levels $A_m \supset A_{m+1}, A_{m+2}, \dots$ *sublevels*, *of* $\boldsymbol{A_m}$.

Definition 1.1.2 (Subscale). A *subscale* of a scale of sets A is a scale of sets B whose levels are subsets of the corresponding levels of A, that is

$$\begin{array}{cccccccccc}
\text{scale} & A & = & A_0 & \supset & A_1 & \supset & A_2 & \supset & \dots \\
 & & & \cup & & \cup & & \cup & & \\
\text{subscale} & B & = & B_0 & \supset & B_1 & \supset & B_2 & \supset & \dots
\end{array}$$

Definition 1.1.3 (Constant scale). The constant scale structure on a set A is the one whose levels $A_m := A$ are all given by A itself.

Definition 1.1.4 (Induced scale $B^{\cap A} \subset A$). A scale structure on a set A induces a scale structure on any subset $B \subset A$, called the *induced scale* or *subscale generated by* $\boldsymbol{B}$, denoted by $B^{\cap A}$. By definition the m^{th} level

$$(B^{\cap A})_m := B_m := B \cap A_m, \qquad m \in \mathbb{N}_0 \tag{1.1.1}$$

is the *part of* $\boldsymbol{B}$ *in level* $\boldsymbol{A_m}$. Observe that $B_\infty := \cap_{m \geq 0} B_m = B \cap A_\infty$.

Note that for an induced scale emptiness $B_\infty = \emptyset$ is possible, even if $A_\infty \neq \emptyset$.

Example 1.1.5 (Not every subscale is an induced scale)**.**

$$
\begin{array}{lccccccccc}
\text{subscale } B \text{ of } A & \{1,2\} & \supset & \{1\} & \supset & \emptyset & \supset & \emptyset & \dots \\
 & \cap & & \cap & & \cap & & & \\
\text{scale } A & \{0,1,2\} & \supset & \{1,2\} & \supset & \{2\} & \supset & \emptyset & \dots \\
 & \cup & & \cup & & \cup & & & \\
\text{induced scale } \{1,2\}^{\cap A} & \{1,2\} & = & \{1,2\} & \supset & \{2\} & \supset & \emptyset & \dots
\end{array}
$$

Definition 1.1.6 (Shifted scale A^k)**.** Forget the first k levels of a scale A and use A_k as the new level zero to obtain the *shifted scale* A^k with levels

$$(A^k)_m := A_{k+m}, \qquad m \in \mathbb{N}_0.$$

We sometimes abbreviate $A^k_m := (A^k)_m$.

Banach scales (sc-Banach spaces)

Definition 1.1.7 (Scale Banach space)**.** A *scale structure* or an *sc-structure* on a Banach space E, is a nested sequence of linear spaces

$$E = E_0 \quad \supset \quad E_1 \quad \supset \quad E_2 \quad \supset \quad \dots$$

called levels such that the following axioms are satisfied.

(`Banach levels`) Each level E_m is a Banach space (coming with a norm $|\cdot|_m := |\cdot|_{E_m}$).

(`compactness`) The inclusions $E_m \overset{I_m}{\hookrightarrow} E_{m-1}$ are compact linear operators for all m.

(`density`) The set of smooth points $E_\infty := \bigcap_{m\in\mathbb{N}_0} E_m$ is dense in each level E_m.

An *sc-Banach space*, also called a *scale Banach space* or a *Banach scale*, is a Banach space E endowed with a scale structure.

Exercise 1.1.8 (sc-direct sum)**.** The Banach space direct sum $E \oplus F$ of two sc-Banach spaces E and F is a Banach scale with respect to the natural levels

$$(E \oplus F)_m := E_m \oplus F_m, \qquad m \in \mathbb{N}_0. \tag{1.1.2}$$

Exercise 1.1.9 (Finite dimensional Banach scales are constant)**.** A finite dimensional Banach space E has the unique sc-structure $E_0 = E_1 = \dots$.

Exercise 1.1.10 (Infinite dimensional Banach scales E)**.** Note that any inclusion operator $E_{m+\ell} \hookrightarrow E_m$ is compact, hence continuous. Show that

(i) every level E_m is a dense subset of each of its superlevel Banach spaces;

(ii) no level E_m is a closed subset of any of its superlevel Banach spaces. Equivalently, every level E_m has a non-empty set complement in each of its superlevel Banach spaces, in symbols

$$E_{m-\ell} \setminus E_m \neq \emptyset$$

whenever $m \in \mathbb{N}$ and $\ell \in \{1, \dots, m\}$.

Definition 1.1.11. A Banach scale is called *reflexive* (resp. *separable*) if every level is a reflexive (resp. separable) Banach space.

Lemma 1.1.12 (Induced nested sequences)**.** *Any subset of an* sc-*Banach space E induces via level-wise intersection a scale of sets; see (1.1.1).*

(closed) *A closed subset $A \subset E$ meets any level E_m in a closed set $A_m = A \cap E_m$. If $A \subset E$ is a closed subspace, then the inclusion $i_m \colon A_m \hookrightarrow A_{m-1}$ is a compact linear operator between Banach spaces.*

(open) *If $U \subset E$ is an open subset, then $U_m = U \cap E_m$ is open in E_m and the set $U_\infty = U \cap E_\infty$ of smooth points is dense in every U_m.*

Proof. The intersection $A \cap E_m = (I_1 \circ \dots \circ I_m)^{-1}(A)$ is the pre-image under a continuous map; analogous for U. `(Compactness)`: Pick a bounded subset B of A_m. Then B is a subset of all four spaces in the diagram

$$\begin{tikzcd}
A_{m-1} \arrow[r, hook, "\iota_{m-1}"] & E_{m-1} \\
A_m \arrow[u, hook, dashed, "i_m"] \arrow[r, hook, "\iota_m"] & E_m \arrow[u, hook, "I_m"']
\end{tikzcd}$$

The closure of B in E_{m-1} is compact since $I_m \circ \iota_m$ is a compact linear operator. But A_{m-1} is a closed subspace of E_{m-1} which contains B. Thus the closure of B is contained in A_{m-1} as well. `(Density)`: Pick $p \in U_\infty = \bigcap_{k \in \mathbb{N}_0} (U \cap E_k) \subset (U_m \cap E_\infty)$. By density of E_∞ in E_m there is a sequence $E_m \ni p_\nu \to p$ in E_m. But $p \in U_m$ and $U_m \subset E_m$ is open. $\square$

Sc-subspaces I

As a closed linear subspace of a Banach space is a Banach space itself under the restricted norm, it is natural to call it a Banach subspace. In view of this the following definition seems natural in the setting of Banach scales.

Definition 1.1.13 (Banach subscale)**.** A *Banach subscale* of a Banach scale E is a Banach scale B whose levels B_m are Banach subspaces of the corresponding levels E_m of E.

On the other hand, we just saw in Lemma 1.1.12 that a closed linear subspace A in an sc-Banach space E generates a nested sequence $A^{\cap E} = (A \cap E_m \subset E_m)_{m=0}^{\infty}$ of Banach subspaces. So it is natural to ask

1) Does the intersection sequence $A \cap E_m$ always form a Banach scale?
Answer: *No*. (Even if $\dim A < \infty$; see Lemma 1.1.16.)

2) Is a Banach subscale $B \subset E$ generated by its top level B? In symbols, is every level B_m given by intersection $B \cap E_m$?
Answer: *Yes*. (See Lemma 1.1.15.)

Definition 1.1.14 (Scale subspaces)**.** An **sc**-*subspace* of an sc-Banach space E is a closed subspace A of E whose intersections with the levels of E form the levels of a Banach subscale of E.[5] Speaking of an sc-subspace A of E implicitly carries the information that A is the Banach subscale of E whose levels are given by

$$A_m := A \cap E_m .$$

Alternatively $A^{\cap E}$ denotes the Banach scale generated by an sc-subspace A.

Lemma 1.1.15. *a) The top level B_0 of a Banach subscale B of a Banach scale E is an* sc-*subspace of E. b) Every* sc-*subspace of E arises this way.*

Proof. a) By `(density)` of the set B_m in the Banach space B_0, the closure $\overline{B_m}^{0}$ with respect to the B_0 norm is the whole space B_0. Hence

$$E_m \cap B_0 = E_m \cap \overline{B_m}^{0} = E_m \cap B_m = B_m$$

where identity two, also three, holds since B_m itself is a closed subspace of the Banach space E_m by axiom `(Banach levels)`. b) By definition an sc-subspace generates a Banach subscale. □

---

[5]The axioms `(Banach levels)` and `(compactness)` are automatically satisfied for *any closed* subspace B of E; see Lemma 1.1.12. The problematic axiom is `(density)`.

Lemma 1.1.16 (Finite dimensional sc-subspaces). *Given a scale Banach space E and a finite dimensional linear subspace $B \subset E$. Then*

$$B \text{ is an sc-subspace of } E \qquad \Leftrightarrow \qquad B \subset E_\infty.$$

The sc-subspace B generates the constant Banach scale with levels $B_m = B$.

Proof. '$\Rightarrow$' The finite dimensional linear subspace $B \cap E_\infty = \bigcap_m (B \cap E_m) = B_\infty$ of B is dense by the (`density`) axiom for the subspace scale generated by B. Thus by finite dimension it is even equal to $B = B \cap E_\infty \subset E_\infty$. '$\Leftarrow$' By assumption $B \subset E_\infty \subset E_m$, thus $B_m := B \cap E_m = B$. So B generates the constant scale which by Exercise 1.1.9 is a Banach scale since $\dim B < \infty$. □

Example 1.1.17 (Closed but not sc). Let $E = L^2([0,1])$ with the, even reflexive, Banach scale structure $E_m := W^{m,2}([0,1])$. Then the characteristic function $\chi = \chi_{[0,1/2]}$ generates a 1-dimensional, thus closed, subspace A of E. Since χ lies in L^2, but not in $W^{m,2}$ for $m \geqslant 1$, the levels $A_m := A \cap E_m = \{0\}$ are trivial for $m \geqslant 1$, hence $A_\infty = \{0\}$ is not dense in $E_0 = L^2$.

Exercise 1.1.18. Infinite dimensional sc-subspaces cannot lie inside E_∞.
[Hint: Given an sc-subspace $A \subset E$, show $A \subset E_\infty \Rightarrow A_\infty = A$, so $A_1 = A$. But $A_1 \subset E_1 \hookrightarrow E$ embeds compactly in E, whereas A is closed in E.]

The finding that for finite dimensional linear subspaces "being located in the set of smooth points" is equivalent to "generating a (constant) Banach subscale" is extremely useful. For instance, this enters the proofs of

- Prop. 1.3.17: Finite dimensional sc-subspaces are sc-complemented;
- Prop. 1.3.20: Finite codimension sc-subspaces are sc-complemented;
- Lemma 1.3.21: Characterization of finite codimensional sc-subspaces.

This list shows that certain classes of scale subspaces have properties analogous to the corresponding class of Banach subspaces.
Suppose A and B are sc-subspaces of an sc-Banach space E. How about the sum $A + B$ and the intersection $A \cap B$?
Is it possible, in general, to endow the sum $A + B$ and the intersection $A \cap B$ with the structure of Banach scales? So it is natural to ask the following.

3) Is the sum $A + B$ of sc-subspaces always an sc-subspace?
Answer: *No.* The sum of two closed subspaces, even in Hilbert space, is not even closed in general;[6] cf. Schochetman, Smith, and Tsui (2001).
Answer: *Yes*, if A and B are finite dimensional.
Answer: *Yes*, if A or B is of finite codimension; see Exercise 1.3.23.

4) Is the intersection $A \cap B$ of sc-subspaces an sc-subspace?
Answer: *Yes*, if A or B is finite dimensional.
Answer: *Yes*, if A and B are of finite codimension; see Exercise 1.3.23.
(General case: In each level E_m the intersection $(A \cap B) \cap E_m = A_m \cap B_m$ is closed. How about density of $(A \cap B)_\infty$ in $A \cap B$?)

1.2 Examples

Throughout $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$ or, likewise, the quotient space $\mathbb{R}/\mathbb{Z}$. It is convenient to think of functions $f : \mathbb{S}^1 \to \mathbb{R}$ as 1-periodic functions on the real line, that is $f : \mathbb{R} \to \mathbb{R}$ such that $f(t + 1) = f(t)$ for every t.

By definition a counterexample is an example with negative sign.

Example 1.2.1 (Not a Banach scale)**.** The vector space $C^k_{\mathrm{bd}}(\mathbb{R})$ of k times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which, together with their derivatives up to order k, are bounded is a Banach space with respect to the C^k norm. However, the scale whose `(Banach levels)` are $E_m := C^m_{\mathrm{bd}}(\mathbb{R})$ satisfies `(density)` since E_∞ is equal to $C^\infty_{\mathrm{bd}}(\mathbb{R})$, but it does not satisfy `(compactness)`. A counterexample is provided by a *bump running to infinity*: Pick a bump, that is a compactly supported function $\chi \geqslant 0$ on $\mathbb{R}$, and set $\chi_\nu := \chi(\cdot - \nu)$. Then the set $C := \{\chi_\nu | \nu \in \mathbb{N}\}$ is bounded in E_1, indeed $\|\chi_\nu\|_{C^1} = \|\chi\|_{C^1} =: c_\chi < \infty$, but there is no convergent subsequence with respect to the C^0 norm, i.e. in E_0.

So non-compactness of the domain obstructs the `(compactness)` axiom. There are two ways to fix this. The obvious one is to use a compact domain; below we illustrate this by choosing the simplest one $\mathbb{S}^1$. Another way is to impose a decay condition when approaching infinity. This works well for domains which are a

[6] The Hilbert space l^2 of square summarizable real sequences contains the closed subspaces $A := \{a \in l^2 \mid a_{2n} = 0 \; \forall n\}$ and $B := \{b \in l^2 \mid b_{2n} = b_{2n-1}/2n \; \forall n\}$. The sum $A + B$ cannot be closed, because it is dense in l^2 (since it contains all sequences of compact support) and $A + B$ is not all of l^2: Write $(1/n)_n \in l^2$ in the form $a + b$ with $a \in A$ and $b \in B$. Then $1/2n = a_{2n} + b_{2n} = b_{2n} = b_{2n-1}/2n$. So $b_{2n-1} = 1$ for all n, hence $b \notin l^2$.

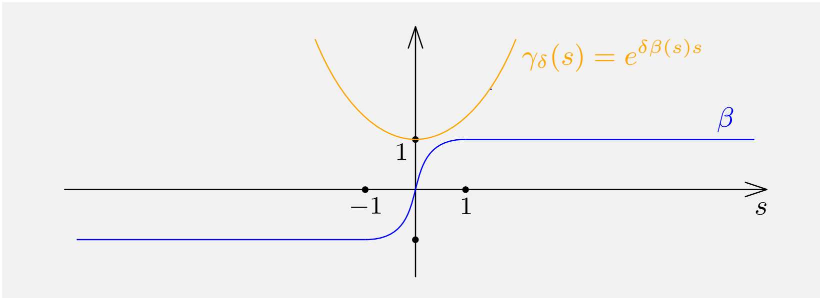

Figure 1.1: Exponential weight function γ_δ and monotone cutoff function β

product of a compact manifold with $\mathbb{R}$. Concerning targets, replacing $\mathbb{R}$ by $\mathbb{R}^n$ makes no difference in the arguments.

Exercise 1.2.2 (The non-reflexive Banach scale $C^k(\mathbb{S}^1)$)**.** Show that the Banach space $C^k(\mathbb{S}^1)$ endowed with the scale structure whose levels are the Banach spaces $E_m := C^{k+m}(\mathbb{S}^1)$ is a separable non-reflexive Banach scale.

[Hint: Concerning (`compactness`) use the Arzelà–Ascoli Theorem A.2.20. For separability see e.g. discussion in Weber (2017b, App. A).]

Example 1.2.3 (Sobolev scales – compact domain)**.** Fix an integer $k \in \mathbb{N}_0$ and a real $p \in [1, \infty)$. The Sobolev space $W^{k,p}(\mathbb{S}^1, \mathbb{R}^n)$ endowed with the scale structure whose levels are the Banach spaces $E_m := W^{k+m,p}(\mathbb{S}^1, \mathbb{R}^n)$ is a Banach scale. These Sobolev scales are separable ($1 \leqslant p < \infty$) and reflexive ($1 < p < \infty$) by Theorem A.3.1.

[Hints: Sobolev embedding theorems and $E_\infty = C^\infty(\mathbb{S}^1, \mathbb{R}^n)$.]

Exercise 1.2.4 (Weighted Sobolev scales – non-compact domain $\mathbb{R}$)**.** Fix a monotone cutoff function $\beta \in C^\infty(\mathbb{R}, [-1, 1])$ with $\beta(s) - 1$ for $s \leqslant -1$ and $\beta(s) = 1$ for $s \geqslant 1$, as illustrated by Figure 1.1. Given a constant $\delta \geqslant 0$, define an exponential weight function by

$$\gamma_\delta(s) := e^{\delta s \beta(s)}.$$

Let $k \in \mathbb{N}_0$ and pick a constant $p \in (1, \infty)$. Check that the set defined by

$$W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n) := \{f \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) \mid \gamma_\delta f \in W^{k,p}(\mathbb{R}, \mathbb{R}^n)\} \tag{1.2.3}$$

is a real vector space on which

$$\|f\|_{W^{k,p}_\delta} := \|\gamma_\delta f\|_{W^{k,p}}.$$

defines a complete norm. Consider a strictly increasing sequence

$$0 = \delta_0 < \delta_1 < \cdots \tag{1.2.4}$$

of reals. Prove that the levels defined by

$$E_m := W^{m,p}_{\delta_m}(\mathbb{R}, \mathbb{R}^n), \quad m \in \mathbb{N}_0$$

form a Banach scale structure on the Banach space $L^p(\mathbb{R}, \mathbb{R}^n)$.

Exercise 1.2.5 (Strictly increasing is necessary)**.** Show that if two weights $\delta_{m-1} = \delta_m$ are equal in (1.2.4), then the `(compactness)` axiom fails.

Exercise 1.2.6 (Reflexivity and separability)**.** Show that the weighted Sobolev space $W^{m,p}_{\delta}(\mathbb{R}, \mathbb{R}^n)$ is a closed subspace of $W^{m,p}(\mathbb{R}, \mathbb{R}^n)$. Conclude that the weighted Sobolev scales in the previous example are separable ($1 \leqslant p < \infty$) and reflexive ($1 < p < \infty$).

Example 1.2.7 (Completion scale – Hölder spaces are not Banach scales)**.** Fix a constant $\mu \in (0, 1)$. The sequence of Hölder spaces $E_m := C^{m,\mu}(\mathbb{S}^1)$ for $m \in \mathbb{N}_0$ satisfies the `(compactness)` axiom by the Arzelà–Ascoli Theorem A.2.20, but the set of smooth points $E_\infty = C^\infty(\mathbb{S}^1)$ is not dense in any level E_m. However, taking the closure of E_∞ in each level produces a Banach scale $\bar{E}_m := \overline{E_\infty}^{k,\mu}$ called the *completion scale*. This works for every nested sequence of Banach spaces that satisfy `(compactness)` as shown by Fabert et al. (2016, Lemma 4.11); they also solve Exercise 1.2.4.

Exercise 1.2.8. For which $p \in [1, \infty]$, if any, is $L^p(\mathbb{S}^1)$ endowed with the levels $E_m := L^{p+m}(\mathbb{S}^1)$ a Banach scale?

Definition 1.2.9 (Weighted Hilbert space valued Sobolev spaces)**.** Let $k \in \mathbb{N}_0$, $p \in (1, \infty)$, and $\delta \geqslant 0$. Suppose H is a separable Hilbert space and define the space $W^{k,p}_{\delta}(\mathbb{R}, H)$ by (1.2.3) with $\mathbb{R}^n$ replaced by H. This is again a Banach space; see Frauenfelder and Weber (2018, Appendix).

Example 1.2.10 (Path spaces for Floer homology)**.** A a monotone unbounded function $f : \mathbb{N} \to (0, \infty)$ is called a *growth function*. Common types of Floer homologies provide such f, order refers to spatial order:

Floer homology	Order	Mapping space	Growth type
Periodic	1st	loop space	$f(\nu) = \nu^2$
Lagrangian	1st	path space	$f(\nu) = \nu^2$
Hyperkähler	1st	$\mathrm{Map}(M^3, \mathbb{R}^{2n})$	$f(\nu) = \nu^{2/3}$
Heat flow	2nd	loop space	$f(\nu) = \nu^4$

Here Periodic and Lagrangian Floer homology refer, respectively, to the elliptic PDEs studied by Floer (1988b, 1989) on the cylinder $\mathbb{R} \times \mathbb{S}^1$ and by Floer (1988a) imposing Lagrangian boundary conditions along the strip $\mathbb{R} \times [0, 1]$. Hyperkähler and Heat flow Floer homology refer to the theories established by Hohloch, Noetzel, and Salamon (2009), respectively, by Weber (2013a,b, 2017a). The heat flow is described by a parabolic PDE that relates to Floer's elliptic PDE; see Salamon and Weber (2006).

Given a constant $p \in (1, \infty)$, let δ_m for $m \in \mathbb{N}_0$ be a sequence as in (1.2.4). Given a growth function f, let $H_m = \ell^2_{f^m}$ be the fractal Hilbert scale on $H = \ell^2$ introduced by Frauenfelder and Weber (2018, Ex. 3.8). Then the Banach space E_m is defined as intersection of $m + 1$ Banach spaces, namely

$$E_m := \bigcap_{i=0}^{m} W^{i,p}_{\delta_m}(\mathbb{R}, H_{m-i}), \quad m \in \mathbb{N}_0.$$

The norm on E_m is the maximum of the $m+1$ individual norms. This is a complete norm. This endows $E = L^p(\mathbb{R}, H)$ with the structure of a Banach scale; see Frauenfelder and Weber (ibid., Thm. 8.6).

1.3 Scale linear theory

1.3.1 Scale linear operators

Definition 1.3.1 (Scale linear operators T and their level operators T_m)**.**

(i) A *scale linear operator* is a linear operator $T\colon E \to F$ between Banach scales which is *level preserving*, that is $T(E_m) \subset F_m$ for every $m \in \mathbb{N}_0$.

(ii) The restriction of a scale linear operator $T\colon E \longrightarrow F$ to a level of E takes values in the corresponding level of F. Hence T viewed as a map between corresponding levels is a linear operator

$$T_m := T|_{E_m} \colon E_m \to F_m, \qquad m \in \mathbb{N}_0$$

between Banach spaces, called the $\boldsymbol{m}^{\mathbf{th}}$ *level operator*.

If a scale linear operator $T\colon E \to F$ is, in addition, a bijective map, then each level operator $T\colon E_m \rightarrowtail F_m$ is injective – but not necessarily surjective. It will be surjective if the inverse linear map $T^{-1}\colon F \to E$ is level preserving: In this case each level operator $(T^{-1})_m\colon F_m \rightarrowtail E_m$ is injective. This proves

Lemma 1.3.2. *Suppose a scale linear operator $T\colon E \to F$ is bijective and its inverse is level preserving. Then every level operator*

$$T_m := T|_{E_m} \colon E_m \rightarrowtail\!\!\!\twoheadrightarrow F_m$$

is a bijective linear map between Banach spaces.

Scale continuous operators

Definition 1.3.3 (sc-operators)**.** A scale linear operator $T\colon E \to F$ is called *scale continuous* or *scale bounded* or of *class* $\mathbf{sc^0}$, if each level operator $T_m \in \mathcal{L}(E_m, F_m)$ is a continuous linear operator between Banach spaces.

$$\begin{tikzcd}
E = E_0 \arrow[rr, "T_0:=T", "\text{continuous}"'] & & F_0 = F \\
\vdots \arrow[u, hook] & & \vdots \arrow[u, hook] \\
E_m \arrow[u, hook] \arrow[rr, "T_m:=T|_{E_m}", "\text{continuous}"'] & & F_m \arrow[u, hook] \\
E_{m+1} \arrow[u, hook, "I_{m+1}", "\text{compact}"'] \arrow[rr, "T_{m+1}:=T|_{E_{m+1}}", "\text{continuous}"'] & & F_{m+1} \arrow[u, hook, "\text{compact}", "J_{m+1}"'] \\
\vdots \arrow[u, hook] & & \vdots \arrow[u, hook] \\
\cap_m E_m =: E_\infty \arrow[u, "\text{dense}"'] \arrow[rr, "T_\infty:=T|_{E_\infty}"] & & F_\infty := \cap_m F_m \arrow[u, "\text{dense}"]
\end{tikzcd}$$

Such T is called an *sc-operator* between Banach scales. In the realm of scale linear operators sc does not abbreviate *scale*, but *s*cale *c*ontinuous. Let $\mathcal{L}_{\mathrm{sc}}(E, F)$ be the *set of* **sc**-*operators* between the Banach scales E and F.

Exercise 1.3.4. Check that $\mathcal{L}_{sc}(E, F)$ is a linear space.

Exercise 1.3.5. Given $\mathcal{L}_{sc}(E, F)$, consider the sequence $(\mathcal{L}_m)_{m\in\mathcal{N}_0}$ of Banach spaces $\mathcal{L}_m := \mathcal{L}(E_m, F_m)$ under the operator norm. Characterize the case in which one has inclusions $\mathcal{L}_{m+1} \subset \mathcal{L}_m$ as a) sets and b) continuous maps between Banach spaces. In b) characterize the case in which c) the set $\mathcal{L}_\infty := \cap_m \mathcal{L}_m$ is dense in each level $\mathcal{L}_m$ and d) every inclusion operator $\mathcal{L}_{m+1} \hookrightarrow \mathcal{L}_m$ is compact.

Definition 1.3.6 (sc-projections)**.** An *sc-projection* is a scale continuous operator P whose level operators P_m are all *projections*, i.e. $P_m \circ P_m = P_m$.

Lemma 1.3.7 (Image and kernel of sc-projections are sc-subspaces)**.** *The image, hence the kernel, of any* $P = P^2 \in \mathcal{L}_{sc}(E)$ *are* sc-*subspaces.*

Proof. As $Q := \mathbb{1} - P$ is an sc-projection whose image is the kernel of P, it suffices to show that the images $R_m := \operatorname{im} P_m = \operatorname{Fix} P_m$ form a Banach subscale of E. The inclusion $R_{m+1} \subset R_m$ holds by $E_{m+1} \subset E_m$. And $R_m = \operatorname{Fix} P_m$ is a closed (linear) subspace of E_m by continuity and linearity of P_m. `(Compactness)` of the inclusion $I_m \colon E_m \hookrightarrow E_{m-1}$, together with $R_m \subset E_m$ being closed, tells that each inclusion $i_m \colon R_m \hookrightarrow R_{m-1}$ takes bounded sets into pre-compact ones. It remains to check `(density)` of $R_\infty = \cap_\ell R_\ell$ in R_m. To see this pick $r_m \in R_m \subset E_m$ and, by density of E_∞ in E_m, pick some in E_m convergent sequence $E_\infty \ni e_\nu \to r_m$. Since $R_m = \operatorname{Fix} P_m$ and by continuity of P_m we get

$$R_m \ni r_m = P_m r_m = \lim_{\nu\to\infty} P_m e_\nu .$$

For each e_ν it holds that

$$P_m e_\nu = P e_\nu \in E_\ell \cap R_\ell = R_\ell \quad \forall \ell .$$

Here the first equality holds since $e_\nu \in E_\infty \subset E_m$ and P_m is the restriction of P to E_m, so $P_m e_\nu = P e_\nu$. But P preserves levels and e_ν lies in every E_ℓ, so $P e_\nu$ lies in every E_ℓ and $P e_\nu = P_\ell e_\nu \in \operatorname{im} P_\ell = R_\ell$. Thus $P e_\nu \in R_\infty$. $\square$

Definition 1.3.8 (sc-isomorphisms)**.** A (linear) *sc-isomorphism* is a bijective sc-operator whose inverse[7] is level preserving.

Exercise 1.3.9. For an sc-isomorphism $T \colon E \to F$ all level operators

$$T_m \in \mathcal{L}(E_m, F_m), \qquad (T^{-1})_m \in \mathcal{L}(F_m, E_m)$$

are continuous bijections with continuous inverses.

[Hint: Bounded inverse theorem, equivalently, open mapping theorem.]

[7] Inverses of bijective sc-operators are not automatically level preserving: Consider the identity operator from the forgetful Banach scale $E_0 \supset E_2 \supset E_3 \dots$ to $E_0 \supset E_1 \supset E_2 \dots$

Scale compact operators include sc$^+$-operators

Definition 1.3.10 (Scale compact operators)**.** An *sc-compact operator* is a scale linear operator $S\colon E \to F$ whose level operators $S_m\colon E_m \to F_m$ are all compact (hence bounded) linear operators between Banach spaces.

Scale compact operators are sc-operators, i.e. elements of $\mathcal{L}_{\mathrm{sc}}$.

Definition 1.3.11 (sc$^+$-operators)**.** Suppose E and F are scale Banach spaces. Recall that F^1 denotes the Banach scale that arises from F by forgetting the 1st level F_0 and taking F_1 as the new level 0. The elements $S \in \mathcal{L}_{\mathrm{sc}}(E, F^1)$ are called **sc$^+$**-*operators* and we use the notation

$$\mathcal{L}^+_{\mathrm{sc}}(E, F) := \mathcal{L}_{\mathrm{sc}}(E, F^1).$$

Remark 1.3.12 (sc$^+$-operators are scale compact)**.** The `(compactness)` axiom not only shows $\mathcal{L}^+_{\mathrm{sc}}(E, F) \subset \mathcal{L}_{\mathrm{sc}}(E, F)$, but also that any sc$^+$-operator $S\colon E \to F$ is sc-compact: This follows from the commutative diagram

$$\begin{tikzcd}
E_m \arrow[r, dashed, "S"] \arrow[dr, "\mathit{bounded}", "S\colon E_m\to(F^1)_m=F_{m+1}"'] & F_m \\
& F_{m+1} \arrow[u, hook, "\text{compact}"']
\end{tikzcd}$$

since the composition of a bounded and a compact linear operator is compact.

Remark 1.3.13 (Are scale compact operators always sc$^+$-operators?)**.**
No: Let E be an infinite dimensional Banach scale. The inclusion $\iota\colon E^1 \to E$ has compact level operators $\iota_m\colon E_{m+1} \to E_m$ and it is an sc$^+$-operator, indeed $\iota \in \mathcal{L}_{\mathrm{sc}}(E^1, E^1) =: \mathcal{L}^+_{\mathrm{sc}}(E^1, E)$. Now forget level one in E^1 and in E, denote the resulting Banach scales by $E^1_{\times 1}$ and $E_{\times 1}$, respectively. All level operators of the inclusion $\iota\colon E^1_{\times 1} \to E_{\times 1}$ are still compact, but ι does not even map level zero $(E^1_{\times 1})_0 = E_1$ to level one $(E_{\times 1})_1 = E_2$, let alone be continuous.

Sc-subspaces II

Direct sum and sc-complements of sc-subspaces

Definition 1.3.14. An sc-subspace F of a Banach scale E is called *sc-complemented* if there is an sc-subspace $G \subset E$ such that every Banach space direct sum of corresponding levels

$$F_m \oplus G_m = E_m$$

is equal to the ambient level E_m. Such G is called an *sc-complement* of F. So the Banach space $F \oplus G$ carries the natural Banach scale structure (1.1.2). Such a pair (F, G) or such direct sum $F \oplus G$ is called an *sc-splitting* of E.

Exercise 1.3.15. Let G be an sc-complement of $F \subset E$. Check that the Banach space $F \oplus G$ together with the natural level structure (1.1.2) indeed satisfies the axioms of a Banach scale.

Exercise 1.3.16 (Sc-projections sc-split)**.** There is an sc-splitting

$$E = \ker P \oplus \operatorname{im} P$$

associated to any sc-projection, i.e. any idempotent $P = P^2 \in \mathcal{L}_{\mathrm{sc}}(E)$.
[Hint: $P_m = (P_m)^2 \in \mathcal{L}(E_m)$ means $E_m = \ker P_m \oplus \operatorname{Fix} P_m$. Lemma 1.3.7.]

Proposition 1.3.17. *Finite dimensional* sc-*subspaces are* sc-*complemented.*

Proof. We recall the proof given in Hofer, Wysocki, and Zehnder (2017, Prop. 1.1). Suppose F is a finite dimensional sc-subspace of a Banach scale E. Then $F \subset E_\infty$ by Lemma 1.1.16 and F generates the constant Banach scale with levels $F_m := F \cap E_m = F$ by Exercise 1.1.9. Pick a basis $e_1, \dots, e_k$ of $F \subset E_\infty$ and let $e_1^*, \dots, e_k^* \in F^*$ be the dual basis. By the Hahn–Banach Theorem A.2.15 any e_i^* extends to a continuous linear functional λ_i on E. The linear operator $P\colon E \to E$ defined by $P(x) := \sum_{i=1}^k \lambda_i(x) e_i$ is continuous and satisfies $P \circ P = P$ by straightforward calculation. Note that the image of P is F, that is $P(E) = F$, and that F is contained in E_∞, hence in every level E_m. This shows that P is level preserving and admits level operators $P_m\colon E_m \to E_m$. By the continuous inclusion $E_m \hookrightarrow E$ the restrictions, still denoted by λ_i, are continuous linear functionals $\lambda_i\colon E_m \hookrightarrow E \to \mathbb{R}$ on every level E_m. By the same arguments as for P every level operator P_m is a continuous linear projection $P_m\colon E_m \to E_m$ with image $P(E_m) = F$. Hence $P \in \mathcal{L}_{\mathrm{sc}}(E)$ is an sc-projection.

Goal. Given the finite dimensional sc-subspace $F \subset E$ (generating the constant Banach scale $F_m = F$), find a closed subspace $G \subset E$ such that

a) $G_m := G \cap E_m$ are the levels of a Banach subscale (G is an sc-subspace);

b) $F_m \oplus G_m = E_m$ for every m.

Solution. The subspace of E defined by $G := (\mathbb{1} - P)E$ is closed since $G = \ker P$ and P is continuous. a) By Lemma 1.1.12 only `(density)` remains to be checked. To see that $G_\infty := \bigcap_m G_m = G \cap E_\infty$ is dense in any level G_m pick $g \in G_m \subset E_m$. By density of E_∞ in E_m choose a sequence $e_\nu \in E_\infty$ that converges in E_m to $g \in E_m$. The sequence $g_\nu := (\mathbb{1} - P_m)e_\nu$ lies in $G \cap E_m =: G_m$ and converges in G_m to g: Indeed $e_\nu - g_\nu = P_m e_\nu = P_m(e_\nu - g)$, since $g \in G_\infty \subset G_m = \ker P_m$, so together with $\|P_m\| \leqslant 1$ we get

$$|g - g_\nu|_m \leqslant |g - e_\nu|_m + |e_\nu - g_\nu|_m \leqslant 2\,|g - e_\nu|_m \to 0, \quad \text{as } \nu \to \infty.$$

b) For $x \in F \cap G = \operatorname{im} P \cap \ker P$ one has $x = Py$ for a $y \in E$, hence $0 = Px = PPy = Py = x$. So the intersection of subspaces $F_m \cap G_m = \{0\}$ is trivial, too. It remains to show the equality $\operatorname{im} P_m + \ker P_m = E_m$, $\forall m$. '$\subset$' Obvious. '$\supset$' Pick $e \in E_m$ and set $f := P_m e$ and $g := e - P_m e$. □

Exercise 1.3.18. Give an example of a finite dimensional subspace F of a Banach scale that is not sc-complemented. [Hint: Pick $f \in C^0(\mathbb{S}^1) \setminus C^1(\mathbb{S}^1)$.]

Quotient Banach scales

If you are not familiar with the quotient construction for Banach spaces, have a look at the neighborhood of Proposition A.2.7 and its proof for definitions and explanations. Understanding that proof helps to prove

Proposition 1.3.19 (Quotient Banach scales)**.** *Let E be a Banach scale and A an* sc-*subspace. Then the quotient Banach space E/A with levels*

$$(E/A)_m := E_m/A_m := \{x + A_m \mid x \in E_m\}, \quad m \in \mathbb{N}_0$$

and inclusions

$$E_{m+1}/A_{m+1} \hookrightarrow E_m/A_m, \quad x + A_{m+1} \mapsto x + A_m \tag{1.3.5}$$

is a Banach scale.

By Proposition A.2.7 the norm on the coset space E_m/A_m defined by

$$\|x + A_m\|_m := d(x, A_m) := \inf_{a \in A_m} |x - a|_m$$

is complete. It is called the *quotient norm* and measures the distance between the coset $x + A$ and the zero coset, the subspace A itself.

Proof. By the sc-subspace assumption on A every level $A_m := A \cap E_m$ is a Banach subspace of E_m, hence the quotient spaces E_m/A_m endowed with the norms $\|\cdot\|_m$ are `(Banach levels)`. To prove that the natural inclusions (1.3.5) are compact linear operators pick a sequence $x_\nu + A_{m+1}$ in the unit ball of E_{m+1}/A_{m+1}. (Note that $x_\nu \in E_{m+1}$.) This means that the distance of each x_ν to the zero coset A_{m+1} of E_{m+1}/A_{m+1} is not larger than 1. Hence for each x_ν there is a point a_ν in the zero coset A_{m+1} at E_{m+1} distance less than 2, that is $|x_\nu - a_\nu|_{m+1} < 2$. What we did is to choose for the given bounded sequence of cosets $x_\nu + A_{m+1} = x_\nu - a_\nu + A_{m+1}$ a sequence of new representatives $x_\nu - a_\nu$ which, most importantly, is bounded in E_{m+1}. By compactness of the inclusion $E_{m+1} \hookrightarrow E_m$ there is a subsequence, still denoted by $x_\nu - a_\nu$, which converges to some element $x \in E_m$. By continuity of the quotient projection $\pi_m \colon E_m \to E_m/A_m$, $x \mapsto x + A_m$, see Proposition A.2.7, we obtain that

$$
\begin{aligned}
\lim_{\nu\to\infty} (x_\nu + A_m) &= \lim_{\nu\to\infty} (x_\nu - a_\nu + A_m) \\
&= \lim_{\nu\to\infty} \pi_m(x_\nu - a_\nu) \\
&= \pi_m\left(\lim_{\nu\to\infty} x_\nu - a_\nu\right) \\
&= \pi_m(x) \\
&= x + A_m .
\end{aligned}
$$

This proves the `(compactness)` axiom. The set of smooth points

$$
(E/A)_\infty := \bigcap_j E_j/A_j = \{x + A_\infty \mid x \in E_\infty\}
$$

is dense in every level E_m/A_m, because the image of a dense subset $E_\infty \subset E_m$ under the continuous surjection $\pi_m \colon E_m \to E_m/A_m$ is dense in the target space by Lemma A.1.23. This proves the `(density)` axiom and Proposition 1.3.19. □

It seems that so far the literature misses out on the analogues for finite <u>co</u>dimensional sc-subspaces of Proposition 1.3.17 (existence of sc-complement for finite dimensional sc-subspaces) and Lemma 1.1.16 (characterization of finite dimensional sc-subspaces). Let's change this.

Proposition 1.3.20. *Finite codimension* sc-*subspaces are* sc-*complemented.*

Of course, the asserted sc-complement C in Proposition 1.3.20 has as dimension the mentioned finite codimension. Hence C carries the constant Banach scale structure and consists of smooth points only; see Lemma 1.1.16.

Proof. Let A be an sc-subspace of a scale Banach space E of finite *codimension* $r = \operatorname{codim} A := \dim E/A$. By closedness and finite codimension the subspace A of E has a topological complement C; cf. Brezis (2011, Prop. 11.6). Since A is the kernel of the quotient projection $\pi\colon E = A \oplus C \twoheadrightarrow E/A$ defined in (A.2.8) we get $\dim C = \dim E/A = r$ for any topological complement.

Recall that a finite dimensional sc-complement of A is an sc-subspace C, endowed with constant levels $C_m = C$, such that

$$A_m \oplus C = E_m, \quad m \in \mathbb{N}_0.$$

Constructing such C inside the vector space E_∞ of smooth points, see Lemma 1.1.16, is equivalent to C being an sc-subspace.

To define C observe that the Banach scale E and the sc-subspace A give rise to the quotient Banach scale in Proposition 1.3.19. Because the top level $(E/A)_0 = E/A$ is of finite dimension r, all sublevels are finite dimensional and therefore the quotient Banach scale is actually constant. Note that

$$E_\infty/A_\infty = E_m/A_m$$

because both sides are of *the same* finite dimension r and there is the natural inclusion $E_\infty/A_\infty \to E_m/A_m$, $\varphi + A_\infty \mapsto \varphi + A_m$. Pick a basis of

$$(E/A)_\infty := \bigcap_m E_m/A_m = E_\infty/A_\infty$$

say $\varphi_1 + A_\infty, \dots, \varphi_r + A_\infty$. Observe that each $\varphi_j \in E_\infty$ and define

$$C := \operatorname{span}\{\varphi_1, \dots, \varphi_r\} \subset E_\infty.$$

We show that C is a topological complement of A_m. To prove $A_m \cap C = \{0\}$, pick $c \in A_m \cap C$. The quotient projection $\pi_m\colon E_m \to E_m/A_m$, $e \mapsto e + A_m$, whose kernel is A_m maps c to the zero coset $0 + A_m$. On the other hand, the only element of C that gets mapped to the zero coset under π_m is $c = 0$. We prove that $A_m + C = E_m$: "$\subset$" Obvious since $A_m \subset E_m$ and $C \subset E_\infty \subset E_m$. "$\supset$" Pick $e \in E_m$ and express $\pi_m(e) = e + A_m \in E_m/A_m$ in terms of the basis $\{\varphi_j + A_m\}_{j=1}^r$, let $c^1, \dots, c^r \in \mathbb{R}$ be the coefficients. Set $c := \sum_{j=1}^r c^j \varphi_j \in C$. Then $\pi_m(c) = \pi_m(e)$, hence $a := e - c$ lies in the kernel of π_m which is A_m. Hence $e = a + c$ is of the desired form. □

A finite codimensional closed subspace of an ordinary Banach space X not only admits a topological complement, but there is even one in each dense subspace X_∞ of X; see e.g. Hofer, Wysocki, and Zehnder (2007, Lemma 2.12) or Brezis (2011, Prop. 11.6). This enters the proof of

Lemma 1.3.21 (Finite codimensional sc-subspaces)**.** *Suppose E is a scale Banach space and A is a linear subspace, then*

$$A \text{ is an sc-subspace of } E \quad \Leftrightarrow \quad A \text{ is closed in } E$$

whenever $A \subset E$ is of finite codimension r.[8]

Proof. An sc-subspace is closed by definition. To prove the reverse implication, let A be a closed subspace of E of finite codimension, say r. By the result mentioned above the subspace A of E admits a topological complement C contained in the dense subset E_∞. In the proof of Proposition 1.3.20 we saw that topological complements satisfy $\dim C = \operatorname{codim} A =: r$.

We need to show that the levels defined by $A_m := A \cap E_m \subset E_m$ satisfy the three axioms of a Banach scale. As A is closed in E, by Lemma 1.1.12 only the `(density)` axiom remains to be shown: density of A_∞ in each A_m.
The inclusion $\underline{C \subset E_\infty}$ means that C is a constant Banach scale by Lemma 1.1.16 and Exercise 1.1.9. Before proving density we show that

$$A_m \oplus C = E_m, \quad m \in \mathbb{N}_0 \tag{1.3.6}$$

is a direct sum of closed subspaces of the Banach space E_m: Firstly, closedness of A_m we already know and C is closed due to its finite dimension. Secondly, trivial intersection $A_m \cap C = \{0\}$ holds true since it even holds for the larger space $A \supset A_m$. Thirdly, we prove $A_m + C = E_m$. "$\subset$" Obvious. "$\supset$" Any $e \in E_m \subset E = A + C$ is of the form $e = a + c$ for some $a \in A$ and $c \in C$. But $a = e - c$ is also in E_m (so we are done), because both e and $c \in \underline{C \subset E_\infty} \subset E_m$ are and E_m is a vector space.

We prove density of A_∞ in A_m. Given $a \in A_m \subset E_m$, by density of E_∞ in E_m there is some in the E_m norm convergent sequence $E_\infty \ni e_\nu \to a \in E_m$. On the other hand, by (1.3.6) there is the direct sum of Banach spaces $E_m = A_m \oplus C$, so e_ν is of the form $e_\nu = a_\nu + c_\nu$ with $(a_\nu, c_\nu) \in A_m \times C$. Clearly $a_\nu - a + c_\nu = e_\nu - a \to 0$ in E_m. But most importantly $a_\nu = e_\nu - c_\nu \in E_\infty$ since the linear space E_∞ contains e_ν and $c_\nu \in \underline{C \subset E_\infty}$. So

$$a_\nu \in (A_m \cap E_\infty) \subset (A \cap E_\infty) = A \cap \bigcap_m E_m = \bigcap_m A \cap E_m = \bigcap_m A_m = A_\infty.$$

Since A_m and C are topological complements of one another, see (1.3.6), the norm in E_m splits in the following sense. By Brezis (2011, Thm. 2.10) there is a constant

[8] A finite codimension subspace in Banach space need not be closed; see e.g. Brezis (2011, Prop. 11.5).

$\mu \geqslant 0$ such that for any element e of E_m the norms of its parts in A_m and in C are bounded above by $\mu|e|_m$. For $e := e_\nu - a = (a_\nu - a) + c_\nu \in A_m + C$ we get that

$$|a_\nu - a|_m + |c_\nu|_m \leqslant 2\mu\, |e_\nu - a|_m \to 0, \quad \text{as } \nu \to \infty.$$

Hence $A_\infty \ni a_\nu \to a \in A_m$ in the E_m norm. This proves Lemma 1.3.21. □

Corollary 1.3.22 (Closed finite codimensional subspaces sc-split E). *Given a scale Banach space E and a finite codimension r subspace A, then*

$$A \text{ is closed in } E \qquad \Leftrightarrow \qquad E = A \oplus C \text{ sc-splits for some } C \subset E_\infty.$$

The sc-*splitting* $E = A \oplus C$ *has levels* $E_m = (A \cap E_m) \oplus C$ *and* $\dim C = r$.

Proof. "⇒" Lemma 1.3.21 and Proposition 1.3.20. "⇐" An sc-subspace is closed by definition. □

Exercise 1.3.23 (Intersection and sum of sc-subspaces). a) If $A, B \subset E$ are finite dimensional sc-subspaces, so are $A \cap B$ and $A + B$.
b) If $A, B \subset E$ are finite codimensional sc-subspaces, so are $A \cap B$ and $A + B$.

[Hints: a) Lemma 1.1.16. b) By Lemma 1.3.21 it suffices to show for $A \cap B$ and for $A + B$ closedness[9] and finite codimension.[10]

1.3.2 Scale Fredholm operators

Definition 1.3.24 (sc-Fredholm operators). An *sc-Fredholm operator* is an sc-operator $T\colon E \to F$ that satisfies the following axioms, namely

(`sc-splittings`) there are sc-splittings $E = K \oplus X$ and $F = Y \oplus C$ such that

(`Ker`) K is the kernel of T and of finite dimension,

(`Coker`) Y is the image of T and C is of finite dimension,

(`sc-isomorphism`) the operator T as a map $T\colon X \to Y$ is an sc-isomorphism.

The *Fredholm index* of T is the integer

$$\operatorname{index} T := \dim K - \dim C = \dim\ker T - \operatorname{codim}\operatorname{im} T.$$

[9] Brezis (ibid., Prop. 11.5): A subspace containing a closed one of finite codimension is closed.
[10] $\operatorname{codim}(A \cap B) \leqslant \operatorname{codim} A + \operatorname{codim} B$ and $\operatorname{codim}(A + B) \leqslant \min\{\operatorname{codim} A, \operatorname{codim} B\}$.

By finite dimension the Banach subscales generated by K and C are constant. So trivially one gets the identities $K = K_\infty$ and $C = C_\infty$. Combined with the equally trivial inclusions $K_\infty \subset E_\infty$ and $C_\infty \subset F_\infty$ they provide the precious information that $K \subset E_\infty$ and $C \subset F_\infty$ consist of smooth points.

Proposition 1.3.25. sc-*Fredholm operators* $T\colon E \to F$ *are* regularizing*: If* T *maps* $e \in E$ *to level* m*, then already* e *was in level* m*; cf (1.3.10).*

Proof. Let $e \in E$ and $Te \in F_m$. But $F_m = T(X_m) \oplus C$, so $Te = Tx + c$ for some $x \in X_m \subset E_m \subset E$ and $c \in C$. As $T(E) \cap C = Y \cap C = \{0\}$ and $e - x \in E$, the identity $T(e-x) = c$ shows that both sides are zero. So $e - x \in \ker T = K = K_\infty \subset E_\infty \subset E_m$. Therefore $e = (e-x) + x \in E_m$. □

Exercise 1.3.26 (Intersection level $Y_m = Y \cap F_m$ is image of level operator T_m)**.** Consider an sc-Fredholm operator $T\colon E = K \oplus X \to F = Y \oplus C$ where the sc-subspace $Y := \operatorname{im} T$ is the image $T(E) = T(X)$. Recall that an sc-subspace Y generates a Banach subscale whose levels are given by intersection $Y \cap F_m$. Show that for sc-Fredholm operators each intersection level is equal to the image of the corresponding level operator $T_m\colon E_m \to F_m$, i.e.

$$Y_m := Y \cap F_m = T(E) \cap F_m = T(E_m) =: \operatorname{im} T_m.$$

[Hint: "$\subset$" Suppose $Te =: y \in F_m$ where $e \in E$.]

Exercise 1.3.27 (Isn't the axiom (`sc-isomorphism`) superfluous?)**.** In view of Exercise 1.3.26 the fourth axiom in Definition 1.3.24 seems to be a consequence of the previous three axioms. Is it?

Exercise 1.3.28. The composition $T \circ T'$ of two sc-Fredholm operators is an sc-Fredholm operator and $\operatorname{index}(T \circ T') = \operatorname{index} T + \operatorname{index} T'$.

Proposition 1.3.29 (Stability of sc-Fredholm property)**.** *Consider an* sc-*Fredholm operator* $T\colon E \to F$ *and an* sc$^+$*-operator* $S\colon E \to F$*, then their sum* $T + S$ *is also an* sc-*Fredholm operator of the same Fredholm index.*

Proof. The sum $T + S\colon E \to F$ is an sc-operator. How about sc-splittings?

Domain splitting. Both T and S provide level operators $E_m \to F_m$ that are Fredholm and compact, respectively. Hence the sum level operators $(T+S)_m = T_m + S_m\colon E_m \to F_m$ are Fredholm for each level m. Note that the kernel K_m of $(T+S)_m$ contains K_{m+1}. To see the reverse inclusion pick $x \in K_m$. Then $Tx = -Sx \in F_{\underline{m+1}}$ by the sc$^+$ nature of S. Thus $x \in E_{m+1}$ by the regularity

Proposition 1.3.25. Hence $x \in K_{m+1}$. Thus $K_m = K_{m+1}$. So $K := K_0 = K_m = K_\infty$ is finite dimensional and $K = K_\infty \subset E_\infty$. Hence K is an sc-subspace by Lemma 1.1.16 and generates a constant Banach scale. By Proposition 1.3.17 the kernel scale K admits an sc-complement X in E. Summarizing, we have

$$E_m = K \oplus X_m, \qquad K = \ker(T_m + S_m) \subset E_\infty$$

for every $m \in \mathbb{N}_0$ and where K does not depend on m.

TARGET SPLITTING. Consider the image $Y := (T+S)(E) = (T+S)(X)$ of the level zero Fredholm operator $T + S\colon E \to F$. But the image of a Fredholm operator is closed and of finite codimension, say r. Hence Y is an sc-subspace of F by Lemma 1.3.21 and admits an r-dimensional sc-complement $C \subset F_\infty$ by Corollary 1.3.22. Summarizing, we have

$$F_m = Y_m \oplus C, \qquad Y_m = \operatorname{im}(T_m + S_m), \quad C \subset F_\infty$$

for every $m \in \mathbb{N}_0$ and where C does not depend on m.

SC-ISOMORPHISM. It is clear that T as a map $T : X \to Y$ is bijective and level preserving with continuous level operators $T_m : X_m \to Y_m$, still injective. But why are these surjective? Exercise. Continuity of the inverse of T_m then follows from the bounded inverse theorem.

FREDHOLM-INDEX. Adding a compact operator, say $S : E \to F$, to a Fredholm operator, say $T : E \to F$, does not change the Fredholm index. This concludes the proof that $T + S$ is an sc-Fredholm operator. □

Scale Fredholm operators – naïve approach through level operators

Intuitively, if not *naively*, an sc-Fredholm operator should be a level preserving linear operator $T\colon E \to F$ between Banach scales whose level operators $T_m\colon E_m \to F_m$ are *Fredholm operators*: Each T_m is linear and continuous, has a finite dimensional kernel $K_m := \ker T_m$, a closed image $Y_m := \operatorname{im} T_m$, and a finite dimensional *cokernel* $\operatorname{coker} T_m := F_m / Y_m$. One calls the integer

$$\operatorname{index} T_m := \dim\ker T_m - \dim\operatorname{coker} T_m$$

the *Fredholm index* of T_m. (As we'll find out, one more condition to come.)

• Firstly, note that the kernels already form a nested sequence $K := K_0 \supset K_1 \supset \dots$ of (by continuity of T_m) closed subspaces $K_m \subset E_m$. Note that $K_m = K \cap E_m$ since $\ker T|_{E_m} = \ker T \cap E_m$. For a Banach scale it only misses the

`(density)` axiom saying that K_∞ is dense in every level K_m.
Before adding a density requirement to the intuitive definition of an sc-Fredholm operator let us investigate its consequences and see if a simpler condition could do the same job. If density holds, then K is an sc-subspace and, by finite dimension, generates the constant Banach scale (see Lemma 1.1.16), still denoted by K and called the *kernel Banach scale*.
Therefore we add to the intuitive definition of sc-Fredholm the requirement

$$\text{all level operators } T_m \text{ have the same kernel } K \tag{1.3.7}$$

in symbols $K := \ker T = \ker T_m \subset E_m\ \forall m$.[11] By Proposition 1.3.17 the kernel sc-subspace $K \subset E_\infty$ admits an sc-complement in E, say X.

• Secondly, the images $Y_m := \operatorname{im} T_m = T(X_m) \supset T(X_{m+1})$ form a nested sequence $Y := Y_0 \supset Y_1 \supset \dots$ of closed subspaces $Y_m \subset F_m$ of finite codimensions r_m. What is missing that the *image scale* $\operatorname{im} T$ with levels $\operatorname{im} T_m$ is a Banach scale is I) `(density)` again, just as in case of the kernel scale. However, this time there is one more thing missing that was automatic for the kernel scale. Namely, we would like to have that II) the image scale is in fact generated by its top level $Y = \operatorname{im} T$, that is we wish that

$$(\operatorname{im} T) \cap F_m = \operatorname{im} T_m \quad \forall m.$$

Suppose I) and II) hold. Namely, the image scale with levels $Y_m := \operatorname{im} T_m$ is a Banach scale and arises by intersection with its top level $Y = \operatorname{im} T$. In other words, the closed finite codimensional subspace $Y = \operatorname{im} T$ is an sc-subspace and the generated Banach subscale has intersection levels $Y \cap F_m = \operatorname{im} T_m$ which are equal to the images of the level operators. Let $r = \dim F/Y$ be the codimension of Y. Then Y admits by Proposition 1.3.20 an r dimensional sc-complement C which necessarily generates the constant Banach scale $C_m = C$. Lemma 1.1.16 tells that $C \subset F_\infty$.

By Corollary 1.3.22 and Lemma 1.3.21 a sufficient condition that $Y = \operatorname{im} T$ is an sc-subspace, thus generating a Banach subscale, is the following which we add as a requirement to the intuitive definition of sc-Fredholm:

$$\text{existence of a topological complement } C \subset F_\infty \text{ of the image of } T. \tag{1.3.8}$$

• Thirdly, to enforce that the intersection levels of the Banach scale generated by $Y = \operatorname{im} T$ coincide with the images of the level operators, i.e.

$$T(X) \cap F_m = T(X_m), \quad m \in \mathbb{N}_0 \tag{1.3.9}$$

[11] Constant dimension $\dim\ker T_m = \dim\ker T$ suffices by the inclusions $K_{m-1} \supset K_m$.

we add to the intuitive definition of sc-Fredholm the requirement

$$\boxed{T(E \setminus E_m) \cap F_m = \emptyset, \quad m \in \mathbb{N}_0} \tag{1.3.10}$$

of *level regularity*. So $e \in E$ and $Te \in F_m$ together imply $e \in E_m$. The next exercise shows that (1.3.10) implies all three conditions (1.3.7–1.3.9).

Exercise 1.3.30. To the naïve notion of sc-Fredholm add (1.3.10) to prove

a) Constancy of kernel scale (1.3.7) holds true. (Thus K is an sc-subspace of finite dimension and therefore K admits an sc-complement X.)

b) Equality of scales (1.3.9) holds true. (That is the image scale Y with levels $T(X_m)$ equals the intersection scale with levels $Y \cap F_m$.)

c) The image scale is a Banach subscale of F generated by its top level $Y = \operatorname{im} T$. (That is Y is an sc-subspace. So (1.3.8) is satisfied by Proposition 1.3.20 and $C \subset F_\infty$ by Lemma 1.1.16.)

d) Each level operator as a map $T_m \colon X_m \to Y_m$ is an isomorphism.

[Hints: a) trivial. b) "$\subset$" easy, "$\supset$" trivial. c) It only remains to show density of Y_∞ in Y. By a) X generates a Banach subscale, so X_∞ is dense in X. Show that $Y_\infty = T(X_\infty)$, then apply Lemma A.1.23. d) Equality (1.3.9).]

Definition 1.3.31 (sc-Fredholm operator – via level operators)**.** An *sc-Fredholm operator* is a level preserving linear operator $T \colon E \to F$ between Banach scales all of whose level operators $T_m \colon E_m \to F_m$ are Fredholm and which satisfies the level regularity condition (1.3.10).

Exercise 1.3.32. Show that Definitions 1.3.24 and 1.3.32 are equivalent.

1.4 Scale differentiability

Motivated by properties of the shift map, see our discussion in the introduction around (0.0.1), the notion of scale differentiability was introduced by Hofer, Wysocki, and Zehnder (2007); see also Hofer, Wysocki, and Zehnder (2010, 2017).

Scale continuous maps – class $\mathbf{sc}^0$

An open subset U of an sc-Banach space E induces via level-wise intersection a nested sequence $U^{\cap E}$ of open subsets $U_m = U \cap E_m$ of the corresponding Banach spaces E_m; cf. Lemma 1.1.12.

Definition 1.4.1. A *partial quadrant* in a Banach scale E is a closed convex subset C of E such that there is an sc-isomorphism $T\colon E \to \mathbb{R}^n \oplus W$, for some n and some sc-Banach space W, satisfying $T(C) = [0,\infty)^n \oplus W$. Note that C necessarily contains the origin 0 of E.
An **sc**-*triple* (U, C, E) consists of a Banach scale E, a partial quadrant $C \subset E$, and a relatively open subset $U \subset E$. Observe that both U and C inherit from E nested sequences of subsets whose levels are the closed subsets $C_m := C \cap E_m \subset E_m$ and the relatively open subsets $U_m := U \cap C_m \subset C_m$.

The notion of partial quadrant is introduced to describe boundaries and corners. At first reading think of $C = E$, so U is an open subset of E.

Definition 1.4.2 (Scale continuity)**.** Let (U, C, E) and (V, D, F) be sc-triples. A map $f : U \to V$ is called *scale continuous* or of *class* $\mathbf{sc}^0$ if

(i) f is level preserving, that is $f(U_m) \subset V_m$ for every m, and

(ii) each restriction viewed as a map $f_m := f|_{U_m} \colon U_m \to V_m$ to level m is continuous. The maps f_m are called *level maps*.

Let us abbreviate terminology as follows.

Convention 1.4.3. If we say *"suppose* $\boldsymbol{f : U \to V}$ *is of class* $\mathbf{sc}^k$*"* it means that f is an sc^k map between sc-triples (U, C, E) and (V, D, F) – suppose at first reading between (U, E, E) and (V, F, F) ;-)

Given Banach scales E and F, an operator $T\colon E \to F$ can have the property of being sc-linear between the Banach scales E and F, i.e. $T \in \mathcal{L}_{\mathrm{sc}}(E, F)$, or it can be *continuous and linear in the usual sense* between the Banach spaces E and F, i.e. $T \in \mathcal{L}(E, F)$. In the latter case, for extra emphasis, we often write E_0 and F_0, instead of E and F, and $T \in \mathcal{L}(E_0, F_0)$.

Definition 1.4.4 (Diagonal maps of height ℓ)**.** Let $f : U \to V$ be an sc^0-map. Pick $\ell \in \mathbb{N}$. View a level map $f_{m+\ell}$ as a map into the higher level V_m

$$\begin{tikzcd}
 & V_m \\
(U^\ell)_m = U_{m+\ell} \arrow[r, "f_{m+\ell}"'] \arrow[ur, dashed, "f^m_{m+\ell} = f|"] & V_{m+\ell} \arrow[u, hook]
\end{tikzcd}$$

to obtain a continuous map $f^m_{m+\ell} = f|\colon U_{m+\ell} \to V_m$ given by restriction of f and called a *diagonal map of height* $\boldsymbol{\ell}$. For simplicity one usually writes $f\colon U_{m+\ell} \to V_m$ and calls it an *induced map*. The collection of all diagonal maps of f of height ℓ is denoted by

$$f^{-\ell} = f|\colon U^\ell \to V^0$$

with level maps $(f^{-\ell})_m = f^m_{m+\ell}$. It is of class sc^0, called the *induced* **sc**-*map of height* $\boldsymbol{\ell}$. If we just say *diagonal map* we mean one of height 1.

Continuously scale differentiable maps – class sc^1

To define scale differentiability let us introduce the notion of tangent bundle. The *tangent bundle* of a Banach scale E is defined as the Banach scale

$$TE := E^1 \oplus E^0.$$

If $A \subset E$ is a subset we denote by $A^k \subset E^k$, as in Definition 1.1.6, the shifted scale of subsets whose levels are given by $(A^k)_m = A_{k+m}$ where $m \in \mathbb{N}_0$.

Definition 1.4.5. The *tangent bundle of an* **sc**-*triple* (U, C, E) is the sc-triple $T(U, C, E) := (TU, TC, TE)$ where[12]

$$TU := U^1 \oplus E^0, \qquad TC := C^1 \oplus E^0, \qquad TE := E^1 \oplus E^0.$$

Note that the levels, for instance of TU, are given by

$$(TU)_m = U_{m+1} \oplus E_m.$$

Definition 1.4.6 (Scale differentiability)**.** Suppose $f\colon U \to V$ is of class sc^0. Then f is called *continuously scale differentiable* or of *class* $\mathbf{sc^1}$ if for every point x in the first sublevel $U_1 \subset U$ there is a bounded linear operator

$$Df(x) \in \mathcal{L}(E_0, F_0), \quad x \in U_1 \tag{1.4.11}$$

between the top level *Banach spaces*, called the **sc**-*derivative of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$ or the sc-*linearization*, such that the following three conditions hold.

[12] The symbol $U^1 \oplus E^0$ actually denotes the subset $U^1 \times E^0$ of the sc-Banach space $E^1 \oplus E^0$ and is just meant to remind us that the ambient Banach space is a direct sum.

`(ptw diff'able)` The upmost diagonal map $f\colon U_1 \to V_0$ is *pointwise* differentiable in the usual sense, notation $df(x) \in \mathcal{L}(E_1, F_0)$; see Definition A.2.22.

`(extension)` The sc-derivative $Df(x)$ extends $df(x)$ from E_1 to E_0, i.e. the diagram

$$\begin{tikzcd} E_0 \arrow[r, dashed, "Df(x)"] & F_0 \\ E_1 \arrow[u, hook, "I_1"] \arrow[ur, "{df(x)\in\mathcal{L}(E_1,F_0),\ x\in U_1}"'] & \end{tikzcd} \tag{1.4.12}$$

commutes.[13] Motivated by the diagram let us call $df(x)$ a *diagonal derivative* if the level index between domain and target drops by 1.

$(Tf$ `is` $\mathrm{sc}^0)$ The *tangent map* $Tf\colon TU \to TV$ defined by

$$Tf(x,\xi) := (f(x), Df(x)\xi)$$

for $(x,\xi) \in U^1 \oplus E^0 = TU$ is of class sc^0.

Remark 1.4.7 (A continuity property of Df)**.** Suppose $f \in \mathrm{sc}^1(U,V)$. By sc^0 there are continuous level maps $f_m = F|\colon U_m \to V_m$, whereas the axiom $(Tf$ `is` $\mathrm{sc}^0)$ requires continuous level maps

$$\begin{aligned} (Tf)_m\colon U_{m+1} \oplus E_m &\to V_{m+1} \oplus F_m \\ (x,\xi) &\mapsto (f(x), Df(x)\xi) \end{aligned} \tag{1.4.13}$$

In particular, for each $m \in \mathbb{N}_0$ the second component map

$$\Phi\colon U_{m+1} \oplus E_m \to F_m, \quad (x,\xi) \mapsto Df(x)\xi \tag{1.4.14}$$

still denoted by Df, is continuous whenever $f \in \mathrm{sc}^1(U,V)$. It is linear in ξ.

Remark 1.4.8 (Continuity in compact-open, but *not in norm*, topology)**.** The compact-open and the norm topologies are reviewed in great detail in Appendix A.1. Continuity of the map $\Phi = Df$ in (1.4.14) means that

$$Df \in C^0\left(U_{m+1}, \mathcal{L}_{\mathrm{c}}(E_m, F_m)\right)$$

[13] So $df(x)\colon E_1 \to F_0$ is compact. This implies $f \in C^1(U_1, V_0)$; see Lemma 1.5.2 (ii).

is continuous whenever the target carries the compact-open topology. Let's refer to this as *horizontal continuity in the compact-open topology*, because both E_m and F_m are of the same level m. It is crucial that the domain has better regularity $m+1$, see Lemma 1.4.12.
In general, continuity is not true in the norm topology, that is with respect to $\mathcal{L}(E_m, F_m)$. The map which prompted the discovery of scale calculus, the shift map (0.0.1), provides a counterexample to continuity of

$$Df : U_{m+1} \to \mathcal{L}(E_m, F_m)$$

for details see e.g. Frauenfelder and Weber (2018, §2).

Things improve drastically if instead of E_m one starts at better regularity E_{m+1}, see Lemma 1.4.12. Now the linear map $Df(x) : E_{m+1} \to F_m$ changes level or "is diagonal" for short. In this case one has norm continuity, that is

$$Df \in C^0\left(U_{m+1}, \mathcal{L}(E_{m+1}, F_m)\right)$$

referred to as *diagonal continuity in the norm topology*.

Remark 1.4.9 (Uniqueness of extension)**.** Since E_1 is dense in the Banach space E_0 the scale derivative $Df(x)$ is uniquely determined by the requirement (1.4.12) to restrict along E_1 to $df(x)$. However, observe that the mere requirement that $f : U_1 \to F_0$ is pointwise differentiable does not guarantee that a bounded extension of $df(x) \in \mathcal{L}(E_1, F_0)$ from E_1 to E_0 exists. Here the B.L.T. Theorem A.2.8 does not help, because the completion of E_1 is E_1 itself.. Existence of such an extension is part of the definition of sc^1.

Exercise 1.4.10. Show that for constant Banach scales E and F, in other words, for finite dimensional normed spaces equipped with the constant scale structure, a map $f : U \to V$ is of class sc^1 iff it is of class C^1.

Exercise 1.4.11. What changes in Exercise 1.4.10 if E or F are constant?

Scale derivative $Df(x)$ induces only some level operators

Lemma 1.4.12 (Level preservation and continuity properties of $Df(x)$)**.** *Let $f : U \to V$ be of class* sc^1 *and $m \in \mathbb{N}_0$. Then the following is true for every point $x \in U$ of* regularity $m+1$*, that is $x \in U_{m+1}$.*

(a) Existence of level operators down to one level above x: *That the* sc*-derivative $Df(x) \in \mathcal{L}(E_0, F_0)$ is level preserving is guaranteed only for levels* $0, \dots, m$.

(b) Continuity of these level operators: *The induced* level operators *are bounded linear operators, in symbols*

$$D_\ell f(x) := Df(x)|_{E_\ell} \in \mathcal{L}(E_\ell, F_\ell), \qquad x \in U_{m+1}, \quad \ell = 0, \dots, m.$$

(c) Horizontal continuity in compact-open topology: *By continuity of the map $\Phi\colon U_{m+1} \oplus E_m \to F_m$ in (1.4.14), still denoted by $D_m f$ or even Df, it holds that $Df \in C^0(U_{m+1}, \mathcal{L}_c(E_m, F_m))$. By linearity of $Df(x)$ this simply means that along any convergent sequence $x_\nu \to x$ in U_{k+1} the scale derivative applied to any individual $\xi \in E_m$ converges, that is*

$$\lim_{\nu\to\infty} \|Df(x_\nu)\xi - Df(x)\xi\|_{F_m} = 0, \qquad \xi \in E_m. \tag{1.4.15}$$

(d) Diagonal continuity in norm topology: *The* sc*-derivative as a map*

$$Df : U_{m+1} \to \mathcal{L}(E_{m+1}, F_m), \quad x \mapsto Df(x)$$

is continuous. Actually $Df = df : U_{m+1} \to \mathcal{L}(E_{m+1}, F_m)$; see (1.5.22).

Corollary 1.4.13. *At smooth points* sc*-derivatives are* sc*-operators, that is*

$$Df(x) \in \mathcal{L}_{\mathrm{sc}}(E, F), \qquad x \in U_\infty.$$

Proof of Lemma 1.4.12. Let $f : U \to V$ be of class sc^1. Pick $x \in U_{m+1}$ and $\ell \in \{0, \dots, m\}$. Hence $x \in U_{\ell+1}$ and so $(x, \xi) \in U_{\ell+1} \oplus E_\ell = (TU)_\ell$ for $\xi \in E_\ell$. The axiom (Tf `is` sc^0) means by definition of sc^0 that every level map $(Tf)_\ell$, see (1.4.13), is continuous. In particular, for fixed $x \in U_{m+1} \subset U_{\ell+1}$ the map between second components $E_\ell \to F_\ell$, $\xi \mapsto Df(x)\xi$, is continuous. This proves (a–b). Since Φ in (1.4.14) is continuous so is $\Phi(\cdot, \xi)\colon U_{m+1} \to F_m$ for each fixed $\xi \in E_m$. This proves (c). Part (d) holds true by Proposition A.2.13 c) for the above map Φ and the compact operator $S := I_{m+1}\colon E_{m+1} \hookrightarrow E_m$. □

Characterization of sc^1 in terms of the scale derivative $Df(x)$

The next lemma and proof are taken from Frauenfelder and Weber (2018).

Lemma 1.4.14 (Characterization of sc^1 in terms of the sc-derivative)**.**
Let $f : U \to V$ be sc^0*. Then f is* sc^1 *iff the following conditions hold:*

(`ptw diff'able`) *(i) The restriction $f : U_1 \to F_0$, that is the top diagonal map, is pointwise differentiable in the usual sense.*

`(extension)` *(ii) Its derivative* $df(x) \in \mathcal{L}(E_1, F_0)$ *at any* $x \in U_1$ *has a continuous extension* $Df(x)\colon E_0 \to F_0$.

`(level operators)` *(iii) The continuous extension* $Df(x)\colon E_0 \to F_0$ *restricts, for all levels* $m \in \mathbb{N}_0$ *and base points* $x \in U_{m+1}$, *to continuous linear operators*

$$D_m f(x) := Df(x)|_{E_m}\colon E_m \to F_m$$

called level operators *such that the corresponding maps*

$$Df|_{U_{m+1} \oplus E_m}\colon U_{m+1} \oplus E_m \to F_m$$

are continuous.

Proof. '⇒' Suppose f is sc^1. Then statements (i) and (ii) are obvious and in statement (iii) the restriction assertion holds by Lemma 1.4.12 part (b), the continuity assertion by part (c).

'⇐' Suppose f is sc^0 and satisfies (i–iii). It remains to show that the tangent map is sc^0, namely, a) level preserving and b) admitting continuous level maps. a) To see that Tf maps $(TU)_m$ to $(TV)_m$ for every $m \in \mathbb{N}_0$, pick $(x, \xi) \in (TU)_m = U_{m+1} \oplus E_m$. Since f is sc^0 we have that $f(x) \in V_{m+1}$. By (iii) we have that $Df(x)\xi \in F_m$. Hence

$$Tf(x, \xi) = (f(x), Df(x)\xi) \in V_{m+1} \oplus F_m = (TV)_m.$$

b) To see that TF as a map $Tf|_{(TU)_m}\colon (TU)_m \to (TV)_m$ is continuous assume $(x_\nu, \xi_\nu) \in (TU)_m = U_{m+1} \oplus E_m$ is a sequence which converges to $(x, \xi) \in (TU)_m$. Because f is sc^0, it follows that

$$\lim_{\nu \to \infty} f(x_\nu) = f(x).$$

Continuity of Df provided by (iii) guarantees that

$$\lim_{\nu \to \infty} Df(x_\nu)\xi_\nu = Df(x)\xi.$$

Therefore

$$\lim_{\nu \to \infty} Tf(x_\nu, \xi_\nu) = \lim_{\nu \to \infty} (f(x_\nu), Df(x_\nu)\xi_\nu) = (f(x), Df(x)\xi) = Tf(x, h).$$

This proves continuity b) and hence the lemma holds. □

Higher scale differentiability – class sc^k

For $k \geqslant 2$ one defines higher continuous scale differentiability sc^k recursively as follows. In the definition of sc^1 one requires a map $f: U \to V$ between open subsets of Banach scales E and F to be sc^0 and then defines a tangent map $F := Tf : TU \to TV$, again between open subsets of Banach scales TE and TF, which among other things is required to be sc^0, too. If the map F itself is of class sc^1, that is if among other things $TF = TTf : TTU \to TTV$ is of class sc^0, one says that f is of class sc^2, and so on.

Definition 1.4.15 (Higher scale differentiability)**.** An sc^1-map $f: U \to V$ is *of class* $\mathbf{sc}^{\boldsymbol{k}}$ if and only if its tangent map $Tf: TU \to TV$ is sc^{k-1}. It is called *sc-smooth*, or of *class* $\mathbf{sc}^{\boldsymbol{\infty}}$, if it is of class sc^k for every $k \in \mathbb{N}$.

An sc^k-map has iterated tangent maps as follows. Recursively one defines the *iterated tangent bundle* as

$$T^{k+1}U := T(T^k U).$$

Let us consider the example T^2U. Recall that for an open subset $U \subset E$ of a Banach scale we set $TU := U^1 \oplus E^0$. Now consider the open subset TU of the Banach scale $TE := E^1 \oplus E^0$ to obtain that

$$\begin{aligned} T^2U := T(TU) :&= (TU)^1 \oplus (TE)^0 \\ &= \left(U^1 \oplus E^0\right)^1 \oplus \left(E^1 \oplus E^0\right)^0 \\ &= U^2 \oplus E^1 \oplus E^1 \oplus E^0. \end{aligned}$$

For f of class sc^k define its *iterated tangent map* $T^k f : T^k U \to T^k V$ recursively as

$$T^k f := T(T^{k-1} f).$$

For example

$$T^2 f : U^2 \oplus E^1 \oplus E^1 \oplus E^0 \to V^2 \oplus F^1 \oplus F^1 \oplus F^0$$

is (as shown in the proof of Lemma 1.4.16 below) given by

$$\begin{aligned} T^2 f(x, \xi, \hat{x}, \hat{\xi}) &= \Big(Tf(x,\xi), D(Tf)|_{(x,\xi)}(\hat{x}, \hat{\xi})\Big) \\ &= \Big(\underbrace{f(x), Df(x)\xi}_{=:Tf(x,\xi)}, \underbrace{Df(x)\hat{x}, D^2 f(x)(\xi, \hat{x}) + Df(x)\hat{\xi}}_{=:D(Tf)_{(x,\xi)}(\hat{x},\hat{\xi})}\Big). \end{aligned} \tag{1.4.16}$$

Here $D^2 f$ is the sc-Hessian of f which we introduce next. The following lemma and proof are taken from Frauenfelder and Weber (2018).

Lemma 1.4.16 (Characterization of sc^2 in terms of the sc-derivative)**.**
Let $f: U \to V$ be sc^1. Then f is sc^2 iff the following conditions hold:

(a) *The restriction $f: U_2 \to V_0$, that is the top diagonal map of height two, is pointwise twice differentiable in the usual sense.*

(b) *Its second derivative $d^2 f(x) \in \mathcal{L}(E_2 \oplus E_2, F_0)$ at any $x \in U_2$ has a continuous extension $D^2 f(x)\colon E_1 \oplus E_1 \to F_0$, the* sc-Hessian of $\boldsymbol{f}$ at $\boldsymbol{x}$.

(c) *The continuous extension $D^2 f(x)\colon E_1 \oplus E_1 \to F_0$ restricts, for all $m \in \mathbb{N}_0$ and $x \in U_{m+2}$, to continuous bilinear maps*

$$D^2_m f(x) := D^2 f(x)|_{E_{m+1} \oplus E_{m+1}} \colon E_{m+1} \oplus E_{m+1} \to F_m$$

such that the corresponding maps

$$D^2_m f: U_{m+2} \oplus E_{m+1} \oplus E_{m+1} \to F_m, \quad (x, \xi_1, \xi_2) \mapsto D^2 f(x)(\xi_1, \xi_2)$$

are continuous.

Proof. '$\Leftarrow$' Suppose $f: U \to F$ is sc^1 and satisfies the three conditions (a-c) of the Lemma. We need to show that f is sc^2 (meaning by definition that $Tf \in \mathrm{sc}^1$). Since f is sc^1 we have a well defined tangent map

$$Tf: TU = U^1 \oplus E^0 \to TF = F^1 \oplus F^0, \quad (x, \xi) \mapsto (f(x), Df(x)\xi),$$

of class sc^0. Suppose that

$$(x, \xi) \in (TU)_1 = U_2 \oplus E_1.$$

Hypotheses (a) and (b) guarantee that the linear map

$$D(Tf)(x, \xi)\colon (TE)_0 = E_1 \oplus E_0 \to (TF)_0 = F_1 \oplus F_0$$

defined for $(\hat{x}, \hat{\xi}) \in E_1 \oplus E_0 = (TE)_0$ by

$$D(Tf)_{(x,\xi)}(\hat{x}, \hat{\xi}) := \big(Df(x)\hat{x}, D^2 f(x)(\xi, \hat{x}) + Df(x)\hat{\xi}\big).$$

is well defined and bounded. To see that this map is the sc-derivative of Tf, see (1.4.11), we need to check the three axioms in the definition of scale differentiability for Tf. Concerning the first two axioms we need to investigate differentiability of the 'diagonal map', i.e. the restriction of $Tf : (TU)_0 \to (TF)_0$ to $(TU)_1$. It suffices to show that

$$\lim_{\|(\hat{x},\hat{\xi})\|_{(TE)_1}\to 0} \frac{\|Tf(x+\hat{x},\xi+\hat{\xi}) - Tf(x,\xi) - D(Tf)_{(x,\xi)}(\hat{x},\hat{\xi})\|_{(TF)_0}}{\|(\hat{x},\hat{\xi})\|_{(TE)_1}} = 0.$$

Since we already know that the first component f of Tf is sc^1 it suffices to check the second component and show that

$$\lim_{\|\hat{x}\|_2+\|\hat{\xi}\|_1\to 0} \frac{\|Df(x+\hat{x})(\xi+\hat{\xi}) - Df(x)(\xi+\hat{\xi}) - D^2f(x)(\xi,\hat{x})\|_0}{\|\hat{x}\|_2+\|\hat{\xi}\|_1} = 0. \tag{1.4.17}$$

We estimate

$$\begin{aligned} &\frac{\|Df(x+\hat{x})(\xi+\hat{\xi}) - Df(x)(\xi+\hat{\xi}) - D^2f(x)(\xi,\hat{x})\|_0}{\|\hat{x}\|_2+\|\hat{\xi}\|_1} \\ \leqslant\ &\frac{\|Df(x+\hat{x})\hat{\xi} - Df(x)\hat{\xi}\|_0}{\|\hat{x}\|_2} \\ &+\frac{\|Df(x+\hat{x})\xi - Df(x)\xi - D^2f(x)(\xi,\hat{x})\|_0}{\|\hat{x}\|_2}. \end{aligned} \tag{1.4.18}$$

Because $D^2f : U_2 \oplus E_1 \oplus E_1 \to F_0$ is continuous by hypothesis (b) there exists an open neighborhood V of x in U_2 and $\delta > 0$ such that for every $y \in V$ and every v and w in B_δ, namely the δ-ball around the origin of E_1, it holds

$$\|D^2f(y)(v,w)\|_0 \leqslant 1.$$

By bilinearity of $D^2f(y)$ for any $v, w \in E_1$ we get the estimate

$$\|D^2f(y)(v,w)\|_0 \leqslant \frac{\|w\|_1\|v\|_1}{\delta^2} \tag{1.4.19}$$

at each $y \in V$. We can assume without loss of generality that V is convex. We rewrite the first term in (1.4.18) as follows

$$\frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\hat{\xi} - Df(x)\hat{\xi}\|_0 = \left\| \int_0^1 D^2f(x+t\hat{x})\Big(\hat{\xi}, \frac{\hat{x}}{\|\hat{x}\|_2}\Big)dt \right\|_0. \tag{1.4.20}$$

From uniform boundedness (1.4.19) we conclude that

$$\lim_{\|\hat{x}\|_2+\|\hat{\xi}\|_1\to 0} \frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\hat{\xi} - Df(x)\hat{\xi}\|_0 \leqslant \lim_{\|\hat{\xi}\|_1\to 0} \frac{c\|\hat{\xi}\|_1}{\delta^2} = 0$$

where $c \geqslant 1$ is a bound for the linear inclusion $E_2 \hookrightarrow E_1$, so $\|\frac{\hat{x}}{\|\hat{x}\|_2}\|_1 \leqslant c$. Hence in view of (1.4.18) in order to show (1.4.17) we are left with showing

$$\lim_{\|\hat{x}\|_2\to 0} \frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\xi - Df(x)\xi - D^2f(x)(\xi,\hat{x})\|_0 = 0. \tag{1.4.21}$$

Fix a constant $\kappa \geqslant 1/\delta^2$ where δ is the constant in (1.4.19). Now choose $\epsilon > 0$. By taking advantage of the fact that E_2 is dense in E_1 we can choose

$$\xi' \in E_2, \qquad \|\xi - \xi'\|_1 \leqslant \frac{\epsilon}{3\kappa c}.$$

Choose $W \subset V$ a convex open neighborhood of x with the property that for every $x + \hat{x} \in W$ it holds that

$$\frac{1}{\|\hat{x}\|_2}\|df(x+\hat{x})\xi' - df(x)\xi' - d^2f(x)(\xi',\hat{x})\|_0 \leqslant \frac{\epsilon}{3}.$$

Suppose that $x + \hat{x} \in W$. We are now ready to estimate

$$\begin{aligned}
&\frac{1}{\|\hat{x}\|_2}\|Df(x+\hat{x})\xi - Df(x)\xi - D^2f(x)(\xi,\hat{x})\|_0 \\
\leqslant\ & \frac{1}{\|\hat{x}\|_2}\|df(x+\hat{x})\xi' - df(x)\xi' - d^2f(x)(\xi',\hat{x})\|_0 \\
&+\left\|\int_0^1 D^2f(x+t\hat{x})\Big(\xi-\xi',\frac{\hat{x}}{\|\hat{x}\|_2}\Big)dt\right\|_0 + \left\|D^2f(x)\Big(\xi-\xi',\frac{\hat{x}}{\|\hat{x}\|_2}\Big)\right\|_0 \\
\leqslant\ & \epsilon.
\end{aligned}$$

To obtain the first inequality we wrote each of the three terms ξ in line one in the form $\xi = \xi' + (\xi - \xi')$, we used that $df = Df$ for diagonal restrictions of f, and we used formula (1.4.20) for $\hat{\xi} = \xi - \xi'$. The second inequality uses, in particular, the estimate (1.4.19) on both D^2f terms. This proves (1.4.21) and therefore the first two axioms of scale differentiability of Tf.

It remains to prove axiom three, namely that the tangent map of Tf, i.e.

$$T^2f = (Tf, D(Tf))\colon T^2U = (TU)^1 \oplus TE \to T^2F = (TF)^1 \oplus TF$$

is sc^0: the map $T^2 f$ must be level preserving and the corresponding level maps

$$(T^2U)_m = U_{m+2} \oplus E_{m+1} \oplus E_{m+1} \oplus E_m \to F_{m+2} \oplus F_{m+1} \oplus F_{m+1} \oplus F_m$$

given by formula (1.4.16) must be continuous for all $m \in \mathbb{N}_0$. For the Tf part both assertions are true, because $Tf \in \mathrm{sc}^1$. Concerning the $D(Tf)$ part there are three terms to be checked. Since $Tf \in \mathrm{sc}^1$ part (iii) of Lemma 1.4.14 applies and asserts that term one exists as a map $Df : U_{m+2} \oplus E_{m+1} \to F_{m+1}$ and is continuous, similarly for the map $Df \circ (\iota, \mathrm{Id}) : U_{m+2} \oplus E_m \to U_{m+1} \oplus E_m \to F_m$ in term three. Concerning term two use hypothesis (c) to see that $D^2 f : U_{m+2} \oplus E_{m+1} \oplus E_{m+1} \to F_m$ is well defined and continuous. This finishes the proof of the implication that under the assumptions (a-c) of the Lemma f is sc^2.

'$\Rightarrow$' For the other implication, namely that if f is sc^2 it satisfies the conditions (a-c) of the Lemma, we point out that by a result of Hofer, Wysocki, and Zehnder Hofer, Wysocki, and Zehnder (2010, Prop. 2.3) it follows that f is actually of class C^2 as a map $f : U_{m+2} \to F_m$ for every $m \in \mathbb{N}_0$. This in particular implies properties (a) and (b). Property (c) is straightforward; cf. proof of Lemma 1.4.14 (iii) based on Lemma 1.4.12 parts (b) and (c). This concludes the proof of Lemma 1.4.16. □

Exercise 1.4.17 (Symmetry of scale Hessian)**.** Show that the *scale Hessian* $\mathrm{Hess}_x f := D^2 f(x) \colon E_1 \oplus E_1 \to F_0$ is symmetric, that is $\mathrm{Hess}_x f(\xi, \eta) = \mathrm{Hess}_x f(\eta, \xi)$ for all $\xi, \eta \in E_1$.
[Hint: The usual second derivative $d^2 f(x) \colon E_2 \oplus E_2 \to F_0$ is symmetric and E_2 is a dense subset of the Banach space E_1.]

Applying the arguments in the proof of Lemma 1.4.16 inductively – Lemma 1.4.14 playing the role of the induction hypothesis – we obtain

Lemma 1.4.18 (Characterizing sc^k by higher sc-derivatives $D^k f(x)$)**.** *Let $k \in \mathbb{N}$ and $f : U \to V$ be sc^{k-1}. Then f is sc^k iff the following conditions hold:*

(i) *The restriction $f : U_k \to V_0$, that is the top diagonal map of height k, is pointwise k times differentiable in the usual sense.*

(ii) *Its k^{th} derivative $d^k f(x) \in \mathcal{L}(E_k \oplus \cdots \oplus E_k, F_0)$ at any $x \in U_k$ has a continuous extension*

$$D^k f(x) \colon \underbrace{E_{k-1} \oplus \cdots \oplus E_{k-1}}_{k \text{ times}} \to F_0.$$

(iii) *The continuous extension* $D^k f(x)\colon E_{k-1} \oplus \dots \oplus E_{k-1} \to F_0$ *restricts, for all* $m \in \mathbb{N}_0$ *and* $x \in U_{m+k}$, *to continuous* k*-fold multilinear maps*

$$D^k_m f(x) := D^k f(x)\colon \underbrace{E_{k-1+m} \oplus \dots \oplus E_{k-1+m}}_{k \text{ times}} \to F_m$$

such that the corresponding maps

$$D^k f|_A\colon A := U_{k+m} \oplus E_{k-1+m} \oplus \dots \oplus E_{k-1+m} \to F_m$$

are continuous.

1.5 Differentiability – Scale vs Fréchet

First we investigate how the new class sc^1 of continuously scale differentiable maps $f : U \to V$ relates to C^1 continuous differentiability in the usual Fréchet sense of all diagonal maps $f : U_{m+1} \to V_m$ of height 1. Then we investigate how the class sc^k of higher scale differentiable maps $f : U \to V$ relates to C^ℓ differentiability of all diagonal maps $f : U_{m+\ell} \to V_m$ of height $\ell \in \{0, \dots, k\}$, hence up to height of at most k. For further details see Hofer, Wysocki, and Zehnder (ibid.).

Maps of class sc^1

Convention 1.5.1 (Topologies)**.** Given Banach spaces E_0 and F_0, the symbol $\mathcal{L}(E_0, F_0)$ denotes the vector space of bounded linear maps $T\colon E_0 \to F_0$ equipped with the (complete) operator norm; see Section A.2.2. The symbol

$$\mathcal{L}_c(E_0, F_0)$$

denotes the same vector space but equipped with the compact-open topology.

Lemma 1.5.2 (Continuity properties of Df and the diagonal differential df)**.** *Let* $f : U \to V$ *be of class* sc^1. *Then the following is true.*

(i) *The map* $U_1 \oplus E_0 \to F_0$, $(x, \xi) \mapsto Df(x)\xi$, *is continuous.*

(ii) *The usual differential* $df : U_1 \to \mathcal{L}(E_1, F_0)$ *of the diagonal map* $f : U_1 \to V_0$ *is continuous, in symbols* $f \in C^1(U_1, V_0)$.

(iii) *Every diagonal map* $f : U_{m+1} \to V_m$ *is of class* C^1. *In other words, its differential, the so-called* diagonal differential

$$df : U_{m+1} \to \mathcal{L}(E_{m+1}, F_m)$$

is a continuous map.

(iv) *At* $x \in U_{m+1}$ *the diagonal derivative* $df(x) \colon E_{m+1} \to F_m$ *in (iii) extends to* E_m *and the extension is the restriction* $Df(x)|_{E_m} \in \mathcal{L}(E_m, F_m)$ *of the* sc-*derivative (1.4.11); cf. Lemma 1.4.12 (b). That is, the diagram*

$$\begin{tikzcd}
E_m \arrow[r, dashed, "Df(x)|_{E_m} =: D_m f(x)"] & F_m \\
E_{m+1} \arrow[u, hook, "I_{m+1}"] \arrow[ur, "df(x)\in\mathcal{L}(E_{m+1},F_m),\ x\in U_{m+1}"'] &
\end{tikzcd} \tag{1.5.22}$$

commutes. As a map $Df : U_{m+1} \oplus E_m \to F_m$ *the level m scale derivative is continuous; cf. (1.4.15).*

Of course, the lemma could be stated more economically, but we enlist the assertions in their order of proof.

Proof. We follow essentially Cieliebak (2018). (i) By assumption f is sc^1, so the induced map $f : U_1 \to V_0$ is pointwise differentiable and for every $x \in U_1$ the usual derivative $df(x) \in \mathcal{L}(E_1, F_0)$ extends from E_1 to a map $Df(x) \in \mathcal{L}(E_0, F_0)$. Moreover, by axiom ($Tf$ `is` $\mathtt{sc}^0$) the map

$$\varphi \colon U_1 \oplus E_0 \to F_0, \quad (x, \xi) \mapsto Df(x)\xi$$

is continuous, cf. (1.4.13), which is assertion (i).

(ii) As the inclusion $S := I_1 \colon E_1 \hookrightarrow E_0$ is compact, continuity of the map

$$U_1 \mapsto \mathcal{L}(E_1, F_0), \quad x \mapsto \varphi(x, S\,\cdot) = Df(x)\cdot = df(x)\cdot$$

holds by Proposition A.2.13 c). We used that $Df(x) = df(x)$ along E_1.

(iii+iv) For $m = 0$ the assertions are true by (i) and (ii) and (ii) will be a key input for the present proof, see Step 1 below, that

a) as a map $f : U_{m+1} \to V_m$ is of class C^1, thereby proving (iii), and

b) its derivative $df(x) = Df(x)|_{E_{m+1}} \colon E_{m+1} \to F_m$ is the sc-derivative $Df(x) \colon E_0 \to V_0$ applied to the elements of E_{m+1} or, equivalently, the restriction to the dense subset E_{m+1} of the level operator $D_m f(x) = Df(x)|_{E_m} \in \mathcal{L}(E_m, F_m)$ which exists by Lemma 1.4.12 (b).

By density $E_{m+1} \subset E_m$ part b) shows that the continuous extension of $df(x) \colon E_{m+1} \to F_m$ to E_m is the level operator $D_m f(x)$. The yet missing continuity assertion in (iv) holds true by (1.4.14). Step 2 below will prove a) and b) which then completes the proof of (iii+iv). Step 1 is just a preliminary.

Step 1. Given $x \in U_1$, let $\xi \in E_1$ be sufficiently small such that the image of the map $\gamma \colon [0,1] \to U_1 \subset E_1$, $t \mapsto x + t\xi$, is contained in U_1. Then

$$f(x+\xi) - f(x) = \int_0^1 \frac{d}{dt} f(x+t\xi)\, dt = \int_0^1 Df(x+t\xi)\xi\, dt.$$

Proof of Step 1. As $f \in C^1(U_1, V_0)$ by (ii), identity one is the integral form of the mean value theorem; see e.g. Lang (1993, XIII Thm. 4.2). Identity two holds since $\frac{d}{dt} f(x+t\xi) = df(x+t\xi)\xi = Df(x+t\xi)\xi$. Equality two is by (1.4.12) – by definition of sc^1 the scale derivative restricted to E_1 is $df(x)$.

In the proof of Step 2 we will use Step 1 for the elements of the subset $U_{m+1} \subset U_1$ and for small $\xi \in E_{m+1} \subset E_1$. For such x and ξ the term $f(x+\xi) - f(x)$ even lies in F_{m+1}, since f is level preserving (it is of class sc^0 by assumption). However, we shall only estimate the F_m norm, as this gives us the opportunity to bring in compactness of the inclusion $S := I_{m+1} \colon E_{m+1} \hookrightarrow E_m$ on the domain side of f.

Step 2. $\forall m$ the map $f \colon U_{m+1} \to V_m$ is of class C^1 with derivative $df = Df$.

Proof of Step 2. Pick $x \in U_{m+1}$ and a non-zero short vector $\xi \in E_{m+1}$ to get

$$
\begin{aligned}
&\frac{1}{|\xi|_{E_{m+1}}} \left| f(x+\xi) - f(x) - Df(x)\xi \right|_{F_m} \\
&= \frac{1}{|\xi|_{E_{m+1}}} \left| \int_0^1 \left(Df(x+t\xi)\xi - Df(x)\xi \right) dt \right|_{F_m} \\
&\leqslant c \int_0^1 \left| Df(x+t\xi) \frac{\xi}{|\xi|_{E_{m+1}}} - Df(x) \frac{\xi}{|\xi|_{E_{m+1}}} \right|_{F_m} dt \\
&\leqslant c \int_0^1 \left\| Df(x+t\xi) - Df(x) \right\|_{\mathcal{L}(E_{m+1}, F_m)} dt \quad \longrightarrow \quad 0 \quad , \text{as } |\xi|_{E_{m+1}} \to 0.
\end{aligned}
$$

Here the equality holds by Step 1. Concerning inequality one note that the path $[0,1] \to F_m$, $t \mapsto Df(x+t\xi)\xi - Df(x)\xi$, is continuous by Lemma 1.4.12 (c) since $x + t\xi \in U_{m+1}$ and $\xi \in E_{m+1} \subset E_m$. So the map is in $\mathrm{L}^1([0,1], F_m)$, hence the norm of the integral is less or equal than the integral along the norm; see e.g. Lang (1993, VI §4 (4)). Inequality two holds by definition of the operator norm.

We prove convergence to zero. This will follow from continuity of the map $Df: U_{m+1} \to \mathcal{L}(E_{m+1}, F_m)$, see Lemma 1.4.12 (d). However, due to infinite dimension it is not just some compactness argument: Let $\xi_\nu \to 0$ be any in E_{m+1} convergent sequence. Then the family of bounded linear operators

$$\mathcal{F} := \{Df(x+t\xi_\nu) \mid \nu \in \mathbb{N}, t \in [0,1]\} \subset \mathcal{L}(E_{m+1}, F_m)$$

generates, for each element $\zeta \in E_{m+1}$, a bounded orbit

$$\mathcal{F}\zeta := \{Df(x+t\xi_\nu)\zeta \mid \nu \in \mathbb{N}, t \in [0,1]\} \subset B_{R(\zeta)} \subset F_m.$$

Indeed by continuity of the map $Df: U_{m+1} \to \mathcal{L}(E_{m+1}, F_m)$, as guaranteed by Lemma 1.4.12 (d), and convergence $\xi_\nu \to 0$ there is a radius $R = R(\zeta)$ such that the whole sequence of elements $Df(x+\xi_\nu)\zeta$ of F_m lies in the ball $B_R \subset F_m$ of radius R and centered at $Df(x)\zeta$. But by convexity of B_R all segments from the center $Df(x)\zeta$ to $Df(x+\xi_\nu)\zeta$ also lie in B_R. The Banach–Steinhaus Theorem A.2.12 then provides a uniform upper bound $c_{\mathcal{F}}$ for the operator norms of all members of $\mathcal{F}$. Now the constant function $g \equiv 2c_{\mathcal{F}}: [0,1] \to [0,\infty)$ is integrable and dominates ($g \geqslant |F_\nu|$) each function

$$F_\nu(t) := \|Df(x+t\xi_\nu) - Df(x)\|_{\mathcal{L}(E_{m+1},F_m)} \leqslant c_{\mathcal{F}} + c_{\mathcal{F}}, \quad t \in [0,1].$$

The pointwise limit $F_\nu(t) \to 0$, as $\nu \to \infty$, is the constant function 0 on $[0,1]$, again by continuity of Df and by continuity of the norm function. Thus the dominated convergence theorem applies, see e.g. Lang (ibid., VI Thm. 5.8), and yields $\lim_\nu \int F_\nu = \int \lim_\nu F_\nu = \int 0 = 0$. This proves convergence to zero.

It remains to prove continuity of the map

$$df: U_{m+1} \to \mathcal{L}(E_{m+1}, F_m), \quad x \mapsto df(x) = \Phi(x)S.$$

Continuity holds by Proposition A.2.13 c) for the by (1.4.13), cf. Lemma 1.4.14, continuous map $\Phi: U_{m+1} \oplus E_m \to F_m$, $(x,\xi) \mapsto Df(x)\xi$, and the compact inclusion $S := I_{m+1}: E_{m+1} \to E_m$. As any $\xi \in E_{m+1}$ lies in E_1, one has

$$\Phi(x)S\xi := Df(x)I_{m+1}\xi = df(x)\xi$$

since the diagram (1.4.12) commutes. This proves Lemma 1.5.2. □

Lemma 1.5.3 (Characterization of sc^1 via diagonal maps being of class C^1). *An* sc^0*-map* $f: U \to V$ *is of class* sc^1 *iff*

(i) *all diagonal maps* $f: U_{m+1} \to V_m$ *are of class* C^1 *and for each of them*

(ii) *the derivative* $df(x) \in \mathcal{L}(E_{m+1}, F_m)$*, at any* $x \in U_{m+1}$*, extends to a bounded linear operator on* E_m *which, moreover, coincides with the restriction* $D_m f(x)\colon E_m \to F_m$ *to* E_m *of the top level extension, namely the* sc*-derivative* $Df(x) \in \mathcal{L}(E_0, F_0)$*;*

(iii) *for every* $m \in \mathbb{N}_0$ *the level* m *scale derivative as a map* $U_{m+1} \oplus E_m \to F_m$*,* $(x, \xi) \mapsto Df(x)\xi$*, is continuous; cf. (1.4.15).*

Proof. '$\Rightarrow$' Lemma 1.5.2. '$\Leftarrow$' Lemma 1.4.14. □

Remark 1.5.4. One can show that for each map $f: U \to V$ of class sc^1 the induced map $f: E_\infty \supset U_\infty \to V_\infty \subset F_\infty$ between Fréchet spaces is of class C^1; cf. Cieliebak (2018, Probl. 5.5).

Maps of class sc^k

It is an immediate consequence of Lemma 1.5.3, together with the identity $(TU)^1 = T(U^1)$, that for an sc^k map $f: U \to V$ one can lift both indices equally and still have an sc^k map, say $f: U^\ell \to V^\ell$.

Lemma 1.5.5 (Lifting indices, Hofer, Wysocki, and Zehnder (2010, Prop. 2.2)). *If* $f: U \to V$ *is an* sc^k*-map, then the induced map* $f: U^1 \to V^1$ *is also of class* sc^k*.*

Proof. Induction over $k \in \mathbb{N}$. Case $k = 1$: This holds true by Lemma 1.5.3 which characterizes sc^1 by some conditions on all [14] diagonal maps $f: U_{m+1} \to V_m$ and their extensions $D_m f(x)$. Replacing U, V by U^1, V^1 means to simply forgetting the two maps for $m = 0$.
Induction step $k \Rightarrow k+1$: Let $f: U \to V$ be sc^{k+1}. By definition this means that f is sc^1 and Tf is sc^k. So by induction hypothesis applied to $Tf \in \mathrm{sc}^k(TU, TV)$ that same map just between shifted spaces, namely $Tf: (TU)^1 \to (TV)^1$, is as well of class sc^k. But $(TU)^1 = T(U^1)$, hence $Tf: T(U^1) \to T(V^1)$ is sc^k. Note that $f: U^1 \to V^1$ is also of class sc^1 as a consequence of the case $k = 1$ applied to $f \in \mathrm{sc}^1(U, V)$. But an sc^1 map, say $f: U^1 \to V^1$, whose tangent map is sc^k is of class sc^{k+1} by Definition 1.4.15. □

[14] saying "*all* diagonal maps $f: U_{m+1} \to V_m$" refers to the set $\{f: U_{m+1} \to V_m\}_{m \in \mathbb{N}_0}$

Lemma 1.5.6 (Necessary and sufficient conditions for sc^k-smoothness)**.** *Let U, V be relatively open subsets of partial quadrants in* sc*-Banach spaces E, F.*

(Necessary) *If $f\colon U \to V$ is sc^k, then all diagonal maps $f\colon U_{m+\ell} \to V_m$ of height ℓ are of class C^ℓ and this is true for all heights from 0 up to k.*

(Sufficient) *Assume that a map $f\colon U \to V$ induces for every level $m \in \mathbb{N}_0$ and every height ℓ between 0 and k a diagonal map $f\colon U_{m+\ell} \to V_m$ which, moreover, is of class $C^{\ell+1}$. Such a map $f\colon U \to V$ is of class sc^{k+1}.*

Sketch of proof. Necessary. Suppose $f \in \mathrm{sc}^k(U,V)$. Firstly, it suffices to prove the case $\ell = k$, because an sc^k map is also an sc^ℓ map for $\ell \in \{0,\dots,k\}$. Secondly, it suffices to prove the case $m = 0$, namely the

Claim. The map $f\colon U_k \to V_0$ is of class C^k.

Given $m \in \mathbb{N}_0$, the claim implies that $f\colon U_{m+k} \to V_m$ is of class C^k and we are done. Indeed by Lemma 1.5.5 the map $f\colon U^m \to V^m$ is also of class sc^k and for this map the claim asserts that $f\colon (U^m)_k \to (V^m)_0$ is of class C^k.

One proves the claim by induction over k. For $k = 0$ the map $f\colon U_0 \to V_0$ is C^0 as $f \in \mathrm{sc}^0$, for $k = 1$ the map $f\colon U_1 \to V_0$ is C^1 by Lemma 1.5.2 (ii).

The induction step $k \Rightarrow k+1$ is very similar in character to the proof of Lemma 1.5.2 (iii+iv) just more technical as one is looking at k-fold derivatives, thus k-multilinear maps. For details see Hofer, Wysocki, and Zehnder (2010, Prop. 2.3).

Sufficient. Proof by induction over k. Case $k = 0$. By assumption there is for each $m \in \mathbb{N}_0$ a C^1 level map $f_m := f|_{U_m}\colon U_m \to V_m$. Together with the restriction i_{m+1} of the linear, hence smooth, embedding $I_{m+1}\colon E_{m+1} \hookrightarrow E_m$ one has a commutative diagram

$$
\begin{tikzcd}
U_m \arrow[r, "f_m \in C^1"] & V_m \\
U_{m+1} \arrow[u, hook, "C^\infty \ni i_{m+1}"] \arrow[ur, dashed, "f = f_m \circ I_{m+1} \in C^1"'] &
\end{tikzcd}
$$

of C^1 maps, in particular, all diagonal maps $f\colon U_{m+1} \to V_m$ are C^1. By the chain rule one gets $df(x) = df_m(x) \circ I_{m+1}$ for $x \in U_{m+1}$. For $m = 0$ and $x \in U_1$ one obtains the sc-derivative $Df(x) := df_0(x)$ and for $x \in U_{m+1}$ restriction to E_m yields that $Df(x)|_{E_m} = df_0(x)|_{E_m} = df_m(x)\colon E_m \to F_m$. Thus (i) and (ii) in Lemma 1.5.3 are satisfied and it remains to check (iii). But this follows by pre-composing the first variable of the continuous map $U_m \oplus E_m \to F_m$, $(x, \xi) \mapsto df_m(x)\xi$, with the continuous embedding $U_{m+1} \hookrightarrow U_m$. The step $k \Rightarrow k+1$ is very technical, see Hofer, Wysocki, and Zehnder (ibid., Prop. 2.4). $\square$

1.6 Chain rule

A key element of calculus, the chain rule, is also available in sc-calculus. This is rather surprising given the fact that the sc-derivative arises by differentiating the diagonal map $f : U_1 \to V_0$ thereby loosing one level, so for a composition one would expect the loss of two levels. However, using `(compactness)` of the embeddings $E_{m+1} \hookrightarrow E_m$ one can avoid loosing two levels.

Theorem 1.6.1 (Chain rule Hofer, Wysocki, and Zehnder (2007, Thm. 2.16))**.** *Suppose $f : U \to V$ and $g : V \to W$ are* sc^1*-maps. Then the composition $g \circ f : U \to W$ is also* sc^1 *and*

$$T(g \circ f) = Tg \circ Tf.$$

Equivalently, in terms of sc*-derivatives it holds that*

$$D(g \circ f)|_x \, \xi = Dg|_{f(x)} \, Df|_x \, \xi, \quad (x, \xi) \in U^1 \oplus E^0 = TU. \tag{1.6.23}$$

Proof. The main principles and tools of the proof have been detailed and referenced in the slightly simpler setting of proving Step 2 in the proof of Lemma 1.5.2 (iii+iv). Fix $x \in U_1$. Because V_1 is an open neighborhood of $f(x)$ in the cone $D_1 \subset F_1$ and because the level map $f : U_1 \to V_1$ is continuous, there is a radius $\delta > 0$ open ball B_δ in E_1 centered at 0 such that $x + B_\delta$ is contained in U_1 and such that the map

$$\phi(t, \xi) := tf(x + \xi) + (1 - t)f(x) \in V_1$$

takes values in V_1 for all $t \in [0, 1]$ and $\xi \in B_\delta$. Because $\underline{g \text{ is sc}^1}$, as a map $g : V_1 \to W_0$ it is of class C^1 by Lemma 1.5.3 (i). Apply the mean value theorem, observing that $\partial_t \phi(t, h) = f(x + \xi) - f(x)$, and add zero to obtain

$$\begin{aligned}
&g(f(x + \xi)) - g(f(x)) - Dg|_{f(x)} Df|_x \xi \\
&= \int_0^1 Dg|_{\phi(t,\xi)} \left(\partial_t \phi(t, \xi) - Df|_x \xi\right) dt \\
&\quad + \int_0^1 \left(Dg|_{\phi(t,\xi)} - Dg|_{f(x)}\right) Df|_x \xi \, dt.
\end{aligned}$$

Divide by $|\xi|_{E_1}$, so the first integral becomes

$$\int_0^1 Dg|_{\phi(t,\xi)}\, h(\xi)\, dt, \qquad h(\xi) := \frac{f(x+\xi) - f(x) - Df|_x\xi}{|\xi|_{E_1}}. \tag{1.6.24}$$

Since f is sc^1 the restriction of $Df|_x \colon E_0 \to F_0$ to E_1 is $df|_x$ whenever $x \in U_1$, see (1.4.12), hence $h(\xi) \to 0$ as $|\xi|_{E_1} \to 0$ by Definition A.2.22 of the Fréchet derivative $df|_x := df(x)$. Now $\phi\colon [0,1] \times B_\delta \to V_1$ is continuous and $\phi(t,\xi) \to f(x)$ uniformly in t as $|\xi|_{E_1} \to 0$. Since g is sc^1 Lemma 1.4.14 (iii) guarantees that the map $V_1 \oplus F_0 \to G_0$, $(y,\eta) \mapsto Dg|_y\eta$, is continuous. Thus $Dg|_{\phi(t,\xi)}h(\xi) \to 0$ uniformly in t as $|\xi|_{E_1} \to 0$. So the integral (1.6.24) vanishes in the limit as $|h|_{E_1} \to 0$.

The second integral divided by $|\xi|_{E_1}$ becomes

$$\int_0^1 \left(Dg|_{\phi(t,\xi)} - Dg|_{f(x)}\right) \frac{Df|_x\xi}{|\xi|_{E_1}}\, dt. \tag{1.6.25}$$

By `(compactness)` of the inclusion $E_1 \hookrightarrow E_0$ and continuity of the sc-derivative $Df|_x \colon E_0 \to F_0$ the set of all $Df|_x\xi/|\xi|_{E_1}$ with $0 \neq \xi \in B_\delta$ has compact closure in F_0.[15] Since the map $V_1 \oplus F_0 \to G_0$, $(y,\eta) \mapsto Dg|_y\eta$, is continuous by Lemma 1.4.14 (iii) – due to g being sc^1 – it follows as above that the integrand in (1.6.25) converges in G_0 to 0 uniformly in t, so the integral (1.6.25) converges in G_0 to 0, both as $|\xi|_{E_1} \to 0$.

This shows that the sc^0 map given by the composition $g \circ f \colon U \to W$ satisfies the first two axioms in Definition 1.4.5 of sc^1. Indeed as a map $g \circ f \colon U_1 \to V_1 \to W_0$ it is pointwise differentiable and at $x \in U_1$ the derivative $d(g \circ f)|_x \colon E_1 \to G_0$ has a continuous extension, namely the composition of bounded linear operators given by $Dg|_{f(x)}Df|_x \colon E_0 \to G_0$. Hence by definition this composition is the sc-derivative $D(g \circ f)|_x$ associated to $g \circ f$. Therefore $T(g \circ f) = Tg \circ Tf \colon TU \to TW$. Because both Tf and Tg are sc^0, so is $T(g \circ f)$. Thus $g \circ f$ also satisfies the third axiom in Definition 1.4.5. So $g \circ f$ is sc^1. $\square$

1.7 Boundary recognition

Let C be a partial quadrant in an sc-Banach space E. Pick a linear sc-isomorphism $T\colon E \to \mathbb{R}^n \oplus W$ with $T(C) = [0,\infty)^n \oplus W$. For $x \in C$ write $Tx =$

[15] images of compact sets under continuous maps are compact

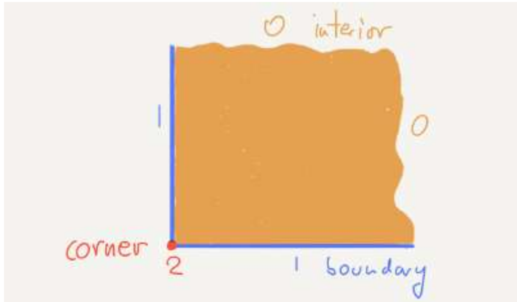

Figure 1.2: Quadrant $C \subset \mathbb{R}^2$ and points of degeneracy index two, one, zero

$(a_1, \dots, a_n, w) \in [0, \infty)^n \oplus W$ and define its *degeneracy index* by

$$d_C(x) := \#\{i \in \{1, \dots, n\} \mid a_i = 0\} \in \mathbb{N}_0. \tag{1.7.26}$$

A point $x \in C$ satisfying $d_C(x) = 0$ is an interior point of C, a boundary point if $d_C(x) = 1$, and a corner point if $d_C(x) \geqslant 2$. See Figure 1.2.

Exercise 1.7.1. The degeneracy index d_C does not depend on the choice of linear sc-isomorphism $T : E \to \mathbb{R}^n \oplus W$.

Theorem 1.7.2 (Invariance under sc^1-diffeomorphisms)**.** *Let* (U, C, E) *and* (V, D, F) *be partial quadrants. Let* $f : U \to V$ *be an* $\mathbf{sc^1}$-diffeomorphism, *that is an* sc^1-*map with an* sc^1*-inverse, then for every* $x \in U$ *one gets equality*

$$d_C(x) = d_D(f(x)).$$

Proof. Hofer, Wysocki, and Zehnder (2007, Thm. 1.19) □

1.8 Sc-manifolds

The new notion of differentiability of maps between the new linear spaces – sc^k differentiability of maps between sc-Banach spaces – allows to carry over the new calculus to topological spaces modeled locally on sc-Banach spaces. This results in a new class of manifolds, called sc^k-manifolds. Their construction parallels the definition of C^k Banach manifolds; see Section A.2.4.

To complement Section A.2.4 (case C^k) we spell out here the smooth case (case sc^∞). Suppose X is a topological space. An **sc**-*chart* $(V, \phi, (U, C, E))$ for

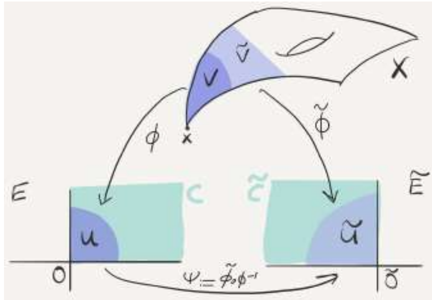

Figure 1.3: Transition map between sc-charts of sc-manifold X

X consists of an sc-triple (U, C, E) and a homeomorphism $\phi\colon X \supset V \to U \subset C$ between open subsets. Two sc-charts are called **sc**-*smoothly compatible* if the transition map (cf. Figure 1.3)

$$\phi \circ \widetilde{\phi}^{-1}\colon \widetilde{E} \supset \widetilde{\phi}(V \cap \widetilde{V}) \to \phi(V \cap \widetilde{V}) \subset E$$

is an **sc**-*smooth diffeomorphism* (invertible sc-smooth map with sc-smooth inverse). An **sc**-*smooth atlas for* $\boldsymbol{X}$ is a collection $\mathcal{A}$ of pairwise sc-smooth compatible Banach sc-charts for X such that the chart domains form a cover $\{V_i\}_i$ of X. Two atlases are called *equivalent* if their union forms an atlas.

Exercise 1.8.1. Let X be a topological space endowed with an sc-smooth atlas $\mathcal{A}$. a) Is it true that X is connected iff it is path connected? b) Show that if X is connected, then all model sc-Banach spaces $\widetilde{E}$ appearing in the charts of $\mathcal{A}$ are (linearly) sc-isomorphic to one and the same sc-Banach space, say E. In this case one says that $(X, \mathcal{A})$ is *modeled on* $\boldsymbol{E}$.

[Hint: b) Given a transition map $\psi\colon E \supset U \to \widetilde{U} \subset \widetilde{E}$ between two sc-charts, observe that U_∞ is a dense subset of U and that sc-derivatives taken at smooth points are sc-operators by Corollary 1.4.13.]

Definition 1.8.2. An **sc**-*manifold* is a paracompact Hausdorff space X, see Definition A.1.21, endowed with an equivalence class of sc-smooth atlases. If all model spaces are sc-Hilbert spaces one speaks of an *Hilbert* **sc**-*manifold*.

Definition 1.8.3 (Sc-smooth maps between sc-manifolds)**.** a) A continuous map $f\colon X \to Y$ between sc-manifolds is called **sc**-*smooth* if for all sc-charts $\phi\colon X \supset V \to C \subset E$ and $\psi\colon Y \subset W \to D \subset F$ the chart representative

$$\psi \circ f \circ \phi^{-1}\colon E \supset C \supset \phi(V \cap f^{-1}(W)) \to D \subset F$$

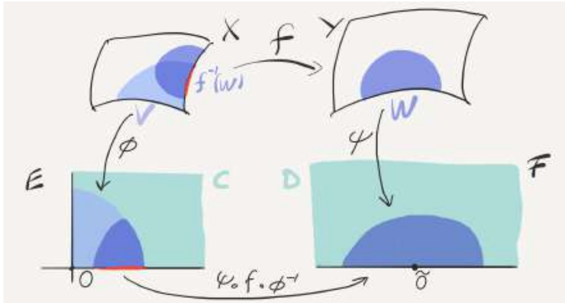

Figure 1.4: Local representative of sc-smooth map between sc-manifolds

is of class sc^∞ as a map from an open subset of the partial quadrant C in the sc-Banach space E into the sc-Banach space F. See Figure 1.4.

b) An **sc**-*diffeomorphism* between sc-manifolds is an invertible sc-smooth map whose inverse is sc-smooth.

Detecting boundaries and corners

Suppose X is an sc-manifold. To define the *degeneracy index* of a point $x \in X$, pick an sc-chart $\phi\colon X \supset V \to C \subset E$ about x and set

$$d_X(x) := d_C(\phi(x)) \in \mathbb{N}_0.$$

By Theorem 1.7.2 the definition *does not depend*[16] on the choice of sc-chart. One calls a point x of degeneracy index $d_X(x) = k$ an *interior point* if $k = 0$, a *boundary point* if $k = 1$, and a *corner point of complexity* $\boldsymbol{k}$ in case $k \geqslant 2$. This is illustrated by Figure 1.2 for $X = C = [0, \infty)^2$.

Levels of sc-manifolds are topological Banach manifolds

A point x of an sc-manifold X is said to be *on level* $\boldsymbol{m}$ if $\phi(x) \in E_m$ lies on level m for some (thus every) sc-chart $\phi\colon X \supset V \to C \subset E$ about x. Indeed the definition does not depend on the choice of chart, even for topological sc-manifolds (those of class sc^0), since any transition map is of class sc^0, hence level preserving (and continuous). *Level* $\boldsymbol{m}$ *of the* **sc**-*manifold* is the set

$$X_m := \{\text{all points of } X \text{ on level } m\}.$$

[16]For M-polyfolds the definition might depend on the choice of chart. The way out will be to take the minimum over all charts.

By continuity of transition maps each level X_m of an sc-manifold is a topological Banach manifold (in general not C^1).

To summarize, an sc-manifold X decomposes into a nested sequence of topological Banach manifolds

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_\infty := \bigcap_{m \geqslant 0} X_m$$

whose intersection X_∞ carries the structure of a smooth Fréchet manifold with boundaries and corners; cf. Cieliebak (2018, §5.3).

Furthermore, each level X_k of an sc-manifold X inherits the structure of an sc-manifold denoted by X^k and called the *shifted* **sc**-*manifold* $\boldsymbol{X^k}$. By definition level m of X^k is level X_{k+m} of X.

Levels of strong sc-manifolds are smooth Banach manifolds

Suppose (U, C, E) and (V, D, F) are sc-triples. The notion of scale differentiability sc^1 is based on usual C^1 differentiability of all diagonal maps of height one. A natural way to strengthen this is to ask all level maps (height zero) to be C^1 (or C^k). Given $k \in \mathbb{N}$ or $k = \infty$, an sc^0 map $f : U \to V$ between sc-triples is called *strongly* $\mathbf{sc}^{\boldsymbol{k}}$ or of *class* $\mathbf{ssc}^{\boldsymbol{k}}$ if all level maps $f_m : U_m \to V_m$ are of class C^k. This means that on each level one works with the usual calculus on Banach spaces. Now one calls a paracompact Hausdorff space X an $\mathbf{ssc}^{\boldsymbol{k}}$*-manifold* if all transition maps are of class ssc^k, that is if they are level-wise C^k.

Important classes of function spaces fit into the framework of strong scale differentiability, for instance loop spaces of finite dimensional manifolds.

Example 1.8.4 (Loop spaces are ssc$^\infty$-manifolds)**.** Let M be a manifold of finite dimension. Then the *loop space*

$$X := W^{1,2}(\mathbb{S}^1, M)$$

that consists of all absolutely continuous maps $x : \mathbb{R} \to M$ of period one, that is $x(t+1) = x(t)$ for every t, is a strongly sc-smooth manifold.

Example 1.8.5. The previous example generalizes to $X := W^{k,p}(N, M)$ where N can be any compact manifold with boundary of finite dimension n and where the numbers $k \in \mathbb{N}$ and $p \in [1, \infty)$ must satisfy the condition $k > n/p$ (assuring continuity of the functions that are the elements of X).

Tangent bundle of sc-manifolds

Let X be an sc-manifold. For an sc-chart $(V, \phi, (U, C, E))$ we shall use the short notation (V, ϕ) with the understanding that $U = \phi(V)$ is an open subset of a partial quadrant C in an sc-Banach space E. Recall that X^1 denotes the sc-manifold that arises from X by forgetting level zero. Let $V^1 \subset X^1$ denote the corresponding scale of levelwise open subsets generated by $V_1 := V \cap X_1$. Now consider tuples (V, ϕ, x, ξ) where (V, ϕ) is an sc-chart of X, the point $x \in V^1$ lies on level one, and $\xi \in E^0$ is a vector in level zero of the sc-Banach space E. Two tuples are called equivalent if the two points $x, \widetilde{x} \in X^1$ are equal and the two vectors correspond to one another through the sc-derivative, in symbols

$$x = \widetilde{x}, \qquad D(\widetilde{\phi} \circ \phi^{-1})|_{\phi(x)} \xi = \widetilde{\xi}.$$

An equivalence class $[V, \phi, x, \xi]$ is called a *tangent vector to the* **sc**-*manifold* at a point x on level one. There is a canonical projection defined on TX, the *set of all tangent vectors at all points of* $\boldsymbol{X_1}$, namely

$$p\colon TX \to X^1, \quad [V, \phi, x, \xi] \mapsto x.$$

Exercise 1.8.6 (Tangent bundle as sc-manifold)**.** Naturally endow the set TX with the structure of an sc-manifold such that the projection $p\colon TX \to X^1$ becomes sc-smooth as a map between sc-manifolds.

[Hint: See Remark A.2.24. Each chart $\phi\colon X \supset V \to E$ of X gives a bijection

$$\Phi := T\phi:\ TV := p^{-1}(V \cap X^1) \to TE = E^1 \oplus E^0, \quad [V, \phi, x, \xi] \mapsto (\phi(x), \xi)$$

onto the open subset $U^1 \oplus E^0$ of $E^1 \oplus E^0$ where $U = \phi(V)$.]

2 Sc-retracts – local models

Let us indoctrinate you right away to the intuition behind the key players O and maps between them. Think of an sc-retract O as a *compressed open set* – the image of some idempotent map $r = r \circ r : U \to U$, called a projection or retraction. Vice versa, think of the open set U as a *decompression* of O. The great variety of possible properties of such O – there can be corners and even jumping dimension – are desirable in applications, because solution spaces to PDEs often exhibit such behavior. In contrast, to do analysis it is desirable that domains of maps are open, so difference quotients, hence derivatives, can be defined. Idempotents $r = r \circ r : U \to U$ combine and provide both of these, somehow contradictory, properties. One uses such O as geometric model space and when it comes to analysis one just decompresses $O = r(U)$ and uses the open set U as domain. For instance, to define differentiability of a function $f : O \to \mathbb{R}$ one *decompresses* the domain O and calls f differentiable if the pre-composition $f \circ r : U \to \mathbb{R}$ is. In such context we often call $f \circ r$ or r itself a *decompression of* $\boldsymbol{f}$. A second highly useful property of images $O = r(U)$ of projections $r : U \to U$ is that any such is precisely the fixed point set $\operatorname{Fix} r = O$ of r.

In Chapter 2 our main source is again Hofer, Wysocki, and Zehnder (2017), together with Cieliebak (2018) and Fabert et al. (2016). Concerning terminology our convention is and was to assign the adjective sc-smooth (or the equivalent symbol sc^∞) to maps that are $k = \infty$ many times continuously scale differentiable. In case of sets, e.g. sc-manifolds or sc-retracts, the "sc" itself already indicates sc-smooth.

Outline of Chapter 2. In the present chapter M-polyfolds[1] are constructed based on the new notion of scale differentiability and locally modeled on rather general topological spaces O which might have corners, even jumping dimension along components, but they will still be accessible to the new weaker form of calculus – sc-calculus. The class of spaces are sc-retracts, generalizing smooth retracts in Banach manifolds. Section 2.1 "Cartan's last theorem" deals with smooth retracts and is the motivation for the generalizations in the following sections. Section 2.2 "Sc-smooth retractions and their images O" provides the local model spaces O for M-polyfolds. A key step is to extend sc-calculus from sc-Banach spaces E to sc-smooth retracts O. Section 2.3 "M-polyfolds and their tangent bundles" defines M-polyfolds, in analogy to manifolds, by patching together local models and asking transition maps to be sc-smooth (in the sense of the extended sc-calculus). Section 2.4 "Strong bundles over M-polyfolds" provides the environment to implement sc-Fredholm sections f. The need for sc^+-sections requires fibers be shiftable in scale by $+1$ leading to double scale structures. In practice f arises as a differential operator of order ℓ leading to asymmetry in base and fiber levels.

Detailed summary of Chapter 2

Section 2.1 "Cartan's last theorem" recalls and proves the surprising result that the image $O = r(U)$ of a smooth idempotent map $r = r \circ r \colon U \to U$ on a Banach manifold, called a *smooth retraction*, is a smooth submanifold.

Section 2.2 "Sc-smooth retractions and their images O" is at the heart of the whole theory. It introduces the local model spaces for M-polyfolds, called **sc**-*retracts* and denoted by (O, C, E), or simply O. These are images O of sc-smooth idempotents $r = r^2 \colon U \to U$, called **sc**-*retractions*, defined on sc-triples (U, C, E). It is useful to observe that image and fixed point set of r coincide and to think of r as a projection onto its fixed point set, in symbols

$$r = r \circ r \colon U \to O := \operatorname{im} r = \operatorname{Fix} r.$$

[1] <u>M</u>-polyfolds are defined analogous to <u>m</u>anifolds, just based on sc-differentiability and more general model spaces. In contrast, polyfolds correspond classically to orbifolds.

While the domain U is a (relatively) open subset of a partial quadrant C in an sc-Banach space E, its image $O = r(U)$ is a projected or compressed version of U. Motivated by continuous retractions one might expect that the compressed set, the sc-retract $O = r(U)$ has non-smooth properties, e.g. jumping dimension or having corners, as illustrated by Figure 2.1. In contrast, the images of in the usual sense smooth retractions on Banach manifolds are smooth Banach submanifolds by Theorem 2.1.1.

One defines **sc**-*smoothness* of a map between sc-retracts

$$f : O \to O'$$

called an **sc**-*smooth retract map* (the future M-polyfold transition maps), if some, hence by Lemma 2.2.4 any, decompression

$$f \circ r : U \to U', \qquad O = r(U)$$

of f is an sc-smooth map in the ordinary sense; see Definition 1.4.15. Given an sc-retract (O, C, E), the tangent map of a decompression $r = r \circ r : U \to U$ of $O = r(U)$ is an sc-smooth retraction itself

$$Tr = (Tr) \circ (Tr) \colon TU = U^1 \oplus E^0 \to TU, \quad (x, \xi) \mapsto (r(x), Dr(x)\xi) .$$

Hence the image

$$TO := Tr(TU) = \operatorname{Fix} Tr \subset O^1 \oplus E^0$$

is an sc-retract (TO, TC, TE) in the tangent sc-triple (TU, TC, TE). Here TO is independent of the choice of the decompression r of O by Lemma 2.2.6. The *tangent bundle of the* **sc**-*retract* $\boldsymbol{O}$ is the natural surjection

$$p \colon TO \to O^1, \quad (x, \xi) \mapsto x.$$

It is an sc-smooth map between sc-retracts. The tangent space at $x \in O^1$

$$T_x O := \operatorname{Fix}[Dr(x) \colon E \to E] \subset E$$

is a Banach subspace, even an sc-subspace for $x \in O_\infty$, by Corollary 1.4.13.

The *tangent map of an* **sc**-*smooth retract map* $f : O \to O'$ is defined as the restriction to TO of the tangent map

$$\begin{aligned} Tf := T(f \circ r)|_{TO} \colon Tr(TU) = TO &\to TO' \\ (x, \xi) &\mapsto (f(x), Df(x)\xi) \end{aligned}$$

of some, hence by Lemma 2.2.10 any, decompression $f \circ r$. Here $f \circ r(x) = f(x)$ since $O = \operatorname{Fix} r$ and $Df(x) := D(f \circ r)|_x$. Section 2.2 on sc-retracts is rounded off by the chain rule for compositions of sc-smooth retract maps.

Section 2.3 "M-polyfolds and their tangent bundles" defines M-polyfolds in analogy to Banach manifolds just using the rather general class of sc-retracts as local models and requiring only scale smoothness of the transition maps. In particular, to define an *M-polyfold* $\boldsymbol{X}$ one starts with a paracompact Hausdorff space X. E.g. sc-manifolds are M-polyfolds ($r = id$ and $O = U$) and so are open subsets of M-polyfolds. Sc-smoothness of maps

$$f : X \to Y$$

between M-polyfolds is defined in terms of local coordinate representatives of f which are required to be sc-smooth retract maps. An M-polyfold X inherits a set scale structure from the local model spaces. Let X_m, level m, consist of all points of X which are mapped in some, hence any, coordinate chart into level m of model space. Each level X_m inherits the structure of an M-polyfold denoted by X^m.

To construct the *tangent bundle* $\boldsymbol{p\colon TX \to X^1}$ one first defines TX as a set and then a natural map p, using the local coordinate charts $\phi\colon V \to O$ of X to define bijections denoted by $T\phi\colon TV \to TO$. Given an atlas $\mathcal{A}$ of X, these bijections induce the collection $\mathcal{B} = \{TV\}_{\phi\in\mathcal{A}}$ of subsets of TX which actually forms a basis of a paracompact Hausdorff topology. Endowing X with that topology the bijections $T\phi$ become homeomorphisms and one gets a natural M-polyfold atlas for TX.

Sub-M-polyfolds. A subset $A \subset X$ of an M-polyfold is a *sub-M-polyfold* if around any point $a \in A$ there is an open neighborhood $V \subset X$ and an sc-smooth retraction $r = r^2\colon V \to V$ such that $A \cap V = r(V) = \operatorname{Fix} r$. Such r is called a *local generator* for the sub-M-polyfold A. Viewed as a map $r : V \to A$ a local generator is sc-smooth and $T_a r(T_a X) = T_a A$ at any point $a \in A \cap V$. At smooth points the tangent space $T_a A$ is sc-complemented in $T_a X$.

Boundaries and corners – tameness. Recall from (1.7.26) that the degeneration index $k = d_C(p)$ of a point p of a partial quadrant C tells whether p is an interior point ($k = 0$), a boundary point ($k = 1$), or a corner point of complexity $k \geqslant 2$. Unfortunately, for points x of M-polyfolds X the degeneration index $d_X(x) := d_C(\phi(x))$ defined in terms of an M-polyfold chart ϕ may depend on the chart; see Figure 2.4. Thus one introduces a new class, the so-called *tame* M-polyfolds, for which there is no dependence on ϕ.

Section 2.4 "Strong bundles over M-polyfolds" provides the environment to implement partial differential operators whose zero sets will represent the moduli spaces which are under investigation in many different geometric analytic situations. Often moduli spaces, hence zero sets, are of finite dimension and are modeled on the kernels of surjective Fredholm operators. To achieve surjectivity in a given geometric PDE scenario one usually perturbs some already present, but inessential, quantity. These perturbations should be related to bounded operators, so the overall Fredholm property is preserved.
Recall from Proposition 1.3.29 that the sc-Fredholm property of a linear map $T\colon E\to F$ is preserved under addition of sc$^+$-operators $S\colon E\to F$. The latter operators are characterized by improving their output quality by one level, that is $S(E_m)\subset F_{m+1}$. As a consequence all level operators $S_m\colon E_m\to F_{m+1}\hookrightarrow F_m$ are compact.

Motivation. Replacing now the linear domain E by an M-polyfold X as domain of a partial differential operator f of order, say ℓ, the task at hand[2] is to construct vector bundles $P\colon Y\to X$ with fibers modeled on an sc-Banach space F, so that the differential operator becomes a section $f\colon X\to Y$. Concerning the implementation of Fredholm properties one has to allow for fiber level shifts by $+1$, that is all fibers $Y_x:=P^{-1}(0)$ should be identifiable with the sc-Banach space $F^0=F$, as well as with the shifted one F^1; cf. Remark 1.3.12. In practice, the level indices m correspond to the degree of differentiability of the level elements. So the domain of f should be $X^{\ell+m}$ in which case f takes values in level m, sometimes even $m+1$. Then one can exploit composition with *compact* embeddings up to level 0; see Remark 2.4.2. This motivates the following asymmetric double scale structure which must be subsequently reduced to two versions of individual scales, in order to be accessible to scale calculus (there is no double scale calculus).

Strong trivial-bundle retracts K – the local models. The *non-symmetric product* $U\triangleright F$ is the subset $U\times F$ of the Banach space $E\oplus F$ endowed with the *double scale*, also called double filtration, defined by

$$(U\triangleright F)_{m,k}:=U_m\oplus F_k,\qquad m\in\mathbb{N}_0,\quad k\in\{0,\dots m+1\}.$$

Projection onto the first component

$$U\triangleright F\to E,\quad (u,\xi)\mapsto u$$

[2]freely borrowed from one of my favorite authors

is called the *strong trivial bundle projection*. However, for sc-calculus one needs one scale structure, not a double scale. Consider the sc-manifolds

$$(U \triangleright F)^{[0]} := U \oplus F$$

and

$$(U \triangleright F)^{[1]} := U \oplus F^1.$$

For $i \in \{0, 1\}$ projection on component one $p = p^{[i]} \colon (U \triangleright F)^{[i]} \to U$ is an sc-smooth map among sc-manifolds called a *strong trivial bundle*. A *strong trivial-bundle retraction* is an idempotent strong trivial bundle map[3]

$$\begin{aligned} R = R \circ R \colon U \triangleright F &\to U \triangleright F \\ (u, \xi) &\mapsto (r(u), \rho_u \xi) \end{aligned}$$

The first component r is necessarily an sc-smooth retraction on U, called *associated base retraction*. Its image $O = r(U)$ is the *associated base retract*. A *strong trivial-bundle retract*[4] $(K, C \triangleright F, E \triangleright F)$ is the image

$$K := R(U \triangleright F) = (\operatorname{Fix} R) \subset (O \triangleright F)$$

of a strong trivial bundle retraction $R = R \circ R$ on $U \triangleright F$ where $O = r(U)$ is the associated base retract. One likewise calls the natural surjection

$$p \colon K \to O, \quad (x, \xi) \mapsto x$$

a *strong trivial-bundle retract*. Call $K := R(U \triangleright F)$ *tame* if R is tame. As a subset of the doubly scaled space $U \triangleright F$ there is an induced double scale

$$\begin{aligned} K_{m,k} &:= K \cap (U_m \oplus F_k) \\ &= \bigcup_{x \in O_m} (\{x\} \oplus \operatorname{Fix}[\rho_x \colon F_k \to F_k]) \end{aligned}$$

for $m \in \mathbb{N}_0$ and $k \in \{0, \dots m+1\}$. The spaces

$$K^{[i]} := K \cap \left(E^0 \oplus F^i\right) = \operatorname{im} R^{[i]} = R(U \triangleright F)^{[i]}, \qquad i = 0, 1$$

[3] i.e. double scale preserving and with $\rho_u \xi := \rho(u, \xi)$ being linear in ξ

[4] 'strong' indicates 'doubly scaled' and the retraction acts on a 'trivial bundle'

with levels $K^{[i]}_m = K_{m,m+i}$ are sc-retracts, hence M-polyfolds. The surjections

$$\begin{aligned} p = p^{[i]}\colon K^{[i]} &\to O, \qquad\qquad i = 0,1 \\ (x,\xi) &\mapsto x \end{aligned}$$

are sc-smooth maps between sc-retracts.

A *section* of a strong trivial-bundle retract $p\colon K \to O$ is a map $s\colon O \to P$ that satisfies $p \circ s = \mathrm{id}_O$. If s is sc-smooth as an sc-retract map

$$s^{[i]}\colon O \to K^{[i]}, \quad x \mapsto \bigl(x, \mathbf{s}^{[i]}(x)\bigr), \qquad \mathbf{s}^{[i]}\colon O \to F^i$$

it is called in case $i = 0$ an **sc**-*section* and in case $i = 1$ an **sc**$^+$-*section*. The map $\mathbf{s}^{[i]}\colon O \to F^i$ is called the *principal part* of the section.

Strong bundles. A *strong bundle over an M-polyfold X* is a continuous surjection $P : Y \to X$ defined on a paracompact Hausdorff space Y such that each pre-image $Y_x := P^{-1}(x)$ is a Banachable space, together with an equivalence class of strong bundle atlases.
As usual, one patches together local model bundles which in our case are the strong trivial-bundle retracts $K = R(U \triangleright F)$ outlined above. A strong bundle atlas for $P : Y \to X$ consists of suitably compatible *strong bundle charts*

$$\bigl(\Phi, P^{-1}(V), (K, C \triangleright F, E \triangleright F)\bigr).$$

Such tuple consists of

- a strong trivial-bundle retract K, that is $p\colon K = (U \triangleright F) \to O$ where $O = r(U)$ is the associated base retract;

- a homeomorphism $\varphi\colon X \supset V \to O$ between an open subset of the base M-polyfold X of Y and the base retract O of K;

- a homeomorphism $\Phi\colon P^{-1}V \to K$ which covers φ in the sense that the diagram

$$\begin{tikzcd} Y \arrow[d, "P"'] & \supset & P^{-1}(V) \arrow[r, "\Phi"] \arrow[d, "P"'] \arrow[dr, phantom, "\circlearrowleft"] & K = R(U \triangleright F) \arrow[d, "p"] \\ X & \supset & V \arrow[r, "\varphi"'] & O = r(U) \end{tikzcd}$$

commutes. Consequently, for every point $v \in V$ the restriction of Φ to $P^{-1}(v)$ takes values in $p^{-1}(\varphi(v))$. It is also required that Φ as a map

$$\Phi\colon Y_v = P^{-1}(v) \xrightarrow{\simeq} p^{-1}(\varphi(v)) = \rho_{\varphi(v)}(F), \qquad \forall v \in V$$

is a continuous linear bijection between fibers.

A strong bundle atlas $\mathcal{A}_X^Y$ for $P\colon Y \to X$ provides a double scale structure on X induced by local charts. As earlier one extracts two individual scale structures and obtains two *induced* **sc**-*bundle atlases* $\mathcal{A}_X^{Y[0]}$ and $\mathcal{A}_X^{Y[1]}$ for sc-bundles[5]

$$P^{[0]}\colon Y^{[0]} \to X, \qquad P^{[1]}\colon Y^{[1]} \to X.$$

A *section* of a strong bundle $P\colon Y \to X$ is a map $s\colon X \to Y$ that satisfies $P \circ s = \mathrm{id}_X$. If s is sc-smooth as a map between M-polyfolds

$$s^{[i]}\colon X \to Y^{[i]}$$

then s is called in case $i = 0$ an **sc**-*section* of $P\colon Y \to X$ and in case $i = 1$ an $\mathbf{sc}^+$-*section* of $P\colon Y \to X$.

2.1 Cartan's last theorem

In the realm of continuous linear operators R on a Banach space E an idempotent $R = R^2$ is called a *projection*. Note that the image $\operatorname{im} R = \operatorname{Fix} R$ is equal to the fixed point set of R. (Both inclusions are immediate, only '$\subset$' uses idempotency.) But the image of a linear operator is a linear subspace and the fixed point set of a continuous map is a closed subset. So the image of a projection is a closed linear subspace which, furthermore, is complemented by the (again due to continuity) closed linear subspace $\ker R$. To summarize

$$R^2 = R \in \mathcal{L}(E) \quad \Rightarrow \quad E = \ker R \oplus \operatorname{im} R = \ker R \oplus \operatorname{Fix} R.$$

More generally, given a topological space X, a continuous idempotent map $r = r \circ r\colon X \to X$ is called a *retraction on* $\boldsymbol{X}$ and the closed subset

$$\operatorname{im} r = \operatorname{Fix} r \subset X$$

is called a *retract of* $\boldsymbol{X}$.

[5] The definition of sc-bundles is sketched around (2.4.7).

Theorem 2.1.1 (Cartan (1986)). *The image of a smooth retraction $r\colon X \to X$ on a Banach manifold is a topologically closed smooth submanifold of X.*

Proof. We follow Cieliebak (2018). Closedness of the set $\operatorname{im} r = \operatorname{Fix} r$ holds by continuity of r. To be a submanifold is a local property. Pick $x \in \operatorname{Fix} r$ and a Banach chart (V, ϕ, E) about x with $\phi(x) = 0$; cf. Section A.2.4. It suffices to show that $\operatorname{Fix} r$ is locally near x the image under a diffeomorphism, say α^{-1}, of an open subset of a linear subspace, say $\operatorname{Fix}[R\colon E \to E]$ for some $R = R^2 \in \mathcal{L}(E)$, of the Banach space E. This takes three steps.

Step 1. (Localize) The retraction r on X descends to a smooth retraction on an open subset $U \subset E$ of the local model Banach space, still denoted by

$$r = r^2\colon E \supset U \to U, \quad r(0) = 0, \quad U := \phi(V \cap r^{-1}(V)).$$

The derivative $R := dr(0) = R^2 \in \mathcal{L}(E)$ is a projection and the maps

$$\alpha, \beta\colon U \to E, \quad \alpha := \beta + R \circ r, \quad \beta := (\mathbb{1} - R) \circ (\mathbb{1} - r)$$

take on the same value $\alpha(0) = \mathbb{1} - R = \beta(0)$ at the origin.

Proof of Step 1. Observe that $V \cap r^{-1}(V)$ is not only an open neighborhood of the fixed point $r(x) = x \in V$, but it is also invariant under r: Indeed

$$r\left(V \cap r^{-1}(V)\right) \subset (r(V) \cap V) \subset \left(V \cap r^{-1}(V)\right) \tag{2.1.1}$$

where both inclusions are immediate, only the second one uses $r \circ r = r$. Hence the local representative $\phi \circ r \circ \phi$, hereafter still denoted by r, is a smooth retraction on U and it maps $0 = \phi(x)$ to itself. The latter fixed point property enters the identity $R = dr|_0 = d(r \circ r)|0 = dr_{r(0)} \circ dr|_0 = R \circ R$.

Step 2. (Local diffeomorphism) The map α conjugates r and R

$$\alpha \circ r = R \circ \alpha\colon E \supset U \to E$$

and it holds that $\alpha(0) = 0$ and $d\alpha(0) = \mathbb{1}$.

Proof of Step 2. The retraction properties of r and R imply the identities

$$\beta \circ r = (\mathbb{1} - R) \circ (\mathbb{1} - r) \circ r = (\mathbb{1} - R) \circ (r - r^2) = 0$$

and

$$R \circ \beta = R \circ (\mathbb{1} - R) \circ (\mathbb{1} - r) = (R - R^2) \circ (\mathbb{1} - r).$$

These two identities imply, respectively, the identities

$$\alpha \circ r = \beta \circ r + R \circ r \circ r = R \circ r$$

and

$$R \circ \alpha = R \circ \beta + R \circ R \circ r = R \circ r.$$

Thus $\alpha \circ r = R \circ \alpha$. Hence $\alpha(0) = \alpha(r(0) = R(\alpha(0)) = R(\mathbb{1} - R) = 0$ and

$$\begin{aligned} d\alpha(0) &= d\left((\mathbb{1} - R) \circ (\mathbb{1} - r) + R \circ r\right)|_0 \\ &= (\mathbb{1} - R) \circ (\mathbb{1} - R) + R \circ R \\ &= \mathbb{1} - 2R + R^2 + R^2 \\ &= \mathbb{1}. \end{aligned}$$

Step 3. (Conjugation to linearization) There is an open subset $W \subset U$ of E such that $\alpha\colon W \to E$ is a diffeomorphism onto its image $\alpha(W)$ and $r(W) \subset W$. Moreover, the linear retraction $R = dr(0)\colon E \to E$ restricts to a smooth retraction on W and coincides with the composition

$$R = \alpha \circ r \circ \alpha^{-1}\colon \alpha(W) \to W \to r(W) \subset W \to \alpha(W).$$

Proof of Step 3. Since $d\alpha(0) = \mathbb{1}$ is invertible there is by the inverse function theorem an open neighborhood $W' \subset U$ of the fixed point $0 \in E$ of α and r such that the restriction $\alpha\colon W' \to E$ is a diffeomorphism onto its image. To obtain, in addition, invariance under r replace W' by $W := W' \cap r^{-1}(W')$. To see this repeat the arguments that led to (2.1.1).

Step 4. (Diffeomorphism to open set in Banach space) Step 3 shows

$$\begin{aligned} \operatorname{Fix}[r\colon W \to W] &= \alpha^{-1}\left(\operatorname{Fix}[R\colon \alpha(W) \to \alpha(W)]\right) \\ &= \alpha^{-1}\left(\alpha(W) \cap \operatorname{Fix}[R\colon E \to E]\right). \end{aligned}$$

Step 4 proves Theorem 2.1.1: Indeed $\alpha(W)$ is an open neighborhood in E of the fixed point 0 of r and Fix R is a (closed) linear subspace of E. So the intersection is an open neighborhood of 0 in the Banach space Fix R. But that intersection is diffeomorphic, under α^{-1}, to the part of Fix r in the open set W. □

2.2 Sc-smooth retractions and their images O

In this section the local model spaces for M-polyfolds are constructed and the maps between them are endowed with an adequate notion of sc-smoothness, namely, sc-smoothness when viewed as maps between decompressed domains. The model spaces are images O of sc-smooth retractions $r = r^2 \colon U \to U$ on sc-triples (U, C, E). It is useful to observe that image and fixed point set of r coincide and to think of r as a projection onto its image

$$r = r \circ r \colon U \to O := \operatorname{im} r = \operatorname{Fix} r.$$

Sc-retracts and sc-smoothness of maps between them

Definition 2.2.1 (Sc-retracts O)**.** An *sc-smooth retraction* on an sc-triple (U, C, E) is an sc-smooth idempotent map $r = r \circ r \colon U \to U$. Note that

$$r \circ r = r \qquad \Leftrightarrow \qquad \operatorname{im} r = \operatorname{Fix} r.$$

$\operatorname{Fix} r \subset U$ is (relatively) closed by continuity of r. An *sc-retract* $O \subset C \subset E$ in a partial quadrant C in a Banach scale E is the image (fixed point set)

$$O = r(U) = \operatorname{Fix} r, \qquad r \circ r = r \colon U \to U$$

of <u>some</u> sc-smooth retraction r whose domain $U \subset C$ is (relatively) open. Usually we abbreviate the notation (O, C, E) of an **sc**-*retract* by simply writing O. As pointed out in Hofer, Wysocki, and Zehnder (2017, before Prop. 2.3), the ambient partial quadrant $C \subset E$ matters, because it is possible that O is an sc-retract with respect to some non-trivial C, but not for $C = E$: Think of (O, E, E) as local models for M-polyfolds in regions without boundary and (O, C, E) as such near boundaries with corners; cf. Hofer, Wysocki, and Zehnder (2010, after Def. 1.13).

Lemma 2.2.2. *If $r \colon U \to U$ is an* sc-*smooth retraction, then all level maps are* continuous *retractions*

$$r_m = r_m \circ r_m \colon U_m \to U_m, \qquad U_m := U \cap E_m$$

and $O_m := O \cap E_m$ is equal to the image $r(U_m)$. In terms of shifted scales

$$O^k = r(U^k) = \operatorname{Fix}\,[r \colon U^k \to U^k], \qquad k \in \mathbb{N}_0. \tag{2.2.2}$$

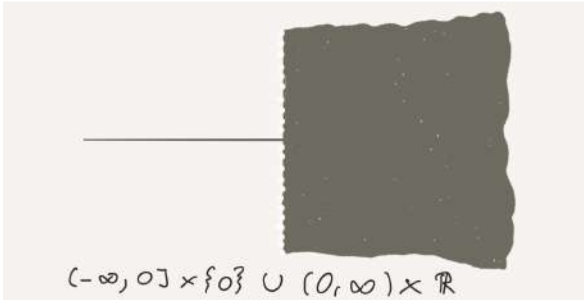

Figure 2.1: Jumping dimension along sc-retract $O = \operatorname{im} r_\pi$ in Section 2.2.2

Proof. To be shown is the equality of sets $O_m = r(U_m)$. '$\subset$' Pick $x \in O \cap E_m \subset U \cap E_m$, then $x = r(x) \in r(U_m)$. '$\supset$' Pick $x \in U_m$, then $r(x) \in r(U) \cap E_m = O \cap E_m$ since $U_m \subset U$ and r is level preserving, respectively. $\square$

Whereas the image of a *smooth* retraction on a Banach manifold is a smooth submanifold by Cartan's last theorem, Theorem 2.1.1, an sc-retract can be connected and nevertheless have pieces of various dimensions; see Figure 2.1. How can one ever do analysis on such spaces? Let's see:

Decompression. To start with, given a map $f\colon O \to O'$ between sc-retracts, one can "decompress" or "unpack" the, possibly "cornered", domain O of the map f into an open set by pre-composing with an sc-smooth retraction $r\colon U \to U$ whose image is $O = r(U)$. Indeed the map $f \circ r\colon U \to O' \subset U'$ has the same image as f, but lies within the reach of sc-calculus since domain and target are (relatively) open subsets of partial quadrants C and C' in sc-Banach spaces.

Definition 2.2.3 (Sc-smooth maps among sc-retracts – decompress domain)**.** A map
$f\colon O \to O'$ between sc-retracts is called an **sc**-*smooth retract map* if the composition $f \circ r\colon U \to U'$ is sc-smooth[6] for some, thus by Lemma 2.2.4 for every, sc-smooth retraction r whose image is $O = r(U)$. Let us refer to such precomposition process as *decompressing (the domain of)* $\boldsymbol{f}$. Sc-smooth retract maps $O \to O'$ are continuous.

Lemma 2.2.4. *Given* sc-*smooth retractions with equal image* $O = r(U) = s(V)$

[6]Sc-smoothness of $f \circ r\colon U \to U'$ implies continuity of $f\colon O \to O'$.

and a map $f : O \to U'$, *then if one of the maps*

$$f \circ r : U \to U', \qquad f \circ s : V \to U'$$

is sc-*smooth, so is the other one.*

Proof. By assumption $\operatorname{im} r = O = \operatorname{Fix} s$ and $\operatorname{im} s = O = \operatorname{Fix} r$, hence

$$s \circ r = r : U \to O, \qquad r \circ s = s : V \to O.$$

By hypothesis s is sc-smooth. If also $f \circ r$ is sc-smooth, so is by the chain rule their composition $(f \circ r) \circ s$. But $f \circ (r \circ s) = f \circ s$. □

Tangent bundle of sc-retracts and tangent map of sc-retract maps

Lemma 2.2.5 (Tangent map of retraction is itself a retraction)**.** *Let* $r : U \to U$ *be an* sc-*smooth retraction on an* sc-*triple* (U, C, E). *Then its tangent map*

$$Tr = (Tr) \circ (Tr) : TU \to TU, \quad (x, \xi) \mapsto (r(x), Dr(x)\xi) \tag{2.2.3}$$

is an sc-*smooth retraction on the* tangent **sc**-triple

$$T(U, C, E) := (TU, TC, TE) := \left(U^1 \oplus E^0, C^1 \oplus E^0, E^1 \oplus E^0\right).$$

Proof. The tangent map of an sc-smooth map is sc-smooth by the iterative definition of sc^k-smoothness; see Definition 1.4.15. It remains to show that

$$\operatorname{im} Tr = \operatorname{Fix} Tr.$$

'$\supset$' A fixed point $x = f(x)$ of a map lies in its image. '$\subset$' An element of $Tr(TU)$ is of the form $(y, \eta) = (r(x), Dr|_x \xi)$ for some $(x, \xi) \in U_1 \oplus E^0$. Hence $r(y) = r \circ r(x) = r(x) = y$ and

$$Dr|_y \eta = Dr|_{r(x)} Dr|_x \xi = D(r \circ r)|_x \xi = Dr|_x \xi = \eta$$

where we used the chain rule (1.6.23). Hence $Tr(y, \eta) = (y, \eta) \in \operatorname{Fix} Tr$. □

Lemma 2.2.6 (Tangent maps of two decompressions have same image)**.** *Assume an* sc-*smooth retract* (O, C, E) *is the image of two* sc-*smooth retractions*

$$r(U) = O = s(V), \qquad r = r \circ r : U \to U, \quad s = s \circ s : V \to V.$$

Then both tangent maps have equal image $Tr(TU) = Ts(TV) \subset (O^1 \oplus E^0)$.

Proof. We need to show $\operatorname{Fix} Tr = \operatorname{Fix} Ts$. '$\subset$' Pick $(x,\xi) \in \operatorname{Fix} Tr = Tr(TU) \subset TU \subset E^1 \oplus E^0$. Then $\xi \in \operatorname{Fix} Dr|_x \subset E$ and with (2.2.2) for r and s we conclude

$$x \in (\operatorname{Fix} r) \cap E^1 = \operatorname{Fix}[r\colon U^1 \to U^1] = O^1 = \operatorname{Fix}[s\colon V^1 \to V^1] \subset V^1.$$

So $(x,\xi) \in V^1 \oplus E^0 = TV$ lies in the domain of Ts and we get that

$$\begin{aligned} Ts(x,\xi) = (s(x), Ds|_x\,\xi) &= \bigl(s(r(x)), Ds|_{r(x)}\, Dr|_x\,\xi\bigr) \\ &= (r(x), D(s\circ r)|_x\,\xi) \\ &= (x, Dr|_x\,\xi) \\ &= (x,\xi). \end{aligned}$$

Here we used twice the identity $s \circ r = r$ which holds since $r(U) = O = \operatorname{Fix} s$ by hypothesis. '$\supset$' Same argument. $\square$

Definition 2.2.7 (Tangent of sc-retract)**.** If $O = r(U)$ is an sc-retract, then

$$TO := Tr(TU) = \operatorname{Fix} Tr \subset \bigl(O^1 \oplus E^0\bigr)$$

is an sc-retract, too. Notation $T(O,C,E) := (TO, TC, TE)$. The definition of TO does not depend on the choice of (r,U) by Lemma 2.2.6.

Lemma 2.2.8 (Tangent bundle of sc-retract O)**.** *The natural projection*

$$p\colon TO \to O^1, \quad (x,\xi) \mapsto x$$

is an open surjective sc*-smooth retract map, cf. Definition 2.2.3, called* tangent bundle of the **sc**-retract **O**. *The pre-image of a point, denoted by*

$$T_xO := p^{-1}(x) \subset E, \qquad x \in O^1$$

is a Banach subspace of E, an sc*-subspace whenever $x \in O_\infty$.*

Proof. Let $O = r(U)$. Then $TO := Tr(TU)$ where $TU := U^1 \oplus E^0$. Hence the first component of Tr is the map r on the domain U^1, see (2.2.3). But $r(U^1) = O^1$ by Lemma 2.2.2 which proves surjectivity of p. The decompression

$$p \circ Tr\colon TU = U^1 \oplus E^0 \to TO \to U^1, \quad (x,\xi) \mapsto (r(x), Dr|_x\,\xi) \mapsto r(x)$$

of p is constant in ξ, and in x it is the map $r\colon U^1 \to U^1$ which is sc-smooth by Lemma 1.5.5, since $r\colon U \to U$ is sc-smooth by assumption. The pre-image

$$T_xO := p^{-1}(x) = \{x\} \oplus \operatorname{Fix}[Dr(x)\colon E \to E] \subset O^1 \oplus E$$

is the fixed point set of a linear operator on the Banach space E and therefore it is a linear subspace. It is a closed linear subspace, because the linear operator is continuous. For simplicity we shall simply write

$$T_x O := p^{-1}(x) = \mathrm{Fix}\,[Dr(x)\colon E \to E] \subset E. \tag{2.2.4}$$

The sc-derivative $Dr(x)\colon E \to E$ at any $x \in O_\infty$ restricts to a continuous linear operator on every level E_m by Corollary 1.4.13. Hence $(p^{-1}(x))_m := p^{-1}(x) \cap E_m$ are the levels of a Banach scale by Exercise 2.2.9. □

Exercise 2.2.9. a) Show that $T_x O$ is an sc-subspace of E whenever $x \in O_\infty$. b) Show that the projection $p\colon TO \to O^1$ is an open map.

[Hint: a) Let $O = r(U)$. Show that $F_m := (p^{-1}(x))_m := p^{-1}(x) \cap E_m$ equals $F_m = \mathrm{Fix}\,[Dr(x)\colon E_m \to E_m]$ and $F = F_0 \supset F_1 \supset \dots$ satisfies the three axioms `(Banach levels)`, `(compactness)`, and `(density)` of a Banach scale. b) First consider the case $C = E$, decompress p.]

Lemma 2.2.10. *Let $f\colon O \to O'$ be an* sc-*smooth retract map. If $r\colon U \to U$ and $s\colon V \to V$ are* sc-*smooth retractions with image O, then the restrictions*

$$T(f \circ r)|_{TO} = T(f \circ s)|_{TO}\colon TO \to TO'$$

a) coincide and b) take values in TO' and c) are sc-*smooth retract maps.*

Proof. a) For $(x, \xi) \in TO = \mathrm{Fix}\, Ts$, as $r \circ s = s$ ($\mathrm{Fix}\, r = O = \mathrm{im}\, s$), we get

$$T(f \circ r)\,(x, \xi) = T(f \circ r)\, Ts(x, \xi) = T(f \circ r \circ s)\,(x, \xi) = T(f \circ s)\,(x, \xi).$$

b) Let $O' = t(W)$, then it suffices to show $\mathrm{im}\, T(f \circ r) \subset \mathrm{Fix}\, Tt$. Observe that $t \circ f = f$ since $\mathrm{im}\, f \subset O' = \mathrm{Fix}\, t$. Hence $(x, \xi) \in TU$ provides a fixed point

$$Tt\,(T(f \circ r)\,(x, \xi)) = T(t \circ f \circ r)\,(x, \xi) = T(f \circ r)\,(x, \xi).$$

c) The decompression of $T(f \circ r)|_{TO}$ given by

$$T(f \circ r) \circ Tr = T(f \circ r \circ r) = T(f \circ r)\colon TU \to TU'$$

is sc-smooth, because $f \circ r\colon U \to U'$ is sc-smooth due to the assumption that $f\colon O \to O'$ is an sc-smooth retract map, see Definition 2.2.3. □

Definition 2.2.11 (Tangent of retract maps via domain decompression)**.** The *tangent map* of an sc-smooth retract map $f\colon O \to O'$ is the restriction

$$Tf := T(f \circ r)|_{TO}\colon Tr(TU) = TO \to TO', \quad (x, \xi) \mapsto (f(x), Df(x)\xi)$$

of the tangent map $T(f \circ r)\colon TU \to TU'$ for a decompression r of $O = r(U)$.

Some remarks are in order. Firstly, by Lemma 2.2.10 the definition of Tf does not depend on the sc-smooth retraction $r\colon U \to U$ with image O. Secondly, concerning component one $f \circ r(x) = f(x)$ since $x \in O^1 \subset O = \operatorname{Fix} r$. Thirdly, component two actually abbreviates

$$Df(x) := D(f \circ r)|_x\colon T_x O \to T_{f(x)} O'.$$

Theorem 2.2.12 (Chain rule for sc-smooth retract maps)**.** *Let $f\colon O \to O'$ and $g\colon O' \to O''$ be* sc-*smooth retract maps. Then the composition $g \circ f\colon O \to O''$ is also a* sc-*smooth retract map and the tangent maps satisfy*

$$T(g \circ f) = Tg \circ Tf\colon TO \to TO''.$$

Proof. Sc-smoothness of the retract maps f and g by definition means sc-smoothness of $f \circ r$ and $g \circ r'$ where $r\colon U \to U$ and $r'\colon U' \to U'$ are sc-smooth retractions with images O and O', respectively. The inclusion $\operatorname{im} f \subset O' = Fix\, r'$ provides the identity $f = r' \circ f$. Hence $(g \circ f) \circ r = (g \circ r') \circ (f \circ r)$ is a composition of two sc-smooth maps, so it is sc-smooth itself by the chain rule for sc-smooth maps, Theorem 1.6.1. By definition of Tf, the chain rule, and $f = r' \circ f$ we get $Tg \circ Tf = T(g \circ r') \circ T(f \circ r)|_{TO} = T(g \circ f \circ r)|_{TO} =: T(g \circ f)$. $\square$

In Section 2.3 the next exercise will be useful a) to show that open subsets of M-polyfolds are M-polyfolds and b) to construct sub-M-polyfold charts.

Exercise 2.2.13. Given an sc-retract (O, C, E), prove the following.

a) Open subsets O' of the sc-retract O are sc-retracts in C.

b) Suppose V is an open subset of O and $s = s \circ s\colon V \to V$ is an idempotent sc-smooth retract map. The image of such s is an sc-retract (o, C, E).

[Hints: Let $O = \operatorname{im}[r\colon U \to U]$. a) How about $U' := r^{-1}(O')$ and $r' := r|_{U'}$? b) Let $o := \operatorname{im} s = \operatorname{Fix} s$, then $o \subset s^{-1}(o) = V \subset O = \operatorname{Fix} r$. How about $U' := r^{-1}(V) = r^{-1}(s^{-1}(o)) \subset U$ and the sc-smooth map $s \circ r\colon U' \to U'$?]

2.2.1 Special case: Splicings and splicing cores

Following Hofer, Wysocki, and Zehnder (2017, Def. 2.18), an **sc**-*smooth splicing on an* **sc**-*Banach space* $\boldsymbol{E}$ consists of the following data. A relatively open neighborhood V of 0 in a partial quadrant $[0,\infty)^\ell \times \mathbb{R}^{d-\ell}$ in $\mathbb{R}^d$ and a family $\{\pi_v\}_{v\in V}$ of sc-projections $\pi_v = \pi_v \circ \pi_v \in \mathcal{L}_{\mathrm{sc}}(E)$ such that the map

$$\pi\colon \mathbb{R}^d \oplus E \supset V \oplus E \to E, \quad (v,f) \mapsto \pi_v f$$

is sc-smooth. Note that in this case each projection π_v restricts to a continuous linear operator $\pi_v|_{E_m} \in \mathcal{L}(E_m)$ on every level. But in the operator norm these operators do not, in general, depend continuously on v. The subset of $\mathbb{R}^d \oplus E$ composed of the images (fixed points) of each projection, i.e.

$$K^\pi := \bigcup_{v\in V} \{v\} \times \operatorname{im}\pi_v = \{(v,f) \in V \oplus E \mid \pi_v f = f\}$$

is called the *splicing core* of the splicing.

Exercise 2.2.14 (Induced sc-smooth retraction)**.** Given an sc-smooth splicing $\{\pi_v\}_{v\in V\subset\mathbb{R}^d}$ on an sc-Banach space E, consider the map given by

$$r_\pi\colon V \oplus E \to V \oplus E, \quad (v,f) \mapsto (v, \pi_v f)\,.$$

Show that the map r_π defines an sc-smooth retraction on the sc-triple $(V \oplus E, ([0,\infty)^\ell \times \mathbb{R}^{d-\ell}) \oplus E, \mathbb{R}^d \oplus E)$ and that its image is the splicing core K^π.

As remarked in Fabert et al. (2016, previous to Def. 5.6), this setup of splicing with finitely many "gluing" parameters covers the sc-retractions relevant for Morse theory and holomorphic curve moduli spaces.

2.2.2 Splicing core with jumping finite dimension

Fix a smooth bump function $\beta \geqslant 0$ supported in $[-1,1]$ and of unit L^2 norm. For $t > 0$ consider the family $\beta_t(s) := \beta(s + e^{1/t})$ of left translates of β by $e^{1/t}$ – huge left translations for t near 0 and almost no translation for $t \sim \infty$. Fix a strictly increasing sequence of reals δ_m starting at $\delta_0 = 0$ and let $E = L^2(\mathbb{R})$ be the sc-Hilbert space whose levels are given by the weighted Sobolev spaces $E_m := W^{m,2}_{\delta_k}(\mathbb{R})$ introduced in Exercise 1.2.4. Consider the family

$$\{\pi_t\colon E \to E\}_{t\in\mathbb{R}}, \qquad \pi_t u := \begin{cases} 0 & , t \leqslant 0 \\ \langle u, \beta_t\rangle\, \beta_t & , t > 0 \end{cases}$$

of linear operators on E. Note that the image of π_t is $\{0\}$ whenever $t \geqslant 0$, whereas for each $t > 0$ the image of π_t is $\mathbb{R}\beta_t$, hence one dimensional.

Exercise 2.2.15. Check that each linear operator $\pi_t \colon E \to E$ is continuous and a projection, that is $\pi_t \circ \pi_t = \pi_t$.

Proposition 2.2.16. *The map* $\pi \colon \mathbb{R} \oplus E \to E$, $(t, f) \mapsto \pi_t f$, *is* sc-*smooth.*

Proof. The result and details of the (hard) proof of sc-smoothness are given in Hofer, Wysocki, and Zehnder (2010, Ex. 1.22 and Lemma 1.23); see also Cieliebak (2018, Prop. 6.8). □

To summarize, the family $\{\pi_t\}_{t\in\mathbb{R}}$ of projections defines an sc-smooth splicing on $E = L^2$. The corresponding splicing core $K^\pi \subset \mathbb{R} \oplus E$ is represented in Figure 2.1 as a subset of $\mathbb{R}^2$ homeomorphic to $K^\pi = \operatorname{im} r_\pi = r_\pi(\mathbb{R} \oplus E)$. Although connected, there are parts of dimension one and two.

2.3 M-polyfolds and their tangent bundles

M-polyfolds are defined analogous to sc-manifolds, just use as local models instead of sc-triples (U, C, E) sc-retracts $O = \operatorname{im}[r = r^2 \colon U \to U]$ in $C \subset E$.

Recall two standard methods to define manifolds. *Method 1* starts with a topological space X, then one defines a collection of homeomorphisms to open sets in model Banach spaces, whose domains are open subsets of X which together cover X. The collection must be suitably compatible on overlaps. *Method 2* starts with *only a set* S, then one defines a collection of bijections between subsets of S onto open subsets of local model Banach spaces, again the domains together must cover X. Now one uses the bijections to define a topology on the set S, essentially by declaring pre-images of open sets in model space to be open sets in S.

In practice one often employs Method 1 to define a manifold X. Then one employs Method 2 in order to define the tangent bundle TX. Namely, *as a set* called TX of equivalence classes whose definition utilizes the manifold charts of X and their tangent maps. The latter are used to define the required bijections that endow the set TX of equivalence classes with a topology.

M-polyfolds and maps between them

Definition 2.3.1. Let X be a topological space. An *M-polyfold chart* $(V, \phi, (O, C, E))$, often abbreviated (V, ϕ, O), consists of

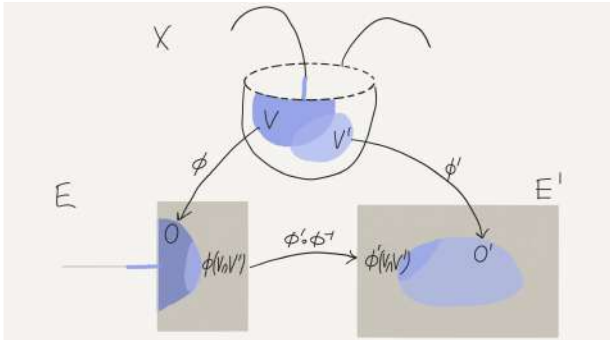

Figure 2.2: Transition map between M-polyfold charts of M-polyfold X

- an open set V in X;
- an sc-retract $O = r(U) = \operatorname{Fix} r$ in a partial quadrant $C \subset E$;
- a homeomorphism $\phi\colon V \to O$ (open sets in sc-retracts are sc-retracts).

Two M-polyfold charts are **sc**-*smoothly compatible* if the *transition map*

$$\psi := \phi' \circ \phi^{-1}\colon O \supset \phi(V \cap V') \to \phi'(V \cap V') \subset O'$$

and its inverse are both sc-smooth retract maps[7] (i.e. sc-smooth after domain decompression). An *M-polyfold atlas for* ***X*** is a collection $\mathcal{A}$ of pairwise sc-smoothly compatible M-polyfold charts $\phi\colon V \to O$ whose domains cover X. Two atlases are called *equivalent* if their union is again an M-polyfold atlas.

Definition 2.3.2. An *M-polyfold* is a paracompact Hausdorff space X endowed with an equivalence class of M-polyfold atlases.

Definition 2.3.3. A map $f\colon X \to X'$ between M-polyfolds is called an **sc**-*smooth M-polyfold map* if every local M-polyfold chart representative

$$\phi' \circ f \circ \phi^{-1}\colon O \supset \phi(V \cap f^{-1}V') \to \phi'(V') \subset O'$$

of f is an sc-smooth retract map. An **sc**-*smooth diffeomorphism* between M-polyfolds is a bijective sc-smooth map between M-polyfolds whose inverse is also sc-smooth.

[7] Indeed open subsets of sc-retracts are sc-retracts by Exercise 2.2.13.

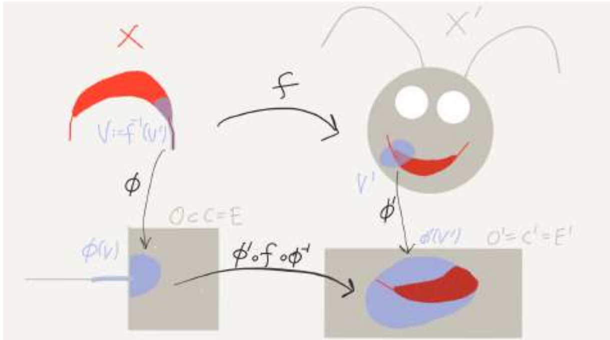

Figure 2.3: Freedom of speech among M-polyfolds – local representative ;-)

Exercise 2.3.4. a) Sc-manifolds are M-polyfolds.
b) Open subsets of M-polyfolds are M-polyfolds.
c) Check that X and X' in Figure 2.3 are M-polyfolds.
d) Use the sc-retract $O = \operatorname{im} r_\pi$ in Figure 2.1 to built further fun M-polyfolds.
e) It is an open problem, Hofer, Wysocki, and Zehnder (2017, Quest. 4.1), whether there is an sc-smooth retract (O, E, E) so that O is homeomorphic to the letter T in $\mathbb{R}^2$.

Definition 2.3.5. One defines *level* $\boldsymbol{m}$ of an M-polyfold X to be the set X_m that consists of all points $x \in X$ which are mapped to level m in some, hence any,[8] M-polyfold chart.

Thus for an M-polyfold X there is the nested sequence of levels

$$X = X_0 \supset X_1 \supset \cdots \supset X_\infty := \bigcap_{m \geqslant 0} X_m .$$

Each level X_m inherits the structure of an M-polyfold, notation X^m, see Hofer, Wysocki, and Zehnder (ibid., p. 21) for charts, and each inclusion $X_{m+1} \hookrightarrow X_m$ is continuous (as a map between topological spaces), see Hofer, Wysocki, and Zehnder (ibid., Lemma 2.1).

Construction of the M-polyfold tangent bundle

THE BASE M-POLYFOLD. Let X be an M-polyfold, in particular, a paracompact Hausdorff space, with M-polyfold atlas $\mathcal{A} = \{(V_i, \phi_i, (O_i, C_i, E_i))\}_{i \in I}$.

[8]transition maps are sc-smooth, thus level preserving

THE TANGENT BUNDLE AS A SET. By definition TX is the set of equivalence classes of tuples $(x, V, \phi, (O, C, E), \xi)$, abbreviated (x, ϕ, ξ), that consist of

- a point $x \in X_1$ on level 1;
- an M-polyfold chart $\phi \colon X \supset V \to O = r(U) \subset C \subset E$ for X about x;
- a tangent vector $\xi \in T_{\phi(x)} O = \operatorname{Fix}[Dr|_{\phi(x)} \in \mathcal{L}(E)]$, see (2.2.4).

Two tuples $(x, V, \phi, O, \xi) \sim (x', V', \phi', O', \xi')$ are said *equivalent* if

$$x = x', \qquad Dr(\phi' \circ \phi^{-1})|_{\phi(x)} \xi = \xi'.$$

THE NATURAL PROJECTION. There is a *natural projection*

$$p \colon TX \to X^1, \quad [x, \phi, \xi] \mapsto x. \tag{2.3.5}$$

The pre-image of any point $x \in X^1$, denoted by

$$T_x X := p^{-1}(x)$$

and called the *tangent space of* $\boldsymbol{X}$ *at* $\boldsymbol{x}$, is a linear space over the reals:

$$\lambda[x, \phi, \xi] + \mu[x, \phi, \eta] := [x, \phi, \lambda\xi + \mu\eta], \qquad \lambda, \mu \in \mathbb{R}.$$

To represent the two input equivalence classes choose the same M-polyfold chart ϕ about x for both of them (choose any two representatives and restrict to the intersection of their domains). This way ξ and η are both in the same vector space, here $T_{\phi(x)} O$, and so adding them makes sense.

THE INDUCED BIJECTIONS. Every M-polyfold chart $\phi \colon V \to O$, say with $O = \operatorname{im} r = \operatorname{Fix} r$ for some sc-smooth retraction r on an sc-triple (U, C, E). Then the map named and defined by

$$T\phi \colon TV := p^{-1}(V \cap X_1) \to TO = \operatorname{Fix} Tr, \quad [x, \phi, \xi] \mapsto (\phi(x), \xi) \tag{2.3.6}$$

is a bijection. The restriction of $T\phi$ to level 1 points $x \in V_1 := V \cap X_1$, i.e.

$$T_x\phi \colon T_x X = T_x V = p^{-1}(x) \to T_{\phi(x)} O, \quad [x, \phi, \xi] \mapsto \xi$$

is a bijection (the identity) on the Banach subspace $T_{\phi(x)} O = \operatorname{Fix} Dr|_{\phi(x)}$ of E; cf. (2.2.4). So $T_x X$ inherits the Banach space structure of $T_{\phi(x)} O$. At smooth points $T_{\phi(x)}O$ is an sc-subspace of E by Exercise 2.2.9, so the linear bijection $T_x\phi$ endows T_xX with the structure of an sc-Banach space.

THE INDUCED TOPOLOGY. Consider the collection $\mathcal{B}$ that consists of all subsets of TX that are pre-images under $T\phi$ of all open subsets in the target space TO, for all M-polyfold charts $\phi\colon V \to O$ of X, in symbols

$$\mathcal{B} := \left\{(T\phi)^{-1}W \mid (V, \phi, O) \in \mathcal{A},\, W \subset TO \text{ open}\right\} \subset 2^{TX}.$$

Exercise 2.3.6. Show that $\mathcal{B}$ is a basis for a topology; cf. Theorem A.1.14.

By definition the *topology on* $\boldsymbol{TX}$ is the topology generated by the basis $\mathcal{B}$: The open sets in TX are arbitrary unions of members of $\mathcal{B}$.

Proposition 2.3.7. *The topology on TX is Hausdorff and paracompact.*

Proof. Hofer, Wysocki, and Zehnder (2017, § 2.6.3) □

Exercise 2.3.8. The map $p\colon TX \to X^1$ in (2.3.5) is continuous and open.

THE M-POLYFOLD CHARTS. For any M-polyfold chart $\phi\colon V \to O$ of X, where $O = r(U)$ say, the bijection $T\phi\colon TV \to TO$ defined by (2.3.6) is an M-polyfold chart for TX:

- $TV \subset TX$ is open by definition of $\mathcal{B}$;
- $TO = Tr(TU)$ is an sc-retract by Definition 2.2.7;
- $T\phi\colon TV \to TO$ is a homeomorphism by definition of $\mathcal{B}$.

Furthermore, if $\phi, \phi' \in \mathcal{A}$ are compatible for X, then $T\phi, T\phi'$ are compatible for TX: We need to show that the map $(T\phi') \circ (T\phi)^{-1}\colon TO \to TO'$ given by

$$\begin{aligned}(\phi(x), \xi) &\mapsto [x, \phi, \xi] = \left[x, \phi', D(\phi' \circ \phi^{-1})_{\phi(x)}\, \xi\right] \\ &\mapsto \left(\phi'(x), D(\phi' \circ \phi^{-1})_{\phi(x)}\, \xi\right) = T(\phi' \circ \phi^{-1})\,(x, \xi)\end{aligned}$$

is an sc-smooth retract map. But $\phi' \circ \phi^{-1}$ is an sc-smooth retract map by the chain rule, Theorem 2.2.12, and so is the tangent map. This shows that an M-polyfold atlas $\mathcal{A}$ for X induces an *M-polyfold atlas for* $\boldsymbol{TX}$, namely

$$T\mathcal{A} := \{(TV, T\phi, (TO, TC, TE)) \mid (V, \phi, (O, C, E)) \in \mathcal{A}\}.$$

Let us then summarize the previous constructions and findings in form of

Theorem 2.3.9. *Let X be an M-polyfold. Then TX is an M-polyfold and*

$$p\colon TX \to X^1$$

is an sc-*smooth map between M-polyfolds.*

M-polyfold tangent maps

Definition 2.3.10. The *tangent map* of an sc-smooth M-polyfold map $f: X \to Y$ is the sc-smooth M-polyfold map defined by

$$Tf: TX \to TY, \quad [x, \phi, \xi] \mapsto \left[f(x), \psi, D(\psi \circ \phi^{-1})|_{\phi(x)}\, \xi\right]$$

where ψ is any M-polyfold chart about $f(x)$.

Exercise 2.3.11. Show that $Tf: TX \to TY$ is sc-smooth as a map between M-polyfolds. Show that for $x \in X_1$ the map

$$T_x f: T_x X \to T_{f(x)} Y, \quad v := [x, \phi, \xi] \mapsto \left[f(x), \psi, D(\psi \circ \phi^{-1})|_{\phi(x)}\, \xi\right]$$

is a continuous linear operator and $T_x f$ is an sc-operator whenever $x \in X_\infty$.

2.3.1 Sub-M-polyfolds

An M-polyfold X is locally modeled on the images O of sc-smooth retractions $r = r^2: E \supset U \to U$ in an sc-Banach space E. Thus it is natural to define a *sub-M-polyfold* of X as a subset $A \subset X$ that is locally the image of an sc-smooth retraction $r = r^2: V \to V$ acting on an open subset V of X.

Definition 2.3.12. A subset $A \subset X$ of an M-polyfold is called a *sub-M-polyfold* if around any point $a \in A$ there is an open neighborhood $V \subset X$ and an sc-smooth retraction $r = r^2: V \to V$ such that $A \cap V = r(V) = \operatorname{Fix} r$. Such r is called a *local generator* for the sub-M-polyfold A.

Proposition 2.3.13. *Suppose $A \subset X$ is a sub-M-polyfold.*

(i) *A sub-M-polyfold A inherits an M-polyfold structure from the ambient X.*

(ii) *The inclusion $\iota: A \hookrightarrow X$ is an* sc-*smooth map between M-polyfolds and a homeomorphism onto its image.*

(iii) *A local generator r for A, viewed as a map $r: V \to A$, is* sc-*smooth and $T_a r(T_a X) = T_a A$ at any point $a \in A \cap V$.*

(iv) *At points $a \in A_\infty$ the tangent space $T_a A$ is* sc-*complemented in $T_a X$.*

Proof. Hofer, Wysocki, and Zehnder (ibid., Prop. 2.6). □

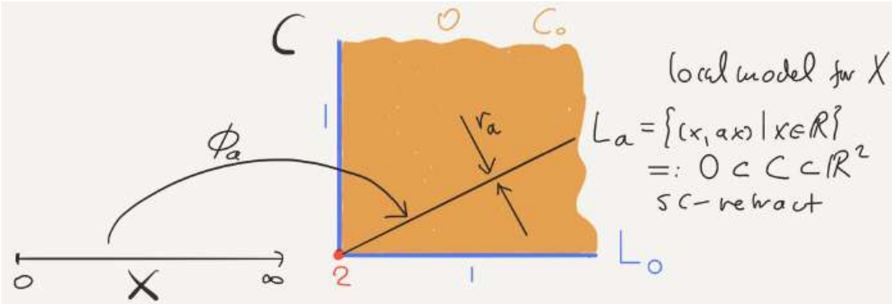

Figure 2.4: Global M-polyfold chart $\phi_a \colon X = [0,\infty) \to O = L_a \subset C$

2.3.2 Boundary and corners – tameness

Unfortunately, on M-polyfolds the degeneracy index of a point, defined through an M-polyfold chart, might depend on the choice of chart, as this example shows: For a real parameter $a \geqslant 0$ define the orthogonal projection

$$r_a = r_a \circ r_a \colon C \to C = [0,\infty)^2, \quad (x,y) \mapsto \frac{x+ay}{1+a^2}\,(1,a)\,.$$

The image of the retraction r_a is the half line $L_a = \{(x,ax) \mid x \geqslant 0\}$ in the quadrant C. On the M-polyfold $X = [0,\infty)$ we choose the global chart $\phi_a \colon X \to O = r_a(C) = L_a \subset C$ shown in Figure 2.4. In this chart the degeneracy index, see Section 1.7, of each point $x \in X$ (also depending on whether $a = 0$ or $a > 0$) is given by

$$d_C\left(\phi_a(x)\right) = \begin{cases} 2 & , x = 0, \\ 1 & , x > 0 \text{ and } a = 0\ (L_0), \\ 0 & , x > 0 \text{ and } a > 0\ (L_a). \end{cases}$$

On the other hand, representing X in the obvious global M-polyfold chart

$$\phi' = \mathrm{id} \colon X \to [0,\infty) = O' = \operatorname{im} r', \quad r' = \mathrm{id}, \quad U' = C' = [0,\infty) \subset \mathbb{R}$$

the degeneracy indices of points are the rather different, but expected, values

$$d_{C'}\left(\phi'(x)\right) = \begin{cases} 1 & , x = 0, \\ 0 & , x > 0. \end{cases}$$

Of course, the discrepancy between d_C and $d_{C'}$ could be caused by incompatibility of charts. However, this is not the case, both transition retract maps are sc-smoothly compatible. Indeed the decompression of $\psi := \phi' \circ {\phi_a}^{-1}$

$$\psi \circ r_a = \phi' \circ {\phi_a}^{-1} \circ r_a \colon C \to r_a(C) = L_a \to O' = [0, \infty), \quad (x, y) \mapsto \frac{x + ay}{1 + a^2}$$

is even C^∞ smooth and so is $\psi^{-1} \circ r' \colon [0, \infty) \to O = L_a \subset C$, $x \mapsto (x, ax)$.

Definition 2.3.14 (Degeneration index on M-polyfolds X)**.** Given a point $x \in X$, just take the minimum

$$d_X(x) := \min_\phi d_C\,(\phi(x))$$

over all M-polyfold charts $\phi \colon V \to O \subset C$ about the point x.

Degeneration index stratification of quadrant – Tameness

To see what went wrong for the chart ϕ_a in the example above note that the quadrant $C = C_0 \cup C_1 \cup C_2$ decomposes into disjoint subsets $C_i := {d_C}^{-1}(i)$, the strata of the *degeneration index stratification*. Now one identifies two problems:

a) The sc-retraction r_a does not preserve the degeneration index strata.

b) The sc-retract $r_a(C) = L_a$ is in a certain sense not transverse to the degeneration index stratification of the quadrant C.

One avoids the problem by giving a name to retractions that do not have the defects a) and b) and then considers only such in theorems.

Definition 2.3.15. An sc-smooth retraction $r \colon U \to U$ on an sc-triple (U, C, E) is called *tame* if

a) the map r preserves the d_C-stratification: $d_C(r(x)) = d_C(x)\ \forall x \in U$;

b) the image of r is transverse to the d_C-stratification: For every smooth point x in the image $r(U_\infty) = O_\infty$ there must be an sc-complement A of the sc-subspace $T_x O = Dr(x)E$ of E, cf. (2.2.4) and Exercise 2.2.9, with $A \subset E_x := T_x C(x)$ where $C(x) := {d_C}^{-1}(d_C(x))$ is the stratum of x.

If in b) above such A exists, then one can choose $A = (\mathbb{1} - Dr(x))E$ by Hofer, Wysocki, and Zehnder (2017, Prop. 2.9). So for tame sc-smooth retractions $r \colon U \to U$ with image O one has the sc-splittings

$$E = H_x \oplus V_x, \quad H_x := T_x O = Dr(x)E, \quad V_x := (\mathbb{1} - Dr(x))E, \quad x \in O_\infty$$

one for each smooth point of $O = r(U)$.

Remark 2.3.16 (Fixed origin)**.** Consider the quadrant $C := [0,\infty)^n$ in $\mathbb{R}^n$ and suppose $r\colon C \to C$ is a *tame* smooth retraction. Then the origin $0 = r(0)$ is fixed by r. Indeed $x = 0$ is the only point in C with $d_C(x) = n$. Moreover, the image $r(C)$ is an open neighborhood of 0 in C; cf. Cieliebak (2018, Problem 6.5).

Definition 2.3.17. An sc-retract (O, C, E) is called *tame* if $O = r(U)$ is the image of a tame sc-smooth retraction r. An M-polyfold is called *tame* if X admits an equivalent M-polyfold atlas modeled on tame sc-smooth retracts.

For tame M-polyfolds X the degeneration index $d_X(x) := d_C(\phi(x))$ of a point $x \in X$ defined via an M-polyfold chart $\phi\colon V \to O \subset C$ about x does not depend on the choice of the chart; see Hofer, Wysocki, and Zehnder (2017, Eq. (2.12)).

2.4 Strong bundles over M-polyfolds

We recall the notion of a vector bundle over a manifold and sketch how to generalize the base to M-polyfolds (bringing in scale and retracts) and accommodate Fredholm sections (bringing in double scales) via sc$^+$-sections.

Motivation and comparison of old and new concepts

The following overview is not meant to be, and is not, rigorous.

Classical vector bundles over manifolds – trivial bundles $U \times F \to U$.
A classical vector bundle over a manifold is locally modeled by trivial bundles $U \times F \to U$. Here U is an open subset of a linear space E, the model space of the manifold, and F is a linear space, the model of the fibers of the vector bundle. Any two local models must be related by a diffeomorphism

$$\Psi\colon U \times F \to \tilde{U} \times F, \quad (u, \xi) \mapsto (\psi(u), \mathcal{T}_u \xi)$$

called a *vector bundle transition map*, whose second component $(u, \xi) \mapsto \mathcal{T}(u, \xi)$ restricts at every point u to a vector space isomorphism $\mathcal{T}_u := \mathcal{T}(u, \cdot)\colon F \to F$. So the building blocks for classical vector bundles are *trivial bundles*

$$U \times F \to U.$$

Sc-bundles over M-polyfolds – trivial-bundle sc-retracts $R(U \oplus F) \to r(U)$.

To define sc-bundles over M-polyfolds one needs to generalize trivial bundles taking into account that now one deals with sc-triples (U, C, E) and sc-Banach spaces F. A useful notation for trivial bundles is $U \oplus F \to U$ which indicates that the set $U \times F$ sits inside the sc-direct sum $E \oplus F$ and thereby inherits the scale structure $(U \oplus F)_m = U_m \oplus F_m$.
Because local models for the base M-polyfold are sc-retracts O, one should replace U by its image $O = r(U)$ under an sc-retraction. It is suggesting to replace the whole space $U \oplus F$ by its image under an sc-retraction

$$R = R \circ R \colon U \oplus F \to U \oplus F, \quad (u, \xi) \mapsto (r(u), \rho_u \xi)$$

for which $F \to F, \xi \mapsto \rho_u \xi := \rho(u, \xi)$ is linear, at any $u \in U$. Such retraction

- produces a local M-polyfold model $O = r(U)$ in the component U and
- also respects the linear structure of the second component F.

A crucial observation is that along $O = r(U) = \operatorname{Fix} r \subset U$ idempotency of R implies that the linear map $\rho_x = (\rho_x)^2 \colon F \to F$ is also idempotent, hence a projection. Choose the identity retraction $R(u, \xi) := (u, \xi)$ and forget scale structures to recover trivial bundles $U \oplus F \to U$, hence classical vector bundles. The building blocks

$$K = R(U \oplus F) \subset U \oplus F \tag{2.4.7}$$

for sc-bundles over M-polyfolds are called *trivial-bundle sc-retracts*. Projection onto the second component provides a surjection

$$p \colon K \to O, \quad (x, \xi) \mapsto x$$

onto an sc-retract, the local model of an M-polyfold. Each pre-image $p^{-1}(x)$ is a closed linear subspace $\rho_x(F) = \operatorname{Fix} \rho_x$ of F.
To summarize, the local models for sc-bundles over M-polyfolds, called *trivial-bundle sc-retracts*, are sc-retracts K in $U \oplus F$ that are families

$$K = R(U \oplus F) = \bigcup_{x \in O} (\{x\} \oplus \operatorname{Fix} \rho_x) \;\to\; O$$

of projection images $\operatorname{Fix} \rho_x$ in F parametrized by local M-polyfold models O. To construct **sc**-*bundles over M-polyfolds* one defines sc-bundle charts as in Definition 2.4.12 disregarding double scales – just replace the double scale symbol $\triangleright$ by the sc-direct sum $\oplus$. Compatibility of charts and sc-bundle atlases are defined as usual.

Definition 2.4.1 (Sc-bundles over M-polyfolds)**.** An *sc-bundle over an M-polyfold* X is an sc-smooth surjection $\pi\colon Y \to X$ between M-polyfolds endowed with an equivalence class of sc-bundle atlases.

Accommodating Fredholm sections: double scale gives two scales $^{[0]}$ *and* $^{[1]}$.
The local model building blocks for strong bundles over M-polyfolds are strong trivial-bundle retracts

$$p^{[i]}\colon K^{[i]} = R(U \triangleright F)^{[i]} \to O = r(U), \qquad i = 0, 1$$

as motivated next.

2.4.1 Strong trivial-bundle retracts - the local models

Throughout F is an sc-Banach space and (U, C, E) an sc-triple, that is U is a relatively open subset of the partial quadrant C in the sc-Banach space E.

Remark 2.4.2 (Motivation for non-symmetric product and shift one)**.** At first sight the introduction of a double scale/filtration in Definition 2.4.3, even an asymmetric one, and its immediate reduction to a single scale in Definition 2.4.4, in two versions though, might be confusing and even appear superfluous given that the two versions inherit their scale structure as subsets of the simple and well known sc-direct sums $U \oplus F$ and $U \oplus F^1$.

a) To perceive the need to shift the vector space part of $U \oplus F$ by one, recall stability of the sc-Fredholm property under addition of sc$^+$-operators; see Proposition 1.3.29.

b) In practice, when implementing a differential operator f of order ℓ the level indices m of U indicate differentiability, simply speaking. So one needs to forget the first ℓ levels and choose U_ℓ, or any sublevel of it, as domain for f. More precisely, one chooses the shifted scale U^ℓ. Then $f\colon (U^\ell)_m \to F_m$ and one can subsequently exploit composition with the compact embeddings $F_m \hookrightarrow F_{m-1} \dots \hookrightarrow F_0$.

Definition 2.4.3 (Non-symmetric product – double scale)**.** The *non-symmetric product* $U \triangleright F$ is the subset $U \times F$ of the Banach space $E \oplus F$ endowed with the *double scale*, also called *double filtration*, defined by[9]

$$(U \triangleright F)_{m,k} := U_m \oplus F_k, \qquad m \in \mathbb{N}_0, \quad k \in \{0, \dots m+1\}.$$

[9]We use the symbol $U \triangleright F$, as opposed to $U \triangleleft F$, since the levels of U are unlimited, any $m \in \mathbb{N}_0$ is allowed, whereas the ones of F depend on m and are restricted to $0, \dots, m+1$.

Non-symmetric products $U \triangleright F$ serve as *total spaces* of strong trivial bundles. Projection onto the first component

$$U \triangleright F \to E, \quad (u, \xi) \mapsto u$$

is called the *strong trivial bundle projection*. However, for sc-calculus one needs one scale structure, not a double scale. To achieve this substitute k by a useful function of m, say $k = m$ or $k = m + 1$.

Definition 2.4.4 (Strong trivial bundles: Two relevant scale structures)**.** Motivated by
Hofer, Wysocki, and Zehnder (2017, §2.5) we denote the sc-manifolds $U \oplus F$ and $U \oplus F^1$ by the symbols

$$(U \triangleright F)^{[0]} := U \oplus F, \qquad (U \triangleright F)^{[1]} := U \oplus F^1.$$

By definition of shifted scales the levels are[10]

$$(U \triangleright F)^{[0]}_m = U_m \oplus F_m, \qquad (U \triangleright F)^{[1]}_m = U_m \oplus F_{m+1}.$$

The projections onto the first component

$$p = p^{[i]} \colon (U \triangleright F)^{[i]} \to U, \quad (u, \xi) \mapsto u, \qquad i = 0, 1$$

are sc-smooth maps between sc-manifolds called *strong trivial bundles*. We often write p for simplicity and because the values do not depend on the choice of shift i for the second component F. If the domain matters we shall write $p^{[i]}$.

Definition 2.4.5 (Morphisms of strong trivial bundles)**.** a) A *strong trivial bundle map* $\Psi \colon U \triangleright F \to \widetilde{U} \triangleright \widetilde{F}$ is a map that *preserves the double scale* and is of the form

$$\Psi(u, \xi) = (\varphi(u), \Gamma(u, \xi))$$

where $\Gamma_u \xi := \Gamma(u, \xi)$ is linear in ξ. Moreover, it is required that both induced maps between sc-manifolds

$$\Psi = \Psi^{[i]} \colon (U \triangleright F)^{[i]} \to (\widetilde{U} \triangleright \widetilde{F})^{[i]}, \quad i = 0, 1$$

are sc-smooth. b) A *strong trivial bundle isomorphism* is an invertible strong trivial bundle map whose inverse is also a strong trivial bundle map.

[10] cf. Definition 1.1.6

It is the previous definition where double scale preservation is required.

Exercise 2.4.6. Check that the second component Γ of a strong trivial bundle map Ψ gives rise to an sc-operator $\Gamma_u \in \mathcal{L}_{\mathrm{sc}}(F, \widetilde{F})$ along the smooth points $u \in U_\infty$.

Definition 2.4.7 (Strong trivial bundle retraction)**.** A *strong trivial bundle retraction* is an idempotent strong trivial bundle map

$$R = R \circ R \colon U \triangleright F \to U \triangleright F, \quad (u, \xi) \mapsto (r(u), \rho_u \xi).$$

The first component r is necessarily an sc-smooth retraction on U, called the *associated base retraction*, whose image $O = r(U)$ is called the *associated base retract*. One calls R *tame* in case the associated base retraction is tame.

Exercise 2.4.8. Let $R(u, \xi) = (r(u), \rho_u \xi)$ be a strong trivial bundle retraction. Check that r is an sc-smooth retraction on U and that $\rho_x \in \mathcal{L}(F)$ is a projection for $x \in \operatorname{Fix} r = r(U) =: O$, even an sc-projection at smooth points, i.e. $x \in O_\infty$.

Definition 2.4.9 (Strong trivial bundle retracts – the local models)**.** A *strong trivial bundle retract*[11] $(K, C \triangleright F, E \triangleright F)$, or simply K, is given by the image

$$K = R(U \triangleright F) = (\operatorname{Fix} R) \subset (O \triangleright F)$$

of a strong trivial bundle retraction $R = R \circ R$ on $U \triangleright F$ where $O = r(U)$ is the associated base retract. One likewise calls the natural surjection

$$p \colon K \to O, \quad (x, \xi) \mapsto x$$

strong trivial bundle retract. For simplicity we identify point pre-images

$$p^{-1}(x) = \{x\} \times K_x, \qquad K_x := \rho_x(F)$$

with the Banach subspace $K_x := \rho_x(F)$ of F, an sc-subspace for smooth points, called the *fiber of* $\boldsymbol{K}$ over x. Call $K = R(U \triangleright F)$ *tame* if R is tame.

Sometimes we write *strong trivial-bundle retract* to emphasize that it is a trivial bundle that gets retracted – and not a bundle retract being trivial in whatever sense. Being a subset of the doubly scaled space $U \triangleright F$ a strong trivial bundle retract K inherits the double scale

$$K_{m,k} := K \cap (U_m \oplus F_k) = \bigcup_{x \in O_m} (\{x\} \oplus \operatorname{Fix}[\rho_x \colon F_k \to F_k])$$

[11] 'strong' indicates 'doubly scaled' and the retraction acts on a 'trivial bundle'

for $m \in \mathbb{N}_0$ and $k \in \{0, \dots m+1\}$. Note that the spaces[12]

$$K^{[i]} := K \cap \left(E^0 \oplus F^i\right) = \operatorname{im} R^{[i]} = R(U \triangleright F)^{[i]}, \qquad i = 0, 1 \qquad (2.4.8)$$

with levels $K_m^{[i]} = K_{m,m+i}$ are sc-retracts, so M-polyfolds. The surjections

$$p = p^{[i]} \colon K^{[i]} \to O, \quad (x, \xi) \mapsto x, \qquad i = 0, 1$$

are both sc-smooth maps between sc-retracts. Indeed by Definition 2.2.3 this requires that some, hence any, decompression, say

$$p \circ R \colon (U \triangleright F)^{[i]} \to K^{[i]} \to O, \quad (u, \xi) \mapsto r(u), \qquad i = 0, 1$$

be sc-smooth. But the associated base retraction r is sc-smooth by assumption.

Definition 2.4.10 (Strong retract maps)**.** A map F of the form

$$F \colon K \to \widetilde{K}, \quad (x, \xi) \mapsto (f(x), \phi_x \xi), \qquad f \colon O \to \widetilde{O}$$

between strong trivial-bundle retracts is called a *strong retract map* if F is linear in the fibers, that is $\phi_x \colon K_x \to \widetilde{K}_{f(x)}$ is linear, if F preserves the double filtrations, and if both induced maps between sc-retracts

$$f^{[i]} \colon K^{[i]} \to \widetilde{K}^{[i]}, \qquad i = 0, 1$$

are sc-smooth (meaning sc-smoothness after decompression).

Definition 2.4.11 (sc- and sc$^+$-sections of strong trivial-bundle retracts)**.** A *section* of a strong trivial bundle $p \colon K \to O$ is a map $s \colon O \to P$ that satisfies $p \circ s = \mathrm{id}_O$. If s is sc-smooth as an sc-retract map

$$s^{[i]} \colon O \to K^{[i]}, \quad x \mapsto \left(x, \mathbf{s}^{[i]}(x)\right), \qquad \mathbf{s}^{[i]} \colon O \to F^i$$

it is called in case $i = 0$ an **sc**-*section* and in case $i = 1$ an **sc**$^{\mathbf{+}}$-*section*. The map $\mathbf{s}^{[i]} \colon O \to F^i$ is called the *principal part* of the section.

Note that a section is sc-smooth iff its principal part is. For simplicity we sometimes omit the superscript$^{[i]}$ if the level shift is clear from the context.

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[12] the symbol $R(U \triangleright F)^{[i]}$ abbreviates $R((U \triangleright F)^{[i]})$

2.4.2 Strong bundles

Throughout F is an sc-Banach space and (U,C,E) an sc-triple, that is U is a relatively open subset of the partial quadrant C in the sc-Banach space E.

Definition 2.4.12 (Strong bundle charts)**.** Let $P: Y \to X$ be a continuous surjection from a paracompact Hausdorff space Y onto an M-polyfold X such that every pre-image $Y_x := P^{-1}(x)$ has the structure of a Banachable space.[13] A *strong bundle chart* for $P: Y \to X$ is a tuple

$$\left(\Phi, P^{-1}(V), (K, C \triangleright F, E \triangleright F)\right)$$

that consists of

- a strong trivial-bundle retract $p\colon K = (U \triangleright F) \to O$ where $O = r(U)$ is the associated base retract;
- a homeomorphism $\varphi\colon V \to O$ between an open subset of the base M-polyfold X of Y and the base retract O of K;
- a homeomorphism $\Phi\colon P^{-1}V \to K$ that covers φ, that is the diagram

$$\begin{tikzcd} Y & \supset & P^{-1}(V) \arrow[r, "\Phi"] \arrow[d, "P"'] \arrow[dr, phantom, "\circlearrowleft"] & K = R(U \triangleright F) \arrow[d, "p"] \\ X \arrow[from=u, "P"'] & \supset & V \arrow[r, "\varphi"'] & O = r(U) \end{tikzcd}$$

  commutes. As a consequence, for every point $v \in V$ the restriction of Φ to $P^{-1}(v)$ takes values in $p^{-1}(\varphi(v))$. One also requires that Φ viewed as a map

$$\Phi\colon Y_v = P^{-1}(v) \xrightarrow{\simeq} p^{-1}(\varphi(v)) = \rho_{\varphi(v)}(F), \qquad \forall v \in V$$

  is a toplinear isomorphism[14] between fibers.[15]

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[13] A *Banachable space* is an equivalence class that consists of all Banach spaces with pairwise equivalent norms.

[14] a continuous linear bijection (making sense although the domain is just Banachable)

[15] Strictly speaking, the target is $\{\varphi(v)\} \times \rho_{\varphi(v)}(F)$.

Definition 2.4.13 (Strong bundle atlases)**.** Two strong bundle charts are called *compatible* if, firstly, the transition map

$$\Psi := \widetilde{\Phi} \circ \Phi^{-1} \colon K \supset \Phi(P^{-1}(V \cap \widetilde{V})) \to \widetilde{\Phi}(P^{-1}(V \cap \widetilde{V})) \subset \widetilde{K}$$

is a strong retract map, thus preserves the double scales, and, secondly, the two induced maps $\Psi^{[0]}$ and $\Psi^{[1]}$ between open subsets of sc-retracts (cf. (2.4.8)), hence M-polyfolds, are sc-smooth diffeomorphisms. A *strong bundle atlas* $\mathcal{A}_X^Y$ consists of pairwise compatible strong bundle charts covering Y. Two such atlases are called equivalent if their union is again a strong bundle atlas.

Definition 2.4.14 (Strong bundles over M-polyfolds)**.** A *strong bundle over an M-polyfold* X is a continuous surjection $P \colon Y \to X$ from a paracompact Hausdorff space equipped with an equivalence class of strong bundle atlases.

Exercise 2.4.15 (A strong bundle provides two M-polyfolds)**.** Check that a strong bundle atlas $\mathcal{A}_X^Y$ for $Y \to X$ naturally provides two M-polyfold atlases $\mathcal{A}_{Y^{[0]}}$ and $\mathcal{A}_{Y^{[1]}}$ for M-polyfolds $Y^{[0]}$ and $Y^{[1]}$, respectively.

Exercise 2.4.16 (A strong bundle provides two sc-bundles)**.** A strong bundle atlas $\mathcal{A}_X^Y$ for $P \colon Y \to X$ naturally provides two *induced* **sc**-*bundle atlases* $\mathcal{A}_X^{Y^{[0]}}$ and $\mathcal{A}_X^{Y^{[1]}}$ for sc-bundles $P^{[0]} \colon Y^{[0]} \to X$ and $P^{[1]} \colon Y^{[1]} \to X$, respectively.

Induced double scale and section types

A strong bundle $P \colon Y \to X$ carries an asymmetric double scale structure $Y_{m,k}$, where $m \in \mathbb{N}_0$ and $k = 0, \dots, m+1$, transmitted from the local models K by the strong bundle charts. Here it enters that the transition maps Ψ are strong retract maps, thus preserve the double scale of the local models K.

Definition 2.4.17 (sc- and sc$^+$-sections of strong bundles)**.** A *section* of a strong bundle $P \colon Y \to X$ is a map $s \colon X \to Y$ that satisfies $P \circ s = \mathrm{id}_X$. If s is sc-smooth as a map between M-polyfolds

$$s^{[i]} \colon X \to Y^{[i]}$$

then s is called in case $i = 0$ an **sc**-*section* of $P \colon Y \to X$ and in case $i = 1$ an **sc$^+$**-*section* of $P \colon Y \to X$.

Pull-back bundle

Suppose $f : Z \to X$ is an sc-smooth map between M-polyfolds and $P : Y \to X$ is a strong bundle over X. The pull-back bundle $P_f : f^*Y \to X$ consists of the subset of $Z \times Y$ defined by

$$f^*Y := \{(z, y) \in Z \times Y \mid P(y) = f(z)\}$$

and projection $P_f(z, y) = P_1(z, y) = z$ onto the first component. Together with projection onto the second component denoted by P_2 the diagram

$$\begin{tikzcd} Z \times Y & \supset & f^*Y \arrow[r, dashed, "P_2"] \arrow[d, dashed, "P_f"', "P_1"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "P"] \\ & & Z \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Exercise 2.4.18 (Induced strong bundle structure)**.** Given an sc-smooth map $f : Z \to X$ between M-polyfolds, show that a strong bundle structure on $P : Y \to X$ induces naturally a strong bundle structure on the pull-back bundle $f^*P : f^*Y \to Z$.

A Background from Topology and Functional Analysis

A.1 Analysis on topological vector spaces

All vector spaces will be over the real numbers $\mathbb{R}$. Let us first repeat

Some basics about sets

The elements of a set S are often called *points*. If a set S contains only finitely many elements it is called *finite*. The number of elements of a finite set is denoted by $|S|$. The set with no element is called the *empty set*, denoted by $\emptyset$ or, in order to indicate the ambient universe S, by $\emptyset_S$. We avoid terminology like *a set of sets*, instead we shall speak of a *family of sets* or of a *collection of sets*. Let 2^S be the collection of all subsets of S. The empty set $\emptyset$ is a subset of any set S, in symbols $\emptyset \subset S$ or $\emptyset \in 2^S$. Our use of $\subset$ allows for equality, otherwise we write $\subsetneq$. For more basics on set theory and logic see e.g. Munkres (2000, Ch. I). See also Ch. I, in particular I.9 on axiomatics, in Dugundji (1966).

Definition A.1.1. Let S be a set. Given a family $\mathcal{A} \subset 2^S$ of subsets A of S, union and intersection of the members of $\mathcal{A}$ are the subsets of S defined by

$$\bigcup \mathcal{A} = \bigcup\{A \mid A \in \mathcal{A}\} := \bigcup_{A\in\mathcal{A}} A := \{x \in S \mid \exists A \in \mathcal{A}\colon x \in A\} \subset S$$

and

$$\bigcap \mathcal{A} = \bigcap\{A \mid A \in \mathcal{A}\} := \bigcap_{A\in\mathcal{A}} A := \{x \in S \mid \forall A \in \mathcal{A}\colon x \in A\} \subset S.$$

Exercise A.1.2. For $\mathcal{A} = \emptyset, \{\emptyset_S\} \subset 2^S$ show $\bigcup\{\emptyset_S\} = \emptyset_S = \bigcap\{\emptyset_S\}$, but

$$\bigcup \emptyset = \emptyset_S, \qquad \bigcap \emptyset = S, \qquad \text{where } \emptyset \subset 2^S.$$

[Hint: Final assertion – empty truth.]

Maps and exponential law

Suppose A, B, C are sets. A *map* $\boldsymbol{f}$ *from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$, in symbols $f\colon A \to B$, is determined by a subset $G(f) \subset A \times B$ such that for each domain element $a \in A$ the set $\{b \in B \mid (a, b) \in G(f)\}$ has precisely 1 element. The unique $b \in B$ such that $(a, b) \in G(f)$ is denoted by $f(a)$ and called the *image* of a under f. The set A is the *domain* of f and B the *codomain* or the *target*. The subset $G(f) \subset A\times B$ is called the *graph* of f. A *function* is a map $f\colon A \to \mathbb{R}$ that takes values in the set of real numbers $\mathbb{R}$.

Let $\mathrm{Map}(A, B)$, or B^A, denote the *set of all maps* from A to B. Motivated by the exponential notation the bijection

$$\Lambda\colon C^{A\times B} \to \left(C^B\right)^A, \quad f \mapsto F, \qquad F(a)(b) := (F(a))\,(b) := f(a,b)$$

is called the *exponential map* or the *exponential law*.

A.1.1 Topological spaces

For an elementary overview see e.g. Munkres (2000, Ch. II), for an exhaustive treatment Dugundji (1966), we also found extremely useful Müger (2016).

Definition A.1.3 (Topology)**.** A *topology* on a set M is a family $\mathcal{T}$ of subsets $U \subset M$, called the *open sets*, such that the following axioms hold.

(i) Both the empty set $\emptyset$ and M itself are open.

(ii) Arbitrary unions of open sets are open.

(iii) Finite intersections of open sets are open.

Such pair $(M, \mathcal{T})$ is called a *topological space*. The complements $\circ U := X \setminus U$ of the open sets form the family of *closed sets*.

Exercise A.1.4. The intersection of a collection of topologies is a topology.

A topology $\mathcal{T}$ on a set M induces on any subset $A \subset M$ a topology $\mathcal{T}^{\cap A}$ which consists of the intersections of A with all the members of the family $\mathcal{T}$ of subsets of M. The topology $\mathcal{T}^{\cap A}$ is called the *subset topology* or the *induced topology* on a subset A. A *subspace* is a subset of a topological space endowed with the subset topology.

Properties of topological spaces that are inherited by subspaces are called *hereditary properties*.

A topological space is called *compact* if every open cover admits a finite subcover. A subset K of a topological space $(M, \mathcal{T})$ is called *compact* if the topology on K induced by $\mathcal{T}$ is compact. A subset is called *pre-compact* if its closure is compact.

One often writes, instead of the pair $(M, \mathcal{T})$, simply M and calls it a topological space. An *open neighborhood* of a subset $P \subset M$ is an open set U that contains P, in symbols $P \subset U \in \mathcal{T}$. Any subset $A \subset M$ that contains an open neighborhood of P is called a *neighborhood* of P. If $P = \{x\}$ is a point set we speak of a neighborhood of a point $x \in M$. It is convenient to write U_x to indicate that a set U contains the point x. With this convention “for any open neighborhood U of x” becomes “*for any open* $\boldsymbol{U_x}$”.

Basis of a given topology

Definition A.1.5 (Basis)**.** Given a sub-collection $\mathcal{C} \subset \mathcal{T}$ of a topology, let

$$\boxed{\mathcal{T}_{\mathcal{C}} := \{\textstyle\bigcup \sigma \mid \sigma \subset \mathcal{C}\} \subset \mathcal{T} \subset 2^M} \qquad \text{(recall } \textstyle\bigcup \sigma \subset M\text{)}$$

be the collection of all unions of elements $C \subset M$ of $\mathcal{C}$. If a sub-collection $\mathcal{B} \subset \mathcal{T}$ satisfies $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, i.e. if all open sets are unions of elements of $\mathcal{B}$, one calls $\mathcal{B}$ a *basis of the topology* $\boldsymbol{\mathcal{T}}$ and says that *the topology is generated by* $\boldsymbol{\mathcal{B}}$.

The elements of a basis $\mathcal{B}$ are called *basic open sets*. Any open set is a union of basic ones. Uniqueness of a basis fails as badly, as existence is trivial: Given $\mathcal{T}$, pick $\mathcal{B} := \mathcal{T}$. Often in practice, the smaller a basis, the better. So a criterion for being a basis is desirable.

Lemma A.1.6. *For a subset $\mathcal{C} \subset \mathcal{T}$ of a topology the following are equivalent:*

(i) The collection $\mathcal{C}$ is a basis of $\mathcal{T}$, in symbols $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$.

(ii) The collection $\mathcal{C}$ is dominated by $\mathcal{T}$ *in the following sense: Each point $x \in U \in \mathcal{T}$ of an open set also lies in a collection member $C \in \mathcal{C}$ that is contained in U, in symbols $x \in C \subset U$.*

Proof. See e.g. Dugundji (1966, p. III.2). □

Definition A.1.7 (Sub-basis)**.** For a sub-collection $\mathcal{S} \subset \mathcal{T}$ of a topology, let

$$\boxed{\mathcal{B}_{\mathcal{S}} := \{\bigcap \sigma \mid \sigma \subset \mathcal{S}, |\sigma| < \infty\} \subset \mathcal{T} \subset 2^M}$$

be the collection of all finite intersections of elements of $\mathcal{S}$. If $\mathcal{B}_{\mathcal{S}}$ is a basis of $\mathcal{T}$, i.e. if all open sets are arbitrary unions of finite intersections of elements of $\mathcal{S}$, one calls $\mathcal{S}$ a *sub-basis of the topology* $\boldsymbol{\mathcal{T}}$ and $\mathcal{B}_{\mathcal{S}}$ the *basis generated by* $\boldsymbol{\mathcal{S}}$.

Definition A.1.8. A topological space is called *second countable* if it admits a countable basis. This property is hereditary.

Definition A.1.9. A subset of a topological space is called *dense* if it meets (has non-empty intersection with) every non-empty open set or, equivalently, if its closure is equal to the whole space. A topological space is called *separable* if it admits a dense sequence (countable subset). Separability is *not* hereditary.

Exercise A.1.10. Show that second countability is hereditary, whereas separability is not, that second countable implies separable and that in metric spaces (endowed with the metric topology $\mathcal{T}_d$) the converse is true, too.

Definition A.1.11 (Local basis)**.** Let $(M, \mathcal{T})$ be a topological space and $x \in M$. A collection $\mathcal{B}(x)$ of open neighborhoods B_x of x is called a *local basis of the topology at* $\boldsymbol{x}$ if every open neighborhood U_x of x contains a member of $\mathcal{B}(x)$, in symbols $U_x \supset B_x \in \mathcal{B}(x)$.

Exercise A.1.12. Let $(M, \mathcal{T})$ be a topological space.

(i) Given a basis $\mathcal{B}$ of $\mathcal{T}$, for every $x \in M$ the family $\mathcal{U}(x) = \{U \in \mathcal{T} \mid x \in U\}$ of all open neighborhoods of x is a local basis of $\mathcal{T}$ at x.

(ii) Vice versa, given for every point x of M a local basis $\mathcal{B}(x)$ for $\mathcal{T}$ at x, show that their union $\mathcal{B} := \bigcup_{x \in M} \mathcal{B}(x) = \{B \mid B \in \mathcal{B}(x),\ x \in M\} \subset \mathcal{T} \subset 2^M$ forms a basis of $\mathcal{T}$.

From sets to topologies

Starting with just a set S, let $\mathcal{C}$ be any collection of subsets of S. The definitions above still provide collections $\mathcal{T}_\mathcal{C}, \mathcal{B}_\mathcal{C} \subset 2^S$. Note that always $\emptyset \in \mathcal{T}_\mathcal{C}$ and $M \in \mathcal{B}_\mathcal{C}$ (pick $\sigma := \emptyset \subset \mathcal{C}$). It is a natural question to ask under what conditions on $\mathcal{C}$ the collections $\mathcal{T}_\mathcal{C}$ or $\mathcal{T}_{\mathcal{B}_\mathcal{C}}$ are topologies on S.

Exercise A.1.13 (Any collection is a sub-basis of <u>some</u> topology)**.** Let S be a set and $\mathcal{S} \subset 2^S$ *any* collection of subsets. Then $\mathcal{T}_{\mathcal{B}_\mathcal{S}}$ is a topology on S, the smallest topology that contains $\mathcal{S}$, and $\mathcal{B}_\mathcal{S}$ is a basis.

[Hints: Let $\mathcal{T}^\mathcal{S}$ be the intersection of all topologies $\mathcal{T}$ containing $\mathcal{S}$ (for example $\mathcal{T} = 2^S$). Show $\mathcal{T}^\mathcal{S} = \mathcal{T}_{\mathcal{B}_\mathcal{S}}$. See e.g. Dugundji (ibid., p. III.3).]

While any collection of subsets of S is a sub-basis of some topology on S, a sufficient condition for being a basis of some topology is the following.

Theorem A.1.14 (Being a basis of <u>some</u> topology)**.** *Given a set S, let $\mathcal{B} \subset 2^S$ be a collection of subsets V of S such that*

(i) $\mathcal{B}$ is a cover of S (the union of all members of $\mathcal{B}$ is S) and

(ii) every point $p \in V_1 \cap V_2$ in an intersection of two $\mathcal{B}$ members simultaneously belongs to a $\mathcal{B}$ member $V_3 \subset V_1 \cap V_2$ contained in the intersection.

Under these conditions $\mathcal{T}_\mathcal{B}$ is a topology on S, the smallest topology containing $\mathcal{B}$, and $\mathcal{B}$ is a basis.

Proof. See e.g. Dugundji (ibid., III Thm. 3.2). □

Exercise A.1.15. Let S be a set. The three collections $\mathcal{S} = \emptyset, \{\emptyset_S\}, \{S\} \subset 2^S$ lead, respectively, to the three bases $\mathcal{B}_\mathcal{S} = \{S\}, \{\emptyset_S, S\}, \{S\} \subset 2^S$ each of which generates the trivial, also called *indiscrete* topology $\mathcal{T}_{\mathcal{B}_\mathcal{S}} = \{\emptyset_S, S\}$.

Here is another method to topologize a set S starting with a family of candidates for local bases, one candidate at each point x of the set. It is a two step process. Firstly, at every point $x \in S$ we wish to specify a collection $\mathcal{B}(x)$ of subsets $V_x \subset S$ in such a way that, secondly, we can construct a unique topology $\mathcal{T}(\mathcal{B})$ on S for which the collection $\mathcal{B}(x)$ will be a local basis at x and this is true for every $x \in S$. Since prior to step two there is no topology, hence no notion of local basis, we call $\mathcal{B}(x)$ a local pre-basis at x.

Definition A.1.16. Let S be a set and $x \in S$. Suppose $\mathcal{B} \subset 2^S$ is the union of a collection of non-empty families $\emptyset \neq \mathcal{B}(x)$ of subsets of S, one family associated to each point x of S, such that the following is satisfied at all points $x, y \in S$.

(1) Every member of $\mathcal{B}(x)$ contains x. $(\emptyset \notin \mathcal{B}(x))$

(2) The intersection $V_1 \cap V_2 \supset V_3$ of any two members of $\mathcal{B}(x)$ contains a $\mathcal{B}(x)$-member V_3.[1] ($\mathcal{B}(x)$ downward directed)

(3) For any $\mathcal{B}(x)$-member V_x each of its points y belongs to a $\mathcal{B}(y)$-member Y_y contained in V_x, i.e. any $V \in \mathcal{B}$ is a union of $\mathcal{B}$-members. $(\mathcal{B} \subset \mathcal{T}(\mathcal{B}))$

The family $\mathcal{B}(x)$ is called a *local pre-basis at* $\boldsymbol{x}$, the union $\mathcal{B} := \bigcup_{x \in M} \mathcal{B}(x)$ of all of them is called a *pre-basis on the set* $\boldsymbol{S}$. The family of subsets

$$\mathcal{T}(\mathcal{B}) := \{U \subset S \mid \text{for every } y \in U \text{ there is a } \mathcal{B}(y) \text{ member } Y_y \subset U\}$$

is called the *topology generated by the pre-basis* $\mathcal{B}$ on the set S.

Exercise A.1.17. a) Under conditions (1) and (2) show that $\mathcal{T}(\mathcal{B})$ is a topology on S.[2] From now on suppose in addition condition (3). b) Show $\mathcal{B} \subset \mathcal{T}(\mathcal{B})$. c) For each point $x \in S$ show that $\mathcal{B}(x)$ is a local basis of $\mathcal{T}(\mathcal{B})$ at x. (Hence $\mathcal{T}_{\mathcal{B}} = \mathcal{T}(\mathcal{B})$ by Exercise A.1.12 (ii), i.e. $\mathcal{B}$ is a basis of $\mathcal{T}(\mathcal{B})$.)

The conditions in Definition A.1.16 are related to the theory of filters; see e.g. Narici and Beckenstein (2011, §1.1.2). See also Narici and Beckenstein (ibid., Thm. 2.3.1).

[1] Note that (2) makes sense since any intersection $V_1 \cap V_2 \ni x$ is non-empty.

[2] While $\mathcal{T}(\mathcal{B})$ under conditions (1) and (2) is already a topology, only in combination with (3) every member of $\mathcal{B}(x)$ will be an open set – a necessary condition for a local basis.

Convergence and continuity

Definition A.1.18 (Convergence)**.** A subset sequence $(x_n) \subset M$ in a topological space is said to *converge* to a point $z \in M$, in symbols $x_n \to z$, if any open neighborhood U_z of z contains all but finitely many of the sequence members.[3]

Definition A.1.19 (Continuity)**.** A map $f : M \to N$ between topological spaces is called *continuous at a point* **x** if the pre-image of any open neighborhood $V_{f(x)}$ of the image point $f(x)$ contains an open neighborhood U_x of x. A *continuous map* is one that is continuous at every point of its domain. Let $C(M, N)$ denote the set of continuous maps from M to N.

Exercise A.1.20. a) A map $f : M \to N$ between topological spaces is continuous at x iff the pre-image of any open neighborhood $V_{f(x)}$ is open.
b) A map f is continuous iff pre-images of open sets are open.

Hausdorffness and paracompactness

A *cover* of a topological space $(M, \mathcal{T})$ is a family of subsets of M whose union is M. The members (elements) of such family are called the sets of the cover or simply the *cover sets*. A cover is called *locally finite* if every point of M admits an open neighborhood which meets (intersects) only finitely many cover sets. A cover is called a *refinement* of another cover if every member of the former is a subset of some member of the latter. A cover $\mathcal{U}$ is called *open* if every cover set is open, in symbols $\mathcal{U} \subset \mathcal{T}$.

Definition A.1.21. A topological space M is called *Hausdorff* or $\boldsymbol{T_2}$ whenever *the topology separates points*: Any two points admit disjoint open neighborhoods. Such a topology is called a *Hausdorff topology*. If the topology separates any two closed sets, then M is called *normal* or $\boldsymbol{T_4}$.

A topological space is called *paracompact* if every open cover $\mathcal{U}$ admits a locally finite open refinement $\mathcal{V}$.

Exercise A.1.22 (Hausdorff property)**.** Show the following.

a) The Hausdorff property (T_2) is hereditary, normality (T_4) is not.[4]

b) In Hausdorff spaces points and, more generally, compact sets are closed. Thus normal implies Hausdorff. $(T_4 \Rightarrow T_2)$

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[3]In symbols, there is $N \in \mathbb{N}$ such that $x_n \subset U_z$ whenever $n \geqslant N$.

[4]However, *closed* subspaces of normal spaces are normal; cf. Müger (2016, Exerc. 8.1.25).

c) In Hausdorff spaces limits are unique:

$$x_n \to y \text{ and } x_n \to z \qquad \Rightarrow \qquad y = z.$$

d) Metric spaces are normal (T_4). (With respect to the metric topology.)

[Hints: a) Counterexample T_4 Müger (2016, Cor. 8.1.47). b) Show the complement of a point is open. c) By contradiction $y \neq z$. d) Müger (ibid., Lemma 8.1.11).]

Whereas already Hausdorff by itself is useful to avoid pathological spaces like a real line with two origins, for a Hausdorff space paracompactness is equivalent to existence of a continuous partition of unity subordinate to any given open cover. For a concise presentation including proofs we recommend Cieliebak (2018, §2.2).

Surjections

Lemma A.1.23. *Let M_∞ be a dense subset of a topological space M. Then the image of M_∞ under any continuous surjection $f\colon M \twoheadrightarrow Y$ is a dense subset $f(M_\infty)$ of the target topological space Y.*

Proof. Suppose by contradiction that there is a non-empty open set $V \subset Y$ disjoint to $f(M_\infty)$. Then the *pre-image*

$$f^{-1}V := \{x \in M \mid f(x) \in V\} \subset 2^M$$

is an open subset of M by continuity of f and non-empty as f is surjective. But

$$f^{-1}V \cap M_\infty = f^{-1}(V \cap f(M_\infty)) = f^{-1}\emptyset = \emptyset$$

which contradicts density of M_∞ in M. $\square$

Compact-open topology

Let $C(M,N)$ be the set of continuous functions between topological spaces M and N. Any pair given by a compact subset $K \subset M$ and an open subset $U \subset N$ determines a collection of continuous functions

$$\mathcal{F}_{K,U} := \{f \in C(M,N) \mid f(K) \subset U\} \in 2^{C(M,N)}. \tag{A.1.1}$$

Let $\mathcal{F} = \{\mathcal{F}_{K,U}\}_{K,U} \subset 2^{C(M,N)}$ be the family of all such collections and denote by $\mathcal{T}_c := \mathcal{T}_{\mathcal{B}_\mathcal{F}}$ the associated topology on the set $C(M,N)$; cf. Exercise A.1.13.

It consists of arbitrary unions of finite intersections of elements of $\mathcal{F}$. One calls $\mathcal{T}_c$ the *compact-open topology on* $\boldsymbol{C(M,N)}$, cf. Narici and Beckenstein (2011, Ex. 2.6.9), notation

$$C_c(M,N) := (C(M,N), \mathcal{T}_c) . \tag{A.1.2}$$

Exercise A.1.24. a) Show that $C_c(M,N)$ is Hausdorff if the target N is.
b) For metric spaces (N,d) convergence in $\mathcal{T}_c$ is equivalent to uniform convergence on compact sets: Show that $f_n \to f$ in $\mathcal{T}_c$ if and only if

$$d_\infty^K(f_n, f) := \sup_{x\in K} d\,(f_n(x), f(x)) \to 0$$

for every compact subset $K \subset M$.

[Hints: a) Dugundji (1966, Ch. XII) or Müger (2016, Lemma 7.9.1). b) Cf. Proposition A.1.59.]

Remark A.1.25 (Only sub-basis)**.** In general, the collections $\mathcal{F}_{K,U}$ do not form a basis for the compact-open topology $\mathcal{T}_c := \mathcal{T}_{\mathcal{B}_\mathcal{F}}$, in symbols $\mathcal{F} \subsetneq \mathcal{B}_\mathcal{F}$. Indeed it is not necessarily true that any non-empty intersection

$$\emptyset \neq \left(\mathcal{F}_{K_1,U_1} \cap \mathcal{F}_{K_2,U_2}\right) \supset \mathcal{F}_{K,U} \neq \emptyset.$$

contains a non-empty family member $\mathcal{F}_{K,U} \in \mathcal{F}$ (let alone one that contains a given point; cf. Theorem A.1.14). Hence $\mathcal{F}$ cannot be a basis: Indeed if $\mathcal{F}$ was a basis, then the non-empty LHS was open, hence a union of members of $\mathcal{F}$ – at least one of which non-empty. We encountered two basis counterexamples on math.stackexchange.com:
Counterexample A. Let $M = N = \{a,b\}$ with the discrete topology $\mathcal{T} = 2^M$ and let $K_1 = U_1 = \{a\}$ and $K_2 = U_2 = \{b\}$. Then $\mathcal{F}_{K_1,U_1} \cap \mathcal{F}_{K_2,U_2} = \{\mathrm{id}_M\}$ contains only one element, the identity map. The inclusion $\mathcal{F}_{K,U} \subset \mathcal{F}_{K_1,U_1} \cap \mathcal{F}_{K_2,U_2}$ implies $K \supset K_1 \cup K_2 = M \neq \emptyset$, hence $K = M$. Thus non-emptiness of $\mathcal{F}_{K,U}$ requires $U \neq \emptyset$. But $\mathcal{F}_{M,U}$ is not a subset of, equivalently equal to, the singleton $\{\mathrm{id}_M\}$ in any of the three possibilities $U = \{a\}, \{b\}, \{a,b\}$.
Counterexample B. $M = N = \mathbb{R}$ with the standard topology. One can show that there are no subsets $K \subset \mathbb{R}$ compact and $U \subset \mathbb{R}$ open such that

$$\emptyset \neq \left(\mathcal{F}_{\{0,1\},(0,1)} \cap \mathcal{F}_{\{1,2\},(0,2)}\right) \supset \mathcal{F}_{K,U} \neq \emptyset$$

by constructing certain continuous functions subject to (non-linear) pointwise constraints.

A.1.2 Topological vector spaces

For topological vector spaces and, most importantly, topologies on the vector space of continuous linear maps between them we recommend the books by Rudin (1991), Schaefer and Wolff (1999, III §3), Narici and Beckenstein (2011, §2.6) (here the additive topological group is investigated first and scalar multiplication is superimposed only from Ch. 4 onward), and Treves (1967). There is a book of counterexamples by Khaleelulla (1982, CH. 2). The present section was originally inspired by the excellent Lecture Notes by Kai Cieliebak (2018).

Definition A.1.26. A *topological vector space* (TVS) is a vector space X endowed with a topology *compatible with the vector space operations* in the sense that both scalar multiplication $\mathbb{R} \times X \to X$ and addition $X \times X \to X$, are continuous maps. Also it is required that points are closed.[5]

Lemma A.1.27. *For a TVS X (without using closedness of points) it holds:*

(i) *The closure of a linear subspace is again a linear subspace.*

(ii) *Given a vector $y \in X$ and a scalar $\alpha \in \mathbb{R}$, translation $y + \cdot\colon X \to X$ and dilation $\alpha\cdot\colon X \to X$ are linear homeomorphisms. Consequence:*

Invariance under translation and dilation. If U is an open subset of X, then so are $x + U$ and tU for all $x \in X$ and $t \in \mathbb{R} \setminus \{0\}$.[6]

(iii) *Any open neighborhood V of* 0 *contains an open neighborhood U of* 0 *which is* symmetric ($U = -U$) *and fits into V "twice"* ($U + U \subset V$).

(iv) *Closed and compact subsets are separated in a strong sense. For any closed set C and any disjoint compact set K there is an open neighborhood U_0 of* 0 *such that the open neighborhoods $C + U_0$ of C and $K + U_0$ of K are still disjoint,*[7] *in symbols* $(C + U_0) \cap (K + U_0) = \emptyset$.

Proof. (i) Narici and Beckenstein (2011, Thm. 4.4.1). (ii) Narici and Beckenstein (ibid., Thm. 4.3.1). (iii) By continuity of addition and as $0 + 0 = 0 \in V$ there are open sets $W \ni 0$ and $\widetilde{W} \ni 0$ with $W + \widetilde{W} \subset V$. The open set $\widetilde{U} := W \cap \widetilde{W}$ satisfies $\widetilde{U} + \widetilde{U} \subset V$. The open set $U := \widetilde{U} \cap -\widetilde{U}$ is symmetric and $U + U \subset \widetilde{U} + \widetilde{U} \subset V$. (iv) Rudin (1991, Thm. 1.10). □

[5] Many books on topological vector spaces do not require closedness of points.
[6] Consequently the open sets containing 0 determine all open sets, hence the topology.
[7] Disjointness remains true if one takes the closure of either $C + U_0$ or of $K + U_0$.

Because of the requirement that points of a TVS are closed, part (iv) of the previous lemma applies to $C = \{x\}$ and $K = \{y\}$ and yields disjoint open neighborhoods of any two points $x \neq y$ of X. This proves

Corollary A.1.28. *A TVS is Hausdorff.*

Definition A.1.29. (i) A subset A of a TVS is called a *bounded set* if for each open neighborhood $U \subset X$ of 0 there is a constant $s > 0$ such that $A \subset tU$ is contained in the rescaled neighborhood *for all*[8] parameters $t > s$.

(ii) A linear map $T\colon X \to Y$ between topological vector spaces is called *bounded* if it takes bounded sets to bounded sets and it is called *compact* if it takes bounded sets to pre-compact sets (compact closure).

Exercise A.1.30 (Bounded sets)**.** Subsets of a bounded set are clearly bounded. If A and B are bounded sets, so are $A \cup B$, $A + B$ and αA whenever $\alpha \in \mathbb{R}$.
[Hint: If you get stuck consult Schaefer and Wolff (1999, I § 5.1).]

Lemma A.1.31. *In a TVS X compact subsets are closed and bounded, whereas the reverse holds iff* $\dim X < \infty$.

Proof. Exercise A.1.22 b) and Rudin (1991, Thm. 1.15 b)). □

Spaces of linear maps as topological vector spaces – $\mathfrak{S}$-topologies

Given topological vector spaces X and Y, the set

$$\mathcal{L}(X, Y)$$

of all continuous linear operators $T\colon X \to Y$ is a vector space under addition of two operators $T, S \in \mathcal{L}(X, Y)$, defined by $(T + S)x := Tx + Sx$, and scalar multiplication with real numbers $\alpha \in \mathbb{R}$, defined by $(\alpha T)x := \alpha Tx$, both whenever $x \in X$.

We will review the standard abstract machinery that produces various topologies on $\mathcal{L}(X, Y)$ for which both operations are continuous, see e.g. Narici and Beckenstein (2011, §11.2) or Schaefer and Wolff (1999, p. III.3). For some of them points T are closed, so the operator space $\mathcal{L}(X, Y)$ endowed with such topology is a TVS: An example is one of the most popular topologies, namely, the compact-open topology or *c-topology* on $\mathcal{L}(X, Y)$. Replacing the family of compact sets by any non-empty family of bounded sets closed under finite unions still guarantees

[8] If $A \subset tU$ for some t, isn't the inclusion automatically true for all larger values of t?

that the generated topology is compatible with addition and scalar multiplication. Hausdorffness might be lost if the sets in the family are not any more compact, but it can be recovered by assumptions on Y, e.g. being normed.
Actually all one needs are topological spaces M and N; cf. Exercise A.1.24. How one arrives at the c-topology by generalizing a natural construction which provides the point-open topology, or *p-topology*, is nicely explained in Müger (2016, § 7.9.1).

Exercise A.1.32. Let $T: X \to Y$ be a linear map between topological vector spaces. (i) Show that T is continuous iff it is *continuous at* $\mathbf{0}$, meaning that the pre-image of every open neighborhood of 0 is open. (ii) Show that continuity implies boundedness of T. (The reverse holds if the domain X is a Fréchet space.)

Let X and Y be topological vector spaces. Let $\mathfrak{S} \subset 2^X$ be a non-empty family of subsets A of X, closed under finite unions, that is

$$A_1, \dots, A_k \in \mathfrak{S} \qquad \Rightarrow \qquad A_1 \cup \dots \cup A_k \in \mathfrak{S}.$$

Examples are the families

$$\mathfrak{S}_{\mathrm{p}} \,/\, \mathfrak{S}_{\mathrm{c}} \,/\, \mathfrak{S}_{\mathrm{b}} = \{\text{all finite-}\underline{\text{p}}\text{oint / }\underline{\text{c}}\text{ompact / }\underline{\text{b}}\text{ounded subsets of } X\}.$$

Definition A.1.33 (Basic collections)**.** For $A \in \mathfrak{S} \subset 2^X$ and any element U of the *family* $\boldsymbol{\mathcal{U}_0}$ *of open neighborhoods of* $\mathbf{0}$ *in* $\boldsymbol{Y}$ consider the collection $\mathcal{B}_{A,U}$ of all continuous linear operators which map A into U, in symbols

$$\mathcal{B}_{A,U} := \{T \in \mathcal{L}(X,Y) \mid T(A) \subset U\} \in 2^{\mathcal{L}(X,Y)}, \quad A \in \mathfrak{S},\ U \in \mathcal{U}_0 \tag{A.1.3}$$

Collections of the form $\mathcal{B}_{A,U}$ are called *basic collections*.

Lemma A.1.34. *a) Any basic collection $\mathcal{B}_{A,U} \ni 0$ contains the zero operator.*
b) Any intersection $\mathcal{B}_{12} := \mathcal{B}_1 \cap \mathcal{B}_2$ of two basic collections contains one, i.e.

$$\mathcal{B}_3 \subset (\mathcal{B}_1 \cap \mathcal{B}_2) \subset \mathcal{L}(X,Y), \quad \mathcal{B}_i := \mathcal{B}_{A_i,U_i} \tag{A.1.4}$$

for some $A_3 \in \mathfrak{S} \subset 2^X$ and some open origin neighborhood $U_3 \in \mathcal{U}_0 \subset 2^Y$.
c) If $U + U \subset V$, then $\mathcal{B}_{A,U} + \mathcal{B}_{A,U} \subset \mathcal{B}_{A,V}$.
d) If $r \in \mathbb{R} \setminus \{0\}$, then $r\mathcal{B}_{A,U} = \mathcal{B}_{r^{-1}A,U} = \mathcal{B}_{A,rU}$.

Proof. Let $T, S \in \mathcal{B}_{A,U}$. a) Obvious. b) $\mathcal{B}_{A_1 \cup A_2, U_1 \cap U_2}$. c) $\forall a \in A$: $(T + S)a = Ta + Sa \in U + U \subset V$. d) $(rT)(r^{-1}A) \subset U$ and $(rT)(A) \subset rU$. $\square$

We denote by $\mathcal{B}(0)$ the *family of all basic collections*,[9] in symbols

$$\mathcal{B}(0) := \{\mathcal{B}_{A,U} \mid A \in \mathfrak{S}, U \in \mathcal{U}_0\} \subset 2^{\mathcal{L}(X,Y)}.$$

The notation reminds us that each element $\mathcal{B}_{A,U}$ of $\mathcal{B}(0)$ contains the zero operator. For $T \in \mathcal{L}(X,Y)$ let $\mathcal{B}(T) := T + \mathcal{B}(0)$ be the translated family. We denote by

$$\mathcal{B} = \mathcal{B}_{\mathfrak{S}}^{\mathcal{U}_0} := \bigcup_{T \in \mathcal{L}(X,Y)} T + \mathcal{B}(0) \quad \subset 2^{\mathcal{L}(X,Y)}$$

the family of all translated basic collections.

Theorem A.1.35. *Let X and Y be topological vector spaces. Let $\mathcal{U}_0 \subset 2^Y$ be the collection of open sets containing the origin of Y. Suppose $\mathfrak{S} \subset 2^X$ is a non-empty family of* <u>*bounded*</u>[10] *subsets A of X which is closed under finite unions. Then the following is true (not using closedness of points in X, Y).*

(loc. basis) *The family $\mathcal{B}(0)$ of all basic collections $\mathcal{B}_{A,U} \subset \mathcal{L}(X,Y)$ forms a local basis at 0 of a topology $\mathcal{T}_{\mathfrak{S}}$ on $\mathcal{L}(X,Y)$ under which addition and scalar multiplication are continuous; cf. Remark A.1.37.*

(basis) *The collection $\mathcal{B} = \mathcal{B}_{\mathfrak{S}}^{\mathcal{U}_0}$ is a basis for the topology $\mathcal{T}_{\mathfrak{S}} = \mathcal{T}_{\mathcal{B}}$ on $\mathcal{L}(X,Y)$.*

The topology $\mathcal{T}_{\mathfrak{S}}$ on $\mathcal{L}(X,Y)$, called $\boldsymbol{\mathfrak{S}}$-topology, *is*

(Hausdorff) *whenever the linear span of $\bigcup \mathfrak{S}$ is dense in X and Y is Hausdorff;*

(loc. convex) *whenever Y is.*

Proof. See e.g. Narici and Beckenstein (2011, Thm. 11.2.2). □

By Exercise A.1.57 any normed vector space Y is a *locally convex TVS*, i.e. a TVS such that any neighborhood of 0 contains a convex[11] one.
In contrast to the basis property of $\mathcal{B}$ in the *linear* setting, recall from Remark A.1.25 that in the general case of topological *spaces* even for the family of *compact* subsets the basic collections do not form a basis, only a sub-basis.

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[9] A collection of non-empty sets such that the intersection of any two of them contains another one is called a *filter base*. So $\mathcal{B}(0)$ is a filter base and so is each translate $\mathcal{B}(x)$.

[10] Boundedness leads to $\mathcal{T}_{\mathfrak{S}}$-continuity of "+" and scalar multiplication on $\mathcal{L}(X,Y)$.

[11] A subset C of a vector space is called a *convex set* if C contains every line segment $\{tx + (1-t)y \mid t \in [0,1]\}$ connecting two of its points $x, y \in C$.

Corollary A.1.36. *Let X and Y be topological vector spaces. If $\mathfrak{S}$ covers X (e.g. if $\mathfrak{S} = \mathfrak{S}_{\mathrm{p}}, \mathfrak{S}_{\mathrm{c}}, \mathfrak{S}_{\mathrm{b}}$), then $\mathcal{L}_{\mathfrak{S}}(X,Y) := (\mathcal{L}(X,Y), \mathcal{T}_{\mathfrak{S}})$ is a TVS.*

The following topologies associated to the indicated families $\mathfrak{S}$ are called

- $\mathcal{L}_{\mathrm{p}}(X,Y) := \mathcal{L}_{\mathfrak{S}_{\mathrm{p}}}(X,Y)$ point-open or *p-topology*
- $\mathcal{L}_{\mathrm{c}}(X,Y) := \mathcal{L}_{\mathfrak{S}_{\mathrm{c}}}(X,Y)$ compact-open or *c-topology*
- $\mathcal{L}_{\mathrm{b}}(X,Y) := \mathcal{L}_{\mathfrak{S}_{\mathrm{b}}}(X,Y)$ bounded-open or *b-topology*

Remark A.1.37 (Continuous vector operations)**.** Suppose X and Y are topological vector spaces. Continuity of addition and scalar multiplication under a $\mathfrak{S}$-topology is equivalent to boundedness of every image $TA \subset Y$ where $T \in \mathcal{L}(X,Y)$ and $A \in \mathfrak{S}$; see e.g. Schaefer and Wolff (1999, III §3.1) or Bourbaki (1987, III §3 Prop. 1).

Exercise A.1.38 (Families of compact sets)**.** For each of the three families $\mathfrak{S} = \mathfrak{S}_{\mathrm{p}}, \mathfrak{S}_{\mathrm{c}}, \mathfrak{S}_{\mathrm{b}}$ show the particular assertion of Theorem A.1.35 that $\mathcal{T}_{\mathfrak{S}}$ is a topology on $\mathcal{L}(X,Y)$ and $\mathcal{B} = \mathcal{B}_{\mathfrak{S}}^{\mathcal{U}_0}$ is a basis – in contrast to Remark A.1.25.
[Hints: Theorem A.1.14 or Exercise A.1.17. Lemma A.1.27 iv).]

Continuity properties

Proposition A.1.39. *Suppose M is a topological space and X, Y, Z are topological vector spaces. Then the following is true.*

a) If the map $\varphi\colon M \times Y \to Z$ is continuous and, moreover, linear in the second variable, then the induced map

$$\Phi\colon M \to \mathcal{L}_{\mathrm{c}}(Y,Z), \quad p \mapsto \varphi(p,\cdot) \tag{A.1.5}$$

is continuous, in symbols $\Phi \in C(M, \mathcal{L}_{\mathrm{c}}(Y,Z))$.

b) If $S\colon X \to Y$ is a compact linear operator, then the induced map

$$\mathcal{L}_{\mathrm{c}}(Y,Z) \to \mathcal{L}_{\mathrm{b}}(X,Z), \quad T \mapsto TS := T \circ S \tag{A.1.6}$$

is continuous.

c) *For Φ and S as in a) and b) the induced map*

$$\Psi\colon M \to \mathcal{L}_b(X,Z), \quad p \mapsto \Phi(p)S \tag{A.1.7}$$

is continuous. (Juxtaposition of linear maps means composition.)

For normed vector spaces X and Y both topological vector spaces $\mathcal{L}_b(X,Y)$ and $\mathcal{L}(X,Y)$ with the operator norm topology coincide by Proposition A.1.59. Operators similar to the one in (A.1.7) are well known in non-linear analysis under the name *Nemitski operators associated to* $\boldsymbol{\varphi}$; see e.g. Ambrosetti and Prodi (1993, §1.2).

Proof of Proposition A.1.39. a) is even true for topological *spaces* M, Y, Z and continuous functions $\varphi\colon M\times Y \to Z$, not necessarily linear in the second variable; see e.g. Dugundji (1966, p. XII.3.1) or Müger (2016, Lemma 7.9.5). Now the conclusion is that Φ is continuous as a map $M \to C_c(Y,Z)$; cf. (A.1.2).
To prove this we must show that for all $p_0 \in M$ and sub-basis elements $\mathcal{F}_{K,V} \subset C_c(Y,Z)$ that contain $\Phi(p_0)$ there is an open neighborhood U_{p_0} of p_0 in M whose image under Φ lies in $\mathcal{F}_{K,V}$, too. Equivalently, we have to show that $\varphi(p_0\times K) \subset V$ implies $\varphi(U_{p_0} \times K) \subset V$ for some open set $p_0 \in U_{p_0} \subset M$. Continuity of φ guarantees an *open* pre-image $\varphi^{-1}(V) \subset M \times Y$ which contains $p_0 \times K$ by assumption. By compactness of K the Slice Lemma, see e.g. Dugundji (1966, p. XI.2.6) or Müger (2016, Prop. 7.5.1), provides an open neighborhood U_{p_0} of $p_0 \in M$ such that the thickening $U_{p_0} \times K$ of $p_0 \times K$ is still contained in $\varphi^{-1}(V)$.

b) Let's show that the pre-image $\Phi^{-1}\mathcal{B}_{A,U}$ of any (open) basis element of the bounded-open topology $\mathcal{T}_b(X,Z)$ is open in $\mathcal{L}_c(Y,Z)$, i.e. contains some basis element $\mathcal{B}'_{K,V} \in \mathcal{T}_c(Y,Z)$ of the compact-open topology. Given $A \subset X$ bounded and $0 \in U \subset Z$ open, note that $\Phi^{-1}\mathcal{B}_{A,U} = \mathcal{B}'_{S(A),U} \supset \mathcal{B}'_{K,U} \in \mathcal{T}_c(Y,Z)$ where by definition the compact set K is the closure of the pre-compact set $S(A) \subset Y$.

c) The composition of continuous maps is continuous. But composing the continuous maps (A.1.5) and (A.1.6) is the map (A.1.7). □

Fréchet and Gâteaux derivative on TVS

Definition A.1.40 (Fréchet derivative on TVS)**.** Suppose $f\colon X \supset U \to Y$ is a map between topological vector spaces defined on an open subset U.

In case $0 \in U$ and $f(0) = 0$ one says that f *has derivative zero at* $\mathbf{0}$ if for each open neighborhood $W_0 \subset Y$ of 0 there is an open neighborhood $V_0 \subset X$ of

0 and a function $o\colon (-1,1) \to \mathbb{R}$ such that

$$\lim_{t\to 0} \frac{o(t)}{t} = 0, \qquad tV_0 \subset U, \qquad f(tV_0) \subset o(t)W_0$$

for every $t \in (-1,1)$.

In general, one calls f *differentiable at* $\boldsymbol{x} \in \boldsymbol{U}$ if there is a continuous linear operator $D\colon X \to Y$ such that the map

$$h \mapsto f(x+h) - f(x) - Dh$$

has derivative zero at 0. In this case $df(x) := D \in \mathcal{L}(X,Y)$ is called the *derivative of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$. If f is differentiable at every point of U one calls f (Fréchet) *differentiable on* $\boldsymbol{U}$. In this case the map

$$f' := df : U \to \mathcal{L}(E,F), \quad x \mapsto df(x)$$

into the vector space of continuous linear maps $\mathcal{L}(X,Y)$ is called the (Fréchet) *differential of* $\boldsymbol{f}$.

By Corollary A.1.36 endowing $\mathcal{L}(X,Y)$ with the topology $\mathcal{T}_{\mathfrak{S}}$ associated to any of the families $\mathfrak{S} = \mathfrak{S}_{\mathrm{p}}, \mathfrak{S}_{\mathrm{c}}, \mathfrak{S}_{\mathrm{b}}$ results in a TVS denoted by $\mathcal{L}_{\mathfrak{S}}(X,Y)$. Hence $df : U \to \mathcal{L}_{\mathfrak{S}}(X,Y)$ is a map between TVS and one defines iteratively the higher order differentials

$$f^{(\ell)} := d^\ell f : U \to \mathcal{L}_{\mathfrak{S}}(E, \mathcal{L}_{\mathfrak{S}}(E, \dots \mathcal{L}_{\mathfrak{S}}(E,F))).$$

For normed vector spaces X and Y the bounded-open topology $\mathcal{T}_{\mathfrak{S}_{\mathrm{p}}}$ and the operator norm topology on $\mathcal{L}(X,Y)$ coincide by Proposition A.1.59 below.

We say that a map $f : X \supset U \to Y$ admits directional derivative at $x \in U$ in direction $\xi \in X$, if there are $\varepsilon > 0$ and $\eta \in Y$ such that the map

$$(-\varepsilon, \varepsilon) \to Y, \quad t \mapsto f(x+t\xi) - f(x) - \eta$$

has derivative zero at 0. In this case $\partial_\xi f(x) := \eta$ is called the derivative of f at x in direction ξ. If the map $\partial f(x)\colon X \to Y$, $\xi \mapsto \partial_\xi f(x)$, is defined for every $\xi \in X$ and is linear and continuous, then f is said *Gâteaux differentiable at* $\boldsymbol{x}$ with *Gâteaux derivative* $\partial f(x) \in \mathcal{L}(X,Y)$.

The (Fréchet) derivative on topological vector spaces enjoys some basic properties such as the chain rule and the fact that Fréchet differentiability implies continuity and Gâteaux differentiability. However, other fundamentals are not available, for instance the implicit function theorem and Cartan's last theorem may fail on TVS; see examples in Cieliebak (2018, §4.2).

A.1.3 Metric spaces

Definition A.1.41. A *metric* on a set M is a function $d : M \times M \to [0, \infty)$ that satisfies the following three axioms whenever $x, y, z \in M$.

(i) $d(x, y) = d(y, x)$ (Symmetry)

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$ (Triangle inequality)

(iii) $d(x, y) = 0 \Leftrightarrow x = y$ (Non-degeneracy)

Such pair (M, d) is called a *metric space*.

The prototype example of a metric d is the distance between two points in euclidean space. Hence a metric is also called a *distance function*. We often use the notation M_d for a metric space, meaning that M is a set endowed with the metric d.

Definition A.1.42. A metric space M_d comes naturally with the *metric topology* $\mathcal{T}_d$ whose basis $\mathcal{B}_d$ consists of the open balls $B^d_x(\varepsilon)$ of all radii $\varepsilon > 0$ about all points x of M_d. A metric space will be automatically endowed with the metric topology, unless mentioned otherwise.

Exercise A.1.43 (Metric topology)**.** Check that the collection $\mathcal{B}_d$ of all open balls in M_d indeed forms a basis for a topology, and not just a sub-basis.

As mentioned earlier, metric spaces are normal, thus Hausdorff. Moreover, second countability (countable basis) is equivalent to separability (dense sequence).

Exercise A.1.44 (Convergent sequence)**.** Check that $x_n \to y \in M_d$, in the sense of Definition A.1.18, if and only if any ε-ball about y contains all but finitely many sequence members x_n, in symbols

$$\forall \varepsilon > 0 \;\exists N \in \mathbb{N} : \quad d(x_n, y) < \varepsilon \quad \forall n \geqslant N.$$

Sequential convergence properties

Proposition A.1.45. *Let Q be a compact topological space and M_d a metric space. Then the compact-open topology $\mathcal{T}_c$ on $C(Q, M_d)$ coincides with the metric topology $\mathcal{T}_{d_\infty}$ associated to the* supremum metric

$$d_\infty(f, g) := \sup_{q \in Q} d(f(q), g(q)), \quad f, g \in C(Q, M_d).$$

Proof. Müger (2016, Prop. 7.9.2). (To show equality of two topologies one shows that the members of a basis, or of a sub-basis, of the first topology are open with respect to the second topology, and vice versa.) □

Exercise A.1.46. If Q is compact, then d_∞ is a metric on $C(Q, M_d)$.
[Hint: If stuck, consult e.g. Müger (ibid., Prop. 2.1.25).]

Convergence $f_\nu \to g$ with respect to $\mathcal{T}_{d_\infty} = \mathcal{T}_{d_\infty}(Q, M_d)$, that is

$$\forall \varepsilon > 0\ \exists \nu_\varepsilon : \quad d\left(f_\nu(q), g(q)\right) \leqslant \varepsilon \quad \text{whenever } \nu \geqslant \nu_\varepsilon \text{ and } q \in Q$$

is called *uniform convergence* on the compact set Q.

Exercise A.1.47. Let N be a topological space. If M_d is a metric space, the compact-open topology $\mathcal{T}_c$ on $C(N, M_d)$ coincides with the topology

$$\bigcap_{Q \subset N \text{ compact}} \mathcal{T}_{d_\infty}(Q, M_d)$$

of *uniform convergence on all compact subsets* K of N.
[Hint: If $N \supset Q$ compact, then $\mathcal{T}_c(N, M_d) \supset \mathcal{T}_c(Q, M_d) = \mathcal{T}_{d_\infty}(Q, M_d)$.]

Definition A.1.48 (Equicontinuous family)**.** Let N be a topological space and M_d a metric space. A family $\mathcal{F} \subset \mathrm{Map}(N, M_d)$ of maps, a priori, continuous or not, is called *equicontinuous* if for every $x \in N$ and every $\varepsilon > 0$ there is an open neighborhood U_x of x such that for all neighborhood elements $x' \in U_x$ and family members $f \in \mathcal{F}$ both values $f(x)$ and $f(x')$ are ε-close, in symbols

$$d\left(f(x), f(x')\right) < \varepsilon, \quad x' \in U_x,\ f \in \mathcal{F}.$$

Exercise A.1.49. The members of an equicontinuous family $\mathcal{F}$ are continuous.

Complete metric spaces – Theorem of Baire and Arzelà–Ascoli

Definition A.1.50. A sequence (x_n) in a metric space M_d is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is a sequence member x_N such that any two subsequent members are within distance ε of one another, in symbols

$$\forall \varepsilon > 0\ \ \exists N \in \mathbb{N} : \quad d(x_n, x_m) < \varepsilon \quad \forall n, m \geqslant N.$$

Exercise A.1.51. Check that every convergent sequence in a metric space is a Cauchy sequence, but the converse is not true.

Definition A.1.52 (Complete metric space)**.** A metric space in which every Cauchy sequence converges is called *complete* and so is the metric.

Exercise A.1.53. Let Q be a compact topological space. Then the metric space $(C(Q, M_d), d_\infty)$ is complete iff the target metric space M_d is complete.
[Hint: If stuck, consult e.g. Müger (ibid., Prop. 3.1.18 and Rmk. 5.2.12).]

Theorem A.1.54 (Baire's Theorem)**.** *Let M_d be a complete metric space and (U_n) a sequence of open and dense subsets. Then the intersection*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in M_d.

Proof. See e.g. Müger (ibid., Thm. 3.3.1). □

Among the many applications of Baire's Theorem are the open mapping theorem and the Banach–Steinhaus Theorem A.2.12, also called the principle of uniform boundedness.

Theorem A.1.55 (Arzelà–Ascoli Theorem)**.** *Let Q be a compact topological space and M_d a complete metric space. Then the following is true. A family*

$$\mathcal{F} \subset C_{\mathrm{c}}(Q, M_d)$$

*of continuous maps is pre-compact (with respect to the supremum metric d_∞) if and only if the family $\mathcal{F}$ is equicontinuous and the $\mathcal{F}$-*orbit *through each domain point $q \in Q$, namely each subset*

$$\mathcal{F}(q) := \{f(q) \mid f \in \mathcal{F}\} \subset M_d, \quad q \in Q$$

is pre-compact.

Proof. See e.g. Müger (ibid., Thm. 7.7.67). □

A.1.4 Normed vector spaces

Definition A.1.56. A *norm* on a (real) vector space X is a function $\|\cdot\| \colon X \to [0, \infty)$ that satisfies the following three axioms

(i) $\|\alpha x\| = |\alpha| \|x\|$ (Homogeneity)

(ii) $\|x + y\| \leqslant \|x\| + \|y\|$ (Triangle inequality)

(iii) $\|x\| = 0 \Leftrightarrow x = 0$ (Non-degeneracy)

for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Such pair $(X, \|\cdot\|)$ is called a *normed vector space*, often just denoted by X. If one drops the requirement $\|x\| = 0 \Rightarrow x = 0$ in (iii), one obtains the definition of a *semi-norm* on X.

The prototype example of a norm $\|\cdot\|$ is the distance of a point in euclidean space from the origin.

Exercise A.1.57 (Normed $\Rightarrow$ metric and TVS with convex basis)**.** Suppose $(X, \|\cdot\|)$ is a normed vector space. Show the following.

a) The definition

$$d_{\|}(x, y) := \|x - y\|, \quad x, y \in X$$

provides a (translation invariant: $d(x + z, y + z) = d(x, y)$) metric on X.
(So normed vector spaces are endowed with a natural topology, the metric topology $\mathcal{T}_{d_{\|}}$. Because metric topologies are Hausdorff, limits are unique.)

b) The vector operations addition and scalar multiplication are continuous.
(So any normed vector space X is a TVS.)

c) Open balls $B_\varepsilon(x)$ of radius $\varepsilon > 0$ centered at $x \in X$ are convex sets. So the natural basis of the topology of a TVS X given by all open balls consists of convex sets.
(So by Theorem A.1.35 the space $\mathcal{L}(X, Y)$ of continuous linear operators between normed vector spaces is a locally convex TVS under the point-open, compact-open, and bounded-open topologies; with respect to the latter it is even normed as we will see.)

[Hints: b) Addition: triangle inequality, scalar multiplication: homogeneity.]

Exercise A.1.58 (The normed vector space $\mathcal{L}(X, Y)$)**.** Let X and Y be normed vector spaces. Recall Definition A.1.29 on boundedness. Show that

a) A linear map $T : X \to Y$ is continuous iff it is bounded iff it maps the open unit ball about 0 into one of finite radius r, in symbols $TB_1 \subset B_r$.

b) Now consider the vector space $\mathcal{L}(X, Y)$ that consists of all bounded linear operators $T : X \to Y$ with addition $T + S : x \mapsto Tx + Sx$ and scalar multiplication $\alpha T : x \mapsto \alpha Tx$ for $\alpha \in \mathbb{R}$. Taking the infimum of all radii $r > 0$ of balls $B_r \supset TB_1$ still containing the image under T of the unit ball defines a norm

$$\begin{aligned} \|\cdot\| = \|\cdot\|_{\mathcal{L}(X,Y)} : \mathcal{L}(X, Y) &\to [0, \infty) \\ T &\mapsto \inf\{r > 0 \mid TB_1 \subset B_r\} \end{aligned}$$

called the *operator norm*. Alternatively, it is given by

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|<1} \|Tx\|.$$

[Hints: a) Cf. Rudin (1991, p. 1.29).]

By Exercise A.1.57 the normed vector space $(\mathcal{L}(X,Y), \|\cdot\|)$ carries a natural metric $d_\|$ and is a locally convex TVS. The metric topology $\mathcal{T}_\| = \mathcal{T}_\|(X,Y)$ is called the *operator norm topology* or the *uniform topology*, also indicated by $\mathcal{L}_\|(X,Y)$.

Convention: Whenever we speak of $\mathcal{L}(X,Y)$ as a normed vector space it is automatically endowed with the operator norm topology.

Proposition A.1.59 (Operator norm topology is bounded-open topology)**.** *For normed vector spaces X and Y the bounded-open and the operator norm topology on $\mathcal{L}(X,Y)$ coincide, in symbols $\mathcal{T}_\mathrm{b} = \mathcal{T}_\|$.*

Proof. Let $\|\cdot\|$ be the operator norm on $\mathcal{L}(X,Y)$. Balls are centered at 0.

$\mathcal{T}_\| \subset \mathcal{T}_\mathrm{b}$: It suffices to show that norm open balls $B_r := \{\|\cdot\| < r\}$ are open with respect to $\mathcal{T}_\mathrm{b}$. This means that B_r must contain together with any element S a whole $\mathcal{T}_\mathrm{b}$-open neighborhood $S + \mathcal{B}_{A,U}$ where $\mathcal{B}_{A,U} = \{T \in \mathcal{L}(X,Y) \mid T(A) \subset U\}$ with $A \subset X$ bounded and $0 \ni U \subset Y$ open; cf. (A.1.3).

To see this abbreviate $s := \|S\| \in [0,r)$ and let A be the closed unit ball in X and U the open ball in Y of radius $\frac{r-s}{2}$. For $T \in \mathcal{B}_{A,U}$ we get

$$\|S+T\| \leqslant \|S\| + \|T\| = s + \sup_{x\in A} \|Tx\|_Y \leqslant s + \frac{r-s}{2} = \frac{r+s}{2} < r.$$

Hence the $\mathcal{T}_\mathrm{b}$-open neighborhood $S + \mathcal{B}_{A,U}$ of T is contained in B_r.

$\mathcal{T}_\mathrm{b} \subset \mathcal{T}_\|$: By translation invariance of both topologies it suffices to show[12] that each element $\mathcal{B}_{A,U} \in \mathcal{B}(0) \subset \mathcal{L}(X,Y)$ of the local basis of $\mathcal{T}_\mathrm{b}$ at 0 contains an open ball $B_r \in \mathcal{T}_\|$ about 0. Given $A \subset X$ bounded and $0 \in U \subset Y$ open, pick open balls $A \subset B_r \subset X$ and $B_\varepsilon \subset U \subset Y$. Then $\|T\|_{\mathcal{L}(X,Y)} < r = \varepsilon/R$ implies that $T \in \mathcal{B}_{A,U}$. Indeed $TA \subset TB_R = RTB_1 \subset RB_{\varepsilon/R} = B_\varepsilon \subset U$. □

Sequential convergence properties

Lemma A.1.60 (Convergence in compact-open topology means convergence of the orbit through each point)**.** *Let X and Y be normed vector spaces. Consider operators $(T_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{L}(X,Y) \ni T$. Then $T_\nu \to T$ in $\mathcal{L}_\mathrm{c}(X,Y)$ iff for each domain element ξ the image sequence $T_\nu\xi$ converges to $T\xi$ in Y.*

[12] We could have localized to 0 already in order to prove $\mathcal{T}_\| \subset \mathcal{T}_\mathrm{b}$ above.

A.2 Analysis on Banach spaces

All vector spaces are over the real numbers. Throughout any linear structure is with respect to the real numbers and, as a rule of thumb, by X and Y we denote normed linear spaces and by E and F Banach spaces. In the context of linear spaces *subspace* means *linear subspace*.

A.2.1 Banach spaces

Definition A.2.1. A *Cauchy sequence* is a sequence x_ν in a normed linear space X such that $\|x_n - x_m\| \to 0$ whenever $n, m \to \infty$. The norm is called *complete* if every Cauchy sequence converges (admits a limit). A linear space E endowed with a complete norm[13] is called a *Banach space*. Any closed linear subspace $F \subset E$ endowed with the norm of E is a Banach space itself, called a *Banach subspace*.

Relevant examples of Banach spaces are enlisted in Theorem A.3.1.

Direct sum and topological complements

Definition A.2.2 (Direct sum)**.** The *direct sum of Banach spaces* $X \oplus Y$ is the set of pairs $\{(x, y) \mid x \in X,\ y \in Y\}$ which is equipped with and complete under the norm $\|(x, y)\| := \|x\| + \|y\|$.[14]

Definition A.2.3 (Banach space complement)**.** A closed subspace X of a Banach space Z is said to be *complemented* if there is a closed subspace Y of Z such that $X \cap Y = \{0\}$ and $X + Y = Z$. In this case we write $X \oplus Y = Z$ and call Y a *Banach space complement* or a *topological complement* of X, one also says that the Banach space X *splits*.

Example A.2.4 (Not every closed subspace is complemented)**.** Consider the Banach space $\ell^\infty := \{x \colon \mathbb{N} \to \mathbb{R}, \nu \mapsto x_\nu, \text{bounded}\}$ of bounded real sequences equipped with the sup norm. The subspace c_0 of sequences that converge to zero is closed, but does not admit a *topological complement*: There is no closed subspace d such that $c_0 \oplus d = \ell^\infty$; see Whitley (1966).

[13] also called a complete normed linear space

[14] Alternatively, use any of the equivalent norms $\|(x, y)\|_p^p := \|x\|^p + \|y\|^p$ for $1 \leqslant p < \infty$ or $\|(x, y)\|_\infty := \max\{\|x\|, \|y\|\}$.

Quotient spaces

Definition A.2.5 (Quotient space). Suppose X is a normed linear space and $A \subset X$ is a closed linear subspace. The *quotient space* of X by A is the set of cosets[15] denoted and defined by

$$X/A := \{x + A \mid x \in X\} \subset 2^X .$$

The function $X/A \to [0, \infty)$ given by the distance of any point representing the coset $x + A$ to the closed subspace A, namely

$$\|x + A\|_{X/A} := d(x, A) := \inf_{a \in A} \|x - a\| = \inf_{y \in x+A} \|y\| ,$$

is called the *quotient norm*. Often we use the shorter notation $\|x + A\|$.

Exercise A.2.6. a) Check that the operations $\alpha(x + A) := \alpha x + A$ for $\alpha \in \mathbb{R}$ and $(a + A) + (b + A) := (a + b) + A$ are well defined on X/A and endow the set of cosets with the structure of a linear space. Here closedness of A is actually not needed. b) Check that $\|x + A\|_{X/A} = \|y + A\|_{X/A}$ whenever $x + A = y + A$ or, equivalently, whenever $x - y \in A$. c) Show that the function $x + A \mapsto \|x + A\|$ is a norm on the linear space X/A.
[Hint: c) Non-degeneracy ($\|x + A\| = 0 \Rightarrow x \in A$) relies on closedness of A.]

Proposition A.2.7 (Quotient Banach spaces). *Suppose E is a Banach space and A is a closed subspace. Then the following is true.*

(i) *The quotient norm on E/A is complete.*

(ii) *The map between Banach spaces defined by*

$$\pi : E \to E/A, \quad x \mapsto x + A \tag{A.2.8}$$

is linear, surjective, continuous, and of norm $\|\pi\| \leqslant 1$ at most one. It is called the projection onto the quotient space E/A.

(iii) *Suppose, in addition, that E is reflexive, then E/A is reflexive.*

Proof. (i) Given a Cauchy sequence $x_\nu + A$ in the coset space E/A, by the Cauchy property it suffices to extract a subsequence that converges to a limit element $e + A$

[15] equivalently, the set of equivalence classes $\{[x] \mid x \in X\}$ where $x \sim y$ if $x - y \in A$

in E/A. Forgetting sequence members, if necessary, there is a subsequence, still denoted by $x_\nu + A$, that satisfies

$$\frac{1}{2^\nu} > \|(x_{\nu+1} + A) - (x_\nu + A)\| = \|(x_{\nu+1} - x_\nu) + A\| := d(x_{\nu+1} - x_\nu, A).$$

Thus there is a sequence of points $a_\nu \in A$ whose distance to $x_{\nu+1} - x_\nu$ satisfies $\|x_{\nu+1} - x_\nu - a_\nu\| < 1/2^\nu$. Consider the partial sum sequence $z_{\nu+1} := a_\nu + \cdots + a_1 \in A$. As the sequence $x_\nu - z_\nu$ is Cauchy in E, indeed

$$\|(x_{\nu+1} - z_{\nu+1}) - (x_\nu - z_\nu)\| = \|x_{\nu+1} - a_\nu - x_\nu\| < 1/2^\nu$$

it admits a limit e in the Banach space E. It follows that the sequence $x_\nu + A$ converges to $e + A$ in E/A and we are done. Indeed

$$\begin{aligned}
\|(x_\nu + A) - (e + A)\| &= \|(x_\nu - e) + A\| \\
&:= \inf_{a \in A} \|x_\nu - e - a\| \\
&\leqslant \|x_\nu - e - z_\nu\| < 1/2^\nu.
\end{aligned}$$

(ii) The map π is linear by definition of addition in the coset space E/A. Surjectivity is obvious. To see continuity and $\|\pi\| \leqslant 1$, given $x \in E$, pick $a = 0 \in A$ to get that $\|\pi(x)\| := \inf_{a \in A} \|x - a\| \leqslant \|x\|$. (iii) Brezis (2011, Prop. 11.11). $\square$

For more details about quotients see e.g. Brezis (ibid., §11.2) or Rudin (1987, §18.14).

A.2.2 Linear operators

Given normed linear spaces X and Y, recall that a linear map $T : X \to Y$ is continuous iff it is continuous at one point iff it is *bounded*; see e.g. Reed and Simon (1980, Thm. I.6). To be bounded means that the *operator norm* of T, defined by

$$\begin{aligned}
\|T\| = \|T\|_{\mathcal{L}(X,Y)} &:= \sup_{\|x\|=1} \|Tx\| \\
&= \inf\{c \geqslant 0 \colon \|Tx\| \leqslant c\,\|x\| \text{ for every } x \in X\}
\end{aligned}$$

is finite. By $\mathcal{L}(X, Y)$ we denote the linear space of continuous linear operators $T : X \to Y$. Juxtaposition $ST : X \to Y \to Z$ denotes composition. The *invertible elements* T of $\mathcal{L}(X, Y)$, that is $TS = \mathbb{1}$ and $ST = \mathbb{1}$ for some (unique)

$S \in \mathcal{L}(Y, X)$, are called *isomorphisms* or *toplinear isomorphisms*[16] to emphasize context. In case of Banach spaces E and F invertible elements of $\mathcal{L}(E, F)$, aka toplinear isomorphisms, aka isomorphisms, are precisely the continuous linear bijections; cf. e.g. Lang (2001, I §2) and Lang (1993, IV §1). (The inverse is continuous by the closed graph theorem.)
Abbreviate $\mathcal{L}(X) := \mathcal{L}(X, X)$. By $\mathcal{L}^k(X, Y)$ we denote the linear space of k-fold multilinear maps $T\colon X \oplus \cdots \oplus X \to Y$. If the norm of Y is complete, then the operator norm is complete, so $\mathcal{L}(X, Y)$ is a Banach space. Thus the dual space $X^* = \mathcal{L}(X, \mathbb{R})$ of a normed linear space is a Banach space.

Unique extension

Theorem A.2.8 (B.L.T. theorem)**.** *Suppose T is a bounded linear map from a normed linear space X to a complete normed linear space F. Then T extends uniquely to a bounded linear map $\tilde{T}$ from the completion of X to F.*

Proof. See e.g. Reed and Simon (1980, Thm. I.7). □

Compact operators and projections

Definition A.2.9 (Compact operator)**.** A linear operator $S\colon X \to Y$ between normed linear spaces is called *compact* if for every bounded sequence in the domain, the image sequence has a convergent subsequence or, equivalently, if it maps bounded sets to *pre-compact sets* (sets whose closure is compact). Compact linear operators are automatically continuous.

Definition A.2.10 (Projection)**.** A continuous linear operator $P\colon X \to X$ is called a *projection* if it is idempotent, in symbols $P \circ P = P$.

Exercise A.2.11 (Continuous projections split)**.** Let E be a Banach space and $P \in \mathcal{L}(E)$ a projection. Then the image $F := \operatorname{im} P$ is closed and complemented by the closed image $G := \operatorname{im} Q$ of the continuous projection $Q := \mathbb{1} - P$, that is

$$E = F \oplus G = \operatorname{im} P \oplus \operatorname{im}(\mathbb{1} - P).$$

[Hint: Kernels of continuous maps are closed and $\operatorname{im} P = \ker Q$ and vice versa.]

[16] A toplinear isomorphism is a continuous linear bijection whose inverse is continuous, too. The notion makes sense in the general context of topological vector spaces.

Principle of uniform boundedness

The Hahn–Banach theorem and the Banach–Steinhaus theorem are two pillars of functional analysis. The latter is also known as the principle of uniform boundedness. Its proof is based on the Baire category theorem which requires a non-empty complete metric space, for instance a Banach space E.

Theorem A.2.12 (Banach–Steinhaus)**.** *Suppose E is a Banach space. Let $\mathcal{F}$ be a family of bounded linear operators $T\colon E \to Y$ to some normed linear space. Suppose that the $\boldsymbol{\mathcal{F}}$-*orbit *through each point $x \in E$, namely each set*

$$\mathcal{F}x := \{Tx : T \in \mathcal{F}\} \subset Y$$

is a bounded subset of Y. Then the operator norm is uniformly bounded along the family $\mathcal{F}$: There is a constant $c_{\mathcal{F}} \geqslant 0$ such that

$$\|T\| = \|T\|_{\mathcal{L}(E,Y)} \leqslant c_{\mathcal{F}} \qquad \forall\, T \in \mathcal{F}.$$

Proof. See e.g. Reed and Simon (1980, Thm. III.9). □

Recall that $\mathcal{L}(E, F)$ carries the operator norm topology. How to utilize the principle of uniform boundedness is illustrated in the proof of

Proposition A.2.13. *Suppose E_1, E_0, F_0 are Banach spaces and $U_1 \subset E_1$ is an open subset. Then the following is true.*

a) Let the map $\Phi\colon U_1 \oplus E_0 \to F_0$, $(x, \eta) \mapsto \Phi(x, \xi) =: \Phi(x)\eta$, be continuous and, moreover, linear in the second variable. Then the induced map

$$U_1 \to \mathcal{L}_c(E_0, F_0), \quad x \mapsto \Phi(x)\cdot \tag{A.2.9}$$

is continuous. (The target carries the compact-open topology.)[17]

b) If $S\colon E_1 \to E_0$ is a compact linear operator, then the induced map

$$\mathcal{L}_c(E_0, F_0) \to \mathcal{L}(E_1, F_0), \quad T \mapsto T \circ S \tag{A.2.10}$$

is continuous. (The target carries the norm topology.)

[17] The target carrying the *compact-open topology* means that a sequence $T_\nu \in \mathcal{L}(E_0, F_0)$ converges to an element $T \in \mathcal{L}(E_0, F_0)$ iff for each domain element ξ the sequence $T_\nu\xi$ converges to $T\xi$ in F_0; see Lemma A.1.60.

c) *For $\Phi\colon U_1 \oplus E_0 \to F_0$ and $S\colon E_1 \to E_0$ as in a) and b) the induced map*

$$U_1 \to \mathcal{L}(E_1, F_0), \quad x \mapsto \Phi(x)S\,\cdot \tag{A.2.11}$$

is continuous.

Proof. a) Proposition A.1.39 a).

b) Given $T \in \mathcal{L}(E_0, F_0)$ and a sequence $T_\nu \in \mathcal{L}(E_0, F_0)$ with $T_\nu\zeta \to T\zeta$ in F_0 for each $\zeta \in E_0$, assume by contradiction that there is a constant $\varepsilon > 0$ and a sequence in E_1 of bounded norm, say $\|\xi_\nu\|_{E_1} = 1$, such that $\|T_\nu S\xi_\nu - TS\xi_\nu\|_{F_0} \geqslant \varepsilon$. Because the linear operator $S\colon E_1 \to E_0$ is compact, there is $\eta \in E_0$ and subsequences, still denoted by T_ν and ξ_ν, such that $S\xi_\nu \to \eta$ in E_0. Hence

$$\begin{aligned}
\lim_{\nu\to\infty} \|T_\nu S\xi_\nu - TS\xi_\nu\|_{F_0} &\leqslant \lim_{\nu\to\infty} \|T_\nu S\xi_\nu - T_\nu\eta\|_{F_0} \\
&\quad + \lim_{\nu\to\infty} \|T_\nu\eta - T\eta\|_{F_0} \\
&\quad + \lim_{\nu\to\infty} \|T\eta - TS\xi_\nu\|_{F_0} \\
&\leqslant \lim_{\nu\to\infty} \|T_\nu\|_{\mathcal{L}(E_0,F_0)} \|S\xi_\nu - \eta\|_{E_0} \\
&\quad + \lim_{\nu\to\infty} \|T_\nu\eta - T\eta\|_{F_0} \\
&\quad + \lim_{\nu\to\infty} \|T\|_{\mathcal{L}(E_0,F_0)} \|\eta - S\xi_\nu\|_{E_0} \\
&= 0.
\end{aligned}$$

Contradiction. Here the two inequalities are obtained by first adding twice zero and applying the triangle inequality, then using the definition of the operator norm. It remains to understand the vanishing of the three limits. For limit three this is obvious and limit two vanishes by hypothesis. Concerning limit one consider the family $\mathcal{F} := \{T_\nu\}_{\nu\in\mathbb{N}} \cup \{T\} \subset \mathcal{L}(E_0, F_0)$. Each $\mathcal{F}$ orbit

$$\mathcal{F}\zeta := \{T_\nu\zeta\}_{\nu\in\mathbb{N}} \cup \{T\zeta\} \subset F_0, \quad \zeta \in E_0$$

is bounded in F_0, even compact, as $T_\nu\zeta \to T\zeta$ by hypothesis. By the Banach–Steinhaus Theorem A.2.12 the family $\mathcal{F}$ is bounded in the operator norm.

c) The composition of continuous maps is continuous. But composing the continuous maps (A.2.9) and (A.2.10) is the map (A.2.11). □

Dual spaces and Reflexivity

Definition A.2.14 (Dual space)**.** Given a normed linear space X, its *dual space* is the Banach space $X^* := \mathcal{L}(X, \mathbb{R})$ of continuous linear functionals $\lambda\colon X \to \mathbb{R}$.

Theorem A.2.15 (Hahn–Banach)**.** *Suppose V is a linear subspace, closed or not, of a Banach space X and $\lambda \in V^*$ is a continuous linear functional on V. Then there is a linear functional $\Lambda \in X^*$ that extends λ and such that*

$$\|\Lambda\|_{X^*} = \sup_{\substack{v \in V \\ \|v\|=1}} |\lambda(v)| =: \|\lambda\|_{V^*} .$$

Proof. See e.g. Brezis (2011, Cor. 1.2). □

Definition A.2.16 (Reflexive)**.** A normed linear space X is called *reflexive* if the canonical isometric linear map $J : X \to (X^*)^*$ given by evaluation

$$J(x)(x^*) := \langle x^*, x \rangle$$

is surjective; see e.g. Bühler and Salamon (2018, §2.4). (Note that any linear isometry is injective.)

Remark A.2.17. We highly recommend Brezis (2011, §3.5).

a) *Kakutani's Theorem*: Reflexivity of a Banach space E is equivalent to compactness of the closed unit ball of E in the weak topology.

b) Closed linear subspaces of reflexive Banach spaces are reflexive.

c) A uniformly convex Banach space, so any Hilbert space, is reflexive.

Example A.2.18 (Non-reflexive Banach spaces)**.**

(i) The closed linear subspace c_0 of the Banach space ℓ^∞ in Example A.2.4 is not reflexive; see e.g. Salamon (2016, Exerc. 4.37). Hence ℓ^∞ is not reflexive either by Remark A.2.17 b).

(I) More generally, let $C^0_{\mathrm{bd}}(X)$ be the space of bounded continuous functions on a locally compact topological space X endowed with the sup norm (e.g. ℓ^∞). Then the Banach space $C^0_{\mathrm{bd}}(X)$ is reflexive iff X is a finite set. See e.g. Conway (1985) (III §11 Exerc. 2 and V §4 Exerc. 3).

(ii) Consequently $C^0(Q)$ is not reflexive for compact manifolds Q of $\dim Q \geqslant 1$. Neither is $C^k(Q)$ for $k \in \mathbb{N}$; this follows by reduction to the case $k = 0$ using the graphs maps of differentials, see e.g. Weber (2017b, App. A).

The following theorem can be viewed as a substitute in the Banach space universe of the orthogonal projections available in the Hilbert space world.

Theorem A.2.19 (Projection theorem for reflexive Banach spaces)**.** *Let E be a reflexive Banach space and $C \subset E$ a closed convex subset. For every $x \in E$ there is an element $y \in C$ which minimizes the distance to x, that is*

$$\|x - y\| = d(x, C) := \inf_{z \in C} \|x - z\| .$$

Proof. The proof uses Kakutani's theorem, see e.g. Brezis (2011, Cor. 3.23). □

Arzelà–Ascoli – convergent subsequences

Theorem A.2.20 (Arzelà–Ascoli Theorem)**.** *Suppose (Q, d) is a compact metric space and $C(Q)$ is the Banach space of continuous functions on Q equipped with the sup norm. Then the following is true. A subset $\mathcal{F}$ of $C(Q)$ is pre-compact if and only if the family $\mathcal{F}$ is* equicontinuous [18] *and* pointwise bounded [19].

For a proof see e.g. Rudin (1991, Thm. A.5) or Salamon (2017, App. C). By Theorem A.2.20 this generalizes to maps taking values in a metric space.

A.2.3 Calculus

An efficient presentation of the Fréchet derivative in Banach spaces E, F is given in §1.1 of Ambrosetti and Prodi (1993) where §2.2 deals with the implicit function theorem (IFT). We follow Lang (1993, PART FOUR).

Fréchet or total derivative $df(x)$

Consider Banach spaces E and F and let U be open in E. One says that *a map* $\boldsymbol{f : U \to F}$ *is differentiable at a point* $\boldsymbol{x}$ of U if there is a continuous linear map $D : E \to F$ and a map ψ defined for all sufficiently small elements h in E and with values in F such that

$$\lim_{h \to 0} \psi(h) = 0$$

and such that f near x is given by the sum

$$f(x + h) = f(x) + Dh + \|h\| \, \psi(h).$$

[18] $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ and $f \in \mathcal{F}$.
[19] $\sup_{f \in \mathcal{F}} |f(x)| < \infty$ for every $x \in Q$.

Set $h = 0$ to see that it makes sense to set $\psi(0) := 0$. Equivalently, denoting $o(h) := \|h\|\,\psi(h)$ the condition becomes

$$0 = \lim_{h\to 0} \frac{\|o(h)\|}{\|h\|} = \lim_{h\to 0} \frac{\|f(x+h) - f(x) - Dh\|}{\|h\|}\,. \tag{A.2.12}$$

Exercise A.2.21. a) Differentiability at x implies continuity at x. b) If $D \in \mathcal{L}(E,F)$ satisfies (A.2.12), then it is uniquely determined by f and x.

Definition A.2.22 (Derivative and differential)**.** Let $f\colon E \supset U \to F$ be differentiable at a point $x \in U$. Then the unique continuous linear operator D satisfying (A.2.12) is called the (Fréchet) *derivative of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$ and denoted by $df(x) := D \in \mathcal{L}(E,F)$. If f is differentiable at every point of U one says that f is *differentiable on* $\boldsymbol{U}$. In this case the map

$$f' := df : U \to \mathcal{L}(E,F), \quad x \mapsto df(x)$$

into the Banach space of continuous linear maps $\mathcal{L}(E,F)$ endowed with the operator norm is called the (Fréchet) *differential of* $\boldsymbol{f}$. If df is continuous one says that $\boldsymbol{f}$ *is of class* $\boldsymbol{C^1}$, in symbols $f \in C^1(U,F)$. Higher derivatives

$$f^{(\ell)} := d^\ell f : U \to \mathcal{L}(E, \mathcal{L}(E, \dots \mathcal{L}(E,F))) \simeq \mathcal{L}^\ell(E,F)$$

are defined iteratively. If they exist and are continuous for $\ell = 0, \dots k$, one says that f is of class C^k. Here $\mathcal{L}^\ell(E,F)$ denotes the Banach space of k-fold multilinear maps $E \oplus \dots \oplus E \to F$. One says that f is a *smooth map*, or of class C^∞, if f is of class C^k for every $k \in \mathbb{N}_0$.

Gâteaux or all-directional derivative $\partial f(x)$

A map $f\colon E \subset U \to F$ between Banach spaces with U open is said *Gâteaux differentiable at* $\boldsymbol{x \in U}$ if for each $\xi \in E$ the *directional derivative*

$$\partial_\xi f(x) := \lim_{t\to 0} \frac{f(x+t\xi) - f(x)}{t}$$

exists and defines a continuous *linear* map $\partial f(x)\colon E \to F$, $\xi \mapsto \partial_\xi f(x)$.

Exercise A.2.23. Show that a) (Fréchet) differentiable implies Gâteaux differentiable, but b) not vice versa.
[Hint: b) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) := 0$ and by $f(u,v) := u^4 v/(u^6 + v^3)$ off the origin. Show that $\partial_\zeta f(0,0) = 0$ for every $\zeta \in \mathbb{R}^2$. So each directional derivative not only exists, but also the map $\zeta \mapsto \partial_\zeta f(0,0)$ is linear. Is f continuous at $(0,0)$?]

A.2.4 Banach manifolds

Roughly speaking, a Banach manifold is a topological space (Hausdorff and paracompact) which is locally modeled on some Banach space such that all transition maps between the local models are differentiable. Differentiability of maps between Banach manifolds is defined in terms of differentiability of the corresponding maps between the local model Banach spaces. We recommend the book by Lang (2001) concerning differential geometry on Banach manifolds.

Suppose X is a topological space and $k \in \mathbb{N}_0$ or $k = \infty$. A *Banach chart* (V, ϕ, E) for X consists of a Banach space E and a homeomorphism $\phi\colon X \supset V \to U \subset E$ between open subsets. Two charts are called ***C^k*** *compatible* if the *transition map*

$$\phi_j \circ \phi_i^{-1}\colon \phi_i(V_i \cap V_j) \to \phi_j(V_i \cap V_j)$$

is a ***C^k*** *diffeomorphism* (an invertible C^k map with C^k inverse). A ***C^k*** *Banach atlas for* ***X*** is a collection $\mathcal{A}$ of pairwise C^k compatible Banach charts for X such that the chart domains form a cover $\{V_i\}_i$ of X. Two atlases are called *equivalent* if their union forms an atlas.

Given such pair $(X, \mathcal{A})$, then X is connected iff it is path connected. Furthermore, for $k \geqslant 1$ connectedness of X implies that all model Banach spaces E_i in the charts of $\mathcal{A}$ are isomorphic to one and the same Banach space E. In this case we say that $(X, \mathcal{A})$ is *modeled on* ***E***.

Remark A.2.24 (Starting from just a set X)**.** Alternatively starting with just *a set* X one can construct a C^k Banach atlas as follows. Choose a collection of bijections (the future coordinate charts)

$$\phi\colon X \supset V \to U \subset E$$

from a subset V of X onto an *open* subset U of a Banach space E. There are two requirements: Firstly, the sets V of all the charts must cover X and, secondly, for each pair of charts the set $\phi(V \cap \widetilde{V})$ must be open in E. The notions C^k compatibility and C^k Banach atlas are unchanged. Given a C^k Banach atlas $\mathcal{A}$, consider the collection $\mathcal{B} \subset 2^X$ of all subsets $\phi^{-1}(U')$ of X where (ϕ, V) runs through all charts of $\mathcal{A}$ and $U' \subset E$ runs through all open subsets of $\phi(V)$. One checks that $\mathcal{B}$ is a basis of a topology and endows X with that topology. Then $\mathcal{A}$ is an atlas on the topological space X in the earlier sense. For an application see Exercise 1.8.6.

Definition A.2.25. A ***C^k*** *Banach manifold* is a paracompact Hausdorff space X endowed with an equivalence class of C^k Banach atlases. If $k = 0$ one speaks of a *topological* and if $k = \infty$ of a *smooth* Banach manifold. We often abbreviate smooth Banach manifold by *Banach manifold*. In case all model spaces are Hilbert spaces one speaks of a *Hilbert manifold*.

Definition A.2.26 (Maps between Banach manifolds)**.** A continuous map $f : X \to Y$ between Banach manifolds is said to be of *class* $\boldsymbol{C^k}$ if for all charts $\phi : X \supset V \to E$ and $\psi : Y \supset W \to F$ the chart representative $\psi \circ f \circ \phi^{-1}$ is of class C^k as a map between open subsets of the Banach spaces E and F.

A.3 Function spaces

Theorem A.3.1 (Properties of L^p and Sobolev spaces)**.**

`(complete)` Fischer–Riesz Theorem*: The spaces* $L^p(\mathbb{R}, \mathbb{R}^n)$ *with norm* $\|\cdot\|_p := \|\cdot\|_{L^p}$ *are Banach spaces whenever* $1 \leqslant p \leqslant \infty$.

`(separable)` *The spaces* $L^p(\mathbb{R}, \mathbb{R}^n)$ *are separable*[20] *for* $1 \leqslant p < \infty$*, but not for* $p = \infty$.

`(reflexive)` *The spaces* $L^p(\mathbb{R}, \mathbb{R}^n)$ *are reflexive for* $1 < p < \infty$*, but not for* $p = 1, \infty$.

(Sobolev spaces) *The Sobolev spaces* $W^{k,p}(\mathbb{R}, \mathbb{R}^n)$ *have equal properties for the same* p.

For proofs of the three properties of L^p see e.g. Theorems 4.8, 4.13, and 4.10, respectively, in Brezis (2011), for $W^{k,p}$ see Brezis (ibid., Prop. 8.1). Concerning Sobolev spaces see also Adams and Fournier (2003).

[20] A topological space is called *separable* if it admits a dense sequence.

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ISBN 978-85-244-0486-3

9 788524 404863