Nonlinear Dispersive Equations on Star Graphs

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Preface

The intention of this book is to provide a self-contained presentation of new approaches in the mathematical studies of nonlinear dispersive evolution equations on metric graphs. We are interested in the nonlinear dynamic generated by two fundamental models with applications in chemistry, engineering, blood pressure waves and several other physical fields: The nonlinear Schrödinger equation and the Korteweg–de Vries equation. It will be in our interest to study the local well-posedness problem of the Cauchy problem, the existence and stability of standing wave and/or stationary solutions in different geometries of the metric graph. Although many results may be found in the literature, in this book we offer a new approach to study this kind of fascinating structures and some new results are established. We also hope with this notes to fill in a little the gap in the literature related to the analytical study of soliton propagation through networks.

This book has also been designed to be instructive as well to be a new source of reference for students and researches interested in nonlinear wave phenomena on quantum graphs. Simplicity and concrete applications are set throughout the book in order to make the material easily assimilated. Also, we hope that it may inspire future projects in this new field of action for the nonlinear dispersive evolution equations.

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We are indebted to many friends and collaborators who gave us the support, encouragements, and suggestions to complete this book.

J. Angulo would like to dedicate this work to his daughter Victoria Mel (por supuesto) and that this may serve as an inspiration in her future academic activity.

M. Cavalcante would like to dedicate this work to his parents Marcos Cavalcante and Maria Silva.

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May 2019

1 Introduction

A quantum graph is a metric graph, i.e., a network-shaped structure of vertices connected by edges, with a linear Hamiltonian operator (such as a Schrödinger-like operator or a Airy-like operator) suitably defined on functions that are supported on the edges. It arises as a simplified models for wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) system that *looks like a thin neighborhood of a graph*. Quantum graph have been used to describe a variety of physical problems and applications, such as in chemistry and engineering (see Berkolaiko and Kuchment 2013; Blank, Exner, and Havlíček 1994; Burioni et al. 2001; Kuchment 2004; Mugnolo 2015, for details and references). Recently, they have attracted much attention in the context of soliton transport in networks and branched structures (see Sobirov, Matrasulov, et al. 2010; Sobirov, Babajanov, and Matrasulov 2017) since wave dynamics in networks can be modeled by nonlinear evolution equations suitably defined on the edges.

Soliton and other nonlinear waves in branched systems appear in different system, for instance, condensed matter, Josephson junction networks, polymers, optics, neuroscience, DNA, blood pressure waves in large arteries or in shallow water equation to describe a fluid network (see Adami and Noja 2013; Ali Mehmeti, von Below, and Nicaise 2001; Berkolaiko, Carlson, et al. 2006; Berkolaiko and Kuchment 2013; Brazhnyi and Konotop 2004; Burioni et al. 2001; Cao and Malomed 1995; Fidaleo 2015; Kogan, Clem, and Kirtley 2000; Kuchment 2004; Mugnolo 2015; Noja 2014, and references therein).

To address these issues, in general the problem is difficult to tackle because both the equation of motion and the geometry are complex. A first direction in the analysis is to look at what happens in a simpler geometry and to examine a linear evolution equation, such as a $\mathcal{Y}$ junction (see Figure 1.1) and models as the linear Schrödinger equation or the linear Korteweg–de Vries equation. In many cases however the nonlinearity can not be neglected, by instance, in fluid system to describe a fluid network.

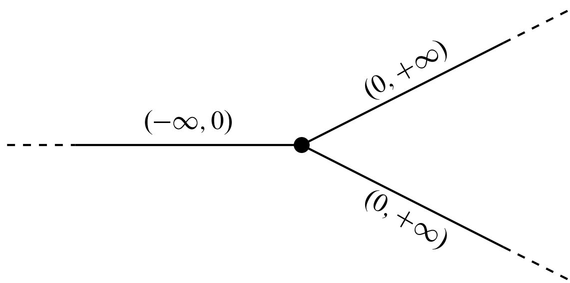

Figure 1.1: $\mathcal{Y}$ junction: a star graph with three edges

Thus, in the last years the study of nonlinear dispersive models on metric graph has attracted a lot of attention of mathematician and physicists. In particular, the prototype of framework (graph-geometry) for description of these phenomena have been a *star graph* $\mathcal{G}$, namely, a metric graph with N half-lines of the form $(0, +\infty)$ connecting at a common vertex $\nu = 0$ (see Figure 1.2), together with a nonlinear equation suitably defined on the edges such as a nonlinear Schrödinger equation or the Benjamin-Bona-Mahony equation (BBM) (see J. L. Bona and Cascaval 2008; Mugnolo and Rault 2014). The sine-Gordon equation is also other basic model which also have been worked on the framework of a $\mathcal{Y}$ junction.

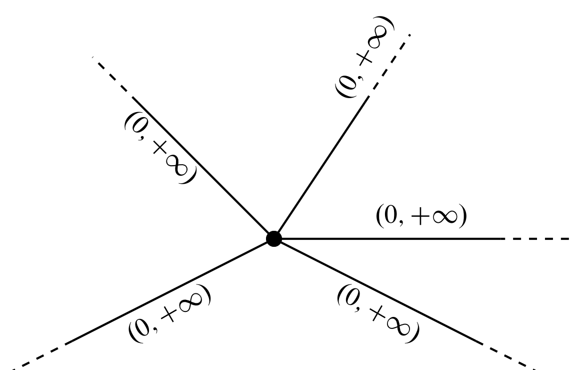

Figure 1.2: Star graph with 5 edges

We note that with the introduction of the nonlinearity in the dispersive model, the network provides a nice field where one can looking for interesting soliton propagation and nonlinear dynamics in general. However, there are few exact analytic study of soliton propagation through networks by the nonlinear flow induced by the equation. Results on the stability or instability mechanism of these profiles are still unclear. One of the objectives of these notes is to provide the reader with several new analytical tools for this study. A central point that makes this analysis a delicate problem is the presence of a vertex where the underlying one-dimensional star graph should bifurcate (or multi-bifurcate in a general metric graph). *We note that not branching angles but the topology of bifurcation is essential*. Indeed, a soliton-profile coming into the vertex along one of the bonds (edge of the graph) shows a complicated motion around the vertex such as reflection and emergence of the radiation there, moreover, in particular one cannot see easily how energy travels across the network. Therefore, the study of the existence and stability of specific soliton-profile will depend heavily on the conditions on the vertex to have a fruitful description of the dynamic of these profiles. For instance, in the case of the following nonlinear (vectorial) Schrödinger model on a star graph $\mathcal{G}$

$$i\partial_t \mathrm{U}(t,x) - \mathcal{A}\mathrm{U}(t,x) + |\mathrm{U}(t,x)|^{p-1}\mathrm{U}(t,x) = 0, \tag{1.1}$$

where $\mathrm{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^N$, $p > 1$, and the nonlinearity acts by components, i.e. $(|\mathrm{U}|^{p-1}\mathrm{U})_j = |u_j|^{p-1}u_j$, the function U has been assumed to satisfy specific boundary conditions such as either Kirchhoff, or δ, or δ'-interaction at the vertex $\nu = 0$, such that the diagonal-matrix Hamiltonian operator

$$\mathcal{A} = \Big(\Big(-\frac{d^2}{dx^2}\Big)\delta ij\Big)$$

remains a self-adjoint operator on $L^2(\mathcal{G})$. For instance, in the case of a δ-interaction we have that $\mathcal{A}$ is a self-adjoint operator on $L^2(\mathcal{G})$ acting as $(\mathcal{A}\mathrm{V})(x) = (-v_j''(x))_{j=1}^N$, $x > 0$, on the domain $D_{\alpha,\delta}(\mathcal{A})$ defined by $\alpha \in \mathbb{R}$ as

$$D_{\alpha,\delta}(\mathcal{A}) := \Big\{\mathrm{V} = (v_j)_1^N \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0), \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\Big\}. \tag{1.2}$$

Other more general coupling conditions at the vertex $\nu = 0$, those set up above, it can be considered in such a way that the dynamics of the quantum system in (1.1) is described also by unitary operators (see Chapter 2). The soliton dynamics for the NLS equation (1.1) with a δ- δ'-interaction, and the free Kirchoff condition at the vertex ($\alpha = 0$ in (1.2)) is studied in Adami, Cacciapuoti, et al. (2014c, 2016) and Angulo and Goloshchapova (2017a, 2018).

Other interest model is that of the Korteweg–de Vries equation (KdV)

$$\partial_t u_{\mathrm{e}}(x,t) = \alpha_{\mathrm{e}}\partial_x^3 u_{\mathrm{e}}(x,t) + \beta_{\mathrm{e}}\partial_x u_{\mathrm{e}}(x,t) + 2u_{\mathrm{e}}(x,t)\partial_x u_{\mathrm{e}}(x,t), \tag{1.3}$$

$x \neq 0$, $t \in \mathbb{R}$, on a metric graph $\mathcal{G}$ with a structure represented by finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$. The half-lines are connected at a unique vertex $\nu = 0$. Here (α_e) and (β_e) are two sequences of real numbers. $\mathcal{G}$ is sometimes also called a star-shaped metric graph (see Figure 1.3).

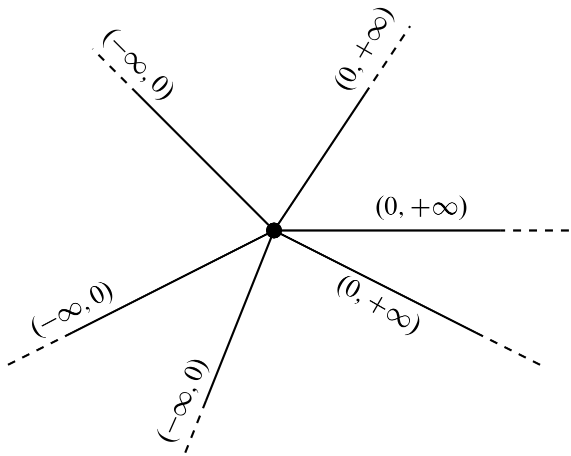

Figure 1.3: A star-shaped metric graph with 6 edges

We recall that the KdV equation was first derived by Korteweg and de Vries (1895) in 1895 as a model for long waves propagating on a shallow water surface. Recently, the KdV equation have been appearing in other context. More precisely, this equation has been used as a model to study blood pressure waves in large arteries. In this way, for example, Chuiko et al. (2016) proposed a new computer model for systolic pulse waves within the cardiovascular system based on the KdV equation. Also, Crépeau and Sorine (2007) showed that some particular solutions of the KdV equation, more exactly, the 2 and 3-soliton well-known solutions, seem to be good candidates to match the observed pressure pulse waves. This new applications for KdV equations suggest your study on star-shaped metric graphs.

We empathize that the study of the Korteweg–de Vries equation in star-shaped metric graphs is relatively underdeveloped. The principal difficulty in studying this model is the fact of differently of the NLS equation (1.1) is not clear which boundary conditions at the vertex $\nu = 0$ should be appropriate for physical applications and an analytical mathematical study. On the mathematical context a result of local well-posedness on the case of specific vertex-conditions on a $\mathcal{Y}$ junction was obtained recently by Cavalcante (2018) (see Chapter 5). On the other side, the non-existence of conserved functionals (energy or charge) for the system makes the study of the dynamics very complicated. One of the main interest of exposition here with regard to the KdV model is to establish a linear instability criterium for stationary profiles on a star-shaped metric graph $\mathcal{G}$ (see Angulo and Cavalcante (2019)). A starting point for the one previously described, it is to determine when the Airy type operator

$$A_0 : (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \to \left(\alpha_{\mathrm{e}} \frac{d^3}{dx^3} u_{\mathrm{e}} + \beta_{\mathrm{e}} \frac{d}{dx} u_{\mathrm{e}}\right)_{\mathrm{e}\in\mathrm{E}} \tag{1.4}$$

being seen as an unbounded operator on a certain Hilbert space, it will have extensions A_{ext} on $L^2(\mathcal{G})$ such that the dynamics induced by the linear evolution problem

$$\begin{cases} z_t = A_{ext} z, \\ z(0) = u_0 \in D(A_{ext}), \end{cases} \tag{1.5}$$

it is given by a C_0-group.

The tools used in the next chapters will be those usual in the study of the dynamics of nonlinear dispersive equations. In a general way, our approach will not be of variational type, so a more local analysis around the objects of our interest will be done. One of the main tools in our study will be one based on the theory of extension for symmetric operators developed by Krein and von Neumann. In this way we will dedicate a section of these notes to recall the basic results of this theory (although several of them are relatively well known) and to see its deep importance in the study of the dynamics of nonlinear dispersive equations on metric graphs.

Now we describe how this book has been divided. In Chapter 2, we give the definitions associated with metric graphs and the class of objects that will be of our study interest here. Chapter 3 provides basic results of extension theory for closed symmetric operators from von Neumann and Krein, and we give many specific applications to linear operators that arise in several places of our exposition. In Chapter 4, we introduce the main models of our study; the nonlinear Schrödinger equation and the Korteweg–de Vries equation on metric graphs. In Chapter 5, we study local well-posedness for the Korteweg–de Vries equation on $\mathcal{Y}$ junction on the Sobolev spaces, with low regularity. In Chapter 6 we study local well-posedness for nonlinear Schrödinger equation on star graph. In Chapter 7, we construct standing waves solutions for nonlinear Schrödinger models on star graphs, stationary solutions for the Korteweg–de Vries equation on star-shaped metric graphs and for the sine-Gordon equation on $\mathcal{Y}$ junction. In Chapter 8 we study the stability of soliton-profile for the Korteweg–de Vries equation on the half-line. Chapter 9 develops a linear instability criterium of stationary solutions for the Korteweg–de Vries model on a star-shaped metric graph and we obtain the linear instability of tail and bump profiles on balanced star-shaped metric graphs. Chapter 10 is dedicated to the stability theory of standing wave solutions for nonlinear Schrödinger models on star graphs. We finish these notes with three appendices. The first one establishes the basic tools of the Theory of distributions. The second appendix we define the classical Sobolev spaces on the half-line and the Bourgain spaces and we describe the fundamental properties of them. Finally, the third appendix contains explanation of the spectrum and resolvent for linear operators. One specific self-contained exposition is given to the Riesz projection and its relation with the decomposition of the spectrum.

2 Metric and Quantum Graphs

In this chapter, we introduce the main framework objects that appear in our study: metric graphs and quantum graphs. We recall that a graph consist of a set of point (vertices) and a set of segments (edges) connecting some of the vertices. More notions and results concerning graph theory can be found in (Berkolaiko and Kuchment 2013).

In the following we collect a few results necessary about metric graph and quantum graph in a self-contained presentation as possible. For further details we refer to (Berkolaiko and Kuchment 2013; Post 2012) and references therein. In a metric graph attention is focused on the edges. Quantum graphs are essentially metric graphs equipped with differential operators.

We start with the following definition.

Definition 2.1. *A discrete graph* $\mathcal{G} \equiv (\mathrm{V}, \mathrm{E}, \partial)$ *consists of a finite or countably infinite set of vertices* $=\{v_i\}$*, a set of adjacent edges at the vertices* $\mathrm{E} = \{e_j\}$*, internal and/or external, and a orientation map* $\partial : \mathrm{E} \to \mathrm{V} \times \mathrm{V}$ *which associates to each internal* e_j *edge the pair* $(\partial_- e_j, \partial_+ e_j)$*, of its initial and terminal vertex, and to an external edge its initial vertex only.*

Each internal edge e_j of the graph can be identified with a finite segment $I_j = [0, \ell_j]$ of the real line, such that 0 correspond to its initial vertex and ℓ_j to its terminal one; each external edge e_j, with the half-line $[0, +\infty)$ (for instance), with 0 corresponding to its initial vertex. This defines a natural topology on $\mathcal{G}$ (the space of union of all edges).

Definition 2.2. *A metric graph is a discrete graph together with the set of edge lengths* $\{\ell_j\}_j$*, equipped with a natural metric, with the distance of two points to be the length of the shortest path in* $\mathcal{G}$ *linking the points.*

Roughly speaking, now we will see the edges of $\mathcal{G}$ not as abstract relations between vertices, but rather as physical "*wires*" or "*networks*" connecting them.

In these notes we will consider two class of metric graphs:

1) **Star graph**: A metric graph $\mathcal{G}$ given by finite number $n \in \mathbb{N}^*$, $n \geqq 3$, of infinite length edges attached to a common vertex, $\nu = 0$, having each edge identified with a copy of the half-line $[0, +\infty)$ (see Figure 1.2).

2) **Star-shaped metric graph**: A metric graph $\mathcal{G}$ with a structure represented by a finite or countable edges attached to a common vertex, $\nu = 0$, having each edge identified with a copy of the half-line $(-\infty, 0]$ or $[0, +\infty)$(see Figure 1.3).

In the case of a star-shaped metric graph is usual to use the notation for the edge's set E as $\mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+$, where E_- represents the collection of negative semi-infinite edges and E_+ represents the collection of positive semi-infinite edges. We will use the notation $|\mathrm{E}_\pm|$ for the number of edges.

A star-shaped metric graph $\mathcal{G}$ with $\mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+$ and $|\mathrm{E}_-| = |\mathrm{E}_+|$ it is called a *balanced star-shaped metric graph*.

By the abuse of language, we will call *a star-shaped metric graph as a star graph also or a star-shaped graph*.

Now, the notation $\mathrm{e} \in \mathrm{E}$ will be taken to mean that e is a edge of $\mathcal{G}$. This identification introduced a coordinate x_e along the edge e.

The reader should note that we do not assume the graph to be embedded in any way into a Euclidean space or a Riemann manifold. In some applications such a natural embedding does exist (*e.g.*, in modeling quantum wires circuits, carbon nanotubes), and in such cases the coordinate along an edge is usually the induced arc length. In some other applications (*e.g.*, in quantum chaos) the graph does not need to be embedded anywhere and can be considered as an abstract complex.

We identify any function u on $\mathcal{G}$ (notation: $\mathrm{u} : \mathcal{G} \to \mathbb{C}$) with a collection $(u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$ of functions u_e defined on the edges e of $\mathcal{G}$. Each u_e can be considered as a function on the interval (finite or semi-infinite) I_e. Thus, we will use the same notation u_e for both the function on the edge e and the function on the interval I_e identified with e.

Definition 2.3. *A Quantum graph is a metric graph equipped with a differential operator* $\mathcal{H}$ *(Hamiltonian), accompanied by "appropriate" vertex conditions.*

Former definition deserve some comments: a metric graph becomes a quantum one after being equipped with an additional structure: assignment of a differential operator acting on each edge of $\mathcal{G}$, denoted by $\mathcal{H}$. In most cases, but not always, $\mathcal{H}$ is required to be self-adjoint. In many (probably most) cases $\mathcal{H}$ acts as the negative second order derivative acting on each edge:

$$u_\mathrm{e}(x) \to -\frac{d^2}{dx^2} u_\mathrm{e}(x),$$

where x is the coordinate along the edge e. Thus, for $\mathcal{G}$ being a star graph determined by N half-lines, $(0,+\infty)$, attached to the common vertex $\nu = 0$, the action of second order derivate can be represented by the diagonal-matrix Schrödinger operator

$$\mathcal{H} = \Big(\Big(-\frac{d^2}{dx^2}\Big)\delta_{i,j}\Big)$$

where $\delta_{i,j}$, $1 \leqslant i, j \leqq N$, it denotes the delta de Kronecker. Thus, for a function $\mathrm{u} : \mathcal{G} \to \mathbb{C}$, $\mathrm{u} = (u_j)_{j=1}^N$, we have the action

$$\mathcal{H}\mathrm{u}(x) = \Big(\Big(-\frac{d^2}{dx^2}u_j(x)\Big)\Big)_{j=1}^N, \quad x > 0.$$

In the next chapter, via the extension theory of symmetric operators, we will determine several vertex conditions for $\mathcal{H}$ becomes a self-adjoint operator. By instance, the domain $D_{\alpha,\delta}(\mathcal{A})$ in (1.2) becomes $\mathcal{H}$ a self-adjoint operator.

The following example shows that the Hamiltonian $\mathcal{H}$ on a quantum graph can be more general. It considers $\mathcal{G}$ to be a star-shaped graph determined by the structure $\mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+$. Thus for a function $\mathrm{u} : \mathcal{G} \to \mathbb{C}$, $\mathrm{u} = (u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$ we obtain the following Airy type operator

$$A_0 : (u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}} \to \Big(\alpha_\mathrm{e}\frac{d^3}{dx^3}u_\mathrm{e} + \beta_\mathrm{e}\frac{d}{dx}u_\mathrm{e}\Big)_{\mathrm{e}\in\mathrm{E}}, \tag{2.1}$$

where $(\alpha_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$ and $(\beta_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$ are two sequences of real numbers.

In the next chapter, via the extension theory of symmetric operators, we will determine several vertex conditions for A_0 becomes a skew-self-adjoint operator. By instance, the domain $D(H_Z)$ in (4.24) becomes $(A_0, D(H_Z))$, in a family of skew-self-adjoint operators for A_0 parametrized by $Z \in \mathbb{R}$.

Now, such as in the classical case of differential operators on a single segment (*i.e.*, a graph with one edge) makes clear that the definition of the quantum graph Hamiltonian is not complete until its domain is described. Our experience shows that the domain description should involve smoothness conditions along the edges and some junctions conditions at the vertices. Moreover, for the self-adjoint property of the Hamiltonian $\mathcal{H}$ will require a more delicate study such as is established in the next chapter.

Next, we give a first step for domain description of a Hamiltonian $\mathcal{H}$ on a quantum graph with regard to some smoothness conditions along the edges. Indeed, the Lebesgue measures on the intervals I_e (being $(-\infty, 0)$ or $(0,+\infty)$) induce a Lebesgue measure on the space $\mathcal{G}$. We introduced the Hilbert space $L^2(\mathcal{G})$ as the space of measurable and square-integrable functions on each edge of $\mathcal{G}$, *i.e.*

$$L^2(\mathcal{G}) = \bigoplus_{\mathrm{e}\in\mathrm{E}} L^2(I_\mathrm{e}), \quad \|\mathrm{u}\|^2_{L^2(\mathcal{G})} = \sum_{e\in\mathrm{E}} \int_{I_\mathrm{e}} |u_\mathrm{e}(x)|^2 dx$$

with $\mathrm{u} = (u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$, where $u_\mathrm{e} \in L^2(I_\mathrm{e})$ is a complex valued function. The inner product

$\langle\cdot,\cdot\rangle$ is the one induced by the usual inner product in $L^2(\mathbb{R})$, *i.e*

$$\langle \mathrm{u}, \mathrm{v}\rangle = \sum_{e\in \mathrm{E}} \int_{I_{\mathrm{e}}} u_{\mathrm{e}}(x)\overline{v_{\mathrm{e}}(x)}dx,$$

with $\mathrm{u} = (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$ and $\mathrm{v} = (v_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$. Analogously, given $1 \leqq p \leqq \infty$ one can defined $L^p(\mathcal{G})$ as the set of functions whose components are elements of $L^p(I_{\mathrm{e}})$ and the corresponding norm for $1 \leqq p < \infty$ by

$$L^p(\mathcal{G}) = \bigoplus_{\mathrm{e}\in\mathrm{E}} L^p(I_{\mathrm{e}}), \quad \|\mathrm{u}\|^p_{L^p(\mathcal{G})} = \sum_{e\in\mathrm{E}} \int_{I_{\mathrm{e}}} |u_{\mathrm{e}}(x)|^p dx,$$

and for $p = \infty$ as

$$L^\infty(\mathcal{G}) = \bigoplus_{\mathrm{e}\in\mathrm{E}} L^\infty(I_{\mathrm{e}}), \quad \|\mathrm{u}\|_{L^\infty(\mathcal{G})} = \sup_{e\in\mathrm{E}} \|u_{\mathrm{e}}\|_{L^\infty(I_{\mathrm{e}})},$$

with $\mathrm{u} = (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$.

The Sobolev space $H^n(\mathcal{G})$, $n \geqq 1$ an integer, it is defined by

$$H^n(\mathcal{G}) = \bigoplus_{\mathrm{e}\in\mathrm{E}} H^n(I_{\mathrm{e}}), \quad \|\mathrm{u}\|^2_{H^n(\mathcal{G})} = \sum_{e\in\mathrm{E}} \|u_{\mathrm{e}}\|^2_{H^n(I_{\mathrm{e}})}$$

where $H^n(I_{\mathrm{e}})$ is the classical Sobolev space on I_{e}. We emphasize that in this definition we are not assuming any condition on the values of the functions at the joint point $\nu = 0$. Moreover, each component u_{e} of u is a continuous function on I_{e}, but for u being see as a function on $\mathcal{G}$ does not need to be continuous at $\nu = 0$.

In order to prove a well-posedness result in Chapter 6 we need to generalize standard one-dimensional Gagliardo-Nirenberg inequality to graphs, i.e

$$\|\mathrm{U}\|_q \leqslant C\|\mathrm{U}'\|^{\frac{1}{2}-\frac{1}{q}} \|\mathrm{U}\|^{\frac{1}{2}+\frac{1}{q}}, \quad q > 2,\ C > 0. \tag{2.2}$$

The proof of (2.2) follows immediately from the analogous estimates for functions of the real line, considering that any function in $H^1(\mathbb{R}^+)$ can be extended to an even function in $H^1(\mathbb{R})$, and applying this reasoning to each component of U.

3 von Neumann and Krein Theory and its Applications

In this chapter we give the basic theory of extension for closed, symmetric operators of von Neumann and Krein. This theory give us one way to construct extensions of a given closed, symmetric operator A densely defined. Two fundamental issues will be established here, which are related to our stability theory for stationary solutions for the Korteweg–de Vries model or standing waves solutions for the nonlinear Schrödinger equation on quantum graphs. The first one issue is about the problem of setting up conditions under which an closed, symmetric operator shall have self-adjoint extension and how to construct all the self-adjoint extensions. The second one issue is concerned about how to estimative the Morse Index of every self-adjoint extension.

3.1 Self-adjoint extensions of symmetric operators

3.1.1 Statement of the problem

One of the fundamental problems in the theory of symmetric operators is to construct all those extensions of a given symmetric operator A which are themselves symmetric operators. A special case of this situation is the problem of setting up conditions under which an operator shall have a self-adjoint extension and to construct all the self-adjoint extensions when these conditions hold.

If B is a symmetric extension of a symmetric operator A, then $A \subset B$ (namely, $D(A) \subset D(B)$ and $Bx = Ax$ for every $x \in D(A)$), and so $B^* \subset A^*$. But B is a symmetric operator, i.e., $B \subset B^*$; and so we have

$$A \subset B \subset B^* \subset A^*, \tag{3.1}$$

i.e., *every symmetric extension of an operator A is a restriction of the operator A^*.*

A symmetric operator A is said to be *maximal* if it has no proper symmetric extension. So, we obtain from (3.1) that every self-adjoint operator A is a maximal, symmetric operator

3.1.2 Deficiency subspaces and deficiency indices of a symmetric operator

For A being a densely defined symmetric operator on a Hilbert space H and A^* its adjoint, we consider the subspaces

$$\mathcal{D}_+ = \ker(A^* - i), \quad \text{and} \quad \mathcal{D}_- = \ker(A^* + i), \tag{3.2}$$

$\mathcal{D}_+$ and $\mathcal{D}_-$ are called the *deficiency subspaces* of A. The pair of numbers n_+, n_-, given by

$$n_+(A) = \dim[\mathcal{D}_+], \quad \text{and} \quad n_-(A) = \dim[\mathcal{D}_-]$$

are called the *deficiency indices* of A.

Theorem 3.1. *Let A be a closed, symmetric operator, then*

$$D(A^*) = D(A) \oplus \mathcal{D}_- \oplus \mathcal{D}_+. \tag{3.3}$$

Therefore, for $u \in D(A^*)$ and $u = x + y + z \in D(A) \oplus \mathcal{D}_- \oplus \mathcal{D}_+$ we have the following complete description of the operator A^*

$$A^* u = Ax + (-i)y + iz. \tag{3.4}$$

Remark 3.1. *The direct sum in (3.3) is not necessarily orthogonal.*

Proof. We start by proving that for

$$x + y + z = 0, \quad x \in D(A), y \in D_-, z \in D_+ \tag{3.5}$$

we obtain, $x = y = z = 0$. Indeed, applying $(A^* - i)$ to both sides of (3.5), we get

$$(A - i)x + (-2i)y = 0. \tag{3.6}$$

Now, since $(ImB)^\perp = ker(B^*)$, we have $(A - i)x \perp y$ and so $(A - i)x = 0$ and $y = 0$. Since i belongs to the resolvent set of the symmetric operator A we also obtain $x = 0$. Therefore, from (3.5) it follows $z = 0$.

For the formula (3.3), since each of the subspaces $D(A)$, D_- and D_+ are contained in $D(A^*)$ it follows

$$D(A) \oplus \mathcal{D}_- \oplus \mathcal{D}_+ \subseteq D(A^*). \tag{3.7}$$

Next we prove the converse relation in (3.7). Since the operator A is closed, $\mathcal{R} \equiv Im(A-i)$ is a closed subspace and so we have the standard decomposition

$$\mathcal{R} \oplus \mathcal{D}_- = H. \tag{3.8}$$

Now, let $u \in D(A^*)$ then from (3.8): $v \equiv (A^* - i)u = v_1 + v_2 \in \mathcal{R} \oplus \mathcal{D}_-$ where

$$v_1 = (A-i)x, \quad v_2 = -2iy, \quad x \in D(A), \quad y \in \mathcal{D}_-. \tag{3.9}$$

Since $A^*x = Ax$ and $A^*y = -iy$ we obtain

$$(A^* - i)u = (A-i)x - 2iy = (A^* - i)(x+y). \tag{3.10}$$

Therefore, for $z \equiv u - (x+y)$ we have $(A^* - i)z = 0$ and so $z \in \mathcal{D}_+$. Hence,

$$u = x + y + z \in D(A) \oplus \mathcal{D}_- \oplus \mathcal{D}_+.$$

□

Corollary 3.1. *A closed, symmetric operator is self-adjoint if and only if its two deficiency spaces are equal to* 0*, i.e.,* $\mathcal{D}_- = \mathcal{D}_+ = \{0\}$.

Proof. From formula (3.3) follows immediately that in this case, and only in this case, $D(A^*) = D(A)$. □

Next we give some generic result that will be used later in Chapters 8 and 9 (see Naĭmark 1969, Chapter 4 for the proof).

Proposition 3.1. *Let A be a closed, symmetric operator.*

1) *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the deficiency spaces of the operators A and $B = \alpha A + \beta I$ have the same dimension.*

2) *For every complex number λ with $Re\lambda > 0$ define*

$$\mathcal{D}_{+,\lambda} = ker(A^* - \lambda I), \quad \mathcal{D}_{-,\lambda} = ker(A^* - \bar{\lambda} I).$$

Then, $dim[\mathcal{D}_{+,\lambda}] = dim[\mathcal{D}_{+,i}] = n_+(A)$, $dim[\mathcal{D}_{-,\bar{\lambda}}] = dim[\mathcal{D}_{-,-i}] = n_-(A)$.

3) *If B is a bounded, self-adjoint operator defined in the whole space H, then the operator A and $A + B$ have the same deficiency indices.*

3.1.3 Construction of the symmetric extensions of a given symmetric operator

The theory to be established in the following represents the heart of the extension theory for symmetric operators developed by Krein and von Neumann . We shall consider only closed, symmetric extensions. We note that every such extension is at the same time an extension of the closure $\bar{A}$ of the operator A; hence, without loss of generality, we will suppose A to be closed, symmetric operator.

The proof of the following result can be found in Reed and Simon (1975).

Theorem 3.2. *The closed, symmetric extension of a given closed, symmetric operator A are in one-to-one correspondence with the set of partial isometries of $\mathcal{D}_+$ in $\mathcal{D}_-$.*

More exactly, if U is such an isometry whose domain of definition $I(U)$ is a closed subspace of $\mathcal{D}_+$ and whose range $R(U)$ is a closed subspace of $\mathcal{D}_-$, then the corresponding closed symmetric extension A_U has domain

$$D(A_U) = \{x + z + Uz : x \in D(A), z \in I(U)\}, \tag{3.11}$$

and the relation

$$A_U(x + z + Uz) = Ax + iz - iUz, \tag{3.12}$$

holds.

Conversely, for each such operator U these formulas determine a certain closed, symmetric extension A_U of the operator A and the deficiency spaces of A_U, $\mathcal{D}_{\pm,U}$ are

$$\mathcal{D}_{+,U} = \mathcal{D}_+ - I(U) = I(U)^\perp, \quad \mathcal{D}_{-,U} = \mathcal{D}_- - R(U) = R(U)^\perp. \tag{3.13}$$

Here $X - Y$ represents the orthogonal complement of Y in X.

The most important case of Theorem 3.2 for us here will be when A_U is a self-adjoint extension of A.

Theorem 3.3. *An extension A_U of a closed, symmetric operator A is self-adjoint if and only if the domain $I(U)$ of the isometric operator U coincides with $\mathcal{D}_+$ and its range $R(U)$ with $\mathcal{D}_-$.*

A closed, symmetric operator A has a self-adjoint extension if and only if its deficiency spaces $\mathcal{D}_+$ and $\mathcal{D}_-$ have the same dimension, i.e. if its deficiency indices are equal $n_+(A) = n_+(A)$.

Proof. By Corollary 3.1, an extension A_U of A is self-adjoint if and only if $\mathcal{D}_{+,U} = \{0\}$ and $\mathcal{D}_{-,U} = \{0\}$, *i.e. if and only if $I(U) = \mathcal{D}_+$ and $R(U) = \mathcal{D}_-$. This finishes the proof.* □

Theorem 3.3 is the basis for our strategy in studying stability properties of stationary waves for the Korteweg–de Vries model or standing waves solutions for the nonlinear Schrödinger equation on metric graphs. Thus, we will construct specific self-adjoint extension of the Laplace operator on start graphs.

We start with a general construction when $\mathcal{D}_+$ and $\mathcal{D}_-$ have the same dimension n, namely, every extension is self-adjoint. We choose in $\mathcal{D}_+$ any orthonormal basis $\mathcal{B}_+ = \{e_1, e_2, \cdots, e_n\}$ and in $\mathcal{D}_-$ an orthonormal basis $\mathcal{B}_- = \{f_1, f_2, \cdots, f_n\}$. Then for $z \in \mathcal{D}_+$ we have

$$z = \sum_{j=1}^{n} \xi_j e_j,$$

and so for every isometric operator U with domain $\mathcal{D}_+$ and the range $\mathcal{D}_-$ is given by the formula

$$Uz = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} u_{jk} \xi_k \Big) f_j,$$

where $u = [u_{ij}]$ is a unitary matrix. Thus, in the case considered, $D(A_U)$ consist of all vectors

$$w = x + \sum_{j=1}^{n} \xi_j e_j + \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} u_{jk} \xi_k \Big) f_j, \quad x \in D(A) \tag{3.14}$$

$$A_U w = Ax + i \sum_{j=1}^{n} \xi_j e_j - i \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} u_{jk} \xi_k \Big) f_j \tag{3.15}$$

We remember that the following conditions are equivalent:

1) u is a unitary matrix .
2) u^* is a unitary matrix .
3) u is invertible with $u^{-1} = u^*$.
4) The columns of u form an orthonormal basis of $\mathbb{C}^n$ with respect to the usual inner product.
5) The rows of u form an orthonormal basis of $\mathbb{C}^n$ with respect to the usual inner product.
6) u is an isometry with respect to the usual norm of $\mathbb{C}^n$.

The following particular case will be very useful in our stability approach in Chapters 9 and 10.

Proposition 3.2. *Let A be a densely defined, closed, symmetric operator in some Hilbert space H with deficiency indices equal $n_\pm(A) = 1$. All self-adjoint extensions A_θ of A may parametrized by a real parameter $\theta \in [0, 2\pi)$ where*

$$D(A_\theta) = \{x + c\phi_+ + ce^{i\theta}\phi_- : x \in D(A), c \in \mathbb{C}\},$$
$$A_\theta(x + c\phi_+ + ce^{i\theta}\phi_-) = Ax + ic\phi_+ - ice^{i\theta}\phi_-,$$

where $A^*\phi_\pm = \pm i\phi_\pm$, and $\|\phi_+\| = \|\phi_-\|$.

Proof. From Theorem 3.3 all the symmetric extension A_U of A are self-adjoint and determined by the unitary operator $U : D_+ = [\phi_+] \to D_- = [\phi_-]$. Since U is represented by a unitary matrix $u = [\omega]$ of order 1×1, we have that $\omega\bar{\omega} = 1$ and so $\omega = e^{i\theta}$, $\theta \in [0, 2\pi)$. This finishes the proof.

□

3.1.4 Self-adjoint extensions for point interactions

Next, we give some examples of self-adjoint extensions associated to point interactions on the line and on star graphs.

Example 1. **δ-point interactions on the line**

Theorem 3.4. *Let $\mathcal{H} = -\frac{d^2}{dx^2}$ be the self-adjoint operator acting in the Hilbert space $H^2(\mathbb{R})$. Then the restriction $A \equiv \mathcal{H}|_{D(A)}$, where*

$$D(A) = \{\psi \in H^2(\mathbb{R}) : \psi(0) = 0\}, \tag{3.16}$$

is a densely defined symmetric operator with deficiency indices equal to 1. *Namely,*

(a) *symmetric: $\langle A\psi, \varphi\rangle = \langle \psi, A\varphi\rangle$ for $\psi, \varphi \in D(A)$;*

(b) *dense: $\overline{D(A)} = L^2(\mathbb{R})$;*

(c) *deficiency elements:*

$$\begin{cases} \text{for } \lambda = i, & g_i(x) \equiv e^{i\sqrt{i}|x|}, \quad Im\sqrt{i} > 0 \\ \text{for } \lambda = -i, & g_{-i} \equiv e^{i\sqrt{-i}|x|}, \quad Im\sqrt{-i} > 0, \end{cases} \tag{3.17}$$

satisfy $g_{\pm i} \in D(A^*)$ and $A^* g_{\pm i} = \pm i g_{\pm i}$. Moreover, we have $n_+(A) = n_-(A) = 1$.

(d) *$D(A^*) = H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R})$ and $A^* = -\frac{d^2}{dx^2}$.*

(e) *All the self-adjoint extension of A can be parametrized by a parameter $Z \in \mathbb{R}$, A_Z, such that*

$$\begin{cases} A_Z u = -\dfrac{d^2}{dx^2} u, \\ D(A_Z) = \{u \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) : \\ \qquad\qquad u'(0+) - u'(0-) = Z u(0+)\}. \end{cases} \tag{3.18}$$

The case $Z = 0$ just leads to the self-adjoint operator $\mathcal{H}$ with domain $H^2(\mathbb{R})$.

Proof. 1) The symmetric property of A follows immediately from that of the definition of the operator $\mathcal{H}$.

2) The operator $\mathcal{H}$ is densely defined and thus for every $f \in L^2(\mathbb{R})$ there exists $\{f_n\} \subset H^2(\mathbb{R})$ such that $\lim_{n\to+\infty} \|f - f_n\| = 0$. The functional δ of Dirac is not a bounded functional on the space $L^2(\mathbb{R})$. Then there exists a sequence $\{\psi_n\} \subset H^2_{per}$ with $\|\psi_n\| = 1$ such that $\delta(\psi_n) = \langle\delta, \psi_n\rangle = \psi_n(0) \to \infty$, as $n \to \infty$. Now, since δ is a bounded linear functional on $H^2(\mathbb{R})$, we can choose this sequence such that

$$\lim_{n\to+\infty} \frac{\langle\delta, f_n\rangle}{\langle\delta, \psi_n\rangle} = 0.$$

Define the sequence $\zeta_n = f_n - \langle\delta, f_n\rangle\psi_n/\langle\delta, \psi_n\rangle$. Then $\{\zeta_n\} \subset D(A)$ and

$$\|\zeta_n - f\| \leqq \|f_n - f\| + \left|\frac{\langle\delta, f_n\rangle}{\langle\delta, \psi_n\rangle}\right| \to 0 \quad \text{as } n \to \infty.$$

Thus, the operator A is densely defined in $L^2(\mathbb{R})$.

3) Since $(\mathcal{H} - i)^{-1} \in B(H^{-2}(\mathbb{R}); L^2(\mathbb{R}))$ and $\delta \in H^{-2}(\mathbb{R})$ we have $g_i \equiv (\mathcal{H} - i)^{-1}\delta \in L^2(\mathbb{R})$, represents the fundamental solution associated to the operator $(\mathcal{H} - i)$. Since $\widehat{\delta}(k) = 1$, for $\psi \in D(A) \subset D(\mathcal{H})$ we obtain

$$\begin{aligned}\langle A\psi, g_i\rangle = \langle \mathcal{H}\psi, (\mathcal{H}-i)^{-1}\delta\rangle &= \int_{-\infty}^{+\infty} k^2\widehat{\psi}(k)\overline{\frac{1}{k^2-i}\widehat{\delta}(k)}dk \\ &= \psi(0) + \int_{-\infty}^{+\infty} \widehat{\psi}(k)\overline{i\,\widehat{g_i}(k)}dk = \langle\psi, i g_i\rangle.\end{aligned} \tag{3.19}$$

So, $g_i \in D(A^*)$ and $A^* g_i = i g_i$. A similar analysis show $g_{-i} \in D(A^*)$ and $A^* g_{-i} = -i g_{-i}$. Lastly, it is well known that $g_{\pm i} \equiv e^{i\sqrt{\pm i}|x|}$, $Im\sqrt{\pm i} > 0$.

The deficiency element g_i is unique (up to multiplication by complex numbers). We introduce the following norm $\|\cdot\|_{2,*}$ in the space $H^2(\mathbb{R})$, which is equivalent to the standard norm in this space,

$$\begin{aligned}\|f\|^2_{2,*} \equiv \|(-\partial_x^2 - i)f\|^2 &= \int_{-\infty}^{+\infty} |(k^2 - i)\widehat{f}(k)|^2 dk \\ &= \langle(\partial_x^4 + 1)^{1/2} f, (\partial_x^4 + 1)^{1/2} f\rangle.\end{aligned} \tag{3.20}$$

Since δ is a bounded linear on $(H^2(\mathbb{R}), \|\cdot\|_{2,*})$, the kernel

$$ker(\delta) = \{f \in H^2(\mathbb{R}) : \delta(f) = f(0) = 0\} = D(A),$$

it is a hyperplane of codimension 1. Next for $h_0 \equiv (\mathcal{H} + i)^{-1} g_i \in H^2(\mathbb{R})$ we have $h_0 \perp ker(\delta)$. In fact, for $f \in ker(\delta)$

$$\langle(\partial_x^4 + 1)^{1/2} h_0, (\partial_x^4 + 1)^{1/2} f\rangle = \int_{-\infty}^{+\infty} \overline{\widehat{f}(k)}dk = \overline{f(0)} = 0.$$

Next, suppose $f_0 \in D(A^*)$ such that $A^* f_0 = i f_0$. Let $\psi \in D(A) \subset D(\mathcal{H})$, then

$$\langle A\psi, f_0\rangle = \langle \psi, A^* f_0\rangle = \langle \psi, i f_0\rangle.$$

Therefore, $\langle (A+i)\psi, f_0\rangle = 0$. Now, we show that for $h_1 \equiv (A+i)^{-1} f_0 \in H^2(\mathbb{R})$ satisfies $h_1 \perp ker(\delta)$. Let $\psi \in ker(\delta)$, then from the above analysis we obtain

$$\begin{aligned}\langle (\partial_x^4+1)^{1/2}\psi, (\partial_x^4+1)^{1/2}h_1\rangle &= \int_{-\infty}^{+\infty}(k^2+i)\widehat{\psi}(k)\overline{\widehat{f_0}(k)}dk \\ &= \langle (A+i)\psi, f_0\rangle = 0.\end{aligned}$$

So, there exists $\lambda \in \mathbb{C}$ such that $f_0 = \lambda g_i$.

4) From Theorem 3.1 we have

$$D(A^*) = D(A) \oplus [\psi_i] \oplus [\psi_{-i}],$$

where $\psi_{\pm i}(x) = \frac{i}{2\sqrt{\pm i}} g_{\pm i}(x)$. Since $D(A) \subset H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R})$ and $g_{\pm i} \in H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R})$ we obtain $D(A^*) \subset H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R})$. Next, let $z \in H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R})$ and $\psi \in D(A)$ then the relation

$$\begin{aligned}\langle A\psi, z\rangle &= \psi(0-)z'(0-) - \psi(0+)z'(0+) + \langle \psi, -z''\rangle \\ &= \langle \psi, -z''\rangle\end{aligned}$$

shows $H^2(\mathbb{R}-\{0\}) \cap H^1(\mathbb{R}) \subset D(A^*)$ and $A^* = -\frac{d^2}{dx^2}$.

5) From Proposition 3.2 every self-adjoint extensions A_θ of A may parametrized by a real parameter $\theta \in [0, 2\pi)$ where

$$D(A_\theta) = \{\phi + c\psi_i + ce^{i\theta}\psi_{-i} : \phi \in D(A), c \in \mathbb{C}\}, \tag{3.21}$$

$$A_\theta\psi(x) = -\frac{d^2}{dx^2}\psi(x), \quad \text{for } x \neq 0. \tag{3.22}$$

Next, we characterize $D(A_\theta)$. Indeed, for

$$\psi = \phi + c\psi_i + ce^{i\theta}\psi_{-i}$$

we obtain that

$$\psi'(0+) - \psi'(0-) = -c(1+e^{i\theta}),$$

with

$$\psi(0+) = c\Big(\frac{i}{2\sqrt{i}} + \frac{i}{2\sqrt{-i}}e^{i\theta}\Big).$$

Therefore, by defining $Z = Z(\theta)$ as

$$Z(\theta) = -\frac{2cos(\frac{\theta}{2})}{cos(\frac{\theta}{2} - \frac{\pi}{4})}, \quad \theta \in [0, 2\pi)$$

we obtain $-\infty < Z \leqq +\infty$ and so

$$\psi'(0+) - \psi'(0-) = Z\psi(0+).$$

This completes the proof of the theorem.

□

Example 2. **δ-point interactions on a star graph**

The following result will be used in the study of the nonlinear stability of standing wave solutions for the nonlinear Schrödinger model (10.1) in Chapter 10.

Theorem 3.5. *Let $\mathcal{G}$ be a star graph determined by N half-lines,* $(0, +\infty)$*, attached to the common vertex $\nu = 0$. The diagonal-matrix Schrödinger operator on $L^2(\mathcal{G})$*

$$\mathrm{L}_0 = \Big(\Big(-\frac{d^2}{dx^2}\Big)\delta_{i,j}\Big)$$

with domain

$$D(\mathrm{L}_0) = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0) = 0, \sum_{j=1}^{N} v_j'(0) = 0\Big\}, \tag{3.23}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(\mathrm{L}_0) = 1$. Moreover, all the self-adjoint extension of L_0 can be parametrized by a parameter $Z \in \mathbb{R}$, L_Z, such that

$$\begin{cases} \mathrm{L}_Z u = \Big(\Big(-\dfrac{d^2}{dx^2}u\Big)\delta_{i,j}\Big), \\ D(\mathrm{L}_Z) = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0), \\ \qquad\qquad \displaystyle\sum_{j=1}^{N} v_j'(0) = Zv_1(0),\ Z \in \mathbb{R}\Big\}. \end{cases} \tag{3.24}$$

Proof. The property of L_0 to be a densely defined symmetric operator follows the same strategy as in the proof of Theorem 3.4. Next we determine that the adjoint operator of $(\mathrm{L}_0, D(\mathrm{L}_0))$ is given by

$$\mathrm{L}_0^* = \mathrm{L}_0, \quad D(\mathrm{L}_0^*) = \big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0)\big\}. \tag{3.25}$$

It is immediate to see that $D(\mathrm{L}_0^*) \subset H^2(\mathcal{G})$ and the action $\mathrm{L}_0^* = \mathrm{L}_0$. Next, by denoting

$$D_0^* := \{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0)\},$$

we easily arrive at $D_0^* \subseteq D(\mathrm{L}_0^*)$. Indeed, for any $\mathrm{U} = (u_j)_{j=1}^N \in D_0^*$ and $\mathrm{V} = (v_j)_{j=1}^N \in D(\mathrm{L}_0)$ denoting $\mathrm{U}^* = \mathrm{L}_0(\mathrm{U}) \in L^2(\mathcal{G})$, we get

$$\begin{aligned}\langle \mathrm{L}_0\mathrm{V}, \mathrm{U}\rangle &= \langle \mathrm{V}, \mathrm{L}_0(\mathrm{U})\rangle + \sum_{j=1}^N \left[-v_j' u_j + v_j u_j'\right]_0^\infty \\ &= \langle \mathrm{V}, \mathrm{L}_0(\mathrm{U})\rangle = \langle \mathrm{V}, \mathrm{U}^*\rangle,\end{aligned}$$

which, by definition of the adjoint operator, means that $\mathrm{U} \in D(\mathrm{L}_0^*)$ or $D_0^* \subseteq D(\mathrm{L}_0^*)$. Let us show the inverse inclusion $D_0^* \supseteq D(\mathrm{L}_0^*)$. Take $\mathrm{U} \in D(\mathrm{L}_0^*)$, then for any $\mathrm{V} \in D(\mathrm{L}_0)$ we have

$$\begin{aligned}\langle \mathrm{L}_0\mathrm{V}, \mathrm{U}\rangle &= \langle \mathrm{V}, \mathrm{L}_0(\mathrm{U})\rangle + \sum_{j=1}^N \left[-v_j' u_j + v_j u_j'\right]_0^\infty \\ &= \langle \mathrm{V}, \mathrm{L}_0^*\mathrm{U}\rangle = \langle \mathrm{V}, \mathrm{L}_0(\mathrm{U})\rangle.\end{aligned}$$

Thus, we arrive at the equality

$$\sum_{j=1}^N \left[-v_j' u_j + v_j u_j'\right]_0^\infty = \sum_{j=1}^N v_j'(0) u_j(0) = 0 \tag{3.26}$$

for any $\mathrm{V} \in D(\mathrm{L}_0)$. Let $\mathrm{W} = (w_j)_{j=1}^N \in D(\mathrm{L}_0)$ such that $w_3'(0) = w_4'(0) = ... = w_N'(0) = 0$. Then for $\mathrm{U} \in D(\mathrm{L}_0^*)$ from (3.26) we get that

$$\sum_{j=1}^N w_j'(0) u_j(0) = w_1'(0) u_1(0) + w_2'(0) u_2(0) = 0. \tag{3.27}$$

Recalling that

$$\sum_{j=1}^N w_j'(0) = w_1'(0) + w_2'(0) = 0$$

and assuming $w_2'(0) \neq 0$, we obtain from (3.27) the equality $u_1(0) = u_2(0)$. Repeating the similar arguments for $\mathrm{W} = (w_j)_{j=1}^N \in D(\mathrm{L}_0)$ such that $w_4'(0) = w_5'(0) = ... = w_N'(0) = 0$, we get $u_1(0) = u_2(0) = u_3(0)$ and so on. Finally, taking $\mathrm{W} = (w_j)_{j=1}^N \in D(\mathrm{L}_0)$ such that $w_N'(0) = 0$, we arrive at $u_1(0) = u_2(0) =$

$\ldots = u_{N-1}(0)$, and consequently $u_1(0) = u_2(0) = \ldots = u_N(0)$. Thus, $\mathrm{U} \in D_0^*$ or $D_0^* \supseteq D(\mathrm{L}_0^*)$, and (3.25) holds.

Now, it is not difficult to see that $\mathcal{D}_\pm = ker(\mathrm{L}_0^* \mp i) = [\mathrm{V}_{\pm i}]$ where

$$\mathrm{V}_{\pm i} = (e^{i\sqrt{\pm i}x})_{j=1}^N, \quad Im\sqrt{\pm i} > 0.$$

Thus, due to Proposition 3.2 every self-adjoint extensions $\mathrm{L}_{0,\theta}$ of L_0 may parametrized by a real parameter $\theta \in [0, 2\pi)$ where

$$D(\mathrm{L}_{0,\theta}) = \left\{\mathrm{F} = \mathrm{F}_0 + c\mathrm{F}_i + ce^{i\theta}\mathrm{F}_{-i} : \mathrm{F}_0 \in D(\mathrm{L}_0), c \in \mathbb{C}\right\}, \tag{3.28}$$

with

$$\mathrm{F}_{\pm i} = (\tfrac{i}{\sqrt{\pm i}} e^{i\sqrt{\pm i}x})_{j=1}^N, \quad Im\sqrt{\pm i} > 0.$$

Now, it is easily seen that for $\mathrm{F} = (F_j)_{j=1}^N \in D(\mathrm{L}_{0,\theta})$, we have

$$\sum_{j=1}^N F_j'(0+) = -Nc(1 + e^{i\theta}), \quad F_j(0) = c\left(e^{i\pi/4} + e^{i(\theta-\pi/4)}\right).$$

From the last equalities it follows that

$$\sum_{j=1}^N F_j'(0) = ZF_1(0), \text{ where } Z = \frac{-N(1 + e^{i\theta})}{\left(e^{i\pi/4} + e^{i(\theta-\pi/4)}\right)} \in \mathbb{R}.$$

This finishes the proof.

□

Example 3. **δ-point interactions on a balanced star graph**

The following result will be used in the study of the stability of stationary solutions for the Korteweg–de Vries model (1.3) on a balanced metric star graph $\mathcal{G}$ in Chapter 9. Thus, we consider $\mathcal{G}$ with a structure represented by the set

$$\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$$

where E_+ and E_- are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively, and $|\mathrm{E}_-| = |\mathrm{E}_+|$ (a balanced star graph). The half lines are connected at a unique vertex $\nu = 0$.

For $u = (u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}}$ we will use the following abbreviations,

$$u(0-) = (u_\mathrm{e}(0-))_{\mathrm{e}\in\mathrm{E}_-}, \quad u'(0-) = (u_\mathrm{e}'(0-))_{\mathrm{e}\in\mathrm{E}_-},$$

similarly for the terms $u(0+)$ and $u'(0+)$, and also

$$(u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}} = (u_{1,-}, \ldots, u_{n,-}, u_{1,+}, \ldots, u_{n,+}).$$

Theorem 3.6. *Let $\mathcal{G}$ be a balanced star graph with a structure represented by the set* $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ *and* $|\mathrm{E}_-| = |\mathrm{E}_+| = n$. *The* $2n \times 2n$*-diagonal-matrix Schrödinger operator on* $L^2(\mathcal{G})$

$$\mathcal{F}_0 = \Big(\Big(-\frac{d^2}{dx^2}\Big)\delta_{i,j}\Big)$$

with domain

$$D(\mathcal{F}_0) = \{u \in H^2(\mathcal{G}) : u(0-) = u(0+) = 0, \\ \sum_{\mathrm{e}\in\mathrm{E}_+} u'_{\mathrm{e}}(0) - \sum_{\mathrm{e}\in\mathrm{E}_-} u'_{\mathrm{e}}(0) = 0\}, \tag{3.29}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(\mathcal{F}_0) = 1$. *Therefore, we have that all the self-adjoint extensions of* $(\mathcal{F}_0, D(\mathcal{F}_0))$ *can be parametrized by* $Z \in \mathbb{R}$, *namely,* $(\mathcal{L}_Z, D(\mathcal{L}_Z))$, *with the action* $\mathcal{L}_Z \equiv \mathcal{F}_0$ *and* $u \in D(\mathcal{L}_Z)$ *if and only if* $u \in \mathcal{C}$,

$$\mathcal{C} = \{(u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in L^2(\mathcal{G}) : u_{1,-}(0-) = ... = u_{n,-}(0-) = u_{1,+}(0+) \\ = u_{2,+}(0+) = ... = u_{n,+}(0+)\} \tag{3.30}$$

and $u \in D_{Z,\delta}$,

$$D_{Z,\delta} = \Big\{u \in H^2(\mathcal{G}) : u(0-) = u(0+), \\ \sum_{\mathrm{e}\in\mathrm{E}_+} u'_{\mathrm{e}}(0+) - \sum_{\mathrm{e}\in\mathrm{E}_-} u'_{\mathrm{e}}(0-) = Znu_{1,+}(0+)\Big\}. \tag{3.31}$$

Proof. The symmetric property of $(\mathcal{F}_0, D(\mathcal{F}_0))$ is immediate. Since,

$$\bigoplus_{\mathrm{e}\in\mathrm{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathrm{e}\in\mathrm{E}_+} C_c^\infty(0, +\infty) \subset D(\mathcal{F}_0)$$

we obtain the density property of $D(\mathcal{F}_0)$. Now, by following the same ideas in the proof of Theorem 3.5, we can see that the adjoint operator $(\mathcal{F}_0^*, D(\mathcal{F}_0^*))$ of $(\mathcal{F}_0, D(\mathcal{F}_0))$ is given by

$$\mathcal{F}_0^* = \mathcal{F}_0, \quad D(\mathcal{F}_0^*) = \{u \in H^2(\mathcal{G}) : u \in \mathcal{C}\}. \tag{3.32}$$

From (3.32) is not difficult to see that the deficiency indices for $(\mathcal{F}_0, D(\mathcal{F}_0))$ are $n_\pm(\mathcal{F}_0) = 1$. Indeed, $\mathcal{D}_+ = ker(\mathcal{F}_0^* - i) = [\Psi_+]$ where for $\Psi_+ = (\Psi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$ we have

$$\text{for } \mathrm{e} \in \mathrm{E}_-,\ \Psi_{\mathrm{e}} = (\frac{i}{k_-}e^{ik_- x}, ..., \frac{i}{k_-}e^{ik_- x}),\ \ x < 0, \tag{3.33}$$

$$\text{for e} \in \text{E}_+, \ \Psi_e = (\frac{i}{k_-}e^{-ik_-x}, ..., \frac{i}{k_-}e^{-ik_-x}), \quad x > 0 \tag{3.34}$$

with $k_-^2 = i$ and $Im(k_-) < 0$. For $\mathcal{D}_- = ker(\mathcal{F}_0^* + i) = [\Psi_-]$ with $\Psi_- = (\Phi_e)_{e \in E}$ defined by

$$\text{for e} \in \text{E}_-, \quad \Phi_e = (\frac{i}{k_+}e^{-ik_+x}, ..., \frac{i}{k_+}e^{-ik_+x}), \quad x < 0, \tag{3.35}$$

$$\text{for e} \in \text{E}_+, \quad \Phi_e = (\frac{i}{k_+}e^{ik_+x}, ..., \frac{i}{k_+}e^{ik_+x}), \quad x > 0 \tag{3.36}$$

with $k_+^2 = -i$ and $Im(k_+) > 0$.

Thus, due to Proposition 3.2 every self-adjoint extensions $\text{F}_{0,\theta}$ of $\mathcal{F}_0$ may be parametrized by a real parameter $\theta \in [0, 2\pi)$ where

$$D(\text{F}_{0,\theta}) = \{u \in H^2(\mathcal{G}) : u = u_0 + c\Psi_- + ce^{i\theta}\Psi_+, \\ u_0 \in D(\mathcal{F}_0), c \in \mathbb{C}\}.$$

Thus, it is easily seen that for $u \in D(\text{F}_{0,\theta})$, we have

$$\sum_{e \in E_+} u_e'(0+) - \sum_{e \in E_-} u_e'(0-) = 2cn(1 - e^{i\theta}), \quad \text{and} \tag{3.37}$$

$$u_{1,+}(0+) = -c(e^{i\frac{\pi}{4}} - e^{i(\theta - \frac{\pi}{4})}). \tag{3.38}$$

From the last equalities it follows that

$$\sum_{e \in E_+} u_e'(0+) - \sum_{e \in E_-} u_e'(0-) = Znu_{1,+}(0+) \tag{3.39}$$

where

$$Z = \frac{-2(1 - e^{i\theta})}{e^{i\frac{\pi}{4}} - e^{i(\theta - \frac{\pi}{4})}} \in \mathbb{R}.$$

This finishes the proof. □

Example 4. **δ'-point interactions on a star graph**

The following result will be used in the study of the nonlinear stability of standing wave solutions for the nonlinear Schrödinger model (10.1) in Chapter 10.

Theorem 3.7. *Let $\mathcal{G}$ be a star graph determined by N half-lines, $(0, +\infty)$, attached to the common vertex $\nu = 0$. The diagonal-matrix Schrödinger operator on $L^2(\mathcal{G})$*

$$\text{L}_0' = \Big(\Big(-\frac{d^2}{dx^2}\Big)\delta_{i,j}\Big)$$

with domain

$$D(\mathrm{L}'_0) = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v'_1(0) = ... = v'_N(0) = 0, \sum_{j=1}^{N} v_j(0) = 0\Big\}, \tag{3.40}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(\mathrm{L}'_0) = 1$. *Moreover, all the self-adjoint extension of* L'_0 *can be parametrized by a parameter* $\lambda \in \mathbb{R}$, L_λ, *such that*

$$\begin{cases} \mathrm{L}_\lambda u = \Big(\Big(-\dfrac{d^2}{dx^2}u\Big)\delta_{i,j}\Big), \\ D(\mathrm{L}_\lambda) = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v'_1(0) = ... = v'_N(0), \\ \qquad\qquad \displaystyle\sum_{j=1}^{N} v_j(0) = \lambda v'_1(0),\ \lambda \in \mathbb{R}\Big\}. \end{cases} \tag{3.41}$$

Proof. The symmetric property of $(\mathrm{L}'_0, D(\mathrm{L}'_0))$ is immediate. Since

$$\bigoplus_{i=1}^{N} C_c^\infty(0, +\infty) \subset D(\mathrm{L}'_0),$$

we obtain the density property of $D(\mathrm{L}'_0)$. The same strategy as in the proof of Theorem 3.5 shows that the adjoint operator of L'_0 is given by the action $(\mathrm{L}'_0)^* = \mathrm{L}'_0$ with domain given by

$$D((\mathrm{L}'_0)^*) = \big\{\mathrm{V} \in H^2(\mathcal{G}) : v'_1(0) = ... = v'_N(0)\big\}. \tag{3.42}$$

Now, it is not difficult to see that $\mathcal{D}_\pm = ker((\mathrm{L}'_0)^* \mp i) = [\mathrm{V}_{\pm i}]$ where

$$\mathrm{V}_{\pm i} = (e^{i\sqrt{\pm i}x})_{j=1}^{N}, \quad Im(\sqrt{\pm i}) > 0.$$

Thus, due to Proposition 3.2 every self-adjoint extensions L_λ of L'_0 may be parametrized by $\lambda \in \mathbb{R}$ determined in (3.41). This finishes the proof. □

3.1.5 Behavior of the spectrum of self-adjoint extensions of a symmetric operator

Many results of the spectral theory of self-adjoint extensions of a symmetric operator will be established in the following. Many of these results are classical from the extension theory of symmetric operator (see Naĭmark 1969). Two main issues related with these ones are related with our stability theory for standing wave solutions for nonlinear Schrödinger equations and stationary solutions for the Korteweg–de Vries equation on metric graphs. The first one is associated with the continuous spectrum of each self-adjoint extension and an estimative for the Morse index of each one of these extensions.

The proof of the following statements can be found in (Naĭmark 1969).

Theorem 3.8. *All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.*

Theorem 3.9. *In the extension of a closed, symmetric operator which has the equal and finite deficiency indices* (m, m) *to a self-adjoint operator, the multiplicity of each of its eigenvalues can increase at most by* m *units; in particular, the new eigenvalues have a multiplicity of at most* m.

Theorem 3.10. *If* A *is a closed, symmetric operator with finite deficiency indices* (m, m), *and if* λ *is a real number belonging to the discrete spectrum of the operator* A, *then the equation* $A^*x = \lambda x$ *has at most* m *linearly independent solutions.*

A symmetric operator A is said to be *semi-bounded from below* if there is a number M such that, for all $x \in D(A)$ we have the inequality

$$\langle Ax, x\rangle \geqq M\|x\|^2.$$

We define a *positive* (or non-negative), symmetric operator as the special case of an operator semi-bounded from below when the number $M = 0$; *i.e.*, a positive, symmetric operator satisfies $\langle Ax, x\rangle \geqq 0$.

Theorem 3.11. *If* A *is a positive, closed, symmetric operator with finite deficiency indices* (n, n), *then the negative part of the spectrum of every self-adjoint extensions of* A *can consist only of a finite number of negative eigenvalues, and the sum of their multiplicities is at most equal to* n.

Next, we give some examples related to the Morse index for the self-adjoint operators of point interaction type established in section 3.1.4. We recall that since these self-adjoint operators may have at most a finite collection of negative eigenvalues, its continuous spectrum coincides with its essential spectrum (see Appendix C).

In subsection 3.1.6 we give a different approach for obtaining the number of negative eigenvalue of self-adjoint extensions for a symmetric operator based in the notion of *Nevanlinna* pairs (see Behrndt and Luger 2010, and reference therein).

Example 1. **Spectrum for δ-point interactions on the line**

Theorem 3.12. *Let* $(A_Z, D(A_Z))$ *be the family of self-adjoint operators defined in* (3.18) *and being the self-adjoint extensions for the symmetric operator* $A \equiv -\frac{d^2}{dx^2}$, *with domain*

$$D(A) = \{\psi \in H^2(\mathbb{R}) : \psi(0) = 0\}, \tag{3.43}$$

and deficiency indices equal to 1. *Then,* $n(A_Z) \leqq 1$ *for all* $Z \in \mathbb{R}$. *For* $Z > 0$, $n(A_Z) = 0$ *and for* $Z < 0$, $n(A_Z) = 1$. *Moreover, the continuous spectrum (essential spectrum) is* $[0, +\infty)$ *for all* Z.

Proof. From the relation $\langle A\psi, \psi\rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \geqq 0$, for $\psi \in D(A)$, we get immediately from Theorem 3.11 that $n(A_Z) \leqq 1$ for all $Z \in \mathbb{R}$. Now, for $\psi \in D(A_Z)$ and $Z > 0$ follows from (3.18)

$$\langle A_Z\psi, \psi\rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx + Z|\psi(0)|^2 \geqq 0$$

and so $n(A_Z) = 0$. Since for $Z < 0$, $\psi_Z(x) = e^{\frac{Z}{2}|x|} \in D(A_Z)$ and satisfies $A_Z\psi_Z = -\frac{Z^2}{4}\psi_Z$, we get $n(A_Z) = 1$.

Lastly, for $Z = 0$ we have that $A_0 = -\frac{d^2}{dx^2}$ is a self adjoint operator with domain $H^2(\mathbb{R})$ and from the spectral theorem (via Fourier transform) we obtain that the continuous spectrum (essential spectrum) is given by $[0, +\infty)$. Then, from Theorem 3.8 we get that all self-adjoint extension for A have continuous spectrum being $[0, +\infty)$. □

Example 2. **Spectrum for δ-point interactions on a star graph**

Theorem 3.13. *Let* $(\mathrm{L}_Z, D(\mathrm{L}_Z))$ *be the family of self-adjoint extensions operators defined in* (3.24) *associated to the symmetric operator* L_0 *defined in Theorem 3.5. Then, for* $Z > 0$, $n(A_Z) = 0$ *and for* $Z < 0$, $n(A_Z) = 1$. *Moreover, the continuous spectrum (essential spectrum) is* $[0, +\infty)$ *for every extension* L_Z.

Proof. For $\mathrm{V} = (v_j)_{j=1}^N \in D(\mathrm{L}_0)$, the relation

$$\begin{aligned}\langle \mathrm{L}_0\mathrm{V}, \mathrm{V}\rangle &= \sum_{j=1}^N v_j'(0+)\overline{v_j(0+)} + \sum_{j=1}^N \int_0^{\infty} |v_j'(x)|^2 dx \\ &= \sum_{j=1}^N \int_0^{\infty} |v_j'(x)|^2 dx \geqq 0\end{aligned}$$

implies immediately from Theorem 3.11 that $n(\mathrm{L}_Z) \leqq 1$ for all $Z \in \mathbb{R}$. Now, for $\mathrm{V} = (v_j)_{j=1}^N \in D(\mathrm{L}_Z)$ and $Z > 0$ follows from (3.24)

$$\langle \mathrm{L}_Z\mathrm{V}, \mathrm{V}\rangle = \sum_{j=1}^N \int_{-\infty}^{\infty} |v_j'(x)|^2 dx + Z|v_1(0+)|^2 \geqq 0$$

and so $n(\mathrm{L}_Z) = 0$. Since for $Z < 0$, and $x > 0$,

$$\Phi_Z = (e^{\frac{Z}{N}x})_{j=1}^N \in D(\mathrm{L}_Z), \quad \text{and} \quad \mathrm{L}_Z\Phi_Z = -\frac{Z^2}{N^2}\Phi_Z$$

we get $n(\mathrm{L}_Z) = 1$.

Next, we note that the self-adjoint operator $\mathrm{L}_{dir} = \mathrm{L}_0$ with homogeneous Dirichlet boundary conditions

$$D(\mathrm{L}_{dir}) = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = \ldots = v_N(0) = 0\Big\}, \tag{3.44}$$

belongs to the family of self-adjoint extension of $(\mathrm{L}_0, D(\mathrm{L}_0))$ given in Theorem 3.5 (it is sufficient to take $Z = +\infty$ in (3.24)). Since $(\mathrm{L}_{dir}, D(\mathrm{L}_{dir})$ posses no point spectrum and it is a positive definite self-adjoint operator, $\sigma_{ess}(\mathrm{L}_{dir}) = \sigma(\mathrm{L}_{dir}) \subset [0, +\infty)$. Now, by the criteria of Weyl (Reed and Simon 1978) we get the reverse inclusion $[0, +\infty) \subset \sigma_{ess}(\mathrm{L}_{dir})$. Then, from Theorem 3.8 we get that all self-adjoint extension for $(\mathrm{L}_0, D(\mathrm{L}_0))$ have continuous spectrum being $[0, +\infty)$.

We recall that since the self-adjoint operator $(\mathrm{L}_Z, D(\mathrm{L}_Z))$ may have at most a finite collection of negative eigenvalues, its continuous spectrum, $\sigma_{ac}(\mathrm{L}_Z)$ coincides with its essential spectrum $\sigma_{ess}(\mathrm{L}_Z)$ and $\sigma(\mathrm{L}_Z) = \sigma_{ess}(\mathrm{L}_Z) \cup \sigma_{disc}(\mathrm{L}_Z)$. This finished the proof. □

Example 3. **Spectrum for δ-point interactions on a balanced star graph**

Theorem 3.14. *Let* $(\mathrm{L}_Z, D(\mathrm{L}_Z))$ *be the family of self-adjoint extensions operators defined in* (3.30)-(3.31) *associated to the symmetric operator* $(\mathcal{F}_0, D(\mathcal{F}_0))$ *defined in Theorem 3.6. Then, for* $Z > 0$, $n(\mathrm{L}_Z) = 0$ *and for* $Z < 0$, $n(\mathrm{L}_Z) = 1$. *Moreover, the continuous spectrum (essential spectrum) is* $[0, +\infty)$ *for every extension* L_Z.

Proof. For $\mathrm{V} = (v_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in D(\mathcal{F}_0)$ we get immediately

$$\langle \mathcal{F}_0 \mathrm{V}, \mathrm{V}\rangle \geqq 0.$$

Moreover, for any $Z \in \mathbb{R}$ and $\mathrm{V} = (v_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in D(\mathrm{L}_Z)$ the relation

$$\langle \mathrm{L}_Z \mathrm{V}, \mathrm{V}\rangle = \sum_{\mathrm{e}\in\mathrm{E}_-} \int_{-\infty}^{0} |v_{\mathrm{e}}'(x)|^2 dx + \sum_{\mathrm{e}\in\mathrm{E}_+} \int_{0}^{+\infty} |v_{\mathrm{e}}'(x)|^2 dx + Zn|v_{1,+}(0+)|^2,$$

it is non-negative for $Z > 0$ and so $n(\mathrm{L}_Z) = 0$. Following a similar reasoning as in the proof of Theorem 3.13, we get the other statements in the Theorem. This finishes the proof. □

Example 4. **Morse index associated to stationary solutions for the KdV model on a balanced start graph**

We consider a balanced metric star graph $\mathcal{G}$ with a structure represented by the set $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ where $\mathrm{E}_+, \mathrm{E}_-$ are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively, and $|\mathrm{E}_+| = |\mathrm{E}_-| = n$. The half-lines are connected at a unique vertex $\nu = 0$.

Suppose $(\phi_{\mathrm{e}})_{\mathrm{e}\in \mathrm{E}}$ is a nontrivial solution of the following set of $2|\mathrm{E}_+| = 2n$ nonlinear elliptic equations

$$-\alpha_{\mathrm{e}}\frac{d^2}{dx^2}\phi_{\mathrm{e}} + \phi_{\mathrm{e}} - \phi_{\mathrm{e}}^2 = 0, \quad \mathrm{e} \in \mathrm{E}, \tag{3.45}$$

where $(\alpha_{\mathrm{e}})_{\mathrm{e}\in \mathrm{E}}$ is a positive sequence of real numbers and $\phi_{\mathrm{e}}(+\infty) = 0$, $\mathrm{e} \in \mathrm{E}_+$, $\phi_{\mathrm{e}}(-\infty) = 0$, $\mathrm{e} \in \mathrm{E}_-$.

We consider the $2n \times 2n$-diagonal-matrix Schrödinger operator

$$\mathcal{E}_0 = \Big(\Big(-\alpha_{\mathrm{e}}\frac{d^2}{dx^2} + 1 - 2\phi_{\mathrm{e}}\Big)\delta_{i,j}\Big). \tag{3.46}$$

Then from Proposition 3.1 we obtain that the symmetric operators $\mathcal{E}_0$ and F_0,

$$\mathrm{F}_0 = \Big(\Big(-\alpha_{\mathrm{e}}\frac{d^2}{dx^2}\Big)\delta_{i,j}\Big), \tag{3.47}$$

with the common domain $D(\mathcal{E}_0) = D(\mathrm{F}_0)$ defined by

$$\begin{aligned} D(\mathcal{E}_0) = \{u \in H^2(\mathcal{G}) : u(0-) = u(0+) = 0,& \\ = \sum_{\mathrm{e}\in \mathrm{E}_+} \alpha_{\mathrm{e}} u'_{\mathrm{e}}(0) - \sum_{\mathrm{e}\in \mathrm{E}_-} \alpha_{\mathrm{e}} u'_{\mathrm{e}}(0) = 0\}.& \end{aligned} \tag{3.48}$$

Thus, we obtain from Proposition 3.1 that $n_\pm(\mathcal{E}_0) = n_\pm(\mathrm{F}_0) = 1$ (see Theorem 3.6). Moreover, all self-adjoint extensions of $(\mathcal{E}_0, D(\mathcal{E}_0))$ can be parametrized by $Z \in \mathbb{R}$, $(\mathrm{E}_Z, D(\mathrm{E}_Z))$, as being $\mathrm{E}_Z = \mathcal{E}_0$ and $D(\mathrm{E}_{Z_0}) = D_{Z,\mathrm{E},\delta} \cap \mathcal{C}$, where $\mathcal{C}$ is defined in (3.30) and

$$\begin{aligned} D_{Z,\mathrm{E},\delta} = \{u \in H^2(\mathcal{G}) : u(0-) = u(0+) =& \\ \sum_{\mathrm{e}\in \mathrm{E}_+} \alpha_{\mathrm{e}} u'_{\mathrm{e}}(0) - \sum_{\mathrm{e}\in \mathrm{E}_-} \alpha_{\mathrm{e}} u'_{\mathrm{e}}(0) = Znu_{1,+}(0+)\}.& \end{aligned} \tag{3.49}$$

Next, by following a similar analysis as in Theorem 3.14, for $\phi_{\mathrm{e}}(\pm\infty) = 0$, $(\phi_{\mathrm{e}}) \in L^\infty(\mathcal{G})$ and Weyl's essential theorem (Reed and Simon 1978) we obtain immediately that $\sigma_{ess}(\mathrm{E}_Z) = [1, +\infty)$.

The following result will be very useful in the study of instability properties of stationary solutions for the Korteweg–de Vries equation (1.3) to be developed in Chapter 9.

Theorem 3.15. *Let* $(\mathrm{E}_Z, D_{Z,\mathrm{E},\delta} \cap \mathcal{C})$ *be the family of self-adjoint extensions operators associated to* $(\mathcal{E}_0, D(\mathcal{E}_0))$ *in* (3.46)-(3.48)*. Suppose that the solution-profiles*

$\phi_e \in L^\infty$ for (3.45) satisfy $\phi_e(\pm\infty) = 0$, $e \in E_\pm$, and for every $e \in E$, $\phi_e'(x) \neq 0$ for $x \neq 0$, with $\phi_e''(0\pm) \neq 0$ if $\phi_e'(0\pm) = 0$. Then, for

$$\sum_{e \in E} \int \phi_e^3(x)dx > 0,$$

we obtain that the Morse index of E_Z is exactly one.

Proof. For $V = (v_e)_{e\in E} \in D(\mathcal{E}_0)$ we will see $\langle \mathcal{E}_0 V, V\rangle \geqq 0$. Indeed, since the relation

$$-\alpha_e \frac{d^2}{dx^2} v_e + v_e - 2\phi_e v_e = -\frac{\alpha_e}{\phi_e'}\frac{d}{dx}\left[(\phi_e')^2 \frac{d}{dx}\left(\frac{v_e}{\phi_e'}\right)\right],$$

holds for every $e \in E$ and $x \neq 0$, it follows from integrating by parts,

$$\begin{aligned}
\langle \mathcal{E}_0 V, V\rangle &= \sum_{e\in E} \alpha_e \int (\phi_e')^2 \left|\frac{d}{dx}\left(\frac{v_e}{\phi_e'}\right)\right|^2 dx \\
&\quad - \sum_{e\in E_-} \alpha_e \frac{v_e}{\phi_e'}\left[(\phi_e')^2 \frac{d}{dx}\left(\frac{v_e}{\phi_e'}\right)\right]_{-\infty}^{0-} \\
&\quad - \sum_{e\in E_+} \alpha_e \frac{v_e}{\phi_e'}\left[(\phi_e')^2 \frac{d}{dx}\left(\frac{v_e}{\phi_e'}\right)\right]_{0+}^{+\infty} \qquad (3.50)\\
&= \sum_{e\in E} \alpha_e \int (\phi_e')^2 \left|\frac{d}{dx}\left(\frac{v_e}{\phi_e'}\right)\right|^2 dx \\
&\quad - \sum_{e\in E_-} \alpha_e v_e^2 \frac{\phi_e''}{\phi_e'}\Big]_{x=0-} - \sum_{e\in E_+} \alpha_e v_e^2 \frac{\phi_e''}{\phi_e'}\Big]_{x=0+}.
\end{aligned}$$

The integral terms in (3.50) are non-negative and equal zero if and only if $V \equiv 0$. Due to the conditions $V(0-) = V(0+) = 0$, $\alpha_e > 0$, and $\phi_e''(0\pm) \neq 0$ if $\phi_e'(0\pm) = 0$, non-integral term vanishes, and we get $\mathcal{E}_0 \geqq 0$.

Due to Theorem 3.11 and $n(\mathcal{E}_0) = 1$, we have that the self-adjoint extensions E_Z of $\mathcal{E}_0$ satisfy $n(E_Z) \leqslant 1$. Next, by taking into account the notation $\Phi = (\phi_e)_{e\in E}$ we have

$$E_Z \Phi = \Psi$$

with $\Psi = (-\phi_e^2)_{e\in E}$, and so we obtain

$$\langle E_Z\Phi, \Phi\rangle = -\sum_{e\in E} \int \phi_e^3(x)dx < 0.$$

Therefore, from minimax principle we arrive at $n(E_Z) \geqq 1$. This finishes the Theorem. □

3.1.6 Nevanlinna pairs and self-adjoint extensions of the Laplacian operator

In the last sections we have seen some of the classical results of the extension theory for symmetric operators developed by von Neumann and Krein, and several applications have been given for the Laplacian operator on metric star graphs where the matching (boundary) conditions at the vertex $\nu = 0$ were of δ-interaction type.

Next, we will see other way to parametrize all self-adjoint realizations L of $-\Delta$ in the $L^2(\mathcal{G})$ space on a metric star graph $\mathcal{G}$ with N half-lines of the form $(0, +\infty)$ attached to the common vertex $\nu = 0$. We will make use of the notion of Nevanlinna pairs given in the next definition (see Kostrykin and Schrader 2006).

Definition 3.1. *A pair $\{A, B\}$ of $N \times N$ matrices is said to be a Nevanlinna pair if;*

(a) $AB^* = BA^*$,

(b) The horizontally concatenated $N \times 2N$ matrix $[A, B]$ has maximal rank N.

Let $u : \mathcal{G} \to \mathbb{C}$ and we write u as a column-vector $u = (u_1, ..., u_N)^t$, each u_j being defined on the interval $(0, +\infty)$. We express the conditions at the vertex $\nu = 0$ as $u(0) = (u_1(0+), ..., u_N(0+))^t$ and $u'(0) = (u'_1(0+), ..., u'_N(0+))^t$. In the following we introduce the Laplacian $-\Delta(A, B)$ with the the domain

$$D(-\Delta(A, B)) = \{u \in H^2(\mathcal{G}) : Au(0) + Bu'(0) = 0\}, \tag{3.51}$$

acting as the second derivate along the edges

$$-\Delta(A, B)u = (-u''_1, ..., -u''_N)^t. \tag{3.52}$$

A crucial result concerning the parametrization of all self-adjoint extensions of the Laplace operator in $L^2(\mathcal{G})$ in terms of the boundary conditions, it was obtained in (Kostrykin and Schrader 2006). Indeed, we have the following proposition.

Proposition 3.3. *Let A, B be $N \times N$ matrices. The next two assertions are equivalent:*

(a) The operator $-\Delta(A, B)$ defined in (3.51)-(3.52) is self-adjoint;

(b) $\{A, B\}$ is a Nevanlinna pair.

The most used type of couplings induced by a Nevanlinna pair $\{A, B\}$ are the following ones :

(1) Kirchhoff-coupling: For V $\in H^2(\mathcal{G})$ such that

$$v_1(0) = ... = v_n(0), \quad \sum_{j=1}^{n} v'_j(0+) = 0;$$

where the Nevanlinna pair $\{A, B\}$ is defined by $A = A_K$, $B = B_K$ as

$$A_K = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, B_K = \begin{pmatrix} 0 & \dots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}. \tag{3.53}$$

We note $AB^* = BA^* = 0$.

(2) δ-coupling: For V $\in H^2(\mathcal{G})$ and $\alpha \in \mathbb{R}$ such that

$$v_1(0) = \dots = v_n(0), \quad \sum_{j=1}^{n} v_j'(0+) = \alpha v_1(0);$$

where the Nevanlinna pair $\{A, B\}$ is defined by $A = A_\delta$, $B = B_\delta$ as

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\alpha}{N} & \frac{\alpha}{N} & \frac{\alpha}{N} & \dots & \frac{\alpha}{N} \end{pmatrix}, B = \begin{pmatrix} 0 & \dots & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ -1 & \dots & -1 \end{pmatrix}. \tag{3.54}$$

(3) δ'-coupling: For V $\in H^2(\mathcal{G})$ and $\lambda \in \mathbb{R}$ such that

$$v_1'(0) = \dots = v_n'(0), \quad \sum_{j=1}^{n} v_j(0+) = \lambda v_1'(0+);$$

where the Nevanlinna pair $\{A, B\}$ is defined by $A = A_{\delta'} = -B_K$, $B = B_{\delta'}$ as

$$B_{\delta'} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\lambda}{N} & \frac{\lambda}{N} & \frac{\lambda}{N} & \dots & \frac{\lambda}{N} \end{pmatrix}. \tag{3.55}$$

The subject of self-adjoint Laplacians on metric graphs has become popular under the name of "quantum graphs", such as was established in Chapter 2 and it is well known for its wide applications in quantum mechanics (see Berkolaiko and Kuchment 2013). We note that most of the literature has been concerned with Hamiltonians $\mathcal{H}$ on metric graphs being self-adjoint operators and so the dynamic of the quantum system

$$z_t = \mathcal{H} z,$$

it is described by unitary operators by the Stone's Theorem.

As we have seen in subsection 3.15, Theorem 3.11 is a power tool for estimating the Morse index of all self-adjoint extensions of a positive, closed symmetric operator with deficiency indices equal and finite. In the case of the Laplacian operator defined by a Nevanlinna pair there is the following nice criterium for determining the Morse index of these self-adjoint realization of the Laplacian in $L^2(\mathcal{G})$. This information is very important for obtaining representations and dispersive estimatives of the unitary group associated to the linear Schrödinger evolution equation

$$\begin{cases} iu_t(t,x) = \Delta(A,B)u(x,t), & t \neq 0, \quad x \in (0,+\infty), \\ u(x,0) = u_0(x) \in D(\Delta(A,B)). \end{cases} \tag{3.56}$$

The proof of the following result can be seen in Behrndt and Luger (2010).

Theorem 3.16. *Let $-\Delta(A,B)$ be a self-adjoint realization of the Laplacian in $L^2(\mathcal{G})$, that is, $\{A,B\}$ is a Nevanlinna pair. The number of negative eigenvalues of $-\Delta(A,B)$ (the Morse index, $n(-\Delta(A,B))$) is given by*

$$n(-\Delta(A,B)) = n_+(AB^*),$$

where $n_+(J)$ represents the number of positive eigenvalues of the matrix J. In particular, $-\Delta(A,B)$ is nonnegative if and only if the matrix AB^* is nonpositive.

Next, we determine the Morse index $n(-\Delta(A,B))$ of the Laplacian $-\Delta(A,B)$ associated to the Nevanlinna pair $\{A,B\}$ given above.

(1) Kirchhoff-coupling: for the pair $\{A_K, B_K\}$ in (3.53) we have that $AB^* = 0$. Then, the Morse index, $n(-\Delta(A_K,B_K)) = 0$ and so $-\Delta(A,B) \geqq 0$.

(2) δ-coupling: for the pair $\{A_\delta, B_\delta\}$ in (3.54) we have

$$A_\delta B_\delta^* = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & -\alpha \end{pmatrix}. \tag{3.57}$$

Then, the Morse index $n(-\Delta(A_\delta,B_\delta)) = 0$ for $\alpha > 0$, and $n(-\Delta(A_\delta,B_\delta)) = 1$ for $\alpha < 0$.

(3) δ'-coupling: for the pair $\{A_{\delta'}, B_{\delta'}\}$ in (3.55) we have

$$A_{\delta'} B_{\delta'}^* = B_{\delta'} A_{\delta'}^* = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & -\lambda \end{pmatrix}. \tag{3.58}$$

Then, the Morse index $n(-\Delta(A_{\delta'}, B_{\delta'})) = 0$ for $\lambda > 0$, and $n(-\Delta(A_{\delta'}, B_{\delta'})) = 1$ for $\lambda < 0$.

4 Basic Models

In this Chapter, we establish some specific models of nonlinear dispersive equations on metric graphs. Special attention is given to non-linear Schrödinger models and the Korteweg–de Vries equation. The main point of the exposition will be transform our metric graphs in quantum graphs for these models. The rich dynamic associated to these models will be the focus of the following Chapters. The local and global well-posedness of the Cauchy problem, the existence and stability of standing waves and/or stationary solutions will be some of our interest.

4.1 Schrödinger models on star graphs

In this section we provide a brief description of point interactions on a star graph $\mathcal{G}$ for the nonlinear Schrödinger model (1.1).

From Theorem 3.5 we have that for the Halmiltonian H_α^δ acting on $\mathcal{G}$ for $\mathrm{V}(x) = (v_j(x))_{j=1}^N$ as

$$(\mathrm{H}_\alpha^\delta \mathrm{V})(x) = (-v_j''(x))_{j=1}^N, \ \ x > 0,$$

and with domain $D(\mathrm{H}_\alpha^\delta) = \mathbb{D}_{\alpha,\delta}$,

$$\mathbb{D}_{\alpha,\delta} := \Big\{ \mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = \ldots = v_N(0), \ \sum_{j=1}^N v_j'(0) = \alpha v_1(0) \Big\}, \tag{4.1}$$

we obtain that $(\mathrm{H}_\alpha^\delta, \mathbb{D}_{\alpha,\delta})$ represents a self-adjoint operator on $L^2(\mathcal{G})$.

In this way, we obtain the following nonlinear Schrödinger equation with δ-interaction on the star graph (quantum graph) $\mathcal{G}$ (NLS-δ equation)

$$i\,\partial_t \mathrm{U} - \mathrm{H}_\alpha^\delta \mathrm{U} + |\mathrm{U}|^{p-1}\mathrm{U} = 0. \tag{4.2}$$

We note from subsection 3.1.6 that $\mathrm{V} \in \mathbb{D}_{\alpha,\delta}$ iff

$$A\mathrm{V}(0) + B\mathrm{V}'(0) = 0,$$

for A and B defined in (3.54).

Model (4.2)-(4.1) has been extensively studied in (Adami, Cacciapuoti, et al. 2014c, 2016) and (Angulo and Goloshchapova 2018). In particular, the authors showed well-posedness of the corresponding Cauchy problem. Moreover, they investigated the existence and the particular form of standing waves, as well as their variational and stability properties (see Chapter 10 below).

The second model we are interested is the nonlinear Schrödinger equation with δ'-interaction on the graph $\mathcal{G}$ (NLS-δ' equation)

$$i\,\partial_t \mathrm{U} - \mathrm{H}_\lambda^{\delta'} \mathrm{U} + |\mathrm{U}|^{p-1}\mathrm{U} = 0, \tag{4.3}$$

it is given by the self-adjoint operator $(\mathrm{H}_\lambda^{\delta'}, \mathbb{D}_{\lambda,\delta'})$ with

$$(\mathrm{H}_\lambda^{\delta'}\mathrm{V})(x) = (-v_j''(x))_{j=1}^N, \;\; x > 0,$$

and

$$\mathbb{D}_{\lambda,\delta'} := \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1'(0) = \ldots = v_N'(0),\; \sum_{j=1}^N v_j(0) = \lambda v_1'(0)\Big\}. \tag{4.4}$$

We note from subsection 3.1.6 that $\mathrm{V} \in \mathbb{D}_{\lambda,\delta'}$ iff

$$A_{\delta'}\mathrm{V}(0) + B_{\delta'}\mathrm{V}'(0) = 0,$$

for $A_{\delta'} = -B_K$ in (3.53) and $B_{\delta'}$ in (3.58), namely, $\{A_{\delta'}, B_{\delta'}\}$ forms a Nevanlinna pair.

To our knowledge such type of δ'-interaction has been few studied for the NLS equation on star graphs. In Chapter 10 we establish some results on the Cauchy problem and the existence and orbital stability of standing wave solutions to (4.3) (see Angulo and Goloshchapova 2018).

More general coupling conditions for the Laplace operator can be considered on a star graph by using a framework based in the Nevanlinna pairs described in subsection 3.1.6.

4.2 Korteweg–de Vries on star graphs

In this section we give a description of interactions on metric star-shaped graphs for the Korteweg–de Vries model (1.3). We start by giving a characterization of all skew-self-adjoint extensions of the Airy operators associated to (1.3) and so we obtain the existence of specific dynamics given by unitary groups. Our strategy will follow the theory recently established in (Mugnolo, Noja, and Seifert 2018).

For sequences of real numbers $(\alpha_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$ and $(\beta_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$, we consider the following Airy operator

$$A_0 : (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \to \left(\alpha_{\mathrm{e}}\frac{d^3}{dx^3}u_{\mathrm{e}} + \beta_{\mathrm{e}}\frac{d}{dx}u_{\mathrm{e}}\right)_{\mathrm{e}\in\mathrm{E}} \tag{4.5}$$

as an unbounded operator on a certain Hilbert space belong to $L^2(\mathcal{G})$, we want to obtain skew-self-adjoint extensions $(A_{ext}, D(A_{ext}))$ of A_0 in such a way that the generated dynamics induced by the linear evolution equation

$$\begin{cases} z_t = A_{ext}z, \\ z(0) = u_0 \in D(A_{ext}), \end{cases} \tag{4.6}$$

it is given by a C_0-unitary group.

Since the Airy operator A_0 is of odd order, changing the sign of each constant α_{e} it is equivalent to exchange the positive and negative half line and so we can choose $\alpha_{\mathrm{e}} > 0$ for every $\mathrm{e} \in \mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+$ without loss of generality.

The following proposition from Mugnolo, Noja, and Seifert (2018) give us an answer about the problem associated to (4.6).

Proposition 4.1. *Let $\mathcal{G}$ be a star graph consisting of finitely many half-lines $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ and let $(\alpha_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$, $(\beta_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$ be two sequences of real numbers with $\alpha_{\mathrm{e}} > 0$ for all $\mathrm{e} \in \mathrm{E}$. Consider the operator A_0 defined in (4.5) with*

$$D(A_0) \equiv \bigoplus_{\mathrm{e}\in\mathrm{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathrm{e}\in\mathrm{E}_+} C_c^\infty(0, +\infty).$$

Then, iA_0 is a densely defined symmetric operator on the Hilbert space

$$L^2(\mathcal{G}) = \bigoplus_{\mathrm{e}\in\mathrm{E}_-} L^2(-\infty, 0) \oplus \bigoplus_{\mathrm{e}\in\mathrm{E}_+} L^2(0, +\infty),$$

with deficiency indices $(2|\mathrm{E}_-| + |\mathrm{E}_+|, |\mathrm{E}_-| + 2|\mathrm{E}_+|)$. Therefore, A_0 has skew-self-adjoint extension on $L^2(\mathcal{G})$ if and only if $|\mathrm{E}_-| = |\mathrm{E}_+|$.

we recall that for $|\mathrm{E}_-| = |\mathrm{E}_+|$, *i.e.* the number of incoming half-lines is the same of outgoing half-lines, the graph $\mathcal{G}$ is *balanced*.

Some comments about the former Proposition deserve to be made which will be very useful in our stability study of stationary solutions for the KdV model.

Remark 4.1. *1) From Proposition 4.1 and from the classical von Neumann-Krein extension theory (see Theorem 3.1 in Chapter 3) the operator* $(A_0, D(A_0))$ *admits a* $9|\mathrm{E}_-|^2$*-parameter family of skew-self-adjoint extension generating each one a unitary dynamics on* $L^2(\mathcal{G})$ *associated to the linear evolution equation (4.6). Moreover, every skew-self-adjoint extension* $(A, D(A))$ *is obtained as a restriction of* $(-A_0^*, D(A_0^*))$ *with* $-A_0^* = A_0$ *and*

$$D(A_0^*) \equiv \bigoplus_{\mathrm{e}\in\mathrm{E}_-} H^3(-\infty, 0) \oplus \bigoplus_{\mathrm{e}\in\mathrm{E}_+} H^3(0, +\infty), \tag{4.7}$$

by Theorem 3.1.

2) From the von-Neumann decomposition Theorem 3.1 follows for the symmetric operator iA_0 *that the domain* $D(A_0^*)$ *can be written as*

$$D(A_0^*) \equiv D(A_0) \oplus ker(A_0^* - I) \oplus ker(A_0^* + I). \tag{4.8}$$

Now, the complete characterization of all skew-self-adjoint extensions of $(A_0, D(A_0))$ is a bit complex and one strategy for finding these was obtained recently by Mugnolo, Noja, and Seifert (2018) via Krein spaces (see also Schubert et al. 2015). The central idea of the process is given in Theorem 3.7 and Theorem 3.8 in (Mugnolo, Noja, and Seifert 2018) where skew-self-adjoint extensions of $(A_0, D(A_0))$ are parametrized through relations between boundary values, a strategy very similar to that established above in the case of the Laplacian operator via Nevanlinna pairs. Here we will use the approach in (Mugnolo, Noja, and Seifert 2018), and for convenience of the reader we briefly explain this one. For abbreviating our notations, for $u = (u_\mathrm{e})_{\mathrm{e}\in\mathrm{E}} \in D(A_0^*)$ we denote

$$u(0-) \equiv (u_\mathrm{e}(0-))_{\mathrm{e}\in\mathrm{E}_-}, \quad \text{and} \quad u(0+) \equiv (u_\mathrm{e}(0+))_{\mathrm{e}\in\mathrm{E}_+}$$

and so we consider the space of vectors boundary values in $\mathbb{C}^{3n}$, $(u(0-), u'(0-), u''(0-))$ and $(u(0+), u'(0+), u''(0+))$, spanning respectively subspaces $\mathbb{G}_-$ and $\mathbb{G}_+$, with $n = |\mathrm{E}_\pm|$. The boundary form of the operator A_0 is easily seen for $u, v \in D(A_0^*)$ to be (where we are identifying a vector with its transpose)

$$\langle A_0^* u, v\rangle + \langle u, A_0^* v\rangle = \left(B_- \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix}, \begin{pmatrix} u(0-) \\ u'(0-) \\ u'(0-) \end{pmatrix}\right)_{\mathbb{G}_-} \tag{4.9}$$

$$- \left(B_+ \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix}, \begin{pmatrix} u(0+) \\ u'(0+) \\ u'(0+) \end{pmatrix}\right)_{\mathbb{G}_+} \tag{4.10}$$

where for $I = I_{n\times n}$ representing the identity matrix of order $n \times n$, we have

$$B_- = \begin{pmatrix} -I\beta_- & 0 & -I\alpha_- \\ 0 & I\alpha_- & 0 \\ -I\alpha_- & 0 & 0 \end{pmatrix}, B_+ = \begin{pmatrix} -I\beta_+ & 0 & -I\alpha_+ \\ 0 & I\alpha_+ & 0 \\ -I\alpha_+ & 0 & 0 \end{pmatrix} \tag{4.11}$$

and $\alpha_\pm = (\alpha_e)^t_{e\in E_\pm}$, $\beta_\pm = (\beta_e)^t_{e\in E_\pm}$. Thus by considering the (indefinite) inner product $\langle\cdot|\cdot\rangle_\pm : \mathbb{G}_\pm \times \mathbb{G}_\pm \to \mathbb{C}$ by

$$\langle x|y\rangle_\pm \equiv (B_\pm x, y)_{\mathbb{G}_\pm}, \qquad x, y \in \mathbb{G}_\pm$$

we obtain that $(\mathbb{G}_\pm, \langle\cdot|\cdot\rangle_\pm)$ are Krein spaces and $\langle\cdot|\cdot\rangle_\pm$ is non-degenerate (for $x \in \mathbb{G}_\pm$ with $\langle x|x\rangle_\pm = 0$ follows $x = 0$). Thus from Theorem 3.8 of Mugnolo, Noja, and Seifert (2018) we have that for a linear operator $L : \mathbb{G}_- \to \mathbb{G}_+$, the operator $(A_L, D(A_L))$ defined by

$$\begin{cases} A_L u = -A_0^* u; \\ u \in D(A_L) \text{ if and only if } u \in D(A_0^*) \text{ and} \\ L(u(0-), u'(0-), u''(0-)) = (u(0+), u'(0+), u''(0+)), \end{cases} \tag{4.12}$$

it is a skew-self-adjoint extension of $(A_0, D(A_0))$ if and only if L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary, namely,

$$\langle Lx|Ly\rangle_+ = (B_+ Lx, Ly)_{\mathbb{G}_+} = \langle x|y\rangle_- = (B_- x, y)_{\mathbb{G}_-}, \tag{4.13}$$

namely, $L^* B_+ L = B_-$. Indeed, For $u, v \in D(A_L)$ it follows from (4.9)

$$\begin{aligned} \langle -A_L u, v\rangle + \langle u, -A_L v\rangle &= \langle A_0^* u, v\rangle + \langle u, A_0^* v\rangle \\ &= \langle u(0-)|v(0-)\rangle_- - \langle u(0+)|v(+)\rangle_+ \\ &= \langle u(0-)|v(0-)\rangle_- - \langle Lu(0-)|Lv(-)\rangle_+. \end{aligned}$$

Then, $(A_L)^* = -A_L$ if and only L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary.

Next, we consider two family of skew-self-adjoint extension of $(A_0, D(A_0))$ which will use in the existence and stability of stationary solutions for the KdV model.

The first one for the case of two half-lines is induced by a singular δ-type interaction at the origin. Thus, our metric star graph $\mathcal{G}$ has a structure represented by the set $E = (-\infty, 0) \cup (0, \infty)$ and since $|E_-| = |E_+| = 1$ follows that the Airy operator $(A_0, D(A_0))$ admits a 9-parameter family of skew-self-adjoint extensions. Moreover, since each $u \in H^3(-\infty, 0) \oplus H^3(0, +\infty)$ can be write as the pair $u = (u_-, u_+)$ we have that the subspaces $\mathbb{G}_-$ and $\mathbb{G}_+$ are given by the triplets

$$(u_-(0-), u_-'(0-), u_-''(0-)) \text{ and } (u_+(0+), u_+'(0+), u_+''(0+)).$$

Proposition 4.2. *Let* $E = (-\infty, 0) \cup (0, \infty)$ *and for* $Z \in \mathbb{R} - \{0\}$ *we define the linear operator* $L_Z : \mathbb{G}_- \to \mathbb{G}_+$ *by*

$$L_Z = \begin{pmatrix} 1 & 0 & 0 \\ Z & 1 & 0 \\ \frac{Z^2}{2} & Z & 1 \end{pmatrix}. \tag{4.14}$$

Then we obtain a family $(A_Z, D(A_Z))$ of skew-self-adjoint extension of $(A_0, D(A_0))$ parametrized by Z and which are defined by

$$\begin{cases} A_Z u = \left(\alpha_e \frac{d^3}{dx^3} u_e + \beta_e \frac{d}{dx} u_e\right)_{e\in E}, \quad u = (u_e)_{e\in E} \\ D(A_Z) = \{u = (u_-, u_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty) : \\ \qquad u_-(0-) = u_+(0+), u'_+(0+) - u'_-(0-) = Zu_-(0-), \\ \qquad \frac{Z^2}{2} u_-(0-) + Zu'_-(0-) = u''_+(0+) - u''_-(0-)\}. \end{cases} \tag{4.15}$$

Moreover, for $\alpha_e = (\alpha_-, \alpha_+)$ and $\beta_e = (\beta_-, \beta_+)$ we need to have $\alpha_- = \alpha_+$ and $\beta_- = \beta_+$. By defining

$$U(x) = \begin{cases} u_-(x), & x < 0, \\ u_+(x), & x > 0 \\ u_-(0-), & x = 0 \end{cases}$$

we obtain that each element in $D(A_Z)$ can be seen as a element in $H^1(\mathbb{R})$.

Proof. From the extension theory framework established above, we see from (4.13) that $L^* B_+ L = B_-$ if and only if $\alpha_- = \alpha_+$ and $\beta_- = \beta_+$. Then the operator $A_Z = A_0$ defined for $u = (u_-, u_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty)$ such that $L(u_-(0-), u'_-(0-), u''_-(0-)) = (u_+(0+), u'_+(0+), u''_+(0+))$ will represent a skew-self-adjoint extension of $(A_0, D(A_0))$. This finishes the proof. □

Proposition 4.2 deserves some comments.

1) It is well know from the theory of extension for the closable symmetric operator $H_0 = -\frac{d^2}{dx^2}$ defined on the space $C_0^\infty(\mathbb{R} - \{0\})$, that all the self-adjoint extensions are completely determined by the family of self-adjoint boundary conditions

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix} = \begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix}, \tag{4.16}$$

with $\psi : \mathbb{R} \to \mathbb{C}$, a, b, c, d and τ satisfying the conditions (see (Albeverio and Kurasov 2000, Theorem 3.2.3) or formula (K.1.2) from Appendix of (Albeverio, Gesztesy, et al. 1988))

$$\{a, b, c, d \in \mathbb{R},\ \tau \in \mathbb{C} : ad - bc = 1,\ |\tau| = 1\}. \tag{4.17}$$

The parameters (4.17) label all the self-adjoint extensions of the closable symmetric operator H_0.

2) The choice $\tau = a = d = 1, b = 0, c = Z, Z \in \mathbb{R} - \{0\}$, in (4.16) corresponds to the so-called δ-interaction of strength Z which gives rise to the family of self-adjoint operators $(H_Z^\delta, D(H_Z^\delta))$ on $L^2(\mathbb{R})$ acting as $(H_Z^\delta v)(x) = -v''(x)$, for $x \neq 0$, on the domain

$$D(H_Z^\delta) \equiv \big\{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : v'(0+) - v'(0-) = Zv(0)\big\}.$$

We note that this is exactly our choice for the first 2×2-matrix block defining the matrix L_Z in (4.14).

3) The other choice of parameters is given by $\tau = a = d = 1, c = 0, b = -\beta, \beta \in \mathbb{R} - \{0\}$ corresponding to the case of so-called δ'-interaction of strength $-\beta$. It gives rise to the family of self-adjoint operators $(H_\beta^{\delta'}, D(H_\beta^{\delta'}))$ on $L^2(\mathbb{R})$ acting as $(H_\beta^{\delta'} v)(x) = -v''(x)$, for $x \neq 0$, on the domain

$$\begin{aligned} D(H_\beta^{\delta'}) \equiv \{v \in H^2(\mathbb{R} \setminus \{0\}) : v(0+) - v(0-) = -\beta v'(0), \\ v'(0+) = v'(0-)\}. \end{aligned} \tag{4.18}$$

Now, the interesting point about this self-adjoint boundary condition lies on that the skew-self-adjoint extensions of $(A_0, D(A_0))$ do not give the existence of non-trivial stationary solutions for (full) Korteweg–de Vries models in (1.3). Indeed, by considering the matrix

$$L_\beta = \begin{pmatrix} 1 & -\beta & 0 \\ 0 & 1 & 0 \\ e & f & g \end{pmatrix}. \tag{4.19}$$

we obtain that the unitary condition $L_\beta^* B_+ L_\beta = B_-$ implies for $\alpha_+ = 0$ that $\alpha_- = 0$ and for $\alpha_+ \neq 0$ that $\alpha_- = 0$.

4) The choice of parameters $\tau = 1, a = \gamma, d = \frac{1}{\tau}, c = b = 0, \gamma \in \mathbb{R} - \{0\}$ corresponding to the case of so-called *dipole*-interaction of strength τ. It gives rise to the family of self-adjoint operators $(H_\gamma, D(H_\gamma))$ on $L^2(\mathbb{R})$ acting as $(H_\gamma v)(x) = -v''(x)$, for $x \neq 0$, on the domain

$$D(H_\gamma) \equiv \{v \in H^2(\mathbb{R} - \{0\}) : \gamma v(0-) = v(0+), v'(0-) = \gamma v(0+)\}.$$

Thus, for

$$L_\gamma = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 \\ e & 0 & \frac{1}{\gamma^3} \end{pmatrix}, \quad e = \frac{\beta_- - \gamma^2 \beta_+}{2\gamma\alpha_+}, \tag{4.20}$$

we obtain the unitary condition $L_\gamma^* B_+ L_\gamma = B_-$ if and only if $\alpha_+ = \gamma^2 \alpha_-$. The constants $\beta_\pm$ can be chosen arbitrary. Then we obtain a family $(A_\gamma, D(A_\gamma))$ of skew-self-adjoint extension of $(A_0, D(A_0))$ parametrized by γ such that $A_\gamma = A_0$ and

$$D(A_\gamma) = \{u = (u_-, u_+) \in H^3(-\infty, 0) \oplus H^3(0, +\infty) :$$
$$\gamma u_-(0-) = u_+(0+), \gamma u_+(0+) = u'_-(0-) \tag{4.21}$$
$$eu_-(0-) + \tfrac{1}{\gamma^3} u''_-(0-) = u''_+(0+)\}.$$

Then, we have only two skew-self-adjoint extension of $(A_0, D(A_0))$, for α_+, α_- positive, $\gamma = \sqrt{\frac{\alpha_+}{\alpha_-}}$ (the continuous dipole-interaction), and for α_+, α_- negative, $\gamma = -\sqrt{\frac{\alpha_+}{\alpha_-}}$ (the discontinuous dipole-interaction).

The second one case of skew-self-adjoint family of extensions for $(A_0, D(A_0))$ is for a balanced metric star graph $\mathcal{G}$ with a structure $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ where $|\mathrm{E}_+| = |\mathrm{E}_-| = n$, $n \geqq 2$, and with a δ-interaction at the vertex. Thus by following the notation above, we have for $I = I_{n\times n}$ being the identity matrix of order $n \times n$, that for

$$B_- = \begin{pmatrix} -\beta I & 0 & -\alpha I \\ 0 & \alpha I & 0 \\ -\alpha I & 0 & 0 \end{pmatrix}, \quad B_+ = \begin{pmatrix} -\beta I & 0 & -\alpha I \\ 0 & \alpha I & 0 \\ -\alpha I & 0 & 0 \end{pmatrix}. \tag{4.22}$$

with $\beta_\pm = (\beta)_{e\in \mathrm{E}}$ and $\alpha_\pm = (\alpha)_{e\in\mathrm{E}}$ are constants sequences, and the matrix $L \equiv L_{3n\times 3n} : \mathbb{G}_- \to \mathbb{G}_+$ of order $3n \times 3n$, $Z \in \mathbb{R}$, defined by

$$L \equiv \begin{pmatrix} I & 0 & 0 \\ ZI & I & 0 \\ \frac{Z^2}{2} I & ZI & I \end{pmatrix}. \tag{4.23}$$

we obtain

$$L^* B_+ L = B_-$$

and so L is $(\mathbb{G}_-, \mathbb{G}_+)$-unitary. Therefore we have the following result.

Proposition 4.3. *Let* $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ *where* $|\mathrm{E}_+| = |\mathrm{E}_-| = n$, $n \geqq 2$. *For* L *defined in* (4.23) *we obtain the following family* $(H_Z, D(H_Z))$ *of skew-self-adjoint extension of* $(A_0, D(A_0))$ *parametrized by* Z *and defined by*

$$\begin{cases} H_Z u = -A_0^* u = A_0 u \\ D(H_Z) = \{u \in D(A_0^*) : L(u(0-), u'(0-), u''(0-)) = \\ \qquad\qquad (u(0+), u'(0+), u''(0+))\}. \end{cases} \tag{4.24}$$

Thus, for each $u = (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in D(H_Z)$ *and from the abbreviations*

$$u(0-) = \quad (u_{\mathrm{e}}(0-))_{\mathrm{e}\in\mathrm{E}_-}, \quad u'(0-) = (u'_{\mathrm{e}}(0-))_{\mathrm{e}\in\mathrm{E}_-},$$

$$u''(0-) = (u''_{\mathrm{e}}(0-))_{\mathrm{e}\in\mathrm{E}_-}$$

(similarly for the terms $u(0+)$, $u'(0+)$ *and* $u''(0\pm)$*), we obtain the following system of boundary conditions*

$$\begin{aligned} u(0-) = u(0+), \quad u'(0+) - u'(0-) = Zu(0-), \\ \tfrac{Z^2}{2}u(0-) + Zu'(0-) = u''(0+) - u''(0-). \end{aligned} \tag{4.25}$$

4.3 sine-Gordon equation on star graphs

Next we consider a metric graph $\mathcal{G}$ with a structure represented by the set $\mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+ = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)$, namely, a $\mathcal{Y}$ junction.

The focus of this section is to give a quantum graph framework for the following vectorial sine-Gordon model

$$\partial_t^2 u_{\mathrm{e}}(x,t) - c_{\mathrm{e}}^2 \partial_x^2 u_{\mathrm{e}}(x,t) + sin(u_{\mathrm{e}}(x,t)) = 0, \quad \mathrm{e} \in \mathrm{E} \tag{4.26}$$

and $(c_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$, a sequence of real numbers. We rewrite the sine-Gordon model as a first-order system for $\mathrm{e} \in \mathrm{E}$,

$$\begin{cases} \partial_t u_{\mathrm{e}} = v_{\mathrm{e}} \\ \partial_t v_{\mathrm{e}} = c_{\mathrm{e}}^2 \partial_x^2 u_{\mathrm{e}} + sin(u_{\mathrm{e}}). \end{cases} \tag{4.27}$$

We consider the following symmetric diagonal-matrix Schrödinger operator on $L^2(\mathcal{G})$

$$\mathcal{J}_0 = \Big(\Big(-c_j^2 \frac{d^2}{dx^2}\Big)\delta_{j,k}\Big)$$

with dense domain

$$\begin{aligned} D(\mathcal{J}_0) = \Big\{ \mathrm{V} = (v_j)_{j=1}^3 \in H^2(\mathcal{G}) : v_1(0) = v_2(0) = v_3(0) = 0, \\ \sum_{j=2}^3 c_j^2 v_j'(0) - c_1^2 v_1'(0) = 0 \Big\}. \end{aligned} \tag{4.28}$$

By following a similar argument as in the proof of Theorem 3.6, we can see that the adjoint operator $(\mathcal{J}_0^*, D(\mathcal{J}_0^*))$ of $(\mathcal{J}_0, D(\mathcal{J}_0))$ is given by

$$\mathcal{J}_0^* = \mathcal{J}_0, \quad D(\mathcal{J}_0^*) = \{(v_j)_{j=1}^3 \in H^2(\mathcal{G}) : v_1(0) = v_2(0) = v_3(0)\}.$$

Thus, it is not difficult to see that the deficiency indices for $(\mathcal{J}_0, D(\mathcal{J}_0))$ are $n_\pm(\mathcal{J}_0) = 1$. Therefore due to Proposition 3.2 every self-adjoint extension $\mathcal{J}_Z$ of $\mathcal{J}_0$ may be parametrized by $Z \in \mathbb{R}$, such that

$$\begin{cases} \mathcal{J}_Z u = \Big(\Big(-c_j^2 \dfrac{d^2}{dx^2} u\Big)\delta_{j,k}\Big), \\ D(\mathcal{J}_Z) = \Big\{(v_j)_{j=1}^3 \in H^2(\mathcal{G}) : v_1(0) = v_2(0) = v_3(0), \\ \qquad\qquad \displaystyle\sum_{j=2}^{3} c_j^2 v_j'(0) - c_1^2 v_1'(0) = Z v_1(0)\Big\}. \end{cases} \tag{4.29}$$

In Chapter 6, section 6.3, we give a novel family of stationary solutions for sine-Gordon equations on the $\mathcal{Y}$ junction describe above.

5 The Korteweg–de Vries Equation on a $\mathcal{Y}$ Junction

In this chapter, we shall study the KdV equation on a star graph $\mathcal{Y} = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)$ with three semi-infinite edges given by one negative half-line and two positives half-lines attached to a common vertex, also known as $\mathcal{Y}$ junction. More precisely, we consider the following problem

$$\begin{cases} u_t + u_{xxx} + u_x u = 0, & (x,t) \in (-\infty, 0) \times (0, T), \\ v_t + v_{xxx} + v_x v = 0, & (x,t) \in (0, +\infty) \times (0, T), \\ w_t + w_{xxx} + w_x w = 0, & (x,t) \in (0, +\infty) \times (0, T), \end{cases} \tag{5.1}$$

with initial conditions

$$u(x,0) = u_0(x), \; v(x,0) = v_0(x) \text{ and } w(x,0) = w_0(x), \tag{5.2}$$

where

$$(u_0, v_0, w_0) \in H^s(\mathbb{R}^-) \times H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+) := H^s(\mathcal{Y}). \tag{5.3}$$

Here, our goal in studying Cauchy problem (5.1)-(5.2) is to obtain results of local well-posedness in Sobolev spaces with low regularity.

5.1 Choices of boundary conditions

In this section, we will choice appropriate boundary conditions for the KdV equation on a $\mathcal{Y}$ junction. Determining the number of boundary conditions necessary for a well-posed problem is a nontrivial issue. As far we know, it's not at all clear which boundary conditions should be appropriate for physical applications, and therefore here we will consider two classes of boundary conditions that are coherent with uniqueness calculations for smooth decaying solutions of a linear version of the Cauchy problem (5.1). In this sense, suppose that $(u(x,t), v(x,t), w(x,t))$ is a smooth decaying solution of a linear version of (5.1), i.e.

$$\begin{cases} u_t + u_{xxx} = 0, & (x,t) \in (-\infty, 0) \times (0, T), \\ v_t + v_{xxx} = 0, & (x,t) \in (0, +\infty) \times (0, T), \\ w_t + w_{xxx} = 0, & (x,t) \in (0, +\infty) \times (0, T), \end{cases} \tag{5.4}$$

with homogeneous initial condition $(u_0, v_0, w_0) = (0,0,0)$. Multiplying the equations in (5.4) by u, v and w respectively, and integrating by parts we obtain

$$\begin{aligned} &\int_{-\infty}^{0} u^2(x,T)dx + \int_0^{+\infty} v^2(x,T)dx + \int_0^{+\infty} w^2(x,T)dx \\ &= \int_0^T (u_x^2(0,t) - v_x^2(0,t) - w_x^2(0,t))dt \\ &\quad - 2\int_0^T u(0,t)u_{xx}(0,t)dt \\ &\quad + 2\int_0^T v(0,t)v_{xx}(0,t)dt + 2\int_0^T w(0,t)w_{xx}(0,t)dt. \end{aligned} \tag{5.5}$$

By analyzing (5.5), we are interested in the boundary conditions for Cauchy problem (5.1)-(5.2) such that the right hand side of (5.5) would have a non positive sign.

For this, we can choice, for example, the following particular boundary conditions

$$u(0,t) = \alpha_2 v(0,t) = \alpha_3 w(0,t), \; t \in (0,T), \tag{5.6}$$

$$u_x(0,t) = \beta_2 v_x(0,t) + \beta_3 w_x(0,t), \; t \in (0,T) \tag{5.7}$$

and

$$u_{xx}(0,t) = \frac{1}{\alpha_2} v_{xx}(0,t) + \frac{1}{\alpha_3} w_{xx}(0,t), \; t \in (0,T), \tag{5.8}$$

where $\alpha_2,\ \alpha_3,\ \beta_2$ and β_3 are real constants satisfying $3\beta_i^2 \leqslant 1$ for $i = 2, 3$ we have that

$$\begin{aligned}
&\int_{-\infty}^{0} u^2(x,T)dx + \int_0^{+\infty} v^2(x,T)dx + \int_0^{+\infty} w^2(x,T)dx \\
&\quad = \int_0^T [(\beta_2 v_x(0,t) + \beta_3 w_x(0,t))^2 - v_x^2(0,t) - w_x^2(0,t)]dt \\
&\qquad + 2\int_0^T u(0,t)(-u_{xx}(0,t) + \frac{1}{\alpha_2} v_{xx}(0,t) + \frac{1}{\alpha_3} w_{xx}(0,t)) \\
&\quad = \int_0^T [(\beta_2^2 - 1)v_x^2(0,t) + 2\beta_2\beta_3 v_x(0,t)w_x(0,t) + (\beta_3^2 - 1)w_x^2(0,t)]dt \\
&\quad \leqslant \int_0^T [(3\beta_2^2 - 1)v_x^2(0,t) + (3\beta_3^2 - 1)w_x^2(0,t)]dt,
\end{aligned} \tag{5.9}$$

where we have used the Cauchy-Schwarz inequality. It follows that $u(x,T) = v(x,T) = w(x,T) = 0$, which implies the uniqueness argument.

In the same way, the following particular boundary conditions

$$u(0,t) = \frac{1}{\alpha_2} v(0,t) + \frac{1}{\alpha_3} w(0,t),\ t \in (0,T), \tag{5.10}$$

$$u_x(0,t) = \beta_2 v_x(0,t) + \beta_3 w_x(0,t),\ t \in (0,T) \tag{5.11}$$

and

$$u_{xx}(0,t) = \alpha_2 v_{xx}(0,t) = \alpha_3 w_{xx}(0,t),\ t \in (0,T), \tag{5.12}$$

where $\alpha_2,\ \alpha_3,\ \beta_2$ and β_3 are real constants satisfying $3\beta_i^2 \leqslant 1$ for $i = 2, 3$ imply the uniqueness argument.

We now define the following two classes of boundary conditions that is coherent with the approach used here, which involves the particular boundary conditions (5.6)-(5.8) and (5.10)-(5.12).

Definition 5.1. *Given $a_2,\ a_3,\ b_2,\ b_3,\ c_2$ and c_3 real constants, we call type 1 boundary conditions for the Cauchy problem* (5.1)-(5.2) *if these satisfy the following boundary conditions at the vertex:*

$$u(0,t) = a_2 v(0,t) = a_3 w(0,t),\ t \in (0,T), \tag{5.13}$$

$$u_x(0,t) = b_2 v_x(0,t) + b_3 w_x(0,t),\ t \in (0,T) \tag{5.14}$$

and

$$u_{xx}(0,t) = c_2 v_{xx}(0,t) + c_3 w_{xx}(0,t),\ t \in (0,T). \tag{5.15}$$

Definition 5.2. *Given $a_2,\ a_3,\ b_2,\ b_3,\ c_2$ and c_3 real constants, we call type 2 boundary conditions for the Cauchy problem* (5.1)-(5.2) *if these satisfy*

$$u(0,t) = a_2 v(0,t) + a_3 w(0,t),\ t \in (0,T), \tag{5.16}$$

$$u_x(0,t) = b_2 v_x(0,t) + b_3 w_x(0,t),\ t \in (0,T) \tag{5.17}$$

and

$$u_{xx}(0,t) = c_2 v_{xx}(0,t) = c_3 w_{xx}(0,t),\ t \in (0,T). \tag{5.18}$$

It is well-known that the trace operator $\gamma_0 : u(x) \mapsto u(0)$ is well-defined on $H^s(\mathbb{R}^+)$ for $s > \frac{1}{2}$. Hence, on the case $s > \frac{1}{2}$ we will assume the following additional condition

$$u_0(0) = a_2 v_0(0) = a_3 w_0(0) \tag{5.19}$$

for initial data for the Cauchy problem (5.1)-(5.2) with type 1 boundary conditions and

$$u_0(0) = a_2 v_0(0) + a_3 w_0(0) \tag{5.20}$$

for type 2 boundary conditions.

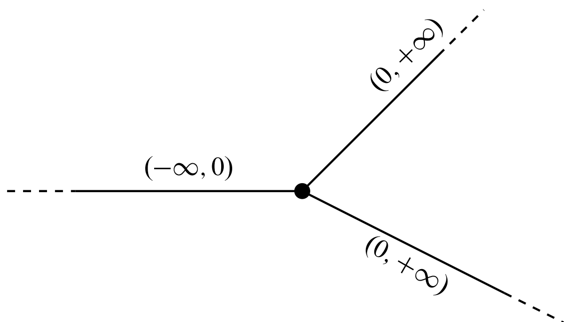

Figure 5.1: A star graph with three edges ($\mathcal{Y}$ junction)

In order to simplify the enunciate of the principal theorem we give an auxiliary definition of some parameters. To do this we fix the following notations,

$$\begin{aligned} d_{\lambda_i} &= 2\sin\left(\frac{\pi}{3}\lambda_i + \frac{\pi}{6}\right),\ e_{\lambda_i} = 2\sin\left(\frac{\pi}{3}\lambda_i - \frac{\pi}{6}\right), \\ \text{and } f_{\lambda_i} &= 2\sin\left(\frac{\pi}{3}\lambda_i - \frac{\pi}{2}\right). \end{aligned} \tag{5.21}$$

We also define the following matrices,

$$\begin{aligned} &\mathrm{M}_1(\lambda_1,\lambda_2,\lambda_3,\lambda_4) := \\ &\begin{bmatrix} d_{\lambda_1} & -a_2 e^{i\pi\lambda_3} & 0 & d_{\lambda_2} \\ d_{\lambda_1} & 0 & -a_3 e^{i\pi\lambda_4} & d_{\lambda_2} \\ e_{\lambda_1} & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & e_{\lambda_2} \\ f_{\lambda_1} & -c_2 e^{i\pi(\lambda_3-2)} & -c_3 e^{i\pi(\lambda_4-2)} & f_{\lambda_2} \end{bmatrix} \end{aligned} \tag{5.22}$$

and

$$\mathrm{M}_2(\lambda_1,\lambda_2,\lambda_3,\lambda_4) :=$$
$$\begin{bmatrix} d_{\lambda_1} & -a_2e^{i\pi\lambda_3} & -a_3e^{i\pi\lambda_4} & d_{\lambda_2} \\ e_{\lambda_1} & -b_2e^{i\pi(\lambda_3-1)} & -b_3e^{i\pi(\lambda_4-1)} & e_{\lambda_2} \\ f_{\lambda_1} & -c_2e^{i\pi(\lambda_3-2)} & f_{\lambda_2} & \\ f_{\lambda_1} & 0 & -c_3e^{i\pi(\lambda_4-2)} & f_{\lambda_2} \end{bmatrix} \tag{5.23}$$

Now, we state principal result of the chapter obtained by Cavalcante (2018).

Theorem 5.1. *Let $-\frac{1}{2} < s < \frac{3}{2}$, with $s \neq \frac{1}{2}$. Assume that u_0, v_0 and w_0 satisfy (5.3).*

(i) For a fixed s suppose that there exists a real constant $\lambda_i(s)$ satisfying

$$\max\{s-1,0\} < \lambda_i(s) < \min\left\{s+\frac{1}{2},\frac{1}{2}\right\} \text{ for } i = 1,2,3,4, \tag{5.24}$$

such that the matrix M_1 defined in (5.22) is invertible. Then there exists a positive time $T > 0$ and a distributional solution (u, v, w) in the space $C([0,T], H^s(\mathcal{Y}))$, for the Cauchy problem (5.1)-(5.2) with type 1 boundary conditions, satisfying the additional compatibility condition (5.19) on the case $\frac{1}{2} < s < \frac{3}{2}$. Furthermore the data-to-solution map $(u_0, v_0, w_0) \mapsto (u, v, w)$ is locally Lipschitz continuous from $H^s(\mathcal{Y})$ to $C([0,T], H^s(\mathcal{Y}))$.

(ii) For a fixed s suppose that there exists a real constant $\lambda_i(s)$ with

$$\max\{s-1,0\} < \lambda_i(s) < \min\left\{s+\frac{1}{2},\frac{1}{2}\right\} \text{ for } i = 1,2,3,4,$$

such that the matrix M_2 defined in (5.23) is invertible. Then there exists a positive time $T > 0$ and a distributional solution (u, v, w) in the space $C([0,T], H^s(\mathcal{Y}))$, for the Cauchy problem (5.1)-(5.2) with type 2 boundary conditions, satisfying the additional compatibility condition (5.20) on the case $\frac{1}{2} < s < \frac{3}{2}$. Furthermore the data-to-solution map $(u_0, v_0, w_0) \mapsto (u, v, w)$ is locally Lipschitz continuous from $H^s(\mathcal{Y})$ to $C([0,T], H^s(\mathcal{Y}))$.

Remark 5.1. *In Theorem 5.1 the indexes λ_i, $i = 1,2,3,4$, are associated to the Duhamel boundary operator classes associated to linear version of the KdV equation and depend of the regularity index s.*

Remark 5.2. *The inversion of the matrix (5.22) and (5.23) condition is necessary in order to reformulate the Cauchy problem (5.1)-(5.2) in an integral version by using the Duhamel boundary forcing operators.*

As the consequence of Theorem 5.1, we can obtain the following result for the special boundary conditions (5.6)-(5.8) and (5.10)-(5.12), which is appropriate for our formal uniqueness calculations associated to the linear version of the KdV equation on a star graph.

Corollary 5.1. *Let* $-\frac{1}{2} < s < \frac{3}{2}$ *with* $s \neq \frac{1}{2}$ *and* $\alpha_2, \alpha_3, \beta_2, \beta_3 \in \mathbb{R}$ *satisfy* $\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2} \neq -1$. *Assume that* u_0, v_0 *and* w_0 *satisfy* (5.3). *Then there exists a positive time* $T > 0$ *and a distributional solution* $(u, v, w) \in C([0, T], H^s(\mathcal{Y}))$ *for the Cauchy problem* (5.1)-(5.2) *with boundary conditions* (5.6)-(5.8)*, and the initial conditions satisfying additional conditions* (5.19) *for* $\frac{1}{2} < s < \frac{3}{2}$. *Furthermore the data-to-solution map* $(u_0, v_0, w_0) \mapsto (u, v, w)$ *is locally Lipschitz continuous from* $H^s(\mathcal{Y})$ *to* $C([0, T], H^s(\mathcal{Y}))$.

Corollary 5.2. *The same result of the Corollary 5.1 is valid for the Cauchy problem* (5.1)-(5.2) *with boundary conditions* (5.10)-(5.12)*, and the initial conditions satisfying additional condition* (5.20) *for* $\frac{1}{2} < s < \frac{3}{2}$.

Remark 5.3. *Note that, in Corollary 5.1 and Corollary 5.2 we don't need of the assumptions* $\beta_i^2 < 1$*,* $i = 2, 3$*, obtained in the previous formal uniqueness calculations for the associated linear problem* (5.4).

The approach used to prove the main result is based on the arguments developed by Cavalcante (2017), Cavalcante and Corcho (2019), Colliander and Kenig (2002), and Holmer (2005, 2006). The main idea to prove Theorem 5.1 is the construction of an auxiliary forced Cauchy problem in all $\mathbb{R}$, analogous to the (5.1); more precisely:

$$\begin{cases} u_t + u_{xxx} + u_x u = \mathcal{T}_1(x)h_1(t) + \mathcal{T}_2(x)h_2(t), & (x,t) \in \mathbb{R} \times (0, T), \\ v_t + v_{xxx} + v_x v = \mathcal{T}_3(x)h_3(t), & (x,t) \in \mathbb{R} \times (0, T), \\ w_t + w_{xxx} + w_x w = \mathcal{T}_4(x)h_4(t), & (x,t) \in \mathbb{R} \times (0, T), \\ u(x,0) = \widetilde{u}_0(x),\ v(x,0) = \widetilde{v}_0(x)\ w(x,0) = \widetilde{w}_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.25}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are distributions supported in a positive half-line $\mathbb{R}^+$, $\mathcal{T}_3$ and $\mathcal{T}_4$ are distributions supported in the negative half-line $\mathbb{R}^-$, $\widetilde{u}_0$, $\widetilde{v}_0$ and $\widetilde{w}_0$ are nice extensions of u_0, v_0 and w_0 in $\mathbb{R}$. The boundary forcing functions h_1, h_2, h_3 and h_4 are selected to ensure that the vertex conditions are satisfied.

The solution of forced Cauchy problem (5.25) satisfying the vertex conditions is constructed using the classical restricted norm method of Bourgain (see Bourgain (1993) and Kenig, Ponce, and Vega (1993)) and the inversion of a Riemann-Liouville fractional integration operator .

Following Cavalcante (2017) and Holmer (2006) we consider the distributions $\mathcal{T}_1 = \frac{x_-^{\lambda_1 - 1}}{\Gamma(\lambda_1)}$, $\mathcal{T}_2 = \frac{x_-^{\lambda_2 - 1}}{\Gamma(\lambda_2)}$, $\mathcal{T}_3 = \frac{x_+^{\lambda_3 - 1}}{\Gamma(\lambda_3)}$ and $\mathcal{T}_4 = \frac{x_+^{\lambda_4 - 1}}{\Gamma(\lambda_4)}$, where

$$\left\langle \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}, \phi \right\rangle = \int_0^{+\infty} \frac{x^{\lambda-1}}{\Gamma(\lambda)} \phi(x) dx, \text{ for Re } \lambda > 0. \tag{5.26}$$

For other values of λ we can define $\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \frac{d^k}{dx} \frac{x^{\lambda+k-1}}{\Gamma(\lambda+k)}$, for any integer k satisfying $k + \text{Re}\lambda > 0$. Finally, we define $\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} = e^{i\pi\lambda} \frac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$.

The crucial point here are the appropriate choices of the parameters λ_i and the functions h_i, for $i = 1, 2, 3, 4$, that will depend on the regularity index s.

Remark 5.4. *We believe that the same approach used to prove Theorem 5.1 can provide similar results for the KdV equation in other star graphs and possibly for other nonlinear dispersive equations. For example, a treatment for the nonlinear Schröndiger equation on a star graphs can be done using the classes of Duhamel boundary operators developed by Holmer (2005) and Cavalcante (2017).*

We denote by $X^{s,b}$ the so called Bourgain spaces associated to linear KdV equation; more precisely, $X^{s,b}$ is the completion of $S'(\mathbb{R}^2)$ with respect to the norm

$$\|w\|_{X^{s,b}(\phi)} = \|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{w}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

To obtain our results we also need define the following auxiliary modified Bougain spaces of Holmer (2006). Let $U^{s,b}$ and V^α the completion of $S'(\mathbb{R}^2)$ with respect to the norms:

$$\|w\|_{U^{s,b}} = \left(\int\int\langle\tau\rangle^{2s/3}\langle\tau-\xi^3\rangle^{2b}|\widehat{w}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

and

$$\|w\|_{V^\alpha} = \left(\int\int\langle\tau\rangle^{2\alpha}|\widehat{w}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}.$$

Fore more details about the Bourgain spaces see Appendix B.

Next nonlinear estimates, in the context of the KdV equation, for $b < \frac{1}{2}$, was derived by Holmer (2006).

Lemma 5.1. *(a) Given $s > -\frac{3}{4}$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$ we have*

$$\left\|\partial_x(v_1v_2)\right\|_{X^{s,-b}} \lesssim \|v_1\|_{X^{s,b}\cap V^\alpha}\|v_2\|_{X^{s,b}\cap V^\alpha}.$$

(b) Given $-\frac{3}{4} < s < 3$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$ we have

$$\left\|\partial_x(v_1v_2)\right\|_{X^{s,-b}} \lesssim \|v_1\|_{X^{s,b}\cap V^\alpha}\|v_2\|_{X^{s,b}\cap V^\alpha}.$$

5.2 The linear versions

5.2.1 Linear group associated to the KdV equation

The linear unitary group $e^{-t\partial_x^3} : S'(\mathbb{R}) \to S'(\mathbb{R})$ associated to the linear KdV equation is defined by

$$e^{-t\partial_x^3}\phi(x) = \left(e^{it\xi^3}\widehat{\phi}(\xi)\right)^\vee(x),$$

that satisfies

$$\begin{cases}(\partial_t+\partial_x^3)e^{-t\partial_x^3}\phi(x,t)=0 & \text{for} \quad (x,t)\in\mathbb{R}\times\mathbb{R},\\ e^{-t\partial_x^3}(x,0)=\phi(x) & \text{for} \quad x\in\mathbb{R}.\end{cases} \tag{5.27}$$

The next estimates were proven by Holmer (2006).

Lemma 5.2. *Let $s\in\mathbb{R}$ and $0<b<1$. If $\phi\in H^s(\mathbb{R})$, then we have*

(a) ***(space traces)***

$$\|e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_t;\,H^s(\mathbb{R}_x)\right)}\lesssim\|\phi\|_{H^s(\mathbb{R})};$$

(b) ***(time traces)***

$$\|\psi(t)\partial_x^j e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x;\,H^{(s+1-j)/3}(\mathbb{R}_t)\right)}\lesssim\|\phi\|_{H^s(\mathbb{R})},\quad j\in\mathbb{N};$$

(c) ***(Bourgain spaces)***

$$\|\psi(t)e^{-t\partial_x^3}\phi(x)\|_{X^{s,b}\cap V^\alpha}\lesssim\|\phi\|_{H^s(\mathbb{R})}.$$

Remark 5.5. *The spaces V^α introduced in (Holmer 2006) give us useful auxiliary norms of the classical Bourgain spaces in order to validate the bilinear estimates associated to the KdV equation for $b<\frac{1}{2}$ (see Lemma 5.1).*

5.2.2 The Duhamel boundary forcing operator associated to the linear KdV equation

Now, we give the properties of the Duhamel boundary forcing operator introduced by Colliander and Kenig (2002), namely

$$\begin{aligned}\mathcal{V}g(x,t)&=3\int_0^t e^{-(t-t')\partial_x^3}\delta_0(x)\mathcal{I}_{-2/3}g(t')dt'\\ &=3\int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathcal{I}_{-2/3}g(t')}{(t-t')^{1/3}}dt',\end{aligned} \tag{5.28}$$

defined for all $g\in C_0^\infty(\mathbb{R}^+)$ and A denotes the Airy function

$$A(x)=\frac{1}{2\pi}\int_\xi e^{ix\xi}e^{i\xi^3}d\xi.$$

From definition of $\mathcal{V}$ it follows that

$$\begin{cases}(\partial_t+\partial_x^3)\mathcal{V}g(x,t)=3\delta_0(x)\mathcal{I}_{-\frac{2}{3}}g(t) & \text{for} \quad (x,t)\in\mathbb{R}\times\mathbb{R},\\ \mathcal{V}g(x,0)=0 & \text{for} \quad x\in\mathbb{R}.\end{cases} \tag{5.29}$$

The proof of the results exhibited in this section was shown by Holmer (2006).

Lemma 5.3. *Let $g \in C_0^\infty(\mathbb{R}^+)$ and consider a fixed time $t \in [0,1]$. Then,*

(a) the functions $\mathcal{V}g(\cdot,t)$ and $\partial_x \mathcal{V}g(\cdot,t)$ are continuous in x for all $x \in \mathbb{R}$. Moreover, they satisfy the spatial decay bounds

$$|\mathcal{V}g(x,t)| + |\partial_x \mathcal{V}g(x,t)| \leqslant c_k \|g\|_{H^{k+1}} \langle x\rangle^{-k} \quad \textit{for all } k \geqslant 0;$$

(b) the function $\partial_x^2 \mathcal{V}g(x,t)$ is continuous in x for all $x \neq 0$ and has a step discontinuity of size $3\mathcal{I}_{\frac{2}{3}} g(t)$ at $x=0$. Also, $\partial_x^2 \mathcal{V}g(x,t)$ satisfies the spatial decay bounds

$$|\partial_x^2 \mathcal{V}g(x,t)| \leqslant c_k \|f\|_{H^{k+2}} \langle x\rangle^{-k} \quad \textit{for all } k \geqslant 0.$$

Since $A(0) = \frac{1}{3\Gamma\left(\frac{2}{3}\right)}$ from (5.28) we have that $\mathcal{V}g(0,t) = g(t)$.

5.2.3 Applications of the operator $\mathcal{V}$

For the convenience to the reader, we present here an application of the operator $\mathcal{V}$ to solve a linear version of the IBVP associated to the KdV equation on the positive half-line, given by Colliander and Kenig (2002). Set

$$v(x,t) = e^{-t\partial_x^3}\phi(x) + \mathcal{V}\big(g - e^{-\cdot\partial_x^3}\phi\big|_{x=0}\big)(x,t), \tag{5.30}$$

where $g \in C_0^\infty(\mathbb{R}^+)$ and $\phi \in S(\mathbb{R})$.

Then from (5.27) and (5.29) we see that v solves the linear problem

$$\begin{cases} (\partial_t + \partial_x^3)v(x,t) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^* \times \mathbb{R}, \\ v(x,0) = \phi(x) & \text{for} \quad x \in \mathbb{R}, \\ v(0,t) = g(t) & \text{for} \quad t \in (0,+\infty), \end{cases} \tag{5.31}$$

in the sense of distributions, and then this would suffice to solve the IBVP on the right half-line associated to linear KdV equation.

Now, we consider the second boundary forcing operator associated to the linear KdV equation:

$$\mathcal{V}^{-1}g(x,t) = \partial_x \mathcal{V}\mathcal{I}_{\frac{1}{3}} g(x,t) = 3\int_0^t A'\left(\frac{x}{(t-t')^{1/3}}\right) \frac{\mathcal{I}_{-\frac{1}{3}} g(t')}{(t-t')^{2/3}} dt'. \tag{5.32}$$

From Lemma 5.3, for all $g \in C_0^\infty(\mathbb{R}^+)$ the function $\mathcal{V}^{-1}g(x,t)$ is continuous in x on $x \in \mathbb{R}$; moreover using that $A'(0) = -\frac{1}{3\Gamma(\frac{1}{3})}$ we get the relation $\mathcal{V}^{-1}g(0,t) = -g(t)$.

Also, the definition of $\mathcal{V}^{-1}g(x,t)$ allows us to ensure that

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{V}^{-1}g(x,t) = 3\delta_0'(x)\mathcal{I}_{-\frac{1}{3}} g(t) & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathcal{V}^{-1}g(x,0) = 0 & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{5.33}$$

in the sense of distributions.

Furthermore, Lemma 5.3 implies that the function $\partial_x \mathcal{V} f(x,t)$ is continuous in x for all $x \in \mathbb{R}$ and, since $A'(0) = -\frac{1}{3\Gamma(\frac{1}{3})}$,

$$\partial_x \mathcal{V} g(0,t) = -\mathcal{I}_{-\frac{1}{3}} g(t). \tag{5.34}$$

Also, $\partial_x \mathcal{V}^{-1} g(x,t) = \partial_x^2 \mathcal{V} \mathcal{I}_{\frac{1}{3}} g(x,t)$ is continuous in x for $x \neq 0$ and has a step discontinuity of size $3\mathcal{I}_{-\frac{1}{3}} g(t)$ at $x = 0$. Indeed,

$$\begin{aligned}
\lim_{x\to 0^+} \partial_x^2 \mathcal{V} g(x,t) &= -\int_0^{+\infty} \partial_y^3 \mathcal{V} g(y,t) dy = \int_0^{+\infty} \partial_t \mathcal{V} g(y,t) dy \\
&= 3\int_0^{+\infty} A(y) dy \int_0^t \partial_t \mathcal{I}_{-\frac{2}{3}} g(t') dt' = \mathcal{I}_{-\frac{2}{3}} g(t),
\end{aligned}$$

then from Lemma 5.3 -(b) we have

$$\lim_{x\to 0^-} \partial_x \mathcal{V}^{-1} g(x,t) = -2\mathcal{I}_{-\frac{1}{3}} g(t) \quad \text{and} \quad \lim_{x\to 0^+} \partial_x \mathcal{V}^{-1} g(x,t) = \mathcal{I}_{-\frac{1}{3}} g(t).$$

Now, for convenience, we give an application of the operator $\mathcal{V}^{-1}$ to solve a IBVP linear associated to the KdV equation on the negative half-line with two boundary conditions given by Holmer (2006). Let $h_1(t)$ and $h_2(t)$ belonging to $C_0^\infty(\mathbb{R}^+)$ we have the relations:

$$\begin{aligned}
\mathcal{V} h_1(0,t) + \mathcal{V}^{-1} h_2(0,t) &= h_1(t) - h_2(t), \\
\lim_{x\to 0^-} \mathcal{I}_{\frac{1}{3}} \partial_x (\mathcal{V} h_1(x,\cdot) + \partial_x \mathcal{V}^{-1} h_2(x,\cdot))(t) &= -h_1(t) - 2h_2(t), \\
\lim_{x\to 0^+} \mathcal{I}_{\frac{1}{3}} \partial_x (\mathcal{V} h_1(x,\cdot) + \partial_x \mathcal{V}^{-1} h_2(x,\cdot))(t) &= -h_1(t) + h_2(t).
\end{aligned}$$

For given $v_0(x)$, $g(t)$ and $h(t)$ we assigned

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} := \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} g - e^{-\cdot\partial_x^3} v_0|_{x=0} \\ \mathcal{I}_{\frac{1}{3}}\left(h - \partial_x e^{-\cdot\partial_x^3} v_0|_{x=0}\right) \end{bmatrix}.$$

Then, taking $v(x,t) = e^{-t\partial_x^3} v_0(x) + \mathcal{V} h_1(x,t) + \mathcal{V}^{-1} h_2(x,t)$ we get

$$\begin{cases}
(\partial_t + \partial_x^3) v(x,t) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^* \times \mathbb{R}, \\
v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}, \\
v(0,t) = g(t) & \text{for} \quad t \in \mathbb{R}, \\
\lim_{x\to 0^-} \partial_x v(x,t) = h(t) & \text{for} \quad t \in \mathbb{R},
\end{cases} \tag{5.35}$$

in the sense of distributions.

5.2.4 The Duhamel boundary forcing operator classes associated to linear KdV equation

In order, to get our results in low regularity (see Remark 5.6), we need to work with two classes of boundary forcing operators in order to obtain the required estimates for the second order derivative of traces. In this way, we define the generalization of operators $\mathcal{V}$ and $\mathcal{V}^{-1}$ given by Holmer (2006).

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -3$ and $g \in C_0^\infty(\mathbb{R}^+)$. Define the operators

$$\mathcal{V}_-^\lambda g(x,t) = \left[\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{V}\big(\mathcal{I}_{-\frac{\lambda}{3}} g\big)(\cdot,t)\right](x)$$

and

$$\mathcal{V}_+^\lambda g(x,t) = \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{V}\big(\mathcal{I}_{-\frac{\lambda}{3}} g\big)(\cdot,t)\right](x),$$

with $\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} = e^{i\pi\lambda}\frac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$. Then, using (5.29) we have that

$$(\partial_t + \partial_x^3)\mathcal{V}_-^\lambda g(x,t) = 3\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}} g(t)$$

and

$$(\partial_t + \partial_x^3)\mathcal{V}_+^\lambda g(x,t) = 3\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}} g(t).$$

The following lemmas state properties of the operators classes $\mathcal{V}_\pm^\lambda$. For the proofs we refer the reader Holmer (2006).

Lemma 5.4. *Let $g \in C_0^\infty(\mathbb{R}^+)$. Then, we have*

$$\mathcal{V}_\pm^{\lambda-2} g = \partial_x^2 \mathcal{V}^\lambda \mathcal{I}_{\frac{2}{3}} g, \quad \mathcal{V}_\pm^{\lambda-1} g = \partial_x \mathcal{V}^\lambda \mathcal{I}_{\frac{1}{3}} g \quad \textit{and} \quad \mathcal{V}_\pm^0 g = \mathcal{V} g.$$

Also, $\mathcal{V}_\pm^{-2} g(x,t)$ has a step discontinuity of size $3g(t)$ at $x = 0$, otherwise for $x \neq 0$, $\mathcal{V}_\pm^{-2} g(x,t)$ is continuous in x. For $\lambda > -2$, $\mathcal{V}_\pm^\lambda g(x,t)$ is continuous in x for all $x \in \mathbb{R}$. For $-2 \leqslant \lambda \leqslant 1$ and $0 \leqslant t \leqslant 1$, $\mathcal{V}_-^\lambda g(x,t)$ satisfies the following decay bounds:

$$|\mathcal{V}_-^\lambda g(x,t)| \leqslant c_{m,\lambda,g}\langle x\rangle^{-m}, \textit{ for all } x \leqslant 0 \textit{ and } m \geqslant 0,$$
$$|\mathcal{V}_-^\lambda g(x,t)| \leqslant c_{\lambda,g}\langle x\rangle^{\lambda-1} \textit{ for all } x \geqslant 0.$$
$$|\mathcal{V}_+^\lambda g(x,t)| \leqslant c_{m,\lambda,g}\langle x\rangle^{-m}, \textit{ for all } x \geqslant 0 \textit{ and } m \geqslant 0,$$

and

$$|\mathcal{V}_+^\lambda g(x,t)| \leqslant c_{\lambda,g}\langle x\rangle^{\lambda-1} \textit{ for all } x \leqslant 0.$$

Lemma 5.5. *For* $\operatorname{Re}\lambda > -2$ *and* $g \in C_0^\infty(\mathbb{R}^+)$ *we have*

$$\mathcal{V}_-^\lambda g(0,t) = 2\sin\left(\frac{\pi}{3}\lambda + \frac{\pi}{6}\right) g(t)$$

and

$$\mathcal{V}_+^\lambda g(0,t) = e^{i\pi\lambda} g(t).$$

Lemma 5.6. *Let* $s \in \mathbb{R}$. *The following estimates are ensured:*

(a) ***(space traces)*** $\|\mathcal{V}_\pm^\lambda g(x,t)\|_{C\left(\mathbb{R}_t;\, H^s(\mathbb{R}_x)\right)} \lesssim \|g\|_{H_0^{(s+1)/3}(\mathbb{R}+)}$ *for all* $s - \frac{5}{2} < \lambda < s + \frac{1}{2}$, $\lambda < \frac{1}{2}$ *and* $\operatorname{supp}(g) \subset [0,1]$.

(b) ***(time traces)*** $\|\psi(t)\partial_x^j \mathcal{V}_\pm^\lambda g(x,t)\|_{C\left(\mathbb{R}_x;\, H_0^{(s+1)/3}(\mathbb{R}_t^+)\right)} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}+)}$ *for all* $-2 + j < \lambda < 1 + j$, *for* $j \in \{0,1,2\}$.

(c) ***(Bourgain spaces)*** $\left\|\psi(t)\mathcal{V}_\pm^\lambda g(x,t)\right\|_{X^{s,b}\cap V^\alpha} \lesssim c\|g\|_{H_0^{(s+1)/3}(\mathbb{R}+)}$ *for all* $s - 1 \leqslant \lambda < s + \frac{1}{2}$, $\lambda < \frac{1}{2}$, $\alpha \leqslant \frac{s-\lambda+2}{3}$ *and* $0 \leqslant b < \frac{1}{2}$.

Remark 5.6. *Note that for* $\lambda = 0$ *the second derivative time traces estimate is not obtained, for this reason we need to work with the family* $\mathcal{V}_\pm^\lambda$. *Also note that the set of regularity where the spaces traces and Bourgain spaces estimates are valid depends of the index* λ, *for example, for* $\lambda = 0$ *we have the Bourgain spaces estimates on the set* $-1/2 < s < 1$.

5.3 The Duhamel inhomogeneous solution operator

The classical inhomogeneous solution operator $\mathcal{K}$ associated to the KdV equation is given by

$$\mathcal{K}w(x,t) = \int_0^t e^{-(t-t')\partial_x^3} w(x,t')dt',$$

that satisfies

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{K}w(x,t) = w(x,t) & \text{for} \quad (x,t) \in \mathbb{R}\times\mathbb{R}, \\ \mathcal{K}w(x,t) = 0 & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{5.36}$$

Now, we summarize some useful estimates for the Duhamel inhomogeneous solution operators $\mathcal{K}$ that will be used later in the proof of the main results and its proof can be seen in (Holmer 2006).

Lemma 5.7. *For all $s \in \mathbb{R}$ we have the following estimates:*

(a) ***(space traces)*** *Let $-\frac{1}{2} < d < 0$, then*

$$\|\psi(t)\mathcal{K}w(x,t)\|_{C\left(\mathbb{R}_t;\, H^s(\mathbb{R}_x)\right)} \lesssim \|w\|_{X^{s,d}}.$$

(b) ***(time traces)*** *Let $-\frac{1}{2} < d < 0$ and $j \in \{0,1,2\}$, then*

$$\left\|\psi(t)\partial_x^j\mathcal{K}w(x,t)\right\|_{C\left(\mathbb{R}_x;\, H^{(s+1)/3}(\mathbb{R}_t)\right)} \lesssim \begin{cases} \|w\|_{X^{s,d}} & \text{if } -1+j \leqslant s \leqslant \frac{1}{2}+j, \\ \|w\|_{X^{s,d}} + \|w\|_{U^{s,d}} & \text{for all } s \in \mathbb{R}. \end{cases}$$

(c) ***(Bourgain spaces estimates)*** *Let $0 < b < \frac{1}{2}$ and $\alpha > 1-b$, then*

$$\|\psi(t)\mathcal{K}w(x,t)\|_{X^{s,b}\cap V^{\alpha}} \lesssim \|w\|_{X^{s,-b}}.$$

Remark 5.7. *We note that the time-adapted Bourgain spaces $U^{k,d}$ used in Lemma 5.7-(c)-(d) are introduced in order to cover the full values of regularity s.*

5.4 Proof of Theorem 5.1

Now, we show the proof of the main result announced of this work. We only prove the part (i) of Theorem 5.1, since the proof of part (ii) is very similar. We follow closely the arguments in (Holmer 2006) (see also Cavalcante 2017; Cavalcante and Corcho 2019). The proof will be divided into five steps.

Step 1. We will first obtain an integral equation that solves Cauchy problem (5.1)-(5.2)**, with type 1 boundary conditions, satisfying** (5.19) **for $\frac{1}{2} < s < \frac{3}{2}$.**

We start rewriting the vertex conditions (5.13), (5.14) and (5.15) in terms of matrices:

$$\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0,t) \\ v(0,t) \\ w(0,t) \end{bmatrix} = 0, \tag{5.37}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_x(0,t) \\ v_x(0,t) \\ w_x(0,t) \end{bmatrix} = 0 \tag{5.38}$$

and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3 \end{bmatrix} \begin{bmatrix} u_{xx}(0,t) \\ v_{xx}(0,t) \\ w_{xx}(0,t) \end{bmatrix} = 0. \tag{5.39}$$

Let $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ nice extensions of u_0, v_0 and w_0, respectively satisfying

$$\|\widetilde{u}_0\|_{H^s(\mathbb{R})} \leqslant c\|u_0\|_{H^s(\mathbb{R}^+)}, \ \|\widetilde{v}_0\|_{H^s(\mathbb{R})} \leqslant c\|v_0\|_{H^s(\mathbb{R}^+)}$$
$$\text{and } \|\widetilde{w}_0\|_{H^s(\mathbb{R})} \leqslant c\|w_0\|_{H^s(\mathbb{R}^+)}.$$

Initially, we look for solutions in the form

$$\begin{aligned} u(x,t) &= \mathcal{V}_-^{\lambda_1}\gamma_1(x,t) + \mathcal{V}_-^{\lambda_2}\gamma_2(x,t) + F_1(x,t), \\ v(x,t) &= \mathcal{V}_+^{\lambda_3}\gamma_3(x,t) + F_2(x,t), \\ w(x,t) &= \mathcal{V}_+^{\lambda_4}\gamma_4(x,t) + F_3(x,t), \end{aligned}$$

where γ_i $(i = 1, 2, 3, 4)$ are unknown functions and

$$\begin{aligned} F_1(x,t) &= e^{it\partial_x^3}\widetilde{u}_0 + \mathcal{K}(uu_x)(x,t), \\ F_2(x,t) &= e^{it\partial_x^3}\widetilde{v}_0 + \mathcal{K}(vv_x)(x,t), \\ F_3(x,t) &= e^{it\partial_x^3}\widetilde{w}_0 + \mathcal{K}(ww_x)(x,t). \end{aligned}$$

By using Lemma 5.5 we see that

$$u(0,t) = 2\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right)\gamma_1(t) + 2\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right)\gamma_2(t) + F_1(0,t), \tag{5.40}$$
$$v(0,t) = e^{i\pi\lambda_3}\gamma_3(t) + F_2(0,t), \tag{5.41}$$
$$w(0,t) = e^{i\pi\lambda_4}\gamma_4(t) + F_3(0,t). \tag{5.42}$$

Now we calculate the traces of first derivative functions. By Lemmas 5.4 and 5.5 we see that

$$\begin{aligned} u_x(0,t) = {} & 2\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right)\mathcal{I}_{-\frac{1}{3}}\gamma_1(t) + 2\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right)\mathcal{I}_{-\frac{1}{3}}\gamma_2(t) \\ & + \partial_x F_1(0,t), \end{aligned} \tag{5.43}$$
$$v_x(0,t) = e^{i\pi(\lambda_3-1)}\mathcal{I}_{-\frac{1}{3}}\gamma_3(t) + \partial_x F_2(0,t), \tag{5.44}$$
$$w_x(0,t) = e^{i\pi(\lambda_4-1)}\mathcal{I}_{-\frac{1}{3}}\gamma_4(t) + \partial_x F_3(0,t). \tag{5.45}$$

In the same way, we calculate the traces of second derivatives functions,

$$u_{xx}(0,t) = 2sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{2}\right)\mathcal{I}_{-\frac{2}{3}}\gamma_1(t) + 2\,sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{2}\right)\gamma_2(t) + \partial_x^2 F_1(0,t), \tag{5.46}$$

$$v_{xx}(0,t) = e^{i\pi(\lambda_3-2)}\mathcal{I}_{-\frac{2}{3}}\gamma_3(t) + \partial_x^2 F_2(0,t), \tag{5.47}$$

$$w_{xx}(0,t) = e^{i\pi(\lambda_4-2)}\mathcal{I}_{-\frac{2}{3}}\gamma_4(t) + \partial_x^2 F_3(0,t). \tag{5.48}$$

Note that by Lemmas 5.4 and 5.5 these calculus are valid for Re $\lambda > 0$.

By substituting (5.40), (5.41) and (5.42) into (5.37); (5.42), (5.43) and (5.44) into (5.38), and (5.46), (5.47) and (5.48) into (5.39) we see that the functions γ_i and indexes λ_i, for $i = 1, 2, 3, 4$, satisfy the expressions

$$\begin{aligned}&\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix}\begin{bmatrix} d_{\lambda_1} & 0 & 0 & d_{\lambda_2} \\ 0 & e^{i\pi\lambda_3} & 0 & 0 \\ 0 & 0 & e^{i\pi\lambda_4} & 0 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} \\ &\qquad = -\begin{bmatrix} 1 & -a_2 & 0 \\ 1 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{bmatrix}\begin{bmatrix} F_1(0,t) \\ F_2(0,t) \\ F_3(0,t) \end{bmatrix},\end{aligned} \tag{5.49}$$

$$\begin{aligned}&\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0. \end{bmatrix}\begin{bmatrix} e_{\lambda_1} & 0 & 0 & e_{\lambda_2} \\ 0 & e^{i(\pi\lambda_3-1)} & 0 & 0 \\ 0 & 0 & e^{i(\pi\lambda_4-1)} & 0 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} \\ &\qquad = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -b_2 & -b_3 \\ 0 & 0 & 0. \end{bmatrix}\begin{bmatrix} \partial_x\mathcal{I}_{\frac{1}{3}} F_1(0,t) \\ \partial_x\mathcal{I}_{\frac{1}{3}} F_2(0,t) \\ \partial_x\mathcal{I}_{\frac{1}{3}} F_3(0,t) \end{bmatrix}\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}&\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3 \end{bmatrix}\begin{bmatrix} f_{\lambda_1} & 0 & 0 & f_{\lambda_2} \\ 0 & e^{i(\pi\lambda_3-2)} & 0 & 0 \\ 0 & 0 & e^{i(\pi\lambda_4-2)} & 0 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} \\ &\qquad = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -c_2 & -c_3. \end{bmatrix}\begin{bmatrix} \partial_x^2\mathcal{I}_{\frac{2}{3}} F_1(0,t) \\ \partial_x^2\mathcal{I}_{\frac{2}{3}} F_2(0,t) \\ \partial_x^2\mathcal{I}_{\frac{2}{3}} F_3(0,t) \end{bmatrix}.\end{aligned} \tag{5.51}$$

It follows that,

$$\begin{bmatrix} d_{\lambda_1} & -a_2 e^{i\pi\lambda_3} & 0 & d_{\lambda_2} \\ d_{\lambda_1} & 0 & -a_3 e^{i\pi\lambda_4} & d_{\lambda_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} = - \begin{bmatrix} F_1(0,t) - a_2 F_2(0,t) \\ F_1(0,t) - a_3 F_3(0,t) \\ 0 \\ 0 \end{bmatrix}. \tag{5.52}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_{\lambda_1} & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & e_{\lambda_2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ \partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \\ 0 \end{bmatrix} \tag{5.53}$$

and

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f_{\lambda_1} & -c_2 e^{i\pi(\lambda_3-2)} & -c_3 e^{i\pi(\lambda_4-2)} & f_{\lambda_2} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t) \end{bmatrix}. \tag{5.54}$$

From (5.52), (5.53) and (5.54) we need to obtain functions γ_i $(i = 1, 2, 3, 4)$ and parameters λ_i $(i = 1, 2, 3, 4)$ satisfying

$$\begin{bmatrix} d_{\lambda_1} & -a_2 e^{i\pi\lambda_3} & 0 & d_{\lambda_2} \\ d_{\lambda_1} & 0 & -a_3 e^{i\pi\lambda_4} & d_{\lambda_1} \\ e_{\lambda_1} & -b_2 e^{i\pi(\lambda_3-1)} & -b_3 e^{i\pi(\lambda_4-1)} & e_{\lambda_2} \\ f_{\lambda_1} & -c_2 e^{i\pi(\lambda_3-2)} & -c_3 e^{i\pi(\lambda_4-2)} & f_{\lambda_2} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \\ \gamma_2 \end{bmatrix} = - \begin{bmatrix} F_1(0,t) - a_2 F_2(0,t) \\ F_1(0,t) - a_3 F_3(0,t) \\ \partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \\ \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t) \end{bmatrix}. \tag{5.55}$$

We denote a simplified notation of (5.55) as

$$\mathrm{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\boldsymbol{\gamma} = \mathrm{F}, \tag{5.56}$$

where $\mathrm{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the first matrix that appears in (5.55), $\boldsymbol{\gamma}$ is the matrix column given by vector $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and F is the last matrix in (5.55). By using the hypothesis of Theorem 5.1 we fix parameters λ_i, for $i = 1, 2, 3, 4$ such that

$$\max\{s-1, 0\} < \lambda_i(s) < \min\left\{s + \frac{1}{2}, \frac{1}{2}\right\}. \tag{5.57}$$

and the matrix $\mathrm{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is invertible.

Step 2. We will define the truncated integral operator and the appropriate functions space.

Given s as in the hypothesis of Theorem 5.1 we fix the parameters λ_i and the functions γ_i $(i = 1, 2, 3, 4)$ chosen as in the Step 1. Let $b = b(s) < \frac{1}{2}$ and $\alpha(b) > 1/2$ such that the estimates given in Lemma 5.1 are valid.

Define the operator

$$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \tag{5.58}$$

where

$$\begin{aligned}
\Lambda_1 u(x,t) &= \psi(t)\mathcal{V}_-^{\lambda_1}\gamma_1(x,t) + \psi(t)\mathcal{V}_-^{\lambda_2}\gamma_2(x,t) + F_1(x,t),\\
\Lambda_2 v(x,t) &= \psi(t)\mathcal{V}_+^{\lambda_3}\gamma_3(x,t) + F_2(x,t),\\
\Lambda_3 w(x,t) &= \psi(t)\mathcal{V}_+^{\lambda_4}\gamma_4(x,t) + F_3(x,t),
\end{aligned}$$

where

$$\begin{aligned}
F_1(x,t) &= \psi(t)(e^{it\partial_x^3}\widetilde{u}_0 + \mathcal{K}(uu_x)(x,t)),\\
F_2(x,t) &= \psi(t)(e^{it\partial_x^3}\widetilde{v}_0 + \mathcal{K}(vv_x)(x,t)),\\
F_3(x,t) &= \psi(t)(e^{it\partial_x^3}\widetilde{w}_0 + \mathcal{K}(ww_x)(x,t)).
\end{aligned}$$

We consider Λ on the Banach space $Z(s) = Z_1(s) \times Z_2(s) \times Z_3(s)$, where

$$\begin{aligned}
Z_i(s) =&\{w \in C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C(\mathbb{R}_x; H^{\frac{s+1}{3}}(\mathbb{R}_t)) \cap X^{s,b} \cap V^{\alpha};\\
&w_x \in C(\mathbb{R}_x; H^{\frac{s}{3}}(\mathbb{R}_t)), w_{xx} \in C(\mathbb{R}_x; H^{\frac{s-1}{3}}(\mathbb{R}_t))\}\ (i = 1, 2, 3),
\end{aligned}$$

with norm

$$\|(u, v, w)\|_{Z(s)} = \|u\|_{Z_1(s)} + \|v\|_{Z_2(s)} + \|w\|_{Z_3(s)},$$

where

$$\begin{aligned}
\|u\|_{Z_i(s)} =&\|u\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} + \|u\|_{C(\mathbb{R}_x;H^{\frac{s+1}{3}}(\mathbb{R}_t))} + \|u\|_{X^{s,b}} + \|u\|_{V^\alpha}\\
&+ \|u_x\|_{C(\mathbb{R}_x;H^{\frac{s}{3}}(\mathbb{R}_t))} + \|u_{xx}\|_{C(\mathbb{R}_x;H^{\frac{s-1}{3}}(\mathbb{R}_t))}.
\end{aligned} \tag{5.59}$$

Step 3. We will prove that the functions $\mathcal{V}_-^{\lambda_1}\gamma_1(x,t)$, $\mathcal{V}_-^{\lambda_2}\gamma_2(x,t)$, $\mathcal{V}_+^{\lambda_3}\gamma_1(x,t)$ and $\mathcal{V}_-^{\lambda_4}\gamma_4(x,t)$ are well defined.

By Lemma (5.6) it suffices to show that these functions are in the closure of the spaces $C_0^\infty(\mathbb{R}^+)$. By using expression (5.55) we see that the functions γ_i $(i=1,2,3,4)$ are linear combinations of the functions $F_1(0,t)-a_2F_2(0,t)$, $F_1(0,t)-a_3F_3(0,t)$, $\partial_x\mathcal{I}_{\frac{1}{3}}F_1(0,t)-b_2\partial_x\mathcal{I}_{\frac{1}{3}}F_2(0,t)-b_3\partial_x\mathcal{I}_{\frac{1}{3}}F_3(0,t)$ and $\partial_x^2\mathcal{I}_{\frac{2}{3}}F_1(0,t)-c_2\partial_x^2\mathcal{I}_{\frac{2}{3}}F_2(0,t)-c_3\partial_x^2\mathcal{I}_{\frac{2}{3}}F_3(0,t)$. Thus, we need to show that the functions $F_i(0,t)$, $\partial_x\mathcal{I}_{\frac{1}{3}}F_i(0,t)$, $\partial_x^2\mathcal{I}_{\frac{2}{3}}F_i(0,t)$ are in appropriate spaces. By using Lemmas 5.2, 5.6, 5.7 and 5.1 we obtain

$$\|F_1(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|u_0\|_{H^s(\mathbb{R}^+)}+\|u\|^2_{X^{s,b}}+\|u\|^2_{Y^\alpha}), \tag{5.60}$$

$$\|F_2(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|v_0\|_{H^s(\mathbb{R}^+)}+\|v\|^2_{X^{s,b}}+\|v\|^2_{Y^\alpha}), \tag{5.61}$$

$$\|F_3(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|w_0\|_{H^s(\mathbb{R}^+)}+\|w\|^2_{X^{s,b}}+\|w\|^2_{Y^\alpha}). \tag{5.62}$$

If $-\frac{1}{2}<s<\frac{1}{2}$ we have that $\frac{1}{6}<\frac{s+1}{3}<\frac{1}{2}$. Thus Lemma B.1 implies that $H^{\frac{s+1}{3}}(\mathbb{R}^+)=H_0^{\frac{s+1}{3}}(\mathbb{R}^+)$. It follows that $F_i(0,t)\in H_0^{\frac{s+1}{3}}(\mathbb{R}^+)$ (for $i=1,2,3$) for $-\frac{1}{2}<s<\frac{1}{2}$.

If $\frac{1}{2}<s<\frac{3}{2}$, then $\frac{1}{2}<\frac{s+1}{3}<\frac{5}{6}$. Using the compatibility condition (5.19) we have that

$$F_1(0,0)-a_2F_2(0,0)=u(0,0)-a_2v(0,0)=u_0(0)-a_2v_0(0)=0,$$

$$F_1(0,0)-a_3F_3(0,0)=u(0,0)-a_3w(0,0)=u_0(0)-a_3w_0(0)=0.$$

Then Lemma B.2 implies

$$\begin{aligned} F_1(0,t)-a_22F_2(0,t)&\in H_0^{\frac{s+1}{3}}(\mathbb{R}^+),\\ F_1(0,t)-a_3F_3(0,t)&\in H_0^{\frac{s+1}{3}}(\mathbb{R}^+) \end{aligned} \tag{5.63}$$

Now using Lemmas 5.2, 5.6, 5.7 and 5.1 we see that

$$\|\partial_xF_1(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leqslant c(\|u_0\|_{H^s(\mathbb{R}^+)}+\|u\|^2_{X^{s,b}}+\|u\|^2_{Y^\alpha}),$$

$$\|\partial_xF_2(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leqslant c(\|v_0\|_{H^s(\mathbb{R}^+)}+\|v\|^2_{X^{s,b}}+\|v\|^2_{Y^\alpha}),$$

$$\|\partial_xF_3(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \leqslant c(\|w_0\|_{H^s(\mathbb{R}^+)}+\|w\|^2_{X^{s,b}}+\|w\|^2_{Y^\alpha}).$$

Since $-\frac{1}{2}<s<\frac{3}{2}$ we have $-\frac{1}{6}<\frac{s}{3}<\frac{1}{2}$, then Lemma B.1 implies that the functions $\partial_xF_i(0,t)\in H_0^{\frac{s}{3}}(\mathbb{R}^+)$, for $i=1,2,3,4$. Then using Lemma A.14 we have that

$$\|\partial_x\mathcal{I}_{\frac{1}{3}}F_1(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|u_0\|_{H^s(\mathbb{R}^+)}+\|u\|^2_{X^{s,b}}+\|u\|^2_{Y^\alpha}),$$

$$\|\partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|^2_{X^{s,b}} + \|v\|^2_{Y^\alpha}),$$

$$\|\partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|^2_{X^{s,b}} + \|w\|^2_{Y^\alpha}).$$

Thus, we have

$$\partial_x \mathcal{I}_{\frac{1}{3}} F_1(0,t) - b_2 \partial_x \mathcal{I}_{\frac{1}{3}} F_2(0,t) - b_3 \partial_x \mathcal{I}_{\frac{1}{3}} F_3(0,t) \in H_0^{\frac{s+1}{3}}(\mathbb{R}^+). \tag{5.64}$$

In the same way we can obtain

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|u\|^2_{X^{s,b}} + \|u\|^2_{Y^\alpha}),$$

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|v_0\|_{H^s(\mathbb{R}^+)} + \|v\|^2_{X^{s,b}} + \|v\|^2_{Y^\alpha}),$$

$$\|\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_3(0,t)\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \leqslant c(\|w_0\|_{H^s(\mathbb{R}^+)} + \|w\|^2_{X^{s,b}} + \|w\|^2_{Y^\alpha}).$$

It follows that

$$\partial_x^2 \mathcal{I}_{\frac{2}{3}} F_1(0,t) - c_2 \partial_x^2 \mathcal{I}_{\frac{2}{3}} F_2(0,t) - c_3 \partial_x \mathcal{I}_{\frac{2}{3}} F_3(0,t) \in H_0^{\frac{s-1}{3}}(\mathbb{R}^+). \tag{5.65}$$

Thus, (5.63), (5.64) and (5.65) imply that the functions $\mathcal{V}_-^{\lambda_1}\gamma_1(x,t)$, $\mathcal{V}_-^{\lambda_2}\gamma_2(x,t)$, $\mathcal{V}_+^{\lambda_3}\gamma_1(x,t)$ and $\mathcal{V}_-^{\lambda_4}\gamma_4(x,t)$ are well defined.

Step 4. We will obtain a fixed point for Λ in a ball of Z.

Using Lemmas A.14, 5.2, 5.6, 5.7 and 5.1 we obtain

$$\begin{aligned} &\|\Lambda(u_2,v_2,w_2) - \Lambda(u_1,v_1,w_1)\|_Z \\ &\quad \leqslant c(\|(u_2,v_2,w_2)\|_Z + \|(u_1,v_1,w_1)\|_Z)\|(u_2,v_2,w_2) - (u_1,v_1,w_1)\|_Z \end{aligned} \tag{5.66}$$

and

$$\begin{aligned} \|\Lambda(u,v,w)\|_Z \leqslant\ & c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|v_0\|_{H^s(\mathbb{R}^+)} + \|w_0\|_{H^s(\mathbb{R}^+)} \\ & + \|u\|^2_{X^{s,b}} + \|u\|^2_{Y^\alpha} + \|v\|^2_{X^{s,b}} + \|v\|^2_{Y^\alpha} + \|w\|^2_{X^{s,b}} + \|w\|^2_{Y^\alpha}). \end{aligned} \tag{5.67}$$

By taking $\|u_0\|_{H^s(\mathbb{R}^+)} + \|v_0\|_{H^s(\mathbb{R}^+)} + \|w_0\|_{H^s(\mathbb{R}^+} < \delta$ for $\delta > 0$ suitable small, we obtain a fixed point $\Lambda(\widetilde{u},\widetilde{v},\widetilde{w}) = (\widetilde{u},\widetilde{v},\widetilde{w})$ in a ball

$$B = \{(u,v,w) \in Z, \|(u,v,w)\|_Z \leqslant 2c\delta\}.$$

It follows that the restriction

$$(u,v,w) = (\widetilde{u}|_{\mathbb{R}^- \times (0,1)}, \widetilde{v}|_{\mathbb{R}^+ \times (0,1)}, \widetilde{w}|_{\mathbb{R}^+ \times (0,1)}) \tag{5.68}$$

solves the Cauchy problem (5.1)-(5.2) with 1 boundary conditions in the sense of distributions.

Existence of solutions for any data in $H^s(\mathcal{Y})$ follows by the standard scaling argument. Suppose we are given data $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ with arbitrary size for the Cauchy problem (5.1)-(5.2) with 1 boundary conditions. For $\lambda << 1$ (to be selected after) define $u_0(x) = \lambda^2\widetilde{u}_0(\lambda x)$, $v_0(x) = \lambda^2\widetilde{v}_0(\lambda x)$ and $w_0(x) = \lambda^2\widetilde{w}_0(\lambda x)$. Taking λ sufficiently small so that

$$\begin{aligned}&\|u_0\|_{H^s} + \|v_0\|_{H^s} + \|w_0\|_{H^s}\\ &\qquad \leqslant \lambda^{\frac{3}{2}}(1+\lambda^s)(\|\widetilde{u}_0\|_{H^s(\mathbb{R}+)} + \|\widetilde{v}_0\|_{H^s(\mathbb{R}+)} + \|\widetilde{w}_0\|_{H^s(\mathbb{R}+)})\\ &\qquad < \delta.\end{aligned}$$

Then using the previous argument, there exists a solution $u(x,t)$ for the problem (5.1)-(5.2), with type 1 boundary conditions, on $0 \leqslant t \leqslant 1$. Then $\widetilde{u}(x,t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-3})$ solves the Cauchy problem for initial data $\widetilde{u}_0, \widetilde{v}_0$ and $\widetilde{w}_0$ on time interval $0 \leqslant t \leqslant \lambda^3$.

Step 5. Proof of locally Lipschitz continuity of map data-to-solution.

Let $\{(u_{0n}, v_{0n}, w_{0,n})\}_{n\in\{1,2\}}$ two initial data in $H^s(\mathcal{Y})$ such that $\|u_{0n}\| + \|v_{0n}\| + \|w_{0n}\| < \delta$, $(i = 1, 2)$ where δ is sufficiently small.

Let (u_n, v_n, w_n) $(n = 1, 2)$ the solution of Cauchy problem (5.1)-(5.2) with 1 boundary condition on the space $C([0, 1] : H^s(\mathcal{Y}))$ with initial data (u_{0n}, v_{0n}, w_{0n}). According to Step 4 the lifespans of these solutions is [0, 1].

By using the arguments used in Step 4 we have that

$$\begin{aligned}&\|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_{Z|_{[0,1]}}\\ &\leqslant c\|(u_{02}, v_{02}, w_{02}) - (u_{01}, v_{01}, w_{01})\|_{H^s(\mathcal{Y})} +\\ &c(\|(u_2, v_2, w_2) + (u_1, v_1, w_1)\|_{Z|[0,1]})\|(u_2, v_2, w_2) - (u_1, v_1, w_1)\|_{Z|[0,1]},\end{aligned}$$

where $Z|_{[0,1]}$ denotes the restrictions of functions of Z in the interval [0, 1]. In a ball of $Z|_{[0,1]}$ we have that

$$\begin{aligned}\|(u_2, v_2, w_2)-&(u_1, v_1, w_1)\|_{Z|_{[0,1]}}\\ &\leqslant c\|(u_{01}, v_{02}, w_{02}) - (u_{01}, v_{01}, w_{01})\|_{H^s(\mathcal{Y})}.\end{aligned} \tag{5.69}$$

which completes the proof for small data assumptions. The local Lipschitz continuity for any data can be showed by a scaling argument.

5.5 Proof of Corollary 5.1

By using Theorem 5.1 given a regularity index s it suffices to get scalars $\lambda_i(s)$ satisfying (5.24) such that the matrix $M(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ given by (5.22) is invertible. These choices of scalars is a crucial point to get Corollary 5.1. We will divide this analysis in 2 cases.

Case 1. Regularity: $-\frac{1}{2} < s < 1$, **for** $s \neq \frac{1}{2}$**.**

Taking $\lambda_i = 0$ for $i = 1, 3, 4$ and $0 < \lambda_2 = \frac{3}{\pi}\epsilon << 1$, a simple computations gives that the determinant of M is given by

$$\begin{aligned} &\det \mathrm{M}\left(0, \frac{3}{\pi}\epsilon, 0, 0\right) \\ &= 2\sqrt{3}\,\alpha_2\,\alpha_3\,\sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0, \end{aligned} \tag{5.70}$$

where we have used the hypothesis of Corollary 5.1 about the parameters α_i and β_i $(i \in \{1, 2\})$, and the fact $0 < \epsilon << 1$. Note that the condition (5.24) given in Theorem 5.1 is not valid for $\lambda = 0$. Then, by a continuity argument we will take the parameters λ_i $(i = 1, 3, 4)$ close to zero. In fact, for fixed $\alpha_2, \alpha_3, \beta_3$ and β_4 satisfying the hypothesis, we have that the function $\lambda \mapsto \det \mathrm{M}(\lambda, \frac{3}{\pi}\epsilon, \lambda, \lambda)$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. It follows that there exists a positive number $\delta(\epsilon) << 1$, depending of ϵ, such that $\det \mathrm{M}(\lambda, \frac{3\epsilon}{\pi}, \lambda, \lambda) \neq 0$, for $0 < \lambda < \delta$.

Thus, given $-\frac{1}{2} < s < 1$ we can choice $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, \frac{3}{\pi}\epsilon, \lambda, \lambda)$ satisfying

$$\begin{aligned} &0 < \frac{3\epsilon}{2\pi} < \min\{s + \frac{1}{2}, \frac{1}{2}\} \\ &0 < \lambda < \min\left\{\delta(\epsilon), s + \frac{1}{2}\right\}. \end{aligned} \tag{5.71}$$

Note that with this choice the all hypothesis of Theorem 5.1 part (i) are valid and the proof of Corollary 5.1 on the **Case 1** is complete.

Case 2. Regularity: $1 \leqslant s < \frac{3}{2}$.

Taking $\lambda_i = \frac{1}{2}$ for $i = 1, 3, 4$ and $0 < \lambda_2 = \frac{1}{2} - \frac{3\epsilon}{\pi}$, for $0 < \epsilon << 1$. A simple calculation shows that the determinant is given by

$$\begin{aligned} &\det \mathrm{M}\left(\frac{1}{2}, \frac{1}{2} - \frac{3\epsilon}{\pi}, \frac{1}{2}, \frac{1}{2}\right) \\ &= 2\sqrt{3}\,\alpha_2\,\alpha_3\,\sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \\ &\neq 0, \end{aligned} \tag{5.72}$$

where we have used the hypothesis $\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2} \neq -1$ and the fact $0 < \epsilon << 1$.

As the estimate condition (5.24) in part (i) of Theorem (5.1) is not valid for $\lambda = \frac{1}{2}$, then we shall make a few perturbation in λ.

For fixed $\alpha_2,\ \alpha_3,\ \beta_3$ and β_4 satisfying the hypothesis, we have that the function $\lambda \mapsto \det \mathrm{M}(\lambda, 1 - \frac{3}{\pi}\epsilon, \lambda, \lambda)$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. It follows that there exists a positive number $\delta(\epsilon) << 1$, depending of ϵ, such that $\det \mathrm{M}(\lambda, \frac{1}{2} - \frac{3}{2}\epsilon, \lambda, \lambda) \neq 0$, for $0 < \frac{1}{2} - \lambda < \delta$.

Thus, given $1 < s < \frac{3}{2}$ we can choice

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda, \frac{1}{2} - \frac{3}{\pi}\epsilon, \lambda, \lambda)$$

satisfying

$$\begin{aligned} & s - 1 < \frac{1}{2} - \frac{3\epsilon}{\pi} < s + \frac{1}{2}, \\ & \max\left\{s - 1, \frac{1}{2} - \delta\right\} < \lambda < s + \frac{1}{2}. \end{aligned} \tag{5.73}$$

This finish the proof of Corollary 5.1.

5.6 Proof of Corollary 5.2

For the regularity $-\frac{1}{2} < s < 1$ the result follows a similar idea of the proof of Corollary 5.1, by using the fact

$$\det \mathrm{M}\left(0, \frac{3}{\pi}\epsilon, 0, 0\right) = 2\sqrt{3}\alpha_2\alpha_3 \sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0,$$

Similarly, the case $1 \leqslant s < \frac{3}{2}$ with $s \neq \frac{1}{2}$ follows from the fact

$$\det \mathrm{M}\left(\frac{1}{2}, \frac{1}{2} - \frac{3\epsilon}{\pi}, \frac{1}{2}, \frac{1}{2}\right) = 2\sqrt{3}\alpha_2\alpha_3 \sin(\epsilon)\left(1 + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_3}{\alpha_3} + \frac{\beta_2}{\alpha_2}\right) \neq 0.$$

6 The Nonlinear Schrödinger Equation on Star Graphs

In this Chapter we study the local and global well-posedness problem on a star graph $\mathcal{G}$ of the initial value problem for the nonlinear Schrödinger models

$$i\partial_t \mathrm{U}(t,x) - \mathcal{A}\mathrm{U}(t,x) + \mathrm{F}(\mathrm{U}(t,x)) = 0, \tag{6.1}$$

for specific choice of $\mathcal{A}$ and the nonlinearity F, in such a way to be used in our study of the stability of standing wave solution in Chapter 10.

The self-adjoint operator $\mathcal{A}$ will be for $\mathrm{V} = (v_j)_{i=1}^N \in \mathcal{G}$ defined as

$$(\mathcal{A}\mathrm{V})(x) = (-v_j''(x))_{i=1}^N, \quad x > 0$$

with $D(\mathcal{A})$ determined by the δ and δ' interactions in (4.1) and (10.39). The nonlinearity being $\mathrm{F}(\mathrm{U}) = |\mathrm{U}|^{p-1}\mathrm{U}$, $p > 1$, and $\mathrm{F}(\mathrm{U}) = \mathrm{U}log|\mathrm{U}|^2$. Thus by the Stone's Theorem we have that the linear flow, $W(t)$, associated to (6.1) is determined by the unitary group on $L^2(\mathcal{G})$,

$$W(t) = e^{-it\mathcal{A}}.$$

To determine a explicit formulation for the group $\{W(t)\}_{t\in\mathbb{R}}$ is not an easy job, because on a star graph we do not have the useful tools of Fourier analysis (Fourier transform), thus we need to use an abstract approach based on the functional calculus of operators (see

Dunford and Schwartz (1988)). Moreover, the boundary conditions on the vertex $\nu = 0$ will produce different behavior of the group.

For more general coupling conditions for the Schrödinger model (6.1) on a star graph such as that given by Nevanlinna pairs (see subsection 3.1.6), local well-posedness theories in either $L^2(\mathcal{G})$ or in the energy space generated by these ones coupling conditions, it can seen in Theorem B and Theorem C of Grecu and Ignat (2019).

6.1 Local well-posedness for the NLS-δ

Next we establish our local well-posedness result in the space

$$\mathcal{E}(\mathcal{G}) = \{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(0) = ... = v_N(0)\},$$

for (6.1) with $\mathrm{F(U)} = |\mathrm{U}|^{p-1}\mathrm{U}$, $p > 1$, $\mathcal{A} = \mathrm{H}_\delta^\alpha$ and $D(\mathrm{H}_\delta^\alpha)$ being defined in (4.1). We note that $\mathcal{E}(\mathcal{G})$ emerges naturally as being the energy space associated to the NLS-δ equation.

Theorem 6.1. *Let $p > 1$. Then for any $\mathrm{U}_0 \in \mathcal{E}(\mathcal{G})$ there exists $T > 0$ such that equation (4.2) has a unique solution $\mathrm{U} \in C([-T, T], \mathcal{E}(\mathcal{G})) \cap C^1([-T, T], \mathcal{E}'(\mathcal{G}))$ satisfying $\mathrm{U}(0) = \mathrm{U}_0$. For each $T_0 \in (0, T)$ the mapping $\mathrm{U}_0 \in \mathcal{E}(\mathcal{G}) \to \mathrm{U} \in C([-T_0, T_0], \mathcal{E}(\mathcal{G}))$, is continuous. In particular, for $p > 2$ this mapping is at least of class C^2. Moreover, for $m \in \mathbb{N}$,*

$$L_m^2(\mathcal{G}) \equiv \{\mathrm{V} \in L^2(\mathcal{G}) : v_1(x) = ... = v_m(x), v_{m+1}(x) = ... = v_N(x)\},$$

and $\mathcal{E}_m(\mathcal{G}) = \mathcal{E}(\mathcal{G}) \cap L_m^2(\mathcal{G})$, we have for $\mathrm{U}_0 \in \mathcal{E}_m(\mathcal{G})$ that $\mathrm{U}(t) \in \mathcal{E}_m(\mathcal{G})$ for all $t \in [-T, T]$.

We divide the proof of Theorem 6.1 in several Lemmas. First, we establish an expression for the unitary group associated to NLS-δ model as well as a commutator property.

Lemma 6.1. *Let $\{e^{-it\mathrm{H}_\delta^\alpha}\}_{t\in\mathbb{R}}$ be the family of unitary operators associated to NLS-δ model (4.2), $\alpha > 0$. Then, for every $\mathrm{V} = (v_j)_{i=1}^N \in H^1(\mathcal{G})$ we have*

$$\partial_x(e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}) = -e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}' + \mathcal{B}(\mathrm{V}'), \tag{6.2}$$

where $\mathcal{B}(\mathrm{V}') = (2e^{it\partial_x^2}\widetilde{v}_j)_{j=1}^N$, with

$$\widetilde{v}_j(x) = \begin{cases} v_j'(x), & x \geqslant 0, \\ 0, & x < 0 \end{cases},$$

and $e^{it\partial_x^2}$ is the unitary group associated with the free Schrödinger operator on $\mathbb{R}$.

Proof. Let $\alpha > 0$. Using functional calculus for unbounded self-adjoint operators and the classical expression for the resolvent of $-\frac{d^2}{dx^2}$ on the positive half-line we get the formulas

$$e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}(x) = \tfrac{i}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau R_{i\tau}\mathrm{V}(x)d\tau, \tag{6.3}$$

where the resolvent $R_\mu \mathrm{V} = (\mathrm{H}_\delta^\alpha + \mu^2 I)^{-1}\mathrm{V}$ has the components

$$(R_\mu \mathrm{V})_j(x) = \widetilde{c}_j e^{-\mu x} + \frac{1}{2\mu}\int_0^\infty v_j(y)e^{-|x-y|\mu}dy. \tag{6.4}$$

The coefficients $\widetilde{c}_j$ are determined by the condition $R_\mu \mathrm{V} \in \mathbb{D}_{\alpha,\delta}$ in (4.1). Thus, from section 3.1 (Nevanlinna pairs) we need to have the relation

$$A\mathrm{V}(0) + B\mathrm{V}'(0) = 0,$$

where A and B are defined in (3.54). Let

$$t_j(\mu) = \frac{1}{2}\int_0^\infty v_j(y)e^{-\mu y}dy,$$

then from (6.4) we get $(\mathrm{R}_\mu \mathrm{V})_j(0) = \widetilde{c}_j + \frac{1}{\mu}t_j(\mu)$ and $\partial_x[(\mathrm{R}_\mu \mathrm{V})_j](0) = -\mu\widetilde{c}_j + t_j(\mu)$. Therefore, $(\widetilde{c}_j)_{j=1}^N$ is the unique solution to the system

$$\mathcal{M}\begin{pmatrix} \widetilde{c}_1 \\ \\ \vdots \\ \\ \widetilde{c}_N \end{pmatrix} = -\frac{1}{\mu}\begin{pmatrix} t_1(\mu) - t_2(\mu) \\ \vdots \\ \\ t_{N-1}(\mu) - t_N(\mu) \\ (\frac{\alpha}{N} - \mu)\sum_{j=1}^N t_j(\mu) \end{pmatrix} \tag{6.5}$$

with

$$\mathcal{M} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ \frac{\alpha}{N}+\mu & \frac{\alpha}{N}+\mu & \frac{\alpha}{N}+\mu & \dots & \frac{\alpha}{N}+\mu \end{pmatrix}. \tag{6.6}$$

Below we find $\mathrm{R}_\mu \mathrm{V}'$. Suppose initially that $v_j \in C_0^\infty(\mathbb{R}^+)$, $1 \leqslant j \leqslant N$, then there are coefficients $\widetilde{d}_j$ such that

$$\begin{aligned}(\mathrm{R}_\mu \mathrm{V}')_j(x) &= \widetilde{d}_j e^{-\mu x} + \frac{1}{2\mu}\int_0^\infty v_j'(y)e^{-\mu|x-y|}dy \\ &= \widetilde{d}_j e^{-\mu x} - \frac{1}{2}\int_0^\infty v_j(y)sign(x-y)e^{-\mu|x-y|}dy,\end{aligned} \tag{6.7}$$

where in the last equality we have used integration by parts. Thus, we obtain $(\mathrm{R}_\mu \mathbb{V}')_j(0) = \widetilde{d}_j + t_j(\mu)$. Moreover, since

$$\partial_x(\mathrm{R}_\mu \mathrm{V}')_j(x) = -\mu \widetilde{d}_j e^{-\mu x} - \frac{1}{2}\int_0^\infty v_j'(y) sign(x-y) e^{-\mu|x-y|}dy,$$

it follows from integration by parts $\partial_x(\mathrm{R}_\mu \mathrm{V}')_j(0) = -\mu\widetilde{d}_j + \mu t_j(\mu)$. Hence from the uniqueness of solution to system (6.6) it follows that $\mathrm{R}_\mu \mathrm{V}' \in \mathbb{D}_{\alpha,\delta}$ iff $\widetilde{d}_j = \mu \widetilde{c}_j$. Therefore, we obtain from (6.4) and the second equality in (6.7)

$$\begin{aligned}\partial_x(\mathrm{R}_\mu \mathrm{V})_j(x) &= -(\mathrm{R}_\mu \mathrm{V}')_j(x) - \int_0^\infty v_j(y) sign(x-y) e^{-\mu|x-y|}dy\\ &= -(\mathrm{R}_\mu \mathrm{V}')_j(x) + \frac{1}{\mu}\int_0^\infty v_j'(y) e^{-\mu|x-y|}dy.\end{aligned}$$

Thus, from representation (6.3) we get

$$\partial_x(e^{-it\mathrm{H}_\alpha^\delta}\mathrm{V}) = -e^{-it\mathrm{H}_\alpha^\delta}\mathrm{V}' + \mathcal{B}(\mathrm{V}'),$$

where

$$(\mathcal{B}(\mathrm{V}'))_j(x) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-it\tau^2}\int_0^\infty v_j'(y) e^{-i\tau|x-y|}dy\,d\tau.$$

Below we find $\mathcal{B}(\mathrm{V}')$. It is well-known that $e^{it\partial_x^2}$ can be represented as $e^{it\partial_x^2}\phi = S_t * \phi$, where $\widehat{S_t}(\xi) = e^{-it\xi^2}$. Since for $t \neq 0$ and $x \in \mathbb{R}$

$$S_t(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-it\tau^2} e^{i\tau x}d\tau = \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{-t}} e^{i\pi/4} e^{i\frac{x^2}{4t}} = \left(\frac{1}{4\pi i t}\right)^{1/2} e^{i\frac{x^2}{4t}},$$

it follows for

$$\phi(x) = \begin{cases} v_j'(x), & x \geqslant 0,\\ 0, & x < 0,\end{cases}$$

that

$$\begin{aligned} I \equiv & \frac{1}{\pi}\int_{-\infty}^\infty e^{-it\tau^2}\int_{-\infty}^\infty \phi(y)\chi_{[0,x]}(y) e^{i\tau(y-x)}dy\,d\tau\\ =& 2\int_{-\infty}^\infty \phi(y)\chi_{[0,+\infty)}(x-y) S_t(x-y)dy = 2(\chi_{[0,+\infty)}S_t) * \phi(x).\end{aligned} \tag{6.8}$$

Similarly,

$$\begin{aligned} II \equiv & \frac{1}{\pi}\int_{-\infty}^\infty e^{-it\tau^2}\int_{-\infty}^\infty \phi(y)\chi_{[x,+\infty)}(y) e^{i\tau(x-y)}dy\,d\tau\\ =& 2(\chi_{(-\infty,0]}S_t) * \phi(x).\end{aligned} \tag{6.9}$$

Thus, from (6.8)-(6.9) we have

$$(\mathcal{B}(\mathrm{V}'))_j(x) = I + II = 2S_t * \phi(x) = 2e^{it\partial_x^2}\phi(x).$$

Hence relation (6.2) follows provided that each component of V has compact support. The general case follows from a density argument. □

Remark 6.1. The case $\alpha < 0$ in Lemma 6.1 deserve a little care when being studied. Indeed, from spectral theory we have for any $\mathrm{V} \in L^2(\mathcal{G})$ that

$$e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V} = e^{-it\mathrm{H}_\delta^\alpha}\mathbb{P}_c\mathrm{V} + e^{-it\mathrm{H}_\delta^\alpha}\mathbb{P}_p\mathrm{V},$$

where $\mathbb{P}_c$ and $\mathbb{P}_p$ are L^2-orthogonal projections onto the subspaces corresponding to the continuous (essential in our case) and the discrete spectral part of H_δ^α. For $\alpha > 0$, we have

$$\sigma_c(\mathrm{H}_\delta^\alpha) = [0,\infty) \quad \text{and} \quad \sigma_p(\mathrm{H}_\delta^\alpha) = \emptyset,$$

therefore $\mathbb{P}_p \equiv 0$ and $\mathbb{P}_c\mathrm{V} = \mathrm{V}$ and so formula (6.2) is obtained. For $\alpha < 0$, we have

$$\sigma_c(\mathrm{H}_\delta^\alpha) = [0,\infty) \quad \text{and} \quad \sigma_p(\mathrm{H}_\delta^\alpha) = \{z_0\} = \{-\frac{\alpha^2}{N^2}\} \tag{6.10}$$

where the corresponding eigenfunction for z_0 is $\mathrm{V}_{z_0}(x) = (e^{\frac{\alpha}{N}x})_{j=1}^N$, and therefore $e^{-it\mathrm{H}_\delta^\alpha}\mathbb{P}_p\mathrm{V} = e^{itz_0^2}(\mathrm{V},\mathrm{V}_{z_0})\mathrm{V}_{z_0}$. In this case the formula (6.3) takes the form

$$e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}(x) = \tfrac{i}{\pi}\int_{-\infty}^{\infty} e^{-it\tau^2}\tau R_{i\tau}\mathrm{V}(x)d\tau + e^{itz_0^2}(\mathrm{V},\mathrm{V}_{z_0})\mathrm{V}_{z_0}(x). \tag{6.11}$$

Then, formula (6.2) is transformed for $\alpha < 0$ and $\mathrm{V} = (v_j)_{i=1}^N \in H^1(\mathcal{G})$ as

$$\partial_x(e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}) = -e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V}' + \mathcal{B}(\mathrm{V}') + e^{itz_0^2}(\mathrm{V},\mathrm{V}_{z_0})\mathrm{V}'_{z_0}. \tag{6.12}$$

The proof of the spectral properties of H_δ^α (continuous spectrum and discrete spectrum characterization) follows from Theorem 3.13.

Lemma 6.2. *The family of unitary operators $\{e^{-it\mathrm{H}_\delta^\alpha}\}_{t\in\mathbb{R}}$ on $L^2(\mathcal{G})$ preserves the space $\mathcal{E}(\mathcal{G})$, i.e. for $\mathrm{V} \in \mathcal{E}(\mathcal{G})$ we have $e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V} \in \mathcal{E}(\mathcal{G})$.*

Proof. Assume $\alpha > 0$. Let $\mathrm{V} \in \mathcal{E}(\mathcal{G})$, then it follows from (6.2) that $e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V} \in H^1(\mathcal{G})$. Further, since $R_\mu\mathrm{V} \in \mathbb{D}_{\alpha,\delta}$, we get from (6.3) the equality $(e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V})_1(0) = \ldots = (e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V})_N(0)$. The case $\alpha < 0$ follows from (6.11) and (6.12) . □

Proof of Theorem 6.1. The local well-posedness result in $\mathcal{E}(\mathcal{G})$ follows from standard arguments of the Banach fixed point theorem applied to non-linear Schrödinger equations (see Cazenave (2003)). We will give the sketch of the proof for the case $\alpha > 0$. Consider the mapping $J_{\mathrm{U}_0} : C([-T,T],\mathcal{E}(\mathcal{G})) \longrightarrow C([-T,T],\mathcal{E}(\mathcal{G}))$ given by

$$J_{\mathrm{U}_0}[\mathrm{U}](t) = e^{-it\mathrm{H}_\delta^\alpha}\mathrm{U}_0 + i\int_0^t e^{-i(t-s)\mathrm{H}_\delta^\alpha}|\mathrm{U}(s)|^{p-1}\mathrm{U}(s)ds,$$

where $e^{-it\mathrm{H}_\delta^\alpha}$ is the unitary group given by (6.3). One needs to show that the mapping J_{U_0} is well-defined (we note immediate that the nonlinearity satisfies the continuity condition at the graph vertex $\nu = 0$). Next, we estimate the nonlinear term $|\mathrm{U}(s)|^{p-1}\mathrm{U}(s)$. Using the one-dimensional Gagliardo-Nirenberg inequality one may show (see formula (2.2))

$$\|\mathrm{U}\|_q \leqslant C\|\mathrm{U}'\|^{\frac{1}{2}-\frac{1}{q}}\|\mathrm{U}\|^{\frac{1}{2}+\frac{1}{q}}, \quad q > 2,\ C > 0. \tag{6.13}$$

Using (6.13), the relation $|(|f|^{p-1}f)'| \leqslant C_0|f|^{p-1}|f'|$ and Hölder's inequality, we obtain for $\mathrm{U} \in H^1(\mathcal{G})$

$$||\mathrm{U}|^{p-1}\mathrm{U}||_{H^1(\mathcal{G})} \leqslant C_1||\mathrm{U}||^p_{H^1(\mathcal{G})}. \tag{6.14}$$

Let $\mathrm{U}_0, \mathrm{U} \in \mathcal{E}(\mathcal{G})$, then from Lemmas 6.1-6.2 and (6.14) it follows that $J_{\mathrm{U}_0}[\mathrm{U}](t) \in \mathcal{E}(\mathcal{G})$. Moreover, using (6.2), (6.14), L^2-unitarity of $e^{-it\mathrm{H}_\delta^\alpha}$ and $e^{it\partial_x^2}$, we get

$$||J_{\mathrm{U}_0}[\mathrm{U}](t)||_{H^1(\mathcal{G})} \leqslant C_2||\mathrm{U}_0||_{H^1(\mathcal{G})} + C_3T\sup_{s\in[0,T]}||\mathrm{U}(s)||^p_{H^1(\mathcal{G})},$$

where the positive constants C_2, C_3 do not depend on U_0. The continuity and contraction property of J_{U_0} are proved in a standard way. Therefore, we obtain the existence of a unique solution to the Cauchy problem associated to (4.2) on $\mathcal{E}(\mathcal{G})$.

Next, we recall that the argument based on the contraction mapping principle above has the advantage that if $F(\mathrm{U},\overline{\mathrm{U}}) = |\mathrm{U}|^{p-1}\mathrm{U}$ has a specific regularity, then it is inherited by the mapping data-solution. In particular, following the ideas in the proof of Angulo and Goloshchapova (2018), we consider for $(\mathrm{V}_0,\mathrm{V}) \in B(\mathrm{U}_0;\epsilon) \times C([-T,T],\mathcal{E}(\mathcal{G}))$ the mapping

$$\Gamma(\mathrm{V}_0,\mathrm{V})(t) = \mathrm{V}(t) - J_{\mathrm{V}_0}[\mathrm{V}](t), \qquad t \in [-T,T].$$

Then $\Gamma(\mathrm{U}_0,\mathrm{U})(t) = 0$ for all $t \in [-T,T]$. For $p-1$ being an even integer, $F(\mathrm{U},\overline{\mathrm{U}})$ is smooth, and therefore Γ is smooth. Hence, using the arguments applied for obtaining the local well-posedness in $\mathcal{E}(\mathcal{G})$ above, we can show that the operator $\partial_\mathrm{V}\Gamma(\mathrm{U}_0,\mathrm{U})$ is one-to-one and onto. Thus, by the Implicit Function Theorem there exists a smooth mapping $\Lambda : B(\mathrm{U}_0;\delta) \to C([-T,T],\mathcal{E}(\mathcal{G}))$ such that $\Gamma(\mathrm{V}_0,\Lambda(\mathrm{V}_0)) = 0$ for all $\mathrm{V}_0 \in B(\mathrm{U}_0;\delta)$. This argument establishes the smoothness property of the mapping data-solution associated to equation (4.3) when $p-1$ is an even integer.

If $p-1$ is not an even integer and $p > 2$, then $F(\mathrm{U},\overline{\mathrm{U}})$ is $C^{[p]}$-function, and consequently the mapping data-solution is of class $C^{[p]}$ (see Linares and Ponce (2009, Remark 5.7)). Therefore, for $p > 2$ we conclude that the mapping data-solution is at least of class C^2.

Next, we show that the unitary group $e^{-it\mathrm{H}_\delta^\alpha}$ preserves the subspace $\mathcal{E}_m(\mathcal{G})$. Indeed, let $\mathrm{V} = (v_j) \in \mathcal{E}_m(\mathcal{G})$, then we obtain $t_1(\mu) = ... = t_m(\mu)$ and $t_{m+1}(\mu) = ... = t_N(\mu)$, where

$$t_j(\mu) = \frac{1}{2}\int_0^\infty v_j(y)e^{-\mu y}dy.$$

Hence, from (6.6) it follows $\widetilde{c}_1 = ... = \widetilde{c}_m$ and $\widetilde{c}_{m+1} = ... = \widetilde{c}_N$. Thus, by (6.3) we get $e^{-it\mathrm{H}_\delta^\alpha}\mathrm{V} \in \mathcal{E}_m(\mathcal{G})$. Lastly, the well-posedness in $\mathcal{E}_m(\mathcal{G})$ follows from the uniqueness of the solution to the Cauchy problem in $\mathcal{E}(\mathcal{G})$ and the invariance of the space $\mathcal{E}_m(\mathcal{G})$ for the unitary group $e^{-it\mathrm{H}_\delta^\alpha}$ shown above. □

The following global well-posedness result for the NLS-δ model is an immediate consequence of Theorem 6.1 and the existence of the conservation of charge and energy , i.e., for $\mathrm{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ the quantities

$$E_\alpha(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}\,|v_1(0)|^2\,,$$

and

$$Q(\mathrm{U}(t)) = ||\mathrm{U}(t)||^2,$$

satisfy $Q(\mathrm{U}(t)) = ||\mathrm{U}_0||^2$ and $E_\alpha(\mathrm{U}(t)) = E_\alpha(\mathrm{U}_0)$, for $t \in [-T, T]$.

Theorem 6.2. *Let $1 < p < 5$. Then for any $\mathrm{U}_0 \in \mathcal{E}(\mathcal{G})$, equation (4.2) has a unique global solution $\mathrm{U} \in C(\mathbb{R}, \mathcal{E}(\mathcal{G})) \cap C^1(\mathbb{R}, \mathcal{E}'(\mathcal{G}))$ satisfying $\mathrm{U}(0) = \mathrm{U}_0$. Similarly for $\mathrm{U}_0 \in \mathcal{E}_k(\mathcal{G})$.*

Remark 6.2. (i) *Using the Sobolev embedding theorem, Gagliardo-Nirenberg inequality (6.13), the above conservation laws, one can see that $E_\alpha : \mathcal{E}(\mathcal{G}) \to \mathbb{R}$ is well defined.*

(ii) *Observe that $E_\alpha \in C^2(\mathcal{E}(\mathcal{G}), \mathbb{R})$ since $p > 1$. This fact allows us to apply the results by Ohta (2011) in our instability analysis in Chapter 6.*

(iii) *The property of the data-solution mapping to be of class C^2 for $p > 2$, it will a tool for showing that the linear instability property of standing wave solution for (4.2) in fact to be nonlinear instability (see Chapter 10).*

6.2 Local well-posedness for the NLS-δ'

Next we establish our local well-posedness result in the space $H^1(\mathcal{G})$ for (6.1) with $\mathrm{F}(\mathrm{U}) = |\mathrm{U}|^{p-1}\mathrm{U}$, $p > 1$, $\mathcal{A} = \mathrm{H}_\lambda^{\delta'}$ and $D(\mathrm{H}_\lambda^{\delta'})$ being defined in (10.39). We note that $H^1(\mathcal{G})$ emerges naturally as being the energy space associated to the NLS-δ' equation. Moreover, this space is the natural framework for studying the orbital stability of standing wave solutions for this model.

First, we establish the following property for the unitary group associated to the NLS-δ' model.

Lemma 6.3. *Let* $\{e^{-it\mathrm{H}_\lambda^{\delta'}}\}_{t\in\mathbb{R}}$ *be the family of unitary operators associated to NLS-*δ' *model. Then for every* $\mathrm{V} \in H^1(\mathcal{G})$ *we have the relation* $\partial_x(e^{-it\mathrm{H}_\lambda^{\delta'}}\mathrm{V}) = -e^{-it\mathrm{H}_\lambda^{\delta'}}\mathrm{V}' + \mathcal{B}(\mathrm{V}')$, *where* $\mathcal{B}(\mathrm{V}') = (2e^{it\partial_x^2}\widetilde{v}_j)_{j=1}^N$, *with*

$$\widetilde{v}_j(x) = \begin{cases} v_j'(x), & x \geqslant 0, \\ 0, & x < 0 \end{cases},$$

and $e^{it\partial_x^2}$ *is the unitary group associated with the free Schrödinger operator on* $\mathbb{R}$.

Proof. The proof repeats the one of Lemma 6.1. The only difference is that δ'-interaction on $\mathcal{G}$ is induced by the following condition

$$\mathrm{V} \in D(\mathrm{H}_\lambda^{\delta'}) \quad \text{iff} \quad A\mathrm{V}(0) + B\mathrm{V}'(0) = 0$$

where A and B are defined by $A = -B_K$ in (3.53) and $B = B_{\delta'}$ in (3.55). □

Theorem 6.3. *Let* $p > 1$. *Then for any* $\mathrm{U}_0 \in H^1(\mathcal{G})$ *there exists* $T > 0$ *such that equation* (4.2) *has a unique solution* $\mathrm{U} \in C([-T,T], H^1(\mathcal{G})) \cap C^1([-T,T], [H^1(\mathcal{G})]')$ *satisfying* $\mathrm{U}(0) = \mathrm{U}_0$. *For each* $T_0 \in (0,T)$ *the mapping* $\mathrm{U}_0 \in H^1(\mathcal{G}) \to \mathrm{U} \in C([-T_0,T_0], H^1(\mathcal{G}))$, *is continuous. In particular, for* $p > 2$ *this mapping is at least of class* C^2.

Moreover, the conservation of energy and charge holds:

$$E_\lambda(\mathrm{U}(t)) = E_\lambda(\mathrm{U}_0), \ \ \textit{and} \ \ Q(\mathrm{U}(t)) = ||\mathrm{U}(t)||^2 = ||\mathrm{U}_0||^2, \ t \in [-T,T],$$

where the energy E_λ *is defined for* $\mathrm{V} = (v_j)_{j=1}^N \in H^1(\mathcal{G})$ *by*

$$E_\lambda(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^N v_j(0)\Big|^2.$$

Consequently, for $1 < p < 5$, *we can choose* $T = +\infty$.

Proof. The prove repeats the one of Theorem 6.1. In particular, it essentially uses Lemma 6.3 and the Banach contraction theorem. □

Remark 6.3. Analogously to the case of NLS-δ equation the following equality holds

$$e^{-it\mathrm{H}_{\delta'}^\lambda}\mathrm{V} = e^{-it\mathrm{H}_{\delta'}^\lambda}\mathbb{P}_c\mathrm{V} + e^{-it\mathrm{H}_{\delta'}^\lambda}\mathbb{P}_p\mathrm{V}.$$

Similarly, for $\lambda > 0$, we have $\sigma_c(\mathrm{H}_{\delta'}^\lambda) = [0,\infty)$ and $\sigma_p(\mathrm{H}_{\delta'}^\lambda) = \emptyset$, therefore $\mathbb{P}_p = 0$. For $\lambda < 0$, $\sigma_c(\mathrm{H}_{\delta'}^\lambda) = [0,\infty)$ and $\sigma_p(\mathrm{H}_{\delta'}^\lambda) = \{z_0\} = \{-\frac{N^2}{\lambda^2}\}$, where the corresponding eigenfunction is $\mathrm{V}_{z_0}(x) = (e^{\frac{N}{\lambda}x})_{j=1}^N$, and therefore $e^{-it\mathrm{H}_{\delta'}^\lambda}\mathbb{P}_p\mathrm{V} = e^{itz_0^2}(\mathrm{V},\mathrm{V}_{z_0})\mathrm{V}_{z_0}$.

The proof of the spectral properties of $\mathrm{H}^{\lambda}_{\delta'}$ can be obtained via the one of Albeverio, Gesztesy, et al. 1988, Theorem 4.3 for the case of the Schrödinger operator with δ'-interaction on the line. But, by using the extension theory in Chapter 3, we can describe the spectrum for $\lambda < 0$ seeing $\mathrm{H}^{\lambda}_{\delta'}$ as the self-adjoint extension of the symmetric non-negative operator $(\mathrm{L}'_0, D(\mathrm{L}'_0))$ defined in Theorem 3.7 with deficiency indices $n_{\pm}(\mathrm{L}'_0) = 1$ and then to apply Theorem 3.11 (see also Example 1 in subsection 3.1.5 and the Nevanlinna pairs approach in subsection 3.1.6).

6.3 Global well-posedness for NLS-log-δ

Next we establish a global well-posedness result for (6.1) with $F(\mathrm{U}) = \mathrm{U}Log|\mathrm{U}|^2$, $\mathcal{A} = \mathrm{H}^{\alpha}_{\delta}$ and $D(\mathrm{H}^{\alpha}_{\delta})$ being defined in (4.1).

In this case the natural space of energy is less immediate than in the two cases above. On $\mathcal{G}$ we define the following weighted Hilbert spaces

$$W^j(\mathcal{G}) = \bigoplus_{j=1}^{N} W^j(\mathbb{R}_+), \quad W^j(\mathbb{R}_+) = \{f \in H^j(\mathbb{R}_+) : x^j f \in L^2(\mathbb{R}_+)\},$$

$$W^j_k(\mathcal{G}) = W^j(\mathcal{G}) \cap L^2_m(\mathcal{G}), \quad j \in \{1, 2\},$$

and the Banach space

$$W(\mathcal{G}) = \bigoplus_{j=1}^{N} W(\mathbb{R}_+), \;\; W(\mathbb{R}_+) = \{f \in H^1(\mathbb{R}_+) : |f|^2 Log|f|^2 \in L^1(\mathbb{R}_+)\}.$$

In particular, $W_{\mathcal{E}}(\mathcal{G}) \equiv W(\mathcal{G}) \cap \mathcal{E}(\mathcal{G})$, $W^1_{\mathcal{E}}(\mathcal{G}) \equiv W^1(\mathcal{G}) \cap \mathcal{E}(\mathcal{G})$, and $W^1_{\mathcal{E},k}(\mathcal{G}) \equiv W^1_{\mathcal{E}}(\mathcal{G}) \cap L^2_k(\mathcal{G})$.

We are interested in the global well-posedness theory for the NLS-log-δ model in the space $W^1_{\mathcal{E}}(\mathcal{G})$ because of our stability theory for the Gaussian tail and bump profiles in (7.18) to be given in Chapter 10.

In (Ardila 2017) the following well-posedness result in $W_{\mathcal{E}}(\mathcal{G})$ was proved (see Angulo and Goloshchapova (2017b)).

Proposition 6.1. *For any* $\mathrm{U}_0 \in W_{\mathcal{E}}(\mathcal{G})$ *there is a unique solution*

$$\mathrm{U} \in C(\mathbb{R}, W_{\mathcal{E}}(\mathcal{G})) \cap C^1(\mathbb{R}, W'_{\mathcal{E}}(\mathcal{G}))$$

of (10.53) *such that* $\mathrm{U}(0) = \mathrm{U}_0$ *and* $\sup_{t\in\mathbb{R}} ||\mathrm{U}(t)||_{W_{\mathcal{E}}(\mathcal{G})} < \infty$. *Furthermore, the conservation of energy and charge holds, that is,*

$$E_{\alpha,\log}(\mathrm{U}(t)) = E_{\alpha,\log}(\mathrm{U}_0), \;\; \textit{and} \;\; Q(\mathrm{U}(t)) = ||\mathrm{U}(t)||^2 = ||\mathrm{U}_0||^2,$$

where the energy $E_{\alpha,\log}$ *is defined for* $\mathrm{V} = (v_j)_{j=1}^N \in W_{\mathcal{E}}(\mathcal{G})$ *by*

$$E_{\alpha,\log}(\mathrm{V}) = \frac{1}{2}||\mathrm{V}'||^2 - \frac{1}{2}\sum_{j=1}^{N}\int_0^\infty |v_j|^2 Log|v_j|^2 dx + \frac{\alpha}{2}|v_1(0)|^2.$$

Using the above result, we obtain well-posedness in $W_{\mathcal{E}}^1(\mathcal{G})$.

Theorem 6.4. *If* $\mathrm{U}_0 \in W_{\mathcal{E}}^1(\mathcal{G})$*, there is a unique solution* $\mathrm{U}(t)$ *of* (10.53) *such that* $\mathrm{U}(t) \in C(\mathbb{R}, W_{\mathcal{E}}^1(\mathcal{G}))$ *and* $\mathrm{U}(0) = \mathrm{U}_0$.

Proof. The proof can be found in (Angulo and Goloshchapova 2018). Basically it follows from Proposition 6.1 and two additional facts. The first one is that $W_{\mathcal{E}}^1(\mathcal{G}) \subset W_{\mathcal{E}}(\mathcal{G})$ (see Angulo and Goloshchapova (2017b, Lemma 3.1)). And the second one is the continuity of the mapping $t \mapsto ||x\mathrm{U}(t)||^2$ on $\mathbb{R}$. □

We note that the proof of Proposition 6.1 and Theorem 6.4 uses a strategy based on regularization of the nonlinear term of the NLS-log-δ and convergence of solutions (see Cazenave (2003)). Therefore the Banach contraction theorem is not used. As will be seen in our study of the orbital stability theory of standing wave solutions for models in (6.1) (Chapter 10, Remarks 10.1 and 10.7), this type of situation can put a “smokescreen” to a full picture about the nonlinear instability problem.

7 Existence of Soliton Profiles on Start Graphs

In this chapter we construct some special solutions for the nonlinear Schrödinger equation, the Korteweg–de Vries equation and the sine Gordon equation. As described before, these equations has important applications Soliton and other nonlinear waves in branched systems appear in different system of condensed matter, Josephson junction networks, polymers, optics, neuroscience, DNA, blood pressure waves in large arteries or in shallow water equation to describe a fluid network.

7.1 Existence of standing waves for NLS models on start graphs

In this section we consider the following vectorial nonlinear Schrödinger equation on $\mathcal{G}$

$$i\partial_t \mathrm{U}(t,x) - \mathcal{A}\mathrm{U}(t,x) + \mathrm{F}(\mathrm{U}(t,x)) = 0, \tag{7.1}$$

where $\mathrm{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{C}^N$, the nonlinearity $\mathrm{F}(\mathrm{U})$ satisfies $\mathrm{F}(e^{i\theta}\mathrm{U}) = e^{i\theta}\mathrm{F}(\mathrm{U})$, $\theta \in [0, 2\pi)$. The star graph $\mathcal{G}$ will be composed by N positive half-lines attached to the common vertex $\nu = 0$, and $\mathcal{A}$ is a self-adjoint operator with $D(\mathcal{A}) \subset L^2(\mathcal{G})$ which represents the coupling conditions in the graph-vertex (see section 3.1).

Next, we consider the so-called *standing wave solutions* for (7.1), i.e. the solutions of the form

$$\mathrm{U}(t,x) = e^{i\omega t}\Phi(x), \tag{7.2}$$

with the profile $\Phi \in D(\mathcal{A})$. By substituting this profile in (7.1) with we arrive to the nonlinear (vectorial) system

$$\mathcal{A}\Phi + \omega\Phi - \mathrm{F}(\Phi) = 0. \tag{7.3}$$

The equality in (7.3) should be understood in a distributional sense.

In the following for specific self-adjoint operators $\mathcal{A}$ and nonlinearity F we determine formulas for the profile Φ. In Chapter 8 we study the stability properties of these solutions.

7.1.1 Standing waves for NLS-δ model

We consider the NLS-δ model in (7.1), namely, $\mathrm{F(U)} = |\mathrm{U}|^{p-1}\mathrm{U}$, $p > 1$, $\mathcal{A} \equiv \mathrm{H}_\alpha^\delta$ with domain $D(\mathrm{H}_\alpha^\delta) = \mathbb{D}_{\alpha,\delta}$ and acting for $\mathrm{V} = (v_j)_{j=1}^N$ as

$$\begin{aligned} &(\mathrm{H}_\alpha^\delta \mathrm{V})(x) = (-v_j''(x))_{j=1}^N, \quad x > 0, \\ &\mathbb{D}_{\alpha,\delta} = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0), \ \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\Big\}. \end{aligned} \tag{7.4}$$

The nonlinearity acts componentwise, i.e. $(|\mathrm{U}|^{p-1}\mathrm{U})_j = |u_j|^{p-1}u_j$.

In (Adami, Cacciapuoti, et al. 2014c) was obtained the following description of all solutions to equation

$$\mathrm{H}_\alpha^\delta \Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0, \tag{7.5}$$

Theorem 7.1. *Let* $[s]$ *denote the integer part of* $s \in \mathbb{R}$, *and* $\alpha \neq 0$. *Then equation* (7.5) *has* $\left[\frac{N-1}{2}\right]+1$ *(up to permutations of the edges of* $\mathcal{G}$*) vector solutions* $\Phi_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$, $m = 0, ..., \left[\frac{N-1}{2}\right]$, *which are given by*

$$\varphi_{m,j}^\alpha(x) = \begin{cases} \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x - a_m\right)\right]^{\frac{1}{p-1}}, & j = 1, .., m; \\ \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + a_m\right)\right]^{\frac{1}{p-1}}, & j = m+1, ..., N, \end{cases} \tag{7.6}$$

where

$$a_m = \tanh^{-1}\left(\frac{\alpha}{(2m-N)\sqrt{\omega}}\right), \ \textit{and} \ \ \omega > \frac{\alpha^2}{(N-2m)^2}. \tag{7.7}$$

Remark 7.1. (i) Note that in the case $\alpha < 0$ vector $\Phi_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has m *bumps* and $N-m$ *tails*. Φ_0^α is called the *N-tail profile*. Moreover, the N-tail profile is the only symmetric (i.e. invariant under permutations of the edges) solution of equation (7.5). In the case $N = 5$ we have three types of profiles: *5-tail profile*, *4-tail/1-bump profile* and *3-tail/2-bump profile*. They are demonstrated on Figure 7.1 (from the left to the right).

(ii) In the case $\alpha > 0$ vector $\Phi_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$ has m *tails* and $N-m$ *bumps* respectively. Φ_0^α is called the *N-bump profile*. For $N = 5$ we have: *5-bump profile, 4-bump/1-tail profile, 3-bump/ 2-tail profile*. They are demonstrated on Figure 7.2 (from the left to the right).

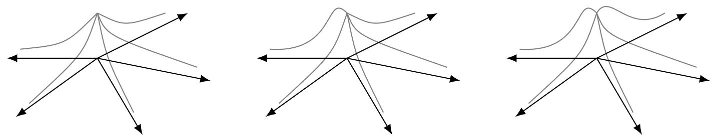

Figure 7.1: Bumps and tails for NLS

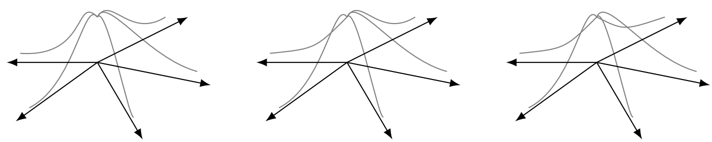

Figure 7.2: Bumps for NLS

Proof. Let $\Phi = (\varphi_j)_{j=1}^N \in D(\mathrm{H}_\alpha^\delta)$ satisfying the vectorial elliptic system (7.5). Thus every component of Φ on every edge must seek $L^2(0,+\infty)$-solution to the equation

$$-\psi'' + \omega\psi - |\psi|^{p-1}\psi = 0, \quad \omega > 0. \tag{7.8}$$

The most general $L^2(0,+\infty)$-solution is $\psi(x) = \sigma\psi_s(x-y)$ with $\sigma \in \mathbb{C}$, $|\sigma| = 1$, $y \in \mathbb{R}$ and

$$\psi_s(x) = \left[\frac{(p+1)\omega}{2}\right]^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\Big(\frac{p-1}{2}\sqrt{\omega}x\Big). \tag{7.9}$$

Therefore, the components φ_j are given by

$$\varphi_j(x) = \sigma_j\psi_s(x-y_j). \tag{7.10}$$

In order to have a solution for (7.5) it is sufficient to impose boundary conditions (7.4). The continuity condition in (7.4) implies $\sigma_1 = \cdots = \sigma_N$ and $y_j = \gamma_j a$ with $\gamma_j = \pm 1$ and $a > 0$. We can consider $\sigma_1 = 1$ without losing generality. Now, we determine γ_j. The second boundary condition in (7.4) rewrites as

$$\tanh\Big(\frac{p-1}{2}\sqrt{\omega}a\Big)\sum_{j=1}^N \gamma_j = \frac{\alpha}{\sqrt{\omega}}. \tag{7.11}$$

Equation (7.11) implies that $\sum_{j=1}^{N}\gamma_j$ must have the same sign of α. Moreover, a choice of the set $\{\gamma_j\}_{j=1}^{N}$, condition (7.11) fixes uniquely a. Now, by referring to the bell shape of the function ψ_s, we say that in the j-th edge there is a *bump* if $y_j > 0$, that is $\gamma_j = 1$; there is a *tail* if $y_j < 0$, that is, if $\gamma_j = -1$. Thus we choose to index the solutions by the number j of bumps. Thus, we obtain a unique solution to (7.11) which we call a_j. In this way we arrive at (7.6) and (7.7). This finishes the proof. □

It was shown by Adami, Cacciapuoti, et al. (2014b) that for $-N\sqrt{\omega} < \alpha < \alpha^* < 0$, the vector tail-solution $\Phi_0^\alpha = (\varphi_{0,j})_{j=1}^{N}$, with $\varphi_{0,j} = \varphi_{0,\alpha}$ for all j and

$$\varphi_{0,\alpha}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\left(\frac{-\alpha}{N\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}} \tag{7.12}$$

it is the ground state associated to (7.5). The parameter α^* guarantees the minimality of the action functional

$$\mathrm{S}_\alpha(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 + \tfrac{\omega}{2}||\mathrm{V}||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}|v_1(0)|^2, \tag{7.13}$$

for $\mathrm{V} = (v_j)_{j=1}^{N} \in \mathcal{E}(\mathcal{G}) = \{\mathrm{V} \in H^1(\mathcal{G}) : v_1(0) = \cdots = v_j(0)\}$, at Φ_0^α with the constraint given by the Nehari manifold

$$\mathcal{N} = \{\mathrm{V} \in \mathcal{E}(\mathcal{G}) \setminus \{0\} : ||\mathrm{V}'||^2 + \omega||\mathrm{V}||^2 - ||\mathrm{V}||_{p+1}^{p+1} + \alpha|v_1(0)|^2 = 0\}.$$

Note that $\Phi_m^\alpha \in \mathcal{N}$ for any m. In (Adami, Cacciapuoti, et al. 2014b) it was proved that for $m \neq 0$ and $\alpha < 0$ we have

$$\mathrm{S}_\alpha(\Phi_0^\alpha) < \mathrm{S}_\alpha(\Phi_m^\alpha) < \mathrm{S}_\alpha(\Phi_{m+1}^\alpha).$$

This fact justifies the name *excited states* for the stationary states Φ_m^α, $m \neq 0$.

For $\alpha > 0$ and any m nothing is known about variational properties of the profiles Φ_m^α. In particular, one can easily verify that

$$\mathrm{S}(\Phi_0^\alpha) > \mathrm{S}(\Phi_m^\alpha) > \mathrm{S}(\Phi_{m+1}^\alpha), \quad m \neq 0.$$

We will see in Chapter 8, Theorems 10.2 and 10.3, that when the profile Φ_m^α has mixed structure (i.e. has bumps and tails), they are "almost always" nonlinearly unstable.

7.1.2 Standing waves for NLS-δ' model

We consider the NLS-δ' model in (7.1), namely, $\mathrm{F}(\mathrm{U}) = |\mathrm{U}|^{p-1}\mathrm{U}$, $p > 1$, $\mathcal{A} \equiv \mathrm{H}_\lambda^{\delta'}$ with domain $D(\mathrm{H}_\lambda^{\delta'}) = \mathbb{D}_{\lambda,\delta'}$ and acting for $\mathrm{V} = (v_j)_{j=1}^{N}$ as

$$\begin{aligned}(\mathrm{H}_\lambda^{\delta'}\mathrm{V})(x) &= (-v_j''(x))_{j=1}^{N}, \quad x > 0,\\ \mathbb{D}_{\lambda,\delta'} &= \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1'(0) = \ldots = v_N'(0),\ \sum_{j=1}^{N} v_j(0) = \lambda v_1'(0)\Big\}.\end{aligned} \tag{7.14}$$

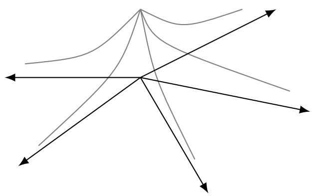

Figure 7.3: 5-tail profile for the NLS-δ' model

Thus, the profile Φ in the standing wave solution (7.2) satisfies the equation

$$\mathrm{H}_\alpha^{\delta'}\Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0, \tag{7.15}$$

The following result establishes a family of continuous tail profiles to equation (7.15) (see figure 7.1.2 below).

Theorem 7.2. *For $\lambda < 0$ and $\omega > \frac{N^2}{\lambda^2}$, the equation (7.15) has a vector solution $\Phi_{\lambda,\delta'} = (\varphi_{\lambda,j})_{j=1}^N$ under the conditions $\varphi_{\lambda,1} = \dots = \varphi_{\lambda,N} \equiv \varphi_{\lambda,\delta'}$ and $N\varphi_{\lambda,j}(0) = \lambda\varphi'_{\lambda,j}(0)$, with*

$$\varphi_{\lambda,\delta'}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\left(\frac{-N}{\lambda\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}. \tag{7.16}$$

Moreover, $\Phi_{\lambda,\delta'} \in \mathbb{D}_{\lambda,\delta'}$

Proof. The proof follows immediately from (7.8), (7.9) and (7.10). □

Remark 7.2. The description of the set of all solutions to the stationary equation (7.15) is unknown. We note that any $L^2(0,+\infty)$-solution to (7.15) has the form

$$\Phi(x) = (\varphi_j(x))_{j=1}^N = (\sigma_j\psi_s(x+x_j))_{j=1}^N,$$

where $\sigma_j \in \mathbb{C}, |\sigma_j| = 1, x_j \in \mathbb{R}$, and ψ_2 defined in (7.9). Hence, denoting $t_j = \tanh(x_j)$, from (7.14) we get the relations

$$\begin{cases} \sigma_1(1-t_1)^{\frac{1}{p-1}}t_1 = \dots = \sigma_N(1-t_N)^{\frac{1}{p-1}}t_N, \\ \sum\limits_{j=1}^N \sigma_j(1-t_j)^{\frac{1}{p-1}} = -\lambda\sqrt{\omega}\sigma_1(1-t_1)^{\frac{1}{p-1}}t_1. \end{cases}$$

In (Adami and Noja 2013), for the case of $\mathcal{G} = \mathbb{R}$ (δ'- interaction on the line), the authors established the existence of two families (odd and asymmetric) of solutions to (7.15). For $N \geqslant 3$, it seems to be very nontrivial problem to determine a complete description of the solutions to (7.15). Observe that in the case of NLS-δ equation the situation is easier since the continuity condition $\varphi_1(0) = ... = \varphi_N(0)$ implies $|\varphi_1'(0)| = ... = |\varphi_N'(0)|$, therefore, $\sigma_1 = ... = \sigma_N$ and $x_j = \pm a$, $a > 0$.

We will see in Chapter 10, Theorem 10.7, the stability properties of the continuous-tail profile $\Phi_{\lambda,\delta'}$ by the flow of the NLS-δ' model.

7.1.3 Standing waves for NLS-log-δ model

We consider the NLS-log-δ model in (7.1), namely, $\mathrm{F(U)} = \mathrm{U}log|\mathrm{U}|$, $\mathcal{A} \equiv \mathrm{H}_\alpha^\delta$ with domain $\mathbb{D}_{\alpha,\delta}$ defined in (7.4).

Thus, the profile Ψ in the standing wave solution (7.2) satisfies the equation

$$\mathrm{H}_\alpha^\delta \Psi + \omega\Psi - \Psi Log|\Psi|^2 = 0, \tag{7.17}$$

The following result establishes all the family of profiles to equation (7.1). The proof is immediate.

Theorem 7.3. *Let $\alpha \neq 0$. Then equation (7.17) has $\left[\frac{N-1}{2}\right] + 1$ vector solutions $\Psi_m^\alpha = (\psi_{m,j}^\alpha)_{j=1}^N$, $m = 0, ..., \left[\frac{N-1}{2}\right]$, given by*

$$\psi_{m,j}^\alpha(x) = \begin{cases} e^{\frac{\omega+1}{2}} e^{-\frac{(x-a_m)^2}{2}}, & j = 1, .., m; \\ e^{\frac{\omega+1}{2}} e^{-\frac{(x+a_m)^2}{2}}, & j = m+1, ..., N, \end{cases} \tag{7.18}$$

where $a_m = \frac{\alpha}{2m-N}$.

We should note that the structure of the profiles that solve (7.17) is similar to the one in the case of NLS-δ equation (see Remark 7.1). It was proved in (Ardila 2017) that for $\alpha < \alpha_{\log}^* < 0$, the vector tail solution $\Psi_0^\alpha = (\psi_{\alpha,\delta})_{j=1}^N$ defined by

$$\psi_{\alpha,\delta} = \psi_{0,j}^\alpha(x) = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{\alpha}{N})^2}{2}} \tag{7.19}$$

is the ground state. The condition $\alpha < \alpha_{Log}^*$ guarantees constrained minimality of the following action functional for $\mathrm{V} \in W_{\mathcal{E}}(\mathcal{G})$,

$$\mathrm{S}_{\alpha,\log}(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 + \tfrac{(\omega+1)}{2}||\mathrm{V}||^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 Log|v_j|^2 dx + \tfrac{\alpha}{2}|v_1(0)|^2, \tag{7.20}$$

by the constraint given by the Nehari manifold $\mathcal{N}$, namely, $V \in \mathcal{N}$ if and only if $V \in W_{\mathcal{E}}(\mathcal{G}) \setminus \{0\}$ and

$$||V'||^2 + \omega||V||^2 - \sum_{j=1}^{N} \int_0^{\infty} |v_j|^2 Log|v_j|^2 dx + \alpha|v_1(0)|^2 = 0.$$

In (Ardila 2017) the author proved that the standing wave $e^{i\omega t}\Psi_{\alpha,\delta}$ is orbitally stable in $W_{\mathcal{E}}(\mathcal{G})$ for $\alpha < \alpha^*_{\log} < 0$ and $\omega \in \mathbb{R}$.

We will see in Chapter 10, Theorem 10.8, the stability properties of the continuous tail and bump profile Ψ^{α}_0, $\alpha \neq 0$, by the flow of the NLS-log-δ model.

7.2 Stationary solutions for the Korteweg–de Vries equation

Next we consider a metric graph $\mathcal{G}$ with a structure represented by the set $E \equiv E_- \cup E_+$ where E_+ and E_- are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively. The half-lines are connected at a unique vertex $\nu = 0$.

The focus of this section is to determined stationary type solutions for the following vectorial KdV model

$$\partial_t u_e(x,t) = \alpha_e \partial_x^3 u_e(x,t) + \beta_e \partial_x u_e(x,t) + 2u_e(x,t)\partial_x u_e(x,t), \tag{7.21}$$

$e \in E = E_- \cup E_+$, and $(\alpha_e)_{e\in E}$, $(\beta_e)_{e\in E}$ are two sequences of real numbers. Thus, we are interested in the existence of solutions of type

$$(u_e(x,t))_{e\in E} = (\phi_e(x))_{e\in E}$$

where for $e \in E_-$ the profile $\phi_e : (-\infty, 0) \to \mathbb{R}$ satisfy $\phi_e(-\infty) = 0$, and for $e \in E_+$ $\phi_e : (0,\infty) \to \mathbb{R}$ satisfy $\phi_e(+\infty) = 0$. The existence of profiles of stationary type, namely, solutions of the following nonlinear elliptic equation

$$\alpha_e \frac{d^2}{dx^2}\phi_e(x) + \beta_e \phi_e(x) + \phi_e^2(x) = 0, \quad e \in E, \tag{7.22}$$

are well know and the profile depend of the soliton associated to the KdV on the full line,

$$\phi_e(x) = c(\alpha_e, \beta_e) sech^2(d(\alpha_e, \beta_e)x + p_e), \quad e \in E. \tag{7.23}$$

For instance, for $\alpha_e > 0$ and $0 > \beta_e$, for each $e \in E$, we can obtain different family of profiles satisfying the conditions $\phi_e(\pm\infty) = 0$, $e \in E_{\pm}$. The specific value of the shift p_e will depend which other (or others) condition(s) imposed on the profile ϕ_e is determined on the vertex of the graph $\nu = 0$.

The following subsections give us examples of specific stationary profiles, and also show the rich variety of these profiles that may emerge for the KdV model on metric star graphs. The main point in our analysis is to determine which stationary solutions $(\phi_e)_{e\in E}$ with ϕ_e defined in (7.23) belong to the domain $D(A_{ext})$ of some skew-self-adjoint extension A_{ext} of the Airy operator A_0 (see (4.5)).

7.2.1 Stationary solutions for a δ-type interaction on two half-lines

Our first example of solutions for (7.21) will be on a start graph $\mathcal{G}$ with a structure represented by the set $E = (-\infty, 0) \cup (0, +\infty)$. We will consider that the profile belongs to the family of skew-self-adjoint extension $(A_Z, D(A_Z))$ of A_0 defined in (4.15). Thus, from Proposition 4.2 we obtain that for $\phi_{e,Z} = (\phi_-, \phi_+) \in D(A_Z)$, $Z \neq 0$, $\alpha_+ = \alpha_- > 0$, $\beta_+ = \beta_- < 0$, the profiles $\phi_\pm$ satisfy the same equation in (7.22) and from (7.23) and for $\omega = -\beta_+ > \frac{Z^2}{4}$ we obtain

$$\phi_+(x) = \frac{3\omega}{2} sech^2\Big(\frac{\sqrt{\omega}}{2}\, x - tanh^{-1}\Big(\frac{Z}{2\sqrt{\omega}}\Big)\Big), \quad x > 0 \tag{7.24}$$

and $\phi_-(x) \equiv \phi_+(-x)$ for $x < 0$. Since $\phi_-(0-) = \phi_+(0+)$ (continuity in zero), we note the condition

$$\phi_+''(0+) - \phi_-''(0-) = \frac{Z^2}{2}\phi_-(0-) + Z\phi_-'(0-) \tag{7.25}$$

in (4.15) is satisfied immediately.

Figures 7.4 and 7.5 below show the profiles of $\phi_\pm$ for $Z \neq 0$. For $Z < 0$ we obtain the so-called *tail* profile on the all line and for $Z > 0$ the so-called *bump* profile on the all line. We note that is not difficult to show that the only stationary solutions (modulo sign) for the KdV model (7.21) and that belong to the domain $D(A_Z)$ in (4.15) are exactly the tail and bump profiles defined by formula (7.24).

We will see in Chapter 9, Theorem 9.3, the linear instability property of the continuous tail and bump profile $\phi_{e,Z}$, $Z \neq 0$.

7.2.2 Stationary solutions for a δ-type interaction on a balanced star graph

In this subsection will be consider the KdV model (7.21) on a metric star graph $\mathcal{G}$ with a structure $E \equiv E_- \cup E_+$ where $|E_+| = |E_-| = n$, $n \geqq 2$. We consider that the stationary profile $(\phi_e)_{e\in E}$ belongs to the family of skew-self-adjoint extension $(H_Z, D(H_Z))$ of A_0 defined in (4.24). For $u = (u_e)_{e\in E} \in D(H_Z)$ we have used the abbreviations

$$u(0-) = (u_e(0-))_{e\in E_-}, u'(0-) = (u_e'(0-))_{e\in E_-}, u''(0-) = (u_e''(0-))_{e\in E_-},$$

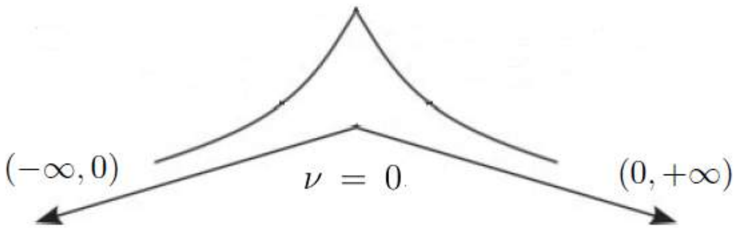

Figure 7.4: (ϕ_-, ϕ_+) for $Z < 0$

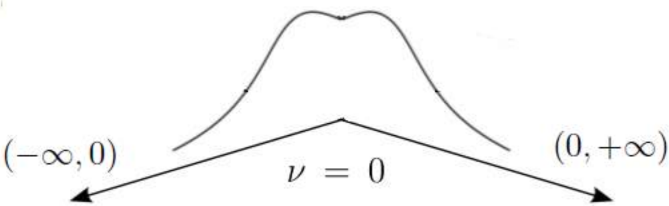

Figure 7.5: (ϕ_-, ϕ_+) for $Z > 0$

(similarly for the terms $u(0+)$, $u'(0+)$ and $u''(0\pm)$). Thus, we obtain the following system of conditions

$$\begin{aligned} u(0-) = u(0+), \quad u'(0+) - u'(0-) = Zu(0-), \\ \frac{Z^2}{2}u(0-) + Zu'(0-) = u''(0+) - u''(0-). \end{aligned} \tag{7.26}$$

Now, we consider the constants sequences $(\alpha_e)_{e\in E} = (\alpha_+)$ and $(\beta_e)_{e\in E} = (\beta_+)$, with $\alpha_+ > 0$ and $\beta_+ < 0$. Then, for $Z \neq 0$ and $-\beta_+ > \frac{Z^2}{4}$ we consider the half-soliton profile ϕ_+ defined in (7.24) and $\phi_-(x) \equiv \phi_+(-x)$ for $x < 0$. We define the constants sequences of functions

$$u_- = (\phi_-)_{e\in E_-}, \; u_+ = (\phi_+)_{e\in E_+},$$

and so $U_Z = (u_-, u_+)$ represents one family of stationary profiles for the KdV model and satisfying the boundary conditions (7.26). The case $Z > 0$, U_Z represents one family of stationary bump profiles in a balanced star graph (see Figure 7.6). The case $Z < 0$, $U_{Z,\omega}$

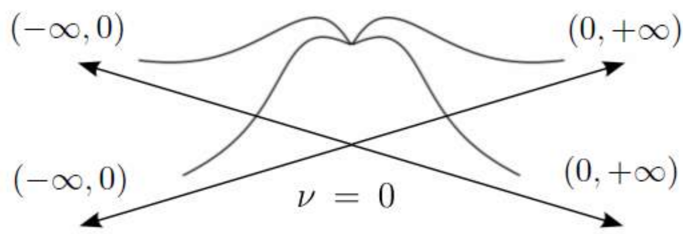

Figure 7.6: Bump profiles in a balanced star graphs with four edges

represents the corresponding family of stationary tail profiles (see Figure 7.7).

We will see in Chapter 9, Theorem 9.4, the linear instability property of the continuous tail and continuous bump profiles U_Z, $Z \neq 0$.

7.3 Stationary solutions for the sine-Gordon equation

Next we consider a metric graph $\mathcal{G}$ with a structure represented by the set $\mathrm{E} = \mathrm{E}_- \cup \mathrm{E}_+ = (-\infty, 0) \cup (0, +\infty) \cup (0, +\infty)$, namely, a $\mathcal{Y}$ junction.

The focus of this section is to determined stationary type solutions for the following vectorial sine-Gordon model

$$\partial_t^2 u_{\mathrm{e}}(x,t) - c_{\mathrm{e}}^2 \partial_x^2 u_{\mathrm{e}}(x,t) + sin(u_{\mathrm{e}}(x,t)) = 0, \quad \mathrm{e} \in \mathrm{E} \tag{7.27}$$

and $(c_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}}$, a sequence of real numbers. We rewrite the sine-Gordon model as a first-order system for $\mathrm{e} \in \mathrm{E}$,

$$\begin{cases} \partial_t u_{\mathrm{e}} = v_{\mathrm{e}} \\ \partial_t v_{\mathrm{e}} = c_{\mathrm{e}}^2 \partial_x^2 u_{\mathrm{e}} + sin(u_{\mathrm{e}}). \end{cases} \tag{7.28}$$

Our stationary type solutions will be

$$(u_{\mathrm{e}}(x,t))_{\mathrm{e}\in\mathrm{E}} = (\phi_{\mathrm{e}}(x))_{\mathrm{e}\in\mathrm{E}}, \text{ and } (v_{\mathrm{e}}(x,t))_{\mathrm{e}\in\mathrm{E}} = (0)_{\mathrm{e}\in\mathrm{E}}. \tag{7.29}$$

Thus, every component satisfies the equation

$$-c_{\mathrm{e}}^2 \phi_{\mathrm{e}}'' + sin(\phi_{\mathrm{e}}) = 0, \tag{7.30}$$

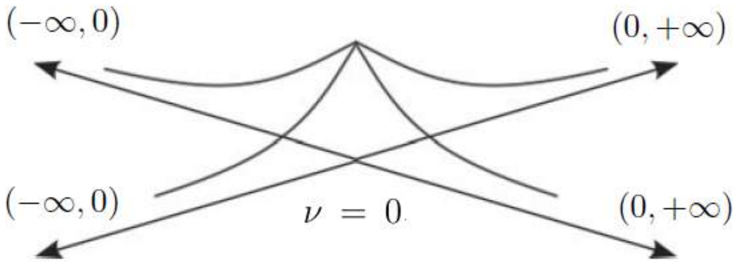

Figure 7.7: Tail profiles in a balanced star graphs with four edges.

it which have a profile depending of the kink-soliton profile for the sine-Gordon on the full line

$$\phi_e(x) = 4arctan\Big(e^{\frac{1}{\sqrt{c_e^2}}(x+y_e)}\Big). \tag{7.31}$$

The specific value of the shift y_e will depend of the conditions $\phi_e(\pm\infty) = 0$, $e \in E_\pm$, and other (or others) condition(s) determined on the vertex of the graph $\nu = 0$.

7.3.1 Stationary solutions for a δ-type interaction on the $\mathcal{Y}$ junction

In this subsection, we consider the stationary profile $(\phi_e(x))_{e\in E}$ belongs to the family of self-adjoint operators $(\mathcal{J}_Z, D(\mathcal{J}_Z))$ in (4.29).

Then, for $(\phi_e)_{e\in E} = (\phi_j)_{j=1}^3 \in D(\mathcal{J}_Z)$, we obtain from (7.31) that

$$\begin{aligned} \phi_1(x) &= 4\text{arctan}\Big(e^{\frac{1}{|c_1|}(x-a_1)}\Big), \qquad x < 0, \\ \phi_i(x) &= 4\text{arctan}\Big(e^{-\frac{1}{|c_i|}(x-a_i)}\Big), \qquad x > 0, \quad i = 2, 3, \end{aligned} \tag{7.32}$$

with $\frac{1}{|c_1|}a_1 = -\frac{1}{|c_2|}a_2 = -\frac{1}{|c_3|}a_3$ by continuity at the vertex $\nu = 0$. The other condition in (4.29) implies the following relation for a_1,

$$-\frac{e^{-\frac{a_1}{|c_1|}}}{1 + e^{-\frac{2a_1}{|c_1|}}} \sum_{j=1}^{3} |c_j| = Z\text{arctan}\Big(e^{-\frac{a_1}{|c_1|}}\Big). \tag{7.33}$$

From (7.33) we deduce that $Z < 0$. Next, from the behavior of the function

$$f(y) = \frac{1 + y^2}{y}\text{arctan}(y), \quad y \geqq 0$$

we obtain that $Z \in (-\sum_{j=1}^{3}|c_j|, 0)$. Moreover, we need to have $a_1 > 0$ providing $Z \in (-\sum_{j=1}^{3}|c_j|, -\frac{2}{\pi}\sum_{j=1}^{3}|c_j|)$ and $a_1 < 0$ providing $Z \in (-\frac{2}{\pi}\sum_{j=1}^{3}|c_j|, 0)$. The case $Z = -\frac{2}{\pi}\sum_{j=1}^{3}|c_j|$ implies $a_1 = 0$. For $a_1 > 0$ we have the typical tail profile (see Figure 7.8). For $a_1 < 0$ we have a "smooth" profile around the vertex $\nu = 0$.

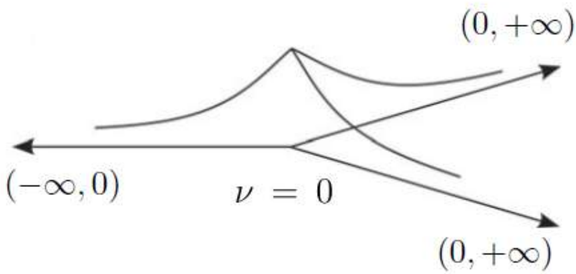

Figure 7.8: Tail profiles in a $\mathcal{Y}$ junction

Remark 7.3. *The study of stability properties for stationary solutions of the sine-Gordon equation 7.27 has been recently done in (Angulo and Plaza 2019).*

8 Stability of KdV Solitons on the Half-Line

Many physical problems arises naturally as initial boundary value problems (IBVP), because of the local character of the corresponding phenomenon (Zabusky and J. 1971). However, the IBVP for the KdV equation has been considerably less studied than the corresponding IVP posed in all $\mathbb{R}$. For example, there are at least two interesting IBVP for KdV still in unbounded domains: the one posed on the right half-line, and a second one posed on the left portion of the line, which we consider in this chapter.

8.1 Unbounded initial boundary value problems

The IBVP for the KdV equation posed on the **right** half-line is the following: for $\mathbb{R}^+ := (0, +\infty)$ and $T > 0$, find a solution u to

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^2) = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+, \\ u(0,t) = f(t), & t \in (0,T), \end{cases} \tag{8.1}$$

while the IBVP for the KdV equation posed on the **left** half-line is the following: for $\mathbb{R}^- := (-\infty, 0)$ and $T > 0$, find a solution u to

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^2) = 0, & (x,t) \in \mathbb{R}^- \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^-, \\ u(0,t) = f(t), & t \in (0,T), \\ \partial_x u(0,t) = f_1(t), & t \in (0,T). \end{cases} \tag{8.2}$$

Both problems differ in the sense that the one on the left half-line needs an additional boundary condition (see Deconinck, Sheils, and Smith (2016) and Holmer (2006)), making this problem more challenging from almost every point of view. As an example, our results differ from (8.1) to (8.2). More in general, for IBVPs, an important issue, both from the mathematical and physical point of view, is the study of the effect of the boundary condition(s) at $x = 0$ on the asymptotic behavior of the solution.

In the recent literature, the mathematical study of IBVPs (8.1) and (8.2) is usually considered in the following setting

$$(u_0, f) \in H^s(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+), \tag{8.3}$$

or

$$(u_0, f, f_1) \in H^s(\mathbb{R}^-) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s}{3}}(\mathbb{R}^+), \tag{8.4}$$

respectively. These assumptions are in some sense sharp because of the following localized smoothing effect for the linear evolution (Kenig, Ponce, and Vega 1993)

$$\|\psi(t)e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x;\, H^{(k+1)/3}(\mathbb{R}_t)\right)} \leqslant c\|\phi\|_{H^k(\mathbb{R})},$$

and

$$\|\psi(t)\partial_x e^{-t\partial_x^3}\phi(x)\|_{C\left(\mathbb{R}_x;\, H^{k/3}(\mathbb{R}_t)\right)} \leqslant c\|\phi\|_{H^k(\mathbb{R})},$$

where $\psi(t)$ is a smooth cutoff function and $e^{-t\partial_x^3}$ denotes the linear homogeneous solution group on $\mathbb{R}$ associated to the linear KdV equation. Therefore, in what follows we will certainly follows both settings (8.3)-(8.4).

8.1.1 Known results for the IBVPS (8.1) and (8.2)

The mathematical study of the IBVP (8.1) began with the work of Ton (1977). He showed existence and uniqueness by assuming that the initial datum u_0 is smooth and the boundary data is $f = 0$. Later, J. Bona and Winther (1983) considered (8.1) and proved global existence and uniqueness solutions in $L^\infty_{loc}(\mathbb{R}^+; H^4(\mathbb{R}^+))$, for data $u_0 \in H^4(\mathbb{R}^+)$ and $f \in H^2_{loc}(\mathbb{R}^+)$. In (J. L. Bona and Winther 1989), they continued the study of (8.1) and proved continuous dependence. Next, Faminskiĭ (1988) considered a generalization of IBVP (8.1) and obtained well-posedness in weighted $H^1(\mathbb{R}^+)$ Sobolev spaces. After this work, J. L. Bona, Sun, and Zhang (2002) obtained *conditional* local well-posedness, in the sense that solutions are only known to be unique if they satisfy some additional auxiliary conditions. This was done for data $u_0 \in H^s(\mathbb{R}^+)$ and $f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ with $s > \frac{3}{4}$. They also proved global well-posedness for $u_0 \in H^s(\mathbb{R}^+)$ and $f \in H^{\frac{3s+7}{12}}(\mathbb{R}^+)$, with $1 \leqslant s \leqslant 3$.

The fundamental contribution of Colliander and Kenig (2002) introduced a more dispersive PDE approach for the generalized Korteweg–de Vries (gKdV) equation posed on $\mathbb{R}^+$, based on writing the original IBVP (8.1) as a superposition of three initial value problems on $\mathbb{R} \times \mathbb{R}$. In particular, for KdV (8.1) this result gives conditional local well-posedness in $L^2(\mathbb{R}^+) \times H^{\frac{1}{3}}(\mathbb{R}^+)$, in which solutions are only known to be unique if they satisfy additional auxiliary conditions. By the same time, Colliander and Kenig (2002) derived a global a priori estimate and for a non-optimal boundary condition $f \in H^{\frac{7}{12}}(\mathbb{R}^+)$, and a conditional global well-posedness was obtained for the case $s = 0$. Recently, Cavalcante (2017) (using some of the Colliander-Kenig techniques) showed conditional local well-posedness for the IVP associated to the KdV equation on a *simple star graph* given by two positive half-lines and a negative half-line attached in a common vertex.

Later, Faminskii (2004) improved the global results of (Colliander and Kenig 2002) and obtained global results by assuming more natural boundary conditions (See Theorem 8.2 below for more details). The local well-posedness of the IBVP (8.1) above $s = -\frac{3}{4}$, which is the critical Sobolev exponent for the KdV initial value problem, was obtained by Holmer (2006) and J. L. Bona, Sun, and Zhang (2006). Surveys describing these results and others are (J. L. Bona, Sun, and Zhang 2002; Fokas, Himonas, and Mantzavinos 2016).

As for the left half-line case, Holmer (2006) obtained local well-posedness in $H^s(\mathbb{R}^+)$ for $s > -\frac{3}{4}$. Then, Faminskii (2007) obtained global well-posedness in $H^s(\mathbb{R}^+)$ for $s \geqslant 0$ for boundary conditions assuming natural conditions.

Another point of view for (8.1) and (8.2) is given by using Inverse Scattering techniques. Fokas (1997) introduced a new approach known as the unified transform method (UTM), which provides a proper generalization of the Inverse Scattering Transform (IST) method for solving IBVPs. For example, it is mentioned in (Fokas and Its 1994) that, under suitable decay and smoothness assumptions, just as in the infinite-line setting, the solution on the right half-line should describe (for large times) a collection of (standard KdV) solitons traveling at constant speeds. These techniques were further improved in (Fokas, Himonas, and Mantzavinos 2016), where well-posedness is proven in Sobolev spaces using the UTM method.

8.1.2 Main result

Before of to state the main result of the chapter, we explain the notion of soliton that we will use. First of all, as far as we know, no exact canonical soliton solution is available for problems (8.1) and (8.2), except for some very particular cases. For example, for a classical soliton in the full line

$$\widetilde{Q}_c = \widetilde{Q}_c(x - ct - x_0),$$

where $\widetilde{Q}_c$ is the classical soliton in $\mathbb{R}$ given by the formula

$$\widetilde{Q}_c(s) = \frac{3c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}s}{2}\right), \tag{8.5}$$

with $c > 0$ as the propagation speed of the wave and x_0 an arbitrary constant.

Let us define the natural half-line soliton as

$$Q_c = \widetilde{Q}_c\big|_{\mathbb{R}+}, \text{ right half-line case,} \tag{8.6}$$

Note that this definition induces a "natural" trace $f(t) = \widetilde{Q}_c(-ct - x_0)$ of Q_c at $x = 0$ in (8.1) (see e.g. Fokas and Lenells (2010) for more details on this point of view). However, this trace assumption strongly depends on the original soliton itself, and because of some energetic conditions, we will need a more suitable boundary condition which will have important consequences on the stability property.

Instead, we will adopt the following approach: given any standard KdV soliton (8.5), restricted to the half-line as in (8.6), and placed far enough from the corner $x = 0$, we will show that this solution is stable in the energy space under perturbations that preserve the zero boundary condition.

Theorem 8.1 (Stability for the right half-line)**.** *Let $c > 0$ be a given constant. There exist constants $\alpha_0, C_0, L_0 > 0$ such that, for all $0 < \alpha < \alpha_0$, and all $L > L_0$, the following is satisfied. Assume that $u_0 \in H^1(\mathbb{R}^+)$ is such that*

$$\begin{aligned} u_0(x = 0) &= 0, \\ \|u_0 - Q_c(\cdot - L)\|_{H^1(\mathbb{R}+)} &< \alpha. \end{aligned} \tag{8.7}$$

Then the solution $u = u(x,t)$ for the IBVP (8.1) with boundary data $u(0,t) = f(t) \equiv 0$, given by Theorem 8.2, satisfies the global estimate

$$\sup_{t\geqslant 0} \|u(t) - Q_c(\cdot - \rho(t) - L)\|_{H^1(\mathbb{R}+)} < C_0(\alpha + e^{-\sqrt{c}L}), \tag{8.8}$$

for a C^1-function $\rho(t) \in \mathbb{R}$ satisfying

$$\sup_{t\geqslant 0} |\rho'(t) - c| < CC_0\alpha, \tag{8.9}$$

for some constant $C > 0$.

This result shows strong stability of KdV soliton for the IBVP (8.1), which was, as far as we know, an open problem, even in the zero boundary condition case. Note that in Theorem 8.1 we do not prove the existence of a soliton solution for (8.1), instead we show that the KdV soliton is the natural candidate to be the standard object appearing in the long-time dynamics, even if it is not an exact solution of the problem.

The proof of this result is in some sense more dynamical than variational, because the half-line border introduces some important restrictions on the dynamics itself, which need to be controlled separately by using a particular extension property, as well as local estimates of the mass and energy of the KdV soliton (see Lemma 8.1), which is only an approximate solution of the problem. We recall that in a general setting, (8.1) has no conserved quantities, but we are still able to find some almost conserved quantities. For that reason, estimate (8.8) in Theorem 8.1 accounts how far the soliton is placed at the initial time.

At the more technical level, we follow the approach introduced by Martel, Merle, and Tsai (2002) for the study of the stability of generalized KdV multi-solitons in the energy space. This approach is based in the introduction of suitable almost conserved quantities and monotonicity properties, which are of proper interest. For the (8.1) case, we follow a simplified version described in (Muñoz 2011), which deal with the case of soliton-like objects of dispersive problems with no exact soliton solution. We will take the advantage of a hidden dissipative mechanism of the model introduced through imposition of the homogeneous boundary condition at $x = 0$, see Lemma 8.1 for more details.

Let us mention that the stability of KdV and more general objects is a large research area lasting for the past thirty years and more. J. L. Bona, Souganidis, and W. A. Strauss (1987), Weinstein (1986),Grillakis, Shatah, and W. Strauss (1987), Martel, Merle, and Tsai (2002) and many others are important references in the field. In the case of KdV multi-solitons, it has also been proved stability even for L^2 perturbations in (Alejo, Muñoz, and Vega 2013). For a simple introduction to subject, we also refer to the monograph of Angulo (2009), see also (Muñoz 2014) for a short review.

Remark 8.1 (On the zero boundary condition)**.** *Note that the condition $u(x = 0, t) = 0$ is assumed because of several important reasons. First of all, from energetic considerations most conserved quantities require the same flux at both sides of the boundary (i.e. zero net flux), and therefore the condition $u(t) \in H^1(\mathbb{R}^+)$ naturally imposes the zero boundary condition at $x = 0$. Another reason to impose this condition is the fact that the exact 2-soliton solution of KdV $U(x, t)$ composed of exactly two large solitons but well-separated at the initial time (one on the right half-line, the other one on the left half-line), is an example of nonzero boundary data for which the corresponding evolution on the right half-line is far from being one soliton and a small perturbation. This shows that $f(t)$ could not be "arbitrary" (not even small for arbitrarily large times, and not even integrable in time probably). However, we believe that the stability property does hold for any $f(t)$ sufficiently small, as far as one can control a second derivative in space, integrated in time. However, this control should require more regularity on the solution, and therefore higher order conserved quantities.*

Remark 8.2 (On the zero boundary condition, 2)**.** *Note that one may think that $u(x=0,t)=0$ for all $t\geqslant 0$ could lead to an odd extension of* (8.1) *to the real line, where we know that KdV has stable solitons. However, this approach fails because KdV does not preserve the oddness property.*[1] *In Section 8.3 (Definition 8.2) we will introduce a new extension of $u(x,t)$ to the real line, which has, as far as we understand, no dynamical meaning, but only a variational purpose. In that sense, this extension seems to be the "least energy extension" for the problem, that is to say, it acts only at the linear spectral level.*

8.1.3 Existence and continuity for the right half-line

For our study of stability, we need of the following result of existence and continuity for the IBVPs (8.1).

Definition 8.1. *For any $T>0$ and $s\geqslant 0$, let $Z^s_T(\mathbb{R}^+)$ be the space given by the functions $u(x,t)$ satisfying*

$$\begin{aligned} &\partial_t^m u \in C([0,T], H^{s-3m}(\mathbb{R}^+)) \ \text{ for any integer } \ 0\leqslant m\leqslant \frac{s}{3}, \\ &\partial_x^l u \in C(\mathbb{R}^+; H^{\frac{s-l+1}{3}}(0,T)) \ \text{ for any integer } \ 0\leqslant l\leqslant s+1. \end{aligned} \tag{8.10}$$

A definition for $Z^s_T(\mathbb{R}^-)$ can be given analogous to that for $Z^s_T(\mathbb{R}^+)$.

Theorem 8.2 (Faminskii (2004))**.** *Consider the IBVP on the right half-line* (8.1)*. Fix a time $T>0$ as in* (8.1)*. Let $u_0\in H^s(\mathbb{R}^+)$ and $f\in H^{\frac{s+1}{3}+\epsilon}(\mathbb{R}^+)$, such that the following conditions are satisfied:*

1. *The regularity $s\geqslant 0$ is such that $\frac{s}{3}-\frac{1}{6}$ is not an integer,*
2. *The parameter $\epsilon>0$ is arbitrary small in the case $s<1$, and ϵ can be taken equals zero in the case $s\geqslant 1$.*
3. *The boundary datum f satisfies the compatibility conditions*

$$f^{(m)}(t=0)=\phi_m(x=0), \quad \text{for any integer} \quad 0\leqslant m<\tfrac{s}{3}-\tfrac{1}{6},$$

where $\phi_0(x):=u_0(x)$ and for $0<m<\frac{s}{3}-\frac{1}{6}$,

$$\phi_m(x):=-\phi_{m-1}'''(x)-\sum_{l=0}^{m-1}\binom{m-1}{l}\phi_l(x)\phi_{m-l-1}'(x). \tag{8.11}$$

[1] Note that the well-known nonlinear Schrödinger equation, which has solitary waves, preserves this oddness condition, therefore the odd extension of a zero boundary data is trivial. Additionally, standard conserved quantities such as mass and energy are conserved under the assumption $u(0,t)=0$, a nice property unfortunately not shared by the KdV dynamics.

Then there exists a solution $u(x,t)$ of the IBVP (8.1) in the space $Z^s_T(\mathbb{R}^+)$. Moreover, the mapping $(u_0, f) \mapsto u$ is Lipschitz continuous on any ball in the norm of the mapping $H^s(\mathbb{R}^+) \times H^{\frac{s+1}{3}+\epsilon}(0,T) \to Z^s_T(\mathbb{R}^+)$.

Remark 8.3. *The previous result is not only important in view of the GWP result, but also because of the compatibility conditions (8.11), that lead to (8.10).*

8.2 Mass and energy estimates

In this section, we obtain several dispersive properties for the solutions of IBVPs (8.1) and (8.2). These properties will involve suitable definitions of mass and energy. Unlike standard KdV on the line, in general mass and energy will not be conserved anymore, but under some additional assumptions, we will be able to prove that, even if they are not precisely conserved, at least they obey suitable estimates.

First, we deal with the case of equation (8.1).

Lemma 8.1. *Consider the following mass and energy functionals*

$$M[u](t) := \frac{1}{2}\int_0^{+\infty} u^2(x,t)dx, \tag{8.12}$$

$$E[u](t) := \int_0^{+\infty} \Big(\frac{1}{2}(\partial_x u)^2(x,t) - \frac{1}{3}u^3(x,t)\Big)dx, \tag{8.13}$$

well-defined according to the initial conditions given. Then the solution $u = u(x,t)$ of the IBVP (8.1) with

$$u(0,t) = 0 \quad \text{for all time } t \geqslant 0, \tag{8.14}$$

and initial datum $u_0 \in H^1(\mathbb{R}^+)$ satisfies, for all $t \geqslant 0$,

$$M[u](t) \leqslant M[u_0], \tag{8.15}$$

and

$$E[u](t) \leqslant E[u_0]. \tag{8.16}$$

Proof. First, we assume that u is sufficiently smooth and decays fast enough. A simple calculation shows that a smooth solution $u(x,t)$ of IBVP (8.1) satisfies the identity

$$\frac{d}{dt}\int_0^{+\infty} u^2(t)dx = -\frac{1}{2}(\partial_x u)^2(0,t) + u(0,t)\Big(\partial_x^2 u + \frac{2}{3}u^2\Big)(0,t). \tag{8.17}$$

Indeed, multiplying the equation in (8.1) by u and integrating on $(0,\infty)$ in x we get

$$\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} u^2(x,t)dx = -\int_0^{+\infty} \partial_x(\partial_x^2 u + u^2)u dx. \tag{8.18}$$

Integrating by parts,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} u^2(x,t)dx &= \int_0^{+\infty} (\partial_x^2 u + u^2)\partial_x u dx - (\partial_x^2 u + u^2)u\Big|_0^\infty \\ &= \int_0^{+\infty} \frac{d}{dx}\left(\frac{1}{2}(\partial_x u)^2 + \frac{u^3}{3}\right)dx \\ &\quad + (\partial_x^2 u(0,t) + u^2(0,t))u(0,t) \\ &= -\frac{1}{2}\partial_x u(0,t)^2 - \frac{1}{3}u(0,t)^3 \\ &\quad + (\partial_x^2 u(0,t) + u^2(0,t))u(0,t) \\ &= -\frac{1}{2}(\partial_x u)^2(0,t) \\ &\quad + u(0,t)\Big(\partial_x^2 u(0,t) + \frac{2}{3}u^2(0,t)\Big).\end{aligned} \tag{8.19}$$

We recall this last estimate again because it will be important for later purposes:

$$\frac{1}{2}\frac{d}{dt}\int_0^{+\infty} u^2(x,t)dx = -\frac{1}{2}(\partial_x u)^2(0,t) + u(0,t)\Big(\partial_x^2 u(0,t) + \frac{2}{3}u^2(0,t)\Big). \tag{8.20}$$

Note that, unless $u(0,t) = 0$, we will have a source term in the mass coming from a second derivative term at $x = 0$. This term is certainly very harmful and difficult to control by using only data in H^1. This fact certainly supports our choice of zero boundary condition on the corner $x = 0$.

Now, after integration in time in (8.17), we obtain

$$\begin{aligned}\int_0^{+\infty} u^2(t)dx &= \int_0^{+\infty} u_0^2 dx - \frac{1}{2}\int_0^t (\partial_x u)^2(0,s)ds \\ &\quad + \int_0^t u(0,s)\Big(\partial_x^2 u + \frac{2}{3}u^2\Big)(0,s)ds.\end{aligned} \tag{8.21}$$

Now we deal with the energy estimate. Indeed, we multiply the equation in (8.1) by $(\partial_x^2 u + u^2)$ and integrate on $(0,\infty)$ in x:

$$\int_0^{+\infty} \partial_t u(\partial_x^2 u + u^2)dx + \frac{1}{2}\int_0^\infty \frac{d}{dx}(\partial_x^2 u + u^2)^2 = 0.$$

Integrating by parts,

$$\int_0^{+\infty} (-\partial_{tx}^2 u\partial_x u + u^2\partial_t u)dx - \partial_t u(0,t)\partial_x u(0,t) - \frac{1}{2}(\partial_x^2 u + u^2)^2(0,t) = 0.$$

Therefore, we obtain a new identity for the energy

$$\frac{d}{dt}\int_0^\infty \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{3}u^3\right)dx = -\partial_t u(0,t)\partial_x u(0,t) - \frac{1}{2}(\partial_x^2 u(0,t) + u^2(0,t))^2. \tag{8.22}$$

This identity essentially says that, unless $\partial_t u(0,t) = 0$, then the energy $E[u]$ has no definite dynamics. Once again, controlling the term $\partial_t u(0,t)$ in (8.22) is hard because it is related through the original equation with third order derivatives in space.

Consequently, replacing the equation (8.1) and integrating in time,

$$\begin{aligned}\int_0^{+\infty} &\Big(\frac{1}{2}(\partial_x u)^2 - \frac{1}{3}u^3\Big)(t)dx \\ &= \int_0^{+\infty}\Big(\frac{1}{2}(\partial_x u_0)^2 - \frac{1}{3}u_0^3\Big)dx \\ &\quad - \frac{1}{2}\int_0^t (\partial_x^2 u(0,s) + u^2(0,s))^2 ds \\ &\quad + \int_0^t \partial_x u(0,s)\partial_x(\partial_x^2 u(x,s) + u^2(x,s))\Big|_{x=0} ds.\end{aligned} \tag{8.23}$$

Now we justify the last mass and energy computations for the case of H^1 data. Assume that $u(x,t)$ is a solution for the IBVP (8.1) with initial data $u_0 \in H^s(\mathbb{R}^+)$, with $u_0(0) = \partial_x^3 u_0(0) = 0$, and boundary data $f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, for a given s satisfying $\frac{7}{2} < s < \frac{11}{2}$ given by Theorem 8.2 (the condition for the third derivative of u_0 comes from the case $m = 1$ in Theorem 8.2). From (8.10) we have that

$$u(0,t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+),\ \partial_x u(0,t) \in H^{\frac{s}{3}}(\mathbb{R}^+),$$
$$\partial_x^2 u(0,t) \in H^{\frac{s-1}{3}}(\mathbb{R}^+) \text{ and } \partial_x^3 u(0,t) \in H^{\frac{s-2}{3}}(\mathbb{R}^+).$$

It follows that for fixed t we have that $\partial_t u(x,t) \in H^{s-3}(\mathbb{R}^+)$ has a well-defined trace at $x = 0$. Hence for homogeneous boundary condition $u(0,t) = f(t) \equiv 0$ we have that

$$0 = \partial_t u(x,t)\big|_{x=0} = -\partial_x(\partial_x^2 u(x,t) + u^2(x,t))\big|_{x=0}. \tag{8.24}$$

Consequently, the identities for the mass (8.21) and the energy (10.5) take the form ($t \geqslant 0$)

$$\int_0^{+\infty} u^2(t)dx = \int_0^{+\infty} u_0^2 dx - \int_0^t \frac{1}{2}(\partial_x u)^2(0,s)ds, \tag{8.25}$$

$$\int_0^{+\infty}\Big(\frac{1}{2}(\partial_x u)^2 - \frac{1}{3}u^3\Big)dx = \int_0^{+\infty}\Big(\frac{1}{2}(\partial_x u_0)^2 - \frac{1}{3}u_0^3\Big)dx - \frac{1}{2}\int_0^t (\partial_{xx} u(0,s) + u^2(0,s))^2. \tag{8.26}$$

From (8.25) and (8.26) we have the following dissipative mechanism for the mass and the energy

$$\int_0^{+\infty} u^2(t)dx \leqslant \int_0^{+\infty} u_0^2 dx,$$

and

$$\int_0^{+\infty} \Big(\frac{1}{2}(\partial_x u)^2 - \frac{1}{3}u^3\Big)dx \leqslant \int_0^{+\infty} \Big(\frac{1}{2}(\partial_x u_0)^2 - \frac{1}{3}u_0^3\Big)dx.$$

Now assume $u_0 \in H^1(\mathbb{R}^+)$. Let $\{u_{0n}\}$ be a bounded sequence in $H^{\frac{7}{2}^+}(\mathbb{R}^+)$ such that $u_{0n}(0) = \partial_x^3 u_{0n}(0) = 0$ and

$$\|u_{0n} - u_0\|_{H^1(\mathbb{R}^+)} \to 0, \text{ when } n \to +\infty. \tag{8.27}$$

It follows from the previous analysis that the identities (8.15) and (8.16) are valid for all u_n. Now letting $n \to +\infty$ and using the continuity of flow data to solution given in Theorem 8.2 the result follows. □

Remark 8.4. *Note that* (8.25) *and* (8.26) *can be recast as hidden trace smoothing effects for bounded in time solutions in the energy space. Indeed, under the boundary value condition* $u(0,t) = 0$ *for all* t*, we have*

$$\int_0^t (\partial_x u)^2(0,s)ds \lesssim \sup_t \|u(t)\|^2_{L^2(0,\infty)},$$

and

$$\int_0^t (\partial_x^2 u)^2(0,s)dx \lesssim \sup_t \|u(t)\|^2_{H^1(0,\infty)}.$$

Now we deal with the left half-line case.

8.3 Start of proof of Theorem 8.1: Extension to the entire line

The proof is based on the classical argument of Weinstein (1986), with some minor changes coming from the fact that we do not work on the whole line, but only on $\mathbb{R}^+$, and the KdV soliton is not an exact solution of the problem by itself. See also (Muñoz 2011, 2014) for a fully explained, similar argument.

The idea behind Weinstein's result is to show a coercivity estimate, which is obtained using spectral properties of a well-chosen linear unbounded operator. In the next sections, we will find a suitable operator for the half-line case, to then extend it to the entire space to make use of the standard Weinstein's theory of stability.

Take $c > 0$ fixed and $L > L_0 > 0$, where L_0 will be taken as large as needed. Assume that (8.7) is satisfied for an initial datum u_0, and for a certain $\alpha < \alpha_0$ to be chosen later.

Let $u(t)$ be the corresponding solution of the IBVP (8.1), with boundary data $u(0,t) \equiv 0$, and given by Theorem 8.2. For $C_0 > 1$, consider the tubular neighborhood

$$\mathcal{M}[C_0] := \Big\{ v \in H^1(\mathbb{R}^+) \ : \ \inf_{\rho_0 \in \mathbb{R}} \|v - Q_c(\cdot - \rho_0 - L)\|_{H^1(\mathbb{R}^+)} < C_0(\alpha + e^{-\frac{\sqrt{c}}{2}L}) \Big\}. \tag{8.28}$$

Note that from (8.7) we have $u_0 \in \mathcal{M}[C_0]$. We want to prove that for L and C_0 large enough, and $\alpha < \alpha_0$ small, $u(t) \in \mathcal{M}[C_0]$ for al $t \geqslant 0$.

Similarly, by the continuity of the KdV flow, we have $u(t) \in \mathcal{M}[C_0]$ for sufficiently small time t. Using a bootstrap argument, we will show the implication

$$t \geqslant 0, \quad u(t) \in \mathcal{M}[C_0] \implies u(t) \in \mathcal{M}[C_0/2], \tag{8.29}$$

which will prove (8.8).

8.3.1 Modulation

By taking α, L smaller, we can ensure the following decomposition argument:

Lemma 8.2 (Modulation)**.** *Assume that $u(t) \in \mathcal{M}[C_0]$ for all $t \geqslant 0$. Then, by taking α_0 smaller and L_0 larger if necessary, there exists $\rho = \rho(t) \in \mathbb{R}$ such that we have the following decomposition:*

$$u(x,t) = Q_c(x - \rho(t) - L) + z(x,t), \tag{8.30}$$

where $z(x,t)$ satisfies, for all $t \geqslant 0$,

$$\int_0^{+\infty} z(x,t) Q_c'(x - \rho(t) - L)dx = 0, \tag{8.31}$$

and $\rho(t)$ satisfies the estimate

$$|\rho'(t) - c| \lesssim \|z(t)\|_{H^1(\mathbb{R}+)} + e^{-\sqrt{c}L}. \tag{8.32}$$

Finally,

$$\|z(0)\|_{H^1(\mathbb{R}+)} \lesssim \alpha + e^{-\sqrt{c}L}, \tag{8.33}$$

with an implicit constant independent of C_0.

Proof. The proof of this result is standard, but since we are working in the case of the half-line, we need some small changes in the proof. We have from (8.29) that for all $t \geqslant 0$,

$$\inf_{\rho_0 \in \mathbb{R}} \|u(t) - Q_c(\cdot - \rho_0 - L)\|_{H^1(\mathbb{R}^+)} < C_0(\alpha + e^{-L}). \tag{8.34}$$

if C_0 is large. Therefore, if we define the functional $\mathcal{F} = \mathcal{F}[v, \rho_0]$ by

$$H^1(\mathbb{R}^+) \times (-L, \infty) \ni (v, \rho_0) \mapsto \int_0^{+\infty} (v(x) - Q_c(x - \rho_0 - L)) Q_c'(x - \rho_0 - L) dx \in \mathbb{R}.$$

it is not difficult to see that it is of class C^1 and $\mathcal{F}[Q_c(\cdot - \rho_0 - L), \rho_0] = 0$ for all $\rho_0 \in (-L, \infty)$. Consequently, since

$$\begin{aligned} \frac{\partial}{\partial \rho_0} \mathcal{F}[v, \rho_0]\Big|_{v = Q_c(\cdot - \rho_0 - L)} &= \int_0^{+\infty} Q_c'^2(x - \rho_0 - L) dx \\ &= \int_{-\rho_0 - L}^{+\infty} Q_c'^2 \geqslant \int_0^{+\infty} Q_c'^2 > 0, \end{aligned}$$

by the Implicit Function Theorem we have that for all $v \in H^1(\mathbb{R}^+)$ such that $\|v - Q_c(\cdot - \rho_0 - L)\|_{H^1(\mathbb{R}^+)} < \delta_0$, there exists $\rho_0 = \rho_0(v) > -L$ for which

$$\int_0^{+\infty} (v(x) - Q_c(x - \rho_0 - L)) Q_c'(x - \rho_0 - L) dx = 0.$$

Using (8.34) for small α and large L if necessary, we have $C_0(\alpha + e^{-L}) < \delta_0$, from which there exists $\rho(t)$ such that for all $t \geqslant 0$,

$$\int_0^{+\infty} z(x,t) Q_c'(x - \rho(t) - L) dx = 0, \quad z(x,t) := u(x,t) - Q_c(x - \rho(t) - L).$$

The rest of the proof is standard, see e.g. (Martel, Merle, and Tsai 2002).

□

Remark 8.5. *From (8.32) we have the lower bound*

$$\rho(t) \geqslant \rho(0) + ct - tC_0(\alpha + e^{-\frac{\sqrt{c}}{2}L}),$$

which for small α and large L ensures that $\rho(t)$ is always an increasing function of time, $t \geqslant 0$. This fact will be used several times through the computations below.

8.3.2 Extension to the whole line

The following step in the proof is a suitable extension of the spectral problem to the whole line. We will see that not every extension is useful, but a mild one will satisfy all the requirements.

Definition 8.2 (Zero extension, right half-line case). *Let $v \in H^1(\mathbb{R}^+)$ such that $v(x = 0) = 0$. We define its (zero) extension $\hat{v}$ as the function*

$$\hat{v}(x) := \begin{cases} v(x) & x \geqslant 0 \\ 0 & x < 0. \end{cases} \tag{8.35}$$

Remark 8.6. *Note that this extension makes sense in $H^1(\mathbb{R}^+)$, and gives a new function $\hat{v} \in H^1(\mathbb{R})$ since $v(x = 0) = 0$ (cf. Lemma B.2). Also, note that $\widetilde{Q}_c$ in (8.5) cannot be considered as the zero extension of Q_c in (8.6). This interesting difference will be important for the stability proof.*

We will apply the extension property to the function $u(t)$ in (8.30). More precisely, for each $t \geqslant 0$, let $\hat{u} = \hat{u}(x,t)$ be the zero extension function of $u(t)$ defined using (8.35). Also, recall $\widetilde{Q}_c(x-\rho(t)-L)$, the natural extension of $Q_c(x-\rho(t)-L)$ obtained reversing (8.6).

For further purposes, let us define

$$\widetilde{z}(x,t) := \hat{u}(x,t) - \widetilde{Q}_c(x-\rho(t)-L). \tag{8.36}$$

Note that both $\widetilde{z}$ and $\widetilde{Q}_c(x-\rho(t)-L)$ obey somehow "natural" extensions, however $\hat{u}$ follows a completely different extension (by zero). More precisely, note that

$$\widetilde{z}(x,t) = -\widetilde{Q}_c(x-\rho(t)-L), \quad x \leqslant 0. \tag{8.37}$$

We have the following useful set of estimates:

Lemma 8.3. *For $\widetilde{z}(t)$ defined in (8.36)-(8.37) and $t \geqslant 0$, we have*

$$\widetilde{z}(t) \in H^1(\mathbb{R}), \tag{8.38}$$

as well as

$$\|\widetilde{z}(t)\|_{H^1(\mathbb{R}^-)} + \|\widetilde{z}(t)\|_{L^\infty(\mathbb{R}^-)} \lesssim e^{-\sqrt{c}|\rho(t)+L|}. \tag{8.39}$$

Finally, we have the global estimate

$$\|\widetilde{z}(t)\|_{H^1(\mathbb{R})} \lesssim \|z(t)\|_{H^1(\mathbb{R}^+)} + e^{-\sqrt{c}|\rho(t)+L|}, \tag{8.40}$$

with implicit constants independent of $t \geqslant 0$ and C_0.

Proof. Direct from (8.37) and (8.5). □

8.4 Almost conserved quantities

Consider the decomposition of the dynamics (8.30). Under the condition $u(t) \in \mathcal{M}[C_0]$ in (8.29), we know that $z(t)$ is a small perturbation of Q_c. Now we prove

Lemma 8.4 (Energy and Mass expansions)**.** *Recall the Mass $M[u]$ and Energy $E[u]$ defined in* (8.12)-(8.13)*. We have*

$$M[u](t) = M[Q_c](t) + \int_0^{+\infty} Q_c z dx + \frac{1}{2}\int_0^{+\infty} z^2 dx, \tag{8.41}$$

where $M[Q_c](t) := M[Q_c(\cdot - \rho(t) - L)]$ and $Q_c = Q_c(\cdot - \rho(t) - L)$. Similarly

$$\begin{aligned} E[u](t) = E[Q_c](t) - \int_0^{+\infty} c Q_c z dx - Q_c(-\rho(t) - L) Q_c'(\rho(t) + L) \\ + \frac{1}{2}\int_0^{+\infty} (\partial_x z)^2 dx - \int_0^{+\infty} Q_c z^2 dx - \frac{1}{3}\int_0^{+\infty} z^3 dx. \end{aligned} \tag{8.42}$$

Here, $E[Q_c](t) := E[Q_c(\cdot - \rho(t) - L)]$. Finally, we have the following combined estimate:

$$\begin{aligned} & E[u](t) + cM[u](t) - E[Q_c](t) - cM[Q_c](t) = \\ & = O(e^{-2\sqrt{c}|\rho(t)+L|}) - \frac{1}{3}\int_0^{+\infty} z^3 dx \\ & + \frac{c}{2}\int_0^{+\infty} z^2 dx + \frac{1}{2}\int_0^{+\infty} (\partial_x z)^2 dx \\ & - \int_0^{+\infty} Q_c z^2 dx. \end{aligned} \tag{8.43}$$

Proof. We compute: by definition of z we see that

$$cM[u](t) = cM[Q_c] + c\int_0^{+\infty} Q_c z dx + \frac{c}{2}\int_0^{+\infty} z^2 dx. \tag{8.44}$$

On the other hand,

$$\begin{aligned} E[u](t) = & E[Q_c] + \int_0^{+\infty} Q_c' z_x dx - \int_0^{\infty} Q_c^2 z dx \\ & + \frac{1}{2}\int_0^{+\infty} z_x^2 dx - \int_0^{\infty} Q_c z^2 dx - \frac{1}{3}\int_0^{+\infty} z^3 dx. \end{aligned} \tag{8.45}$$

Integrating by parts and using (8.5) we see that

$$\begin{aligned}\int_0^{+\infty}(Q_c'z_x - Q_c^2 z)dx &= \int_0^{+\infty}(-Q_c'' - Q_c^2)z\,dx - Q_c'(-\rho(t)-L)z(0,t)\\ &= \int_0^{+\infty}(-Q_c'' - Q_c^2)z\,dx \\ &\quad + Q_c'(-\rho(t)-L)(f(t) - Q_c(-\rho(t)-L)),\end{aligned}.$$

therefore

$$\int_0^{+\infty}(Q_c'z_x - Q_c^2 z)dx = -\int_0^{+\infty} cQ_c z\,dx - Q_c(\rho(t)+L)Q_c'(\rho(t)+L). \quad (8.46)$$

Combining (8.45), (8.46) and (8.44) we get

$$\begin{aligned}&E[u](t) + cM[u](t) - (E[Q_c](t) + cM[Q_c](t)) =\\ &= -Q_c(-\rho(t)-L)Q_c'(-\rho(t)-L) - \frac{1}{3}\int_0^{+\infty} z^3 dx\\ &\quad + \frac{c}{2}\int_0^{+\infty} z^2 dx + \frac{1}{2}\int_0^{+\infty} z_x^2 dx\\ &\quad - \int_0^{+\infty} Q_c z^2 dx.\end{aligned} \quad (8.47)$$

Note now that we easily have the pointwise estimate

$$|Q_c(-\rho(t)-L)Q_c'(-\rho(t)-L)| \leqslant Ce^{-2\sqrt{c}|\rho(t)+L|}.$$

Replacing this last estimate in (8.47), we get the desired bound (8.43). The proof is complete. □

Let us continue with the proof. Let $\tilde{z}$ as in (8.36). We have from (8.43),

$$\begin{aligned}&E[u](t) + cM[u](t) - E[Q_c](t) - cM[Q_c](t) =\\ &= O(e^{-2\sqrt{c}|\rho(t)+L|}) - \frac{1}{3}\int_0^{+\infty} z^3 dx\\ &\quad + \frac{c}{2}\int_{\mathbb{R}} \tilde{z}^2 dx + \frac{1}{2}\int_{\mathbb{R}}(\partial_x\tilde{z})^2 dx - \int_{\mathbb{R}} \widetilde{Q}_c\tilde{z}^2 dx\\ &\quad - \frac{c}{2}\int_{-\infty}^0 \tilde{z}^2 dx - \frac{1}{2}\int_{-\infty}^0(\partial_x\tilde{z})^2 dx + \int_{-\infty}^0 \widetilde{Q}_c\tilde{z}^2 dx.\end{aligned}$$

Using (8.37) and (8.39), we have

$$\left| -\frac{c}{2}\int_{-\infty}^0 \tilde{z}^2 dx - \frac{1}{2}\int_{-\infty}^0 (\partial_x\tilde{z})^2 dx + \int_{-\infty}^0 \widetilde{Q}_c\tilde{z}^2 dx \right| \lesssim e^{-2\sqrt{c}|\rho(t)+L|}.$$

Consequently,

$$
\begin{aligned}
&E[u](t) + cM[u](t) - E[Q_c](t) - cM[Q_c](t) = \\
&= O(e^{-2\sqrt{c}|\rho(t)+L|}) - \frac{1}{3}\int_0^{+\infty} z^3 dx \\
&\quad + \frac{c}{2}\int_{\mathbb{R}} \widetilde{z}^2 dx + \frac{1}{2}\int_{\mathbb{R}} (\partial_x \widetilde{z})^2 dx - \int_{\mathbb{R}} \widetilde{Q}_c \widetilde{z}^2 dx.
\end{aligned} \tag{8.48}
$$

Now we need some control on the terms $E[Q_c](t)$ and $M[Q_c](t)$. Note that these terms are not conserved in time (because Q_c in (8.5) does not satisfy the zero boundary condition at $x = 0$), and we cannot use the Lemma 8.1 and (8.15). However, with some standard procedure we can get independent estimates.

Lemma 8.5. *Recall the terms $M[Q_c]$ and $E[Q_c]$ in (8.12)-(8.13) and $Q_c = Q_c(\cdot - \rho(t) - L)$. We have, for all $t \geqslant 0$,*

$$
\begin{aligned}
M[Q_c](t) &\gtrsim M(Q_c)(0) - e^{-2\sqrt{c}(L+\rho(t))} - e^{-2\sqrt{c}(L+\rho(0))}. \\
E[Q_c](t) &\gtrsim E(Q_c)(0) - e^{-2\sqrt{c}(L+\rho(t))} - e^{-2\sqrt{c}(L+\rho(0))}.
\end{aligned} \tag{8.49}
$$

Moreover, for all $t \geqslant 0$,

$$
\begin{aligned}
-\Big(E[Q_c](t) + cM[Q_c](t)\Big) \lesssim &- \Big(E(Q_c)(0) + cM[Q_c](0)\Big) \\
&+ e^{-2\sqrt{c}(L+\rho(t))} + e^{-2\sqrt{c}(L+\rho(0))}.
\end{aligned} \tag{8.50}
$$

Proof. We easily have

$$
\begin{aligned}
M[Q_c](t) &= \frac{1}{2}\int_{\mathbb{R}} \widetilde{Q}_c^2(x-\rho(t)-L)dx - \frac{1}{2}\int_{-\infty}^0 \widetilde{Q}_c^2(x-\rho(t)-L)dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \widetilde{Q}_c^2(x-\rho(0)-L)dx - \frac{1}{2}\int_{-\infty}^0 \widetilde{Q}_c^2(x-\rho(t)-L)dx \\
&= M[Q_c](0) + \frac{1}{2}\int_{-\infty}^0 \widetilde{Q}_c^2(x-\rho(0)-L)dx \\
&\quad - \frac{1}{2}\int_{-\infty}^0 \widetilde{Q}_c^2(x-\rho(t)-L)dx.
\end{aligned}
$$

Thus the first estimate follows by Lemma 8.3. On the other hand, $E[Q_c](t)$ can be easily estimated by

$$
\begin{aligned}
E[Q_c](t) &= \int_{\mathbb{R}} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t) - \frac{1}{3}\widetilde{Q}_c^3(t)\Big)dx - \int_{-\infty}^{0} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t) - \frac{1}{3}\widetilde{Q}_c^3(t)\Big)dx \\
&= \int_{\mathbb{R}} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t=0) - \frac{1}{3}\widetilde{Q}_c^3(t=0)\Big)dx \\
&\quad - \int_{-\infty}^{0} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t) - \frac{1}{3}\widetilde{Q}_c^3(t)\Big)dx \\
&= E[Q_c](0) + \int_{-\infty}^{0} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t=0) - \frac{1}{3}\widetilde{Q}_c^3(t=0)\Big)dx \\
&\qquad - \int_{-\infty}^{0} \Big(\frac{1}{2}\widetilde{Q}_c'^2(t) - \frac{1}{3}\widetilde{Q}_c^3(t)\Big)dx.
\end{aligned}
$$

Recall $\widetilde{Q}_c = \widetilde{Q}_c(x - \rho(t) - L)$, where $\widetilde{Q}_c$ is given in (8.5). It follows naturally from Lemma 8.3 that for each $t \geqslant 0$,

$$
E(Q_c)(t) \gtrsim E(Q_c)(0) - e^{-2\sqrt{c}(L+\rho(t))} - e^{-2\sqrt{c}(L+\rho(0))}.
$$

This proves the last estimate in (8.49). □

Now, combining (8.48) and (8.50), we obtain that

$$
\begin{aligned}
&E[u](t) + cM[u](t) \\
&\quad = \frac{1}{2}\Big(\int_{\mathbb{R}} (\partial_x \widetilde{z})^2 dx + c\int_{\mathbb{R}} \widetilde{z}^2 dx - 2\int_{\mathbb{R}} \widetilde{Q}_c \widetilde{z}^2 dx\Big) \\
&\qquad - \frac{1}{3}\int_{0}^{+\infty} z^3 dx - O(e^{-2\sqrt{c}(\rho(t)+L)} + e^{-2\sqrt{c}(\rho(0)+L)}).
\end{aligned}
\tag{8.51}
$$

On the other hand, Lemma 8.1 implies

$$
M[u](t) \leqslant M[u_0] \ \text{ and } \ E[u](t) \leqslant E[u_0], \tag{8.52}
$$

so that

$$
E[u](t) + cM[u](t) - (E[u](0) + cM[u](0)) \leqslant 0.
$$

Therefore, from this last inequality, (8.51), (8.40) and (8.33),

$$\begin{aligned}\int_{\mathbb{R}} \Big((\partial_x \tilde{z})^2 + c\tilde{z}^2 - 2\widetilde{Q}_c\tilde{z}^2\Big)dx &\lesssim \|\tilde{z}(0)\|^2_{H^1(\mathbb{R})} + \int_0^{+\infty} |z|^3 dx \\ &\quad + e^{-2\sqrt{c}(L+\rho(t))} + e^{-2\sqrt{c}(L+\rho(0))} \\ &\lesssim \|z(0)\|^2_{H^1(\mathbb{R}+)} + \|z(t)\|^3_{H^1(\mathbb{R}+)} \\ &\quad + e^{-2\sqrt{c}(L+\rho(t))} + e^{-2\sqrt{c}(L+\rho(0))} \\ &\lesssim \alpha^2 + \|z(t)\|^3_{H^1(\mathbb{R}+)} \\ &\quad + e^{-2\sqrt{c}(L+\rho(t))} + e^{-2\sqrt{c}(L+\rho(0))}.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}} \Big((\partial_x \tilde{z})^2 + c\tilde{z}^2 - 2\widetilde{Q}_c\tilde{z}^2\Big)dx \lesssim \alpha^2 + \|z(t)\|^3_{H^1(\mathbb{R}+)} + e^{-2\sqrt{c}L}. \tag{8.53}$$

The purpose of the next paragraph is to get a suitable lower bound on the term

$$\int_{\mathbb{R}} \Big((\partial_x \tilde{z})^2 + c\tilde{z}^2 - 2\widetilde{Q}_c\tilde{z}^2\Big)dx =: \int_{\mathbb{R}} \tilde{z}\mathcal{L}\tilde{z}dx,$$

where

$$\mathcal{L}\tilde{z} := -\partial_x^2\tilde{z} + c\tilde{z} - 2\widetilde{Q}_c\tilde{z}. \tag{8.54}$$

Note that we have reduced the problem on the half-line to an extended spectral problem on the whole line, and where we have good estimates on the left half-line portion of $\tilde{z}(t)$.

8.5 End of proof of Theorem 8.1

Let us start out with the following easy estimate:

Claim 8.1. *We have, for all $t \geqslant 0$,*

$$\left|\int_{-\infty}^0 \tilde{z}(x,t)\widetilde{Q}_c'(x-\rho(t)-L)dx\right| \lesssim e^{-2\sqrt{c}|\rho(t)+L|}. \tag{8.55}$$

Proof. Direct from Lemma 8.3 and (8.39). □

From (J. L. Bona, Souganidis, and W. A. Strauss 1987; Weinstein 1986) (see also Muñoz 2014, for more details), we have the standard coercivity estimate valid for each $\tilde{z} \in H^1(\mathbb{R})$, $Q_c = Q_c(x-\rho(t)-L)$ and $\mathcal{L}$ as in (8.54):

$$\int_{\mathbb{R}} \tilde{z}\mathcal{L}\tilde{z}dx \gtrsim \|\tilde{z}\|^2_{H^1(\mathbb{R})} - \left|\int_{\mathbb{R}} \tilde{z}(x,t)\widetilde{Q}_c'(x-\rho(t)-L)dx\right|^2 - \left|\int_{\mathbb{R}} \tilde{z}(x,t)\widetilde{Q}_c(x-\rho(t)-L)dx\right|^2.$$

(Note that both quadratic reminder terms above are not zero in our case, but both are very small.) Using Lemma 8.2 and (8.55) , we have

$$\begin{aligned}\int_{\mathbb{R}} \tilde{z}\mathcal{L}\tilde{z}dx &\gtrsim \|\tilde{z}\|^2_{H^1(\mathbb{R})} - \left|\int_{\mathbb{R}} \tilde{z}(x,t)\widetilde{Q}_c(x-\rho(t)-L)dx\right|^2 - e^{-2\sqrt{c}|\rho(t)+L|} \\ &\gtrsim \|z\|^2_{H^1(\mathbb{R}+)} - \left|\int_0^{+\infty} z(x,t)Q_c(x-\rho(t)-L)dx\right|^2 - e^{-2\sqrt{c}|\rho(t)+L|}.\end{aligned}$$

On the other hand we have

$$\int_0^{+\infty} z^2(x,t)dx = \int_0^{+\infty} u^2(x,t)dx + M[Q_c](t) - 2\int_0^{+\infty} z(x,t)Q_c(x-\rho(t)-L)dx.$$

Using this last expression, (8.15) and (8.49) we obtain

$$\left|\int_0^{+\infty} z(x,t)Q_c(x-\rho(t)-L)dx\right| \lesssim \|z(0)\|_{H^1(\mathbb{R}+)} + \|z(t)\|^2_{H^1(\mathbb{R}+)} + e^{-2\sqrt{c}L}. \quad (8.56)$$

Therefore, for α small and L large,

$$\int_{\mathbb{R}} \tilde{z}\mathcal{L}\tilde{z}dx \gtrsim \|z\|^2_{H^1(\mathbb{R}+)} - C(\alpha^2 + e^{-2\sqrt{c}L}).$$

Combining this last estimate (8.43) and (8.53), we obtain

$$\|z(t)\|^2_{H^1(\mathbb{R}+)} \lesssim \alpha^2 + e^{-2\sqrt{c}L} + \|z(t)\|^3_{H^1(\mathbb{R}+)}, \quad (8.57)$$

with constants independent of C_0, which implies $u(t) \in \mathcal{M}[C_0/2]$ for C_0, L large enough, and α small.

9 Instability for the KdV Equation on Star Graphs

In this chapter, we establish a general linear instability criterium of stationary solutions for the vectorial KdV model

$$\partial_t u_e(x,t) = \alpha_e \partial_x^3 u_e(x,t) + \beta_e \partial_x u_e(x,t) + 2u_e(x,t)\partial_x u_e(x,t), \tag{9.1}$$

$x \neq 0,\ t \in \mathbb{R}$, on a metric star graph $\mathcal{G}$ with a structure represented by the set $E \equiv E_- \cup E_+$ where E_- and E_+ are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively. The half-lines are connected at a unique vertex $\nu = 0$. Here $(\alpha_e)_{e\in E}$ and $(\beta_e)_{e\in E}$ are two sequences of real numbers.

Thus, we are interested in the dynamics generated by the flow of the KdV model (9.1) around solutions of stationary type, namely,

$$(u_e(x,t))_{e\in E} = (\phi_e(x))_{e\in E}$$

where for $e \in E_-$ the profile $\phi_e : (-\infty, 0) \to \mathbb{R}$ satisfy $\phi_e(-\infty) = 0$, and for $e \in E_+$ $\phi_e : (0, \infty) \to \mathbb{R}$ satisfy $\phi_e(+\infty) = 0$. The existence of profiles of stationary type for the KdV, namely, solutions of the following nonlinear elliptic equation

$$\alpha_e \frac{d^2}{dx^2}\phi_e(x) + \beta_e \phi_e(x) + \phi_e^2(x) = 0, \tag{9.2}$$

on every semi-infinite edge are well know and will depend of the profile of the classical soliton associated to the KdV on the full line and for specific conditions on α_e and β_e (see Chapter 7-section 7.2).

Our linear instability criterium of stationary profiles for (9.1) will be based on a spectral study of linear operators of self-adjoint type associated with the profile of the stationary solution and therefore specific boundary conditions on that profiles will be necessary to be impose at the vertex of the graph. A starting point for our study is that the Airy type operator

$$A_0 : (u_e)_{e\in E} \to \left(\alpha_e \frac{d^3}{dx^3}u_e + \beta_e \frac{d}{dx}u_e\right)_{e\in E} \tag{9.3}$$

have extensions A_{ext} on $L^2(\mathcal{G})$ such that the dynamics induced by the linearized KdV model (9.1)

$$\begin{cases} z_t = A_{ext} z, \\ z(0) = u_0 \in D(A_{ext}), \end{cases} \tag{9.4}$$

it is given by a C_0-group, $z(t) = e^{tA_{ext}}u_0$ (see Chapter 4-section 4.2). In this point the theory in (Mugnolo, Noja, and Seifert 2018; Schubert et al. 2015) give us that properties of the induced dynamics can be obtained by studying boundary operators in the corresponding boundary space induced by the vertex of the graph.

9.1 Linearized equation for KdV on a start graph

Let $(A_{ext}, D(A_{ext}))$ be a extension for the Airy operator A_0 in (9.3) on $L^2(\mathcal{G})$, such that the dynamics induced by the linear evolution problem (9.4) is given by a C_0-group.

Suppose for $(\phi_e)_{e\in E} \in D(A_L)$ we have that $(\widetilde{u}_e(x,t))_{e\in E} = (\phi_e(x))_{e\in E}$ is a nontrivial solution of (9.1), thus we obtain the following set of $|E_+| + |E_-|$ nonlinear equations

$$\alpha_e \frac{d^3}{dx^3}\phi_e + \beta_e \frac{d}{dx}\phi_e + 2\phi_e \frac{d}{dx}\phi_e = 0, \quad e \in E. \tag{9.5}$$

Then, since $\phi_e(\pm\infty) = 0$ we obtain for $e \in E$ that each component of the stationary solution satisfies the elliptic equation (9.2).

Next, we suppose for $e \in E$, that u_e satisfy formally equality in (9.1) and we define

$$v_e(x,t) \equiv u_e(x,t) - \phi_e(x). \tag{9.6}$$

Then, for $(v_e)_{e\in E} \in D(A_{ext})$ we have for each $e \in E$ the equation

$$\partial_t v_e = \alpha_e \partial_x^3 v_e + \beta_e \partial_x v_e + 2\partial_x(v_e\phi_e) + \partial_x(v_e^2), \tag{9.7}$$

Thus, we have that the system (abusing the notation)

$$\partial_t v_e(x,t) = \alpha_e \partial_x^3 v_e(x,t) + \beta_e \partial_x v_e(x,t) + 2\partial_x(v_e(x,t)\phi_e(x)), \tag{9.8}$$

represents the linearized equation for (9.1) around $(\phi_{\mathrm{e}}(x))_{\mathrm{e}\in\mathrm{E}}$.

Our objective in the following will be to give sufficient conditions for obtaining that the trivial solution $v_{\mathrm{e}} \equiv 0$, $\mathrm{e} \in \mathrm{E}$, it is unstable by the linear flow of (9.8). More exactly, we are interested in finding a *growing mode solution* of (9.8) with the form

$$v_{\mathrm{e}}(x,t) = e^{\lambda t}\psi_{\mathrm{e}} \ \text{ and } \ \mathrm{Re}(\lambda) > 0.$$

In other words, we need to solve the formal system for $\mathrm{e} \in \mathrm{E}$,

$$\lambda\psi_{\mathrm{e}} = -\partial_x \mathcal{L}_{\mathrm{e}}\psi_{\mathrm{e}}, \qquad \mathcal{L}_{\mathrm{e}} = -\alpha_{\mathrm{e}}\partial_x^2 - \beta_{\mathrm{e}} - 2\phi_{\mathrm{e}}, \tag{9.9}$$

with $\psi_{\mathrm{e}} \in D(\partial_x \mathcal{L}_{\mathrm{e}})$.

Next, we write our eigenvalue problem in (9.9) in an Hamiltonian matrix form and we establish formally the meaning of this eigenvalue problem. Indeed, we made the following abbreviations: for $\psi = (\psi_-, \psi_+)$ with $\psi_- = (\psi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_-}$ and $\psi_+ = (\psi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_+}$, we write $(\mathcal{L}_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} = (\mathcal{L}_1, \mathcal{L}_2)$ where

$$\begin{aligned} \mathcal{L}_1\psi_- &= (-\alpha_{\mathrm{e}}\partial_x^2\psi_{\mathrm{e}} - \beta_{\mathrm{e}}\psi_{\mathrm{e}} - 2\phi_{\mathrm{e}}\psi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_-}, \\ \mathcal{L}_2\psi_+ &= (-\alpha_{\mathrm{e}}\partial_x^2\psi_{\mathrm{e}} - \beta_{\mathrm{e}}\psi_{\mathrm{e}} - 2\phi_{\mathrm{e}}\psi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_+}. \end{aligned} \tag{9.10}$$

Thus, the eigenvalue problem in (9.9) can be written in a Hamiltonian vectorial form

$$\lambda \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\partial_x \mathcal{L}_- & 0 \\ 0 & -\partial_x \mathcal{L}_+ \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \equiv NE \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{9.11}$$

by identifying $\mathcal{L}_-$ as a $\mathrm{E}_- \times \mathrm{E}_-$-diagonal matrix defined for

$$(\alpha_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_-} = (\alpha_{1,-}, \ldots, \alpha_{n,-}), \ \ (\beta_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_-} = (\beta_{1,-}, \ldots, \beta_{n,-}),$$

and $(\phi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_-} = (\phi_{1,-}, \ldots, \phi_{n,-})$ as

$$\mathcal{L}_- = \mathrm{diag}\Big(-\alpha_{1,-}\frac{d^2}{dx^2} - \beta_{1-} - 2\phi_{1,-}, \ldots, -\alpha_{n,-}\frac{d^2}{dx^2} - \beta_{n,-} - 2\phi_{n,-}\Big), \tag{9.12}$$

and $\mathcal{L}_+$ being a $\mathrm{E}_+ \times \mathrm{E}_+$-diagonal matrix being define similarly for $(\alpha_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_+}$, $(\beta_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_+}$ and $(\phi_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}_+}$. Thus, we have that N and E in (9.11) are $(|\mathrm{E}_-| + |\mathrm{E}_+|) \times (|\mathrm{E}_-| + |\mathrm{E}_+|)$-diagonal matrix defined by

$$N = \begin{pmatrix} -\partial_x I_- & 0 \\ 0 & -\partial_x I_+ \end{pmatrix}, \quad E = \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}, \tag{9.13}$$

with $I_\pm$ being the $\mathrm{E}_\pm \times \mathrm{E}_\pm$-identity matrix.

Next, if we denote by $\sigma(NE)$ the "spectrum" of NE, namely, $\lambda \in \sigma(NE)$ if there is a $\psi \neq 0$ satisfying $NE\psi = \lambda\psi$. The later discussion suggests the utility of the following definition:

Definition 9.1. *The stationary vector solution* $(\phi_e)_{e\in E} \in D(A_{ext})$ *is said to be spectrally stable for model* (9.1) *if the spectrum of* NE, $\sigma(NE)$, *satisfy* $\sigma(NE) \subset i\mathbb{R}$. *Otherwise, the stationary solution* $(\phi_e)_{e\in E}$ *is said to be spectrally unstable.*

It is standard to show that $\sigma(NE)$ is symmetric with respect to the real and imaginary axes (see, for instance, (Grillakis, Shatah, and W. Strauss 1990, Lemma 5.6)) providing that N is skew-symmetric and E self-adjoint. Hence it is equivalent to say that $(\phi_e)_{e\in E} \in D(A_{ext})$ is spectrally unstable if $\sigma(NE)$ contains point λ with $\mathrm{Re}(\lambda) > 0$.

9.2 Linear instability criterium

In this section we establish a linear instability criterium of stationary solutions for the KdV model (9.1) on a star graph $\mathcal{G}$ with a structure represented by the set $E \equiv E_- \cup E_+$ where E_- and E_+ are finite or countable collections of semi-infinite edges e parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively. The half-lines are connected at a unique vertex $\nu = 0$.

From (9.11) our eigenvalue problem to solve is reduced to,

$$NE\psi = \lambda\psi, \quad Re(\lambda) > 0, \quad \psi \in D(E). \tag{9.14}$$

Next, we establish our theoretical framework and assumptions for obtaining a nontrivial solution to problem in (9.14):

S_1) Let $(A_{ext}, D(A_{ext}))$ be a extension of $(A_0, D(A_0))$ such that the solution of the linearized KdV model (9.4) is given by a C_0-group.

S_2) Suppose $0 \neq \phi = (\phi_e)_{e\in E} \in D(A_{ext})$ such that $(\widetilde{u}_e(x,t))_{e\in E} = (\phi_e(x))_{e\in E}$ is a stationary solution for the KdV model (9.1).

S_3) Let E be the matrix-operator in (9.13) defined on a domain $D(E) \subset L^2(\mathcal{G})$ on which E is self-adjoint and such that we have the property $D(A_{ext}) \subset D(E)$.

S_4) Since for every $u \in D(A_{ext})$ we have $Eu \in D(N) = H^1(\mathcal{G})$, we suppose $\langle NEu, \phi\rangle = 0$ for every $u \in D(A_{ext})$.

S_5) Suppose $E : D(E) \to L^2(\mathcal{G})$ is invertible with Morse index $n(E)$ such that:

- a) for $n(E) = 1$, $\sigma(E) = \{\lambda_0\} \cup J_0$ with $J_0 \subset [r_0, +\infty)$, for $r_0 > 0$, and $\lambda_0 < 0$,
- b) for $n(E) = 2$, $\sigma(E) = \{\lambda_1, \lambda_2\} \cup J$ with $J \subset [r, +\infty)$, for $r > 0$, and $\lambda_1, \lambda_2 < 0$. Moreover, for $\Phi_1, \Phi_2, \in D(E) - \{0\}$ with $E\Phi_i = \lambda_i\Phi_i$ $(i = 1, 2)$ we have $\langle N\phi, \Phi_1\rangle \neq 0$ or $\langle N\phi, \Phi_2\rangle \neq 0$.

S_6) For $\psi \in D(E)$ with $E\psi = \phi$, we have $\langle \psi, \phi\rangle \neq 0$.

S_7) Suppose the operator $N : D(N) \cap D(E) \to L^2(\mathcal{G})$ is a skew-symmetric operator. We recall that N on $D(N)$ is always one-to-one.

We note immediately that the following matrix-operator relation

$$NE\psi = A_{ext}\psi + \operatorname{diag}((2\partial_x(\phi_e\psi))\delta_{ij}), \quad 1 \leqq i, j \leqq |\mathrm{E}_-| + |\mathrm{E}_+|,$$

together with assumption S_1), $\phi, \phi' \in L^\infty(\mathcal{G})$ and from semigroup theory (see (Pazy 1983)) imply that the linear Hamiltonian equation

$$\frac{d}{dt}u(t) = NEu(t) \tag{9.15}$$

generates a C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ on $L^2(\mathcal{G})$.

Some of the former assumptions deserve specific comments which will be very useful in the development of our stability theory.

Remark 9.1.

1) By depending of the context we identify $u = (u_-, u_+) \in L^2(\mathcal{G})$ as a element in $\Pi_{i=1}^n L^2(-\infty, 0) \times \Pi_{i=1}^m L^2(0, +\infty)$, with $n = |\mathrm{E}_-|$ and $m = |\mathrm{E}_+|$ or as $(n+m) \times 1$-matrix column as in (9.15).

2) For a balanced graph, namely, $n = m$, and $f = (f_-, f_+) \in D(N)$ with $f_-(0-) = f_+(0+)$ we can identify f as a element of $\bigoplus_{\mathrm{e}\in\mathrm{E}_+} H^1(\mathbb{R})$ in the obvious way.

3) In contrast to the classical stability theories for solitary waves solutions on all line, in the case of a star graph we have in general that $N\phi \notin D(E)$ (see Lemma 9.1 below). But from (9.5) we will have always that (see (9.12))

$$\mathcal{L}_+\phi_+'(x) = 0, \ \ for \ \ x > 0, \quad \mathcal{L}_-\phi_-'(x) = 0 \ \ for \ \ x < 0,$$

where we are writing $(\phi_e)_{e\in\mathrm{E}} = (\phi_-, \phi_+)$, with $\phi_- = (\phi_e)_{e\in\mathrm{E}_-}$ and $\phi_+ = (\phi_e)_{e\in\mathrm{E}_+}$.

4) From Proposition 4.2 (the case of two half-lines) and $\phi_\pm$ being either the tail or the bump profiles in (7.24), we have for $\phi = (\phi_-, \phi_+)$ that the second part of assumption S_4), $\langle NEu, \phi\rangle = 0$ for every $u \in D(A_L)$, it is true in the case of a δ-interaction. Indeed, for $u = (u_-, u_+) \in D(A_Z)$ defined in (4.15) follows from integration by parts (without loss *of generality we consider $\alpha_- = \alpha_+ = 1$ and $\beta_- = \beta_+ = -1$ in (9.5))*

$$\begin{aligned}
&\int_{-\infty}^{0} \partial_x(\partial_x^2 u_-)\phi_- dx + \int_0^{+\infty} \partial_x(\partial_x^2 u_+)\phi_+ dx \\
&= -\int_{-\infty}^{0} u_-\phi_-''' dx - \int_0^{\infty} u_+\phi_+''' dx \\
&+[u_-''(0-) - u_+''(0+)]\phi_+(0+) + u_+'(0+)\phi_+'(0+) - u_-'(0-)\phi_-'(0-) \\
&= -\int_{-\infty}^{0} u_-\phi_-''' dx - \int_0^{\infty} u_+\phi_+''' dx + [-\tfrac{Z^2}{2}u_-(0-) - Zu_-'(0-)]\phi_+(0+) \\
&+Zu_-'(0-)\phi_+(0+) + Zu_-(0-)\phi_+'(0+) \\
&= -\int_{-\infty}^{0} u_-\phi_-''' dx - \int_0^{\infty} u_+\phi_+''' dx + u_-(0-)[Z\phi_+'(0+) - \tfrac{Z^2}{2}\phi_+(0+)] \\
&= -\int_{-\infty}^{0} u_-\phi_-''' dx - \int_0^{\infty} u_+\phi_+''' dx,
\end{aligned} \tag{9.16}$$

where in the equality we use the "even-property" of (ϕ_-, ϕ_+), namely, $\phi_+'(0+) = \frac{Z}{2}\phi_+(0+)$. Next, since $u_-(0-) = u_+(0+)$ and $\phi_-(0-) = \phi_+(0+)$ we obtain

$$\begin{aligned}
&\int_{-\infty}^{0} \partial_x(u_- - 2\phi_- u_-)\phi_- dx + \int_0^{+\infty} \partial_x(u_+ - 2\phi_+ u_+)\phi_+ dx \\
&= -\int_{-\infty}^{0} u_-(1 - 2\phi_-)\phi_-' dx - \int_0^{+\infty} u_+(1 - 2\phi_+)\phi_+' dx.
\end{aligned} \tag{9.17}$$

Thus from (9.16) and (9.17) we obtain for $u \in D(A_L)$

$$\begin{aligned}
\langle NEu, \phi\rangle &= \langle -\partial_x \mathcal{L}_- u_-, \phi_-\rangle + \langle -\partial_x \mathcal{L}_+ u_+, \phi_+\rangle \\
&= \int_{-\infty}^{0} u_-(-\phi_-''' + \phi_-' - 2\phi_-\phi_-')dx \\
&+ \int_0^{+\infty} u_+(-\phi_+''' + \phi_+' - 2\phi_+\phi_+')dx = 0.
\end{aligned} \tag{9.18}$$

5) From Proposition 4.2 we see that our assumption S_3) in the case of a δ-interaction for two half-line is not empty. Indeed, for $E = diag(\mathcal{L}_1, \mathcal{L}_2)$, with $\phi_\pm$ being either the tail or the bump profiles in (7.24) and with

$$\begin{aligned}
D(E) \;&= \{u \in H^2(-\infty, 0) \oplus H^2(0, +\infty) : u_-(0-) = u_+(0+), \\
&\quad \text{and } \; u_+'(0+) - u_-'(0-) = Zu_-(0-)\},
\end{aligned}$$

we have the self-adjoint property of E and $D(A_L) \subset D(E)$ (see remarks after Proposition 4.2). Moreover, assumption S_7) is immediately satisfied in this case.

Next, we give the preliminaries for establishing our instability criterium in Theorem 9.1 below. We start by considering the orthogonal projection $Q : L^2(\mathcal{G}) \to L^2(\mathcal{G})$

$$Q(u) = u - \langle u, \phi\rangle \frac{\phi}{\|\phi\|^2} \tag{9.19}$$

associated to the nontrivial stationary solution ϕ, and we consider $X_2 = Q(L^2(\mathcal{G})) = \{f \in L^2(\mathcal{G}) : f \perp \phi\} = [\phi]^\perp$. We also define the closed skew-adjoint operator $N_0 : D(N_0) \subset X_2 \to X_2$, $D(N_0) \equiv D(N) \cap D(A_{ext}) \cap X_2$, for $f \in D(N_0)$ by

$$N_0 f \equiv QNf = Nf - \langle Nf, \phi\rangle \frac{\phi}{\|\phi\|^2} \tag{9.20}$$

and the reduced self-adjoint operator for E, $F : D(F) \to X_2$, $D(F) = D(E) \cap X_2$ by

$$Ff \equiv QEf = Ef - \langle Ef, \phi\rangle \frac{\phi}{\|\phi\|^2}. \tag{9.21}$$

We note that N_0 is no necessarily one-to-one. Now, for $f \in D(NE) \cap X_2 = D(A_{ext}) \cap X_2$ ($Ef \in D(N)$), assumptions S_4) and S_7) we get the relation

$$\begin{aligned} N_0 Ff &= NEf - \langle Ef, \phi\rangle \tfrac{N\phi}{\|\phi\|^2} - \langle NEf - \langle Ef, \phi\rangle \tfrac{N\phi}{\|\phi\|^2}, \phi\rangle \tfrac{\phi}{\|\phi\|^2} \\ &= NEf - \langle Ef, \phi\rangle \tfrac{N\phi}{\|\phi\|^2}. \end{aligned} \tag{9.22}$$

Our first result is the following,

Proposition 9.1. $N_0 F : D(N_0 F) \subset X_2 \to X_2$, $D(N_0 F) = D(A_{ext}) \cap X_2 \subset D(E) \cap X_2$, *it is the infinitesimal generator of a strongly continuous* C_0*-group of operators* $S_0(t)$ *in the space* X_2.

Proof. We divide the proof in two steps:

a) Define $C = QNQEQ : D(C) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G})$, $D(C) = D(A_{ext})$. Then for $f \in D(A_{ext})$

$$\begin{aligned} Cf &= NEf - \langle f, \phi\rangle \tfrac{NE\phi}{\|\phi\|^2} - \langle Ef, \phi\rangle \tfrac{N\phi}{\|\phi\|^2} + \langle f, \phi\rangle \tfrac{\langle E\phi, \phi\rangle}{\|\phi\|^2} \tfrac{N\phi}{\|\phi\|^2} \\ &= NEf - Bf \end{aligned} \tag{9.23}$$

where $B : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ defined by

$$Bf = \langle f, \phi\rangle \frac{NE\phi}{\|\phi\|^2} + \langle f, E\phi\rangle \frac{N\phi}{\|\phi\|^2} - \langle f, \phi\rangle \frac{\langle E\phi, \phi\rangle}{\|\phi\|^2} \frac{N\phi}{\|\phi\|^2},$$

it is a bounded operator. Here was used that E is a self-adjoint operator on $D(E) \supseteq D(A_{ext})$. Thus, from the theory of semigroups (see (Pazy 1983)) C generates a strongly continuous C_0-group of operators $S_1(t)$ on $L^2(\mathcal{G})$. Since C commutes with Q, $S_1(t)$ also commutes with Q.

b) Define $S_0(t) : X_2 \to X_2$ by $S_0(t) = QS_1(t)$. Then S_0 is a strongly continuous C_0-group of linear operators on X_2 and it is not difficult to see that its infinitesimal generator is $N_0 F$.

This finishes the Proposition. □

Next, we have the following basic assumption.

(H) There is a real number η, satisfying $\eta > 0$, such that the operator $F : D(F) \to X_2$, $D(F) = D(E) \cap X_2$, it is invertible with exactly one negative eigenvalue and all other eigenvalues are contained in $[\eta, +\infty)$.

Our instability criterium for stationary solutions of the KdV model (9.1) on star graphs is the following:

Theorem 9.1. *Suppose the assumptions $S_1) - S_7)$ above and the basic assumption* (H)*, then the operator NE has a real positive and a real negative eigenvalue.*

The proof of Theorem 9.1 is based in ideas from Lopes (2002) and from the following result on closed convex cone (Krasnosel'skiĭ 1964, Chapter 2).

Theorem 9.2. *Let K be a closed convex cone of a Hilbert space $(X, \|\cdot\|)$ such that there are a continuous linear functional Φ and a constant $a > 0$ such that $\Phi(u) \geqq a\|u\|$ for any $u \in K$. If $T : X \to X$ is a bounded linear operator that leaves K invariant, then T has an eigenvector in K associated to a nonnegative eigenvalue.*

Proof. Next we give a sketch of the proof, for more details we suggest the reader to see Angulo and Cavalcante (2019). We divide our analysis in several steps:

(1) The operator $N_0E : D(N_0E) \subset X_2 \to X_2$ has a real positive and a real negative eigenvalue. Indeed, from assumption (H) we consider $\psi \in D(F) = D(E) \cap X_2$, $\|\psi\| = 1$ and $\lambda_0 < 0$ such that $F\psi = \lambda_0\psi$. We define,

$$K = \{z \in D(F) : \langle Fz, z\rangle \leqq 0, \text{ and } \langle z, \psi\rangle \geqq 0\}$$

then K is a nonempty closed convex cone in $L^2(\mathcal{G})$. Moreover, by using a density argument we can see that this cone is invariant under the group $S_0(t)$. Indeed, for $f \in K$ and smooth enough we obtain that the reduced hamiltonian equation

$$\begin{cases} \dot{z} = N_0Fz \\ z(0) = f \end{cases} \tag{9.24}$$

has solution $z(t) = S_0(t)f$ that such for all t,

$$\frac{d}{dt}\langle Fz(t), z(t)\rangle = \langle FN_0Fz(t), z(t)\rangle + \langle Fz(t), N_0Fz(t)\rangle = 0,$$

where we use the self-adjoint property of F and the skew-symmetric property of N_0. Then for all t,

$$\langle Fz(t), z(t)\rangle = \langle Ff, f\rangle \leqq 0.$$

Next, we suppose $\langle f, \psi \rangle > 0$ and that there is t_0 such that $\langle S_0(t_0)f, \psi \rangle < 0$. Then by continuity of the flow $t \to S_0(t)f$ there is $\tau \in (0, t_0)$ with $\langle S_0(\tau)f, \psi \rangle = 0$. Now, from assumption (H) we have from the spectral theorem for self-adjoint operators the orthogonal decomposition for $f_\tau = S_0(\tau)f$

$$f_\tau = \sum_{i=1}^{m} a_i h_i + g, \quad g \perp h_i, \text{ for all } i,$$

where $Fh_i = \lambda_i h_i$, $\|h_i\| = 1$, $\lambda_i \in \sigma_d(F)$ with $\lambda_i \geqq \eta$, and $\langle Fg, g \rangle \geqq \theta \|g\|^2$, $\theta > 0$. Therefore

$$0 \geqq \langle Ff_\tau, f_\tau \rangle \geqq \eta \sum_{i=1}^{m} a_i^2 + \theta \|g\|^2 \geqq 0.$$

Thus, it follows $g = 0$ and $a_i = 0$ for i. Therefore, $S_0(\tau)f = 0$ and since $S_0(t)$ is a group we obtain $f = 0$ and so $\langle f, \psi \rangle = 0$ which is a contradiction. Now we suppose $\langle f, \psi \rangle = 0$, then the former analysis shows $f = 0$ and so $S_0(t)f \equiv 0$ for all t. It shows the invariance of K by $S_0(t)$.

From semigroup's theory, we have for μ large the following integral representation of the resolvent

$$Tz = (\mu I_d - N_0 F)^{-1}(z) = \int_0^\infty e^{-\mu t} S_0(t) z \, dt$$

and by the former analysis it also leaves K invariant.

Next, we consider the continuous linear functional $\Phi : L^2(\mathcal{G}) \to \mathbb{R}$ by $\Phi(z) = \langle z, \psi \rangle$ and we will see that there is $a > 0$ such that $\Phi(z) \geqq a\|z\|$ for any $z \in K$. Indeed, suppose for $\|g\| = 1$, $\langle g, \psi \rangle = \gamma > 0$ and $\langle Fg, g \rangle \leqq 0$. Since $ker(\Phi)$ is a hyperplane we obtain $g = z + \gamma\psi$ with $\langle z, \psi \rangle = 0$. So, $-\lambda\gamma^2 \geqq \langle Fz, z \rangle$. Now, from the orthogonal decomposition

$$z = \sum_{i=1}^{m} \langle z, h_i \rangle h_i + g, \quad g \perp h_i, \text{ for all } i,$$

follows for $\eta, \theta > 0$, $\langle Fz, z \rangle \geqq min\{\eta, \theta\}(1 - \gamma^2)$. Then,

$$\langle g, \psi \rangle = \gamma \geqq \sqrt{\frac{min\{\eta, \theta\}}{-\lambda + min\{\eta, \theta\}}} \equiv a > 0.$$

Therefore, from Theorem 9.2 there are an $\alpha \geqq 0$ and a nonzero element $\omega_0 \in K$ such that

$$(\mu I - N_0 F)^{-1}(\omega_0) = \alpha\omega_0.$$

It is immediate that $\alpha > 0$ and so $N_0 F\omega_0 = \zeta\omega_0$ with

$$\zeta = \frac{\mu\alpha - 1}{\alpha}.$$

Next we see that $\zeta \neq 0$. Suppose that $\zeta = 0$, then from (9.22) and the injectivity of N we obtain

$$E\omega_0 = \langle E\omega_0, \phi\rangle \frac{\phi}{\|\phi\|^2}.$$

From assumption S_5), let $\psi \in D(E)$ with $E\psi = \phi$, then since E is invertible follows

$$\omega_0 = \frac{\langle E\omega_0, \phi\rangle}{\|\phi\|^2}\psi \ \text{ and } \ 0 = \langle \omega_0, \phi\rangle = \frac{\langle E\omega_0, \phi\rangle}{\|\phi\|^2}\langle \psi, \phi\rangle.$$

Since $\langle \psi, \phi\rangle \neq 0$ follows $\langle E\omega_0, \phi\rangle = 0$. Hence $E\omega_0 = 0$ and so $\omega_0 = 0$, which is a contradiction. Then, $N_0 F$ has a nonzero real eigenvalue ζ.

Now, since $\sigma(N_0 F) = -\sigma(N_0 F)$ we have $-\zeta$ also belongs to $\sigma(N_0 F)$. Thus from Theorem 5.8 of (Grillakis, Shatah, and W. Strauss 1990), the essential spectrum of $N_0 F$ lies on the imaginary axis and then $-\zeta$ is an eigenvalue of $N_0 F$ and this proves the claim.

(2) Thus, for $\omega_0 \in D(N_0 F)$, $\omega_0 \neq 0$, and $\zeta > 0$ we have,

$$NE\omega_0 = \langle E\omega_0, \phi\rangle \frac{N\phi}{\|\phi\|^2} + \zeta\omega_0. \tag{9.25}$$

Next we consider the following cases:

a) Suppose $\langle E\omega_0, \phi\rangle = 0$, then $NE\omega_0 = \zeta\omega_0$ and the proof of the criterium finishes.

b) Suppose $r \equiv \frac{1}{\|\phi\|^2}\langle E\omega_0, \phi\rangle \neq 0$ and Assumption $S5$) with $n(E) = 2$. Let

$$u = \omega_0 + a\Phi_1 + b\Phi_2, \quad E\Phi_i = \lambda_i\Phi_i, \ 1 \leqq i \leqq 2,$$

with $\|\Phi_i\| = 1$, $\Phi_1 \perp \Phi_2$. We will find $a, b \in \mathbb{R}$, not both zero, such that

$$NEu = \zeta u, \quad u \neq 0.$$

Thus, we obtain initially the relation

$$rN\phi + a\lambda_1 N\Phi_1 + b\lambda_2 N\Phi_2 = a\zeta\Phi_1 + b\zeta\Phi_2. \tag{9.26}$$

Therefore, from the skew-symmetric property of N we obtain the system

$$\begin{cases} a\zeta + b\lambda_2\langle N\Phi_1, \Phi_2\rangle = r\langle N\phi, \Phi_1\rangle \\ a\lambda_1\langle N\Phi_1, \Phi_2\rangle - \zeta b = -r\langle N\phi, \Phi_2\rangle. \end{cases} \tag{9.27}$$

Thus, since the determinant of the coefficients is different of zero

$$\zeta^2 + \lambda_1\lambda_2\langle N\Phi_1, \Phi_2\rangle^2 \neq 0$$

$r \neq 0$ and from Assumption $S5$), we obtain a nontrivial solution for (9.27). Next we see $u \neq 0$. Indeed, suppose $u = 0$. Then, from relation $\omega_0 = -a\Phi_1 - b\Phi_2$ and by substituting in (9.25) we obtain the relation

$$a\lambda_1 r\langle N\phi, \Phi_1\rangle + b\lambda_2 r\langle N\phi, \Phi_2\rangle = 0. \tag{9.28}$$

Then, by using system (9.27) in (9.28) we arrive to the relation $\zeta(a^2\lambda_1 + b^2\lambda_2) = 0$, it which is a contradiction.

c) Suppose $n(E) = 1$ in Assumption $S5$). Then from Theorem 9.2 applied to $T = (\mu I - NE)^{-1}$, μ large, implies that NE has a real positive and a real negative eigenvalue. This finishes the proof.

□

9.2.1 One application of Theorem 9.1

Suppose that assumptions $S_1) - S_7)$ above hold and for ψ such that $E\psi = \phi$ we have $\langle\psi, \phi\rangle < 0$. Then assumption (H) is true. Indeed, from assumption S_5) we obtain that F is invertible. Next, let $\lambda_1, \lambda_2 < 0$, $\Phi_1, \Phi_2 \in D(E)$ with $E\Phi_i = \lambda_i\Phi_i$, $\Phi_1 \perp \Phi_2$. Suppose that for some i we have $\Phi_i \perp \phi$, then $\Phi_i \in D(F)$ and

$$F(\Phi_i) = E\Phi_i = \lambda_i\Phi_i$$

and so $n(F) \geqq 1$. Now suppose that for all i, $\langle\Phi_i, \phi\rangle \neq 0$, then there are $a, b \in \mathbb{R} - \{0\}$ such that $\langle a\Phi_1 + b\Phi_2, \phi\rangle = 0$, and

$$\langle F(a\Phi_1 + b\Phi_2), a\Phi_1 + b\Phi_2\rangle = \lambda_1 a^2\|\Phi_1\|^2 + \lambda_2 b^2\|\Phi_2\|^2 < 0.$$

Then via min-max principle we also have $n(F) \geqq 1$. Next, suppose that $n(F) = 2$ and consider $z_1, z_2 \in X_2$, $z_1 \perp z_2$, $\mu_1, \mu_2 < 0$, and $Fz_i = \mu_i z_i$. Then we get

$$\langle Ez_i, z_i\rangle = \mu_i\|z_i\|^2 < 0, \quad \text{and} \quad \langle Ez_1, z_2\rangle = 0.$$

Moreover, since $\psi \notin X_2$ follows that set $\{\psi, z_1, z_2\} \subset E$ is linearly independent and we have the relations

$$\langle Ez_i, \psi\rangle = \langle z_i, \phi\rangle = 0, \quad \text{and,} \quad \langle E\psi, \psi\rangle = \langle\phi, \psi\rangle < 0.$$

Therefore $\langle E(\alpha\psi + \beta z_1 + \theta z_2), \alpha\psi + \beta z_1 + \theta z_2\rangle < 0$ and so $n(E) \geqq 3$, it which is not true. Then $n(F) = 1$ and all other eigenvalues (and the remain of the spectrum) are contained in $[\eta, +\infty)$. Thus, from Theorem 9.1 follows that NE has a real positive and a real negative eigenvalue.

9.3 Linear instability of tail and bump on two half-lines

The focus of this section is to apply the linear instability criterium in Theorem 9.1 to the KdV on a star graph with two half-lines and a δ-interaction-type at the vertex $\nu = 0$. Our main result is the following,

Theorem 9.3. *For $Z \neq 0$, $\alpha_- = \alpha_+ > 0$, $\beta_- = \beta_+ < 0$, $-\beta_+ > \frac{Z^2}{4}$, let $\phi_Z \equiv (\phi_-, \phi_+) \in D(A_Z)$ defined for $\phi_+(x)$ by the formula (7.24) with $x > 0$ and $\phi_-(x) = \phi_+(-x)$ for $x < 0$. We consider the following family of stationary solutions for the Korteweg–de Vries model (9.1) on the star graph $\mathcal{G}$ with $\mathrm{E} = (-\infty, 0) \cup (0, +\infty)$,*

$$U(x,t) = (\phi_-(x), \phi_+(x)), \quad t \in \mathbb{R}.$$

Then, this family of tail ($Z < 0$) and bump ($Z > 0$) profiles are linearly unstable.

Next, we consider the cases $\alpha_- = \alpha_- = 1$, $\beta_- = \beta_+ = -1$ and $1 > \frac{Z^2}{4}$, without loss of generality. From Proposition 4.2, assumption S_1) is filled by $(A_Z, D(A_Z))$ defined in (4.15). The linear eigenvalue problem to be solve (9.14) for $\lambda > 0$, it is determined by the matrices N, E in (9.13) with the Schrödinger operators

$$\mathcal{L}_\pm = -\frac{d^2}{dx^2} + 1 - 2\phi_\pm.$$

The domain for $E = E_Z$ is given in $H^2(\mathcal{G}) = H^2(-\infty, 0) \oplus H^2(0, +\infty)$ for $Z \in \mathbb{R}$ by

$$D(E_Z) = \{(u_-, u_+) \in H^2(\mathcal{G}) : u_-(0-) = u_+(0+), \\ u'_+(0+) - u'_-(0-) = Zu_-(0-)\}, \tag{9.29}$$

and so $(E_Z, D(E_Z))$ represents a self-adjoint family of operators for each $Z \in \mathbb{R}$ with $D(A_Z) \subset D(E_Z)$ (assumption S_3)). From Remark 4.2-item 2) we have assumption S_4). Assumption S_7) is immediate by continuity.

The following lemma implies that E_Z is invertible (assumption S_5)).

Lemma 9.1. *For every $Z \neq 0$ we have $ker(E_Z) = \{0\}$. Moreover, since $\sigma_{ess}(E_Z) = [1, +\infty)$ we obtain $E_Z : D(E_Z) \to L^2(\mathbb{R})$ is invertible.*

Proof. Let $u = (u_-, u_+) \in D(E_Z)$, $E_Z u = 0$. Since $\mathcal{L}_\pm \phi'_\pm = 0$, we need to have $u_-(x) = a\phi'_-(x)$, $x < 0$, and $u_+(x) = b\phi'_+(x)$, $x > 0$ (see Berezin and Shubin 1991). From the continuity property at zero, $\phi'_+(0+) = -\phi'_-(0-)$ and $\phi''_+(0+) = \phi''_-(0-)$ we have that

$$a = -b, \quad \text{and} \quad -2a\phi''_+(0+) = Zu_+(0+) = Zu_-(0-) = Za\phi'_-(0-). \tag{9.30}$$

Suppose $a \neq 0$. Then, by using condition (7.25) and (9.30) we have $\phi''_+(0+) = \frac{Z^2}{4}\phi_+(0+)$, then from (7.22) we obtain

$$1 - \phi(0+) = \frac{Z^2}{4} \Longrightarrow Z^2 = 4$$

which does not happen. So $a = b = 0$ and $u \equiv 0$.

Next, by Weyl's theorem see Theorem XIII.14 of Reed and Simon 1978, the essential spectrum of E_Z coincides with $[1, +\infty)$. Then E_Z is an invertible operator. This finishes the proof. □

Lemma 9.2. *For $Z > 0$ we have $n(E_Z) = 2$ and for $Z < 0$ that $n(E_Z) = 1$.*

Proof. Our strategy is to use perturbation theory. For this purpose we define the self-adjoint operator on $L^2(\mathbb{R})$

$$\mathcal{L}_0 = -\frac{d^2}{dx^2} + 1 - 2\phi_0, \quad D(\mathcal{L}_0) = H^2(\mathbb{R}) \tag{9.31}$$

where ϕ_0 denotes the soliton for the KdV equation on the full line,

$$\phi_0(x) = \frac{3}{2} sech^2\left(\frac{1}{2}x\right), \quad x \in \mathbb{R}. \tag{9.32}$$

From classical Sturm-Liouville Theory (see Berezin and Shubin 1991)

$$ker(\mathcal{L}_0) = [\phi_0'], \quad n(\mathcal{L}_0) = 1, \quad \sigma_{ess}(\mathcal{L}_0) = [1, +\infty)$$

Now, we consider the domain

$$D(E_0) = \{(u_-, u_+) \in H^2(\mathcal{G}) : u_-(0-) = u_+(0+), u_-'(0-) = u_+'(0+)\}. \tag{9.33}$$

on which the following "limit" operator E_0 is self-adjoint

$$E_0 = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - 2\phi_{0,-} & 0 \\ 0 & -\frac{d^2}{dx^2} + 1 - 2\phi_{0,+} \end{pmatrix}, \tag{9.34}$$

with $\phi_{0,-} = \phi_0|_{(-\infty,0)}$ and $\phi_{0,+} = \phi_0|_{(0,+\infty)}$. Thus, by considering the following unitary operator $\mathcal{U} : D(E_0) \to H^2(\mathbb{R})$ defined for $u = (u_-, u_+) \in E_0$ by $\mathcal{U}(u) = \widetilde{u} \in H^2(\mathbb{R})$ where

$$\widetilde{u} = \begin{cases} u_-(x), & x < 0 \\ u_+(x), & x > 0 \\ u_+(0+), & x = 0, \end{cases} \tag{9.35}$$

we obtain $\sigma(E_0) = \sigma(\mathcal{L}_0)$ and $\lambda \in \sigma_{disc}(E_0)$ if and only if $\lambda \in \sigma_{disc}(\mathcal{L}_0)$ with the same multiplicity. Moreover, $\sigma_{ess}(E_0) = [1, +\infty)$. Therefore, $ker(E_0) = [\Phi_0']$, $\Phi_0 = (\phi_{0,-}, \phi_{0,+})$, and $n(E_0) = 1$.

The theory of analytic perturbation will be to follow our strategy in the study of the instability property of ϕ_Z (see Angulo and Goloshchapova 2017a, 2018; Le Coz et al. 2008) for an application of this strategy in the study of standing wave solutions for the nonlinear Schrödinger equation on the all line and on star graphs). Therefore we will give a sketch of the main points of these analysis. Indeed,

i) It is not difficult to see the convergence $\phi_Z = (\phi_-, \phi_+) \to \Phi_0$, as $Z \to 0$, in $H^1(\mathcal{G})$.

ii) The family $\{E_Z\}_{Z\in\mathbb{R}}$ represents a real-analytic family of self-adjoint operators of type (B) in the sense of Kato (1966).

iii) Since E_Z converges to E_0 as $Z \to 0$ in the generalized sense, we obtain from Theorem IV-3.16 from Kato (1966) and from Kato-Rellich Theorem ((Reed and Simon 1978), Theorem XII.8) the existence of two analytic functions Ω, Π defined in a neighborhood of zero with $\Omega : (-Z_0, Z_0) \to \mathbb{R}$ and $\Pi : (-Z_0, Z_0) \to L^2(\mathcal{G})$ such that $\Omega(0) = 0$ and $\Pi(0) = \Phi_0'$. For all $Z \in (-Z_0, Z_0)$, $\Omega(Z)$ is the simple isolated second eigenvalue of E_Z, and $\Pi(Z)$ is the associated eigenvector for $\Omega(Z)$. Moreover, Z_0 can be chosen small enough to ensure that for $Z \in (-Z_0, Z_0)$ the spectrum of E_Z in $L^2(\mathcal{G})$ is positive, except at most the first two eigenvalues.

iv) If λ is an simple eigenvalue for E_Z then the eigenfunction associated is either even or odd. Therefore since $N\Phi_0$ is odd we have $\Pi(Z) \in H^2(\mathbb{R})$ and it is a odd function for $Z \in (-\infty, \infty)$. Thus we obtain that

$$\langle N\phi_Z, \Pi(Z)\rangle \neq 0, \quad Z \in \mathbb{R}. \tag{9.36}$$

Indeed, since $\lim_{Z\to 0}\langle N\phi_Z, \Pi(Z)\rangle = \|N\Phi_0\|^2 > 0$, we have for Z small property (9.36). Thus, an continuation argument shows (9.36).

v) From Taylor's theorem there exists $0 < Z_1 < Z_0$ such that $\Omega(Z) > 0$ for any $Z \in (-Z_1, 0)$, and $\Omega(Z) < 0$ for any $Z \in (0, Z_1)$. Thus, in the space $L^2(\mathcal{G})$ for Z small, we have $n(E_Z) = 1$ as $Z < 0$, and $n(E_Z) = 2$ as $Z > 0$.

vi) Recall that $ker(E_Z) = \{0\}$ for $Z \neq 0$. Thus, we define Z_∞ by

$$Z_\infty = \sup\{\widetilde{Z} > 0 : E_Z \text{ has exactly two negative eigenvalues for all } Z \in (0, \widetilde{Z})\}. \tag{9.37}$$

Item iv) above implies that Z_∞ is well defined and $Z_\infty \in (0, \infty]$. We claim that $Z_\infty = \infty$. Suppose that $Z_\infty < \infty$. Let $M = n(E_{Z_\infty})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(E_{Z_\infty})$, and all the negative eigenvalues of E_{Z_∞} belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to E_{Z_∞}.

Next, from item ii) above follows that there is $\epsilon > 0$ such that for $Z \in [Z_\infty - \epsilon, Z_\infty + \epsilon]$ we have $\Gamma \subset \rho(E_Z)$ and for $\xi \in \Gamma$, $Z \to (E_Z - \xi I_d)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $Z \to P(Z)$ given by

$$P(Z) = -\frac{1}{2\pi i}\oint_\Gamma (E_Z - \xi I_d)^{-1} d\xi$$

implies that (see Lemma C.3 in Appendix C)

$$\dim(Im(P(Z))) = \dim(Im(P(Z_\infty))) = M$$

for all $Z \in [Z_\infty - \epsilon, Z_\infty + \epsilon]$. Next, by definition of Z_∞, $E_{Z_\infty - \epsilon}$ has two negative eigenvalues, and $M = 2$, hence E_Z has two negative eigenvalues for $Z \in (0, Z_\infty + \epsilon]$, which contradicts with the definition of Z_∞. Therefore, $Z_\infty = \infty$.

Analogously we can prove that $n(E_Z) = 1$ in the case $Z < 0$. This finishes the proof.

□

The following lemma shows assumption S_6) in the case $n(E) = 2$. Initially, the profiles $\phi_\pm$ in (7.24) represent a differentiable family of stationary solutions a one-parameter $\omega = -\beta_+ > 0$. Thus we will denote it dependence as $\phi_\omega = (\phi_{-,\omega}, \phi_{+,\omega})$. From (9.5) we obtain after derivation in ω that

$$\mathcal{L}_\pm\Big(\frac{d}{d\omega}\phi_{\pm,\omega}\Big) \equiv \Big(-\frac{d^2}{dx^2} + \omega - 2\phi_{\pm,\omega}\Big)\Big(\frac{d}{d\omega}\phi_{\pm,\omega}\Big) = -\phi_{\pm,\omega}. \tag{9.38}$$

Next, by denoting $\psi_\omega = (-\frac{d}{d\omega}\phi_{-,\omega}, -\frac{d}{d\omega}\phi_{+,\omega})$ is not difficult to see that $\psi_\omega \in D(E_Z)$ and so we can assure that for Z, ω fixed, that the expression $E_Z\psi_\omega = \phi_\omega$ makes sense.

Lemma 9.3. *Let $Z \neq 0$. The smooth curve of profiles $\omega \in (\frac{Z^2}{4}, +\infty) \to \phi_\omega \in D(E_Z)$ with formula (7.24) satisfies for $\psi_\omega \equiv -\frac{d}{d\omega}\phi_\omega$ the relations*

$$E_Z\psi_\omega = \phi_\omega, \quad \textit{and}, \quad \langle\psi_\omega, \phi_\omega\rangle < 0. \tag{9.39}$$

Proof. From Proposition 3.19 in Angulo and Goloshchapova (2018) (item (ii), $p = 2$) we have for every $Z \in \mathbb{R}$, the relation $\langle\frac{d}{d\omega}\phi_\omega, \phi_\omega\rangle > 0$ and so $\langle\psi_\omega, \phi_\omega\rangle < 0$. This finishes the proof. □

Proof of Theorem 9.3. Let $Z \neq 0$. From Lemmas 9.1, 9.2 and 9.3, and (9.36), follows from Theorem 9.1 that the profiles of type tail and bump for the KdV are linear unstable. This finishes the proof. □

9.4 Linear instability of tail and bump on balanced star graphs

We consider the KdV model (9.1) on a metric star graph $\mathcal{G}$ with a structure $\mathrm{E} \equiv \mathrm{E}_- \cup \mathrm{E}_+$ where $|\mathrm{E}_+| = |\mathrm{E}_-| = n$, $n \geqq 2$, and with a δ-interaction at the vertex. Thus, from Proposition (4.3) we consider the skew-self-adjoint family $(H_Z, D(H_Z))$ of extensions for $(A_0, D(A_0))$ defined in (4.24). Thus, for for $u = (u_e)_{e\in E} \in D(H_Z)$ we obtain the following system of conditions

$$\begin{aligned} u(0-) = u(0+), \quad u'(0+) - u'(0-) = Zu(0-),\\ \frac{Z^2}{2}u(0-) + Zu'(0-) = u''(0+) - u''(0-). \end{aligned} \tag{9.40}$$

Now, for $Z > 0$, $0 > \beta_+$ and $-\beta_+ > \frac{Z^2}{4}$ we consider the half-soliton profile ϕ_+ defined in (7.24) and $\phi_-(x) \equiv \phi_+(-x)$ for $x < 0$. We define the constants sequences of functions

$$u_- = (\phi_-)_{e\in E_-},\ u_+ = (\phi_+)_{e\in E_+},$$

and so $U_{Z,\omega} = (u_-, u_+)$ represents a family of stationary bump profiles for the KdV model in (9.1) (see 7.5) and satisfying the boundary conditions (9.40). The case $Z < 0$, $U_{Z,\omega}$ represents the corresponding family of stationary tail profiles (see Figure 7.4).

With the notations above, the main result of this section is the following.

Theorem 9.4. *Let $Z \neq 0$. For $\alpha_+ > 0$ and $0 > \beta_+$, $\omega = -\beta_+ > \frac{Z^2}{4}$, we consider the profiles $\phi_\pm$ in* (7.24). *Define $U_{Z,\omega} = (\phi_e)_{e\in E} \in D(H_Z)$ with $\phi_e = \phi_-$ for $e \in E_-$ and $\phi_e = \phi_+$ for $e \in E_+$. Then,*

$$\Phi_{Z,\omega}(x,t) = U_{Z,\omega}(x)$$

defines a family of linearly unstable stationary solutions for the Korteweg–de Vries model (9.1).

The linear instability of the *continuous tail and bump profile $U_{Z,\omega}$*, $Z \neq 0$, it will be a consequence of Theorems 9.1 with a framework determined by the space $D(H_Z) \cap \mathcal{C}$ where

$$\begin{aligned} \mathcal{C} = \{(u_e)_{e\in E} \in L^2(\mathcal{G}) : u_{1,-}(0-) = \dots = u_{n,-}(0-) = u_{1,+}(0+)\\ = \dots = u_{n,+}(0+)\}. \end{aligned} \tag{9.41}$$

Thus, by following the notation in section 5 ($(\alpha_e)_{e\in E} = (1)_{e\in E}$, $(\beta_e)_{e\in E} = (-1)_{e\in E}$, without loss of generality) we start our analysis by considering the $2n \times 2n$-matrix derivate operator N in (9.13) and the $2n \times 2n$-matrix Schrödinger operator

$$\mathcal{E}_Z = \begin{pmatrix} \mathcal{L}_{Z,-} & 0 \\ 0 & \mathcal{L}_{Z,+} \end{pmatrix}. \tag{9.42}$$

with

$$\mathcal{L}_{Z,\pm} = \mathrm{diag}\Big(-\frac{d^2}{dx^2} + 1 - 2\phi_\pm, \dots, -\frac{d^2}{dx^2} + 1 - 2\phi_\pm\Big), \tag{9.43}$$

being $n \times n$-diagonal matrices.

From Theorem 3.6 and from Krein-von Neumann extension theory $\mathcal{E}_Z$ is self-adjoint with domain $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C} \subset H^2(\mathcal{G})$ with

$$u \in D_{Z,\delta} \Leftrightarrow u(0-) = u(0+), \sum_{\mathrm{e}\in \mathrm{E}_+} u'_{\mathrm{e}}(0+) - \sum_{\mathrm{e}\in \mathrm{E}_-} u'_{\mathrm{e}}(0-) = Znu_{1,+}(0+). \tag{9.44}$$

It is immediate from (9.40) that $D(H_Z) \cap \mathcal{C} \subset D(\mathcal{E}_Z)$ and so assumption S_3) holds. From Remark 4.2-item 2) we obtain again assumption S_4). Assumption S_7) is immediate by the definition of $D(H_Z) \cap \mathcal{C}$. Moreover, from Angulo and Cavalcante (2019) we have that subspace $D(H_Z) \cap \mathcal{C}$ is invariant by the unitary group $\{W(t)\}_{t\in\mathbb{R}}$ generated by H_Z.

The proof of the following result follows the same strategy as in Lemma 9.1.

Lemma 9.4. *Let $Z \neq 0$ and the operator $\mathcal{E}_Z : D(\mathcal{E}_Z) \to L^2(\mathcal{G})$ defined in (9.42) with $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C}$. Then, $\mathcal{E}_Z$ is invertible with $\sigma_{ess}(\mathcal{E}_Z) = [1, +\infty)$.*

Proposition 9.2. *Let $\mathcal{E}_Z : D(\mathcal{E}_Z) \to L^2(\mathcal{G})$ defined in (9.42) with $D(\mathcal{E}_Z) = D_{Z,\delta} \cap \mathcal{C}$. Define the following closed subspace on $L^2(\mathcal{G})$,*

$$L^2_n(\mathcal{G}) = \{u = (u_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} : u_{\mathrm{e}} = f, \textit{for all } \mathrm{e} \in \mathrm{E}_-, u_{\mathrm{e}} = g, \textit{ for all } \mathrm{e} \in \mathrm{E}_+\}$$

Then, $n(\mathcal{E}_Z|_{L^2_n(\mathcal{G})}) = 2$, for $Z > 0$, and $n(\mathcal{E}_Z|_{L^2_n(\mathcal{G})}) = 1$, for $Z < 0$.

The proof of Proposition 9.2 follows from the perturbation theory and the extension theory of symmetric operator (Angulo and Cavalcante 2019). We note that in the case $Z < 0$ (tail case) can be given an argument based exclusively in the extension theory of symmetric operator. Moreover, in this case is obtained that $n(\mathcal{E}_Z) = 1$ in $L^2(\mathcal{G})$.

The proof of Proposition 9.2 will be divide in several lemmas.

Lemma 9.5. *Define the self-adjoint matrix Schrödinger operator in $L^2(\mathcal{G})$ with Kirchhoff's type condition at $\nu = 0$*

$$\mathcal{E}_0 = \begin{pmatrix} \mathcal{L}_{0,-} & 0 \\ 0 & \mathcal{L}_{0,+} \end{pmatrix} \tag{9.45}$$

with

$$\mathcal{L}_{0,\pm} = diag\Big(-\frac{d^2}{dx^2} + 1 - 2\phi_0, ..., -\frac{d^2}{dx^2} + 1 - 2\phi_0\Big), \tag{9.46}$$

being $n \times n$-diagonal matrices, ϕ_0 the soliton defined in (9.32), and

$$D(\mathcal{E}_0) = \{u \in H^2(\mathcal{G}) : u(0-) = u(0+), \sum_{\mathrm{e}\in \mathrm{E}_+} u'_{\mathrm{e}}(0+) - \sum_{\mathrm{e}\in \mathrm{E}_-} u'_{\mathrm{e}}(0-) = 0\}. \tag{9.47}$$

1) *In the space $L^2_n(\mathcal{G})$ we have $ker(\mathcal{E}_0) = [\Phi'_0]$, where $\Phi'_0 = (\phi'_0)_{\mathrm{e}\in\mathrm{E}}$.*
2) *The operator $(\mathcal{E}_0, D(\mathcal{E}_0))$ has one simple negative eigenvalue in $L^2(\mathcal{G})$. Moreover, we have also $n(\mathcal{E}_0|_{L^2_n(\mathcal{G})}) = 1$.*

3) The rest of the spectrum of $\mathcal{E}_0$ is positive and bounded away from zero.

Proof. The proof of item 1) follows from a similar analysis as in Lemma 9.1. Indeed, let $v = (v_e)_{e\in E} \in ker(\mathcal{E}_0) \cap L^2_n(\mathcal{G})$, then

$$-v_e'' + v_e - 2\phi_0 v_e = 0, \quad e \in E. \tag{9.48}$$

Then, $v_e = c_e\phi_0'$ for $e \in E$ and so $v_e(0-) = v_e(0+) = 0$. Now, since $v \in L^2_n(\mathcal{G})$, we obtain for e $\in E_-$ that $v_e = c_0\phi_0'$ with $c_0 = c_e$, and for e $\in E_+$ that $v_e = c_1\phi_0'$ with $c_1 = c_e$. Then from (9.47) we obtain $nc_1\phi_0''(0) = nc_0\phi_0''(0)$. Therefore, $v = c_0\Phi_0'$.

For item 2), we used Theorem 3.6, von Neumann and Krein extension theory and Theorem 3.11. This finishes the proof. □

Remark 9.2. *We observe that, when we deal with deficiency indices, the operator $\mathcal{E}_0$ is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of $\mathcal{E}_0$ acting on real-valued functions.*

Combining Lemma 9.5 and the framework of the perturbation theory as in Lemma 9.2 (see (Angulo and Goloshchapova 2017a)) we obtain the following Lemma. We note that for $u_- = (\phi_-)_{e\in E_-}$, $u_+ = (\phi_+)_{e\in E_+}$, it is not difficult to see the convergence $U_{Z,\omega} = (u_-, u_+) \to \Phi_0 = (\phi_0)_{e\in E}$, as $Z \to 0$, in $H^1(\mathcal{G}) \cap L^2_n(\mathcal{G})$.

Lemma 9.6. *There exist $Z_0 > 0$ and two analytic functions $\Theta : (-Z_0, Z_0) \to \mathbb{R}$ and $\Upsilon : (-Z_0, Z_0) \to L^2_n(\mathcal{G})$ such that*

(i) *$\Theta(0) = 0$ and $\Upsilon(0) = \Phi_0'$, where $\Phi_0' = (\phi_0')_{e\in E}$.*

(ii) *For all $Z \in (-Z_0, Z_0)$, $\Theta(Z)$ is the simple isolated second eigenvalue of $\mathcal{E}_Z$ in $L^2_n(\mathcal{G})$, and $\Upsilon(Z)$ is the associated eigenvector for $\Theta(Z)$.*

(iii) *Z_0 can be chosen small enough to ensure that for $Z \in (-Z_0, Z_0)$ the spectrum of E_Z in $L^2_n(\mathcal{G})$ is positive, except at most the first two eigenvalues.*

(iv) *Since $\lim_{Z\to 0}\langle NU_{Z,\omega}, \Pi(Z)\rangle = \|N\Phi_0\|^2 > 0$ we obtain that*

$$\langle NU_{Z,\omega}, \Pi(Z)\rangle \neq 0, \tag{9.49}$$

at least for Z small. Thus, an continuation argument shows (9.49) for all Z.

By using the Taylor's theorem and by following a similar argument as in Proposition 3.9 in (Angulo and Goloshchapova 2017a) we establish how the perturbed second eigenvalue moves depending on the sign of Z.

Proposition 9.3. *There exists $0 < Z_1 < Z_0$ such that $\Theta(Z) > 0$ for any $Z \in (-Z_1, 0)$, and $\Theta(Z) < 0$ for any $Z \in (0, Z_1)$. Thus, in the space $L^2_n(\mathcal{G})$ for Z small, we have $n(\mathcal{E}_Z) = 1$ as $Z < 0$, and $n(\mathcal{E}_Z) = 2$ as $Z > 0$.*

Proof of Proposition 9.2. From Proposition 9.3 we have for Z small that $n(\mathcal{E}_Z) = 1$ as $Z < 0$, and $n(\mathcal{E}_Z) = 2$ as $Z > 0$. Thus for counting the Morse index of $\mathcal{E}_Z$ for any Z we use a classical continuation argument based on the Riesz-projection as in step vi)-proof of Lemma 9.2- and Lemma 9.4. This finishes the proof. □

The following lemma shows assumption S_6) (case $Z > 0$). Similarly to the case of two half-lines we have the smooth curve $\omega \to \phi_\omega = (\phi_{-,\omega}, \phi_{+,\omega})_{\mathrm{e}\in\mathrm{E}}$, for $\omega = -\beta_+ > 0$ such that $\mathcal{L}_{Z,\pm}\left(\frac{d}{d\omega}\phi_{\pm,\omega}\right) = -\phi_{\pm,\omega}$. Thus we obtain the following result.

Lemma 9.7. *The smooth curve of profiles* $\omega \in (\frac{Z^2}{4}, +\infty) \to \phi_\omega \in D(\mathcal{E}_Z) \cap L^2_n(\mathcal{G})$ *satisfies for* $\psi_\omega \equiv -\frac{d}{d\omega}\phi_\omega$ *the relations*

$$\mathcal{E}_Z \psi_\omega = \phi_\omega, \quad \textit{and}, \quad \langle \psi_\omega, \phi_\omega \rangle < 0. \tag{9.50}$$

Proof of Theorem 9.4. Let $Z \neq 0$. From Lemmas 9.4 and 9.7, Proposition 9.2, relation (9.49) and Theorem 9.1 we obtain the linear instability property of the profiles tail and bump $U_{Z,\omega}$ for the KdV model (9.1). This finishes the proof. □

Remark 9.3. *1) Theorem 9.3 about the linear instability of the tail and bump profiles on two-half-lines for KdV model, it shows the delicate dynamic of these profiles. We note that in the case of the nonlinear Schrödinger equation on all the line, the dynamic of these type of profiles is well known and it is completely different (Angulo and Goloshchapova 2018; Angulo and Ponce 2013; Fukuizumi and Jeanjean 2008; Fukuizumi, Ohta, and Ozawa 2008; Goodman, Holmes, and Weinstein 2004; Le Coz et al. 2008, see).*

2) The existence and stability of other families of stationary solutions profiles for the KdV model (9.1) and their generalizations (for instance, the modified KdV) defined on a different graph geometry is being the goal of some works in progress. As well as, the study of the local well-posedness for the Cauchy problem.

10 (In)Stability for the NLS equation on Star Graphs

In this chapter we study the nonlinear stability of standing waves solutions for the following vectorial nonlinear Schrödinger equation on a star graph $\mathcal{G}$,

$$i\partial_t \mathrm{U}(t,x) - \mathcal{A}\mathrm{U}(t,x) + |\mathrm{U}(t,x)|^{p-1}\mathrm{U}(t,x) = 0, \quad x > 0 \tag{10.1}$$

where $\mathrm{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}^N$, and $p > 1$. The nonlinearity acts componentwise, i.e. $(|\mathrm{U}|^{p-1}\mathrm{U})_j = |u_j|^{p-1}u_j$. Here, we will consider the star graph $\mathcal{G}$ being composed by N positive half-lines attached to the common vertex $\nu = 0$, and $\mathcal{A}$ is a self-adjoint operator with $D(\mathcal{A}) \subset L^2(\mathcal{G})$ which represents the coupling conditions in the graph-vertex (see section 4.1).

In the case of $\mathcal{A}$ representing the Laplace operator with boundary conditions of type δ, we have that $\mathcal{A} \equiv \mathrm{H}_\alpha^\delta$ with domain $D(\mathrm{H}_\alpha^\delta) = \mathbb{D}_{\alpha,\delta}$ is acting for $\mathrm{V} = (v_j)_{j=1}^N$ as

$$\begin{aligned} &(\mathrm{H}_\alpha^\delta \mathrm{V})(x) = (-v_j''(x))_{j=1}^N, \quad x > 0, \\ &\mathbb{D}_{\alpha,\delta} = \Big\{ \mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = \ldots = v_N(0),\ \sum_{j=1}^N v_j'(0) = \alpha v_1(0) \Big\}. \end{aligned} \tag{10.2}$$

We recall that quantum graphs (metric graphs equipped with a Hamiltonian linear evolution equation) have been a very developed subject in the last couple of decades.

They give simplified models in mathematics, physics, chemistry, and engineering, when one considers propagation of waves of various type through a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin neighborhood of a graph. Moreover, equation (10.1) models propagation, for instance, through junctions in nonlinear optics, Bose-Einstein condensates (BEC) on networks (see Berkolaiko and Kuchment 2013; Brazhnyi and Konotop 2004; Burioni et al. 2001; Cacciapuoti, Finco, and Noja 2017; Cao and Malomed 1995; Fidaleo 2015; Kuchment 2004; Mugnolo 2015; Noja 2014, and references therein).

The analysis of the behavior of NLS equation on general networks is not yet fully developed, but it is currently growing (see Adami, Cacciapuoti, et al. (2014b), Angulo and Goloshchapova (2017a, 2018), Ardila (2017), Banica and Ignat (2014), and Noja (2014) and references therein).

Various recent analytical works (see Adami, Cacciapuoti, et al. 2014b; Angulo and Goloshchapova 2017a, 2018; Noja 2014, and references therein) deal with special solutions of (10.1) called *standing wave solutions*, i.e. the solutions of the form (see Chapter 6)

$$\mathrm{U}(t,x) = e^{i\omega t}\Phi(x), \tag{10.3}$$

with the profile Φ satisfying specific coupling conditions in the vertex $\nu = 0$. In the case of a δ-interaction condition, namely, $\Phi \in \mathbb{D}_{\alpha,\delta}$ (see (10.2)) in (Adami, Cacciapuoti, et al. 2014b) was obtained a complete description of the profiles Φ for any $\alpha \in \mathbb{R}$ such as was established in Theorem 7.1 of Chapter 7. Here, we are interested in the stability investigation of all that $[\frac{N-1}{2}] + 1$ standing wave solutions in each case of $\alpha < 0$ and $\alpha > 0$, respectively.

Since our stability approach is based in the classical theory of Grillakis, Shatah, and W. Strauss (1990), in the next section we will give the corresponding stability framework by convenience of the reader.

10.1 Stability framework for the NLS on star graphs

The NLS model (10.1) is invariant under the rotation-symmetry of the group $T(\theta)\Psi = e^{i\theta}\Psi$, for any $\theta \in [0, 2\pi)$, namely, if U is a solution of (10.1) then $e^{i\theta}\mathrm{U}$ is also a solution. Thus, the standing wave solutions in (10.3) can be write as $\mathrm{U}(t,x) = T(\omega t)\Phi(x)$. We note that the classical translation-symmetry does not hold on $\mathcal{G}$. Thus, we have the following orbital stability definition.

Definition 10.1. *The standing wave* $\mathrm{U}(t,x) = e^{i\omega t}\Phi(x)$ *for model (10.1) is said to be orbitally stable in a Hilbert space* X *if for any* $\varepsilon > 0$ *there exists* $\eta > 0$ *with the following property: if* $\mathrm{U}_0 \in X$ *satisfies* $||\mathrm{U}_0 - \Phi||_X < \eta$*, then the solution* $\mathrm{U}(t)$ *of (10.1) with* $\mathrm{U}(0) = \mathrm{U}_0$ *exists for any* $t \in \mathbb{R}$ *and*

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} ||\mathrm{U}(t) - e^{i\theta}\Phi||_X < \varepsilon.$$

Otherwise, the standing wave $\mathrm{U}(t,x) = e^{i\omega t}\Phi(x)$ *is said to be orbitally unstable in* X*.*

In particular, for the NLS model (10.1) with a boundary condition of δ-type (see (10.2)) the space X coincides with the continuous energy-space $\mathcal{E}(\mathcal{G})$,

$$\mathcal{E}(\mathcal{G}) = \{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(0) = ... = v_N(0)\}. \tag{10.4}$$

Next, we assume the existence of a $C^2(X, \mathbb{R})$-conserved functional $E : X \to \mathbb{R}$ (interpreted as the "energy" in certain applications) i.e., for $\mathrm{V} = (v_j)_{j=1}^N \in X$

$$E(\mathrm{U}(t)) = E(\mathrm{U}_0), \quad \text{for } \ t \in [-T, T], \tag{10.5}$$

and $Q : L^2(\mathcal{G}) \to \mathbb{R}$ (interpreted as the "charge" in certain applications) defined by $Q(\mathrm{V}) = ||\mathrm{V}||^2$ also a conserved functional, i.e., $Q(\mathrm{U}(t)) = ||\mathrm{U}(t)||^2$, for $t \in [-T, T]$. Moreover, we also assume that E is invariant under T (obviously Q satisfies this property); that is

$$E(T(\theta)\mathrm{V}) = E(\mathrm{V}), \quad \text{for } \ \theta \in [0, 2\pi), \quad \mathrm{V} \in X.$$

Now, by substituting the standing wave profile in (10.3) with $\Phi \in D(\mathcal{A})$ we arrive to the nonlinear system

$$\mathcal{A}\Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0. \tag{10.6}$$

The equality in (10.6) should be understood in a distributional sense. We suppose that the vector Φ is a critical point of the action functional $\mathrm{S} = E + \omega Q$. For a stability study of Φ a main information will be given by the the second variation of S at Φ, $\mathrm{S}''(\Phi)$. We suppose that for $\mathrm{U} = \mathrm{U}_1 + i\mathrm{U}_2$ and $\mathrm{V} = \mathrm{V}_1 + i\mathrm{V}_2$, where the vector functions $\mathrm{U}_j, \mathrm{V}_j,\ j \in \{1, 2\}$, are assumed to be real valued, we have the following equality

$$\mathrm{S}''(\Phi)(\mathrm{U}, \mathrm{V}) = \langle \mathrm{L}_1\mathrm{U}_1, \mathrm{V}_1\rangle + \langle \mathrm{L}_2\mathrm{U}_2, \mathrm{V}_2\rangle, \tag{10.7}$$

here $\langle\cdot,\cdot\rangle$ represents for us the inner product in $L^2(\mathcal{G})$, and L_i are self-adjoint operators with $D(\mathrm{L}_i) = D(\mathcal{A}) \subset L^2(\mathcal{G})$.

Formally $\mathrm{S}''(\Phi)$ can be considered as a self-adjoint $2N \times 2N$ matrix operator

$$\mathrm{H} = \begin{pmatrix} \mathrm{L}_1 & 0 \\ 0 & \mathrm{L}_2 \end{pmatrix}. \tag{10.8}$$

Next, we suppose the existence of C^1 in ω standing wave solutions for (10.6), $\omega \in J \subset \mathbb{R} \to \Phi_\omega$. Define

$$p(\omega_0) = \begin{cases} 1 & \text{if } \partial_\omega||\Phi_\omega||^2 > 0 \text{ at } \omega = \omega_0, \\ 0 & \text{if } \partial_\omega||\Phi_\omega||^2 < 0 \text{ at } \omega = \omega_0. \end{cases}$$

Lastly, we suppose the well-posedness of the associated Cauchy problem for (10.1) in the energy space X. The next stability/instability result follows from (Grillakis, Shatah, and W. Strauss 1990).

Theorem 10.1. *Let $n(\mathrm{H})$ be the number of negative eigenvalues of* H *(the Morse index). Suppose also that*

1) $ker(\mathrm{L}_2) = span\{\Phi_\omega\}$,

2) $ker(\mathrm{L}_1) = \{0\}$,

3) *the Morse index of* L_1 *and* L_2 *consists of a finite number of negative eigenvalues (counting multiplicities),*

4) *the rest of the spectrum of* L_1 *and* L_2 *is positive and bounded away from zero. Then the following assertions hold.*

(i) *If* $n(\mathrm{H}) = p(\omega) = 1$, *then the standing wave* $e^{i\omega t}\Phi_\omega$ *is orbitally stable in the energy space* X.

(ii) *If* $n(\mathrm{H}) - p(\omega)$ *is odd, then the standing wave* $e^{i\omega t}\Phi_\omega$ *is orbitally unstable in the energy space* X.

Remark 10.1. The instability part of the above theorem needs some additional comments.

(i) It is known from (Grillakis, Shatah, and W. Strauss 1990) that when $n(\mathrm{H}) - p(\omega)$ is odd, we obtain only spectral instability of $e^{i\omega t}\Phi_\omega$. To obtain orbital instability due to (Grillakis, Shatah, and W. Strauss 1990, Theorem 6.1), it is sufficient to show estimate (6.2) in (Grillakis, Shatah, and W. Strauss 1990) for the semigroup e^{tA} generated by

$$A = \begin{pmatrix} 0 & \mathrm{L}_2 \\ -\mathrm{L}_1 & 0 \end{pmatrix}.$$

In the case of Schrödinger models on star graphs it is not clear how to prove estimate (6.2).

(ii) When $n(\mathrm{H}) = 2$ (which usually happens in many applications), we can apply the results by Ohta (2011, Corollary 3 and 4) to get the instability part of the above Theorem. We note that in this case the orbital instability follows without using spectral instability.

(iii) Generally, to imply the orbital instability from the spectral one, the approach by Henry, Perez, and Wreszinski (1982) can be used (see Theorem 2). The key point of this method is to use the fact that the mapping data-solution associated to the model is of class C^2. We note in particular, for the NLS-δ (and NLS-δ') model (10.1)-(10.2) the mapping data-solution is of class C^2 as $p > 2$ (see Theorem 6.1). The approach by Henry, Perez, and Wreszinski (1982) have been applied successfully in (Angulo, Lopes, and Neves 2008) and (Angulo and Natali 2016) for models of KdV-type.

10.2 Stability theory for the NLS-δ on star graphs

In this section we study the orbital stability of the standing wave $\mathrm{U}(t,x) = e^{i\omega t}\Phi(x)$ of the Schrödinger model (10.1) for the case of $\mathcal{A} = \mathrm{H}_\alpha^\delta$ defined in (10.2) (henceforth, the NLS-δ equation) with the profile $\Phi \equiv \Phi_m^\alpha,\ m = 0, ..., \left[\frac{N-1}{2}\right]$ determined by the formulas in Theorem 7.1. We recall that in the case $\alpha < 0$, vector Φ_m^α has m *bumps* and $N - m$

tails (see Figure 7.1). Φ_0^α is the *N-tail profile* which is the only symmetric (i.e. invariant under permutations of the edges) solution of equation

$$\mathrm{H}_\alpha^\delta \Phi + \omega \Phi - |\Phi|^{p-1}\Phi = 0. \tag{10.9}$$

In the case $\alpha > 0$, vector Φ_m^α has m *tails* and $N-m$ *bumps* respectively (see Figure 7.2). Φ_0^α is the *N-bump profile* which is the only symmetric solution of equation (10.9).

We will investigate orbital stability in the energy space $X = \mathcal{E}(\mathcal{G})$ defined in (10.4). Thus the functional $E_\alpha : \mathcal{E}(\mathcal{G}) \to \mathbb{R}$ defined for $\mathrm{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$ by

$$E_\alpha(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}\,|v_1(0)|^2\,. \tag{10.10}$$

is well defined by the Sobolev embedding theorem and Gagliardo-Nirenberg inequality (6.13). Thus, by using Theorem 6.1 (continuous dependence property) follows that E_α is a conservation law for the NLS-δ and so as $Q(\mathrm{V}) = \|\mathrm{V}\|^2$. Moreover, for the action $\mathrm{S}_\alpha = E_\alpha + \omega Q$ follows from (10.9) the critical point property of Φ_m^α, $\mathrm{S}'_\alpha(\Phi_m^\alpha) = 0$, for any $\alpha \neq 0$ and $m = 0, ..., \left[\frac{N-1}{2}\right]$. Also, for $\Phi_m^\alpha = (\varphi_{m,j})_{j=1}^N$, we consider the following two self-adjoint diagonal matrix operators

$$\begin{aligned} \mathrm{L}_{1,m,\alpha} &= \Big(\Big(-\frac{d^2}{dx^2} + \omega - p(\varphi_{m,j})^{p-1}\Big)\delta_{i,j}\Big),\\ \mathrm{L}_{2,m,\alpha} &= \Big(\Big(-\frac{d^2}{dx^2} + \omega - (\varphi_{m,j})^{p-1}\Big)\delta_{i,j}\Big),\\ D(\mathrm{L}_{1,m,\alpha}) &= D(\mathrm{L}_{2,m,\alpha}) = \mathbb{D}_{\alpha,\delta}, \end{aligned} \tag{10.11}$$

where $\delta_{i,j}$ is the Kronecker symbol. The operators $\mathrm{L}_{i,m,\alpha}$ are associated with the second variation $\mathrm{S}''_\alpha(\Phi_m^\alpha)$ and satisfy the relation in (10.7).

It was shown in (Adami, Cacciapuoti, et al. 2014b) that for $-N\sqrt{\omega} < \alpha < \alpha^* < 0$, the vector tail-solution $\Phi_0^\alpha = (\varphi_{0,j})_{j=1}^N$, with $\varphi_{0,j} = \varphi_{0,\alpha}$ for all j and

$$\varphi_{0,\alpha}(x) = \Big[\frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\Big(\frac{-\alpha}{N\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p-1}} \tag{10.12}$$

it is the ground state. The parameter α^* above originates from the variational problem associated with equation (10.9), and it guarantees the minimality of the action functional

$$\mathrm{S}_\alpha(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 + \tfrac{\omega}{2}||\mathrm{V}||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{\alpha}{2}|v_1(0)|^2, \tag{10.13}$$

for $\mathrm{V} = (v_j)_{j=1}^N \in \mathcal{E}(\mathcal{G})$, at Φ_0^α with the constraint given by the Nehari manifold

$$\mathcal{N} = \{\mathrm{V} \in \mathcal{E}(\mathcal{G}) \setminus \{0\} : ||\mathrm{V}'||^2 + \omega||\mathrm{V}||^2 - ||\mathrm{V}||_{p+1}^{p+1} + \alpha|v_1(0)|^2 = 0\}.$$

We will see below that when the profile Φ_m^α has mixed structure (i.e. has bumps and tails), they are "almost always" unstable. More exactly, for the space

$$L^2_m(\mathcal{G}) = \{\mathrm{V} = (v_j)_{j=1}^N \in L^2(\mathcal{G}) : v_1(x) = ... = v_m(x),\\ v_{m+1}(x) = ... = v_N(x),\ x > 0\}, \tag{10.14}$$

and $\mathcal{E}_m(\mathcal{G}) = \mathcal{E}(\mathcal{G}) \cap L^2_m(\mathcal{G})$, we obtain the following orbital stability/instability of the excited states. The case of tail and bump profiles ($m = 0$) will be study separately.

Theorem 10.2. *Let $\alpha \neq 0$, $m \in \{1, ..., [\frac{N-1}{2}]\}$, and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Let also the profile Φ_m^α be defined by (7.6), we consider the spaces $\mathcal{E} = \mathcal{E}(\mathcal{G})$ and $\mathcal{E}_m = \mathcal{E}_m(\mathcal{G})$. Then the following assertions hold.*

(i) Let $\alpha < 0$, then

1) for $1 < p \leqslant 5$ the standing wave $e^{i\omega t}\Phi_m^\alpha$ is orbitally unstable in $\mathcal{E}$;

2) for $p > 5$ there exists $\omega_m^* > \frac{\alpha^2}{(N-2m)^2}$ such that the standing wave $e^{i\omega t}\Phi_m^\alpha$ is orbitally unstable in $\mathcal{E}$ as $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \omega_m^*)$.

(ii) Let $\alpha > 0$, then

1) for $1 < p \leqslant 3$ the standing wave $e^{i\omega t}\Phi_m^\alpha$ is orbitally stable in $\mathcal{E}_m$;

2) for $3 < p < 5$ there exists $\hat{\omega}_m > \frac{\alpha^2}{(N-2m)^2}$ such that the standing wave $e^{i\omega t}\Phi_m^\alpha$ is orbitally unstable in $\mathcal{E}$ as $\omega \in (\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m)$, and $e^{i\omega t}\Phi_m^\alpha$ is orbitally stable in $\mathcal{E}_m$ as $\omega \in (\hat{\omega}_m, \infty)$;

3) for $p \geqslant 5$ the standing wave $e^{i\omega t}\Phi_m^\alpha$ is orbitally unstable in $\mathcal{E}$.

In the case of $p > 5$, $\alpha < 0$, and $\omega > \omega_m^*$ our approach does not provide any information about the stability of the excited states Φ_m^α. The proof of Theorem 10.2 is based on the extension theory of symmetric operators, the analytic perturbations theory, and Weinstein-Grillakis-Shatah-Strauss approach established in Theorem 10.1.

Next we establish the results of stability for the cases of tail and bump profiles.

Theorem 10.3. *Let $\alpha \neq 0$ and Φ_0^α be defined by (10.12).*

(1) **Bump case***: Let $\alpha > 0$, $1 < p < 5$, and $\omega > \frac{\alpha^2}{N^2}$. Then the following assertions hold.*

(i) If $1 < p \leqslant 3$, then $e^{i\omega t}\Phi_0^\alpha$ is orbitally unstable in $\mathcal{E}(\mathcal{G})$.

(ii) If $3 < p < 5$, then there exists $\omega_2 > \frac{\alpha^2}{N^2}$ such that $e^{i\omega t}\Phi_0^\alpha$ is orbitally unstable in $\mathcal{E}(\mathcal{G})$ for $\omega > \omega_2$.

(2) **Tail case**: *Let* $\alpha < 0$ *and* $\omega > \frac{\alpha^2}{N^2}$. *Then the following assertions hold.*

   (i) *If* $1 < p \leqslant 5$, *then* $e^{i\omega t}\Phi_0^\alpha$ *is orbitally stable in* $\mathcal{E}(\mathcal{G})$.

   (ii) *If* $p > 5$, *then there exists* $\omega_1 > \frac{\alpha^2}{N^2}$ *such that* $e^{i\omega t}\Phi_0^\alpha$ *is orbitally stable in* $\mathcal{E}(\mathcal{G})$ *for* $\omega < \omega_1$, *and* $e^{i\omega t}\Phi_0^\alpha$ *is orbitally unstable in* $\mathcal{E}(\mathcal{G})$ *for* $\omega > \omega_1$.

In the bump case, we have that for $p \geqslant 5$ and $3 < p < 5$ (with $\omega \in (\frac{\alpha^2}{N^2}, \omega_2)$) that our method does not provide any information about orbital stability of $e^{i\omega t}\Phi_0^\alpha$.

The proof of Theorems 10.2 and Theorems 10.3 will be developed in the following subsections. We start with the conditions in Theorem 10.1.

10.2.1 Kernel of operators $\mathrm{L}_{i,m,\alpha}$, $i = 1, 2$, in (10.11)

Let the profile $\Phi_m \equiv \Phi_m^\alpha$ be defined by (7.6), including the case $m = 0$ (tail and bump profiles), and we consider the domain $\mathbb{D}_{\alpha,\delta}$ in (10.2).

Proposition 10.1. *Let* $\alpha \neq 0$, $m \in \left\{0, 1, ..., \left[\frac{N-1}{2}\right]\right\}$ *and* $\omega > \frac{\alpha^2}{(N-2m)^2}$. *Then the following assertions hold for* $\mathrm{L}_{i,\alpha} = \mathrm{L}_{i,m,\alpha}$.

(i) $\ker(\mathrm{L}_{2,\alpha}) = span\{\Phi_m\}$ *and* $\mathrm{L}_{2,\alpha} \geqslant 0$.

(ii) $\ker(\mathrm{L}_{1,\alpha}) = \{0\}$.

(iii) *The positive part of the spectrum of the operators* $\mathrm{L}_{i,\alpha}$, $i = 1, 2$, *is bounded away from zero.*

Proof. (i) It is clear that $\Phi_m = (\varphi_{m,j})_{j=1}^N \in \ker(\mathrm{L}_{2,\alpha})$. To show the equality $\ker(\mathrm{L}_{2,\alpha}) = span\{\Phi_m\}$ let us note that any $\mathrm{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - \varphi_{m,j}^{p-1} v_j = \frac{-1}{\varphi_{m,j}}\frac{d}{dx}\left[\varphi_{m,j}^2 \frac{d}{dx}\left(\frac{v_j}{\varphi_{m,j}}\right)\right], \quad x > 0.$$

Thus, for $\mathrm{V} \in \mathbb{D}_{\alpha,\delta}$ we obtain

$$\begin{aligned}\langle \mathrm{L}_{2,\alpha}\mathrm{V}, \mathrm{V}\rangle &= \sum_{j=1}^N \int_0^\infty (\varphi_{m,j})^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_{m,j}}\right)\right|^2 dx \\ &+ \sum_{j=1}^N \left[-v_j' v_j + |v_j|^2 \frac{(\varphi_{m,j})'}{\varphi_{m,j}}\right]_0^\infty = \sum_{j=1}^N \int_0^\infty (\varphi_{m,j})^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_{m,j}}\right)\right|^2 dx \\ &+ \sum_{j=1}^N \left[v_j'(0)v_j(0) - |v_j(0)|^2 \frac{\varphi_{m,j}'(0)}{\varphi_{m,j}(0)}\right].\end{aligned}$$

Using boundary conditions (10.2), we get

$$\sum_{j=1}^{N}\left[v_j'(0)v_j(0)-|v_j(0)|^2\frac{(\varphi_{m,j})'(0)}{\varphi_{m,j}(0)}\right]=\alpha|v_1(0)|^2$$
$$+\sqrt{\omega}|v_1(0)|^2\left[\sum_{j=1}^{m}\tanh(-a_k)+\sum_{j=m+1}^{N}\tanh(a_k)\right]$$
$$=\alpha|v_1(0)|^2+\sqrt{\omega}|v_1(0)|^2(N-2m)\frac{\alpha}{(2m-N)\sqrt{\omega}}=0,$$

which induces $(\mathrm{L}_{2,\alpha}\mathrm{V},\mathrm{V}\rangle \geqq 0$. Moreover, since $(\mathrm{L}_{2,\alpha}\mathrm{V},\mathrm{V}\rangle=0$ if and only if $\mathrm{V}=c\Phi_m$ we obtain immediate that $\ker(\mathrm{L}_{2,\alpha})=span\{\Phi_m\}$ and $\mathrm{L}_{2,\alpha}\geqq 0$.

(ii) Concerning the kernel of $\mathrm{L}_{1,\alpha}$, the only $L^2(\mathbb{R}_+)$-solution of the equation

$$-v_j''+\omega v_j-p\varphi_{m,j}^{p-1}v_j=0$$

is $v_j=\varphi_{m,j}'$ up to a factor (see Berezin and Shubin (1991)). Thus, any element of $\ker(\mathrm{L}_{1,\alpha})$ has the form $\mathrm{V}=(v_j)_{j=1}^N=(c_j\varphi_{m,j}')_{j=1}^N$, $c_j\in\mathbb{R}$. Continuity condition $v_1(0)=\ldots=v_N(0)$ induces that $c_1=\ldots=c_N$, i.e.

$$v_j(x)=c\begin{cases}-\varphi_{m,j}', & j=1,..,m;\\ \varphi_{m,j}', & j=m+1,\ldots,N\end{cases},\quad c\in\mathbb{R}.$$

Condition $\sum\limits_{j=1}^{N}v_j'(0)=\alpha v_j(0)$ is equivalent to the equality

$$c\left(\frac{\omega(1-p)}{2}+\frac{p-1}{2}\frac{\alpha^2}{(N-2m)^2}\right)=0.$$

The last one induces that either $\omega=\frac{\alpha^2}{(N-2m)^2}$ (which is impossible) or $c=0$, and therefore $\mathrm{V}\equiv 0$.

(iii) By Weyl's theorem (see (Reed and Simon 1978)) the essential spectrum of $\mathrm{L}_{i,\alpha}$ coincides with $[\omega,\infty)$. Thus, there can be only finitely many isolated eigenvalues in $(-\infty,\omega')$ for any $\omega'<\omega$. Then (iii) follows easily. □

10.2.2 Morse index for $\mathrm{L}_{1,m,\alpha}$ in (10.11) with $m\neq 0$

Let $L^2_m(\mathcal{G})$ be defined in (10.14) and consider the matrix operator H defined in (10.8) associated with operators $\mathrm{L}_{1,m,\alpha}\equiv\mathrm{L}_{1,\alpha}$ in (10.11). The main theorem of this subsection is the following.

Theorem 10.4. *Let $\alpha\neq 0$, $m\in\left\{1,\ldots,\left[\frac{N-1}{2}\right]\right\}$ and $\omega>\frac{\alpha^2}{(N-2m)^2}$. Then the following assertions hold.*

(i) *If* $\alpha < 0$, *then* $n(\mathrm{H}) = 2$ *in* $L^2_m(\mathcal{G})$, *i.e.* $n(\mathrm{H}|_{L^2_m(\mathcal{G})}) = 2$.

(ii) *If* $\alpha > 0$, *then* $n(\mathrm{H}) = 1$ *in* $L^2_m(\mathcal{G})$, *i.e.* $n(\mathrm{H}|_{L^2_m(\mathcal{G})}) = 1$.

The proof of Theorem 10.4 will be based on the perturbation analytic theory and the extension theory of symmetric operators developed in Chapter 3. For this purpose let us define the following self-adjoint matrix Schrödinger operator on $L^2(\mathcal{G})$ with Kirchhoff condition at $\nu = 0$

$$\begin{aligned} \mathrm{L}_1^0 &= \left(\left(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\right)\delta_{i,j}\right), \\ D(\mathrm{L}_1^0) &= \left\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = \dots = v_N(0),\ \sum_{j=1}^N v_j'(0) = 0\right\}, \end{aligned} \tag{10.15}$$

where φ_0 represents the half-soliton solution for the classical NLS model,

$$\varphi_0(x) = \left[\frac{(p+1)\omega}{2}\,\mathrm{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}x\right)\right]^{\frac{1}{p-1}}. \tag{10.16}$$

We note that the self-adjoint property of $(\mathrm{L}_1^0, D(\mathrm{L}_1^0))$ is a consequence of Theorem 3.5 with $Z = 0$ and from the property that $B \equiv \omega - p\varphi_0^{p-1}$ is a bounded operator (see Proposition 3.1). Next, from definition of the profiles Φ_m in (7.6) it follows

$$\Phi_m = \Phi_m^\alpha \to \Phi_0, \quad \text{as} \quad \alpha \to 0, \quad \text{on} \quad H^1(\mathcal{G}),$$

where $\Phi_0 = (\varphi_0, \dots, \varphi_0)$. As we intend to study negative spectrum of $\mathrm{L}_{1,\alpha}$, we first need to describe spectral properties of L_1^0 (which is "limit value" of $\mathrm{L}_{1,\alpha}$ as $\alpha \to 0$).

Theorem 10.5. *Let* L_1^0 *be defined by* (10.15) *and* $m \in \{1, \dots, [\frac{N-1}{2}]\}$. *Then*

(i) $\ker(\mathrm{L}_1^0) = span\{\hat{\Phi}_{0,1}, \dots, \hat{\Phi}_{0,N-1}\}$, *where*

$$\hat{\Phi}_{0,j} = (0, \dots, 0, \underset{\mathrm{j}}{\varphi_0'}, \underset{\mathrm{j+1}}{-\varphi_0'}, 0, \dots, 0).$$

(ii) *In the space* $L^2_m(\mathcal{G})$ *we have* $\ker(\mathrm{L}_1^0) = span\{\widetilde{\Phi}_{0,m}\}$, *i.e. for any* m *we have* $\ker(\mathrm{L}_1^0|_{L^2_m(\mathcal{G})}) = span\{\widetilde{\Phi}_{0,m}\}$, *where*

$$\widetilde{\Phi}_{0,m} = \left(\underset{1}{\tfrac{N-m}{m}\varphi_0'}, \dots, \underset{\mathrm{m}}{\tfrac{N-m}{m}\varphi_0'}, \underset{\mathrm{m+1}}{-\varphi_0'}, \dots, \underset{\mathrm{N}}{-\varphi_0'}\right). \tag{10.17}$$

(iii) *The operator* L_1^0 *has one simple negative eigenvalue in* $L^2(\mathcal{G})$, *i.e.* $n(\mathrm{L}_1^0) = 1$. *Moreover,* L_1^0 *has one simple negative eigenvalue in* $L^2_m(\mathcal{G})$ *for any* m, *i.e.* $n(\mathrm{L}_1^0|_{L^2_m(\mathcal{G})}) = 1$.

(iv) *The positive part of the spectrum of* L_1^0 *is bounded away from zero.*

Proof. The proof repeats the one of Theorem 3.6 in (Angulo and Goloshchapova 2018). We give it for self-contentedness.

(i) The only $L^2(\mathbb{R}_+)$-solution to the equation

$$-v_j'' + \omega v_j - p\varphi_0^{p-1} v_j = 0$$

is $v_j = \varphi_0'$ (up to a factor). Thus, any element of $\ker(\mathrm{L}_1^0)$ has the form $\mathrm{V} = (v_j)_{j=1}^N = (c_j\varphi_0')_{j=1}^N$, $c_j \in \mathbb{R}$. It is easily seen that continuity condition is satisfied since $\varphi_0'(0) = 0$. Condition $\sum\limits_{j=1}^{N} v_j'(0) = 0$ gives rise to $(N-1)$-dimensional kernel of L_1^0. It is obvious that functions $\hat{\Phi}_{0,j}$, $j = 1, ..., N-1$ form basis there.

(ii) Arguing as in the previous item, we can see that $\ker(\mathrm{L}_1^0)$ is one-dimensional in $L_m^2(\mathcal{G})$, and it is spanned on $\widetilde{\Phi}_{0,m}$.

(iii) By following a similar analysis as in the proof of Theorem 3.5 and Proposition 3.1, we have that the symmetric operator $(\mathrm{L}_0^0, D(\mathrm{L}_0^0))$ with

$$\mathrm{L}_0^0 = \left(\left(-\frac{d^2}{dx^2} + \omega - p\varphi_0^{p-1}\right)\delta_{i,j}\right),$$

and

$$D(\mathrm{L}_0^0) = \left\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0) = 0,\ \sum_{j=1}^{N} v_j'(0) = 0\right\},$$

has deficiency indices $n_\pm(\mathrm{L}_0^0) = 1$. Moreover, $(\mathrm{L}_1^0, D(\mathrm{L}_1^0))$ in (10.15) belongs to the one-parameter family of self-adjoint extension of the symmetric operator $(\mathrm{L}_0^0, D(\mathrm{L}_0^0))$.

Let us show that operator L_0^0 is non-negative on $D(\mathrm{L}_0^0)$. First, note that every component of the vector $\mathrm{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - p\varphi_0^{p-1} v_j = \frac{-1}{\varphi_0'}\frac{d}{dx}\left[(\varphi_0')^2 \frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right], \quad x > 0.$$

Using the above equality and integrating by parts, we get for $\mathrm{V} \in D(\mathrm{L}_0^0)$ the equality

$$\begin{aligned}\langle \mathrm{L}_0^0\mathrm{V}, \mathrm{V}\rangle &= \sum_{j=1}^{N}\int_0^\infty (\varphi_0')^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right|^2 dx + \sum_{j=1}^{N}\left[-v_j'v_j + v_j^2\frac{\varphi_0''}{\varphi_0'}\right]_0^\infty \\ &= \sum_{j=1}^{N}\int_0^\infty (\varphi_0')^2 \left|\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right|^2 dx \geqslant 0,\end{aligned}$$

where the non-integral term becomes zero by the boundary conditions for V and the fact that $x = 0$ is the first-order zero for φ_0' (i.e. $\varphi_0''(0) \neq 0$). Indeed,

$$\begin{aligned}\sum_{j=1}^{N}\left[-v_j' v_j + v_j^2 \frac{\varphi_0''}{\varphi_0'}\right]_0^{\infty} &= -\sum_{j=1}^{N} \lim_{x\to 0+} \frac{2v_j(x)v_j'(x)\varphi_0''(x) + v_j^2(x)\varphi_0'''(x)}{\varphi_0''(x)} \\ &= 0.\end{aligned}$$

Since $n_{\pm}(\mathrm{L}_0^0) = 1$, by Theorem 3.11 follows that $n(\mathrm{L}_1^0) \leqslant 1$. Taking into account that

$$\langle \mathrm{L}_1^0 \Phi_0, \Phi_0\rangle = -(p-1)||\Phi_0||_{p+1}^{p+1} < 0,$$

we arrive at $n(\mathrm{L}_1^0) = 1$. Finally, since $\Phi_0 \in L^2_m(\mathcal{G})$ for any m, we have $n(\mathrm{L}_1^0|_{L^2_m(\mathcal{G})}) = 1$.

(iv) Follows from Weyl's theorem. □

Remark 10.2. *Observe that, when we deal with deficiency indices, the operator* L_0^0 *is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of* L_1^0 *acting on real-valued functions.*

Theorem 10.5 give us a good framework for applying tools from analytic perturbation theory on space $L^2_m(\mathcal{G})$ for operator L_1^0 and so the main point in the analysis will be determine which is the direction that the simple eigenvalue zero for L_1^0 will jump, to the right or to the left (we recall from Proposition 10.1 that $\ker(\mathrm{L}_{1,\alpha})$ is trivial for any $\alpha \neq 0$).

We start our analytic perturbation theory framework with other one characterization of the self-adjoint operators (10.11). Indeed, for $\mathrm{U}, \mathrm{V} \in \mathcal{E}$ written like real and imaginary parts $\mathrm{U} = \mathrm{U}_1 + i\mathrm{U}_2$ and $\mathrm{V} = \mathrm{V}_1 + i\mathrm{V}_2$, then it is easily seen that $S''(\Phi_m)(\mathrm{U}, \mathrm{V})$ can be formally rewritten as

$$S''(\Phi_m)(\mathrm{U}, \mathrm{V}) = B^{\alpha}_{1,m}(\mathrm{U}_1, \mathrm{V}_1) + B^{\alpha}_{2,m}(\mathrm{U}_2, \mathrm{V}_2). \tag{10.18}$$

Here bilinear forms $B^{\alpha}_{1,m}$ and $B^{\alpha}_{2,m}$ are defined for $\mathrm{F} = (f_j)_{j=1}^N, \mathrm{G} = (g_j)_{j=1}^N \in \mathcal{E}$ by

$$\begin{aligned} B^{\alpha}_{1,m}(\mathrm{F}, \mathrm{G}) &= \sum_{j=1}^{N} \int_0^{\infty} (f_j' g_j' + \omega f_j g_j - p(\varphi_{m,j})^{p-1} f_j g_j)dx + \alpha f_1(0) g_1(0), \\ B^{\alpha}_{2,k}(\mathrm{F}, \mathrm{G}) &= \sum_{j=1}^{N} \int_0^{\infty} (f_j' g_j' + \omega f_j g_j - (\varphi_{m,j})^{p-1} f_j g_j)dx + \alpha f_1(0) g_1(0). \end{aligned} \tag{10.19}$$

Next, we determine the self-adjoint operators associated with the forms $B^{\alpha}_{j,m}$ in order to establish a self-contained analysis. First note that the forms $B^{\alpha}_{j,m}$, $j \in \{1, 2\}$, are bilinear bounded from below and closed. Thus, there appear self-adjoint operators $\mathbb{L}_{1,m,\alpha}$

and $\mathbb{L}_{2,m,\alpha}$ associated (uniquely) with $B^{\alpha}_{1,m}$ and $B^{\alpha}_{2,m}$ by the First Representation Theorem (see Chapter VI, Section 2.1 Kato 1966), namely,

$$\begin{array}{ll}\mathbb{L}_{j,m,\alpha}\mathrm{V} = \mathrm{W}, & j \in \{1,2\},\\ D(\mathbb{L}_{j,m,\alpha}) = \{\mathrm{V} \in \mathcal{E} : \exists \mathrm{W} \in L^2(\mathcal{G}) \ s.t. \ \forall \mathrm{Z} \in \mathcal{E} & \\ \qquad\qquad B^{\alpha}_{j,m}(\mathrm{V},\mathrm{Z}) = \langle \mathrm{W},\mathrm{Z}\rangle\}. & \end{array} \tag{10.20}$$

In the following theorem we describe the operators $\mathbb{L}_{1,m,\alpha}$ and $\mathbb{L}_{2,m,\alpha}$ in more explicit form and its relation with the operators $\mathrm{L}_{i,m,\alpha}$, $j = 1, 2$, in (10.11).

Theorem 10.6. *The operators $\mathbb{L}_{1,m,\alpha}$ and $\mathbb{L}_{2,m,\alpha}$ defined by (10.20) are given on the domain $D(\mathbb{L}_{j,m,\alpha}) = \mathbb{D}_{\alpha,\delta}$ by*

$$\mathbb{L}_{1,m,\alpha} = \mathrm{L}_{1,m,\alpha}, \quad \mathbb{L}_{2,m,\alpha} = \mathrm{L}_{2,m,\alpha}$$

Proof. Since the proof for $\mathbb{L}_{2,m,\alpha}$ is similar to the one for $\mathbb{L}_{1,m,\alpha}$, we deal with $\mathbb{L}_{1,m,\alpha}$. Let $B^{\alpha}_{1,m} = B^{\alpha} + B_{1,m}$, where $B^{\alpha} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ and $B_{1,m} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ are defined by

$$B^{\alpha}(\mathrm{U},\mathrm{V}) = \sum_{j=1}^{N} \int_0^{\infty} u'_j v'_j \, dx + \alpha u_1(0) v_1(0),$$

$$B_{1,m}(\mathrm{U},\mathrm{V}) = \sum_{j=1}^{N} \int_0^{\infty} (\omega - p(\varphi_{m,j})^{p-1}) u_j v_j \, dx.$$

We denote by $\mathbb{L}^{\alpha}$ (resp. $\mathbb{L}_{1,m}$) the self-adjoint operator on $L^2(\mathcal{G})$ associated (by the First Representation Theorem) with B^{α} (resp. $B_{1,m}$). Thus,

$$\begin{array}{l}\mathbb{L}^{\alpha}\mathrm{V} = \mathrm{W},\\ D(\mathbb{L}^{\alpha}) = \{\mathrm{V} \in \mathcal{E} : \exists \mathrm{W} \in L^2(\mathcal{G}) \ s.t. \ \forall \mathrm{Z} \in \mathcal{E},\ B^{\alpha}(\mathrm{V},\mathrm{Z}) = \langle \mathrm{W},\mathrm{Z}\rangle\}.\end{array}$$

The operator $\mathbb{L}^{\alpha}$ belong to the family of self-adjoint extensions in (3.24) of the symmetric operator $(\mathrm{L}_0, D(\mathrm{L}_0))$ defined in Theorem 3.5. Indeed, initially we see $\mathrm{L}_0 \subset \mathbb{L}^{\alpha}$. Let $\mathrm{V} \in D(\mathrm{L}_0)$ and we consider $\mathrm{W} = (-v''_j(x))_{j=1}^N \in L^2(\mathcal{G})$. Then for every $\mathrm{Z} \in \mathcal{E}$ we have $B^{\alpha}(\mathrm{V},\mathrm{Z}) = \langle \mathrm{W},\mathrm{Z}\rangle$. Thus, $\mathrm{V} \in D(\mathbb{L}^{\alpha})$ and $\mathbb{L}^{\alpha}\mathrm{V} = \mathrm{W} = (-v''_j(x))_{j=1}^N$, which yields the claim. Therefore $(\mathbb{L}^{\alpha}, D(\mathbb{L}^{\alpha}))$ is a self-adjoint extension for $(\mathrm{L}_0, D(\mathrm{L}_0))$ and so Theorem 3.5 implies the existence of $Z \in \mathbb{R}$ such that $D(\mathbb{L}^{\alpha}) = D(\mathrm{L}_Z)$.

Finally, we need to prove that $Z = \alpha$. Take $\mathrm{V} \in D(\mathbb{L}^{\alpha})$, with $\mathrm{V}(0) = (v_j(0))_{j=1}^N \neq 0$, then we obtain

$$\langle \mathbb{L}^{\alpha}\mathrm{V},\mathrm{V}\rangle = \sum_{j=1}^{N} \int_0^{\infty} (v'_j)^2 dx + Z(v_1(0))^2,$$

which should be equal to $B^\alpha(\mathrm{V},\mathrm{V}) = \sum_{j=1}^{N}\int_0^\infty (v_j')^2dx + \alpha(v_1(0))^2$ for all $\mathrm{V}\in\mathcal{E}$. Therefore, $Z=\alpha$.

Next, we have that $\mathbb{L}_{1,m}$ is the self-adjoint extension of the following multiplication operator

$$\mathbb{L}_{0,m}\mathrm{V} = \Big((\omega - p(\varphi_{m,j})^{p-1})v_j(\cdot)\Big)_{j=1}^{N}, \quad D(\mathbb{L}_{0,m}) = \mathcal{E}.$$

Indeed, for $\mathrm{V}\in D(\mathbb{L}_{0,m})$, $\mathrm{V}=(v_j)_{j=1}^N$, we define

$$\mathrm{W} = \Big((\omega - p(\varphi_{m,j})^{p-1})v_j(\cdot)\Big)_{j=1}^{N} \in L^2(\mathcal{G}).$$

Then for every $\mathrm{Z}\in\mathcal{E}$ we get $B_{1,m}(\mathrm{V},\mathrm{Z}) = \langle \mathrm{W},\mathrm{Z}\rangle$. Thus, $\mathrm{V}\in D(\mathbb{L}_{1,m})$ and $\mathbb{L}_{1,m}\mathrm{V} = \mathrm{W} = \Big((\omega - p(\varphi_{m,j})^{p-1})v_j(\cdot)\Big)_{j=1}^{N}$. Hence, $\mathbb{L}_{0,m}\subseteq\mathbb{L}_{1,m}$. Since $\mathbb{L}_{0,m}$ is self-adjoint, $\mathbb{L}_{1,m}=\mathbb{L}_{0,m}$. The Theorem is proved. □

The following lemma states the analyticity of the family of operators $\mathrm{L}_{1,m,\alpha}$.

Lemma 10.1. *As a function of α, $(\mathrm{L}_{1,m,\alpha})$ is real-analytic family of self-adjoint operators of type (B) in the sense of Kato.*

Proof. By Theorem 10.6 and (Kato 1966, Theorem VII-4.2), it suffices to prove that the family of bilinear forms $(B^\alpha_{1,m})$ defined in (10.19) is real-analytic of type (B). Indeed, it is immediate that it is bounded from below and closed. Moreover, the decomposition of $B^\alpha_{1,m}$ into B^α and $B_{1,m}$, implies that $\alpha\to(B^\alpha_{1,m}\mathrm{V},\mathrm{V})$ is analytic. □

Combining Lemma 10.1 and Theorem 10.5, in the framework of the perturbation theory we obtain the following proposition.

Proposition 10.2. *Let $m\in\{1,\dots,[\frac{N-1}{2}]\}$. Then there exist $\alpha_0>0$ and two analytic functions $\lambda_m:(-\alpha_0,\alpha_0)\to\mathbb{R}$ and $\mathbb{F}_m:(-\alpha_0,\alpha_0)\to L^2_m(\mathcal{G})$ such that*

- (i) *$\lambda_m(0)=0$ and $\mathbb{F}_m(0)=\widetilde{\Phi}_{0,m}$, where $\widetilde{\Phi}_{0,m}$ is defined by (10.17).*
- (ii) *For all $\alpha\in(-\alpha_0,\alpha_0)$, $\lambda_m(\alpha)$ is the simple isolated second eigenvalue of $\mathrm{L}_{1,m,\alpha}$ in $L^2_m(\mathcal{G})$, and $\mathbb{F}_m(\alpha)$ is the associated eigenvector for $\lambda_m(\alpha)$.*
- (iii) *α_0 can be chosen small enough to ensure that for $\alpha\in(-\alpha_0,\alpha_0)$ the spectrum of $\mathrm{L}_{1,m,\alpha}$ in $L^2_m(\mathcal{G})$ is positive, except at most the first two eigenvalues.*

Proof. Using the structure of the spectrum of the operator L^0_1 given in Theorem 10.5(ii)–(iv), we can separate the spectrum $\sigma(\mathrm{L}^0_1)$ in $L^2_k(\mathcal{G})$ into two parts $\sigma_0=\{\lambda^0_1,0\}$, $\lambda^0_1<0$, and σ_1 by a closed curve Γ (for example, a circle), such that σ_0 belongs to the inner domain of Γ and σ_1 to the outer domain of Γ (note that $\sigma_1\subset(\epsilon,+\infty)$ for $\epsilon>0$). Next, Lemma 10.1 and the analytic perturbations theory imply that $\Gamma\subset\rho(\mathrm{L}_{1,m,\alpha})$ for sufficiently small $|\alpha|$, and $\sigma(\mathrm{L}_{1,m,\alpha})$ is likewise separated by Γ into two parts, such that the part of $\sigma(\mathrm{L}_{1,m,\alpha})$ inside Γ consists of a finite number of eigenvalues with total multiplicity (algebraic) two. Therefore, we obtain from the Kato-Rellich Theorem (see (Reed and Simon 1978, Theorem XII.8)) the existence of two analytic functions $\lambda_m, \mathbb{F}_m$ defined in a neighborhood of zero such that the items (i), (ii) and (iii) hold. □

Now we investigate how the perturbed second eigenvalue moves depending on the sign of α.

Proposition 10.3. *There exists $0 < \alpha_1 < \alpha_0$ such that $\lambda_k(\alpha) < 0$ for any $\alpha \in (-\alpha_1, 0)$, and $\lambda_m(\alpha) > 0$ for any $\alpha \in (0, \alpha_1)$. Thus, in $L^2_m(\mathcal{G})$ for α small, we have $n(\mathrm{L}_{1,m,\alpha}) = 2$ as $\alpha < 0$, and $n(\mathrm{L}_{1,m,\alpha}) = 1$ as $\alpha > 0$.*

Proof. From Taylor's theorem we have the following expansions

$$\lambda_m(\alpha) = \lambda_{0,m}\alpha + O(\alpha^2) \quad \text{and} \quad \mathbb{F}_m(\alpha) = \widetilde{\Phi}_{0,m} + \alpha\mathbb{F}_{0,m} + O(\alpha^2), \tag{10.21}$$

where $\lambda_{0,m} = \lambda_m'(0) \in \mathbb{R}$ and $\mathbb{F}_{0,m} = \partial_\alpha \mathbb{F}_m(\alpha)|_{\alpha=0} \in L^2_k(\mathcal{G})$. The desired result will follow if we show that $\lambda_{0,m} > 0$. We compute for $\mathrm{L}_{1,m,\alpha} = \mathrm{L}_{1,\alpha}$, $\langle \mathrm{L}_{1,\alpha}\mathbb{F}_m(\alpha), \widetilde{\Phi}_{0,m}\rangle$ in two different ways.

Note that for $\Phi_m = \Phi_m^\alpha$ defined by (7.6) we have

$$\begin{aligned} &\Phi_m(\alpha) = \Phi_0 + \alpha\mathbb{G}_{0,m} + O(\alpha^2), \\ &\mathbb{G}_{0,m} = \partial_\alpha \Phi_m(\alpha)|_{\alpha=0} = \tfrac{2}{(p-1)(N-2m)\omega}\Big(\underset{1}{\varphi_0'}, \dots, \underset{m}{\varphi_0'}, \underset{m+1}{-\varphi_0'}, \dots, \underset{N}{-\varphi_0'}\Big). \end{aligned} \tag{10.22}$$

From (10.33) we obtain

$$\langle \mathrm{L}_{1,\alpha}\mathbb{F}_m(\alpha), \widetilde{\Phi}_{0,m}\rangle = \lambda_{0,m}\alpha||\widetilde{\Phi}_{0,m}||_2^2 + O(\alpha^2). \tag{10.23}$$

By $\mathrm{L}_1^0\widetilde{\Phi}_{0,m} = 0$ and (10.33) we get

$$\begin{aligned} \mathrm{L}_{1,\alpha}\widetilde{\Phi}_{0,k} &= p\left((\Phi_0)^{p-1} - (\Phi_m)^{p-1}\right)\widetilde{\Phi}_{0,k} \\ &= -\alpha p(p-1)(\Phi_0)^{p-2}\mathbb{G}_{0,m}\widetilde{\Phi}_{0,m} + O(\alpha^2). \end{aligned} \tag{10.24}$$

The operations in the last equality are componentwise. Equations (10.36), (10.34), and $\widetilde{\Phi}_{0,m} \in D_\alpha$ induce

$$\begin{aligned} &\langle \mathrm{L}_{1,\alpha}\mathbb{F}_m(\alpha), \widetilde{\Phi}_{0,m}\rangle = \langle \mathbb{F}_m(\alpha), \mathrm{L}_{1,\alpha}\widetilde{\Phi}_{0,m}\rangle \\ &= -\langle \widetilde{\Phi}_{0,m}, \alpha p(p-1)(\Phi_0)^{p-2}\mathbb{G}_{0,m}\widetilde{\Phi}_{0,m}\rangle + O(\alpha^2) \\ &= -2\alpha p\tfrac{N-m}{m\omega}\int_0^\infty (\varphi_0')^3\varphi_0^{p-2}dx + O(\alpha^2). \end{aligned} \tag{10.25}$$

Finally, combining (10.37) and (10.35), we obtain

$$\lambda_{0,m} = -\frac{2p(N-m)}{m\omega||\widetilde{\Phi}_{0,m}||_2^2} \int_0^\infty (\varphi_0')^3 \varphi_0^{p-2} dx + O(\alpha).$$

It follows that $\lambda_{0,m}$ is positive for sufficiently small $|\alpha|$ (due to negativity of φ_0' on $\mathbb{R}_+$), which in view of (10.33) ends the proof. □

Now we can count the number of negative eigenvalues of $\mathrm{L}_{1,m,\alpha}$ in $L_m^2(\mathcal{G})$ for any α, using a classical continuation argument based on the Riesz projection (see Appendix C).

Proposition 10.4. *Let $m \in \{1, ..., [\frac{N-1}{2}]\}$ and $\omega > \frac{\alpha^2}{(N-2m)^2}$. Then the following assertions hold.*

(*i*) *If $\alpha > 0$, then $n(\mathrm{L}_{1,m,\alpha}|_{L_m^2(\mathcal{G})}) = 1$.*

(*ii*) *If $\alpha < 0$, then $n(\mathrm{L}_{1,m,\alpha}|_{L_m^2(\mathcal{G})}) = 2$.*

Proof. We consider the case $\alpha < 0$. Recall that $\ker(\mathrm{L}_{1,m,\alpha}) = \{0\}$ by Proposition 10.1. Define α_∞ by

$$\alpha_\infty = \inf\{\widetilde{\alpha} < 0 : \mathrm{L}_{1,m,\alpha} \text{ has exactly two negative eigenvalues for all } \alpha \in (\widetilde{\alpha}, 0)\}. \tag{10.26}$$

Proposition 10.3 implies that α_∞ is well defined and $\alpha_\infty \in [-\infty, 0)$. We claim that $\alpha_\infty = -\infty$. Suppose that $\alpha_\infty > -\infty$. Let $M = n(\mathrm{L}_{1,m,\alpha_\infty})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathrm{L}_{1,m,\alpha_\infty})$, and all the negative eigenvalues of $\mathrm{L}_{1,m,\alpha_\infty}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to $\mathrm{L}_{1,m,\alpha_\infty}$.

Next, from Lemma 10.1 it follows that there is $\epsilon > 0$ such that for $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$ we have $\Gamma \subset \rho(\mathrm{L}_{1,m,\alpha})$ and for $\xi \in \Gamma$, $\alpha \to (\mathrm{L}_{1,m,\alpha} - \xi)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $\alpha \to P(\alpha)$ given by

$$P(\alpha) = -\frac{1}{2\pi i} \oint_\Gamma (\mathrm{L}_{1,m,\alpha} - \xi)^{-1} d\xi$$

implies that (see Lemma C.3 in Appendix C)

$$\dim(Ran P(\alpha)) = \dim(Ran P(\alpha_\infty)) = M, \quad \text{for all } \alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon].$$

Next, by definition of α_∞, $\mathrm{L}_{1,m,\alpha_\infty+\epsilon}$ has two negative eigenvalues and $M = 2$, hence $\mathrm{L}_{1,m,\alpha}$ has two negative eigenvalues for $\alpha \in (\alpha_\infty - \epsilon, 0)$, which contradicts with the definition of α_∞. Therefore, $\alpha_\infty = -\infty$. □

10.2.3 Slope analysis for $m \neq 0$

In this subsection we evaluate $p(\omega)$ defined in section 9.1.

Proposition 10.5. *Let* $\alpha \neq 0$, $m \in \{1, ..., [\frac{N-1}{2}]\}$, *and* $\omega > \frac{\alpha^2}{(N-2m)^2}$. *Let also* $J_m(\omega) = \partial_\omega ||\Phi_m^\alpha||_2^2$. *Then the following assertions hold*

(*i*) *Let* $\alpha < 0$, *then*

1) *for* $1 < p \leqslant 5$, *we have* $J_m(\omega) > 0$;
2) *for* $p > 5$, *there exists* ω_m^* *such that* $J_m(\omega_m^*) = 0$, *and* $J_m(\omega) > 0$ *for* $\omega \in \left(\frac{\alpha^2}{(N-2m)^2}, \omega_m^*\right)$, *while* $J_m(\omega) < 0$ *for* $\omega \in (\omega_m^*, \infty)$.

(*ii*) *Let* $\alpha > 0$, *then*

1) *for* $1 < p \leqslant 3$, *we have* $J_m(\omega) > 0$;
2) *for* $3 < p < 5$, *there exists* $\hat{\omega}_m$ *such that* $J_m(\hat{\omega}_m) = 0$, *and* $J_m(\omega) < 0$ *for* $\omega \in \left(\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m\right)$, *while* $J_m(\omega) > 0$ *for* $\omega \in (\hat{\omega}_m, \infty)$;
3) *for* $p \geqslant 5$, *we have* $J_m(\omega) < 0$.

Proof. Recall that $\Phi_m^\alpha = (\varphi_{m,j}^\alpha)_{j=1}^N$, where $\varphi_{m,j}^\alpha$ is defined by (7.6). Changing variables we have

$$\int_0^\infty (\varphi_{m,j}^\alpha(x))^2 dx = G(\omega) \begin{cases} \int_{\frac{-\alpha}{(2m-N)\sqrt{\omega}}}^{1} (1-t^2)^{\frac{2}{p-1}-1} dt, & j = 1, ..., m; \\ \int_{\frac{\alpha}{(2m-N)\sqrt{\omega}}}^{1} (1-t^2)^{\frac{2}{p-1}-1} dt, & j = m+1, ..., N. \end{cases}$$

with $G(\omega) = \left(\frac{p+1}{2}\right)^{\frac{2}{p-1}} \frac{2}{p-1} \omega^{\frac{2}{p-1}-\frac{1}{2}}$. Therefore, we obtain

$$||\Phi_m^\alpha||^2 = G(\omega) P(\omega) \tag{10.27}$$

with

$$P(\omega) = m \int_{\frac{-\alpha}{(2m-N)\sqrt{\omega}}}^{1} (1-t^2)^{\frac{2}{p-1}-1} dt + (N-m) \int_{\frac{\alpha}{(2m-N)\sqrt{\omega}}}^{1} (1-t^2)^{\frac{2}{p-1}-1} dt. \tag{10.28}$$

Thus we get,

$$\begin{aligned} J_m(\omega) &= C\omega^{\frac{7-3p}{2(p-1)}} \tfrac{5-p}{p-1} P(\omega) - C\omega^{\frac{7-3p}{2(p-1)}} \tfrac{\alpha}{\sqrt{\omega}} \left(1 - \tfrac{\alpha^2}{(N-2m)^2\omega}\right)^{\frac{3-p}{p-1}} \\ &= C\omega^{\frac{7-3p}{2(p-1)}} \widetilde{J}_m(\omega), \end{aligned} \tag{10.29}$$

where $C = \frac{1}{p-1}\left(\frac{p+1}{2}\right)^{\frac{2}{p-1}} > 0$ and

$$\widetilde{J}_m(\omega) = \tfrac{5-p}{p-1} P(\omega) - \tfrac{\alpha}{\sqrt{\omega}} \left(1 - \tfrac{\alpha^2}{(N-2m)^2\omega}\right)^{\frac{3-p}{p-1}}.$$

Thus,

$$\widetilde{J}'_m(\omega) = -\tfrac{\alpha}{\omega^{3/2}} \tfrac{3-p}{p-1} \left(1 - \tfrac{\alpha^2}{(N-2m)^2\omega}\right)^{\frac{3-p}{p-1}} \left[1 + \tfrac{\alpha^2}{(N-2m)^2\omega - \alpha^2}\right]. \tag{10.30}$$

(i) Let $\alpha < 0$. It is immediate that $J_m(\omega) > 0$ for $1 < p \leqslant 5$ which yields 1). Consider the case $p > 5$. It is easily seen that

$$\lim_{\omega \to \frac{\alpha^2}{(N-2m)^2}} \widetilde{J}_m(\omega) = \infty, \quad \lim_{\omega\to\infty} \widetilde{J}_m(\omega) = \tfrac{5-p}{p-1} N \int_0^1 (1-t^2)^{\frac{3-p}{p-1}} dt < 0.$$

Moreover, from (10.52) it follows that $\widetilde{J}'_m(\omega) < 0$ for $\omega > \frac{\alpha^2}{(N-2m)^2}$ and consequently $J_m(\omega)$ is strictly decreasing. Therefore, there exists a unique $\omega^*_m > \frac{\alpha^2}{(N-2m)^2}$ such that

$$\widetilde{J}_m(\omega^*_m) = J_m(\omega^*_m) = 0,$$

consequently $J_m(\omega) > 0$ for $\omega \in \left(\frac{\alpha^2}{(N-2m)^2}, \omega^*_m\right)$ and $J_m(\omega) < 0$ for $\omega \in (\omega^*_m, \infty)$, and the proof of $(i) - 2)$ is completed.

(ii) Let $\alpha > 0$. It is easily seen that $\widetilde{J}_m(\omega) < 0$ for $p \geqslant 5$, thus, 3) holds. Let $1 < p < 5$. It can be easily verified that

$$\lim_{\omega\to+\infty} \widetilde{J}_m(\omega) = \frac{5-p}{p-1} N \int_0^1 (1-t^2)^{\frac{3-p}{p-1}} dt > 0, \tag{10.31}$$

and

$$\lim_{\omega \to \frac{\alpha^2}{(N-2m)^2}} \widetilde{J}_m(\omega) = \begin{cases} \frac{5-p}{p-1}(N-k) \int_{-1}^1 (1-t^2)^{\frac{3-p}{p-1}} dt > 0, & p \in (1,3], \\ -\infty, & p \in (3,5). \end{cases} \tag{10.32}$$

Let $1 < p \leqslant 3$, using the fact that $\widetilde{J}_m'(\omega) < 0$ we get from (10.31)-(10.32) the inequality $J_m(\omega) > 0$, and $(ii) - 1)$ holds. Let $3 < p < 5$, then $\widetilde{J}_m'(\omega) > 0$, therefore, from (10.31)-(10.32) it follows that there exists $\hat{\omega}_m > \frac{\alpha^2}{(N-2m)^2}$ such that $\widetilde{J}_m(\hat{\omega}_m) = J_m(\hat{\omega}_m) = 0$, moreover, $J_m(\omega) < 0$ for $(\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m)$, and $J_m(\omega) > 0$ for $(\hat{\omega}_m, \infty)$, i.e. $(ii) - 2)$ is proved. □

10.2.4 Proof of Theorem 10.2

In this subsection we proof our main Theorem 10.2 via the framework established in section 8.1 above.

Proof. From Theorem 6.1 we obtain the local well-posed in $\mathcal{E}$ and $\mathcal{E}_m$ of the Cauchy problem for (10.1) in the case of a δ-interaction at the vertex $\nu = 0$.

(i) Let $\alpha < 0$. Due to Theorem 10.4, we have $n(\mathrm{H}) = 2$ in $L^2_m(\mathcal{G})$. Therefore, by Proposition 10.10(i) we obtain

$$n(\mathrm{H}^\alpha_m) - p(\omega) = 1$$

for $1 < p \leqslant 5, \omega > \frac{\alpha^2}{(N-2m)^2}$, and for $p > 5, \omega \in (\frac{\alpha^2}{(N-2m)^2}, \omega^*_m)$. Thus, from Theorem 10.1 we get the assertions $(i) - 1)$ and $(i) - 2)$ in $\mathcal{E}_m$. Since $\mathcal{E}_m \subset \mathcal{E}$, we get the results in $\mathcal{E}$.

(ii) Let $\alpha > 0$. Due to Theorem 10.4, we have $n(\mathrm{H}) = 1$ in $L^2_m(\mathcal{G})$. Therefore, by Proposition 10.10(ii) we obtain

$$n(\mathrm{H}) - p(\omega) = 1$$

for $p \geqslant 5, \omega > \frac{\alpha^2}{(N-2m)^2}$ and $3 < p < 5, \omega \in (\frac{\alpha^2}{(N-2m)^2}, \hat{\omega}_m)$. Therefore, we obtain instability of $e^{i\omega t}\Phi^\alpha_m$ in $\mathcal{E}_m$ and consequently in $\mathcal{E}$. From the other hand, for $1 < p \leqslant 3, \omega > \frac{\alpha^2}{(N-2m)^2}$ and $3 < p < 5, \omega \in (\hat{\omega}_m, \infty)$, we have

$$n(\mathrm{H}) - p(\omega) = 0,$$

which yields stability of $e^{i\omega t}\Phi^\alpha_m$ in $\mathcal{E}_m$. Thus, (ii) is proved. This finishes the proof. □

Remark 10.3. *From Remark 10.1 we recall that when $n(\mathrm{H}) - p(\omega)$ is odd, we obtain initially from (Grillakis, Shatah, and W. Strauss 1990) only spectral instability of $e^{i\omega t}\Phi^\alpha_m$. To conclude orbital instability from spectral instability we use from Theorem 6.1 that the mapping data-solution is of class C^2 for $p > 2$.*

10.2.5 Morse index for $\mathrm{L}_{1,m,\alpha}$ in (10.11) with $m = 0$

The main result of this subsection associated to the tail and bump profiles $\Phi_0^\alpha = (\varphi_{0,\alpha})_{j=1}^N$ defined in (10.12) is the following.

Proposition 10.6. *Let* $m \in \{1, ..., [\frac{N-1}{2}]\}$, $\alpha \neq 0$ *and* $\omega > \frac{\alpha^2}{N^2}$. *Then for* $\mathrm{L}_{1,0,\alpha} = \mathrm{L}_{1,\alpha}$

(*i*) *for* $\alpha > 0$, $n(\mathrm{L}_{1,\alpha}) = 2$ *in* $L^2_m(\mathcal{G})$, *i.e.,* $n(\mathrm{L}_{1,\alpha}|_{L^2_(\mathcal{G})}) = 2$,

(*ii*) *for* $\alpha < 0$, $n(\mathrm{L}_{1,\alpha}) = 1$ *in* $L^2_m(\mathcal{G})$, *i.e.,* $n(\mathrm{L}_{1,\alpha}|_{L^2_m(\mathcal{G})}) = 1$.

Proof. The proof of Proposition 10.6 follows the same strategy as in the prove of Proposition 10.4. We note that Theorem 10.5, Theorem 10.6, Lemma 10.1 and Proposition 10.2 remain true. Thus, there exist $\alpha_0 > 0$ and two analytic functions $\mu : (-\alpha_0, \alpha_0) \to \mathbb{R}$ and $\mathbb{F} : (-\alpha_0, \alpha_0) \to L^2_m(\mathcal{G})$ such that

(*i*) $\mu(0) = 0$ and $\mathbb{F}(0) = \widetilde{\Phi}_{0,m}$, where $\widetilde{\Phi}_{0,m}$ is defined by (10.17).

(*ii*) For all $\alpha \in (-\alpha_0, \alpha_0)$, $\mu(\alpha)$ is the simple isolated second eigenvalue of $\mathrm{L}_{1,\alpha}$ in $L^2_m(\mathcal{G})$, and $\mathbb{F}(\alpha)$ is the associated eigenvector for $\mu(\alpha)$.

(*iii*) α_0 can be chosen small enough to ensure that for $\alpha \in (-\alpha_0, \alpha_0)$ the spectrum of $\mathrm{L}_{1,\alpha}$ in $L^2_m(\mathcal{G})$ is positive, except at most the first two eigenvalues.

Thus, it is enough to investigate how the perturbed second eigenvalue $\mu(\alpha)$ moves depending on the sign of α. Next we will see that in the space $L^2_m(\mathcal{G})$ for α small, we have $n(\mathrm{L}_{1,\alpha}) = 1$ as $\alpha < 0$, and $n(\mathrm{L}_{1,\alpha}) = 2$ as $\alpha > 0$. Indeed, from Taylor's theorem we have the following expansions

$$\mu(\alpha) = \mu_0\alpha + O(\alpha^2) \quad \text{and} \quad \mathbb{F}(\alpha) = \widetilde{\Phi}_{0,m} + \alpha\mathbb{F}_0 + \mathbb{O}(\alpha^2), \tag{10.33}$$

where $\mu_0 = \mu'(0) \in \mathbb{R}$, $\mathbb{F}_0 = \partial_\alpha \mathbb{F}(\alpha)|_{\alpha=0} \in L^2_m(\mathcal{G})$, and $\widetilde{\Phi}_{0,m}$ is defined by (10.17). The desired result will follow if we show that $\mu_0 < 0$. We compute $\langle \mathrm{L}_{1,\alpha}\mathbb{F}(\alpha), \widetilde{\Phi}_{0,m}\rangle$ in two different ways.

In what follows, we will use the following decomposition for $\mathrm{K}(\alpha) \equiv \Phi_0^\alpha = (\varphi_{0,\alpha})_{j=1}^N$ defined by (10.12) around $\alpha = 0$

$$\mathrm{K}(\alpha) = \Phi_0 + \alpha \mathrm{G}_0 + \mathrm{O}(\alpha^2) \tag{10.34}$$

where

$$\mathrm{G}_0 = \partial_\alpha(\mathrm{K}(\alpha)|_{\alpha=0} = \tfrac{-2}{(p-1)N\omega} \left(\varphi_0'\right)_{j=1}^N .$$

From (10.33) we obtain

$$\langle \mathrm{L}_{1,\alpha}\mathbb{F}(\alpha), \widetilde{\Phi}_{0,m}\rangle = \mu_0\alpha ||\widetilde{\Phi}_{0,m}||^2 + O(\alpha^2). \tag{10.35}$$

By $\mathrm{L}_{1,0}\widetilde{\Phi}_{0,m} = 0$ and (10.33), we get

$$\begin{aligned}\mathrm{L}_{1,\alpha}\widetilde{\Phi}_{0,m} &= p\left((\Phi_0)^{p-1} - (\Phi_0^{\alpha})^{p-1}\right)\widetilde{\Phi}_{0,m} \\ &= -\alpha p(p-1)(\Phi_0)^{p-2}\mathrm{G}_0\widetilde{\Phi}_{0,m} + \mathrm{O}(\alpha^2).\end{aligned} \tag{10.36}$$

The operations in the last equality are componentwise. Equations (10.36) and (10.34) induce

$$\begin{aligned}\langle \mathrm{L}_{1,\alpha}\mathbb{F}(\alpha), \widetilde{\Phi}_{0,m}\rangle &= -\langle \widetilde{\Phi}_{0,m}, \alpha p(p-1)(\Phi_0)^{p-2}\mathrm{G}_0\widetilde{\Phi}_{0,m}\rangle + O(\alpha^2) \\ &= \tfrac{2\alpha p(N-m)}{m\omega}\int_0^{\infty}(\varphi_0')^3\varphi_0^{p-2}dx + O(\alpha^2).\end{aligned} \tag{10.37}$$

Finally, combining (10.37) and (10.35), we obtain for $m \in \{1, ..., N-1\}$

$$\mu_0 = \frac{2p(N-m)}{m\omega||\widetilde{\Phi}_{0,k}||^2}\int_0^{\infty}(\varphi_0')^3\varphi_0^{p-2}dx + O(\alpha).$$

It follows that μ_0 is negative for sufficiently small $|\alpha|$ (due to the negativity of φ_0' on $\mathbb{R}_+$).

Thus by using an argument based on the Riesz projection we obtain the statements $n(\mathrm{L}_{1,\alpha}|_{L^2_m(\mathcal{G})}) = 2$ for any $\alpha > 0$ and also $n(\mathrm{L}_{1,\alpha}|_{L^2_m(\mathcal{G})}) = 1$ for any $\alpha < 0$. □

The item (ii) in Proposition 10.6 can be obtained without an perturbation analysis and to be generalized via the extension theory for symmetric operators. That is the objective of the following.

Proposition 10.7. *Let $\alpha < 0$ and $\omega > \frac{\alpha^2}{N^2}$. Then $n(\mathrm{L}_{1,\alpha}) = 1$ in $L^2(\mathcal{G})$.*

Proof. We need to repeat the arguments of the proof of Theorem 10.5-(iii) (i.e. L_1^0 has to be replaced by $\mathrm{L}_{1,\alpha}$, and Φ_0 by Φ_0^{α}). Namely, $(\mathrm{L}_{1,\alpha}, \mathbb{D}_{\alpha,\delta})$ has to be considered as the family of self-adjoint extensions of the non-negative symmetric operator

$$\begin{aligned}\mathrm{L}_0^0 &= \left(\left(-\tfrac{d^2}{dx^2} + \omega - p(\varphi_{0,\alpha})^{p-1}\right)\delta_{i,j}\right), \\ D(\mathrm{L}_0^0) &= \left\{\mathrm{V} \in H^2(\mathcal{G}) : v_1(0) = ... = v_N(0) = 0, \sum_{j=1}^{N} v_j'(0) = 0\right\},\end{aligned}$$

with deficiency indices $n_{\pm}(\mathrm{L}_0^0) = 1$. Note that since $\alpha < 0$, we have $\varphi_{0,\alpha}'(x) < 0$ for $x \geqslant 0$ and so the following basic equality is well defined for all $x \geqslant 0$,

$$-v'' + \omega v - p\varphi_{0,\alpha}^{p-1}v = \frac{-1}{\varphi_{0,\alpha}'}\frac{d}{dx}\left[(\varphi_{0,\alpha}')^2\frac{d}{dx}\left(\frac{v}{\varphi_{0,\alpha}'}\right)\right]. \tag{10.38}$$

Therefore, for $\mathrm{V} = (v_j)_{i=1}^N \in D(\mathrm{L}_0^0)$ the equality

$$\langle \mathrm{L}_0^0\mathrm{V}, \mathrm{V}\rangle = \sum_{j=1}^{N}\int_0^{\infty}(\varphi_{0,\alpha}')^2\left|\frac{d}{dx}\left(\frac{v_j}{\varphi_{0,\alpha}'}\right)\right|^2 dx \geqslant 0,$$

it is immediate.

By Theorem 3.11 follows that $n(\mathrm{L}_{1,\alpha}) \leqslant 1$. Taking into account that $\Phi_0^\alpha \in D(\mathrm{L}_{1,\alpha})$,

$$\langle \mathrm{L}_{1,\alpha}\Phi_0^\alpha, \Phi_0^\alpha\rangle = -(p-1)||\Phi_0^\alpha||_{p+1}^{p+1} < 0,$$

we arrive at $n(\mathrm{L}_{1,\alpha}) = 1$ in $L^2(\mathcal{G})$. This finishes the proof. □

Remark 10.4. *The extension symmetric approach in the proof of Proposition 10.7, by using the symmetric operator* $(\mathrm{L}_0^0, D(\mathrm{L}_0^0))$*, it is not clear that can be used in the case of bump profiles* ($\alpha > 0$)*. Although equality* (10.38) *is still right, at least for* $x > 0$ *and* $x \neq z_0$ *with* $\varphi'_{0,\alpha}(z_0) = 0$*, it is not obvious that the quadratic form* $\langle \mathrm{L}_0^0\mathrm{V}, \mathrm{V}\rangle$ *continues being non-negative.*

10.2.6 Slope analysis for $m = 0$

The proof of the following Proposition follows the same ideas as the case $m \neq 0$ (see Angulo and Goloshchapova (2018)).

Proposition 10.8. *Let* $\omega > \frac{\alpha^2}{N^2}$ *and* $J(\omega) = \partial_\omega||\Phi_0^\alpha||^2$*. Then the following assertions hold.*

(i) *Let* $\alpha < 0$*, then*

1) *for* $1 < p \leqslant 5$*, we have* $J(\omega) > 0$*;*

2) *for* $p > 5$*, there exists* ω_1 *such that* $J(\omega_1) = 0$*, and* $J(\omega) > 0$ *for* $\omega \in \left(\frac{\alpha^2}{N^2}, \omega_1\right)$*, while* $J(\omega) < 0$ *for* $\omega \in (\omega_1, \infty)$*.*

(ii) *Let* $\alpha > 0$*, then*

1) *for* $1 < p \leqslant 3$*, we have* $J(\omega) > 0$*;*

2) *for* $3 < p < 5$*, there exists* ω_2 *such that* $J(\omega_2) = 0$*, and* $J(\omega) < 0$ *for* $\omega \in \left(\frac{\alpha^2}{N^2}, \omega_2\right)$*, while* $J(\omega) > 0$ *for* $\omega \in (\omega_2, \infty)$*;*

3) *for* $p \geqslant 5$*, we have* $J(\omega) < 0$*.*

10.2.7 Proof of Theorem 10.3

In this subsection we proof Theorem 10.3 via the framework established in section 8.1 above.

Proof. (1) For $\alpha > 0$, from Proposition 10.6-(i), Proposition 10.1, and Proposition 10.8 -(ii), we obtain

$$n(\mathrm{H}|_{L^2_m(\mathcal{G})}) - p(\omega) = 1$$

as $p \in (1, 3]$, $\omega > \frac{\alpha^2}{N^2}$, and $p \in (3, 5)$, $\omega > \omega_2$. Thus, from Theorem 10.1 we get orbital instability of $e^{i\omega t}\Phi_0^\alpha$ in $\mathcal{E}_m(\mathcal{G})$ and consequently in $\mathcal{E}(\mathcal{G})$.

(2) For $\alpha < 0$, from Proposition 10.6-(ii), Proposition 10.1, Proposition 10.8 -(i), we obtain the orbital stability $e^{i\omega t}\Phi_0^\alpha$ for $1 < p \leqq 5$ and any $\omega > \frac{\alpha^2}{N^2}$ and for $p > 5$ and $\omega \in (\frac{\alpha^2}{N^2}, \omega_1)$. Moreover, applying the approach by Henry, Perez, and Wreszinski (1982) (see Remark 10.1 and Theorem 6.1) we may deduce the orbital instability of $e^{i\omega t}\Phi_0^\alpha$ from the spectral one for $p > 5$ and $\omega > \omega_1$.

This finishes the proof. □

10.3 Stability theory for the NLS-δ' on star graphs

In this section we study the orbital stability of standing wave $\mathrm{U}(t, x) = e^{i\omega t}\Phi(x)$ of the Schrödinger model (10.1) for the case of the self-adjoint operator $\mathcal{A} = \mathrm{H}_\lambda^{\delta'}$ (see Theorem 3.7) acting as

$$(\mathrm{H}_\lambda^{\delta'}\mathrm{V})(x) = (-v_j''(x))_{j=1}^N, \; x > 0,$$

on the domain $D(\mathrm{H}_\lambda^{\delta'}) = \mathbb{D}_{\lambda,\delta'}$, where

$$\mathbb{D}_{\lambda,\delta'} := \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1'(0) = \ldots = v_N'(0), \; \sum_{j=1}^N v_j(0) = \lambda v_1'(0)\Big\}. \tag{10.39}$$

In particular, we study the orbital stability of standing wave with the particular N-tail profile $\Phi_{\lambda,\delta'} = (\varphi_{\lambda,j})_{j=1}^N$ under the conditions $\varphi_{\lambda,1} = \ldots = \varphi_{\lambda,N} \equiv \varphi_{\lambda,\delta'}$ and $N\varphi_{\lambda,j}(0) = \lambda\varphi_{\lambda,j}'(0)$. Thus we need to have the profile

$$\varphi_{\lambda,\delta'}(x) = \Big[\frac{(p+1)\omega}{2}\,\mathrm{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}x + \tanh^{-1}\Big(\frac{-N}{\lambda\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p-1}}, \tag{10.40}$$

with $\omega > \frac{N^2}{\lambda^2}$ and $\lambda < 0$. Moreover, $\Phi_{\lambda,\delta'} \in \mathbb{D}_{\lambda,\delta'}$ and satisfies the stationary equation

$$\mathrm{H}_\lambda^{\delta'}\Phi + \omega\Phi - |\Phi|^{p-1}\Phi = 0.$$

We will investigate orbital stability of $\mathrm{U}(t, x) = e^{i\omega t}\Phi_{\lambda,\delta'}$ in the energy space $X = H^1(\mathcal{G})$ inside the framework established in section 8.1.

Next, it consider the following two self-adjoint matrix operators

$$\begin{aligned} \mathrm{L}_{1,\lambda} &= \Big(\Big(-\frac{d^2}{dx^2} + \omega - p(\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{i,j}\Big), \\ \mathrm{L}_{2,\lambda} &= \Big(\Big(-\frac{d^2}{dx^2} + \omega - (\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{i,j}\Big), \end{aligned} \tag{10.41}$$

with $D(\mathrm{L}_{1,\lambda}) = D(\mathrm{L}_{2,\lambda}) = \mathbb{D}_{\lambda,\delta'}$. Here $\delta_{i,j}$ is the Kronecker symbol. These operators are associated in a standard way with the second derivative of the following action functional

$$\mathrm{S}_\lambda(\mathrm{V}) = \tfrac{1}{2}||\mathrm{V}'||^2 - \tfrac{1}{p+1}||\mathrm{V}||_{p+1}^{p+1} + \tfrac{1}{2\lambda}\Big|\sum_{j=1}^{N} v_j(0)\Big|^2 + \tfrac{\omega}{2}||\mathrm{V}||^2,$$

where $\mathrm{V} = (v_j)_{j=1}^N \in H^1(\mathcal{G})$. Namely,

$$(\mathrm{S}_\lambda)''(\Phi_{\lambda,\delta'})(\mathrm{U},\mathrm{V}) = \langle \mathrm{L}_{1,\lambda}\mathrm{U}_1, \mathrm{V}_1\rangle + \langle \mathrm{L}_{2,\lambda}\mathrm{U}_2, \mathrm{V}_2\rangle$$

with $\mathrm{U} = \mathrm{U}_1 + i\mathrm{U}_2$ and $\mathrm{V} = \mathrm{V}_1 + i\mathrm{V}_2$. Next, we consider the form $(\mathrm{S}_\lambda)''(\Phi_{\lambda,\delta'})$ as a linear operator

$$\mathrm{H}_\lambda = \begin{pmatrix} \mathrm{L}_{1,\lambda} & 0 \\ 0 & \mathrm{L}_{2,\lambda} \end{pmatrix}. \qquad (10.42)$$

The energy functional E_λ defined by (6.3) belongs to $C^2(H^1(\mathcal{G}),\mathbb{R})$ and so the analog of stability/instability Theorem 10.1 is true for $e^{i\omega t}\Phi_{\lambda,\delta'}$.

The following is the orbital stability/instability properties of the continuous tail profiles $\Phi_{\lambda,\delta'}$.

Theorem 10.7. *Let $\lambda < 0$, and $\omega > \frac{N^2}{\lambda^2}$. Let also $\Phi_{\lambda,\delta'}$ be defined by (10.40), and the space $H^1_{eq}(\mathcal{G})$ be defined by*

$$H^1_{eq}(\mathcal{G}) = \{(v_j)_{j=1}^N \in H^1(\mathcal{G}) : v_1(x) = \ldots = v_N(x),\ x > 0\}.$$

Then the following assertions hold.

(i) Let $1 < p \leqslant 5$.

1) If $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $e^{it\omega}\Phi_{\lambda,\delta'}$ is orbitally stable in $H^1(\mathcal{G})$.

2) If $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ and N is even, then $e^{it\omega}\Phi_{\lambda,\delta'}$ is orbitally unstable in $H^1(\mathcal{G})$.

(ii) Let $p > 5$ and $\omega \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Then there exists $\omega^* > \frac{N^2}{\lambda^2}$ such that $e^{it\omega}\Phi_{\lambda,\delta'}$ is orbitally unstable in $H^1(\mathcal{G})$ for $\omega > \omega^*$, and $e^{it\omega}\Phi_{\lambda,\delta'}$ is orbitally stable in $H^1_{eq}(\mathcal{G})$ for $\omega < \omega^*$.

The relative position of ω^* and $\frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is discussed in Remark 10.6. In the case $N = 2$ the above result coincides with Proposition 6.9(1) (partially) and Theorem 6.11 in (Adami and Noja 2013).

10.3.1 Spectrum of operators $\mathrm{L}_{i,\lambda}$, $i = 1, 2$, in (10.41)

Next we give the description of the spectrum of the operators $\mathrm{L}_{1,\lambda}$ and $\mathrm{L}_{2,\lambda}$ defined in (10.41).

Proposition 10.9. *Let $\lambda < 0$ and $\omega > \frac{N^2}{\lambda^2}$, then the following results hold.*

(i) $\ker(\mathrm{L}_{2,\lambda}) = span\{\Phi_{\lambda,\delta'}\}$, *and* $\mathrm{L}_{2,\lambda} \geqslant 0$.

(ii) *If* $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $\ker(\mathrm{L}_{1,\lambda}) = \{0\}$, *and* $n(\mathrm{L}_{1,\lambda}) = 1$.

(iii) *If* $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $n(\mathrm{L}_{1,\lambda}) = 1$, *and the kernel of* $\mathrm{L}_{1,\lambda}$ *is given by* $\ker(\mathrm{L}_{1,\lambda}) = span\{\hat{\Phi}_{\lambda,1}, .., \hat{\Phi}_{\lambda,N-1}\}$, *where*

$$\hat{\Phi}_{\lambda,j} = (0, .., 0, \underset{\text{j}}{\varphi'_{\lambda,\delta'}}, \underset{\text{j+1}}{-\varphi'_{\lambda,\delta'}}, 0, .., 0). \tag{10.43}$$

(iv) *If* $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, *then* $\ker(\mathrm{L}_{1,\lambda}) = \{0\}$, *and* $n(\mathrm{L}_{1,\lambda}) \leqslant N$. *Moreover, for N even in the space*

$$L^2_{\frac{N}{2}}(\mathcal{G}) = \{\mathrm{V} = (v_j)_{j=1}^N \in L^2(\mathcal{G}) : v_1(x) = ... = v_N(x),\ x > 0\},$$

we have $n(\mathrm{L}_{1,\lambda}|_{L^2_{\frac{N}{2}}(\mathcal{G})}) = 2$.

(v) *The rest of the spectrum of* $\mathrm{L}_{1,\lambda}$ *and* $\mathrm{L}_{2,\lambda}$ *is positive and bounded away from zero.*

Proof. (i) It is clear that $\Phi_{\lambda,\delta'} \in \ker(\mathrm{L}_{2,\lambda})$. To show the equality $\ker(\mathrm{L}_{2,\lambda}) = span\{\Phi_{\lambda,\delta'}\}$ let us note that any $\mathrm{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - (\varphi_{\lambda,\delta'})^{p-1} v_j = \frac{-1}{\varphi_{\lambda,\delta'}}\frac{d}{dx}\Big[\varphi^2_{\lambda,\delta'}\frac{d}{dx}\Big(\frac{v_j}{\varphi_{\lambda,\delta'}}\Big)\Big], \quad x > 0. \tag{10.44}$$

Thus, for $\mathrm{V} \in \mathbb{D}_{\lambda,\delta'}$, we obtain from (10.44), (10.39), and (10.40)

$$\langle \mathrm{L}_{2,\lambda}\mathrm{V}, \mathrm{V}\rangle = \sum_{j=1}^N \int_0^\infty \varphi^2_{\lambda,\delta'}\Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi_{\lambda,\delta'}}\Big)\Big]^2 dx + R_{\lambda,N}, \tag{10.45}$$

where

$$R_{\lambda,N} = \frac{1}{\lambda}\Big[\sum_{j=1}^N v_j(0)\Big]^2 - \frac{N}{\lambda}\sum_{j=1}^N v_j^2(0). \tag{10.46}$$

The term $R_{\lambda,N}$ if positive for $\lambda < 0$ by Jensen's inequality applied to $f(x) = x^2$. Thus, $\langle \mathrm{L}_{2,\lambda}\mathrm{V}, \mathrm{V}\rangle \geqq 0$ for $\mathrm{V} \in \mathbb{D}_{\lambda,\delta'} \cap [span\{\Phi_{\lambda,\delta'}\}]^\perp$ which proves (i).

(ii) Concerning the kernel of $\mathrm{L}_{1,\lambda}$, we recall that the only $L^2(\mathbb{R}_+)$-solution of the equation

$$-v_j'' + \omega v_j - p(\varphi_{\lambda,\delta'})^{p-1} v_j = 0$$

is given by $v_j = \varphi'_{\lambda,\delta'}$ (up to a factor). Thus, any element of $\ker(\mathrm{L}_{1,\lambda})$ has the form

$$\mathrm{V} = (v_j)_{j=1}^N = (c_j \varphi'_{\lambda,\delta'})_{j=1}^N, \; c_j \in \mathbb{R}.$$

If $v_1'(0) = ... = v_N'(0) \neq 0$, then by (10.39) we get $c_1 = ... = c_N \neq 0$, and consequently $N\varphi'_{\lambda,\delta'}(0) = \lambda \varphi''_{\lambda,\delta'}(0)$. Therefore, $\omega = \frac{N^2}{\lambda^2}$, which is impossible. Otherwise, the condition $v_j'(0) = 0$ implies that $\varphi''_{\lambda,\delta'}(0) = 0$, which is equivalent to the identity

$$\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}.$$

Thus, we get that $c_1 = ... = c_N = 0$ and $\mathrm{V} \equiv 0$ for $\omega \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

The proof of the equality $n(\mathrm{L}_{1,\lambda}) = 1$ for $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is similar to the one in the case of the operator $\mathbb{L}_{1,0}$ defined by (10.15). Namely, denoting

$$\boldsymbol{l}_\lambda = \Big(\Big(-\frac{d^2}{dx^2} + \omega - p(\varphi_{\lambda,\delta'})^{p-1}\Big)\delta_{i,j}\Big), \tag{10.47}$$

we define the following symmetric operator $\mathrm{L}_0' = \boldsymbol{l}_\lambda$ with

$$D(\mathrm{L}_0') = \Big\{\mathrm{V} \in H^2(\mathcal{G}) : v_1'(0) = ... = v_N'(0) = 0, \; \sum_{j=1}^N v_j(0) = 0\Big\}.$$

From Theorem 3.7 follows that $\mathrm{L}_{1,\lambda}$ is the family of self-adjoint extension of L_0'. Let us show that the operator L_0' is non-negative. First, note that any $\mathrm{V} = (v_j)_{j=1}^N \in H^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j - p(\varphi_{\lambda,\delta'})^{p-1} v_j = \frac{-1}{\varphi'_{\lambda,\delta'}}\frac{d}{dx}\Big[(\varphi'_{\lambda,\delta'})^2 \frac{d}{dx}\Big(\frac{v_j}{\varphi'_{\lambda,\delta'}}\Big)\Big], \quad x > 0.$$

Using the above equality and integrating by parts, we get for $\mathrm{V} \in D(\mathrm{L}_0')$

$$\langle \mathrm{L}_0'\mathrm{V}, \mathrm{V}\rangle = \sum_{j=1}^N \int_0^\infty (\varphi'_{\lambda,\delta'})^2 \Big[\frac{d}{dx}\Big(\frac{v_j}{\varphi'_{\lambda,\delta'}}\Big)\Big]^2 dx - \sum_{j=1}^N v_j^2(0)\frac{\varphi''_{\lambda,\delta'}(0)}{\varphi'_{\lambda,\delta'}(0)}.$$

Taking into account that

$$-v_j^2(0)\frac{\varphi''_{\lambda,\delta'}(0)}{\varphi'_{\lambda,\delta'}(0)} = v_j^2(0)\frac{\lambda\omega}{2N}\Big(p - 1 - (p+1)\frac{N^2}{\lambda^2\omega}\Big), \tag{10.48}$$

we get non-negativity of L_0' for $\omega \leqslant \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. By extension theory follows that $n(\mathrm{L}_{1,\lambda}) \leqq 1$. Moreover, due to

$$\langle \mathrm{L}_{1,\lambda}\Phi_{\lambda,\delta'}, \Phi_{\lambda,\delta'}\rangle = -(p-1)||\Phi_{\lambda,\delta'}||_{p+1}^{p+1} < 0,$$

we finally arrive at $n(\mathrm{L}_{1,\lambda}) = 1$, and (ii) is proved.

(iii) From the proof of item (ii) we induce that $n(\mathrm{L}_{1,\lambda}) = 1$, and the kernel of $\mathrm{L}_{1,\lambda}$ is nonempty as $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Moreover, we know that any element of the kernel has the form $\mathrm{V} = (v_j)_{j=1}^N = (c_j\varphi'_{\lambda,\delta'})_{j=1}^N$, $c_j \in \mathbb{R}$, and it is necessary that $v_1'(0) = ... = v_N'(0) = 0$. Hence, the condition

$$\lambda v_1'(0) = \sum_{j=1}^N v_j(0) = 0 \tag{10.49}$$

gives rise to $(N-1)$-dimensional kernel of $\mathrm{L}_{1,\lambda}$. Since the functions $\hat{\Phi}_{\lambda,j}$, $1 \leqslant j \leqslant N-1$, defined in (10.43) are linearly independent and satisfy the condition (10.49), they form the basis in $\ker(\mathrm{L}_{1,\lambda})$, and (iii) is proved.

(iv) The identity $\ker(\mathrm{L}_{1,\lambda}) = \{0\}$ was shown in (ii). To show the inequality $n(\mathrm{L}_{1,\lambda}) \leqslant N$ we introduce the following minimal symmetric operator $\mathrm{L}_{\min} = \boldsymbol{l}_\lambda$ with

$$D(\mathrm{L}_{\min}) = \left\{\mathrm{V} \in H^2(\mathcal{G}) : \begin{array}{c} v_1'(0) = ... = v_N'(0) = 0, \\ v_1(0) = ... = v_N(0) = 0 \end{array}\right\}, \tag{10.50}$$

where $\boldsymbol{l}_\lambda$ is defined in (10.47). The operator $\mathrm{L}_{1,\lambda}$ is self-adjoint extension of $\mathrm{L}_{\min}$. From the formula (10.48) it follows that $\mathrm{L}_{\min}$ is a non-negative operator. It is obvious that $\mathrm{L}_{\min}^* = \boldsymbol{l}_\lambda$, $D(\mathrm{L}_{\min}^*) = H^2(\mathcal{G})$. Then, due to the von Neumann formula (for $\mathrm{L}_{\min}$ acting on complex-valued functions)

$$D(\mathrm{L}_{\min}^*) = D(\mathrm{L}_{\min}) \oplus span\{\mathrm{V}_i^1, .., \mathrm{V}_i^N\} \oplus span\{\mathrm{V}_{-i}^1, .., \mathrm{V}_{-i}^N\},$$

where

$$\mathrm{V}_{\pm i}^j = (0, ..., \underset{\mathrm{j}}{e^{i\sqrt{\pm i}x}}, 0, ..., 0), \quad Im(\sqrt{\pm i}) > 0,$$

and consequently $n_\pm(\mathrm{L}_{\min}) = N$. By extension theory, $n(\mathrm{L}_{1,\lambda}) \leqslant N$.

Let N be even. It is easily seen that $n_\pm(\mathrm{L}_{\min}) = 2$ in $L^2_{\frac{N}{2}}(\mathcal{G})$. Indeed,

$$D(\mathrm{L}_{\min}^*) = D(\mathrm{L}_{\min}) \oplus span\{\widetilde{\mathrm{V}}_i^1, \widetilde{\mathrm{V}}_i^2\} \oplus span\{\widetilde{\mathrm{V}}_{-i}^1, \widetilde{\mathrm{V}}_{-i}^2\},$$

where

$$\widetilde{\mathrm{V}}_{\pm i}^1 = (\underset{1}{e^{i\sqrt{\pm i}x}}, ..., \underset{\mathrm{N/2}}{e^{i\sqrt{\pm i}x}}, \underset{\mathrm{N/2+1}}{0}, ..., \underset{\mathrm{N}}{0}),$$

and

$$\widetilde{\mathrm{V}}_{\pm i}^2 = (\underset{1}{0}, ..., \underset{\mathrm{N/2}}{0}, \underset{\mathrm{N/2+1}}{e^{i\sqrt{\pm i}x}}, ..., \underset{\mathrm{N}}{e^{i\sqrt{\pm i}x}}).$$

By extension theory, we get $n(\mathrm{L}_{1,\lambda}) \leqslant 2$ in $L^2_{\frac{N}{2}}(\mathcal{G})$.

Next, we introduce the following quadratic form $\mathbb{F}_{1,\lambda}$ associated with the operator $\mathrm{L}_{1,\lambda}$

$$\mathbb{F}_{1,\lambda}(\mathrm{V}) = ||\mathrm{V}'||^2 + \omega||\mathrm{V}||^2 - p\sum_{j=1}^{N}\int_0^{\infty}(\varphi_{\lambda,\delta'})^{p-1}|v_j|^2dx + \tfrac{1}{\lambda}\Big|\sum_{j=1}^{N}v_j(0)\Big|^2,$$

with $D(\mathbb{F}_{1,\lambda}) = H^1(\mathcal{G})$. Let

$$\Phi_\lambda^- = (\underset{1}{\varphi'_{\lambda,\delta'}}, ..., \underset{\mathrm{N}/2}{\varphi'_{\lambda,\delta'}}, \underset{\mathrm{N}/2+1}{-\varphi'_{\lambda,\delta'}}, ..., \underset{\mathrm{N}}{-\varphi'_{\lambda,\delta'}}),$$

then integrating by parts we obtain

$$\begin{aligned}\mathbb{F}_{1,\lambda}(\Phi_\lambda^-) &= N\int_0^{\infty}\varphi'_{\lambda,\delta'}\left(-\varphi'''_{\lambda,\delta'} + \omega\varphi'_{\lambda,\delta'} - p(\varphi_{\lambda,\delta'})^{p-1}\varphi'_{\lambda,\delta'}\right)dx\\ &\quad - N\varphi'_{\lambda,\delta'}(0)\varphi''_{\lambda,\delta'}(0)\\ &= \tfrac{N^2}{2\lambda}\omega\left[\left(\tfrac{(p+1)\omega}{2}\right)\left(1-\tfrac{N^2}{\lambda^2\omega}\right)\right]^{\frac{2}{p-1}}\left(p-1-(p+1)\tfrac{N^2}{\lambda^2\omega}\right),\end{aligned}$$

which is negative for $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. Since $\langle \mathrm{L}_{1,\lambda}\Phi_{\lambda,\delta'}, \Phi_{\lambda,\delta'}\rangle < 0$, we get by orthogonality of Φ_λ^- and $\Phi_{\lambda,\delta'}$

$$\mathbb{F}_{1,\lambda}(s\Phi_{\lambda,\delta'} + r\Phi_\lambda^-) = |s|^2F^{\lambda}_{1,\omega}(\Phi_{\lambda,\delta'}) + |r|^2F^{\lambda}_{1,\omega}(\Phi_\lambda^-) < 0.$$

Thus, we obtain that $\mathbb{F}_{1,\lambda}$ is negative on two-dimensional subspace $\mathcal{M} = span\{\Phi_{\lambda,\delta'}, \Phi_\lambda^-\}$. Therefore, by minimax principle, we get $n(\mathrm{L}_{1,\lambda}) \geqslant 2$. The assertion (iv) is proved. The proof of item (v) is standard and relies on Weyl's theorem. This finishes the proof of the Proposition. □

10.3.2 Slope analysis

Next we study the sign of $\partial_\omega||\Phi_{\lambda,\delta'}||^2$.

Proposition 10.10. *Let* $\omega > \frac{N^2}{\lambda^2}$, $\lambda < 0$, *and* $J(\omega) = \partial_\omega||\Phi_{\lambda,\delta'}||^2$.

(i) *If* $1 < p \leqslant 5$, *then* $J(\omega) > 0$.

(ii) *If* $p > 5$, *then there exists* ω^* *such that* $J(\omega^*) = 0$, *and* $J(\omega) > 0$ *for* $\omega \in (\frac{N^2}{\lambda^2}, \omega^*)$, *while* $J(\omega) < 0$ *for* $\omega \in (\omega^*, \infty)$.

Proof. Recall that $\Phi_{\lambda,\delta'} = (\varphi_{\lambda,\delta'})_{j=1}^N$, where $\varphi_{\lambda,\delta'}$ is defined by (10.40), we have via change of variables

$$\int_0^\infty (\varphi_{\lambda,\delta'}(x))^2 dx = \left(\frac{p+1}{2}\right)^{\frac{2}{p-1}} \frac{2\omega^{\frac{2}{p-1}-\frac{1}{2}}}{p-1} \int_{\frac{N}{|\lambda|\sqrt{\omega}}}^1 (1-t^2)^{\frac{2}{p-1}-1} dt.$$

From the last equality, we get

$$J(\omega) = C\omega^{\frac{7-3p}{2(p-1)}} J_1(\omega), \qquad C = \tfrac{N}{p-1}\left(\frac{p+1}{2}\right)^{\frac{2}{p-1}}, \tag{10.51}$$

where

$$J_1(\omega) = \tfrac{5-p}{p-1} \int_{\frac{N}{|\lambda|\sqrt{\omega}}}^1 (1-t^2)^{\frac{3-p}{p-1}} dt + \tfrac{N}{|\lambda|\sqrt{\omega}}(1-\tfrac{N^2}{\lambda^2\omega})^{\frac{3-p}{p-1}}.$$

Thus,

$$J_1'(\omega) = \tfrac{N}{|\lambda|\omega^{3/2}}\tfrac{3-p}{p-1}\left[\left(1-\tfrac{N^2}{\lambda^2\omega}\right)^{\frac{3-p}{p-1}} + \tfrac{N^2}{\lambda^2\omega}\left(1-\tfrac{N^2}{\lambda^2\omega}\right)^{-\frac{2(p-2)}{p-1}}\right]. \tag{10.52}$$

It is immediate that $J(\omega) > 0$ for $1 < p \leqslant 5$. Consider the case $p > 5$. It is easily seen

$$\lim_{\omega\to+\infty} J_1(\omega) = \frac{5-p}{p-1}\int_0^1 (1-t^2)^{\frac{3-p}{p-1}} dt < 0, \qquad \lim_{\omega\to\frac{N^2}{\lambda^2}} J_1(\omega) = \infty.$$

Moreover, from (10.52) it follows that $J_1'(\omega) < 0$ for $\omega > \frac{N^2}{\lambda^2}$, and consequently $J_1(\omega)$ is strictly decreasing. Therefore, there exists a unique $\omega^* > \frac{N^2}{\lambda^2}$ such that $J_1(\omega^*) = J(\omega^*) = 0$, consequently $J(\omega) > 0$ for $\omega \in (\frac{N^2}{\lambda^2}, \omega^*)$, and $J(\omega) < 0$ for $\omega \in (\omega^*, \infty)$. □

10.3.3 Proof of Theorem 10.7

In this subsection we proof Theorem 10.7 via the framework established in section 10.1 above.

Proof. (i) 1) Combining Theorem 6.3, Theorem 10.1 (adapted to the case of the NLS-δ' equation), Proposition 10.9 (items (i), (ii) and (v)), and Proposition 10.10-(i), we get stability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1(\mathcal{G})$.

2) Combining Theorem 10.1, Proposition 10.9 (items (i), (iv) and (v)), and Proposition 10.10-(i), we get orbital instability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1_{\frac{N}{2}}(\mathcal{G})$ (compare with Remark 10.1-(ii)). We note that well-posedness of the Cauchy problem associated with equation

(4.3) in $H^1_{\frac{N}{2}}(\mathcal{G})$ follows from the uniqueness of the solution to the Cauchy problem in $H^1(\mathcal{G})$ and the fact that the group $e^{-it\mathrm{H}^{\delta'}_\lambda}$ preserves the space $H^1_{\frac{N}{2}}(\mathcal{G})$. Finally, instability in the smaller space $H^1_{\frac{N}{2}}(\mathcal{G})$ induces instability in all $H^1(\mathcal{G})$.

(ii) Relative position of ω^* and $\omega = \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ is not clear (see Remark 10.6), which complicates the analysis in the framework of Theorem 10.1. But we can overcome this difficulty restricting the operator $\mathrm{L}_{1,\lambda}$ onto the space $L^2_{eq}(\mathcal{G})$. Moreover, we introduce $H^1_{eq}(\mathcal{G}) = H^1(\mathcal{G}) \cap L^2_{eq}(\mathcal{G})$. We note that $H^1_{eq}(\mathcal{G})$ is also preserved by the group $e^{-it\mathrm{H}^{\delta'}_\lambda}$ (see (Angulo and Goloshchapova 2018)) .

Recall that $\mathrm{L}_{1,\lambda}$ is the self-adjoint extension of the minimal symmetric operator $\mathrm{L}_{\min}$ defined by (10.50). It is easily seen that the operator $\mathrm{L}_{\min}|_{L^2_{eq}(\mathcal{G})}$ satisfies $\mathcal{D}_\pm(\mathrm{L}_{\min}|_{L^2_{eq}(\mathcal{G})}) = span\{(e^{i\sqrt{\pm i}x})_{j=1}^N\}$. The last equality, implies by extension theory and relation

$$\langle \mathrm{L}_{1,\lambda}\Phi_{\lambda,\delta'}, \Phi_{\lambda,\delta'}\rangle < 0,$$

with $\Phi_{\lambda,\delta'} \in L^2_{eq}(\mathcal{G})$, that $n(\mathrm{L}_{1,\lambda}|_{L^2_{eq}(\mathcal{G})}) = 1$.

Without loss of generality we can assume that $\omega^* \neq \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$. All our forthcoming conclusions about orbital stability are based on Theorem 10.1 for the spaces $H^1(\mathcal{G})$ and $H^1_{eq}(\mathcal{G})$, Remark 10.1, Theorem 6.3, Proposition 10.9, and Proposition 10.10. Consider 2 cases.

1. Suppose that $\omega^* < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

Let $\omega < \omega^* < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathrm{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and we have $\partial_\omega||\Phi_{\lambda,\delta'}||^2 > 0$. Therefore, $e^{i\omega t}\Phi_{\lambda,\delta'}$ is orbitally stable in $H^1(\mathcal{G})$, and hence in $H^1_{eq}(\mathcal{G})$.

If $\omega^* < \omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathrm{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and $\partial_\omega||\Phi_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1(\mathcal{G})$.

Let $\omega > \frac{N^2}{\lambda^2}\frac{p+1}{p-1} > \omega^*$. Then $n(\mathrm{L}_{1,\lambda}|_{L^2_{eq}(\mathcal{G})}) = 1$ and also $\partial_\omega||\Phi_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1_{eq}(\mathcal{G})$ and consequently in $H^1(\mathcal{G})$.

2. Suppose that $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$.

If $\omega < \frac{N^2}{\lambda^2}\frac{p+1}{p-1} < \omega^*$, then $n(\mathrm{L}_{1,\lambda}) = 1$ in $L^2(\mathcal{G})$ and $\partial_\omega||\Phi_{\lambda,\delta'}||^2 > 0$, consequently $e^{i\omega t}\Phi_{\lambda,\delta'}$ is orbitally stable in $H^1(\mathcal{G})$, and therefore in $H^1_{eq}(\mathcal{G})$.

If $\frac{N^2}{\lambda^2}\frac{p+1}{p-1} < \omega < \omega^*$, then $n(\mathrm{L}_{1,\lambda}|_{L^2_{eq}(\mathcal{G})}) = 1$ and $\partial_\omega||\Phi_{\lambda,\delta'}||^2 > 0$, which induces stability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1_{eq}(\mathcal{G})$.

Let $\omega > \omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$, then $n(\mathrm{L}_{1,\lambda}|_{L^2_{eq}(\mathcal{G})}) = 1$ and $\partial_\omega||\Phi_{\lambda,\delta'}||^2 < 0$, which induces orbital instability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1_{eq}(\mathcal{G})$ and consequently in $H^1(\mathcal{G})$.

Summarizing all the cases, we get for $\omega > \omega^*$ nonlinear instability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in $H^1(\mathcal{G})$, and for $\omega < \omega^*$ stability of $e^{i\omega t}\Phi_{\lambda,\delta'}$ at least in $H^1_{eq}(\mathcal{G})$. This finishes the proof. □

Remark 10.5. (i) It is worth mentioning that the orbital instability result follows easily for $2 < p < 5$ from the spectral instability using the fact that the mapping data-solution for (4.3) is of class C^2 (see Theorem 6.3 and Remark 10.1-(iii)).

(ii) Observe that for $p > 5$ the orbital instability results are obtained via classical approach by Grillakis, Shatah, and W. Strauss (1987) without using spectral instability. Otherwise, the orbital instability can be deduced from the spectral one since for $p > 5$ the mapping data-solution for (4.3) is of class C^2.

Remark 10.6. Note that the integral appearing in (10.51) (via change of variables) is related to the incomplete Beta function

$$B\Big(y; \frac{1}{2}, b\Big) = \int_0^y x^{-\frac{1}{2}}(1-x)^{b-1}dx,$$

with $b = \frac{2}{p-1}$. Using basic numerical simulations, one can show that for $p = 6, 7, ...,$ relation $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ holds. By the continuity of the function J as a function of p, we get the relation $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ in the neighborhood of every integer $p > 5$.

We conjecture that $\omega^* > \frac{N^2}{\lambda^2}\frac{p+1}{p-1}$ holds for any $p > 5$. This conjecture by Theorem 10.1 implies the following stability properties of $e^{i\omega t}\Phi_{\lambda,\delta'}$ in the case $p > 5$:

(i) if $\omega \in (\frac{N^2}{\lambda^2}, \frac{N^2}{\lambda^2}\frac{p+1}{p-1})$, then $e^{i\omega t}\Phi_{\lambda,\delta'}$ is stable in $H^1(\mathcal{G})$;

(ii) if $\omega \in (\frac{N^2}{\lambda^2}\frac{p+1}{p-1}, \omega^*)$ and N is even, then $e^{i\omega t}\Phi_{\lambda,\delta'}$ is unstable in $H^1(\mathcal{G})$.

(iii) In the case $\omega = \omega^*$ and $p > 5$ we conjecture due to (Ohta 2011, Corollary 2) that the standing wave $e^{i\omega t}\Phi_{\lambda,\delta'}$ is unstable.

10.4 Stability theory for the NLS-log-δ on a star graph

In this section we consider the following NLS equation with logarithmic nonlinearity on star graph $\mathcal{G}$ (NLS-log-δ),

$$i\partial_t \mathrm{U} - \mathrm{H}_\alpha^\delta \mathrm{U} + \mathrm{U} Log|\mathrm{U}|^2 = 0, \tag{10.53}$$

with the δ-interaction operator H_α^δ defined in (10.2).

We are interested here in the stability properties of the standing wave $e^{i\omega t}\Psi_{\alpha,\delta} \in \mathbb{D}_{\alpha,\delta}$ where $\Psi_{\alpha,\delta} = (\psi_{\alpha,\delta})_{j=1}^N$ is of Gaussian type

$$\psi_{\alpha,\delta}(x) = e^{\frac{\omega+1}{2}} e^{-\frac{(x-\frac{\alpha}{N})^2}{2}}, \quad \alpha \neq 0,\ \omega \in \mathbb{R}. \tag{10.54}$$

Our main result is the following.

Theorem 10.8. *Let $\omega \in \mathbb{R}$, and $\Psi_{\alpha,\delta} = (\psi_{\alpha,\delta})_{j=1}^N$ be defined by (10.54). Then the standing wave $e^{i\omega t}\Psi_{\alpha,\delta}$ is orbitally stable in $W^1_{\mathcal{E}}(\mathcal{G})$ for any $\alpha < 0$, and $e^{i\omega t}\Psi_{\alpha,\delta}$ is spectrally unstable for any $\alpha > 0$.*

The strategy of the proof of Theorem 10.8 is analogous to the one in the previous case of the NLS model (10.1) with power nonlinearity. In particular, we will use the adapted (weaker) version of the stability/instability Theorem 10.1 (to the specific Gaussian profile $\Psi_{\alpha,\delta}$ and the space $W^1_{\mathcal{E}}(\mathcal{G})$).

Consider the following two harmonic oscillator self-adjoint matrix operators with domain $D(\mathbb{T}_{1,\alpha}) = D(\mathbb{T}_{2,\alpha}) = \mathbb{D}^{\log}_{\alpha,\delta}$ defined by

$$
\begin{aligned}
\mathrm{T}_{1,\alpha} &= \Big(\Big(-\frac{d^2}{dx^2} + (x - \tfrac{\alpha}{N})^2 - 3\Big)\delta_{i,j}\Big),\\
\mathrm{T}_{2,\alpha} &= \Big(\Big(-\frac{d^2}{dx^2} + (x - \tfrac{\alpha}{N})^2 - 1\Big)\delta_{i,j}\Big),\\
\mathbb{D}^{\log}_{\alpha,\delta} &:= \Big\{\mathrm{V} \in W^2(\mathcal{G}) : v_1(0) = \ldots = v_N(0), \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\Big\},
\end{aligned}
\tag{10.55}
$$

where $\delta_{i,j}$ is the Kronecker symbol. These operators are associated with $\mathrm{H}_{\alpha,\log} := (\mathrm{S}_{\alpha,\log})''(\Psi_{\alpha,\delta})$, where

$$
\mathrm{S}_{\alpha,\log}(\mathrm{V}) = E_{\alpha,Log}(\mathrm{V}) + \frac{\omega+1}{2}\|\mathrm{V}\|^2
$$

and in a standard way,

$$
\mathrm{H}_{\alpha,\log} = \begin{pmatrix} \mathrm{T}_{1,\alpha} & 0 \\ 0 & \mathrm{T}_{2,\alpha} \end{pmatrix}.
$$

Noting that for any ω, $\partial_\omega ||\Psi_{\alpha,\delta}||^2 > 0$, $E_{\alpha,\log} \in C(W^1_{\mathcal{E}}(\mathcal{G}), \mathbb{R})$ (see Angulo and Goloshchapova (2018, Proposition 2.3)), and combining Grillakis, Shatah, and W. Strauss (1987, Theorem 3.5) with Grillakis, Shatah, and W. Strauss (1990, Theorem 5.1), we can formulate the stability/instability theorem for the NLS-log-δ equation.

Theorem 10.9. *Let $\alpha \neq 0$, and $n(\mathrm{H}_{\alpha,\log})$ be the number of negative eigenvalues of $\mathrm{H}_{\alpha,\log}$. Suppose also that*

1) $\ker(|\mathrm{T}_{2,\alpha}) = span\{\Psi_{\alpha,\delta}\}$,

2) $\ker(\mathrm{T}_{1,\alpha}) = \{0\}$,

3) *the negative spectrum of $\mathrm{T}_{1,\alpha}$ and $\mathrm{T}_{2,\alpha}$ consists of a finite number of negative eigenvalues (counting multiplicities),*

4) *the rest of the spectrum of $\mathrm{T}_{2,\alpha}$ and $\mathrm{T}_{1,\alpha}$ is positive and bounded away from zero.*

Then the following assertions hold.

(i) *If $n(\mathrm{H}_{\alpha,\log}) = 1$, then the standing wave $e^{i\omega t}\Psi_{\alpha,\delta}$ is orbitally stable in $W^1_{\mathcal{E}}(\mathcal{G})$.*

(ii) *If $n(\mathrm{H}_{\alpha,\log}) = 2$ in $L^2_k(\mathcal{G})$, then the standing wave $e^{i\omega t}\Psi_{\alpha,\delta}$ is spectrally unstable.*

Remark 10.7. (i) By saying $e^{i\omega t}\Psi_{\alpha,\delta}$ is spectrally unstable we mean that the spectrum of the linear part

$$\mathrm{A}_{\alpha,\log} = \begin{pmatrix} 0 & \mathrm{T}_{2,\alpha} \\ -\mathrm{T}_{1,\alpha} & 0 \end{pmatrix}$$

of the linearization of the NLS-log-δ equation around $\Psi_{\alpha,\delta}$ contains an eigenvalue with positive real part.

(ii) In item (ii) of Theorem 10.9 we affirm only spectral instability since we can't apply neither Ohta (2011, Corollary 3 and 4) (since we don't know if $E_{\alpha,\log} \in C^2(W^1_{\mathcal{E}}(\mathcal{G}), \mathbb{R})$), nor Henry, Perez, and Wreszinski (1982, Theorem 2 Remark, Section 2) (since we don't know if the mapping data-solution associated to the NLS-log-δ equation is at least of class C^2 around $\Psi_{\alpha,\delta}$) to prove orbital instability (see Remark 10.1 above). We recall that global well-posedness Theorem 6.4 was not obtained via the Banach Contraction Principle.

10.4.1 Spectrum of operators $\mathrm{T}_{i,\alpha}$, $i = 1, 2$, in (10.55)

Next we study the spectral properties of $\mathrm{T}_{1,\alpha}$ and $\mathrm{T}_{2,\alpha}$. To investigate the spectrum of the operator $\mathrm{T}_{1,\alpha}$ we will use the perturbation theory analogously to the previous case of the NLS-δ equation with power nonlinearity. In particular, define the following self-adjoint Schrödinger operator on $L^2(\mathcal{G})$ with Kirchhoff condition at $\nu = 0$

$$\mathrm{T}_{1,0} = \Big(\Big(-\frac{d^2}{dx^2} + x^2 - 3\Big)\delta_{i,j}\Big), \tag{10.56}$$

$$D(\mathrm{T}_{1,0}) = \Big\{\mathrm{V} \in W^2(\mathcal{G}) : v_1(0) = ... = v_N(0),\ \sum_{j=1}^{N} v_j'(0) = 0\Big\}.$$

As above $\mathrm{T}_{1,\alpha}$ "tends" to $\mathrm{T}_{1,0}$ for $\alpha \to 0$. In the next Theorem we describe the spectral properties of $\mathrm{T}_{1,0}$.

Theorem 10.10. *Let* $\mathrm{T}_{1,0}$ *be defined by* (10.56) *and* $k \in \{1, ..., N-1\}$. *Then the following assertions hold*

(i) $\ker(T_{1,0}) = span\{\hat{\Psi}_{0,1}, ..., \hat{\Psi}_{0,N-1}\}$, *where*

$$\hat{\Psi}_{0,j} = (0, ..., 0, \underset{\mathrm{j}}{\psi_0'}, \underset{\mathrm{j+1}}{-\psi_0'}, 0, ..., 0), \qquad \psi_0(x) = e^{-\frac{x^2}{2}}.$$

(ii) *In the space* $L^2_k(\mathcal{G})$ *we have* $\ker(\mathrm{T}_{1,0}) = span\{\widetilde{\Psi}_{0,k}\}$, *where*

$$\widetilde{\Psi}_{0,k} = \Big(\underset{1}{\tfrac{N-k}{k}\psi_0'}, ..., \underset{\mathrm{k}}{\tfrac{N-k}{k}\psi_0'}, \underset{\mathrm{k+1}}{-\psi_0'}, ..., \underset{\mathrm{N}}{-\psi_0'}\Big), \tag{10.57}$$

i.e. $\ker(\mathrm{T}_{1,0}|_{L^2_k(\mathcal{G})}) = span\{\widetilde{\Psi}_{0,k}\}$.

(iii) *The operator* $\mathrm{T}_{1,0}$ *has one simple negative eigenvalue, i.e. we have* $n(\mathrm{T}_{1,0}) = 1$. *Moreover, operator* $\mathrm{T}_{1,0}$ *has one simple negative eigenvalue in* $L^2_k(\mathcal{G})$*, i.e. we have* $n(\mathrm{T}_{1,0}|_{L^2_k(\mathcal{G})}) = 1$.

(iv) *The spectrum of* $\mathrm{T}_{1,0}$ *is discrete.*

Proof. The proof of items (i)-(ii) repeats the one of Theorem 10.5 (i)-(ii).

(iii) We will follow the ideas of the proof of item (iii) of Theorem 10.5 and Lemma 4.11 in (Angulo and Goloshchapova 2017b) (see Chapter 3). Denote

$$\boldsymbol{t}_0 = \Big(\Big(-\frac{d^2}{dx^2} + x^2 - 3\Big)\delta_{i,j}\Big).$$

First, one needs to show that the operator T_0 acting as $\mathrm{T}_0 = \boldsymbol{t}_0$ on

$$D(\mathrm{T}_0) = \Big\{\mathrm{V} \in W^2(\mathcal{G}) : v_1(0) = ... = v_N(0) = 0, \sum_{j=1}^{N} v_j'(0) = 0\Big\}.$$

is non-negative. The proof follows from the identity

$$-v_j'' + (x^2 - 3)v_j = \frac{-1}{\psi_0'}\frac{d}{dx}\Big[(\psi_0')^2 \frac{d}{dx}\Big(\frac{v_j}{\psi_0'}\Big)\Big], \quad x > 0,$$

for any $\mathrm{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$.

Next we need to prove that $n_\pm(\mathrm{T}_0) = 1$. We use the ideas of the proof of Angulo and Goloshchapova (2017b, Lemma 4.11). First, we establish the scale of Hilbert spaces associated with the self-adjoint non-negative operator $\mathrm{T} = \boldsymbol{t}_0 + 3I$ defined on

$$D(\mathrm{T}) = \Big\{\mathrm{V} \in W^2(\mathcal{G}) : v_1(0) = ... = v_N(0), \sum_{j=1}^{N} v_j'(0) = 0\Big\}.$$

Define for $s \geqslant 0$ the space

$$\mathfrak{H}_s(\mathrm{T}) = \Big\{\mathrm{V} \in L^2(\mathcal{G}) : \|\mathrm{V}\|_{s,2} = \Big\|(\mathrm{T} + I)^{s/2}\mathrm{V}\Big\| < \infty\Big\}.$$

The space $\mathfrak{H}_s(\mathrm{T})$ with norm $\|\cdot\|_{s,2}$ is complete. The dual space of $\mathfrak{H}_s(\mathrm{T})$ is denoted by $\mathfrak{H}_{-s}(\mathrm{T}) = \mathfrak{H}_s(\mathrm{T})'$. The norm in the space $\mathfrak{H}_{-s}(\mathrm{T})$ is defined by the formula

$$\|\mathrm{V}\|_{-s,2} = \|(\mathrm{T} + I)^{-s/2}\mathrm{V}\|.$$

The spaces $\mathfrak{H}_s(\mathrm{T})$ form the following chain

$$... \subset \mathfrak{H}_2(\mathrm{T}) \subset \mathfrak{H}_1(\mathrm{T}) \subset L^2(\mathcal{G}) = \mathfrak{H}_0(\mathrm{T}) \subset \mathfrak{H}_{-1}(\mathrm{T}) \subset \mathfrak{H}_{-2}(\mathrm{T}) \subset ...$$

The norm in the space $\mathfrak{H}_1(\mathrm{T})$ can be calculated as follows

$$\|\mathrm{V}\|_{1,2}^2 = ((\mathrm{T}+I)^{1/2}\mathrm{V}, (\mathrm{T}+I)^{1/2}\mathrm{V}) = \sum_{j=1}^{N}\int_0^\infty \left(|v_j'(x)|^2 + |v_j(x)|^2 + x^2|v_j(x)|^2\right) dx.$$

Therefore, we have the embedding $\mathfrak{H}_1(\mathrm{T}) \hookrightarrow H^1(\mathcal{G})$ and, by the Sobolev embedding, $\mathfrak{H}_1(\mathrm{T}) \hookrightarrow L^\infty(\mathcal{G})$. From the former remark we obtain that the functional $\delta_1 : \mathfrak{H}_1(\mathrm{T}) \to \mathbb{C}$ acting as $\delta_1(\mathrm{V}) = v_1(0)$ belongs to $\mathfrak{H}_1(\mathrm{T})' = \mathfrak{H}_{-1}(\mathrm{T})$, and consequently $\delta_1 \in \mathfrak{H}_{-2}(\mathrm{T})$. Therefore, it follows that the restriction $\hat{\mathrm{T}}_0$ of the operator T onto the domain

$$D(\hat{\mathrm{T}}_0) = \{\mathrm{V} \in D(\mathrm{T}) : \delta_1(\mathrm{V}) = v_1(0) = 0\} = D(\mathrm{T}_0)$$

is a densely defined symmetric operator with equal deficiency indices $n_\pm(\hat{\mathrm{T}}_0) = 1$. By extension theory we obtain that the operators $\hat{\mathrm{T}}_0$ and T_0 have equal deficiency indices. Therefore, $n(\mathrm{T}_{1,0}) \leqslant 1$. Since $\mathrm{T}_{1,0}\Psi_0 = -2\Psi_0$, where $\Psi_0 = (\psi_0)_{j=1}^N$, we get $n(\mathrm{T}_{1,0}) = 1$. As $\Psi_0 \in L^2_k(\mathcal{G})$ for any k, we get $n(\mathrm{T}_{1,0}|_{L^2_k(\mathcal{G})}) = 1$.

(iv) With slight modifications we can repeat the proof of Berezin and Shubin (1991, Theorem 3.1, Chapter II) to show that the spectrum of $\mathrm{T}_{1,0}$ is discrete by $\lim_{x\to+\infty}(x^2-3) = +\infty$, i.e.

$$\sigma(\mathrm{T}_{1,0}) = \sigma_p(\mathrm{T}_{1,0}) = \{\mu_{0,j}\}_{j\in\mathbb{N}}.$$

In particular, we have the following distribution of the eigenvalues

$$\mu_{0,1} < \mu_{0,2} < \cdots < \mu_{0,j} < \cdots,$$

with $\mu_{0,j} \to +\infty$ as $j \to +\infty$. □

Proposition 10.11. *Let $k \in \{1, ..., N-1\}$, $\alpha \neq 0$, and $\Psi_{\alpha,\delta}$ be defined by* (10.54)*. Then*

(i) $\ker(\mathrm{T}_{2,\alpha}) = span\{\Psi_{\alpha,\delta}\}$ *and* $\mathrm{T}_{2,\alpha} \geqslant 0$,

(ii) $\ker(\mathrm{T}_{1,\alpha}) = \{0\}$,

(iii) *for $\alpha > 0$, $n(\mathrm{T}_{1,\alpha}) = 2$ in $L^2_k(\mathcal{G})$, i.e. $n(\mathrm{T}_{1,\alpha}|_{L^2_k(\mathcal{G})}) = 2$,*

(iv) *for $\alpha < 0$, $n(\mathrm{T}_{1,\alpha}) = 1$ in $L^2(\mathcal{G})$,*

(v) *the spectrum of the operators $\mathrm{T}_{1,\alpha}$ and $\mathrm{T}_{2,\alpha}$ in $L^2(\mathcal{G})$ is discrete.*

Proof. (i) We only need to note that any $\mathrm{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + ((x - \tfrac{\alpha}{N})^2 - 1)v_j = \frac{-1}{\psi_{\alpha,\delta}}\frac{d}{dx}\left[\psi_{\alpha,\delta}^2 \frac{d}{dx}\left(\frac{v_j}{\psi_{\alpha,\delta}}\right)\right], \quad x > 0.$$

(ii) The proof is standard. It is sufficient to note that any vector from the kernel of $\mathrm{T}_{1,\alpha}$ has the form $\mathrm{V} = (v_j)_{j=1}^N$, where $v_j = c_j \psi'_{\alpha,\delta}\ c_j \in \mathbb{R}$.

(iii) The proof of this item is analogous to the one of the item (iii) of Proposition 10.4. It suffices to note that for the operator $\mathrm{T}_{1,\alpha}$ the coefficient μ_0 in the decomposition (10.33) is negative. Indeed, (see the proof of Proposition 4.17 in (Angulo and Goloshchapova 2017b))

$$\mu_0 = -\frac{2(N-k)}{k||\widetilde{\Psi}_{0,k}||^2} \int_0^\infty x(\psi_0')^2 dx + O(\alpha),$$

where $\widetilde{\Psi}_{0,k}$ is defined by (10.57).

(iv) To show the equality in the whole space $L^2(\mathcal{G})$, we need to repeat the arguments of the proof of Theorem 10.10-(iii) (i.e. $\mathrm{T}_{1,0}$ has to be replaced by $\mathrm{T}_{1,\alpha}$, and Ψ_0 by $\Psi_{\alpha,\delta}$).

(v) The proof follows from (Berezin and Shubin 1991, Chapter II, Theorem 3.1). □

Proof. **[Proof of Theorem 10.8.]** Combining Theorem 6.4, Theorem 10.9, Proposition 10.11, we get orbital stability of $e^{i\omega t}\Psi_{\alpha,\delta}$ in $W^1_{\mathcal{E}}(\mathcal{G})$ for $\alpha < 0$ and spectral instability of $e^{i\omega t}\Psi_{\alpha,\delta}$ for $\alpha > 0$. □

A Distributions

In this Appendix, we will give some basic notions about distributions.

Definition A.1. *Let $X \subset \mathbb{R}^n$. A linear form $u : C_c(\mathbb{R}^n) \to \mathbb{C}$ is called a distribution, if for every compact set $K \subset X$, there is a real number $C \geqslant 0$ and a nonnegative integer N such that*

$$|\langle u, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup \partial^\alpha \phi \tag{A.1}$$

for all $\phi \in C_c^\infty(X)$ with supp$\phi \subset X$. The vector space of distributions on X is denoted by $\mathcal{D}'(X)$.

Inequalities such as (A.1) are called semi-norm estimates.
The vector space structure of $\mathcal{D}'(X)$ is defined in the natural way:

$$\begin{aligned} &\langle u + v, \phi \rangle = \langle u, \phi \rangle + \langle v, \phi \rangle, \ u, v \in \mathcal{D}'(X), \ \phi \in C_c^\infty(X) \\ &\langle cu, \phi \rangle = c \langle u, \phi \rangle, \ u \in \mathcal{D}'(X), \ c \in \mathbb{C}, \ \phi \in C_c^\infty(X) \end{aligned} \tag{A.2}$$

A simple, but important example of a distribution is the following.

Example A.1. *Let $X \subset \mathbb{R}^n$ be an open set, and let $f \in C^0(X)$. We have that*

$$\langle f, \phi \rangle = \int_X f \phi dx \tag{A.3}$$

is a distribution.

The fact of f be a distribution follows from

$$\left|\int_X f\phi dx\right| \leqslant \sup|\phi| \int_K |f| \mathrm{d}x, \quad \phi \in C_c^\infty(K),$$

where $K \subset X$ is a compact set and

$$C_c^\infty(K) = \{\phi : \phi \in C_c^\infty(\mathbb{R}^n), \quad \operatorname{supp}\phi \subset K\}.$$

Remark A.1. *Note that the function locally integrable are also distributions.*

Another basic example is the Dirac distribution.

Example A.2. *Let X an open set of $\mathbb{R}^n$ and y a point of X, then the $\delta_y \in \mathcal{D}'(X)$ is defined by*

$$\langle \delta_y, \phi \rangle = \phi(y), \ \phi \in C_c^\infty(X). \tag{A.4}$$

Now, we define the notion of convergence to zero in $C_c^\infty(X)$.

Definition A.2. *Let $X \subset \mathbb{R}^n$ an open set. A sequence $\{\phi_j\}_{j\in\mathbb{N}}$ in $C_c^\infty(X)$ is said to converge (or tend) to zero in $C_c^\infty(X)$ if*

(i) the supports of the ϕ_j are contained in a fixed compact subset K of X;

(ii) for each multi-index α, the $\partial^\alpha \phi$ converge to zero uniformly as $j \to \infty$.

Definition A.3. *Let $X \subset \mathbb{R}^n$ be an open set, and let $u \in \mathcal{D}'(X)$. The support of u, written as supp u, is the complement of the set*

$$\{x \in X; \ u = 0 \text{ on a neighbourhood of } x\}.$$

Note that this is a closed set of X. For example, the support of δ_0 is the origin. It follow that $\delta = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Proposition A.1. *Let $u \in \mathcal{D}'$ and $\phi \in C_c^\infty(X)$. If the supports of u and ϕ are disjoints, then $\langle u, \phi \rangle = 0$.*

Now, we define the notion of convergence of distributions.

Definition A.4. *Let $X \subset \mathbb{R}^n$ an open set, and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of distributions on X. The sequence is said to converge in $\mathcal{D}'(X)$ to u in $\mathcal{D}'(X)$ if*

$$\lim_{n\to+\infty} \langle u_j, \phi \rangle = \langle u, \phi \rangle \text{ for all } \phi \in C_c^\infty(X). \tag{A.5}$$

The following classical example of convergence of functions in $\mathcal{D}'$ is a direct consequence of dominated convergence theorem.

Example A.3. *If* $\{f_j\}_{j\in\mathbb{N}}$ *is a sequence which converges almost everywhere to a function* f*, and there is a function* $g \in L^1_{loc}(X)$ *such that* $|f_j| \leqslant g$ *for all* j*, then* $f \in L^1_{loc}(X)$ *and* $f_j \to f$ *in* $\mathcal{D}'(X)$ *as* $j \to \infty$.

Remark A.2. *Note that this includes the simple case of a sequence of continuous functions which converges uniformly on compact sets.*

Now, we define the derivative of distributions.

If $u \in C^1(X)$, then the distribution which is equal to the derivative $\partial_{x_i} u$, where $i = 1, 2, ..., n$, is

$$\langle \partial_{x_i} u, \phi \rangle = \int \phi \partial_{x_i} u dx = -\int u \partial_{x_i} \phi dx, \quad \phi \in C_c^\infty(X),$$

by an integration. One can write this as

$$\langle \partial_i u, \phi \rangle = -\langle u, \partial_i \phi \rangle, i = 1, \ldots, n, \quad \phi \in C_c^\infty(X). \tag{A.6}$$

Note that, the expression (A.6) makes sense for any $u \in \mathcal{D}'(X)$ and it is not difficult to show that $\partial_{x_i} u$ is a distribution if u is a distribution.

Definition A.5. *Let* $u \in \mathcal{D}'(X)$*. The distributions* $\partial_{x_i} u$ *defined by* (A.6)*, are called the first order derivatives of* u*.*

We empathize that a distribution has derivatives of all orders. For one can iterate (A.6) to obtain

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle, \quad \phi \in C_c^\infty(X)$$

.

Example A.4. *Take* $X = \mathbb{R}$*. The Heavside function* H *is given by*

$$H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leqslant 0. \end{cases} \tag{A.7}$$

To compute his derivative, one has

$$\langle \partial H, \phi \rangle = -\langle H, \partial \phi \rangle = -\int_0^\infty \partial_x \phi(x) dx = \phi(0), \quad \phi \in C_c^\infty(\mathbb{R}).$$

Hence, $\partial_x H = \delta$.

A.1 Tempered distributions

Now, we give the definition of a special class of distributions.

Definition A.6. *The space of tempered distributions $S'(\mathbb{R}^n)$ is the subspace of $\mathcal{D}'(\mathbb{R}^n)$ consisting of distributions which extend to continuous linear forms on the Schwartz space $S(\mathbb{R}^n)$.*

We shall define Fourier transform on $S'(\mathbb{R}^n)$.

Definition A.7. *The Fourier transform of $u \in S'(\mathbb{R}^n)$ is the distribution $\hat{u} \in S'(\mathbb{R}^n)$ given by*

$$\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle, \ \phi \in S(\mathbb{R}^n). \tag{A.8}$$

For $\operatorname{Re} \alpha > 0$, the tempered distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function by

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \ f \right\rangle := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} f(t) dt,$$

where Γ denotes the classical gamma function (see Stein and Shakarchi 2003).

Example A.5. *The simplest example is the Fourier transform of Dirac distribution on $\mathbb{R}^n$. Indeed an easy calculation gives*

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \langle 1, \phi \rangle, \quad \phi \in S'(\mathbb{R}^n) \, (\mathrm{R}^n) \, .$$

For more details about the tempered distributions we refer the reader (Friedlander 1998).

A.2 A distribution obtained by an analytical continuation

Now, we will define a distribution, by analytic continuation, that was used to study the KdV equation on a $\mathcal{Y}$ junction on the Chapter 5.

Take $X = \mathbb{R}$ and define a function $x \mapsto x_+$ by setting

$$x_+ = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x \leqslant 0. \end{cases} \tag{A.9}$$

If $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda > 0$, then $x \mapsto x_+^{\lambda-1}$ is locally integrable and so determines a distribution,

$$\left\langle x_+^{\lambda-1}, \phi \right\rangle = \int_0^{\infty} x^{\lambda-1} \phi(x) dx, \quad \phi \in C_c^{\infty}(\mathbb{R}). \tag{A.10}$$

By using a theorem of Lebesgue integral we can differentiate with respect to λ under the integral sign. The second member of A.10 is C^1 and satisfies the Cauchy-Riemann equations, hence is analytic on $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$. Thus (A.10) gives an analytic function from $\mathbb{C}^+$ to $\mathcal{D}'(\mathbb{R})$.

We easily have

$$\partial x_+^{\lambda} = \lambda x_+^{\lambda-1}, \text{ when Re } \lambda > 0 \tag{A.11}$$

. Let us now define the distribution $x^{\lambda-1}$ by setting

$$x_+^{\lambda-1} = \frac{1}{\lambda(\lambda+1)\dots(\lambda+k-1)}\partial^k x_+^{\lambda+k-1}.$$

where k is a nonnegative integer chosen so that Re $k + \lambda > 0$. Explicitly, one has

$$\left\langle x_+^{\lambda-1}, \phi \right\rangle = \frac{(-1)^k}{\lambda(\lambda+1)\dots(\lambda+k-1)}\int_0^{\infty} x^{\lambda+k-1}\partial^k \phi(x)dx,$$

for $\phi \in C_c^{\infty}(\mathbb{R})$.

Note that, by partial integration this coincides with (A.10) when Re $\lambda > 0$.

The gamma function has the same poles that the distribution $x_+^{\lambda-1}$, and its residues at the point $\lambda = -k$ is $(-1)^k/k!$ (see Stein and Shakarchi 2003). Define $E_\lambda \in \mathcal{D}'$ by

$$E_\lambda = x_+^{\lambda-1}/\Gamma(\lambda), \quad \lambda \in \Omega; \quad E_{-k} = \partial^k \delta, \quad k = 0, 1, \dots$$

Then $\lambda \mapsto E_\lambda$ is analytic in Ω and continuous in $\mathbb{C}$. An elementary argument shows that E_λ is analytic on $\mathbb{C}$ (see Stein and Shakarchi 2003). Note that one has, by (A.11)

$$\partial_x E_\lambda = E_{\lambda-1}. \tag{A.12}$$

Can be proved that $E_\lambda \in S'(\mathbb{R})$ for $\lambda \in \mathbb{C}$ and your Fourier transform is given by

$$\widehat{E_\lambda} = e^{-\frac{1}{7}2i\pi\lambda}(\xi - i0)^{-\lambda}.$$

A.3 Riemann-Liouville fractional integral

The method used on Chapter 5 for constructing solutions of certain Cauchy problems for the KdV equation on star graphs exploits properties of a fractional integration operator whose properties are described in this section.

For Re $\alpha > 0$, integration by parts implies that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k\left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)}\right)$$

for all $k \in \mathbb{N}$. This expression allows to extend the definition, in the sense of distributions, of $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ to all $\alpha \in \mathbb{C}$.

The integration of an appropriate contour yields us

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\right)^{\widehat{}}(\tau) = e^{-\frac{1}{2}\pi\alpha}(\tau - i0)^{-\alpha}, \tag{A.13}$$

where $(\tau - i0)^{-\alpha}$ is the distributional limit.

Definition A.8. *If $f \in C_0^\infty(\mathbb{R}^+)$, we define*

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f.$$

Thus, for $\operatorname{Re}\alpha > 0$,

$$\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

and notice that

$$\mathcal{I}_0 f = f, \quad \mathcal{I}_1 f(t) = \int_0^t f(s)ds, \quad \mathcal{I}_{-1} f = f' \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}.$$

The following results state important properties of the Riemann-Liouville fractional integral operator . The proof of them can be found in (Holmer 2006).

Lemma A.1. *If $f \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+)$ for all $\alpha \in \mathbb{C}$.*

Lemma A.2. *If $0 \leqslant \alpha < \infty$, $s \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R})$, then we have*

$$\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \leqslant c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)} \tag{A.14}$$

and

$$\|\varphi \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \leqslant c_\varphi \|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}. \tag{A.15}$$

For more details on the distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ we refer the book (Friedlander 1998).

B Some Function Spaces

In this appendix, we give the principal definitions and properties of the functions spaces used in this book.

B.1 Sobolev spaces

We start by defining the L^2-based Sobolev spaces on the positive half-line.

Definition B.1. *For $s \geqslant 0$ we say that $\phi \in H^s(\mathbb{R}^+)$ if exists $\tilde{\phi} \in H^s(\mathbb{R})$ such that $\phi = \tilde{\phi}|_{\mathbb{R}+}$. In this case we set $\|\phi\|_{H^s(\mathbb{R}+)} := \inf_{\tilde{\phi}} \|\tilde{\phi}\|_{H^s(\mathbb{R})}$. For $s \geqslant 0$ define*

$$H_0^s(\mathbb{R}^+) = \Big\{\phi \in H^s(\mathbb{R}^+);\, supp(\phi) \subset [0, +\infty)\Big\}.$$

For $s < 0$, define $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$ as the dual space of $H_0^{-s}(\mathbb{R}^+)$ and $H^{-s}(\mathbb{R}^+)$, respectively.

Now, in a similar we define the Sobolev spaces posed on a star graphs given by two positive half-lines and a negative half-line, that was used in Chapter 5.

Definition B.2. *We define the usual Sobolev spaces for functions defined on the junction $\mathcal{Y}$ as*

$$H^s(\mathcal{Y}) = H^s(\mathbb{R}^-) \times H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+). \tag{B.1}$$

Also define

$$C_0^\infty(\mathbb{R}^+) = \left\{\phi \in C^\infty(\mathbb{R});\ \mathrm{supp}(\phi) \subset [0, +\infty)\right\}$$

and $C_{0,c}^\infty(\mathbb{R}^+)$ as those members of $C_0^\infty(\mathbb{R}^+)$ with compact support. We recall that $C_{0,c}^\infty(\mathbb{R}^+)$ is dense in $H_0^s(\mathbb{R}^+)$ for all $s \in \mathbb{R}$. A definition for $H^s(\mathbb{R}^-)$ and $H_0^s(\mathbb{R}^-)$ can be given analogous to that for $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$.

The following results summarize useful properties of the Sobolev spaces on the half-line. For the proofs we refer the reader (Colliander and Kenig 2002).

Lemma B.1. *For all $f \in H^s(\mathbb{R})$ with $-\frac{1}{2} < s < \frac{1}{2}$ we have*

$$\|\chi_{(0,+\infty)} f\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}.$$

Lemma B.2. *If $\frac{1}{2} < s < \frac{3}{2}$ the following statements are valid:*

(a) $H_0^s(\mathbb{R}^+) = \left\{f \in H^s(\mathbb{R}^+);\ f(0) = 0\right\}$,

(b) If $f \in H^s(\mathbb{R}^+)$ with $f(0) = 0$, then $\|\chi_{(0,+\infty)} f\|_{H_0^s(\mathbb{R}^+)} \lesssim \|f\|_{H^s(\mathbb{R}^+)}$.

Lemma B.3. *If $f \in H_0^s(\mathbb{R}^+)$ with $s \in \mathbb{R}$, we then have*

$$\|\psi f\|_{H_0^s(\mathbb{R}^+)} \lesssim \|f\|_{H_0^s(\mathbb{R}^+)}.$$

Remark B.1. *In Lemmas B.1, B.2 and B.3 all the constants c only depend on s and ψ.*

Fore more details about the Sobolev spaces on the half-line, we refer the reader the work of Colliander and Kenig (2002).

B.2 Bourgain spaces

We next briefly review the main ingredients of the Bourgain method (Bourgain 1993) in its simplest form, which will be sufficient for our purpose (see (Ginibre 1996) for a more detailed pedagogical account), in order to locate precisely the estimates that are required on the nonlinear interaction. We consider the case of a single equation

$$i u_t = \phi(-i\nabla)u + f(u), \tag{B.2}$$

where ϕ is a real function. The Cauchy problem for (B.2) with initial data $u(0) = u_0$ is rewritten in a standard way as the integral equation

$$\begin{aligned} u(t) &= U(t)u_0 - i\int_0^t U\left(t - t'\right) f\left(u\left(t'\right)\right) dt' \\ &\equiv U(t)u_0 - iU *_R f(u), \end{aligned} \tag{B.3}$$

where $U(t) = \exp[-it\phi(-i\nabla)]$ is the unitary group that solves the underlying linear equation and $*_R$ denotes the retarded convolution in time. One wants to use function space norms defined in terms of the space time Fourier transform of u while solving the Cauchy problem locally in time in some interval $[-T, T]$. For that purpose, it is convenient to introduce a time cut off in (B.3). Let $\psi_1 \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ be even, with $0 \leqslant \psi \leqslant 1$ and

$$\psi_1(t) = \begin{cases} 1, & \text{if } |t| \leqslant 1 \\ 0, & \text{if } |t| > 2. \end{cases} \tag{B.4}$$

We also, define $\psi_T(t) = \psi_1(t/T)$, for $0 < T \leqslant 1$.

One replaces the equation (B.3) by the cut off equation

$$\begin{aligned} u(t) &= \psi_1(t)U(t)u_0 - i\psi_T(t)\int_0^t U\left(t-t'\right) f\left(u\left(t'\right)\right) dt' \\ &\equiv \psi_1(t)U(t)u_0 - i\psi_T(t)U. *_R f(u), \end{aligned} \tag{B.5}$$

Solving the equation (B.5) for all $t \in \mathbb{R}$ solves the equation (B.3) locally in time for $|t| \leqslant T$, so that T will be the time of local resolution of (B.3). The basic spaces X where to solve the equation (B.5) are defined as spaces of functions u such that $U(t)u$ belongs to some classical (in the present case Sobolev) space H

$$\|u\|_X = \|U(-t)u\|_H. \tag{B.6}$$

The immediate effect of the choice (B.6) is to eliminate the free evolution $U(t)u_0$ from the linear estimates. Clearly,

$$\|\psi_1(t)U(t)u_0\|_X = \|\psi_1 u_0\|_H. \tag{B.7}$$

In the present case, we shall primarily take for H the simplest Sobolev spaces $H = H^{s,b}$. The corresponding spaces X defined by (B.6) will be denoted $X^{s,b}$. In that case the equality (B.7) becomes

$$\left\|\psi_1 U(t)u_0; X^{s,b}\right\| = \left\|\psi_1 u_0; H^{s,b}\right\| = \left\|\psi_1; H^b\right\| \|u_0; H^s\| \tag{B.8}$$

While the estimate of the nonlinear part is given by the following lemma.

Lemma B.4. *Let* $-\frac{1}{2} < b' \leqslant 0 \leqslant b \leqslant b' + 1$ *and* $T \in [0, 1]$. *Then for* $F \in X^{s,b'}(\phi)$ *we have that*

$$\begin{aligned} \left\|\psi_1(t)W_\phi(t)\omega_0\right\|_{X^{s,b}(\phi)} &\leqslant C\,\|\omega_0\|_{H^*} \\ \left\|\psi_T(t)\int_0^t W_\phi\left(t-t'\right)F\left(t',\cdot\right)dt'\right\|_{X^{s,b}(\phi)} &\leqslant CT^{1-b+b'}\|F\|_{X^{s,b}(\phi)} \end{aligned} \tag{B.9}$$

Finally, we fix the notation in context of KdV equations. Denote by $X^{s,b}$ the so called Bourgain spaces associated to linear KdV equation; more precisely, $X := X^{s,b}$ is the completion of $S'(\mathbb{R}^2)$ with respect to the norm

$$\|w\|_{X^{s,b}(\phi)} = \|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{w}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

C Spectrum and the Riesz Projection

In this Appendix we introduce the basic definitions of the spectrum and resolvent for linear operators. The Riesz projection is defined and its relation with the decomposition of the spectrum is also established.

C.1 The spectrum

Let $(X, \|\cdot\|)$ be a Banach space on $\mathbb{C}$ and A a linear operator on X with domain $D(A) \subset X$, $A : D(A) \subset X \to X$. Associated with A we have two linear subspaces;

a) The image (or range) of A:

$$Ran(A) = \{y \in X : y = Ax, \text{ for some } x \in D(A)\}.$$

Sometimes we use $Ran(A) = Im(A)$.

b) The kernel of A: $ker(A) = \{x \in D(A) : Ax = 0\}$.

Let $Y \subset X$. We denote by I_Y the identity function on Y.

Definition C.1. *A linear operator $B : Ran(A) \to D(A)$ is called the inverse of A if $BA = I_{D(A)}$ and $AB = I_{Ran(A)}$.*

Remark C.1. *The operator B in the above definition is denoted by A^{-1}.*

Theorem C.1. *A linear operator A has an inverse if and only if $ker(A) = \{0\}$.*

Proof. Suppose $Ker(A) = \{0\}$. Then for every $y \in Ran(A)$ there is a unique $x \in D(A)$ such that $Ax = y$. Therefore we can define a operator $A^{-1} : Ran(A) \to D(A)$ by $A^{-1}y \equiv x$, where $Ax = y$. It is easy to see that A^{-1} is well defined and linear. Moreover, since $A^{-1}y = A^{-1}(Ax) = x$ for all $x \in D(A)$ we have $A^{-1}A = I_{D(A)}$. Lastly, since for $y \in Ran(A)$, $AA^{-1}y = Ax = y$, we obtain $AA^{-1} = I_{Ran(A)}$.

Now, suppose that $A^{-1} : Ran(A) \to D(A)$ exists. Then for $x \in ker(A)$ we have $x = A^{-1}(Ax) = 0$. Hence, $ker(A) = \{0\}$. It finishes the Theorem. □

The last Theorem shows one condition for the existence of the inverse A^{-1} of A as a well defined function, and for our interest it is few information. More precisely, we would like that A^{-1} is a bounded linear operator on X, namely, $Ran(A) = X$ and there exists $K > 0$ such that

$$\|A^{-1}f\| \leqq K\|f\|, \quad \text{for all } f \in D(A^{-1}) = Ran(A) = X. \tag{C.1}$$

Definition C.2. *A linear operator $A : D(A) \subset X \to X$ is invertible if A has an bounded inverse defined on X.*

Remark C.2. *If A is invertible, then A^{-1} is unique.*

Example C.1. *Let $X = C([0,1]) = \{f : [0,1] \to \mathbb{C} : f$ is continuous$\}$ with the norm $\|f\| = sup_{x\in[0,1]}|f(x)|$. Then is easy to see that $(X, \|\cdot\|)$ is a Banach space. We define the linear operator $A : X \to X$ by*

$$(Af)(t) = \int_0^t f(s)ds, \quad t \in [0,1].$$

Then we have the following properties;

1) *A has an inverse: let $f \in X$ such that $Af = 0$, then the Fundamental Theorem of the Calculus implies that $f(t) = 0$ for all $t \in [0,1]$. Hence, $ker(A) = \{0\}$.*

2) *Determination of A^{-1}: Let $g \in Ran(A)$, then there exists a unique $f \in X$ such that*

$$(Af)(t) = \int_0^t f(s)ds = g(t), \quad \text{for } t \in [0,1].$$

Then, since f is integrable we obtain that g is absolutely continuous and $g'(t) = f(t)$ for all $t \in [0,1]$. Moreover, $g(0) = 0$. Therefore, we have

$$\begin{cases} (A^{-1}g)(t) = \dfrac{dg}{dt}(t), \quad \text{for } t \in [0,1], \\ D(A^{-1}) = \{g \in X : g \text{ is absolutely continuous}, \\ \qquad\qquad g' \in X \text{ and } g(0) = 0\}. \end{cases} \tag{C.2}$$

3) *A is not invertible: Suppose that A^{-1} defined in (C.2) can be extended as a bounded linear operator on X and satisfying (C.1). Next, we consider the sequence $g_n(t) = t^n$, $n = 1, 2, 3, \cdots$, and $t \in [0, 1]$. Then we obtain that $g_n \in D(A^{-1})$ with $\|g_n\| = 1$ for all $n \geqq 1$, and*

$$\|A^{-1} g_n\| = \|g_n'\| = n \leqq K.$$

It which is a contradiction.

By using the operator A in the example above, we consider the following eigenvalue problem for $\lambda \in \mathbb{C}$:

$$\begin{cases} Af = \lambda f, \\ f \in X - \{0\}. \end{cases} \tag{C.3}$$

Next we show that the operator A has not eigenvalues. Suppose $\lambda = 0$, then the equation $Af = 0$ has the unique solution $f \equiv 0$. Now, for $\lambda \neq 0$ we obtain the linear differential equation of first order $\lambda f'(t) = f(t)$, for $t \in [0, 1]$, it which has the general solution $f(t) = ce^{\frac{1}{\lambda}t}$. Hence, since $f(0) = 0$ we obtain that $f(t) \equiv 0$.

By considering the property of the operator $A - \lambda I_X$, for $\lambda \in \mathbb{C}$, to be invertible, we obtain one decomposition disjoint of $\mathbb{C}$ in two sets that characterize many deep properties of the operator A.

Definition C.3. *Let A be a linear operator on X with domain $D(A)$.*

1) *The spectrum of A, $\sigma(A)$, is the set of all points $\lambda \in \mathbb{C}$ such that $A - \lambda I_X$ is not invertible.*

2) *The resolvent set of A, $\rho(A)$, is the set of all points $\lambda \in \mathbb{C}$ such that $A - \lambda I_X$ is invertible. If $\lambda \in \rho(A)$, the inverse of $A - \lambda I_X$ is called the resolvent of A in λ and it is denoted by*

$$R_A(\lambda) = (A - \lambda I_X)^{-1}.$$

Remark C.3. *By Definition C.3 we have: $\mathbb{C} = \rho(A) \cup \sigma$ and $\rho(A) \cap \sigma(A) = \emptyset$. In the following we will denote the identity operator on X, I_X, only by I.*

Definition C.4. *Let $\mathcal{O}$ be an open subset of $\mathbb{C}$ and $\mathcal{L}(X)$ be the set of bounded operators on X. A mapping $\lambda \in \mathcal{O} \to B(\lambda) \in \mathcal{L}(X)$ is called analytic (in norm) in $\lambda_0 \in \mathcal{O}$, if there are operators $B_n \in \mathcal{L}(X)$ and $\delta > 0$ such that*

$$B(\lambda) = \sum_{n=0}^{\infty} B_n(\lambda - \lambda_0)^n, \quad \text{for } \lambda \in \mathcal{O} \text{ and } |\lambda - \lambda_0| < \delta.$$

Remark C.4. *By using the uniform boundedness principle (or Banach-Steinhaus theorem) and the Cauchy integral formula, the Definition C.4 has the following equivalence statements for X being a Hilbert space with inner product $\langle \cdot, \cdot \rangle$:*

a) *the mapping $\lambda \in \mathcal{O} \to B(\lambda) \in \mathcal{L}(X)$ is analytic in λ_0,* *b) the mapping* $\lambda \to B(\lambda)u \in X$ *is analytic in* λ_0 *for all* $u \in X$,

c) the mapping $\lambda \to \langle B(\lambda)u, v\rangle$ *is analytic in* λ_0 *for all* $u, v \in X$.

The proof of the following classical result can be found in Reed and Simon (1980).

Lemma C.1. *Let* $T \in \mathcal{L}(X)$ *with* $\|T\| = sup_{\|y\|=1}\|T(y)\| < 1$. *Then,* $I - T$ *is invertible and its inverse is given by the following absolutely convergent Neumann series*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

In other words, $\lim_{N\to\infty} \sum_{k=0}^{N} T^k$ *converges in norm to* $(I - T)^{-1}$.

Theorem C.2. *Let* A *be a linear operator on* X *with domain* $D(A)$. *Then,*

1) $\rho(A)$ *is a open set of* $\mathbb{C}$.

2) $\sigma(A)$ *is a closed set of* $\mathbb{C}$.

3) If $\rho(A) \neq \emptyset$, *the mapping* $\lambda \in \rho(A) \to R_A(\lambda)$ *is analytic.*

Proof. 1) Let $\lambda_0 \in \rho(A)$ and $\epsilon < \frac{1}{\|R(\lambda_0)\|}$. Then the open disk $D(\lambda_0; \epsilon)$ is contained in $\rho(A)$. Indeed, for $\lambda \in D(\lambda_0; \epsilon)$, the relation

$$A - \lambda I = [I - R(\lambda_0)(\lambda - \lambda_0)](A - \lambda_0 I) \tag{C.4}$$

and Lemma C.5 imply that $\lambda \in \rho(A)$. Therefore, $\rho(A)$ is an open set of $\mathbb{C}$.

2) It follows immediately from item 1).

3) Let $\lambda_0 \in \rho(A)$ and $\lambda \in D(\lambda_0; \epsilon)$, with ϵ being chosen as in the proof of item 1). Then relation (C.4) and Lemma C.5 imply

$$R(\lambda) = R(\lambda_0) \sum_{k=0}^{\infty} (A - \lambda_0)^{-k} (\lambda - \lambda_0)^k = \sum_{k=0}^{\infty} (A - \lambda_0)^{-k-1} (\lambda - \lambda_0)^k.$$

It finishes the proof. □

Remark C.5. *a) The spectrum of a linear bounded operator* $A : X \to X$, $\sigma(A)$, *is not empty and compact. Moreover,* $\sigma(A) \subset \{z \in \mathbb{C} : |z| \leqq \|A\|\}$. *Indeed, suppose* $\sigma(A) = \emptyset$. *Then the mapping* $\lambda \in \rho(A) = \mathbb{C} \to R(\lambda)$ *is entire. Next, we consider* λ *such that* $|\lambda| > \|A\|$. *Then from the following relation*

$$R(\lambda) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} (\lambda^{-1} A)^k,$$

we have that

$$\|R(\lambda)\| \leqq \frac{1}{|\lambda| - \|A\|}.$$

Therefore, $\lim_{|\lambda|\to\infty} \|R(\lambda)\| = 0$ *and so* $\|R(\lambda)\| \leqq M$ *for all* $\lambda \in \mathbb{C}$. *Then from the vectorial Liouville Theorem (Theorem C.12 we have that* $R(\lambda) = 0$ *for all* λ. *But our argument above shows that for* λ *sufficiently large we have* $\lambda \in \rho(A)$. *Therefore* $\sigma(A) \neq \emptyset$.

b) The spectrum of a unbounded linear operator $A : D(A) \subset X \to X$ *can be empty or all* $\mathbb{C}$. *Indeed, it consider* $X = C([0,1])$ *and the following two linear operators* A_1 *and* A_2*:*

- $D(A_1) = \{f \in C([0,1]) : f' \in C([0,1])\}, \quad A_1 f = f'$,
- $D(A_2) = \{f \in C([0,1]) : f' \in D(A_1), f(0) = 0\}$ *with* $A_2 f = f'$.

Then following a similar argument as in Example 4.1.1 above, we have that A_1 *and* A_2 *are unbounded operators on* X. *Now, since* $e^{\lambda t} \in ker(A_1 - \lambda)$ *we obtain immediately that* $\sigma(A_1) = \mathbb{C}$. *Next we show that* $\sigma(A_2) = \emptyset$ *and* $\rho(A_2) = \mathbb{C}$. *Indeed, it follows from Example 4.1.1 that for all* λ *we have* $ker(\lambda - A_2) = \{0\}$. *Next, we show that* $\lambda - A_2$ *has a bounded inverse com domain all* X. *For that, let* $g \in X$ *and we determine* $f \in D(A_2)$ *such that* $(\lambda - A_2)f = g$. *The classical theory of edo's shows that* f *has the formula*

$$f(t) = e^{\lambda t} f(0) - \int_0^t e^{\lambda(t-s)} g(s)ds, \quad t \in [0,1].$$

Then, for $f(0) = 0$ *we have that* $f \in D(A_2)$ *and* $\lambda - A_2$ *has an inverse given for all* $g \in X$ *by*

$$(\lambda - A_2)^{-1} g(t) = -\int_0^t e^{\lambda(t-s)} g(s)ds, \quad t \in [0,1].$$

Now we show that $(\lambda - A_2)^{-1} : X \to D(A_2)$ *is bounded: for* $g \in X$ *we obtain easily that*

$$\|(\lambda - A_2)^{-1} g\| \leqq sup_{t\in[0,1]} e^{|\Re(\lambda)|} \|g\|.$$

Therefore, $\rho(A_2) = \mathbb{C}$.

Next we establish some basic properties of the resolvent $R_A(\lambda) = (A - \lambda)^{-1}$, $\lambda \in \rho(A)$ (see Reed and Simon (1980)).

Theorem C.3. *For* $\mu, \lambda \in \rho(A)$ *we have:*

1) $R(\lambda)R(\mu) = R(\mu)R(\lambda)$.

2) *The first identity of the resolvent:*

$$R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu).$$

3) *The second identity of the resolvent: for* $\lambda \in \rho(B)$

$$R_A(\lambda) - R_B(\lambda) = R_A(\lambda)(A - B)R_B(\lambda) = R_B(\lambda)(B - A)R_A(\lambda).$$

There are three reasons for $A - \lambda$ not to be invertible,

a) $ker(A - \lambda) \neq \{0\}$;

b) $ker(A - \lambda) = \{0\}$ and $Ran(A - \lambda)$ is dense. Then $A - \lambda$ has a inverse densely defined but it is not bounded.

c) $ker(A - \lambda) = \{0\}$ and $Ran(A - \lambda)$ is not dense. In this case, $(A - \lambda)^{-1}$ exists and it can be bounded on $Ran(A - \lambda)$, but it is not densely defined and so it can not be uniquely extended to a bounded operator on X.

Definition C.5 (Classification of the spectrum). *Let* $A : D(A) \to X$ *be a linear operator. Then,*

1) *If* $\lambda \in \sigma(A)$ *satisfies that* $ker(A - \lambda) \neq \{0\}$*, then* λ *is called an eigenvalue of* A*, and every* $u \in ker(A-\lambda)$, $u \neq 0$*, it is called an eigenvector of* A *for* λ *and satisfies* $Au = \lambda u$*. The dimension of the linear subspace* $ker(A - \lambda)$*,* $dim(ker(A - \lambda))$ *,is called the geometric multiplicity of* λ*.*

2) *The discrete spectrum of* A*,* $\sigma_d(A)$*, is the set of all eigenvalues of* A *with finite (algebraic) multiplicity and that are isolated points of* $\sigma(A)$*.*

3) *The essential spectrum of* A*,* $\sigma_{ess}(A)$*, is defined as the complement of* $\sigma_d(A)$ *in* $\sigma(A)$*;* $\sigma_{ess}(A) = \sigma(A) - \sigma_d(A)$*. Therefore,* $\sigma(A) = \sigma_d(A) \cup \sigma_{ess}(A)$*.*

Remark C.6. *If* $dim(X) < \infty$*, the only reason for* $A - \lambda$ *not to be invertible is that* $A - \lambda$ *is not injector. Then,* $\sigma(A) = \sigma_d(A)$*. We recall that the algebraic multiplicity of* λ*,* $\mu(\lambda)$*, is the multiplicity of* λ *as being a root of the characteristic equation* $F(\lambda) = det(A - \lambda)$*. It is well known that the geometric multiplicity of* λ*,* $m(\lambda) = dim(ker(A - \lambda))$*, satisfies* $m(\lambda) \leqq \mu(\lambda)$*.*

C.2 Linear operators on Hilbert spaces

The existence of a inner product on a Hilbert space have many consequences for the structure of linear operators defined on it. One of the most important is the existence of the adjoint operator defined on the same Hilbert space. We recall that on a Banach space the adjoint operator is defined on its dual, it which in general is different of the initial space. By the Representation Theorem of Riesz, the dual space of a Hilbert space is identified with itself.

Definition C.6. *Let $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space. For M being a closed subspace of H, we define the orthogonal complement of M, denoted by $M^{\perp}$, as the set*

$$M^{\perp} = \{x \in H : \langle x, m\rangle = 0, \text{ for all } m \in M\}.$$

Remark C.7. *$M^{\perp}$ can be defined if M is not a closed subspace.*

The proof of the following Proposition is immediate.

Proposition C.1. *Let M be a closed subspace of H. Then,*

1) *$M^{\perp}$ is a closed subspace of H (therefore it is a Hilbert space).*
2) *$M \cap M^{\perp} = \{0\}$.*
3) *$M^{\perp\perp} = M$.*

Remark C.8. *if M is not a closed subspace we still have that $M^{\perp}$ is a closed subspace and we have $M^{\perp\perp} = \overline{M}$. Therefore, a subspace $M \subset H$ is dense in H if $M^{\perp} = \{0\}$.*

Theorem C.4 (The Projection Theorem). *Let $M \subset H$ a closed subspace. Then, H is the direct sum of M and $M^{\perp}$, $H = M \oplus M^{\perp}$. Therefore, every $x \in H$ has a unique decomposition in the form*

$$x = y + z, \quad y \in M, z \in M^{\perp}.$$

Proof. Let $x \in H$ and we consider $y \in M$ the unique vector such that $d(x, M) = \|x-y\|$. Then we have that $z \equiv x - y$ satisfies $z \in M^{\perp}$. Indeed, for $t \in \mathbb{R}$ and $m \in M$ we obtain the relation

$$d^2(x, M) \leqq \|x - (y + tm)\|^2 = \|z\|^2 + t^2\|m\|^2 - 2t\Re\langle z, m\rangle.$$

Then for all $t \in \mathbb{R}$ we obtain, $t^2\|m\| - 2t\Re\langle z, m\rangle \geqq 0$. Therefore, $\langle z, m\rangle = 0$, otherwise, for $m \neq 0$ and $t = \frac{\Re\langle z,m\rangle}{\|m\|^2}$ we have

$$\frac{(\Re\langle z, m\rangle)^2}{\|m\|^2} - 2\frac{(\Re\langle z, m\rangle)^2}{\|m\|^2} = -\frac{(\Re\langle z, m\rangle)^2}{\|m\|^2} < 0.$$

□

C.2.1 Adjoint operators on Hilbert spaces

Let $A \in \mathcal{L}(H)$. For $y \in H$ we define the linear functional $f_y : H \to \mathbb{C}$ by $f_y(x) = \langle Ax, y\rangle$. Then, we have the following;

a) f is bounded: By Cauchy-Schwartz, $|f_y(x)| \leqq \|Ax\|\|y\| \leqq \|A\|\|x\|\|y\|$. Then

$$\|f_y\| = \sup_{\|x\|=1} |f_y(x)| \leqq \|A\|\|y\|.$$

b) By Riesz representation theorem there is a unique $y^* \in H$ such that for $x \in H$

$$\langle Ax, y\rangle = f_y(x) = \langle x, y^*\rangle.$$

Moreover, $\|f_y\| = \|y^*\|$.

c) We define the operator $A^* : H \to H$ by $A^* y = y^*$. Hence, for $x \in H$ we have

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

Moreover, A^* is linear and satisfies $\|A^* y\| \leqq \|A\|\|y\|$. Therefore, $A^* \in \mathcal{L}(H)$ with $\|A^*\| \leqq \|A\|$.

d) The relation $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$, it will imply that $\|A\| \leqslant \|A^*\|$. Therefore, $\|A\| = \|A^*\|$.

From the items above we have the following definition.

Definition C.7. *Let $A \in \mathcal{L}(H)$. Then the bounded linear operator $A^* : H \to H$ defined by the relation*

$$\langle Ax, y\rangle = \langle x, A^* y\rangle, \quad \textit{for } x, y \in H,$$

it is called the adjoint operator associated to A.

The proof of the following Proposition is immediate.

Proposition C.2. *Let $A, B \in \mathcal{L}(H)$. Then we have,*

1) $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$, for $\alpha \in \mathbb{C}$.

2) $A^{**} = A$, $(AB)^* = B^* A^*$.

3) $ker(A) = (ImA^*)^{\perp}$ $((ker(A))^{\perp} = \overline{ImA^*})$, $ker(A^*) = (ImA)^{\perp}$ $((ker(A^*))^{\perp} = \overline{ImA}$.

Definition C.8. *$A \in \mathcal{L}(H)$ is called self-adjoint if $A = A^*$.*

From Proposition C.2 we have immediate the following theorem.

Theorem C.5. *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then $ker(A)^{\perp} = \overline{ImA}$. Therefore, $H = ker(A) \oplus \overline{ImA}$.*

There is a very important class of self-adjoint operators in the spectral theory of linear operators, the orthogonal projections.

Definition C.9. *Let $M \subset H$ a closed subspace. A operator $P : H \to H$ is called the orthogonal projection on M if*

$$P(m + y) = m, \quad \textit{for all} \quad m \in M \textit{ and } y \in M^{\perp}.$$

The proof of the following proposition is immediate from Definition C.9.

Proposition C.3. *Let $M \subset H$ be a non-trivial closed subspace and $P : H \to H$ the orthogonal projection on M. Then,*

1) *P is a bounded linear operator with $\|P\| = 1$.*

2) *$ImP = M$, $kerP = M^{\perp}$, and $Pm = m$ for all $m \in M$.*

3) *$I - P$ is the orthogonal projection on $M^{\perp}$ with $K(I - P) = M$.*

4) *For $x \in H$, we have $Px = y$ where y is the unique vector satisfying $d(x, M) = \|x - y\|$.*

Definition C.10. *$P \in \mathcal{L}(H)$ is called a projection if $P^2 = P$.*

Theorem C.6. *$P \in \mathcal{L}(H)$ is a orthogonal projection if and only if $P^2 = P$ and $P^* = P$.*

Proof. Suppose P being a orthogonal projection on a closed subspace $M \subset H$. Then for $x = m + y \in H = M \oplus M^{\perp}$, we have $P(Px) = Pm = Px$, hence $P^2 = P$. Moreover, for $z = m_1 + y_1 \in H = M \oplus M^{\perp}$ we have

$$\langle Px, y\rangle = \langle m, y_1\rangle = \langle m + y, Pz\rangle = \langle x, Pz\rangle,$$

and so $P^* = P$.

Suppose $P^2 = P$ and $P^* = P$. Let $M = ImP$. Since $M = ker(I - P)$ we have that M is closed. Then $M^{\perp} = (ImP)^{\perp} = ker(P^*) = ker(P)$. Next we see that P is a orthogonal projection on M. Let $m \in M$ and $y \in M^{\perp}$, then

$$P(m + y) = P(m) + P(y) = P(m) = m$$

since for $m = P(r)$ we have $P(m) = P^2(r) = P(r) = m$. It finishes the Theorem. □

The proof of the following Theorem is immediate.

Theorem C.7. *Let P be a projection on H. Then,*

1) *$ImP = ker(I - P)$. Therefore, ImP is closed.*

2) *All $v \in H$ has a unique decomposition in the form*

$$v = x + y, \quad x \in kerP \text{ and } Py = y.$$

Therefore, H has the decomposition (not necessary orthogonal) $H = kerP \oplus ImP$.

C.2.2 Adjoint of unbounded operators

In this subsection we define the adjoint operator of an unbounded operator $A : D(A) \subset H \to H$. We start with the definition of a closed operator.

Definition C.11. *Let $(X, \|\cdot\|)$ be a Banach space and A a linear operator on X with domain $D(A) \subset X$. A is called a closed operator if*

- *for $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to x$, and*
- *$Ax_n \to y$,*

we have that $x \in D(A)$ and $Ax = y$.

Remark C.9. *a) If $A \in \mathcal{L}(H)$ then A is closed.*

b) If $A : D(A) \subset X \to X$ is invertible then A is closed.

c) The graph of A, $G(A)$, is defined as the following linear subspace of $X \times X$

$$G(A) = \{(x, Ax) : x \in D(A)\}.$$

Then it is easy to show that A is closed if and only if $G(A)$ is closed in $X \times X$.

d) For $x \in D(A)$ we define the norm (called the graph-norm on A)

$$\|x\|_A = (\|x\|^2 + \|Ax\|^2)^{1/2}.$$

Then, if A is closed we have that $(D(A), \|\cdot\|_A)$ is a Banach space and so we have that $A : (D(A), \|\cdot\|_A) \to X$ is a bounded linear operator. The basic importance of the graph-norm, $\|\cdot\|_A$, on $D(A)$ is that well known results for bounded operators can be established to the case of closed operators. For instance, from the closed graph theorem we obtain the following deep result: if $A : D(A) \subset H \to H$ is a closed linear operator with $ker(A) = \{0\}$ and $Im A = H$ then A is invertible, namely, $A^{-1} : H \to D(A)$ is bounded.

Example C.2. *The operator $A_2 f = f'$ defined in Remark C.5 satisfies $\sigma(A_2) = \emptyset$. Hence A_2 is invertible and therefore it is a closed operator.*

Definition C.12. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $A : D(A) \subset H \to H$ a linear operator with domain $D(A)$ dense in X. The adjoint of A, A^*, is defined by*

$$\begin{cases} D(A^*) = \{y \in H : \exists z \in H \ \text{ such that } \ \langle Ax, y\rangle = \langle x, z\rangle, \ \forall x \in D(A)\}, \\ A^* y = z \end{cases} \tag{C.5}$$

Remark C.10. *z in Definition C.12 is unique satisfying $\langle Ax, y\rangle = \langle x, z\rangle$ for all $x \in D(A)$.*

The proof of the following properties of A^* are immediate.

Proposition C.4. *We have the following properties of A^*.*

1) *A^* is a closed operator.*

2) *If A is invertible then A^* is invertible with $(A^*)^{-1} = (A^{-1})^*$.*

3) *$(ImA)^{\perp} = ker(A^*)$.*

4) *H has the following orthogonal decomposition,*

$$H = ker(A^*) \oplus \overline{ImA}.$$

Definition C.13. *A linear operator $A : D(A) \subset H \to H$ densely defined, it is called self-adjoint if $A^* = A$. In other words, $D(A) = D(A^*)$ and $\langle Ax, y \rangle = \langle x, Ay \rangle$, for $x, y \in D(A)$.*

Remark C.11. *For A self-adjoint we have:*

a) *A is closed.*

b) *Let A be to be invertible. Then A^{-1} is self-adjoint.*

c) *$(ImA)^{\perp} = ker(A)$. Therefore, $H = ker(A) \oplus \overline{ImA}$.*

d) *$R(\lambda; A)^* = R(\overline{\lambda}; A)$.*

The next Theorem gives a characterization of the spectrum of unbounded self-adjoint operator. Initially we give the following definition.

Definition C.14. *Let A be a linear operator $A : D(A) \subset H \to H$. The residual spectrum of A is defined as*

$$\sigma_{res}(A) = \{\lambda \in \mathbb{C} : ker(A - \lambda) = \{0\}, \text{ and } Im(A - \lambda) \text{ is not dense}\}.$$

Theorem C.8. *Let A be a self-adjoint operator. Then,*

1) *All eigenvalues of A are real.*

2) *$\sigma(A) \subset \mathbb{R}$.*

3) *$\sigma_{res}(A) = \emptyset$.*

4) *Eigenfunctions associated to different eigenvalues are orthogonal.*

Proof. The proof of item 1) is immediate. Next, we only establish Item 2) because Items 3) and 4) follows immediately from it. So, we will see that for $z = \lambda + i\mu$, with $\mu \neq 0$, we obtain $z \in \rho(A)$. The following inequality

$$\|(A - z)u\|^2 = \|(A - \lambda)u\|^2 + \mu^2\|u\|^2 \geqq \mu^2\|u\|^2, \quad \mu \in D(A),$$

show that $A - z$ is one-to-one, $Im(A - z)$ is closed and $(A - z)^{-1} : Im(A - z) \to D(A)$ is bounded. Next we show that $Im(A - z) = H$. Indeed, suppose that it is not true, then by Theorem C.4 there exists $f \neq 0$ such that $f \in Im(A - z)^{\perp}$. Hence, for all $x \in D(A)$ follows that

$$\langle Ax, f\rangle = \langle x, \overline{z} f\rangle$$

and therefore $f \in D(A^*) = D(A)$ and $Af = \overline{z} f$. Hence item 1) implies that $\mu = 0$. This contradiction shows that $z \in \rho(A)$ and the proof is complete. □

Corollary C.1. *1) If $A = A^*$ and there exists $M > 0$ such that*

$$\|(A - \lambda)x\| \geqq M\|u\| \quad \textit{for all} \;\; x \in D(A),$$

then $\lambda \in \rho(A)$. Moreover, the disc $D(\lambda; M)$ is contained in $\rho(A)$.

2) If $A = A^*$ then for $z \in \mathbb{C}$, with $Imz \neq 0$, we have $\|R(z; A)\| \leqq \frac{1}{|Imz|}$.

The proof of the following theorem can be found in Reed and Simon (1980).

Theorem C.9. *For $A = A^*$ and $\lambda \in \rho(A)$ we have*

$$\|R(\lambda; A)\| \leqq \frac{1}{dist(\lambda; \sigma(A))} = \sup\left\{\frac{1}{|\lambda - \mu|} : \mu \in \sigma(A)\right\}.$$

C.3 Riesz projection

We start with some results of the classical complex analysis for the case of vector-valued mappings (with values in a Banach space), more exactly, the vectorial version of the Cauchy and Liouville Theorems.

The proof of the following existence theorem can be prove by following the same ideas as in the classical complex theory (see Theorem 1.4 in (Conway 1978)).

Theorem C.10. *Let $(Z, \|\cdot\|)$ be a Banach space and $\gamma : [a, b] \to \mathbb{C}$ a simple closed path by part with $G(\gamma) = \{\gamma(t) : t \in [a, b]\} \equiv \Gamma$ and $g : \Gamma \to Z$ continuous. Then there exist $I \in Z$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for $P = \{t_0 = a < t_1 < \cdots < t_m = b\}$ being a partition of $[a, b]$ with $\|P\| = max\{|t_k - t_{k-1}| : 1 \leqq k \leqq m\} < \delta$ we have*

$$\left\| I - \sum_{k=1}^{m} g(\gamma(s_k))[\gamma(t_k) - \gamma(t_{k-1})] \right\| < \epsilon$$

for every chosen of $s_k \in [t_{k-1}, t_k]$.

I is called the Riemann-Stieltjes integral of g on Γ and it is denoted by

$$I = \int_\Gamma g(\lambda) d\lambda.$$

Remark C.12. *a) The value of I in Theorem C.10 is independent of the path γ such that $G(\gamma) = \Gamma$.*

b) By definition, I is the uniform limit on all chose of partitions P of $[a, b]$ and on all chosen s_i. Moreover,

$$I = \int_\Gamma g(\lambda) d\lambda = \lim_{m\to\infty} \sum_{i=1}^{m} g(\lambda_i)\Delta_i = \int_a^b g(\gamma(t)) d\gamma(t)$$
$$= \int_a^b g(\gamma(t))\gamma'(t) dt$$

c) For $F : Z \to \mathbb{C}$ a bounded linear functional, we have

$$F(\int_\Gamma g(\lambda)d\lambda) = \int_\Gamma F(g(\lambda))d\lambda.$$

d) Let $Z = \mathcal{L}(X)$ with X a Hilbert space and $g : \Gamma \to Z$ continuous. Then, we have in this case that I in Theorem C.10 is a bounded linear operator such that for all $x \in X$

$$Ix = \Big(\int_\Gamma g(\lambda)d\lambda\Big)x = \int_\Gamma g(\lambda)x d\lambda.$$

Moreover, for X_i Banach spaces and $A : X_1 \to X$ and $B : X \to X_2$ bounded linear operators, we have

$$BIA = \int_\Gamma Bg(\lambda)A d\lambda.$$

Theorem C.11 (Vectorial Cauchy Integral Theorem). *Let Ω be a non-empty set in $\mathbb{C}$ and $g : \Omega \to Z$ an analytic mapping. Then, for Γ being a simple closed path such that Γ and its inner domain are contained in Ω we have that $\int_\Gamma g(\lambda)d\lambda = 0$.*

Proof. Let $y \equiv \int_\Gamma g(\lambda)d\lambda$ and $f : Z \to \mathbb{C}$ a bounded linear functional. Then $f \circ g : \Omega \to \mathbb{C}$ is an analytic mapping. Hence by the classical Cauchy Integral Theorem

$$f(y) = \int_\Gamma f(g(\lambda))d\lambda = 0.$$

Therefore, the Hahn-Banach theorem implies that $y = 0$. □

Theorem C.12 (Vectorial Liouville Theorem). *Let $g : \mathbb{C} \to Z$ be an analytic mapping such that $\|g(\lambda)\| \leqq M$ for all $\lambda \in \mathbb{C}$. Then, the mapping g is constant.*

Proof. Let $f : Z \to \mathbb{C}$ be a bounded linear functional. Then $f \circ g : \mathbb{C} \to \mathbb{C}$ is an analytic mapping such that $|(f \circ g)(z)| \leqq \|f\|\|g(z)\| \leqq \|f\|M$ for all $z \in \mathbb{C}$ see Stein and Shakarchi 2003. Therefore, from the classical Liouville Theorem we have $f(g(z)) \equiv z_0$ for all $z \in \mathbb{C}$ (see Stein and Shakarchi 2003). Let $y_0 \in Z$ such that $f(y_0) = z_0$, then $f(g(z) - y_0) = 0$. Therefore, the Hahn-Banach theorem implies that $g(z) = y_0$ for all $z \in \mathbb{C}$. □

Riesz-projections

Theorem C.13 (Riesz Projection). *Let A be a closed linear operator on a Hilbert space H. Let $\lambda \in \sigma(A)$ an isolated point such that for $\epsilon > 0$ we have $\overline{D(\lambda;\epsilon)} \cap \sigma(A) = \{\lambda\}$. Let $\Gamma_\epsilon = \partial D(\lambda;\epsilon) = \{\mu \in \mathbb{C} : |\mu - \lambda| = \epsilon\}$ such that $\Gamma_\epsilon \cap \sigma(A) = \emptyset$. Then,*

1) *For all r such that $0 < r < \epsilon$*

$$P_\lambda = -\frac{1}{2\pi i}\oint_{\Gamma_r}(A-\mu)^{-1}d\mu$$

there exists and it is independent of r.

2) *$P_\lambda^2 = P_\lambda$. Hence P_λ is a projection (it is not necessarily orthogonal).*

3) *The subspaces $G_\lambda = ImP_\lambda$ and $F_\lambda = kerP_\lambda$ satisfy:*

   i) *G_λ and F_λ are closed complementary subspaces (they are not necessarily orthogonal),*

$$H = G_\lambda \oplus F_\lambda, \qquad G_\lambda \cap F_\lambda = \{0\}.$$

   ii) *G_λ and F_λ are A-invariant:*

$$\begin{cases} G_\lambda \subset D(A) \text{ and } AG_\lambda \subset G_\lambda, \\ F_\lambda \cap D(A) \text{ is dense in } F_\lambda, \text{ and } A[F_\lambda \cap D(A)] \subset F_\lambda. \end{cases}$$

   iii) *$A|_{G_\lambda} : G_\lambda \to G_\lambda$ is bounded.*

   iv) *$\sigma(A|_{G_\lambda}) = \{\lambda\}$ and $\sigma(A|_{F_\lambda \cap D(A)}) = \sigma(A) - \{\lambda\}$.*

4) *$G_\lambda = ImP_\lambda \supseteq ker(A-\lambda)$.*

5) *If A is self-adjoint, then P_λ is a orthogonal projection on $ker(A-\lambda)$ [$ImP_\lambda = ker(A-\lambda)$].*

Remark C.13. *P_λ defined in Theorem C.13 is called the Riesz projection for A and λ (see Hislop and Sigal (1996), Reed and Simon (1978)).*

Proof. 2). Let $s \in (r, \epsilon)$, then from the first resolvent identity (Theorem C.3), from the continuity of the mapping $(\mu, \nu) \in \Gamma_r \times \Gamma_s \to (\mu-\nu)^{-1} R(\nu)$ and from the index theorem of a curve with respect to a point, we have

$$\begin{aligned} P_\lambda^2 &= \frac{1}{(2\pi i)^2} \oint_{\Gamma_r} \oint_{\Gamma_s} R(\mu) R(\nu) d\nu d\mu \\ &= -\frac{1}{2\pi i} \oint_{\Gamma_r} R(\mu) \Big(\frac{1}{2\pi i} \oint_{\Gamma_s} \frac{d\nu}{\nu - \mu} \Big) d\nu - \frac{1}{(2\pi i)^2} \oint_{\Gamma_s} R(\nu) \Big(\oint_{\Gamma_r} \frac{d\mu}{\mu - \nu} \Big) d\nu \\ &= -\frac{1}{2\pi i} \oint_{\Gamma_r} R(\mu) d\mu = P_\lambda. \end{aligned}$$

1). Let $\delta \in (0, r)$ and we consider for $x \in H$ the element

$$P_{\lambda,\delta} x \equiv y = -\frac{1}{2\pi i} \oint_{\Gamma_\delta} R(\mu) x d\mu.$$

Next, let Ω be a open set contained in $D(\lambda; \epsilon) - \{\lambda\}$ such that $\Gamma_\delta, \Gamma_r \subset \Omega$. For f a bounded linear functional on H, we define the mapping $G : \Omega \to \mathbb{C}$ by $G(\mu) = f(R(\mu)x)$. Therefore, since G is analytic it follows from the Cauchy Integral Theorem that

$$\oint_{\Gamma_r} f(R(\mu)x) d\mu = \oint_{\Gamma_\delta} f(R(\mu)x) d\mu.$$

Hence we obtain that $f(P_{\lambda,r} x) = f(y)$, for all $f \in H^*$. Therefore, $P_{\lambda,r} x = y = P_{\lambda,\delta} x$ for all $x \in H$, and so $P_{\lambda,r} = P_{\lambda,\delta}$.

3) Since P_λ is a projection we have immediately that G_λ and F_λ are closed complementary subspaces. Now we see that $G_\lambda \subset D(A)$ and $AG_\lambda \subset G_\lambda$. Let $\psi \in G_\lambda$, then since $\psi = P_\lambda \psi$ it follows from the closedness of A that $\psi \in D(A)$ and

$$A\psi = -\frac{1}{2\pi i} \oint_{\Gamma_r} AR(\mu)\psi d\mu = -\frac{1}{2\pi i} \oint_{\Gamma_r} [\psi + \mu R(\mu)\psi] d\mu, \tag{C.6}$$

where we are used the relation $AR(\mu) = I + \mu R(\mu)$. Now, since A and its resolvent $R(\mu)$ commute we obtain from (C.6) that for all $\psi \in G_\lambda$, $A\psi = AP_\lambda\psi = P_\lambda A\psi$. We note that the later analysis show also that P_λ and A commute on $D(A)$, namely, for all $x \in D(A)$ we have $AP_\lambda x = P_\lambda Ax$.

The prove that $F_\lambda \cap D(A)$ is dense in F_λ is immediately. Next, for $\psi \in F_\lambda \cap D(A)$ we see that $A\psi \in F_\lambda$. Indeed, the equality

$$(I - P_\lambda) A\psi = A\psi - AP_\lambda \psi = A\psi,$$

implies that $P_\lambda A\psi = 0$.

3) $-iii$) We obtain from (C.6) that for $\psi \in G_\lambda$

$$\|A\psi\| \leqq \frac{1}{2\pi} \oint_{\Gamma_r} [\|\psi\| + M\|\psi\|] |d\mu| = \frac{1}{2\pi} (1 + M) \ell(\Gamma_r) \|\psi\|,$$

where $M = \max_{\mu\in\Gamma_r}|\mu|\|R(\mu)\| < \infty$ because of Γ_r being compact and the mapping $\mu \in \Gamma_r \to |\mu|\|R(\mu)\|$ continuous.

3) $-iv$) Let $\nu \notin \Gamma_r$ and define the following bounded linear operator $S(\nu) : H \to H$ by

$$S(\nu) = -\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{1}{\nu-\mu}R(\mu)d\mu. \tag{C.7}$$

Similarly as the prove of item 3) we obtain that for all $x \in H$, $S(\nu)x \in D(A)$. Next we prove that G_λ and F_λ are invariant by $S(\nu)$. Indeed, since $R(\gamma)P_\lambda = P_\lambda R(\gamma)$ for all $\gamma \in \rho(A)$ we have that $S(\nu)$ commute with P_λ. Therefore we obtain that $S(\nu)G_\lambda \subset G_\lambda$ and $S(\nu)F_\lambda \subset F_\lambda$. In fact, for $\psi \in G_\lambda$ we have $P_\lambda S(\nu)\psi = S(\nu)P_\lambda\psi = S(\nu)\psi$, and for $\psi \in F_\lambda$ we obtain $(I - P_\lambda)S(\nu)\psi = S(\nu)\psi$ and so $P_\lambda S(\nu)\psi = 0$.

Now, since A and its resolvent commute we obtain from the relation $S(\nu)(A-\nu) = (A-\nu)S(\nu)$ that

$$S(\nu)(A-\nu) = -\frac{1}{2\pi i}\Big(\oint_{\Gamma_r}\frac{1}{\nu-\mu}d\mu\Big)I + \frac{1}{2\pi i}\oint_{\Gamma_r}R(\mu)d\mu. \tag{C.8}$$

Therefore,

$$S(\nu)(A-\nu) = (A-\nu)S(\nu) = \begin{cases} I - P_\lambda, & \text{for } \nu \text{ inside of } \Gamma_r \\ -P_\lambda, & \text{for } \nu \text{ outside of } \Gamma_r. \end{cases} \tag{C.9}$$

Next we see that $\sigma(A|_{G_\lambda}) \subset \{\lambda\} \equiv \sigma_0$. Suppose that $\nu \in \sigma(A|_{G_\lambda})$ and $\nu \neq \lambda$. Then for r small enough we can choose ν outside of Γ_r and so for $x \in G_\lambda$ we have

$$(A-\nu)S(\nu)x = S(\nu)(A-\nu)x = -x.$$

Now, since $S(\nu)G_\lambda \subset G_\lambda$ we obtain that $A-\nu : G_\lambda \to G_\lambda$ is one to one and onto and so by the closed graph theorem $A-\nu$ is invertible and $(A-\nu|_{G_\lambda})^{-1} = S(\nu)|_{G_\lambda}$. Therefore $\nu \in \rho(A|_{G_\lambda})$ which is a contradiction. Similarly, we show that $\sigma(A|_{F_\lambda\cap D(A)}) \subset \sigma(A)-\{\lambda\} \equiv \sigma_1$. By denoting $N_\lambda = F_\lambda \cap D(A)$ we show now that $\sigma(A) \subset \sigma(A|_{G_\lambda}) \cup \sigma(A|_{N_\lambda})$. Let $\nu \in \sigma(A)$ and suppose that $\nu \notin [\sigma(A|_{G_\lambda})\cup\sigma(A|_{N_\lambda})$. Then we have that $A-\mu : G_\lambda \to G_\lambda$ and $A-\mu : N_\lambda \to F_\lambda$ are one to one and onto. Since $H = G_\lambda \oplus F_\lambda$ then $A-\mu : D(A) \to H$ is one to one and onto and so $\mu \in \rho(A)$. The later analysis shows that $\sigma(A) \subset \sigma(A|_{G_\lambda}) \cup \sigma(A|_{N_\lambda}) \subset \sigma_0 \cup \sigma_1 = \sigma(A)$. Then $\sigma(A) = \sigma_0 \cup \sigma_1 = \sigma(A|_{G_\lambda}) \cup \sigma(A|_{N_\lambda})$, where we obtain that $\sigma_0 = \sigma(A|_{G_\lambda})$ and $\sigma_1 = \sigma(A) - \{\lambda\} = \sigma(A|_{F_\lambda\cap D(A)})$.

4) Let $\psi \in ker(A-\lambda)$, then for all $\mu \in \Gamma_r$ we have that $(A-\mu)\psi = (\lambda-\mu)\psi$ implies that $R(\mu)\psi = \frac{1}{\lambda-\mu}\psi$. Next we see that $P_\lambda\psi = \psi$. Indeed,

$$P_\lambda\psi = -\frac{1}{2\pi i}\oint_{\Gamma_r}R(\mu)\psi d\mu = -\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{1}{\lambda-\mu}\psi d\mu = \psi.$$

5) Let $r > 0$ and $\mu = \lambda + re^{i\theta}$, with $\theta \in [0, 2\pi]$ and $\lambda \in \mathbb{R}$. Initially, we have that P_λ is a orthogonal projection ($P_\lambda^* = P_\lambda$). Indeed, since A is self-adjoint we have for $x \in H$

that the relation

$$P_\lambda x = -\frac{1}{2\pi}\int_{-\pi}^{\pi} R(\lambda + re^{i\theta})re^{i\theta}x d\theta,$$

implies immediately

$$P_\lambda^* x = -\frac{1}{2\pi}\int_{-\pi}^{\pi} R(\lambda + re^{-i\theta})re^{-i\theta}x d\theta,$$

and so the change of variable $\theta \to -\theta$ implies that $P_\lambda x = P_\lambda^* x$. Next we show that $ImP_\lambda \subset ker(A-\lambda)$, namely, $(A-\lambda)P_\lambda = 0$. Indeed, by using the relation $(A-\lambda)R(\mu) = I + (\mu-\lambda)R(\mu)$ we obtain

$$(A-\lambda)P_\lambda = -\frac{1}{2\pi i}\oint_{\Gamma_r}[I + (\mu-\lambda)R(\mu)]d\mu = -\frac{1}{2\pi i}\oint_{\Gamma_r}(\mu-\lambda)R(\mu)d\mu.$$

Next, we consider $B_\lambda = D(\lambda; r) - \{\lambda\}$ and the analytic mapping $f(\mu) = (\mu-\lambda)R(\mu)$ for $\mu \in B_\lambda$. Then for all $\mu \in \Gamma_r$ follows from (C.9) that

$$|\mu-\lambda|\|R(\mu)\| \leqq \frac{|\mu-\lambda|}{d(\mu,\sigma(A))}.$$

So, by choosing r small enough such that λ is the nearest point from $\sigma(A)$ to Γ_r, we obtain $d(\mu,\sigma(A)) \geqq d(\Gamma_r,\sigma(A)) = r$, that for all $\mu \in \Gamma_r$. Hence,

$$\frac{|\mu-\lambda|}{d(\mu,\sigma(A))} \leqq \frac{|\mu-\lambda|}{d(\Gamma_r,\sigma(A))} = 1.$$

Then, we obtain that $f : B_\lambda \to H$ is uniformly bounded and therefore the Riemann removable singularities theorem (extended to the vectorial case) implies that f can be extended to a analytic function on $D(\lambda; r)$. Hence, The Vectorial Cauchy Integral Theorem implies that

$$\oint_{\Gamma_r} f(\mu)d\mu = 0.$$

This complete the proof of the theorem. □

Definition C.15. *Let A be a closed linear operator $A : D(A) \subset H \to H$.*

1) *A point $\lambda \in \sigma(A)$ is called discrete (or an eigenvalue of A of finite type) if*
   - *λ is a isolated point of $\sigma(A)$ and,*
   - *The Riesz projection P_λ determined by λ is finite dimensional: $dim(ImP_\lambda) < \infty$.*
2) *If $dim(ImP_\lambda) = 1$, λ is called a simple or non-degenerate eigenvalue for A.*

3) *The discrete spectrum of A,* $\sigma_d(A)$*, is the set of all eigenvalues of A of finite type.*

Remark C.14. *a) The number* $dim(ImP_\lambda)$ *is called the algebraic multiplicity of* λ*. The vectors in* ImP_λ *are called generalized eigenvectors of A, in the sense that there is* $\eta > 0$*, the algebraic multiplicity of* λ*, such that* $(A-\lambda)^n\psi = 0$ *for* $\psi \in ImP_\lambda$*.*

b) The number $dim(ker(A-\lambda))$ *is called the geometric multiplicity of* λ

c) Since $G_\lambda = ImP_\lambda$ *is finite dimensional and* $\{\lambda\} = \sigma(A|_{G_\lambda})$ *(by Theorem C.13) we obtain that* λ *is an eigenvalue of A.*

d) If $G_\lambda = 1$ *then* $G_\lambda = [\psi]$*. Moreover, by c) there exists* $f \neq 0$ *such that* $f \in ker(A-\lambda) \subset G_\lambda$*. Then* $f = \gamma\psi$*,* $\gamma \in \mathbb{C}$*. Therefore* $[\psi] \subset ker(A-\lambda) \subset [\psi]$*. Hence, in this case we obtain that* $ImP_\lambda = ker(A-\lambda)$*.*

The next theorem give us a good tool for locating some part of the discrete spectrum of a closed operator.

Theorem C.14. *Let A be a closed linear operator such that* $\Gamma_r = \{\mu \in \mathbb{C} : |\mu - \lambda| = r\} \subset \rho(A)$*. Then,*

1) The bounded linear operator

$$P = -\frac{1}{2\pi i}\oint_{\Gamma_r}(A-\mu)^{-1}d\mu$$

it is a projection.

2) If $n = dim(ImP) < \infty$*, then A has at most n points of its spectrum in* $\{\mu \in \mathbb{C} : |\mu - \lambda| < r\}$ *and each is discrete.*

3) If $n = 1$*, there is exactly one spectral point in* $\{\mu \in \mathbb{C} : |\mu - \lambda| < r\}$ *and it is nondegenerate.*

Proof. 1) The proof of Theorem C.13-2) carries through without change to prove that P is a projection and the proof of 3) implies that $G = ImP$ and $F = kerP$ are closed complementary invariant subspaces.

2) Let $A_1 = A|_G : G \to G$. Then A_1 has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ ($k \leqq n$). Now, for $A_2 = A|_{F\cap D(A)}$, we see that if ν satisfies $|\nu - \lambda| < r$ then $\nu \notin \sigma(A_2)$. Indeed, let $S(\nu)$ be the bounded linear operator defined in (C.8), then $S(\nu)F \subset F$ and from (C.9) we have $S(\nu)(A-\nu) = (A-\nu)S(\nu) = I - P$. Hence for $x \in F \cap D(A)$, we have $S(\nu)(A-\nu)x = x$ and for $x \in F$, $(A-\nu)S(\nu)x = x$. Then, $A - \nu : F \cap D(A) \to F$ is one-to-one and onto. Therefore, $A - \nu$ is invertible and so $\nu \notin \sigma(A_2)$.

Now, since $H = G \oplus F$ we obtain from the later analysis that $\nu \in \rho(A)$ for ν such that $|\nu - \lambda| < r$ if and only if $\nu \in \rho(A_1)$.

Next we show that $\lambda_1, \lambda_2, \cdots, \lambda_k$ belong to the open disc $D(\lambda; r)$. Indeed, since λ_i are eigenvalues of A we have that $\lambda_i \notin \Gamma_r$. By considering $\psi_i \in G - \{0\}$ such that $A\psi = \lambda_i \psi$, we obtain from the relation $(A - \mu)\psi = (\lambda_i - \mu)\psi$ for every $\mu \in \Gamma_r$, that

$$\psi_i = P\psi = -\frac{1}{2\pi i}\oint_{\Gamma_r}(A-\mu)^{-1}\psi d\mu = \frac{1}{2\pi i}\Big(\oint_{\Gamma_r}\frac{1}{\mu-\lambda_i}d\mu\Big)\psi_i.$$

Therefore, since $\psi_i \neq 0$ we need to have that $\lambda_i \in D(\lambda; r)$. Hence, from the analysis above we obtain $\sigma(A) \cap D(\lambda; r) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}_{(k \leqq n)} = \sigma(A_1) \cap D(\lambda; r)$, and so A has at most n points of its spectrum in $D(\lambda; r)$. Now we see that each λ_i $(i \leqq n)$ is discrete. Let $\Theta_\epsilon = \partial D(\lambda_i; \epsilon) \subset D(\lambda; r)$ and define

$$P_{\lambda_i} = -\frac{1}{2\pi i}\oint_{\Theta_\epsilon}(A-\gamma)^{-1}d\gamma.$$

Then, from the index theorem of a curve with respect to a point, we obtain $P_{\lambda_i}P = PP_{\lambda_i} = P_{\lambda_i}$ and so $ImP_{\lambda_i} \subset ImP$ and so $dim(ImP_{\lambda_i}) \leqq dim(ImP)$.

3) It follows immediately from 2). □

We finish this Appendix with two "innocent results" from linear algebra, it which have deep consequences in perturbation theory.

Lemma C.2. *Let M and N be two subspaces of H with $dimM > dimN$. Then there is $x \in M$ such that $x \perp N$.*

Proof. By considering a subspace of M, if necessary, we can assume that $dimM$ and $dimN$ are finite. We consider orthonormal bases $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$, $m > n$, for M and N. Let $x = \sum_{i=1}^m a_i x_i$, and we solve homogeneous linear system

$$\langle x, y_j\rangle = \sum_{i=1}^m a_i\langle x_i, y_j\rangle = 0, \quad \text{for } j = 1, ..., n.$$

Since the matrix $A = [\langle x_i, y_j\rangle]$ is an $n \times m$ matrix with $n < m$, the homogeneous system of linear equations $AX = 0$ has a non-trivial solution $X = (a_1, a_2, ..., a_m)^t$. Therefore, there is $x \in M - \{0\}$ such that $\langle x, y_j\rangle = 0$, for $j = 1, ..., n$. □

Lemma C.3. *Let $P, Q : H \to H$ be projections. If $dim(ImP) \neq dim(ImQ)$, then $\|P - Q\| \geqq 1$.*

Proof. Suppose $dim(ImP) < dim(ImQ)$. Let $F = Ker(P)$ and $E = ImQ$. Then, from Theorems C.7 and C.4 we have the relations

$$H = F \oplus ImP, \quad \text{and} \quad H = F \oplus F^{\perp}.$$

Therefore, there is a linear isomorphism $\varphi : F^{\perp} \to ImP$. Then, we deduce that $dim(F^{\perp}) = dim(ImP) < dim(E)$. Hence, from Lemma C.2 there is $x \in E \cap F^{\perp\perp} = E \cap F, x \neq 0$. Then the relations $Qx = x$ and $Px = 0$ imply

$$\|(P - Q)x\| = \|x\|.$$

Hence $\|P - Q\| \geqq 1$. □

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