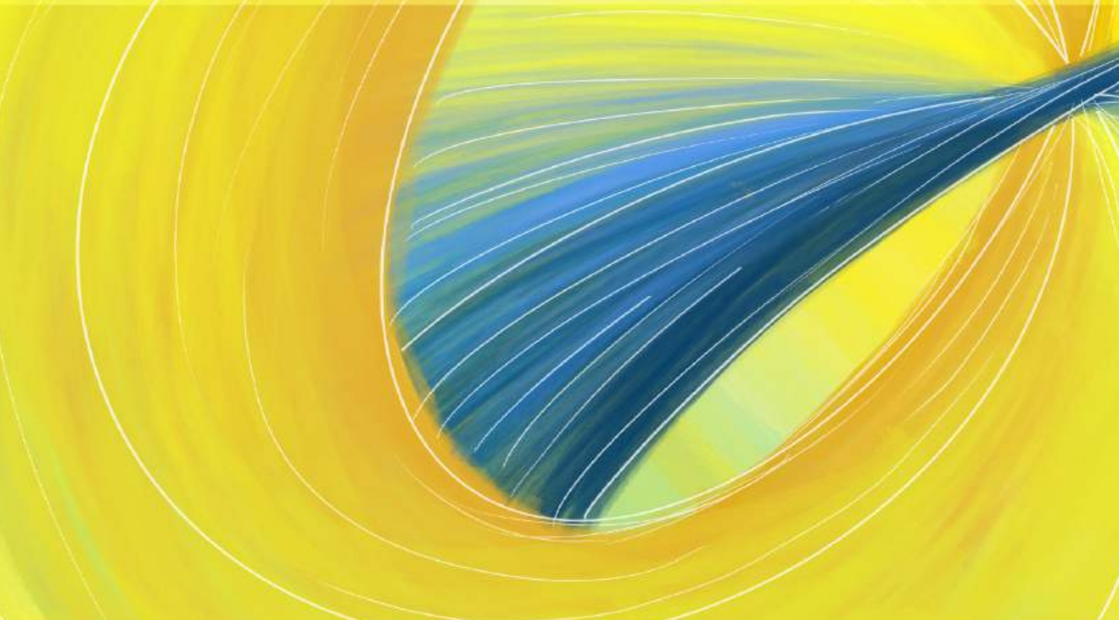
A dynamical system approach for Lane–Emden type problems



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Preface

The idea of this book and course originated from the experience of a joint project started in 2019, where the first two authors were visiting the university of Rome La Sapienza under the invitation of Prof. Filomena Pacella. At first glance, working with Dynamical Systems seemed challenging, mostly because it was not the area of expertise for any of us. But we developed a huge amount of research, linked to our previous experience, and ended up with a rather good knowledge on the theory. In these notes we present in a simple form the essential tools in Ordinary Differential Equations and Dynamical Systems to solve Partial Differential Equations of fully nonlinear type in the radial regime.

The book is aimed for graduate students interested in Partial Differential Equations and/or Dynamical Systems. We tried to gather a self contained and detailed analysis on the subject, in addition to several references, in such a way to ease the experience of the reader.

Here we exploit the classification of radial positive solutions for a class of fully nonlinear problems. Our approach is entirely based on the analysis of the dynamics induced by an autonomous quadratic system, which is obtained after a suitable transformation. This method allows us to treat both regular and singular solutions in a unified way. It applies to define critical exponents, from which existence and nonexistence of solutions are completely characterized.

It is our goal to enable the reader to identify all trajectories produced by the dynamical system and translate it into positive radial solutions of the corresponding second order partial differential equations problem. We will deal with solutions in a ball, the whole space, exterior domains, and annuli.

We would like to thank the nice environment promoted by the Math Department of La Sapienza University of Rome, as well as the opportunity given by the Organizing Committee of 33rd Brazilian Colloquium of Mathematics, in particular Prof. Carolina Araújo. We also acknowledge the support provided by the editorial board of IMPA, in special Paulo Ney de Souza for continuous assistance and attention in the production of the text.

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Contents

Introduction

In this book we study existence, uniqueness, nonexistence, and classification of radial positive solutions for some nonlinear problems, subject to a Lane–Emden coupling with Hénon type power weight, and driven by fully nonlinear operators.

Let us recall some history on the development of these problems. Semilinear equations like

$$-\Delta u(x) = f(|x|, u(x)) \quad \text{in } \mathbb{R}^N, \; N \geqslant 3, \tag{0.0.1}$$

have long been studied, in special by arising in the context of Astrophysics. Here Δ is the standard Laplacian operator in the Euclidean space $\mathbb{R}^N$,

$$\Delta u \; = \; \textstyle\sum_{i=1}^N e_i \; , \quad \{e_i\}_{i=1}^N = \mathrm{spec}(D^2 u),$$

which is just the sum of the eigenvalues of the Hessian $D^2 u$ whenever u is a C^2 function. In some cases, the admissibility of stationary and spherically symmetric stellar dynamic models is equivalent to the solvability of an equation in the form (0.0.1) when $N = 3$, see (Batt, Faltenbacher, and Horst 1986; Mercuri and Moreira dos Santos 2019; Yi Li 1993) and references therein.

In particular, the Lane–Emden equation in the space

$$-\Delta u(x) = |u(x)|^{p-1} u(x) \quad \text{in} \quad \mathbb{R}^3, \;\; p > 1.$$

refers to the description of certain self-gravitating spherically symmetric stellar systems.

Jonathan Homer Lane (American, 1819–1880) and Robert Emden (Swiss, 1862–1940) were two astrophysicists who lead similar mathematical investigations on stellar structures modeled by polytropic fluids. These were the first stellar models studied in the end of the 19 century (Lane 1870), although are still useful for understanding the basic structure of astrophysical objects. For instance, they apply to stars possessing a spherical symmetry, not varying with time, and without no internal motions. As in (Meier 2012, Section 5.2.4.1), polytropic stellar structures can be derived from Newtonian fluid equations, in which three conservation laws yield to the second order differential equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}u}{\mathrm{d}r}\right)+u^p=0,$$

such that u satisfies a polynomial relation in the form $\rho = \rho_c\, u^p$, where ρ_c is the central mass density, and p stands for the polytropic index. More general equations like

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^\gamma\,\frac{\mathrm{d}u}{\mathrm{d}r}\right)+r^\sigma u^p=0,$$

are also called Emden–Fowler in the literature, see the survey (Wong 1975).

The General Relativity theory of Albert Einstein has made impressive and even apocalyptic predictions about the space-time structure of the universe, among them the existence of black holes. Since the important work of the German astronomer Karl Schwarzschild, over the past decades astrophysicists and mathematicians have been devoted to understand their properties, from the ones which possess the same mass as stars up to recent photographed discoveries of supermassive objects lying in the Milky Way's center, by culminating at the Physics Nobel prize awards in 2019, see (Hawking and Ellis 1973; Meier 2012; Overbye and Taylor 2020; Peebles 1972; Rhode 2007; The Event Horizon Telescope Collaboration et al. 2019; Wald 1984).

A related model problem is the so called Hénon equation,

$$-\Delta u(x) = |x|^a\,|u(x)|^{p-1}u(x) \quad \text{in} \quad \mathbb{R}^3. \tag{0.0.2}$$

Michel Hénon (French, 1931-2013) was a mathematician and astronomer who developed an analysis of the stability of spherical steady state stellar systems numerically, in what concerns the concentric shell model, see (Hénon 1973). In the case $a < 0$ this equation is also known as Hardy-Hénon (see for instance (Phan and Souplet 2012)), because its relation with the Hardy inequality by referring to the mathematician Godfrey Harold Hardy (English, 1877-1947).

On the other hand, in this text we will be interested in second order problems with fully nonlinear nature, which means the operator depends on the second derivatives entry in a nonlinear way. In particular, we will look at the following Pucci extremal operators

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \Lambda \textstyle\sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = \lambda \textstyle\sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

for $0 < \lambda < \Lambda$, which is a weighted sum of the eigenvalues $\{e_i\}_{i=1}^N$ of the Hessian D^2u whenever u is a C^2 function. They are named so due to the mathematician Carlo Pucci (Italian, 1925-2003), see (Pucci 1966). They are extremal operators, in the sense that $\mathcal{M}^+_{\lambda,\Lambda}(X)$ is the supremum of all linear linear operators in the form $\mathrm{tr}(AX)$ over all matrices with eigenvalues between λ and Λ, while $\mathcal{M}^-_{\lambda,\Lambda}$ is the infimum one, see the next chapter for details. In this sense, they define the whole class of fully nonlinear uniformly elliptic operators, see (Caffarelli and Cabré 1995). Hence they play, in the fully nonlinear context, the same role as the Laplace operator in the linear case.

To make matters precise, we study positive radial solutions of the following class of fully nonlinear elliptic equations involving the Pucci's operators,

$$\mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) + |x|^a u^p = 0, \quad u > 0 \;\text{ in } \Omega, \tag{0.0.3}$$

where $a > -1$, $p > 1$. The set $\Omega \in \mathbb{R}^N$, $N \geqslant 3$, is a radial domain such as $\mathbb{R}^N$, a ball B_R of radius $R > 0$ centered at the origin, the exterior of B_R, or an annulus.

We deal with both regular and singular solutions u of (0.0.3) which are C^2 for $r > 0$. In the singular case Ω will be either $\mathbb{R}^N \setminus \{0\}$ or $B_R \setminus \{0\}$, and we assume the condition

$$\lim_{r\to 0} u(r) = +\infty, \quad r = |x|. \tag{0.0.4}$$

Finally, whenever Ω has a boundary, we prescribe on it the Dirichlet condition

$$u = 0 \text{ on } \partial\Omega, \qquad \text{or} \qquad u = 0 \text{ on } \partial\Omega \setminus \{0\} \;\text{ under (0.0.4)}.$$

Our solutions are understood in the classical sense out of 0 and they are of class C^1 up to 0, since $a > -1$, as shows our Proposition 3.1.7 ahead.

Let us have in mind the so called dimension-like numbers $\tilde{N}_\pm$ as

$$\tilde{N}_+ = \tfrac{\lambda}{\Lambda}(N-1)+1, \qquad \tilde{N}_- = \tfrac{\Lambda}{\lambda}(N-1)+1\,,$$

whenever $\tilde{N}_+ > 2$.

A general existence result in bounded domains Ω, not necessarily radial, was obtained in (Quaas and Sirakov 2006) under the condition

$$1 < p \leqslant \tfrac{\tilde{N}_+}{\tilde{N}_+ - 2} \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda}\,, \qquad 1 < p \leqslant \tfrac{\tilde{N}_-}{\tilde{N}_- - 2} \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda}$$

These intervals come from the optimal range for existence of supersolutions to $\mathcal{M}^\pm$ when $a = 0$, see Theorem 1.2.8 in the next chapter.

When the Pucci's operators reduce to the Laplacian (i.e. for $\lambda = \Lambda$, and $\tilde{N}_\pm = N$) the previous exponents are equal to $\frac{N}{N-2}$ which is known as the Serrin exponent. They do not provide optimal bounds in terms of solutions of (0.0.3) when $a = 0$, which is clear by considering the semilinear case.

Nevertheless, as far as the radial setting is concerned, critical exponents which represent the threshold for the existence of solutions to (0.0.3) can be defined when $N \geqslant 3$. They were introduced for $a = 0$ by Felmer and Quaas in the seminal work (Felmer and Quaas 2003) for establishing existence and classification of radial positive solutions in $\mathbb{R}^N$. These are also the watershed for existence and nonexistence of positive solutions in the ball. In the case the dimension is $N = 2$ no critical exponent exists for the Laplacian or for the $\mathcal{M}^+_{\lambda,\Lambda}$ operator, while it can still be defined for the $\mathcal{M}^-_{\lambda,\Lambda}$ case, see (Pacella and Stolnicki 2021a). However here we only consider the dimensions $N \geqslant 3$.

Note that every positive solution in the ball when $a = 0$ is radial, by (Da Lio and Sirakov 2007), while this is not true in general for $a \neq 0$, even in the semilinear case. In fact, for the Dirichlet problem associated to the standard Hénon equation (0.0.2), in (Willem 2002) it was obtained the existence of a radial solution, in addition to a least energy solution which is not radially symmetric. Moreover, the power weight $|x|^a$ as the parameter a gets large induces symmetry breaking and concentration phenomena, see (Mercuri and Moreira dos Santos 2019; Wang 2006; Yan 2009).

When $\lambda = \Lambda$ the corresponding critical exponents are the same, both in radial and nonradial settings; see (Caffarelli, Gidas, and Spruck 1989) for $a = 0$, and (Gladiali, Grossi, and Neves 2013) for $a \neq 0$. The identification of critical exponents in the nonradial case for fully nonlinear operators, in turn, remains open.

For $p > 1$ and $a > -1$, we set

$$\alpha = \frac{2+a}{p-1}.$$

In the study of the standard Hénon equation (0.0.2), two classes of radial positive solutions are important: the fast decaying and the slow decaying ones. Namely,

$$\lim_{r\to\infty} r^{N-2}u(r) = c, \quad \text{and} \quad \lim_{r\to\infty} r^{\alpha}u(r) = c,$$

respectively, for some $c > 0$ whenever $N > 2$. It is known that fast decaying solutions exist at the critical exponent

$$p_{\Delta}^{a} = \frac{N+2+2a}{N-2},$$

while slow decaying solutions emerge for $p > p_{\Delta}^{a}$.

In the aforementioned work (Felmer and Quaas 2003) it was shown for the operators $\mathcal{M}^{\pm}$, when $a = 0$, and $\lambda < \Lambda$, the existence of critical exponents $p_{\pm}^{*}$ as far as $\tilde{N}_{+} > 2$. They play the role of p_{Δ}^{a} for Laplacian in the sense of being the threshold for existence and nonexistence of regular solutions in $\mathbb{R}^N$, see Theorem 1.2.9 in the next chapter. A new class of solutions was also detected, namely pseudo-slow decaying solutions, which satisfy

$$c_1 = \liminf_{r\to\infty} r^{\alpha}u(r) < \limsup_{r\to\infty} r^{\alpha}u(r) = c_2 \ ,$$

for some $0 < c_1 < c_2$, see Definition 1.2.3.

In the case of the operator $\mathcal{M}^{+}$ the authors also made precise the range of the exponent p for which pseudo-slow decaying solutions exist. The proof of this result in (Felmer and Quaas 2003) is involved. It is a combination of the Emden–Fowler phase plane analysis and the Coffman Kolodner technique. The latter consists in differentiating the solution with respect to the exponent p, and then studying a related nonhomogeneous differential equation, from which they derive the behavior of the solutions for p on both right and left hand sides of $p_{\pm}^{*}$, as well as the uniqueness of the exponent p for which a fast decaying solution exists.

The existence of a critical exponent unveils an important feature of the Pucci's operators. It reflects some intrinsic properties of these operators and induces concentration phenomena besides of energy invariance, see (Birindelli et al. 2018), as it happens in the classical semilinear case.

In this book we show how to derive critical exponents, for both regular and singular solutions, of the respective weighted problem (0.0.3), in light of our recent paper (Maia, Nornberg, and Pacella 2020). In what concerns regular solutions our results are similar to those in (Felmer and Quaas 2003), a little bit improved, but with the difference that we exploit much more the strength provided by the dynamical system itself, which makes the proofs simpler. This approach has further been used in (Pacella and Stolnicki 2021a) to refine the bounds given in (Felmer and Quaas 2003) for the critical exponents even more.

Another advantage of our approach is that it treats in a unified way several kind of solutions to (0.0.3). We refer the reader to Figures 4.1 to 4.7 where, for a given value of the exponent p, all the orbits of the system corresponding to different types of solutions of (0.0.3) are displayed simultaneously. Our proofs do not involve energy functions, except for showing a center configuration which appears at the critical exponents, in addition to a particular existence result in annuli for weighted equations.

We highlight that the strategy of introducing an associated quadratic system to treat radial Lane–Emden problems, with or without weight, has long been managed, see for instance (Chicone and Tian 1982; Wong 1975) and references therein. We will follow more closely the ideas applied in (Bidaut-Véron and Giacomini 2010). Finally, we point out that the quadratic system approach can be extended to treat systems with Lane–Emden configuration, in the spirit of (Bidaut-Véron and Giacomini 2010). In the fully nonlinear context, it is the subject of our recent work (Maia, Nornberg, and Pacella 2021).

The text is organized as follows. In Chapter 1 we introduce notations and give an overview of the prerequisites on Dynamical Systems and second order partial differential equations that will be used throughout the text. In particular, in Section 1.2.1 we write down equation (0.0.3) in the radial form. In Chapter 2 we introduce the quadratic system associated to (0.0.3), and study its intrinsic flow properties. In Chapter 3 we classify the different solutions of (0.0.3) in terms of orbits of the corresponding dynamical systems. In the last chapter, Sections 4.1 and 4.2 are devoted to the proofs of the main results for the Pucci $\mathcal{M}^+$ and $\mathcal{M}^-$ operators, respectively.

1 Preliminaries

In this chapter we review some preliminary facts. From ODEs we recall some results for two dimensional first order autonomous systems such as the stable and unstable manifold theorem, Poincaré–Bendixson theorem, Dulac's type criteria, and extension of local trajectories. From PDEs we establish the radial form of (0.0.3) and discuss its difficulties. We also revisit some comparison principles, Hopf lemma, symmetry properties of solutions, and state some theorems mentioned in the Introduction.

1.1 ODEs overview

We first recall some standard definitions from the theory of dynamical systems. Consider the system of ordinary differential equations (ODEs for short),

$$\dot{x} = F(x), \quad \text{where} \quad x = (X, Z), \quad F(x) := (f(x), g(x)). \tag{1.1.1}$$

with $\dot{x} = (\dot{X}, \dot{Z})$. Solutions of ODE problems in the form (1.1.1) will be called trajectories or orbits, and they are differentiable in the variable t in their interval of definition.

Here and onward in the text we deal with F as a locally Lipschitz function on the variable x. In particular, it implies the so called existence and uniqueness property of trajectories in the sense: given any point x_0, there exists a unique trajectory $\tau(t)$ solving (1.1.1) and passing through the point x_0 at time $t = 0$, i.e. such that $\dot{\tau}(t) = F(\tau(t))$ and $\tau(0) = x_0$. Moreover, such orbits are continuous with respect to the initial condition x_0, and with respect to possible additional parameters involved in the problem, see (Hale 1980). The latter will appear in terms of the exponent p in (0.0.3).

Definition 1.1.1. A stationary point Q of (1.1.1) is a zero of the vector field F. If σ_1 and σ_2 are the eigenvalues of the Jacobian matrix $DF(Q)$, then Q is hyperbolic if both σ_1, σ_2 have nonzero real parts. If this is the case, Q is a *source* if $\mathrm{Re}(\sigma_1), \mathrm{Re}(\sigma_2) > 0$, and a *sink* if $\mathrm{Re}(\sigma_1), \mathrm{Re}(\sigma_2) < 0$; Q is a *saddle point* if $\mathrm{Re}(\sigma_1) < 0 < \mathrm{Re}(\sigma_2)$.

Next we recall an important result from the theory of dynamical systems which describes the local stable and unstable manifolds near saddle points of the system (1.1.1); see (Hale and Koçak 1991, theorems 9.29, 9.35). Here the usual theory for autonomous planar systems applies since each stationary point Q possesses a neighborhood which is strictly contained in R_λ^+ or R_λ^- where the vector field F is C^1.

Proposition 1.1.2. *Let Q be a saddle point of* (1.1.1). *Then the local stable (resp. unstable) manifold at Q is locally a C^1 graph over the stable (resp. unstable) line of the linearized vector field. In this case, if the linearized system has a stable line direction L, then there exists exactly two trajectories τ_1 and τ_2 arriving at Q which admit the same tangent at the point $Q = \overline{\tau}_1 \cap \overline{\tau}_2$ given by L. Analogously there are only two trajectories coming out from Q with the same property.*

We sometimes use the following notation to describe the limit of trajectories in the phase plane.

Definition 1.1.3 (α and ω limits)**.** We call α-limit of the orbit τ, and we denote it by $\alpha(\tau)$, as the set of limit points of $\tau(t)$ as $t \to -\infty$. Similarly one defines $\omega(\tau)$ i.e. the ω-limit of τ at $+\infty$.

We recall the classical theorem of Poincaré–Bendixson (see Theorems 12.1 and 12.5 in (Hale and Koçak 1991)), which classifies all possible α and ω limits of autonomous dynamical systems in the plane.

Theorem 1.1.4 (Poincaré–Bendixson)**.** *Suppose that $\dot{x} = F(x)$, $x(0) = x_0$, is a planar system with the existence and uniqueness property of trajectories. Assume the system has a finite number of stationary points. If the orbit τ is bounded forward in time, then one of the following is true:*

- $\omega(\tau) = \bar{x}$ *for a stationary point* $\bar{x}$*, such that* $\tau(t) \to \bar{x}$ *as* $t \to +\infty$*;*

- $\omega(\tau) = \theta$*, where* θ *is a periodic orbit such that either the trajectory itself is periodic* $\theta = \tau$*, or the trajectory spirals with increasing forward time toward* θ *on one side of* θ*;*

- $\omega(\tau)$ *consists of stationary points and periodic orbits whose* α *and* ω *limits are the stationary points.*

The statement is similar for trajectories bounded backward in time with respect to the α *limit.*

In order to exclude the existence of periodic orbits, the following criteria of Bendixson and Dulac (Theorems 12.8 and 12.9 in (Hale and Koçak 1991)) are useful.

Theorem 1.1.5. *Let* D *be a simply connected open of* $\mathbb{R}^2$*. Set* $x = (X, Z)$*, and* $F = (f, g)$*. If either:*

- (Bendixson′s criterium) $\mathrm{div} F = \partial_X f + \partial_Z g$ *is of constant sign and not identically zero in* D*;*

  or

- (Dulac′s criterium) $\mathrm{div}(BF) = \partial_X(Bf) + \partial_Z(Bg)$ *is of constant sign and not identically zero in* D*, where* $B(x)$ *is a real-valued* C^1 *function in* D*;*

then $\dot{x} = F(x)$ *has no periodic orbit lying entirely in the region* D*.*

Here, B is called a Dulac's function, and if $B \equiv 1$ it recovers the Bendixson criterium.

We finish the section with a result in (Hale 1980, p.18) about extensions of maximal interval of existence of trajectories.

Proposition 1.1.6. *Assume* $F = F(x)$ *is defined and locally Lipschitz continuous in* $[0, C_0]$ *for some* $C_0 > 0$*. If a certain solution* $x(t)$ *of* $\dot{x} = F(x)$ *satisfies* $|x(t)| \leqslant C < C_0$ *for all values of* $t \geqslant t_0$ *in which* $x(t)$ *is defined, then necessarily* $x(t)$ *is defined for all* $t \geqslant t_0$*.*

1.2 PDEs overview

In this section we obtain the ODE problems associated to the second order PDE problem (0.0.3), which are satisfied by Pucci's operators in the radial regime. We also recall some known theorems and change of variables from PDEs theory which will be fundamental to understand the difficulties originated both from the fully nonlinear nature of the problem and from the weighted power term. We split the sections into subsections to better situate the reader each time a check up on the notation or result is necessary.

1.2.1 Pucci's operators in the radial form

We start the section by recalling the definition of Pucci's extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ for $0 < \lambda \leqslant \Lambda$,

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) := \sup_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AX), \quad \mathcal{M}^{-}_{\lambda,\Lambda}(X) := \inf_{\lambda I \leqslant A \leqslant \Lambda I} \operatorname{tr}(AX),$$

where A, X are $N \times N$ symmetric matrices, and I is the identity matrix. Equivalently, if we denote by $\{e_i\}_{1 \leqslant i \leqslant N}$ the eigenvalues of X, we can define the Pucci's operators as

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \tag{1.2.1}$$
$$\mathcal{M}^{-}_{\lambda,\Lambda}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i.$$

From now on we will drop writing the parameters λ, Λ in the notations for the Pucci's operators.

Definition 1.2.1 (Radial function)**.** We say that u, defined in $\Omega \subset \mathbb{R}^N$, is a radial function with respect to the point $x_0 \in \Omega$ if there exists φ such that $u(x) = \varphi(r)$ where $r = |x - x_0|$.

Throughout the text we will consider $x_0 = 0$ and, with an abuse of notation, most of the times we write simply $u(x)$ and $u(r)$ for $r = |x|$ interchangeably.

Lemma 1.2.2. *If u is a radial C^2 function, the eigenvalues of the Hessian matrix D^2u are given by u'' which is simple, and $\frac{u'(r)}{r}$ with multiplicity $N - 1$.*

Proof. We write $u(x) = \varphi(r)$, with $x = (x_1, \ldots, x_N)$. Let us prove that

$$\operatorname{spec}(D^2u)(x) = \left(\tfrac{\varphi'(r)}{r}, \ldots, \tfrac{\varphi'(r)}{r}, \varphi''(r)\right).$$

We have $u_{x_i}(x) = \varphi'(|x|)\frac{x_i}{|x|}$, and so $Du(x) = \frac{\varphi'(|x|)}{|x|}x$ with $|Du(x)| = |\varphi'(r)|$. Also, $u_{x_i x_j}(x) = \varphi''(r)\frac{x_j x_i}{r^2} + \varphi'(r)\frac{\delta_{ij}}{r} - \varphi'(r)\frac{x_i x_j}{r^3}$.

For the vector $\xi = (\xi_1, \ldots, \xi_N)$, we denote $\xi \otimes \xi = (\xi_i \xi_j)_{i,j=1}^N$. We claim that

$$\operatorname{spec}(\xi \otimes \xi) = (0, \ldots, 0, |\xi|^2). \tag{1.2.2}$$

Indeed, observe that if $A := \xi \otimes \zeta = \xi\,\zeta^T$ and $\mathbb{R}^N$ is the direct sum of V and W, for $V = \operatorname{span}\zeta$, $W = \operatorname{span}\{w_i\}_{i=1}^{n-1}$, then $A\xi = \xi\,(\zeta^T\xi) = (\xi \cdot \zeta)\,\xi$ and $Aw_i = 0$. Hence $\xi \cdot \zeta$ is an eigenvalue associated to the eigenvector ξ, while 0 is an eigenvalue with multiplicity $N - 1$.

Now it is just a question of applying (1.2.2) to the matrix

$$D^2u(x) = \varphi''(r)\tfrac{x \otimes x}{|x|^2} + \tfrac{\varphi'(r)}{r}\left(I - \tfrac{x \otimes x}{|x|^2}\right).$$

■

Next, we define the Lipschitz functions

$$m_+(s) = \begin{cases} \lambda s & \text{if } s \leqslant 0 \\ \Lambda s & \text{if } s > 0 \end{cases} \quad \text{and} \quad M_+(s) = \begin{cases} s/\lambda & \text{if } s \leqslant 0 \\ s/\Lambda & \text{if } s > 0; \end{cases} \tag{1.2.3}$$

$$m_-(s) = \begin{cases} \Lambda s & \text{if } s \leqslant 0 \\ \lambda s & \text{if } s > 0 \end{cases} \quad \text{and} \quad M_-(s) = \begin{cases} s/\Lambda & \text{if } s \leqslant 0 \\ s/\lambda & \text{if } s > 0. \end{cases} \tag{1.2.4}$$

The equations $\mathcal{M}^+(D^2u) + r^a u^p = 0$ and $\mathcal{M}^-(D^2u) + r^a u^p = 0$, for $r \neq 0$, in radial coordinates for positive solutions then become, respectively,

$$u'' = M_+(-r^{-1}(N-1)\,m_+(u') - r^a u^p), \quad u > 0; \tag{P_+}$$

$$u'' = M_-(-r^{-1}(N-1)\,m_-(u') - r^a u^p), \quad u > 0, \tag{P_-}$$

which are understood in the maximal interval where u is positive.

We recall the definition of the dimension-like numbers

$$\widetilde{N}_+ = \tfrac{\lambda}{\Lambda}(N-1)+1, \qquad \widetilde{N}_- = \tfrac{\Lambda}{\lambda}(N-1)+1. \tag{1.2.5}$$

For instance, in terms of positive solutions, when $u'' \leqslant 0$ and $u' < 0$, (P_+) is reduced to

$$-u''(r) - \tfrac{N-1}{r}u'(r) = r^a \tfrac{u^p(r)}{\lambda},$$

where the left hand side is the standard radial Laplacian operator. Meanwhile for $u'' > 0$ and $u' < 0$, (P_+) reads as

$$-u''(r) - \tfrac{\widetilde{N}_+-1}{r}u'(r) = r^a \tfrac{u^p(r)}{\Lambda},$$

where, in turn, the LHS is the Laplacian in the possible noninteger dimension $\widetilde{N}_+$ from (1.2.5). Now, when $u' > 0$ then $u'' < 0$, and so (P_+) becomes

$$-u''(r) - \tfrac{\widetilde{N}_--1}{r}u'(r) = r^a \tfrac{u^p(r)}{\lambda},$$

for which we recover the Laplacian operator in the possible noninteger dimension $\widetilde{N}_-$ from (1.2.5).

Analogously one treats the problems involving the operator $\mathcal{M}^-$. In this case it appears the Laplacian operator in dimensions N and $\widetilde{N}_-$ when $u' < 0$, while $\widetilde{N}_+$ takes place when $u' > 0$.

Therefore, our system is the union of equations driven by different Laplacian-like operators, in dimensions N, $\widetilde{N}_+$ or $\widetilde{N}_-$. This explains the difficulty in dealing with fully nonlinear operators.

1.2.2 Types of decay and blow-up for solutions

In this section we define the decay and blow-up properties for the solutions of (P_+) and (P_-) we will be interested in.

Definition 1.2.3. Let u be a solution of (P_+) or (P_-) defined for all $r \geqslant r_0$, for some $r_0 \geqslant 0$. Set $\alpha = \frac{2+a}{p-1}$. Then u is said to be:

(i) *fast decaying* if there exists $c > 0$ such that

$$\lim_{r\to\infty} r^{\widetilde{N}-2}u(r) = c\,,$$

where $\widetilde{N}$ is either $\widetilde{N}_+$ if the operator is $\mathcal{M}^+$ or $\widetilde{N}_-$ for $\mathcal{M}^-$ in (1.2.5);

(ii) *slow decaying* if there exists $c > 0$ such that

$$\lim_{r\to\infty} r^{\alpha} u(r) = c\,,$$

(iii) *pseudo-slow decaying* if there exist constants $0 < c_1 < c_2$ such that

$$c_1 = \liminf_{r\to\infty} r^{\alpha} u(r) < \limsup_{r\to\infty} r^{\alpha} u(r) = c_2.$$

The definitions (i) and (ii) are classical from the theory of Lane–Emden equations. In turn (iii) was introduced in (Felmer and Quaas 2003) and is peculiar of the fully nonlinear case. It corresponds to solutions oscillating at $+\infty$ by changing concavity infinitely many times.

Definition 1.2.4. Let u be a solution of (P_+) or (P_-) defined for $r \in (0, r_0)$ for some $r_0 > 0$, and such that $\lim_{r\to\infty} u(r) = 0$. Then the singular solution u is said to be:

(i) $(\widetilde{N} - 2)$*–blowing up* if there exists $c > 0$ such that

$$\lim_{r\to 0} r^{\widetilde{N}-2} u(r) = c,$$

where $\widetilde{N}$ is either $\widetilde{N}_+$ if the operator is $\mathcal{M}^+$ or $\widetilde{N}_-$ for $\mathcal{M}^-$ in (1.2.5);

(ii) α*–blowing up* if there exists $c > 0$ such that

$$\lim_{r\to 0} r^{\alpha} u(r) = c\,,$$

with α as in (1.2.6);

(iii) *pseudo–blowing up* if there exist constants $0 < c_1 < c_2$ such that

$$c_1 = \liminf_{r\to 0} r^{\alpha} u(r) < \limsup_{r\to 0} r^{\alpha} u(r) = c_2.$$

We highlight that Definition 1.2.4 (iii) corresponds to a type of solutions which change concavity infinitely many times in a neighborhood of zero. The existence of such a type of singular solution was already detected for a more general class of uniformly elliptic equations, for values of the exponent p close to the critical one, see (Felmer and Quaas 2006, Section 6).

1.2.3 Known results

We start the section by recalling some maximum and comparison principles from Proposition 2.1 in (Quaas 2004).

Proposition 1.2.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$. If $\mathcal{M}^{\pm}(D^2u) \geqslant \mathcal{M}^{\pm}(D^2v)$ in Ω, and $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

The next proposition is the Hopf lemma from (Bardi and Da Lio 1999), see also (Sirakov 2017) for a more general context.

Proposition 1.2.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is such that $\mathcal{M}^{\pm}(D^2u) \geqslant 0$, $u > 0$ in Ω, with $u = 0$ on $\partial\Omega$, then $\partial_\nu u > 0$ on $\partial\Omega$, where ν is the inward unit normal derivative.*

The following is a radial symmetry result of positive solutions for equations driven by the Pucci's operators from (Da Lio and Sirakov 2007).

Proposition 1.2.7. *Let B be a ball in $\mathbb{R}^N$, and $p > 1$. Every positive solution of $-\mathcal{M}^{\pm}(D^2u) = u^p$ in B, with $u = 0$ on ∂B, is radially symmetric and strictly decreasing with the radius.*

In the unbounded domain setting, additional conditions on decay are needed to establish symmetry (Gidas, Ni, and Nirenberg 1981; Li 1991). On the other hand, this result fails to be true in the presence of a positive weight $a > 0$. More generally, in the Laplacian case symmetry breaking occurs when $a \to +\infty$, see (Mercuri and Moreira dos Santos 2019).

We finish the section by recalling some important theorems in the theory of fully nonlinear equations regarding existence of radial solutions, which were previously mentioned in the Introduction.

Theorem 1.2.8 (Cutrì and Leoni 2000)**.** *If $u \geqslant 0$ is a solution of $\mathcal{M}^{\pm}(D^2u) + |x|^a u^p \leqslant 0$ in $\mathbb{R}^N$ with $p \leqslant \frac{\tilde{N}_{\pm}+a}{\tilde{N}_{\pm}-2}$ then $u \equiv 0$ in $\mathbb{R}^N$.*

Even though this theorem can be stated in terms of viscosity solutions, we prefer to skip it in the present text since we are concerned just with classical solutions. Other extensions and related results can be found in (Armstrong and Sirakov 2011; Armstrong, Sirakov, and Smart 2011).

Theorem 1.2.9 (Theorems 1.1 and 1.2 in (Felmer and Quaas 2003))**.** *Consider the problem $\mathcal{M}^{\pm}(D^2u) + u^p \leqslant 0$ in $\mathbb{R}^N$, $\tilde{N}_+ > 2$, and $\lambda < \Lambda$. Then there exist critical exponents p_+^*, p_-^* satisfying the bounds*

$$\max\left\{\frac{\tilde{N}_+}{\tilde{N}_+-2}, \frac{N+2}{N-2}\right\} < p_+^* < \frac{\tilde{N}_++2}{\tilde{N}_+-2} \quad \textit{and} \quad \frac{\tilde{N}_-+2}{\tilde{N}_--2} < p_-^* < \frac{N+2}{N-2},$$

such that the following holds for $\Omega = \mathbb{R}^N$:

(i) if $p \in (1, p_\pm^*)$ there is no nontrivial radial solution;

(ii) if $p = p_\pm^*$ there exists a unique fast decaying radial solution;

(iii) If $p > p_\pm^*$ there exists a unique radial solution, which is either slow decaying or pseudo-slow decaying.

In (i) and (ii) uniqueness is meant up to scaling.

In addition, in the case of $\mathcal{M}^+$ the authors showed that pseudo-slow decaying solutions exist in the range

$$p \in \left(p_+^*, \frac{\tilde{N}_++2}{\tilde{N}_+-2}\right].$$

The respective question of pseudo-slow decaying solutions for the operator $\mathcal{M}^-$ was left open there. Here we will give a partial result for this question, see also the improvements done more recently in (Pacella and Stolnicki 2021b).

1.2.4 Weighted notation

Let us fix some notations, depending on the number $a > -1$ which characterizes the weight in (3.4.1):

$$p_\pm^{p,a} = \frac{\tilde{N}_\pm + 2a + 2}{\tilde{N}_\pm - 2}, \; p_\pm^{s,a} = \frac{\tilde{N}_\pm + a}{\tilde{N}_\pm - 2}, \; p_\Delta^a = \frac{N + 2 + 2a}{N - 2}, \; \alpha = \frac{2 + a}{p - 1}. \tag{1.2.6}$$

Now we recall the usual change of variables which transforms Hénon type problems into non weighted ones when $\lambda = \Lambda$ in (Bidaut-Véron and Giacomini 2010; Clément, de Figueiredo, and Mitidieri 1996; Gladiali, Grossi, and Neves 2013).

Remark 1.2.10 (Change of variable for the Laplacian operator)**.** Assume u solves the following Hénon (weighted) equation

$$u'' + \frac{N-1}{r}u' = -r^a|u|^{p-1}u.$$

Set $A = \frac{2}{p-1}$, and define

$$U(s) = \kappa^A u(r), \quad \text{where } r = s^\kappa, \quad \kappa = \frac{\hat{N}-2}{N-2},$$

for $\hat{N} = \frac{2(N+a)}{2+a}$. That is, $\kappa = \frac{\hat{N}}{N+a} = \frac{2}{2+a}$.

Since $U'(s) = \kappa^{A+1} u'(r) s^{\kappa-1}$ and $U''(s) = \kappa^{A+2} u''(r) s^{2\kappa-2} + \kappa^{A+1}(\kappa - 1)s^{\kappa-2}u'(r)$, with $\frac{\mathrm{d}r}{\mathrm{d}s} = \kappa s^{\kappa-1}$, we compute

$$\begin{aligned} U''(s) + \tfrac{\hat{N}-1}{s}U'(s) &= \kappa^{A+2}u''(r)s^{2\kappa-2} + \kappa^{A+1}(\kappa-1)s^{\kappa-2}u'(r) + (\hat{N}-1)\kappa^{A+1}s^{\kappa-2}u'(r) \\ &= \kappa^{A+2}s^{2\kappa-2}\left\{u''(r) + \tfrac{N-1}{s^\kappa}u'\right\} = -\kappa^{A+2}s^{2\kappa-2}r^a|u|^{p-1}u \\ &= -\kappa^{Ap}|u|^{p-1}u = -|U|^{p-1}U \end{aligned}$$

by using the definition of A, $\kappa = \frac{2}{2+a}$, and

$$\kappa - 1 + \hat{N} - 1 = \frac{\hat{N}-2}{N-2} + \hat{N} - 2 = \frac{\hat{N}-2}{N-2}(N-1) = \kappa(N-1),$$

Hence, U becomes a solution of a (non weighted) Lane–Emden equation.

When $\lambda = \Lambda$, the classification result in Theorem 1.2.9 can be immediately extended for weighted equations using the above change of variable.

However, this change does not seem to work for Pucci's operators in the general case $\lambda \neq \Lambda$, because it depends on the dimension. Recall that radial Pucci operators in the radial form change their expressions accordingly to the regions where solutions change monotonicity and concavity, by assuming radial Laplacian forms in dimensions N and $\widetilde{N}_\pm$ from (1.2.5), see Section 1.2.1.

One of main goals will be to obtain classification results when $\lambda < \Lambda$. As we already said we will improve Theorem 1.2.9 a bit in the sense of regular solutions. But we go much further by exploiting also singular, annular, and exterior domain solutions.

2 The dynamical system

In this chapter we introduce the dynamical system associated to the problems

$$u'' = M_+(-r^{-1}(N-1)\, m_+(u') - r^a u^p), \quad u > 0; \qquad (P_+)$$

$$u'' = M_-(-r^{-1}(N-1)\, m_-(u') - r^a u^p), \quad u > 0, \qquad (P_-)$$

for $m_\pm$ and $M_\pm$ given by (1.2.3) and (1.2.4). We also investigate their first local and global properties. Recall that the solutions of (P_+) and (P_-) correspond to positive radial solutions of the PDE problem

$$-\mathcal{M}^\pm(D^2u) = |x|^a u^p \ , \quad u > 0, \qquad (2.0.1)$$

with $p > 1$ and $a > -1$, in some radial domain Ω.

2.1 The new variables

In this section we define some new variables which allow to transform the radial fully nonlinear equations into a quadratic dynamical system.

Let u be a positive solution of (P_+) or (P_-). Thus we can define the new functions

$$X(t) = -\frac{ru'(r)}{u(r)}, \qquad Z(t) = -\frac{r^{1+a}\,u^p(r)}{u'(r)} \qquad \text{for } t = \ln(r), \tag{2.1.1}$$

whenever $r > 0$ is such that $u(r) \neq 0$ and $u'(r) \neq 0$.

We consider the phase plane $(X, Z) \in \mathbb{R}^2$. Since we are studying positive solutions, the points $(X(t), Z(t))$ belong to the first quadrant when $u' < 0$; or to the third quadrant when $u' > 0$. We denote the first and third quadrants by $1Q$, $3Q$ respectively, i.e.

$$1Q = \{(X, Z) \in \mathbb{R}^2 : X, Z > 0\}, \quad 3Q = \{(X, Z) \in \mathbb{R}^2 : X, Z < 0\}.$$

As a consequence of this monotonicity, the problems (P_+) and (P_-) become in $1Q$:

$$\text{for } \mathcal{M}^+ : \qquad u'' = M_+(-\lambda r^{-1}(N-1)u' - r^a u^p), \quad u > 0 \quad \text{in } 1Q, \tag{2.1.2}$$

$$\text{for } \mathcal{M}^- : \qquad u'' = M_-(-\Lambda r^{-1}(N-1)u' - r^a u^p), \quad u > 0 \quad \text{in } 1Q. \tag{2.1.3}$$

On the other hand, since $u' > 0$ implies $u'' < 0$, one finds out in $3Q$:

$$\text{for } \mathcal{M}^+ : \qquad \lambda u'' = -\Lambda r^{-1}(N-1)u' - r^a u^p, \quad u > 0 \quad \text{in } 3Q, \tag{2.1.4}$$

$$\text{for } \mathcal{M}^- : \qquad \Lambda u'' = -\lambda r^{-1}(N-1)u' - r^a u^p, \quad u > 0 \quad \text{in } 3Q. \tag{2.1.5}$$

Observe that from $u'' = M_+(-\lambda(N-1)\frac{u'}{r} - r^a u^p)$ in $1Q$, and $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = e^t = r$, we get

$$\begin{aligned}
\dot{X}(t) &= -r\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u'}{u}\right) - \dot{r}\frac{u'}{u} = -r^2\,\frac{u''u - (u')^2}{u^2} - r\frac{u'}{u} \\
&= -\frac{r^2}{u}\left(-\frac{u'}{r}\right) M_+\left(\lambda(N-1) + r^{1+a}\frac{u^p}{u'}\right) + \left(-r\frac{u'}{u}\right)^2 - r\frac{u'}{u} \\
&= -XM_+(\lambda(N-1) - Z) + X^2 + X,
\end{aligned}$$

since $u' < 0$ in $1Q$ and M_+ is 1-homogeneous. Analogously one computes $\dot{X}$ in $3Q$ and $\dot{Z}$. Thus, in terms of the functions (2.1.1), we derive the following autonomous dynamical systems:

$$\text{in } 1Q, \quad \begin{cases} \dot{X} &= X\,[X + 1 - M_+(\lambda(N-1) - Z)], \\ \dot{Z} &= Z\,[1 + a - pX + M_+(\lambda(N-1) - Z)]; \end{cases} \tag{2.1.6}$$

$$\text{in } 3Q, \quad \dot{X} = X\,[X - (\tilde{N}_- - 2) + Z/\lambda], \ \ \dot{Z} = Z\,[\tilde{N}_- + a - pX - Z/\lambda], \tag{2.1.7}$$

corresponding to (2.1.2), (2.1.4) for $\mathcal{M}^+$, where the dot $\dot{}$ stands for $\frac{\mathrm{d}}{\mathrm{d}t}$. Similarly one has

$$\text{in } 1Q, \quad \begin{cases} \dot{X} &= X\,[X + 1 - M_-(\Lambda(N-1) - Z)], \\ \dot{Z} &= Z\,[1 + a - pX + M_-(\Lambda(N-1) - Z)]; \end{cases} \tag{2.1.8}$$

$$\text{in } 3Q, \quad \dot{X} = X\,[X - (\tilde{N}_+ - 2) + Z/\Lambda], \ \ \dot{Z} = Z\,[\tilde{N}_+ + a - pX - Z/\Lambda], \tag{2.1.9}$$

associated to (2.1.3), (2.1.5) for $\mathcal{M}^-$.

We stress that (2.1.6), (2.1.8) correspond to positive, decreasing solutions of (P_+), (P_-). We will see in Chapter 3 that this holds for regular and singular solutions of (2.0.1) in $\mathbb{R}^N$ or in a ball.

On the other hand, given any trajectory $\tau = (X, Z)$ of (2.1.6)-(2.1.9) either in $1Q$ or $3Q$, we define

$$u(r) = r^{-\alpha}(X(t)\,Z(t))^{\frac{1}{p-1}}, \qquad \text{where } r = e^t. \tag{2.1.10}$$

Then we deduce

$$\begin{aligned} u'(r) &= -\alpha r^{-\alpha-1}(XZ)^{\frac{1}{p-1}} + \frac{r^{-\alpha}}{p-1}(XZ)^{\frac{1}{p-1}-1}\,\frac{\dot{X}Z + X\dot{Z}}{r} \\ &= -Xr^{-\alpha-1}(XZ)^{\frac{1}{p-1}} = -\frac{Xu(r)}{r}, \end{aligned}$$

from which we recover (2.1.1). Since $X \in C^1$, then $u \in C^2$. From this, one immediately sees that u satisfies either (P_+) or (P_-) from the respective equations for $\dot{X}$, $\dot{Z}$ in the dynamical system.

An important role in the study of our problem is played by the lines $\ell_\pm$, defined by

$$\begin{aligned} \ell_+ &= \{(X,Z):\ Z = \lambda(N-1)\} \cap 1Q \quad \text{for } \mathcal{M}^+, \\ \ell_- &= \{(X,Z):\ Z = \Lambda(N-1)\} \cap 1Q \quad \text{for } \mathcal{M}^-. \end{aligned} \tag{2.1.11}$$

For each of the two systems (2.1.6) and (2.1.8) respectively, the lines $\ell_\pm$ splits $1Q$ into two regions, up and down:

$$\begin{aligned} R^+_\lambda &= \{(X,Z) : Z > \lambda(N-1)\} \cap 1Q, \\ R^-_\lambda &= \{(X,Z) : Z < \lambda(N-1)\} \cap 1Q, \end{aligned} \tag{2.1.12}$$

for the operator $\mathcal{M}^+$, and

$$\begin{aligned} R^+_\Lambda &= \{(X,Z) : Z > \Lambda(N-1)\} \cap 1Q, \\ R^-_\Lambda &= \{(X,Z) : Z < \Lambda(N-1)\} \cap 1Q, \end{aligned} \tag{2.1.13}$$

for $\mathcal{M}^-$.

In terms of $(P_\pm)$, $\ell_\pm$ is the line where a decreasing solution u changes concavity in the sense that, when $(X(t), Z(t)) \in R^+_\lambda$ (or R^+_Λ) then the corresponding solution u through the transformation (2.1.10) is concave, while for $(X(t), Z(t)) \in R^-_\lambda$ (or R^-_Λ), u is convex. Hence, these regions are essential to determine the precise expressions of (P_+) and (P_-) according to M_+ and M_- in (1.2.3), (1.2.4). For instance, in view of what was discussed in Section 1.2.1, problem (P_+) in $1Q$ assumes the form of either a standard radial Laplacian operator in dimension N, or a Laplacian in the possible noninteger dimension $\tilde{N}_+$, see (1.2.5). Analogously one treats $\mathcal{M}^-$. Note that in $3Q$ we always obtain Laplacian operators in dimensions $\tilde{N}_-$ (for $\mathcal{M}^+$) and $\tilde{N}_+$ (for $\mathcal{M}^-$), see (2.1.4), (2.1.5), and Section 1.2.1. As we said, our system is then the union of equations involving Laplacian-like operators.

We stress that Lane–Emden–Hénon problems for Laplacian operators were already studied in (Bidaut-Véron and Giacomini 2010) in terms of the dynamical system (2.1.6) in the case $\lambda = \Lambda = 1$ subject to the transformation (2.1.1).

At this stage it is worth observing that the systems (2.1.6) and (2.1.8) are continuous on $\ell_\pm$. More than that, the right hand sides are locally Lipschitz functions of X, Z, so the usual ODE theory applies, as mentioned in Section 1.1. That is, one recovers existence, uniqueness, and continuity with respect to initial data as well as continuity with respect to the parameter p. Any trajectory $\tau = (X, Z)$ is automatically differentiable in t in each quadrant.

Remark 2.1.1 (The study of concavity)**.** Let us see what happens with an arbitrary trajectory τ in the plane (X, Z) when it intercepts the concavity line $\ell_\pm$ in $1Q$. We consider the operator $\mathcal{M}^+$ and the respective line $Z = \lambda(N-1)$.

We start noticing that $\dot{X}$ is a continuous function of t such that $\dot{X} = X(X+1) > 0$ on ℓ_+. Hence, if $\dot{X}(t_0) > 0$ then $t \mapsto X(t)$ is strictly increasing near t_0. By the inverse function theorem one may write t as a C^1 function of X in a neighborhood of $X(t_0)$. So $Z(t) = Z(X)$ there, namely

$$Z(t) = \frac{e^{(2+a)t} u^{p-1}(t)}{X(t)} = \frac{e^{(2+a)t(X)} u^{p-1}(t(X))}{X} = Z(X).$$

Therefore, to analyze concavity properties of u one needs to look at $\frac{\mathrm{d}Z}{\mathrm{d}X}$ on ℓ_+. Note that

$$\frac{\mathrm{d}Z}{\mathrm{d}X} = \frac{\dot{Z}}{\dot{X}} = \frac{\lambda(N-1)\,(1+a-pX)}{X\,(X+1)} \quad \text{on} \quad \ell_+ \,. \tag{2.1.14}$$

This expression is positive for $X < \frac{1+a}{p}$, and it is negative for $X > \frac{1+a}{p}$.

A concavity change from R_λ^+ to R_λ^- verifies $X \geqslant \frac{1+a}{p}$. It must be transversal if it occurs when $X > \frac{1+a}{p}$. In this case Z is strictly decreasing with respect to X; and so the corresponding solution u'' changes sign from $-$ to $+$.

Now we infer that a change of concavity does not happen at $X = \frac{1+a}{p}$. Indeed, if $\tau = (X, Z)$ changed concavity at t_0 such that $\tau(t_0) = (\frac{1+a}{p}, \lambda(N-1))$, then $\partial_X Z(\frac{1+a}{p}) = \dot{Z}(t_0) = 0$. Since we are assuming τ crosses ℓ_+ transversely, we have $\lim_{X \to \frac{1+a}{p}^-} \partial_X Z(t) < 0$, while $\lim_{X \to \frac{1+a}{p}^+} \partial_X Z(t) = 0$. But this contradicts the fact that τ is a differentiable trajectory.

In what concerns trajectories which start in R_λ^+, the first (strict) change of concavity must occur when $X > \frac{1+a}{p}$. Indeed, Z decreases strictly at such point with respect to X when moving from R_λ^+ to R_λ^- in $1Q$. If a second concavity change occurs, then $\frac{\mathrm{d}Z}{\mathrm{d}X} > 0$, which is only admissible when $X < \frac{1+a}{p}$, see also Proposition 2.2.2(1) ahead.

The same reasoning applies to the operator $\mathcal{M}^-$ with respect to the line ℓ_-.

2.2 Local analysis

In this section and onward in the text it will be convenient to write the dynamical systems (2.1.6) and (2.1.8) in terms of the following ODE first order autonomous equation

$$\dot{x} = F(x), \quad \text{where} \quad x = (X, Z), \quad F(x) := (f(x), g(x)). \tag{2.2.1}$$

with $\dot{x} = (\dot{X}, \dot{Z})$. For instance, in the case of the operator $\mathcal{M}^+$, then f, g are given by

$$f(x) = \begin{cases} X(X - (N-2) + \frac{Z}{\lambda}) & \text{in } R_\lambda^+ \\ X(X - (\tilde{N}_+ - 2) + \frac{Z}{\Lambda}) & \text{in } R_\lambda^- \end{cases}$$

and

$$g(x) = \begin{cases} Z(N + a - pX - \frac{Z}{\lambda}) & \text{in } R_\lambda^+ \\ Z(\tilde{N}_+ + a - pX - \frac{Z}{\Lambda}) & \text{in } R_\lambda^-. \end{cases}$$

2.2.1 Stationary lines and points

We start this section investigating the sets where $\dot{X} = 0$ and $\dot{Z} = 0$. Let us focus our analysis on $1Q$, since the only stationary point on the boundary of $3Q$ is the origin.

We observe that both X and Z axes are invariant by the flow. In particular, each quadrant is an invariant set for the dynamics. Moreover, let us keep in mind the following segments in the plane (X, Z). For the system (2.1.6), we define

$$\ell_1^+ = \{\, (X, Z) : \; Z = \Lambda(\tilde{N}_+ - 2) - \Lambda X \,\} \cap 1Q \tag{2.2.2}$$

which is the set where $\dot{X} = 0$ and $X > 0$; also

$$\ell_2^+ = \ell_{2+}^+ \; \cup \; \ell_{2-}^+ \tag{2.2.3}$$

with $\ell_{2+}^+ = \{(X, Z) : Z = \lambda(N + a - pX)\} \cap R_\lambda^+$, $\ell_{2-}^+ = \{(X, Z) : Z = \Lambda(\tilde{N}_+ + a - pX)\} \cap R_\lambda^-$, which is the set where $\dot{Z} = 0$ and $Z > 0$; see Figures 2.1 to 2.3.

Notice that ℓ_1^+ is a segment entirely contained in R_λ^-, since there are no other points in $1Q$ where $\dot{X} = 0$ in the interior of the region R_λ^+. Moreover, (2.2.3) is the union of two segments which join at the point $(\frac{1+a}{p}, \lambda(N-1)) \in \ell_+ \cap \overline{\ell_2^+}$, see Figure 2.1. The analogous sets for $\mathcal{M}^-$ are

$$\ell_1^- = \{\, (X, Z) : \; Z = \lambda(\tilde{N}_- - 2) - \lambda X \,\} \cap 1Q \tag{2.2.4}$$

which is the set where $\dot{X} = 0$ and $X > 0$ (contained in R_Λ^-); and

$$\ell_2^- = \ell_{2+}^- \cup \ell_{2-}^- \tag{2.2.5}$$

with $\ell_{2+}^- = \{(X, Z) : Z = \Lambda(N_- + a - pX)\} \cap R_\Lambda^+$, $\ell_{2-}^- = \{(X, Z) : Z = \lambda(\widetilde{N}_- + a - pX)\} \cap R_\Lambda^-$, which is the set where $\dot{Z} = 0$ and $Z > 0$.

Lemma 2.2.1. *The stationary points of the dynamical systems* (2.1.6)–(2.1.9) *are:*

$$\text{for } \mathcal{M}^+: \quad O = (0,0), \quad N_0 = (0, \lambda N + \lambda a), \quad A_0 = (\widetilde{N}_+ - 2, 0),$$
$$M_0 = (X_0, Z_0),$$

where $X_0 = \alpha$, *and* $Z_0 = \Lambda(\widetilde{N}_+ - p\alpha + a) = \Lambda(\widetilde{N}_+ - 2 - \alpha)$, *see Figure 2.1;*

$$\text{for } \mathcal{M}^-: \quad O = (0,0), \quad N_0 = (0, \Lambda N + \Lambda a), \quad A_0 = (\widetilde{N}_- - 2, 0),$$
$$M_0 = (X_0, Z_0),$$

where $X_0 = \alpha$ *and* $Z_0 = \lambda(\widetilde{N}_- - p\alpha + a) = \lambda(\widetilde{N}_- - 2 - \alpha)$.

Proof. We just show the $\mathcal{M}^+$ case. First notice that the system does not admit stationary points in $3Q$ nor on the line ℓ_+. In the region $\overline{R_\lambda^+}$ we have already seen that $\dot{X} = 0$ implies $X = 0$, since ℓ_1^+ does not intersect R_λ^+. By $\dot{Z} = 0$ we obtain $Z = \lambda(N + a - pX)$ since $Z \neq 0$ in R_λ^+. Hence we reach the equilibrium point N_0. In $\overline{R_\lambda^-}$, from $\dot{X} = 0$ we have either $X = 0$ or $Z = \Lambda(\widetilde{N}_+ - 2 - X)$, while by $\dot{Z} = 0$ we deduce that either $Z = 0$ or $Z = \Lambda(\widetilde{N}_+ + a - pX)$. Therefore we obtain the points O, A_0, M_0, and $(0, \Lambda(\widetilde{N}_+ + a))$. However, the latter does not belong to $\overline{R_\lambda^-}$ as long as $a > -1$. ■

Next we analyze the directions of the vector field F in (2.2.1) on the X, Z axes, on the concavity line $\ell_\pm$, and also on $\ell_1^\pm$, $\ell_2^\pm$; see (2.1.11), and (2.2.2)–(2.2.5).

Proposition 2.2.2. *The systems* (2.1.6) *and* (2.1.8) *enjoy the following properties (see Figures 2.1 to 2.3):*

(1) Every trajectory of (2.1.6) *in* $1Q$ *crosses the line* ℓ_+ *transversely except at the point*

$$P = \left(\tfrac{1+a}{p}, \lambda(N - 1)\right).$$

Moreover, it passes from R_λ^+ *to* R_λ^- *if* $X > \frac{1+a}{p}$*, while it moves from* R_λ^- *to* R_λ^+ *if* $X < \frac{1+a}{p}$*. The vector field at* P *always points to the right. A similar statement holds for* $\mathcal{M}^-$ *via the system* (2.1.8) *considering respectively* ℓ_-, $(\frac{1+a}{p}, \Lambda(N-1))$, R_Λ^+, R_Λ^-*;*

(2) The flow induced by (2.2.1) *on the* X *axis points to the left for* $X \in (0, \widetilde{N}_\pm - 2)$*, and to the right when* $X > \widetilde{N}_\pm - 2$*. On the* Z *axis it moves up between* O *and* N_0*, and down above* N_0*;*

(3) The vector field F *on the line* $\ell_1^\pm$ *is parallel to the* Z *axis whenever* $X \neq \alpha$*. It points up if* $X < \alpha$*, and down if* $X > \alpha$*. Further, on the set* $\ell_2^\pm$ *the vector field* F *is parallel to the* Z *axis for* $X \neq \alpha$*. It moves to the right if* $X < \alpha$*, and to the left if* $X > \alpha$*.*

Proof. (1) We just observe that $\dot{X} = X(X+1) > 0$, and $\dot{Z} = Z(1 + a - pX)$ on $\ell_\pm$.

(2) For instance consider $\mathcal{M}^+$. Since the X axis is contained in R_λ^-, then $\dot{X} = X(X - (\widetilde{N}_+ - 2))$ which is positive for $X < \widetilde{N}_+ - 2$ and negative for $X > \widetilde{N}_+ - 2$. Now, $\dot{Z} = Z(N + a - Z/\lambda)$ in R_λ^+ is positive if $Z < \lambda(N+a)$ and negative for $Z > \lambda(N+a)$. On the other hand, $\dot{Z} = Z(\widetilde{N}_+ + a - Z/\Lambda) > 0$ in R_λ^-, since $\Lambda(\widetilde{N}_+ + a) > \lambda(N-1)$ for $a > -1$.

(3) Notice that $\dot{Z} = (p-1)Z(\alpha - X)$ on $\ell_1^\pm$ and $\dot{X} = (p-1)X(\alpha - X)$ on $\ell_2^\pm$. Both are positive quantities for $X < \alpha$, and negative when $X > \alpha$. ∎

Remark 2.2.3. An orbit can only reach the point P in Proposition 2.2.2 (1) from R_λ^-, see Remark 2.1.1. See also the vector field at P in Figures 2.1 to 2.3.

The next proposition gathers the crucial dynamics at each stationary point.

Proposition 2.2.4 ($\mathcal{M}^\pm$)**.** *The following properties are verified for the systems* (2.1.6) *and* (2.1.8)*,*

1. *The origin* O *is a saddle point. The stable and unstable directions of the linearized system are the* X *and* Z *axes respectively;*

2. N_0 *is a saddle point. The tangent unstable direction is parallel to the line*

$$Z = \frac{-p\lambda(N+a)}{N+2+2a} X \quad \text{if the operator is } \mathcal{M}^+,$$
$$Z = \frac{-p\Lambda(N+a)}{N+2+2a} X \quad \text{for } \mathcal{M}^-;$$

3. *A_0 is a saddle point for $p > p_\pm^{s,a}$. The linear stable direction is parallel to the line*

$$Z = \frac{-p(\tilde{N}_+-2)+2+a}{\tilde{N}_+-2}\Lambda X \quad \text{in the case of } \mathcal{M}^+,$$
$$Z = \frac{-p(\tilde{N}_--2)+2+a}{\tilde{N}_--2}\lambda X \quad \text{for } \mathcal{M}^-,$$

while the unstable tangent direction lies on the X axis. For $p < p_\pm^{s,a}$ A_0 is a source;

At $p = p_\pm^{s,a}$ A_0 coincides with M_0 and belongs to the X axis. In this case, it is not a hyperbolic stationary point.

4. *For $p < p_\pm^{s,a}$ M_0 belongs to the fourth quadrant. Also, $M_0 \in 1Q \Leftrightarrow p > p_\pm^{s,a}$ in which case:*
   (i) *M_0 is a source if $p_\pm^{s,a} < p < p_\pm^{p,a}$;*
   (ii) *M_0 is a sink for $p > p_\pm^{p,a}$;*
   (iii) *M_0 is a center at $p = p_\pm^{p,a}$.*

Proof. The dynamics at each stationary point depends upon the linearization of the system (2.1.6). Since the point N_0 belongs to R_λ^+ where the system corresponds to the Hénon equation for the standard Laplacian, we could just refer to (Bidaut-Véron and Giacomini 2010) for the local analysis of N_0, as long as $p > p_\pm^{p,a}$. The other points O, N_0, A_0 instead belong to R_λ^- where the system now corresponds to the Hénon equation for the Laplacian in dimension $\tilde{N}_\pm$. In this last case some variations with respect to (Bidaut-Véron and Giacomini 2010) are needed.

The linearization for $\mathcal{M}^+$, $\tilde{N} = \tilde{N}_+$, is given by

$$L(X,Z) = \begin{pmatrix} 2X-(N-2)+\frac{Z}{\lambda} & \frac{X}{\lambda} \\ -pZ & N+a-pX-\frac{2Z}{\lambda} \end{pmatrix} \quad \text{in } R_\lambda^+,$$

$$L(X,Z) = \begin{pmatrix} 2X-(\tilde{N}-2)+\frac{Z}{\Lambda} & \frac{X}{\Lambda} \\ -pZ & \tilde{N}+a-pX-\frac{2Z}{\Lambda} \end{pmatrix} \quad \text{in } R_\lambda^-.$$

For instance, at $N_0 = (0, \lambda(N+a))$ and $A_0 = (\tilde{N}-2, 0)$ one has

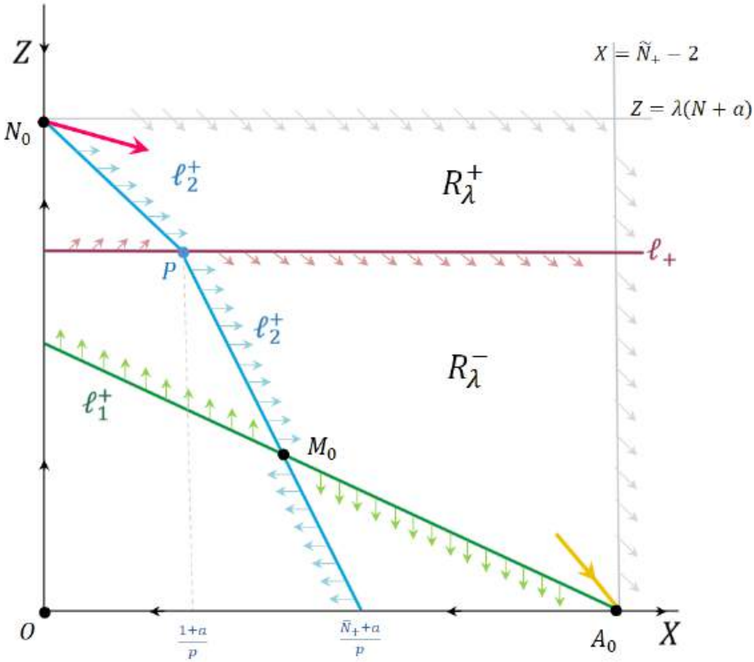

Figure 2.1: The flow behavior in $1Q$ for $\mathcal{M}^+$ when $p > p_+^{s,a}$.

$$L(N_0) = \begin{pmatrix} 2+a & 0 \\ -p\lambda(N+a) & -N-a \end{pmatrix},$$
$$L(A_0) = \begin{pmatrix} \tilde{N}-2 & \frac{\tilde{N}-2}{\Lambda} \\ 0 & \tilde{N}+a-p(\tilde{N}-2) \end{pmatrix}.$$

The eigenvalues for N_0 are 2 and $-N-a$, while for A_0 are $\sigma_1 = \tilde{N}_\pm - 2$ and $\sigma_2 = \tilde{N}_\pm + a - p(\tilde{N}-2)$. Recall that $M_0 = (X_0, Z_0)$, where $X_0 = \alpha = \frac{a+2}{p-1}$ and $Z_0 = \Lambda(\tilde{N} - p\alpha + a) = \Lambda(\tilde{N} - 2 - \alpha)$,

$$L(M_0) = \begin{pmatrix} \alpha & \frac{\alpha}{\Lambda} \\ -p\Lambda(\tilde{N} - p\alpha + a) & -(\tilde{N} - p\alpha + a) \end{pmatrix}.$$

In order to analyze the eigenvalues of $L(M_0)$ one needs to look at the roots of the equation

$$\sigma^2 + \sigma\left(\frac{Z_0}{\Lambda} - X_0\right) + X_0(p-1)\frac{Z_0}{\Lambda} = 0.$$

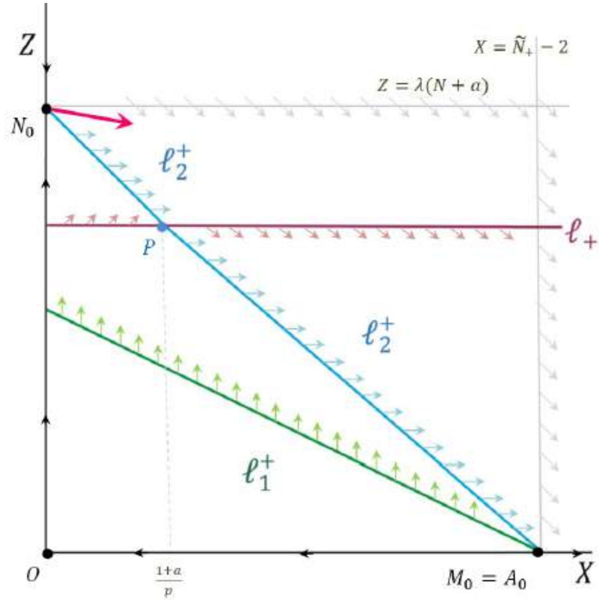

Figure 2.2: The flow behavior in $1Q$ for $\mathcal{M}^+$ at $p = p_+^{s,a}$.

They are given by $2\sigma_\pm = X_0 - \frac{Z_0}{\Lambda} \pm \sqrt{\Delta}$, where $\Delta = \left(\frac{Z_0}{\Lambda} - X_0\right)^2 - 4(2 + a)\frac{Z_0}{\Lambda}$.

Note that $X_0 = \frac{Z_0}{\Lambda}$ is equivalent to $\alpha = \frac{\tilde{N}-2}{2}$, i.e. $p = p_+^{p,a}$. In this case $\mathrm{Re}(\sigma_\pm) = 0$ and the roots are purely imaginary. Moreover, $X_0 > \frac{Z_0}{\Lambda} \Leftrightarrow p < p_+^{p,a}$, and $X_0 < \frac{Z_0}{\Lambda} \Leftrightarrow p > p_+^{p,a}$.

If $\mathrm{Im}(\sigma_\pm) \neq 0$, this already determines the sign of $\mathrm{Re}(\sigma_\pm)$. Assume then $\mathrm{Im}(\sigma_\pm) = 0$ i.e. $\sigma_\pm \in \mathbb{R}$. Observe that $\Delta < (X_0 - \frac{Z_0}{\Lambda})^2$ as far as M_0 stays in $1Q$. This yields $\sigma_- > 0$ for $p_+^{s,a} < p < p_+^{p,a}$ (so $\sigma_\pm > 0$ and M_0 is a source); while $\sigma_+ < 0$ if $p > p_+^{p,a}$ (so $\sigma_\pm < 0$ and M_0 is a sink).

It is possible to prove that M_0 is a saddle point in the fourth quadrant when $1 < p < p_+^{s,a}$. However, this would correspond to solutions of the absorption problem $\mathcal{M}^+ u - u^p = 0$. See (Bidaut-Véron and Giacomini 2010) in the case of the Laplacian operator. ■

On the other hand, in both cases, some deeper analysis is required when $p \leqslant$

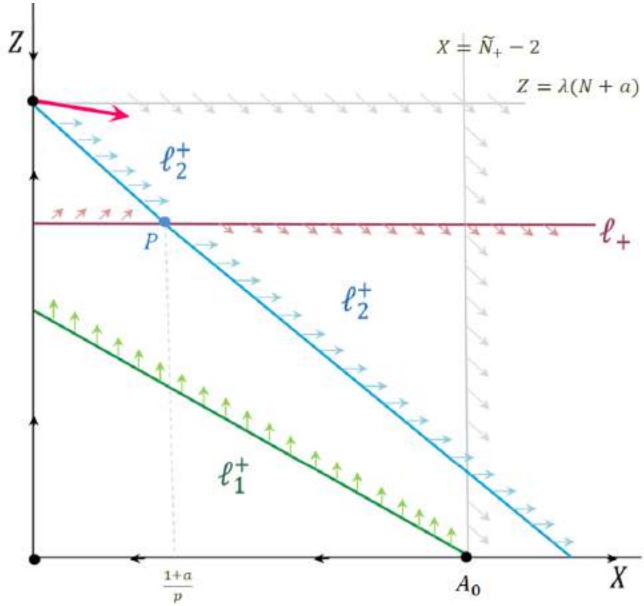

Figure 2.3: The flow behavior in $1Q$ for $\mathcal{M}^+$ for $1 < p < p_+^{s,a}$.

$p_{\pm}^{p,a}$. We will treat this case in Proposition 4.1.9 by using the dynamics of the system. The proof $p = p_{\pm}^{p,a}$ will be presented in Section 2.2.3 ahead, which we restate in Proposition 2.2.7 there.

2.2.2 Local uniqueness

A local uniqueness result follows directly from Propositions 1.1.2 and 2.2.4.

Proposition 2.2.5. *For every $p > 1$ there is a unique trajectory coming out from N_0 at $-\infty$, which we denote by Γ_p. Further, for $p > p_{\pm}^{s,a}$ there exists a unique trajectory arriving at A_0 at $+\infty$ that we denote by Υ_p. In terms of Definition 1.1.3,*

$$\text{for all } p > 1, \qquad \Gamma_p \quad \text{is such that} \quad \alpha(\Gamma_p) = N_0; \tag{2.2.6}$$

$$\text{for all } p > p_{\pm}^{s,a}, \qquad \Upsilon_p \quad \text{is such that} \quad \omega(\Upsilon_p) = A_0. \tag{2.2.7}$$

Remark 2.2.6. Notice that these trajectories uniquely determine the global unstable and stable manifolds of the stationary points N_0 and A_0 respectively. In particular, by Proposition 1.1.2 they are graphs of functions in a neighborhood of the stationary points in their respective ranges of p. The tangent directions at N_0, A_0 are displayed together in Figure 2.1 for $p > p_{\pm}^{s,a}$. In fact, they both belong to the region where $\dot{X} > 0$ and $\dot{Z} < 0$ in their respective ranges of p.

Alternatively, we can obtain the uniqueness of the tangent lines for regular and fast decaying trajectories by using directly the properties of the flow. Take in what follows the operator $\mathcal{M}^+$.

We first consider a trajectory $\tau = (X, Z)$ leaving N_0. We know that $\dot{X} > 0$ in R_λ^+. So, by the inverse function theorem, $t = t(X)$ is a C^1 function in a neighborhood of $X = 0$, and so is Z. The limit $L = L_p$ coincides with the limit of secant lines passing through the point N_0, i.e.

$$L := \partial_X Z\,(0) = \lim_{X\to 0} \tfrac{Z(X)-Z(0)}{X-0} = \lim_{X\to 0} \tfrac{Z-\lambda(N+a)}{X}.$$

In order to calculate this limit, we compute it in R_λ^+,

$$\begin{aligned} L = \lim_{X\to 0} \tfrac{\dot{Z}}{\dot{X}} &= \lim_{X\to 0} \frac{Z}{X - N + 2 + \frac{Z}{\lambda}} \lim_{X\to 0} \frac{N + a - pX - \frac{Z}{\lambda}}{X} \\ &= \frac{\lambda(N+a)}{2+a}(-p - \frac{L}{\lambda}), \end{aligned}$$

and so $L = -\frac{p\lambda(N+a)}{N+2+2a}$.

Analogously, for a trajectory arriving at A_0 one takes into account

$$\iota = \lim_{X\to \tilde{N}_+ - 2} \partial_X Z(X) = \lim_{X\to \tilde{N}_+ - 2} \tfrac{Z}{X-(\tilde{N}_+-2)}.$$

Observe that Z is a strictly decreasing function of X from this moment on. Near the point A_0 in the region R_λ^- we pass to limits under $X \to \tilde{N}_+ - 2$ to obtain

$$\begin{aligned} \iota = \lim \tfrac{\dot{Z}}{\dot{X}} &= \lim \frac{\tilde{N}_+ + a - pX - \frac{Z}{\Lambda}}{X} \lim \frac{Z}{X - (\tilde{N}_+ - 2) + \frac{Z}{\Lambda}} \\ &= \frac{\tilde{N}_+ + a - p(\tilde{N}_+ - 2)}{\tilde{N} - 2} \lim \frac{1}{\frac{X-(\tilde{N}_+-2)}{Z} + \frac{1}{\Lambda}}. \end{aligned}$$

From this one derives that $\iota = 0$ if and only if $p = p_+^{s,a}$. Thus $\iota = \frac{-p(\tilde{N}_+-2)+2+a}{\tilde{N}_+-2}\Lambda$ if $p \neq p_+^{s,a}$.

These illustrate the conclusions of Proposition 1.1.2 in terms of $\partial_X Z(X)$.

2.2.3 Center configuration

In this section we show that M_0 is a center at $p = p_{\pm}^{p,a}$. This peculiar configuration propagates to the whole region below the concavity line $\ell_{\pm}$.

Let us consider the energy functional E of the operator $\mathcal{M}^+$ in the region R_λ^-, which is a slight variation of the energy of the Laplacian operator in dimension $\tilde{N}_+$ treated in (Bidaut-Véron and Giacomini 2010),

$$E(t,X,Z) = e^{t(\tilde{N}_+-2-2\alpha)}\,X(XZ)^\alpha\left\{\frac{X}{2}+\frac{Z}{\Lambda(p+1)}-\frac{\tilde{N}_+}{p+1}\right\} \quad \text{in } R_\lambda^-\cup\ell_+$$

understood as natural extension up to ℓ_+. We stress that this is the only place where energy considerations will be used in the text. In terms of u, the energy functional E for $\mathcal{M}^+$ reads as

$$E(r) = E(r,u) = r^{\tilde{N}_+}\left(\frac{(u')^2}{2}+\frac{1}{\Lambda}\frac{r^a u^{p+1}}{p+1}\right)+\frac{\tilde{N}_+}{p+1}\,uu'r^{\tilde{N}_+-1} \quad \text{if } u''\geqslant 0.$$

Of course these two expressions are equivalent after the transformation (2.1.1). Moreover,

$$E'(r) = r^{\tilde{N}_++a-1}(u')^2\left(\frac{\tilde{N}_++a}{p+1}-\frac{\tilde{N}_+-2}{2}\right) \quad \text{if } u''\geqslant 0,$$

and so the following monotonicity holds in $R_\lambda^-\cup\ell_+$:

$$\dot{E}<0 \ \text{ if } p>p_+^{p,a}, \quad \dot{E}=0 \ \text{ if } p=p_+^{p,a}, \quad \dot{E}>0 \ \text{ if } p<p_+^{p,a}. \tag{2.2.8}$$

Now we investigate the precise behavior of the trajectories close to M_0 at $p = p_{\pm}^{p}$. Here, $\lambda\leqslant\Lambda$ and the result gives an alternative proof in the case of the Laplacian operator $\lambda=\Lambda=1$ in (Bidaut-Véron 1989).

Proposition 2.2.7. *M_0 is a center when $p = p_{\pm}^{p,a}$.*

Proof. We present the proof for $\mathcal{M}^+$; for $\mathcal{M}^-$ it is the same in light of Section 4.2. Let $\tau=(X,Z)$ be an orbit contained in $R_\lambda^-\cup\ell_+$. Let us show that τ is periodic. To simplify notation let $a=0$, $\tilde{N}=\tilde{N}_+$. The energy of τ on the line ℓ_2^+ is given by

$$E_{|_{\ell_2^+\cap R_\lambda^-}} = E^-_{|_{\ell_2^+}}(X) = -\frac{\Lambda^\alpha}{\tilde{N}}\,X^{\alpha+2}\,(\tilde{N}-pX)^\alpha \quad \text{at } p=p_+^p=p_+^{p,0}.$$

Since the energy is a constant function of t when $p = p_+^p$ with $\frac{\alpha+2}{\alpha} = p$, then

$$(\tilde{N} - pX)X^p \equiv c > 0 \quad \text{on } \ell_2^+. \tag{2.2.9}$$

Now we may translate the information from (2.2.9) in terms of the function h defined as

$$h(X) = (\tilde{N} - pX)X^p, \quad \text{for } X \in [1/p, \tilde{N}/p], \quad \text{where } p = p_+^p,$$

for which (2.2.9) represents its level curves. The domain $[1/p, \tilde{N}/p]$ entails the behavior of h in the respective interval delimited by ℓ_2^+ on R_λ^-, up to the boundary.

Let us analyze the function h; it is positive at $1/p$, and equals to zero at $\tilde{N}/p$. Since

$$h'(X) = pX^p(\tfrac{\tilde{N}}{X} - 1 - p), \quad \text{with } \tfrac{\tilde{N}}{1+p} = \tfrac{\tilde{N}-2}{2} = \alpha, \quad p = p_+^p,$$

then h is increasing when $X < \alpha$, decreasing for $X > \alpha$, and it assumes the positive maximum value $h(\alpha) = \alpha^{p+1} := c_\infty$ at $X = \alpha$. Moreover, note that $h(1/p) = (\tilde{N} - 1)(1/p)^p + := c_1$.

Here h is a polynomial function which prescribes the value of the energy on $\ell_2^+ \cap R_\lambda^-$, namely $h = -E^{1/\alpha} \equiv c$. For any $k \in \mathbb{N}$ with $c_k \in [c_1, c_\infty)$, the line $h \equiv c_k$ intersects the graph of h at exactly two points X_1^k, X_2^k such that $X_1^k < \alpha < X_2^k$. Also, they satisfy

$$c_k = h(X_1^k) = h(X_2^k) \to h(\alpha) = c_\infty \tag{2.2.10}$$

when $X_i^k \to \alpha$ as $k \to +\infty$, $i = 1, 2$. Furthermore, the line $h = c_\infty$ intersects the graph of h only once at the point $X = \alpha$.

In our phase plane context, this means that any trajectory τ contained in $R_\lambda^- \cup \ell_+$ bisects the line ℓ_2^+ at exactly two points $P_1 = (X_1, Z_1)$, $P_2 = (X_2, Z_2)$, with $X_1 < \alpha$, $X_2 > \alpha$. By Proposition 2.2.2 (3) the flow moves horizontally on ℓ_2^+, namely to the right for $X < \alpha$, and to the let when $X > \alpha$.

Observe that ℓ_2^+ is a transversal section to the flow, on which any trajectory approaching M_0 must pass across, either in the past or in the future. Hence, the trajectory τ has to be closed, by moving clockwise. Since this dynamics is realized for any trajectory contained on $R_\lambda^- \cup \ell_+$, and (2.2.10) holds, in particular any trajectory close to M_0 is periodic, so M_0 is a center. ■

2.3 Periodic orbits

Recall the Poincaré–Bendixson theorem (Hale and Koçak 1991) for planar autonomous systems which says that the only admissible ω and α limits of bounded trajectories are either a stationary point or a periodic orbit. We have already been acquainted to the stationary points, so we are left to study the periodic orbits as limit sets for our trajectories.

More precisely, in this section we investigate in which intervals of p the dynamical systems (2.1.6) and (2.1.8) allow the existence of periodic orbits.

When $\lambda = \Lambda$, the exponents defined in (1.2.6) are such that

$$p_\Delta^a = \frac{N+2+2a}{N-2} = p_\pm^{p,a}, \qquad p_\pm^{s,a} = \frac{N+a}{N-2},$$

while when $\lambda \neq \Lambda$ we have the following ordering

$$p_-^{p,a} < p_\Delta^a < p_+^{p,a}.$$

Theorem 2.3.1 (Dulac's criterion)**.**

- *If* $\lambda = \Lambda$, *the system* (2.1.6) *does not admit periodic orbits for* $p \neq p_\Delta^a$.
- *Let* $\lambda < \Lambda$. *For* $\mathcal{M}^+$ *there are no periodic orbits of* (2.1.6) *when* $1 < p \leqslant p_\Delta^a$ *or* $p > p_+^{p,a}$. *For* $\mathcal{M}^-$ *no periodic orbits of* (2.1.8) *exist if* $1 < p < p_-^{p,a}$ *or* $p \geqslant p_\Delta^a$. *Moreover,*
  - *(i) there are no periodic orbits strictly contained in* $R_\lambda^+ \cup \ell_+$ *(resp.* $(R_\Lambda^+ \cup \ell_-)$*), for any* $p > 1$*;*
  - *(ii) periodic orbits contained in* $R_\lambda^- \cup \ell_+$ *(resp.* $R_\Lambda^- \cup \ell_-$*) are admissible only at* $p = p_\pm^{p,a}$*. Also, no periodic orbits at* $p_\pm^{p,a}$ *can cross the concavity line* $\ell_\pm$ *twice.*

As in (Bidaut-Véron and Giacomini 2010), Theorem 2.3.1 immediately implies the whole classification when $\lambda = \Lambda$, as we illustrate in Proposition 3.4.3 in the next chapter.

Proof. Define $\varphi(X, Z) = X^\alpha Z^\beta$, where $\beta = \frac{3-p}{p-1}$ and α as in (1.2.6). Set

$$\Phi(X, Z) = \partial_X(\varphi f) + \partial_Z(\varphi g),$$

with f and g defined in (2.2.1).

When $\lambda = \Lambda$, as in (Bidaut-Véron and Giacomini 2010) we have

$$\begin{aligned}
\Phi(X,Z) &= \partial_X(\varphi f) + \partial_Z(\varphi g)\\
&= X^\alpha Z^\beta\,[\alpha(X - N + 2 + \tfrac{Z}{\lambda}) + \beta(N + a - pX - \tfrac{Z}{\lambda}) + (2-p)X + 2 - \tfrac{Z}{\lambda}]\\
&= \varphi(X,Z)(p-1)^{-1}\,[-p(N-2) + (N+2+2a)],
\end{aligned}$$

this expression is positive if $1 < p < p^a_\Delta$ and is negative if $p > p^a_\Delta$.

If $\lambda < \Lambda$, for the operator $\mathcal{M}^+ = \mathcal{M}^+_{\lambda,\Lambda}$ we have

$$\Phi(X,Z) = \begin{cases} X^\alpha Z^\beta\,[\alpha(X - (N-2) + \tfrac{Z}{\lambda}) + \beta(N + a - pX - \tfrac{Z}{\lambda}) \\ \qquad\qquad + (2-p)X + 2 - \tfrac{Z}{\lambda}] \ \text{ in } R^+_\lambda, \\ X^\alpha Z^\beta\,[\alpha(X - (\tilde{N}_+ - 2) + \tfrac{Z}{\Lambda}) + \beta(\tilde{N}_+ + a - pX - \tfrac{Z}{\Lambda}) \\ \qquad\qquad + (2-p)X + 2 - \tfrac{Z}{\Lambda}] \ \text{ in } R^-_\lambda, \end{cases}$$

$$= \begin{cases} \varphi(X,Z)(p-1)^{-1}\,[-p(N-2) + (N+2+2a)] & \text{in } R^+_\lambda, \\ \varphi(X,Z)(p-1)^{-1}\,[-p(\tilde{N}_+ - 2) + (\tilde{N}_+ + 2 + 2a)] & \text{in } R^-_\lambda. \end{cases}$$

Both expressions are positive if $1 < p < \min(p^{p,a}_+, p^a_\Delta) = p^a_\Delta$; and both are negative if $p > \max(p^{p,a}_+, p^a_\Delta) = p^{p,a}_+$.

Anyway one concludes by the same argument as in the classical Bendixson–Dulac criterion, see also (González-Melendez and Quaas 2017, Theorem 3.1). Indeed, the vector field $F = (f, g)$ is Lipschitz continuous in (X, Z), so Green's area formula for the domain D enclosed by a periodic trajectory applies. For $\lambda = \Lambda$,

$$\int_{\partial D} \varphi\,\{f\,\mathrm{d}Z - g\,\mathrm{d}X\} = \int_D \Phi(X,Z)\,\mathrm{d}X\mathrm{d}Z.$$

The RHS is nonzero whenever $p \neq p^a_\Delta$, but the LHS is zero because

$$\mathrm{d}X = f\,\mathrm{d}t, \qquad \mathrm{d}Z = g\,\mathrm{d}t. \tag{2.3.1}$$

Now, for $\lambda < \Lambda$,

$$\begin{aligned}
\int_{\partial D} \varphi\,\{f\,\mathrm{d}Z - g\,\mathrm{d}X\} &= \int_D \Phi(X,Z)\,\mathrm{d}X\mathrm{d}Z\\
&= \int_{R^+_\lambda \cap D} \Phi(X,Z)\,\mathrm{d}X\mathrm{d}Z + \int_{R^-_\lambda \cap D} \Phi(X,Z)\,\mathrm{d}X\mathrm{d}Z.
\end{aligned} \tag{2.3.2}$$

The RHS is nonzero for $p \in (1, p_\Delta^a) \cup (p_+^{p,a}, \infty)$, but the LHS is zero by (2.3.1). Further, at p_Δ^a one has $\Phi = 0$ in R_λ^+ and so the first integral in (2.3.2) (in the RHS) vanishes, while the second one is positive. For $\mathcal{M}^-$ the computations are similar by using that $\min(p_-^{p,a}, p_\Delta^a) = p_-^{p,a}$ and $\max(p_-^{p,a}, p_\Delta^a) = p_\Delta^a$.

Next we look at the interval $[p_\Delta^a, p_+^{p,a}]$ for $\mathcal{M}^+$ when $\lambda < \Lambda$. Note that Poincaré–Bendixson theorem guarantees the existence of a stationary point in the domain D inside a periodic orbit. Since the only admissible stationary point in the interior of $1Q$ is $M_0 \in R_\lambda^-$ for $p > p_+^{s,a}$, while for $p \leqslant p_+^{s,a}$ M_0 is not an option (see Figures 2.2 and 2.3), then (i) follows.

To prove (ii) let us observe that if a periodic orbit is contained in $R_\lambda^- \cup \ell_+$ then by Proposition 2.2.2 (1) it may intersect the line ℓ_+ only at one point, namely the point P. Hence we can repeat the previous argument, neglecting the integral expression in R_λ^+. Then we get that there are no periodic orbits in $R_\lambda^- \cup \ell_+$ for every $p \neq p_+^{p,a}$. To finish, if a periodic orbit existed which crossed twice the line ℓ_+ at $p_+^{p,a}$, then the first integral of (2.3.2) (in the RHS) would be positive, while the second one is equal to zero because $\Phi = 0$ in R_λ^-. The case for $\mathcal{M}^-$ and $p_-^{p,a}$ is analogous. ■

Notably Dulac's criterion brings out the critical exponents p_Δ^a and $p_\pm^{p,a}$. They correspond to critical exponents for the two Laplacian operators Δ_N and $\Delta_{\widetilde{N}_\pm}$, in dimensions N and $\widetilde{N}_\pm$.

Other limit cycles θ are admissible by the dynamical system as far as they cross $\ell_\pm$ twice. They do appear for $\mathcal{M}\pm$ as we shall see in Sections 4.1 and 4.2. This happens because Dulac's criterion is inconclusive in a whole interval when $\lambda < \Lambda$. Formally, the Pucci problem opens space for new periodic orbits in order to appropriately glue both Laplacian operators.

Remark 2.3.2. The result of Theorem 2.3.1 has been recently improved in (Pacella and Stolnicki 2021b) giving new bounds for the range of p for which periodic orbits exist. The proof in (Pacella and Stolnicki 2021b) is technically complicated, therefore we prefer to state and prove only Theorem 2.3.1. The same applies to the case of the operator $\mathcal{M}_{\lambda,\Lambda}^-$.

2.4 A priori bounds and blow-up

We prove ahead important bounds for trajectories of (2.1.6) or (2.1.8) which are defined for all t in intervals of type $(\hat{t}, +\infty)$ or $(-\infty, \hat{t})$.

By Poincaré–Bendixson theorem, if a trajectory of (2.1.6) or (2.1.8) does not converge to a stationary point neither to a periodic orbit, either forward or backward in time, then it necessarily blows up in that direction. In the next propositions we prove that a blow up may only occur in finite time. The first result is obtained in the first quadrant.

Proposition 2.4.1. *Let τ be a trajectory of* (2.1.6) *or* (2.1.8) *in* $1Q$*, with* $\tau(t) = (X(t), Z(t))$ *defined for all* $t \in (\hat{t}, +\infty)$*, for some* $\hat{t} \in \mathbb{R}$*. Then*

$$X(t) < \widetilde{N}_{\pm} - 2, \quad \textit{for all } \ t \geqslant \hat{t}. \tag{2.4.1}$$

If instead, τ *is defined for all* $t \in (-\infty, \hat{t})$*, for some* $\hat{t} \in \mathbb{R}$*, then*

$$\begin{aligned} Z(t) &< \lambda(N + a) \textit{ in the case of } \mathcal{M}^+, \\ Z(t) &< \Lambda(N + a) \textit{ for } \mathcal{M}^-, \quad \textit{for all } \ t \leqslant \hat{t}. \end{aligned} \tag{2.4.2}$$

In particular, if a global trajectory is defined for all $t \in \mathbb{R}$ *in* $1Q$ *then it is contained in the box* $(0, \widetilde{N}_+ - 2) \times (0, \lambda(N + a))$ *in the case of* $\mathcal{M}^+$*, while it stays in* $(0, \widetilde{N}_- - 2) \times (0, \Lambda(N + a))$ *for* $\mathcal{M}^-$.

Proof. Let us first prove (2.4.1) when the operator is $\mathcal{M}^+$, $\widetilde{N}_+ \leqslant N$. Arguing by contradiction we assume that for some $t_1 \geqslant \hat{t}$ we have $X(t_1) \geqslant \widetilde{N}_+ - 2$. Notice that $\dot{X} > 0$ on the half line $L^+ = \{(X, Z) : X = \widetilde{N}_+ - 2\} \cap 1Q$, see (2.2.2). Therefore $X(t) > X(t_1) \geqslant \widetilde{N}_+ - 2$ for all $t > t_1$.

We claim that $X(t) \to +\infty$ as $t \to +\infty$. To see this, first notice that Z is bounded from t_1 on, since $\dot{Z} < 0$ to the right of L^+, see (2.2.3). If we had $X(t) \leqslant C$ for some $C > 0$ for $t \geqslant t_1$, then τ would be a bounded trajectory from t_1 on. Then by Poincaré–Bendixson theorem it should converge to a stationary point as $t \to +\infty$. Notice that periodic orbits are not allowed to the right of L^+ by the direction of the vector field, see Figure 2.1. This proves the claim, since no stationary points exist on the right of L^+.

Thus, we can pick a time t_2 such that $X(t_2) > N - 2 \geqslant \widetilde{N}_+ - 2$. Again by monotonicity, $X(t) > \widetilde{N}_+ - 2$ for all $t \geqslant t_2$.

Then we have two cases: either the trajectory τ reaches the region R_λ^- for some $t_3 \geqslant t_2$, or it stays in R_λ^+ for all time. If the first holds, then $\tau(t)$ remains there for all $t \geqslant t_3$, since $\dot{Z} < 0$ to the right of L^+, see Figure 2.1. Observe that the first equation in (2.1.6) yields $\frac{\dot{X}}{X[X-(\widetilde{N}_+-2)]} \geqslant 1$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{X(t)-\widetilde{N}_+ +2}{X(t)}\right) = \frac{\dot{X}}{X-(\widetilde{N}_+-2)} - \frac{\dot{X}}{X} = \frac{(\widetilde{N}_+-2)\dot{X}}{X[X-(\widetilde{N}_+-2)]} \geqslant \widetilde{N}_+ - 2 \tag{2.4.3}$$

for all $t \geqslant t_3$. Therefore, by integrating (2.4.3) in the interval $[t_3, t]$ we get

$$\frac{X-(\widetilde{N}_+-2)}{X} \frac{X(t_3)}{X(t_3)-(\widetilde{N}_+-2)} \geqslant e^{(\widetilde{N}_+-2)(t-t_3)}$$

and so

$$X(t) \geqslant \frac{\widetilde{N}_+ - 2}{1 - ce^{(\widetilde{N}_+-2)(t-t_3)}} \quad \text{where} \quad c = 1 - \frac{\widetilde{N}_+-2}{X(t_3)} \in (0,1). \tag{2.4.4}$$

In particular, X blows up in the finite time $t_1 = t_3 + \frac{\ln(1/c)}{\widetilde{N}_+-2}$.

If instead τ stays in R^+_λ from t_2 on, then the same computations developed with N in place of $\widetilde{N}_+$ imply, using the first equation in (2.1.6), that X blows up in finite time. Both ways one gets a contradiction.

Let us now prove (2.4.2) for $\mathcal{M}^+$. Notice that $\dot{Z} < 0$ in the region above the line $Z = \lambda(N+a)$ which is contained in R^+_λ, see Figure 2.1. Now, if $Z = \lambda(N+a)$ occurs at some point for the orbit τ, then in particular there is some t_0 such that $Z > \lambda(N+a)$ for all $t \leqslant t_0$, thus $\dot{Z} \leqslant Z(N+a-Z/\lambda)$. In particular τ remains in the region R^+_λ up to the time t_0. Moreover,

$$\frac{\lambda(N+a)\dot{Z}}{\lambda(N+a)-Z} = \frac{\dot{Z}}{Z} - \frac{\dot{Z}}{Z-\lambda(N+a)} = \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{Z(t)}{Z(t)-\lambda(N+a)}\right)$$

for all $t \leqslant t_0$. Integration in $[t, t_0]$ as before gives us that the trajectory blows up in finite time.

The proof of (2.4.1) and (2.4.2) for the operator $\mathcal{M}^-$ is analogous if one uses $\widetilde{N}_- \geqslant N$. ∎

Now, a similar argument as in Proposition 2.4.1 allows us to characterize all the orbits in $3Q$.

Proposition 2.4.2 (The flow in $3Q$). *Every orbit of* (2.1.7) *or* (2.1.9) *in* $3Q$ *blows up in finite time, backward and forward. The vector field in there always point to the right and down, with* $\dot{X} > 0$ *and* $\dot{Z} < 0$.

Proof. Recall that in $3Q$ we have $X, Z < 0$. Let us consider $\mathcal{M}^+$. Hence, by the first equation in (2.1.7) one gets $\dot{X} \geqslant X(X - (\widetilde{N}_- - 2))$, which is positive. Similarly, by the second equation in (2.1.9) one figures out that $\dot{Z} \leqslant Z(\widetilde{N}_- + a - Z/\lambda)$, which is now negative. Then integration as in (2.4.3), (2.4.4) gives us the result. For $\mathcal{M}^-$ it is analogous. ∎

Now we are able to characterize all the admissible types of blow-up for trajectories of (2.1.6) or (2.1.8) in the first quadrant.

Proposition 2.4.3 (Blow-up types in $1Q$). *Let u be a positive solution of (P_+) or (P_-) in an interval (R_1, R_2), $0 < R_1 < R_2$, and $\tau = (X, Z)$ be a corresponding trajectory of* (2.1.6) *or* (2.1.8) *lying in $1Q$ through the transformation* (2.1.1). *Then the following holds:*

(i) there exists $r_1 \in (R_1, R_2)$ such that $u'(r_1) = 0 \Leftrightarrow$ there exists $t_1 \in \mathbb{R}$ such that $Z(t) \to +\infty$ as $t \to t_1^+$. In addition, $X(t) \to 0$ as $t \to t_1^+$;

(ii) there exists $r_2 \in (R_1, R_2)$ such that $u(r_2) = 0 \Leftrightarrow$ there exists $t_2 \in \mathbb{R}$ such that $X(t) \to +\infty$ as $t \to t_2^-$. Further, $Z(t) \to 0$ as $t \to t_2^-$.

Moreover, no other blow-up types other than those of (i) and (ii) are admissible for τ in $1Q$.

Proof. Let us first observe that u and u' can never be zero at the same point r_1. Otherwise, by the uniqueness of the initial value problem we would have $u \equiv 0$ in a neighborhood of r_1, which is not possible by the strong maximum principle.

(i) Assume that there exists $r_1 > 0$ such that $u'(r_1) = 0$. Thus $u(r_1) > 0$ and by (2.1.1) it is easy to deduce the limits of $X(t)$ and $Z(t)$ as $t \to t_1^+$, for $t_1 = \ln(r_1)$. Vice versa if $Z(t) \to +\infty$ as $t \to t_1^+$, by (2.1.1) we immediately get $u'(r) \to 0$ as $r \to r_1 = e^{t_1}$, because u is continuous in (R_1, R_2). This in turn gives that $X(t) \to 0$ as $t \to t_1^+$, and no other asymptote parallel to the Z axis is admissible.

(ii) Suppose that $u(r_2) = 0$ for some $r_2 > R_1$. Then $u'(r_2) < 0$ and by (2.1.1) we easily obtain the behavior of X and Z as $t \to t_2^-$, where $t_2 = \ln(r_2)$. Vice versa if $X(t) \to +\infty$ as $t \to t_2^-$ then necessarily $u(r) \to 0$ as $r \to r_2 = e^{t_2}$, because u' is continuous in (R_1, R_2). Thus $Z(t) \to 0$ as $t \to t_2^-$ as before.

The arguments above also show that, in finite time, no other blow-up types are possible for τ in $1Q$. Indeed, as soon as X or Z tends to infinity, then u or u' vanishes at a positive radius. Recall that a blow up in infinite time is not admissible by Proposition 2.4.1. ■

Corollary 2.4.4. *Let u be a solution of (P_+) or (P_-), and τ be a corresponding trajectory of* (2.1.6) *or* (2.1.8) *starting above the line $\ell_\pm$ in $1Q$. Then u changes concavity at least once.*

Proof. Consider the $\mathcal{M}^+$ operator, for $\mathcal{M}^-$ is the same. If u never changed concavity, then $\tau = (X, Z)$ would remain inside the region R_λ^+ for all time. By

Lemma 2.2.1 and Theorem 2.3.1 there are no stationary points or periodic orbits in R_λ^+. Recall that $\dot{X} > 0$, $\dot{Z} < 0$ in R_λ^+, see Figure 2.1. Then τ must blow up at a finite forward time $\hat{t}$ such that $X(t) \to +\infty$ and $Z(t) \to Z_1$ as $t \to \hat{t}$, for some $Z_1 > \lambda(N-1) > 0$. But this blow-up is not admissible by Proposition 2.4.3. ■

Remark 2.4.5 (Blow-up in $3Q$)**.** Every orbit $\tau = (X, Z)$ of (2.1.7) or (2.1.9) in $3Q$ verifies

$$X(t) \to 0, \quad Z(t) \to -\infty \text{ as } t \to t_1^-$$

for some $t_1 \in \mathbb{R}$ such that $r_1 = e^{t_1}$ and $u'(r_1) = 0$. Moreover,

$$X(t) \to -\infty, \quad Z(t) \to 0 \text{ as } t \to t_3^+$$

for some $t_3 \in \mathbb{R}$ where $r_3 = e^{t_3}$ and $u(r_3) = 0$.

3 Classification

In this chapter we classify the solutions of the second order equations (P_+) and (P_-) and we show that this induces a classification of the orbits of the dynamical systems (2.1.6)-(2.1.9). We investigate four kinds of solutions of (P_+) and (P_-): regular solutions, singular solutions, and solutions in annuli and exterior domains.

In the end of the chapter we state the main theorems conceived by the dynamical system analysis.

3.1 Regular solutions

Let us consider the following initial value problem:

$$\begin{cases} u'' = M_\pm\left(-r^{-1}(N-1)\, m_\pm(u') - r^a|u|^{p-1}u\right), \\ u(0) = \gamma, \quad u'(0) = 0, \qquad \gamma > 0, \end{cases} \tag{3.1.1}$$

where $M_\pm$ and $m_\pm$ are defined in (1.2.3), (1.2.4).

Definition 3.1.1. By *regular solution* we mean a solution $u = u_p$ of (3.1.1) which is twice differentiable for $r > 0$, and C^1 up to 0.

Remark 3.1.2. Any regular radial solution of the differential equation in (P_+) or (P_-) must satisfy $u'(0) = 0$. Indeed, from the definition of radial function $u(x)$ is symmetric with respect to any axis passing through the origin so that $\nabla u(0) = 0$.

We denote by R_p, with $R_p \leqslant +\infty$, the radius of the maximal interval $[0, R_p)$ where u is positive.

Hence, in such interval u is a solution of $(P_\pm)$. Obviously, if $R_p = +\infty$ then u corresponds to a radial positive solution of (2.0.1) in $\mathbb{R}^N$. When $R_p < +\infty$ it gives a positive solution of the Dirichlet problem (2.0.1) in the ball B_{R_p}, with $u = 0$ on ∂B_{R_p}.

Remark 3.1.3. Given a regular positive solution $u = u_p$ in $[0, R_p)$ satisfying (3.1.1) for some $\gamma > 0$, then the rescaled function $v(r) = \tau u(\tau^{\frac{1}{\alpha}} r)$, for α as in (1.2.6) and $\tau > 0$, is still a positive solution of the same equation in $[0, \tau^{-\frac{1}{\alpha}} R_p)$ with initial value $v(0) = \tau\gamma$, see also (Felmer and Quaas 2003, Lemma 2.3).

If u is defined in the whole interval $[0, +\infty)$, thus there is a family of regular solutions obtained via $v = v_\tau$, for all $\tau > 0$. In this case we say that u is *unique* up to scaling.

On the other hand, a solution in the ball of radius R_p automatically produces a solution for an arbitrary ball, by properly choosing the parameter $\tau > 0$.

Remark 3.1.4. Note that, by rescaling a fast decaying solution, we get infinitely many fast decaying solutions which give a different value for the constant c in Definition 1.2.3(i), see (3.1.7) ahead. The same happens for the $(\tilde{N} - 2)$–blowing up solution in Definition 1.2.4(i). Instead it is easy to see that for the slow decaying solutions or α–blowing up solutions in Definitions 1.2.3(ii) and 1.2.4(ii), the constant c is independent of the rescaling, see (3.1.8) again.

Now, using the transformation (2.1.1) our goal is to characterize the regular solutions of (P_+) or (P_-) as trajectories of the dynamical systems (2.1.6) and (2.1.8) in the first quadrant.

Proposition 3.1.5. *Let $u = u_p$ be any positive regular solution of (P_+) (resp. (P_-)). Then the corresponding trajectory belongs to $1Q$ and is the unique trajectory of* (2.1.6) (*resp.* (2.1.8)) *whose α-limit is N_0.*

Proof. The proof is the same for both operators $\mathcal{M}^\pm$. The solution $u = u_p$ satisfies $\lim_{r\to 0} u(t) = u(0) = \gamma$ and $\lim_{r\to 0} u'(t) = u'(0) = 0$, for some $\gamma > 0$. In terms of the trajectory (X, Z), by the definition of X in (2.1.1) we easily find

$$\lim_{t\to -\infty} X(t) = 0. \tag{3.1.2}$$

Moreover, in the simpler case when $a = 0$ we have

$$\lim_{t\to-\infty} Z(t) = \lim_{r\to 0} \frac{-ru^p(r)}{u'(r)} = -\gamma^p \lim_{r\to 0} \frac{r}{u'(r)-u'(0)} = -\frac{\gamma^p}{u''(0)} \in (0,+\infty),$$

since it is easier to check from the equation that $u''(0) < 0$. When $a \neq 0$ we need some other argument to show that $Z(t)$ has a finite limit as $t \to -\infty$. First let us show that

$$\text{there exists } R_1 > 0 \text{ such that } u' \neq 0 \text{ for all } r \in (0, R_1). \tag{3.1.3}$$

If this was not true, then there would exist a sequence of positive radii $r_n \to 0$ such that $u'(r_n) = 0$. By the mean value theorem this yields the existence a sequence $s_n \in (r_n, r_{n+1})$ such that $u''(s_n) = 0$. Thus, since u' cannot be identically zero in a neighborhood of 0 by the equation (P_+), then u changes infinitely many times its concavity in a neighborhood of 0.

In terms of the dynamical system, say (2.1.6) for $\mathcal{M}^+$, this means that the respective trajectory intersects the line ℓ_+ (see (2.1.11)) more than once as $t \to -\infty$. In particular it should pass from R_λ^+ to R_λ^- infinitely many times, which, by Proposition 2.2.2 (1), may only occur at $X(s_n) > \frac{1+a}{p}$ for $a > -1$. This contradicts the fact that $X(s_n) \to 0$ for large n from (3.1.2).

By (3.1.3) we have that $Z(t)$ is well defined in some interval $(-\infty, \hat{t})$ so that (2.4.2) in Proposition 2.4.1 yields $Z(t) < \lambda(N + a)$ for all $t < \hat{t}$. Hence $\lim_{t\to-\infty} Z(t) < +\infty$. Note that the trajectory cannot belong to $3Q$ when blow-up in finite time occurs both backward and forward, by Proposition 2.4.2. Moreover, it cannot converge to O by Propositions 2.2.2 (2) and 2.2.4(1). Indeed, the unstable manifold at O is on the Z axis which cannot correspond to the solution u in any interval $(0, r)$. Hence it converges to N_0, independently of the initial datum $\gamma > 0$. ■

Remark 3.1.6. Thus, by Propositions 3.1.5 and 2.2.5 one concludes that a regular solution u_p, corresponds to the unique trajectory Γ_p labeled in (2.2.6), for all $p > 1$. Here, Γ_p is defined in a maximal interval $[0, T_p)$, $T_p = \ln R_p \leqslant +\infty$.

Note that the fact that Γ_p does not depend on the initial datum of u_p is not a surprise since we already observed that the change of initial datum is equivalent to rescaling the radius. This, in turn, is equivalent to shifting the time for the systems (2.1.6), (2.1.8), which does not produce any change in the trajectory since the system is autonomous.

We now prove the monotonicity and concavity properties of the regular solutions, deriving them directly by the dynamical systems (2.1.6) or (2.1.8), and not from the second order ODEs.

Proposition 3.1.7. *All regular solutions u of (P_+) or (P_-) are concave in an interval $(0, r_0)$ for some $r_0 > 0$ and change concavity at least once. Moreover, they are decreasing as long as they remain positive. In addition,*

$$u'(r) = O(-r^{1+a}), \quad u''(r) = O(-r^a) \quad \textit{as } r \to 0,$$

whenever $a > -1$.

Proof. Let us consider the $\mathcal{M}^+$ case; for $\mathcal{M}^-$ is the same. By Proposition 3.1.5 the corresponding trajectory Γ_p starts at $-\infty$ from the stationary point N_0 and enters the region R_λ^+ which is above the concavity line ℓ_+; see Proposition 2.2.4 (item 2) and Remark 2.2.6. Then we immediately deduce that u is concave near $r = 0$ and changes concavity at least once; see Corollary 2.4.4. Next, since $1Q$ is invariant by the flow and corresponds to positive decreasing solutions of (P_+) we get the monotonicity claim. For the asymptotics one computes

$$-\gamma^p \lim_{r\to 0} \frac{r^{1+a}}{u'(r)} = \lim_{t\to-\infty} Z(t) = \lambda(N + a),$$

from which

$$\lim_{r\to 0} \frac{u''}{r^a} = -\frac{\gamma^p}{\lambda} \frac{a + 1}{N + a},$$

which concludes the proof. ■

Given an exponent $p > 1$, for a regular solution u_p of (P_+) or (P_-), which is positive in $[0, +\infty)$, Definition 1.2.3 holds according to its behavior at infinity. Taking into account that u_p is unique, up to rescaling, as in (Felmer and Quaas 2003) we define the following sets:

$$\begin{aligned} \mathcal{F} &= \{\, p > 1 : \; u_p \; \text{ is fast decaying}\,\}; \\ \mathcal{S} &= \{\, p > 1 : \; u_p \; \text{ is slow decaying}\,\}; \\ \mathcal{P} &= \{\, p > 1 : \; u_p \; \text{ is pseudo-slow decaying}\,\}. \end{aligned} \tag{3.1.4}$$

Then we add the set

$$\mathcal{C} = \{\, p > 1 : \; (P_\pm) \text{ has a solution } u_p \text{ with } u_p(R_p) = 0 \,\}, \tag{3.1.5}$$

where, as before, R_p is the radius of the maximal positivity interval for u_p. We characterize the previous sets in terms of the orbits of the dynamical systems (2.1.6) or (2.1.8).

Proposition 3.1.8. *In terms of the dynamical systems* (2.1.6) *or* (2.1.8)*, the previous sets can be equivalently defined as follows:*

$$\begin{aligned}
\mathcal{F} &= \{\, p > 1 : \ \omega(\Gamma_p) = A_0 \,\}; \\
\mathcal{S} &= \{\, p > 1 : \ \omega(\Gamma_p) = M_0 \,\}; \\
\mathcal{P} &= \{\, p > 1 : \ \omega(\Gamma_p) \text{ is a periodic orbit around } M_0 \,\}; \\
\mathcal{C} &= \{\, p > 1 : \ \lim_{t\to T} X(t) = +\infty \text{ and } \lim_{t\to T} Z(t) = 0 \text{ for some } T > 0 \,\},
\end{aligned} \tag{3.1.6}$$

where $\Gamma_p(t) = (X(t), Z(t))$ is as in Remark 3.1.6. In particular,

$$(1, +\infty) = \mathcal{C} \cup \mathcal{F} \cup \mathcal{P} \cup \mathcal{S}.$$

Proof. The proof is the same for both operators $\mathcal{M}^{\pm}$, i.e. for both systems (2.1.6) and (2.1.8).

In the case of the sets $\mathcal{F}, \mathcal{S}$ and $\mathcal{P}$, u_p (as in Proposition 3.1.5) is positive in $(0, +\infty)$ which implies that Γ_p is defined for all $t \in \mathbb{R}$. By Proposition 2.4.1 this trajectory is bounded, and so by Poincaré–Bendixson theorem it converges as $r \to +\infty$ either to a stationary point or to a periodic orbit. In the first case only A_0 and M_0 are admissible. Moreover, via the transformation (2.1.1),

$$\omega(\Gamma_p) = A_0 \quad \Leftrightarrow \quad \lim_{r\to+\infty} u(r) r^{\tilde{N}_{\pm}-2} = C_{p,\gamma} \quad \text{for some } C_{p,\gamma} > 0, \tag{3.1.7}$$

and $u = u_p$ has fast decay at $+\infty$. On the other hand u is slow decaying at $+\infty$ when

$$\omega(\Gamma_p) = M_0 \quad \Leftrightarrow \quad \lim_{r\to+\infty} u(r) r^{\alpha} = C_p \quad \text{with } C_p = (X_0 Z_0)^{\frac{1}{p-1}}, \tag{3.1.8}$$

where $M_0 = (X_0, Z_0)$ is given explicitly in Lemma 2.2.1.

Indeed, (3.1.8) comes from the identity $X(t)Z(t) = r^{2+a} u^{p-1}$ for all $t \in \mathbb{R}$. On the other hand,

$$\lim_{t\to+\infty} X(t) = \tilde{N}_{\pm} - 2$$

is equivalent to

$$\frac{d}{dr}\ln(u(r)) = \frac{u'(r)}{u(r)} \sim -\frac{\tilde{N}_\pm - 2}{r} \quad \text{as } r \to +\infty.$$

Then, integration in $[r_0, r]$ for a fixed large r_0 implies (3.1.7) with

$$C_{p,\gamma} = u(r_0)\, r_0^{\tilde{N}_\pm - 2}, \quad u = u_{p,\gamma}.$$

Now, by rescaling, the function $v = v_\tau$ in Remark 3.1.3 satisfies

$$\lim_{t\to+\infty} v(r) r^{\tilde{N}_\pm - 2} = \tau^{1-\frac{\tilde{N}_\pm - 2}{\alpha}} C_{p,\gamma} \text{ under (3.1.7);}$$

$$\lim_{r\to+\infty} v(r) r^{\alpha} = C_p \text{ under (3.1.8).}$$

Thus C_p is independent of the initial value $\gamma > 0$ in (3.1.1).

Finally, assume that $\omega(\Gamma_p)$ is a periodic orbit θ. Note that the region inside θ is bounded, and by Poincaré–Bendixson theorem it must contain M_0. Using again $XZ = r^{2+a} u^{p-1}$ one defines

$$c_1^{p-1} := \inf_{t\in\mathbb{R}} \{X(t)Z(t) : (X, Z) \in \theta\},$$

$$c_2^{p-1} = \sup_{t\in\mathbb{R}} \{X(t)Z(t) : (X, Z) \in \theta\}.$$

Therefore we deduce

$$\omega(\Gamma_p) = \theta \quad \Leftrightarrow \quad 0 < c_1 = \underline{\lim}_{r\to+\infty} u(r) r^{\alpha} < \overline{\lim}_{r\to+\infty} u(r) r^{\alpha} = c_2. \qquad (3.1.9)$$

Now we consider the set $\mathcal{C}$. The corresponding trajectory Γ_p cannot be defined for all time since $u(R_p) = 0$. So it must blow up at the finite time $T_p = \ln(R_p)$ by Proposition 2.4.3

Vice versa if $\mathcal{F}, \mathcal{S}, \mathcal{P}, \mathcal{C}$ are defined in terms of the property of the trajectory Γ_p of the dynamical system then they give exactly the same sets as defined for u_p, by using (2.1.1) and (2.1.10), and arguing in a similar way. ■

Remark 3.1.9 ($\mathcal{C}$ is open)**.** When $p \in \mathcal{C}$, the trajectory Γ_p crosses the line $L = \{(X, Z), X = \tilde{N}_+ - 2\}$ and next blows up in finite time. This property is preserved for p' close to p.

3.2 Singular solutions

As mentioned in Introduction, by singular solution we mean a radial solution $u = u(|x|) = u(r)$ of (2.0.1) in a domain $\Omega \setminus \{0\}$ (and hence a solution of $(P_\pm)$) satisfying $\lim_{r\to 0} u(r) = +\infty$. It may be either positive for all $r \in (0, +\infty)$, or be equal to zero at a certain radius $R > 0$. In the first case it produces a solution in $\mathbb{R}^N \setminus \{0\}$, while in the latter a solution in $B_R \setminus \{0\}$.

In terms of the systems (2.1.6), (2.1.8), this means that the corresponding trajectory, say Σ_p, will be defined either in $\mathbb{R}$ or in an interval $(-\infty, T)$, for some $T < \infty$. Under the latter, as in the characterization of $\mathcal{C}$ in (3.1.6) we have that Σ_p blows up forward in a finite time $T < +\infty$. Otherwise, by Proposition 2.4.1 the global trajectory Σ_p is contained in the box

$$\mathcal{Q}^+ = (0, \tilde{N}_+ - 2) \times (0, \lambda(N + a)) \text{ for } \mathcal{M}^+;$$

$$\mathcal{Q}^- = (0, \tilde{N}_- - 2) \times (0, \Lambda(N + a)) \text{ for } \mathcal{M}^-.$$

Then the α and ω limits can be either a periodic orbit or a stationary point.

We point out that Σ_p cannot converge to N_0, neither backward nor in forward time, because the stable direction at N_0 is the Z axis, while the unstable direction corresponds to the regular trajectory Γ_p, for all $p > 1$. So all possible α and ω limits of Σ_p are M_0, A_0, or a periodic orbit.

By the analysis of the stationary points M_0 and A_0, and of the periodic orbits given in Section 2.3, the α and ω limits of Σ_p depend on the exponent p. Then a classification of the singular solutions, according to Definition 1.2.4 can be easily formulated in terms of the dynamical systems (2.1.6), (2.1.8), analogously to Proposition 3.1.8. Obviously if Σ_p is defined in $\mathbb{R}$ they are also classified according to the behavior at $+\infty$, as in Definition 1.2.3. Here we just emphasize that, as for the regular solutions, the so called pseudo–blowing up solutions, see Definition 1.2.4 (iii), may only occur at the values of p for which Σ_p has a periodic orbit as α-limit.

3.3 Annuli and exterior domain solutions

By solution in annulus or exterior domain solution we mean a solution u of (P_+) or (P_-) defined in an interval $[\alpha, \rho]$, for $\alpha \in (0, +\infty)$ and $\rho \leqslant +\infty$, and verifying the Dirichlet condition $u(\mathfrak{a}) = 0$. We look at the initial value problem

$$\begin{cases} u'' = M_{\pm}\left(-r^{-1}(N-1)\, m_{\pm}(u') - r^{a}|u|^{p-1}u\right), \\ \quad u(\mathfrak{a}) = 0, \quad u'(\mathfrak{a}) = \delta, \qquad \delta > 0, \end{cases} \tag{3.3.1}$$

The equations (P_+), (P_-), together with (3.3.1) were studied in (Galise, Iacopetti, and Leoni 2020), (Galise, Leoni, and Pacella 2017), and more recently in (Maia and Nornberg 2021) for weighted equations. In this section we follow the sketch in (Maia and Nornberg 2021).

For any $p > 1$ and for each $\delta > 0$ there exists a unique solution $u = u_\delta$ defined in a maximal interval $(\mathfrak{a}, \rho_\delta)$ where u is positive, $\mathfrak{a} < \rho_\delta \leqslant +\infty$.

If $\rho_\delta = +\infty$ we get a positive radial solution in the exterior of the ball $B_{\mathfrak{a}}$. In the second case, a positive solution in the annulus $(\mathfrak{a}, \rho_\delta)$ is produced. Note that equations (3.3.1) are not invariant by rescaling.

Our first main result in this section is the following existence in annuli, whose proof will be enlightening to illustrate the existence of exterior domain solutions of (3.3.1) from the shooting parameter point of view.

Theorem 3.3.1. *For any $p > 1$, and $0 < \mathfrak{a} < \mathfrak{b} < +\infty$, the problem* (3.4.1) *has both a positive and a negative radial solution in the annulus*

$$\Omega = A_{\mathfrak{a},\mathfrak{b}} = \{x \in \mathbb{R}^N \ : \mathfrak{a} < |x| < \mathfrak{b}\}.$$

Note that the positive solutions obtained in Theorem 3.3.1 may not be radial, since the Gidas–Ni–Nirenberg type symmetry result of (Da Lio and Sirakov 2007) does not hold for annular domains.

Remark 3.3.2. All results obtained for $\delta > 0$ will be also true for $\delta < 0$. Indeed, negative shootings for an operator F can be seen as positive shootings for the operator G defined as $G(x, X) = -F(x, -X)$, which is still elliptic and satisfies all the properties we considered so far. In particular, the negative solutions of $\mathcal{M}^+$ are positive solutions of $\mathcal{M}^-$ in the same domain, and vice versa.

Remark 3.3.3. All zeros of u are simple, that is, it does not exist a point $r_1 > \mathfrak{a}$ with $u(r_1) = u'(r_1) = 0$. This follows by the ODE existence and uniqueness with respect to the initial conditions at r_1, since the function $h(s) = r^a|s|^{p-1}s$ is locally Lipschitz continuous as long as $p > 1$ and $r > \mathfrak{a}$. Indeed, its derivative $h'(s) = p\, r^a |s|^{p-1}$ is globally bounded for $r \in [\mathfrak{a}, r_1 + \varepsilon]$ and $v \equiv 0$ is also a solution.

Remark 3.3.4. The function u has at most one extremum between two consecutive zeros: a maximal point when $u > 0$, and a minimum point when $u < 0$. Indeed, at a critical point r_0, i.e. $u'(r_0) = 0$, we have $m(u'')(r_0) = -|u|^{p-1}(r_0)\,u(r_0)$ and so $u''(r_0)$ has the opposite sign of $u(r_0)$. By Remark 3.3.3, zeros $r_1 > \alpha$ with $u(r_1) = u'(r_1) = 0$ are not admissible.

Proposition 3.3.5. *For each $\delta > 0$, and u_δ solution of* (3.3.1)*, we set*

$$\mathcal{E}_\sigma(r) = \frac{1}{2r^a}(u')^2 + \frac{1}{\sigma(p+1)}|u|^{p+1} \quad \textit{for } \sigma > 0. \tag{3.3.2}$$

Then the energy function

$$\mathcal{E}(r) = \begin{cases} \mathcal{E}_\Lambda(r) & \textit{if } uu' > 0 \\ \mathcal{E}_\lambda(r) & \textit{if } uu' < 0 \end{cases}$$

is piecewise monotone decreasing in $\{u' \neq 0\}$ whenever $\widetilde{N}_+ \geqslant 3/2$.

Proof. To fix the ideas let $\delta > 0$ and the operator $\mathcal{M}^+$. For simplicity, we write

$$u'' + \frac{r^a}{m_{u''}}|u|^{p-1}u = -\frac{m_{u'}}{m_{u''}}\frac{N-1}{r}\,u'$$

where m_s is the step function defined through $m_s s = m_+(s)$, for $s = u'(r)$ or $s = u''(r)$, whenever $u'' \neq 0$. Here $m_+(s)$ is the Lipschitz function given in (1.2.3).

Set $\hat{N} - 1 := \frac{m_{u'}}{m_{u''}}(N-1)$ which is either $N-1$, $\widetilde{N}_+ - 1$ or $\widetilde{N}_+ - 1$, whenever $u'' \neq 0$. We have $\sigma = \Lambda \geqslant m_{u''}$ when $uu' > 0$; while $\sigma = \lambda \leqslant m_{u''}$ when $uu' < 0$. Anyways it yields $\frac{uu'}{\sigma} \leqslant \frac{uu'}{m_{u''}}$, then

$$\begin{aligned} \mathcal{E}'_\sigma(r) &= -\tfrac{a}{2}r^{-a-1}(u')^2 + r^{-a}u'u'' + \tfrac{1}{\sigma}\,|u|^{p-1}uu' \\ &\leqslant -\tfrac{a}{2}r^{-a-1}(u')^2 + r^{-a}u'\,\{u'' + \tfrac{r^a}{m_{u''}}\,|u|^{p-1}u\} \\ &= -r^{-a-1}(u')^2(\tfrac{a}{2} + \hat{N} - 1) < 0 \end{aligned}$$

whenever $u'' \neq 0$ and $u' \neq 0$ and $2(\hat{N} - 1) + a > 0$. The latter is ensured for instance when $a > -1$ and $\widetilde{N}_+ \geqslant 3/2$. Note that at a point r_0 where $u''(r_0) = 0$ it happens that $u'(r_0)$ has the opposite sign of $u(r_0)$, and they cannot be both equal to zero by Remark 3.3.3. ∎

Proposition 3.3.6. *Let $p > 1$ be fixed and assume $\widetilde{N}_+ \geqslant 3/2$. For any $\delta > 0$, the local solution u_δ of* (3.3.1) *is extended to the whole interval* $[\alpha, +\infty)$.

Proof. To fix the ideas we consider operator $\mathcal{M}^+$. By Proposition 3.3.5 we know that the energy function $\mathcal{E}(r)$ is piecewise differentiable and continuous, being strictly decreasing in each interval of consecutive extrema. Note that at the zeros of u, the energy is continuous and strictly decreasing (recall that $u' \neq 0$ at a zero, see Remark 3.3.3).

We first claim that if u is oscillatory, that is, if u has infinitely many zeros $\rho_k > \alpha$, then necessarily $\rho_k \to +\infty$, and so u is automatically defined for all r. In fact, if $\rho_k \to \bar{\rho} \in (0, +\infty)$, then by the mean value theorem there would exist a sequence $s_k \in (\rho_k, \rho_{k+1})$ such that $u'(s_k) = 0$, and so $u(\bar{\rho}) = u'(\bar{\rho}) = 0$, which is impossible (see Remark 3.3.3).

We then may assume that u has a finite number of zeros. In particular, it has a finite number of critical points τ_k by Remark 3.3.4. In this case, as in (Kajikiya 2001) our goal is to show that the energy (3.3.2) is bounded. This will imply that u is defined for all time by Proposition 1.1.6. By the monotonicity of the energy,

$$\mathcal{E}(\alpha) = \mathcal{E}_\Lambda(\alpha) > \mathcal{E}_\Lambda(\tau_1) = \tfrac{\lambda}{\Lambda}\mathcal{E}_\lambda(\tau_1) > \tfrac{\lambda}{\Lambda}\mathcal{E}_\lambda(\rho_1) > \ldots C\mathcal{E}(r) \quad \text{for all } r \in I,$$

where I is the maximal interval of definition for u. The claim is proved since (3.3.2) bounded gives an upper bound for $|u|$ and $|u'|$ for all $r \in I$. ■

Remark 3.3.7. We have already seen that regular solutions have corresponding trajectories issued from N_0, and by the dynamical system either exist for all time or blow-up in finite time. In this latter case, they give positive regular solutions in a ball, and are further extended to the whole line (by means of sign changing solutions). In fact, positive solutions in a ball B_R are such that $\delta = u'(R) < 0$ by Hopf lemma. Then we apply Proposition 3.3.6 and Remark 3.3.2, whenever $\tilde{N}_+ \geqslant 3/2$. The same holds for negative solutions in a ball.

From the dynamical system we obtain a complete characterization of monotonicity for solutions u_δ of (3.3.1) as follows. Specially in this section we keep the notation in (Galise, Leoni, and Pacella 2017) for $\tau = \tau_\delta$ as a radius (and not for trajectories as in the rest of the text).

Lemma 3.3.8. *For any $\delta > 0$ such that u_δ is a positive solution of* (3.3.1) *in* $[\alpha, \rho]$, *with $\rho = \rho_\delta \leqslant +\infty$, there exists a unique number $\tau = \tau(\delta)$ with $\tau \in (\alpha, \rho)$, such that*

$$u'(r) > 0 \;\; \textit{for } r \in [\alpha, \tau), \qquad u'(\tau) = 0, \qquad u'(r) < 0 \;\; \textit{for } r \in (\tau, \rho].$$

Proof. Let us observe that a first critical point exists for u. To see this we look at the dynamical system driven by X, Z. In this case, the behavior at the third quadrant $X, Z < 0$ is given by $\dot{X} > 0$ and $\dot{Z} < 0$, with a blow up at finite time T such that $u'(e^T) = 0$ by Remark 2.4.5, and so $\tau = e^T$. The uniqueness of τ follows by Remark 3.3.4. ■

Remark 3.3.9. Alternatively, for the existence of τ in Lemma 3.3.8 one could have argued via the second order PDE problem in the following way: if u were strictly increasing and concave for all $r > \mathfrak{a}$ (see Proposition 3.3.6), then u would be positive, increasing and concave for all $r > \mathfrak{a}$. In this case it would be allowed to use the change of variables in Remark 1.2.10 to transform the weighted problem into a non weighted one, and apply the proof of Lemma 2.1 in (Galise, Leoni, and Pacella 2017).

If $\rho_\delta = +\infty$ then $\lim_{r\to\infty} u_\delta(r) = 0$. This comes from the a priori bounds in Proposition 2.4.1. Thus, for any $\delta > 0$ either $\rho_\delta = +\infty$ and $\lim_{r\to\infty} u_\delta(r) = 0$, or there exists some $\rho_\delta < +\infty$ such that $u(\rho_\delta) = 0$. Moreover, by continuous dependence on the initial data, the function $\delta \mapsto \rho_\delta$ is continuous in a neighborhood of any $\delta > 0$ where $\rho_\delta < +\infty$ whenever $p > 1$.

We shall omit the dependence on the parameter $\delta > 0$ whenever it is clear from the context.

Proposition 3.3.10. *For any pair $\delta > 0$, and u of (3.3.1), the energy functions*

$$E_\lambda(r) = r^{2(\tilde{N}_- - 1)+a}\, \mathcal{E}_\lambda(r) \quad in \ [\mathfrak{a}, \tau]$$
$$E_\Lambda(r) = r^{2(\tilde{N}_- - 1)+a}\, \mathcal{E}_\Lambda(r) \quad in \ [\tau, \rho]$$

are monotone increasing, where $\mathcal{E}_\sigma$ is given in (3.3.2) for $\sigma \in \{\lambda, \Lambda\}$.

Proof. Let us consider the operator $\mathcal{M}^+$. We recall that in the interval $[\mathfrak{a}, \tau]$ we have $u' \geqslant 0$, $u'' \leqslant 0$, and

$$u''u' + \tfrac{r^a}{\lambda} u^p u' = -\tfrac{(\tilde{N}_- - 1)}{r}(u')^2.$$

On the other hand, in $[\tau, \rho]$ we have $u' \leqslant 0$, and

$$u''u' + \tfrac{r^a}{\Lambda} u^p u' \geqslant u''u' + \tfrac{r^a}{m_{u''}} u^p u' = -\tfrac{(\hat{N}-1)}{r}(u')^2 \geqslant -\tfrac{(\tilde{N}_- - 1)}{r}(u')^2,$$

where $(m_{u''}, \hat{N})$ is either (λ, N) or $(\Lambda, \tilde{N}_+)$.

Set $\sigma = \lambda$ if $r \in [\mathfrak{a}, \tau]$ and $\sigma = \Lambda$ if $r \in [\tau, \rho]$. In any case, for $A = 2(\widetilde{N}_- - 1)$ we obtain

$$E'_\sigma(r) = A r^{A-1}\left\{\tfrac{1}{2}(u')^2 + \tfrac{r^a}{\sigma(p+1)} u^{p+1}\right\} + r^A\left\{u''u' + \tfrac{r^a}{\sigma} u^p u' + \tfrac{a r^{a-1}}{\sigma(p+1)} u^{p+1}\right\}$$
$$\geqslant r^{A-1}(u')^2\left\{\tfrac{A}{2} - (\widetilde{N}_- - 1)\right\} \geqslant 0$$

since $2(\widetilde{N}_- - 1) + a \geqslant 0$, which holds for $a > -1$ and $N \geqslant 2$. ■

The idea for proving Theorem 3.3.1 is to show that, for any given $+\infty > \mathfrak{b} > \mathfrak{a} > 0$, there exists a parameter $\delta > 0$ such that $\rho_\delta = \mathfrak{b}$ in addition to $u(\mathfrak{b}) = 0$.

From now on we start analyzing the behavior of the solutions u_δ as δ approaches the extremum values 0 and $+\infty$.

Lemma 3.3.11. *When $\delta \to 0$ then we have $u(\tau_\delta) \to 0$ and $\rho_\delta \to +\infty$.*

Proof. By Proposition 3.3.5 we have $\mathcal{E}_\Lambda(r) \leqslant \mathcal{E}_\Lambda(\mathfrak{a})$ for all $r \leqslant \tau$, that is,

$$\tfrac{r^a}{p+1} u^{p+1}(r) \leqslant \tfrac{\Lambda}{2}\,\delta^2 \quad \text{for all } r \in [\mathfrak{a}, \tau], \tag{3.3.3}$$

since $uu' \geqslant 0$ in $[\mathfrak{a}, \tau]$. In particular, at $r = \tau = \tau_\delta$,

$$u^{p+1}(\tau_\delta) \leqslant \tfrac{\Lambda(p+1)}{2\mathfrak{a}^a}\,\delta^2 \to 0 \quad \text{when } \delta \to 0.$$

Next we write the equation for u in $[\mathfrak{a}, \tau]$ as $(u' r^{\widetilde{N}_- - 1})' = -\tfrac{r^a}{\lambda} u^p\, r^{\widetilde{N}_- - 1}$, and so integrating from $\mathfrak{a}$ to τ produces

$$0 = u'(\tau)\,\tau^{\widetilde{N}_- - 1} = \delta\,\mathfrak{a}^{\widetilde{N}_- - 1} - \frac{1}{\lambda}\int_{\mathfrak{a}}^{\tau} s^{\widetilde{N}_- - 1 + a}\, u^p\,. \tag{3.3.4}$$

By combining the estimate for u in (3.3.3) and equality (3.3.4) we obtain

$$\delta = \frac{1}{\lambda \mathfrak{a}^{\widetilde{N}_- - 1}}\int_{\mathfrak{a}}^{\tau} s^{\widetilde{N}_- - 1 + a}\, u^p \leqslant \frac{C}{\widetilde{N}_- + a}\,\delta^{\frac{2p}{p+1}}\,\tau^{\widetilde{N}_- + a}$$

and so

$$\tau_\delta^{\widetilde{N}_- + a} \geqslant \frac{C_0}{\delta^{\frac{p-1}{p+1}}} \to +\infty \quad \text{as } \delta \to 0\,.$$

In particular, $\rho_\delta \geqslant \tau_\delta \to +\infty$ as $\delta \to 0$. ■

Lemma 3.3.12. *When $\delta \to +\infty$ then $\rho_\delta \to \mathfrak{a}$ and $u(\tau_\delta) \to +\infty$. Moreover, for every $C_0 > 0$ there exists a positive constant c_0 depending only on $C_0, \mathfrak{a}, N, p, \lambda, \Lambda$ such that*

$$\delta \leqslant C_0 \quad \textit{implies} \quad \rho_\delta \geqslant \mathfrak{a} + c_0.$$

Proof. We denote $A_{\mathfrak{r},\mathfrak{s}} = B_{\mathfrak{r}} \setminus \overline{B}_{\mathfrak{s}}$ for any $\mathfrak{r} > \mathfrak{s}$ and fix the operator $\mathcal{M}^+$ (the case for $\mathcal{M}^-$ will be analogous).

Step 1) $u(\tau_\delta) \to +\infty$ when $\delta \to \infty$.

Assume by contradiction that there exists a sequence $\delta_k \to \infty$ with respective solutions $u_k = u_{\delta_k}$ of (3.3.1), with $\tau_k = \tau_{\delta_k}$, $\rho_k = \rho_{\delta_k}$, and $u_k \leqslant M$ for all k.

Since $E_\lambda(r) \geqslant E_\lambda(\mathfrak{a})$ for all $r \in [\mathfrak{a}, \tau_k]$ by Proposition 3.3.10, then

$$\tau_k^{2(\widetilde{N}_- -1)+a}\, u_k^{p+1}(\tau_k) \geqslant \tfrac{\lambda(p+1)}{2} \mathfrak{a}^{2(\widetilde{N}_- -1)}\, \delta_k^2 \to +\infty. \tag{3.3.5}$$

Since $u_k \leqslant M$, then $\tau_k \to \infty$ as $k \to \infty$. In particular, $\tau_k \geqslant \mathfrak{a} + 1$ for large k. Take $\varepsilon \in (0,1)$ with $\varepsilon \leqslant \frac{\mathfrak{a}}{\widetilde{N}_- -1}$ and $r \in [\mathfrak{a}, \mathfrak{a}+\varepsilon] \subset [\mathfrak{a}, \tau_k]$. Then we use Taylor expansion of u_k at the point $\mathfrak{a}$ to write

$$u_k(r) = u_k(\mathfrak{a}) + u_k'(\mathfrak{a})(r-\mathfrak{a}) + \tfrac{1}{2}\, u_k''(c_k)(r-\mathfrak{a})^2, \quad \text{for some } c_k \in (\mathfrak{a}, r). \tag{3.3.6}$$

Now we notice that

$$\tfrac{\widetilde{N}_- -1}{c_k} u_k'(c_k) \leqslant \tfrac{\widetilde{N}_- -1}{\mathfrak{a}} \delta_k \quad \text{since } \ u_k'(c_k) \in (0, \delta_k).$$

Moreover, since u_k' is decreasing in $(\mathfrak{a}, r)$, we have $u_k'(c_k) \leqslant \delta_k$ and so, by the second order PDE in (P_+) and the fact that u_k is increasing in $(\mathfrak{a}, r)$, we deduce

$$u_k''(c_k) = -\tfrac{\widetilde{N}_- -1}{c_k} u_k'(c_k) - \tfrac{c_k^a}{\lambda} u_k^p(c_k) \geqslant -\tfrac{\widetilde{N}_- -1}{\mathfrak{a}} \delta_k - \tfrac{(\mathfrak{a}+1)^a}{\lambda} u_k^p(r).$$

Putting this estimate into (3.3.6) one finds

$$u_k(r) \geqslant \delta_k(r-\mathfrak{a}) - \tfrac{\widetilde{N}_- -1}{2\mathfrak{a}} \delta_k (r-\mathfrak{a})^2 - \tfrac{(\mathfrak{a}+1)^a}{2\lambda} u_k^p(r)\,(r-\mathfrak{a})^2.$$

Finally, by evaluating it at $r = a + \varepsilon$ it yields

$$u_k(\mathfrak{a}+\varepsilon) + \tfrac{(\mathfrak{a}+1)^a \varepsilon^2}{2\lambda}\, u_k^p(\mathfrak{a}+\varepsilon) \geqslant \delta_k \varepsilon \left\{1 - \tfrac{\widetilde{N}_- -1}{2\mathfrak{a}} \varepsilon\right\} \geqslant \tfrac{1}{2} \delta_k \varepsilon \quad \text{for sufficiently large } k,$$

since $\frac{\widetilde{N}_- -1}{2\mathfrak{a}} \varepsilon \leqslant \frac{1}{2}$. But this is impossible since $\delta_k \to +\infty$ and u_k is bounded. This shows Step 1.

Step 2) $\rho_\delta \to \alpha$ as $\delta \to +\infty$.

We first show that $\tau_\delta \to \alpha$ as $\delta \to +\infty$. This will be a consequence of Step 1 and the estimate

$$u_\delta^{\frac{p-1}{2}}(\tau_\delta) \leqslant C\,(\tau_\delta - \alpha)^{-1}. \tag{3.3.7}$$

In order to prove (3.3.7), we write for $r \in [\alpha, \tau]$,

$$-\tfrac{1}{2}(u'(r)^2)' \geqslant -u''u' - \tfrac{\tilde{N}_- -1}{r}(u')^2 \geqslant \tfrac{r^a}{\lambda} u^p u' \geqslant \tfrac{\alpha^a}{\lambda} u^p u',$$

and by integrating it in $[r, \tau]$, for $r \in [\alpha, \tau)$, one gets

$$u'(r) \geqslant C\,\{u^{p+1}(\tau) - u^{p+1}(r)\}^{1/2}.$$

Another integration in $[\alpha, \tau]$ yields

$$\int_\alpha^\tau \frac{u'\,\mathrm{d}r}{\sqrt{u^{p+1}(\tau)-u^{p+1}(r)}} \geqslant C_{\lambda,p} \int_\alpha^\tau \mathrm{d}r = C_{\lambda,p}\,(\tau - \alpha).$$

By using $s = u(r)$ and $u'\mathrm{d}r = \mathrm{d}s$ we get

$$C_{\lambda,p}\,(\tau - a) \leqslant \int_0^{u(\tau)} \frac{\mathrm{d}s}{\sqrt{u^{p+1}(\tau)-s^{p+1}}} = \frac{1}{u^{\frac{p+1}{2}}(\tau)} \int_0^1 \frac{u(\tau)\,\mathrm{d}\sigma}{\sqrt{1-\sigma^{p+1}}} = \frac{C}{u^{\frac{p-1}{2}}(\tau)}$$

by taking $\sigma = \frac{s}{u(\tau)}$ and $\mathrm{d}\sigma = \frac{\mathrm{d}s}{u(\tau)}$, from which we deduce (3.3.7).

Now it is enough to prove that

$$\lim_{\delta\to\infty} \frac{\rho_\delta}{\tau_\delta} = 1.$$

If not, then there exists $\epsilon > 0$ and a sequence $\delta_k \to \infty$, with $\rho_k = \rho_{\delta_k} \leqslant +\infty$ and $\tau_k = \tau_{\delta_k}$ such that $\rho_k > (1+\epsilon)\tau_k$ for the solutions $u_k = u_{\delta_k}$ of (3.3.1). In particular, u_k is positive and decreasing in the interval $[\tau_k, (1+\epsilon)\tau_k]$.

For $r \in (\tau_k, (1+\epsilon)\tau_k]$ we consider the annulus $A_k = A_{\tau_k, r}$ where u_k solves

$$-\mathcal{M}^\pm(D^2 u_k) \geqslant t_k |x|^a u_k \quad \text{in } A_k\,, \quad u_k > 0 \text{ in } A_k,$$

where

$$t_k = \min_{A_k} u_k^{p-1} = u_k^{p-1}(r).$$

Now, by the definition of first eigenvalue $\lambda_1^+(D) = \lambda_1^+(\mathcal{M}^+, D)$ for the fully nonlinear Lane–Emden equation driven by $\mathcal{M}^+$ in the domain D with respect to the weight $|x|^a$ (see (Busca, Esteban, and Quaas 2005; Moreira dos Santos et al. 2020; Quaas and Sirakov 2008)), we have

$$u_k^{p-1}(r) \leqslant \lambda_1^+(A_k), \quad \text{for all } r \in (\tau_k\,, (1+\epsilon)\tau_k). \tag{3.3.8}$$

Note that the following scaling holds

$$\lambda_1^+(A_{\varsigma,\,\varsigma(1+\epsilon)}) = \frac{1}{\varsigma^{2+a}}\,\lambda_1^+(A_{1,1+\epsilon}), \quad \text{for all } \varsigma > 0. \tag{3.3.9}$$

In fact, if λ_1^+, ϕ_1^+ are a positive eigenvalue and eigenfunction for the operator $\mathcal{M}^+$ with weight $|x|^a$ in $A_{1,1+\epsilon}$ i.e.

$$\mathcal{M}^+(D^2\phi_1^+) + \lambda_1^+|x|^a\phi_1^+ = 0, \quad \phi_1^+ > 0 \quad \text{in } A_{1,1+\epsilon}, \quad \phi_1^+ = 0 \quad \text{on } \partial A_{1,1+\epsilon}$$

then μ_1^+, ψ_1^+, where $\mu_1^+ = \lambda_1^+\varsigma^{-2-\alpha}$ and $\psi_1^+(x) = \phi_1^+(\frac{x}{\varsigma})$ are a positive eigenvalue and eigenfunction in $A_{\varsigma,\varsigma(1+\epsilon)}$ for $\mathcal{M}^+$ with weight $|x|^a$.

Then, by combining (3.3.8) and (3.3.9) one finds

$$u_k^{p-1}(r) \leqslant \frac{1}{\alpha^{2+a}}\,\lambda_1^+(A_{1,1+\frac{\epsilon}{2}}), \quad \text{for all } r \in [(1+\tfrac{\epsilon}{2})\tau_k, (1+\epsilon)\tau_k]. \tag{3.3.10}$$

Using $E_\Lambda(\tau_k) \leqslant E_\Lambda(r)$ for $r \in [\tau_k, \rho_k)$, it comes

$$\begin{aligned} r^{2(\tilde{N}_- -1)}\left\{\frac{r^a}{\Lambda(p+1)}u_k^{p+1}(r) + \tfrac{1}{2}(u_k')^2(r)\right\} &\geqslant \frac{\tau_k^{2(\tilde{N}_- -1)+a}}{\Lambda(p+1)}\,u_k^{p+1}(\tau_k) \\ &\geqslant \frac{\alpha^{2(\tilde{N}_- -1)+a}}{\Lambda(p+1)}\,u_k^{p+1}(\tau_k). \end{aligned} \tag{3.3.11}$$

Since $\tau_k \to \alpha$ as $k \to +\infty$ then

$$r \leqslant (1+\epsilon)\tau_k \leqslant (1+\epsilon)(\alpha+1) \quad \text{for large } k.$$

Now, by putting the latter and (3.3.10) into (3.3.11) we derive

$$(u_k')^2(r) \geqslant J_k\,,$$

where

$$J_k := C_{\epsilon,p,N,\Lambda}\{(\alpha+1)^{-2(\tilde{N}_- -1)}\alpha^{2(\tilde{N}_- -1)+a}\,u_k(\tau_k)^{p+1} - (\alpha+1)^a\alpha^{\frac{-(2+a)(p+1)}{p-1}}\}$$

and $J_k \to +\infty$ as $k \to \infty$ by Step 1. Hence

$$-u_k'(r) \geqslant J_k^{1/2} \to +\infty \text{ as } k \to \infty, \quad \text{for all } r \in [(1+\tfrac{\epsilon}{2})\tau_k\,, (1+\epsilon)\tau_k].$$

Via integration we get

$$\begin{aligned} u_k((1+\tfrac{\epsilon}{2})\tau_k) &\geqslant u_k((1+\tfrac{\epsilon}{2})\tau_k) - u_k((1+\epsilon)\tau_k) \\ &= -\int_{(1+\epsilon/2)\tau_k}^{(1+\epsilon)\tau_k} u_k'(r)\mathrm{d}r \geqslant \tfrac{\epsilon\tau_k}{2} J_k^{1/2} \to +\infty \end{aligned}$$

which contradicts (3.3.10).

Step 3) $\delta \leqslant C_0$ implies $\rho_\delta \geqslant \mathfrak{a} + c_0$.

Let us prove the contrapositive, that is, if $\rho_\delta \to \mathfrak{a}$ then $\delta \to +\infty$.

As in Step 2, if $s = \max_{A_{\mathfrak{a},\rho}} u^{p-1} = u^{p-1}(\tau)$ then u solves

$$-\mathcal{M}^\pm(D^2 u) \leqslant |x|^a |u|^{p-1} u \leqslant s\,|x|^a u \quad \text{in } A_{\mathfrak{a},\rho}\,, \quad u = 0 \quad \text{on } \partial A_{\mathfrak{a},\rho}\,.$$

Now, by the maximum principle for the fully nonlinear equation through the characterization of the first eigenvalue in (Busca, Esteban, and Quaas 2005; Quaas and Sirakov 2008) (see also (Moreira dos Santos et al. 2020) for the weighted version) yields

$$u^{p-1}(\tau) \geqslant \lambda_1^+(\mathcal{M}^+, A_{\mathfrak{a},\rho}). \tag{3.3.12}$$

In fact, if we had $s < \lambda_1^+(\mathcal{M}^+, A_{\mathfrak{a},\rho})$ then by the mentioned maximum principle we would obtain $u \leqslant 0$ in $A_{\mathfrak{a},\rho}$ which is impossible.

Using the scaling for the eigenvalue in (3.3.9), (3.3.3), and (3.3.12), we derive

$$\lambda_1^+(\mathcal{M}^+, A_{1,\rho/\mathfrak{a}}) = \mathfrak{a}^{2+a}\, \lambda_1^+(\mathcal{M}^+, A_{\mathfrak{a},\rho}) \leqslant \mathfrak{a}^{2+a}\, \big(\tfrac{A(p+1)}{2\mathfrak{a}^a}\, \delta^2 \big)^{\frac{p-1}{p+1}}.$$

Again by the scaling as in Step 2, $\lambda_1^+(\mathcal{M}^+, D) \to +\infty$ as $|D| \to 0$. Then $\rho_\delta \to \mathfrak{a}$ implies $\delta \to +\infty$. As a consequence, the ratio $\rho_\delta/\mathfrak{a}$ remains bounded away from 1 whenever δ is bounded from above. ■

Proof of Theorem 3.3.1. We fix the annulus $A_{\mathfrak{a},\mathfrak{b}}$ for some $0 < \mathfrak{a} < \mathfrak{b}$. For every $\delta > 0$, recall that u_δ is the unique radial solution of the initial value problem (3.3.1) defined for all $r > \mathfrak{a}$ (by Proposition 3.3.6), with a maximal radius of positivity given by $\rho_\delta \in (\mathfrak{a}, +\infty]$. Here, $u(\rho_\delta) = 0$ if $\rho_\delta < +\infty$, while $u(r) \to 0$ as $r \to +\infty$ is $\rho_\delta = +\infty$.

The mapping $\delta \to \rho_\delta$ is continuous by ODE continuous dependence on initial data. In particular, the set

$$\mathcal{D} = \mathcal{D}(p) := \{\, \delta \in (0, +\infty) : \; \rho_\delta < +\infty \,\} \tag{3.3.13}$$

is open. By Lemma 3.3.12, $\mathcal{D}$ is nonempty and contains an open neighborhood of $+\infty$.

Let $\delta^* = \delta^*(p)$ be the infimum of the unbounded connected component of $\mathcal{D}$. Since $\mathcal{D}$ is open, if $\delta^* > 0$ then $\rho_{\delta^*} = +\infty$. If $\delta^* = 0$ then $\lim_{\delta\to 0} \rho_\delta \geqslant \lim_{\delta\to 0} \tau_\delta = +\infty$ by Lemma 3.3.11.

The function $\delta \mapsto \rho_\delta$ is well defined and leads the interval $(\delta^*, +\infty)$ onto $(\mathfrak{a}, +\infty)$ by the second part of Lemma 3.3.12. Then there exists $\delta > 0$ such that $\rho_\delta = \mathfrak{b}$. The existence of negative solutions follows by Remark 3.3.2. ■

Remark 3.3.13. For the non weighted case $a = 0$, in (Galise, Iacopetti, and Leoni 2020) it was shown that $\delta^* = \inf \mathcal{D}$ for all p, that is $\mathcal{D} = (\delta^*, +\infty)$ is an open interval. Moreover, they prove there that at δ^* only a fast decaying solution is admissible.

Now we would like to describe the trajectories of the dynamical systems (2.1.6)-(2.1.8) which correspond to u_δ through the variables X, Z in (2.1.1).

Proposition 3.3.14. *Let $p > 1$ and $u_\delta = u_{\delta,p}$ be a positive solution of (P_+) (resp. (P_-)) satisfying* (3.3.1)*. Then there exists a unique trajectory $\Xi_{\delta,p}$ in $1Q$ for the system* (2.1.6) *(resp.* (2.1.8)*) which blows up backward in a finite time t_δ. More precisely, if $\Xi_{\delta,p}(t) = (X(t), Z(t))$ then*

$$\lim_{t\to t_\delta^+} Z(t) = +\infty \quad \textit{and} \quad \lim_{t\to t_\delta^+} X(t) = 0, \tag{3.3.14}$$

where $t_\delta = \ln(\tau_\delta)$, τ_δ given in Lemma 3.3.8. The trajectory $\Xi_{\delta,p}$ corresponds, after the transformation (2.1.1) *to the restriction of u_δ to the interval $I_\delta = (\tau_\delta, \rho_\delta)$, with $\rho_\delta \leqslant +\infty$.*

Proof. The proof works for both operators $\mathcal{M}^\pm$. We fix p and δ. By Lemma 3.3.8, u_δ is positive and decreasing in the interval I_δ. Then, after (2.1.1), to u_δ corresponds a unique trajectory $\Xi_\delta = \Xi_{\delta,p}$ contained to $1Q$ and is defined for all $t \in (t_\delta, \ln(\rho_\delta))$. Thus, by Proposition 2.4.3 we get (3.3.14). ■

When $\rho_\delta = +\infty$ we can classify the solutions accordingly to their behavior at $+\infty$, i.e. u_δ is fast, slow, or pseudo-slow decaying via the limits (i)-(iii) as $r \to +\infty$ in Definition 1.2.3.

3.4 Main results

In this section we state the main theorems produced by the dynamical system approach we are studying. The proofs will be presented in Chapter 4.

We recall our second order PDE problems

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) + |x|^a u^p = 0, \quad u > 0 \text{ in } \Omega, \tag{3.4.1}$$

where $a > -1$, $p > 1$.

In the singular case Ω is either $\mathbb{R}^N \setminus \{0\}$ or $B_R \setminus \{0\}$, and we assume the condition

$$\lim_{r\to 0} u(r) = +\infty, \quad r = |x|. \tag{3.4.2}$$

Whenever Ω has a boundary, we prescribe the Dirichlet condition

$$u = 0 \text{ on } \partial\Omega, \qquad \text{or} \qquad u = 0 \text{ on } \partial\Omega \setminus \{0\} \text{ under (3.4.2)}. \tag{3.4.3}$$

Theorem 3.4.1 ($\mathcal{M}^+$ regular solutions)**.** *Assume $\tilde{N}_+ > 2$, and $\lambda < \Lambda$. Then there exists a critical exponent p^*_{a+} such that*

$$\max\{p_+^{s,a}, p_\Delta^a\} < p^*_{a+} < p_+^{p,a}, \tag{3.4.4}$$

and the following assertions hold:

(i) if $p \in (1, p^*_{a+})$ there is no nontrivial radial solution of (3.4.1) *in the whole $\mathbb{R}^N$, while for any $R > 0$ there exists a unique radial solution in the ball B_R;*

(ii) if $p = p^*_{a+}$ there exists a unique fast decaying radial solution of (3.4.1) *in $\mathbb{R}^N$;*

(iii) if $p \in (p^*_{a+}, p_+^{p,a}]$ there is a unique pseudo-slow decaying radial solution to (3.4.1) *in $\mathbb{R}^N$;*

(iv) if $p > p_+^{p,a}$ there exists a unique slow decaying radial solution of (3.4.1) *in $\mathbb{R}^N$;*

(v) if $p > p^*_{a+}$ there is no nontrivial solution to (3.4.1), (3.4.3) *when Ω is a ball.*

In (ii)–(iv) uniqueness is meant up to scaling.

Theorem 3.4.2 ($\mathcal{M}^-$ regular solutions). *If $\lambda < \Lambda$, then there exists a critical exponent p^*_{a-} satisfying*

$$p_-^{p,a} < p^*_{a-} < p^a_\Delta, \tag{3.4.5}$$

and there exists $\varepsilon > 0$ such that:

(i) if $p \in (1, p^*_{a-})$ there is no nontrivial radial solution of (3.4.1) *in the whole $\mathbb{R}^N$, while for any $R > 0$ there exists a unique radial solution of the Dirichlet problem* (3.4.1), (3.4.3) *in B_R;*

(ii) if $p = p^*_{a-}$ there exists a unique fast decaying radial solution of (3.4.1) *in $\mathbb{R}^N$;*

(iii) if $p \in (p^*_{a-}, p^a_\Delta - \varepsilon]$ there is a unique pseudo-slow or slow decaying radial solution of (3.4.1) *in $\mathbb{R}^N$;*

(iv) if $p > p^a_\Delta - \varepsilon$ there exists a unique slow decaying radial solution of (3.4.1) *in $\mathbb{R}^N$;*

(v) if $p > p^*_{a-}$ there is no nontrivial solution to (3.4.1), (3.4.3) *when Ω is a ball.*

In (ii)–(iv) uniqueness is meant up to scaling.

In the sequel we illustrate the power of the classification results via Theorem 2.3.1 when $\lambda = \Lambda$, inspired by the analysis in (Bidaut-Véron and Giacomini 2010).

Proposition 3.4.3 (The case of the Laplacian). *p^a_Δ is the critical exponent for the Laplacian operator, in the sense of Theorems 3.4.1 and 3.4.2 with $p^*_{a+} = p^*_{a-} = p^a_\Delta$ in which $\mathcal{P} = \emptyset$.*

Proof. We first observe that, by Theorem 2.3.1, there are no periodic orbits of the system (2.1.6) (which coincides with the system (2.1.8)) when $p \neq p^a_\Delta$. In particular,

$$((1, p^a_\Delta) \cup (p^a_\Delta, +\infty)) \cap \mathcal{P} = \emptyset.$$

Step 1) If $p > p^a_\Delta$ then $p \in \mathcal{S}$.

For $p > p^a_\Delta$, M_0 is a sink by Proposition 2.2.4. Let us show that $p \notin \mathcal{C} \cup \mathcal{F}$. If $p \in \mathcal{C}$, then Γ_p crosses the line $L := \{(X, Z), X = N - 2\}$, and blows up in finite time. Then the region D enclosed by Γ_p, L and the X, Z axes is a bounded domain from which an orbit can only leave D forward in time through L. Thus, an orbit arriving at $M_0 \in D$ cannot go anywhere in backward time, giving a contradiction with Poincaré–Bendixon theorem.

If instead $p \in \mathcal{F}$, then the bounded set whose boundary is given by Γ_p together with the X and Z axes, is invariant and contains M_0. Again the orbits arriving at M_0 cannot escape in backward time. Therefore $p \in \mathcal{S}$.

Step 2) For $p \in (1, p_\Delta^a)$ we have $p \in \mathcal{C}$.

If $1 < p < \frac{N+a}{N-2}$, then A_0 is a source and $M_0 \notin 1Q$. No periodic orbits exist in this case as we already known.

At the Serrin exponent $p = \frac{N+a}{N-2}$ we point out that the proof of Theorem 2.3.1 also shows the nonexistence of homoclinics (i.e. orbits τ with $\omega(\tau) = \alpha(\tau) = A_0 = M_0$). Therefore, if we had $\omega(\Gamma_p) = A_0$ then the orbits which come out from A_0 could not go anywhere, see also Proposition 4.1.9.

Now, if $\frac{N+a}{N-2} < p < p_\Delta^a$ then M_0 is a source by Proposition 2.2.4. The trajectory Γ_p cannot be bounded, otherwise it could only converge to A_0 as $t \to +\infty$. As in the proof of Proposition 4.1.5 this would produce a contradiction, because the region D enclosed by Γ_p, and the X, Z axes would be an invariant set from which any trajectory issued from M_0 cannot exit.

In any case Γ_p blows up in finite time, and so $p \in \mathcal{C}$.

Step 3) $p_\Delta^a \in \mathcal{F}$.

First, $p_\Delta^a \notin \mathcal{C}$ since $\mathcal{C}$ is open. Moreover, by the center configuration around M_0 proved in Proposition 2.2.7 we deduce that $p_\Delta^a \notin \mathcal{S}$. We only need to exclude the case $p \in \mathcal{P}$. If we had $p \in \mathcal{P}$, then for $p < p_\Delta^a$ close to p_Δ^a, a regular trajectory Γ_p would need to cross the line $\ell_1 = \ell_1^\pm$ by continuity of the ODE problem with respect to the parameter p. However, since $p \in \mathcal{C}$ we know that these solutions do not cross it. This concludes the proof. ■

For $\lambda < \Lambda$, in the $\mathcal{M}^-$ case our Theorem 3.4.1 slightly improves the corresponding one in (Felmer and Quaas 2003) for $a = 0$. Our point (iv) of Theorem 3.4.2 shows that, for p near p_Δ^a, only a slow decaying solution is allowed to exist. As stated in the Introduction, the bounds (3.4.4) and (3.4.5) regarding the range of p's for which pseudo-slow decaying solutions exist were further refined in (Pacella and Stolnicki 2021b), see Remark 2.3.2.

The proofs of the previous theorems rely entirely on a careful analysis of an autonomous quadratic dynamical system that we obtain after a suitable transformation, see Chapter 2. It was used in (Bidaut-Véron and Giacomini 2010) to study the

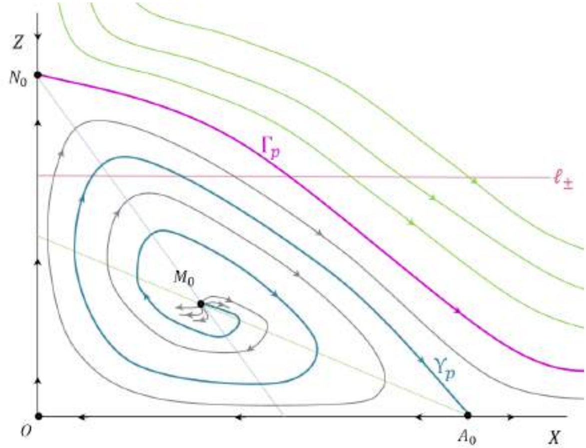

Figure 3.1: The dynamical system configuration in the case $\lambda = \Lambda$ when $p_{\pm}^{s,a} < p < p_{\Delta}^{a}$. The stationary set $\ell_2^{\pm}$ in this case is a smooth line.

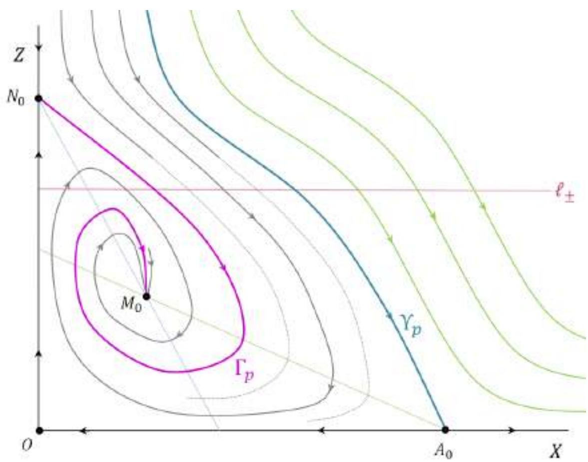

Figure 3.2: The dynamics at $\lambda = \Lambda$ when $p > p_{\Delta}^{a}$. The regular trajectory is always slow decaying.

classical semilinear Lane–Emden system. Once the correspondence between the radial solution of (3.4.1) and the orbits of the dynamical systems (2.1.6) and (2.1.8) is made (see Chapter 3), all existence and classification results of Theorems 3.4.1

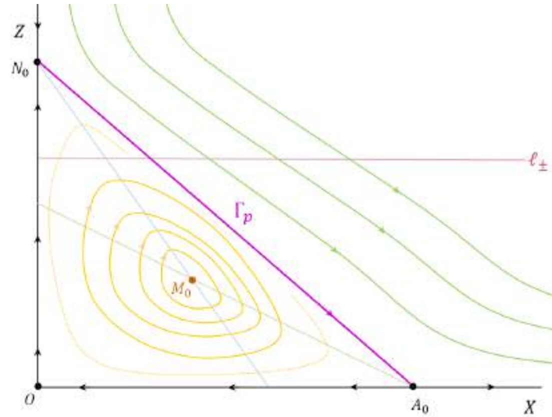

Figure 3.3: The case $\lambda = \Lambda$ at $p = p^a_\Delta$. The regular trajectory is a line connecting the points A_0 and N_0.

and 3.4.2 are derived by studying the stationary points and the flow lines of these systems. In particular, the uniqueness of the critical exponent and the behavior of the solutions, by varying the exponent p, are obtained as a direct consequence of the properties of the vector fields which define the dynamical systems.

Note that these systems are derived from the Pucci fully nonlinear equations and are piecewise C^1. This, in particular, allows the presence of several periodic orbits which produce regular and singular solutions with different features like pseudo-slow decay or pseudo–blowing up behavior at infinity or at the origin.

One reason why our approach is quite simple is that the most relevant sets which determine the flow generated by (2.1.6) and (2.1.8) are just straight lines; see Figures 2.1 to 2.3. Moreover, the presence of the weight $|x|^a$ in (3.4.1) does not produce additional difficulties, while it could be complicated via the method of (Felmer and Quaas 2003).

On the other hand, by the same analysis of the dynamics induced by (2.1.6) and (2.1.8) we also get the classification of singular solutions of (3.4.1) in a punctured ball or in $\mathbb{R}^N \setminus \{0\}$.

Remark 3.4.4. For all $p > p^{s,a}_+$ in the case of $\mathcal{M}^+$, resp. $p > p^{s,a}_-$ for $\mathcal{M}^-$, the function $u_p(r) = C_p r^{-\alpha}$, C_p as in (3.1.8), is a singular solution of (3.4.1) in $\mathbb{R}^N \setminus \{0\}$. We call it *trivial singular solution*.

Theorem 3.4.5 ($\mathcal{M}^+$ singular solutions)**.** *Assuming $\tilde{N}_+ > 2$ and $\lambda < \Lambda$, for* (3.4.1)–(3.4.2) *it holds:*

(i) for any $p \leqslant p_+^{s,a}$ *there is no singular radial solution in* $\mathbb{R}^N \setminus \{0\}$*, while for each* $R > 0$ *there are infinitely many* $(\tilde{N}_+ - 2)$*–blowing up radial solutions of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(ii) if $p_+^{s,a} \leqslant p_\Delta^a$ *then for any* $p \in (p_+^{s,a}, p_\Delta^a]$ *there is a unique* α*–blowing up radial solution in* $\mathbb{R}^N \setminus \{0\}$ *with fast decay at* $+\infty$*. Also, for any* $R > 0$ *there are infinitely many* α*–blowing up radial solutions of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(iii) for each $p \in (p_\Delta^a+, p_{a+}^*)$ *there exists a unique singular radial solution in* $\mathbb{R}^N \setminus \{0\}$ *with fast decay at* $+\infty$*. Moreover, for any* $R > 0$ *there exist infinitely many singular radial solutions of the Dirichlet problem* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(iv) if $p = p_{a+}^*$ *there exist infinitely many pseudo–blowing up radial solutions in* $\mathbb{R}^N \setminus \{0\}$ *with pseudo-slow decay at* $+\infty$*, and infinitely many* α*–blowing up in* $\mathbb{R}^N \setminus \{0\}$ *with pseudo-slow decay at* $+\infty$*. Also, there is no singular radial solution of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(v) if $p \in (p_{a+}^*, p_+^{p,a})$ *there are infinitely many* α*–blowing up radial solutions in* $\mathbb{R}^N \setminus \{0\}$ *with pseudo-slow decay at* $+\infty$*, and there is a pseudo–blowing up radial solution with pseudo-slow decay at* $+\infty$*. Further, there is no singular radial solution of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(vi) if $p \in [p_+^{p,a}, +\infty)$ *there are no nontrivial singular radial solutions, cf. Remark* 3.4.4*.*

Here, uniqueness in $\mathbb{R}^N \setminus \{0\}$ *is meant up to scaling.*

Theorem 3.4.6 ($\mathcal{M}^-$ singular solutions)**.** *If* $\lambda < \Lambda$*, for the problem* (3.4.1)–(3.4.2) *we have:*

(i) if $p \leqslant p_-^{s,a}$ *there is no singular radial solution in the whole* $\mathbb{R}^N \setminus \{0\}$*, while for any* $R > 0$ *there are infinitely many* $(\tilde{N}_- - 2)$*–blowing up radial solutions of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(ii) for each $p \in (p_-^{s,a}, p_-^{p,a})$ *there exists a unique* α*–blowing up radial solution in* $\mathbb{R}^N \setminus \{0\}$ *with fast decay at* $+\infty$*. Further, for any* $R > 0$ *there exist infinitely many* α*–blowing up radial solutions of the Dirichlet problem* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(iii) *for any* $p \in (p_-^{p,a}, p_{a-}^*)$ *there are infinitely many pseudo–blowing up radial solutions in* $\mathbb{R}^N \setminus \{0\}$*. Among them there is a unique fast decaying, a pseudo-slow decaying, and infinitely many with slow decay at* $+\infty$*. Moreover, for each* $R > 0$ *there exist infinitely many pseudo–blowing up radial solutions of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(iv) *if* $p = p_{a-}^*$ *there exist infinitely many pseudo–blowing up radial solutions in* $\mathbb{R}^N \setminus \{0\}$*. Among them there are infinitely many with slow-decay at* $+\infty$*, and infinitely many pseudo-slow decaying at* $+\infty$*. Further, there is no singular radial solution of* (3.4.1)–(3.4.3) *in* $B_R \setminus \{0\}$*;*

(v) *there exists* $\varepsilon > 0$ *such that for* $p \in [p_\Delta^a - \varepsilon, +\infty)$ *no nontrivial singular radial solution exists.*

Here, uniqueness in $\mathbb{R}^N \setminus \{0\}$ *is meant up to scaling.*

Our results on singular solutions are obtained by complementing the analysis of the flow lines of the dynamical systems (2.1.6) and (2.1.8). To the best of our knowledge they are the first global classification results on singular solutions found for this class of fully nonlinear equations. In (Felmer and Quaas 2003, Remark 3.2) it is pointed out that, in the case $a = 0$, periodic orbits of the Emden–Fowler system would produce singular solutions, while in (Felmer and Quaas 2006, Theorem 6.3) the existence of singular solutions is proved near the critical exponent.

For the critical exponents $p_{a\pm}^*$, our dynamical systems (2.1.6) and (2.1.8) furnish infinitely many periodic orbits. On the other hand, for $p_\pm^{p,a}$ infinitely many periodic orbits appear which do not correspond to C^2 solutions for $r > 0$, see Remark 4.1.10. For $p \in (p_{a-}^*, p_\Delta^a - \varepsilon)$ the existence of singular solutions cannot be deduced directly from the dynamical system approach.

Let us underline the fact that obtaining periodic orbits is in general a very difficult task in the theory of dynamical systems. Even in the very particular case of a polynomial autonomous system this question is not completely understood, see (Chicone and Tian 1982; Hale and Koçak 1991).

Finally, as a byproduct of the study of regular radial solutions of (3.4.1), either in $\mathbb{R}^N$ or in a ball, we easily get the range of the exponents p for which a positive radial solution of the Dirichlet problem in the exterior of a ball does not exist. Indeed, we get the following result.

Theorem 3.4.7. *Let* $p > 1$*. Then there are no radial solutions of*

$$\mathcal{M}^\pm(D^2u) + |x|^a u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N \setminus B_R, \; u = 0 \text{ on } \partial B_R \qquad (3.4.6)$$

if $p \leqslant p^*_{a\pm}$ for each $R > 0$.

In the case of $a = 0$, Theorem 3.4.7 has been recently proved in (Galise, Iacopetti, and Leoni 2020) with different arguments which rely both on the study of the second order ODE and on the analysis of the Emden–Fowler system. Their work presents a complete picture of existence and nonexistence of solutions for distinct intervals for the values of the parameter p. However, through our arguments we get their nonexistence result by a considerably shorter proof. Indeed, we will see in Sections 4.1 and 4.2 that the result of Theorem 3.4.7 becomes a straightforward consequence of the characterization of the critical exponents $p^*_{a\pm}$ in terms of the associated quadratic system we consider. Let us point out that in (Galise, Iacopetti, and Leoni 2020) also the existence and classification of the solutions of (3.4.6) are provided when $a = 0$. Alternatively, this could be done through our methods. Since this is not the main goal of our research we just refer to Section 4.1 for further comments.

Remark 3.4.8. Let $\mathcal{D}(p)$ be the set defined in (3.3.13), and $\delta^*(p)$ the infimum of the unbounded connected component of $\mathcal{D}(p)$. By the dynamical system approach (see the main theorems), we automatically deduce $\delta^*(p) = 0 = \inf \mathcal{D}(p)$ for all $p \leqslant p^*_{a\pm}$ (since no exterior domain solutions exist in this range), while $\delta^*(p) > 0$ when $p > p^*_{a\pm}$ (here we have exactly one fast decaying exterior domain solution, and infinitely many slow decaying exterior domain solutions).

Remark 3.4.9. Positive exterior domain solutions are not allowed to exist in the range where positive solutions of the ball do (check in the figures). That is, the existence of a blow-up regular solution creates an invariant set through which exterior domain trajectories cannot pass, and in turn remain bounded forward in time by Proposition 2.4.1, forward in time. Moreover, negative exterior domain solutions of $\mathcal{M}^+$ are the positive exterior domain solutions of $\mathcal{M}^-$. In particular, the solutions produced by the shooting method (3.1.1) when $p < \min\{p^*_{a+}, p^*_{a-}\} = p^*_{a-}$, which are sign-changing solutions for all $r > 0$ by Proposition 3.3.6, need to oscillate indefinitely. We refer to (Galise, Iacopetti, Leoni, and Pacella 2020) for more details on sign-changing solutions to fully nonlinear operators.

4 The flow study

This chapter is devoted to the proofs of the theorems in the preceding chapter. We split our analysis in two cases, accordingly to the operators $\mathcal{M}^+$ and $\mathcal{M}^-$. The first Section 4.1 concerning $\mathcal{M}^+$ gathers the main tools of our approach, while in Section 4.2 we just complement by inserting the differences from the previous case.

4.1 The $\mathcal{M}^+$ case

In this section we study the solutions of the equations involving the Pucci $\mathcal{M}^+$ operator. Hence we refer to the dimension-like parameter $\tilde{N}_+$ and the relevant exponents for $\mathcal{M}^+$ defined in (1.2.5) and (1.2.6), as well as their ordering:

$$\max\{\, p_+^{s,a},\ p_\Delta^a \} \leqslant p_+^{p,a}.$$

4.1.1 Some properties of regular trajectories

We first consider the case of a regular solution of (P_+) whose corresponding trajectory for the system (2.1.6) will be denoted by $\Gamma_p = \Gamma_p(t)$ as in Section 3.1.

We also keep the other notations already introduced, in particular for the sets $\mathcal{F}, \mathcal{S}, \mathcal{P}, \mathcal{C}$ defined in (3.1.4), (3.1.5), and Proposition 3.1.8.

Lemma 4.1.1. *For any $p > 1$, with $\Gamma_p = (X_p, Z_p)$, we have:*

(i) if Γ_p reaches the line ℓ_1^+ (see (2.2.2)) at some t_0 with $X_p(t_0) \geqslant \alpha$, then $p \in \mathcal{S} \cup \mathcal{P}$, i.e. the corresponding solutions u_p of (P_+) are either slow decaying or pseudo-slow decaying. In the latter case Γ_p crosses ℓ_1^+ and ℓ_2^+ (see (2.2.3)) infinitely many times;

(ii) if Γ_p does not intersect the line ℓ_1^+, then it intersects the concavity line ℓ_+ exactly once. Moreover, $\dot{X}_p > 0$ and $\dot{Z}_p < 0$ for all time. In particular this happens for $p \in \mathcal{F} \cup \mathcal{C}$.

Proof. We recall that Γ_p starts at $-\infty$ from the stationary point N_0 and must cross the concavity line ℓ_+ at least once, see Proposition 3.1.7.

(i) If Γ_p reaches ℓ_1^+ for $X_p(t_0) = \alpha$ then clearly $\lim_{t\to+\infty} \Gamma_p(t) = M_0$, whenever M_0 belongs to $1Q$ (see Figure 2.1). If instead $X_p(t_0) > \alpha$, by taking into account Proposition 2.2.2 (3) (see again Figure 2.1) and that Γ_p cannot self intersect, we have that Γ_p is contained in a bounded region from which it cannot leave. Thus, by Poincaré–Bendixson theorem the ω-limit of Γ_p is either M_0 or a periodic orbit θ which contains M_0 in its interior. In the latter case Γ_p goes around θ clockwise according to the direction of the vector field, intersecting ℓ_1^+ and ℓ_2^+ infinitely many times.

(ii) If Γ_p does not intersect ℓ_1^+ then it cannot turn back and cross the concavity line ℓ_+ another time because of the direction of the flow. Moreover, it can neither intersect nor be tangent to the line ℓ_2^+ where $\dot{Z} = 0$, since a C^1 trajectory of (2.1.6) may only intersect the line ℓ_2^+ transversely by passing from left to right, see Proposition 2.2.2 (3) and Figure 2.1. This fact and item (i) conclude the final assertion. ∎

The next proposition is crucial to study the behavior of Γ_p for different values of p.

Proposition 4.1.2. *Assume that $p_1 \in \mathcal{F} \cup \mathcal{C}$, and let Γ_{p_2} be any regular trajectory with $p_2 \neq p_1$. Then Γ_{p_1} and Γ_{p_2} can never intersect.*

Proof. Both Γ_{p_1} and Γ_{p_2} have their α-limit at the stationary point N_0 which is a saddle point. By Proposition 2.2.4(2) the tangent unstable directions for Γ_{p_1} and

Γ_{p_2} at N_0 are given, respectively, by

$$Z = -\tfrac{p_1\lambda(N+a)}{N+2+2a} X \quad \text{and} \quad Z = -\tfrac{p_2\lambda(N+a)}{N+2+2a} X. \tag{4.1.1}$$

Assume by contradiction that $\Gamma_{p_1}(t) = (X_1(t), Z_1(t))$ and $\Gamma_{p_2}(t) = (X_2(t), Z_2(t))$ intersect. Let us denote by Q the first intersection point. Since the dynamical system (2.1.6) is autonomous, one may assume that the intersection happens at the same time for both trajectories, i.e. $Q = (X_1(t_0), Z_1(t_0)) = (X_2(t_0), Z_2(t_0))$.

To fix the ideas assume $p_1 < p_2$. Then, by (4.1.1), at least in a neighborhood of N_0, Γ_{p_1} is above Γ_{p_2} because $X \to 0^+$ (from the right). Moreover, from (2.1.6) and Lemma 4.1.1(ii) we have

$$\dot{X}_1(t_0) = \dot{X}_2(t_0) > 0, \quad \dot{Z}_2(t_0) < \dot{Z}_1(t_0) < 0, \tag{4.1.2}$$

since only $\dot{Z}$ depends on p. In particular Γ_{p_1} remains above Γ_{p_2} after intersecting. Thus the two trajectories must have the same tangent at the point Q, which is not possible by (4.1.2). The case $p_2 < p_1$ is analogous. ∎

From the previous results we immediately get that a fast decaying solution can exist for only one value of p.

Corollary 4.1.3. *There exists at most one p in the interval $(p_+^{s,a}, +\infty)$ such that $p \in \mathcal{F}$.*

Proof. Assume by contradiction that $p_1, p_2 \in \mathcal{F}$ for some $p_+^{s,a} < p_1 < p_2$. This means that the corresponding trajectories Γ_{p_1} and Γ_{p_2} both come out from N_0 at $-\infty$ and converge to A_0 at $+\infty$. We have already observed by (4.1.1) that Γ_{p_1} stays above Γ_{p_2} in a neighborhood of N_0.

On the other hand, since A_0 is a saddle point for $p > p_+^{s,a}$, looking at the linear stable directions given by Proposition 2.2.4(3), we have that Γ_{p_1} and Γ_{p_2} arrive at A_0 with a reversed order; i.e. Γ_{p_2} is above Γ_{p_1}. This is because $X \to (\widetilde{N}_+ - 2)$ from the left.

Hence, Γ_{p_1} and Γ_{p_2} should intersect, but this is not possible by Proposition 4.1.2. ∎

Another important consequence of Proposition 4.1.2 is the following result.

Corollary 4.1.4. *Let $p_0 \in \mathcal{F}$, $p_0 > p_+^{s,a}$, then $p \in \mathcal{C}$ for $p_+^{s,a} < p < p_0$, and $p \in \mathcal{P} \cup \mathcal{S}$ for $p > p_0$.*

Proof. If $p_0 \in \mathcal{F}$ then Γ_{p_0} cannot intersect any other regular orbit Γ_p for $p \neq p_0$, by Proposition 4.1.2. This means that if Γ_p is above or below Γ_{p_0} in a neighborhood of N_0, it remains so for all time. Moreover, $p \notin \mathcal{F}$ for $p \neq p_0$ by Corollary 4.1.3.

Thus, if $p < p_0$, Γ_p lies above Γ_{p_0} and so cannot converge to $M_0 = M_0(p)$ neither to a periodic orbit around it, since the line ℓ_1^+ is below Γ_{p_0}. Notice that ℓ_1^+ does not depend on p. So $p \in \mathcal{C}$.

On the other hand, if $p > p_0$ then Γ_p lies below Γ_{p_0} and therefore cannot cross the line $L = \{(X, Z) : X = \widetilde{N}_+ - 2\}$ in order to blow up in finite time. Hence $p \notin \mathcal{C}$ and so must be in $\mathcal{P} \cup \mathcal{S}$. ■

4.1.2 The critical exponent

Our goal here is to define and characterize the critical exponent which will be proved to have all properties listed in Theorem 3.4.1.

We start by showing that $\mathcal{S}$ and $\mathcal{C}$ contain the intervals $(p_+^{p,a}, +\infty)$ and $(1, p_\Delta)$ respectively.

Proposition 4.1.5. *If $p > p_+^{p,a}$ then $p \in \mathcal{S}$.*

Proof. In case $p > p_+^{p,a}$, by Theorem 2.3.1 we know that there are no periodic orbits of the system (2.1.6), hence $p \notin \mathcal{P}$. Moreover, M_0 is a sink by Proposition 2.2.4. Let us show that $p \notin \mathcal{C} \cup \mathcal{F}$.

If $p \in \mathcal{C}$, then Γ_p crosses the line $L := \{(X, Z), X = \widetilde{N}_+ - 2\}$, and blows up in finite time. Then the region D enclosed by Γ_p, L and the X, Z axes form a bounded domain from which an orbit can only leave D forward in time through L. Thus, an orbit arriving at $M_0 \in D$ cannot go anywhere in backward time, giving a contradiction with Poincaré–Bendixon theorem.

If instead $p \in \mathcal{F}$, then the bounded set whose boundary is given by Γ_p together with the X and Z axes, is invariant and contains M_0. Again the orbits arriving at M_0 cannot escape in backward time. Therefore $p \in \mathcal{S}$. ■

Proposition 4.1.6. *For $p \in (1, \max\{p_+^{s,a}, p_\Delta^a\})$ it holds that $p \in \mathcal{C}$.*

Proof. If $1 < p < p_+^{s,a}$, then A_0 is a source and $M_0 \notin 1Q$. In particular there are no periodic orbits contained in $1Q$. Hence $p \notin \mathcal{F} \cup \mathcal{S} \cup \mathcal{P}$, so if $\max\{p_+^{s,a}, p_\Delta^a\} = p_+^{s,a}$ the proof is complete.

Assume $p_+^{s,a} < p_\Delta^a$. Then, at $p = p_+^{s,a}$ no periodic orbits are allowed by Theorem 2.3.1, whose proof also shows nonexistence of homoclinics at $A_0 = M_0$

(i.e. orbits τ with $\omega(\tau) = \alpha(\tau) = A_0$). Therefore, if we had $\omega(\Gamma_p) = A_0$ then the orbits which come out from A_0 could not go anywhere. Alternatively, see Proposition 4.1.9. In particular, $p_+^{s,a} \notin \mathcal{F} \cup \mathcal{S} \cup \mathcal{P}$.

Finally, if $p_+^{s,a} < p < p_\Delta^a$, then M_0 is a source by Proposition 2.2.4. The trajectory Γ_p cannot be bounded, otherwise it could only converge to A_0 as $t \to +\infty$. As in the proof of Proposition 4.1.5 this would produce a contradiction, because the region D enclosed by Γ_p, and the X, Z axes would be an invariant set from which any trajectory issued from M_0 cannot exit.

In any case Γ_p blows up in finite time, and so $p \in \mathcal{C}$. ■

By Propositions 4.1.5 and 4.1.6 we have that the set $\mathcal{C}$ is nonempty and bounded from above. Therefore we define

$$p_{a+}^* = \sup \mathcal{C} \tag{4.1.3}$$

and obviously

$$p_{a+}^* \in [\max\{p_+^{s,a}, p_\Delta^a\}, p_+^{p,a}].$$

From now on we refer to p_{a+}^* as the critical exponent for the Pucci operator $\mathcal{M}^+$ with weight $|x|^a$. The next result characterizes p_{a+}^*.

Theorem 4.1.7. *The number p_{a+}^* defined in (4.1.3) belongs to $\mathcal{F}$. Thus it is the only exponent in the equation (P_+) for which there exists a unique, up to scaling, fast decaying solution.*

Moreover, if $\lambda < \Lambda$, then $p_{a+}^* \neq p_\Delta^a$, $p_{a+}^* \neq p_+^{p,a}$, and (3.4.4) holds. Further, $\mathcal{P} = (p_{a+}^*, p_+^{p,a}]$, and for any $p \in \mathcal{P}$ the corresponding trajectory Γ_p crosses the concavity line ℓ_+ infinitely many times.

Proof. First, $p_{a+}^* \notin \mathcal{C}$ i.e. $\mathcal{C}$ does not have a maximum because $\mathcal{C}$ is open, see Remark 3.1.9. By Proposition 2.2.4 we know that M_0 is a source for every $p \in [p_\Delta^a, p_+^{p,a})$; and M_0 is a center at $p = p_+^{p,a}$. Whence $p \notin \mathcal{S}$ for all $p \in [p_\Delta^a, p_+^{p,a}]$, and in particular $p_{a+}^* \notin \mathcal{S}$. On the other side, if $p_{a+}^* \in \mathcal{P}$ then $\Gamma_{p_{a+}^*}$ would cross the line ℓ_1^+ by Lemma 4.1.1(ii). Thus, by continuity with respect to p, the trajectory Γ_p should also cross ℓ_1^+ for p close to p_{a+}^*. But every trajectory Γ_p for $p \in \mathcal{C}$ does not cross ℓ_1^+, by Lemma 4.1.1. Therefore $p_{a+}^* \notin \mathcal{P}$.

Hence p_{a+}^* belongs to $\mathcal{F}$ and the trajectory $\Gamma_{p_{a+}^*}$ together with the X and Z axes enclose a bounded invariant region D which contains M_0 in its interior. Since M_0 is a source for $p \in [p_\Delta^a, p_+^{p,a})$, and a center for $p = p_+^{p,a}$, the set

D contains periodic orbits which cross the line ℓ_+ twice. Indeed, the flow is subjected to Poincaré–Bendixson theorem, see Figure 4.6. This implies that p^*_{a+} can be neither p^a_Δ nor $p^{p,a}_+$ if $\lambda < \Lambda$, by Theorem 2.3.1. In fact, at p^a_Δ there are no periodic orbits at all, while at $p^{p,a}_\pm$ no periodic orbits cross ℓ_+ twice.

Note that we obtain (3.4.4) as long as $p^a_\Delta \geqslant p^{s,a}_+$. If $p^a_\Delta < p^{s,a}_+$ we still need to prove that $p^*_{a+} \neq p^{s,a}_+$. For instance this follows by the known Liouville results recalled in Theorem 1.2.8. Alternatively, a proof of this fact is accomplished in Proposition 4.1.9 which give nonexistence of entire positive solutions for $p = p^{s,a}_+$.

Next, by Corollary 4.1.4 we get $(p^*_{a+}, p^{p,a}_+] = \mathcal{P}$, since we have already observed that $[p^a_\Delta, p^{p,a}_+) \cap \mathcal{S} = \emptyset$. By the definition of $\mathcal{P}$, the corresponding trajectory Γ_p goes around a periodic orbit θ. By Theorem 2.3.1, if $p < p^{p,a}_+$ then θ must necessarily intersect both R^+_λ and R^-_λ, while for $p = p^{p,a}_+$ the maximal periodic orbit θ_0 does not intersect R^+_λ.

We claim that θ_0 is tangent to ℓ_+ at the point $P = (\frac{1+a}{p}, \lambda(N-1)) \in \ell_+ \cap \ell^+_2$ when $p = p^{p,a}_+$. If this was not the case, then $\Gamma = \Gamma_{p^{p,a}_+}$ would belong to the region R^-_λ for all $t \in I = [T, +\infty)$ for some $T > 0$. Let us consider the restriction of Γ to I, namely τ. Since τ is a part of a trajectory for the Laplacian operator in dimension $\widetilde{N}_+$, we may follow τ backward in time as a trajectory of the respective Laplacian-like dynamical system. However, the characterization of $p^{p,a}_+$ as the critical exponent there immediately contradicts the existence of τ. Indeed, at the critical exponent only periodic trajectories are admissible around M_0, see for instance the proof of Proposition 2.2.7.

Thus, in both cases Γ_p for $p \in \mathcal{P}$ must cross the concavity line ℓ_+ infinitely many times. ■

Remark 4.1.8 ($\Gamma_p = \Upsilon_p$)**.** The critical exponent p^*_{a+} is the unique value of p for which Γ_p and Υ_p coincide, see Proposition 2.2.5.

Proof of Theorem 3.4.1. One establishes the conclusion of Theorem 3.4.1 by combining (4.1.3), Corollary 4.1.4 and Propositions 4.1.5 and 4.1.6, together with Theorem 4.1.7. ■

4.1.3 Singular and exterior domain solutions

Here we show how the analysis of the regular trajectories performed in the previous sections almost completely determines the behavior of the other orbits of the dynamical system (2.1.6).

Let us start by considering singular solutions. When $p \leqslant p_+^{s,a}$ we saw in Proposition 4.1.6 that $p \in \mathcal{C}$. On the other hand, there is not a unique trajectory arriving at the stationary point A_0 as in Proposition 2.2.5. Indeed, for $p < p_+^{s,a}$, A_0 is a source and M_0 belongs to the fourth quadrant, see Proposition 2.2.4. The case $p = p_+^{s,a}$ is a bit more involved. The point $A_0 = M_0$ is not a hyperbolic point, and we complement its local study in what follows.

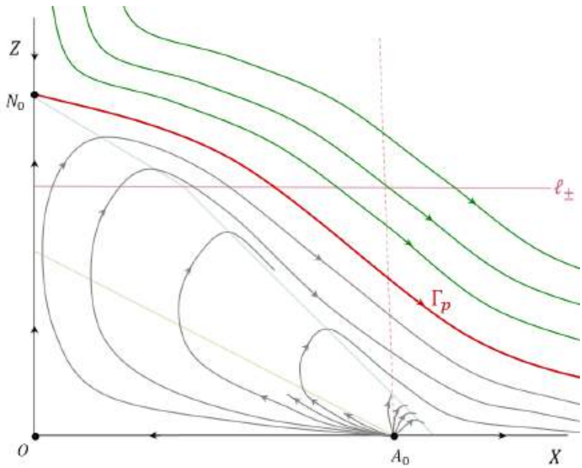

Figure 4.1: Case $p < p_\pm^{s,a}$ for $\mathcal{M}^\pm$; A_0 is a source and M_0 belongs to the fourth quadrant. Below Γ_p trajectories corresponding to infinitely many $(\tilde{N}_\pm - 2)$–blowing up solutions in a ball are shown, while above Γ_p there are orbits corresponding to solutions in an annulus.

Proposition 4.1.9. *At $p = p_+^{s,a}$, there exist infinitely many unstable orbits issued from $M_0 = A_0$ below the line ℓ_1^+. They move clockwise and blow up in finite forward time.*

Proof. At $p = p_+^{s,a}$ the eigenvalues of $A_0 = M_0$ are $(\tilde{N}_+ - 2)$ and 0. In particular, A_0 is not hyperbolic and Proposition 1.1.2 no longer applies. The linear direction corresponding to $(\tilde{N}_+ - 2)$ lies on the X axis, while the one corresponding to 0 coincides with the line ℓ_1^+. However, through the flow analysis in Proposition 2.2.2 (3) (see Figure 2.2) it is easy to conclude that M_0 has infinitely many repulsive directions between these two lines. In this case, the orbits are issued from A_0, with respective tangent lines between the X axis and the line ℓ_1^+.

To see this let us first observe that ℓ_1^+ and ℓ_2^+ intersect at A_0. Then note that $\dot{X} > 0$ in the region above ℓ_1^+. On the other hand, an orbit coming out from A_0 needs to increase its Z values, so staying below ℓ_2^+. If it started between ℓ_1^+ and ℓ_2^+, then it should initially decrease its X values, which gives a contradiction. Hence the only way to come out from A_0 is below the line ℓ_1^+.

We have already deduced in the proof of Proposition 4.1.6 that periodic orbits at $p_+^{s,a}$ are not admissible if $p_+^{s,a} < p_\Delta^a$ by Dulac's criterion (Theorem 2.3.1). However, this is true even if $p^{s,a} \geqslant p_\Delta^a$ by the flow direction, see Figure 2.2. Indeed, the region $\dot{X}, \dot{Z} < 0$ does not intersect $1Q$. By the same reason, the trajectories near $M_0 = A_0$ move clockwise, by intersecting both lines ℓ_1^+ and ℓ_2^+ exactly once.

To conclude we infer that the behavior of the flow on the lines ℓ_1^+ and ℓ_2^+ does not allow any orbit to reach $A_0 = M_0$ in forward time. Assume on the contrary that there exists a homoclinic orbit τ with $\omega(\tau) = \alpha(\tau) = A_0$. In this case τ creates a bounded invariant region D such that any orbit inside D is also homoclinic, by Poincaré–Bendixson theorem. Fix a point $Q_0 \in R_\lambda^- \cap \ell_2^+ \cap D$, and consider the unique trajectory τ_0 passing through this point at time $t = 0$. By construction, τ_0 lies entirely in the region R_λ^-. However, the proof of Theorem 2.3.1(ii), applied to the region D_0 enclosed by the trajectory τ_0, yields a contradiction with the fact that $p_+^{s,a} \neq p_+^{p,a}$. ■

It is interesting that when $p \leqslant p_+^{s,a}$ these results give a simple proof of some Liouville theorems in (Cutrì and Leoni 2000) (namely Theorem 1.2.8), concerning radial solutions. The same holds for $\mathcal{M}^-$, as we shall see in Section 4.2.

Remark 4.1.10. For all $p > p_+^{s,a}$, as already mentioned in Introduction, there exists a singular trivial solution given by $u_p = C_p r^{-\alpha}$, C_p as in (3.1.8). This corresponds to the stationary trajectory $\Sigma_p \equiv M_0$. Moreover, any periodic orbit of the dynamical system (2.1.6) which intersects the concavity line ℓ_+ twice corresponds to a classical pseudo–blowing up solution for the problem (P_+). Instead, in the case $p = p_+^{p,a}$, periodic orbits around the center configuration of M_0 lying entirely in the region R_λ^- (see Proposition 2.2.7) cannot correspond to C^2 solutions, since they oscillate between the two functions $c_1 r^{-\alpha}$ and $c_2 r^{-\alpha}$ indefinitely for some $0 < c_1 < c_2$, without never changing convexity. We stress that these two types of solutions originated from periodic orbits do exist in the case of the Laplacian in dimensions N and $\widetilde{N}_+$, for the critical exponents p_Δ^a and $p_+^{p,a}$ respectively, see Theorem 6.1(iii) in (Bidaut-Véron 1989).

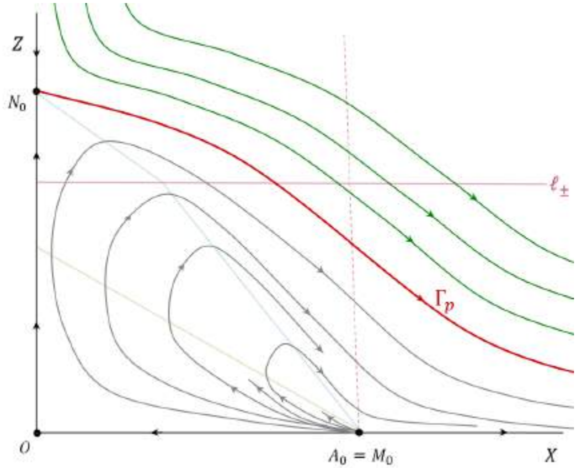

Figure 4.2: Case $p = p_{\pm}^{s,a}$: $p \in \mathcal{C}$ and $A_0 = M_0$ has infinitely many unstable directions. Below Γ_p are the orbits corresponding to infinitely many $(\widetilde{N}_{\pm} - 2)$–blowing up solutions in a ball.

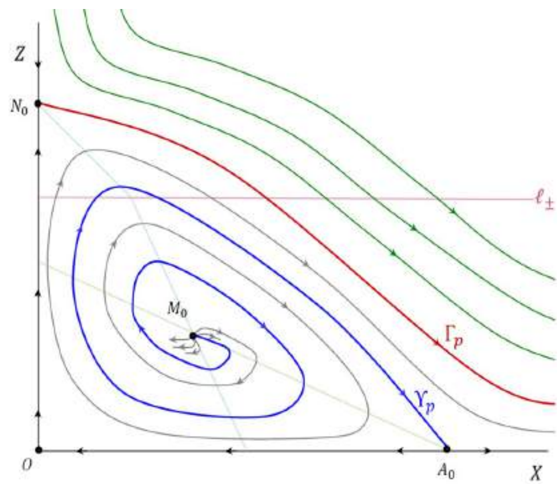

Figure 4.3: Case $p > p_{\pm}^{s,a}$, $p \in \mathcal{C}$ without periodic orbits.

Lemma 4.1.11. *If $p > p^*_{a+}$ then Υ_p (see Proposition 2.2.5) blows up in finite backward time. In particular, Υ_p does not correspond to a singular solution for*

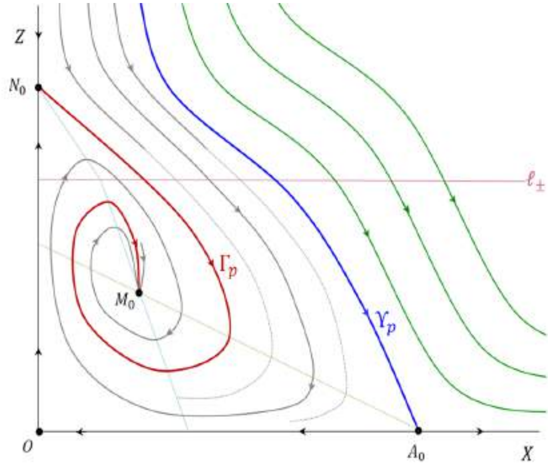

Figure 4.4: Case $p > p_{\pm}^{s,a}$, $p \in \mathcal{S}$ without periodic orbits.

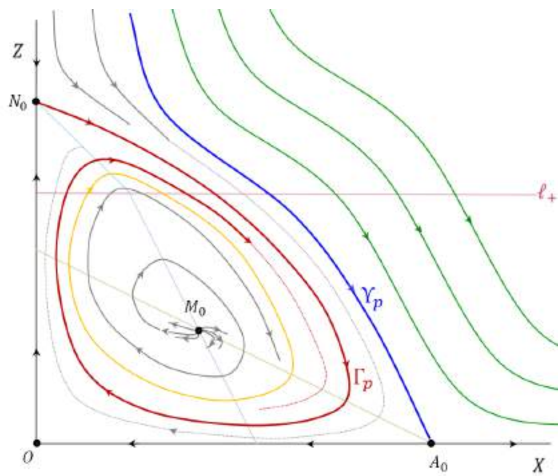

Figure 4.5: Case $p \in \mathcal{P}$ for $\mathcal{M}^+$, $p \in (p_{a+}^*, p_+^{p,a})$. Here M_0 is a source. The orbits inside the displayed periodic orbit correspond to infinitely many α–blowing up solutions with pseudo-slow decay at $+\infty$. All trajectories above Γ_p correspond to solutions either in the exterior of a ball or in an annulus.

any $p > p_{a+}^*$.

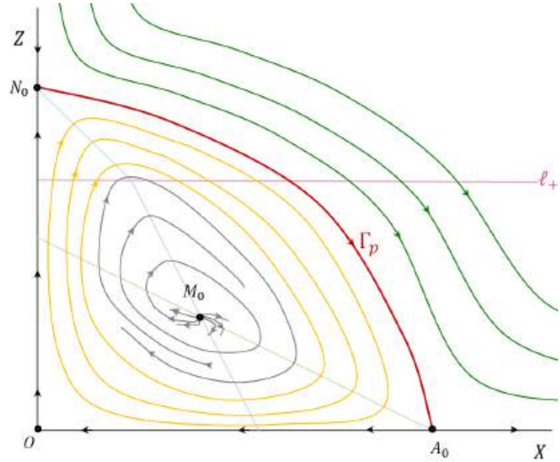

Figure 4.6: Case $p = p^*_{a+} \in \mathcal{F}$. Here M_0 is a source and $\Gamma_p = \Upsilon_p$, see Remark 4.1.8. There are infinitely many α–blowing up solutions with pseudo-slow decay (inside the minimal periodic orbit), and infinitely many pseudo–blowing up solutions with pseudo-slow decay (periodic orbits). Moreover, there are no solutions in the exterior of a ball.

Proof. If $p \in (p^*_{a+}, p^{p,a}_+]$ we have $p \in \mathcal{P}$, and there exists a maximal periodic orbit θ_p around M_0 such that $\omega(\Gamma_p) = \theta_p$. If $\alpha(\Upsilon_p) = \theta_p$, then Γ_p and Υ_p would cross somewhere. Indeed, this comes from the fact that the stable linear tangent direction at A_0 is above the line ℓ_2^+ (see Figure 2.1), and the vector field on ℓ_2^+ points down for $X > \alpha$, by Proposition 2.2.2 (3). Obviously crossings are not admissible by uniqueness of the ODE problem. On the other side, for $p > p^{p,a}_+$, M_0 is a sink and periodic orbits are not allowed by Theorem 2.3.1. Thus, in both cases, using Proposition 2.4.1, we get that Υ_p blows up backward in finite time. ■

Proof of Theorem 3.4.5. (i) First we recall that for $p \in (1, p^{s,a}_+]$ there are no periodic orbits, by Theorem 2.3.1 and the proof of Proposition 4.1.9. Moreover, we know by Corollary 4.1.4 and Theorem 4.1.7 that the regular trajectory Γ_p blows up forward in finite time. Hence, Γ_p together with the line $L = \{(X, Z) : X = \tilde{N}_+ - 2\}$ and the X, Z axes, create a bounded region D from which an orbit of (2.1.6) may only leave through L.

Thus, if $p < p^{s,a}_+$, any trajectory issued from A_0 (which is a source by Proposition 2.2.4) crosses the line L and then blows up in finite time. If $p = p_+^{s,a}$ the same holds, by Proposition 4.1.9. In both cases there are infinitely many such trajectories which correspond to singular solutions in an interval $(0, R)$, $R > 0$, see Proposition 2.4.3. They are $(\widetilde{N}_+ - 2)$–blowing up, cf. (3.1.7) with the ω-limit exchanged by α-limit. Therefore there cannot be singular solutions in $\mathbb{R}^N \setminus \{0\}$ for this range of p.

(ii)-(iii) For $p \in (p_+^{s,a}, p_{a+}^*)$, as in (i), the trajectory Γ_p, the line L and the X, Z axes determine a bounded region D from which an orbit may only leave through L. Recall that for these values of p, A_0 is a saddle point and M_0 is a source, see Proposition 2.2.4. Thus, the unique orbit Υ_p arriving at A_0 (see Proposition 2.2.5) can either converge to M_0 or to a periodic orbit around M_0, backward in time. If $p \leqslant p_\Delta^a$ there are no periodic orbits (Theorem 2.3.1), so Υ_p corresponds to α–blowing up solution of (P_+); in particular this is the case for each $p \in (p_+^{s,a}, p_\Delta^a]$ if $p_+^{s,a} \leqslant p_\Delta^a$.

If in turn $p \in (p_\Delta^a, p_{a+}^*)$ there could be periodic orbits around M_0, so that Υ_p corresponds to either a pseudo–blowing up or a α–blowing up solution of (P_+).

All the other orbits coming out from M_0 or from a periodic orbit θ around M_0 (whenever such θ exists) must necessarily leave D by crossing the line L in forward time, and therefore blow up in finite time. This gives infinitely many singular solutions of (P_+); they are either α–blowing up or pseudo–blowing up in intervals $(0, R)$, $R > 0$.

If θ exists, we have in addition infinitely many orbits τ issued from M_0 and converging to a minimal periodic orbit (which is θ if the system has only one limit cycle). Each τ crosses infinitely many times the line ℓ_+ when $t \to +\infty$, and so corresponds to a α–blowing up solution of (P_+) with pseudo-slow decay at $+\infty$ as in (3.1.9). On the other hand, a periodic orbit itself in this range of p's crosses ℓ_+ twice, so corresponds to a pseudo–blowing up solution to (P_+), see Remark 4.1.10; they are pseudo-slow decaying and change concavity infinitely many times both as $r \to 0$ and $r \to +\infty$.

(iv) When $p = p_{a+}^*$, the regular trajectory $\Gamma_{p_{a+}^*}$ together with the X and Z axes delimit an invariant set D containing M_0. Since M_0 is a source, we have already seen that there exists a periodic orbit around M_0; say θ is the minimal one. We then infer that there exist infinitely many periodic orbits in the region $D \setminus \operatorname{int}(\theta)$, at least in a neighborhood of ∂D, see Figure 4.6. Indeed, the existence of a maximal periodic orbit θ_0 inside D would create a bounded region $D \setminus \operatorname{int}(\theta_0)$ in which the orbits issued from θ_0 could not go anywhere, thus violating Poincaré–Bendixson theorem.

(v) When $p \in (p^*_{a+}, p^{p,a}_+)$, we have that M_0 is a source and there exists a minimal periodic orbit θ around M_0. In this case, θ crosses the line ℓ_+ twice since $p < p^{p,a}_+$, see Theorem 2.3.1. Thus, all trajectories issued from M_0 converge to θ in forward time. These and the periodic orbits give us singular solutions as in the last part of the proof of (iii). Finally, note that no singular solutions converge to A_0 by Lemma 4.1.11, so the assertion holds.

(vi) By Propositions 2.2.4 and 2.2.7 and Theorem 2.3.1 we have that for $p = p^{p,a}_+$ the stationary point M_0 is a center while for $p > p^{p,a}_+$ M_0 is a sink without periodic orbits. These and the fact that A_0 is a saddle point, whose unstable manifold is the X axis, imply that no singular nontrivial solutions are admissible. ■

Finally we consider the case of exterior domain solutions, proving Theorem 3.4.7 for $\mathcal{M}^+$. The proof for $\mathcal{M}^-$ turns out to be the same.

In Section 3.3 we have observed that a solution u of (3.3.1) necessarily satisfies the monotonicity in Lemma 3.3.8. Hence the corresponding trajectory Ξ_p blows up backward in finite time, see Proposition 3.3.14. Thus, to prove Theorem 3.4.7 it is enough to show that for $p \in (1, p^*_{a+}]$ there are no orbits of the dynamical system (2.1.6) defined in $(T, +\infty)$ for some $T > 0$ with this kind of blow-up behavior.

Proof of Theorem 3.4.7. By the definition and properties of the critical exponent in Sections 4.1.1 and 4.1.2, we know that $p \in \mathcal{C}$ for $p \in (1, p^*_{a+})$, while $p^*_{a+} \in \mathcal{F}$. In the first case the regular trajectory Γ_p together with the X and Z axes and the line $L = \{(X, Z) : X = \tilde{N}_+ - 2\}$ bound a region D from which any trajectory can only escape in forward time through L. In the second case Γ_p and the X and Z axes enclose a bounded invariant region D. In both cases the closure of D contains the points M_0 (for $p \geqslant p^{s,a}_+$) and A_0.

By contradiction assume that a radial solution of (3.4.6) exists. Then, by Proposition 3.3.14 the corresponding trajectory Ξ_p is defined in an interval $(t_\delta, +\infty]$ for some $t_\delta > -\infty$, and blows at t_δ satisfying (3.3.14). Since it does not blow up in forward time, by Proposition 2.4.1 (see (2.4.1)) and Poincaré–Bendixson theorem the ω-limit $\omega(\Xi_p)$ is either M_0 (if $p > p^{s,a}_+$), or A_0, or a periodic orbit around M_0. In any case Ξ_p should cross Γ_p which is not possible. ■

Remark 4.1.12. It is proved in (Galise, Iacopetti, and Leoni 2020), when $a = 0$, that for every $p \in (p^*_{a+}, +\infty)$ both a fast decaying and infinitely many slow or pseudo-slow decaying solutions of (3.4.6) exist. In terms of our quadratic system (2.1.6) this could be proved using Lemma 4.1.11 for the fast decaying solutions, or studying the trajectories arriving at M_0 or at a periodic orbit for the slow or pseudo-slow decaying solutions. However, since the proof of (Galise, Iacopetti, and Leoni 2020) easily extends to the case $a \neq 0$, we prefer to omit the details.

Note that with the analysis of the trajectories blowing up backward in finite time one can only get the existence of a solution u satisfying the conclusion of Lemma 3.3.8 at some radius $\mu = \tau_\delta > 0$. Then the solution should be continued (in $3Q$) to reach a positive radius $\rho_0 > \mathfrak{a}$ where $u(\rho_0) = 0$, so to verify the Dirichlet problem in the exterior of a ball. This is possible by using a shooting argument from μ, as done for instance in (Galise, Iacopetti, and Leoni 2020, proof of Theorem 6.1).

4.2 The $\mathcal{M}^-$ case

In this section we analyze the complementary case for the operator $\mathcal{M}^-$. Recall its respective dimension-like parameter from (1.2.5) satisfying $\tilde{N}_- \geqslant N$. The main difference with the case of $\mathcal{M}^+$ is the reverse ordering of the exponents p^a_Δ and $p^{p,a}_-$ from (1.2.6), with $p^{s,a}_- \leqslant p^{p,a}_- \leqslant p^a_\Delta$. Also, if $\lambda < \Lambda$ then the stationary point M_0 is a sink in the interval $(p^{p,a}_-, p^a_\Delta)$, see Proposition 2.2.4 (4).

We start by pointing out that all properties stated for $\mathcal{M}^+$ in Section 4.1.1 also hold for $\mathcal{M}^-$. In particular, one gets that the set $\mathcal{F}$ possesses at most one point, which splits the interval $(1, +\infty)$ into two components $\mathcal{C}$ and $\mathcal{P} \cup \mathcal{S}$ accordingly to Corollary 4.1.4. Moreover, each of these components is nonempty, since one verifies, as in Propositions 4.1.5 and 4.1.6, the following result.

Proposition 4.2.1. *If $p > p^a_\Delta$ then $p \in \mathcal{S}$, and for $p < p^{p,a}_-$ it holds that $p \in \mathcal{C}$.*

This allows us to define, as in Section 4.1, the critical exponent p^*_{a-} as follows

$$p^*_{a-} = \sup \mathcal{C}.$$

Then, by Proposition 4.2.1,

$$p^*_{a-} \in [\, p^{p,a}_-,\ p^a_\Delta \,]$$

and, as for $\mathcal{M}^+$, we call it the critical exponent for $\mathcal{M}^-$. Next we show that $p^*_{a-} \in \mathcal{F}$ and p^*_{a-} is in the interior of the previous interval.

Theorem 4.2.2. *The critical exponent p^*_{a-} belongs to $\mathcal{F}$. Thus it is the only exponent in the equation (P_-) for which there exists a unique, up to scaling, fast decaying solution.*

Moreover, if $\lambda < \Lambda$, then (3.4.5) holds and there exists $\varepsilon > 0$ such that

$$(p^a_\Delta - \varepsilon, +\infty) \subset \mathcal{S}.$$

Proof. Obviously $p^*_{a-} \notin \mathcal{C}$ because $\mathcal{C}$ is open, see Remark 3.1.9. Moreover, p^*_{a-} cannot belong to $\mathcal{P}$; otherwise $\Gamma_{p^*_{a-}}$ should cross the line ℓ_1^- by Lemma 4.1.1(ii), while Γ_p for $p \in \mathcal{C}$ never does it.

Finally we show that $p^*_{a-} \notin \mathcal{S}$. Indeed, if this was the case then $p^*_{a-} > p^{p,a}_-$ because M_0 is a center at $p^{p,a}_-$, see Proposition 2.2.7. Hence $M_0 = M_0(p)$ is a sink for every p in a neighborhood $I_\varepsilon = (p^*_{a-} - \varepsilon, p^*_{a-} + \varepsilon)$ for some $\varepsilon > 0$, by Proposition 2.2.4 (4). Then there exists a maximal ball B_{η_p} centered at $M_0(p)$ with the property that any trajectory τ_p entering in B_{η_p} satisfies $\omega(\tau_p) = M_0(p)$, see (Hale and Koçak 1991). Since we are assuming that $p^*_{a-} \in \mathcal{S}$ then $\omega(\Gamma_{p^*_{a-}}) = M_0(p^*_{a-})$. By the continuity of the dynamical system with respect to the parameter p, also $\omega(\Gamma_p) = M_0(p)$ for $p \in I_\varepsilon$ (up to diminishing ε). But this contradicts the definition of p^*_{a-}, since Γ_q blows up in finite time when $q \in \mathcal{C}$.

Hence, $p^*_{a-} \in \mathcal{F}$. The proof that p^*_{a-} cannot be $p^{p,a}_-$ nor p^a_Δ is the same as the one for $\mathcal{M}^+$, see Theorem 4.1.7. It relies on Theorem 2.3.1 which states, in particular, that there are no periodic orbits of (2.1.8) for p^a_Δ. This also proves that the regular trajectory $\Gamma_{p^a_\Delta}$ converges to $M_0 = M_0(p^a_\Delta)$, so that $p^a_\Delta \in \mathcal{S}$. Next, a continuity argument as in the first part of this proof shows that $p \in (p^a_\Delta - \varepsilon, p^a_\Delta)$ also belongs to $\mathcal{S}$ for sufficiently small $\varepsilon > 0$. Consequently, for such p's it does not exist a periodic orbit around M_0, and the proof is complete. ■

Proof of Theorem 3.4.2. All previous results obtained for $\mathcal{M}^-$ prove the theorem. In particular, the statements (iii)–(iv) follow from Theorem 4.2.2. ■

We finish the section with the proof of Theorem 3.4.6 about singular solutions.

Proof of Theorem 3.4.6. It is enough to prove (iii) and (v), since the proof of the other items are the same as for Theorem 3.4.5. Let us analyze the whole interval $p \in (p^{p,a}_-, p^a_\Delta]$.

Recall that M_0 is a sink whenever $p > p^{p,a}_-$. Therefore, there is no trajectory coming out from M_0 in the range $p \in (p^{p,a}_-, p^a_\Delta]$. In particular, $\alpha(\Upsilon_p)$ is never M_0 in this range of p.

For $p \in (p^{p,a}_-, p^*_{a-})$ we have $p \in \mathcal{C}$. In this case the regular trajectory Γ_p and the line $L = \{(X, Z) : X = \widetilde{N}_- - 2\}$, together with the X and Z axes, create a bounded region from which any orbit may only leave forward in time through L; recall that the flow is going out on L, see Figure 2.1. Therefore, Poincaré–Bendixson theorem implies the existence of a periodic orbit θ_p around M_0 such that $\alpha(\Upsilon_p) = \theta_p$. This immediately determines four types of nontrivial positive pseudo–blowing up solutions of (3.4.1)–(3.4.2) (in the case of $\mathcal{M}^-$):

(1) a fast decaying solution corresponding to the trajectory Υ_p;

(2) solutions with slow decay, whose corresponding orbits lie inside a minimal periodic orbit θ_0 around M_0; here θ_0 crosses ℓ_- twice due to Theorem 2.3.1;

(3) solutions of the Dirichlet problem in $B_R \setminus \{0\}$, such that the corresponding orbits are issued from θ_p and blow up in finite forward time;

(4) pseudo-slow decaying solutions, which correspond to the periodic orbits.

All of these singular solutions change concavity infinitely many times in a neighborhood of $r = 0$. Further, there are infinitely many solutions of types (2) and (3); see Figure 4.7. Thus, (iii) holds.

To prove (v), let us recall that at p^a_Δ no periodic orbits are admissible by Theorem 2.3.1. Also, by Theorem 4.2.2 there exists $\varepsilon > 0$ such that $p \in \mathcal{S}$ for all $p \in (p_\Delta - \varepsilon, +\infty)$. Now, arguing as in Lemma 4.1.11 one sees that Υ_p blows up in finite backward time for $p > p^*_{a-}$, so (v) is proved. ■

Remark 4.2.3. In the case of $\mathcal{M}^-$ the existence of singular solutions in the range $(p^*_{a-}, p^a_\Delta - \varepsilon)$ is not guaranteed, though solutions as in the cases (2)–(4) in the proof above are admissible.

Concerning the exterior domain solutions, we have already observed in Section 4.1.3 that the proof of Theorem 3.4.7 is the same for both operators $\mathcal{M}^\pm$.

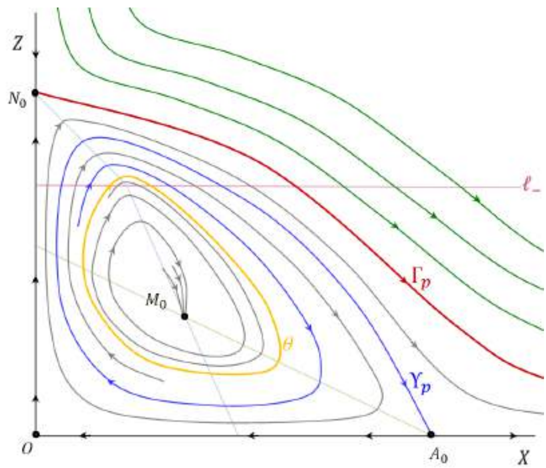

Figure 4.7: Case $p \in (p_{-}^{p,a}, p_{a-}^{*})$ for $\mathcal{M}^-$, M_0 is a sink. There are infinitely many pseudo–blowing up solutions: a unique fast decaying (given via the orbit Υ_p); a pseudo-slow decaying (periodic orbit θ); infinitely many in a ball (outside θ); infinitely many slow decaying (inside θ).

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