A friendly invitation to Fourier analysis on polytopes

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Preface

Figure 1: Joseph Fourier

What is a Fourier transform? Why is it so useful? How can we apply Fourier transforms and Fourier series - which were originally used by Fourier to study heat diffusion - in order to better understand topics in discrete and combinatorial geometry, number theory, and sampling theory?

To begin, there are some useful analogies: imagine that you are drinking a milkshake (lactose free), and you want to know the ingredients of your tasty drink. You would need to filter out the shake into some of its most basic components. This decomposition into its basic ingredients may be thought of as a sort of "Fourier transform of the milkshake". Once we understand each of the ingredients, we will also be able to restructure these ingredients in new ways, to form many other types of tasty goodies. To move the analogy back into mathematical language, the milkshake represents a function, and each of its basic ingredients represents for us the basis of sines and cosines; we may also think of a basic ingredient more compactly as a complex exponential $e^{2\pi inx}$, for some $n \in \mathbb{Z}$. Composing these basic ingredients together in a new way represents a Fourier series.

Mathematically, one of the most basic kinds of milkshakes is the indicator function of the unit interval, and to break it down into its basic components, mathematicians, Engineers, Computer scientists, and Physicists have used the sinc function (since the 1800's):

$$\operatorname{sinc}(z) := \frac{\sin(\pi z)}{\pi z}$$

with great success, because it happens to be the Fourier transform of the unit interval $[-\frac{1}{2}, \frac{1}{2}]$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi izx} dx = \operatorname{sinc}(z),$$

as we will compute shortly in identity (2.5). Somewhat surprisingly, comparatively little energy has been given to some of its higher dimensional extensions, namely those extensions that arise naturally as Fourier transforms of polytopes.

One motivation for this book is to better understand how this 1-dimensional function – which has proved to be extremely powerful in applications – extends to higher dimensions. Namely, we will build various mathematical structures that are motivated by the question:

What is the Fourier transform of a polytope?

Of course, we will ask "how can we apply it"? An alternate title for this book might have been:

We're taking Poisson summation and Fourier transforms of polytopes for a very long ride....

Historically, sinc functions were used by Shannon (as well as Hardy, Kotelnikov, Nyquist, and Whittaker) when he published his seminal work on sampling theory and information theory.

In the first part of this book, we will learn how to use the technology of Fourier transforms of polytopes in order to build the (Ehrhart) theory of integer point enumeration in polytopes, to prove some of Minkowski's theorems in the geometry of numbers, and to understand when a polytope tiles Euclidean space by translations.

In the second portion of this book, we give some applications to active research areas which are sometimes considered more applied, including the sphere packing problem, and the angle polynomial of a polytope.

There are also current research developments of the material developed here, to the learning of deep neural networks. In many applied scientific areas, in particular radio astronomy, computational tomography, and magnetic resonance imaging, a frequent theme is the reconstruction of a function from knowledge of its Fourier transform. Somewhat surprisingly, in various applications we only require very partial/sparse knowledge of its Fourier transform in order to reconstruct the required function, which may represent an image or a signal.

There is a rapidly increasing amount of research focused in these directions in recent years, and it is therefore time to put many of these new findings in one place, making them accessible to a general scientific reader. The fact that the sinc function is indeed the Fourier transform of the 1-dimensional line segment $[-\frac{1}{2}, \frac{1}{2}]$, which is a 1-dimensional polytope, gives us a first hint that there is a deeper link between the geometry of a polytope and the analysis of its Fourier transform.

Indeed one reason that sampling and information theory, as initiated by Claude Shannon, works so well is precisely because the Fourier transform of the unit interval has this nice form, and even more so because of the existence of the Poisson summation formula.

The approach we take here is to gain insight into how the Fourier transform of a polytope can be used to solve various specific problems in discrete geometry, combinatorics, optimization, and approximation theory:

(a) Analyze tilings of Euclidean space by translations of a polytope

(b) Give wonderful formulas for volumes of polytopes

(c) Compute discrete volumes of polytopes, which are combinatorial approximations to the continuous volume

(d) Introduce the geometry of numbers, via Poisson summation

(e) Optimize sphere packings, and get bounds on their optimal densities

Let's see at least one direction that quickly motivates the study of Fourier transforms. In particular, we often begin with simple-sounding problems that arise naturally in combinatorial enumeration, discrete and computational geometry, and number theory.

Throughout, an **integer point** is any vector $v := (v_1, \ldots, v_d) \in \mathbb{R}^d$, all of whose coordinates v_j are integers. In other words, v belongs to the integer lattice $\mathbb{Z}^d$. A **rational point** is a point m whose coordinates are rational numbers, in other words $m \in \mathbb{Q}^d$. We define the **Fourier transform** of a function $f(x)$:

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \tag{1}$$

defined for all $\xi \in \mathbb{R}^d$ for which the latter integral converges, and where we use the standard inner product $\langle a, b \rangle := a_1 b_1 + \cdots + a_d b_d$. We will also use the notation $\mathcal{F}(f)$ for the Fourier transform of f, which is useful in some typographical contexts, for example when considering $\mathcal{F}^{-1}(f)$.

Now we can introduce one of the main objects of study in this book, the **Fourier transform of a polytope** $\mathcal{P}$, defined by:

$$\hat{1}_{\mathcal{P}}(\xi) := \int_{\mathbb{R}^d} 1_{\mathcal{P}}(x) e^{-2\pi i \langle \xi, x \rangle} dx = \int_{\mathcal{P}} e^{-2\pi i \langle \xi, x \rangle} dx, \tag{2}$$

where the function $1_{\mathcal{P}}(x)$ is the **indicator function** of $\mathcal{P}$, defined by

$$1_{\mathcal{P}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{P} \\ 0 & \text{if not.} \end{cases}$$

Thus, the words "Fourier transform of a polytope $\mathcal{P}$" will always mean the Fourier transform of the indicator function of $\mathcal{P}$.

The **Poisson summation formula**, named after Siméon Denis Poisson, tells us that for any "sufficiently nice" function $f : \mathbb{R}^d \to \mathbb{C}$ we have:

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$$

In particular, if we were to naively set $f(n) := 1_{\mathcal{P}}(n)$, the indicator function of a polytope $\mathcal{P}$, then we would get:

$$\sum_{n \in \mathbb{Z}^d} 1_{\mathcal{P}}(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\mathcal{P}}(\xi), \tag{3}$$

which is technically false in general due to the fact that the indicator function $1_{\mathcal{P}}$ is a discontinuous function on $\mathbb{R}^d$.

However, this technically false statement is very useful! We make this claim because it helps us build intuition for the more rigorous statements that are true, and which we study in later chapters. For applications to discrete geometry, we are interested in the number of integer points in a closed convex polytope $\mathcal{P}$, namely $|\mathcal{P} \cap \mathbb{Z}^d|$. The combinatorial-geometric quantity $|\mathcal{P} \cap \mathbb{Z}^d|$ may be regarded as a **discrete volume** for $\mathcal{P}$. From the definition of the indicator function of a polytope, the left-hand-side of (3) counts the number of integer points in $\mathcal{P}$, namely we have by definition

$$\sum_{n \in \mathbb{Z}^d} 1_{\mathcal{P}}(n) = |\mathcal{P} \cap \mathbb{Z}^d|. \tag{4}$$

On the other hand, the right-hand-side of (3) allows us to compute this discrete volume of $\mathcal{P}$ in a new way. This is great, because it opens a wonderful window of computation for us in the following sense:

$$|\mathcal{P} \cap \mathbb{Z}^d| = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\mathcal{P}}(\xi). \tag{5}$$

We notice that for the $\xi = 0$ term, we have

$$\hat{1}_{\mathcal{P}}(0) := \int_{\mathbb{R}^d} 1_{\mathcal{P}}(x) e^{-2\pi i \langle 0, x \rangle} dx = \int_{\mathcal{P}} dx = \text{vol}(\mathcal{P}), \tag{6}$$

and therefore the discrepancy between the continuous volume of $\mathcal{P}$ and the discrete volume of $\mathcal{P}$ is

$$|\mathcal{P} \cap \mathbb{Z}^d| - \text{vol}(\mathcal{P}) = \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_{\mathcal{P}}(\xi), \tag{7}$$

showing us very quickly that indeed $|\mathcal{P} \cap \mathbb{Z}^d|$ is a discrete approximation to the classical Lebesgue volume $\text{vol}(\mathcal{P})$, and pointing us to the task of finding ways to evaluate the transform $\hat{1}_P(\xi)$. From the trivial but often very useful identity

$$\hat{1}_{\mathcal{P}}(0) = \text{vol}(\mathcal{P}),$$

we see another important motivation for this book: the Fourier transform of a polytope is a very **natural extension of volume**. Computing the volume of a polytope $\mathcal{P}$ captures a bit of information about $\mathcal{P}$, but we also lose a lot of information.

On the other hand, computing the Fourier transform of a polytope $\hat{1}_{\mathcal{P}}(\xi)$ uniquely determines $\mathcal{P}$, so we do not lose any information at all. Another way of saying this is that the Fourier transform of a polytope is a **complete invariant**. In other words, it is a fact of life that

$$\hat{1}_{\mathcal{P}}(\xi) = \hat{1}_{\mathcal{Q}}(\xi) \text{ for all } \xi \in \mathbb{R}^d \iff \mathcal{P} = \mathcal{Q}.$$

Combinatorially, there are brilliant identities (notably the Brion identities) that emerge between the Fourier and Laplace transforms of a given polytope, and its facets and vertex tangent cones.

In Statistics, the moment generating function of any probability distribution is given by a Fourier transform of the indicator function of the distribution, hence Fourier transforms arise very naturally in Statistical applications. At this point, a natural glaring question naturally comes up:

$$\text{How do we } \textbf{compute} \text{ the Fourier transform of a polytope } \hat{1}_{P}(\xi)? \tag{8}$$

And how do we use such computations to help us understand the important "error" term

$$\sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_{\mathcal{P}}(\xi)$$

that came up naturally in (7) above?

There are many applications of the theory that we will build up. Often, we find it instructive to sometimes give an informal proof first, because it brings the intuitive ideas to the foreground, allowing the reader to gain an overview of the steps. Then, later on, we revisit the same intuitive proof again, making it rigorous.

The Poisson summation formula is one of our main stars, and has a relatively easy proof. But it constitutes a very first step for many of our explorations. It may even be said that, from this perspective, the Poisson summation formula is to combinatorial analysis as a microscope is to our vision. It enhances our ability to see mathematical facts, and often in a surprisingly simple way. So it's a question of what we do with these tools - where do we point them?

A word about **prerequisites** for this book: **Linear Algebra** is always very useful! A couple of calculus courses would be helpful as well, with perhaps a touch of real analysis. We will assume some familiarity with the basic definitions of polytopes and their faces, although we also include some of these requisite definitions as well. There are many excellent texts that introduce the student to the classical language of polytopes, in particular the two classic books Ziegler (1995), and Grünbaum (2003b).

For an easy introduction to the interactions between polytopes and lattice point enumeration, the reader is invited to consult "Computing the continuous discretely: integer point enumeration in polytopes", by M. Beck and Robins (2015).

The level here is aimed at advanced undergraduates and beginning graduate students in various fields, and in particular Mathematics, Computer Science, Electrical Engineering, and Physics.

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June 2021

1 Tiling a rectangle with little rectangles

Ripping up carpet is easy – *tiling* is the issue.
– Douglas Wilson

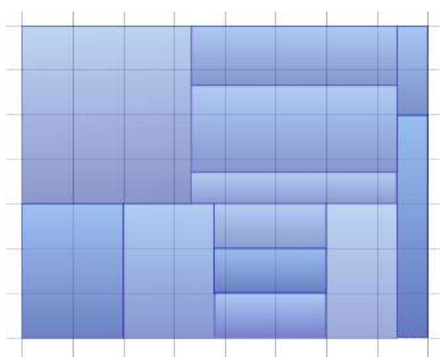


Figure 1.1: A rectangle tiled by nice rectangles

1.1 Intuition

To warm up, we begin with a simple tiling problem in the plane. A rectangle will be called **nice** if at least one of its sides is an integer. We prove a classical fact about tiling a rectangle with nice rectangles, and the idea is to focus on the **method** of the simple proof.

This proof brings to the foreground an important idea: by simply taking a Fourier transform of a body B, we immediately get interesting geometric consequences for B. In particular, we will see throughout this book various ways in which the Fourier transform of a geometric body is a natural extension of its volume, sometimes in a continuous way, and sometimes in a discrete way. So in order to study relationships between volumes of bodies, it is very natural and useful to play with their Fourier transforms.

1.2 Nice rectangles

The tilings that we concern ourselves with, in this small chapter, are composed of smaller rectangles, all of which have their sides parallel to the axes, and all of which are nice.

Theorem 1.1. *Suppose we tile a fixed rectangle $\mathcal{R}$ with smaller, nice rectangles. Then $\mathcal{R}$ is a nice rectangle.*

There are at least 14 different known proofs Wagon (1987) of Theorem 1.1. Here we give a proof that uses very basic Fourier tools, from first principles, motivating the chapters that follow.

Proof. Suppose that the rectangle $\mathcal{R}$ is tiled with smaller rectangles $\mathcal{R}_1, \ldots, \mathcal{R}_N$, as in Figure 1.1. Using inclusion–exclusion with respect to the lower-dimensional components of the rectangles, we have

$$1_{\mathcal{R}}(x) = \sum_{k=1}^{N} 1_{\mathcal{R}_k}(x) \tag{1.1}$$

$$+ \sum (\pm \text{ indicator functions of lower-dimensional polytopes}), \tag{1.2}$$

where the notation $1_S(x)$ always means we are using indicator functions. To ease the reader into the computations, we recall that the Fourier transform of the indicator function of any rectangle $R := [a,b] \times [c,d]$ is, by definition:

$$\hat{1}_{\mathcal{R}}(\xi) := \int_{\mathbb{R}^2} 1_{\mathcal{R}}(x) e^{-2\pi i \langle \xi, x\rangle} dx = \int_a^b \int_c^d e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)} dx_1 dx_2. \quad (1.3)$$

Now we may formally take the Fourier transform of both sides of (1.1). In other words we simply multiply both sides of (1.1) by an exponential function and then integrate both sides over $\mathbb{R}^2$:

$$\hat{1}_{\mathcal{R}}(\xi) = \sum_{k=1}^{N} \hat{1}_{\mathcal{R}_k}(\xi). \quad (1.4)$$

In (1.4), we have used the fact that a 2-dimensional integral over a 1-dimensional line segment always vanishes, due to the fact that a line segment has measure 0 relative to the 2-dimensional measure of the 2-dimensional transform. Let's compute one of these integrals, over a generic rectangle $\mathcal{R}_k := [a_1, a_2] \times [b_1, b_2]$:

$$\hat{1}_{\mathcal{R}_k}(\xi) := \int_{\mathbb{R}^2} 1_{\mathcal{R}_k}(x) e^{-2\pi i \langle x, \xi\rangle} dx = \int_{\mathcal{R}_k} e^{-2\pi i \langle x, \xi\rangle} dx \quad (1.5)$$

$$= \int_{b_1}^{b_2} \int_{a_1}^{a_2} e^{-2\pi i \langle x, \xi\rangle} dx \quad (1.6)$$

$$= \int_{a_1}^{a_2} e^{-2\pi i \xi_1 x_1} dx_1 \int_{b_1}^{b_2} e^{-2\pi i \xi_2 x_2} dx_2 \quad (1.7)$$

$$= \frac{e^{-2\pi i \xi_1 a_2} - e^{-2\pi i \xi_1 a_1}}{-2\pi i \xi_1} \cdot \frac{e^{-2\pi i \xi_2 b_2} - e^{-2\pi i \xi_2 b_1}}{-2\pi i \xi_2} \quad (1.8)$$

$$= \frac{1}{(-2\pi i)^2} \frac{e^{-2\pi i(\xi_1 a_1 + \xi_2 b_1)}}{\xi_1 \xi_2} (e^{-2\pi i \xi_1 (a_2 - a_1)} - 1)(e^{-2\pi i \xi_2 (b_2 - b_1)} - 1), \quad (1.9)$$

valid for all $(\xi_1, \xi_2) \in \mathbb{R}^2$ except for the union of the two lines $\xi_1 = 0$ and $\xi_2 = 0$. Considering the latter formula for the Fourier transform of a rectangle, we make the following leap of faith:

Claim. Suppose that $\mathcal{R}$ is a rectangle whose sides are parallel to the axes. Then $\mathcal{R}$ is a nice rectangle if and only if

$$\hat{1}_{\mathcal{R}}(\xi) = 0, \text{ for all } \xi \in \mathbb{Z}^2 - \{0\}. \quad (1.10)$$

To prove the claim, we consider the last equality (1.9). We see that $\hat{1}_{\mathcal{R}_k}(\xi) = 0$ if and only if

$$(e^{-2\pi i \xi_1 (a_2 - a_1)} - 1)(e^{-2\pi i \xi_2 (b_2 - b_1)} - 1) = 0, \quad (1.11)$$

which is equivalent to having either

$$e^{-2\pi i \xi_1 (a_2 - a_1)} = 1, \text{ or } e^{-2\pi i \xi_2 (b_2 - b_1)} = 1.$$

But we know that due to Euler, $e^{2\pi i \theta} = 1$ if and only if $\theta \in \mathbb{Z}$ (Exercise 1.1), so we have

$$\hat{1}_{\mathcal{R}}(\xi) = 0 \iff \xi_1(a_2 - a_1) \in \mathbb{Z} \text{ or } \xi_2(b_2 - b_1) \in \mathbb{Z}. \tag{1.12}$$

Now, if $\mathcal{R}$ is a nice rectangle, then one of its sides is an integer, say $a_1 - a_2 \in \mathbb{Z}$ without loss of generality. Therefore $\xi_1(a_2 - a_1) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^2$, and by (1.12), we see that $\hat{1}_{\mathcal{R}}(\xi) = 0$ for all $\xi \in \mathbb{Z}^2$. Conversely, if we assume that $\hat{1}_{\mathcal{R}}(\xi) = 0$ for all $\xi \in \mathbb{Z}^2$, then in particular $\hat{1}_{\mathcal{R}}(1, 1) = 0$, which tells us by (1.12) that either $1 \cdot (a_2 - a_1) \in \mathbb{Z}$ or $1 \cdot (b_2 - b_1) \in \mathbb{Z}$, proving the claim.

Now, by hypothesis, each little rectangle $\mathcal{R}_k$ is a nice rectangle, so by the claim above it satisfies $\hat{1}_{\mathcal{R}_k}(\xi) = 0$ for all $\xi \in \mathbb{Z}^2 - \{0\}$. Returning to (1.4), we see that therefore $\hat{1}_{\mathcal{R}}(\xi) = \sum_{k=1}^{N} \hat{1}_{\mathcal{R}_k}(\xi) = 0$, for all $\xi \in \mathbb{Z}^2 - \{0\}$, and using the claim again (the converse part of it this time), we see that $\mathcal{R}$ must be nice. $\square$

The proof of Theorem 1.1 was straightforward and elegant, motivating the use of Fourier transforms of polytopes in the ensuing chapters. The claim 1.10 offers an intriguing beginning for deeper investigations - it tells us that we can convert a geometric statement about tiling to a purely analytic statement about the vanishing of a certain integral transform. Later, in Theorem 4.5 we will see that this small initial success of claim 1.10 is part of a larger theory. This is the beginning of a beautiful friendship...

Notes

(a) This little chapter was motivated by the lovely article written by Stan Wagon (1987), which gives 14 different proofs of this result. The article Wagon (ibid.) is important because it shows how tools from one field can leak into another field, and thus may lead to important discoveries in the future.

(b) In a related direction, we might wonder which polygons, and more generally which polytopes, tile Euclidean space by translations with a lattice. It turns out (Theorem 4.5) that this question is equivalent to the statement that the Fourier transform of $\mathcal{P}$ vanishes on a (dual) lattice.

(c) In the context of the Hilbert space of functions $L^2([0,1])$, Exercise 1.3 is one step towards showing that the set of exponentials $\{e_n(x)\}_{n\in\mathbb{Z}}$ is a basis for $L^2([0,1])$. Namely, the identity above shows that these basis elements are orthogonal to each other - their inner product $\langle e_a, e_b\rangle := \int_0^1 e_a(x)\overline{e_b(x)}dx$ vanishes for integers $a \neq b$. Thus, the identity of Exercise 1.3 is often called the orthogonality relations for exponentials, over $L^2([0,1])$. To show that they *span* the space of functions in $L^2([0,1])$ is a bit harder, but see Travaglini (2014) for details.

(d) The question in Exercise 1.14 for $\mathbb{Z}$ was originally asked by Paul Erdős in 1951, and has an affirmative answer. This question also has higher-dimensional analogues:

> Suppose we give a partition of the integer lattice $\mathbb{Z}^d$ into a finite, disjoint union of translated sublattices. Is it always true that at least two of these sublattices are translates of each other?

The answer is known to be false for $d \geqslant 3$, but is still unsolved for $d = 2$ (see Feldman, Propp, and Robins (2011), Borodzik, Nguyen, and Robins (2016)).

Exercises

1.1. ♣ *Show that if $x \in \mathbb{C}$, then $e^{2\pi i x} = 1$ if and only if $x \in \mathbb{Z}$.*

1.2. *Show that $|e^z| \leqslant e^{|z|}$, for all complex numbers $z \in \mathbb{C}$.*

1.3. ♣ *Here we prove the **orthogonality relations for the exponential functions** defined by $e_n(x) := e^{2\pi i n x}$, for each integer n. Recall that the complex conjugate of any complex number $x + iy$ is defined by*

$$\overline{x+iy} := x - iy,$$

so that $\overline{e^{i\theta}} := e^{-i\theta}$ for real θ. Prove that for all integers a, b:

$$\int_0^1 e_a(x)\overline{e_b(x)}dx = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if not.} \end{cases} \tag{1.13}$$

1.4. *Here the reader may gain some practice with integrals that use complex valued integrands $f(x) := u(x) + iv(x)$. We recall for the reader the following definition:*

$$\int_{\mathbb{R}^d} f(x)dx := \int_{\mathbb{R}^d} (u(x) + iv(x))\, dx := \int_{\mathbb{R}^d} u(x)dx + i \int_{\mathbb{R}^d} v(x)dx, \tag{1.14}$$

a linear combination of two real-valued integrals. Recalling that by definition,

$$\hat{1}_{[0,1]}(\xi) := \int_{[0,1]} e^{-2\pi i \xi x} dx,$$

show directly from the Equation (1.14) that for any nonzero $\xi \in \mathbb{R}$, we have

$$\int_{[0,1]} e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi} - 1}{-2\pi i \xi}.$$

Notes. Another way of thinking about this exercise is that it extends the 'Fundamental theorem of calculus' to complex valued functions in a rather easy way. The antiderivative of the integrand $f(x) := e^{-2\pi i \xi x}$ is $F(x) := \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}$, and we are saying that it is ok to use it in place of the usual antiderivative in Calculus 1 - it is consistent with Equation (1.14). In the future, we generally do not have to break up complex integrals into their real and imaginary parts, because we can make use of the fact that antiderivatives of complex valued functions are often simple, such as the one in this example.

We also note that throughout the book we do not have to integrate a function of a complex variable, because the **domains of our integrands**, as well as the measure we are using throughout this book, in order to integrate, is always over real Euclidean space $\mathbb{R}^d$, which is still Calculus 1.

1.5. ♣ *We recall that the N 'th roots of unity are by definition the set of N complex solutions to $z^N = 1$, and are given by the set $\{e^{2\pi i k/N} \mid k = 0, 1, 2, \ldots, N-1\}$ of points on the unit circle. Prove that the sum of all of the N 'th roots of unity vanishes. Precisely, fix any positive integer $N \geqslant 2$, and show that*

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i k}{N}} = 0.$$

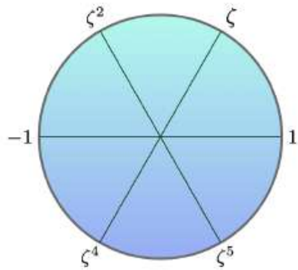


Figure 1.2: The 6'th roots of unity, with $\zeta := e^{\frac{2\pi i}{6}}$. Geometrically, Exercise 1.5 tells us that their center of mass is the origin.

1.6. *Prove that, given positive integers M, N, we have*

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i k M}{N}} = \begin{cases} 1 & \text{if } N \mid M \\ 0 & \text{if not.} \end{cases}$$

Notes. This result is sometimes referred to as "the harmonic detector" for detecting when a rational number $\frac{M}{N}$ is an integer; that is, it assigns a value of 1 to the sum if $\frac{M}{N} \in \mathbb{Z}$, and it assigns a value of 0 to the sum if $\frac{M}{N} \notin \mathbb{Z}$.

1.7. ♣ *Here we prove the* ***orthogonality relations for roots of unity****. Namely, fix any two nonnegative integers a, b, and prove that*

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i k a}{N}} e^{-\frac{2\pi i k b}{N}} = \begin{cases} 1 & \text{if } a \equiv b \mod N \\ 0 & \text{if not.} \end{cases} \tag{1.15}$$

Notes. In a later chapter on Euclidean lattices (Chapter 5), we will see that the identity 1.15 is a special case of the more general orthogonality relations for characters on lattices. From this perspective, this exercise is the orthogonality relations on the finite cyclic group $\mathbb{Z}/N\mathbb{Z}$. There are more general orthogonality relations for characters of group representations, which play an important role in Number Theory.

1.8. *Show that for any positive integer n, we have*

$$n = \prod_{k=1}^{n-1} (1 - \zeta^k),$$

where $\zeta := e^{2\pi i/n}$.

1.9. *An N'th root of unity is called a* **primitive root of unity** *if it is not a k'th root of unity for some smaller positive integer* $k < N$. *Show that the primitive N'th roots of unity are precisely the numbers* $e^{2\pi i k/N}$ *for which* $\gcd(k, N) = 1$.

1.10. *The Möbius* μ*-function is defined by:*

$$\mu(n) := \begin{cases} (-1)^{\text{number of prime factors of } n} & \text{if } n > 1 \\ 1 & \text{if } n = 1. \end{cases}$$

Prove that the sum of all of the primitive N'th roots of unity is equal to the Möbius μ*-function, evaluated at* N:

$$\sum_{\gcd(k,N)=1} e^{\frac{2\pi i k}{N}} = \mu(N). \tag{1.16}$$

1.11. *Here the reader needs to know a little bit about the quotient of two groups (this is one of the few exercises that assumes group theory). We prove that the group of 'real numbers mod 1' under addition, is isomorphic to the unit circle, under multiplication of complex numbers. Precisely, we can define* $h : \mathbb{R} \to S^1$ *by* $h(x) := e^{2\pi i x}$.

(a) *We recall the definition of the kernel of a map, namely* $ker(h) := \{x \in \mathbb{R} \mid h(x) = 1\}$. *Show that* $ker(h) = \mathbb{Z}$.

(b) *Using the first isomorphism Theorem for groups, show that* $\mathbb{R}/\mathbb{Z}$ *is isomorphic to the unit circle* S^1.

1.12. *Using gymnastics with roots of unity, we recall here a very classical solution to the problem of finding the roots of a cubic polynomial.*

(a) *Let* $\omega := e^{2\pi i/3}$, *and show that we have the polynomial identity:*

$$(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b) = x^3 - 3abx + a^3 + b^3.$$

(b) *Using the latter identity, solve the cubic polynomial:* $x^3 - px + q = 0$ *by substituting* $p = 3ab$ *and* $q = a^3 + b^3$.

1.13. *Thinking of the function* $\sin(\pi z)$ *as a function of a complex variable* $z \in \mathbb{C}$, *show that its zeros are precisely the set of integers* $\mathbb{Z}$.

1.14. *In 1951, Paul Erdős asked: "Can the set* $\mathbb{Z}_{>0}$ *of all positive integers be partitioned (that is, written as a disjoint union) into a finite number of arithmetic progressions, such that no two of the arithmetic progressions will have the same common difference?"*

Suppose that we have a list of these disjoint arithmetic progressions (at least two of them, by assumption), each with its common difference a_k:

$$\{a_1 n + b_1 \mid n \in \mathbb{Z}\}, \dots, \{a_N n + b_N \mid n \in \mathbb{Z}\},$$

where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_N$. *Prove that in any such partitioning of the integers, there are at least two arithmetic progressions that have the same maximal* a_N.

(see also Exercise 5.26 *for an extension to lattices in* $\mathbb{R}^d$*)*

2 Examples that nourish the theory

A pint of example is worth a gallon of advice

– Anonymous

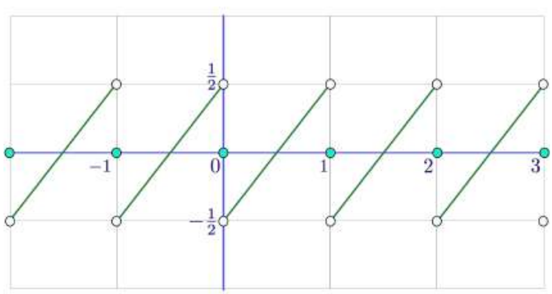


Figure 2.1: The first periodic Bernoulli polynomial $P_1(x)$, sometimes called the sawtooth function, which turns out to be one of the building blocks of integer point enumeration in polytopes

2.1 Intuition

One way to think about the Fourier transform of a polytope $\mathcal{P} \subset \mathbb{R}^d$ is that it simultaneously captures all of the moments of $\mathcal{P}$, thereby uniquely defining $\mathcal{P}$. Here we begin concretely by computing some Fourier transforms of various polytopes in dimensions 1 and 2, as well as the Fourier transforms of some simple families of polytopes in dimension d as well.

This chapter is "fast and loose", showing the reader quickly some of the interesting facts, from an intuitive perspective. The ensuing chapters will fill in the details for the intuitive proofs of this chapter, going into the necessary subtleties and details.

The 2-dimensional computations will get the reader more comfortable with the basics. In later chapters, once we learn a little more theory, we will return to these families of polytopes and compute some of their Fourier transforms in general.

We also see, from small examples, that the Bernoulli polynomials immediately enter into the picture, forming natural building blocks. In this chapter we compute Fourier transforms without thinking too much about convergence issues, to let the reader run with the ideas. But commencing with the next chapter, we will be more rigorous when using Poisson summation, and with convergence issues.

2.2 Dimension 1 – the classical sinc function

We begin by computing the classical 1-dimensional example of the Fourier transform of the symmetrized unit interval $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$:

$$\begin{aligned}
\hat{1}_{\mathcal{P}}(\xi) &:= \int_{\mathbb{R}} 1_{\mathcal{P}}(x)\, e^{-2\pi i x\xi} dx && (2.1)\\
&= \int_{[-\frac{1}{2},\frac{1}{2}]} e^{-2\pi i x\xi} dx && (2.2)\\
&= \frac{e^{-2\pi i (\frac{1}{2})\xi} - e^{-2\pi i (\frac{-1}{2}\xi)}}{-2\pi i\xi} && (2.3)\\
&= \frac{\cos(-\pi\xi) + i\sin(-\pi\xi) - (\cos(\pi\xi) + i\sin(\pi\xi))}{-2\pi i\xi} && (2.4)\\
&= \frac{\sin(\pi\xi)}{\pi\xi}, && (2.5)
\end{aligned}$$

valid for all $\xi \neq 0$. The latter function is also known as the **sinc function**. We notice that $\xi = 0$ is a removable singularity, so that we may define the continuous (and in fact smooth) sinc-function by

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x}, & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{2.6}$$

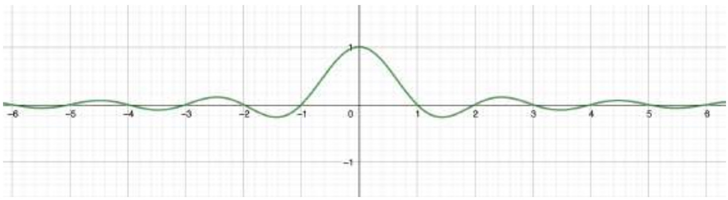


Figure 2.2: The function $\operatorname{sinc}(x)$, which is Fourier transform of the 1-dimensional polytope $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]$.

Next, we introduce the inverse Fourier transform , or as it is often called, the **Fourier inversion formula**: property:

$$(\mathcal{F} \circ \mathcal{F})(f)(\xi) = f(-\xi), \tag{2.7}$$

an extremely useful tool (see Travaglini (2014)).

Example 2.1. A famous and historically somewhat tricky integral formula for the sinc function is the following fact:

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) dx := \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} dx = 1. \tag{2.8}$$

The careful reader notices that the latter integrand is not absolutely convergent, which means that $\int_{-\infty}^{\infty} \left| \frac{\sin(\pi x)}{\pi x} \right| dx = \infty$. So we have to specify what we really mean by the identity (2.8). The rigorous claim is:

$$\lim_{N \to \infty} \int_{-N}^{N} \frac{\sin(\pi x)}{\pi x} dx = 1.$$

We will proceed informally at the moment, using (2.1), together with the inverse Fourier transform (2.7) (The reason for the informality is that usually the Fourier inversion formula applies to functions that are absolutely integrable, and whose FT is also absolutely integrable, but let's do it anyway!) Applying the inverse Fourier transform, this identity will now become somewhat trivial to prove. We saw above that the Fourier transform of the indicator function of the interval $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$ is:

$$\mathcal{F}(1_{\mathcal{P}})(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}, \tag{2.9}$$

so that

$$\mathcal{F}\left(\frac{\sin(\pi\xi)}{\pi\xi}\right) = (\mathcal{F} \circ \mathcal{F})(1_{\mathcal{P}})(\xi) = 1_{\mathcal{P}}(-\xi). \tag{2.10}$$

Using the definition of the Fourier transform, the latter identity is:

$$\int_{\mathbb{R}} \frac{\sin(\pi x)}{\pi x} e^{-2\pi i \xi x} dx = 1_{\mathcal{P}}(\xi), \tag{2.11}$$

and now evaluating both sides at $\xi = 0$ gives us (2.8). $\square$

Moving to dimension d, we can extend this example in a natural way to all Fourier pairs of functions, $\{f(x), \hat{f}(\xi)\}$, as follows. We recall that $\hat{g}(0) := \int_{\mathbb{R}^d} g(x)dx$, using the definition of the transform. If we make the substitution $g := \hat{f}$, then by Fourier inversion we have $\hat{g}(x) = (\mathcal{F} \circ \mathcal{F})(f)(x) = f(-x)$, which immediately implies that:

$$\int_{\mathbb{R}^d} \hat{f}(x)dx = f(0). \tag{2.12}$$

To summarize, Example 2.1 is simply identity (2.12) with $f(x) := 1_{[-\frac{1}{2}, \frac{1}{2}]}(x)$.

Another very useful fact about the Fourier transform of a polytope is that it is an entire function, meaning that it is differentiable everywhere. This differentiability is already observable in the sinc function above, with its removable singularity at the origin.

Lemma 2.1. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional polytope. Then $\hat{1}_{\mathcal{P}}(\xi)$ is an entire function of $\xi \in \mathbb{C}^d$.*

Proof. Because $\mathcal{P}$ is compact, we can safely differentiate under the integral sign (this is a special case of Lebesgue's Dominated Convergence Theorem). Namely, for any coordinate variable ξ_k, we have: $\frac{d}{d\xi_k} \int_{\mathcal{P}} e^{-2\pi i \langle \xi, x\rangle} dx = \int_{\mathcal{P}} \frac{d}{d\xi_k} e^{-2\pi i \langle \xi, x\rangle} dx = 2\pi i \int_{\mathcal{P}} x_k e^{-2\pi i \langle \xi, x\rangle} dx$, and it is clear that all possible derivatives exist in this manner, because the integrand is infinitely smooth. $\square$

We also have the very fortuitous fact that the Fourier transform of any polytope $\mathcal{P} \subset \mathbb{R}^d$ is a complete invariant, in the following sense.

Lemma 2.2. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a polytope. Then $\hat{1}_{\mathcal{P}}(\xi)$ uniquely determines $\mathcal{P}$. In other words, given any two polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$, we have*

$$\hat{1}_{\mathcal{P}}(\xi) = \hat{1}_{\mathcal{Q}}(\xi) \textit{ for all } \xi \in \mathbb{R}^d \iff \mathcal{P} = \mathcal{Q}.$$

Proof. (outline) If $\mathcal{P} = \mathcal{Q}$, it is clear that $\hat{1}_{\mathcal{P}}(\xi) = \hat{1}_{\mathcal{Q}}(\xi)$ for all $\xi \in \mathbb{R}^d$. Conversely, suppose that $\hat{1}_{\mathcal{P}}(\xi) = \hat{1}_{\mathcal{Q}}(\xi)$ for all $\xi \in \mathbb{R}^d$. Using Fourier inversion, given by (2.7), we may take the Fourier transform of both sides of the latter equation to get $1_{\mathcal{P}}(-\xi) = 1_{\mathcal{Q}}(-\xi)$, for all $\xi \in \mathbb{R}^d$. $\square$

2.3 Bernoulli polynomials

We introduce the Bernoulli polynomials, which turn out to be a sort of "glue" between discrete geometry and number theory, as we will see throughout the book. The Bernoulli polynomials are defined via the following generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}. \tag{2.13}$$

It's fruitful to sometimes restrict the Bernoulli polynomials to the unit interval $[0, 1]$, and then periodize them. In other words, using

$$\{x\} := x - \lfloor x \rfloor,$$

the fractional part of x, we may define the n'th periodic Bernoulli polynomial:

$$P_n(x) := B_n(\{x\}), \tag{2.14}$$

for $n \geqslant 2$. Since $P_n(x)$ is periodic on $\mathbb{R}$ with period 1, it has a Fourier series, and in fact:

$$P_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z} - \{0\}} \frac{e^{2\pi i k x}}{k^n}, \tag{2.15}$$

valid for $x \in \mathbb{R}$ (Exercise 2.9).

When $n = 1$, we have the first Bernoulli polynomial

$$P_1(x) := x - \lfloor x \rfloor - \frac{1}{2},$$

which is very special (see Figure 2.1). For one thing, it is the only periodic Bernoulli polynomial that is not continuous everywhere, and we note that its Fourier series does not converge absolutely, although it is quite appealing:

$$P_1(x) = -\frac{1}{2\pi i} \sum_{k \in \mathbb{Z}-\{0\}} \frac{e^{2\pi i k x}}{k}, \tag{2.16}$$

valid for all $x \notin \mathbb{Z}$. Hence special care must be taken with $P_1(x)$. Exercises 2.4 to 2.17 illustrate some of the important properties of these polynomials. Exercise 2.30 provides a rigorous proof of the convergence of (2.16).

Example 2.2. The first few Bernoulli polynomials are:

$$B_0(x) = 1 \tag{2.17}$$

$$B_1(x) = x - \frac{1}{2} \tag{2.18}$$

$$B_2(x) = x^2 - x + \frac{1}{6} \tag{2.19}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \tag{2.20}$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \tag{2.21}$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \tag{2.22}$$

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42} \tag{2.23}$$

The **Bernoulli numbers** are defined to be the constant terms of the Bernoulli polynomials:

$$B_k := B_k(0).$$

The first few Bernoulli numbers are:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}.$$

It follows quickly from the Equation (2.13) above that for odd $k \geqslant 3$, $B_k = 0$ (Exercise 2.15). From the generating function 2.13 the Bernoulli numbers are defined via

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2.24}$$

□

Historically, the first appearance of the Bernoulli polynomials occurred while Jakob Bernoulli tried to compute sums of powers of integers. In particular, Bernoulli showed that:

$$\sum_{k=1}^{n-1} k^{d-1} = \frac{B_d(n) - B_d}{d},$$

for all integers $d \geqslant 1$ and $n \geqslant 2$ (Exercise 2.8). An interesting identity that allows us to compute the Bernoulli numbers recursively rather quickly is:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0,$$

valid for all $n \geqslant 1$ (Exercise 2.17).

Some of the most natural, and beautiful, Fourier series arise naturally from the periodized Bernoulli polynomials. The following intuitive application of the Poisson summation formula already suggests an initial connection between periodized Bernoulli polynomials and Fourier transforms of polytopes - even in dimension 1.

Example 2.3 (Intuitive Poisson summation)**.** In this example we allow ourselves to be completely intuitive, and unrigorous at this moment, but often such arguments are useful in pointing us to their rigorous counterparts. Consider the 1-dimensional polytope $\mathcal{P} := [a, b]$, and restrict attention to the case of $a, b \notin \mathbb{Z}$. If we could use the Poisson summation formula, applied to the function $1_{\mathcal{P}}(x)$, we would get:

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} 1_{\mathcal{P}}(n) &= \sum_{\xi\in\mathbb{Z}} \hat{1}_{\mathcal{P}}(\xi) \\
&= \hat{1}_{\mathcal{P}}(0) + \sum_{\xi\in\mathbb{Z}-\{0\}} \frac{e^{-2\pi i\xi b} - e^{-2\pi i\xi a}}{-2\pi i\xi} \\
&= (b-a) - \frac{1}{2\pi i} \sum_{\xi\in\mathbb{Z}-\{0\}} \frac{e^{-2\pi i\xi b}}{\xi} + \frac{1}{2\pi i} \sum_{\xi\in\mathbb{Z}-\{0\}} \frac{e^{-2\pi i\xi a}}{\xi} \\
&= (b-a) + \frac{1}{2\pi i} \sum_{\xi\in\mathbb{Z}-\{0\}} \frac{e^{2\pi i\xi b}}{\xi} - \frac{1}{2\pi i} \sum_{\xi\in\mathbb{Z}-\{0\}} \frac{e^{2\pi i\xi a}}{\xi} \\
&= (b-a) - \left(\{b\} - \frac{1}{2}\right) + \left(\{a\} - \frac{1}{2}\right) \\
&= b - \{b\} - (a - \{a\}) = \lfloor b \rfloor - \lfloor a \rfloor.
\end{aligned}
$$

Since we already know how to evaluate the LHS of Poisson summation above, namely that $\sum_{n\in\mathbb{Z}} 1_{\mathcal{P}}(n) = \#\{\mathbb{Z} \cap \mathcal{P}\} = \lfloor b \rfloor - \lfloor a \rfloor$, we have confirmed that Poisson summation has given us here the correct formula. □

The reason that the intuitive argument in Example 2.3 is not rigorous, is that in order to plug a function f into Poisson summation, f and its Fourier transform $\hat{f}$ must both satisfy some growth conditions at infinity. We will see such conditions later, in Section 3.4. Once we learn how to use Poisson summation, we will return to this example (see Example 8.3).

We recall that a series $\sum_{n\in\mathbb{Z}} a_n$ is said to **converge absolutely** if $\sum_{n\in\mathbb{Z}} |a_n|$ converges. It's easy to see that the series in (2.16) for $P_1(x)$ does not converge absolutely. Such convergent series that do not converge absolutely are called **conditionally convergent**.

To prove rigorously that the conditionally convergent series (2.16) does in fact converge, see Exercises 2.26, 2.27, 2.29 and 2.30, which include the Abel summation formula, and the Dirichlet convergence test (although extremely useful, we will not use them that much in the ensuing chapters).

2.4 The cube, and its Fourier transform

Perhaps the easiest way to extend the Fourier transform of the unit interval is to consider the d-dimensional unit cube

$$\square := \left[-\frac{1}{2}, \frac{1}{2}\right]^d .$$

What is its Fourier transform? When we compute a Fourier transform of a function f, we will say that $\{f, \hat{f}\}$ is a **Fourier pair**. We have seen that $\{1_{[-\frac{1}{2},\frac{1}{2}]}(x), \operatorname{sinc}(\xi)\}$ is a Fourier pair in dimension 1.

Example 2.4. Due to the fact that the cube is the direct product of line segments, it follows that the ensuing integral can be separated into a product of integrals, and so it is the product of 1-dimensional transforms:

$$\begin{aligned}
\hat{1}_\square(\xi) &= \int_{\mathbb{R}^d} 1_\square(x) e^{-2\pi i \langle x, \xi \rangle} dx && (2.25)\\
&= \int_\square e^{-2\pi i (x_1\xi_1 + \cdots + x_d\xi_d)} dx && (2.26)\\
&= \prod_{k=1}^d \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x_k \xi_k} dx_k && (2.27)\\
&= \prod_{k=1}^d \frac{\sin(\pi \xi_k)}{\pi \xi_k}, && (2.28)
\end{aligned}$$

valid for all $\xi \in \mathbb{R}^d$ such that none of their coordinates vanishes. So here we have the Fourier pair

$$\left\{ 1_\square(x), \prod_{k=1}^d \frac{\sin(\pi \xi_k)}{\pi \xi_k} \right\}.$$

In general, though, polytopes are not a direct product of lower-dimensional polytopes, so we will need to develop more tools to compute their Fourier transforms. $\square$

2.5 The simplex, and its Fourier transform

Another basic building block for polytopes is the **standard simplex**, defined by

$$\Delta := \left\{ x \in \mathbb{R}^d \mid x_1 + \cdots + x_d \leqslant 1, \text{ and all } x_k \geqslant 0 \right\}. \tag{2.29}$$

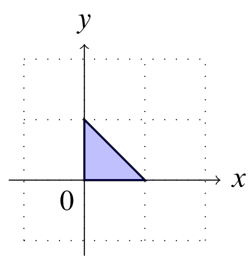


Figure 2.3: The standard simplex in $\mathbb{R}^2$

Example 2.5. Just for fun, let's compute the Fourier transform of $\triangle$ for $d = 2$, via brute-force. We may use the following parametrization (called a hyperplane description) for this standard triangle:

$$\Delta = \{(x, y) \mid x + y \leqslant 1, \text{ and } x \geqslant 0, y \geqslant 0\}.$$

Hence, we have:

$$
\begin{aligned}
\hat{1}_{\triangle}(\xi_1,\xi_2) &:= \int_{\triangle} e^{-2\pi i\left(x\xi_1+y\xi_2\right)}\,dx\,dy \\
&= \int_0^1 \int_{y=0}^{y=1-x} e^{-2\pi i\left(x\xi_1+y\xi_2\right)}\,dy\,dx \\
&= \int_0^1 e^{-2\pi i x\xi_1}\left[\frac{e^{-2\pi i y\xi_2}}{-2\pi i\xi_2}\Big|_{y=0}^{y=1-x}\right]dx \\
&= \frac{1}{-2\pi i\xi_2}\int_0^1 e^{-2\pi i x\xi_1}\left(e^{-2\pi i(1-x)\xi_2}-1\right)dx \\
&= \frac{1}{-2\pi i\xi_2}\int_0^1 \left(e^{-2\pi i x(\xi_1-\xi_2)}e^{-2\pi i\xi_2}-e^{-2\pi i x\xi_1}\right)dx \\
&= \frac{1}{(-2\pi i)^2}\frac{e^{-2\pi i\xi_2}}{\xi_2(\xi_1-\xi_2)}\left(e^{-2\pi i(\xi_1-\xi_2)}-1\right)-\frac{1}{(-2\pi i)^2}\frac{e^{-2\pi i\xi_1}-1}{\xi_1\xi_2} \\
&= \frac{1}{(-2\pi i)^2}\left[\frac{e^{-2\pi i\xi_1}-e^{-2\pi i\xi_2}}{\xi_2(\xi_1-\xi_2)}-\frac{e^{-2\pi i\xi_1}-1}{\xi_1\xi_2}\right].
\end{aligned}
$$

We may simplify further by noticing the rational function identity

$$
\frac{e^{-2\pi i\xi_1}}{\xi_2(\xi_1-\xi_2)}-\frac{e^{-2\pi i\xi_1}}{\xi_1\xi_2}=\frac{e^{-2\pi i\xi_1}}{\xi_1(\xi_1-\xi_2)},
$$

giving us the symmetric function of (ξ_1,ξ_2):

$$
\hat{1}_{\triangle}(\xi_1,\xi_2)=\frac{1}{(-2\pi i)^2}\left[\frac{e^{-2\pi i\xi_1}}{\xi_1(\xi_1-\xi_2)}+\frac{e^{-2\pi i\xi_2}}{\xi_2(\xi_2-\xi_1)}+\frac{1}{\xi_1\xi_2}\right]. \tag{2.30}
$$

□

We need the concept of a **convex set** $X \subset \mathbb{R}^d$, defined by the property that for any two points $x, y \in X$, the line segment joining them also lies in X. In other words:

$$
\{\lambda x+(1-\lambda)y \mid 0 \leqslant \lambda \leqslant 1\} \subset X,\ \forall x,y \in X.
$$

Given any finite set of points $S := \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^d$, we can also form the set of all **convex linear combinations** of S by defining

$$\mathrm{conv}(S) := \left\{ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_N v_N \mid \sum_{k=1}^{N} \lambda_k = 1, \text{ where all } \lambda_k \geqslant 0 \right\}. \tag{2.31}$$

Given any set $U \subset \mathbb{R}^d$ (which is not restricted to be finite), we define the **convex hull** of U as the set of convex linear combinations, taken over all finite subsets of U, and denoted by $\mathrm{conv}(U)$.

We define a **polytope** as the convex hull of any finite set of points in $\mathbb{R}^d$. This definition of a polytope is called its **vertex description**.

We define a k**-simplex** $\Delta \subset \mathbb{R}^d$ as the convex hull of a finite set of points $\{v_1, v_2, \ldots, v_{k+1}\}$:

$$\Delta := \mathrm{conv}\{v_1, v_2, \ldots, v_{k+1}\},$$

where $0 \leqslant k \leqslant d$, and $v_2 - v_1, v_3 - v_1, \ldots, v_{k+1} - v_1$ are linearly independent vectors in $\mathbb{R}^d$. The points $v_1, v_2, \ldots, v_{k+1}$ are called the vertices of Δ, and this object is one of the basic building-blocks of polytopes, especially when triangulating a polytope.

The simplex Δ is a k-dimensional polytope, sitting in $\mathbb{R}^d$. When $k = d$, the dimension of Δ equals the dimension of the ambient space $\mathbb{R}^d$ - see Figure 2.4.

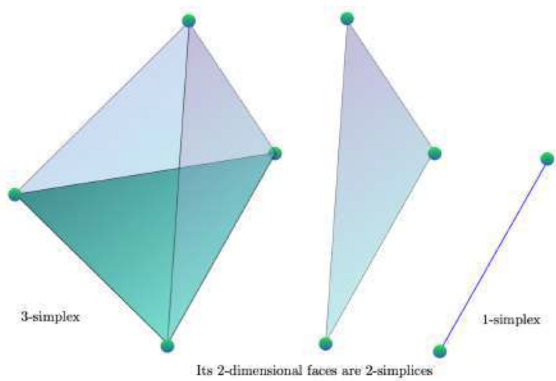


Figure 2.4: A 3-simplex and its faces, which are lower-dimensional simplices as well

We have already computed the Fourier transform of a particular 2-simplex, in (2.30).

More generally, let's compute the Fourier transform of any 2-simplex in $\mathbb{R}^2$. In order to handle a general triangle, let Δ be any triangle in the plane, with vertices

$$v_1 := \binom{a_1}{b_1},\, v_2 := \binom{a_2}{b_2},\, v_3 := \binom{a_3}{b_3}.$$

Can we reduce the computation of $\hat{1}_\Delta$ to our already known formula for $\hat{1}_{\triangle}$, given by (2.30)? We first notice (after a cup of coffee) that we can map any triangle in the plane to the standard triangle, by using a linear transformation followed by a translation:

$$\Delta = M(\triangle) + v_3, \tag{2.32}$$

where M is the 2×2 matrix whose columns are $v_1 - v_3$ and $v_2 - v_3$. We are now ready to compute the Fourier transform of a general triangle Δ:

$$\hat{1}_\Delta(\xi) = \int_\Delta e^{-2\pi i \langle \xi, x\rangle} dx = \int_{M(\triangle)+v_3} e^{-2\pi i \langle \xi, x\rangle} dx.$$

Making the substitution $x := My + v_3$, with $y \in \triangle$, we have $dx = |\det M| dy$, and so

$$\begin{aligned}
&\int_{M(\triangle)+v_3} e^{-2\pi i \langle \xi, x\rangle} dx = |\det M| \int_{\triangle} e^{-2\pi i \langle \xi, My+v_3\rangle} dy \\
&= |\det M| e^{-2\pi i \langle \xi, v_3\rangle} \int_{\triangle} e^{-2\pi i \langle M^T\xi, y\rangle} dy \\
&= |\det M| e^{-2\pi i \langle \xi, v_3\rangle} \hat{1}_{\triangle}(M^T\xi) \\
&= |\det M| e^{-2\pi i \langle \xi, v_3\rangle} \hat{1}_{\triangle}\big(\langle v_1 - v_3, \xi\rangle, \langle v_2 - v_3, \xi\rangle\big) \\
&= |\det M| e^{-2\pi i \langle \xi, v_3\rangle} \frac{1}{(-2\pi i)^2} \left[\frac{e^{-2\pi i z_1}}{z_1(z_1 - z_2)} + \frac{e^{-2\pi i z_2}}{z_2(z_2 - z_1)} + \frac{1}{z_1 z_2} \right],
\end{aligned}$$

where we've used our formula (2.30) for the FT of the standard triangle (thereby bootstrapping out way to the general case) with $z_1 := \langle v_1 - v_3, \xi\rangle$, and $z_2 := \langle v_2 - v_3, \xi\rangle$. Substituting these values into the latter expression, we finally arrive at the FT of our general triangle Δ:

$$\begin{aligned}
\hat{1}_\Delta(\xi) = \frac{|\det M|}{(-2\pi i)^2} \Bigg[&\frac{e^{-2\pi i \langle v_1, \xi\rangle}}{\langle v_1 - v_3, \xi\rangle\langle v_1 - v_2, \xi\rangle} + \frac{e^{-2\pi i \langle v_2, \xi\rangle}}{\langle v_2 - v_3, \xi\rangle\langle v_2 - v_1, \xi\rangle} \\
&+ \frac{e^{-2\pi i \langle \xi, v_3\rangle}}{\langle v_3 - v_1, \xi\rangle\langle v_3 - v_2, \xi\rangle} \Bigg].
\end{aligned} \tag{2.33}$$

We can notice in equation (2.33) many of the same patterns that had already occurred in Example 2.8. Namely, the Fourier transform of a triangle has denominators that are products of linear forms in ξ, and it is a finite linear combination of rational functions multiplied by complex exponentials.

Also, in the particular case of equation (2.33), $\hat{1}_\Delta(\xi)$ is a symmetric function of v_1, v_2, v_3, as we might have expected.

Using exactly the same ideas that were used in equation (2.33), it is possible to prove (by induction on the dimension) that the Fourier transform of a general d-dimensional simplex $\Delta \subset \mathbb{R}^d$ is:

$$\hat{1}_\Delta(\xi) = (\operatorname{vol} \Delta) d! \sum_{j=1}^{N} \frac{e^{-2\pi i \langle v_j, \xi \rangle}}{\prod_{k=1}^{d} \langle v_j - v_k, \xi \rangle} [k \neq j], \tag{2.34}$$

where the vertex set of $\mathcal{P}$ is $\{v_1, \ldots, v_N\}$ (Exercise 2.25), and in fact the same formula persists for all complex $\xi \in \mathbb{C}^d$ such that the products of linear forms in the denominators do not vanish.

However, looking back at the computation leading to (2.33), and the corresponding computation which would give (2.34), the curious reader might be thinking:

"There must be an easier way!"

But never fear - indeed there is. So even though at this point the computation of $\hat{1}_\Delta(\xi)$ may be a bit laborious (but still interesting), computing the Fourier transform of a general simplex will become quite easy once we will revisit it in a later chapter (see Theorem 6.1).

2.6 Stretching and translating

The perspicacious reader may have noticed that in order to arrive at the formula (2.33) above for the FT of a general triangle, we exploited the fact that the Fourier transform interacted peacefully with the linear transformation M, and with the translation by the vector v. Is this true in general?

Indeed it is, and we record these thoughts in the following two lemmas, which will become our bread and butter for future computations. In general, given any invertible linear transformation $M : \mathbb{R}^d \to \mathbb{R}^d$, and any function $f : \mathbb{R}^d \to \mathbb{C}$ whose FT (Fourier transform) exists, we have the following useful interaction between Fourier transforms and linear transformations.

Lemma 2.3 (Stretch)**.**

$$\widehat{(f \circ M)}(\xi) = \frac{1}{|\det M|} \hat{f}(M^{-T}\xi) \tag{2.35}$$

Proof. By definition, we have $\widehat{(f \circ M)}(\xi) := \int_{\mathbb{R}^d} f(Mx)e^{-2\pi i\langle \xi, x\rangle}dx$. We perform the change of variable $y := Mx$, implying that $dy = |\det M| dx$, so that:

$$\begin{aligned}
\widehat{(f \circ M)}(\xi) &= \frac{1}{|\det M|} \int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle \xi, M^{-1}y\rangle}dy \\
&= \frac{1}{|\det M|} \int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle M^{-T}\xi, y\rangle}dy \\
&= \frac{1}{|\det M|} \hat{f}(M^{-T}\xi).
\end{aligned}$$

□

What about translations? They are even simpler.

Lemma 2.4 (Translate)**.** *For any translation $T(x) := x + v$, where $v \in \mathbb{R}^d$ is a fixed vector, we have*

$$\widehat{(f \circ T)}(\xi) = e^{2\pi i\langle \xi, v\rangle} \hat{f}(\xi). \tag{2.36}$$

Proof. By definition, we have $\widehat{(f \circ T)}(\xi) = \int_{\mathbb{R}^d} f(Tx)e^{-2\pi i\langle \xi, x\rangle}dx$, so that performing the simple change of variable $y = Tx := x + v$, we have $dy = dx$ this time. The latter integral becomes

$$\begin{aligned}
\widehat{(f \circ T)}(\xi) &= \int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle \xi, y-v\rangle}dy \\
&= e^{2\pi i\langle \xi, v\rangle} \int_{\mathbb{R}^d} f(y)e^{-2\pi i\langle \xi, y\rangle}dy := e^{2\pi i\langle \xi, v\rangle} \hat{f}(\xi).
\end{aligned}$$

□

In general, any function $\phi : \mathbb{R}^d \to \mathbb{C}$ of the form

$$\phi(x) = Mx + v, \tag{2.37}$$

where M is a fixed linear transformation and $v \in \mathbb{R}^d$ is a fixed vector, is called an **affine transformation**. For example, we've already seen in (2.32) that the right triangle ◺ was mapped to the more general triangle Δ by an affine transformation. So the latter two lemmas allow us to compose Fourier transforms very easily with affine transformations.

Example 2.6. Considering any measurable set $B \subset \mathbb{R}^d$, let's translate B by a fixed vector $v \in \mathbb{R}^d$, and compute $\hat{1}_{B+v}(\xi)$.

We note that because $1_{B+v}(\xi) = 1_B(\xi - v)$, the translate lemma applies, but with a minus sign. That is, we can use $T(x) := x - v$ and $f := 1_B$ to get:

$$\hat{1}_{B+v}(\xi) = (\widehat{1_B \circ T})(\xi) = e^{-2\pi i \langle \xi, v \rangle} \hat{1}_B(\xi). \tag{2.38}$$

$\square$

2.7 The parallelepiped, and its Fourier transform

Now that we know how to compose the FT with rigid motions (translations and linear transformations), we can easily find the FT of any parallelepiped in $\mathbb{R}^d$ by using our formula for the Fourier transform of the unit cube $\square := \left[-\frac{1}{2}, \frac{1}{2}\right]^d$, which we derived in Example 2.4:

$$\hat{1}_{\square}(\xi) = \prod_{k=1}^{d} \frac{\sin(\pi \xi_k)}{\pi \xi_k}, \tag{2.39}$$

for all $\xi \in \mathbb{R}^d$ such that all the coordinates of ξ do not vanish. First, we translate the cube $\square$ by the vector $(\frac{1}{2}, \cdots, \frac{1}{2})$, to obtain

$$C := \square + \left(\frac{1}{2}, \cdots, \frac{1}{2}\right) = [0, 1]^d.$$

It's straightforward to compute its FT as well (Exercise 2.2), by using Lemma 2.4, the 'translate' lemma:

$$\hat{1}_C(\xi) = \frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \xi_k}}{\xi_k}. \tag{2.40}$$

Next, we define a d-dimensional **parallelepiped** $\mathcal{P} \subset \mathbb{R}^d$ as an affine image of the unit cube. In other words, any parallelepiped has the description

$$\mathcal{P} = M(C) + v,$$

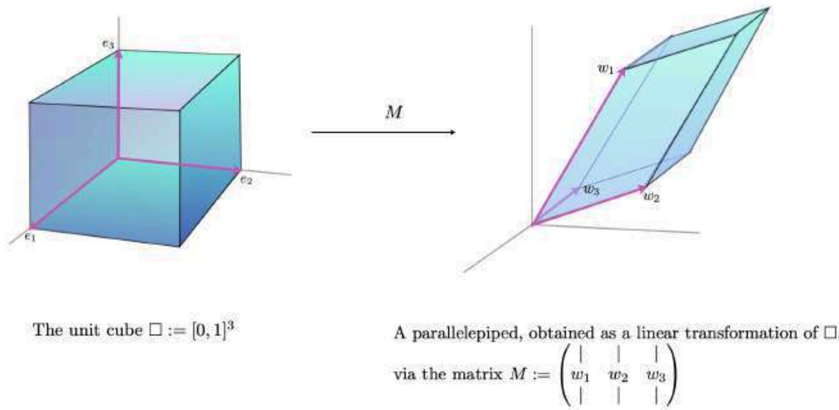


Figure 2.5: Mapping the unit cube to a parallelepiped

for some linear transformation M, and some translation vector v. Geometrically, the cube is stretched and translated into a parallelepiped.

For the sake of concreteness, will will first set $v := 0$ and compute the Fourier transform of $\mathcal{P} := M(C)$, where we now give M as a $d \times d$ invertible matrix whose columns are $w_1, w_2, \ldots, w_d$. Because the cube C may be written as a convex linear combination of the basis vectors e_j, we see that $\mathcal{P}$ may be written as a convex linear combination of $Me_j = w_j$. In other words, we see that the parallelepiped $\mathcal{P}$ has the equivalent vertex description:

$$\mathcal{P} = \{\sum_{k=1}^{d} \lambda_k w_k \mid \text{all } \lambda_k \in [0,1]\}.$$

To review the basics, let's compute the FT of our parallelepiped $\mathcal{P}$ from first principles:

$$\hat{1}_{\mathcal{P}}(\xi) := \int_{\mathcal{P}} e^{-2\pi i\langle \xi, x\rangle} dx = \int_{M(C)} e^{-2\pi i\langle \xi, x\rangle} dx \tag{2.41}$$

$$= |\det M| \int_C e^{-2\pi i\langle \xi, My\rangle} dy \tag{2.42}$$

$$= |\det M| \int_C e^{-2\pi i\langle M^T\xi, y\rangle} dy := |\det M|\, \hat{1}_C(M^T\xi) \tag{2.43}$$

$$= \frac{|\det M|}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i\langle w_k, \xi\rangle}}{\langle w_k, \xi\rangle}. \tag{2.44}$$

where in the third equality we used the substitution $x := My$, with $y \in C$, yielding $dx = |\det M| dy$. In the last equality, we used our known formula (2.40) for the FT of the cube C, together with the elementary linear algebra fact that the k'th coordinate of $M^T\xi$ is given by $\langle w_k, \xi\rangle$.

Finally, for a general parallelepiped $Q := \mathcal{P} + v$, so that by definition

$$Q = \{v + \sum_{k=1}^{d} \lambda_k w_k \mid \text{ all } \lambda_k \in [0,1]\}.$$

Noting that $1_{\mathcal{P}+v}(\xi) = 1_{\mathcal{P}}(\xi - v)$, we compute the Fourier transform of Q by using the 'translate lemma' (Lemma 2.4), together with formula (2.44) for the Fourier transform of $\mathcal{P}$:

$$\hat{1}_Q(\xi) = e^{-2\pi i\langle \xi, v\rangle} \frac{|\det M|}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i\langle w_k, \xi\rangle}}{\langle w_k, \xi\rangle}, \tag{2.45}$$

for all $\xi \in \mathbb{R}^d$, except for those ξ that are orthogonal to one of the w_k (which are edge vectors for Q).

Example 2.7. A straightforward computation shows that if we let $v := -\frac{w_1 + \cdots + w_d}{2}$, then $Q := \{v + \sum_{k=1}^{d} \lambda_k w_k \mid \text{ all } \lambda_k \in [0,1]\}$ is symmetric about the origin, in the sense that $x \in Q \iff -x \in Q$. In other words, the center of mass of this new Q is now the origin. Geometrically, we've translated the previous parallelepiped by using half its 'body diagonal'. For such a parallelepiped Q, centered at the origin, formula (2.45) above gives

$$\hat{1}_Q(\xi) = e^{2\pi i \langle \xi, \frac{w_1+\cdots+w_d}{2} \rangle} \frac{|\det M|}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \langle w_k, \xi \rangle}}{\langle w_k, \xi \rangle} \tag{2.46}$$

$$= \frac{|\det M|}{(2\pi i)^d} \prod_{k=1}^{d} \frac{e^{\pi i \langle w_k, \xi \rangle} - e^{-\pi i \langle w_k, \xi \rangle}}{\langle w_k, \xi \rangle} \tag{2.47}$$

$$= \frac{|\det M|}{(2\pi i)^d} \prod_{k=1}^{d} \frac{(2i)\sin(\pi \langle w_k, \xi \rangle)}{\langle w_k, \xi \rangle} \tag{2.48}$$

$$= |\det M| \prod_{k=1}^{d} \frac{\sin(\pi \langle w_k, \xi \rangle)}{\pi \langle w_k, \xi \rangle}. \tag{2.49}$$

To summarize, for a parallelepiped that is symmetric about the origin, we have the Fourier pair

$$\left\{ 1_Q(x),\ |\det M| \prod_{k=1}^{d} \frac{\sin(\pi \langle w_k, \xi \rangle)}{\pi \langle w_k, \xi \rangle} \right\}.$$

We could have also computed the latter FT by beginning with our known Fourier transform (2.39) of the cube $\square$, composing the FT with the same linear transformation M of (2.41), and using the 'stretch' lemma, so everything is consistent. $\square$

2.8 The cross-polytope

Another natural convex body in $\mathbb{R}^2$ is the cross-polytope

$$\Diamond_2 := \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; |x_1| + |x_2| \leqslant 1 \right\}. \tag{2.50}$$

In dimension d, the **cross-polytope** $\Diamond_d$ can be defined similarly by its **hyperplane description**

$$\Diamond_d := \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; |x_1| + |x_2| + \cdots + |x_d| \leqslant 1 \right\}. \tag{2.51}$$

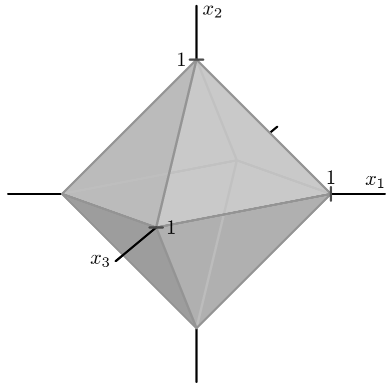


Figure 2.6: The cross-polytope $\Diamond$ in $\mathbb{R}^3$ (courtesy of David Austin)

The cross-polytope is also, by definition, the unit ball in the L_1-norm on Euclidean space, and from this perspective a very natural object. In $\mathbb{R}^3$, the cross-polytope $\Diamond_3$ is often called an **octahedron**.

In this section we only work out the 2-dimensional case of the Fourier transform of the crosspolytope, In Chapter 6, we will work out the Fourier transform of any d-dimensional cross-polytope, $\hat{1}_{\Diamond_d}$, because we will have more tools at our disposal.

Nevertheless, it's instructive to compute $\hat{1}_{\Diamond_2}$ via brute-force for $d = 2$ here, in order to gain some practice. First we define

$$\text{conv(S)} := \text{The } \textbf{convex hull} \text{ of any set } S \subset \mathbb{R}^d, \tag{2.52}$$

the smallest convex set in $\mathbb{R}^d$ that contains S.

Example 2.8. Using the definition of the Fourier transform, we first compute the FT of the 2-dimensional cross polytope:

$$\hat{1}_{\Diamond_2}(\xi) := \int_{\Diamond_2} e^{-2\pi i \langle \xi, x \rangle} dx. \tag{2.53}$$

In $\mathbb{R}^2$, we may write $\Diamond_2$ as a union of the following 4 triangles:

$$\begin{aligned}
\Delta_1 &:= \mathrm{conv}(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right))\\
\Delta_2 &:= \mathrm{conv}(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right))\\
\Delta_3 &:= \mathrm{conv}(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right))\\
\Delta_4 &:= \mathrm{conv}(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)).
\end{aligned}$$

Since these four triangles only intersect in lower-dimensional subsets of $\mathbb{R}^2$, the 2-dimensional integral vanishes on such lower dimensional subsets, and we have:

$$\hat{1}_{\Diamond_2}(\xi) = \hat{1}_{\Delta_1}(\xi) + \hat{1}_{\Delta_2}(\xi) + \hat{1}_{\Delta_3}(\xi) + \hat{1}_{\Delta_4}(\xi). \tag{2.54}$$

Recalling from equation (2.30) of Example 2.5 that the Fourier transform of the standard simplex Δ_1 is

$$\hat{1}_{\Delta_1}(\xi) = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{\xi_1\xi_2} + \frac{e^{-2\pi i\xi_1}}{(-\xi_1+\xi_2)\xi_1} + \frac{e^{-2\pi i\xi_2}}{(\xi_1-\xi_2)\xi_2}\right), \tag{2.55}$$

we can compute $\hat{1}_{\Delta_2}(\xi)$, by reflecting Δ_2 about the x_2-axis (the Jacobian of this transformation is 1), and using our known computation (2.55) for the transform of Δ_1:

$$\begin{aligned}
\hat{1}_{\Delta_2}(\xi_1,\xi_2) &:= \int_{\Delta_2} e^{-2\pi i(x_1\xi_1+x_2\xi_2)}dx\\
&= \int_{\Delta_1} e^{-2\pi i(-x_1\xi_1+x_2\xi_2)}dx\\
&= \int_{\Delta_1} e^{-2\pi i(x_1(-\xi_1)+x_2\xi_2)}dx\\
&= \hat{1}_{\Delta_1}(-\xi_1,\xi_2)).
\end{aligned}$$

Similarly, we have $\hat{1}_{\Delta_3}(\xi_1,\xi_2) = \hat{1}_{\Delta_1}(-\xi_1,-\xi_2)$, and $\hat{1}_{\Delta_4}(\xi_1,\xi_2) = \hat{1}_{\Delta_1}(\xi_1,-\xi_2)$.

Hence we may continue the computation from Equation (2.54) above, putting all the pieces back together:

$$
\begin{aligned}
\hat{1}_{\Diamond_2}(\xi) &= \hat{1}_{\Delta_1}(\xi_1,\xi_2) + \hat{1}_{\Delta_1}(-\xi_1,\xi_2) + \hat{1}_{\Delta_1}(-\xi_1,-\xi_2) + \hat{1}_{\Delta_1}(\xi_1,-\xi_2) \qquad (2.56)\\
&= \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{\xi_1\xi_2} + \frac{-\, e^{2\pi i \xi_1}}{(-\xi_1+\xi_2)\xi_1} + \frac{-\, e^{2\pi i \xi_2}}{(\xi_1-\xi_2)\xi_2} \right) \qquad (2.57)\\
&\quad + \left(\frac{1}{2\pi i}\right)^2 \left(\frac{-1}{\xi_1\xi_2} + \frac{e^{-2\pi i \xi_1}}{(\xi_1+\xi_2)\xi_1} + \frac{e^{2\pi i \xi_2}}{(\xi_1+\xi_2)\xi_2} \right) \qquad (2.58)\\
&\quad + \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{\xi_1\xi_2} + \frac{e^{-2\pi i \xi_1}}{(\xi_1-\xi_2)\xi_1} + \frac{e^{-2\pi i \xi_2}}{(-\xi_1+\xi_2)\xi_2} \right) \qquad (2.59)\\
&\quad + \left(\frac{1}{2\pi i}\right)^2 \left(\frac{-1}{\xi_1\xi_2} + \frac{e^{2\pi i \xi_1}}{(\xi_1+\xi_2)\xi_1} + \frac{e^{-2\pi i \xi_2}}{(\xi_1+\xi_2)\xi_2} \right) \qquad (2.60)\\
&= -\frac{1}{4\pi^2} \left(\frac{\cos(2\pi\xi_1)}{(\xi_1-\xi_2)\xi_1} + \frac{\cos(2\pi\xi_2)}{(-\xi_1+\xi_2)\xi_2} + \frac{\cos(2\pi\xi_1)}{(\xi_1+\xi_2)\xi_1} + \frac{\cos(2\pi\xi_2)}{(\xi_1+\xi_2)\xi_2} \right) \qquad (2.61)\\
&= -\frac{1}{2\pi^2} \left(\frac{\cos(2\pi\xi_1) - \cos(2\pi\xi_2)}{(\xi_1+\xi_2)(\xi_1-\xi_2)} \right). \qquad (2.62)
\end{aligned}
$$

□

There is another fundamental relationship between the cross-polytope and the cube $\mathcal{P} := [-1,1]^d$. To see this relationship, we define, for any polytope $\mathcal{P} \subset \mathbb{R}^d$, its **dual polytope**:

$$
\mathcal{P}^* := \left\{ x \in \mathbb{R}^d \;\middle|\; \langle x, y\rangle \leqslant 1, \text{ for all } y \in \mathcal{P} \right\}. \qquad (2.63)
$$

It is an easy fact (Exercise 2.23) that the cross-polytope and the cube $\mathcal{P} := [-1,1]^d$ are dual to each other, as in Figure 2.7 below.

2.9 Observations and questions

Now we can make several observations about all of the formulas that we found so far, for the Fourier transforms of various polytopes. For the 2-dimensional cross-

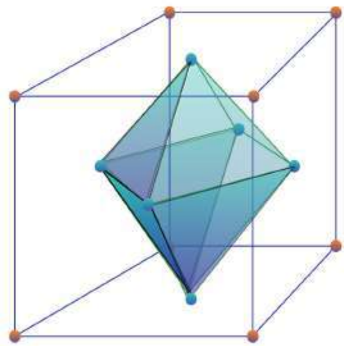


Figure 2.7: The cube and the cross-polytope are duals of each other

polytope, we found that

$$\hat{1}_{\Diamond_2}(\xi) = -\frac{1}{2\pi^2}\left(\frac{\cos(2\pi\xi_1) - \cos(2\pi\xi_2)}{(\xi_1+\xi_2)(\xi_1-\xi_2)}\right). \tag{2.64}$$

(a) It is real-valued for all $\xi \in \mathbb{R}^2$, and this is due to the fact that $\Diamond_2$ is symmetric about the origin (see Section 4.5).

Question 1. *Is it true that* any *symmetric property of a polytope $\mathcal{P}$ is somehow mirrored by a corresponding symmetric property of its Fourier transform?*

Although this question is not well-defined at the moment (it depends on how we define 'symmetric property'), it does sound exciting, and we can morph it into a few well-defined questions later.

(b) The only apparent singularities of the FT in (2.64) (though they are in fact removable singularities) are the two lines $\xi_1 - \xi_2 = 0$ and $\xi_1 + \xi_2 = 0$, and these two lines are *perpendicular* to the facets of $\Diamond_2$, which is not a coincidence (see Chapter 10).

(c) It is always true that the Fourier transform of a polytope is an entire function, by Lemma 2.1, so that the apparent singularities in the denominator $(\xi_1 + \xi_2)(\xi_1 - \xi_2)$ of (2.64) must be removable singularities!

(d) The denominators of all of the FT's so far are always products of **linear forms** in ξ.

Question 2. *Is it true that the Fourier transform of any polytope is always a finite sum of rational functions times an exponential, where the denominators of the rational functions are always products of linear forms?*

Answer: (spoiler alert) Yes! It's too early to prove this here, but we will do so in Theorem 6.2.

(e) We may retrieve the volume of $\Diamond_2$ by letting ξ_1 and ξ_2 tend to zero (Exercise 2.21), as always. Doing so, we obtain

$$\lim_{\xi \to 0} \hat{1}_{\Diamond_2}(\xi) = 2 = \text{Area}(\Diamond_2).$$

Notes

(a) Another way to compute $1_{\Diamond}(\xi)$ for the 2-dimensional cross-polytope $\Diamond$ is to begin with the square $[-\frac{1}{2}, \frac{1}{2}]^2$ and apply a rotation of the plane by $\pi/4$, followed by a simple dilation. Because we know that linear transformations interact in a very elegant way with the FT, this method gives an alternate approach for the Example 2.8 in $\mathbb{R}^2$.

However, this method no longer works for the cross-polytope in dimensions $d \geqslant 3$, where it is not (yet) known if there is a simple way to go from the FT of the cube to the FT of the cross-polytope.

More generally, one may ask:

Question 3. *is there a nice relationship between the FT of a polytope $\mathcal{P}$ and the FT of its dual?*

As far as we know, this question is completely open.

(b) We note that $P_1(x)$ is defined to be equal to 0 at the integers, because its Fourier series naturally converges to the mean of the discontinuity of the function, at each integer.

(c) It has been known since the work of Riemann that the Bernoulli numbers occur as special values of the Riemann zeta function (see Exercise 3.6). Similarly, the Hurwitz zeta function, defined for each fixed $x > 0$ by

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

has a meromorphic continuation to all of $\mathbb{C}$, and its special values at the negative integers are the Bernoulli polynomials $B_n(x)$ (up to a multiplicative constant).

(d) There are sometimes very unusual (yet useful) formulations for the Fourier transform of certain functions. Ramanujan (1915) discovered the following remarkable formula for the Fourier transform of the Gamma function:

$$\int_{\mathbb{R}} |\Gamma(a+iy)| e^{-2\pi i \xi y} \, dy = \frac{\sqrt{\pi}\, \Gamma(a)\Gamma(a+\frac{1}{2})}{\cosh(\pi\xi)^{2a}}, \tag{2.65}$$

valid for $a > 0$. For example with $a := \frac{1}{2}$, in the language of this chapter we have the Fourier pair $\{|\Gamma(\frac{1}{2}+iy)|, \frac{\pi}{\cosh(\pi\xi)}\}$.

Exercises

Problems worthy of attack prove their worth by fighting back. – Paul Erdős

2.1. ♣ *Show that the Fourier transform of the closed interval* $[a, b]$ *is:*

$$\hat{1}_{[a,b]}(\xi) = \frac{e^{-2\pi i \xi a} - e^{-2\pi i \xi b}}{2\pi i \xi},$$

for $\xi \neq 0$.

2.2. *Show that the Fourier transform of the unit cube $C := [0,1]^d \subset \mathbb{R}^d$ is:*

$$\hat{1}_C(\xi) = \frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1 - e^{-2\pi i \xi_k}}{\xi_k}, \tag{2.66}$$

valid for all $\xi \in \mathbb{R}^d$, except for the union of hyperplanes defined by $H := \{x \in \mathbb{R}^d \mid \xi_1 = 0 \text{ or } \xi_2 = 0 \dots \text{ or } \xi_d = 0\}$.

2.3. *Suppose we are given two polynomials $p(x)$ and $q(x)$, of degree d. If there are $d+1$ distinct points $\{z_1, \dots, z_{d+1}\}$ in the complex plane such that $p(z_k) = q(z_k)$ for $k = 1, \dots, d+1$, show that the two polynomials are identical. (Hint: consider $(p-q)(z_k)$)*

2.4. *To gain some facility with generating functions, show by a brute-force computation with Taylor series that the coefficients on the right-hand-side of equation (2.13), which are called $B_n(x)$ by definition, must in fact be polynomials in x.*

In fact, your direct computations will show that for all $n \geqslant 1$, we have

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, x^k,$$

where B_j is the j'th Bernoulli number.

2.5. ♣ *Show that for all $n \geqslant 1$, we have*

$$B_n(1-x) = (-1)^n B_n(x).$$

2.6. ♣ *Show that for all $n \geqslant 1$, we have*

$$B_n(x+1) - B_n(x) = n x^{n-1}.$$

2.7. ♣ *Show that for all $n \geqslant 1$, we have*

$$\frac{d}{dx} B_n(x) = n B_{n-1}(x).$$

2.8. ♣ *Prove that:*

$$\sum_{k=1}^{n-1} k^{d-1} = \frac{B_d(n) - B_d}{d},$$

for all integers $d \geqslant 1$ and $n \geqslant 2$.

2.9. ♣ *Show that the periodic Bernoulli polynomials $P_n(x) := B_n(\{x\})$, for all $n \geqslant 2$, have the following Fourier series:*

$$P_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k^n}, \tag{2.67}$$

valid for all $x \in \mathbb{R}$. For $n \geqslant 2$, these series are absolutely convergent. We note that from the definition above, $B_n(x) = P_n(x)$ when $x \in (0, 1)$.

2.10. *Show that the greatest integer function $\lfloor x \rfloor$ (often called the 'floor function') enjoys the property:*

$$\sum_{k=0}^{N-1} \left\lfloor x + \frac{k}{N} \right\rfloor = \lfloor N x \rfloor,$$

for all $x \in \mathbb{R}$, and all positive integers N, and that in the same range we also have

$$\sum_{k=0}^{N-1} \left\{ x + \frac{k}{N} \right\} = \{N x\}.$$

2.11. *Show that the Bernoulli polynomials enjoy the following identity, proved by Joseph Ludwig Raabe in 1851:*

$$B_n(N x) = N^{n-1} \sum_{k=0}^{N-1} B_n\left(x + \frac{k}{N}\right),$$

for all $x \in \mathbb{R}$, all positive integers N, and for each $n \geqslant 1$.

Notes. Such formulas, in these last two exercises, are also called "multiplication Theorems", and they hold for many other functions, including the Gamma function, the dilogarithm, the Hurwitz zeta function, and many more.

2.12. ♣ *Here we give a different method for defining the Bernoulli polynomials, based on the following three properties that they enjoy:*

1. $B_0(x) = 1$.
2. *For all* $n \geqslant 1$, $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$.
3. *For all* $n \geqslant 1$, *we have* $\int_0^1 B_n(x)dx = 0$.

Show that the latter three properties imply the original defining property of the Bernoulli polynomials (2.13).

2.13. *Here is a more explicit, useful recursion for computing the Bernoulli polynomials. Show that*

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = n x^{n-1},$$

for all $n \geqslant 2$.

2.14. *Use the previous exercise, together with the known list the first* 6 *Bernoulli polynomials that appear in* Equation (2.23), *to compute* $B_7(x)$.

2.15. *Show that for odd* $k \geqslant 3$, *we have* $B_k = 0$.

2.16. *Show that the even Bernoulli numbers alternate in sign. Precisely,* $(-1)^{n+1} B_{2n} \geqslant 0$ *for each positive integer* n.

2.17. *Show that the Bernoulli numbers enjoy the recursive property:*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0,$$

for all $n \geqslant 1$.

2.18. *Show that the Bernoulli numbers enjoy the following asymptotics:*

$$B_{2n} \sim 2\frac{(2n)!}{(2\pi)^{2n}}$$

as $n \to \infty$*. Here we are using the usual notation for asymptotic functions, namely that* $f(n) \sim g(n)$ *as* $n \to \infty$ *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} \to 1$.

2.19. ♣ *Show that the following integrals converge and have the closed forms:*

$$\int_{-\infty}^{\infty} \cos(x^2)dx = \sqrt{\frac{\pi}{2}}, \tag{2.68}$$

$$\int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\frac{\pi}{2}}. \tag{2.69}$$

Notes. These integrals are called Fresnel integrals, and they are related to the Cornu spiral, which was created by Marie Alfred Cornu. Marie used the spiral as a tool for computing diffraction patterns that arise naturally in optics.

2.20. *Prove the following Gamma function identity, using the sinc function:*

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{\Gamma(1+x)\Gamma(1-x)},$$

for all $x \notin \mathbb{Z}$.

Notes. This identity is often called Euler's reflection formula.

2.21. ♣ *Using the formula for the Fourier transform of the 2-dimensional cross-polytope* $\Diamond$*, derived in the text, namely*

$$\hat{1}_{\Diamond}(\xi) = -\frac{1}{2\pi^2}\left(\frac{\cos(2\pi\xi_1) - \cos(2\pi\xi_2)}{\xi_1^2 - \xi_2^2}\right),$$

find the area of $\Diamond$ *by letting* $\xi \to 0$ *in the latter formula.*

2.22. *There are (at least) two different ways of periodizing a given function $f : \mathbb{R} \to \mathbb{C}$ with respect to $\mathbb{Z}$. First, we can define $F_1(x) := f(\{x\})$, so that F_1 is periodic on $\mathbb{R}$ with period 1. Second, we may also define $F_2(x) := \sum_{n\in\mathbb{Z}} f(x+n)$, which is also a periodic function on $\mathbb{R}$ with period 1.*

Find an integrable (meaning that $\int_{\mathbb{R}} f(x)dx$ converges) function f for which $F_1 \neq F_2$, as functions.

Notes. In Chapter 3, we will see that the latter function $F_2(x) := \sum_{n\in\mathbb{Z}} f(x+n)$ captures a lot more information about f, and often captures all of f as well.

2.23. ♣ *Show that the d-dimensional cross-polytope $\Diamond$ and the cube $\square := [-1,1]^d$ are dual to each other.*

2.24. *Prove the following 2-dimensional integral formula:*

$$\int_{\substack{\lambda_1,\lambda_2\geqslant 0\\ \lambda_1+\lambda_2\leqslant 1}} e^{A\lambda_1}e^{B\lambda_2}d\lambda_1 d\lambda_2 = \frac{Be^A - Ae^B}{AB(A-B)} + \frac{1}{AB}, \tag{2.70}$$

valid for all $A, B \in \mathbb{C}$ such that $AB(A-B) \neq 0$.

2.25. *Using the ideas of Equation (2.33), prove (by induction on the dimension) that the Fourier transform of a general d-dimensional simplex $\Delta \subset \mathbb{R}^d$ is given by:*

$$\hat{1}_{\Delta}(\xi) = (\mathrm{vol}\,\Delta)d!\sum_{j=1}^{N} \frac{e^{-2\pi i\langle v_j,\xi\rangle}}{\prod_{1\leqslant k\leqslant d}\langle v_j - v_k, \xi\rangle}[k \neq j], \tag{2.71}$$

for all $\xi \in \mathbb{R}^d$, where the vertex set of $\mathcal{P}$ is $\{v_1, \ldots, v_N\}$.

2.26 (Abel summation by parts)**.** ♣ *Here we prove the straightforward but very useful technique of Niels Abel, called* ***Abel summation by parts****. Suppose we are given two sequences $\{a_n\}_{n=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$. We define the finite partial sums $B_n := \sum_{k=1}^{n} b_k$. Then we have*

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1}), \tag{2.72}$$

for all $n \geqslant 2$.

Notes. Using the forward difference operator, it's easy to recognize identity (2.72) as a discrete version of integration by parts.

2.27 (Dirichlet's convergence test). ♣ *Suppose we are given a real sequence $\{a_n\}_{n=1}^{\infty}$, and a complex sequence $\{b_n\}_{n=1}^{\infty}$, such that*

(a) $\{a_n\}$ is monotonically decreasing to 0, and

(b) $|\sum_{k=1}^{n} b_k| \leqslant M$, for some positive constant M, and all $n \geqslant 1$.

Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

2.28. *Prove that for all $x \in \mathbb{R} - \mathbb{Z}$, we have the following important identity, called the "Dirichlet kernel", after Peter Gustav Lejeune Dirichlet:*

$$\sum_{k=-n}^{n} e^{2\pi ikx} = \frac{\sin\left(2\pi x(n+\frac{1}{2})\right)}{\sin(\pi x)}. \tag{2.73}$$

2.29. *For any fixed $x \in \mathbb{R} - \mathbb{Z}$, show that we have the bound on the following exponential sum:*

$$\left|\sum_{k=1}^{n} e^{2\pi ikx}\right| \leqslant \frac{1}{|\sin(\pi x)|}. \tag{2.74}$$

2.30. ♣ *Prove that $\sum_{m=1}^{\infty} \frac{e^{2\pi ima}}{m}$ converges, given any fixed $a \in \mathbb{R} - \mathbb{Z}$.*

Notes. We see that, although $\sum_{m=1}^{\infty} \frac{e^{2\pi ima}}{m}$ does not converge absolutely, Abel's summation formula (2.72) gives us

$$\sum_{k=1}^{n} \frac{e^{2\pi ika}}{k} = \frac{1}{n}\sum_{r=1}^{n} e^{2\pi ira} + \sum_{k=1}^{n-1}\left(\sum_{r=1}^{k} e^{2\pi ira}\right)\frac{1}{k(k+1)},$$

and the latter series ***does converge absolutely****. So we see that Abel summation transforms one series (that barely converges at all) into another series that converges more rapidly.*

3 Tools of the trade: Fourier analysis

"... Fourier's great mathematical poem."

[Referring to Fourier's mathematical theory of the conduction of heat]

– William Thomson Kelvin

3.1 Intuition

Because we will use tools from Fourier analysis throughout, we introduce them here as an outline of the field, with the goal of *applying* them to the discrete geometry of polytopes, lattices, and their interactions.

In this chapter we develop, and sometimes quote, the basic and necessary tools of Fourier analysis, so that we may tackle problems in the enumerative combinatorics of polytopes, in number theory, and in some other fields. We emphasize that the Poisson summation formula allows us to **discretize integrals**, in a sense that will be made precise in later chapters.

One pattern that the reader may have already noticed, among all of the examples of Fourier transforms of polytopes computed thus far, is that each of them is

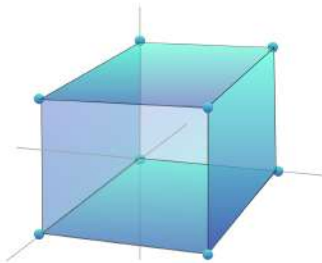


Figure 3.1: The unit cube $\square := [0,1]^3$, in $\mathbb{R}^3$, which tiles the space by translations. Which other polytopes tile by translations? How can we make mathematical use of such tilings? In particular, can we give an explicit basis of exponentials for functions defined on $\square$?

a linear combination of a very special kind of rational function of ξ, multiplied by a complex exponential that involves a vertex of the polytope:

$$\hat{1}_{\mathcal{P}}(\xi) = \sum_{k=1}^{M} \frac{1}{\prod_{j=1}^{d} \langle \omega_{j,k}(v_k), \xi \rangle} e^{2\pi i \langle v_k, \xi \rangle}, \tag{3.1}$$

where the vertices of $\mathcal{P}$ are $v_1, \ldots, v_N$, where $M \geqslant N$. We observed that in all of our examples thus far, the denominators are in fact products of linear forms, as in (3.1). We will be able to see some of the more precise geometric structure for these products of linear forms, which come from the edges of the polytope, once we learn more about Fourier–Laplace transforms of cones.

It is rather astounding that every single fact about a given polytope $\mathcal{P}$ is somehow hiding inside these **rational-exponential** functions given by (3.1).

3.2 Introduction

In the spirit of bringing the reader very quickly up to speed, regarding the applications of Fourier analytic tools, we outline the basics of the field, and only prove some of them. Nowadays, there are many good texts on Fourier analysis, see Note (a).

One of the most useful tools for us is the Poisson summation formula, so we will indeed prove it in some generality. As we will see, the Fourier transform is a very friendly creature, allowing us to travel back and forth between the "space domain" and the "frequency domain" to obtain many useful results. The readers who are already familiar with basics of Fourier analysis may easily skip this chapter without impeding their understanding of the rest of the book. For the rest of this chapter, we sometimes recall some standard facts, and the reader is invited to look at the books mentioned above for their relevant proofs. We also note that this chapter is only meant as an intuitive introduction to Fourier analysis, so although we think about the Hilbert space $L^2([0,1]^d)$, we will often not dwell on when a function belongs to L^1, or $L^2 \cap L^1$, or $L^1 \cap C^1$, etc.

First, we introduce the space of **square integrable** functions on the cube $\square := [0,1]^d$:

$$L^2(\square) := \{f : [0,1]^d \to \mathbb{C} \mid \int_{[0,1]^d} |f(x)|^2 dx < \infty\}.$$

The latter function space is in fact a **Hilbert space**, because it is endowed with the following natural inner product:

$$\langle f, g\rangle := \int_{[0,1]^d} f(x)\overline{g(x)}dx,$$

relative to which it is a complete metric space. The norm of a function $f \in L^2(\square)$ is, by definition:

$$\|f\| := \sqrt{\langle f, f\rangle} = \sqrt{\int_{[0,1]^d} |f(x)|^2 dx}.$$

Because $L^2(\square)$ is a Hilbert space, we have the Cauchy–Schwarz inequality here:

$$\int_\square f(x)\overline{g(x)}dx \leqslant \left(\int_\square |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\square |g(x)|^2 dx\right)^{\frac{1}{2}}, \tag{3.2}$$

for all $f, g \in L^2(\square)$, with equality if and only if $f(x) = Cg(x)$ for some constant C (Exercise 3.2). It is in this space, $L^2(\square)$, that we build Fourier series.

The fact that $L^2(\square)$ is a Hilbert space means that it is a very friendly space, due to the extra structure that exists here, namely the ability to use inner products to measure distance between functions. In any Hilbert space, one of the most useful inequalities is the Cauchy–Schwarz inequality:

$$\langle u, v\rangle \leqslant \|u\|\|v\|, \tag{3.3}$$

with equality if and only if u, v are linearly independent. At this point the curious reader might wonder 'are there any other inner products on $\mathbb{R}^d$', besides the usual inner product $\langle x, y\rangle := \sum_{k=1}^{d} x_k y_k$? A classification of all inner products on Euclidean space is given in Exercise 3.13.

There are (at least) two other function spaces that come up very naturally as well. First, the space of **square integrable functions** on $\mathbb{R}^d$ is defined by:

$$L^2(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} \mid \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty\}.$$

Second, the space of **integrable functions** on Euclidean space is often used, and is defined by:

$$L^1(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} \mid \int_{\mathbb{R}^d} |f(x)| dx < \infty\}.$$

There are many fascinating facts about all of these functions spaces, including the fact that $L^1(\mathbb{R}^d)$ is not a Hilbert space, as we can easily show by exhibiting a counterexample to the Cauchy–Schwarz inequality, as follows.

Example 3.1. We claim that the Cauchy–Schwarz inequality (which is one of the most useful inequalities) is in fact false in $L^1(\mathbb{R})$. If the Cauchy–Schwarz inequality was true here, we would have

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx \leqslant \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |g(x)|^2 dx\right)^{\frac{1}{2}}$$

for all functions $f, g \in L^1(\mathbb{R})$. As a counterexample, let

$$f(x) := 1_{(0,1)}(x)\frac{1}{\sqrt{x}}.$$

It's easy to see that $f \in L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} 1_{(0,1)}(x)\frac{1}{\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{x}}dx = \frac{1}{2}.$$

But $\int_{\mathbb{R}} f(x) \cdot f(x)dx = \int_0^1 \frac{1}{x}dx$ diverges, so that we do not have a Cauchy–Schwarz inequality in $L^1(\mathbb{R})$, because here both the left-hand-side and the right-hand-side of such an inequality do not even converge.

However, if at least one of the functions f, g above is bounded on $\mathbb{R}$, then we do have a Cauchy–Schwartz inequality in $L^2(\mathbb{R})$. □

Some books prefer to use the language of "the torus", which is the unit cube $\square$ but with opposite faces identified, or "glued", but here we only need the unit cube.

For any $k := (k_1, \dots, k_d) \in \mathbb{Z}^d_{\geqslant 0}$, we can define the multivariable differential operator

$$D_k := \frac{\partial}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}.$$

Example 3.2. In $\mathbb{R}^1$, this is the usual k'th derivative: $D_k f(x) := \frac{d}{dx^k} f(x)$. In $\mathbb{R}^2$, for example, we have $D_{(1,7)} f(x) := \frac{\partial}{\partial x_1 \partial x_2^7} f(x)$. $\square$

The **order** of the differential operator D_k is by definition $|k| := k_1 + \cdots + k_d$. To define spaces of differentiable functions, we call a function $f : \mathbb{R}^d \to \mathbb{C}$ a C^m-function if all partial derivatives $D_k f$ of order $|k| \leqslant m$ exists and are continuous. We denote the collection of all such C^m-functions on Euclidean space by $C^m(\mathbb{R}^d)$. When considering **infinitely differentiable functions on Euclidean space**, we denote this space by $C^\infty(\mathbb{R}^d)$.

Finally, we mention another basic ingredient, which holds for all of the function spaces above, the **triangle inequality for integrals**:

$$\left| \int_S f(x)dx \right| \leqslant \int_S |f(x)|dx, \tag{3.4}$$

for any measurable subset $S \subset \mathbb{R}^d$. It follows that if f is bounded on $\mathbb{R}^d$, say $f(x) \leqslant M$ for all $x \in \mathbb{R}^d$, then

$$\left| \int_S f(x)dx \right| \leqslant \int_S |f(x)|dx \leqslant \text{measure(S)} \cdot \text{M}, \tag{3.5}$$

a simple but very useful bound for many applications. We will also use some norms on Euclidean space, and the two most common norms here are the 1-norm $\|x\|_1 := |x_1| + \cdots + |x_d|$, and the 2-norm $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$. Among the many norm relations, we mention one elementary but interesting relation between two norms on $\mathbb{R}^d$:

$$\|x\|_1 \leqslant \sqrt{n}\, \|x\|_2,$$

for all vectors $x \in \mathbb{R}^d$ (Exercise 3.1).

3.3 Orthogonality

For the unit cube $\square := [0,1]^d$, there is a natural inner product for the space of square-integrable functions $V := L^2(\square)$, as we mentioned above, and this inner product is defined by:

$$\langle f, g\rangle := \int_{[0,1]^d} f(x)\overline{g(x)}dx. \tag{3.6}$$

We focus on a particularly useful sequence of exponential functions, which turn out to be a basis for V. For each $n \in \mathbb{Z}$, we define $e_n : \mathbb{R}^d \to \mathbb{C}$ by:

$$e_n(x) := e^{2\pi i \langle n,x\rangle}.$$

This countable collection of exponentials turns out to form a complete basis for V. The orthogonality is the first step, which we prove next, and we simply quote the more difficult fact that they also span all of V.

Theorem 3.1 (Orthogonality relations for the exponentials $e_n(x)$).

$$\int_{[0,1]^d} e_n(x)\overline{e_m(x)}dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if not.} \end{cases} \tag{3.7}$$

Proof. Because of the geometry of the cube, we can proceed in this case by separating the variables, so that for $n \neq m$ we compute:

$$\begin{aligned}\int_{[0,1]^d} e_n(x)\overline{e_m(x)}dx &= \int_{[0,1]^d} e^{2\pi i\langle n-m,x\rangle}dx \\ &= \prod_{k=1}^{d} \frac{e^{2\pi i(n_k-m_k)}-1}{2\pi i(n_k-m_k)} = 0,\end{aligned}$$

because each $n_k - m_k \in \mathbb{Z}$. □

Let's see how we can expand (certain) functions in a Fourier series, as well as find a formula for their series coefficients, in a "footloose and carefree" way for just a moment - i.e. throwing to the wind all caution regarding convergence.

Given that the sequence of exponential functions $\{e_n(x)\}_{n\in\mathbb{Z}^d}$ forms a basis of $V := L^2(\Box)$, we know from Linear Algebra that any function $f \in V$ may be written in terms of this basis:

$$f(x) = \sum_{n\in\mathbb{Z}^d} a(n)e_n(x). \tag{3.8}$$

How do we compute the Fourier coefficients a_n? Let's go through the intuitive process here, ignoring convergence issues. Well, again by Linear Algebra, we take the inner product of both sides with a fixed basis element $e_k(x)$:

$$\begin{aligned}
\langle f(x), e_k(x)\rangle &= \langle \sum_{n\in\mathbb{Z}^d} a(n)e_n(x), e_k(x)\rangle \\
&= \sum_{n\in\mathbb{Z}^d} a(n)\langle e_n(x), e_k(x)\rangle \\
&= \sum_{n\in\mathbb{Z}^d} a(n)\delta(n,k) \\
&= a(k),
\end{aligned}$$

where we've used the orthogonality relations, Theorem 3.1 above, in the third equality. Therefore, it must be the case that

$$\begin{aligned}
a(k) &= \langle f(x), e_k(x)\rangle \\
&:= \int_{[0,1]^d} f(x)\overline{e^{2\pi i\langle k,x\rangle}}dx \\
&= \int_{[0,1]^d} f(x)e^{-2\pi i\langle k,x\rangle}dx,
\end{aligned}$$

also called the **Fourier coefficients** of f.

In other words, the Fourier coefficients of f are just the projection of the function f onto the basis of exponentials. We record here a rigorous version of the latter intuitive arguments, which constitutes the basis for the classical result that a periodic function on $\mathbb{R}^d$, belonging to $L^2(\Box)$, has a pointwise convergent Fourier series.

Theorem 3.2 (Fourier series for periodic functions). *Let $F : \mathbb{R}^d \to \mathbb{C}$ be a periodic function, continuous on $[0,1)^d$, and furthermore let $F \in L^2(\Box)$. Then F*

has the pointwise convergent ***Fourier series****:*

$$F(x) = \sum_{n \in \mathbb{Z}^d} a(n) e^{2\pi i \langle n, x \rangle}, \tag{3.9}$$

which holds for all $x \in (0,1)^d$*. In addition, the* ***Fourier coefficients*** *of* F *have the integral formula:*

$$a(n) = \int_{[0,1]^d} F(u) e^{-2\pi i \langle n, u \rangle} du, \tag{3.10}$$

for all $n \in \mathbb{Z}^d$*.*

We note that the Fourier coefficients above are integrals over the unit cube $[0,1]^d$. We will encounter below, in the Poisson summation formula, a similar integral, but over all of $\mathbb{R}^d$. Such an integral comes up naturally during the typical "unfolding" procedure of the proof of Poisson summation, and will be called a Fourier transform (more on this in Section 3.6). We therefore define the **Fourier transform** of f, for all $x \in \mathbb{R}^d$, by the integral:

$$\hat{f}(x) = \int_{\mathbb{R}^d} f(u) e^{-2\pi i \langle x, u \rangle} du. \tag{3.11}$$

For the real line, we have the following refined version of Theorem 3.2. We use the standard notation $f(t^+) := \lim_{\epsilon \to 0} f(t+\epsilon)$, and $f(t^-) := \lim_{\epsilon \to 0} f(t-\epsilon)$, where ϵ is always chosen to be positive.

Theorem 3.3. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a periodic function, with fundamental domain* $[-\frac{1}{2}, \frac{1}{2})$*, and piecewise smooth on* $\mathbb{R}$*. Then, for each* $t \in \mathbb{R}$*, we have*

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n t} = \frac{f(t^+) + f(t^-)}{2}. \tag{3.12}$$

3.4 The Schwartz space, and nice functions

We recall that our definition of a 'nice function' was any function $f : \mathbb{R}^d \to \mathbb{C}$ for which the Poisson summation formula holds. Here we give various sufficient conditions for a function f to be nice. A **Schwartz function** $f : \mathbb{R} \to \mathbb{C}$ is defined as any function $f \in C^\infty(\mathbb{R})$ that satisfies the following growth condition: for all integers $a, k \geqslant 0$,

$$|x^a \frac{d}{dx^k} f(x)| \text{ is bounded on } \mathbb{R}. \tag{3.13}$$

For $\mathbb{R}^d$, we can define **Schwartz functions** similarly: they are infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that for all vectors $a, k \in \mathbb{Z}^d_{\geqslant 0}$ we have:

$$|x^a D_k f(x)| \text{ is bounded on } \mathbb{R}^d, \tag{3.14}$$

where $x^a := x_1^{a_1} \cdots x_d^{a_d}$ is the standard multi-index notation. Intuitively, a Schwartz function decreases 'at infinity' faster than any polynomial function. We also define the **Schwartz space** $S(\mathbb{R}^d)$ to be set of all Schwartz functions $f : \mathbb{R}^d \to \mathbb{C}$. We recall again that the Fourier transform is defined by $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(u) e^{-2\pi i \langle \xi, u \rangle} du$.

Theorem 3.4. *The Fourier transform maps the Schwartz space $S(\mathbb{R}^d)$ one-to-one, onto itself. (See Exercise 3.12)*

Example 3.3. The Gaussian function $G_t(x) := e^{-t||x||^2}$ is a Schwartz function, for each fixed $t > 0$.

We first consider $\mathbb{R}^1$, where we note that the 1-dimensional Gaussian is a Schwartz function, as follows. We observe that

$$x^a \frac{d}{dx^k} G_t(x) = H_n(x) G_t(x),$$

where $H_n(x)$ is a univariate polynomial (Exercise 3.15). Because $\lim_{x\to\infty} \frac{H_n(x)}{e^{t||x||^2}} = 0$, $G_t(x)$ is a Schwartz function. Now we note that the product of Schwartz functions if again a Schwartz function, and hence the d-dimensional Gaussian, $G_t(x) := e^{-t||x||^2} = \prod_{k=1}^d e^{-tx_k^2}$, a product of 1-dimensional Gaussians, is a Schwartz function. □

3.5 The inverse Fourier transform

We will often use the fundamental fact that the Fourier transform is invertible, and more precisely that its inverse has the following form.

Theorem 3.5. *Given a function $f : \mathbb{R}^d \to \mathbb{C}$, with both $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, we have*

$$(\mathcal{F} \circ \mathcal{F}) f(x) = f(-x), \tag{3.15}$$

for all $x \in \mathbb{R}^d$.

Identity 3.15 tells us that the inverse Fourier transform $\mathcal{F}^{-1}$ exists, so if we compose both sides of 3.15 with $\mathcal{F}^{-1}$ (and replace x by $-x$), we get the alternative, perhaps more familiar form of the inversion formula:

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(u) e^{2\pi i \langle u, x\rangle} du. \tag{3.16}$$

3.6 Poisson Summation

We formally introduce the Poisson summation formula, one of the most useful tools in analytic number theory, and in discrete / combinatorial geometry. There are many different families of sufficient conditions that a function f can satisfy, in order for Poisson summation to be applicable to f.

Theorem 3.6 (The Poisson Summation Formula for $\mathbb{Z}^d$). *Given a* Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$, *we have*

$$\sum_{n \in \mathbb{Z}^d} f(n + x) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{2\pi i \langle \xi, x\rangle}, \tag{3.17}$$

valid for all $x \in \mathbb{R}^d$.

Proof. If we let $F(x) := \sum_{n \in \mathbb{Z}^d} f(n + x)$, then we notice that F is periodic on $\mathbb{R}^d$, with the cube $[0, 1)^d$ as a fundamental domain. The argument is easy: fixing any $m \in \mathbb{Z}^d$ we get

$$F(x + m) = \sum_{n \in \mathbb{Z}^d} f(n + x + m) = \sum_{k \in \mathbb{Z}^d} f(x + k) = F(x),$$

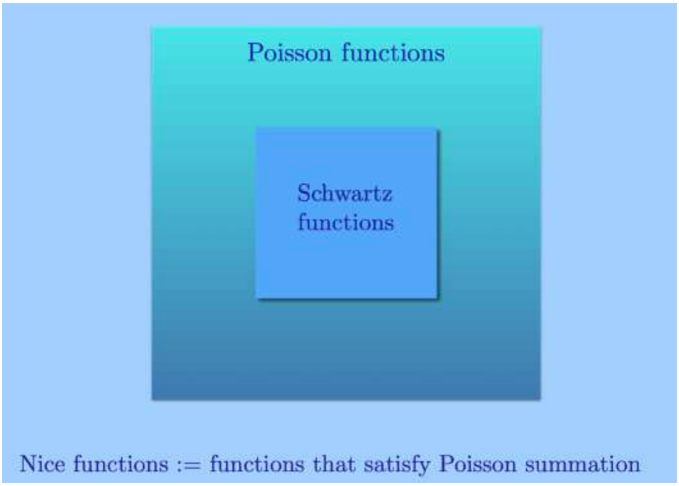


Figure 3.2: Spaces of functions for Poisson summation

because $\mathbb{Z}^d + m = \mathbb{Z}^d$. By Theorem 3.2, the periodic function F has a Fourier series, so let's compute it:

$$F(x) := \sum_{k \in \mathbb{Z}^d} a(k) e^{2\pi i \langle k, x \rangle},$$

where $a(k) = \int_{[0,1)^d} F(u) e^{2\pi i \langle k,u \rangle} du$ for each fixed $k \in \mathbb{Z}^d$. Let's see what happens if we massage $a(k)$ a bit:

$$\begin{aligned} a(k) &:= \int_{[0,1)^d} F(u) e^{-2\pi i \langle k,u \rangle} du \\ &:= \int_{[0,1)^d} \sum_{n \in \mathbb{Z}^d} f(n+u) e^{-2\pi i \langle k,u \rangle} du \\ &:= \sum_{n \in \mathbb{Z}^d} \int_{[0,1)^d} f(n+u) e^{-2\pi i \langle k,u \rangle} du, \end{aligned}$$

where the interchange of summation and integral is allowed because f is a Schwartz function, and hence the Dominated convergence theorem applies. Now, we fix an $n \in \mathbb{Z}^d$ in the outer sum, and make the change of variable in the integral: $n + u := w$, so that $du = dw$.

A critical step in this proof is the fact that as u varies over the cube $[0,1)^d$, $n+u$ varies over all of $\mathbb{R}^d$ because we have a tiling of Euclidean space by the unit cube: $[0,1)^d + \mathbb{Z}^d = \mathbb{R}^d$. We note that under this change of variable, $e^{-2\pi i\langle k,u\rangle} = e^{-2\pi i\langle k,w-n\rangle} = e^{-2\pi i\langle k,w\rangle}$, because $k,n \in \mathbb{Z}^d$ and hence $e^{2\pi i\langle k,n\rangle} = 1$. Thus, we finally have:

$$a(k) = \int_{\mathbb{R}^d} f(w)e^{-2\pi i\langle k,w\rangle}dw := \hat{f}(k),$$

so that $F(x) = \sum_{k\in\mathbb{Z}^d} a(k)e^{2\pi i\langle k,x\rangle} = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{2\pi i\langle k,x\rangle}$. $\square$

We define a function $f : \mathbb{R}^d \to \mathbb{C}$ to be a **nice function** if Poisson summation (3.17) holds for f pointwise. Figure 3.2 suggests a simple containment relation between some of these function spaces, as we can (and will) easily prove.

Theorem 3.7 (Poisson, 1837). *Let $f \in L^2(\mathbb{R}^d)$, and suppose there exist positive constants δ, C such that for all $x \in \mathbb{R}^d$:*

$$|f(x)| < \frac{C}{(1+|x|)^{d+\delta}} \quad \text{and} \quad |\hat{f}(x)| < \frac{C}{(1+|x|)^{d+\delta}}. \tag{3.18}$$

Then f is a nice function.

For a proof of this important theorem, due to Poisson himself, see Stein and Weiss (1971). We therefore call the space of functions that satisfy the hypotheses (3.18), the **Poisson space** of functions.

We observe that if a function f is a Schwartz function, then using the fact that the Fourier transform maps $S(\mathbb{R}^d)$ bijectively onto itself, we see that $\hat{f}$ also satisfies the same growth conditions, and $\hat{f}$ is therefore another Schwartz function. Hence both f and $\hat{f}$ decay faster than any polynomials, and in particular they are both in the Poisson space, as Figure 3.2 suggested. There are other families of nice functions in the literature that impose other restrictions, such as 'functions of bounded variation', and 'absolutely continuous functions'. These other sufficient conditions for the validity of Poisson summation depend on the application at hand.

There are a few things to notice about the classical, and pretty proof of Equation (3.17). The first is that we began with any square-integrable function f defined on all of $\mathbb{R}^d$, and forced a periodization of it, which we called F. This is known as the "folding" part of the proof. Then, at the end of the proof, there is the "unfolding" process, where we sum an integral over the lattice, and it transforms into a single integral over $\mathbb{R}^d$.

The second thing we notice is that the integral over $\mathbb{R}^d$, which by definition is called the "Fourier transform of f" (3.11), appears quite naturally, due to the tiling of $\mathbb{R}^d$ by the unit cube $[0, 1)^d$. Hopefully there will now be no confusion as to the difference between the integral over the cube, and the integral over $\mathbb{R}^d$, both appearing together in this proof. In fact Poisson summation may also be thought of as bringing together Fourier integrals with Fourier series.

Very frequently, in various applications, we will have occasion to set $x = 0$ in (3.17) above, yielding the more symmetric relation:

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{\xi\in\mathbb{Z}^d} \hat{f}(\xi). \tag{3.19}$$

There is a more general version of the Poisson summation formula, Theorem 3.8 below, which holds for any lattice, and which follows rather quickly from the Poisson summation formula above. Given any full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$, we define its **dual lattice** to be $\mathcal{L}^* := M^{-T}(\mathbb{Z}^d)$, where M^{-T} is the inverse transpose matrix of the real matrix M (see Section 5.6 for more on dual lattices).

As we've seen in Lemma 2.3, Fourier Transforms behave beautifully under compositions with any linear transformation. We will use this fact again in the proof of the following extension of Poisson summation, which holds for all lattices and is quite standard.

Theorem 3.8 (The Poisson Summation Formula for lattices). *Given a function $f : \mathbb{R}^d \to \mathbb{C}$, belonging to the Poisson space as defined in Theorem 3.7, we have*

$$\sum_{n\in\mathcal{L}} f(n+x) = \frac{1}{\det\mathcal{L}} \sum_{m\in\mathcal{L}^*} \hat{f}(m)e^{2\pi i\langle x,m\rangle}, \tag{3.20}$$

valid for all $x \in \mathbb{R}^d$.

Proof. Any lattice (full-rank) may be written as $\mathcal{L} := M(\mathbb{Z}^d)$, so that $\det\mathcal{L} := |\det M|$. Using the Poisson summation formula (3.17), with the change of variable

$n = Mk$, with $k \in \mathbb{Z}^d$, we have:

$$\begin{aligned}
\sum_{n\in\mathcal{L}} f(n) &= \sum_{k\in\mathbb{Z}^d} (f\circ M)(k) \\
&= \sum_{\xi\in\mathbb{Z}^d} \widehat{(f\circ M)}(\xi) \\
&= \frac{1}{|\det M|} \sum_{\xi\in\mathbb{Z}^d} \hat{f}(M^{-T}\xi) \\
&= \frac{1}{\det\mathcal{L}} \sum_{m\in\mathcal{L}^*} \hat{f}(m).
\end{aligned}$$

where in the third equality we used the elementary Lemma 2.3. □

As before, we have the useful special case:

$$\sum_{n\in\mathcal{L}} f(n) = \frac{1}{\det\mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{f}(\xi). \tag{3.21}$$

As an afterthought, it turns out that this latter special case also implies the general case, namely Theorem 3.8 (Exercise 3.18).

A traditional application of the Poisson summation formula is the quick derivation of the functional equation of the theta function. We first define the Gaussian function by:

$$G_t(x) := t^{-\frac{d}{2}} e^{-\frac{\pi}{t}||x||^2}, \tag{3.22}$$

for each fixed $t > 0$, and for all $x \in \mathbb{R}^d$. Two immediately interesting properties of the Gaussian are:

$$\int_{\mathbb{R}^d} G_t(x)dx = 1,$$

and

$$\hat{G}_t(m) = e^{-\pi t||m||^2}, \tag{3.23}$$

properties which are important in Statistics as well (Exercises 3.19 and 3.20). Each fixed ϵ gives us one Gaussian function and Intuitively, as $\epsilon \to 0$, this sequence of Gaussians approaches the "Dirac delta function" at the origin, which is really known as a "generalized function", or "distribution" (see the Notes).

Example 3.4. The classical theta function (for the integer lattice) is defined by:

$$\theta(t) = \sum_{n\in\mathbb{Z}^d} e^{-\pi t||m||^2}. \tag{3.24}$$

We claim that it has the functional equation

$$\theta\left(\frac{1}{t}\right) = t^{\frac{d}{2}}\theta(t), \tag{3.25}$$

for all $t > 0$. To see this, we apply the Poisson summation formula (3.19) with $f(x) := G_t(x)$, using our knowledge of its transform, from (3.23):

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} G_t(n) &= \sum_{\xi\in\mathbb{Z}^d} \hat{G}_t(\xi) \\ &= \sum_{\xi\in\mathbb{Z}^d} e^{-\pi t||\xi||^2} := \theta(t).\end{aligned}$$

Since $\sum_{n\in\mathbb{Z}^d} G_t(n) := t^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} e^{-\frac{\pi}{t}||n||^2} := t^{-\frac{d}{2}}\theta\left(\frac{1}{t}\right)$, (3.25) is proved. □

3.7 Convolution

For $f, g \in L^1(\mathbb{R}^d)$, we define their **convolution** by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy,$$

for all $x \in \mathbb{R}^d$ for which the integral makes sense. Intuitively - and it'll take a few formulas to make the following sentence more rigorous - "convolution is how 'waves' in the frequency space like to multiply". Although there is no Cauchy–Schwarz inequality in $f, g \in L^1(\mathbb{R}^d)$ (Example 3.1), we do have the following useful and standard facts.

Lemma 3.1. *For $f, g \in L^1(\mathbb{R}^d)$, their convolution $(f * g)(x)$ exists for almost all $x \in \mathbb{R}^d$.* □

Lemma 3.2.
$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi),$$
for all $f, g \in L^2(\mathbb{R}^d)$. □

Example 3.5. When $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$, the convolution of $1_{\mathcal{P}}$ with itself is drawn in Figure 3.3. We can already see that this convolution is a continuous function, hence a little smoother than the discontinuous function $1_{\mathcal{P}}$. Using Lemma 3.2 we have
$$\widehat{(1_{\mathcal{P}} * 1_{\mathcal{P}})}(\xi) = \hat{1}_{\mathcal{P}}(\xi)\hat{1}_{\mathcal{P}}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2.$$

We've used equation 2.1 in the last equality, for the Fourier transform of our interval $\mathcal{P}$ here. Considering the graph in Figure 3.4, for the Fourier transform of the convolution $(1_{\mathcal{P}} * 1_{\mathcal{P}})$, we see that this positive function is already much more tightly concentrated near the origin, as compared with $\text{sinc}(x) := \hat{1}_{\mathcal{P}}(\xi)$.

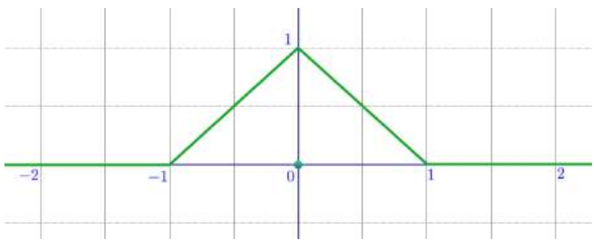


Figure 3.3: The function $(1_{\mathcal{P}} * 1_{\mathcal{P}})(x)$, with $\mathcal{P} := \left[-\frac{1}{2}, \frac{1}{2}\right]$

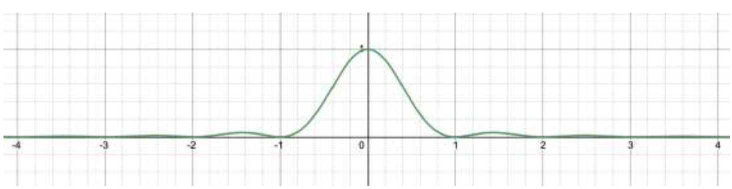


Figure 3.4: The Fourier transform $\widehat{(1_{\mathcal{P}} * 1_{\mathcal{P}})}(\xi)$, which is equal to the smooth function $\left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 := \text{sinc}^2(\xi)$.

□

Lemma 3.2 means that convolution of functions in the space domain corresponds to the usual multiplication of functions in the frequency domain (and vice versa). It is easy to check that the convolution product is commutative, distributive and associative, meaning that $f * g = g * f$, $f * (g + h) = f * g + f * h$, and $f * (g * h) = (f * g) * h$, respectively, for all $f, g, h \in L^1(\mathbb{R}^d)$.

Another useful intuitive idea is that convolution is a kind of averaging process, and that the convolution of two functions becomes a little smoother than either one of them.

For our applications, when we consider the indicator function $1_{\mathcal{P}}(x)$ for a polytope $\mathcal{P}$, then this function is not continuous on $\mathbb{R}^d$, so that the Poisson summation formula does not necessarily hold for it. But if we consider the convolution of $1_{\mathcal{P}}(x)$ with a Gaussian, for example, then we arrive at the C^∞ function

$$(1_{\mathcal{P}} * G_t)(x),$$

for which the Poisson summation does hold. In the sequel, we will use the latter convolved function in tandem with Poisson summation to study "solid angles".

One of the main results in Fourier analysis is the **Plancherel Theorem**, which tells us that the Fourier transform is an isometry of metric spaces. In other words, the transform preserves norms of functions: $\|\hat{f}\| = \|f\|$.

Theorem 3.9 (Plancherel I)**.** *Let $f \in L^2(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

Proof. We let $g(x) := \overline{f(-x)}$, so that

$$\begin{aligned} \hat{g}(\xi) &= \int_{\mathbb{R}^d} \overline{f(-x)} e^{-2\pi i \langle x, \xi \rangle} dx \\ &= \overline{\int_{\mathbb{R}^d} f(-x) e^{2\pi i \langle x, \xi \rangle} dx} \\ &= \overline{\hat{f}(\xi)}. \end{aligned}$$

We define $h := f * g$, and by Lemma 3.2 we have $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$, so that $\hat{h}(\xi) = \|\hat{f}(\xi)\|^2$.

Now, $h(0) := \int_{\mathbb{R}^d} f(0 - x) g(x) dx = \int_{\mathbb{R}^d} f(-x)\overline{f(-x)} dx = \int_{\mathbb{R}^d} |f(x)|^2$.

On the other hand, $h(0) = \int_{\mathbb{R}^d} \hat{h}(\xi) d\xi = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi$. We therefore have

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

□

More generally, we have the following extended version of Plancherel's Theorem, which has an essentially identical proof (Exercise 3.21).

Theorem 3.10 (Plancherel II)**.** *Let $f, g \in L^2(\mathbb{R}^d)$. Then $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$. In other words,*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} dx = \int_{\mathbb{R}^d} \hat{f}(x)\overline{\hat{g}(x)} dx. \tag{3.26}$$

□

Example 3.6. The sinc function, which we recall is defined by

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x}, & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

plays an important role in many fields, and here we will glimpse another aspect of its importance, as an application of Plancherel's identity (3.26) above. Let's show that

$$\int_{\mathbb{R}} \operatorname{sinc}(x-n)\operatorname{sinc}(x-m) dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if not.} \end{cases} \tag{3.27}$$

Using Plancherel, with $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$, we have

$$\begin{aligned} \int_{\mathbb{R}} \operatorname{sinc}(x-n)\operatorname{sinc}(x-m) dx &= \int_{\mathbb{R}} \mathcal{F}(\operatorname{sinc}(x-n))(\xi)\overline{\mathcal{F}(\operatorname{sinc}(x-m))(\xi)} d\xi \\ &= \int_{\mathbb{R}} 1_{\mathcal{P}}(\xi) e^{2\pi i \xi n} 1_{\mathcal{P}}(\xi)\overline{e^{2\pi i \xi m}} d\xi \\ &= \int_{\mathcal{P}} e^{2\pi i \xi (n-m)} d\xi \\ &= \delta(n, m), \end{aligned}$$

by the orthogonality of the exponentials over $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$. (Exercise 1.3)

So we see that the collection of functions $\{\text{sinc}(x-n) \mid n \in \mathbb{Z}\}$ forms an orthogonal collection of functions in the Hilbert space $L^2(\mathbb{R})$, relative to its norm. It turns out that when one studies Shannon's sampling theorem, these translated sinc functions are in fact a basis for the vector subspace of $L^2(\mathbb{R})$ that consists of 'bandlimited functions'. □

Returning now to the Hilbert space $L^2(\square)$, we have the following useful result for periodic functions, known as **Parseval's identity** .

Theorem 3.11 (Parseval's identity). *Suppose that $f, g \in L^2(\square)$. We expand both in their Fourier series: $f(x) = \sum_{n\in\mathbb{Z}^d} a_n e^{2\pi i\langle n,x\rangle}$, $g(x) = \sum_{n\in\mathbb{Z}^d} b_n e^{2\pi i\langle n,x\rangle}$. Then*

$$\int_\square f(u)\overline{g(u)}du = \sum_{n\in\mathbb{Z}^d} a_n\overline{b_n}. \tag{3.28}$$

In particular, when $f = g$, we obtain

$$\int_\square |f(u)|^2 du = \sum_{n\in\mathbb{Z}^d} |a_n|^2. \tag{3.29}$$

Proof. The proof is a straightforward application of the orthogonality relations. We compute:

$$\begin{aligned}
\int_\square f(u)\overline{g(u)}du &= \int_\square \sum_{n\in\mathbb{Z}^d} a_n e^{2\pi i\langle n,u\rangle} \sum_{m\in\mathbb{Z}^d} \overline{b_n} e^{-2\pi i\langle m,u\rangle} du \\
&= \sum_{n,m\in\mathbb{Z}^d} a_n\overline{b_m} \int_\square e^{2\pi i\langle n,u\rangle} e^{-2\pi i\langle m,u\rangle} du \\
&= \sum_{n,m\in\mathbb{Z}^d} a_n\overline{b_m}\, [n=m] \\
&= \sum_{n\in\mathbb{Z}^d} a_n\overline{b_n},
\end{aligned}$$

using the orthogonality relations, Theorem 3.1, in the penultimate equality. □

Parseval's identity (3.29) is used very frequently in almost every branch of science. The following is a simple Number-theoretic application.

Example 3.7. Let $f(x) := P_1(x) := \{x\} - \frac{1}{2}$, the first periodic Bernoulli polynomial. Then as we have seen, f has the Fourier series

$$f(x) = -\frac{1}{2\pi i} \sum_{k \in \mathbb{Z}-\{0\}} \frac{e^{2\pi i k x}}{k},$$

for all $x \notin \mathbb{Z}$. So here we have $a_k = \frac{-1}{2\pi i} \frac{1}{k}$, by definition of f. Using Parseval's identity (3.29) $\int_0^1 |f(u)|^2 du = \sum_{n \in \mathbb{Z}} |a_n|^2$, we compute:

$$\sum_{n \in \mathbb{Z}} |a_n|^2 = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}-\{0\}} \frac{1}{n^2} = \frac{1}{2\pi^2} \sum_{n \geqslant 1} \frac{1}{n^2},$$

while

$$\int_0^1 |f(u)|^2 du = \int_0^1 \left(\{x\} - \frac{1}{2}\right)^2 dx = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}.$$

Therefore $\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

In a very similar manner one can evaluate the Riemann zeta function at all positive even integers $2k$ (using the Fourier series for the periodic Bernoulli polynomial $P_k(x)$) (Exercise 3.6) . □

3.8 More useful properties

It is natural to wonder about the asymptotic decay rate of the Fourier coefficients of a function f. In this direction we have the **Riemann–Lebesgue lemma**, as follows.

Lemma 3.3. *Let $f \in L^1(\mathbb{R}^d)$. Then:*

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0.$$

□

We collect here a few standard and useful properties of the Fourier transform.

Lemma 3.4. *Suppose we are given any functions $f, g \in L^1(\mathbb{R}^d)$. Then the following properties hold:*

- *Let $T : \mathbb{R} \to \mathbb{R}$ be the 'translation by h' function, defined by $T(x) := x+h$, for all $x \in \mathbb{R}$, and a fixed $h \in \mathbb{R}$. Then:*

$$\mathcal{F}(f \circ T)(m) = \hat{f}(m)e^{2\pi i h m}.$$

- *Conversely, we also have:*

$$\mathcal{F}\left(f(x)e^{-2\pi i x h}\right)(m) = \hat{f}(m+h).$$

- *Suppose that $f \in L^1(\mathbb{R})$. Then*

$$\mathcal{F}\left(f'\right)(m) = 2\pi i m \hat{f}(m).$$

- *Conversely, if $f \in L^1(\mathbb{R})$ and $xf(x) \in L^1(\mathbb{R})$, then*

$$\mathcal{F}\left(-2\pi i x f(x)\right)(m) = \left(\frac{d}{dx}\hat{f}\right)(m).$$

□

(See Exercise 3.10).

3.9 Approximate identity

It is a sad fact of life that there is no identity in $L^1(\mathbb{R}^d)$ for the convolution product - in other words, there is no function $h \in L^1(\mathbb{R}^d)$ such that

$$f * h = f \tag{3.30}$$

for all $f \in L^1(\mathbb{R}^d)$.

Why is that? Suppose there was such a function - then taking the Fourier transform of both sides of (3.30), we would also have $\hat{f}\,\hat{h} = \hat{f}$ for all $f \in \mathcal{L}^1(\mathbb{R}^d)$. Picking an f whose transform is nowhere zero, we can divide both sides by $\hat{f}$ now, to conclude that $\hat{h} = 1$. Now taking inverse Fourier transforms, we solve for h, getting

$$h(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle x,\xi\rangle} dx, \tag{3.31}$$

an extremely interesting integral that unfortunately diverges. So there is no function that plays the role of an identity for the convolution product, but see Note (c). But it turns out that we can get close! Here is how we may do it, and as a consequence we will be able to rigorously apply the Poisson summation formula to a wider class of functions, including smoothed versions of the indicator function of a polytope.

An **approximate identity** is any sequence of functions

$$\phi_n(x) := n^d \phi(nx), \tag{3.32}$$

which is defined for any integrable function $\phi : \mathbb{R}^d \to \mathbb{C}$ with the additional property that

$$\int_{\mathbb{R}^d} \phi(x)dx = 1.$$

It's easy to show that the latter two equalities imply that

$$\int_{\mathbb{R}^d} \phi_n(x)dx = 1,$$

for all $n \geqslant 1$ (Exercise 3.22). So scaling ϕ by these n's has the effect of squeezing ϕ so that it is becomes concentrated near the origin, while maintaining a total mass of 1. Intuitively, then, a sequence of such ϕ_n functions approach the "Dirac delta-function" at the origin (which is a distribution, not a function).

There are many families of functions that give an approximate identity. In practice, we will seldom have to specify exactly which sequence ϕ_n we pick, because we will merely use the existence of such a sequence to facilitate the use of Poisson summation. Returning now to the motivation of this section, we can recover the next-best-thing to an identity for the convolution product, as follows.

Theorem 3.12. *Suppose we are given a function $f \in L^1(\mathbb{R}^d)$, which is continuous at a point $p \in \mathbb{R}^d$. Then for any approximate identity ϕ_n, assuming that $f * \phi$ exists, we have*

$$\lim_{n\to\infty} (f * \phi_n)(p) = f(p). \tag{3.33}$$

Proof. We begin by massaging the convolution product:

$$\begin{aligned}
(\phi_n * f)(p) &:= \int_{\mathbb{R}^d} \phi_n(x) f(p-x)dx \\
&= \int_{\mathbb{R}^d} \phi_n(x)\Big(f(p-x) - f(p) + f(p)\Big)dx \\
&= \int_{\mathbb{R}^d} \phi_n(x)\Big(f(p-x) - f(p)\Big)dx + f(p)\int_{\mathbb{R}^d} \phi_n(x)dx \\
&= f(p) + \int_{\mathbb{R}^d} \phi_n(x)\Big(f(p-x) - f(p)\Big)dx,
\end{aligned}$$

using the assumption that $\int_{\mathbb{R}^d} \phi_n(x)dx = 1$. Using the definition of $\phi_n(x) := n^d\phi(nx)$, and making a change of variable $u = nx$ in the latter integral, we have:

$$(\phi_n * f)(p) := f(p) + \int_{\mathbb{R}^d} \phi(u)\Big(f(p - \frac{1}{n}u) - f(p)\Big)du.$$

In the second part of the proof, we will show that as $n \to \infty$, the latter integral tends to zero. We will do this in two steps, first bounding the tails of the integral in a neighborhood of infinity, and then bounding the integral in a neighborhood of the origin.

Step 1. Given any $\epsilon > 0$, we note that the latter integral converges, so the 'tails are arbitrarily small'. In other words, there exists an $r > 0$ such that

$$\Big|\int_{\|u\|>r} \phi(u)\Big(f(p - \frac{1}{n}u) - f(p)\Big)du\Big| < \epsilon.$$

Step 2. Now we want to bound $\int_{\|u\|<r} \phi(u)\Big(f(p - \frac{1}{n}u) - f(p)\Big)du$. We will use the fact that $\int_{\mathbb{R}^d} |\phi(u)|du = M$, a constant. Also, by continuity of f at p, we can pick an n sufficiently large, such that:

$$|f(p - \frac{1}{n}u) - f(p)| < \frac{\epsilon}{M},$$

when $\|\frac{1}{n}u\| < r$. Putting all of this together, and using the triangle inequality for integrals, we have the bound

$$\begin{aligned}
&\Big|\int_{\|u\|<r} \phi(u)\Big(f(p - \frac{1}{n}u) - f(p)\Big)du\Big| \\
&\qquad \leqslant \int_{\|u\|<r} |\phi(u)||f(p - \frac{1}{n}u) - f(p)|du < \epsilon.
\end{aligned}$$

Therefore, as $n \to \infty$, we have $(\phi_n * f)(p) \longrightarrow f(p)$. □

We note that a point of discontinuity of f, Theorem 3.12 may be false even in dimension 1, as the next example shows.

Example 3.8. *Let $f(x) := 1_{[0,1]}(x)$, which is discontinuous at $x = 0$ and $x = 1$. We claim that for $p = 1$, for example, we have*

$$\lim_{n\to\infty} (f * \phi_n)(p) = \frac{1}{2} f(p),$$

so that the result of Theorem 3.12 does not hold at this particular p, because p lies on the boundary of the 1-dimensional polytope $[0,1]$. When $p \in \operatorname{int}([0,1])$, however, Theorem 3.12 does hold. □

3.10 A practical Poisson summation formula

In practice, we want to apply Poisson summation to indicator functions $1_{\mathcal{P}}$ of polytopes and convex bodies. With this in mind, it's useful for us to have our own, home cooked version of Poisson summation that is made for this culinary purpose.

Throughout, we fix any compactly supported function $\varphi \in L^2(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} \varphi(x)dx = 1$, and we set $\varphi_\epsilon(x) := \frac{1}{\epsilon^d}\varphi(\frac{x}{\epsilon})$, for each $\epsilon > 0$.

Theorem 3.13. *Let $f(x) \in L^2(\mathbb{R}^d)$ be a compactly supported function, and suppose that for each $x \in \mathbb{R}^d$, we have:*

$$f(x) = \lim_{\varepsilon\to 0^+} \{\varphi_\varepsilon * f(x)\}. \tag{3.34}$$

Then, for each $\varepsilon > 0$, we have

$$\sum_{m\in\mathbb{Z}^d} \left|\widehat{\varphi}(\varepsilon m)\, \widehat{f}(m)\right| < +\infty$$

and for each $x \in \mathbb{R}^d$, we have

$$\sum_{n\in\mathbb{Z}^d} f(n+x) = \lim_{\varepsilon\to 0^+} \left\{ \sum_{m\in\mathbb{Z}^d} \widehat{\varphi}(\varepsilon m)\, \widehat{f}(m)\, e^{2\pi i\langle m,x\rangle} \right\}. \tag{3.35}$$

We note that by our assumption on f, the left-hand-side of equation (3.35) is a finite sum.

For a detailed proof of Theorem 3.13, see Brandolini et al. 2021.

Notes

(a) There are some wonderful introductory books that develop the subject thoroughly from first principles, such as the books by Stein and Shakarchi (2003) and Giancarlo Travaglini (2014). The reader is also encouraged to read more advanced but fundamental introductions to Fourier analysis, in particular the books by Mark Pinsky (2002), Edward Charles Titchmarsh (1986), and Stein and Weiss (1971). In addition, the book by Audrey Terras (1999) is an excellent introduction to Fourier analysis on finite groups, with applications. A more informal introduction to Fourier analysis, focusing on various applications, is given by Brad Osgood (2019).

(b) There are some "elementary" techniques that we will use, from the calculus of a complex variable, but which require essentially no previous knowledge in this field. In particular, suppose we have two analytic functions $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$, such that $f(z_k) = g(z_k)$ for a convergent sequence of complex numbers $z_k \to L$, where L is any fixed complex number. Then $f(z) = g(z)$ for all $z \in \mathbb{C}$.

The same conclusion is true even if the hypothesis is relaxed to the assumption that both f and g are meromorphic functions, as long as the sequence and its limit stay away from the poles of f and g.

(c) The "Dirac delta function" is part of the theory of "generalized functions" and may be intuitively defined by the full sequence of Gaussians $G_t(x) := t^{-\frac{d}{2}} e^{-\frac{\pi}{t}||x||^2}$. The observation that there is no identity for the convolution product on $\mathbb{R}^d$ is a clear motivation for a theory of generalized functions, beginning with the Dirac delta function. Another intuitive way of "defining" the Dirac delta function is:

$$\delta_0(x) := \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if not,} \end{cases}$$

even though this is not, strictly speaking, a function. But in the sense of distributions (i.e. generalized functions), we have $\lim_{\to 0} G_t(x) = \delta_0(x)$.

More rigorously, the δ-function belongs to a theory of distributions that was developed by Laurent Schwartz in the 1950's and by S. L. Sobolev in 1936, where we can think of generalized functions as linear functionals on the space of all bump functions on $\mathbb{R}^d$.

Such generalized functions were originally used by the Physicist Paul Dirac in 1920, before the rigorous mathematical theory was even created for it, in order to better understand quantum mechanics.

(d) Of great practical importance, and historical significance, a **bump function** is defined as any infinitely smooth function on $\mathbb{R}^d$, which is compactly supported. In other words, a bump function enjoys the following properties:

- ϕ has compact support on $\mathbb{R}^d$.
- $\phi \in C^\infty(\mathbb{R}^d)$.

Bump functions are also called **test functions**, and if we consider the set of all bump functions on $\mathbb{R}^d$, under addition, we get a vector space V, whose dual vector space is called the space of **distributions on** $\mathbb{R}^d$.

(e) The cotangent function, appearing in some of the exercises below, is the unique *meromorphic function* that has a simple pole at every integer, with residue 1 (up to multiplication by an entire function).

(f) A deeper exploration into projections and sections of the unit cube in $\mathbb{R}^d$ can be found in "The cube - a window to convex and discrete geometry", by Chuanming Zong (2006). In his book Alexander Koldobsky (2005), gives a thorough introduction to sections of convex bodies, intersection bodies, and the Busemann–Petty problem.

Exercises

In theory, there is no difference between theory and practice. But in practice.....there is!
– Walter J. Savitch

3.1. ♣ *Recalling that the L^2-norm is defined by $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$, and the L^1-norm is defined by $\|x\|_1 := |x_1| + \cdots + |x_d|$, we have the following elementary norm relations.*

(a) *Show that $\|x\|_2 \leqslant \|x\|_1$, for all $x \in \mathbb{R}^d$.*

(b) *On the other hand, show that we have* $\|x\|_1 \leqslant \sqrt{d}\ \|x\|_2$, *for all* $x \in \mathbb{R}^d$.

3.2. ♣ *Show that the Cauchy–Schwarz inequality holds in the Hilbert space* $L^2(\square)$*:*

$$\int_{\square} f(x)\overline{g(x)}dx \leqslant \left(\int_{\square} |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\square} |g(x)|^2 dx\right)^{\frac{1}{2}}, \tag{3.36}$$

for all $f, g \in L^2(\square)$, *with equality if and only if* $f(x) = Cg(x)$ *for some constant* C

3.3. *We know that the functions* $u(t) := \cos t = \frac{e^{it}+e^{-it}}{2}$ *and* $v(t) := \sin t = \frac{e^{it}-e^{-it}}{2i}$ *are natural, partly because they parametrize the unit circle:* $u^2 + v^2 = 1$. *Here we see that there are other similarly natural functions, parametrizing the hyperbola.*

(a) *Show that the following functions parametrize the hyperbola* $u^2 - v^2 = 1$*:*

$$u(t) := \frac{e^t + e^{-t}}{2}, \quad v(t) := \frac{e^t - e^{-t}}{2}.$$

(This is the reason that the function $\cosh t := \frac{e^t+e^{-t}}{2}$ *is called the hyperbolic cosine, and the function* $\sinh t := \frac{e^t-e^{-t}}{2}$ *is called the hyperbolic sine)*

(b) *The hyperbolic cotangent is defined as* $\coth t := \frac{\cosh t}{\sinh t} = \frac{e^t+e^{-t}}{e^t-e^{-t}}$. *Using Bernoulli numbers, show that* $t \coth t$ *has the Taylor series:*

$$t \coth t = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} t^{2n}.$$

3.4. *Fix* $t > 0$, *and let* $f(x) := e^{-2\pi t|x|}$, *for all* $x \in \mathbb{R}$. *Show that* f *is a Schwarz function on the real line.*

3.5. ♣ *We continue with the same function as in the previous exercise,* $f(x) := e^{-2\pi t|x|}$.

(a) *Show that* $\hat{f}(\xi) = \frac{t}{\pi}\frac{1}{\xi^2+t^2}$, *for all* $\xi \in \mathbb{R}$.

(b) *Using Poisson summation, show that:*

$$\frac{t}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^2+t^2} = \sum_{m\in\mathbb{Z}} e^{-2\pi t|m|}.$$

3.6. ♣ *Here we evaluate the Riemann zeta function at the positive even integers, by using the previous exercise.*

(a) *Show that*

$$\sum_{n\in\mathbb{Z}} e^{-2\pi t|n|} = \frac{1+e^{-2\pi t}}{1-e^{-2\pi t}} := \coth(\pi t),$$

for all $t > 0$.

(b) *Show that the cotangent function has the following (well-known) partial fraction expansion:*

$$\pi\cot(\pi x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2},$$

valid for any $x \in \mathbb{R} - \mathbb{Z}$.

(c) *Let* $0 < t < 1$. *Show that*

$$\frac{t}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^2+t^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)\,t^{2m-1},$$

where $\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$ *is the Riemann zeta function, initially defined by the latter series, which is valid for all* $s \in \mathbb{C}$ *with* $Re(s) > 1$.

(d) *Finally, we may quickly evaluate the Riemann zeta function at all even integers, as follows. We recall the definition of the Bernoulli numbers, namely:*

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{m \geqslant 1} \frac{B_{2m}}{2m!} z^{2m}.$$

Prove that for all $m \geqslant 1$,

$$\zeta(2m) = \frac{(-1)^{m+1}}{2} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Thus, for example, using the first 3 Bernoulli numbers, we have: $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, and $\zeta(6) = \frac{\pi^6}{945}$.

3.7. *For each $n \geqslant 1$, let $T_n(x) = \cos(nx)$. For example, $T_2(x) = \cos(2x) = 2\cos^2(x) - 1$, so $T_2(x) = 2u^2 - 1$, a polynomial in $u := \cos x$.*

(a) *Show that for all $n \geqslant 1$, $T_n(x)$ is a polynomial in $\cos x$.*

(b) *Can you write $x^n + \frac{1}{x^n}$ as a polynomial in the variable $x + \frac{1}{x}$? Would your answer be related to the polynomial $T_n(x)$? What's the relationship in general? For example, $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$.*

Notes. The polynomials $T_n(x)$ are very important in applied fields such as approximation theory, because they have many useful extremal properties. They are called Chebyshev polynomials .

3.8. *The hyperbolic secant is defined by*

$$\operatorname{sech}(\pi x) := \frac{2}{e^{\pi x} + e^{-\pi x}}, \text{ for } x \in \mathbb{R}.$$

Show that $\operatorname{sech}(\pi x)$ *is its own Fourier transform:*

$$\mathcal{F}(\operatorname{sech})(\xi) = \operatorname{sech}(\xi),$$

for all $\xi \in \mathbb{R}$.

3.9. *Using the previous exercise, conclude that*

$$\int_{\mathbb{R}} \frac{1}{e^{\pi x} + e^{-\pi x}} dx = \frac{1}{2}.$$

3.10. ♣ *Here we gain more practice in handling general Fourier transforms and their basic properties. Throughout, we assume that the Fourier transform of f exists, where $f \in L^1(\mathbb{R})$, and if necessary you may also assume that $xf(x) \in L^1(\mathbb{R})$, and that f, $\hat{f}$ are continuously differentiable.*

(a) *Show that*

$$\mathcal{F}\left(f(x)e^{-2\pi i x h}\right)(m) = \hat{f}(m+h),$$

for all $m \in \mathbb{R}$.

(b) *Show that*

$$\mathcal{F}\left(f'\right)(m) = 2\pi i m \hat{f}(m),$$

for all $m \in \mathbb{R}$.

(c) *Conversely, show that:*

$$\mathcal{F}\Big(-2\pi i x f(x)\Big)(m) = \left(\frac{d}{dx}\hat{f}\right)(m),$$

for all $m \in \mathbb{R}$.

3.11. ♣ *Prove that:*

$$\int_0^1 P_1(ax)P_1(bx)dx = \frac{1}{12\,\mathrm{gcd}^2(\mathrm{a},\mathrm{b})}.$$

for all positive integers a, b. Here $P_1(x) := x - \{x\} - \frac{1}{2}$ is the first periodic Bernoulli polynomial.

Notes. This integral is called a ***Franel integral****, and there is a substantial literature about related integrals.*

3.12. ♣ *Let $f : \mathbb{R} \to \mathbb{C}$ belong to the Schwarz class of functions on $\mathbb{R}$, denoted by $S(\mathbb{R})$. Show that $\hat{f} \in S(\mathbb{R})$ as well.*

3.13. ♣ *Here we answer the very natural question "What are the other inner products on $\mathbb{R}^d$, besides the usual inner product $\langle x, y \rangle := \sum_{k=1}^{d} x_k y_k$?"*

The fact is that all inner products on $\mathbb{R}^d$ are related to each other using positive definite matrices, as follows. We recall from Linear Algebra that a symmetric matrix is called positive definite if all of its eigenvalues are positive. Prove that the following two conditions are equivalent:

1. *$\langle x, y \rangle$ is an inner product on $\mathbb{R}^d$.*
2. *$\langle x, y \rangle := x^T M y$, for some positive definite matrix M.*

3.14. *For any positive real numbers $a < b < c < d$, define*

$$f(x) := 1_{[a,b]}(x) + 1_{[c,d]}(x).$$

Can you find a, b, c, d such that $\hat{f}(\xi)$ is nonzero for all $\xi \in \mathbb{R}$?

3.15. ♣ *Show that for the Gaussian $G_t(x) := e^{-tx^2}$, we have*

$$\frac{d}{dx^k} G_t(x) = H_k(x,t) G_t(x),$$

for all positive integers k, where $H_k(x,t)$ is a polynomial in t and x (Here t is a fixed positive constant).

Notes. The polynomial $H_n(x,t)$ is closely related to the important sequence of **Hermite polynomials**.

3.16. ♣ *Show that the only eigenvalues of the linear operator $f \to \hat{f}$ are $\{1, -1, i, -i\}$, and show that each of these eigenvalues is achieved by some function. You can assume that f belongs to $L^2(\mathbb{R}^d)$.*

3.17. ♣ *For this exercise, we call a function* $f : \mathbb{R}^d \to \mathbb{C}$ ***integrable*** *if* $\int_{\mathbb{R}^d} |f(x)| dx$ *converges, and* f *is called* **summable** *if* $\sum_{n \in \mathbb{Z}^d} |f(n)|$ *converges. If* f *is piecewise continuous on* $\mathbb{R}^d$*, show that* f *is summable if and only if* f *is integrable.*

3.18. ♣ *Show that the special case of Poisson summation, 3.21, implies the general case, Theorem 3.8.*

3.19. ♣ *We define the Gaussian, for each fixed* $\epsilon > 0$*, and for all* $x \in \mathbb{R}^d$*, by*

$$G_\epsilon(x) := \frac{1}{\epsilon^{\frac{d}{2}}} e^{-\frac{\pi}{\epsilon}||x||^2}. \tag{3.37}$$

Show that:

$$\int_{\mathbb{R}^d} G_\epsilon(x) dx = 1.$$

3.20. ♣ *Show that, for all* $m \in \mathbb{R}^d$*, the Fourier transform of the Gaussian* $G_\epsilon(x)$ *is:*

$$\hat{G}_\epsilon(m) = e^{-\pi\epsilon||m||^2}.$$

3.21. ♣ *For all* $f, g \in S(\mathbb{R}^d)$*, show that* $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$*.*

3.22. ♣ *Given any approximate identity sequence* ϕ_ϵ*, as defined in* (3.32)*, show that for each* $\epsilon > 0$*,*

$$\int_{\mathbb{R}^d} \phi_\epsilon(x) dx = 1.$$

3.23. *We define the* **ramp function** $x_+ := \max\{x, 0\}$*, which may also be written as*

$$x_+ := \begin{cases} x & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Show that the ramp also has the representation

$$x_+ = \frac{x + |x|}{2}, \tag{3.38}$$

for all $x \in \mathbb{R}$.

Notes. This ramp function is quite common nowadays in machine learning, and has a strong presence in approximation theory.

3.24. *(hard)* ♣ *Here we show that there exist compactly supported functions, (which are also radial) whose Fourier transform is* **strictly positive** *on all of* $\mathbb{R}^d$. *Let* $\phi : \mathbb{R}^d \to \mathbb{R}$ *be defined by*

$$\phi(x) := \int_0^\infty \left(1 - \frac{\|x\|^2}{t}\right)_+^\lambda (1 - \sqrt{t})_+ dt$$

where the constant $\lambda \geqslant \frac{d-1}{2}$ *if* $d \geqslant 2$, *and for dimension* $d = 1$, *we set* $\lambda = \frac{1}{2}$. *Using properties of* J*-Bessel functions, show that* $\hat{\phi}(\xi) > 0$ *for all* $\xi \in \mathbb{R}^d$. *(see Buhmann 1998 for more details)*

4 The geometry of numbers – Minkowski meets Siegel

> "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."
>
> – Hermann Minkowski

4.1 Intuition

To see a wonderful and fun application of Poisson summation, we give a relatively easy proof of Minkowski's fundamental theorem, in the Geometry of Numbers. Minkowski's theorem gives the existence of an integer point inside symmetric bodies in $\mathbb{R}^d$, once we know their volume is sufficiently large.

In fact we first prove a more powerful identity which is a classical result of Carl Ludwig Siegel (Theorem 4.3), yielding an identity between Fourier transforms of convex bodies and their volume. Our proof of this identity of Siegel uses Poisson summation, applied to the convolution of an indicator function with itself.

The geometry of numbers is an incredibly beautiful field, and too vast to en-

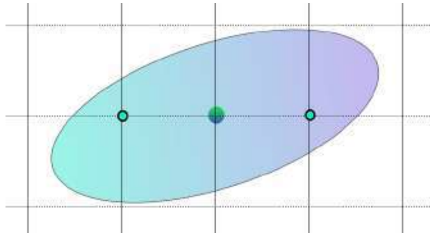


Figure 4.1: A convex, symmetric body in $\mathbb{R}^2$, with area bigger than 4, containing two nonzero integer points.

compass in just one chapter (see Note (c)). We hope this chapter, a small bite of a giant fruit, gives the reader motivation to pursue the interactions between convex bodies and lattices even further.

4.2 Minkowski's convex body Theorem

Minkowski initiated the field that we call today 'the geometry of numbers', around 1890. To begin, we define a **body** in $\mathbb{R}^d$ as a bounded, closed set. Most of the time, it is useful to work with convex bodies that enjoy the following symmetry. We call a body B **centrally symmetric**, also called **symmetric about the origin**, if for all $\mathbf{x} \in \mathbb{R}^d$ we have

$$\mathbf{x} \in \mathcal{P} \iff -\mathbf{x} \in \mathcal{P}. \tag{4.1}$$

a body Q is called **symmetric** if some translation of Q is centrally symmetric. For example, the ball $\{x \in \mathbb{R}^d \mid \|x\| \leqslant 1\}$ is centrally symmetric, and for any fixed nonzero $v \in \mathbb{R}^d$, the translated ball $\{x \in \mathbb{R}^d \mid \|x - v\| \leqslant 1\}$ is symmetric, but not centrally symmetric.

Theorem 4.1 (Minkowski's convex body Theorem for $\mathbb{Z}^d$)**.** *Let B be a d-dimensional convex body in $\mathbb{R}^d$, symmetric about the origin.*

If $\operatorname{vol} B > 2^d$*, then* B *must contain a nonzero integer point in its interior.*

Sometimes this very useful result of Minkowski is stated in its contrapositive form: Let $B \subset \mathbb{R}^d$ be any convex, symmetric body about the origin. If the only integer point in the interior of B is the origin, then $\operatorname{vol} B \leqslant 2^d$.

It is natural, and straightforward, to extend this result to any lattice $\mathcal{L} := M(\mathbb{Z}^d)$, by simply applying the linear transformation M to both the integer lattice, and to the convex body B. The conclusion is the following, which is the version that we will prove as a consequence of Siegel's Theorem 4.3.

Theorem 4.2 (Minkowski's convex body Theorem for a lattice $\mathcal{L}$)**.** *Let B be a d-dimensional convex body in $\mathbb{R}^d$, symmetric about the origin, and let $\mathcal{L}$ be a (full rank) lattice in $\mathbb{R}^d$.*

$$\textit{If } \operatorname{vol} B > 2^d (\det \mathcal{L}), \textit{ then } B \textit{ must contain}$$
$$\textit{a nonzero point of } \mathcal{L} \textit{ in its interior.}$$

Proof. The proof appears below - see 4.24. □

This very important initial result of [Minkowski (1968)], proved in 1889, has found applications in algebraic number theory, diophantine analysis, combinatorial optimization, and other fields. In the next section we show that Minkowski's result (4.2) follows as a special case of Siegel's formula.

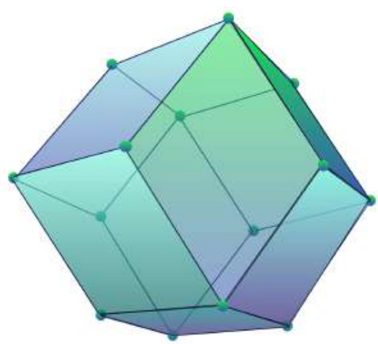


Figure 4.2: The Rhombic dodecahedron, a 3-dimensional symmetric polytope that tiles $\mathbb{R}^3$ by translations, and is another extremal body for Minkowski's convex body Theorem.

4.3 Siegel's generalization of Minkowski, a Fourier transform identity for convex bodies

An important construction in the geometry of numbers is the **Minkowski sum** of convex bodies. Given two convex bodies $K, L \subset \mathbb{R}^d$, their Minkowski sum is defined by

$$K + L := \{x + y \mid x \in K, y \in L\}.$$

Another related construction, appearing in some of the results below, is

$$K - L := \{x - y \mid x \in K, y \in L\},$$

In this connection, a very useful gadget is the **Minkowski symmetrized body** of K, defined by

$$\frac{1}{2}K - \frac{1}{2}K. \tag{4.2}$$

All of the resulting bodies above turn out to be convex (Exercise 4.2).

Example 4.1. Consider the following 3 line segments in $\mathbb{R}^2$:

$$\mathrm{conv}\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}, \mathrm{conv}\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)\}, \mathrm{conv}\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\3\end{smallmatrix}\right)\}.$$

The Minkowski sum of these three line segments is the hexagon whose vertices are $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\3\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}4\\3\end{smallmatrix}\right)$. Notice that once we graph it, in Figure 4.3, the graph is hinting to us that this body is a projection of a 3-dimensional cube, and indeed this turns out to be always true for Minkowski sums of line segments. □

Another natural geometric notion is the **dilation of a convex body** by a positive real number t:

$$tB := \{tx \mid x \in B\},$$

The most basic version of Siegel's theorem is the following identity, which assumes that a convex body K is symmetric about the origin.

Theorem 4.3 (Siegel)**.** *Let B be any d-dimensional convex body in $\mathbb{R}^d$, symmetric about the origin, and suppose that the only integer point in the interior of B is the origin. Then*

$$2^d = \mathrm{vol}\, B + \frac{4^d}{\mathrm{vol}\, B} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}B}(\xi) \right|^2. \tag{4.3}$$

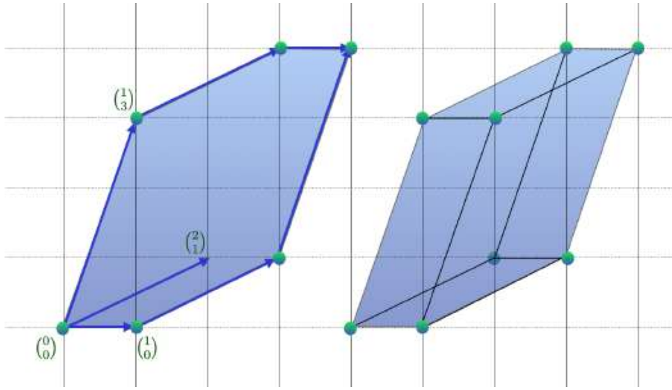


Figure 4.3: The Minkowski sum of 3 line segments in the plane.

We will prove a slight extension of the latter Siegel formula (4.3), namely (4.4) below, which applies to bodies that are not necessarily symmetric about the origin. Our proof here consists of yet another application of Poisson summation.

Theorem 4.4 (Siegel's formula, for a general body K, and a lattice $\mathcal{L}$)**.** *Let K be any bounded, measurable subset of $\mathbb{R}^d$, not necessarily symmetric. Let $B := \frac{1}{2}K - \frac{1}{2}K$ be the symmetrized body of K (hence B is convex), and suppose that the only integer point in the interior of B is the origin. Then*

$$2^d = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2. \tag{4.4}$$

More generally, if we replace the lattice $\mathbb{Z}^d$ by any full-rank lattice $\mathcal{L}$, and assume that the only lattice point of $\mathcal{L}$ in the interior of B is the origin, then we have:

$$2^d \det \mathcal{L} = \operatorname{vol} K + \frac{4^d}{\operatorname{vol} K} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2. \tag{4.5}$$

Proof. We start with the function

$$f(x) := \left(1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K} \right)(x), \tag{4.6}$$

which is continuous on $\mathbb{R}^d$, and we plug f into Poisson summation (3.19) :

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{\xi\in\mathbb{Z}^d} \hat{f}(\xi). \tag{4.7}$$

We first compute the left-hand-side of Poisson summation, using the definition of f:

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) 1_{-\frac{1}{2}K}(n-y)dy, \tag{4.8}$$

and now the fact that $y \in \frac{1}{2}K$ and $n - y \in -\frac{1}{2}K$ implies that the integer point $n \in \frac{1}{2}K - \frac{1}{2}K$. But by hypothesis $\frac{1}{2}K - \frac{1}{2}K$ contains the origin as its *only* interior integer point, so the left-hand-side of Poisson contains only one term, namely the $n = 0$ term:

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) 1_{-\frac{1}{2}K}(n-y)dy \tag{4.9}$$

$$= \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) 1_{-\frac{1}{2}K}(-y)dy \tag{4.10}$$

$$= \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y)dy \tag{4.11}$$

$$= \operatorname{vol}\left(\frac{1}{2}K\right) = \frac{\operatorname{vol} K}{2^d}. \tag{4.12}$$

On the other hand, the right-hand-side of Poisson summation gives us:

$$\sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\frac{1}{2}K}(\xi) \hat{1}_{-\frac{1}{2}K}(\xi) \tag{4.13}$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \langle \xi, x \rangle} dx \int_{-\frac{1}{2}K} e^{2\pi i \langle \xi, x \rangle} dx \tag{4.14}$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \langle \xi, x \rangle} dx \int_{\frac{1}{2}K} e^{2\pi i \langle -\xi, x \rangle} dx \tag{4.15}$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \langle \xi, x \rangle} dx \, \overline{\int_{\frac{1}{2}K} e^{2\pi i \langle \xi, x \rangle} dx} \tag{4.16}$$

$$= \sum_{\xi \in \mathbb{Z}^d} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 \tag{4.17}$$

$$= \left| \hat{1}_{\frac{1}{2}K}(0) \right|^2 + \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2 \tag{4.18}$$

$$= \frac{\mathrm{vol}^2 K}{4^d} + \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2, \tag{4.19}$$

the main point of the last computation being that the lattice sum contains only nonnegative summands. So we have the identity:

$$\frac{\mathrm{vol}\, K}{2^d} = \frac{\mathrm{vol}^2 K}{4^d} + \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2, \tag{4.20}$$

yielding

$$2^d = \mathrm{vol}\, K + \frac{4^d}{\mathrm{vol}\, K} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2. \tag{4.21}$$

Finally, to prove the stated extension to all lattices $\mathcal{L}$, we use the following, more general, form of Poisson summation, valid for any lattice $\mathcal{L}$:

$$\sum_{n \in \mathcal{L}} f(n) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{f}(\xi). \tag{4.22}$$

All the steps of the proof above are identical, except for the factor of $\frac{1}{\det \mathcal{L}}$, so that we arrive at

$$\frac{\operatorname{vol} K}{2^d} = \frac{\operatorname{vol}^2 K}{4^d \det \mathcal{L}} + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2, \tag{4.23}$$

and finally the required identity of Siegel for arbitrary lattices. □

The proof of Minkowski's convex body Theorem for lattices, namely Theorem 4.2 above, now follows immediately.

Proof of Theorem 4.2. [Minkowski's convex body Theorem for a lattice $\mathcal{L}$] Applying Siegel's Theorem 4.4 to the symmetric body $B := K$, in the statement of Siegel's theorem, we see that the lattice sum on the right-hand-side of identity (4.4) contains only non-negative terms. It follows that we immediately get the analogue of Minkowski's result for a given symmetric body B and a lattice $\mathcal{L}$, in its contrapositive form. Namely, if the only lattice point of $\mathcal{L}$ in the interior of B is the origin, then

$$2^d \det \mathcal{L} \geqslant \operatorname{vol} B. \tag{4.24}$$

□

In fact, we can easily extend Minkowski's Theorem 4.2, using the same ideas of the latter proof, by using Siegel's Theorem 4.4 so that it applies to non-symmetric bodies as well (but there's a small 'catch' - see Exercise 4.11).

4.4 Tiling and multi-tiling Euclidean space by translations of polytopes

First, we give a 'spectral' equivalence for being able to tile Euclidean space by a single polytope, using only translations by a lattice. It will turn out that the equality case of Minkowski's convex body Theorem is characterized precisely by the polytopes that tile $\mathbb{R}^d$ by translations. These bodies are called extremal bodies.

More generally, we would like to also consider the notion of multi-tiling, as follows. We say that a polytope $\mathcal{P}$ **k-tiles $\mathbb{R}^d$ by using a set of translations $\mathcal{L}$** if

$$\sum_{n \in \mathcal{L}} 1_{\mathcal{P}+n}(x) = k, \tag{4.25}$$

for all $x \in \mathbb{R}^d$, except those points x that lie on the boundary of $\mathcal{P}$ or its translates under $\mathcal{L}$ (and of course these exceptions form a set of measure 0 in $\mathbb{R}^d$). In other words, $\mathcal{P}$ is a k-tiling body if almost every $x \in \mathbb{R}^d$ is covered by exactly k translates of $\mathcal{P}$. We notice that this definition of k-tiling allows for overlaps between the various translates of $\mathcal{P}$.

Other synonyms for k-tilings in the literature are **multi-tilings** of $\mathbb{R}^d$, or **tiling at level** k. When $\mathcal{L}$ is a lattice, we will say that such a k-tiling is **periodic**. A common research theme is to search for tilings which are not necessarily periodic, but this is a difficult problem in general. The classical notion of tiling, such that there are no overlaps between the interiors of any two tiles, corresponds here to the case $k = 1$.

Theorem 4.5. *Fix any integer $k \geqslant 1$. A polytope $\mathcal{P}$ k-tiles $\mathbb{R}^d$ by translations with a lattice $\mathcal{L}$ if and only if the following two conditions hold:*

1. $\hat{1}_{\mathcal{P}}(\xi) = 0$*, for all nonzero $\xi \in \mathcal{L}^*$, the dual lattice.*
2. $k = \frac{\operatorname{vol} \mathcal{P}}{\det \mathcal{L}}$.

Proof. We begin with the definition of k-tiling, so that by assumption

$$\sum_{n \in \mathcal{L}} 1_{\mathcal{P}+n}(x) = k, \tag{4.26}$$

for all $x \in \mathbb{R}^d$ except those points x that lie on the boundary of $\mathcal{P}$ or its translates under $\mathcal{L}$ (and of course these exceptions form a set of measure 0 in $\mathbb{R}^d$). A trivial but useful observation is that

$$1_{\mathcal{P}+n}(x) = 1 \iff 1_{\mathcal{P}}(x - n) = 1,$$

so we can rewrite the defining identity (4.26) as $\sum_{n \in \mathcal{L}} 1_{\mathcal{P}}(x - n) = k$. Now we notice that the left-hand-side is a periodic function of x, namely

$$F(x) := \sum_{n \in \mathcal{L}} 1_{\mathcal{P}}(x - n)$$

is periodic in x with $\mathcal{L}$ as its set of periods. This is easy to see: if we let $l \in \mathcal{L}$, then $F(x + l) = \sum_{n \in \mathcal{L}} 1_{\mathcal{P}}(x + l - n) = \sum_{m \in \mathcal{L}} 1_{\mathcal{P}}(x + m) = F(x)$, because the lattice $\mathcal{L}$ is invariant under a translation by any vector that belongs to it.

The following 'intuitive proof' would in fact be rigorous if we were allowed to use 'generalized functions', but since we do not use them in this book, we label this part of the proof as 'intuitive', and we then give a rigorous proof, using functions rather than generalized functions.

[Intuitive proof] By the fundamental Theorem of Fourier series, we may expand F into its Fourier series, and by Poisson summation we know that its Fourier coefficients are the following:

$$\sum_{m\in\mathcal{L}} 1_{\mathcal{P}}(x+m) = \frac{1}{\det\mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{1}_{\mathcal{P}}(\xi) e^{2\pi i\langle\xi,x\rangle}, \tag{4.27}$$

By our multi-tiling assumption (4.26), we have $\sum_{n\in\mathcal{L}} 1_{\mathcal{P}+n}(x) = k$. So comparing the latter assumption with (4.27), we have two Fourier series that represent the same function, so by uniqueness of Fourier series, the corresponding coefficients on both sides must be equal to each other. Equating the constant terms, we have

$$k = \frac{1}{\det\mathcal{L}} \hat{1}_{\mathcal{P}}(0) = \frac{\operatorname{vol}\mathcal{P}}{\det\mathcal{L}},$$

while equating all of the other terms we see that $\hat{1}_{\mathcal{P}}(\xi) = 0$ for all $\xi \in \mathcal{L}^*$.

[Rigorous proof] In order to apply Poisson summation, it is technically necessary to replace $1_{\boldsymbol{P}}(x)$ by a smoothed version of it, in (4.27). Because this process is so common and useful in applications, this proof is instructive. So we let ϕ_n be any approximate identity, as in (3.33), and we will also assume that ϕ_n is supported on $\mathcal{P}$. Now we apply our 'practical Poisson summation' 3.35 to $1_{\mathcal{P}} * \phi_n$:

$$\sum_{m\in\mathcal{L}} (1_{\mathcal{P}} * \phi_n)(x+m) = \frac{1}{\det\mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{1}_{\mathcal{P}}(\xi)\hat{\phi}_n(\xi) e^{2\pi i\langle\xi,x\rangle}, \tag{4.28}$$

Using Theorem 3.12, and the fact that the summands on the left-hand-side of (4.28) are compactly supported, we may take the limit as $n \to \infty$ inside the finite sum on the LHS of (4.28), to obtain $\sum_{m\in\mathcal{L}} 1_{\mathcal{P}}(x+m)$, which is equal to the constant k, by assumption. Summarizing, we have:

$$k = \sum_{m\in\mathcal{L}} 1_{\mathcal{P}}(x+m) = \lim_{n\to\infty} \frac{1}{\det\mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{1}_{\mathcal{P}}(\xi)\hat{\phi}_n(\xi)\, e^{2\pi i\langle\xi,x\rangle}. \tag{4.29}$$

Due to the fact that $1_{\mathcal{P}}$ and ϕ_n are both supported on $\mathcal{P}$, we may drop the limit, as long as we evaluate the RHS of (4.29) for all sufficiently large values of n. Using the uniqueness of Fourier series, we may compare coefficients on both sides of

$$k = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{1}_{\mathcal{P}}(\xi) \hat{\phi}_n(\xi)\, e^{2\pi i \langle \xi, x \rangle}. \tag{4.30}$$

Comparing the constant terms, and using $\hat{\phi}_n(0) = 1$, we obtain $k = \frac{1}{\det \mathcal{L}} \hat{1}_{\mathcal{P}}(0)\hat{\phi}_n(0) = \frac{\operatorname{vol} \mathcal{P}}{\det \mathcal{L}}$, as claimed. Comparing all other terms, we obtain $0 = \hat{1}_{\mathcal{P}}(\xi)\hat{\phi}_n(\xi)\, e^{2\pi i \langle \xi, x \rangle}$. If we knew that $\hat{\phi}_n(\xi) \neq 0$ for any $\xi \in \mathcal{L}^*$, we would be done, for then we would have $\hat{1}_{\mathcal{P}}(\xi) = 0$ for all nonzero $\xi \in \mathcal{L}^*$. In fact, using Exercise 3.24, we can allow our approximate identity ϕ to enjoy an additional constraint: $\hat{\phi}(\xi) > 0$ *for all* $\xi \in \mathbb{R}^d$. □

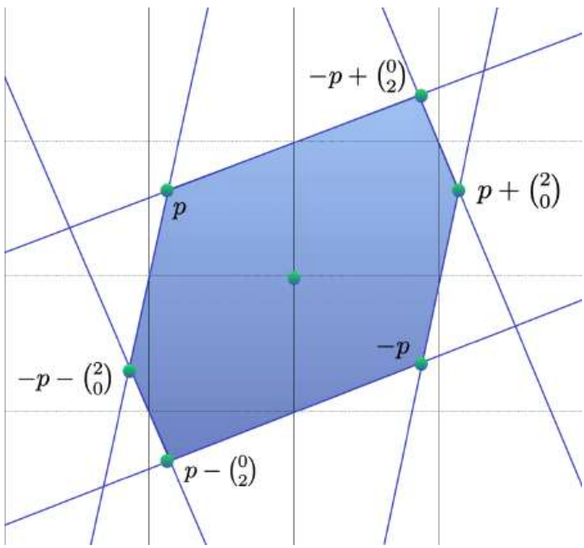


Figure 4.4: An extremal body in $\mathbb{R}^2$, relative to the integer lattice, which is a hexagon. It has area 4, and no integer points in its interior. We also get a 2-parameter family of such extremal bodies, parametrized by the point $p \in \mathbb{R}^2$ in the figure. It is clear from the picture that this family of extremal bodies consists of either symmetric hexagons, or symmetric quadrilaterals.

In 1905, Minkowski gave necessary conditions for a polytope $\mathcal{P}$ to tile $\mathbb{R}^d$ by translations. Later, Venkov and independently McMullen found sufficient conditions as well, culminating in the following fundamental result.

Theorem 4.6 (Minkowski–Venkov–McMullen). *A polytope $\mathcal{P}$ tiles $\mathbb{R}^d$ by translations if and only if the following 3 conditions hold:*

1. *$\mathcal{P}$ is a symmetric polytope.*

2. *The facets of $\mathcal{P}$ are symmetric polytopes.*

3. *Fix any face $F \subset \mathcal{P}$ of codimension 2, and project $\mathcal{P}$ onto the 2-dimensional plane that is orthogonal to the $(d-2)$-dimensional affine span of F. Then this projection is either a parallelogram, or a centrally symmetric hexagon.*

An **extremal body** is a convex body K for which we have equality in Minkowski's convex body Theorem:

$$\text{vol}\, K = 2^d (\det \mathcal{L}).$$

If we just look at Equation (4.4) a bit more closely, we quickly get a nice corollary that arises by combining Theorem 4.5 and Siegel's Theorem 4.3. Namely, equality occurs in Minkowski's convex body theorem if and only if K tiles $\mathbb{R}^d$ by translations. Precisely, we have the following consequence.

Theorem 4.7 (Extremal bodies). *Let K be any convex, centrally symmetric subset of $\mathbb{R}^d$, and fix a full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$. Suppose that the only point of $\mathcal{L}$ in the interior of K is the origin. Then:*

$2^d \det \mathcal{L} = \text{vol}\, K \iff \frac{1}{2}K$ *tiles $\mathbb{R}^d$ by translations with the lattice $\mathcal{L}$.*

Proof. By Siegel's formula (4.5), we have

$$2^d \det \mathcal{L} = \text{vol}\, K + \frac{4^d}{\text{vol}\, K} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2. \tag{4.31}$$

Therefore, the assumption $2^d \det \mathcal{L} = \text{vol}\, K$ holds $\iff$

$$0 = \frac{4^d}{\text{vol}\, K} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2, \tag{4.32}$$

$\iff$ all of the non-negative summands $\hat{1}_{\frac{1}{2}K}(\xi) = 0$, for all nonzero $\xi \in \mathcal{L}^*$. Now we would like to use Theorem 4.5 to show the required tiling equivalence, namely that $\frac{1}{2}K$ tiles $\mathbb{R}^d$ by translations with the lattice $\mathcal{L}$. We have already verified condition (a) of Theorem 4.5, applied to the body $\frac{1}{2}K$, namely that $\hat{1}_{\frac{1}{2}K}(\xi) = 0$, for all nonzero $\xi \in \mathcal{L}^*$. To verify condition (b) of Theorem 4.5, we notice that because $\operatorname{vol}\left(\frac{1}{2}K\right) = \frac{1}{2^d}\operatorname{vol} K$, it follows that $2^d \det \mathcal{L} = \operatorname{vol} K$ is equivalent to $1 = \frac{\operatorname{vol}\left(\frac{1}{2}K\right)}{\det \mathcal{L}}$, so that we may apply Theorem 4.5 with $\mathcal{P} := \frac{1}{2}K$, and with the multiplicity $k := 1$. $\square$

There is an extension of Theorem 4.6, the Minkowski–Venkov–McMullen result, to multi-tilings.

Theorem 4.8. *Gravin, Robins, and Shiryaev (2012) If a polytope $\mathcal{P}$ multi-tiles $\mathbb{R}^d$ by translations with a discrete set of vectors, then*

1. *$\mathcal{P}$ is a symmetric polytope.*

2. *The facets of $\mathcal{P}$ are symmetric polytopes.*

In the case that $\mathcal{P} \subset \mathbb{R}^d$ is a rational polytope, meaning that all the vertices of $\mathcal{P}$ have rational coordinates, the latter two necessary conditions for multi-tiling become sufficient conditions as well Gravin, Robins, and Shiryaev (ibid.).

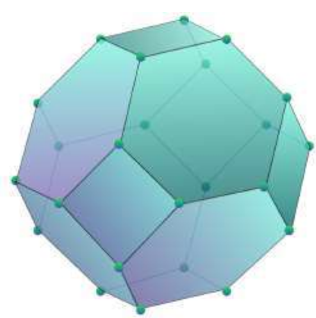


Figure 4.5: The truncated Octahedron, one of the 3-dimensional polytopes that tiles $\mathbb{R}^3$ by translations.

4.5 More about centrally symmetric polytopes

It's both fun and instructive to begin by seeing how very simple Fourier methods can give us deeper insight into the geometry of symmetric polytopes. The reader may glance at the definitions above, in (4.1).

Example 4.2. Consider the cross-polytope $\Diamond \subset \mathbb{R}^3$, defined in Chapter 2. This is a centrally symmetric polytope, but each of its facets is *not* a symmetric polytope, because its facets are triangles. $\square$

If *all* of the k-dimensional faces of a polytope $\mathcal{P}$ are symmetric, for $1 \leqslant k \leqslant d$, then $\mathcal{P}$ is called a **zonotope**. Zonotopes form an extremely important class of polytopes, and have various equivalent formulations.

Lemma 4.1. *A polytope $\mathcal{P} \subset \mathbb{R}^d$ is a zonotope if and only if it has one of the following properties.*

(a) $\mathcal{P}$ is a projection of some n-dimensional cube.

(b) $\mathcal{P}$ is the Minkowski sum of a finite number of line segments.

A projection here means any affine transformation of $\mathcal{P}$, where the rank of the associated matrix may be less than d.

Zonotopes have been very useful in the study of tilings (Ziegler (1995), M. Beck and Robins (2015)). For instance, in dimension 3, the only polytopes that tile $\mathbb{R}^3$ by translations with a lattice are zonotopes, and there is a list of 5 of them (up to an isomorphism of their face posets), called the **Fedorov solids**, and drawn in Figure 4.7 (also see our Note (d) below).

By definition, any zonotope is a symmetric polytope, but the converse is not true; for example, the cross-polytope is symmetric, but it has triangular faces, which are not symmetric, so the cross-polytope is not a zonotope.

Example 4.3. A particular embedding of the truncated octahedron $\mathcal{P}$, drawn in Figure 4.5, is given by the convex hull of the set of 24 vertices defined by all permutations of $(0, \pm 1, \pm 2)$. We note that this set of vertices can also be thought of as the orbit of just the one point $(0, 1, 2) \in \mathbb{R}^3$ under the hyperoctahedral group (see Chen and Guo (2014) for more on the hyperoctahedral group). It turns out that this truncated octahedron $\mathcal{P}$ tiles $\mathbb{R}^3$ by translations with a lattice (Exercise 4.5). $\square$

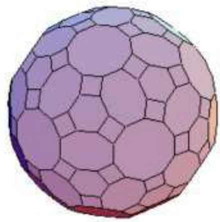


Figure 4.6: A more complex 3-dimensional zonotope, which does not tile $\mathbb{R}^3$ by translations.

As the following Lemma shows, it is easy to detect/prove whether or not S is centrally symmetric by just observing whether or not its Fourier transform is real-valued.

Lemma 4.2. *A (measurable) set $S \subset \mathbb{R}^d$ is centrally symmetric if and only if*

$$\hat{1}_S(\xi) \in \mathbb{R}, \text{ for all } \xi \in \mathbb{R}^d.$$

Proof. Suppose that the set S is centrally symmetric. Then we have

$$\overline{\hat{1}_S(\xi)} := \overline{\int_S e^{2\pi i \langle \xi, x\rangle} dx} = \int_S e^{-2\pi i \langle \xi, x\rangle} dx \tag{4.33}$$

$$= \int_{-S} e^{2\pi i \langle \xi, x\rangle} dx \tag{4.34}$$

$$= \int_S e^{2\pi i \langle \xi, x\rangle} dx := \hat{1}_S(\xi), \tag{4.35}$$

$$\tag{4.36}$$

showing that the complex conjugate of $\hat{1}_S$ is itself, hence that it is real-valued. Conversely, suppose that $\hat{1}_S(\xi) \in \mathbb{R}$, for all $\xi \in \mathbb{R}^d$. We use the fact that the

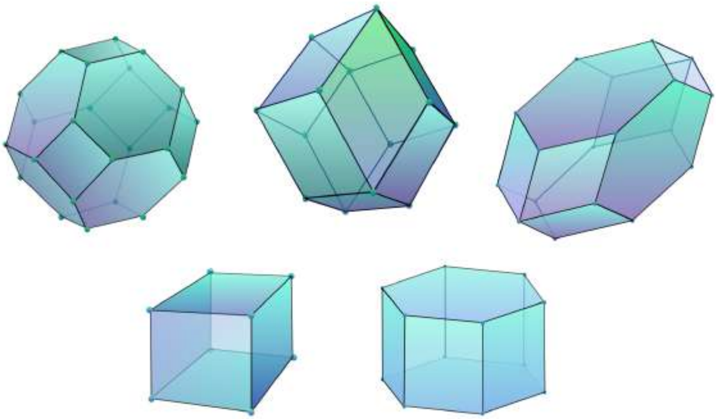


Figure 4.7: The Fedorov solids, the only 3-dimensional polytopes that tile $\mathbb{R}^3$ by translations. All 5 of them are zonotopes, and they are also extreme bodies for Minkowski's convex body theorem. The top three, from left to right, are: the Truncated octahedron, the Rhombic dodecahedron, and the Hexarhombic dodecahedron. The bottom two are the cube and the hexagonal prism.

Fourier transform is invertible:

$$(\mathcal{F} \circ \mathcal{F})(1_S)(x) = 1_S(-x), \tag{4.37}$$

for all $x \in \mathbb{R}^d$. To show that S is centrally symmetric, it suffices to show that $1_{-S}(x) = 1_S(x)$, for all $x \in \mathbb{R}^d$. Further, by 4.37, it now suffices to show that $\hat{1}_{-S}(\xi) = \hat{1}_S(\xi)$, for all $\xi \in \mathbb{R}^d$. We therefore compute:

$$\begin{aligned}
\hat{1}_{-S}(\xi) := \int_{-S} e^{2\pi i\langle \xi, x\rangle} dx &= \overline{\int_S e^{-2\pi i\langle \xi, x\rangle} dx} && (4.38)\\
&= \overline{\int_S e^{2\pi i\langle \xi, y\rangle} dy} && (4.39)\\
&:= \overline{\hat{1}_S(\xi)} && (4.40)\\
&= \hat{1}_S(\xi), && (4.41)
\end{aligned}$$

for all $\xi \in \mathbb{R}^d$, where we have used the assumption that $\hat{1}_S(\xi)$ is real-valued in the last equality. $\square$

Example 4.4. The interval $\mathcal{P} := [-\frac{1}{2}, \frac{1}{2}]$ is a symmetric polytope, and indeed we can see that its Fourier transform $\hat{1}_{\mathcal{P}}(\xi)$ is real-valued, namely we have $\hat{1}_{\mathcal{P}}(\xi) = \operatorname{sinc}(\xi)$, as we saw in equation (2.6). $\square$

Example 4.5. The cross-polytope $\Diamond_2$ is a symmetric polytope, and as we verified in dimension 2, equation (2.56), its Fourier transform $1_{\Diamond_2}(\xi)$ is real-valued. $\square$

Alexandrov (1933), and independently Shephard, proved the following remarkable fact.

Theorem 4.9 (Alexandrov and Shephard)**.** *Let P be any real, d-dimensional polytope, with $d \geqslant 3$. If all of the facets of P are centrally symmetric, then P is centrally symmetric.* $\square$

Example 4.6. The converse to the latter result is clearly false, as demonstrated by the cross-polytope in dimension $d > 2$: it is centrally symmetric, but its facets are not symmetric because they are simplices and we know that no simplex (of dimension $\geqslant 2$) is symmetric (Exercise 10.9). $\square$

Later, Peter McMullen discovered the following wonderful generalization of Theorem 4.9.

Theorem 4.10 (McMullen)**.** *Let P be any real, d-dimensional polytope, with $d \geqslant 3$, and fix any positive integer k with $2 \leqslant k \leqslant d-1$.*

If all the k-dimensional faces of $\mathcal{P}$ are symmetric, then all $(k+j)$-dimensional faces of $\mathcal{P}$ are symmetric as well (including $\mathcal{P}$ itself), for all $1 \leqslant j \leqslant d-k$. $\square$

So the Alexandrov–Shephard Theorem 4.9 above follows as a special case of McMullen's theorem by setting $k = d - 1$.

One might wonder what happens if we 'discretize the volume' of a symmetric body K, by counting integer points, and then ask for an analogue of Minkowski Theorem 4.1. In fact, Minkowski already had a result about this too (and he had so many beautiful ideas that it's hard to put them all in one place!). We give Minkowski's own elegant and short proof.

Theorem 4.11 (Minkowski, 1910). *Let $K \subset \mathbb{R}^d$ be any d-dimensional, convex, symmetric set. If the only integer point in the interior of K is the origin, then*

$$\left| K \cap \mathbb{Z}^d \right| \leqslant 3^d. \tag{4.42}$$

Proof. We define the map $\phi : \mathbb{Z}^d \to (\mathbb{Z}/3\mathbb{Z})^d$, by reducing each coordinate modulo 3. Now we claim that when restricted to the set $K \cap \mathbb{Z}^d$, our map ϕ is $1-1$. The statement of the theorem follows directly from this claim. So let $x, y \in K \cap \mathbb{Z}^d$, and suppose $\phi(x) = \phi(y)$. Then, by definition of the map ϕ, we have

$$n := \frac{1}{3}(x - y) \in \mathbb{Z}^d, \tag{4.43}$$

Now we define C to be the **interior** of the convex hull of $x, -y$, and 0. Because K is symmetric, and $x, y \in K$, we know that $-y \in K$ as well, so that $C \subset K^o$. Now using the convexity of C, we also see that $n \in C$, because n is a non-trivial convex linear combination of $0, x, -y$.

Therefore $n \in K^o$ as well. Altogether, $n \in K^o \cap \mathbb{Z}^d = \{0\}$, which forces $n = 0$. Therefore $x - y = 0$. □

Theorem 4.11 is often called "Minkowski's 3^d theorem". An immediate and natural question is: which bodies account for the 'equality case'? One direction is easy to see - if K is the integer cube whose vertices are $\pm e_1 \pm e_2 \pm \cdots \pm e_d$, then it is clear that K is symmetric about the origin, and the only integer point in its interior is the origin. In addition, $\operatorname{vol} K = 2^d$, and K contains 3^d integer points. It is a bit surprising, perhaps, that only in 2012 was it proved that this integer cube is the only case of equality in Minkowski's 3^d theorem Draisma, McAllister, and Nill (2012).

In a different direction, it turns out that the volume of the Minkowski symmetrized body $\frac{1}{2}K - \frac{1}{2}K$, which appeared quite naturally in some of the proofs above, can be related in a rather precise manner to the volume of K itself, via the Brunn–Minkowski inequality Gruber (2007). The consequence, known as the Rogers–Shephard inequality, is as follows:

$$\operatorname{vol} K \leqslant \operatorname{vol}\left(\frac{1}{2}K - \frac{1}{2}K\right) \leqslant \binom{2n}{n} \operatorname{vol} K, \tag{4.44}$$

where equality on the left holds $\iff$ K is a symmetric body, and equality on the right holds $\iff$ K is a simplex (see Cassels (1997)).

Notes

(a) Siegel's original proof of Theorem 4.3 used Parseval's identity, but the spirit of the two proofs is similar.

(b) In Exercise 4.4 below, we see three equivalent conditions for a 2-simplex to be unimodular. In higher dimensions, a d-simplex will not satisfy all three conditions, and hence this exercise shows one important 'breaking point' between 2-dimensional and 3-dimensional discrete geometry.

(c) There are a growing number of interesting books on the geometry of numbers. One encyclopedic text that contains many other connections to the geometry of numbers is the book by Peter Gruber (2007). Two other excellent and classic introductions are Siegel (1989), and Cassels (1997). An expository introduction to some of the elements of the Geometry of numbers, at a level that is even appropriate for high school students, is given by Olds, Lax, and Davidoff (2000). For upcoming books, the reader may also consult Martin Henk's lecture notes 'Introduction to geometry of numbers' Henk (n.d.), and the book 'Geometry of Numbers' Fukshansky and Garcia (n.d.).

(d) The Fedorov solids are depicted, and explained via the modern ideas of Conway and Sloan, in an excellent expository article by David Austin (2013). For a view into the life and work of Evgraf Stepanovich Fedorov, as well as a fascinating account of how Fedorov himself thought about the 5 parallelohedra, the reader may consult the article by Senechal and Galiulin (1984). The authors of Senechal and Galiulin (ibid.) also discuss the original book of Fedorov, called *An Introduction to the Theory of Figures*, published in 1885, which is now considered a pinnacle of modern crystallography. Fedorov later became one of the great crystallographers of his time.

(e) The field of multi-tiling is still growing. One of the first important papers in this field was by Mihalis Kolountzakis (2000), who related the multi-tiling problem to a famous technique known as the idempotent theorem, and thereby proved that if we have a multi-tiling in $\mathbb{R}^2$ with any discrete set of translations, then we also have a multi-tiling with a finite union of lattices. A recent advance is an equivalence between multi-tiling and certain Hadwiger-type invariants, given by Lev and Liu (2019). Here the authors show as well that for a generalized polytope $\mathcal{P} \subset \mathbb{R}^d$ (not necessarily convex or connected), if $\mathcal{P}$ is spectral, then $\mathcal{P}$ is equidecomposable by translations to a cube of equal volume.

Another natural question in multi-tiling, which is still open, is the following:

Question 4. *Suppose that $\mathcal{P}$ multi-tiles with a discrete set of translations D. Do we really need the set D of translates of $\mathcal{P}$ to be a very complicated discrete set, or is it true that just a finite union of lattices suffices? Even better, perhaps one lattice always suffices?*

In this direction, Liu proved recently that if we assume that $\mathcal{P}$ multi-tiles with a finite union of lattice, then $\mathcal{P}$ also multi-tiles with a single lattice Liu (2018). This is big step in the direction of answering Question 4 in general. An earlier, and smaller step, was taken in Gravin, Kolountzakis, et al. (2013), where the authors answered part of Question 4 in $\mathbb{R}^3$, reducing the search from an arbitrary discrete set of translations, to translations by a finite union of lattices. Taken together, the latter two steps imply that in $\mathbb{R}^3$ (and in $\mathbb{R}^2$), any multi-tiling with a discrete set of translations also occurs with just a one lattice.

In a different direction, the work of Gennadiy Averkov (2021) analyzes the equality cases for an extension of Minkowski's theorem, relating those extremal bodies to multi-tilers. In Yang and Zong (2019), the authors show that the smallest k for which we can obtain a nontrivial k-tiling in $\mathbb{R}^2$ is $k = 5$, and the authors characterize those 5-tiling bodies, showing in particular that if a convex polygon is a 5-tiler, then it must be either an octagon, or a decagon.

Question 5. *In $\mathbb{R}^d$, what is the smallest integer k such that there exists a d-dimensional polytope $\mathcal{P}$ that k-tiles $\mathbb{R}^d$ by translations?*

(f) We say that a body $\mathcal{P}$ (any bounded, measurable subset of $\mathbb{R}^d$) is 'spectral' if the function space $L^2(\mathcal{P})$ possesses an orthonormal, complete basis of exponentials. There is a fascinating and vast literature about such spectral bodies, relating them to tiling, and multi-tiling problems. One of the most interesting and natural questions in this direction is the following conjecture, by Bent Fuglede.

The Fuglede conjecture tells us that $\mathcal{P}$ is spectral $\iff$ $\mathcal{P}$ tiles $\mathbb{R}^d$ by translations. In the case that $\mathcal{P}$ is convex, one might hope that more is true. Indeed, in 2003 Alex Iosevich, Nets Katz, and Terry Iosevich, N. Katz, and Tao (2003) proved that the Fuglede conjecture is true for all convex domains in $\mathbb{R}^2$. In 2021, this conjecture was proved for all convex domains (which must necessarily be polytopes by an additional simple argument), in the work of Lev and Matolcsi (2019).

In a related direction, Grepstad and Lev (2014) showed that for any bounded, measurable subset $S \subset \mathbb{R}^d$, if S multi-tiles with by translations with a discrete set, then S has a Riesz basis of exponentials.

(g) We have seen in Theorem 4.5 that the zero set of the Fourier transform of a polytope (also known as the null set of a polytope) is very important, in that it gave us a necessary and sufficient condition for multi-tiling. But the zero set of the FT also gives more information, and an interesting application of the information content in the zero set is the Pompeiu problem. The Pompeiu problem is an ancient problem (defined in 1929 by Pompeiu) that asks the following: which bodies $\mathcal{P} \in \mathbb{R}^d$ are uniquely characterized by the collection of their integrals over $\mathcal{P}$, and over all rigid motions of $\mathcal{P}$? An equivalent formulation is the following.

Question 6. *Does the vanishing of all of the integrals*

$$\int_{\sigma(\mathcal{P})} f(x)dx = 0, \tag{4.45}$$

taken over all rigid motions σ that include translations, imply that $f = 0$?

A body $\mathcal{P} \subset \mathbb{R}^d$, for which the answer to the question above is affirmative, is said to have the Pompeiu property. It is still an open problem, in general dimension, whether all convex bodies have the Pompeiu property. It is known, by the work of Brown, Schreiber, and Taylor (1973) that $\mathcal{P}$ has the Pompeiu property $\iff$ the collection of Fourier transforms $\hat{1}_{\sigma(\mathcal{P})}(z)$, taken over all rigid motions σ of $\mathbb{R}^d$, have a common zero z. It was also known that all polytopes have the Pompeiu property. Recently, in Machado and Robins (2021), Fabrício Machado and SR showed that the null set of a polytope does not contain (almost all) circles, and as a consequence we get a simple new proof that all polytopes have the 'Pompeiu property'.

Exercises

4.1. *Suppose that in $\mathbb{R}^2$, we are given a symmetric, convex body K of area 4, which contains only the origin. Prove that B must tile $\mathbb{R}^2$ by translations.*

4.2. ♣ *Given a convex d-dimensional body $K \subset \mathbb{R}^d$, prove that $K - K$ is convex, and that $K - K$ is symmetric about the origin.*

4.3. ♣ *The support of a function f is defined here as*

$$\operatorname{supp}(f) := \{x \in \mathbb{R}^d \mid f(x) \neq 0\}. \tag{4.46}$$

Suppose that we are given two convex bodies $A, B \subset \mathbb{R}^d$. Show that

$$\operatorname{supp}(1_A * 1_B) = A + B,$$

where the addition is the Minkowski addition of sets.

Note. Many books often use the closure of the set (4.46) as the definition of the support of f, but for our purposes the definition above is sufficient.

4.4. ♣ *Suppose we have a triangle Δ whose vertices v_1, v_2, v_3 are integer points. Prove that the following properties are equivalent:*

(a) Δ has no other integer points inside or on its boundary.

(b) $Area(\Delta) = \frac{1}{2}$.

(c) Δ is a unimodular triangle - i.e. $v_3 - v_1$ and $v_2 - v_1$ form a basis for $\mathbb{Z}^2$.

(Hint: You might begin by "doubling" the triangle to form a parallelogram.)

4.5. *Show that the truncated octahedron, defined in Example 4.3, tiles $\mathbb{R}^3$ by using only translations with a lattice. Which lattice can you use for this tiling?*

4.6. *Show that in $\mathbb{R}^d$, an integer simplex Δ is unimodular if and only if $\operatorname{vol}(\Delta) = \frac{1}{d!}$.*

4.7. *Find in $\mathbb{R}^3$, an integer simplex Δ that has no other integer points inside or on its boundary (other than its vertices of course), but such that Δ is not a unimodular simplex.*

4.8. *Prove that for any polytope $\mathcal{P}$, $\hat{1}_{\mathcal{P}}$ is not a Schwartz function.*

4.9. *Define $f(x) := a \sin x + b \cos x$, for constants $a, b \in \mathbb{R}$. Show that the maximum value of f is $\sqrt{a^2 + b^2}$, and occurs when $\tan x = \frac{a}{b}$.*

4.10. *Find an example of a symmetric polygon $\mathcal{P} \subset \mathbb{R}^2$ that multi-tiles (nontrivially) with multiplicity $k = 5$.*

(A trivial multi-tiling for $\mathcal{P}$ is by definition a multi-tiling that uses $\mathcal{P}$, with some multiplicity $k > 1$, but such that there also exists a 1-tiling (classical) using the same $\mathcal{P}$)

4.11. *Here we use Siegel's Theorem 4.4 to give an extension of Minkowski's classical Theorem 4.2 for bodies K that are not necessarily symmetric.*

Namely, let K be any bounded, measurable subset of $\mathbb{R}^d$ (so K is not necessarily symmetric), with a positive d-dimensional measure. Let $B := \frac{1}{2}K - \frac{1}{2}K$ be the symmetrized body of K (hence B is a convex symmetric body). Let $\mathcal{L}$ be a (full rank) lattice in $\mathbb{R}^d$. Prove the following statement:

> **If** $\operatorname{vol} K > 2^d (\det \mathcal{L})$**, then** B **must contain a nonzero point of** $\mathcal{L}$ **in its interior.**

Notes. We note that the positive conclusion of the existence of a nonzero integer point holds only for the symmetrized body B, with no guarantees for any integer points in K.

5 An introduction to Euclidean lattices

Lattices quantify the idea of periodic structures.

– Anonymous

Less is more.....more or less.

– Ludwig Mies van der Rohe

5.1 Intuition

We introduce Euclidean lattices here, which may be thought of intuitively as regularly-spaced points in $\mathbb{R}^d$, with some hidden number-theoretic structure. Another intuitive way to think of lattices is that they are one of the most natural ways to **discretize Euclidean space**. A lattice in $\mathbb{R}^d$ is also the most natural extension of an infinite set of equally-spaced points on the real line. In the real-world, lattices come up very naturally when we study crystals, for example.

It is perhaps not surprising that number theory comes in through study of the integer lattice $\mathbb{Z}^d$, as it is the d-dimensional extension of the integers $\mathbb{Z}$. Moreover, whenever we study almost any periodic behavior, lattices naturally come up,

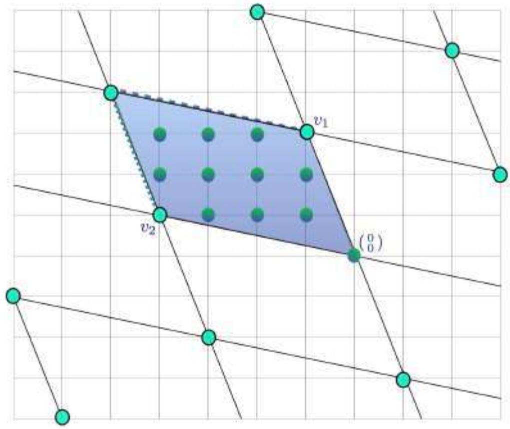


Figure 5.1: A fundamental parallelepiped (half-open), for a lattice $\mathcal{L}$, generated by the vectors v_1 and v_2.

essentially from the definition of **periodicity** in Euclidean space. And of course, where there are lattices, there are Fourier series, as we also saw in Chapter 3.

5.2 Introduction to lattices

Definition 5.1. *A* **lattice** *is defined by the integer linear span of a fixed set of linearly independent vectors* $\{v_1, \ldots, v_m\} \subset \mathbb{R}^d$*:*

$$\mathcal{L} := \left\{ n_1 v_1 + \cdots + n_m v_m \in \mathbb{R}^d \;\middle|\; \textit{all } n_j \in \mathbb{Z} \right\}. \tag{5.1}$$

The most common lattice is the **integer lattice**

$$\mathbb{Z}^d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; \text{all } x_j \in \mathbb{Z} \right\}.$$

However, we often encounter different types of lattices, occurring very naturally in practice, and it is natural to ask how they are related to each other. The first thing we might notice is that, by Equation (5.1), a lattice may also be written as follows:

$$\mathcal{L} := \left\{ \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_m \\ | & | & \dots & | \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix} \;\middle|\; \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix} \in \mathbb{Z}^m \right\} := M(\mathbb{Z}^m), \tag{5.2}$$

where by definition, M is the $d \times m$ matrix whose columns are the vectors $v_1, \dots, v_m$. This set of basis vectors $\{v_1, \dots, v_m\}$ is called a **basis** for the lattice $\mathcal{L}$, and m is called the **rank** of the lattice $\mathcal{L}$. In this context, we also use the notation $\operatorname{rank}(\mathcal{L}) = m$.

We will call M a **basis matrix** for the lattice $\mathcal{L}$. But there are always infinitely many other bases for $\mathcal{L}$ as well, and Lemma 5.6 below shows how they are related to each other.

Most of the time, we will be interested in **full-rank** lattices, which means that $m = d$; however, sometimes we will also be interested in lattices that have lower rank, and it is important to understand them. The **determinant** of a full-rank lattice $\mathcal{L} := M(\mathbb{Z}^d)$ is defined by

$$\det \mathcal{L} := |\det M|.$$

It is easy to prove that this definition is independent of the choice of basis matrix M, which is the content of Lemma 5.6 below.

Example 5.1. In $\mathbb{R}^1$, we have the integer lattice $\mathbb{Z}$, but we also have lattices of the form $r\mathbb{Z}$, for any real number r. It's easy to show that any lattice in $\mathbb{R}^1$ is of this latter type (Exercise 5.5). For example, if $r = \sqrt{2}$, then all integer multiples of $\sqrt{2}$ form a 1-dimensional lattice. □

Example 5.2. In $\mathbb{R}^2$, consider the lattice $\mathcal{L}$ generated by the two integer vectors $v_1 := \binom{-1}{3}$ and $v_2 := \binom{-4}{1}$, drawn in Figure 5.1. A different choice of basis for the same lattice $\mathcal{L}$ is $\{\binom{-3}{-2}, \binom{-8}{-9}\}$, drawn in Figure 5.2. We note that $\det \mathcal{L} = 11$, and indeed the areas of both half-open parallelepipeds equals 11. □

A **fundamental parallelepiped** for a lattice $\mathcal{L}$, whose basis is given by $\{v_1, \dots, v_m\}$, is defined by :

$$D := \left\{ \lambda_1 v_1 + \dots + \lambda_m v_m \;\middle|\; \text{all } 0 \leqslant \lambda_k < 1 \right\}, \tag{5.3}$$

also known as a **half-open parallelepiped**.

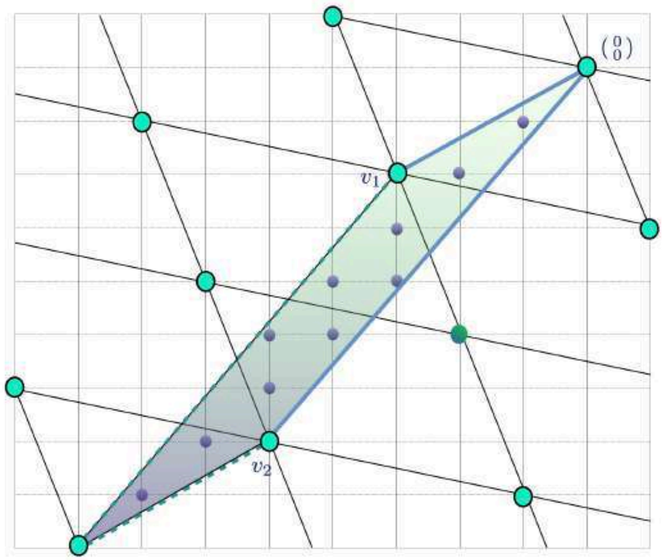


Figure 5.2: A second fundamental parallelepiped for the same lattice $\mathcal{L}$ as in Figure 5.1

Given any lattice $\mathcal{L} \subset \mathbb{R}^d$, and any fixed fundamental parallelepiped D of $\mathcal{L}$, we have the pleasant property that D tiles $\mathbb{R}^d$ by translations with vectors from $\mathcal{L}$. More precisely, this useful fact tells us that any $x \in \mathbb{R}^d$ may be written **uniquely** as

$$x = n + f, \tag{5.4}$$

where $n \in \mathcal{L}$, and $f \in D$ (Exercise 5.1).

How do we define the determinant of a general lattice $\mathcal{L}$ of rank r? We can start by observing how the (squared) lengths of vectors in $\mathcal{L}$ behave w.r.t. a given basis of $\mathcal{L}$:

$$\|x\|^2 = \left\langle \sum_{j=1}^{r} c_j v_j, \sum_{k=1}^{r} c_k v_k \right\rangle = \sum_{1\leqslant j,k\leqslant r} c_j c_k \langle v_j, v_k\rangle := c^T M^T M c, \tag{5.5}$$

where $M^T M$ is an $r \times r$ matrix whose columns are basis vectors of $\mathcal{L}$. With this as motivation, we define:

$$\det \mathcal{L} := \sqrt{M^T M}, \tag{5.6}$$

called the **determinant of the lattice** $\mathcal{L}$. This definition coincides, as it turns out, with the Lebesgue measure of any fundamental parallelepiped of $\mathcal{L}$ (Exercise 5.15).

Given two lattices $\mathcal{L} \subset \mathbb{R}^d$, and $\mathcal{M} \subset \mathbb{R}^d$, such that

$$\mathcal{L} \subseteq \mathcal{M},$$

we say that $\mathcal{L}$ **is a sublattice of** $\mathcal{M}$. For example, Figure 5.3 shows a rank 1 sublattice of the integer lattice $\mathbb{Z}^2$, together with its determinant. But even sublattices that have the same rank are very interesting, and quite useful in applications.

Given a sublattice $\mathcal{L}$ of $\mathcal{M}$, both of the same rank, a crucial idea is to think of all of the translates of $\mathcal{L}$ by an element of the coarser lattice $\mathcal{M}$, which we call:

$$\mathcal{M}/\mathcal{L} := \left\{\mathcal{L} + m \;\middle|\; m \in \mathcal{M}\right\}. \tag{5.7}$$

Each such translate $\mathcal{L} + m$ is called a **coset** of $\mathcal{L}$ in $\mathcal{M}$, and the collection of all of these cosets, namely $\mathcal{M}/\mathcal{L}$, is called the **quotient lattice** (or quotient group).

Theorem 5.1. *Let $\mathcal{L} \subseteq \mathcal{M}$ be any two lattices of the same rank. Then*

1. $\frac{\det \mathcal{L}}{\det \mathcal{M}}$ *is an integer.*
2. *The positive integer $\frac{\det \mathcal{L}}{\det \mathcal{M}}$ is equal to the number of cosets of $\mathcal{L}$ in $\mathcal{M}$. In other words, $|\mathcal{M}/\mathcal{L}| = \frac{\det \mathcal{L}}{\det \mathcal{M}}$.*

Example 5.3. Let $\mathcal{M} := \mathbb{Z}^d$, and $\mathcal{L} := 2\mathbb{Z}^d$, the sublattice consisting of vectors all of whose coordinates are even integers. So $\mathcal{L} \subset \mathcal{M}$, and the quotient lattice $\mathcal{M}/\mathcal{L}$ consists of the sets $\left\{2\mathbb{Z}^d + n \;\middle|\; n \in \mathbb{Z}^d\right\}$. It is (almost) apparent that the number of elements of the latter set is exactly 2^d, so in our new notation we have $\left|\mathbb{Z}^d/2\mathbb{Z}^d\right| = 2^d$.

We may also think of this quotient lattice $\mathbb{Z}^d/2\mathbb{Z}^d$ as the discrete unit cube, namely $\{0, 1\}^d$, a common object in theoretical computer science. □

5.3 Discrete subgroups – an alternate definition of a lattice

The goal here is to give another useful way to define a lattice. The reader does not need any background in group theory, because the ideas here are self-contained, given some background in basic linear algebra.

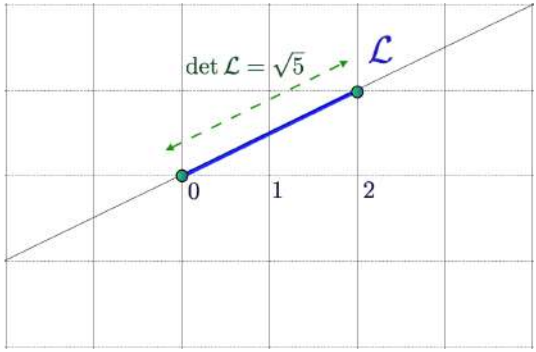


Figure 5.3: A sublattice $\mathcal{L} \subset \mathbb{Z}^2$ of rank 1, which has just one basis vector. Here $\mathcal{L}$ has a 1-dimensional fundamental parallelepiped, showing that $\det \mathcal{L} = \sqrt{v^T v} = \sqrt{5}$.

Definition 5.2. *We define a* **discrete subgroup** *as a set $S \subset \mathbb{R}^d$, together with the operation of vector addition between all of its elements, which enjoys the following two properties.*

(a) **[The subgroup property]** *If $x, y \in S$, then $x - y \in S$.*

(b) **[The discrete property]** *There exists a positive real number $\delta > 0$, such that the distance between any two distinct points of S is at least δ.*

In particular, it follows from Definition 5.2 (a) that the zero vector must be in S, because for any $x \in S$, it must be the case that $x - x \in S$. The distance alluded to in Definition 5.2 (b) is the usual Euclidean distance function, which we denote here by $\|x - y\|_2 := \sqrt{\sum_{k=1}^{d} (x_k - y_k)^2}$.

Example 5.4. The lattice $\mathbb{Z}^d$ is a discrete subgroup of $\mathbb{R}^d$. In dimension 1, the lattice $r\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$, for any fixed $r > 0$. Can we think of discrete subgroups that are not lattices? The answer is given by Lemma 5.1 below. □

The magic here is the following very useful way of going back and forth between this new notion of a discrete subgroup of $\mathbb{R}^d$, and our Equation (5.1) of a lattice. The idea of using this alternate Definition 5.2, as opposed to our previous Definition 5.1 of a lattice, is that it gives us a **basis-free** way of proving and discovering facts about lattices.

Lemma 5.1. *$\mathcal{L} \subset \mathbb{R}^d$ is a lattice $\iff$ $\mathcal{L}$ is a discrete subgroup of $\mathbb{R}^d$.* □

(For a proof see Gruber 2007).

Example 5.5. Given any two lattices $\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{R}^d$, let's show that $S := \mathcal{L}_1 \cap \mathcal{L}_2$ is also a lattice. First, any lattice contains the zero vector, and it may be the case that their intersection consists of only the zero vector. For any vectors $x, y \in S$, we also have $x, y \in \mathcal{L}_1$, and $x, y \in \mathcal{L}_2$, hence by the subgroup property of $\mathcal{L}_1$ and of $\mathcal{L}_2$, we know that both $x - y \in \mathcal{L}_1$, and $x - y \in \mathcal{L}_2$. In other words, $x - y \in \mathcal{L}_1 \cap \mathcal{L}_2 := S$. To see why the discrete property of Definition 5.2 holds here, we just notice that since $x - y \in \mathcal{L}_1$, we already know that $|x - y| > \delta_1$, for some $\delta_1 > 0$; similarly, because $x - y \in \mathcal{L}_2$, we know that $|x - y| > \delta_2$ for some $\delta_2 > 0$. So we let $\delta := \min(\delta_1, \delta_2\}$, and we have shown that S is a discrete subgroup of $\mathbb{R}^d$. By Lemma 5.1, we see that S is a lattice.

If we had used Equation (5.1) of a lattice to show that S is indeed a lattice, it would require us to work with bases, and the proof would be longer and less transparent, for this problem. □

Example 5.6. Consider the following discrete set of points in $\mathbb{R}^d$:

$$A_{d-1} := \left\{ x \in \mathbb{Z}^d \mid \sum_{k=1}^{d} x_k = 0 \right\},$$

for any $d \geqslant 2$, as depicted in Figure 5.4. Is A_d a lattice? Using the definition 5.1 of a lattice, it is not obvious that A_d is a lattice, because we would have to exhibit a basis, but it turns out that the following set of vectors may be shown to be a basis:

$$\{e_2 - e_1, e_3 - e_1, \cdots e_d - e_1\},$$

and hence A_d is a sublattice of $\mathbb{Z}^d$, of rank $d - 1$ (Exercise 5.10).

Just for fun, we will use Lemma 5.1 to show that A_d is indeed a lattice. To verify the subgroup property of Definition 5.2 (a) suppose that $x, y \in A_d$. Then by definition we have $\sum_{k=1}^d x_k = 0$ and $\sum_{k=1}^d y_k = 0$. So $\sum_{k=1}^d (x_k - y_k) = 0$, implying that $x - y \in A_d$.

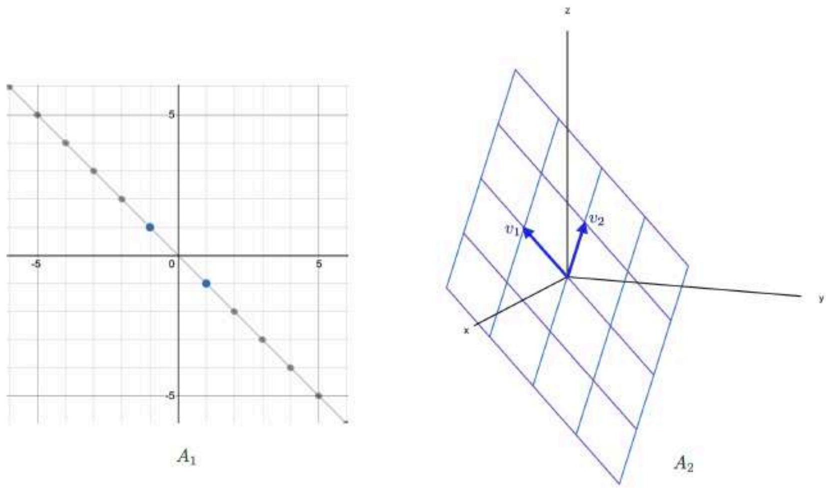


Figure 5.4: The lattice A_1, and the lattice A_2, with basis $\{v_1, v_2\}$

To verify the discrete property of Definition 5.2 (b) suppose we are given two distinct points $x, y \in A_d$. We can first compute their "cab metric" distance function, in other words the L^1-norm defined by

$$\|x - y\|_1 := |x_1 - y_1| + \cdots + |x_d - y_d|,$$

By assumption, there is at least one coordinate where x and y differ, say the k'th coordinate. Then

$$\|x - y\|_1 := |x_1 - y_1| + \cdots + |x_d - y_d| \geqslant 1,$$

because all of the coordinates are integers, and $x_k \neq y_k$ by assumption. Since the L^1-norm and the L^2-norm are only off by $\sqrt{d}$ (by Exercise 3.1), we have:

$$\sqrt{d}\,\|x - y\|_2 \geqslant \|x - y\|_1 \geqslant 1,$$

so the property 5.2 (b) is satisfied with $\delta := \frac{1}{\sqrt{d}}$, and we've shown that A_d is a lattice. □

We note that the lattices A_d defined in Example 5.6 are very important in many fields of Mathematics, including Lie algebras (root lattices), Combinatorial geometry, and Number theory.

5.4 Lattices defined by congruences

In this section we develop some of the theory in a concrete manner. A classic example of a lattice defined by an auxiliary algebraic construction is the following. Suppose we are given a constant integer vector $(c_1, \dots, c_d) \in \mathbb{Z}^d$, where we further assume that $\gcd(c_1, \dots, c_d) = 1$. Let

$$C := \left\{ x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d \equiv 0 \mod N \right\}, \tag{5.8}$$

where N is a fixed positive integer.

Is C a lattice? Indeed, we can see that C is a lattice by first checking Definition 5.2 (a). For any $x, y \in C$, we have $c_1 x_1 + \cdots + c_d x_d \equiv 0 \mod N$ and $c_1 y_1 + \cdots + c_d y_d \equiv 0 \mod N$. Subtracting these two congruences gives us $c_1(x_1 - y_1) + \cdots + c_d(x_d - y_d) \equiv 0 \mod N$, so that $x - y \in C$. The verification of Definition 5.2 (b) if left to the reader, and its logic is similar to Example 5.6.

There is even a simple formula for the volume of a fundamental parallelepiped for C:

$$\det C = N, \tag{5.9}$$

as we prove below, in (5.20). Perhaps we can solve an easier problem first. Consider the **discrete hyperplane** defined by:

$$H := \left\{ x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = 0 \right\},$$

Is H a lattice? We claim that H itself is indeed a lattice (also known as a rank-$(d-1)$ sublattice of $\mathbb{Z}^d$), and since its verification is similar to the arguments above, this is Exercise 5.19.

The fundamental parallelepiped (which is $(d-1)$-dimensional) of H also has a wonderful formula, as follows. First, we recall a general fact (from Calculus/analytic geometry) about hyperplanes, namely that the distance δ between any two parallel hyperplanes $c_1 x_1 + \cdots + c_d x_d = k_1$ and $c_1 x_1 + \cdots + c_d x_d = k_2$ is given by

$$\delta = \frac{|k_1 - k_2|}{\sqrt{c_1^2 + \cdots + c_d^2}}. \tag{5.10}$$

(see Exercise 5.3)

Lemma 5.2. *For any discrete hyperplane*

$$H := \left\{x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = 0\right\},$$

we have:

$$\det H = \sqrt{c_1^2 + \cdots + c_d^2}. \tag{5.11}$$

Proof. We first fix a basis $\{v_1, \ldots, v_{d-1}\}$ for the $(d-1)$-dimensional sublattice defined by $H := \left\{x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = 0\right\}$. We adjoin to this basis one new vector, namely any integer vector w that translates H to its nearest discrete hyperplane companion $H + w$, where we define

$$H + w := \left\{x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = 1\right\}.$$

It's easy to see that there are no integer points strictly between these two hyperplanes H and $H + w$, and so the parallelepiped $\mathcal{P}$ formed by the edge vectors $v_1, \ldots, v_{d-1}, w$ is a fundamental domain for $\mathbb{Z}^d$, hence has volume 1.

On the other hand, we may also calculate the volume of $\mathcal{P}$ by multiplying the volume of its base times its height, using (5.10):

$$\begin{aligned} 1 = \operatorname{vol} \mathcal{P} &= (\text{volume of the base of } \mathcal{P})(\text{height of } \mathcal{P}) && (5.12)\\ &= (\det H) \cdot \delta && (5.13)\\ &= (\det H) \frac{1}{\sqrt{c_1^2 + \cdots + c_d^2}}, && (5.14) \end{aligned}$$

and so $\det H = \sqrt{c_1^2 + \cdots + c_d^2}$. □

To get started, it follows directly from the Equation (5.8) of C that we may write the lattice C as a countable, disjoint union of translates of H:

$$C := \left\{x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = kN, \text{ where } k = 1, 2, 3, \ldots\right\}. \tag{5.15}$$

To be concrete, let's work out some examples.

Example 5.7. Using Lemma 5.2, we can easily compute the determinant of the A_d lattice from Example 5.6:

$$\det A_d = \sqrt{1 + 1 + \cdots + 1} = \sqrt{d}.$$

Example 5.8. As in Figure 5.5, consider the set of all integer points $(m, n) \in \mathbb{R}^2$ that satisfy

$$2m + 3n \equiv 0 \mod 4.$$

In this case the related hyperplane is the line $2x + 3y = 0$, and the solutions to the latter congruence may be thought of as a union of discrete lines:

$$C = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \mid 2x + 3y = 4k, \text{ and } k \in \mathbb{Z} \right\}.$$

In other words, our lattice C, a special case of (5.8), can in this case be visualized

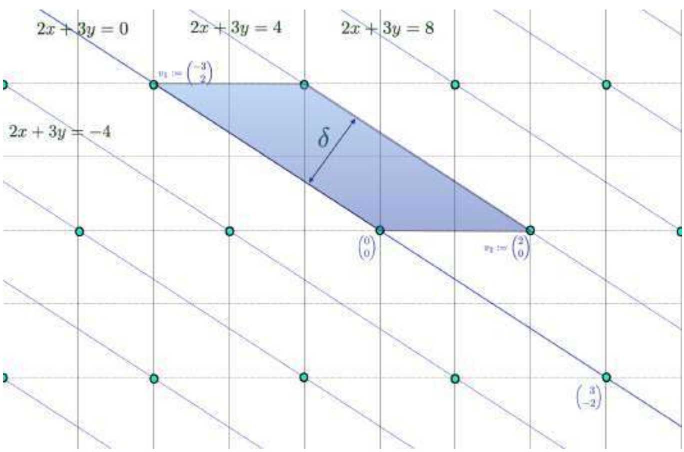


Figure 5.5: The lattice of Example 5.8

in Figure 5.5 as a disjoint union of discrete lines. If we denote the distance between any two of these adjacent discrete lines by δ, then using (5.10) we have:

$$\delta = \frac{4}{\sqrt{3^2 + 2^2}}. \tag{5.16}$$

Finally, the determinant of our lattice C here is the area of the shaded parallelepiped:

$$\det C = \delta\sqrt{3^2 + 2^2} = 4. \tag{5.17}$$

□

Eager to prove the volume relation $\det C = N$, we can use the ideas of Example 5.8 as a springboard for this generalization. Indeed, Example 5.8 and the proof of Lemma 5.2 both suggest that we should compute the volume of a fundamental parallelepiped $\mathcal{P}$, for the lattice C (as opposed to the lattice $\mathbb{Z}^d$), by using a fundamental domain for its base, and then by multiplying its volume by the height of $\mathcal{P}$.

Lemma 5.3. *Given a constant integer vector* $(c_1, \ldots, c_d) \in \mathbb{Z}^d$*, with* $\gcd(c_1, \ldots, c_d) = 1$*, let*

$$C := \left\{ x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d \equiv 0 \mod N \right\}, \tag{5.18}$$

where N *is a fixed positive integer. Then* C *is a lattice, and it has the determinant:*

$$\det C = N.$$

Proof. We fix a basis $\{v_1, \ldots, v_{d-1}\}$ for the $(d-1)$-dimensional sublattice defined by $H := \left\{ x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = 0 \right\}$, and we adjoin to this basis one new vector, namely any integer vector w that translates H to its nearest discrete hyperplane companion

$$H + w := \left\{ x \in \mathbb{Z}^d \mid c_1 x_1 + \cdots + c_d x_d = N \right\}.$$

Together, the set of vectors $\{v_1, \ldots, v_{d-1}, w\}$ form the edge vectors of a fundamental parallelepiped $\mathcal{P}$ for the lattice C, whose hight δ is the distance between these two parallel hyperplanes H and $H + w$. Using (5.10), we can may calculate the volume of $\mathcal{P}$ (which is by definition equal to $\det C$) by multiplying the volume of its 'base' times its 'height':

$$\begin{aligned} \det C &= (\text{volume of the base of } \mathcal{P})(\text{height of } \mathcal{P}) = (\det H)\delta && (5.19)\\ &= (\det H)\frac{N}{\sqrt{c_1^2 + \cdots + c_d^2}} = N, && (5.20) \end{aligned}$$

using the fact that $\det H = \sqrt{c_1^2 + \cdots + c_d^2}$ from Lemma 5.2. $\square$

5.5 The Gram matrix

There is another very natural matrix that we may use to study lattices, which we can motivate as follows. Suppose we are given any basis for a lattice $\mathcal{L} \subset \mathbb{R}^d$, say $\beta := \{v_1, \ldots, v_r\}$, where $1 \leqslant r \leqslant d$. By definition $\mathcal{L} = M(\mathbb{Z}^d)$, and $\operatorname{rank}(\mathcal{L}) = r$, where the columns of M are defined by the basis vectors from β, and so M is a $d \times r$ matrix. We can therefore represent any $x \in \mathbb{R}^d$ uniquely in terms of the basis β like this:

$$x = c_1 v_1 + \cdots + c_r v_r, \tag{5.21}$$

and the squared length of x is:

$$\|x\|^2 = \left\langle \sum_{j=1}^{r} c_j v_j, \sum_{k=1}^{r} c_k v_k \right\rangle = \sum_{1 \leqslant j,k \leqslant r} c_j c_k \langle v_j, v_k \rangle := c^T M^T M c, \tag{5.22}$$

where $c := (c_1, \ldots, c_r)^T$ is the coefficient vector defined by (5.21).

It's therefore very natural to focus on the matrix $M^T M$, whose entries are the inner products $\langle v_j, v_k \rangle$ of all the basis vectors of the lattice $\mathcal{L}$, so we define

$$G := M^T M,$$

called a **Gram matrix** for $\mathcal{L}$. It's clear from the computation above in (5.22) that G is positive definite. Although G does depend on which basis of $\mathcal{L}$ we choose, it is an elementary fact that $\det G$ is independent of the basis of $\mathcal{L}$.

Because we are always feeling the urge to learn more Linear Algebra, we would like to see why any real symmetric matrix B is the Gram matrix of some set of vectors. To see this, we apply the Spectral Theorem: $B = PDP^T$, for some orthogonal matrix P and a diagonal matrix D with nonnegative diagonal elements. So we can write $B = (P\sqrt{D})(P\sqrt{D})^T := M^T M$, where we defined the matrix $M := (P\sqrt{D})^T$, so that the columns of M are the vectors whose corresponding dot products form the symmetric matrix B, and now B is a Gram matrix.

To review some more linear algebra, suppose we are given a real symmetric matrix A. We recall that such a matrix is called **positive definite** if in addition we have the positivity condition

$$x^T A x > 0,$$

for all $x \in \mathbb{R}^d$. Equivalently, all of the eigenvalues of A are positive. The reason is easy: $Ax = \lambda x$ for a non-zero vector $x \in \mathbb{R}^d$ implies that

$$x^T A x := \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \|x\|^2,$$

so that $x^T Ax > 0$ if and only if $\lambda > 0$. In the sequel, if we only require a symmetric matrix A that enjoys the property $x^T Ax \geqslant 0$ for all $x \in \mathbb{R}^d$, then we call such a matrix **positive semidefinite**.

Also, for a full-rank lattice, we see that $B := M^T M$ will be positive definite if and only if M is invertible, so that the columns of M are a basis. Since a positive definite matrix is symmetric by definition, we've proved:

Lemma 5.4. *Suppose we are given a real symmetric matrix B. Then:*

(a) B is positive definite if and only if it is the Gram matrix of a basis.

(b) B is positive semidefinite if and only if it is the Gram matrix of some set of vectors.

□

What about reconstructing a lattice, knowing only one of its Gram matrices? This is almost possible to accomplish, up to an orthogonal transformation, as follows.

Lemma 5.5. *Suppose that G is an invertible matrix, whose spectral decomposition is $G = PDP^T$. Then*

$$G = X^T X, \tag{5.23}$$

if and only if

$$X = Q(\sqrt{D}P^T),$$

where Q is an orthogonal matrix.

Proof. The assumption $G = X^T X$ guarantees that G is symmetric and has positive eigenvalues, so by the Spectral Theorem we have:

$$G = PDP^T,$$

where D is a diagonal matrix consisting of the positive eigenvalues of G, and P is an orthogonal matrix consisting of eigenvectors of G. Setting $X^T X = PDP^T$, we must have

$$I = X^{-T} PDP^T X^{-1} = (X^{-T} P\sqrt{D})(X^{-T} P\sqrt{D})^T, \tag{5.24}$$

where we define $\sqrt{D}$ to be the diagonal matrix whose diagonal elements are the positive square roots of the eigenvalues of G. From 5.24, it follows that $X^{-T} P\sqrt{D}$ is an orthogonal matrix, let's call it Q^{-T}. Finally, $X^{-T} P\sqrt{D} = Q^{-T}$ implies that $X = Q\sqrt{D}P^T$. □

So Lemma 5.5 allows us to reconstruct a lattice $\mathcal{L}$, up to an orthogonal transformation, by only knowing one of its Gram matrices. To better understand lattices, we need the **unimodular group**, which we write as $\mathrm{SL}_d(\mathbb{Z})$, under matrix multiplication:

$$\mathrm{SL}_d(\mathbb{Z}) := \{M \mid M \text{ is a } d \times d \text{ integer matrix, with } |\det M| = 1\}. \tag{5.25}$$

The elements of $\mathrm{SL_d}(\mathbb{Z})$ are called **unimodular matrices**.

Example 5.9. Some typical elements of $\mathrm{SL}_2(\mathbb{Z})$ are

$$S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), T := \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \text{ and } \; -I := \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right),$$

so we include matrices with determinant equal to -1 as well as 1. □

Any lattice $\mathcal{L}$ has infinitely many fundamental parallelepipeds and (Exercise 5.14) it is a nice fact that they are all images of one another by the unimodular group. Now, suppose a lattice $\mathcal{L}$ is defined by two different basis matrices: $\mathcal{L} = M_1(\mathbb{Z}^d)$ and $\mathcal{L} = M_2(\mathbb{Z}^d)$. Is there a nice relationship between M_1 and M_2?

Lemma 5.6. *If a full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$ is defined by two different basis matrices M_1, and M_2, then*

$$M_1 = M_2 U,$$

where $U \in \mathrm{SL_d}(\mathbb{Z})$, a unimodular matrix.

In particular, $\det \mathcal{L}$ *is independent of the choice of basis matrix M.*

Proof. By hypothesis, we know that the columns of M_1, say $v_1, \dots, v_d$, form a basis of $\mathcal{L}$, and that the columns of M_2, say $w_1, \dots, w_d$, also form a basis of $\mathcal{L}$. So we can begin by writing each fixed basis vector v_j in terms of all the basis vectors w_k:

$$v_j = \sum_{k=1}^{d} c_{j,k} w_k,$$

for each $j = 1, \dots, d$, and for some $c_{j,k} \in \mathbb{Z}$. We may collect all d of these identities into matrix form:

$$M_1 = M_2 C,$$

where C is the integer matrix whose entries are defined by the integer coefficients $c_{j,k}$ above. Conversely, we may also write each basis vector w_j in terms of the basis vectors v_k: $w_j = \sum_{k=1}^{d} d_{j,k} v_k$, for some $d_{j,k} \in \mathbb{Z}$, getting another matrix identity:

$$M_2 = M_1 D.$$

Altogether we have

$$M_1 = M_2 C = (M_1 D)C,$$

and since M_1^{-1} exists by assumption, we get $DC = I$, the identity matrix. Taking determinants, we see that

$$|\det D||\det C| = 1,$$

and since both C and D are integer matrices, they must belong to $\mathrm{SL_d}(\mathbb{Z})$, by definition. Finally, because, because a unimodular matrix U has $|\det U| = 1$, we see that any two basis M_1, M_2 matrices satisfy $|\det M_1| = |\det M_2|$. □

Using similar techniques, it is not hard to show the following fact (Exercise 5.13).

Theorem 5.2. *Fix a full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$. The group of one-to-one, onto, linear transformations from $\mathcal{L}$ to itself is equal to the unimodular group $SL_d(\mathbb{Z})$.*

Such linear transformations that occur in Theorem 5.2 are also known as **linear automorphisms** of the lattice.

5.6 Dual lattices

Every lattice $\mathcal{L} := M(\mathbb{Z}^d)$ has a **dual lattice**, which we have already encountered in the Poisson summation formula for arbitrary lattices. The dual lattice of $\mathcal{L}$ was defined by:

$$\mathcal{L}^* = M^{-T}(\mathbb{Z}^d). \tag{5.26}$$

But there is another way to define the dual lattice, which is coordinate-free:

$$\mathcal{L}^* := \left\{ x \in \mathbb{R}^d \mid \langle x, n \rangle \in \mathbb{Z}, \text{ for all } n \in \mathcal{L} \right\}. \tag{5.27}$$

Lemma 5.7. *The two definitions above, (5.26) and (5.27), are equivalent.*

Proof. We let $A := \mathcal{L}^* := M^{-T}(\mathbb{Z}^d)$, and $B := \{x \in \mathbb{R}^d \mid \langle x, n\rangle \in \mathbb{Z}, \text{ for all } n \in \mathcal{L}\}$. We first fix any $x \in A$. To show $x \in B$, we fix any $n \in \mathcal{L}$, and we now have to verify that $\langle x, n\rangle \in \mathbb{Z}$. By assumption, $x = M^{-T}m$ for some $m \in \mathbb{Z}^d$, and $n = Mk$, for some $k \in \mathbb{Z}^d$. Therefore

$$\langle x, n\rangle = \langle M^{-T}m, n\rangle = \langle m, M^{-1}n\rangle = \langle m, k\rangle \in \mathbb{Z},$$

because both $m, k \in \mathbb{Z}^d$. So we have $A \subset B$. For the other direction, suppose that $y \in B$, so by definition

$$\langle y, n\rangle \in \mathbb{Z}, \text{ for all } n \in \mathcal{L}. \tag{5.28}$$

We need to show that $y = M^{-T}k$ for some $k \in \mathbb{Z}^d$, which is equivalent to $M^T y \in \mathbb{Z}^d$. Noticing that the k'th element of $M^T y$ is $\langle n, y\rangle$ with n belonging to a basis of $\mathcal{L}$, we are done, by (5.28). Therefore $A = B$. □

Example 5.10. Let $\mathcal{L} := r\mathbb{Z}^d$, the integer lattice dilated by a positive real number r. It's dual lattice is $\mathcal{L}^* = \frac{1}{r}\mathcal{L}$, because a basis for $\mathcal{L}$ is $M := rI$, implying that a basis matrix for $\mathcal{L}^*$ is $M^{-T} = \frac{1}{r}I$. We also notice that $\det \mathcal{L} = r^d$, while $\det \mathcal{L}^* = \frac{1}{r^d}$. □

A fundamental relation between a full-rank lattice and its dual follows immediately from definition 5.26: $\det(\mathcal{L}^*) := \det(M^{-T}) = \frac{1}{\det M} = \frac{1}{\det \mathcal{L}}$, which we record as:

$$(\det \mathcal{L})(\det \mathcal{L}^*) = 1. \tag{5.29}$$

If we consider any integer sublattice of $\mathbb{Z}^d$, say $\mathcal{L} \subset \mathbb{Z}^d$, together with its dual lattice $\mathcal{L}^*$ in the same space, some interesting relations unfold between them. Let's consider an example.

Example 5.11. In $\mathbb{R}^2$, let $\mathcal{L} := \{m\binom{1}{1} + n\binom{1}{4} \mid m, n \in \mathbb{Z}\}$, a lattice with $\det \mathcal{L} = 3$ that is depicted in Figure 5.6 by the larger green balls. Its dual lattice is

$$\mathcal{L}^* := \left\{\frac{1}{3}\left(a\binom{4}{-1} + b\binom{-1}{1}\right) \mid a, b \in \mathbb{Z}\right\},$$

whose determinant equals $\frac{1}{3}$, and is depicted in Figure 5.6 by the smaller orange balls. So $\mathcal{L}$ is a coarser lattice than $\mathcal{L}^*$.

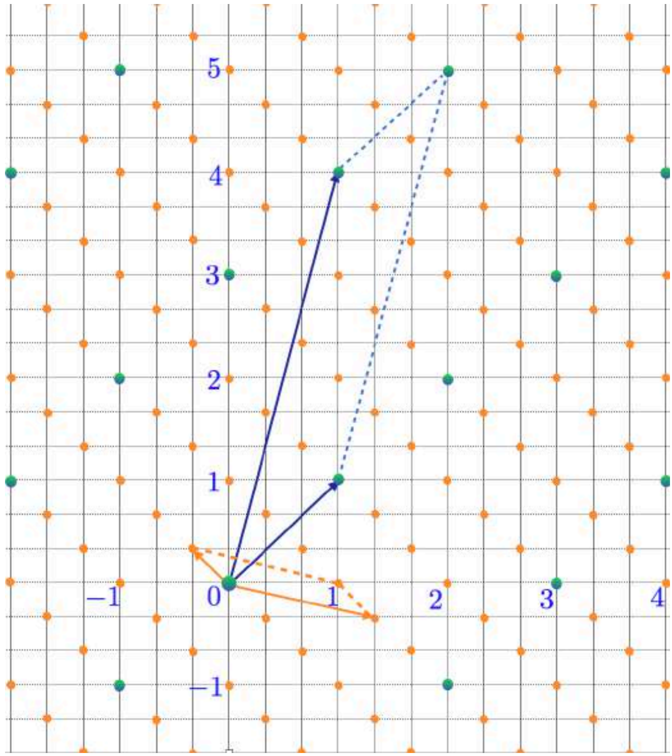


Figure 5.6: The lattice of Example 5.11

We can verify that the relation (5.29) works for this example: $\det \mathcal{L}^* = \frac{1}{3} = \frac{1}{\det \mathcal{L}}$. We also notice that $\mathcal{L}$ is a sublattice of $\mathcal{L}^*$. We may notice here that $\mathcal{L}^*/\mathcal{L}$ forms a finite group of order $9 = (\det \mathcal{L})^2$, which is equal to the number of cosets of the coarser lattice $\mathcal{L}$ in the finer lattice $\mathcal{L}^*$.

□

The dual lattice also appears as the kernel of a certain map, as follows. Suppose that for each point $n \in \mathcal{L}$, we define a function called a **character** of $\mathcal{L}$:

$$\chi_n(y) := e^{2\pi i \langle n, y \rangle}, \tag{5.30}$$

whose domain is the whole space $\mathbb{R}^d$. To see a connection with the dual lattice $\mathcal{L}^*$, we may consider the simultaneous kernel of all of these functions taken together:

$$\text{Ker} := \{x \in \mathbb{R}^d \mid \chi_n(x) = 1, \text{ for all } n \in \mathcal{L}\}.$$

It's clear that $\text{Ker} = \mathcal{L}^*$, because $e^{2\pi i z} = 1$ if and only if $z \in \mathbb{Z}$.

Now let's consider the collection of all of these characters:

$$G_{\mathcal{L}} := \{\chi_n \mid n \in \mathcal{L}\}. \tag{5.31}$$

If we multiply these character together by defining $\chi_n \chi_m := \chi_{n+m}$, then $G_{\mathcal{L}}$ forms a group, called the **group of characters** of $\mathcal{L}$. To see that this multiplication makes sense, we can compute:

$$(\chi_n \chi_m)(x) = e^{2\pi i \langle n,x\rangle} e^{2\pi i \langle m,x\rangle} = e^{2\pi i \langle n+m,x\rangle} = \chi_{n+m}(x).$$

Even more is true: $G_{\mathcal{L}}$ is isomorphic, as a group, to the lattice $\mathcal{L}$ (Exercise 5.11) via the map $n \to \chi_n$. Intuitively, one of the great benefits of the group of characters is that by using the magic of just two-dimensional complex numbers, we can study high-dimensional lattices.

Example 5.12. *For the integer lattice $\mathbb{Z}^d$, its group of characters is composed of the following functions, by definition:*

$$\chi_n(x) := e^{2\pi i \langle n,x\rangle},$$

for each $n \in \mathbb{Z}^d$. □

5.7 The successive minima of a lattice, and Hermite's constant

To warm up, we recall a very classical inequality of Hadamard, giving a bound on determinants. Intuitively, Hadamard's inequality tells us that if we keep all the lengths of the sides of a parallelepiped constant, and consider all possible parallelepipeds $\mathcal{P}$ with these fixed side lengths, then the volume of $\mathcal{P}$ is maximized exactly when $\mathcal{P}$ is rectangular.

Theorem 5.3 (Hadamard's inequality)**.** *Given a non-singular matrix M, over the reals, whose column vectors are $v_1, \ldots, v_d$, we have:*

$$|\det M| \leqslant \|v_1\| \|v_2\| \cdots \|v_d\|,$$

with equality if and only if all of the v_k's are pairwise orthogonal.

Proof. We use the following matrix decomposition from Linear Algebra: $M = QR$, where Q is an orthogonal matrix, $R := [r_{i,j}]$ is an upper-triangular matrix, and $r_{kk} > 0$ (this decomposition is a well-known consequence of the Gram-Schmidt process applied to the columns of M). So now we know that $|\det Q| = 1$, and $\det R = \prod_{k=1}^{d} r_{kk}$, and it follows that

$$|\det M| = |\det Q \det R| = \det R = \prod_{k=1}^{d} r_{kk}.$$

Let's label the columns of Q by Q_k, and the columns of R by R_k. We now consider the matrix $M^T M = R^T Q^T Q R = R^T R$. Comparing the diagonal elements on both sides of $M^T M = R^T R$, we see that $\|v_k\|^2 = \|R_K\|^2$. But we also have $\|R_K\|^2 \geqslant r_{kk}^2$, so that $\|v_k\| \geqslant r_{kk}$. Altogether we have

$$|\det M| = \prod_{k=1}^{d} r_{kk} \leqslant \prod_{k=1}^{d} \|v_k\|. \tag{5.32}$$

The case of equality occurs if and only if $\|R_K\|^2 = r_{kk}^2$ for all $1 \leqslant k \leqslant d$, which means that R is a diagonal matrix. Thus, we have equality in inequality 5.32 if and only if $M^T M = R^T R$ is a diagonal matrix, which means that the columns of M are mutually orthogonal. □

A very important characteristic of a lattice $\mathcal{L}$ is the **length of its shortest nonzero vector**:

$$\lambda_1(\mathcal{L}) := \min \left\{ \|v\| \;\middle|\; v \in \mathcal{L} - \{0\} \right\}.$$

Every lattice has at least two shortest nonzero vectors, because if $v \in \mathcal{L}$, then $-v \in \mathcal{L}$. Thus, when we use the words 'its shortest vector', we always mean that we are free to make a choice between any of its vectors that have the same shortest, nonzero length. A natural question, which has many applications, is "how short is the shortest nonzero vector in $\mathcal{L}$, as we somehow vary over all normalized lattices $\mathcal{L}$?"

Example 5.13. We define the following lattice in $\mathbb{R}^2$:

$$\mathcal{L} := \left\{ m \begin{pmatrix} 102 \\ 11 \end{pmatrix} + n \begin{pmatrix} 200 \\ 16 \end{pmatrix} \;\middle|\; m, n \in \mathbb{Z} \right\}.$$

What is the shortest nonzero vector in this lattice $\mathcal{L}$? Without using any fancy Theorems, we might still try simple subtraction, sort of mimicking the Euclidean algorithm. So for example, we might try $\binom{200}{16} - 2\binom{102}{11} = \binom{-4}{-6}$, which is pretty short. So we seem to have gotten lucky - we found a relatively short vector. But here comes the impending question:

Question 7. *How do we know whether or not this is really the shortest nonzero vector in our lattice $\mathcal{L}$? Can we find an even shorter vector in $\mathcal{L}$?*

This is not easy to answer in general, and we need to learn a bit more theory even to approach it in $\mathbb{R}^2$. Moreover, in dimensions $d \geqslant 3$, the corresponding problem of finding a shortest nonzero vector in any given lattice is terribly difficult - it is considered to be one of the most difficult problems in computational number theory. □

To capture the notion of the second-smallest vector in a lattice, and third-smallest vector, etc, we begin by imagining balls of increasing radii, centered at the origin, and we can (at least theoretically) keep track of how they intersect $\mathcal{L}$.

Let B_r be the ball of radius r, centered at the origin. For each fixed j, with $1 \leqslant j \leqslant d$, let r be the smallest positive real number such that B_r contains at least j linearly independent lattice points of $\mathcal{L}$. This value of r is called $\lambda_j(\mathcal{L})$, the **j'th successive minima** of the lattice.

Another way of saying the same thing is:

$$\lambda_j(\mathcal{L}) := \inf\left\{r > 0 \mid \dim\left(\text{span}\left(\mathcal{L} \cap B_r\right)\right) \geqslant j\right\}. \tag{5.33}$$

By definition, we have $|\lambda_1(\mathcal{L})| \leqslant |\lambda_2(\mathcal{L})| \leqslant \cdots \leqslant |\lambda_d(\mathcal{L})|$.

Example 5.14. For $\mathcal{L} := \mathbb{Z}^d$, the shortest nonzero vector has length $\lambda_1(\mathbb{Z}^d) = 1$, and the successive minima for $\mathbb{Z}^d$ all have the same value. One choice for their corresponding vectors is $v_1 := \mathbf{e_1}, \ldots, v_d := \mathbf{e_d}$, the standard basis vectors. □

Example 5.15. In $\mathbb{R}^2$, there is a very special lattice, sometimes called the **hexagonal lattice**, or the **Eisenstein lattice**:

$$\mathcal{L} := \left\{ m\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} + n\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid m, n \in \mathbb{Z} \right\}.$$

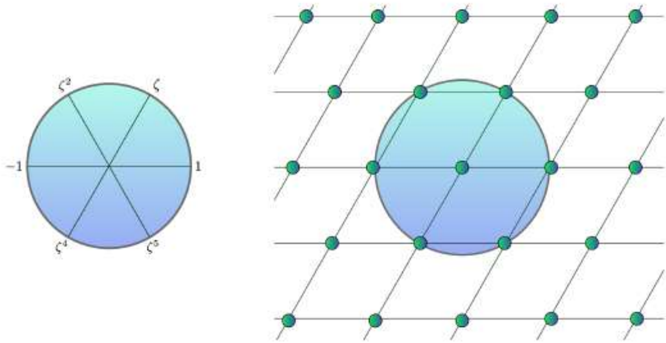


Figure 5.7: The Eisenstein lattice, also known as the hexagonal lattice

This lattice has $\det \mathcal{L} = \frac{\sqrt{3}}{2}$ and is generated by the 6'th roots of unity (Exercise 5.6). Given the basis above, we see that here we have $\lambda_1(\mathcal{L}) = \lambda_2(\mathcal{L}) = 1$. It also turns out to be an extremal lattice in the sense that it (more precisely a dilate of it) is the lattice that achieves Hermite's constant γ_2, below, over all lattices in $\mathbb{R}^2$. (Exercise 5.7). □

Example 5.16. Let's define the following family of 2-dimensional lattices. For each $t > 0$, we let

$$M := \begin{pmatrix} e^t & 0 \\ & e^{-t} \end{pmatrix}, \text{ and we let } \mathcal{L}_t := M(\mathbb{Z}^d),$$

so that we get a parametrized family of lattices. While all of the lattices in this family have $\det \mathcal{L} = 1$, their shortest nonzero vectors approach 0 as $t \to \infty$, since $\lambda_1(\mathcal{L}_t) = e^{-t}$. So we see that it does not necessarily make sense to talk about the shortest nonzero vector among a collection of lattices, but it will make sense to consider a "max-min problem" of this type (Hermite's constant (5.34) below). □

For each dimension d, we define **Hermite's constant** as follows:

$$\gamma_d := \max \left\{ \lambda_1(\mathcal{L})^2 \mid \mathcal{L} \text{ is a full-rank lattice in } \mathbb{R}^d, \text{ with } \det \mathcal{L} = 1 \right\}. \tag{5.34}$$

In words, Hermite's constant is retrieved by varying over all normalized lattices in $\mathbb{R}^d$, which have determinant 1, picking out the smallest squared norm of any nonzero vector in each lattice, and then taking the maximum of these smallest norms. In a later chapter, on sphere packings, we will see an interesting interpretation of Hermite's constant in terms of the densest lattice packing of spheres.

We next give a simple bound, in Theorem 5.4 below, for the shortest nonzero vector in a lattice and hence for Hermite's constant. But first we need to give a simple lower bound for the volume of the unit ball, in Lemma 5.8. Curiously, Hermite's constant γ_d is only known precisely for $1 \leqslant d \leqslant 8$, and $d = 24$, as of this writing.

Lemma 5.8.

$$\mathrm{vol}(B_r) \geqslant \left(\frac{2r}{\sqrt{d}}\right)^d.$$

Proof. The cube $C := \{x \in \mathbb{R}^d \mid \text{ all } |x_k| \leqslant \frac{r}{\sqrt{d}}\}$ is contained in the ball B_r: if $x \in C$ then $\sum_{k=1}^d x_k^2 \leqslant d\left(\frac{r}{\sqrt{d}}\right)^2 = r^2$. So the volume of the ball B_r is greater than the volume of the cube, which is equal to $\left(\frac{2r}{\sqrt{d}}\right)^d$. □

The following result of Minkowski give a bound for the shortest nonzero vector in a lattice.

Theorem 5.4 (Minkowski)**.** *Suppose that $\mathcal{L} \subset \mathbb{R}^d$ is a full-rank lattice. Then the shortest nonzero vector $v \in \mathcal{L}$ satisfies*

$$\|v\| \leqslant \sqrt{d}\,(\det \mathcal{L})^{\frac{1}{d}}. \tag{5.35}$$

In other words, $\lambda_1(\mathcal{L}) \leqslant \sqrt{d}\,(\det \mathcal{L})^{\frac{1}{d}}$.

Proof. The idea is to apply Minkowski's convex body Theorem 4.2 to a ball of sufficiently large radius. Let $r := \lambda_1(\mathcal{L})$ be the length of the shortest nonzero vector in $\mathcal{L}$, and consider the open ball B_r of radius r. By definition, B_r does not contain any lattice points of $\mathcal{L}$. So by Minkowski's convex body Theorem, and Lemma 5.8,

$$\mathrm{vol}(B_r) \leqslant \left(\frac{2\lambda_1(\mathcal{L})}{\sqrt{d}}\right)^d \leqslant 2^d \det \mathcal{L}.$$

It follows that $\lambda_1(\mathcal{L}) \leqslant \sqrt{d}\,(\det \mathcal{L})^{\frac{1}{d}}$, proving the claim.

□

Despite the latter bound (5.35) on the shortest nonzero vector in a lattice, there are currently no known efficient algorithms to find such a vector, and it is thought to be one of the most difficult problems we face today. In practice, researchers often use the LLL algorithm to find a 'relatively short' vector in a given lattice, and the same algorithm even finds a relatively short basis for $\mathcal{L}$.

While we may not know explicitly all of the short vectors in a given lattice, it is often still useful to construct an ellipsoid that is based on the successive minima of a lattice, as we do below. In the spirit of reviewing basic concepts from Linear Algebra, an **ellipsoid** centered at the origin is defined by

$$\{x \in \mathbb{R}^d \mid \sum_{j=1}^{d} \frac{\langle x, b_j \rangle^2}{c_j^2} = 1\}, \tag{5.36}$$

for some fixed orthonormal basis $\{b_1, \ldots, b_d\}$ of $\mathbb{R}^d$. Here the vectors b_j are called the **principal axes** of the ellipsoid, and the c_j's are the lengths along the principal axes of the ellipsoid. A more geometric way of defining an ellipsoid (which turns out to be equivalent to our definition above) is by applying a linear transformation M to the unit sphere S^{d-1} in $\mathbb{R}^d$ (Exercise 5.20).

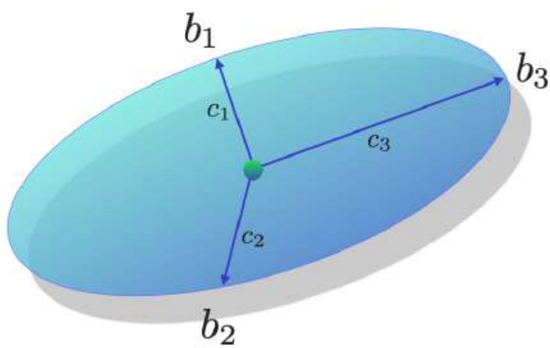


Figure 5.8: An ellipsoid in $\mathbb{R}^3$.

Lemma 5.9. *Corresponding to the successive minima of a full-rank lattice $\mathcal{L}$, we have d linearly independent vectors $v_1, \ldots, v_d$, so that by definition $\|v_k\| :=$*

$\lambda_k(\mathcal{L})$. We apply the Gram-Schmidt algorithm to this set of vectors $\{v_1, \ldots, v_d\}$, obtaining a corresponding orthonormal basis $\{b_1, \ldots, b_d\}$ for $\mathbb{R}^d$.

Now, we define the following open ellipsoid body (a d-dimensional body in $\mathbb{R}^d$):

$$E := \{x \in \mathbb{R}^d \mid \sum_{k=1}^{d} \frac{\langle x, b_k\rangle^2}{{\lambda_k}^2} < 1\}, \tag{5.37}$$

whose axes are the b_k's, and whose radii are the $\lambda_k := \lambda_k(\mathcal{L})$. We claim that E does not contain any lattice points of $\mathcal{L}$.

Proof. We fix any vector $v \in \mathcal{L}$. Let $1 \leqslant k \leqslant d$ be the maximal index such that $\lambda_k(\mathcal{L}) \leqslant \|v\|$. We may write $v = \sum_{j=1}^d \langle v, b_j\rangle b_j$, so that $\|v\|^2 = \sum_{j=1}^d \langle v, b_j\rangle^2$.

Now v must lie in $\operatorname{span}\{v_1, \ldots v_k\} = \operatorname{span}\{b_1, \ldots b_k\}$, for some $1 \leqslant k \leqslant d$. Hence we may write $v = \sum_{j=1}^d \langle v, b_j\rangle b_j = \sum_{j=1}^k \langle v, b_j\rangle b_j$, so that $\|v\|^2 = \sum_{j=1}^k |\langle v, b_j\rangle|^2$. We now check if v is contained in E:

$$\sum_{j=1}^{d} \frac{\langle v, b_j\rangle^2}{{\lambda_j}^2} = \sum_{j=1}^{k} \frac{\langle v, b_j\rangle^2}{{\lambda_j}^2} \geqslant \frac{1}{{\lambda_k}^2} \sum_{j=1}^{k} \langle v, b_j\rangle^2 = \frac{\|v\|^2}{{\lambda_k}^2} \geqslant 1,$$

so that $v \notin E$. □

More generally, we have the following refinement of Theorem 5.4, which gives us a bound on the first d shortest (nonzero) vectors in a lattice.

Theorem 5.5 (Minkowski)**.** *The successive minima of a full-rank lattice $\mathcal{L}$ enjoy the property:*

$$\Big(\|\lambda_1(\mathcal{L})\| \cdots \|\lambda_d(\mathcal{L})\|\Big)^{\frac{1}{d}} \leqslant \sqrt{d}\Big(\det \mathcal{L}\Big)^{\frac{1}{d}}.$$

Proof. Using Lemma 5.9, the ellipsoid E contains no lattice points belonging to $\mathcal{L}$, so that by Minkowski's convex body Theorem, we have $\operatorname{vol} E \leqslant 2^d \det \mathcal{L}$. We also know that

$$\operatorname{vol} E = \Big(\prod_{j=1}^{d} \lambda_j\Big) \operatorname{vol} B_1 \geqslant \left(\prod_{j=1}^{d} \lambda_j\right) \left(\frac{2}{\sqrt{d}}\right)^d.$$

Altogether, we have

$$2^d \det \mathcal{L} \geqslant \operatorname{vol} E \geqslant \left(\prod_{j=1}^{d} \lambda_j\right) \left(\frac{2}{\sqrt{d}}\right)^d,$$

arriving at the desired inequality. $\square$

We notice that $(\|\lambda_1(\mathcal{L})\| \cdots \|\lambda_d(\mathcal{L})\|)^{\frac{1}{d}} \geqslant \|\lambda_1(\mathcal{L})\|$, because $\|\lambda_1(\mathcal{L})\| \leqslant \|\lambda_k(\mathcal{L})\|$ for all indices $1 < k \leqslant d$. We therefore see that Theorem 5.5 is indeed a refinement of Theorem 5.4.

Example 5.17. The E_8 lattice is defined by

$$E_8 := \{(x_1, x_2, \cdots x_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 \mid \sum_{k=1}^{8} x_k \equiv 0 \mod 2\}. \tag{5.38}$$

It turns out that the E_8 lattice gives the optimal solution to the sphere packing problem, as well as the optimal solution for the kissing number problem in $\mathbb{R}^8$. $\square$

5.8 Hermite normal form

We call a lattice $\mathcal{L}$ an **integral lattice** if $\mathcal{L} \subset \mathbb{Z}^d$. Further, we may recall that any lattice $\mathcal{L} \subset \mathbb{R}^d$ has infinitely many bases, so it may seem impossible at first to associate a single matrix with a given lattice. However, there is an elegant way to do this, as follows.

Example 5.18. Suppose we are given a lattice $\mathcal{L}$ as the integral span of the vectors

$$v_1 := \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right), v_2 := \left(\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right),$$

which clearly has determinant 8. Then any integer linear combinations of v_1 and v_2 is still in $\mathcal{L}$. In particular, mimicking Gaussian elimination, we place v_1 and v_2 as rows of a matrix, and row-reduce over the integers:

$$\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & -8 \\ 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix},$$

where at each step we performed row operations (over $\mathbb{Z}$) that did not change the lattice. Hence we have a reduced basis for $\mathcal{L}$, consisting of $\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 8 \end{smallmatrix}\right)$.

We notice that the resulting matrix is upper-triangular, with positive integers on the diagonal, nonnegative integers elsewhere, and in each column the diagonal element is the largest element in that column.

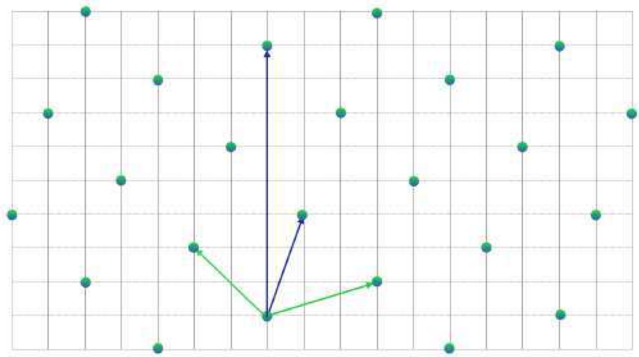


Figure 5.9: The lattice $\mathcal{L}$ of Example 5.18, depicted by the bold green points, and showing the original basis $\{v_1, v_2\}$ of $\mathcal{L}$, and the Hermite-reduced basis of $\mathcal{L}$

There is another way to interpret the matrix reductions above, by using unimodular matrices, as follows. The first reduction step can be accomplished by the multiplication on the left by a unimodular matrix:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Similarly, each step in the reduction process can be interpreted by multiplying on the left by some new unimodular matrix, so that at the end of the process we have a product of unimodular matrices times our original matrix $\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}$. Because a product of unimodular matrices is yet another unimodular matrix, we can see that we arrived at a reduction of the form:

$$U \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix},$$

where U is a unimodular matrix. □

The point of Example 5.18 is that a similar matrix reduction persists for all integer lattices, culminating in the following result, which just hinges on the fact that $\mathbb{Z}$ has a division algorithm.

Theorem 5.6. *Given an invertible integer $d \times d$ matrix M, there exists a unimodular matrix U with $UM = H$, such that H satisfies the following conditions:*

1. $[H]_{i,j} = 0$ *if* $i > j$.
2. $[H]_{i,i} > 0$, *for each* $1 \leqslant i \leqslant d$.
3. $0 \leqslant [H]_{i,j} < [H]_{i,i}$, *for each* $i > j$.

Property 3 tells us that each diagonal element $[H]_{i,i}$ *in the i'th column of* H *is the largest element in the i'th column.*

Moreover, the matrix H *is the only integer matrix that satisfies the above conditions.* □

The matrix H in Theorem 5.6 is called the **Hermite normal form** of M. To associate a unique matrix to a given integral full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$, we first choose any basis of $\mathcal{L}$, and we then construct a $d \times d$ integer matrix M whose rows are the basis vectors that we chose. We then apply Theorem 5.6 to M, arriving at an integer matrix H whose rows are another basis of $\mathcal{L}$, called the **Hermite-reduced basis**.

Corollary 5.1. *There is a one-to-one correspondence between full-rank integral lattices in* $\mathbb{R}^d$ *and integer* $d \times d$ *Matrices in their Hermite Normal Form.* □

5.9 The Voronoi cell of a lattice

The **Voronoi cell** of a lattice $\mathcal{L}$ is defined by

$$\mathcal{V}(\mathcal{L}) := \left\{ x \in \mathbb{R}^d \mid \|x\| \leqslant \|x - v\|, \text{ for all } v \in \mathcal{L} \right\}. \tag{5.39}$$

In other words, the Voronoi cell of a lattice $\mathcal{L}$ is the set of all point in space that are closer to the origin than to any other lattice point in $\mathcal{L}$. Because the origin wins the battle of minimizing this particular distance function, it is also possible to construct the Voronoi cell by using half-spaces. Namely, for each $v \in \mathcal{L}$, we define the half-space

$$H_v := \{x \in \mathbb{R}^d \mid \|x\| \leqslant \|x - v\|\},$$

and observe that the Voronoi cell is also given by

$$\mathcal{V}(\mathcal{L}) = \bigcap_{v \in \mathcal{L} - \{0\}} H_v.$$

Example 5.19. The D_n lattice is defined by

$$D_n := \{x \in \mathbb{Z}^n \mid \sum_{k=1}^{n} x_k \equiv 0 \mod 2\},$$

and is often called the "checkerboard" lattice. In particular, in $\mathbb{R}^4$, the D_4 lattice turns out to be a fascinating object of study.

The Voronoi cell of D_4 is called the 24-**cell**, as depicted in Figure 5.10, a 4-dimensional polytope with some wonderful properties. It is one of the few polytopes that is self-dual. It is also an example of a polytope $\mathcal{P}$ in the lowest possible dimension d (namely $d = 4$) such that $\mathcal{P}$ tiles $\mathbb{R}^d$ by translations, and yet $\mathcal{P}$ is not a zonotope.

By (5.20) above, we see that $\det D_4 = 2$. □

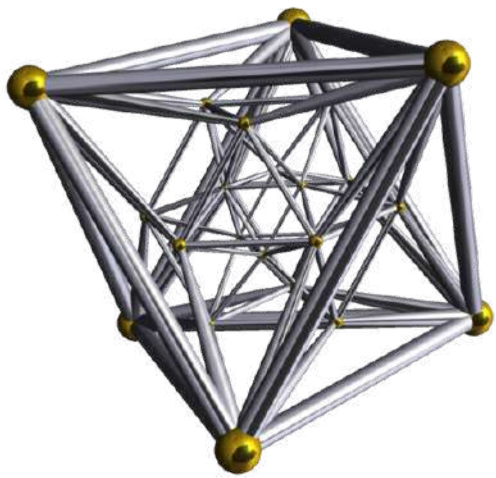


Figure 5.10: The Voronoi cell of the D_4 lattice in $\mathbb{R}^4$, known as the 24-cell.

A fascinating open problem is the Voronoi conjecture, named after the Ukrainian mathematician George Voronoi, who formulated it in 1908:

Conjecture 1 (Voronoi)**.** *A polytope $\mathcal{P}$ tiles $\mathbb{R}^d$ by translations if and only if $\mathcal{P}$ is the Voronoi cell of some lattice $\mathcal{L}$, or $\mathcal{P}$ is affinely equivalent to to such a Voronoi cell.*

Example 5.20. For the lattice $A_n \subset \mathbb{R}^{n+1}$ defined in Example 5.6, its Voronoi cell turns out to have beautiful and important properties: $A_2 \subset \mathbb{R}^3$ is a hexagon, $A_3 \subset \mathbb{R}^4$ is a truncated octahedron (one of the Fedorov solids), and so on (Conway and Sloane Conway and Sloane 1999).

5.10 Quadratic forms and lattices

The study of lattices is in a strong sense equivalent to the study of positive definite quadratic forms, over integer point inputs, for the following simple reason. Any positive definite quadratic form $f : \mathbb{R}^d \to \mathbb{R}$ is defined by $f(x) := x^T Ax$, where A is a positive definite matrix, so the image of the integer lattice under f is $\{x^T Ax \mid x \in \mathbb{Z}^d\}$. On the other hand, any full-rank lattice in $\mathbb{R}^d$ is given by $\mathcal{L} := M(\mathbb{Z}^d)$, for some real (non-singular) matrix M. By definition, this implies that the square of the norm of any vector in $\mathcal{L}$ has the following shape: $\|v\|^2 = v^T v = x^T M^T Mx$, for some $x \in \mathbb{Z}^d$. We notice that $M^T M$ in the last identity is positive definite.

We may summarize this discussion as follows. Given any lattice $\mathcal{L} := M(\mathbb{Z}^d)$, we have

$$\left\{\|v\|^2 \mid v \in \mathcal{L}\} = \{x^T Ax \mid x \in \mathbb{Z}^d\right\},$$

where $A := M^T M$ is positive definite.

So the distribution of the (squared) norms of all vectors in a given lattice is equivalent to the image of $\mathbb{Z}^d$ under a positive definite quadratic form.

Interestingly, despite this equivalence, for an arbitrary given lattice $\mathcal{L}$ it is not known in general whether the knowledge of the norms of all vectors in $\mathcal{L}$ uniquely determines the lattice $\mathcal{L}$. In very small dimensions it is true, but for dimensions $\geqslant 4$ there are some counterexamples due to A. Schiemann, and John Conway.

The above equivalence between lattices in $\mathbb{R}^d$ and quadratic forms is straightforward but often useful, because it allows both algebraic and analytic methods to come to bear on important problems involving lattices.

Notes

(a) Kurt Mahler was one of the main contributors to the development of the Geometry of Numbers. We mention here one of his more advanced results, involving limits of lattices, called Mahler's compactness theorem (also known as Mahler's selection theorem). So far we worked with one lattice at a time, but it turns out to be fruitful to work with infinite sets of lattices.

Theorem 5.7 (Mahler)**.** *Fix $\rho > 0, C > 0$. Then any infinite sequence of lattices $\mathcal{L} \subset \mathbb{R}^d$ such that*

$$\min\{\|x\| \in \mathcal{L} - \{0\}\} \geqslant \rho, \text{ and } \det \mathcal{L} \leqslant C,$$

has an infinite convergent subsequence of lattices.

In other words, Mahler realized that among all lattices of volume 1, if a sequence of lattices diverges, then it must be true that the lengths of the shortest nonzero vectors of these lattice tend to zero. To complete the story, we should define what it means for a sequence of lattices $\{\mathcal{L}_n\}_{n=1}^{\infty}$ to converge to a fixed lattice L. One way to define this convergence is to say that there exists a sequence of bases β_n of the lattices $\mathcal{L}_n$ that converge to a basis β of L, in the sense that the j'th basis vector of β_n converges to the j'th basis vector of β.

(b) There is a well-known meme in Mathematics: "Can one hear the shape of a drum?", which is the title of Mark Kac's famous paper regarding the desire to discern the shape of a drum from its 'frequencies'. An analogous question for lattices, studied by John Conway, is "which properties of quadratic forms are determined by their representation numbers?". For further reading, there is the lovely little book by Conway called "The sensual quadratic form", which draws connections between quadratic forms and many different fields of Mathematics Conway 1997.

Of course, no library is complete without the important and biblical "Sphere Packings, Lattices and Groups", by John H. Conway and Neil Sloane Conway and Sloane 1999.

(c) The idea of periodicity, as embodied by any lattice in $\mathbb{R}^d$, also occurs on other manifolds, besides Euclidean space. If we consider a closed geodesic on a manifold, then it's intuitively clear that as we flow along that geodesic, we have a periodic orbit along that geodesic. One important family of manifolds where this type of periodicity occurs naturally is the family of Hyperbolic manifolds. Following the philosophy that 'if we have periodicity, then we have Fourier-like series', we discover that there is also an hyperbolic analogue of the Poisson summation formula, known as the Selberg trace formula, and this type of number theory has proved extremely fruitful.

(d) A strong bound for Hermite's constant in dimension d was given by Blichfeldt Blichfeldt 1929:

$$\gamma_d \leqslant \left(\frac{2}{\pi}\right) \Gamma\left(2 + \frac{d}{2}\right)^{\frac{2}{d}}.$$

(e) The family of diagonal matrices in Example 5.16 is very important in the study of homogeneous dynamics, because it acts by multiplication on the left, on the space of all lattices that have $\det \mathcal{L} = 1$. This fascinating action is sometimes called the "modular flow", and was studied intensively by Etienne Ghys. A beautiful result in this direction is that the periodic orbits of the modular flow are in bijection with the conjugacy classes of hyperbolic elements in the modular group $SL_2(\mathbb{Z})$, and furthermore that these periodic orbits produce incredible knots in the complement of the trefoil knot.

(f) Gauss initiated the systematic study of finding the minimum value of positive definite, binary quadratic forms $f(x, y) := ax^2 + 2bxy + y^2$, over all integer inputs $(x, y) \in \mathbb{Z}^2$. Gauss' theory is also known as a reduction theory for positive definite binary quadratic forms, and is now a popular topic that can be found in many standard Number Theory books. By the discussion above, in Section 5.10, it is clear that minimizing positive definite quadratic forms is closely related to finding a vector of smallest length in a lattice, and to finding a "fat" fundamental parallelepiped for the lattice.

(g) It is clear that because lattices offer a very natural way to discretize $\mathbb{R}^d$, they continue to be of paramount importance to modern research. In particular, the theory of modular forms, with linear (Hecke) operators that are defined using lattices, is crucial for modern number theory. Euclidean lattices are also the bread-and-butter of crystallographers.

Exercises

5.1. ♣ *Given a lattice $\mathcal{L} \subset \mathbb{R}^d$, and any fundamental parallelepiped D of $\mathcal{L}$, show that any $x \in \mathbb{R}^d$ may be written uniquely as $x = p + n$, where $n \in \mathcal{L}$, and $p \in D$.*

Notes. This exercise gives a rigorous version of the statement "A fundamental parallelepiped of a lattice $\mathcal{L}$ tiles the space by translations with $\mathcal{L}$".

5.2. ♣

1. *Prove that* $(\mathbb{Z}^d)^* = \mathbb{Z}^d$.
2. *More generally, prove that for any lattice* $\mathcal{L} \subset \mathbb{R}^d$, *we have* $(\mathcal{L}^*)^* = \mathcal{L}$.

5.3. ♣ *Show that the distance* δ *between any two parallel hyperplanes* $c_1x_1 + \cdots + c_dx_d = k_1$ *and* $c_1x_1 + \cdots + c_dx_d = k_2$ *is given by*

$$\delta = \frac{|k_1 - k_2|}{\sqrt{c_1^2 + \cdots + c_d^2}}.$$

5.4. *Given an integer point* $n \in \mathbb{Z}^d$, *we call the set of all integer multiples of* n *a* **lattice line** *(also known as a rank-*1 *sublattice). Suppose we are given a full-rank, rational lattice* $\mathcal{L} \subset \mathbb{R}^d$ *(so that it has a rational basis matrix). Suppose, in addition, that we are also given a fixed lattice line* l_1.
Prove or disprove: the lattice $\mathcal{L}$ *and the lattice line* l_1 *always intersect in another lattice line:*

$$\mathcal{L} \cap l_1 = l_2,$$

where l_2 *is another lattice line in* $\mathbb{Z}^d$.

5.5. ♣ *Let* $\mathcal{L}$ *be a lattice in* $\mathbb{R}^1$. *Show that* $\mathcal{L} = r\mathbb{Z}$ *for some real number* r.

5.6. *The* **hexagonal lattice** *is the* 2*-dimensional lattice defined by*

$$\mathcal{L} := \{m + n\omega \mid m, n \in \mathbb{Z}\}, \text{ where } \omega := e^{2\pi i/3}.$$

Prove that $\det \mathcal{L} = \frac{\sqrt{3}}{2}$, *and give a description of the dual lattice to the hexagonal lattice.*

5.7 (hard)**.** *Show that the hexagonal lattice attains the minimal value for Hermite's constant in* $\mathbb{R}^2$, *namely* $\gamma_2^2 = \frac{2}{\sqrt{3}}$.

5.8. *Let $\mathcal{L} \subset \mathbb{R}^2$ be any rank 2 lattice. Show that there exists a basis $\beta := \{v, w\}$ of $\mathcal{L}$ such that the angle θ_β between v and w satisfies*

$$\frac{\pi}{3} \leqslant \theta_\beta \leqslant \frac{\pi}{2}.$$

5.9. *Suppose that M is a $d \times d$ matrix, all of whose d^2 elements are bounded by B. Show that $|\det M| \leqslant B^d d^{\frac{d}{2}}$.*

(Hint: consider Hadamard's inequality 5.3)

Notes. It follows from this exercise that if all of the elements of M are ± 1, then $|\det M| \leqslant d^{\frac{d}{2}}$. Such matrices are important in combinatorics and are called Hadamard matrices. It is known that if $d > 2$, then Hadamard matrices only exists when $4 \mid d$. But for each $d = 4m$, it is not known whether a $d \times d$ Hadamard matrix exists, except for very small cases.

5.10. ♣ *Show that the following set of vectors is a basis for A_d:*

$$\{e_2 - e_1,\ e_3 - e_1, \cdots,\ e_d - e_1\},$$

where the e_j are the standard basis vectors. Hence A_d is a rank-$(d-1)$ sublattice of $\mathbb{Z}^d$, by definition.

5.11. ♣ *Recall that $G_{\mathcal{L}}$ is the group of characters of the lattice $\mathcal{L}$, under the usual multiplication of complex numbers, and that the lattice $\mathcal{L}$ is a group under the usual operation of vector addition. Show that they are isomoprhic as groups: $G_{\mathcal{L}} \simeq \mathcal{L}$.*

5.12. ♣ *Here we prove the* **orthogonality relations for characters of a lattice***, for sublattices of $\mathbb{Z}^d$. Fix a sublattice $\mathcal{L}$ of $\mathbb{Z}^d$, and let D be a fundamental parallelepiped for $\mathcal{L}$. Using the notation in Exercise 5.11, prove that for any two characters $\chi_a, \chi_b \in G_{\mathcal{L}}$, we have:*

$$\frac{1}{\det \mathcal{L}} \sum_{n \in D \cap \mathbb{Z}^d} \chi_a(n)\overline{\chi_b(n)} = \begin{cases} 1 & \text{if } \chi_a = \chi_b \\ 0 & \text{if not.} \end{cases} \tag{5.40}$$

5.13. ♣ *Prove Theorem 5.2.*

5.14. ♣ *Prove that any two fundamental parallelepipeds (as defined in the text) of $\mathcal{L}$, say D_1 and D_2, must be related to each other by an element of the unimodular group:*

$$D_1 = M(D_2),$$

for some $M \in SL_d(\mathbb{Z})$.

5.15. ♣ *Given a sublattice $\mathcal{L} \subset \mathbb{R}^d$ of rank r, show that our definition of its determinant, namely $\det \mathcal{L} := \sqrt{M^T M}$, conincides with the Lebesgue measure of any of its fundamental parallelepipeds.*

(Here M is a $d \times r$ matrix whose columns are basis vectors of $\mathcal{L}$)

5.16. *Show that a set of vectors $v_1, \ldots, v_m \in \mathbb{R}^d$, where $1 \leqslant m \leqslant d$, are linearly independent $\iff$ their Gram matrix is nonsingular.*

5.17. *Prove that for any given lattice $\mathcal{L} \subset \mathbb{R}^2$, any set of shortest nonzero vectors for $\mathcal{L}$ generate the lattice $\mathcal{L}$. (As a reminder, the first two shortest nonzero vectors may be equal in length)*

5.18. *Find a lattice $\mathcal{L} \subset \mathbb{R}^5$ such that any set of five shortest nonzero vectors of $\mathcal{L}$ do not generate the lattice $\mathcal{L}$.*

5.19. ♣ *Consider the* **discrete hyperplane** *defined by:*

$$H := \left\{x \in \mathbb{Z}^d \;\middle|\; c_1 x_1 + \cdots + c_d x_d = 0\right\},$$

Show that H is a lattice (also known as a rank-$(d-1)$ sublattice of $\mathbb{Z}^d$).

5.20. *Here we give the details for* (5.36)*, the definition of an ellipsoid in* $\mathbb{R}^d$*. Starting over again, we fix an orthonormal basis* $\{b_1, \dots, b_d\}$ *for* $\mathbb{R}^d$*, and we define the following matrix:*

$$M := \begin{pmatrix} | & | & \cdots & | \\ c_1 b_1 & c_2 b_2 & \cdots & c_d b_d \\ | & | & \cdots & | \end{pmatrix},$$

where the c_k*'s are positive scalars. We now apply the linear transformation* M *to the unit sphere* $S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\|^2 = 1\}$ *in* $\mathbb{R}^d$*, and we recall what this means. Now we define the*

$$\mathit{Ellipsoid}_M := M(S^{d-1}),$$

a $(d-1)$*-dimensional object. In the spirit of review, we recall the definition* $M(S^{d-1}) := \{u \in \mathbb{R}^d \mid u = Mx, x \in S^{d-1}\}$.

(a) *Show that*

$$\mathit{Ellipsoid}_M = \{x \in \mathbb{R}^d \mid \sum_{j=1}^{d} \frac{\langle x, b_j \rangle^2}{c_j^2} = 1\}. \tag{5.41}$$

(b) *We recall that the unit ball in* $\mathbb{R}^d$ *is defined by*

$$B^d := \{x \in \mathbb{R}^d \mid \|x\|^2 \leqslant 1\}.$$

Show that for the open ellipsoid body E *(a* d*-dimensional object), as defined in* (5.37)*, we have the* d*-dimensional volume formula:*

$$\mathrm{vol}(E) = \mathrm{vol}(B^d) \prod_{j=1}^{d} c_j.$$

5.21. *We will use the equation* (5.41) *definition of an ellipsoid, from above. We can extend the previous exercise in the following way. Let* A *be* **any** $d \times d$ *real matrix, and look at the action of* A *on the unit sphere* $S^{d-1} \subset \mathbb{R}^d$*. Suppose that* $rank(A) = r$*. Show:*

(a) *If* $r = d$*, then* $A(S^{d-1})$ *is a* d*-dimensional ellipsoid, defined by an equation of the form* (5.41)*.*

(b) *If* $r < d$*, then* $A(S^{d-1})$ *is an* r*-dimensional ellipsoid.*

5.22. *Suppose that A is a positive definite, real matrix. Solve for (i.e. characterize) all matrices X that are the 'square roots' of A:*

$$A = X^2.$$

5.23. *Suppose that a certain 2-dimensional lattice $\mathcal{L}$ has a Gram matrix*

$$G := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Reconstruct $\mathcal{L}$ (i.e. find a basis for $\mathcal{L}$), up to an orthogonal transformation.

5.24. *Find a 2 by 2 matrix M that enjoys one of the properties of a positive semidefinite matrix, namely that $x^T M x \geqslant 0$, for all $x \in \mathbb{R}^2$, but such that M is not symmetric.*

5.25. *Show that any real 2 by 2 matrix A is positive definite if and only if both* trace(A) > 0 *and* $\det A > 0$.

5.26. *(hard) Erdős' question, given in Exercise 1.14, possesses a natural extension to dimension d. Suppose that the integer lattice $\mathbb{Z}^d$ is partitioned into a disjoint union of a finite number of integer sublattices, say:*

$$\mathbb{Z}^d = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_N.$$

Is it true that there are at least two integer sublattices, say $\mathcal{L}_j, \mathcal{L}_k$, that enjoy the property that $\mathcal{L}_k = \mathcal{L}_j + v$, for some integer vector v?

Prove that in $\mathbb{R}^3$, we can find a partition of $\mathbb{Z}^3$ into 4 integer sublattices, such that no two of them are integer translates of one another. Using an easy extension to $d > 3$, also show that the answer to the question above is 'no', if $d \geqslant 3$.

Notes. This problem remains unsolved in dimension $d = 2$ Feldman, Propp, and Robins 2011.

6 The Fourier transform of a polytope: vertex description

See in **nature** the cylinder, the sphere, the cone.

– Paul Cézanne

6.1 Intuition

Here we introduce the basic tools for computing precise expressions for the Fourier transform of a polytope. To compute transforms here, we assume that we are given the vertices of a polytope $\mathcal{P}$, together with the local geometric information at each vertex of $\mathcal{P}$, namely its neighboring vertices in $\mathcal{P} \subset \mathbb{R}^d$. It turns out that computing the Fourier–Laplace transform of the tangent cone at each vertex of $\mathcal{P}$ completely characterizes the Fourier transform of $\mathcal{P}$.

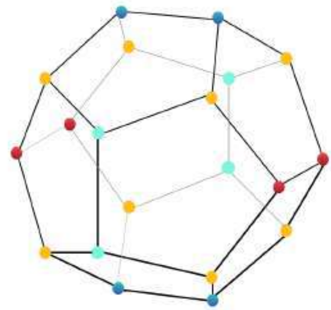


Figure 6.1: The Dodecahedron in $\mathbb{R}^3$, an example of a simple polytope. In Exercise 6.6, we compute its Fourier–Laplace transform by using Theorem 6.1 below.

One of the basic results here, called the discrete version of Brion's Theorem (6.5), may be viewed as an extension of the finite geometric sum, from dimension 1 to dimension d. Some basic families of polytopes are introduced, including simple polytopes and their duals, which are simplicial polytopes. These families of polytopes play an important role in the development of Fourier analysis on polytopes.

6.2 Tangent cones, and an amazing formula of Brion

We begin by defining a **simple polytope**, which is a d-dimensional polytope $\mathcal{P}$ such that each of its vertices has exactly d edges emanating from it. In other words, the associated graph of vertices and edges of $\mathcal{P}$, also called the **edge graph** of $\mathcal{P}$, is a d-regular graph.

Example 6.1. The 3-dimensional dodecahedron, in Figure 6.1, is a simple polytope. Its edge graph, which is always a planar graph for a convex polytope, consists of 20 vertices, 30 edges, and 12 faces.

Example 6.2. The d-dimensional cube $[0, 1]^d$ is a simple polytope. Its dual polytope, which is the cross-polytope $\Diamond$ (see (2.8)), is not a simple polytope for $d \geqslant 3$.

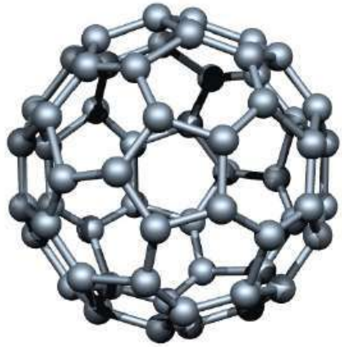


Figure 6.2: The C60 Carbon molecule, also known as a buckyball, is another example of a simple polytope. The nickname "buckyball' came from Buckminster Fuller, who used this molecule as a model for many other tensegrity structures. (the graphic is used with permission from Nanografi, at https://phys.org/news/2015-07-scientists-advance-tunable-carbon-capture-materials.html)

Example 6.3. Any d-dimensional simplex Δ is a simple polytope. In fact, any k-dimensional face of the simplex Δ is also a simplex, and hence a simple polytope of lower dimension.

One of our most important concepts here is the definition of a **cone** $\mathcal{K}$ in $\mathbb{R}^d$, which is the set of all non-negative real linear combinations of a finite set of fixed vectors $\{w_1, \ldots, w_N\} \subset \mathbb{R}^d$:

$$\mathcal{K} := \left\{ \sum_{k=1}^{N} \lambda_k w_k \mid \lambda_k \geqslant 0 \right\}, \tag{6.1}$$

which is a cone whose edge vectors are among the $w_1, \ldots, w_N$. If the vectors $w_1, \ldots, w_N$ span a k-dimensional subspace of $\mathbb{R}^d$, we say that the cone $\mathcal{K}$ has dimension k.

A **pointed cone** $\mathcal{K}$ is a cone that enjoys the property that it lies entirely in some half-space $\{x \in \mathbb{R}^d \mid \langle x, \mathbf{n} \rangle > 0\}$, where $\mathbf{n}$ is a normal vector to the hyperplane

$\langle x, \mathbf{n}\rangle = 0$. A pointed cone has an apex, which we will generally denote by v, and we will denote such a cone by $\mathcal{K}_v$. When a k-dimensional cone $\mathcal{K} \subset \mathbb{R}^d$ has exactly k linearly independent edge vectors $w_1, \dots w_k \in \mathbb{R}^d$, we call such a cone a **simplicial cone**.

An n-dimensional polytope $\mathcal{P} \subset \mathbb{R}^d$ is called **simplicial** if each facet of $\mathcal{P}$ is a simplex. Equivalently:

(a) Each facet of $\mathcal{P}$ is a simplex.

(b) Each facet of $\mathcal{P}$ has exactly n vertices.

(c) Each k-dimensional face of $\mathcal{P}$ has exactly $k+1$ vertices, for $0 \leqslant k \leqslant n-1$.

It is a fun exercise to show that any simplicial cone is always a pointed cone (Exercise 6.10). By contrast with the notion of a simplicial polytope, we have the following 'dual' family of polytopes.

An n-dimensional polytope $\mathcal{P} \subset \mathbb{R}^d$ is called **simple** if each tangent cone of $\mathcal{P}$ is a simplicial cone. Equivalently:

(a) Each vertex of $\mathcal{P}$ is contained in exactly n of its edges.

(b) Each vertex of $\mathcal{P}$ is contained in exactly n of its facets.

(c) Each k-dimensional face of $\mathcal{P}$ is contained in exactly $d-k$ facets, for all $k \geqslant 0$.

Example 6.4. The tetrahedron, octahedron, and icosahedron are all simplicial polytopes, while the tetrahedron, cube, and dodecahedron are all simple polytopes.

It is a nice exercise to show that the only polytopes which are both simple and simplicial are either simplices, or 2-dimensional polygons (Exercise 6.12).

One might ask: are the facets of a simple polytope necessarily simplicial polytopes? An example helps here. The 120-cell is a 4-dimensional polytope whose 3-dimensional boundary is composed of 120 dodecahedra (Schleimer and Segerman (2015)). The 120-cell is a simple polytope, but because all of its facets are dodecahedra, it does not have any simplicial facets. □

As becomes apparent after comparing the notion of a simple polytope with that of a simplicial polytope, these two types of polytopes are indeed dual to each other, in the sense of duality that we've already encountered in definition (2.63) (see Grünbaum (2003b) for a thorough study of this duality). The duality above, between simple and simplicial polytopes, suggests a stronger connection between our geometric structures thus far, and the combinatorics inherent in the partially ordered set of faces of $\mathcal{P}$, for example counting the number of k-dimensional faces in $\mathcal{P}$, and in the dual polytope $\mathcal{P}^*$. Indeed, Grünbaum put it elegantly:

> "In my opinion, the most satisfying way to approach the definition of polyhedra is to distinguish between the combinatorial structure of a polyhedron, and the geometric realizations of this combinatorial structure." Grünbaum (2003a)

One of the important steps for us is to work with the Fourier transform of a cone, and then build some Theorems that allow us to simplify many geometric computations, by using the Frequency domain on the Fourier transform side. One of the most important results in this direction, which has many consequences, is Theorem 6.1 below, due to Brion.

We may define the **tangent cone** of each face $\mathcal{F} \subset \mathcal{P}$ as follows. First pick any point $\mathbf{v}$ in the relative interior of the face $\mathcal{F} \subset \mathcal{P}$. Then the tangent cone of $\mathcal{F}$ may be defined by:

$$\mathcal{K}_{\mathcal{F}} = \{\mathbf{v} + \mathbf{x} \mid \mathbf{v} \in \mathcal{F}, \text{and } \mathbf{v} + \epsilon\mathbf{x} \in \mathcal{P}, \text{for all sufficiently small } \epsilon > 0\}. \quad (6.2)$$

The tangent cone is also known as the **cone of feasible directions**. Intuitively, we can imagine standing at the point v, belonging to the relative interior of the face F, and looking in the direction of all points that belong to P. Now we take the union of all of these directions.

In the case that that face F is a vertex of $\mathcal{P}$, we call this tangent cone a **vertex tangent cone**. In this very important special case, the vertex tangent cone $\mathcal{K}_v$ at v may also be generated by the edge vectors $v_j - v$, where $[v, v_j]$ is an edge of $\mathcal{P}$:

$$\mathcal{K}_v = \left\{ \sum_{k=1}^{N} \lambda_k (v_k - v) \mid \begin{array}{c} \text{all } \lambda_k \geqslant 0, \text{ and} \\ v_k \text{ are neighboring vertices of } v \end{array} \right\} \quad (6.3)$$

a construction we will sometimes use in practice. The tangent cone of an edge of a 3-dimensional convex polytope is an infinite wedge containing the whole line passing through that edge, while the tangent cone of a vertex (for a convex polytope) never contains a whole line. Equation (6.2) above is nice, because it even makes sense for non-convex polytopes, whose vertices are often difficult to even define. One definition for the vertices of non-convex polytopes appears in Akopyan, Bárány, and Robins (2017), although there are other definitions that give a different set of vertices in this non-convex case.

Example 6.5. For the unit cube $\square := [0,1]^d$, the tangent cone at the vertex v which is the origin is

$$\mathcal{K}_v = \{\lambda_1 \mathbf{e_1} + \lambda_2 \mathbf{e_2} + \lambda_3 \mathbf{e_3} + \cdots + \lambda_d \mathbf{e_d} \mid \lambda_k \geqslant 0\},$$

which also happens to be the **positive orthant** $\mathbb{R}^d_{\geqslant 0}$. On the other hand, the tangent cone of $\square$ at the vertex $v = (1, 0, \ldots, 0)$ is:

$$\mathcal{K}_v = v + \{\lambda_1(-\mathbf{e_1}) + \lambda_2 \mathbf{e_2} + \lambda_3 \mathbf{e_3} + \cdots + \lambda_d \mathbf{e_d} \mid \lambda_k \geqslant 0\},$$

where $\mathbf{e_j}$ is the standard unit vector along the j'th axis. □

Example 6.6. To relate some of these definitions, consider a d-dimensional simplex $\Delta \subset \mathbb{R}^d$. Located at each of its vertices $v \in \Delta$, we have a tangent cone K_v, as in (6.3), and here K_v is a simplicial cone. The simplex Δ is both a simple polytope and a simplicial polytope. □

Brion proved the following extremely useful result, concerning the Fourier–Laplace transform of a *simple polytope* $\mathcal{P}$. To describe the result, we note that for each vertex v of $\mathcal{P}$, if we fix the d edges $w_1, \ldots, w_d$ that emanate from v (not necessarily unit vectors), and put them as columns of a matrix M_v, then we define

$$\det \mathcal{K}_v := |\det M_v|, \tag{6.4}$$

the absolute value of the determinant of the ensuing matrix. The determinant $\det \mathcal{K}_v$ clearly depends on our choice of edge vectors $w_1, \ldots, w_d$, but it is straightforward that the quotient $\frac{\det \mathcal{K}_v}{\prod_{k=1}^d \langle w_k(v), z\rangle}$ does not depend on the choice of edge vectors (Exercise 6.1).

The following basic and important result gives a formula for the Fourier transform of any d-dimensional simple polytope $\mathcal{P} \subset \mathbb{R}^d$.

Theorem 6.1 (Brion's theorem - the continuous form, 1988). *Let $\mathcal{P} \subset \mathbb{R}^d$ be a simple, d-dimensional polytope. For each vertex tangent cone $\mathcal{K}_v$ of $\mathcal{P}$, we fix a set of edge vectors of $\mathcal{K}_v$, say $w_1(v), w_2(v), \ldots, w_d(v)$. Then*

$$\int_{\mathcal{P}} e^{-2\pi i \langle u, z\rangle}\, du = \left(\frac{1}{2\pi i}\right)^d \sum_{v \text{ a vertex of } \mathcal{P}} \frac{e^{-2\pi i \langle v, z\rangle} \det \mathcal{K}_v}{\prod_{k=1}^d \langle w_k(v), z\rangle} \tag{6.5}$$

for all $z \in \mathbb{C}^d$ such that the denominators on the right-hand side do not vanish.
□

6.3 Fourier-Laplace transforms of cones

First, let's understand what it means to consider the complex vector $z \in \mathbb{C}^d$ in the transform above: $z := x + iy$, with $x, y \in \mathbb{R}^d$.

Our inner product $\langle u, z \rangle := u_1 z_1 + \cdots + u_d z_d$ is always the usual inner product on $\mathbb{R}^d$, defined without using the Hermitian inner product here. In other words, we simply use the usual inner product on $\mathbb{R}^d$, and then formally substitute complex numbers z_k into it. This means, by definition, that

$$\int_{\mathcal{P}} e^{-2\pi i \langle u,z \rangle}\, du = \int_{\mathcal{P}} e^{-2\pi i \langle u,x+iy \rangle} \tag{6.6}$$

$$= \int_{\mathcal{P}} e^{-2\pi i \langle u,x \rangle} e^{2\pi \langle u,y \rangle}\, du, \tag{6.7}$$

so that we have an extra useful real factor of $e^{2\pi \langle u,y \rangle}$ that makes the integral converge quite rapidly over unbounded domains, provided that $\langle u, y \rangle < 0$. If we set $y = 0$, then it's clear that we retrieve the usual Fourier transform of $\mathcal{P}$, while if we set $x = 0$, we get a new integral, which we call the **Laplace transform** of $\mathcal{P}$. For a generic $z \in \mathbb{C}^d$, the integral $\hat{1}_{\mathcal{P}}(z) := \int_{\mathcal{P}} e^{-2\pi i \langle u,z \rangle}\, du$ is called the **Fourier–Laplace transform** of $\mathcal{P}$.

One reason we need the flexibility of the full Fourier–Laplace transform is the fact that for a cone $\mathcal{K}$, its usual Fourier transform diverges. But if we allow a complex variable $z \in \mathbb{C}^d$, then the integral does converge on a restricted domain. Namely, the Fourier–Laplace transform of a cone $\mathcal{K}$ is defined by:

$$\hat{1}_{\mathcal{K}}(z) := \int_{\mathcal{K}} e^{-2\pi i \langle u,z \rangle}\, du,$$

for a certain set of $z \in \mathbb{C}^d$, but we can easily understand its precise domain of convergence. For an arbitrary cone $\mathcal{K} \subset \mathbb{R}^d$, we define its **dual cone** by:

$$\mathcal{K}^* := \{y \in \mathbb{R}^d \mid \langle y, u \rangle < 0 \text{ for all } u \in \mathcal{K}\},$$

which is an open cone. Let's start with dimension 1.

Example 6.7. Given the 1-dimensional cone $\mathcal{K}_0 := \mathbb{R}_{\geqslant 0}$, we compute its Fourier–

Laplace transform:

$$\int_{\mathcal{K}_0} e^{-2\pi i u z}\, du = \int_0^{\infty} e^{-2\pi i u z}\, du = = \frac{1}{-2\pi i z} e^{-2\pi i u(x+iy)}\Big|_{u=0}^{u=\infty}$$
$$= \frac{1}{-2\pi i z} e^{-2\pi i u x} e^{2\pi u y}\Big|_{u=0}^{u=\infty}$$
$$= \frac{1}{-2\pi i z}(0-1) = \frac{1}{2\pi i}\frac{1}{z},$$

valid for all $z := x + iy \in \mathbb{C}$ such that $y < 0$. We note that for such a fixed complex z, $|e^{-2\pi i u z}| = e^{2\pi u y}$ is a rapidly decreasing function of $u \in \mathbb{R}$. □

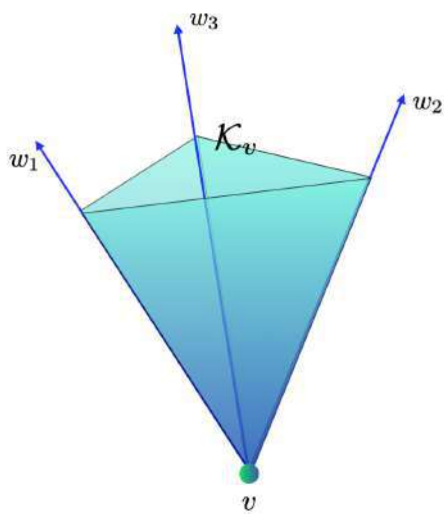


Figure 6.3: A simplicial, pointed cone in $\mathbb{R}^3$, with apex v and edge vectors w_1, w_2, w_3

Now let's work out the Fourier–Laplace transform of a d-dim'l cone whose apex is the origin.

Lemma 6.1. *Let $\mathcal{K} \subset \mathbb{R}^d$ be a simplicial, d-dimensional cone, with apex at the origin. If the edges of $\mathcal{K}$ are labelled $w_1, \ldots, w_d$, then*

$$\hat{1}_K(z) := \int_{\mathcal{K}} e^{-2\pi i \langle u, z\rangle}\, du = \frac{1}{(2\pi i)^d} \frac{\det \mathcal{K}}{\prod_{k=1}^d \langle w_k, z\rangle}.$$

In addition, the domain of convergence for the latter integral is $\{z := x + iy \in \mathbb{C}^d \mid y \in \mathcal{K}^*\}$.

Proof. We first compute the Fourier–Laplace transform of the positive orthant $\mathcal{K}_0 := \mathbb{R}^d_{\geqslant 0}$, with a complex vector $z = x + iy \in \mathbb{C}^d$:

$$\hat{1}_{\mathcal{K}_0}(z) := \int_{\mathcal{K}_0} e^{-2\pi i \langle z,u\rangle} du \tag{6.8}$$

$$= \int_{\mathbb{R}_{\geqslant 0}} e^{-2\pi i z_1 u_1} du_1 \cdots \int_{\mathbb{R}_{\geqslant 0}} e^{-2\pi i z_d u_d} du_d \tag{6.9}$$

$$= \prod_{k=1}^{d} \frac{0-1}{-2\pi i z_k} = \left(\frac{1}{2\pi i}\right)^d \frac{1}{z_1 z_2 \cdots z_d}. \tag{6.10}$$

Next, the positive orthant $\mathcal{K}_0$ may be mapped to the cone $\mathcal{K}$ by a linear transformation. Namely, we may use the matrix M whose columns are defined to be the edges of $\mathcal{K}$, so that by definition $\mathcal{K} = M(\mathcal{K}_0)$. Using this mapping, we have:

$$\begin{aligned}
\hat{1}_{\mathcal{K}}(z) &:= \int_{\mathcal{K}} e^{-2\pi i \langle z,u\rangle} du \\
&= |\det M| \int_{\mathcal{K}_0} e^{-2\pi i \langle z, Mt\rangle} dt \\
&= |\det M| \int_{\mathcal{K}_0} e^{-2\pi i \langle M^T z, t\rangle} dt \\
&= \left(\frac{1}{2\pi i}\right)^d \frac{|\det M|}{\prod_{k=1}^d \langle w_k, z\rangle}.
\end{aligned}$$

where in the second equality we've made the substitution $u = Mt$, with $t \in \mathcal{K}_0, u \in \mathcal{K}$, and $du = |\det M| dt$. In the final equality, we used equation (6.10) above, noting that the k'th element of the vector $M^T z$ is $\langle w_k, z\rangle$, and we note that by definition $|\det M| = \det \mathcal{K}$. □

Example 6.8. Given the 2-dimensional cone

$$\mathcal{K} := \{\lambda_1 \binom{1}{5} + \lambda_2 \binom{-3}{2} \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geqslant 0}\},$$

we compute its Fourier–Laplace transform, and find its domain of convergence. By Lemma 6.1,

$$\hat{1}_{\mathcal{K}}(z) := \int_{\mathcal{K}} e^{-2\pi i \langle u, z\rangle}\, du = \frac{1}{(2\pi i)^2} \frac{17}{(z_1 + 5z_2)(-3z_1 + 2z_2)},$$

valid for all $z = \binom{z_1}{z_2} := x + iy$ such that $y \in \mathcal{K}^*$. Here the dual cone is given here by
$\mathcal{K}^* = \{\lambda_1 \binom{5}{-1} + \lambda_1 \binom{-2}{-3} \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geqslant 0}\}$. □

To compute the Fourier–Laplace transform of a simplicial cone $\mathcal{K}$ whose apex is $v \in \mathbb{R}^d$, we may first compute the transform of the translated cone $\mathcal{K}_0 := \mathcal{K} - v$, whose apex is at the origin, using the previous lemma. We can then use the fact that the Fourier transform behaves in a simple way under translations, namely

$$\hat{1}_{\mathcal{K}+v}(z) = e^{-2\pi i \langle z, v\rangle}\, \hat{1}_{\mathcal{K}}(z),$$

to obtain the following result (Exercise 6.3). We recall that $\det \mathcal{K}_v$ was defined by (6.4).

Corollary 6.1. *Let $\mathcal{K}_v \subset \mathbb{R}^d$ be a* ***simplicial*** *d-dimensional cone, whose apex is $v \in \mathbb{R}^d$. Then*

$$\hat{1}_{\mathcal{K}_v}(z) := \int_{\mathcal{K}_v} e^{-2\pi i \langle u, z\rangle}\, du = \frac{1}{(2\pi i)^d} \frac{e^{-2\pi i \langle v, z\rangle} \det \mathcal{K}_v}{\prod_{k=1}^d \langle w_k, z\rangle}, \tag{6.11}$$

a rational-exponential function.

For a general cone $\mathcal{K}$, we can always triangulate it into simplicial cones [De Loera, Rambau, and Santos (2010)], and apply the previous Corollary to each simplicial piece, obtaining the following structural result.

Corollary 6.2. *Let $\mathcal{K}_v \subset \mathbb{R}^d$ be any d-dimensional cone, whose apex is $v \in \mathbb{R}^d$. Then*

$$\hat{1}_{\mathcal{K}_v}(z) := \int_{\mathcal{K}_v} e^{-2\pi i \langle u, z\rangle}\, du = \frac{e^{-2\pi i \langle v, z\rangle}}{(2\pi i)^d} \sum_{j=1}^{M} \frac{\det \mathcal{K}_j}{\prod_{k=1}^d \langle w_{j,k}, z\rangle}, \tag{6.12}$$

a rational-exponential function. Here $\mathcal{K}_{\boldsymbol{v}} = \cup_{j=1}^M \mathcal{K}_j$, a triangulation of $\mathcal{K}_{\boldsymbol{v}}$ into simplicial subcones.

There is an extension of Brion's Theorem 6.1 for the Fourier transform of a polytope, which allows us to drop the assumption that $\mathcal{P}$ is a simple polytope, and is due to Barvinok. This extension has the merit that it is structurally the same as Theorem 6.1, for all real polytopes. However, for a non-simple polytope, the question of computing efficiently the Fourier–Laplace transforms of all of its tangent cones becomes unwieldy, as far as we know (and this is related to the $P \neq NP$ problem).

The following fundamental result is the main Theorem in this chapter. When we write $\hat{1}_{\mathcal{K}_v}(z)$ below, by definition we mean its expression as a rational-exponential function, which is the meromorphic continuation of the initial integral definition of the same object.

Theorem 6.2 (Fourier–Laplace transform of any real polytope). *Let $\mathcal{P} \subset \mathbb{R}^d$ be any d-dimensional polytope, and let N be the number of its vertices. For each vertex v of $\mathcal{P}$, we consider the tangent cone $\mathcal{K}_v$ of $\mathcal{P}$, and fix a set of edges of $\mathcal{K}_v$, say $w_1(v), w_2(v), \ldots, w_d(v) \in \mathbb{R}^d$. Then*

$$\int_{\mathcal{P}} e^{-2\pi i \langle u, z\rangle}\, du = \hat{1}_{\mathcal{K}_{v_1}}(z) + \cdots + \hat{1}_{\mathcal{K}_{v_N}}(z),$$

for all $z \in \mathbb{C}^d$.

Proof. (See Section 6.5). □

Example 6.9. Let's work out a 2-dim'l example, using Fourier–Laplace transforms of tangent cones. We will find the rational-exponential function for the Fourier–Laplace transform of the triangle Δ, whose vertices are defined by $v_1 := \binom{0}{0}$, $v_2 := \binom{a}{0}$, and $v_3 := \binom{0}{b}$, with $a > 0, b > 0$.

First, the tangent cone at the vertex $v_1 := \binom{0}{0}$ is simply the nonnegative orthant in this case, with edge vectors $w_1 = \binom{1}{0}$ and $w_2 = \binom{0}{1}$. Its determinant, given these two edge vectors, is equal to 1. Its Fourier–Laplace transform is

$$\int_{\mathcal{K}_{v_1}} e^{-2\pi i \langle x, z\rangle}\, dx = \frac{1}{(2\pi i)^2}\frac{1}{z_1 z_2}, \tag{6.13}$$

and note that here we must have both $\Im(z_1) > 0$ and $\Im(z_2) > 0$ in order to make the integral converge. Here we use the standard notation $\Im(z)$ is the imaginary part of z.

The second tangent cone at vertex v_2 has edges $w_1 = \binom{-a}{b}$ and $w_2 = \binom{0}{-b}$ (recall that we don't have to normalize the edge vectors at all). Its determinant has absolute value equal to ab, and its Fourier–Laplace transform is

$$\int_{\mathcal{K}_{v_2}} e^{-2\pi i\langle x,z\rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{-2\pi i a z_1}}{(-az_1+bz_2)(-az_1)}, \tag{6.14}$$

and here the integral converges only for those z for which $\Im(-az_1+bz_2) > 0$ and $\Im(-az_1) > 0$.

Finally, the third tangent cone at vertex v_3 has edges $w_1 = \binom{a}{-b}$ and $w_2 = \binom{0}{-b}$. Its determinant has absolute value equal to ab, and its Fourier–Laplace transform is

$$\int_{\mathcal{K}_{v_3}} e^{-2\pi i\langle x,z\rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{-2\pi i b z_2}}{(az_1-bz_2)(-bz_2)}. \tag{6.15}$$

and here the integral converges only for those z for which $\Im(az_1-bz_2) > 0$ and $\Im(-bz_2) > 0$.

We can again see quite explicitly the disjoint domains of convergence in this example, so that there is not even one value of $z \in \mathbb{C}^2$ for which all three Fourier–Laplace transforms of all the tangent cones converge simultaneously. Despite this apparent shortcoming, Brion's identity (6.1) still tells us that we may somehow still add these local contributions of the integrals at the vertices combine to give us a formula for the Fourier–Laplace transform of the triangle:

$$\begin{aligned}\hat{1}_\Delta(z) &:= \int_\Delta e^{-2\pi i\langle x,z\rangle} dx \\ &= \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_1z_2} + \frac{-b\, e^{-2\pi i a z_1}}{(-az_1+bz_2)z_1} + \frac{-a\, e^{-2\pi i b z_2}}{(az_1-bz_2)z_2}\right),\end{aligned}$$

which is *now magically valid* for all generic $(z_1, z_2) \in \mathbb{C}^2$; in other words, it is now valid for all $(z_1, z_2) \in \mathbb{C}^2$ except those values which make the denominators vanish. □

Corollary 6.3. *The Fourier–Laplace transform of any real polytope $\mathcal{P} \subset \mathbb{R}^d$, with vertex set V, is a rational-exponential function of the form:*

$$\int_{\mathcal{P}} e^{-2\pi i\langle u,z\rangle}\, du = \sum_{v\in V} \frac{e^{-2\pi i\langle v,z\rangle}}{(2\pi i)^d} \sum_{j=1}^{M_v} \frac{\det \mathcal{K}_{v,j}}{\prod_{k=1}^d \langle w_{j,k}, z\rangle},$$

for all $z \in \mathbb{C}^d$, except for those values of z for which any of the denominators vanish.

Proof. We recall [see De Loera, Rambau, and Santos 2010] that we may triangulate any cone into simplicial cones. Using this algorithm, we can triangulate any tangent cone $\mathcal{K}_v$ into smaller simplicial cones $K_j(v)$, noting that the identity $1_{\mathcal{K}_v} = 1_{\mathcal{K}_1(v)} + \cdots + 1_{\mathcal{K}_{M(v)}}(v)$ implies the Fourier identity

$$\hat{1}_{\mathcal{K}_v} = \hat{1}_{\mathcal{K}_1(v)} + \cdots + \hat{1}_{\mathcal{K}_{M(v)}}(v).$$

Hence Theorem 6.2 gives us:

$$\begin{aligned}
\int_{\mathcal{P}} e^{-2\pi i \langle u, z\rangle}\, du &= \sum_{v\in V} \hat{1}_{\mathcal{K}_v}(z) \\
&= \sum_{v\in V} \sum_{j=1}^{M(v)} \hat{1}_{\mathcal{K}_j(v)} \\
&= \sum_{v\in V} \sum_{j=1}^{M(v)} \frac{e^{-2\pi i \langle v, z\rangle}}{(2\pi i)^d} \frac{\det \mathcal{K}_j(v)}{\prod_{k=1}^{d} \langle w_{j,k}(v), z\rangle},
\end{aligned}$$

where the last step follows from Corollary 6.1. □

6.4 The Brianchon–Gram identity

Brion's theorem is particularly useful because whenever we are given a polytope in terms of its local data at the vertices - including the tangent cone's edges at each vertex - we can easily write down the Fourier transform of a simple polytope, by Theorem 6.1. Consequently, we will be able to give a clean formulation for its volume as well, as we will soon see in Theorem 6.4.

The following combinatorial identity may be thought of as a geometric inclusion–exclusion principle, and is quite general, holding true for any convex polytope, simple or not. For proofs of the following result see M. Beck and Robins (2015), for example.

Theorem 6.3 (Brianchon–Gram identity). *Let $\mathcal{P}$ be any convex polytope. Then*

$$1_{\mathcal{P}} = \sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{dim \mathcal{F}} 1_{\mathcal{K}_F}, \tag{6.16}$$

where the sum takes place over all faces of $\mathcal{P}$, including $\mathcal{P}$ itself. □

It turns out that the Brianchon–Gram relations (6.16) can be shown to be equivalent to the **Euler–Poincaré relation** (Exercise 6.9) for the face-numbers of a convex polytope, which says that

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} + (-1)^d f_d = 1. \tag{6.17}$$

Here f_k is the number of faces of $\mathcal{P}$ of dimension k.

Example 6.10. If we let $\mathcal{P}$ be a 2-dim'l polygon (including its interior of course) with V vertices, then if must also have V edges, and exactly 1 face, so that (6.17) tells us that $V - V + 1 = 1$, which is not very enlightening, but true. □

Example 6.11. If we let $\mathcal{P}$ be a 3-dim'l polytope with V vertices, E edge, and F facets, then (6.17) tells us that $V - E + F - 1 = 1$, which retrieves Euler's well known formula

$$V - E + F = 2$$

for the Euler characteristic of 3-dimensional polytopes. □

6.5 Proof of Theorem 6.1

In this section we provide a proof of Brion's identity, namely Theorem 6.1, from first principles. This new proof uses some of the Fourier techniques that we've developed so far. Since this book offers a friendly approach, we first give a short outline of the straightforward ideas of the proof:

Step 1. We begin with the Brianchon-Gram identity (a standard first step) involving the indicator functions of all of the tangent cones of $\mathcal{P}$.

Step 2. We now multiply both sides of the Brianchon-Gram identity (6.16) with the function $e^{2\pi i \langle x, \xi \rangle - \epsilon \|x\|^2}$, where we fix an $\epsilon > 0$, and then we will integrate over all $x \in \mathbb{R}^d$. Using these integrals, due to the damped Gaussians for each fixed $\epsilon > 0$, we are able to keep the *same domain of convergence* for all of our ensuing functions.

Usually, the relevant Laplace integrals over the vertex tangent cones have disjoint domains of convergence, lending the feeling that something magical is going on with the disjoint domains of convergence. Getting around this problem was exactly the motivation for this proof.

Step 3. Now we let $\epsilon \to 0$ and prove that the limits of each integral gives us something meaningful. Using integration by parts, we prove that for any vertex tangent cone $\mathcal{K}$ the corresponding integral $\int_{\mathcal{K}} e^{2\pi i \langle x,\xi\rangle - \epsilon \|x\|^2} dx$ converges, as $\epsilon \to 0$, to the desired exponential-rational function. In an analogous but easier manner, we will also prove that the corresponding integral over a non-pointed cone (which includes all faces of positive dimension) converges to zero.

On to the rigorous details of the proof. We will make use of the following technical, but crucial Lemma. We favor a slightly longer but clearer expositional proof over a shorter, more obscure proof.

Lemma 6.2. *Let $\mathcal{K}_v$ be a d-dim'l simplicial pointed cone, with apex v, and edge vectors $w_1, \dots, w_d \in \mathbb{R}^d$. Then*

$$\lim_{\epsilon \to 0+} \int_{\mathcal{K}_v} e^{2\pi i \langle x,\xi\rangle - \epsilon \|x\|^2} dx = \left(\frac{-1}{2\pi i}\right)^d \frac{e^{2\pi i \langle v,z\rangle} |\det \mathcal{K}_v|}{\prod_{k=1}^d \langle w_k(v), \xi\rangle}, \tag{6.18}$$

for all $\xi \in \mathbb{C}^d$ such that $\prod_{k=1}^d \langle w_k(v), \xi\rangle \neq 0$.

Proof. We begin by noticing that we may prove the conclusion in the case that $v = 0$, the origin, and for simplicity write $\mathcal{K}_v := \mathcal{K}$ in this case. First we make a change of variables, mapping the simplicial cone $\mathcal{K}$ to the nonnegative orthant $\mathbb{R}^d_{\geqslant 0}$ by the matrix M^{-1}, where M is the d by d matrix whose columns are precisely the vectors w_k. Thus, in the integral of (6.18), we let $x := My$, with $y \in \mathbb{R}^d_{\geqslant 0}$, so that $dx = |\det M|\, dy$. Recalling that by definition $\det \mathcal{K} = \det M$, We have

$$\int_{\mathcal{K}} e^{2\pi i \langle x,\xi\rangle - \epsilon ||x||^2} dx = |\det \mathcal{K}| \int_{\mathbb{R}^d_{\geqslant 0}} e^{2\pi i \langle My,\xi\rangle - \epsilon ||My||^2} dy. \tag{6.19}$$

It is sufficient to therefore show the following limiting identity:

$$\lim_{\epsilon \to 0+} \int_{\mathbb{R}^d_{\geqslant 0}} e^{2\pi i \langle My,\xi\rangle - \epsilon ||My||^2} dy = \left(\frac{-1}{2\pi i}\right)^d \frac{1}{\prod_{k=1}^d \langle w_k(v), \xi\rangle}. \tag{6.20}$$

To see things very clearly, we first prove the $d = 1$ case. Here we must show that

$$\lim_{\epsilon \to 0+} \int_0^\infty e^{2\pi i x\xi - \epsilon x^2} dx = \frac{-1}{2\pi i \xi}, \tag{6.21}$$

for all $\xi \in \mathbb{C}$, and we see that even this 1-dimensional case is interesting. We proceed with integration by parts by letting $dv := e^{2\pi i x\xi} dx$ and $u := e^{-\epsilon x^2}$, to get

$$\int_0^\infty e^{2\pi i x\xi - \epsilon x^2} dx = e^{-\epsilon x^2} \frac{e^{2\pi i x\xi}}{2\pi i \xi}\Big|_{x=0}^{x=+\infty} - \int_0^\infty \frac{e^{2\pi i x\xi}}{2\pi i \xi}(-2\epsilon x)e^{-\epsilon x^2} dx \tag{6.22}$$

$$= \frac{-1}{2\pi i \xi} + \frac{\epsilon}{\pi i \xi} \int_0^\infty x e^{2\pi i x\xi - \epsilon x^2} dx \tag{6.23}$$

$$= \frac{-1}{2\pi i \xi} + \frac{\sqrt{\epsilon}}{\pi i \xi} \int_0^\infty e^{2\pi i \frac{u}{\sqrt{\epsilon}}\xi} u e^{-u^2} du \tag{6.24}$$

where we've used the substitution $u := \sqrt{\epsilon} x$ in the last equality (6.24). We now notice that

$$\lim_{\epsilon \to 0} \int_0^\infty e^{2\pi i \frac{u}{\sqrt{\epsilon}}\xi} u e^{-u^2} du = \lim_{w \to \infty} \hat{g}(w),$$

where $g(u) := u e^{-u^2} 1_{[0,+\infty]}(u)$, and where $w := \frac{1}{\sqrt{\epsilon}}\xi$. Using the Riemann–Lebesgue lemma 3.3, we know that

$$\lim_{w \to \infty} \hat{g}(w) = 0,$$

completing the proof of the 1-dimensional case.

We now proceed with the general case, which just uses the 1-dimensional idea above several times. To prove (6.20), we first fix the variables $y_2, \ldots, y_d$ and perform integration by parts on y_1 first. Thus, we let

$$dv_1 := e^{2\pi i \langle My, \xi\rangle} dy_1 = e^{2\pi i \langle y, M^t \xi\rangle} dy_1 = e^{2\pi i (y_1 \langle w_1, \xi\rangle + \cdots + y_d \langle w_d, \xi\rangle)} dy_1, \tag{6.25}$$

thought of as a function of only y_1. Carrying out the integration in the variable y_1, we have $v_1 = e^{2\pi i \langle y, M^t \xi\rangle} / (2\pi i \langle w_1, \xi\rangle)$. We let $u_1 := e^{-\epsilon ||My||^2}$, also thought of as a function of y_1 alone. We have $du_1 = -\epsilon L(y) e^{-\epsilon ||My||^2} dy_1$, where $L(y)$

is a real polynomial in y, with coefficients from M. Integrating by parts in the variable y_1 now gives us

$$\int_{\mathbb{R}^d_{\geqslant 0}} e^{2\pi i\langle My,\xi\rangle - \epsilon||My||^2} dy = \int_{\mathbb{R}^{d-1}_{\geqslant 0}} dy_2 \cdots dy_d \left[u_1 v_1 \Big|_0^\infty - \int_0^\infty v_1 du_1 \right]$$

$$= \int_{\mathbb{R}^{d-1}_{\geqslant 0}} dy_2 \cdots dy_d \left[\frac{e^{2\pi i\langle y, M^t\xi\rangle - \epsilon||My||^2}}{2\pi i\langle w_1, \xi\rangle} \Big|_{y_1=0}^{y_1=\infty} + \right.$$

$$\left. + \frac{\epsilon}{2\pi i\langle w_1,\xi\rangle} \int_0^\infty L(y) e^{2\pi i\langle y, M^t\xi\rangle - \epsilon||My||^2} dy_1 \right]$$

$$= \int_{\mathbb{R}^{d-1}_{\geqslant 0}} \frac{-e^{2\pi i\langle \mathbf{t}, M^t\xi\rangle - \epsilon||Mt||^2}}{2\pi i\langle w_1,\xi\rangle} dt +$$

$$+ \frac{\sqrt{\epsilon}}{2\pi i\langle w_1,\xi\rangle} \int_{\mathbb{R}^d_{\geqslant 0}} L(y) e^{2\pi i\langle y, M^t\xi\rangle - ||My||^2} dy$$

$$= \frac{-1}{2\pi i\langle w_1,\xi\rangle} \int_{\mathbb{R}^{d-1}_{\geqslant 0}} e^{2\pi i\langle t, M^t\xi\rangle - \epsilon||Mt||^2} dt +$$

$$+ \frac{\sqrt{\epsilon}}{2\pi i\langle w_1,\xi\rangle} \int_{\mathbb{R}^d_{\geqslant 0}} L(y) e^{2\pi i\langle y, M^t\xi\rangle - ||My||^2} dy,$$

where we've used $t := (y_2, \ldots, y_d)$. Repeating exactly the same process of integration by parts, we see that we get a sum of d terms, where the first term does not contain any ϵ factors, and all the other terms do contain ϵ factors in the numerators, with positive exponents. Therefore, when we complete the d-many integration by parts iteratively, and finally let ϵ tend to zero, only the leading term remains, namely $\left(\frac{-1}{2\pi i}\right)^d \frac{1}{\prod_{k=1}^d \langle w_k,\xi\rangle}$. We've shown that (6.20) is true. □

Proof. (of Theorem 6.1) We begin with the Brianchon Gram identity:

$$1_{\mathcal{P}} = \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{dim\mathcal{F}} 1_{K_F}. \tag{6.26}$$

We fix any $\xi \in \mathbb{R}^d$, and any $\epsilon > 0$. Multiplying both sides of (6.26) by $e^{2\pi i\langle x,\xi\rangle - \epsilon\|x\|^2}$, and integrate over all $x \in \mathbb{R}^d$, we have:

$$\int_{\mathbb{R}^d} 1_{\mathcal{P}}(x) e^{2\pi i\langle x,\xi\rangle - \epsilon\|x\|^2} dx = \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{dim\mathcal{F}} \int_{\mathbb{R}^d} 1_{K_F}(x) e^{2\pi i\langle x,\xi\rangle - \epsilon\|x\|^2} dx. \tag{6.27}$$

Equivalently,

$$\int_{\mathcal{P}} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx = \sum_{\mathcal{F}\subseteq\mathcal{P}} (-1)^{dim\mathcal{F}} \int_{\mathcal{K}_F} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx. \tag{6.28}$$

For each fixed $\epsilon > 0$, all integrands in (6.28) are Schwartz functions, and so all of the integrals in the latter identity now converge absolutely (and rapidly). We identify two types of tangent cones that may occur on the right-hand side of (6.28), for each face $\mathcal{F} \subseteq \mathcal{P}$.

Case 1. When $\mathcal{F} = v$, a vertex, we have the vertex tangent cone $\mathcal{K}_v$: these are the tangent cones that exist for each vertex of $\mathcal{P}$. It is a standard fact that all of these vertex tangent cones are pointed cones. Letting $\epsilon \to 0$, and calling on Lemma 6.2, we obtain the required limit for $\int_{\mathcal{K}_v} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx$.

Case 2. When $\mathcal{F}$ is not a vertex, we have the tangent cone $\mathcal{K}_{\mathcal{F}}$, and it is a standard fact that in this case $\mathcal{K}_{\mathcal{F}}$ always contains a line. Another standard fact in the land of polytopes is that each tangent cone in this case may be written as $\mathcal{K}_{\mathcal{F}} = \mathbb{R}^k \oplus \mathcal{K}_p$, the direct sum of a copy of Euclidean space with a pointed cone $\mathcal{K}_p$ for any point $p \in \mathcal{F}$. (as a side-note, it is also true that $\dim \mathcal{F} = k + \dim(\mathcal{K}_p)$).

We would like to show that for all faces $\mathcal{F}$ that are not vertices of $\mathcal{P}$, the associated integrals tend to 0:

$$\int_{\mathcal{K}_F} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx \to 0,$$

as $\epsilon \to 0$. Indeed,

$$\int_{\mathcal{K}_F} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx = \int_{\mathbb{R}^k\oplus\mathcal{K}_p} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx \tag{6.29}$$

$$= \int_{\mathbb{R}^k} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx \int_{\mathcal{K}_p} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx. \tag{6.30}$$

The integral $\int_{\mathbb{R}^k} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx$ is precisely the usual Fourier transform of a Gaussian, which is known to be the Gaussian $G_\epsilon(x) := \epsilon^{-d/2} e^{-\frac{\pi}{\epsilon}\|x\|^2}$ by Exercise 3.20. It is apparent that for any fixed nonzero value of $x \in \mathbb{R}^d$, we have $\lim_{\epsilon\to 0} G_\epsilon(x) = 0$. Finally, by Lemma 6.2 again, the integral $\int_{\mathcal{K}_p} e^{2\pi i\langle x,\xi\rangle-\epsilon\|x\|^2} dx$ is finite, because $\mathcal{K}_p$ is another pointed cone. Therefore the product of the integrals in (6.30) tends to zero, completing the proof. □

6.6 An application of transforms to the volume of a simple polytope, and its moments

The following somewhat surprising formula for the volume of a simple polytope gives us a very rapid algorithm for computing volumes of simple polytopes. For more general polytopes, we note that it is an NP-hard problem Bárány (2008) to compute volumes of polytopes for general dimension. Nevertheless, there are various other families of polytopes whose volumes possess tractable algorithms.

Theorem 6.4 (Lawrence (1991)). *Suppose $\mathcal{P} \subset \mathbb{R}^d$ is a simple, d-dimensional polytope. For a vertex tangent cone $\mathcal{K}_v$ of $\mathcal{P}$, fix a set of edges of the cone, say $w_1(v), w_2(v), \ldots, w_d(v) \in \mathbb{R}^d$. Then*

$$\operatorname{vol} \mathcal{P} = \frac{(-1)^d}{d!} \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\langle v, z \rangle^d \det \mathcal{K}_v}{\prod_{k=1}^d \langle w_k(v) \cdot z \rangle} \tag{6.31}$$

for all $z \in \mathbb{C}^d$ such that the denominators on the right-hand side do not vanish. More generally, for any integer $k \geqslant 0$, we have the ***moment formulas****:*

$$\int_{\mathcal{P}} \langle x, z \rangle^k dx = \frac{(-1)^d k!}{(k+d)!} \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\langle v, z \rangle^{k+d} \det \mathcal{K}_v}{\prod_{m=1}^d \langle w_m(v) \cdot z \rangle}. \tag{6.32}$$

Proof. We begin with Brion's identity (6.5), and we substitute $z := t z_0$ for a fixed complex z_0, and any positive real value of t:

$$\int_{\mathcal{P}} e^{-2\pi i \langle u, z_0 \rangle t} \, du = \left(\frac{1}{2\pi i} \right)^d \sum_{v \text{ a vertex of } \mathcal{P}} \frac{e^{-2\pi i \langle v, z_0 \rangle t} \det \mathcal{K}_v}{t^d \prod_{m=1}^d \langle w_m(v), z_0 \rangle}.$$

Now we expand both sides in their Taylor series about $t = 0$. The left-hand-side becomes:

$$\int_{\mathcal{P}} \sum_{k=0}^{\infty} \frac{1}{k!} (-2\pi i \langle u, z_0 \rangle t)^k \, du =$$

$$= \left(\frac{1}{2\pi i} \right)^d \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\sum_{j=0}^{\infty} \frac{1}{j!} (-2\pi i \langle v, z_0 \rangle t)^j \det \mathcal{K}_v}{t^d \prod_{m=1}^d \langle w_m(v), z_0 \rangle}$$

Integrating term by term on the left-hand side, we get:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}(-2\pi i)^k \int_{\mathcal{P}} \langle u, z_0\rangle^k \, du =$$

$$= \left(\frac{1}{2\pi i}\right)^d \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\det \mathcal{K}_v}{\prod_{m=1}^d \langle w_m(v), z_0\rangle} \sum_{j=0}^{\infty} \frac{t^{j-d}}{j!}(-2\pi i)^j \langle v, z_0\rangle^j .$$

Comparing the coefficients of t^k on both sides, we have:

$$\frac{(-2\pi i)^k}{k!} \int_{\mathcal{P}} \langle u, z_0\rangle^k \, du =$$

$$= \left(\frac{1}{2\pi i}\right)^d \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\det \mathcal{K}_v}{\prod_{m=1}^d \langle w_m(v), z_0\rangle} \frac{1}{(k+d)!}(-2\pi i)^{k+d} \langle v, z_0\rangle^{k+d},$$

and simplifying, we arrive at the moment formulas:

$$\int_{\mathcal{P}} \langle u, z_0\rangle^k \, du = (-1)^d \frac{k!}{(k+d)!} \sum_{v \text{ a vertex of } \mathcal{P}} \frac{\langle v, z_0\rangle^{k+d} \det \mathcal{K}_v}{\prod_{m=1}^d \langle w_m(v), z_0\rangle}.$$

In particular, when $k = 0$, we get the volume formula (6.31). □

6.7 Brion's theorem - the discrete form

Example 6.12 (Finite geometric sums)**.** Consider the 1-dimensional polytope $\mathcal{P} := [a, b]$, where $a, b \in \mathbb{Z}$. The problem is to compute the finite geometric series:

$$\sum_{n \in \mathcal{P} \cap \mathbb{Z}} e^{2\pi i n z} = \sum_{a \leqslant n \leqslant b} q^n,$$

where we've set $q := e^{2\pi i z}$. Of course, we already know that it possesses a 'closed form' of the type:

$$\sum_{a \leqslant n \leqslant b} q^n = \frac{q^{b+1} - q^a}{q-1} \tag{6.33}$$

$$= \frac{q^{b+1}}{q-1} - \frac{q^a}{q-1}, \tag{6.34}$$

because we already recognize this formula for a **finite geometric sum**. On the other hand, anticipating the discrete form of Brion's theorem below, we first compute the discrete sum corresponding to the vertex tangent cone at the vertex a, namely $\sum_{a \leqslant n} q^n$:

$$q^a + q^{a+1} + \cdots = \frac{q^a}{1-q}. \tag{6.35}$$

Now we compute the the sum corresponding to the vertex tangent cone at vertex b, namely $\sum_{n \leqslant b} q^n$:

$$q^b + q^{b-1} + \cdots = \frac{q^b}{1-q^{-1}} = \frac{q^{b+1}}{q-1}. \tag{6.36}$$

Summing these two contributions, one from each vertex tangent cone, we get:

$$\frac{q^a}{1-q} + \frac{q^{b+1}}{q-1} = \sum_{a \leqslant n \leqslant b} q^n,$$

by the finite geometric sum identity, thereby verifying Theorem 6.5 for this example. This example shows that Brion's Theorem 6.5 (the discrete version) may be thought of as a d-dimensional extension of the finite geometric sum.

But something is still very wrong here - namely, identity (6.35) converges for $|q| < 1$, while identity (6.36) converges only for $|q| > 1$, so there is not even one value of q for which the required identity (6.34) is true. So how can we make sense of these completely **disjoint domains of convergence** ?! □

To resolve these conundrums, there is an extremely useful result of Michel Brion Brion (1988) that comes to the rescue. We will discretize the continuous form of Brion's Theorem 6.1, using the Poisson summation formula, to arrive at a very useful, discrete form of Brion's Theorem (also due to Brion) using very different methods.

To this discrete end, we define the **integer point transform** of a rational polytope $\mathcal{P}$ by

$$\sigma_{\mathcal{P}}(z) := \sum_{n \in \mathcal{P} \cap \mathbb{Z}^d} e^{\langle n, z \rangle}.$$

We similarly define the **integer point transform of a rational cone** $\mathcal{K}_v$ by the rational-exponential function that is initially defined by the series

$$\sigma_{\mathcal{K}_v}(z) := \sum_{n \in \mathcal{K}_v \cap \mathbb{Z}^d} e^{\langle n, z \rangle}. \tag{6.37}$$

First, we need a slightly technical Lemma, whose proof is straightforward and therefore relegated to Exercise 6.11.

Lemma 6.3. *Let $\mathcal{K}_v$ be a rational cone, with apex at v. We pick any approximate identity ϕ_ϵ that is compactly supported, and we define:*

$$R_{\mathcal{K}}(z) := \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d \cap \operatorname{int} \mathcal{K}_v} \left(e^{2\pi i \langle x, z \rangle} * \phi_\epsilon \right)(n).$$

*Then $R_{\mathcal{K}}(z)$ is a rational-exponential function of z, valid for almost all $z \in \mathbb{C}^d$, and is the **meromorphic continuation of the series** defined by*

$$\sigma_{\operatorname{int} K_v}(z) := \sum_{n \in \mathbb{Z}^d \cap \operatorname{int} \mathcal{K}_v} e^{2\pi i \langle n, z \rangle}.$$

□

It turns out that the continuous form of Brion's theorem, namely Theorem 6.1, can be used to prove the discrete form of Brion's theorem, namely Theorem 6.5 below.

Theorem 6.5 (Brion's theorem - the discrete form, 1988). *Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational, d-dimensional polytope, and let N be the number of vertices of $\mathcal{P}$. For each vertex v of $\mathcal{P}$, we consider the open vertex tangent cone* $\operatorname{int} \mathcal{K}_v$ *of* $\operatorname{int} \mathcal{P}$*, the interior of $\mathcal{P}$. Then*

$$\sigma_{\operatorname{int} \mathcal{P}}(z) = \sigma_{\operatorname{int} \mathcal{K}_{v_1}}(z) + \cdots + \sigma_{\operatorname{int} \mathcal{K}_{v_N}}(z). \tag{6.38}$$

for all $z \in \mathbb{C}^d - S$, where S is the hyperplane arrangement defined by the (removable) singularities of all of the transforms $\hat{1}_{\mathcal{K}_{v_j}}(z)$.

Proof. We will use the continuous version of Brion, namely Theorem 6.1, together with the Poisson summation formula, to deduce the discrete version here. In a sense, the Poisson summation formula allows us to discretize the integrals.

Step 1. [Intuition] To begin, in order to motivate the real process, we will use Poisson summation on a function $1_{\mathcal{P}}(n) e^{2\pi i \langle n, z \rangle}$ that "doesn't have the right" to be used in Poisson summation, because it is discontinuous. But this first step brings the intuition to the foreground. Then, in Step 2, we will literally "smooth" out the lack of rigor in Step 1, making everything rigorous.

$$\begin{aligned}
\sum_{n\in\mathcal{P}\cap\mathbb{Z}^d} e^{2\pi i\langle n,z\rangle} &:= \sum_{n\in\mathbb{Z}^d} 1_{\mathcal{P}}(n)e^{2\pi i\langle n,z\rangle} \\
&= \sum_{\xi\in\mathbb{Z}^d} \hat{1}_{\mathcal{P}}(z+\xi) \\
&= \sum_{\xi\in\mathbb{Z}^d} \left(\hat{1}_{\mathcal{K}_{v_1}}(z+\xi) + \cdots + \hat{1}_{\mathcal{K}_{v_1}}(z+\xi)\right) \\
&= \sum_{\xi\in\mathbb{Z}^d} \hat{1}_{\mathcal{K}_{v_1}}(z+\xi) + \cdots + \sum_{\xi\in\mathbb{Z}^d} \hat{1}_{\mathcal{K}_{v_N}}(z+\xi) \\
&= \sum_{n\in\mathbb{Z}^d} 1_{\mathcal{K}_{v_1}}(n)e^{2\pi i\langle n,z\rangle} + \cdots + \sum_{n\in\mathbb{Z}^d} 1_{\mathcal{K}_{v_N}}(n)e^{2\pi i\langle n,z\rangle} \\
&:= \sum_{n\in\mathbb{Z}^d\cap\mathcal{K}_{v_1}} e^{2\pi i\langle n,z\rangle} + \cdots + \sum_{n\in\mathbb{Z}^d\cap\mathcal{K}_{v_N}} e^{2\pi i\langle n,z\rangle},
\end{aligned}$$

where we have used the Poisson summation formula in the second and fifth equalities. The third equality used Brion's Theorem 6.1.

Step 2 [Rigorous proof]. To make Step 1 rigorous, we pick any compactly supported approximate identity ϕ_ϵ, and form a smoothed version of the function in step 1. Namely we let

$$f_\epsilon(x) := (1_{\mathcal{P}}(x)e^{2\pi i\langle x,z\rangle}) * \phi_\epsilon(x),$$

so that now we are allowed to apply Poisson summation to f_ϵ, because it is a Schwartz function. Recalling Theorem 3.12, we know that at a point $x\in\mathbb{R}^d$ of continuity of $1_{\mathcal{P}}(x)e^{2\pi i\langle x,z\rangle}$, we have

$$\lim_{\epsilon\to 0} f_\epsilon(x) = 1_{\mathcal{P}}(x)e^{2\pi i\langle x,z\rangle}.$$

To proceed further, it is therefore natural to consider points $x\in\operatorname{int}\mathcal{P}$, the interior of $\mathcal{P}$, because $1_{\mathcal{P}}$ is continuous there, while it is not continuous on the boundary of $\mathcal{P}$. To recap, we have so far the equalities

$$\begin{aligned}
\sum_{n \in \operatorname{int} \mathcal{P} \cap \mathbb{Z}^d} e^{2\pi i \langle n, z\rangle} &:= \sum_{n \in \mathbb{Z}^d} 1_{\operatorname{int} \mathcal{P}}(x) e^{2\pi i \langle x, z\rangle} \\
&= \sum_{n \in \operatorname{int} \mathcal{P} \cap \mathbb{Z}^d} \lim_{\epsilon \to 0} f_\epsilon(n) \\
&= \lim_{\epsilon \to 0} \sum_{n \in \operatorname{int} \mathcal{P} \cap \mathbb{Z}^d} f_\epsilon(n),
\end{aligned}$$

where we've used the fact that f_ϵ is compactly supported, because it is the convolution of two compactly supported functions. So the exchange above, of the sum with the limit, is trivial because the sum is finite. With this in mind, the Poisson summation formula, applied to the Schwarz function f_ϵ, gives us:

$$\begin{aligned}
\sum_{n \in \operatorname{int} \mathcal{P} \cap \mathbb{Z}^d} e^{2\pi i \langle n, z\rangle} &= \lim_{\epsilon \to 0} \sum_{n \in \operatorname{int} \mathcal{P} \cap \mathbb{Z}^d} f_\epsilon(n) \\
&= \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \left(1_{\operatorname{int} \mathcal{P}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon \right)(n) \\
&= \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} \mathcal{F}\big((1_{\operatorname{int} \mathcal{P}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon \big)(\xi) \\
&= \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\operatorname{int} \mathcal{P}}(z + \xi) \hat{\phi}_\epsilon(\xi) \\
&= \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \left(\hat{1}_{\operatorname{int} \mathcal{K}_{v_1}}(z + \xi) + \cdots + \hat{1}_{\operatorname{int} \mathcal{K}_{v_1}}(z + \xi) \right) \hat{\phi}_\epsilon(\xi) \\
&= \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \mathcal{F}\big((1_{\operatorname{int} \mathcal{K}_{v_1}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon \big)(\xi) + \cdots + \\
&\qquad + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \mathcal{F}\big((1_{\operatorname{int} \mathcal{K}_{v_N}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon \big)(\xi) \\
&= \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{\operatorname{int} \mathcal{K}_{v_1}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon(n) + \cdots + \\
&\qquad + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} (1_{\operatorname{int} \mathcal{K}_{v_N}}\, e^{2\pi i \langle x, z\rangle}) * \phi_\epsilon(n) \\
&= \sigma_{\operatorname{int} \mathcal{K}_{v_1}}(z) + \cdots + \sigma_{\operatorname{int} \mathcal{K}_{v_N}}(z),
\end{aligned}$$

using Lemma 6.3 in the last step. □

Notes

(a) There is a large literature devoted to triangulations of cones, polytopes, and general point-sets, and the reader is invited to consult the excellent and encyclopedic book on triangulations, by De Loera, Rambau, and Santos (2010).

(b) The notion of a **random polytope** has a large literature as well, and although we do not go into this topic here, one classic survey paper is by Imre Bárány (2008).

(c) The attempt to extend Ehrhart theory to non-rational polytopes, whose vertices have some irrational coordinates, is ongoing. The pioneering papers of Randol (1969) and Randol (1997) extended integer point counting to algebraic polytopes, meaning that their vertices are allowed to have coordinates that are algebraic numbers. Recently, a growing number of papers are considering all real dilates of a rational polytope, which is still rather close to the Ehrhart theory of rational polytopes.

In this direction, it is natural to ask how much more of the geometry of a given polytope $\mathcal{P}$ can be captured by counting integer points in all of its positive real dilates. Suppose we translate a d-dimensional integer polytope $\mathcal{P} \subset \mathbb{R}^d$ by an integer vector $n \in \mathbb{Z}^d$. The standard Ehrhart theory gives us an invariance principle, namely the equality of the Ehrhart polynomials for $\mathcal{P}$ and $\mathcal{P} + n$: $L_{\mathcal{P}+n}(t) = L_{\mathcal{P}}(t)$, for all *integer* dilates $t > 0$. However, when we allow t to be a positive real number, it is no longer true that $L_{\mathcal{P}+n}(t) = L_{\mathcal{P}}(t)$ for all $t > 0$. In fact, these two Ehrhart functions are so different in general, that by the recent breakthrough [See Royer (2017a), Royer (2017b), Royer (2017c)], it's even possible to uniquely reconstruct the polytope $\mathcal{P}$, if we know all the counting quasi-polynomials $L_{\mathcal{P}+n}(t)$ for all integer translates $n \in \mathbb{Z}^d$. In other words, the work of Royer (2017a) shows that for two rational polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$, the equality $L_{\mathcal{P}+n}(t) = L_{\mathcal{Q}+n}(t)$ holds for all integer translates $n \in \mathbb{Z}^d \iff \mathcal{P} = \mathcal{Q}$. It is rather astounding that just by counting integer points in sufficiently many translates of $\mathcal{P}$, we may completely reconstruct the whole polytope $\mathcal{P}$ uniquely. It was further demonstrated in Royer (2017b) that such an idea also works if we replace a polytope by any symmetric convex body. Now it is therefore natural to try to prove the following extended question.

Question 8. Suppose we are given polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$. Can we always find a finite subset $S \subset \mathbb{Z}^d$ (which may depend on $\mathcal{P}$ and Q) such that

$$L_{\mathcal{P}+n}(t) = L_{Q+n}(t) \text{ for all } n \in S \iff \mathcal{P} = Q?$$

Exercises

6.1. ♣ *Although* $\det \mathcal{K}_v$ *depends on the choice of the length of each edge of* $\mathcal{K}_v$*, show that the ratio* $\frac{|\det \mathcal{K}_v|}{\prod_{k=1}^d \langle w_k(v), z\rangle}$ *remains invariant if we replace each edge* $w_k(v)$ *of a simplicial cone by a constant positive multiple of it, say* $\alpha_k w_k(v)$.

6.2. *Consider the regular hexagon* $\mathcal{P} \subset \mathbb{R}^2$*, whose vertices are the 6'th roots of unity.*

(a) Compute the area of $\mathcal{P}$ *using Theorem 6.4.*

(b) Compute all of the moments of $\mathcal{P}$*, as in Theorem 6.4.*

6.3. ♣ *Prove Corollary 6.1 for a simplicial cone* $\mathcal{K}_v$*, whose apex is* v*, by translating a cone whose vertex is at the origin, to get:*

$$\hat{1}_{\mathcal{K}_v}(z) := \int_{\mathcal{K}_v} e^{-2\pi i \langle u,z\rangle}\, du = \frac{1}{(2\pi i)^d} \frac{e^{-2\pi i \langle v,z\rangle} \det \mathcal{K}_v}{\prod_{k=1}^d \langle w_k, z\rangle}.$$

6.4. *Consider the following* 3*-dimensional polytope* $\mathcal{P}$*, whose vertices are as follows:*

$$\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,0),\ (0,0,1)\}.$$

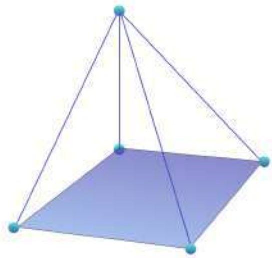
"a pyramid over a square". Compute its Fourier–Laplace transform $\hat{1}_{\mathcal{P}}(z)$.

6.5. *We recall that the* 3*-dimensional cross-polytope (also called an octahedron) was defined by*

$$\lozenge := \left\{(x_1, x_2, x_3) \in \mathbb{R}^d \mid |x_1| + |x_2| + |x_3| \leqslant 1\right\}.$$

Compute the Fourier–Laplace transform of $\lozenge$ *by using Theorem 6.2.*

(Here not all of the tangent cones are simplicial cones, but we may triangulate each vertex tangent cones into simplicial cones).



6.6. *Here we will find the Fourier transform of a dodecahedron $\mathcal{P}$, centered at the origin. Suppose we fix the following* 20 *vertices of* $\mathcal{P}$*:*

$$\{(\pm 1,\ \pm 1,\ \pm 1),\ (0,\ \pm\phi,\ \pm\frac{1}{\phi}),\ (\pm\frac{1}{\phi},\ 0,\ \pm\phi),\ (\pm\phi,\ \pm\frac{1}{\phi},\ 0)\},$$

where $\phi := \frac{1+\sqrt{5}}{2}$. *It turns out that* $\mathcal{P}$ *is a simple polytope. Compute its Fourier–Laplace transform using Theorem 6.1.*

Notes. All of the vertices of $\mathcal{P}$ *given here can easily be seen to lie on a sphere* S *of radius* $\sqrt{3}$, *and this is a regular embedding of the dodecahedron. It is also true (though a more difficult fact) that these* 20 *points maximize the volume of any polytope whose* 20 *vertices lie on the surface of this sphere* S.

6.7. *We define the* 3*-dimensional polytope*

$$\mathcal{P} := \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (\mathrm{a}, \mathrm{b}, \mathrm{c})\},$$

where we fix real numbers $a, b, c, > 0$. *Compute* $\hat{1}_{\mathcal{P}}(z)$, *by first computing the Fourier–Laplace transforms of its tangent cones.*

(Note. Here not all of the vertex tangent cones are simplicial cones).

6.8. *This exercise extends Exercise 6.4 to* $\mathbb{R}^d$*, as follows. Consider the* d*-dimensional polytope* $\mathcal{P}$*, called a "pyramid over a cube", defined by the convex hull of the unit cube* $[0,1]^{d-1} \subset \mathbb{R}^{d-1}$*, with the point* $(0,0,\ldots,0,1) \in \mathbb{R}^d$. *Compute its Fourier–Laplace transform* $\hat{1}_{\mathcal{P}}(z)$.

Figure 6.4: A climbing wall in Sweden, made up of Dodecahedrons, showing one of their real life applications

6.9. ♣

(a) *Show that the Brianchon–Gram relations* (6.16) *imply the Euler–Poincaré relation for the face-numbers of a convex polytope* $\mathcal{P}$*:*

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} + (-1)^d f_d = 1, \tag{6.39}$$

where f_k *is the number of faces of* $\mathcal{P}$ *of dimension* k.

(b) *(hard) Conversely, given a* d*-dimensional polytope* $\mathcal{P} \subset \mathbb{R}^d$*, show that the Euler–Poincaré relation above implies the Brianchon–Gram relations:*

$$1_{\mathcal{P}}(x) = \sum_{\mathcal{F} \subset \mathcal{P}} (-1)^{dim \mathcal{F}} 1_{\mathcal{K}_F}(x),$$

for all $x \in \mathbb{R}^d$.

6.10. ♣ *Suppose we are given a* d*-dimensional simplicial cone* $K \subset \mathbb{R}^d$ *(so be definition* K *has exactly* d *edges). Show that* K *must be pointed.*

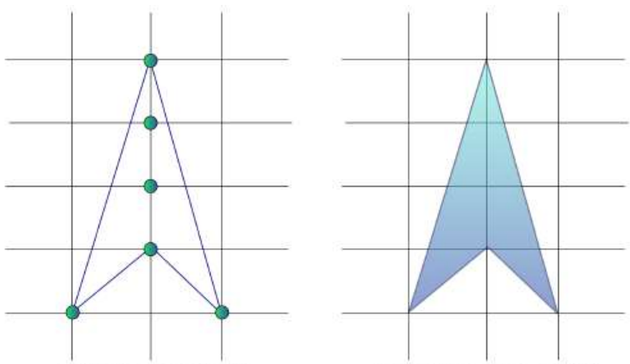
6.11. ♣ *Prove* Lemma 6.3.

6.12. ♣ *Show that the only polytopes that are both simple and simplicial are either simplices, or 2-dimensional polygons.*

7 Counting integer points in polytopes

How wonderful that we have met with a paradox. Now we have some hope of making progress.

– Niels Bohr



A discrete volume for $\mathcal{P}$ The continuous volume of $\mathcal{P}$

7.1 Intuition

A basic question in discrete geometry is "how do we discretize volume?" One method of discretizing the volume of $\mathcal{P}$ is to count the number of integer points in $\mathcal{P}$. Even in $\mathbb{R}^2$, this question may be highly non-trivial, depending on the arithmetic properties of the vertices of $\mathcal{P}$. Ehrhart first considered integer dilations of a fixed, integer polytope $\mathcal{P}$, and defined:

$$L_{\mathcal{P}}(t) := |\mathbb{Z}^d \cap t\mathcal{P}|, \tag{7.1}$$

where $t\mathcal{P}$ is the t'th dilate of $\mathcal{P}$, and t is a positive integer. Ehrhart showed that $L_{\mathcal{P}}(t)$ is a polynomial in the positive integer parameter t, known as the **Ehrhart polynomial** of $\mathcal{P}$.

A different method of discretizing volume is achieved by placing a small sphere at each integer point, computing the proportion of that sphere that intersects $\mathcal{P}$, and summing these weights at all integer points - we call this discretized volume the "angle polynomial", which was also analyzed by Ehrhart, and developed by I.G. Macdonald.

More generally, given a function $f : \mathbb{R}^d \to \mathbb{C}$, we may sum the values of f at all integer points and observe how close this sum gets to the integral of f over $\mathcal{P}$. This approach is known as Euler–Maclaurin summation over polytopes, and is a current and exciting topic of a growing literature (see Note (f) below).

7.2 Computing integer points in polytopes via the discrete Brion Theorem

Here we present a proof of Ehrhart's result, using Brion's Theorem 6.5, the discrete form, which we now recall. When all the vertices of a polytope $\mathcal{P}$ have rational coordinates, we call $\mathcal{P}$ a **rational polytope** .

Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational, d-dimensional polytope, and let N be the number of its vertices. For each vertex v of $\mathcal{P}$, we consider the vertex tangent cone $\mathcal{K}_v$ of $\mathcal{P}$. Once we dilate $\mathcal{P}$ by t, each vertex v of $\mathcal{P}$ gets dilated to become tv, and so each of the vertex tangent cone $\mathcal{K}_v$ of $\mathcal{P}$ simply get shifted to the corresponding vertex tangent cone $\mathcal{K}_{tv}$ of $t\mathcal{P}$. Thus, we have

$$\sum_{n \in t\mathcal{P}\cap\mathbb{Z}^d} e^{2\pi i\langle n,z\rangle} = \sum_{n\in\mathcal{K}_{tv_1}\cap\mathbb{Z}^d} e^{2\pi i\langle n,z\rangle} + \cdots + \sum_{n\in\mathcal{K}_{tv_N}\cap\mathbb{Z}^d} e^{2\pi i\langle n,z\rangle}, \tag{7.2}$$

for all $z \in \mathbb{C}^d - S$, where S is the hyperplane arrangement defined by the (removable) singularities of all of the transforms $\hat{1}_{\mathcal{K}_{v_j}}(z)$. To simplify notation, we may absorb the constant $2\pi i$ into the complex vector z by replacing z by $\frac{1}{2\pi i} z$, so that we may assume without loss of generality that Brion's Theorem 6.5 has the form

$$\sum_{n \in t\mathcal{P} \cap \mathbb{Z}^d} e^{\langle n, z \rangle} = \sum_{n \in \mathcal{K}_{tv_1} \cap \mathbb{Z}^d} e^{\langle n, z \rangle} + \cdots + \sum_{n \in \mathcal{K}_{tv_N} \cap \mathbb{Z}^d} e^{\langle n, z \rangle}. \tag{7.3}$$

Now we notice that when $z = 0$, the left-hand-side gives us precisely

$$\sigma_{\mathcal{P}}(0) := \sum_{n \in t\mathcal{P} \cap \mathbb{Z}^d} 1 := |\mathbb{Z}^d \cap t\mathcal{P}|,$$

which is good news - it is the Ehrhart polynomial $L_{\mathcal{P}}(t)$, by definition. The bad news is that $z = 0$ is a singularity of the right-hand-side of (7.2). But then again, there is more good news - we already saw in the previous chapter that it is a removable singularity. So we may let $z \to 0$ to see what happens.

In order to obtain precise formulas for the integer point transform of a cone, we first need to better understand the fundamental (half-open) parallelepipeds of simplicial cones.

Lemma 7.1. *Let D be any half-open integer parallelepiped in $\mathbb{R}^d$. Then:*

$$\#\{\mathbb{Z}^d \cap D\} = \operatorname{vol} D.$$

Example 7.1. We compute the integer point transform of the **standard triangle** in the plane, using Brion's Equation (7.3). Namely, for the standard triangle

$$\Delta := \operatorname{conv}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right),$$

as depicted in Figure 7.1, we find $\sigma_\Delta(z)$. By definition, the integer point transform of its vertex tangent cone $\mathcal{K}_{v_1}$ is

$$\begin{aligned}
\sigma_{\mathcal{K}_{v_1}}(z) := \sum_{n \in \mathcal{K}_{v_1} \cap \mathbb{Z}^d} e^{\langle n, z \rangle} &= \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\langle n_1 \binom{1}{0} + n_2 \binom{0}{1}, z \rangle} \\
&= \sum_{n_1 \geqslant 0} e^{n_1 z_1} \sum_{n_2 \geqslant 0} e^{n_2 z_2} \\
&= \frac{1}{(1 - e^{z_1})(1 - e^{z_2})}.
\end{aligned}$$

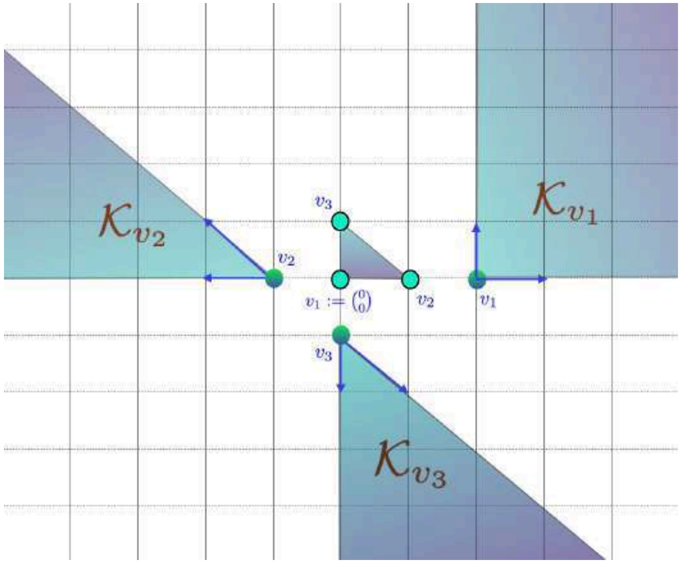


Figure 7.1: The standard triangle, with its vertex tangent cones

For the vertex tangent cone $\mathcal{K}_{v_2}$, we have

$$\begin{aligned}
\sigma_{\mathcal{K}_{v_2}}(z) &:= \sum_{n \in \mathcal{K}_{v_2} \cap \mathbb{Z}^d} e^{\langle n, z \rangle} = \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\left\langle \binom{1}{0} + n_1 \binom{-1}{0} + n_2 \binom{-1}{1}, z \right\rangle} \\
&= e^{z_1} \sum_{n_1 \geqslant 0} e^{n_1(-z_1)} \sum_{n_2 \geqslant 0} e^{n_2(-z_1+z_2)} \\
&= \frac{e^{z_1}}{(1-e^{-z_1})(1-e^{-z_1+z_2})}.
\end{aligned}$$

Finally, for the vertex tangent cone $\mathcal{K}_{v_3}$, we have

$$\begin{aligned}
\sigma_{\mathcal{K}_{v_3}}(z) &:= \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\langle \binom{0}{1} + n_1 \binom{0}{-1} + n_2 \binom{1}{-1}, z \rangle} \\
&= e^{z_2} \sum_{n_1 \geqslant 0} e^{n_1(-z_2)} \sum_{n_2 \geqslant 0} e^{n_2(z_1 - z_2)} \\
&= \frac{e^{z_2}}{(1 - e^{-z_2})(1 - e^{z_1 - z_2})}.
\end{aligned}$$

Altogether, using 7.3 we have

$$\begin{aligned}
\sigma_{\mathcal{P}}(z) &= \sigma_{\mathcal{K}_{v_1}}(z) + \sigma_{\mathcal{K}_{v_2}}(z) + \sigma_{\mathcal{K}_{v_3}}(z) && (7.4) \\
&= \frac{1}{(1 - e^{z_1})(1 - e^{z_2})} + \frac{e^{z_1}}{(1 - e^{-z_1})(1 - e^{-z_1 + z_2})} + && (7.5) \\
&\qquad + \frac{e^{z_2}}{(1 - e^{-z_2})(1 - e^{z_1 - z_2})}. && (7.6)
\end{aligned}$$

$\square$

Example 7.2. We find a formula for the Ehrhart polynomial $L_{\mathcal{P}}(t) := |\mathbb{Z}^2 \cap t\mathcal{P}|$ of the standard triangle, continuing the computation of the previous example. It turns out, as we show in the section that follows, that the method we use here is universal - it can always be used to find the Ehrhart polynomial of any rational polytope.

In this example we are lucky in that we may use brute-force to compute it, since the number of integer points in the t-dilate of $\mathcal{P}$ may be computed along the diagonals:

$$L_{\mathcal{P}}(t) = 1 + 2 + 3 + \cdots + (t+1) = \frac{(t+1)(t+2)}{2} = \frac{1}{2}t^2 + \frac{3}{2}t + 1.$$

Now we can confirm this lucky answer with our brand new machine, as follows. Using (7.3), and the formulation (7.6) from the previous example, we have the integer point transform for the dilates of $\mathcal{P}$:

$$
\begin{aligned}
\sum_{n\in t\Delta\cap\mathbb{Z}^d} e^{\langle n,z\rangle} &= \sum_{n\in\mathcal{K}_{tv_1}\cap\mathbb{Z}^d} e^{\langle n,z\rangle} + \sum_{n\in\mathcal{K}_{tv_1}\cap\mathbb{Z}^d} e^{\langle n,z\rangle} + \sum_{n\in\mathcal{K}_{tv_3}\cap\mathbb{Z}^d} e^{\langle n,z\rangle} \qquad (7.7)\\
&= \frac{1}{(1-e^{z_1})(1-e^{z_2})} + \frac{e^{tz_1}}{(e^{-z_1}-1)(e^{-z_1+z_2}-1)} \qquad (7.8)\\
&\quad + \frac{e^{tz_2}}{(e^{-z_2}-1)(e^{z_1-z_2}-1)} \qquad (7.9)\\
&:= F_1(z)+F_2(z)+F_3(z), \qquad (7.10)
\end{aligned}
$$

where we have defined F_1, F_2, F_3 by the last equality. We can let $z \to 0$ along almost any direction, but it turns out that we can simplify our computations by taking advantage of the symmetry of this polytope, so we will pick $z = \binom{x}{-x}$, which will simplify our computations (Note (e)). Here is our plan:

(a) We pick $z := \binom{x}{-x}$.

(b) We expand all three meromorphic functions F_1, F_2, F_3 in terms of their Laurent series in x, giving us Bernoulli numbers.

(c) Finally, we let $x \to 0$, to retrieve the constant term (which will be a polynomial function of t) of the resulting Laurent series.

To expand $F_1(z), F_2(z), F_3(z)$ in their Laurent series, we recall the Equation (2.24) of the Bernoulli numbers in terms of their generating function, namely $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$:

$$
\begin{aligned}
F_1(x,-x) &= \frac{-1}{x^2} \sum_{m\geqslant 0} B_m \frac{x^m}{m!} \sum_{n\geqslant 0} B_n \frac{(-x)^n}{n!}\\
&= \frac{-1}{x^2}\left(1-\frac{x}{2}+\frac{x^2}{6}+O(x^3)\right)\left(1+\frac{x}{2}+\frac{x^2}{6}+O(x^3)\right)\\
&= \frac{-1}{x^2} - \frac{1}{3} + O(x)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
F_2(x,-x) &= \frac{1 + tx + \frac{t^2}{2!}x^2 + O(x^3)}{2x^2}\left(1 + \frac{x}{2} + \frac{x^2}{6} + O(x^3)\right)\cdot \\
&\qquad\qquad\qquad \cdot\left(1 + \frac{(2x)}{2} + \frac{(2x)^2}{6} + O(x^3)\right) \\
&= \frac{1}{2x^2} + \frac{3}{4x} + \frac{2}{3} + \frac{t}{2x} + \frac{3t}{4} + \frac{t^2}{4} + O(x)
\end{aligned}
$$

Now, by symmetry we see that $F_3(x,-x) = F_2(-x,x)$, so that by (7.10) and the latter expansions, we finally have:

$$
\begin{aligned}
\sum_{n \in t\Delta \cap \mathbb{Z}^d} e^{\langle n, \left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)\rangle} &= F_1(x,-x) + F_2(x,-x) + F_2(-x,x) \\
&= 1 + \frac{3}{2}t + \frac{1}{2}t^2 + O(x).
\end{aligned}
$$

Letting $z := \left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right) \to 0$ in the latter computation, we retrieve the (Ehrhart) polynomial answer:

$$
\sum_{n \in t\Delta \cap \mathbb{Z}^d} 1 = L_\Delta(t) = 1 + \frac{3}{2}t + \frac{1}{2}t^2,
$$

as desired. □

7.3 Examples, examples, examples....

Example 7.3. We work out the integer point transform $\sigma_{\mathcal{K}}(z)$ of the cone

$$
\mathcal{K} := \{\lambda_1 \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) + \lambda_2 \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geqslant 0}\},
$$

Drawn in the figures below. We note that here $\det \mathcal{K} = 5$, and that there are indeed 5 integer points in D, its half-open fundamental parallelepiped.

We may 'divide and conquer' the integer point transform $\sigma_{\mathcal{K}}(z)$, by breaking it up into 5 infinite series, one for each integer point in D, as follows:

$$
\sigma_{\mathcal{K}}(z) := \sum_{n \in \mathcal{K} \cap \mathbb{Z}^d} e^{\langle n, z\rangle} := \sum_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} + \sum_{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} + \sum_{\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)} + \sum_{\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)} + \sum_{\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)},
$$

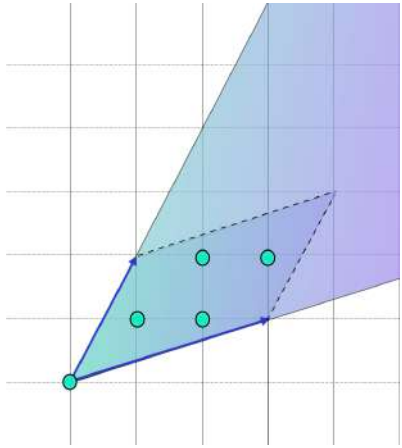


Figure 7.2: The 5 integer points in a fundamental parallelepiped D, of $\mathcal{K}$.

where

$$
\begin{aligned}
\sum_{\binom{1}{1}} &:= \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\langle \binom{1}{1} + n_1 \binom{3}{1} + n_2 \binom{1}{2}, z \rangle} \\
&= e^{\langle \binom{1}{1}, z \rangle} \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\langle n_1 \binom{3}{1} + n_2 \binom{1}{2}, z \rangle} \\
&= e^{\langle \binom{1}{1}, z \rangle} \sum_{n_1 \geqslant 0} e^{n_1 \langle \binom{3}{1}, z \rangle} \sum_{n_2 \geqslant 0} e^{n_2 \langle \binom{1}{2}, z \rangle} \\
&= \frac{e^{z_1 + z_2}}{(1 - e^{3z_1 + z_2})(1 - e^{z_1 + 2z_2})},
\end{aligned}
$$

and similarly we have

$$
\sum_{\binom{2}{1}} = \frac{e^{2z_1 + z_2}}{(1 - e^{3z_1 + z_2})(1 - e^{z_1 + 2z_2})},
$$

$$
\sum_{\binom{2}{2}} = \frac{e^{2z_1 + 2z_2}}{(1 - e^{3z_1 + z_2})(1 - e^{z_1 + 2z_2})},
$$

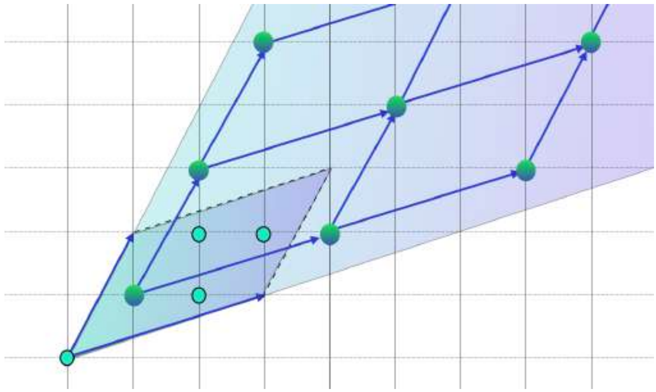


Figure 7.3: The point $\binom{1}{1}$ in D, with its images in $\mathcal{K}$ under translations by the edge vectors of $\mathcal{K}$.

$$\sum_{\binom{3}{2}} = \frac{e^{3z_1+2z_2}}{(1-e^{3z_1+z_2})(1-e^{z_1+2z_2})},$$

and finally

$$\sum_{\binom{0}{0}} = \frac{1}{(1-e^{3z_1+z_2})(1-e^{z_1+2z_2})}.$$

To summarize, we have the following expression:

$$\sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} e^{\langle n,z\rangle} = \frac{1+e^{z_1+z_2}+e^{2z_1+z_2)}+e^{2z_1+2z_2}+e^{3z_1+2z_2}}{(1-e^{3z_1+z_2})(1-e^{z_1+2z_2})}.$$

Equivalently, we may use multinomial notation: let $q_j := e^{z_j}$, so that by definition

$$q_1{}^{n_1}\cdots q_d{}^{n_d} := e^{z_1 n_1}\cdots e^{z_d n_d} = e^{\langle z,n\rangle}.$$

It is common to use the following shorthand multinomial definition: $q^n := q_1{}^{n_1}\cdots q_d{}^{n_d}$. With this multinomial notation, we have

$$\sum_{n\in\mathcal{K}\cap\mathbb{Z}^d} q^n = \frac{1+q_1q_2+q_1{}^2q_2+q_1{}^2q_2{}^2+q_1^3q_2^2}{(1-q_1^3q_2)(1-q_1q_2^2)}.$$

$\square$

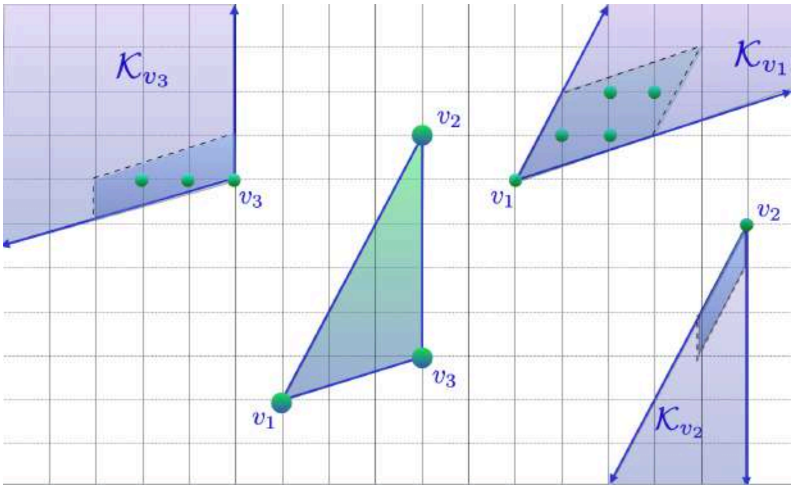


Figure 7.4: A triangle with vertices v_1, v_2, v_3, and its vertex tangent cones

Example 7.4. *Here we will compute the integer point transform of the triangle Δ defined by the convex hull of the points $\binom{0}{0}, \binom{3}{1}, \binom{3}{6}$, as shown in Figure 7.4 below. We first compute the integer point transforms of all of its tangent cones. For the vertex v_1, we already computed the integer point transform of its tangent cone in the previous example.*

For the vertex v_2, we notice that its vertex tangent cone is a unimodular cone, because $|\det \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}| = 1$. *Its integer point transform is:*

$$\begin{aligned}
\sigma_{\mathcal{K}_{v_2}}(z) :&= \sum_{n \in \mathcal{K}_{v_2} \cap \mathbb{Z}^d} e^{\langle n,\, z\rangle} = \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\langle \binom{3}{6} + n_1 \binom{0}{-1} + n_2 \binom{-1}{-2}, z\rangle} \\
&= e^{3z_1+6z_2)} \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{n_1(-z_2)} e^{n_2(-z_1-2z_2)} \\
&= \frac{e^{3z_1+6z_2}}{(1-e^{-z_2})(1-e^{-z_1-2z_2})}.
\end{aligned}$$

Equivalently, using the notation from Example 7.3 above,

$$\sigma_{\mathcal{K}_{v_2}}(z) := \sum_{n \in \mathcal{K}_{v_2} \cap \mathbb{Z}^d} q^n = \frac{q_1^3 q_2^6}{(1-q_2^{-1})(1-q_1^{-1}q_2^{-2})}.$$

For vertex v_3, the computation is similar to vertex tangent cone $\mathcal{K}_{v_1}$, and we have:

$$\begin{aligned}
\sigma_{\mathcal{K}_{v_3}}(z) &:= \sum_{n \in \mathcal{K}_{v_3} \cap \mathbb{Z}^d} e^{\langle n,\, z\rangle} = \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{\left\langle \binom{3}{1} + n_1 \binom{-3}{-1} + n_2 \binom{0}{1},\, z\right\rangle} \\
&= e^{3z_1+z_2} \sum_{n_1 \geqslant 0, n_2 \geqslant 0} e^{(-3z_1-z_2)n_1} e^{2\pi i (z_2) n_2} \\
&= e^{3z_1+z_2} \frac{1+e^{-z_1}+e^{-2z_1}}{(1-e^{3z_1+z_2})(1-e^{z_2})} \\
&= \frac{e^{3z_1+z_2}+e^{2z_1+z_2}+e^{z_1+z_2}}{(1-e^{3z_1+z_2})(1-e^{z_2})} \\
&= \frac{q_1^3 q_2 + q_1^2 q_2 + q_1 q_2}{(1-q_1^{-3}q_2^{-1})(1-q_2)}.
\end{aligned}$$

Finally, putting all of the three vertex tangent cone contributions together, using (7.2), we get:

$$\begin{aligned}
\sigma_{\Delta}(z) &= \sigma_{\mathcal{K}_{v_1}}(z) + \sigma_{\mathcal{K}_{v_2}}(z) + \sigma_{\mathcal{K}_{v_3}}(z) \\
&= \frac{1+q_1q_2+{q_1}^2q_2+{q_1}^2{q_2}^2+q_1^3q_2^2}{(1-q_1^3q_2)(1-q_1q_2^2)} + \frac{q_1^3q_2^6}{(1-q_2^{-1})(1-q_1^{-1}q_2^{-2})} \\
&+ \frac{q_1^3q_2+q_1^2q_2+q_1q_2}{(1-q_1^{-3}q_2^{-1})(1-q_2)}.
\end{aligned}$$

□

As these examples suggest, there is a thread that they all share, namely that their numerators are polynomials that encode the integer points inside a fundamental parallelepiped Π which sits at the vertex of each vertex tangent cone. The proof is fairly easy - we only need to put several geometric series together. Let's formalize this.

First, given any d-dimensional simplicial cone $\mathcal{K} \subset \mathbb{R}^d$, with edge vectors $\omega_1, \ldots, \omega_d$, we define the **fundamental parallelepiped** of the cone $\mathcal{K}$ by:

$$\Pi := \{\lambda_1 \omega_1 + \cdots + \lambda_d \omega_d \mid \text{all } 0 \leqslant \lambda_j < 1\}, \tag{7.11}$$

a half-open parallelepiped. In the same way that we've encoded integer points in polytopes, we can encode the integer points in Π by defining

$$\sigma_\Pi(z) := \sum_{n \in \mathbb{Z}^d \cap \Pi} e^{\langle z, n \rangle}.$$

For rational cones K_v, it turns out that their integer point transforms, defined in (6.37), have a pretty structure theorem (for one proof see M. Beck and Robins (2015)).

Theorem 7.1. *Given a d-dimensional simplicial cone $\mathcal{K}_v \subset \mathbb{R}^d$, with apex $v \in \mathbb{R}^d$, and with d linearly independent integer edge vectors $\omega_1, \ldots, \omega_d \in \mathbb{Z}^d$, we have:*

$$\sum_{n \in \mathcal{K}_v \cap \mathbb{Z}^d} e^{\langle n, z \rangle} = \frac{\sigma_{\Pi + v}(z)}{\prod_{k=1}^d \left(1 - e^{\langle \omega_k, z \rangle}\right)}. \tag{7.12}$$

7.4 The Ehrhart polynomial of an integer polytope

Generalizing Example 7.2, Ehrhart theory shows that for any integer polytope $\mathcal{P} \subset \mathbb{R}^d$, the discrete volume $L_{\mathcal{P}}(t)$ is a polynomial in t, for positive integer values of t.

Theorem 7.2 (Ehrhart)**.** *For an integer polytope $\mathcal{P} \subset \mathbb{R}^d$, its discrete volumes $L_{\mathcal{P}}(t)$ and $L_{\operatorname{int} \mathcal{P}}(t)$ are both polynomials in t, for all positive integer values of the dilation parameter t.*

More generally, Ehrhart also proved that for a rational polytope $\mathcal{P} \subset \mathbb{R}^d$, the discrete volume $L_{\mathcal{P}}(t)$ is a **quasi-polynomial** in the positive integer parameter t, which means by definition that

$$L_{\mathcal{P}}(t) = c_d t^d + c_{d-1}(t) t^{d-1} + \cdots + c_1(t) t + c_0(t), \tag{7.13}$$

where each $c_j(t)$ is a periodic function of $t \in \mathbb{Z}_{>0}$.

Theorem 7.3 (Ehrhart). *For a rational polytope $\mathcal{P} \subset \mathbb{R}^d$, its discrete volumes $L_{\mathcal{P}}(t)$ and $L_{\text{int}\,\mathcal{P}}(t)$ are both quasi-polynomials in t, for all positive integer values of the dilation parameter t.*

For proofs of Theorem 7.2 and Theorem 7.3, see M. Beck and Robins 2015.

7.5 Unimodular polytopes

A d-dimensional integer simplex Δ is called a **unimodular simplex** if Δ is the modular image of the standard simplex Δ_{standard}, the convex hull of the points $\{0, \mathbf{e_1}, \ldots, \mathbf{e_d}\} \subset \mathbb{R}^d$, where $\mathbf{e_k} := (0, \ldots, 0, 1, 0, \ldots, 0)$ is the standard unit vector pointing in the direction of the positive axis x_k.

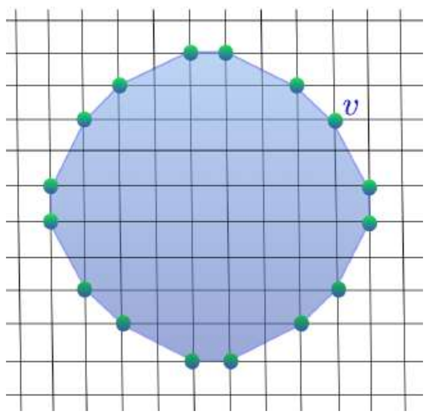


Figure 7.5: A unimodular polygon - each vertex tangent cone is a unimodular cone. It is clear from the construction in the Figure that we can form arbitrarily large unimodular polygons.

Example 7.5. Let $\Delta := \text{conv}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$, their convex hull. Then Δ is a unimodular simplex, because the unimodular matrix $\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$ maps the standard simplex Δ_{standard} to Δ.

It is not difficult to show that the tangent cone of a unimodular simplex possesses edge vectors that form a lattice basis for $\mathbb{Z}^d$. Thus, it is natural to define a

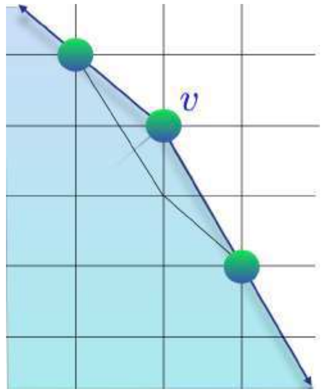


Figure 7.6: A unimodular cone at v, appearing as one of the vertex tangent cones in Figure 7.5. We notice that its half-open fundamental parallelepiped, with vertex at v, does not contain any integer points other than v.

unimodular cone $\mathcal{K} \subset \mathbb{R}^d$ as a simplicial cone, possessing the additional property that its d edge vectors form a lattice basis for $\mathbb{Z}^d$.

Example 7.6. We consider the polygon $\mathcal{P}$ in Figure 7.5. An easy verification shows that each of its vertex tangent cones is unimodular. For example, focusing on the vertex v, we see from Figure 7.6, that its vertex tangent cone is $\mathcal{K}_v := v + \{\lambda_1 \binom{1}{-2} + \lambda_2 \binom{-1}{1} \mid \lambda_1, \lambda_2 \geqslant 0\}$. $\mathcal{K}_v$ is a unimodular cone, because the matrix formed by the its two edges $\binom{1}{-2}$ and $\binom{-1}{1}$ is a unimodular matrix. $\square$

More generally, a simple, integer polytope is called a **unimodular polytope** if each of its vertex tangent cones is a unimodular cone. Unimodular polytopes are the first testing ground for many conjectures in discrete geometry and number theory. Indeed, we will see later that the number of integer points in a unimodular polytope, namely $|\mathbb{Z}^d \cap \mathcal{P}|$, admits a simple and computable formula, if we are given the local tangent cone information at each vertex. By contrast, it is in general thought to be quite difficult to compute the number of integer points $|\mathbb{Z}^d \cap \mathcal{P}|$, even for (general) simple polytopes, a problem that belongs to the NP-hard class of problems.

Lemma 7.2. *Suppose we have two integer polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$, which are unimodular images of each other:*

$$\mathcal{P} = U\mathcal{Q},$$

for some unimodular matrix U. Then $L_{\mathcal{P}}(t) = L_{\mathcal{Q}}(t)$, for all $t \in \mathbb{Z}_{\geqslant 0}$.

7.6 Rational polytopes and quasi-polynomials

The following properties for the floor function, the ceiling function, and the fractional part function may be useful. Often it's very useful to include the following indicator function, for the full set of integers, as well:

$$1_{\mathbb{Z}}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases},$$

the indicator function for $\mathbb{Z}$. For all $x \in \mathbb{R}$, we have:

(a) $\lceil x \rceil = -\lfloor -x \rfloor$

(b) $1_{\mathbb{Z}}(x) = \lfloor x \rfloor - \lceil x \rceil + 1$

(c) $\{x\} + \{-x\} = 1 - 1_{\mathbb{Z}}(x)$

(d) $\lceil x \rceil = x + 1 - \{x\} - 1_{\mathbb{Z}}(x)$

(e) Let $m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}$. Then $\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lceil \frac{n}{m} \right\rceil$.

(Exercise 7.11)

Example 7.7. Let's find the integer point enumerator $L_{\mathcal{P}}(t) := |\mathbb{Z} \cap t\mathcal{P}|$ of the rational line segment $\mathcal{P} := [\frac{1}{3}, 1]$. Proceeding by brute-force, for $t \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} L_{\mathcal{P}}(t) &= \left| \left[\frac{t}{3}, t \right] \cap \mathbb{Z} \right| = \lfloor t \rfloor - \left\lceil \frac{t}{3} \right\rceil + 1 && (7.14) \\ &= t + \left\lfloor -\frac{t}{3} \right\rfloor + 1 && (7.15) \\ &= t + -\frac{t}{3} - \left\{ -\frac{t}{3} \right\} + 1 && (7.16) \\ &= \frac{2}{3}t - \left\{ -\frac{t}{3} \right\} + 1, && (7.17) \end{aligned}$$

a periodic function on $\mathbb{Z}$ with period 3. Here we used property (a) in the third equality. In fact, here we may let t be any positive real number, and we still obtain the same answer, in this 1-dimensional case.

Now we will compare this to a new computation, but this time from the perspective of the vertex tangent cones. For the cone $\mathcal{K}_{tv_1} := [\frac{t}{3}, +\infty)$, we can parametrize the integer points in this cone by $\mathcal{K}_{tv_1} \cap \mathbb{Z} = \{\lceil \frac{t}{3} \rceil, \lceil \frac{t}{3} \rceil + 1, \dots\}$, so that

$$\sigma_{\mathcal{K}_{tv_1}}(z) = e^{\lceil \frac{t}{3} \rceil z} \sum_{n \geqslant 0} e^{nz} = e^{\lceil \frac{t}{3} \rceil z} \frac{1}{1 - e^{z}}.$$

For the cone $\mathcal{K}_{tv_2} := (-\infty, t]$, we can parametrize the integer points in this cone by $\mathcal{K}_{tv_2} \cap \mathbb{Z} = \{t, t-1, \dots\}$, so that

$$\sigma_{\mathcal{K}_{tv_2}}(z) = e^{t \cdot z} \sum_{n \leqslant 0} e^{nz} = e^{tz} \frac{1}{1 - e^{-z}}.$$

So by the discrete Brion Theorem (which is here essentially a finite geometric sum), we get:

$$\begin{aligned}
\sum_{n \in [\frac{t}{3}, t]} e^{nz} &= e^{\lceil \frac{t}{3} \rceil z} \frac{1}{1 - e^{z}} + e^{tz} \frac{1}{1 - e^{-z}} \\
&= -\left(1 + \left\lceil \frac{t}{3} \right\rceil z + \left\lceil \frac{t}{3} \right\rceil^2 \frac{z^2}{2!} + \cdots \right) \left(\frac{1}{z} - \frac{1}{2} + \frac{1}{12} z + \cdots \right) \\
&+ \left(1 + (t+1)z + (t+1)^2 \frac{z^2}{2!} + \cdots \right) \left(\frac{1}{z} - \frac{1}{2} + \frac{1}{12} z + \cdots \right) \\
&= \frac{1}{2} - \left\lceil \frac{t}{3} \right\rceil + (t+1) - \frac{1}{2} + O(z) \longrightarrow t - \left\lceil \frac{t}{3} \right\rceil + 1,
\end{aligned}$$

as $z \to 0$, recovering the same answer 7.14 above. □

Example 7.8. Let's find the integer point enumerator $L_{\mathcal{P}}(t) := |\mathbb{Z}^2 \cap t\mathcal{P}|$ of the rational triangle

$$\mathcal{P} := \operatorname{conv}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right).$$

First we will proceed by brute-force (which does not always work well), and then we will use the machinery of 7.3.

For the brute-force method, we need to consider separately the even integer dilates and the odd integer dilates. Letting $t = 2n$ be a positive even integer, it's clear geometrically that

$$\begin{aligned} L_{\mathcal{P}}(t) &:= |\mathbb{Z}^2 \cap 2n\mathcal{P}| = 1 + 2 + \cdots + n \\ &= \frac{n(n+1)}{2} = \frac{\frac{t}{2}(\frac{t}{2}+1)}{2} \\ &= \frac{1}{8}t^2 + \frac{1}{4}t. \end{aligned}$$

On the other hand, if $t = 2n - 1$, then we notice that we never have an integer point on the diagonal face of $\mathcal{P}$, so that in this case we get:

$$L_{\mathcal{P}}(t) := |\mathbb{Z}^2 \cap (2n-1)\mathcal{P}| = 1+2+\cdots+n = \frac{\frac{t+1}{2}(\frac{t+1}{2}+1)}{2} = \frac{1}{8}t^2 + \frac{1}{2}t + \frac{3}{8}.$$

Alternatively, we may also rederive the same answer by using the Brion identity 7.3. We can proceed as in Example 7.2. The only difference now is that the vertex tangent cones have rational apices, so to parametrize the integer points in $\mathcal{K}_{tv_3} \cap \mathbb{Z}^d$, we can still use the same edge vectors, but now we have the rational vertex $v_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, so that $\{n \in \mathcal{K}_{tv_3} \cap \mathbb{Z}^d\} = \{\begin{pmatrix} 0 \\ \frac{t}{2} \end{pmatrix} + n_1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + n_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid n_1, n_2 \in \mathbb{Z}_{\geqslant 0}\}$. We invite the reader to complete this alternate derivation of the Ehrhart quasi-polynomial $L_{\mathcal{P}}(t)$ in this case. □

7.7 Ehrhart reciprocity

There is a wonderful, and somewhat mysterious, relation between the Ehrhart polynomial of the (closed) polytope $\mathcal{P}$, and the Ehrhart polynomial of its interior, called $\operatorname{int}\mathcal{P}$. We recall our convention that all polytopes are, by definition, closed polytopes. We first compute $L_{\mathcal{P}}(t)$, for positive integers t, and once we have this polynomial in t, we formally replace t by $-t$. So by definition, we form $L_{\mathcal{P}}(-t)$ algebraically, and then embark on a search for its new combinatorial meaning.

Theorem 7.4 (Ehrhart reciprocity). *Given a d-dimensional rational polytope $\mathcal{P} \subset \mathbb{R}^d$, let*

$$L_{\operatorname{int}\mathcal{P}}(t) := |\mathbb{Z}^d \cap \operatorname{int}\mathcal{P}|,$$

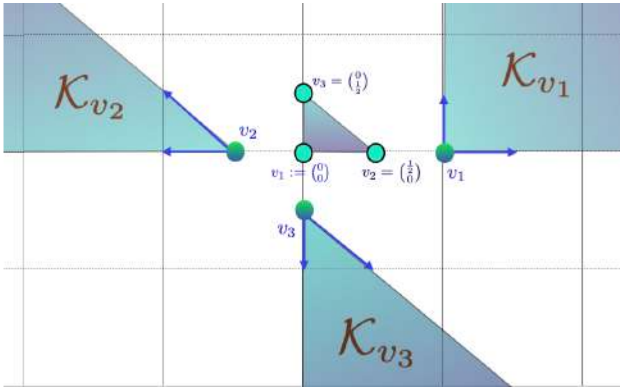


Figure 7.7: A rational triangle, which happens to be a **rational dilate** of the standard simplex.

the integer point enumerator of its interior.

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\operatorname{int}\mathcal{P}}(t), \tag{7.18}$$

for all $t \in \mathbb{Z}$.

Offhand, it seems like 'a kind of magic', and indeed Ehrhart reciprocity is one of the most elegant geometric inclusion–exclusion principles we have. Some examples are in order.

Example 7.9. For the unit cube $\square := [0,1]^d$, we can easily compute $L_\square(t) = (t+1)^d = \sum_{k=0}^{d} \binom{d}{k} t^k$. For the open cube $\operatorname{int}\square$, we can also easily compute

$$\begin{aligned} L_{\operatorname{int}\square}(t) = (t-1)^d &= \sum_{k=0}^{d} \binom{d}{k} t^k (-1)^{d-k} \\ &= (-1)^d \sum_{k=0}^{d} \binom{d}{k} (-t)^k \\ &= (-1)^d L_\square(-t), \end{aligned}$$

using the known computation of $L_\square(t) = (t+1)^d$. $\square$

Example 7.10. For the standard simplex Δ, we consider its t-dilate, given by

$$t\Delta := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{k=1}^{d} x_i \leqslant t, \text{ and all } x_k \geqslant 0\}.$$

We can easily compute its Ehrhart polynomial, by using combinatorics. We need to find the number of nonnegative integer solutions to

$$x_1 + \cdots + x_d \leqslant t,$$

which is equal to $L_\Delta(t)$, for a fixed positive integer t. We can introduce a 'slack variable' z, to transform the latter inequality to an equality: $x_1 + \cdots + x_d + z = t$, where $0 \leqslant z \leqslant t$. By a very classical and pretty argument, (involving placing t balls into urns that are separated by d walls) this number is equal to $\binom{t+d}{d}$ (Exercise 7.12). So we found that

$$L_\Delta = \binom{t+d}{d} = \frac{(t+d)(t+d-1)\cdots(t+1)}{d!}, \tag{7.19}$$

a degree d polynomial, valid for all positive integers t.

What about the interior of Δ? Here we need to find the number of **positive** integer solutions to $x_1 + \cdots + x_d < t$, for each positive integer t. It turns out that by a very similar argument as above (Exercise 7.13), the number of positive integer solutions is $\binom{t-1}{d} = L_{\operatorname{int}\Delta}(t)$. So is it really true that

$$(-1)^d \binom{d-t}{d} = \binom{t-1}{d}\,?$$

Let's compute, substituting $-t$ for t in (7.19) to get:

$$\begin{aligned} L_\Delta(-t) = \binom{-t+d}{d} &= \frac{(-t+d)(-t+d-1)\cdots(-t+1)}{d!} \\ &= (-1)^d \frac{(t-d)(t-d+1)\cdots(t-1)}{d!} \\ &= (-1)^d \binom{t-1}{d} = (-1)^d L_{\operatorname{int}\Delta}(t), \end{aligned}$$

confirming that Ehrhart reciprocity works for the standard simplex as well. □

Notes

(a) Ehrhart theory has a fascinating history, commencing with the fundamental work of Ehrhart (1962), Ehrhart (1967), Ehrhart (1977), in the 1960's. Danilov (1978) made a strong contribution to the field, but after that the field of Ehrhart theory laid more or less dormant, until it was rekindled by Jamie Pommersheim in 1993 J. E. Pommersheim (1993), giving it strong connections to Toric varieties. Alexander Barvinok (1994) gave the first polynomial-time algorithm for counting integer points in polytopes in fixed dimension.

In recent years, Ehrhart theory has enjoyed an enthusiastic renaissance (see the books Barvinok (2008), M. Beck and Robins (2015), and Fulton (1993)). For more relations with combinatorics, the reader may enjoy reading Chapter 4 of the classic book "Enumerative Combinatorics", by R. P. Stanley (2012).

(b) Regarding the computational complexity of counting integer points in polytopes, Alexander Barvinok settled the problem in Barvinok (1994) by showing that for a fixed dimension d, there is a polynomial-time algorithm, as a function of the 'bit capacity' of any given rational polytope $\mathcal{P} \subset \mathbb{R}^d$, for counting the number of integer points in $\mathcal{P}$.

(c) It is also true that for integer polytopes which are not necessarily convex (for example simplicial complexes), the integer point enumerator makes sense as well. In this more general context, the constant term of the corresponding integer point enumerator equals the (reduced) Euler characteristic of the simplicial complex.

(d) For more information about the rapidly expanding field of Euler–MacLaurin summation over polytopes, a brief (and by no means complete) list of paper in this direction consists of the work by Berline and Vergne (2007), Baldoni, Berline, and Vergne (2008), Garoufalidis and J. Pommersheim (2012), Brandolini, Colzani, Travaglini, and Robins Brandolini et al. (2020), Karshon, Sternberg, and Weitsman (Karshon, Sternberg, and Weitsman (2003), Karshon, Sternberg, and Weitsman (2007)), and very recently Fischer and J. Pommersheim (2021).

(e) The trick used in Example 7.2 of picking the particular vector $z := (x, -x)$, which turns out to simplify the computations a lot, is due to Michel Faleiros.

(f) In a future version of this book, we will also delve into Dedekind sums, which arise very naturally when considering the Fourier series of certain rational-exponential functions. To define a general version of these sums, let $\mathcal{L}$ be a d-dimensional lattice in $\mathbb{R}^d$, let $w_1, \ldots, w_d$ be linearly independent vectors from $\mathcal{L}^*$, and let W be a matrix with the w_j's as columns. For any d-tuple $e = (e_1, \ldots, e_d)$ of positive integers, define $|e| := \sum_{j=1}^{k} e_j$. Then, for all $x \in \mathbb{R}^d$, a lattice Dedekind sum is defined by

$$L_{\mathcal{L}}(W, e; x) := \lim_{\epsilon \to 0} \frac{1}{(2\pi i)^{|e|}} \sum_{\substack{\xi \in \mathcal{L} \\ \langle w_j, \xi \rangle \neq 0, \forall j}} \frac{e^{-2\pi i \langle x, \xi \rangle}}{\prod_{j=1}^{k} \langle w_j, \xi \rangle^{e_j}} e^{-\pi \epsilon \|\xi\|^2}. \tag{7.20}$$

Gunnells and Sczech (2003) have an interesting reduction theorem for these sums, giving a polynomial-time complexity algorithm for them, for fixed dimension d.

Exercises

7.1. *Consider the* 1*-dimensional polytope* $\mathcal{P} := [a, b]$*, for any* $a, b \in \mathbb{Z}$*.*

(a) *Show that the Ehrhart polynomial of* $\mathcal{P}$ *is* $L_{\mathcal{P}}(t) = (b - a)t + 1$*.*

(b) *Find the Ehrhart quasi-polynomial* $L_{\mathcal{P}}(t)$ *for the rational segment* $\mathcal{Q} := [\frac{1}{3}, \frac{1}{2}]$*.*

7.2. *We recall that the* d*-dimensional cross-polytope was defined by*

$$\Diamond := \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid |x_1| + |x_2| + \cdots + |x_d| \leqslant 1 \right\}.$$

(a) *For* $d = 2$*, find the Ehrhart polynomial* $L_{\Diamond}(t)$*.*

(b) *For* $d = 3$*, find the Ehrhart polynomial* $L_{\Diamond}(t)$*.*

7.3. *Using the same notation for the* d*-dimensional cross-polytope* $\Diamond$ *as above, compute the Ehrhart polynomial* $L_{\Diamond}(t)$ *in* $\mathbb{R}^d$*.*

7.4. *Let $d = 2$, and consider the cross-polytope $\Diamond \subset \mathbb{R}^2$. Find the Ehrhart quasi-polynomial $L_{\mathcal{P}}(t)$ for the rational polygon $\mathcal{P} := \frac{1}{2}\Diamond$.*

7.5. *Suppose Δ is the standard simplex in $\mathbb{R}^d$. Show that the first d dilations of Δ do not contain any integer points in their interior:*

$$t(\operatorname{int}\Delta) \cap \mathbb{Z}^d = \phi,$$

for $t = 1, 2, \ldots, d$. In other words, show that

$$L_{\operatorname{int}\mathcal{P}}(1) = L_{\operatorname{int}\mathcal{P}}(2) = \cdots = L_{\operatorname{int}\mathcal{P}}(d) = 0.$$

Conclude that the same statement is true for any unimodular simplex.

7.6. *Here we show that the Bernoulli polynomial $B_d(t)$, is essentially equal to the Ehrhart polynomial $L_{\mathcal{P}}(t)$ for the "Pyramid over a cube" (as defined in Exercise 6.4). We recall the definition: let $C := [0,1]^{d-1}$ be the $d-1$-dimensional cube, considered as a subset of $\mathbb{R}^d$, and let $\mathbf{e_d}$ be the unit vector pointing in the x_d-direction. Now we define $\mathcal{P} := \operatorname{conv}\{C, \mathbf{e_d}\}$, a pyramid over the unit cube. Show that its Ehrhart polynomial is*

$$L_{\mathcal{P}}(t) = \frac{1}{d}(B_d(t+2) - B_d),$$

for $t \in \mathbb{Z}_{>0}$.

7.7. *For any integer d-dimensional (convex) polytope $\mathcal{P} \subset \mathbb{R}^d$, show that*

$$\operatorname{vol}\mathcal{P} = \frac{(-1)^d}{d!}\left(1 + \sum_{k=1}^{d}\binom{d}{k}(-1)^k L_{\mathcal{P}}(k)\right), \tag{7.21}$$

which can be thought of as a generalization of Pick's formula to $\mathbb{R}^d$.

Note. *Using iterations of the forward difference operator $\Delta f(n) := f(n+1) - f(n)$, the latter identity may be thought of a* ***discrete analogue*** *of the d'th derivative of the Ehrhart polynomial. This idea in fact gives another method of proving* (7.21).

7.8. *Show that the convolution of the indicator function $1_{\mathcal{P}}$ with the heat kernel G_ϵ, as in equation (8.7), is a Schwartz function.*

7.9. *Consulting Figure 7.5:*

(a) *Find the integer point transform of the unimodular polygon in the Figure.*

(b) *Find the Ehrhart polynomial $L_{\mathcal{P}}(t)$ of the integer polygon $\mathcal{P}$ from part (a).*

7.10. ♣ *Show that (8.2) is equivalent to the following definition, using balls instead of spheres. Recall that the unit ball in $\mathbb{R}^d$ is define by $B^d := \{x \in \mathbb{R}^d \mid \|x\| \leqslant 1\}$, and similarly the ball of radius ϵ, centered at $x \in \mathbb{R}^d$, is denoted by $B^d(x,\epsilon)$. Show that for all sufficiently small ϵ, we have*

$$\frac{\mathrm{vol}(S^{d-1}(x,\epsilon) \cap \mathcal{P})}{\mathrm{vol}(S^{d-1}(x,\epsilon))} = \frac{\mathrm{vol}(B^d(x,\epsilon) \cap \mathcal{P})}{\mathrm{vol}(B^d(x,\epsilon))}.$$

7.11. *Here we gain some practice with 'floors', 'ceilings', and 'fractional parts'. First, we recall that by definition, the fractional part of any real number x is $\{x\} := x - \lfloor x \rfloor$. Next, we recall the indicator function of $\mathbb{Z}$, defined by: $1_{\mathbb{Z}}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$.*

Show that:

(a) $\lceil x \rceil = -\lfloor -x \rfloor$

(b) $1_{\mathbb{Z}}(x) = \lfloor x \rfloor - \lceil x \rceil + 1$

(c) $\{x\} + \{-x\} = 1 - 1_{\mathbb{Z}}(x)$

(d) $\lceil x \rceil = x + 1 - \{x\} - 1_{\mathbb{Z}}(x)$

(e) *Let $m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}$. Then* $\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lceil \frac{n}{m} \right\rceil$.

7.12. ♣ *Show that the number of nonnegative integer solutions $x_1, \ldots, x_d, z \in \mathbb{Z}_{\geqslant 0}$ to*

$$x_1 + \cdots + x_d + z = t,$$

with $0 \leqslant z \leqslant t$, equals $\binom{t+d}{d}$.

7.13. ♣ *Show that for each positive integer t, the number of* **positive** *integer solutions to $x_1 + \cdots + x_d < t$ is equal to $\binom{t-1}{d}$.*

7.14. *We define the rational triangle whose vertices are*

$$(0,0), (1, \frac{N-1}{N}), (N, 0),$$

where $N \geqslant 2$ is a fixed integer. Prove that the Ehrhart quasi-polynomial is in this case

$$L_{\mathcal{P}}(t) = \frac{p-1}{2}t^2 + \frac{p+1}{2}t + 1,$$

for all $t \in \mathbb{Z}_{>}0$.

Notes. *So we see here a phenomenon known as 'period collapse', where we expect a quasi-polynomial behavior, with some nontrivial period, but in fact we observe a strict polynomial.*

7.15. *Here we show that the Ehrhart polynomial $L_{\mathcal{P}}(t)$ remains invariant under the full unimodular group $SL_d(\mathbb{Z})$. In particular, show that:*

(a) *Every element of $SL_d(\mathbb{Z})$ acts on the integer lattice $\mathbb{Z}^d$ bijectively.*

(b) *Let $\mathcal{P}$ be an integral polytope, and let $Q := M(\mathcal{P})$, where $M \in SL_d(\mathbb{Z})$. Thus, by definition $\mathcal{P}$ and Q are unimodular images of each other. Prove that*

$$L_{\mathcal{P}}(t) = L_Q(t),$$

for all $t \in \mathbb{Z}_{>0}$.

(c) *Is the converse of part (b) true?*

8 The angle polynomial of a polytope

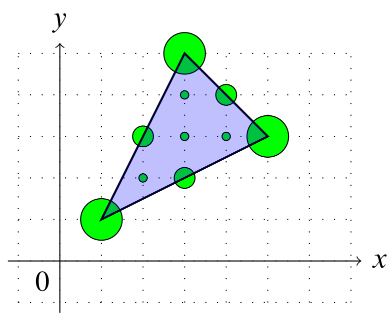


Figure 8.1: A different discrete volume, called the angle polynomial of a polytope $\mathcal{P}$. Here we sum local angle weights, relative to $\mathcal{P}$, at all integer points.

8.1 Intuition

There are infinitely many ways to discretize the classical notion of volume, and here we offer a second path, using 'local solid angles'. Given a rational polytope $\mathcal{P}$, we will place small spheres at all integer points in $\mathbb{Z}^d$, and compute the proportion of the local intersection of each small sphere with $\mathcal{P}$. This discrete, finite sum, gives us a new method of discretizing the volume of a polytope, and it turns out to be a more symmetric way of doing so. To go forward, we first discuss how to extend the usual notion of 'angle' to higher dimensions, and then use Poisson summation again to pursue the fine detail of this new discrete volume.

8.2 What is an angle in higher dimensions?

The question of how an angle in two dimensions extends to higher dimensions is a basic one in discrete geometry. A natural way to extend the notion of an angle is to consider a cone $\mathcal{K} \subset \mathbb{R}^d$, place a sphere centered at the apex of $\mathcal{K}$, and then compute the proportion of the sphere that intersects $\mathcal{K}$. This intuition is captured more rigorously by the following integral:

$$\omega_{\mathcal{K}} = \int_{\mathcal{K}} e^{-\pi \|x\|^2} dx. \tag{8.1}$$

called the **solid angle of the cone** $\mathcal{K}$.

The literature contains other synonyms for solid angles, arising in different fields, including the **volumetric moduli** Gourion and Seeger (2010), and the **volume of a spherical polytope** M. Beck and Robins (2015), Desario and Robins (2011), Diaz, Le, and Robins (2016).

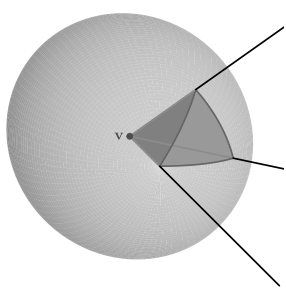


Figure 8.2: A solid angle in $\mathbb{R}^3$ - note the equivalence with the area of the geodesic triangle on the sphere.

We can easily show that the latter definition of a solid angle is equivalent to the volume of a spherical polytope, using polar coordinates in $\mathbb{R}^d$, as follows. We denote the unit sphere by

$$S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\}.$$

Using the fact that the Gaussians give a probability distribution, namely $\int_{\mathbb{R}^d} e^{-\pi||x||^2} dx = 1$ (which we know by Exercise 3.19), we have

$$\begin{aligned}
\omega_{\mathcal{K}} &= \frac{\int_{\mathcal{K}} e^{-\pi\|x\|^2} dx}{\int_{\mathbb{R}^d} e^{-\pi\|x\|^2} dx} = \frac{\int_0^\infty e^{-\pi r^2} r^{d-1} dr \int_{S^{d-1}\cap\mathcal{K}} d\theta}{\int_0^\infty e^{-\pi r^2} r^{d-1} dr \int_{S^{d-1}} d\theta} \\
&= \frac{\int_{S^{d-1}\cap\mathcal{K}} d\theta}{\int_{S^{d-1}} d\theta} \\
&= \frac{\operatorname{vol}\left(\mathcal{K} \cap S^{d-1}\right)}{\operatorname{vol}\left(S^{d-1}\right)},
\end{aligned}$$

the normalized volume of a spherical polytope defined by the intersection of the cone $\mathcal{K}$ with the unit sphere. We used polar coordinates in the second equality above: $x = (r, \theta)$, with $r \geqslant 0$, $\theta \in \mathcal{S}^{d-1}$. The Jacobian in the change of variables is $dx = r^{d-1} dr d\theta$.

We note that when K is replaced by all of Euclidean space, the integral 8.1 becomes $\int_{\mathbb{R}^d} e^{-\pi||x||^2} dx = 1$, confirming that we do indeed have the proper normalization with $\omega_{\mathcal{K}} = 1$ if and only if $\mathcal{K} = \mathbb{R}^d$.

Example 8.1. If $\mathcal{K}$ is a half-space, then $\omega_{\mathcal{K}} = \frac{1}{2}$. If $\mathcal{K} := \mathbb{R}^d_{\geqslant 0}$, the positive orthant, then

$$\omega_{\mathcal{K}} = \int_{\mathbb{R}^d_{\geqslant 0}} e^{-\pi||x||^2} dx = \left(\int_{\mathbb{R}_{\geqslant 0}} e^{-\pi u^2} du \right)^d = \frac{1}{2^d}.$$

□

Next, given any polytope $\mathcal{P} \subset \mathbb{R}^d$, we can define a local solid angle relative to $\mathcal{P}$, at any point $x \in \mathbb{R}^d$. The normalized **solid angle** fraction that a d-dimensional polytope $\mathcal{P}$ subtends at any point $x \in \mathbb{R}^d$ is defined by

$$\omega_{\mathcal{P}}(x) = \lim_{\epsilon \to 0} \frac{\operatorname{vol}(S^{d-1}(x, \epsilon) \cap \mathcal{P})}{\operatorname{vol}(S^{d-1}(x, \epsilon))}. \tag{8.2}$$

Here, $\omega_{\mathcal{P}}(x)$ measures the fraction of a small $(d-1)$-dimensional sphere $S^{d-1}(x,\epsilon)$ centered at x, that intersects the polytope $\mathcal{P}$. We will use the standard notation for the interior of a convex body, namely $\operatorname{int}(\mathcal{P})$, and for the boundary of a convex body, namely $\partial\mathcal{P}$. As a sidenote, we mention that balls and spheres can be used interchangeably in this definition, meaning that the fractional weight given by (8.2) is the same using either method (see Exercise 7.10).

It follows from the definition of a solid angle that $0 \leqslant \omega_{\mathcal{P}}(x) \leqslant 1$, for all $x \in \mathbb{R}^d$, and that

$$\omega_{\mathcal{P}}(x) = \begin{cases} 1 & \text{if } x \in \operatorname{int}(\mathcal{P}) \\ 0 & \text{if } x \notin \mathcal{P}. \end{cases}$$

But when $x \in \partial\mathcal{P}$, we have $\omega_{\mathcal{P}}(x) > 0$. For example, if x lies in a codimension two face of $\mathcal{P}$, then $\omega_{\mathcal{P}}(x)$ is the fractional dihedral angle subtended by $\mathcal{P}$ at x.

Returning to discrete volumes, Ehrhart and Macdonald analyzed a different discrete volume for any polytope $\mathcal{P}$. Namely, for each positive integer t, define the finite sum

$$A_{\mathcal{P}}(t) := \sum_{n\in\mathbb{Z}^d} \omega_{t\mathcal{P}}(n), \tag{8.3}$$

where $t\mathcal{P}$ is the t'th dilation of the polytope $\mathcal{P}$. In other words, $A_{\mathcal{P}}(1)$ is a new discrete volume for $\mathcal{P}$, obtained by placing at each integer point $n \in \mathbb{Z}^d$ the weight $\omega_{t\mathcal{P}}(x)$, and summing all of the weights.

Example 8.2. In Figure 8.1, the solid angle sum of the polygon Δ is

$$A_{\Diamond}(1) = \theta_1 + \theta_2 + \theta_3 + 3\frac{1}{2} + 4 = 6.$$

Here the θ_j's are the three angles at the vertices of Δ. □

Using purely combinatorial methods, Macdonald showed that for any integer polytope $\mathcal{P}$, and for **positive integer values** of t,

$$A_{\mathcal{P}}(t) = (\operatorname{vol}\mathcal{P})t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + \cdots + \begin{cases} a_1 t & \text{if } d \text{ is odd,} \\ a_2 t^2 & \text{if } d \text{ is even.} \end{cases} \tag{8.4}$$

We will call $A_{\mathcal{P}}(t)$ the **angle-polynomial** of $\mathcal{P}$, for integer polytopes $\mathcal{P}$ and positive integer dilations t. However, when these restrictions are lifted, the sum still captures crucial geometric information of $\mathcal{P}$, and we will simply call it the (solid) angle-sum of $\mathcal{P}$.

We define the **heat kernel**, for each fixed positive ϵ, by

$$G_\epsilon(x) := \epsilon^{-\frac{d}{2}} e^{-\frac{\pi}{\epsilon}\|x\|^2}, \tag{8.5}$$

for all $x \in \mathbb{R}^d$. By Exercises 3.19 and 3.20, we know that $\int_{\mathbb{R}^d} G_\epsilon(x)dx = 1$ for each fixed ϵ, and that

$$\hat{G}_\epsilon(\xi) = e^{-\epsilon\pi\|\xi\|^2}. \tag{8.6}$$

The convolution of the indicator function $1_{\mathcal{P}}$ by the heat kernel G_ϵ will be called the **Gaussian smoothing** of $1_{\mathcal{P}}$:

$$(1_{\mathcal{P}} * G_\epsilon)(x) := \int_{\mathbb{R}^d} 1_{\mathcal{P}}(y)G_\epsilon(x-y)dy = \int_{\mathcal{P}} G_\epsilon(y-x)dy \tag{8.7}$$

$$= \epsilon^{-\frac{d}{2}} \int_{\mathcal{P}} e^{-\frac{\pi}{\epsilon}\|y-x\|^2} dy, \tag{8.8}$$

a C^∞ function of $x \in \mathbb{R}^d$, and in fact a Schwartz function (Exercise 7.8). The following Lemma provides a first crucial link between the discrete geometry of a local solid angle and the convolution of $1_{\mathcal{P}}$ with a Gaussian-based approximate identity.

Lemma 8.1. *Let $\mathcal{P}$ be a full-dimensional polytope in $\mathbb{R}^d$. Then for each point $x \in \mathbb{R}^d$, we have*

$$\lim_{\epsilon\to 0}(1_{\mathcal{P}} * G_\epsilon)(x) = \omega_P(x). \tag{8.9}$$

Proof. We have

$$\begin{aligned}(1_{\mathcal{P}} * G_\epsilon)(x) &= \int_{\mathcal{P}} G_\epsilon(y-x)dy \\ &= \int_{u\in P-x} G_\epsilon(u)du = \int_{\frac{1}{\sqrt{\epsilon}}(P-x)} G_1(v)dv.\end{aligned}$$

In the calculation above, we make use of the evenness of G_ϵ in the second equality. The substitutions $u = y - x$ and $v = u/\sqrt{\epsilon}$ are also used. Following those substitutions, we change the domain of integration from P to the translation $P - x$, and to the dilation of $P - x$ by the factor $\frac{1}{\sqrt{\epsilon}}$.

Now, when ϵ approaches 0, $\frac{1}{\sqrt{\epsilon}}(P - x)$ tends to the cone K at the origin, subtended by $P - x$. The cone K is in fact a translation of the tangent cone of P at x. Thus, we arrive at

$$\lim_{\epsilon \to 0} (1_{\mathcal{P}} * G_\epsilon)(x) = \int_K G_1(v) dv = \omega_K(0) = \omega_P(x).$$

$\square$

Putting things together, the Equation (8.3) and Lemma 8.1 above tell us that

$$A_{\mathcal{P}}(t) = \sum_{n \in \mathbb{Z}^d} \omega_{tP}(x) = \sum_{n \in \mathbb{Z}^d} \lim_{\epsilon \to 0} (1_{t\mathcal{P}} * G_\epsilon)(n). \tag{8.10}$$

We would like to interchange a limit with an infinite sum over a lattice, so that we may use Poisson summation, and although this is subtle in general, it's possible to carry out here, because the summands are rapidly decreasing.

Lemma 8.2. *Let $\mathcal{P}$ be a full-dimensional polytope in $\mathbb{R}^d$. Then*

$$A_{\mathcal{P}}(t) = \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{t\mathcal{P}} * G_\epsilon)(n). \tag{8.11}$$

Proof. Exercise 8.11. $\square$

We now apply the Poisson summation formula to the Schwartz function $f(x) := (1_{\mathcal{P}} * G_\epsilon)(x)$:

$$\begin{aligned}
A_P(t) &= \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_{t\mathcal{P}} * G_\epsilon)(n) && (8.12)\\
&= \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{t\mathcal{P}}(\xi) \hat{G}_\epsilon(\xi) && (8.13)\\
&= \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{t\mathcal{P}}(\xi)\, e^{-\epsilon\pi \|\xi\|^2} && (8.14)\\
&= t^d \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\mathcal{P}}(t\xi)\, e^{-\epsilon\pi \|\xi\|^2} && (8.15)\\
&= t^d\, \hat{1}_{\mathcal{P}}(0) + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_{\mathcal{P}}(t\xi)\, e^{-\epsilon\pi \|\xi\|^2} && (8.16)\\
&= t^d (\operatorname{vol} \mathcal{P}) + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_{\mathcal{P}}(t\xi)\, e^{-\epsilon\pi \|\xi\|^2}, && (8.17)
\end{aligned}$$

where we used the fact that Fourier transforms interact nicely with linear transformations of the domain:

$$\hat{1}_{t\mathcal{P}}(\xi) = \int_{t\mathcal{P}} e^{-2\pi i\langle \xi, x\rangle} dx = t^d \int_{\mathcal{P}} e^{-2\pi i \langle \xi, ty\rangle} dy =$$
$$= t^d \int_{\mathcal{P}} e^{-2\pi i \langle t\xi, y\rangle} dy = t^d \hat{1}_{\mathcal{P}}(t\xi).$$

We also used the simple change of variable $x = ty$, with $y \in \mathcal{P}$, implying that $dx = t^d dy$, as well as the Fourier transform formula for the heat kernel (8.6).

Altogether, we now have:

$$A_{\mathcal{P}}(t) = t^d(\operatorname{vol}\mathcal{P}) + t^d \lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d - \{0\}} (\hat{1}_{\mathcal{P}}(t\xi) * G_\epsilon)(n), \tag{8.18}$$

suggesting a polynomial-like behavior for the angle polynomial $A_{\mathcal{P}}(t)$. To prove that for integral values of t we do indeed have a polynomial in t, we may use the following useful little relation between solid angle sums and integer point sums. We recall that for any polytope $\mathcal{F}$, the integer point enumerator for the relative interior of $\mathcal{F}$ was defined by $L_{\operatorname{int}\mathcal{F}}(t) := |\mathbb{Z}^d \cap \operatorname{int}\mathcal{F}|$.

For each face $\mathcal{F} \subseteq \mathcal{P}$, we also define the d-dimensional **solid angle of the face** $\mathcal{F}$ by picking any point x inside the relative interior of $\mathcal{F}$ and denoting

$$\omega_{\mathcal{P}}(\mathcal{F}) := \omega_{\mathcal{P}}(x).$$

Lemma 8.3. *Let $\mathcal{P}$ be a d-dimensional polytope in $\mathbb{R}^d$. Then we have*

$$A_{\mathcal{P}}(t) = \sum_{\mathcal{F} \subseteq \mathcal{P}} \omega_{\mathcal{P}}(\mathcal{F}) L_{\operatorname{int}\mathcal{F}}(t). \tag{8.19}$$

Proof. The polytope $\mathcal{P}$ is the disjoint union of its relatively open faces $\mathcal{F} \subseteq \mathcal{P}$, and similarly the dilated polytope $t\mathcal{P}$ is the disjoint union of its relatively open faces $t\mathcal{F} \subseteq t\mathcal{P}$. We therefore have:

$$A_{\mathcal{P}}(t) = \sum_{n \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(n) = \sum_{\mathcal{F} \subseteq \mathcal{P}} \sum_{n \in \mathbb{Z}^d} \omega_{t\mathcal{P}}(n) 1_{\operatorname{int}(t\mathcal{F})}(n).$$

But by definition each $\omega_{t\mathcal{P}}(n)$ is constant on the relatively open face $\operatorname{int}(t\mathcal{F})$ of $t\mathcal{P}$, and we denoted this constant by $\omega_{\mathcal{P}}(\mathcal{F})$. Altogether, we have:

$$A_{\mathcal{P}}(t) = \sum_{\mathcal{F} \subseteq \mathcal{P}} \omega_{\mathcal{P}}(\mathcal{F}) \sum_{n \in \mathbb{Z}^d} 1_{\operatorname{int}(t\mathcal{F})}(n) := \sum_{\mathcal{F} \subseteq \mathcal{P}} \omega_{\mathcal{P}}(\mathcal{F}) L_{\operatorname{int}\mathcal{F}}(t).$$

□

As a quick application, we can prove that the angle polynomial $A_{\mathcal{P}}(t)$ is indeed a polynomial, for positive integer values of t.

Theorem 8.1. *Given an integer polytope $\mathcal{P} \subset \mathbb{R}^d$, the discrete volume $A_{\mathcal{P}}(t)$ is a polynomial in t, for integer values of the dilation parameter t.*

Proof. By Ehrhart's Theorem 7.2, we know that for each face $\mathcal{F} \subseteq \mathcal{P}$, $L_{\text{int}\,\mathcal{F}}(t)$ is a polynomial function of t, for positive integers t. By Lemma Lemma 8.3, we see that $A_{\mathcal{P}}(t)$ is a finite linear combination of polynomials, with constant coefficients, and is therefore a polynomial in t. □

The next step will be to use our knowledge of the Fourier transform of the polytope $\mathcal{P}$, on the right-hand-side of (8.18), for which even a 1-dimensional example is interesting.

Example 8.3. Let's compute the angle polynomial of the 1-dim'l polytope $\mathcal{P} := [a, b]$, with $a, b \in \mathbb{R}$. We will use our knowledge of the 1-dimensional Fourier transform of an interval, from Exercise 2.1, to compute:

$$A_P(t) = (b-a)t + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \hat{1}_{\mathcal{P}}(t\xi)\, e^{-\epsilon\pi\xi^2} \tag{8.20}$$

$$= (b-a)t + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \left(\frac{e^{-2\pi i t\xi b} - e^{-2\pi i t\xi a}}{-2\pi i \xi} \right) e^{-\epsilon\pi\xi^2} \tag{8.21}$$

$$= (b-a)t + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \frac{e^{-2\pi i t b\xi - \epsilon\pi\xi^2}}{-2\pi i \xi} - \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \frac{e^{-2\pi i t a\xi - \epsilon\pi\xi^2}}{-2\pi i \xi} \tag{8.22}$$

Throughout this example, all series converge absolutely (and quite rapidly) due to the existence of the Gaussian damping factor $e^{-\epsilon\pi\xi^2}$. Let's see what happens when we specialize the vertices a or b. Perhaps we can solve for these new limits?

case 1. $a, b \in \mathbb{Z}$. This is the case of an integer polytope, which in this case is an interval in $\mathbb{R}^1$. Because we are restricting attention to integer dilates t, and since $a, b, \xi \in \mathbb{Z}$, we have $e^{-2\pi i t\xi b} = e^{-2\pi i t\xi a} = 1$. Therefore

$$A_P(t) = (b-a)t + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \left(\frac{e^{-2\pi i t\xi b} - e^{-2\pi i t\xi a}}{-2\pi i \xi} \right) e^{-\epsilon\pi\xi^2} \tag{8.23}$$

$$= (b-a)t + 0. \tag{8.24}$$

We arrive at

$$A_P(t) = (b-a)t,$$

so that the solid angle sum $A_P(1)$ is exactly the length of the interval we considered. We may compare this discrete volume with the other discrete volume, namely the Ehrhart polynomial of this interval: $L_{[a,b]}(t) = (b-a)t + 1$.

case 2. $a = 0, b \notin \mathbb{Z}$. Here one of the two series in (8.22) is:

$$\sum_{\xi \in \mathbb{Z}-\{0\}} \frac{e^{-2\pi i t a\xi - \epsilon\pi\xi^2}}{-2\pi i \xi} = \sum_{\xi \in \mathbb{Z}-\{0\}} \frac{e^{-\epsilon\pi\xi^2}}{-2\pi i \xi} = 0,$$

because the summand is an odd function of ξ. But we already know by direct computation that in this case $A_{[0,b]}(t) = \frac{1}{2} + \lfloor bt \rfloor$, we can solve for the other limit:

$$\frac{1}{2} + \lfloor bt \rfloor = bt + \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \left(\frac{e^{-2\pi i t\xi b}}{-2\pi i \xi} \right) e^{-\epsilon\pi\xi^2}$$

So this simple example has given us a nice theoretical result. We record this rigorous proof above as Lemma 8.4 below, after relabelling $bt := x \in \mathbb{R}$. □

Lemma 8.4. *For any $x \in \mathbb{R}$, we have*

$$\frac{1}{2\pi i} \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}-\{0\}} \frac{e^{-2\pi i x\xi - \epsilon\pi\xi^2}}{\xi} = x - \lfloor x \rfloor - \frac{1}{2}.$$

Theorem 8.2. *Let $\mathcal{P}$ be an integer polygon. Then the angle polynomial of $\mathcal{P}$ is:*

$$A_{\mathcal{P}}(t) = (\text{area}\mathcal{P})\text{t}^2,$$

for all positive integer dilations t.

It turns out that this result, for $A_{\mathcal{P}}(1)$, is easily equivalent to the well-known Pick's formula for an integer polygon.

Theorem 8.3 (Pick's formula, 1899)**.** *Let $\mathcal{P}$ be an integer polygon. Then*

$$\text{area}\mathcal{P} = \mathrm{I} + \frac{1}{2}\mathrm{B} - 1,$$

where I is the number of interior integer points in $\mathcal{P}$, and B is the number of boundary integer points in $\mathcal{P}$.

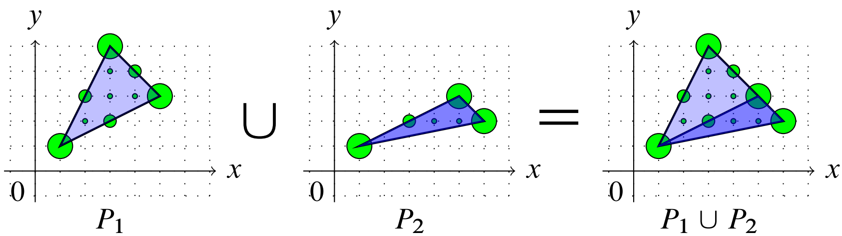


Figure 8.3: Additive property of the angle polynomial

There is also a way to characterize the polytopes that k-tile $\mathbb{R}^d$ by translations, using solid angle sums. In [Gravin, Robins, and Shiryaev (2012, Theorem 6.1)] we have the following characterization.

Theorem 8.4. *A polytope P k-tiles $\mathbb{R}^d$ by integer translations if and only if*

$$\sum_{\lambda \in \mathbb{Z}^d} \omega_{P+v}(\lambda) = k,$$

for every $v \in \mathbb{R}^d$.

8.3 The Gram relations for solid angles

How does our elementary school identity, giving us the sum of the angles of a triangle, extend to higher dimensions? We describe the extension here, mainly due to Gram. First, for each face F of a polytope $\mathcal{P} \subset \mathbb{R}^d$, we define the **solid angle of** F, as follows. Fix any $x_0 \in \operatorname{int} F$, and let

$$\omega_F := \omega_P(x_0).$$

We notice that this definition is independent of x_0, as long as we restrict x_0 to the relative interior of F.

Example 8.4. *If $\mathcal{P}$ is the d-dimensional cube $[0,1]^d$, then each of its facets F has $\omega_F = \frac{1}{2}$. However, it is a fact that for the cube, a face of dimension k has a solid angle of $\frac{1}{2^{d-k}}$ (Exercise 8.9). In particular a vertex v of this cube, having dimension 0, has solid angle $\omega_v = \frac{1}{2^d}$.* □

Theorem 8.5 (Gram relations)**.** *Given any d-dimensional polytope $P \subset \mathbb{R}^d$, we have*

$$\sum_{F \subset \mathcal{P}} (-1)^{\dim F} \omega_F = 0.$$

Proof. □

Example 8.5. Let's see what the Gram relations tell us in the case of a triangle Δ. For each edge E of Δ, placing a small sphere at a point in the interior of E means half of it is inside Δ and half of it is outside of Δ, so that $\omega_E = \frac{1}{2}$. Next, each vertex of Δ has a solid angle equal to the usual (normalized) angle $\theta(v)$ at that vertex. Finally Δ itself has a solid angle of 1, because picking a point p in the interior of Δ, and placing a small sphere centered at p, the whole sphere will be contained in Δ. Putting it all together, the Gram relations read:

$$\begin{aligned}
0 &= \sum_{F \subset \Delta} (-1)^{\dim F} \omega_F \\
&= (-1)^0(\theta(v_1) + \theta(v_2) + \theta(v_3)) + (-1)^1 \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) + (-1)^2 \cdot 1 \\
&= \theta(v_1) + \theta(v_2) + \theta(v_3) - \frac{1}{2},
\end{aligned}$$

which looks familiar! We've retrieved our elementary school knowledge, namely that the three angles of a triangle sum to π radians. So the Gram relations really are an extension of this fact. □

What about $\mathbb{R}^3$? Another example is in order.

Example 8.6. Let's see what hidden secrets lie behind the Gram relations for the standard simplex $\Delta \subset \mathbb{R}^3$. At the origin $v_0 = 0$, the tangent cone is the positive orthant, so that $\omega(v_0) = \frac{1}{8}$. The other 3 vertices all "look alike", in the sense that their tangent cones are all isometric, and hence have the same solid angle ω_v. What about the edges? In general, it's a fact that the solid angle of an edge equals the dihedral angle between the planes of its two bounding facets (Exercise 8.10). There are two types of edges here, as in the figure. For an edge E which lies on the boundary of the skew facet, we have the dihedral angle $\cos\phi = \langle \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\rangle = \frac{1}{\sqrt{3}}$, so that $\omega_E = \phi = \cos^{-1}\frac{1}{\sqrt{3}}$. It's straightforward that for the other type of edge, each of those 3 edges has a solid angle of $\frac{1}{4}$. Putting it all together, we see that

$$\begin{aligned} 0 &= \sum_{F \subset \Delta} (-1)^{\dim F}\omega_F \\ &= (-1)^0\left(\frac{1}{8} + 3\omega_v\right) + (-1)^1\left(3\frac{1}{4} + 3\cos^{-1}\frac{1}{\sqrt{3}}\right) + (-1)^2\frac{1}{2}\cdot 4 + (-1)^3\cdot 1. \end{aligned}$$

Solving for ω_v, we get $\omega_v = \cos^{-1}\frac{1}{\sqrt{3}} - \frac{1}{8}$. So we were able to compute the solid angle of at a vertex of Δ in $\mathbb{R}^3$, using the Gram relations, together with a bit of symmetry. □

Notes

(a) Let's compare and contrast the two notions of discrete volumes that we have encountered so far. For a given rational polytope $\mathcal{P}$, we notice that the Ehrhart quasi-polynomial $L_{\mathcal{P}}(t)$ is invariant when we map $\mathcal{P}$ to any of its unimodular images. That is, any rational polytope in the whole orbit of the unimodular group $\mathrm{SL_d}(\mathbb{Z})(\mathcal{P})$ has the same discrete volume $L_{\mathcal{P}}(t)$. This is false for the second discrete volume $A_{\mathcal{P}}(t)$ - it is not invariant under the modular group (Exercise 8.8). But $A_{\mathcal{P}}(t)$ is invariant under the large finite group of the isometries of $\mathbb{R}^d$ that preserve the integer lattice (known as the hyperoctahedral group).

So we see that $A_{\mathcal{P}}(t)$ is more sensitive to the particular embedding of $\mathcal{P}$ in space, and it can therefore distinguish between more rational polytopes. It also has the advantage of being a much more symmetric polynomial, with half as many coefficients that occur in the Ehrhart polynomial of integer polytopes.

However, $L_{\mathcal{P}}(t)$ has its advantages as well - to compute any *local summand* for $A_{\mathcal{P}}(t) := \sum_{n\in\mathbb{Z}^d} \omega_{tP}(x)$ requires finding the volume of a local spherical polytope, while to compute a *local summand* for $L_{\mathcal{P}}(t) := \sum_{n\in\mathbb{Z}^d} 1$ is quite easy: it is equal to 1.

But as we have seen, computing the full global sum for $A_{\mathcal{P}}(t)$ turns out to have its own simplifications and symmetries.

Exercises

8.1. *Let* $\mathcal{K} = \{\lambda_1 \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) + \lambda_2 \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right) + \lambda_3 \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \mid \lambda_1, \lambda_2, \lambda_3 \geqslant 0\}$, *a simplicial cone. Show that the solid angle of* $\mathcal{K}$ *is* $\omega_{\mathcal{K}} = \frac{1}{48}$.

8.2. *We recall the 2-dimensional cross-polytope*

$$\Diamond := \left\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| \leqslant 1\right\}.$$

Find, from first principles, the angle quasi-polynomial for the rational polygon $\mathcal{P} := \frac{1}{3}\Diamond$, *for all integer dilations of* $\mathcal{P}$.

8.3. *We recall that the 3-dimensional cross-polytope was defined by*

$$\Diamond := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1| + |x_2| + |x_3| \leqslant 1\right\}.$$

Compute the angle polynomial of $A_{\Diamond}(t)$.

8.4. *We recall that the d-dimensional cross-polytope was defined by*

$$\Diamond := \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid |x_1| + |x_2| + \cdots + |x_d| \leqslant 1\right\}.$$

Compute the angle polynomial of $A_{\Diamond}(t)$.

8.5. *Let $\mathcal{P}$ be an integer zonotope. Prove that the angle polynomial of $\mathcal{P}$ is*

$$A_{\mathcal{P}}(t) = (\operatorname{vol}\mathcal{P})t^d,$$

valid for all positive integers t.

8.6. *Let $\mathcal{P}$ be a rational interval $[\frac{a}{c}, \frac{b}{d}]$. Compute the angle quas-ipolynomial $A_{\mathcal{P}}(t)$ here.*

8.7. *Define the rational triangle Δ whose vertices are*

$$(0,0), \left(1, \frac{N-1}{N}\right), (N,0),$$

where $N \geqslant 2$ is a fixed integer. Find the angle quasi-polynomial $A_{\Delta}(t)$.

8.8. ♣ *For each dimension d, find an example of a rational polytope $\mathcal{P} \subset \mathbb{R}^d$ and a unimodular matrix $U \in \mathrm{SL_d}(\mathbb{Z})$, such that the angle quasi-polynomials $A_{\mathcal{P}}(t)$ and $A_{U(\mathcal{P})}(t)$ are not equal to each other for all $t \in \mathbb{Z}_{>0}$.*

8.9. ♣ *For the cube $\square := [0,1]^d$, show that any face $F \subseteq \square$ that has dimension k has a solid angle $\omega_F = \frac{1}{2^{d-k}}$.*

8.10. ♣ *Show that the solid angle ω_E of an edge E (1-dimensional face) of a polytope equals the dihedral angle between the hyperplanes defined by its two bounding facets. (Hint: use the unit normal vectors for both facets)*

8.11. ♣ *Prove Lemma 8.2.*

8.12. *Using the Gram relations, namely Theorem 8.5, compute the solid angle at any vertex of the following regular tetrahedron:*

$$T := \operatorname{conv}\left\{ \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \right\}.$$

9 Sphere packings

The problem of packing, as densely as possible, an unlimited number of equal nonoverlapping circles in a plane was solved millions of years ago by the bees, who found that the best arrangement consists of circles inscribed in the hexagons of the regular tessellation.
– H. S. M. Coxeter

There is geometry in the humming of the strings. There is music in the spacing of the spheres. – Pythagoras

9.1 Intuition

The sphere packing problem traces its roots back to Kepler, and it asks for a packing of solid spheres in Euclidean space that achieves the maximum possible density. In all of the known cases, such optimal configurations - for the centers of the spheres - form a lattice. It's natural, therefore, that Fourier analysis comes into the picture. We prove here a result of Cohn and Elkies, from 2003, which is a beautiful application of Poisson summation, and gives upper bounds for the maximum densities of sphere packings in $\mathbb{R}^d$.

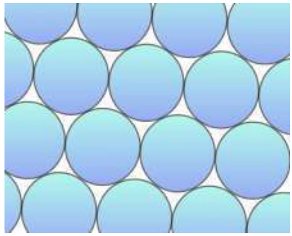


Figure 9.1: A lattice sphere packing, using the Eisenstein lattice, which gives the densest packing in 2 dimensions.

At this point it may be wise to define carefully all of the terms - what is a packing? what is density? Who was Kepler?

9.2 Definitions

A **sphere packing** in $\mathbb{R}^d$ is any arrangement of spheres of fixed radius $r > 0$ such that no two interiors overlap, so we do not preclude the possibility that the spheres may touch one another at some points on their boundary.

A **lattice packing** is a sphere packing with the property that the centers of the spheres form a lattice $\mathcal{L} \subset \mathbb{R}^d$. Relaxing this restriction a little, a **periodic packing** is a sphere packing with a lattice $\mathcal{L}$, together with a finite collection of its translates, say $\mathcal{L} + v_1, \ldots, \mathcal{L} + v_N$, such that the differences $v_i - v_j \notin L$. This means that the centers of the spheres may be placed at any points belonging to the disjoint union of $\mathcal{L}$, together with its N translates.

The density of any sphere packing is intuitively the proportion of Euclidean space covered by the spheres, in an asymptotic sense, but rather than go into these technical asymptotic details, we will simply define a density function for lattice packings and for general periodic packings, as follows. Given a lattice packing, with the lattice $\mathcal{L} \subset \mathbb{R}^d$, and with spheres of radius r, we define its **lattice packing density** by

$$\Delta(\mathcal{L}) := \frac{\operatorname{vol}\left(B^d(\frac{r}{2})\right)}{\det \mathcal{L}}, \tag{9.1}$$

where $B^d(\frac{r}{2})$ is a ball of radius $\frac{r}{2}$. This lattice packing corresponds to placing a sphere of radius $\frac{r}{2}$ at each lattice point of $\mathcal{L}$, guaranteeing that the spheres do not

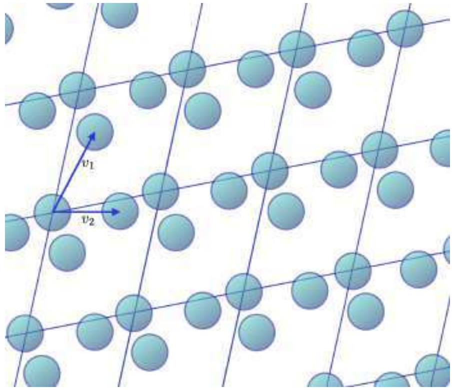


Figure 9.2: A periodic packing, which is not a lattice packing, with two translates of the same lattice.

overlap.

More generally, given a period packing with a lattice $\mathcal{L}$ and a set of translates $v_1, \ldots, v_N$, we define its **periodic packing density** by

$$\Delta_{periodic}(\mathcal{L}) := \frac{N \operatorname{vol}\left(B^d(\frac{r}{2})\right)}{\det \mathcal{L}}, \tag{9.2}$$

corresponding to placing a sphere of radius $\frac{r}{2}$ at each point of $\mathcal{L}$, and also at each point of its translates $\mathcal{L} + v_1, \ldots, \mathcal{L} + v_N$. It's not hard to prove that the latter Equation (9.2) matches our intuition that any fixed fundamental parallelepiped of $\mathcal{L}$ intersects this configuration of spheres in a set whose measure is exactly $N \operatorname{vol}\left(B^d(\frac{r}{2})\right)$ (Exercise 9.1).

Henceforth, we use the words 'packing density' to mean 'periodic packing density', and we always restrict attention to periodic packings - see the Notes for technical remarks involving any sphere arrangement, and why periodic packings are sufficient.

We define the **sphere packing problem** as follows:

> What is the maximum possible packing density, in any periodic packing of spheres?

In other words, the problem asks us to find the maximum density $\Delta_{periodic}\mathcal{L}$, among all lattices (and their finite collections of translates). The sphere packing problem also asks us to find, if possible, the lattice that achieves this optimal density.

Many other questions naturally arise: is such a lattice unique in each dimension? Are there examples of dimensions d for which a single lattice does not suffice, in order to obtain the maximum sphere packing, but for which we do in fact need to use some lattice translates of a fixed lattice? The sphere packing problem is a very important problem in Geometry, Number theory, Coding theory, and information theory.

9.3 Upper bounds for sphere packings via Poisson summation

Here we give an exposition of the groundbreaking result of Henry Cohn and Noam Elkies on the sphere packing problem. This result sets up the machinery for finding certain **magical functions** f, as defined in Theorem 9.1 below, that allow us to give precise upper bounds on $\Delta_{periodic}\mathcal{L}$. The main tool is Poisson summation again, for arbitrary lattices.

Theorem 9.1 (Cohn–Elkies)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a nice function, not identically zero, which enjoys the following three conditions:*

1. $f(x) \leqslant 0$*, for all* $\|x\| \geqslant r$*, where* $r > 0$ *is some fixed real constant.*
2. $\hat{f}(\xi) \geqslant 0$*, for all* $\xi \in \mathbb{R}^d$*.*
3. $f(0) > 0$*, and* $\hat{f}(0) > 0$*.*

Then the periodic packing density of any d-dimensional sphere packing has the upper bound

$$\Delta_{periodic}(\mathcal{L}) \leqslant \frac{f(0)}{\hat{f}(0)} \operatorname{vol} B^d \left(\frac{r}{2}\right).$$

Proof. Suppose we have a periodic packing with spheres of radius r, a lattice $\mathcal{L}$, and translation vectors $v_1, \dots, v_N$, so that by definition the packing density is $\delta := \frac{N \operatorname{vol}\left(B^d\left(\frac{r}{2}\right)\right)}{\det \mathcal{L}}$.

By Poisson summation, we have

$$\sum_{n \in \mathcal{L}} f(n+v) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{f}(\xi) e^{2\pi i \langle v, \xi \rangle}, \tag{9.3}$$

converging absolutely for all $v \in \mathbb{R}^d$. Now form the following sums, and rearrange them:

$$\sum_{1\leqslant i\leqslant j\leqslant N} \sum_{n\in\mathcal{L}} f(n+v_i-v_j) = \frac{1}{\det \mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{f}(\xi) \sum_{1\leqslant i\leqslant j\leqslant N} e^{2\pi i\langle v_i-v_j,\xi\rangle} \tag{9.4}$$

$$= \frac{1}{\det \mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{f}(\xi) \Bigg| \sum_{1\leqslant k\leqslant N} e^{2\pi i\langle v_k,\xi\rangle} \Bigg|^2. \tag{9.5}$$

Now, every summand on the right-hand-side of (9.5) is nonnegative, because by the second assumption of the Theorem, we have $\hat{f} \geqslant 0$, so that the whole series can be bounded from below by its constant term, which for $\xi = 0$ gives us the bound $\frac{\hat{f}(0)N^2}{\det \mathcal{L}}$.

On the other hand, considering the vectors $n + v_i - v_j$ on the left-hand-side of (9.4), suppose we have $\|n + v_i - v_j\| \geqslant r$. Then the first hypothesis of the Theorem guarantees that $f(n+v_i-v_j) \leqslant 0$. If we have $\|n+v_i-v_j\| < r$, then the vector $n+v_i-v_j$ is contained in the sphere of radius r, centered at the origin, but this means that it must be the zero vector: $n + v_i - v_j = 0$. By assumption, the difference between any two translations $v_i - v_j$ is never a nonzero element of $\mathcal{L}$, so now we have $v_i = v_j$, which implies that $n = 0$. We conclude that the only positive contribution from the left-hand-side of (9.4) is the $n = 0$ term, and so the left-hand-side of (9.4) has an upper bound of $Nf(0)$.

Altogether, Poisson summation gave us the bound:

$$\begin{aligned} Nf(0) &\geqslant \Bigg| \sum_{1\leqslant i\leqslant j\leqslant N} \sum_{n\in\mathcal{L}} f(n+v_i-v_j) \Bigg| \\ &= \frac{1}{\det \mathcal{L}} \sum_{\xi\in\mathcal{L}^*} \hat{f}(\xi) \Bigg| \sum_{1\leqslant k\leqslant N} e^{2\pi i\langle v_k,\xi\rangle} \Bigg|^2 \\ &\geqslant \frac{\hat{f}(0)N^2}{\det \mathcal{L}}. \end{aligned}$$

Simplifying, we have

$$\frac{f(0)}{\hat{f}(0)} \geqslant \frac{N}{\det \mathcal{L}} := \frac{\Delta_{periodic}(\mathcal{L})}{\operatorname{vol}\left(B^d(\frac{r}{2})\right)},$$

where the last equality follows from the definition of $\Delta_{periodic}(\mathcal{L})$. □

Example 9.1 (The trivial bound)**.** Let $\mathcal{L}$ be a full-rank lattice in $\mathbb{R}^d$, whose shortest nonzero vector has length $r > 0$. We define the function

$$f(x) := 1_K(x) * 1_K(x),$$

where K is the ball of radius $\frac{r}{2}$, centered at the origin. We claim that f satisfies all of the conditions of Theorem 9.1. Indeed, by the convolution Theorem,

$$\hat{f}(\xi) = (\widehat{1_K * 1_K})(\xi) = \left(\hat{1}_K(\xi)\right)^2 \geqslant 0,$$

for all $\xi \in \mathbb{R}^d$, verifying condition 2. Condition 1 is also easy to verify, because the support of f is equal to the Minkowski sum (by Exercise 4.3) $K + K = 2K$, a sphere of radius r. It follows that f is identically zero outside a sphere of radius r. For condition 3, by the definition of convolution we have $f(0) = \int_{\mathbb{R}^d} 1_K(0 - x)1_K(x)dx = \int_{\mathbb{R}^d} 1_K(x)dx = \mathrm{vol}(K) > 0$. Finally, $\hat{f}(0) = \left(\hat{1}_K(0)\right)^2 = \mathrm{vol}^2(K) > 0$.

By the Cohn–Elkies Theorem 9.1, we know that the packing density of such a lattice is therefore bounded above by

$$\frac{f(0)}{\hat{f}(0)} \mathrm{vol}\, B^d\left(\frac{r}{2}\right) = \frac{\mathrm{vol}(K)}{\mathrm{vol}^2(K)} \mathrm{vol}(K) = 1,$$

the trivial bound. So we don't get anything interesting, but all this tells us is that our particular choice of function f above was a poor choice, as far as density bounds are concerned. We need to be more clever in picking our magical f. □

Although it is far from trivial to find magical functions f that satisfy the hypothesis of the Cohn–Elkies Theorem, and simultaneously give a strong upper bound, there has been huge success recently in finding exactly such functions - in dimensions 8 and 24. These recent magical functions gave the densest sphere packings in these dimensions, knocking off the whole sphere packing problem in dimensions 8 and 24.

This exciting story continues today, and we mention some of the recent spectacular applications of the Cohn–Elkies Theorem, initiated recently by Maryna Viazovska for $\mathbb{R}^8$, and then extended by a large joint effort from Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska, for $\mathbb{R}^{24}$. Here is a synopsis of some of their results.

Theorem 9.2. *The lattice E_8 is the densest periodic packing in $\mathbb{R}^8$. The Leech lattice is the densest periodic packing in $\mathbb{R}^{24}$. In addition, these lattices are unique, in the sense that there do not exist any other periodic packings that achieve the same density.*

At the moment, the provably densest packings are known only in dimensions $1, 2, 3, 8$, and 24. Each dimension seems to require slightly different methods, and sometimes wildly different methods, such as $\mathbb{R}^3$. For $\mathbb{R}^3$, the sphere packing problem was solved by Hales, and before Hales' proof, it was an open problem since the time of Kepler. Somewhat surprisingly, the sphere packing problem is still open in all other dimensions.

In $\mathbb{R}^4$, it is very tempting to think of the lattice D_4 as a possible candidate for the densest lattice sphere packing in $\mathbb{R}^4$, but this is still unknown.

9.4 Transforms of balls in Euclidean space

Whenever considering packing or tiling by a convex body B, we have repeatedly seen that taking the Fourier transform of the body, namely $\hat{1}_B$, is very natural, especially from the perspective of Poisson summation. Thus, it is also natural here to consider the FT of a ball in $\mathbb{R}^d$. While we are at it, let's dilate the unit ball by $c > 0$, defining $B_c := \left\{x \in \mathbb{R}^d \mid \|x\| \leqslant c\right\}$.

To compute the Fourier transform of 1_{B_c}, a very classical computation, we first define the **Bessel function** J_p of order p (Epstein (2008), page 147), which comes up naturally here:

$$J_p(x) := \frac{(x/2)^p}{\Gamma\left(p + \frac{1}{2}\right)\sqrt{\pi}} \int_0^{\pi} e^{ix\cos\varphi} \sin^{2p}(\varphi)\, d\varphi. \tag{9.6}$$

Lemma 9.1. *The Fourier transform of B_c, the ball of radius c in $\mathbb{R}^d$ centered at the origin, is*

$$\hat{1}_{B_c}(\xi) := \int_{B_c} e^{-2\pi i\langle \xi, x\rangle} dx = \left(\frac{c}{\|\xi\|}\right)^{d/2} J_{d/2}\big(2\pi c\|\xi\|\big).$$

Proof. Taking advantage of the inherent rotational symmetry of the ball, and also using the fact that the Fourier transform of a radial function is again radial (Exercise 9.4), we have:

$$\hat{1}_{B_c}(\xi) = \hat{1}_{B_c}(0, \ldots, 0, \|\xi\|),$$

for all $\xi \in \mathbb{R}^d$. With $c = 1$ for the moment, we therefore have:

$$\hat{1}_{B_1}(\xi) = \int_{\|x\| \leqslant 1} e^{-2\pi i x_d \|\xi\|}\, dx_1 \ldots dx_d,$$

Now we note that for each fixed x_d, the function being integrated is constant and the integration domain for the variables $x_1, \ldots, x_{d-1}$ is a $(d-1)$-dimensional ball of radius $(1 - x_d^2)^{1/2}$. Using the classical fact that the volume of this ball is $(1 - x_d^2)^{\frac{d-1}{2}} \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}$ (see Exercise 9.5), we have

$$\begin{aligned}
\hat{1}_{B_1}(\xi) &= \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \int_{-1}^{1} e^{-2\pi i x_d \|\xi\|} (1 - x_d^2)^{\frac{d-1}{2}}\, dx_d \\
&= \frac{\pi^{\frac{d}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2}\right)} \int_0^{\pi} e^{2\pi i \|\xi\| \cos\varphi} \sin^d \varphi\, d\varphi.
\end{aligned}$$

Using the definition (9.6) of the J-Bessel function, we get

$$\hat{1}_{B_1}(\xi) = \|\xi\|^{-\frac{d}{2}} J_{\frac{d}{2}}(2\pi \|\xi\|),$$

and consequently

$$\hat{1}_{B_c}(\xi) = \left(\frac{c}{\|\xi\|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi c \|\xi\|). \qquad \square$$

We call a function $f : \mathbb{R}^d \to \mathbb{C}$ **radial** if it is invariant under all rotations of $\mathbb{R}^d$. In other words, we have the definition

$$f \text{ is radial} \iff f \circ \sigma = f,$$

for all $\sigma \in SO_d(\mathbb{R})$, the orthogonal group. Another way of describing a radial function is to say that the function f is constant on each sphere that is centered at the origin, so that intuitively a radial function only depends on the norm of its input.

A very useful fact in various applications of Fourier analysis (in particular medical imaging) is that the Fourier transform of a radial function is again a radial function, and in fact we've already used this fact in the proof of Lemma 9.1.

Now we may notice that if we have a magical function f that enjoys all three hypotheses of the Cohn–Elkies Theorem 9.1, then it is easy to see that $f \circ \sigma$ also satisfies the same hypotheses, for any $\sigma \in SO_d(\mathbb{R})$ (Exercise 9.7), and therefore we may take radial functions as candidates for magical functions as well.

Notes

(a) Each dimension d appears to have a separate theory for sphere packings. This intuition is sometimes tricky to conceptualize, but there are facts that help us do so. For example, it is a fact that the Gram matrix (see 5.22) of a lattice $\mathcal{L} \subset \mathbb{R}^d$ consists entirely of integers, with even diagonal elements $\iff$ d is divisible by 8. For this reason, it turns out that the theta series of a lattice possesses certain functional equations (making it a modular form) if and only if $8 \mid d$, which in turn allows us to build some very nice related 'magical' functions f that are sought-after in Theorem 9.1, at least for $d = 8$ and $d = 24$ so far.

(b) Johannes Kepler (1571–1630) was a German astronomer and mathematician. Kepler's laws of planetary motion motivated Sir Isaac Newton to develop further the theory of gravitational attraction and planetary motion. Kepler conjectured that the densest packing of sphere is given by the "face-centered cubic" packing. It was Gauss (1831) who first proved that, if we assume the packing to be a lattice packing, then Kepler's conjecture is true. In 1998 Thomas Hales (using an approach initiated by Fejes Tóth (1953)), gave an unconditional proof of the Kepler conjecture.

(c) It is also possible, of course, to pack other convex bodies. One such variation is to pack regular tetrahedra in $\mathbb{R}^3$. The fascinating article by Lagarias and Zong (2012) gives a nice account of this story.

(d) Regarding lower bounds for the optimal density of sphere packings, Keith Ball (1992) discovered the following lower bound in all dimensions:

$$\Delta_{periodic}(\mathcal{L}) \geqslant \frac{(n-1)}{2^{n-1}}\zeta(n),$$

where $\zeta(s)$ is the Riemann zeta function. Recently, Akshay Venkatesh (2013) has given an interesting improvement (over the known lower bounds) by a multiplicative constant. For all sufficiently large dimensions, this is an improvement by a factor of at least $10,000$.

Exercises

9.1. *Given a periodic lattice packing, with a lattice $\mathcal{L} \subset \mathbb{R}^d$, show that any fixed fundamental parallelepiped of $\mathcal{L}$ intersects the union of all the spheres in a set of measure N vol $\left(B^d\left(\frac{r}{2}\right)\right)$. Thus, we may compute the density of a periodic sphere packing by just considering the portions of the spheres that lie in one fundamental parallelepiped.*

9.2. *Suppose we pack equilateral triangles in the plane, by using only translations of a fixed equilateral triangle. What is the maximum packing density of such a packing? Do you think it may be the worst possible density among translational packings of any convex body in $\mathbb{R}^2$?*

9.3. *Here we show that the integer lattice is in fact a very poor choice for sphere packing.*

(a) *Compute the packing density of the integer lattice $\mathbb{Z}^2$.*

(b) *Compute the packing density of the integer lattice $\mathbb{Z}^d$.*

9.4. *Show that the Fourier transform of a radial function $f : \mathbb{R}^d \to \mathbb{C}$ is another radial function.*

9.5. *Show that the volume of the unit ball $B_1 := \left\{x \in \mathbb{R}^d \mid \|x\| \leqslant 1\right\}$ is:*

$$\operatorname{vol} B^d = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}.$$

9.6. *Show that the volume of the unit sphere* $S^{d-1} := \left\{x \in \mathbb{R}^d \mid \|x\| = 1\right\}$ *is:*

$$\operatorname{vol} S^{d-1} = \frac{d\,\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}.$$

9.7. *Show that if we have a magical function* f *that enjoys all* 3 *hypotheses of Theorem 9.1, then* $f \circ \sigma$ *also satisfies the same hypotheses, for any orthogonal transformation* $\sigma \in SO_d(\mathbb{R})$.

10 The Fourier transform of a polytope

> Like a zen koan, Stokes' Theorem tells us that in the end, what happens on the outside is purely a function of the change within.
>
> –Keenan Crane

10.1 Intuition

The divergence Theorem is a multi-dimensional version of "integration by parts", a very useful tool in 1-dimensional calculus. When we apply the divergence Theorem, described below, to a polytope, we obtain a kind of combinatorial version of the divergence Theorem, allowing us to transfer some of the complexity of computing the Fourier transform of a polytope to the complexity of computing corresponding Fourier transforms of its facets. This kind of game can be iterated, yielding interesting geometric identities and results for polytopes, as well as for discrete volumes of polytopes.

In the process, we also obtain another useful way to compute the Fourier transform of a polytope in its own right.

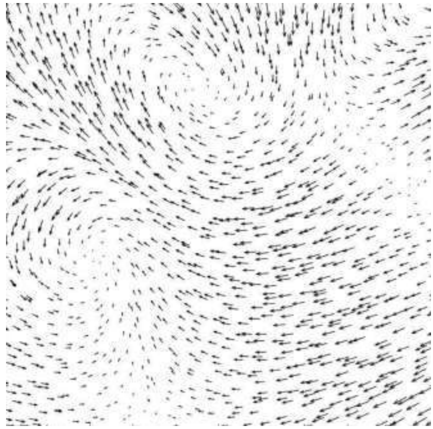


Figure 10.1: A real vector field in $\mathbb{R}^2$

10.2 The divergence Theorem, and a combinatorial divergence theorem for polytopes

To warm up, we recall the Divergence Theorem, with some initial examples. A **vector field** on Euclidean space is a function $F : \mathbb{R}^d \to \mathbb{C}^d$ that assigns to each point in $\mathbb{R}^d$ another vector in $\mathbb{C}^d$, which we will denote by

$$F(x) := (F_1(x), F_2(x), \ldots, F_d(x)) \in \mathbb{C}^d.$$

If F is a continuous (respectively, smooth) function, we say that F is a **continuous vector field** (respectively, **smooth vector field**). If all of the coordinate functions F_j are real-valued functions, we say that we have a **real vector field**.

We define the **divergence** of F at each $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$ by

$$\operatorname{div} F(x) := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_d}{\partial x_d},$$

assuming that F is a smooth vector field (or at least a differentiable vector field). This divergence of F is a measure of the local change (sink versus source) of the vector field at each point $x \in \mathbb{R}^d$. Given a surface $S \subset \mathbb{R}^d$, and an outward pointing unit normal vector $\mathbf{n}$, defined at each point $x \in S$, we also define the **flux** of the vector field F across the surface S by

$$\int_S F \cdot \mathbf{n}\, dS,$$

where dS denotes the Lebesgue measure of the surface S, and where the dot product $F \cdot \mathbf{n}$ is the usual inner product $\langle F, \mathbf{n}\rangle := \sum_{k=1}^{d} F_k n_k$. We will apply the divergence Theorem (which is technically a special case of Stokes' Theorem) to a polytope $\mathcal{P} \subset \mathbb{R}^d$, and its $(d-1)$-dimensional bounding surface $\partial\mathcal{P}$. Intuitively, the divergence Theorem tells us that the total divergence of a vector field F inside a manifold is equal to the total flux of F across its boundary.

Theorem 10.1 (The Divergence Theorem)**.** *Let $M \subset \mathbb{R}^d$ be a piecewise smooth manifold, and let F be a smooth vector field. Then*

$$\int_M divF(x)dx = \int_S F \cdot \boldsymbol{n}\, dS. \tag{10.1}$$

Example 10.1. Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional polytope, containing the origin, with defining facets $G_1, \ldots, G_N$. Define the real vector field

$$F(x) := x,$$

for all $x \in \mathbb{R}^d$. First, we can easily compute here the divergence of F, which turns out to be constant:

$$\mathrm{div} F(x) = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_d}{\partial x_d} = \frac{\partial x_1}{\partial x_1} + \cdots + \frac{\partial x_d}{\partial x_d} = d.$$

If we fix any facet G of $\mathcal{P}$ then, due to the piecewise linear structure of the polytope, every point $x \in G$ has the same constant outward pointing normal vector to F, which we call $\mathbf{n}_G$. Computing first the left-hand-side of the divergence Theorem, we see that

$$\int_P \mathrm{div} F(x)dx = d\int_P dx = (\mathrm{vol}\,\mathcal{P})d. \tag{10.2}$$

Computing now the right-hand-side of the divergence Theorem, we get

$$\int_S F \cdot \mathbf{n}\, dS = \int_{\partial\mathcal{P}} \langle x, \mathbf{n}\rangle\, dS = \sum_{k=1}^{N} \int_{G_k} \langle x, \mathbf{n}_G\rangle\, dS.$$

Now it's easy to see that the inner product $\langle x, n_G\rangle$ is constant on each facet $G \subset \mathcal{P}$, namely it is the distance from the origin to G (Exercise 10.3), denoted by $\mathrm{dist}(G)$. So we now have

$$\int_{\partial\mathcal{P}} F \cdot n \, dS = \sum_{k=1}^{N} \int_{G_k} \langle x, \mathbf{n}_{G_k}\rangle dS$$
$$= \sum_{k=1}^{N} \mathrm{dist}(G_k) \int_{G_k} dS = \sum_{k=1}^{N} \mathrm{dist}(G_k) \,\mathrm{vol}\, G_k,$$

so that altogether we the following conclusion from the divergence Theorem:

$$\mathrm{vol}\,\mathcal{P} = \frac{1}{d} \sum_{k=1}^{N} \mathrm{dist}(G_k) \,\mathrm{vol}\, G_k. \tag{10.3}$$

known as "the pyramid formula" for a polytope, a classical result in Geometry, which also has a very easy geometrical proof (Exercise 10.1). □

Example 10.2. Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional polytope with defining facets $G_1, \dots, G_N$, and outward pointing unit vectors $n_{G_1}, \dots, n_{G_N}$. We fix any constant vector $\lambda \in \mathbb{C}^d$, and we consider the **constant vector field**

$$F(x) := \lambda,$$

defined for all $x \in \mathbb{R}^d$. Here the divergence of F is $\mathrm{div} F(x) = 0$, because F is constant, and so the left-hand-side of Equation (10.1) gives us

$$\int_P \mathrm{div} F(x) dx = 0.$$

Altogether, the divergence Theorem gives us:

$$0 = \int_{\partial\mathcal{P}} F \cdot \mathbf{n} \, dS = \sum_{k=1}^{N} \int_{G_k} \langle \lambda, \mathbf{n}_{G_k}\rangle dS$$
$$= \sum_{k=1}^{N} \langle \lambda, \mathbf{n}_{G_k}\rangle \int_{G_k} dS$$
$$= \langle \lambda, \sum_{k=1}^{N} \mathrm{vol}(G_k) \mathbf{n}_{G_k}\rangle,$$

and because this holds for any constant vector λ, we can conclude that

$$\sum_{k=1}^{N} \mathrm{vol}(G_k)\mathbf{n}_{G_k} = 0. \tag{10.4}$$

Identity (10.4) is widely known as the **Minkowski relation** for polytopes. There is a marvelous converse to the latter relation, given by Minkowski as well, for any convex polytope. [See Theorem 10.7] □

Now we fix $\xi \in \mathbb{R}^d$ and apply the divergence Theorem to the vector field

$$F(x) := e^{-2\pi i\langle x,\xi\rangle}\xi. \tag{10.5}$$

First,

$$\begin{aligned}
\mathrm{div} F(x) &= \frac{\partial\left(e^{-2\pi i\langle x,\xi\rangle}\xi_1\right)}{\partial x_1} + \cdots + \frac{\partial(e^{-2\pi i\langle x,\xi\rangle}\xi_d)}{\partial x_d} \\
&= (-2\pi i\xi_1^2)e^{-2\pi i\langle x,\xi\rangle} + \cdots + (-2\pi i\xi_d^2)e^{-2\pi i\langle x,\xi\rangle} \\
&= -2\pi i\|\xi\|^2 e^{-2\pi i\langle x,\xi\rangle}.
\end{aligned}$$

So here the divergence Theorem gives us

$$\int_{x\in P} -2\pi i||\xi||^2 e^{-2\pi i\langle x,\xi\rangle}dx = \int_{\partial P} e^{-2\pi i\langle x,\xi\rangle}\langle \xi, \mathbf{n}\rangle\, dS,$$

which gives us another version of the Fourier transform of a polytope:

$$\begin{aligned}
\hat{1}_{\mathcal{P}}(\xi) &:= \int_{x\in P} e^{-2\pi i\langle x,\xi\rangle}dx \\
&= \frac{1}{-2\pi i\|\xi\|^2}\int_{\partial P}\langle \xi, \mathbf{n}\rangle e^{-2\pi i\langle x,\xi\rangle}dS \\
&= \frac{1}{-2\pi i\|\xi\|^2}\int_{G_1}\langle \xi, \mathbf{n}_{G_1}\rangle e^{-2\pi i\langle x,\xi\rangle}dS + \cdots + \\
&\qquad\qquad + \frac{1}{-2\pi i\|\xi\|^2}\int_{G_N}\langle \xi, \mathbf{n}_{G_N}\rangle e^{-2\pi i\langle x,\xi\rangle}dS \\
&= \frac{\langle \xi, \mathbf{n}_{G_1}\rangle}{-2\pi i\|\xi\|^2}\hat{1}_{G_1}(\xi) + \cdots + \frac{\langle \xi, \mathbf{n}_{G_N}\rangle}{-2\pi i\|\xi\|^2}\hat{1}_{G_N}(\xi),
\end{aligned}$$

where we used the fact that the boundary $\partial\mathcal{P}$ of a polytope is a finite union of $(d-1)$-dimensional polytopes (its facets), to write $\int_{\partial P} = \int_{G_1} + \cdots + \int_{G_N}$, a sum of integrals over the N facets of $\mathcal{P}$. We've arrived at the following formula for the Fourier transform of $\mathcal{P}$:

Lemma 10.1. *Given any d-dimensional polytope $\mathcal{P} \subset \mathbb{R}^d$, with outward pointing normal vector n_G to each facet G of $\mathcal{P}$, its Fourier transform has the form*

$$\hat{1}_{\mathcal{P}}(\xi) = \frac{1}{-2\pi i} \sum_{G \subset \partial P} \frac{\langle \xi, \boldsymbol{n}_G \rangle}{||\xi||^2} \hat{1}_G(\xi) \tag{10.6}$$

for all nonzero $\xi \in \mathbb{R}^d$. Here the integral that defines each $\hat{1}_G$ is taken with respect to Lebesgue measure that matches the dimension of the facet $G \subset \partial P$.

To simplify the notation that will follow, we can also the **Iverson bracket** notation, defined as follows. Suppose we have any boolean property $P(n)$, where $n \in \mathbb{Z}^d$; that is, $P(n)$ is either true or false. Then the Iverson bracket $[P]$ is defined by:

$$[P] = \begin{cases} 1 & \text{if P is true} \\ 0 & \text{if P is false} \end{cases} \tag{10.7}$$

Now we may refine (10.6) above as follows:

$$\hat{1}_{\mathcal{P}}(\xi) = \operatorname{vol}\mathcal{P}\,[\xi = 0] + \frac{1}{-2\pi i} \sum_{G \subset \partial P} \frac{\langle \xi, \mathbf{n}_G \rangle}{||\xi||^2} \hat{1}_G(\xi)\ [\xi \neq 0]. \tag{10.8}$$

Later, after Theorem 10.2 below, we will return to the Iverson bracket, and be able to use it efficiently. To proceed further, we need to define the **affine span** of a face F of $\mathcal{P}$:

$$\operatorname{aff}(F) := \left\{ \sum_{j=1}^{k} \lambda_j v_j \mid k > 0, v_j \in F, \lambda_j \in \mathbb{R}, \text{ and } \sum_{j=1}^{k} \lambda_j = 1 \right\}. \tag{10.9}$$

In other words, we may think of the affine span of a face F of $\mathcal{P}$ as follows. We first translate F so that this translate, call if F_0, contains the origin. Then we take all real linear combinations of points of F_0, obtaining a vector subspace of $\mathbb{R}^d$, which we call the **linear span** of F. Another way to describe the linear span of a face F of $\mathcal{P}$ is:

$$\operatorname{lin}(F) := \{x - y \mid x, y \in F\}.$$

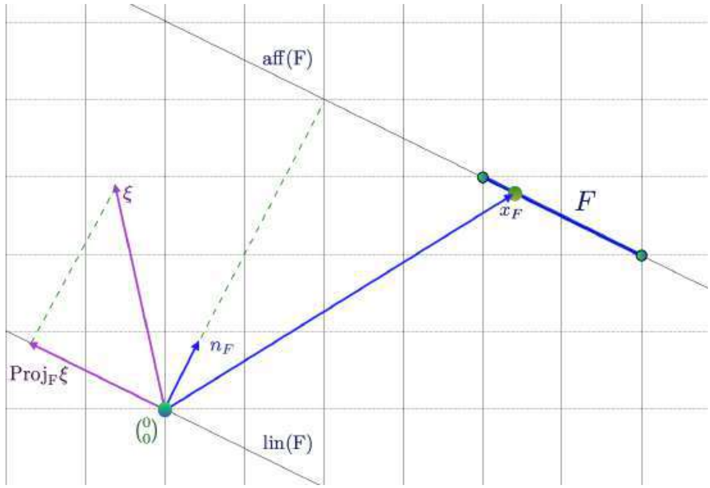


Figure 10.2: The affine span of a face F, its linear span , and the projection of ξ onto F. Here we note that the distance from the origin to F is $\sqrt{20}$.

Finally, we translate this subspace $\mathrm{lin}(F)$ back using the same translation vector, to obtain $\mathrm{aff}(F)$ (see Figure 10.2).

Example 10.3. The affine span of two points in $\mathbb{R}^d$ is the unique line in $\mathbb{R}^d$ passing through them. The affine span of three points in $\mathbb{R}^d$ is the unique 2-dimensional plane passing through them. The affine span of a k-dimensional polytope $F \subset \mathbb{R}^d$ is a translate of a k-dimensional vector subspace of $\mathbb{R}^d$. Finally, the affine span of a whole d-dimensional polytope $\mathcal{P} \subset \mathbb{R}^d$ is all of $\mathbb{R}^d$. □

In formalizing (10.6) further, we will require the notion of the projection of any point $\xi \in \mathbb{R}^d$ onto the linear span of any face $F \subseteq \mathcal{P}$, which we abbreviate by $\mathrm{Proj}_F\,\xi$:

$$\mathrm{Proj}_F\,\xi := \mathrm{Proj}_{\mathrm{lin}(F)}(\xi). \tag{10.10}$$

(see Figure 10.2) We will also need the following elementary fact. Let F be any k-dimensional polytope in $\mathbb{R}^d$, and fix the outward-pointing unit normal to F, calling it $\mathbf{n}_F$. It is straightforward to show that if we take any point $x_F \in \mathcal{F}$, then $\langle x_F, \mathbf{n}_F\rangle$ is the distance from the origin to F. Therefore, if $\mathrm{Proj}_F\,\xi = 0$, then a straightforward computation shows that $\langle \xi, x_F\rangle = \|\xi\|\mathrm{dist}(F)$ (Exercise 10.3).

We now record the reasoning above, leading to (10.6), more formally but with one slight extension: we replace $\mathcal{P}$ by any k-dimensional face F of $\mathcal{P}$, as follows.

Theorem 10.2 (Combinatorial Divergence Theorem). *Let F be a polytope in $\mathbb{R}^d$, where $1 \leqslant \dim F \leqslant d$. For each facet $G \subseteq F$, we let $\boldsymbol{n}(G, F)$ be the unit normal vector to G, pointing outwards from F. Then for each $\xi \in \mathbb{R}^d$, we have:*

(a) If $Proj_F \xi = 0$, then we have

$$\hat{1}_F(\xi) = (\text{vol } F)e^{-2\pi i \|\xi\| dist(F)}. \tag{10.11}$$

(b) If $Proj_F \xi \neq 0$, then

$$\hat{1}_F(\xi) = \frac{1}{-2\pi i} \sum_{G \subset \partial F} \frac{\langle Proj_F \xi, \boldsymbol{n}(G, F)\rangle}{||Proj_F \xi||^2} \hat{1}_G(\xi). \tag{10.12}$$

□

We notice that, as before, we are getting rational-exponential functions for the Fourier transform of a polytope, but now we are using the facets of $\mathcal{P}$ when applying Theorem 10.2 to $\mathcal{P}$ itself.

We are now set up to iterate this process, defined by Theorem 10.2, reapplying it to each facet $G \subset \partial\mathcal{P}$. Let's use the Iverson bracket, defined in (10.7), and apply the combinatorial divergence Theorem 10.2 to $\mathcal{P}$ twice:

$$\begin{aligned}
\hat{1}_{\mathcal{P}}(\xi) &= \text{vol}\,\mathcal{P}\,[\xi = 0] + \frac{1}{-2\pi i} \sum_{F_1 \subset \partial P} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle}{||\xi||^2}\,[\xi \neq 0]\,\hat{1}_{F_1}(\xi) \\
&= \text{vol}\,\mathcal{P}\,[\xi = 0] + \frac{1}{-2\pi i} \sum_{F_1 \subset \partial P} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle}{||\xi||^2}[\xi \neq 0] \cdot \\
&\qquad \cdot \Big((\text{vol}\,F_1)e^{-2\pi i\langle \xi, x\rangle}\,[\text{Proj}_{F_1}\xi = 0] + \\
&\qquad\qquad + \frac{1}{-2\pi i} \sum_{F_2 \subset \partial F_1} \frac{\langle \text{Proj}_{F_2}\xi, \mathbf{n}(F_2, F_1)\rangle}{||\text{Proj}_{F_2}\xi||^2} \hat{1}_{F_2}(\xi)[\text{Proj}_{F_1}\xi \neq 0]\Big) \\
&= \text{vol}\,\mathcal{P}\,[\xi = 0] + \\
&\qquad + \frac{1}{-2\pi i} \sum_{F_1 \subset \partial P} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle(\text{vol}\,F_1)e^{-2\pi i\langle \xi, x\rangle}}{||\xi||^2}\,[\xi \neq 0][\text{Proj}_{F_1}\xi = 0] + \\
&+ \frac{1}{(-2\pi i)^2} \sum_{F_1 \subset \partial P} \sum_{F_2 \subset \partial F_1} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle}{||\xi||^2} \frac{\langle \text{Proj}_{F_2}\xi, \mathbf{n}(F_2, F_1)\rangle}{||\text{Proj}_{F_2}\xi||^2} \cdot \\
&\qquad\qquad \cdot \hat{1}_{F_2}(\xi)\,[\xi \neq 0][\text{Proj}_{F_1}\xi \neq 0]
\end{aligned}$$

It is an easy fact that the product of two Iverson brackets is the Iverson bracket of their intersection: $[P][Q] = [P \text{ and } Q]$ (Exercise 10.11). Hence, if we define

$$F^\perp := \{x \in \mathbb{R}^d \mid \langle x, y\rangle = 0 \text{ for all } y \in \operatorname{lin} F\},$$

Then we see that $\mathcal{P}^\perp = \{0\}$, and we can rewrite the latter identity as

$$\begin{aligned}\hat{1}_{\mathcal{P}}(\xi) &= \operatorname{vol}\mathcal{P}\,[\xi \in \mathcal{P}^\perp] + \\ &+ \frac{1}{-2\pi i} \sum_{F_1 \subset \partial P} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle (\operatorname{vol} F_1) e^{-2\pi i \langle \xi, x\rangle}}{||\xi||^2} \, [\xi \in F_1^\perp - \mathcal{P}^\perp] \\ &+ \frac{1}{(-2\pi i)^2} \sum_{F_1 \subset \partial P} \sum_{F_2 \subset \partial F_1} \frac{\langle \xi, \mathbf{n}_{F_1}\rangle}{||\xi||^2} \frac{\langle \operatorname{Proj}_{F_2} \xi, \mathbf{n}(F_2, F_1)\rangle}{||\operatorname{Proj}_{F_2}\xi||^2} \hat{1}_{F_2}(\xi)\, [\xi \notin F_1^\perp].\end{aligned}$$

In order to keep track of the iteration process, we will introduce another bookkeeping device. The **face poset** of a polytope $\mathcal{P}$ is defined to be the partially ordered set (poset) of all faces of $\mathcal{P}$, ordered by inclusion, including $\mathcal{P}$ and the empty set.

Example 10.4. Consider a 2-dimensional polytope $\mathcal{P}$ that is a triangle. We have the following picture for the face poset $\mathfrak{F}_P$ of $\mathcal{P}$, as in Figure 10.3. It turns out that if we consider a d-simplex $\mathcal{P}$, then its face poset $\mathfrak{F}_P$ has the structure of a "Boolean poset", which is isomorphic to the edge graph of a $(d+1)$-dimensional cube.

We only have to consider rooted chains in the face poset $\mathfrak{F}_P$, which means chains whose root is P. The only appearance of non-rooted chains are in the following definition. If G is a facet of F, we attach the following weight to any (local) chain (F, G), of length 1, in the face poset of P:

$$W_{(F,G)}(\xi) := \frac{-1}{2\pi i} \frac{\langle \operatorname{Proj}_F(\xi), \mathbf{n}(G, F)\rangle}{\| \operatorname{Proj}_F(\xi)\|^2}. \tag{10.13}$$

Note that these weights are functions of ξ rather than constants. Moreover, they are all homogeneous of degree -1. Let $\mathbf{T}$ be any rooted chain in $\mathfrak{F}_P$, given by

$$T := (P \to F_1 \to F_2, \ldots, \to F_{k-1} \to F_k),$$

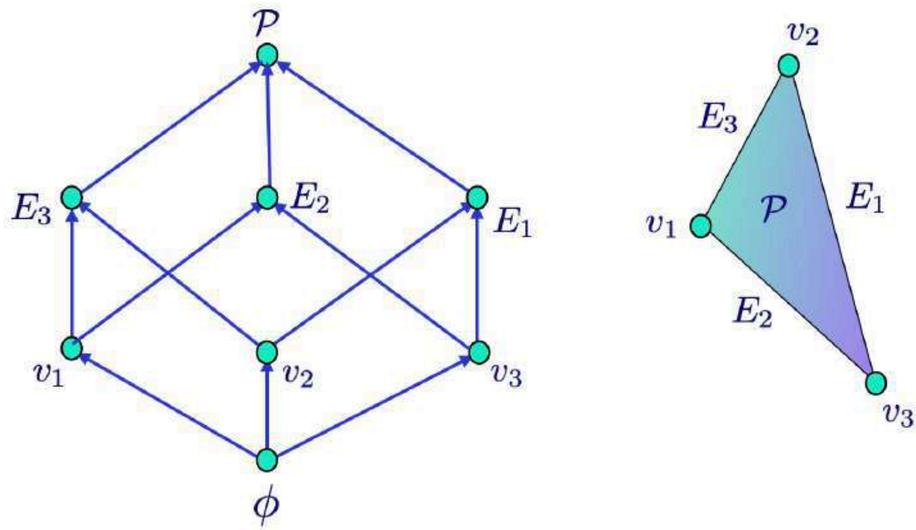


Figure 10.3: The face poset of a triangle

so that by definition $\dim(F_j) = d - j$. We define the **admissible set** $S(\mathbf{T})$ of the rooted chain $\mathbf{T}$ to be the set of all vectors $\xi \in \mathbb{R}^d$ that are orthogonal to the linear span of F_k but not orthogonal to the linear span of F_{k-1}. In other words,

$$\begin{aligned} S(\mathbf{T}) &:= \{\xi \in \mathbb{R}^d \mid \xi \perp \operatorname{lin}(F_k), \text{ but } \xi \not\perp \operatorname{lin}(F_{k-1})\} \\ &= \{\xi \in \mathbb{R}^d \mid \xi \in F_k^\perp - F_{k-1}^\perp\}. \end{aligned}$$

Finally, we define the following weights associated to any such rooted chain $\mathbf{T}$:

(a) The rational weight $\mathcal{R}_{\mathbf{T}}(\xi) = \mathcal{R}_{(P \to \dots \to F_{k-1} \to F_k)}(\xi)$ is defined to be the product of weights associated to all the rooted chains $\mathbf{T}$ of length 1, times the Hausdorff volume of F_k (the last node of the chain $\mathbf{T}$). It is clear from this definition that $\mathcal{R}_{\mathbf{T}}(\xi)$ is a homogeneous rational function of ξ.

(b) The exponential weight $\mathcal{E}_{\mathbf{T}}(\xi) = \mathcal{E}_{(P \to \dots \to F_{k-1} \to F_k)}(\xi)$ is defined to be the evaluation of $e^{-2\pi i \langle \xi, x\rangle}$ at any point x on the face F_k:

$$\mathcal{E}_{\mathbf{T}}(\xi) := e^{-2\pi i \langle \xi, x_0\rangle}, \tag{10.14}$$

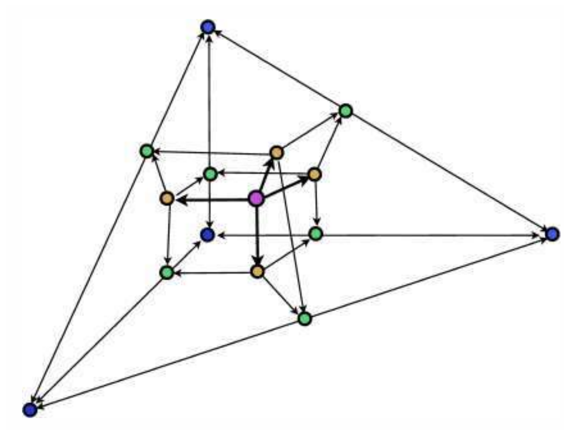


Figure 10.4: A symbolic depiction of the face poset $\mathfrak{F}_P$, where P is a 3-dimensional tetrahedron. Here the points and arrows are drawn suggestively, as a directed graph. We can see all the rooted chains, beginning from a symbolic vertex in the center, marked with the color purple. The rooted chains that terminate with the yellow vertices have length 1, those that terminate with the green vertices have length 2, and those that terminate with the blue vertices have length 3.

for any $x_0 \in F_k$. We note that the inner product $\langle \xi, x_0 \rangle$ does not depend on the position of $x_0 \in F_k$.

(c) The total weight of a rooted chain T is defined by the rational-exponential function

$$W_{\mathbf{T}}(\xi) = W_{(P \to \ldots \to F_{k-1} \to F_k)}(\xi) := \mathcal{R}_{\mathbf{T}}(\xi)\mathcal{E}_{\mathbf{T}}(\xi)\mathbf{1}_{S(\mathbf{T})}(\xi), \qquad (10.15)$$

where $\mathbf{1}_{S(\mathbf{T})}(\xi)$ is the indicator function of the admissible set $S(\mathbf{T})$ of $\mathbf{T}$.

By repeated applications of the combinatorial divergence Theorem 10.2, we now have a description of the Fourier transform of P as the sum of weights of all the rooted chains of the face poset $\mathfrak{F}_P$, as follows.

Theorem 10.3.

$$\hat{1}_P(\xi) = \sum_{\mathbf{T}} W_{\mathbf{T}}(\xi) = \sum_{\mathbf{T}} \mathcal{R}_{\mathbf{T}}(\xi)\mathcal{E}_{\mathbf{T}}(\xi)\mathbf{1}_{S(\mathbf{T})}(\xi), \qquad (10.16)$$

valid for any fixed $\xi \in \mathbb{R}^d$.

For a detailed proof, see Diaz, Le, and Robins (2016).

Using this explicit description of the Fourier transform of a polytope, we will see an application of it in the following section, for the coefficients of Macdonald's angle quasi-polynomial. In the process, equation (10.16), which gives an explicit description of the Fourier transform of a polytope, using the facets and lower-dimensional faces of $\mathcal{P}$, will become even more explicit with some examples.

10.3 Generic frequencies versus special frequencies

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$, we call a vector $\xi \in \mathbb{R}^d$ a **generic frequency** (relative to $\mathcal{P}$) if ξ is not orthogonal to any face of $\mathcal{P}$. All other $\xi \in \mathbb{R}^d$ are orthogonal to some face F of $\mathcal{P}$, and are called **special frequencies**. Let's define the following hyperplane arrangement, given by the finite collection of hyperplanes orthogonal to any edge of $\mathcal{P}$:

$$\mathcal{H} := \{x \in \mathbb{R}^d \mid \langle x, F_1 \rangle = 0, \text{ for any 1-dimensional edge } F_1 \text{ of } \mathcal{P}\}.$$

Then it is clear that the special frequencies are exactly those vectors that lie in the hyperplane arrangement $\mathcal{H}$. So we see from Theorem 10.3 that for a generic frequency ξ, we have

$$\hat{1}_P(\xi) = \sum_{\mathbf{T}:P\to\ldots\to F_1\to F_0} \mathcal{R}_{\mathbf{T}}(\xi) e^{-2\pi i\langle \xi, F_0\rangle}, \tag{10.17}$$

where the F_0 faces are the vertices of $\mathcal{P}$. In other words, for generic frequencies, all of our rooted chains in the face poset of $\mathcal{P}$ go all the way to the vertices. The special frequencies, however, are more complex. But we can collect the special frequencies in 'packets', giving us the following result.

Theorem 10.4 (Coefficients for Macdonald's angle quasi-polynomial)**.** *Diaz, Le, and Robins (ibid.) Let P be a d-dimensional rational polytope in $\mathbb{R}^d$, and let t be a positive real number. Then we have the quasi-polynomial*

$$A_P(t) = \sum_{i=0}^{d} a_i(t) t^i,$$

where, for $0 \leqslant i \leqslant d$,

$$a_i(t) := \lim_{\epsilon\to 0+} \sum_{\xi\in\mathbb{Z}^d\cap S(\mathbf{T})} \sum_{l(\mathbf{T})=d-i} \mathcal{R}_{\mathbf{T}}(\xi)\mathcal{E}_{\mathbf{T}}(t\xi)\, e^{-\pi\epsilon\|\xi\|^2}, \tag{10.18}$$

where $l(\mathbf{T})$ is the length of the rooted chain $\mathbf{T}$ in the face poset of P, $\mathcal{R}_{\mathbf{T}}(\xi)$ is the rational function of ξ defined above, $\mathcal{E}_{\mathbf{T}}(t\xi)$ is the complex exponential defined in (10.14) *above, and $\mathbb{Z}^d \cap S(\mathbf{T})$ is the set of all integer points that are orthogonal to the last node in the chain T, but not to any of its previous nodes.* □

For a detailed proof, see Diaz, Le, and Robins (ibid.).

We call the coefficients $a_i(t)$ the **quasi-coefficients** of the solid angle sum $A_P(t)$. As a consequence of Theorem 10.4, it turns out that there is a closed form for the codimension-1 quasi-coefficient, which extends the previously known special cases of this coefficient. We recall our first periodic Bernoulli polynomial, from (2.14):

$$P_1(x) := \begin{cases} x - \lfloor x\rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases} \tag{10.19}$$

where $\lfloor x\rfloor$ is the integer part of x.

Theorem 10.5. *Diaz, Le, and Robins (2016) Let $\mathcal{P}$ be any real polytope. Then the codimension-1 quasi-coefficient of the solid angle sum $A_P(t)$ has the following closed form:*

$$a_{d-1}(t) = - \sum_{\substack{F \text{ a facet of } \mathcal{P} \\ with\ v_F \neq 0}} \frac{\operatorname{vol}(F)}{\|v_F\|} P_1(\langle v_F, x_F \rangle t), \tag{10.20}$$

where v_F is the primitive integer vector which is an outward-pointing normal vector to F, x_F is any point lying in the affine span of F, and t is any positive real number. □

We note that, rather surprisingly, the latter formula shows in particular that for any real polytope $\mathcal{P}$, the quasi-coefficient $a_{d-1}(t)$ is always a periodic function of $t > 0$, with a period of 1. Although it is not necessarily true that for any real polytope the rest of the quasi-coefficients $a_k(t)$ are periodic functions of t, it is true that in the case of rational polytopes, the quasi-coefficients are periodic functions of all real dilations t, as we show below.

We recall that zonotopes are projections of cubes or, equivalently, polytopes whose faces (of all dimensions) are symmetric. We also recall the result of Alexandrov and Shephard (Theorem 4.9) from Chapter 4: If all the facets of $\mathcal{P}$ are symmetric, then $\mathcal{P}$ must be symmetric as well. The following result appeared in Barvinok and J. E. Pommersheim (1999), and here we give a different proof, using the methods of this chapter.

Theorem 10.6. *Suppose $\mathcal{P}$ is a d-dimensional integer polytope in $\mathbb{R}^d$ all of whose facets are centrally symmetric. Then*

$$A_{\mathcal{P}}(t) = (\operatorname{vol} \mathcal{P}) t^d,$$

for all positive integers t.

Proof. We recall the formula for the solid angle polynomial $A_{\mathcal{P}}(t)$.

$$A_{\mathcal{P}}(t) = \lim_{\epsilon \to 0^+} \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{t\mathcal{P}}(\xi) e^{-\pi\epsilon \|\xi\|^2}. \tag{10.21}$$

The Fourier transform of the indicator function of a polytope may be written as follows, after one application of the combinatorial divergence formula.

$$\hat{1}_{t\mathcal{P}}(\xi) = t^d \operatorname{vol}(\mathcal{P})[\xi = 0] + \left(\frac{-1}{2\pi i}\right) t^{d-1} \sum_{\substack{F \subseteq \mathcal{P} \\ \dim F = d-1}} \frac{\langle \xi, \mathbf{n}_F \rangle}{\|\xi\|^2} \hat{1}_F(t\xi)[\xi \neq 0], \tag{10.22}$$

where we sum over all facets F of $\mathcal{P}$. Plugging this into (10.21) we get

$$A_{\mathcal{P}}(t) - t^d \operatorname{vol}(\mathcal{P}) =$$
$$= \left(\frac{-1}{2\pi i}\right) t^{d-1} \lim_{\epsilon \to 0^+} \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{-\pi\epsilon\|\xi\|^2}}{\|\xi\|^2} \sum_{\substack{F \subseteq \mathcal{P} \\ \dim F = d-1}} \langle \xi, \mathbf{n}_F \rangle \hat{1}_F(t\xi) \tag{10.23}$$

Thus, if we show that the latter sum over the facets vanishes, then we are done.

The assumption that all facets of $\mathcal{P}$ are centrally symmetric implies that $\mathcal{P}$ itself is also centrally symmetric, by Theorem 4.9. We may therefore combine the facets of $\mathcal{P}$ in pairs of opposite facets F and F'. We know that $F' = F + c$, where c is an integer vector, using the fact that the facets are centrally symmetric.

Therefore, since $\mathbf{n}'_F = -\mathbf{n}_F$, we have

$$\begin{aligned} \langle \xi, \mathbf{n}_F \rangle \hat{1}_F(t\xi) + \langle \xi, -\mathbf{n}_F \rangle \hat{1}_{F+c}(t\xi) &= \\ &= \langle \xi, \mathbf{n}_F \rangle \hat{1}_F(t\xi) - \langle \xi, \mathbf{n}_F \rangle \hat{1}_F(t\xi) e^{-2\pi i \langle t\xi, c \rangle} \\ &= \langle \xi, \mathbf{n}_F \hat{1}_F(t\xi) \left(1 - e^{-2\pi i \langle t\xi, c \rangle}\right) = 0, \end{aligned}$$

because $\langle t\xi, c \rangle \in \mathbb{Z}$ when both $\xi \in \mathbb{Z}^d$ and $t \in \mathbb{Z}$. We conclude that the entire right-hand side of (10.23) vanishes, and we are done. □

Fourier analysis can also be used to give yet more general classes of polytopes that satisfy the formula $A_{\mathcal{P}}(t) = (\operatorname{vol} \mathcal{P}) t^d$, for positive integer values of t (See also Machado and Robins (2019), Deligne, Tabachnikov, and Robins (2014)).

At the moment, the following problem is still open.

Question 9. *Classify all integer polytopes $\mathcal{P}$ whose angle polynomial satisfies*

$$A_{\mathcal{P}}(t) = (\operatorname{vol} \mathcal{P}) t^d,$$

for all positive integers t.

There is a wonderful result of Minkowski that gives a converse to the relation (10.4), as follows.

Theorem 10.7 (The Minkowski problem for polytopes). *Suppose that $u_1, \ldots, u_k \in \mathbb{R}^d$ are unit vectors that do not lie in a hyperplane. Suppose further that we are given positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$ that satisfy the relation*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0.$$

Then there exists a polytope $\mathcal{P} \subset R^d$, with facet normals $u_1, \ldots, u_k \in \mathbb{R}^d$, and facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$. Moreover, this polytope $\mathcal{P}$ is unique, up to translations. □

There is a large body of work, since the time of Minkowski, that is devoted to extensions of Minkowski's Theorem 10.7, to other convex bodies, as well as to other manifolds.

Finally, we briefly mention that the angle polynomial $A_{\mathcal{P}}(t)$ also possesses the following fascinating functional equation (For a proof of Theorem 10.8, and an extension of it, see Desario and Robins (2011)).

Theorem 10.8 (Functional equation for the angle polynomial). *Given a d-dimensional rational polytope $\mathcal{P} \subset \mathbb{R}^d$, we have*

$$A_{\mathcal{P}}(-t) = A_{\mathcal{P}}(t),$$

for all $t \in \mathbb{Z}$. □

Notes

(a) We could also define another useful vector field, for our combinatorial divergence Theorem, besides our vector field in equation (10.5). Namely, if we define $F(x) := e^{2\pi i \langle x, \xi \rangle} \lambda$, for a fixed $\lambda \in \mathbb{C}^d$, then we would get the analogous combinatorial divergence formula as shown below in (Exercise 10.4), and such vector fields have been used, for example, by Alexander Barvinok (1992) in an effective way. To the best of our knowledge, the first researcher to use iterations of Stokes' formula to obtain lattice point asymptotics was Burton Randol (1984).

Exercises

10.1. ♣ *We define the distance from the origin to F, denoted by dist(F), as the length of the shortest vector of translation between aff(F) and lin(F) (resp. the affine span of F and the linear span of F, defined in* (10.9)*). Figure 10.2 shows what can happen in such a scenario.*

(a) *Suppose that we consider a facet F of a given polytope $\mathcal{P} \subset \mathbb{R}^d$, and we let $\boldsymbol{n}_F$ be the unit normal vector to F. Show that the function*

$$x_F \to \langle x_F, \boldsymbol{n}_F \rangle$$

is constant for $x_F \in F$, and is in fact equal to the distance from the origin to F. In other words, show that

$$\langle x, \boldsymbol{n}_F \rangle = dist(F).$$

(b) *Show that if $Proj_F \xi = 0$, then $\langle \xi, x_F \rangle = \|\xi\| dist F$.*

10.2. *Here we prove the elementary geometric formula for a pyramid over a polytope. Namely, suppose we are given a $(d-1)$-dimensional polytope $\mathcal{P}$, lying in the vector space defined by the first $d-1$ coordinates. We define a pyramid over $\mathcal{P}$, of height $h > 0$, as the d-dimensional polytope defined by*

$$Pyr(\mathcal{P}) := \operatorname{conv}\{\mathcal{P},\ h \cdot e_d\},$$

where $e_d := (0, 0, \dots, 0, 1) \in \mathbb{R}^d$. Show that

$$\operatorname{vol} Pyr(\mathcal{P}) = \frac{h}{d} \operatorname{vol} \mathcal{P}.$$

10.3. ♣ *Prove the Pyramid formula,* (10.3) *in Example 10.1, for a d-dimensional polytope $\mathcal{P}$ which contains the origin, but now using just elementary geometry:*

$$\operatorname{vol} \mathcal{P} = \frac{1}{d} \sum_{k=1}^{N} dist(G_k) \operatorname{vol} G_k, \tag{10.24}$$

where the G_k's are the facets of $\mathcal{P}$, and dist(G_k) is the distance from the origin to G_k.

Figure 10.5: The meaning of Minkowski's relation in dimension 2 - see Exercise 10.8

10.4. ♣ *Show that if we use the alternative vector field* $F(x) := e^{-2\pi i \langle x, \xi \rangle} \lambda$ *in equation* (10.5)*, with a constant nonzero vector* $\lambda \in \mathbb{C}^d$*, then we get:*

$$\hat{1}_{\mathcal{P}}(\xi) = \frac{1}{-2\pi i} \sum_{G \subset \partial P} \frac{\langle \lambda, \boldsymbol{n}_G \rangle}{\langle \lambda, \xi \rangle} \hat{1}_G(\xi), \tag{10.25}$$

valid for all nonzero $\xi \in \mathbb{R}^d$*. Note that one advantage of this formulation of the Fourier transform of* $\mathcal{P}$ *is that each summand in the right-hand-side of* (10.25) *is free of singularities, assuming the vector* λ *has a nonzero imaginary part.*

10.5. *Show that the identity* (10.25) *of Exercise 10.4 is equivalent to the vector identity:*

$$\xi \hat{1}_{\mathcal{P}}(\xi) = \frac{1}{-2\pi i} \sum_{G \subset \partial P} \boldsymbol{n}_G \hat{1}_G(\xi),$$

valid for all $\xi \in \mathbb{R}^d$.

10.6. *Show that the result of Exercise 10.5 quickly gives us the Minkowski relation* (10.4)*:*

$$\sum_{\textit{facets } \boldsymbol{G} \textit{ of } \boldsymbol{P}} \operatorname{vol}(G) \boldsymbol{n}_G = 0.$$

10.7. *Continuing Exercise 10.4, show that by iterating this particular version of the Fourier transform of a polytope* $\mathcal{P}$*,* k *times, we get:*

$$\hat{1}_{\mathcal{P}}(\xi) = \frac{1}{(-2\pi i)^k} \sum_{G_k \subset G_{k-1} \subset \cdots G_1 \subset \partial P} \prod_{j=1}^{k} \frac{\langle \lambda, \boldsymbol{n}_{G_j, G_{j-1}} \rangle}{\langle \lambda, Proj_{G_{j-1}} \xi \rangle} \hat{1}_{G_k}(\xi), \tag{10.26}$$

valid for all nonzero $\xi \in \mathbb{R}^d$*, and where we sum over all chains* $G_k \subset G_{k-1} \subset \cdots G_1$ *of length* k *in the face poset of* $\mathcal{P}$*, with* $codim(G_j) = j$.

10.8. *Show that in the case of polygons in* $\mathbb{R}^2$*, the Minkowski relation* (10.4) *has the meaning that the sum of the pink vectors in* Figure 10.5 *sum to zero. In other words, the geometric interpretation of the Minkowski relation in dimension* 2 *is that the sum of the boundary (pink) vectors wind around the boundary and close up perfectly.*

10.9. ♣ *Let's consider a simplex* $\Delta \subset \mathbb{R}^d$ *whose dimension satisfies* $2 \leqslant \dim \Delta \leqslant d$. *Show that* Δ *is not a symmetric body.*

10.10. *Let* $F \subset \mathbb{R}^d$ *be a centrally symmetric, integer polytope of dimension* k. *Show that the distance from the origin to* F *is always a half-integer or an integer. In other words, show that*

$$dist(F) \in \frac{1}{2}\mathbb{Z}.$$

(See Exercise 10.1 *above for the definition of distance of* F *to the origin)*

10.11. ♣ *To get more practice with the Iverson bracket, defined in* (10.7)*, show that:*

(a) $[P \text{ and } Q] = [P][Q]$.

(b) $[P \text{ or } Q] = [P] + [Q] - [P][Q]$.

11 Solutions and hints

There are no problems, just pauses between ideas.

– David Morrell, Brotherhood of the Rose

Chapter 1

Exercise 1.1 By Euler, we have $1 = e^{i\theta} = \cos\theta + i\sin\theta$, which holds if and only if $\cos\theta = 1$, and $\sin\theta = 0$. The latter two conditions hold simultaneously if and only if $\theta \in 2\pi k$, with $k \in \mathbb{Z}$.

Exercise 1.2 Let $z := a + bi$, so that $|e^z| = |e^{a+bi}| = |e^a||e^{bi}| = e^a \cdot 1 \leqslant e^{\sqrt{a^2+b^2}} = e^{|z|}$.

Exercise 1.3 In case $a \neq b$, we have

$$\int_0^1 e_a(x)\overline{e_b(x)}dx = \int_0^1 e^{2\pi i(a-b)x}dx = \frac{e^{2\pi i(a-b)}}{2\pi i(a-b)} - 1 = 0,$$

because we know that $a - b \in \mathbb{Z}$. In case $a = b$, we have

$$\int_0^1 e_a(x)\overline{e_a(x)}dx = \int_0^1 dx = 1.$$

Exercise 1.5 Let $S := \sum_{k=0}^{N-1} e^{\frac{2\pi ik}{N}}$, and note that we may write

$$S = \sum_{k \bmod N} e^{\frac{2\pi ik}{N}}.$$

Now, pick any m such that $e^{\frac{2\pi im}{N}} \neq 1$. Consider

$$\begin{aligned} e^{\frac{2\pi im}{N}} S &= \sum_{k \bmod N} e^{\frac{2\pi i(k+m)}{N}} \\ &= \sum_{n \bmod N} e^{\frac{2\pi in}{N}} = S, \end{aligned}$$

so that $0 = (e^{\frac{2\pi im}{N}} - 1)S$, and since by assumption $e^{\frac{2\pi im}{N}} \neq 1$, we have $S = 0$.

Exercise 1.6 We use the finite geometric series: $1 + x + x^2 + \cdots + x^{N-1} = \frac{x^N-1}{x-1}$. Now, if $N \nmid M$, then $x := e^{\frac{2\pi iM}{N}} \neq 1$, so we may substitute this value of x into the finite geometric series to get:

$$\begin{aligned} \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi ikM}{N}} &= \frac{e^{\frac{2\pi iMN}{N}} - 1}{e^{\frac{2\pi iM}{N}} - 1} \\ &= \frac{0}{e^{\frac{2\pi iM}{N}} - 1} = 0. \end{aligned}$$

On the other hand, if $N \mid M$, then $\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi ikM}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} 1 = 1$.

Exercise 1.8 We begin with the factorization of the polynomial $x^n - 1 = \prod_{k=1}^{n}(x - \zeta^k)$, with $\zeta := e^{2\pi i/n}$. Dividing both sides by $x - 1$, we obtain $1 + x + x^2 + \cdots + x^{n-1} = \prod_{k=1}^{n-1}(x - \zeta^k)$. Now substituting $x = 1$, we have $n = \prod_{k=1}^{n-1}(1 - \zeta^k)$.

Exercise 1.7

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i k a}{N}} e^{-\frac{2\pi i k b}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i k(a-b)}{N}}.$$

Therefore, using Exercise 1.6, we see that the latter sum equals 1 exactly when $N \mid a-b$, and vanishes otherwise.

Exercise 1.4 By definition,

$$\begin{aligned}\int_{[0,1]} e^{-2\pi i \xi x} dx &:= \int_{[0,1]} \cos(2\pi\xi x)dx + i\int_{[0,1]} \sin(2\pi\xi x)dx \\ &= \frac{\sin(2\pi\xi)}{2\pi\xi} + i\frac{-\cos(2\pi\xi)+1}{2\pi\xi} \\ &= \frac{i\sin(2\pi\xi)}{2\pi i\xi} + \frac{\cos(2\pi\xi)-1}{2\pi i\xi} \\ &= \frac{e^{2\pi i\xi}-1}{2\pi i\xi}.\end{aligned}$$

Exercise 1.9 Suppose to the contrary, that a primitive N'th root of unity is of the form $e^{2\pi i m/N}$, where $\gcd(m,N) > 1$. Let $m_1 := \frac{m}{\gcd(m,N)}$, and $k := \frac{N}{\gcd(m,N)}$, so that by assumption both m_1 and k are integers. Thus $e^{2\pi i m/N} = e^{2\pi i m_1/k}$, a k'th root of unity, with $k < N$, a contradiction.

Exercise 1.13 We recall Euler's identity:

$$e^{iw} = \cos w + i \sin w,$$

which is valid for all $w \in \mathbb{C}$. Using Euler's identity first with $w := \pi z$, and then with $w := -\pi z$, we have the two identities $e^{\pi i z} = \cos \pi z + i \sin \pi z$, and $e^{-\pi i z} = \cos \pi z - i \sin \pi z$. Subtracting the second identity from the first, we have

$$\sin(\pi z) = \frac{1}{2i}\left(e^{\pi i z} - e^{-\pi i z}\right).$$

Now it's clear that $\sin(\pi z) = 0 \iff e^{\pi i z} = e^{-\pi i z} \iff e^{2\pi i z} = 1 \iff z \in \mathbb{Z}$, by Exercise 1.1.

Exercise 1.14 We will assume, to the contrary, that we only have one arithmetic progression with a common difference of a_N, the largest of the common differences. We hope to obtain a contradiction.

To each arithmetic progression $\{a_k n + b_k \mid n \in \mathbb{Z}\}$, we associate the generating function

$$f_k(q) := \sum_{a_k n + b_k \geqslant 0,\ n \in \mathbb{Z}} q^{a_k n + b_k},$$

where $|q| < 1$, in order to make the series converge. The hypothesis that we have a tiling of the integers by these N arithmetic progressions translates directly into an identity among these generating functions:

$$\sum_{a_1 n + b_1 \geqslant 0,\ n \in \mathbb{Z}} q^{a_1 n + b_1} + \cdots + \sum_{a_N n + b_N \geqslant 0,\ n \in \mathbb{Z}} q^{a_N n + b_N} = \sum_{n=0}^{\infty} q^n.$$

Next, we use the fact that we may rewrite each generating function in a 'closed form' of the following kind, because they are geometric series: $f_k(q) := \sum_{a_k n + b_k \geqslant 0,\ n \in \mathbb{Z}}$ $\frac{q^{b_k}}{1-q^{a_k}}$. Thus, we have:

$$\frac{q^{b_1}}{1-q^{a_1}} + \cdots + \frac{q^{b_N}}{1-q^{a_N}} = \frac{1}{1-q}.$$

Now we make a 'pole-analysis' by observing that each rational function $f_k(q)$ has poles at precisely all of the k'th roots of unity. The final idea is that the 'deepest' pole, namely $e^{\frac{2\pi i}{N}}$, cannot cancel with any of the other poles. To make this idea precise, we isolate the only rational function that has this pole (by assumption):

$$\frac{q^{b_N}}{1-q^{a_N}} = \frac{1}{1-q} - \left(\frac{q^{b_1}}{1-q^{a_1}} + \cdots + \frac{q^{b_{N-1}}}{1-q^{a_{N-1}}} \right).$$

Finally, we let $q \to e^{\frac{2\pi i}{N}}$, to get a finite number on the right-hand-side, and infinity on the left-hand-side of the latter identity, a contradiction.

Chapter 2

Exercise 2.1 If $\xi = 0$, we have $\hat{1}_{[a,b]}(0) := \int_a^b e^0 dx = b - a$. If $\xi \neq 0$, we can compute the integral:

$$\begin{aligned}\hat{1}_{[a,b]}(\xi) &:= \int_a^b e^{-2\pi i \xi x} dx \\ &= \frac{e^{-2\pi i \xi b} - e^{-2\pi i \xi a}}{-2\pi i \xi}.\end{aligned}$$

Exercise 2.2 Beginning with the definition of the Fourier transform of the unit cube $[0, 1]^d$, we have:

$$\begin{aligned}\hat{1}_{\square}(\xi) &= \int_{\square} e^{2\pi i \langle x, \xi \rangle} dx \\ &= \int_0^1 e^{2\pi i \xi_1 x_1} dx_1 \int_0^1 e^{2\pi i \xi_2 x_2} dx_2 \cdots \int_0^1 e^{2\pi i \xi_d x_d} dx_d \\ &= \frac{1}{(-2\pi i)^d} \prod_{k=1}^{d} \frac{e^{-2\pi i \xi_k} - 1}{\xi_k},\end{aligned}$$

valid for all $\xi \in \mathbb{R}^d$, except for the finite union of hyperplanes defined by $H := \{x \in \mathbb{R}^d \mid \xi_1 = 0 \text{ or } \xi_2 = 0 \ldots \text{ or } \xi_d = 0\}$.

Exercise 2.4 To see that the generating function definition of the Bernoulli polynomials in fact gives polynomials, we first write the Taylor series of the following two analytic functions:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$$

$$e^{xt} = \sum_{j=0}^{\infty} \frac{x^j t^j}{j!}.$$

Multiplying these series together by brute-force gives us:

$$\frac{t}{e^t-1}e^{xt} = \left(\sum_{k=0}^{\infty}\frac{B_k}{k!}t^k\right)\left(\sum_{j=0}^{\infty}\frac{x^j}{j!}t^j\right) \tag{11.1}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j+k=n}\frac{B_k}{k!}\frac{x^j}{j!}\right)t^n \tag{11.2}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!}\right)t^n. \tag{11.3}$$

The coefficient of t^n on the LHS is by definition $\frac{1}{n!}B_n(x)$, and by uniqueness of Taylor series, this must also be the coefficient on the RHS, which is seen here to be a polynomial in x. In fact, we see more, namely that

$$\frac{1}{n!}B_n(x) = \sum_{k=0}^{n}\frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!},$$

which can be written more cleanly as $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$.

Exercise 2.5 Commencing with the generating function definition of the Bernoulli polynomials, Equation (2.13), we replace x with $1-x$ in order to observe the coefficients $B_k(1-x)$:

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{B_k(1-x)}{k!}t^k &= \frac{te^{t(1-x)}}{e^t-1} \\
&= \frac{te^te^{-tx}}{e^t-1} \\
&= \frac{te^{-tx}}{1-e^{-t}} \\
&= \frac{-te^{-tx}}{e^{-t}-1} \\
&= \sum_{k=0}^{\infty}\frac{B_k(x)}{k!}(-t)^k,
\end{aligned}$$

where the last equality follows from the definition of the same generating function, namely Equation (2.13), but with the variable t replaced by $-t$. Comparing the coefficient of t^k on both sides, we have $B_k(1-x) = (-1)^k B_k(x)$.

Exercise 2.6 To show that $B_n(x+1) - B_n(x) = nx^{n-1}$, we play with:

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{B_k(x+1)}{k!}t^k - \frac{B_k(x)}{k!}t^k\right) &= \frac{te^{t(x+1)}}{e^t-1} - \frac{te^{t(x)}}{e^t-1} \\
&= e^t\frac{te^{tx}}{e^t-1} - \frac{te^{t(x)}}{e^t-1} \\
&= (e^t-1)\frac{te^{tx}}{e^t-1} \\
&= te^{tx} \\
&= \sum_{k=0}^{\infty}\frac{x^k}{k!}t^{k+1} \\
&= \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}t^k \\
&= \sum_{k=1}^{\infty}\frac{kx^{k-1}}{k!}t^k.
\end{aligned}$$

Therefore, again comparing the coefficients of t^k on both sides, we arrive at the required identity.

Exercise 2.7 We need to show that $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$. Well,

$$\sum_{k=0}^{\infty} \frac{d}{dx}\frac{B_k(x)}{k!}t^k = \frac{d}{dx}\frac{te^{tx}}{e^t - 1}$$
$$= t\sum_{k=0}^{\infty} \frac{B_k(x)}{k!}t^k$$
$$= \sum_{k=0}^{\infty} \frac{B_k(x)}{k!}t^{k+1}$$
$$= \sum_{k=1}^{\infty} \frac{B_{k-1}(x)}{(k-1)!}t^k$$
$$= \sum_{k=1}^{\infty} k\frac{B_{k-1}(x)}{k!}t^k,$$

so that comparing the coefficient of t^k on both sides, the proof is complete.

Exercise 2.27 Considering the partial sum $S_n := \sum_{k=1}^{n} a_k b_k$, we know by Abel summation that

$$S_n = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1}),$$

for each $n \geqslant 2$, where $B_n := \sum_{k=1}^{n} b_k$. By assumption, $|B_n| := |\sum_{k=1}^{n} b_k| \leqslant M$, and the a_k's are going to 0, so we see that the first part of the right-hand-side approaches zero, namely: $|a_n B_n| := |a_n||\sum_{k+1}^{n} b_k| \to 0$, as $n \to \infty$.

Next, we have

$$|\sum_{k=1}^{n-1} B_k(a_k - a_{k+1})| \leqslant \sum_{k=1}^{n-1} |B_k||a_k - a_{k+1}| \leqslant M\sum_{k=1}^{n-1} |a_k - a_{k+1}|$$
$$= M\sum_{k=1}^{n-1}(a_k - a_{k+1}),$$

where the last equality holds because by assumption the a_k's are decreasing. But the last finite sum equals $-Ma_n + Ma_1$, and we have $\lim_{n\to\infty}(-Ma_n + Ma_1) = Ma_1$, a finite limit.

Therefore $\sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$ converges absolutely, and so S_n converges, as desired.

Exercise 2.29 We fix $x \in \mathbb{R} - \mathbb{Z}$, and let $z := e^{2\pi i x}$, which lies on the unit circle, and by assumption $z \neq 1$. Then

$$\left|\sum_{k=1}^{n} e^{2\pi i k x}\right| = \left|\sum_{k=1}^{n} z^k\right| = \left|\frac{z^{n+1}-1}{z-1}\right| \leqslant \frac{2}{z-1}, \tag{11.4}$$

because $|z^{n+1} - 1| \leqslant |z^{n+1}| + 1 = 2$. We also have

$$|z-1|^2 = |e^{2\pi i x} - 1||e^{-2\pi i x} - 1| = |2 - 2\cos(2\pi x)| = 4\sin^2(\pi x),$$

so that we have the equality $\left|\frac{2}{z-1}\right| = \left|\frac{1}{\sin(\pi x)}\right|$. Altogether, we see that

$$\left|\sum_{k=1}^{n} e^{2\pi i k x}\right| \leqslant \frac{1}{|\sin(\pi x)|}. \tag{11.5}$$

Exercise 2.30 We fix $a \in \mathbb{R} - \mathbb{Z}$ and need to prove that $\sum_{m=1}^{\infty} \frac{e^{2\pi i m a}}{m}$ converges. Abel's summation formula (2.72) gives us

$$\sum_{k=1}^{n} \frac{e^{2\pi i k a}}{k} = \frac{1}{n}\sum_{r=1}^{n} e^{2\pi i r a} + \sum_{k=1}^{n-1}\Big(\sum_{r=1}^{k} e^{2\pi i r a}\Big)\frac{1}{k(k+1)},$$

so that

$$\sum_{k=1}^{\infty} \frac{e^{2\pi i k a}}{k} = \sum_{k=1}^{\infty}\Big(\sum_{r=1}^{k} e^{2\pi i r a}\Big)\frac{1}{k(k+1)}.$$

and the latter series in fact converges absolutely.

Chapter 3

Exercise 3.1 The first part follows from the fact that $\sqrt{a^2+b^2} \leqslant |a| + |b|$, which is clear by squaring both sides. For the second part, we use the Cauchy–Schwarz inequality, with the two vectors $x := (a_1, \ldots, a_d)$ and $(1, 1, \ldots, 1)$:

$$\|x\|_1 := |a_1| \cdot 1 + \cdots + |a_d| \cdot 1 \leqslant \sqrt{a_1^2 + \cdots + a_d^2}\sqrt{1 + \cdots + 1} = \sqrt{d}\ \|x\|_2,$$

which also shows that we obtain equality if and only if $(a_1, \dots, a_d)$ is a scalar multiple of $(1, 1, \dots, 1)$.

Chapter 4

Exercise 4.9 We use the Cauchy–Schwartz inequality:

$$\left\langle \binom{a}{b}, \binom{\sin x}{\cos x} \right\rangle^2 := (a \sin x + b \cos x)^2 \leqslant \left(a^2 + b^2\right)\left(\sin^2 x + \cos^2 x\right) \\ = a^2 + b^2.$$

By the equality condition of Cauchy–Schwartz, we see that the maximum is obtained when the two vectors are linearly dependent, which gives $\tan x = \frac{a}{b}$.

Chapter 5

Exercise 5.20 It's easy to see that the inverse matrix for M is

$$M^{-1} := \begin{pmatrix} | & | & \cdots & | \\ \frac{1}{c_1} b_1 & \frac{1}{c_2} b_2 & \cdots & \frac{1}{c_d} b_d \\ | & | & \cdots & | \end{pmatrix}^T .$$

The image of the unit sphere under the matrix M is, by definition:

$$\begin{aligned} M(S^{d-1}) &:= \{u \in \mathbb{R}^d \mid u = Mx, x \in S^{d-1}\} \\ &= \{u \in \mathbb{R}^d \mid M^{-1}u \in S^{d-1}\} \\ &= \{u \in \mathbb{R}^d \mid \frac{1}{c_1^2} \langle b_1, u \rangle^2 + \cdots + \frac{1}{c_d^2} \langle b_d, u \rangle^2 = 1\}, \end{aligned}$$

using our description of M^{-1} above.

For part (b), we begin with the definition of volume, and we understand that we want to compute the volume of the region $M(B^d) := \{u \in \mathbb{R}^d \mid u = My, \text{ with } \|y\| \leqslant 1\}$.

$$\begin{aligned} \mathrm{vol}(Ellipsoid_M) &:= \int_{M(B^d)} du \\ &= |\det M| \int_{B^d} dy \\ &= |\det M| \, \mathrm{vol}(B^d). \end{aligned}$$

using the change of variable $u = My$, with $y \in B^d$. We also used the Jacobian, which gives $du = |\det M| dy$.

Finally, we note that the matrix $M^T M$ is a diagonal matrix, with diagonal entries c_k^2, due to the fact that the b_k's form an orthonormal basis. Thus we use: $|\det M|^2 = |\det M^T M| = \prod_{k=1}^d c_k^2$, so taking the positive square root, we arrive at $|\det M| = \prod_{k=1}^d c_k$, because all of the c_k's are positive by assumption.

Chapter 7

Exercise 7.6 Here $\mathcal{P} := \text{conv}\{C, \mathbf{e_d}\}$, where C is the $(d-1)$-dimensional unit cube $[0,1]^{d-1}$. To compute the Ehrhart polynomial $\mathcal{L}_{\mathcal{P}}(t)$ here, we use the fact that a 'horizontal' slice of $\mathcal{P}$, meaning a slice parallel to C, and orthogonal to e_d, is a dilation of C. Thus, each of these slices counts the number of points in a k-dilate of C, as k varies from 0 to $t+1$. Summing over these integer dilations of C, we have

$$\mathcal{L}_{\mathcal{P}}(t) = \sum_{k=0}^{t+1} (t+1-k)^{d-1} = \sum_{k=0}^{t+1} k^{d-1} = \frac{1}{d}(B_d(t+2) - B_d),$$

where the last step holds thanks to Exercise 2.8.

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