

# **Limit Cycles, Abelian Integral and Hilbert's Sixteenth Problem**



Publicações Matemáticas

**Limit Cycles, Abelian Integral and  
Hilbert's Sixteenth Problem**

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31<sup>o</sup> Colóquio Brasileiro de Matemática

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*Dedicated to the memory of Elon Lages Lima*



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# Preface

The main goal of the present book is to collect old and recent developments in direction of Hilbert's sixteenth problem. The main focus has been on limit cycles arising from perturbations of Hamiltonian systems and the study of corresponding Abelian and iterated integrals. The second author acknowledges the hospitality of IMPA and the financial support of CNPq during the preparation of this text.

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*May 2017*  
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# Chapter 1

## Introduction

### 1.1 Hilbert's sixteenth problem

The main aim of these notes is to introduce the reader with some problems arising from the centennial Hilbert's 16th problem, H16 for short. It appears in the famous list of Hilbert's problems which he proposed in the International Congress of Mathematicians, 1901 in Paris, see [45]. Our aim is not to collect all the developments and theorems in direction of H16 (for this see for instance [55]), but to present a way of breaking the problem in many pieces and observing the fact that even such partial problems are extremely difficult to treat. Our point of view is algebraic and we want to point out that the both real and complex algebraic geometry would be indispensable for a systematic approach to the H16. Here is Hilbert's announcement of the problem:

16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the  $n$ -th order can have has been determined by Harnack. There arises the further question as to the relative position

of the branches in the plane. As to curves of the 6-th order, I have satisfied myself by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4-th order in three dimensional space can really have.

In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré boundary cycles (cycles limits) for a differential equation of the first order and degree of the form

$$dy/dx = Y/X,$$

where X and Y are rational integral functions of the n-th degree in x and y. Written homogeneously, this is

$$X(ydz/dt - zdy/dt) + Y(zdx/dt - xdz/dt) + Z(xdy/dt - ydx/dt) = 0$$

where X, Y, and Z are rational integral homogeneous functions of the n-th degree in x, y, z, and the latter are to be determined as functions of the parameter t. <sup>1</sup>

In the modern literature, one usually finds the differential equation

$$\begin{cases} \dot{x} = P_n(x, y) \\ \dot{y} = Q_n(x, y) \end{cases}, \quad (1.1)$$

where  $P_n$  and  $Q_n$  are real polynomials in  $x, y$  with degrees  $\leq n$ . The second half of the famous Hilbert's 16th problem which is related to the system (1.1) can be stated as follows:

---

<sup>1</sup>Translation taken from  
<http://aleph0.clarku.edu/~djoyce/hilbert/problems.html>

**Problem 1.1.** *What is the maximum number, denoted by  $H(n)$ , of limit cycles for all possible  $P_n$  and  $Q_n$ ? And how about the possible relative positions of limit cycles of the system (1.1).*

A limit cycle  $\gamma$  of the system (1.1) is an isolated closed solution in the set of all closed solutions, i.e. there exists a neighborhood  $V$  of  $\gamma$  such that  $\gamma$  is the only closed solution contained in  $V$ . The number  $H(n)$  is commonly called the *Hilbert number*.

For instance, the differential equation

$$\begin{cases} \dot{x} = 2y + \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + 0.9y \end{cases} \quad (1.2)$$

has a limit cycle which is depicted in Figure 1.1. One of the goals

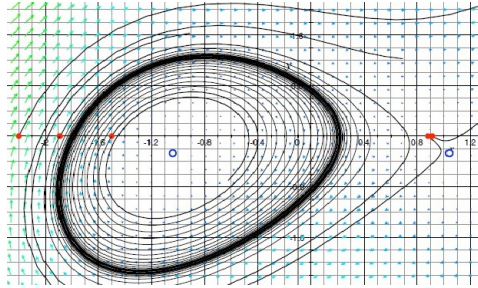


Figure 1.1: A limit cycle crossing  $(x, y) \sim (-1.79, 0)$

of the present text is to explain the fact that Hilbert's 16th problem is a combination of many unsolved difficult problems. In many applications the number and positions of limit cycles are important to understand the dynamical behavior of the system. We observe that the problem is trivial for  $n = 1$ , because a linear system has not limit cycles, so we assume that  $n \geq 2$ . The Hilbert's 16th problem in the case  $n = 2$  is still open.

## 1.2 Other variations of Hilbert's 16th problem

The first main challenge in direction of Hilbert's 16 problem is to prove that  $H(n)$  is finite.

**Problem 1.2. (Dulac problem)** *For a given integer  $n$ , the number of limit cycles of the planar differential system given by (1.1) is finite?*

This problem is also known as the *finiteness problem*. As far as we know the notion of limit cycles appeared in the works of Poincaré [95] during the years 1891 and 1897. He proved that a polynomial differential system given by (1.1) without saddle connections has finitely many limit cycles. The finiteness problem was firstly studied by Dulac in 1923, see [17], but later remarked that the proof is incomplete. In 1985 Bamón ([7], [8]) proved this individual finiteness property for quadratic case ( $n = 2$ ). In ([51], [23]) Yu. Ilyashenko and J. Ecalle around 80's published independently, new proofs on the individual finiteness theorem, filling up the gap in Dulac's paper. The finiteness problem is the most general fact established so far in connection with the Hilbert's 16th Problem. For more information with respect to the Hilbert's 16th problem and for other details the reader can consult the survey article of Yu. Ilyashenko [55], the survey article of J. Li [58] or the book of C. Christopher and C. Li [15]. As S. Smale said in [102] on the problems of XXI century: "except for the Riemann hypothesis, the second part of the Hilbert's 16th problem seems to be the most elusive of Hilbert's problems".

Bifurcation theory is related to Hilbert's 16th Problem. Indeed, the function of number of limit cycles of the equation has points of discontinuity corresponding to equations whose perturbations generate limit cycles via bifurcations. Limit cycles may bifurcate from polycycles. A *polycycle* is a connected finite union of singular points and solution orbits of the system. In general, a polycycle form a kind of polygon, see Figure 1.2 and the cyclicity of the polycycle in a family of differential equations is the maximal number of limit cycles that may bifurcate from the polycycle in this family. The following theorem is called non-accumulation theorem

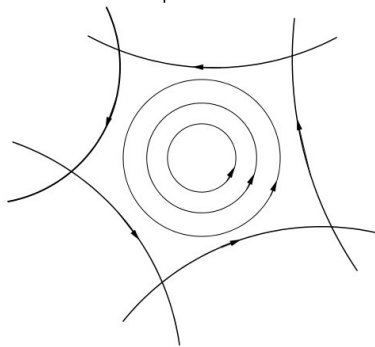


Figure 1.2: A polycycle of the system (1.1)

**Theorem 1.1. (Ilyashenko-Écalle [51],[23])** *For an arbitrary analytic vector field in a two dimensional real analytic surface, limit cycles cannot accumulate on a polycycle.*

The next step is to understand the *uniform finiteness*, i. e.

$$H(n) < \infty, \quad \forall n \in \mathbb{N}.$$

**Problem 1.3. (Existential Hilbert problem)** *Is it true that for any  $n \in \mathbb{N}$ , the number of limit cycles of (1.1) is bounded by a constant that only depends on the degree  $n$ ?*

In 1988 R. Roussarie [100] proposed a program to prove the uniform finiteness by reducing this problem, via compactification of the system and of the parameter space, to the problem of proving the finite cyclicity of limit periodic sets. Dumortier, Roussarie and Rousseau in [18] started this program for the quadratic case and listed 121 graphics as all limit periodic sets. First several cases were studied in the same paper by applying standard tools of bifurcation theory and others of such graphics have been studied by many authors in different papers. The remaining graphics are more degenerates and the study of them is more difficult.

**Problem 1.4. (Constructive Hilbert problem)** *Is it possible to find an explicit upper bound for the Hilbert number  $H(n)$ ?*

A solution to each problem mentioned above imply a solution to the precedent problems. Only the Dulac's problem has been solved completely. There exist analytical counterparts associated with Dulac and existential problems, respectively. In this context, an analytic (finite) family of vector fields is a family of vector fields that has a finite number of parameters which depend analytically on these parameters. More details for the analytic cases may be seen in [101]. Other problems associated to the Hilbert's 16th problem are:

**Problem 1.5.** *Has a family of analytic vector fields in the two dimension sphere  $\mathbb{S}^2$  a finite number of limit cycles?*

**Problem 1.6.** *For a family of analytic vector field in the two dimensional sphere  $\mathbb{S}^2$ , there exists a uniform bound of the number of limit cycles that only depends on the parameters set.*

Let  $(X_\lambda)_\lambda$  be a  $\mathcal{C}^1$ -family of vector fields on a surface  $M$  and  $\lambda$  belong to a parameter space in  $\mathbb{R}^\Lambda$ .

**Definition 1.1.** A *limit periodic set* for  $(X_\lambda)_\lambda$  is a compact non-empty subset  $\Gamma$  in  $M$ , such that there exists a sequence  $(\lambda_k)_{k \in \mathbb{N}}$ ,  $\lambda_k \rightarrow \lambda_0$  in the parameter space, and for each  $\lambda_k$ , the vector field  $X_{\lambda_k}$  has a limit cycle  $\Gamma_{\lambda_k}$  with the following property:

$$\text{dist}_H(\Gamma, \Gamma_{\lambda_k}) \rightarrow 0,$$

where  $k \rightarrow \infty$  in the Hausdorff topology of the space  $\mathcal{C}(M)$  of all non-empty compact subset of  $M$ .

The structure of a limit periodic set admits a simple geometric description in the sense that a limit periodic set is either a polycycle (a union of connected singular points and regular curves) or contains an arc of non-isolated singularities of the vector field  $X_0$  (see [24]).

A more precise definition of the number of limit cycles which bifurcate from a limit periodic set is given by:

**Definition 1.2. (Roussarie [101])** Let  $\Gamma$  be a limit periodic set of  $\mathcal{C}^1$ -family  $X_\lambda$ , defined at some value  $\lambda_0$ . Denote by  $\text{dist}_H$  the induced Hausdorff distance and by  $d$  a distance defined in the parameter



set. For each  $\varepsilon, \delta > 0$  define

$$N(\delta, \varepsilon) = \text{Sup}\{\text{number of limit cycles } \gamma \text{ of } X_\lambda: \\ \text{dist}_H(\gamma, \Gamma) \leq \varepsilon; d(\lambda, \lambda_0) \leq \delta\}$$

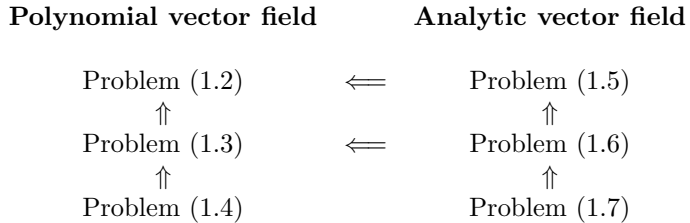
then the *cyclicity* of the germ  $(X_\lambda, \Gamma)$  is define by

$$\text{Cycl}(X_\lambda, \Gamma) = \text{Inf}_{\varepsilon, \delta} N(\delta, \varepsilon).$$

As indicated in the definition, this bound  $\text{Cycl}(X_\lambda, \Gamma)$  depends only on the germ of  $X_\lambda$  along the limit periodic set  $\Gamma$ .

**Problem 1.7. (Finite cyclicity)** *An arbitrary limit periodic set of an analytic family of vector fields in the 2-sphere  $\mathbb{S}^2$  has a finite cyclicity in this family.*

The diagram below shows the relationship between different versions of the Hilbert's 16th problem:



### 1.3 Infinitesimal Hilbert's 16th problem

In this section we give a weak version of the Hilbert's 16th problem which was proposed by V. I. Arnold (see [4]). For the relation of this with limit cycles see Chapter 3. We consider a real polynomial  $H$  of degree  $n + 1$  in the plane  $\mathbb{R}^2$ . We denote by  $\gamma(t)$  a continuous family of ovals defined in the level curves  $\{(x, y) \in \mathbb{R}^2 : H(x, y) = t\}$ , see for instance Figure 1.3.

Consider  $\omega = p(x, y)dy - q(x, y)dx$  with  $p, q \in \mathbb{R}[x, y]$  a polynomial 1-form where the maximum degree of  $p$  and  $q$  is  $m \geq 2$  and the

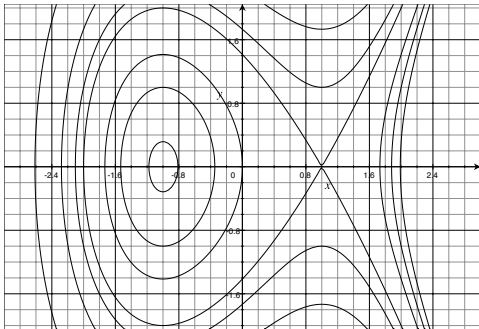


Figure 1.3: Ovals in the level curves of  $H = y^2 - x^3 + 3x$ .

integral

$$I(t) = \int_{\gamma(t)} \omega \quad (1.3)$$

over the continuous family of ovals  $\gamma(t)$ .

**Definition 1.3.** A real (complete) *Abelian integral* associated to a Hamiltonian  $H$  and to a polynomial 1-form  $\omega = p(x, y)dy - q(x, y)dx$  with  $H, p, q \in \mathbb{R}[x, y]$  is a multivalued function  $I(t)$  on the real variable  $t$  defined by integration of  $\omega$  over a real oval  $\gamma(t)$  of the algebraic curve  $\{(x, y) \in \mathbb{R}^2 : H(x, y) = t\}$ .

Arnold proposed the following problem.

**Problem 1.8. (Infinitesimal Hilbert's 16th problem)** For fixed integers  $n$  and  $m$  find the maximum number  $Z(n, m)$  of isolated real zeros of an Abelian integral defined as in (1.3).

In general, the Abelian integral (1.3) is a multivalued function, in the sense that there might exist several ovals lying on the same level curve  $\{H^{-1}(t)\}$ . Note that the maximum number  $Z(n, m)$  must be a uniform bound with respect to all possible Hamiltonian  $H$  with all possible families of ovals  $\gamma(t)$  and arbitrary polynomial 1-forms  $\omega$ . The maximum bound must depend only on the degrees  $n$  and  $m$ . The function  $I(t)$ , given by the Abelian integral (1.3), is in general,

the first approximation, with respect to small parameter  $\varepsilon$ , of the displacement function defined on a segment transversal to the families of ovals  $\gamma(t)$  for the small perturbations of the Hamiltonian systems associated to  $H$ .

Fix any two integer numbers  $n$  and  $m$  and consider all Hamiltonian systems of degree  $n$  and all polynomial 1-forms of degree  $m$ . The existence of uniform upper bounds for the number of isolated zeros of the Abelian integrals (1.3), was proved in 1984. This result is the most general result related to the infinitesimal Hilbert's 16th problem.

**Theorem 1.2. (Khovansky-Varchenko [57], [105])** *For any integer numbers  $n$  and  $m$ , the maximum number of isolated zeros of the function  $I(t)$  defined by (1.3) is bounded by a constant  $N = N(n, m)$  uniformly over all Hamiltonian of degree  $n$  and all polynomial 1-forms of degree  $m$ .*

Theorem 1.2 is an existential statement, giving no information on the number  $Z(n, m)$ . The problem to find isolated zeros of Abelian integrals is usually called the weak (or tangential, infinitesimal) Hilbert's 16th problem, and the number  $\tilde{Z}(n) = Z(n + 1, n)$  can be chosen as a lower bound of the Hilbert number  $H(n)$ . The connection between zeros of Abelian integrals (1.3) and limit cycles is given in Chapter 3.

**Theorem 1.3. (Binyamini, D. Novikov and S. Yakovenko, [6])** *We have*

$$\tilde{Z}(n) \leq 2^{2^{Poly(n)}},$$

where  $Poly(n) = O(n^p)$  stands for an explicit polynomially growing term with the exponent  $p$  not exceeding 61.

This is the first explicit uniform bound for the number of isolated zeros of Abelian integrals  $I(t)$  with  $m = n - 1$ . In order to prove this result, the authors use the complexification of the Abelian integrals and they reduce the weak Hilbert's 16th problem to a question about zeros of solutions to an integrable Pfaffian system subject to a condition on its monodromy. They use the fact that Abelian integrals of a given degree are horizontal sections of a regular flat meromorphic connection (the Gauss-Manin connection) with a quasiunipotent

monodromy group. Based on the above result, the same authors also made the following conjecture:

**Conjecture 1.1.**

$$Z(n, m) = 2^{2^{Poly(n)}} + O(m); \text{ as } n, m \rightarrow \infty.$$

In Chapter 3 we discuss the relation of zeros of an Abelian integral  $I(t)$  defined by (1.3) with small polynomial perturbations of linear Hamiltonian systems of type

$$dH + \varepsilon\omega = 0, \tag{1.4}$$

where the Hamiltonian function  $H$  and the polynomial 1-form  $\omega$  are defined as in the infinitesimal Hilbert's 16th problem and  $\varepsilon$  is a small real parameter. The integral in (1.3) is analytic on  $t$ , and its analytic continuation is a multivalued function and branched in a finite set of atypical values. We consider the analytic continuation of  $I(t)$  in the complex domain, this extension allows us to use tools from complex algebraic geometry (vanishing ovals, monodromy, etc).

## Chapter 2

# Polynomial differential equations in the plane

In this chapter we introduce the basic notions on the qualitative theory of differential equations, and in particular, the limit cycles of polynomial differential equations in  $\mathbb{R}^2$ .

### 2.1 Preliminary notions

What we want to study is the following ordinary differential equation:

$$X : \begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}, \quad (2.1)$$

where  $P, Q$  are polynomials in  $x, y$  with coefficients in  $\mathbb{R}$  of degrees at most  $d$  and the derivatives  $\dot{x} = \frac{dx}{dt}$  and  $\dot{y} = \frac{dy}{dt}$ . We will assume that  $P$  and  $Q$  do not have common factors. The variables  $x, y$  are called the *dependent variables* and  $t$  is the *independent variable* of the system (2.1), usually  $t$  is also called the *time*. The system (2.1) is called *autonomous*, because it does not depend on the variable  $t$ .

We recall that solutions of the system (2.1) are differentiable maps

$\varphi : I \rightarrow \mathbb{R}^2$  such that

$$\frac{d\varphi}{dt}(t) = X(\varphi(t))$$

for every  $t \in I$ . These are the trajectories of the vector field:

$$X(x, y) := P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

We will also write the vector field as  $X = (P, Q)$ .

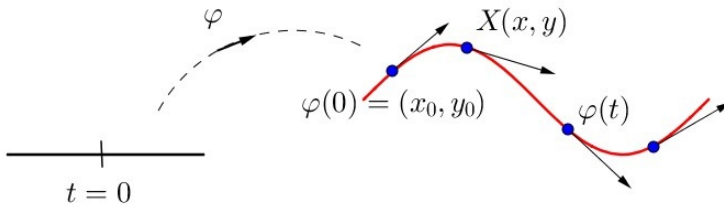


Figure 2.1: An integral curve

Let us first recall the first basic theorem of ordinary differential equations.

**Theorem 2.1.** For  $p \in \mathbb{R}^2$  if  $X(p) \neq 0$ , there is a unique analytic function

$$\varphi : (\mathbb{R}, 0) \rightarrow \mathbb{R}^2$$

such that

$$\varphi(0) = p, \quad \dot{\varphi} = X(\varphi(t)).$$

*Proof.* Let us write formally

$$\varphi(t) = \sum_{i=0}^{\infty} \varphi_i t^i, \quad \varphi_i \in \mathbb{R}^2, \quad \varphi(0) := p$$

and substitute it in  $\dot{\varphi}(t) = X(\varphi(t))$ . It turns out that if  $X(p) \neq 0$  then  $\varphi_i$  can be written in a unique way in terms of  $\varphi_j$ ,  $j < i$ . This

guaranties the existence of a unique formal  $\varphi$ . As an exercise, the reader may recover the proof of convergence of  $\varphi$  from classical books on ordinary differential equations.  $\square$

The trajectories of the following differential equations are depicted in Figure 2.2:

$$(a) : \begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad (b) : \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}, \quad (c) : \begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases},$$

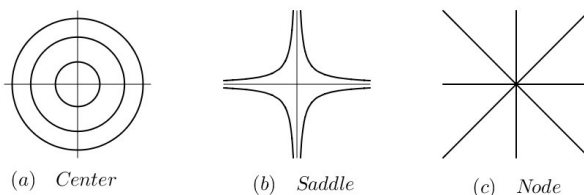


Figure 2.2: Trajectories of some linear systems

The collection of all the images of solutions of the system (2.1) gives us an analytic singular foliation  $\mathcal{F} = \mathcal{F}(X)_{\mathbb{R}} = \mathcal{F}(X) = \mathcal{F}_{\mathbb{R}}$  in  $\mathbb{R}^2$ . Therefore, when we are talking about a foliation we are not interested in the parametrization of its leaves (trajectories). It is left to the reader to verify that for a polynomial  $R \in \mathbb{R}[x, y]$  the foliation associated to  $X$  and  $RX$  in  $\mathbb{R}^2 \setminus \{R = 0\}$  are the same.

From the beginning we assume that  $P$  and  $Q$  have no common factors. Being interested only on the foliation  $\mathcal{F}(X)$ , we may write the system (2.1) in the form

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)},$$

or

$$\omega = 0, \quad \text{where } \omega = P(x, y)dy - Q(x, y)dx.$$

In the second case, we use the notation  $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}} = \mathcal{F}(\omega)$ . In this case the foliation  $\mathcal{F}$  is characterized by the fact that the 1-form  $\omega$  restricted to the leaves of  $\mathcal{F}$  is identically zero.

**Definition 2.1.** The *singular set* of the foliation  $\mathcal{F}(P(x, y)dy - Q(x, y)dx)$  is defined in the following way:

$$\text{Sing}(\mathcal{F}) = \text{Sing}(\mathcal{F})_{\mathbb{R}} := \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = Q(x, y) = 0\}.$$

The points  $p \in \text{Sing}(\mathcal{F})$  (respectively  $p \notin \text{Sing}(\mathcal{F})$ ) are called the *singularities* of  $\mathcal{F}$  or *singular points* (respectively *regular points*). By our assumption  $\text{Sing}(\mathcal{F})$  is a finite set of points. The leaves of  $\mathcal{F}$  near a point  $p \in \text{Sing}(\mathcal{F})$  may be complicated. By Bezout theorem we have

$$\#\text{Sing}(\mathcal{F}) \leq \deg(P) \deg(Q).$$

The upper bound can be reached, for instance by the differential equation  $\mathcal{F}(Pdy - Qdx)$ , where

$$P = (x - 1)(x - 2) \cdots (x - d), \quad Q = (y - 1)(y - 2) \cdots (y - d').$$

## 2.2 Phase portraits of linear systems

Let  $p \in \text{Sing}(\mathcal{F})$  be a singular point in the plane for the system (2.1), without loss of generality, to be at the origin. The associated linearized system at the origin is given by calculating the Jacobian matrix  $J_{(0,0)}$ , where

$$J_{(x,y)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} + O(2),$$

where  $O(2)$  represent terms of degree 2 or higher in  $x$  and  $y$ . The singular point is said to be *non-degenerate* if the determinant of  $J_{(0,0)}$  is non-zero.

**Definition 2.2.** The system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad (2.2)$$

where  $A = J_{(0,0)}$  is called the *linear planar system*.



**Definition 2.3.** A *phase portraits* of the vector field given by (2.1) is the set of all trajectories (solution or orbits) of the system.

The phase portraits of linear system (2.2) with  $\Delta = \det(A) \neq 0$  are well known. If  $\Delta < 0$  we have a *saddle point*; if  $\Delta > 0$  and  $T = \text{trace}(A) = 0$  we have a *center*, if  $\Delta > 0$  and  $T^2 - 4\Delta < 0$  we have a *focus*; and if  $\Delta > 0$  and  $T^2 - 4\Delta > 0$  we have a *node*. The eigenvalues of  $A$  are

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}.$$

The trajectories of linear systems under the condition  $\Delta = \det(A) \neq 0$  are depicted in Figure 2.3

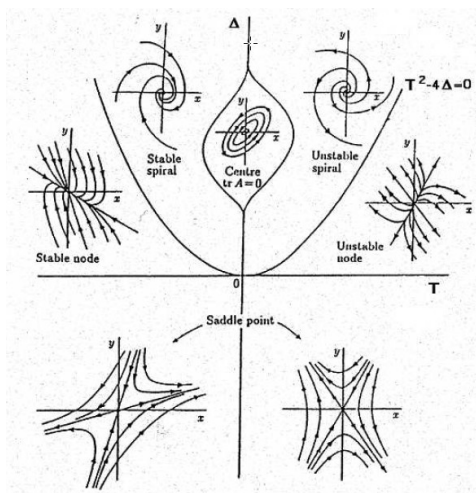


Figure 2.3: Phase portraits of linear systems

We recall that a singular point  $p$  is *hyperbolic* if the eigenvalues of the linear part of the system at  $p$  have non-zero real part. By the Hartman-Grobman theorem in a small neighborhood of the hyperbolic singular point, the system (2.1) is topologically equivalent to its linear part (2.2). That is, as long as the linear part do not give a

center. An interesting problem is when a singular point whose linear part gives a center, really is a center of system (2.1). This problem is known as *center focus problem*, see Chapter 5.

Suppose that the polynomial system (2.1) has a non-degenerate singular point at the origin. If a singular point is either a center or focus, then we can bring the vector field to the form

$$\begin{cases} \dot{x} = -y + \lambda x + p(x, y) \\ \dot{y} = x + \lambda y + q(x, y) \end{cases}, \quad (2.3)$$

by a linear transformation, where  $p$  and  $q$  are polynomials without constant or linear terms. The case when  $\lambda = 0$  corresponds to a weak focus or a center.

The linear part of the system

$$\begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases} \quad (2.4)$$

at the origin gives a center point, but for nonlinear system we have

$$\frac{d}{dt}(x^2 + y^2) = 2(x^4 + y^4) \quad (2.5)$$

and so trajectories travel away from the origin, and the system has therefore an unstable focus there.

## 2.3 Limit cycles

**Definition 2.4.** A *limit cycle* of a planar vector field given by (2.1) is an isolated periodic trajectory (isolated compact leaf of the corresponding foliation).

In other words, a periodic trajectory of a vector field is a limit cycle, if it has an annular neighborhood free from other periodic trajectories.

An example with one limit cycle is provided by the differential equation:

**Example 2.1.** Consider the following Hamiltonian system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2). \end{cases} \quad (2.6)$$

It is easy to see that the annulus  $S$  bounded by the circles with radii  $1/2$  and  $2$ , respectively, contains no singular points of the system. Moreover  $S$  is positively invariant. To prove this fact, we consider the normal vector  $N(x, y) = (x, y)$  on  $\partial S$  and compute the interior product of  $N$  and the vector field corresponding to the differential equation. In fact, the interior product

$$x^2(1 - x^2 - y^2) + y^2(1 - x^2 - y^2) = (x^2 + y^2)(1 - x^2 - y^2)$$

is positive on the circle with radius  $1/2$  and negative on the circle with radius  $2$ . Therefore,  $S$  is positively invariant and there is at least one limit cycle in  $S$ .

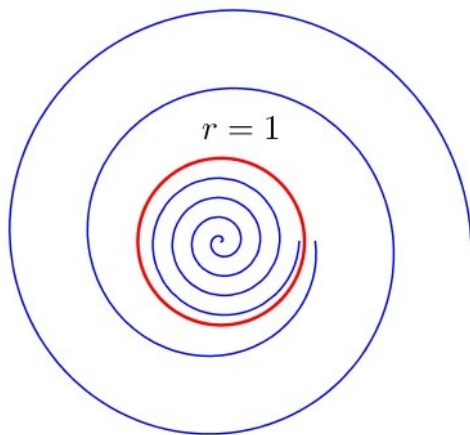


Figure 2.4: One limit cycle at  $r = 1$

By changing to polar coordinates  $(r, \theta)$ , the system (2.6) is transformed in the system

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1$$

and its flow is given by

$$\phi_t(r, \theta) = \left( \left( \frac{r^2 e^{2t}}{1 - r^2 + r^2 e^{2t}} \right)^{1/2}, \theta + t \right).$$

Note that  $\phi_t(1, \theta) = (1, \theta + t)$  and, in particular,  $\phi_{2\pi}(1, \theta) = (1, \theta + 2\pi)$ . Thus, the unit circle in the plane is a periodic orbit with period  $2\pi$ , see Figure 2.4.

□

The following example is a case, where there exists an infinite number of limit cycles.

**Example 2.2.** In the  $\mathcal{C}^\infty$  context we consider the system

$$\begin{cases} \dot{x} = -y + xf(x, y) \\ \dot{y} = x + yf(x, y) \end{cases} \quad (2.7)$$

with

$$f(x, y) = \sin\left(\frac{1}{x^2 + y^2}\right)e^{-1/(x^2 + y^2)}.$$

The system (2.7) has an infinite number of limit cycles,  $x^2 + y^2 = 1/n\pi$ , for  $n \in \mathbb{Z}_+$  accumulating at the origin. If we use  $\sin(x^2 + y^2)$  instead of  $\sin(\frac{1}{x^2 + y^2})$  in the definition of  $f$  we obtain a differential equation which has an infinite number of limit cycles accumulating at infinity.

For polynomial systems (or analytic systems), this situation does not occur. A singular point whose linear part give a center is either asymptotically stable, asymptotically unstable or it is a center. This can be most easily seen by computing the return map at the origin, which we will define in the next chapter.

# Chapter 3

## Differential equations and Abelian integrals

The principal aim in this chapter is to establish the relation between a planar polynomial differential system and Abelian integrals, in particular, we will explain the relation between the zeros of these integrals and the number of limit cycles of the corresponding perturbed planar Hamiltonian vector fields.

### 3.1 Hamiltonian deformation

Consider a perturbed planar Hamiltonian vector field

$$X_\varepsilon : \begin{cases} \dot{x} = H_y(x, y) + \varepsilon P(x, y, \varepsilon) \\ \dot{y} = -H_x(x, y) + \varepsilon Q(x, y, \varepsilon) \end{cases}, \quad (3.1)$$

we suppose that  $H$ ,  $P$  and  $Q$  are real polynomials in the variables  $x, y$  moreover,  $P$  and  $Q$  depend analytically on a small real parameter  $\varepsilon$ . Suppose that for  $\varepsilon = 0$  the Hamiltonian system  $X_0 = H_y(x, y) \frac{\partial}{\partial x} - H_x(x, y) \frac{\partial}{\partial y}$  has at least one center point. Some particular perturbed Hamiltonian systems of (3.1) were studied by Petrov [91], Horozov and Iliev [46], Gavrilov and Iliev [29], Mardešić, Pelletier and Jebrane [70, 71], Movasati [77], Uribe [103], Pontigo [96],

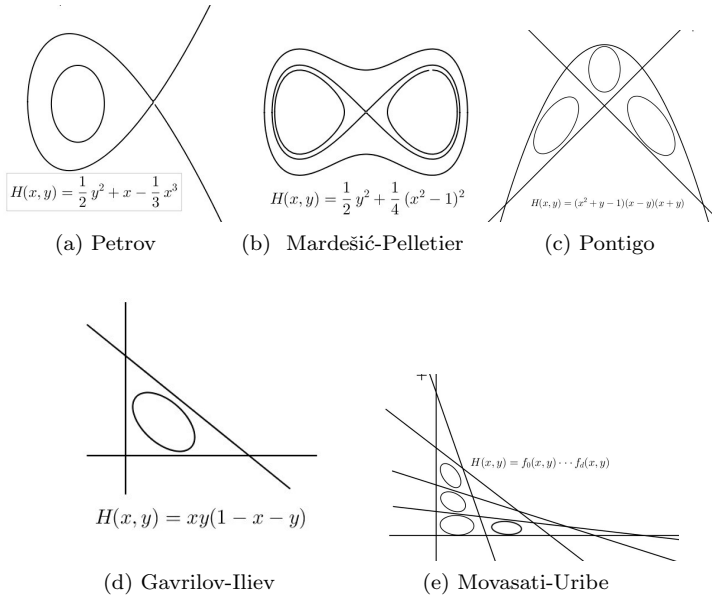


Figure 3.1: Some Hamiltonian functions with at least one center

etc. The corresponding graphics of some Hamiltonians  $H$  are depicted in Figure 3.1.

Since  $dH$  vanishes on the level curves of  $H$ , in particular, a periodic orbit must be a compact oval of some level curve and hence all nearby leaves must be also closed, so for  $\varepsilon = 0$  the Hamiltonian system  $X_0$  cannot have limit cycles. Under the condition  $\varepsilon \neq 0$ , usually, the periodic orbits of  $X_0$  are broken and knowledge of isolated periodic orbits that persist (limit cycles) by perturbation is the aim of study in this chapter. Here the problem is to ask for a maximum number of limit cycles which bifurcate from the period annulus of the center for the perturbed system (3.1). Suppose that there is a family of ovals,  $\gamma(t) \subset H^{-1}(t)$ , continuously depending on a parameter  $t$  defined in a maximal open interval  $\Sigma = (a, b)$  and we consider the

Abelian integral

$$I(t) = \int_{\gamma(t)} Q(x, y, 0)dx - P(x, y, 0)dy. \quad (3.2)$$

It is clear that all  $\gamma(t)$  filling up an annulus for  $t \in \Sigma$ , are periodic orbits of the Hamiltonian system  $X_0$ .

## 3.2 Poincaré first return map

Assume that for a certain open interval  $\Sigma \subset \mathbb{R}$ , the level sets given by  $\{H(x, y) = t, \quad t \in \Sigma\}$  of the Hamiltonian  $H$  contain a continuous family of ovals. An oval is a smooth simple closed curve which is free of critical points of  $H$ . Such a family is called a period annulus of the unperturbed system. In general, the end points of  $\Sigma$  are critical levels of the Hamiltonian function  $H$  that correspond to centers, saddle-loops or infinity.

**Definition 3.1.** For  $\Sigma \subset \mathbb{R}$ , we define the *Poincaré map*  $\mathcal{P}_\varepsilon : \Sigma \rightarrow \Sigma$  as the first return map of the solutions of (3.1) on  $\Sigma$ , i.e. for each point  $t \in \Sigma$  belonging to a specific solution, the Poincaré map gives us the first point where the solution intersects  $\Sigma$  in the positive direction of flow, see Figure 3.2.

Note that for  $\varepsilon = 0$  the Poincaré map is the identity map on  $\Sigma$  and for  $\varepsilon \neq 0$ , in general, the periodic orbits of the system (3.1) are obtained in the fixed points of the Poincaré map, moreover if this fixed points are isolated in the periodic orbits set then the periodic orbits are limit cycles of the system (3.1). In general, given a Hamiltonian system as in (3.1), it is not trivial to compute the Poincaré map. Figure 3.3 shows graphics of Poincaré maps with 0, 1 and 2 limit cycles. The corresponding difference is the *displacement function* defined by  $\Delta_\varepsilon(t) = \mathcal{P}_\varepsilon(t) - t$ , has a representation as a power series in the variable  $\varepsilon$ :

$$\Delta_\varepsilon(t) = \varepsilon M_1(t) + \varepsilon^2 M_2(t) + \dots \quad (3.3)$$

which is convergent for small  $\varepsilon$ .

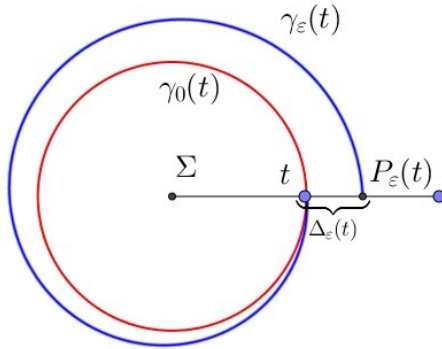


Figure 3.2: The Poincaré first return map

**Definition 3.2.** The functions  $M_\ell(t)$ , obtained in (3.3) are called the *Poincaré-Pontryagin functions* or *Melnikov functions* of order  $\ell$  with  $\ell = 1, 2, 3, \dots$

Note that  $\Delta_0(t) \equiv 0$ , so we assume that the function  $\Delta_\varepsilon(t)$  is not identically zero. Therefore, there exists a positive integer  $k$  such that  $M_1(t) = M_2(t) = \dots = M_{k-1}(t) = 0$  and  $M_k(t) \neq 0$ . We will call  $M_k(t)$  the *principal Poincaré-Pontryagin function* and say that  $k$  is its order. It is also called the *generating function* in [29], *Melnikov function* in [47] and *variation function* in [107].

The principal Poincaré-Pontryagin function of order one is always an Abelian integral.

**Proposition 3.1. (Poincaré-Pontryagin [94], [97])** *We have*

$$\Delta_0(t) \equiv 0, \quad \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \Delta_\varepsilon(t) = - \int_{\gamma(t)} P(x, y) dy - Q(x, y) dx. \quad (3.4)$$

*Proof.* Firstly, note that the number of zeros of the displacement function is independent of the choice of the transversal segment  $\Sigma$ . Let  $t$  be a regular value of the Hamiltonian function  $H$  on the transversal section  $\Sigma$ . The solution curve  $\gamma_\varepsilon(t)$  is an oriented segment of



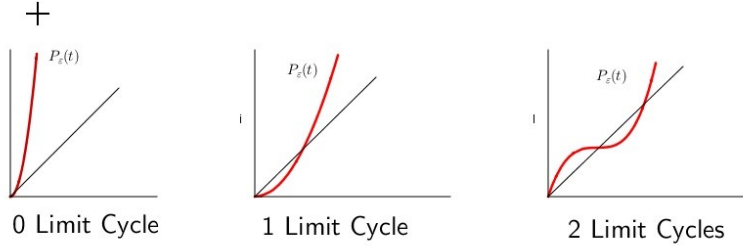


Figure 3.3: The Poincaré maps with limit cycles

the leaf between the initial point  $t \in \Sigma$  and the next intersection  $\mathcal{P}_\varepsilon(t) \in \Sigma$ , see Figure 3.2. As  $t$  is a value of  $H$ , the displacement map  $\Delta_\varepsilon(t)$  is the difference of the values  $H$  at the endpoints of  $\gamma_\varepsilon(t)$ , hence

$$\Delta_\varepsilon(t) = \int_{\gamma_\varepsilon(t)} dH.$$

The expression  $dH + \varepsilon(P(x, y)dy - Q(x, y)dx)$  vanishes on  $\gamma_\varepsilon(t)$ , therefore the integral above is equal to the  $-\varepsilon(P(x, y)dy - Q(x, y)dx)$  along  $\gamma_\varepsilon(t)$  and since  $\gamma_\varepsilon(t)$  converges uniformly to the closed curve  $\gamma(t) = \gamma_0(t) \subset \{H = t\}$  as  $\varepsilon \rightarrow 0$ , we conclude that

$$\Delta_\varepsilon(t) = -\varepsilon \int_{\gamma(t)} P(x, y)dy - Q(x, y)dx + o(\varepsilon); \quad \varepsilon \rightarrow 0$$

with  $o(\varepsilon)$  uniform and analytic on  $t$  and  $\varepsilon$ . This yields the formula (3.4) for the derivative.  $\square$

The following example is taken from [101] and shows a simple computation for  $M_1(t)$ , the first non-zero term of the displacement function  $\Delta_\varepsilon(t)$ .

**Example 3.1. (Calculating  $M_1(t)$ )** We consider the one parameter family  $X_\varepsilon$  defined by

$$X_\varepsilon : \begin{cases} \dot{x} &= y + \varepsilon x \\ \dot{y} &= -x - x^2 + \varepsilon y. \end{cases} \quad (3.5)$$

The dual form of the vector fields  $X_\varepsilon$  is:

$$\omega_\varepsilon = dH + \varepsilon\omega$$

where  $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3$  and  $\omega = -ydx + xdy$ . Observe that  $(0, 0)$  and  $(-1, 0)$  are the critical points of  $X_0$  and  $\gamma(t)$  are periodic orbits for  $t \in ]0, 1/6[$ , we have by Proposition 3.1 that the first

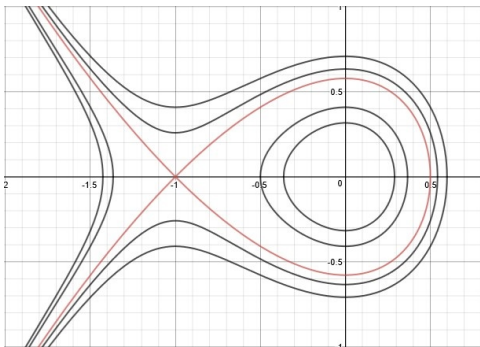


Figure 3.4: Levels of the cubic Hamiltonian  $H$

Poincaré-Pontryagin term  $M_1(t)$  of the displacement function is given by

$$M_1(t) = - \int_{\gamma(t)} -ydx + xdy = 2 \int_{\gamma(t)} ydx.$$

Note that  $M_1(t) \neq 0$  for  $t \in ]0, 1/6[$ , due to  $\int_{\gamma(t)} ydx$  is the area of interior region of the curve  $\gamma(t)$ .

Note that in the development (3.3), the first term of the displacement map is obtained for  $k = 1$  and this function gives always an Abelian integral, however for  $k > 1$  the principal Poincaré-Pontryagin function is not necessary an Abelian integrals. For the Hamiltonian vector field (3.1) in the case of quadratic systems the principal term of the displacement map were calculated by many authors, see for instance [47], [103], [113], [79].

To answer the question over a maximum bound of the number of isolated period orbits for the perturbed system (3.1) we consider the following definition and result which can be also seen in [61].

**Definition 3.3.** If there exist  $t^* \in \Sigma$  and  $\varepsilon^*$  such that the system (3.1) has a limit cycles  $\gamma(t, \varepsilon)$  for  $0 < |\varepsilon| < \varepsilon^*$  and  $\gamma(t, \varepsilon)$  tends to  $\gamma(t^*)$  as  $\varepsilon \rightarrow 0$  then we will say that  $\gamma(t, \varepsilon)$  bifurcates from  $\gamma(t^*)$ . We say that a limit cycle  $\gamma(t, \varepsilon)$  bifurcates from the annulus  $\bigcup_{t \in \Sigma} \gamma(t)$ .

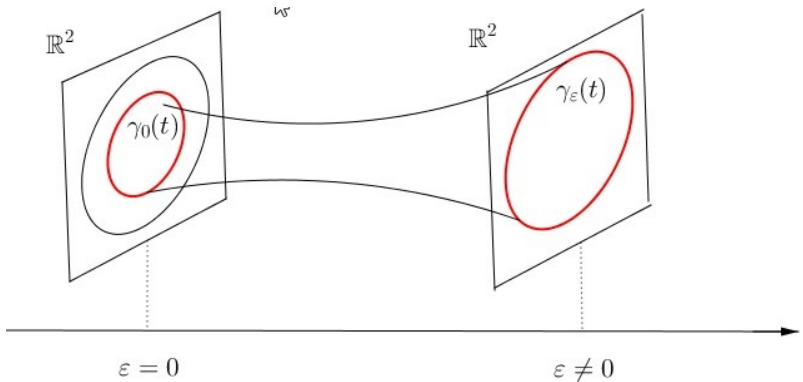


Figure 3.5: Bifurcation of a limit cycles

**Theorem 3.1.** *We suppose that  $M_1(t)$  is not identically zero for  $t \in \Sigma$ . The following statements hold.*

1. *If the system (3.1) has a limit cycle bifurcating from  $\gamma(t^*)$ , then  $M_1(t^*) = 0$ .*
2. *If there exists  $t^* \in \Sigma$  such that  $M_1(t^*) = 0$  and  $M_1'(t^*) \neq 0$ , then the system (3.1) has a unique limit cycle bifurcating from  $\gamma(t^*)$ , moreover, this limit cycle is hyperbolic.*
3. *If there exists  $t^* \in \Sigma$  such that  $M_1(t^*) = M_1'(t^*) = \dots = M_1^{(k-1)}(t^*) = 0$  and  $M_1^{(k)}(t^*) \neq 0$ , then the system (3.1) has at most  $k$  limit cycles bifurcating from the same  $\gamma(t^*)$ , taking into account the multiplicities of the limit cycles.*

4. *The total number (counting the multiplicities) of the limit cycles of the system (3.1) bifurcating from the annulus  $\bigcup_{t \in \Sigma} \gamma(t)$  is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of the Abelian integral  $M_1(t)$  for  $t \in \Sigma$ .*

*Proof.* We will transcribe the proof of this theorem that may be found in [15].

1. Suppose that a limit cycle  $\gamma(t, \varepsilon)$  of  $X_\varepsilon$  bifurcates from  $\gamma(t^*)$ . By Proposition 3.1, there exists  $\varepsilon^* > 0$  and  $t_\varepsilon \rightarrow t^*$  as  $\varepsilon \rightarrow 0$ , such that

$$\Delta(t_\varepsilon, \varepsilon) = \varepsilon(M_1(t_\varepsilon) + \varepsilon\phi(t_\varepsilon, \varepsilon)) \equiv 0, \quad 0 < |\varepsilon| < \varepsilon^*.$$

Dividing by  $\varepsilon$  on both sides, and taking limit as  $\varepsilon \rightarrow 0$ , we obtain  $M_1(t^*) = 0$ .

2. Suppose that there exists a  $t^* \in \Sigma$  such that  $M_1(t^*) = 0$  and  $M_1'(t^*) \neq 0$ . Since we consider limit cycles for small  $\varepsilon > 0$ , instead of the displacement function  $\Delta(t, \varepsilon)$  we may study the zeros of  $\tilde{\Delta}(t, \varepsilon) = \Delta(t, \varepsilon)/\varepsilon$ . By Proposition 3.1, we have

$$\tilde{\Delta}(t, \varepsilon) = M_1(t) + \varepsilon\phi(t, \varepsilon),$$

where  $\phi$  is analytic and uniformly bounded in a compact region near  $(t^*, 0)$ . Since  $\tilde{\Delta}(t^*, 0) = M_1(t^*) = 0$  and  $\tilde{\Delta}_t(t^*, 0) = M_1'(t^*) \neq 0$ , by the implicit function theorem, we find an  $\varepsilon^* > 0$  and  $\eta^* > 0$  and a unique function  $t = t(\varepsilon)$  defined in  $U^* = \{(t, \varepsilon) : |t - t^*| \leq \eta^*, |\varepsilon| \leq \varepsilon^*\}$ , such that  $t(0) = t^*$  and  $\tilde{\Delta}(t(\varepsilon), \varepsilon) \equiv 0$  for  $(t, \varepsilon) \in U^*$ . Hence, the unique  $t(\varepsilon)$  gives a unique limit cycles  $\gamma(t, \varepsilon)$  of the system (3.1) for each small  $\varepsilon$ .

3. Assume that there exists a  $t^* \in \Sigma$  such that  $M_1(t^*) = M_1'(t^*) = \dots = M_1^{(k-1)}(t^*) = 0$  and  $M_1^{(k)}(t^*) \neq 0$ . We need to show that there exists  $\delta > 0$  and  $\eta > 0$  such that for any  $(t, \varepsilon) \in U = \{|t - t^*| < \eta; |\varepsilon| < \delta\}$ , the displacement function  $\Delta(t, \varepsilon)$  has at most  $k$  zeros in  $\Sigma$ , taking into account their multiplicities. Suppose the contrary, then for any integer  $j$  there exist  $\varepsilon_j > 0$

and  $\eta_j > 0$ ,  $\varepsilon_j \rightarrow 0$  and  $\eta \rightarrow 0$  as  $j \rightarrow \infty$ , such that for any  $\varepsilon_j$  the function  $\Delta(t, \varepsilon_j)/\varepsilon_j$  has the least  $(k+1)$  zeros for  $|t-t^*| < \eta_j$ . By using the Rolle Theorem we find a  $t_j$  such that  $|t_j - t^*| < \eta_j$  and

$$M_1^{(k)}(t_j) + \varepsilon_j \frac{\partial^k}{\partial t^k} \phi(t_j, \varepsilon_j) = 0,$$

which implies  $M_1^{(k)}(t^*) = 0$  by taking limit as  $j \rightarrow \infty$ , leading to a contradiction.

4. For any small  $\sigma > 0$ , we consider the number of limit cycles bifurcating from  $t \in \Sigma = [a+\sigma, b-\sigma]$  for small  $\varepsilon$ . As  $Z(n, m) < \infty$ , we suppose that this number is the uniform bound. We take the maximum of this number as  $\sigma \rightarrow 0$ , then we get the cyclicity of the period annulus.

□

The following example is classical and shows the existence of a limit cycle that appear by a deformation of the harmonic oscillator given by the Hamiltonian  $H(x, y) = \frac{1}{2}(x^2 + y^2)$ , see also [15],[26].

**Example 3.2.** Consider the Van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0. \quad (3.6)$$

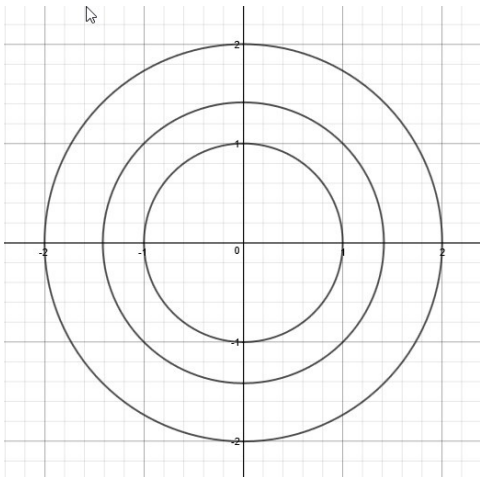
The differential equation (3.6) is equivalent to the system

$$X_\varepsilon : \begin{cases} \dot{x} &= y \\ \dot{y} &= -x - \varepsilon(1 - x^2)y. \end{cases} \quad (3.7)$$

When  $\varepsilon = 0$ , the system (3.7) is a Hamiltonian vector field with a family of periodic orbits defined by

$$\gamma(t) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = t, t > 0\}.$$

Using the polar coordinate, we have the principal Poincaré-Pontryagin

Figure 3.6: Ovals of the quadratic Hamiltonian  $H$ 

function of first order given by

$$\begin{aligned} M_1(t) &= - \int_{\gamma(t)} (1-x^2)y dx = \int_0^{2\pi} (1-t^2 \cos^2 \theta)t^2 (-\sin^2 \theta) d\theta \\ &= \pi t^2 \left( \frac{t^2}{4} - 1 \right). \end{aligned}$$

The value  $t = 0$  corresponds to the singular point of the system (3.7), moreover  $t = 2$  is the only positive value such that  $M_1(2) = 0$  and  $M_1'(2) = 4\pi$ . Applying Theorem 3.1 we conclude that for a small  $\varepsilon$  the system (3.7) has a unique limit cycle which is hyperbolic bifurcating from the annulus periodic.  $\square$

**Example 3.3.** Consider the Hamiltonian  $H(x, y) = \frac{1}{2}(x^2 + y^2)$ . In the complex coordinates  $z = x + iy$ , we have  $H(z, \bar{z}) = \frac{1}{2}z\bar{z}$ . One can easily verify that the polynomial 1-form  $\omega = A(z, \bar{z})dz + B(z, \bar{z})d\bar{z}$  has identically vanishing integral over the circles  $\{H = t\}$  if and only if the differential  $d\omega = (-A_{\bar{z}} + B_z)dz \wedge d\bar{z}$  contains no monomial

terms of the form  $(z\bar{z})^k dz \wedge d\bar{z}$ . Any other monomial can obviously be represented in the form (3.9):

$$z^i \bar{z}^j dz \wedge d\bar{z} = dG \wedge dH, \quad \text{with } G = \frac{z^i \bar{z}^j}{i-j}.$$

Suppose firstly that  $\omega = A(z, \bar{z})dz + B(z, \bar{z})d\bar{z}$  with  $A, B$  polynomials in  $z$  and  $\bar{z}$ . From Stokes theorem we have

$$\begin{aligned} \int_{\{H=t\}} \omega &= \int_{\{H=t\}} A(z, \bar{z})dz + B(z, \bar{z})d\bar{z} \\ &= \int_{\{H \leq t\}} d(A(z, \bar{z})dz + B(z, \bar{z})d\bar{z}) \\ &= \int_{\{H \leq t\}} \left( \frac{\partial A}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial B}{\partial z} dz \wedge d\bar{z} \right) \\ &= \int_{\{H \leq t\}} \left( -A_{\bar{z}} + B_z \right) dz \wedge d\bar{z}. \end{aligned}$$

As  $A$  and  $B$  are polynomial functions, then the derivative function  $-A_{\bar{z}} + B_z$  is also polynomial, so we only need to calculate integrals of the form  $\int_{\{H \leq t\}} z^k \bar{z}^l dz \wedge d\bar{z}$ , for every  $k, l$ . If we consider the change  $z = Re^{i\theta}$ , then

$$\begin{aligned} \int_{\{H \leq t\}} z^k \bar{z}^l dz \wedge d\bar{z} &= \int_0^{\sqrt{2t}} \int_0^{2\pi} (-2iR) R^{k+l} e^{i(k-l)\theta} dR \wedge d\theta \\ &= -2i \int_0^{\sqrt{2t}} R^{k+l+1} dR \int_0^{2\pi} e^{i(k-l)\theta} d\theta. \end{aligned}$$

The above integral is zero for  $k \neq l$  and  $-2\pi \frac{(2t)^{k+1}}{k+1}$  for  $k = l$ . Therefore, the first part of this example holds. The inverse is obtained from a direct calculation and it is left to the reader. We observe that

for  $k \neq l$  the monomial terms satisfy

$$\begin{aligned}
 a_{kl} z^k \bar{z}^l dz \wedge d\bar{z} &= \frac{a_{kl}}{k-l} d\left(z^k \bar{z}^l (\bar{z} dz + z d\bar{z})\right) \\
 &= \frac{a_{kl}}{k-l} d\left(z^k \bar{z}^l (2dH)\right) \\
 &= d\left(2 \frac{a_{kl}}{k-l} z^k \bar{z}^l (dH)\right) \\
 &= d(g(z, \bar{z}) dH) \\
 &= dg \wedge dH.
 \end{aligned}$$

where  $g(z, \bar{z}) = \frac{2}{k-l} a_{kl} z^k \bar{z}^l$  □

**Example 3.4.** We take the ordinary differential equation

$$\begin{cases} \dot{x} = 2y + \varepsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \varepsilon sy \end{cases}, \quad (3.8)$$

which is a perturbation of the Hamiltonian in Figure 1.3. If

$$\int_{\gamma(0)} \left(\frac{x^2}{2} dy - sy dx\right) = 0$$

or equivalently

$$s := \frac{-\int \int_{\Delta_0} x dx \wedge dy}{\int \int_{\Delta_0} dx \wedge dy} = \frac{5 \Gamma(\frac{5}{12}) \Gamma(\frac{13}{12})}{7 \Gamma(\frac{7}{12}) \Gamma(\frac{11}{12})} \sim 0.9025,$$

where  $\Delta_0$  is the bounded open set in  $\mathbb{R}^2$  with the boundary  $\gamma(0)$ , then for  $\varepsilon$  near to 0, the system (3.8) has a limit cycle near  $\gamma(0)$ . Here,  $\Gamma$  is the classical Gamma function. For  $\varepsilon = 1$  and  $s = 0.9$  such a limit cycle still exists and it is depicted in Figure 1.1. For the discussion of Abelian integrals which are computable in terms of the values of the Gamma function see [85].

### 3.3 Principal Poincaré-Pontryagin function

Let  $\omega_\varepsilon = dH + \varepsilon\omega$  be a one parameter family of polynomial 1-forms in the plane  $\mathbb{R}^2$  and  $\gamma(t)$  be a continuous family of ovals in the level curves  $\{H(x, y) = t\}$ .



**Definition 3.4.** We will say that the Hamiltonian function  $H$  satisfy the *Françoise's condition/star condition*, if for all polynomial 1-form  $\omega$  we have the following

$$\int_{\gamma(t)} \omega \equiv 0 \implies \omega = G(x, y)dH + dF(x, y), \quad \text{for some } G, F \in \mathbb{C}[x, y] \quad (3.9)$$

and hence  $d\omega = dG \wedge dH$ .

**Definition 3.5.** A polynomial  $H \in \mathbb{C}[x, y]$  of degree  $n + 1$  is called *ultra-Morse* if the function  $H$  has  $n^2$  non-degenerate critical points with different critical values and the homogeneous part of  $H$  of higher degree is a product of  $n + 1$  linear distinct factors.

The assumption on critical points and critical values defines a class special of Morse polynomials, this motivates the term. Ultra-Morse polynomials form a Zariski open set in the space of all polynomials of given degree.

**Theorem 3.2. (Exactness Theorem, [52, 53, 98, 31]).** *The ultra-Morse polynomial satisfies the Françoise's condition.*

Note that for the Abelian integral given by (1.3) with  $\gamma(t)$  a family of ovals of a real ultra-Morse polynomial  $H$  of degree  $(n + 1) \geq 3$  and  $\omega$  a polynomial 1-form of degree no greater than  $n$ , if  $I(t) \equiv 0$  then the form  $\omega$  is exact, i.e.  $\omega = df$  for some polynomial  $f$ .

In this section we will consider the problem of characterizing the principal Poincaré-Pontryagin function of order  $k$  when  $M_1(t) = M_2(t) = \cdots M_{k-1}(t) \equiv 0$  for  $t \in \Sigma$ . We recall that the displacement function has an expansion given by (3.3) and as a direct consequence, there exists a positive integer  $k$  such that the displacement function is given by

$$\Delta_\varepsilon(t) = \varepsilon^k M_k(t) + o(\varepsilon^{(k+1)}), \quad (3.10)$$

where  $\varepsilon$  is small. The question is the following: if  $M_1(t) \equiv 0$ , then how to compute the term of second order  $M_2(t)$  in (3.10) and so on? The computation of higher terms in the displacement function is relatively simple, J. P. Françoise in [25] present an algorithm for calculating the principal term  $M_k(t)$  associated to (3.10), see also

[108]. This algorithm assumes the condition in Definition 3.4. In [27] the authors determine that any Poincaré-Pontryagin function of higher term can be written as a sum of an iterated integral and of a combination of all previous Poincaré-Pontryagin functions and their derivatives, in this case the nullity condition of the previous terms is not required.

Using the condition (3.9), the principal term of second order in the displacement function can be expressed as an Abelian integral. This expression is given in the following lemma:

**Lemma 3.1.** *Let  $H$  be a polynomial function that the condition (3.9) holds and  $\omega$  is a polynomial 1-form, assume that  $M_1(t) \equiv 0$ , then the principal Poincaré-Pontryagin function of order two is given by the Abelian integral*

$$M_2(t) = \int_{\gamma(t)} G\omega. \quad (3.11)$$

*Proof.* Recall that in the development of the displacement map we have

$$M_2(t) = \left. \frac{d^2}{d\varepsilon^2} \right|_{\varepsilon=0} \Delta_\varepsilon(t).$$

Suppose that  $M_1(t) = 0$ , i.e. the integral  $\int_{\gamma(t)} \omega = 0$ , by (3.9) there are two polynomials  $G, F \in \mathbb{R}[x, y]$  such that  $\omega = GdH + dF$ , this implies that

$$(1 - \varepsilon G)(dH + \varepsilon\omega) = d(H + \varepsilon F) - \varepsilon^2 G\omega.$$

Integrating the last equality over  $\gamma(t, \varepsilon)$ , we obtain that

$$\Delta_\varepsilon(t) = \int_{\gamma(t, \varepsilon)} dH = -\varepsilon \int_{\gamma(t, \varepsilon)} dF + \varepsilon^2 \int_{\gamma(t, \varepsilon)} G\omega.$$

We note that  $\int_{\gamma(t, \varepsilon)} dF = O(\varepsilon^2)$  and using the condition that the first term  $M_1(t) = 0$ , we obtain

$$\Delta_\varepsilon(t) = \varepsilon^2 \int_{\gamma(t)} G\omega + O(\varepsilon^3),$$

so, we have

$$M_2(t) = \int_{\gamma(t)} G\omega.$$

This completes the proof.  $\square$

The following example is taken from [47] and the author gives a small deformation of a Hamiltonian system associated with a Hamiltonian  $H$  of degree three in which  $M_1(t) = 0$  and  $M_2(t) \neq 0$ .

**Example 3.5. (Calculating  $M_2(t)$ )** Consider the family of vector fields  $X_\varepsilon$  defined by

$$X_\varepsilon : \begin{cases} \dot{x} &= y + \varepsilon(y^2 + xy - x^2) \\ \dot{y} &= -x - x^2 + 2\varepsilon xy. \end{cases} \quad (3.12)$$

The dual form associated to (3.12) is:

$$\omega_\varepsilon = dH + \varepsilon\omega$$

where

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3$$

and

$$\omega = -2xydx + (y^2 + xy - x^2)dy.$$

We observe that the first term of displacement function is

$$\begin{aligned} M_1(t) &= - \int_{\gamma(t)} -2xydx + (y^2 + xy - x^2)dy \\ &= - \int_{\gamma(t)} d(yx^2 + \frac{y^3}{3}) + \int_{\gamma(t)} yxdy, \end{aligned}$$

by symmetry conditions, we have

$$\int_{\gamma(t)} yxdy = \int_{\gamma(t)} x(dH - (x + x^2)dx) = 0.$$

It can be clearly seen that

$$yxdy = G(x, y)dH + dF(x, y),$$

for some  $G$  and  $F$  and hence  $M_1(t) = 0$ . From Lemma (3.1), the principal Poincaré-Pontryagin function is of second order and given by:

$$M_2(t) = \int_{\gamma(t)} x\omega = \int_{\gamma(t)} -2x^2ydx + x(y^2 + xy - x^2)dy. \quad (3.13)$$

By a direct calculation, we know that  $M_2(t)$  is not zero and can be rewritten as a combination of integrals

$$M_2(t) = -\frac{2}{33}tI_0(t) + \frac{20}{11}I_1(t),$$

where  $I_k(t) = \int_{\gamma(t)} x^k y dx$ ;  $k = 0, 1$ . □

We recall now how to compute the principal Poincaré-Pontryagin function of higher order. Consider the perturbation of type

$$\omega_\varepsilon = dH + \varepsilon\omega_1 + \dots + \varepsilon^{k+1}\omega_{k+1} + o(\varepsilon^{k+1}). \quad (3.14)$$

Therefore, the first term of displacement map is given by the expression  $M_1(t) = -\int_{\gamma(t)} \omega_1$ , where  $\gamma(t)$  is a family of periodic orbits of the unperturbed Hamiltonian system. The recurrence formula that allows to obtain the higher order terms of the Poincaré-Pontryagin function is given by J.P. Françoise in [25], see also S. Yakovenko in [108]. We consider a more general version obtained by J.C. Poggiale in [93] for any 1-parameter family. The proof essentially follows from R. Roussarie in [101].

**Theorem 3.3. (Higher order approximation)** *Assume that  $H$  satisfies Françoise's condition,  $M_1(t) = \dots = M_k(t) = 0$  and  $M_{k+1}(t) \neq 0$ , then*

$$M_{k+1}(t) = \int_{\gamma(t)} \left( \sum_{i=1}^k g_i \omega_{k+1-i} - \omega_{k+1} \right), \quad (3.15)$$

where  $g_i$  with  $i = 1, \dots, k$  are polynomial functions and are defined inductively by

$$\omega_i - g_i dH = \sum_{j=1}^{i-1} g_j \omega_{k-j} + dR_i$$

*Proof.* The proof can be done by induction in  $k$ . Let  $\Sigma$  be the transverse section to the periodic solutions of (3.14), see Figure 3.2. The case  $k = 1$  is proved in Proposition 3.1. The hypothesis of induction is as follow: there exist polynomial functions  $g_i$  with  $i = 1, \dots, k$  such that

$$M_j(t) = \int_{\gamma(t)} \left( \sum_{i=1}^{j-1} g_i \omega_{j-i} - \omega_j \right) \equiv 0.$$

Using this relation for  $j = k$  and as  $H$  satisfy the Françoise condition, we find the polynomials  $g_k$  and  $R_k$  such that

$$- \sum_{i=1}^{k-1} g_i \omega_{k-i} + \omega_k = g_k dH + dR_k,$$

and so

$$\omega_k - g_k dH = \sum_{i=1}^{k-1} g_i \omega_{k-i} + dR_k,$$

this proves that the functions  $g_k$  and  $R_k$  are constructed by recurrence. Now, a direct expansion gives

$$\left( 1 - \sum_{i=1}^k g_i \varepsilon^i \right) \omega_\varepsilon = d \left( H - \sum_{i=1}^k \varepsilon^i R_i \right) + \varepsilon^{k+1} \left( \omega_{k+1} - \sum_{i=1}^k g_i \omega_{k+1-i} \right) + O(\varepsilon^{k+2}).$$

Let  $\gamma_\varepsilon(t)$  be a path in the leaf of  $\omega_\varepsilon$  through  $t$  which connects  $t$  to  $\mathcal{P}_\varepsilon(t)$  (first return Poincaré) along the path  $\gamma(t)$ . Integrating the above equality over the path  $\gamma_\varepsilon(t)$ , we have

$$\begin{aligned} & H(\mathcal{P}_\varepsilon(t)) - H(t) \\ & + \left( \sum_{i=1}^k R_i \varepsilon^i \right)_t^{\mathcal{P}_\varepsilon(t)} + \varepsilon^{k+1} \int_{\gamma_\varepsilon(t)} \left( \sum_{i=1}^k g_i \omega_{k+1-i} + \omega_{k+1} \right) + O(\varepsilon^{k+2}) = 0. \end{aligned}$$

We observe that  $\int_{\gamma_\varepsilon(t)} = \int_{\gamma(t)} + O(\varepsilon)$  and so by putting zero the coefficient of  $\varepsilon^{k+1}$  in the above formula we get the equality

$$M_{k+1}(t) = \int_{\gamma(t)} \left( \sum_{i=1}^k g_i \omega_{k+1-i} - \omega_{k+1} \right).$$

□

In the linear case  $\omega_\varepsilon = dH + \varepsilon\omega_1$  the recurrence formula is done by J.P. Françoise in [25] and

$$M_{k+1}(t) = (-1)^k \int_{\gamma(t)} g_k \omega_1$$

where the  $g_i$ 's are given inductively by:

$$g_i dH + dR_i = -g_{i-1} \omega_1.$$

Theorem 3.3 says that if  $H$  satisfies the Françoise's condition then the principal Poincaré-Pontryagin Function is always an Abelian integral.

There are cases where the Françoise's condition is not satisfied, however it is also possible to use a recursion formula, but the functions  $g_i$  and  $R_i$  may not be polynomials. In these cases, the principal Poincaré-Pontryagin function is not in general an Abelian integral, see [47], [103], [104], [90], [98]. It turns out to be an iterated integral, see [83, 35] and the references therein. We will discuss this in Chapter 6.

### 3.4 Estimate of the number of zeros of Abelian integrals

In this section, the purpose is to exhibit a list of principal results in estimation of the number of zeros of Abelian integrals. In the first chapter of this notes we saw some important results over the maximum bound for the number of limit cycles of polynomial vector fields of order  $n$ . As we have already discussed this problem is known as the finiteness Hilbert problem. Our interest is to consider the infinitesimal Hilbert's 16th problem, where the estimation of limit cycles is given by the estimation of the number of zeros of Abelian integrals, we recall that this problem was proposed by V. I. Arnold in [4].

We consider the Hamiltonian function  $H$  in the variables  $x, y$  of degree  $n + 1 \geq 2$  and the continuous family of ovals  $\gamma(t) \subset \{(x, y) : H(x, y) = t\}$  for  $t \in \Sigma$ . The transversal segment  $\Sigma$  to  $\gamma(t)$  is parameterized using the level value  $t$  of  $H$  and finally we consider a polynomial 1-form  $\omega = P(x, y)dy - Q(x, y)dx$ , where  $P$  and  $Q$  are

polynomials in  $\mathbb{R}[x, y]$  and  $\max\{\deg(P), \deg(Q)\} = m$ , for  $m \geq 2$ . Recall the principal problem presented as Problem (1.8) in the first chapter. For any fixed integers  $m$  and  $n$ , find the maximum number  $Z(n, m)$  of isolated zeros of the Abelian integral

$$I(t) = \int_{\gamma(t)} \omega.$$

The main interest is  $\tilde{Z}(n) := Z(n+1, n)$ .

For the case of quadratic systems with at least one center are always integrable, which can be classified into the following five classes: Hamiltonian ( $Q_3^H$ ), reversible ( $Q_3^R$ ), generalized Lotka-Volterra ( $Q_3^{LV}$ ), codimension four ( $Q_4$ ) and the Hamiltonian triangle, see [113]. The same statement in the complex domain goes back to H. Dulac, see [17].

**Definition 3.6.** A quadratic integrable system is said to be generic if it belongs to one of the first four classes and does not belong to other integrable classes. Otherwise, it is called degenerate.

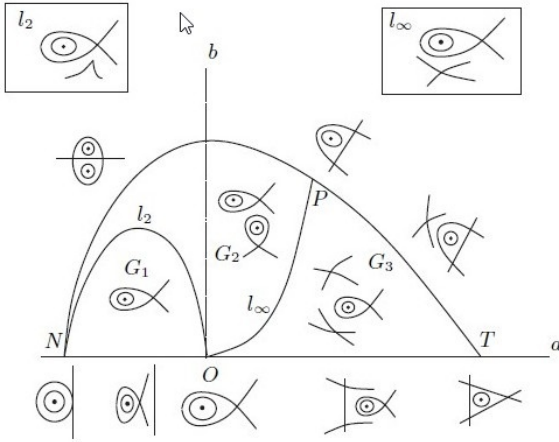
A more degenerate situation, in the quadratic systems, is the Hamiltonian triangle. Quadratic perturbations of this Hamiltonian type were studied in detail by Iliev, Gavrilov, Zolađec and Uribe (see [48], [29], [113], [103]), for multi-parameter perturbations of Hamiltonian triangle see also [72]. Horozov and Iliev proved in [46] that any cubic Hamiltonian, with at least one period annulus contained in its level curves, can be transformed into the following normal form,

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3, \quad (3.16)$$

where  $a, b$  are parameters lying in the region

$$\Omega = \{(a, b) \in \mathbb{R}^2 : -\frac{1}{2} \leq a \leq 1; \quad 0 \leq b \leq (1-a)\sqrt{1+2a}\}.$$

If  $(a, b) \in \Omega \setminus \partial\Omega$ , then the vector field associated to  $X_H$  is called generic and degenerate if  $(a, b) \in \partial\Omega$ . The Figure 3.7 is taken of [46] and show the phase portraits of Hamiltonian vector fields  $X_H$  for different values of  $a, b$  that belong the set  $\Omega$ . In the generic cases, i.e. if  $H$  is a real cubic polynomial with four distinct critical values,

Figure 3.7: Cubic Hamiltonian  $H$ 

then all quadratic vector fields sufficiently close to the Hamiltonian vector field, have at most two limit cycles. The proof of this statement is a product of a long chain of efforts by Gavrilov [32], Horozov and Iliev [46], Li and Zhang ([110],[59]) Markov [74]. Li and Llibre in [60] give a unified proof in quadratic perturbation of the generic quadratic Hamiltonian using the real domain, combining geometric and analytical methods, and using deformation arguments. The main result in quadratic perturbation of cubic generic Hamiltonian is:

**Theorem 3.4.**  $\tilde{Z}(2) = 2$ .

As it was commented by S. Yakovenko on the paper of Gavrilov [32], Theorem 3.4 concludes a long line of research and is a remarkable achievement. The problem is reduced to an investigation of Abelian integrals of quadratic 1-forms over the level curves of  $H$ . Such integrals constitute a three-parameter linear family, and the question on zeros occurring in this family reduces to a question on the geometry of a planar (or rather projective) centroid curve relative to different straight lines.

For degenerate cases, the first order term of the asymptotic de-



velopment of the displacement function is not always an Abelian integral, hence it does not give information about the cyclicity of the period annulus. Higher order approximations must be considered. Iliev in [49] gives formulas (called second- or third-order Melnikov function) to determine the cyclicity for some degenerate cases. The cyclicity of the period annulus is 3 for the Hamiltonian triangle case [48], and is 2 for all other seven cases (see [14], [28], [47], [111], [112]).

Consider the Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 + P_m(x) \quad (3.17)$$

where  $P_m$  is a polynomial function in the variable  $x$  of degree  $m$ . For  $m = 1$  the level curves of  $H$  are rational curves and have no oval. For  $m = 2$  we have

**Theorem 3.5.** *For  $H(x, y) = \frac{1}{2}(x^2 + y^2)$  and polynomial perturbations of degree  $n$ , the function  $I(t)$  defined in (3.4) has at most  $[(n-1)/2]$  zeros except the trivial zero at  $t = 0$ , which corresponds to the origin.*

**Definition 3.7.** For  $m = 3, 4$  the Hamiltonian  $H$  is called *elliptic*.

Petrov in [91], [92] studied perturbation of the elliptic Hamiltonian. Mardešić in [68] determined the lowest upper bound for the number of limit cycles of small nonconservative polynomial deformations of degree  $n$  of the elliptic Hamiltonian cubic.

**Theorem 3.6. (Mardešić [68])** *Let  $H$  the elliptic Hamiltonian cubic and  $\omega$  a polynomial 1-form of degree  $n$ , the number of zeros of the function  $I(t)$  defined in (3.4) is at most  $n - 1$ .*

For the elliptic Hamiltonian of degree four there exist five types of continuous families of ovals in level curves of  $H$  (the normal form  $H$  in this case depends on three parameters), these families are called: the truncated pendulum, the saddle loop, the global center, the cuspidal loop and the figure eight loop. Petrov, Liu, Dumortier-Li, Iliev-Perko, Li-Mardešić-Roussarie (see [66], [19], [20], [21], [22], [62]) have studied the number of zeros associated to these Hamiltonians. One of the recent results given by Gavrilov and Iliev in [30] for cubic perturbation

of elliptic Hamiltonian vector fields of degree three establishes that the period annuli of  $X_0$ , the associated principal Poincaré-Pontryagin function can produce for sufficiently small  $\varepsilon$ , at most 5, 7 or 8 zeros in the interior eight loop case, the saddle loop case, and the exterior eight loop case, respectively. In the interior eight loop case the bound is exact, while in the saddle-loop case we provide examples of Hamiltonian fields which produce 6 small amplitude limit cycles.

**Definition 3.8.** For  $m \geq 5$  the Hamiltonian  $H$  is called *hyperelliptic*. This is because the complex level curves are of genus  $\geq 2$ .

The function  $P_m$  is called the *potential* and the system associated to  $H$  is a Newtonian system of type

$$\ddot{x} - \frac{\partial P_m}{\partial x} = 0.$$

The Abelian integral associated to this system is called hyperelliptic integral. Finding an explicit upper bound of the number of zeros is extremely hard. The only general result is:

**Theorem 3.7. (Novikov-Yakovenko [89])** *For any real polynomial  $P_m(x)$  of degree  $m \geq 5$  and any polynomial 1-form  $\omega$  of degree  $n$ , the number of real ovals  $\gamma \subset \{(x, y) \in \mathbb{R}^2 : y^2 + P_m(x) = t\}$  yielding an isolated zero of the integral  $I(t) = \int_{\gamma} \omega$ , is bounded by a primitive recursive function  $B(m, n)$  of two integer variables  $m$  and  $n$ , provided that all critical values of  $P_m$  are real.*

Finally, another problem related to hyperelliptic Hamiltonian is

**Theorem 3.8. (Moura [75])** *Let  $H = y^2 - x^{n+1} - \tilde{H}(x)$  be a hyperelliptic Hamiltonian with  $\tilde{H}$  a polynomial of degree  $n - 1$  of Morse type. Let  $\omega$  a polynomial 1-form such that  $\deg \omega < \deg H = n + 1$ . If  $I(t) = \int_{\gamma(t)} \omega \neq 0$ , then*

$$\text{ord}_{t=t_0} I(t) \leq n - 1 + \frac{n(n-1)}{2},$$

where  $t_0$  is a regular value of  $H$ .

For upper bounds on the multiplicity of abelian integrals see [69, 81].

### 3.5 Arnold-Hilbert problem in dimension zero

Let  $f \in \mathbb{C}[z]$  be a polynomial of degree  $m$  and in one variable  $z$ , and  $n_1, n_2, \dots, n_m$  be integers satisfying  $n_1 + n_2 + \dots + n_m = 0$ . Define a zero cycle  $\gamma(t)$  by the formula

$$\gamma(t) = n_1 z_1 + \dots + n_m z_m$$

where  $z_1(t), \dots, z_m(t)$  denote the algebraic functions satisfying the equation  $f(z(t)) = t$ .

**Definition 3.9.** For every polynomial  $g \in \mathbb{C}[z]$ , define the integral  $I(t)$  of dimension zero by the formula

$$I(t) = \int_{\gamma(t)} g = n_1 g(z_1(t)) + \dots + n_m g(z_m(t))$$

In the following we will only consider the special case

$$\gamma(t) = z_1 - z_2$$

which we call it a simple cycle. The infinitesimal Hilbert's 16th problem in dimension zero is then to study the number  $Z(m, n)$  of zeros of the algebraic function  $I(t)$ , where  $\deg(g) \leq n$ . This problem has been studied firstly by one of the authors. The following bound is obtained.

**Theorem 3.9. (Gavrilov-Movasati [34])** *We have*

$$n - 1 - \left\lfloor \frac{n}{m} \right\rfloor \leq Z(m, n) \leq \frac{(m-1)(n-1)}{2}. \quad (3.18)$$

The lower bound in this inequality is given by the dimension of the vector space of integrals

$$V_n = \left\{ \int_{\gamma(t)} g, \deg g \leq n \right\},$$

where  $f$  is a fixed general polynomial of degree  $m$ , while the upper bound is a reformulation of Bezout's theorem. When  $m = 3$  we get  $Z(m, m-1) = 1$ . The space of integrals  $V_n$  is Chebyshev, possibly with some accuracy.

**Definition 3.10.** Recall that  $V_n$  is said to be Chebyshev with accuracy  $c$  if every  $I \in V_n$  has at most  $\dim V_n - 1 + c$  zeros in the domain  $D$ .

The exact description of the number  $Z(n+1, n)$  is not yet known. If we do not put any restriction on  $z_1$  and  $z_2$  the sharp upper bound is  $n(n-1)/2$ . For instance take  $f(z) = (z-1)(z-2)\dots(z-n-1)$  and  $g(z) = (z-1)(z-2)\dots(z-n)$ . In the image of  $f$  we can find  $n$  intervals with mentioned property and the main problem is: How many of those zeros can be grouped in one of such intervals. On the other hand, Alvarez, Bravo and Mardešić in [2] study the vanishing integrals on zero dimensional cycles.

**Theorem 3.10.** *Given  $f$  and  $g$  in  $\mathbb{C}(z)$ , the following conditions are equivalent*

(a)  $\int_{\gamma(t)} g \equiv 0$  for every (simple) cycles  $\gamma(t)$ .

(b) There exists  $g_0 \in \mathbb{C}(z)$  such that  $g = g_0 \circ f$ .

For a proof see [34]. The tangent center problem asks one to find all polynomials  $g$  such that the algebraic function  $I(t)$  vanishes identically. This problem, first formulated and studied in [34, 3], has finally been solved by Gavrilov and Pakovich in [36].

# Chapter 4

## Complexification of Abelian integrals

In this chapter we will explain some fundamental properties of the principal Poincaré-Pontryagin function. A natural context for the study of this function is the complex plane. The main tool is Picard-Lefschetz theory of Hamiltonian differential equations. For this we mainly use [5] and [29]. The principal term in the asymptotic development of the displacement map associated to small one parameter polynomial deformation of polynomial Hamiltonian vector fields has an analytic continuation and the real zeros of this principal term correspond to limit cycles bifurcating from the periodic orbits of the Hamiltonian flow.

### 4.1 Picard-Lefschetz theory

In this section we introduce basic concepts of Picard-Lefschetz theory, such as fibration, vanishing cycles, monodromy and variation operators, the Picard-Lefschetz formula, etc.

Let  $H : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a polynomial function in the complex variables  $x$  and  $y$ . There exist a finite set  $C$ , called atypical values of  $H$ ,

such that the map

$$H : \mathbb{C}^2 \setminus H^{-1}(C) \longrightarrow \mathbb{C} \setminus C, \quad (4.1)$$

is a locally trivial fibration, see for instance [11]. This follows from Ehresmann's fibration theorem applied to a compactification and desingularization of the indeterminacy singular points of  $H$ , see for instance [85]. This fibration is generally called the Milnor Fibration. Moreover, a canonical connection is given in the the cohomology group fibers which is called the Gauss-Manin connection. In [44] it is proved that the set of atypical values is the union of the set  $\Delta_H$  of critical values of  $H$  and the set  $\Delta_\infty$  of infinitive values. These are values for which the Euler Poincaré characteristic of the fiber is different from the characteristic of the general fiber.

**Definition 4.1.** A polynomial  $H \in \mathbb{C}[x, y]$  with  $\deg(H) = d$  is said to be *transversal to infinity*, if its principal homogeneous part  $L(x, y) = \sum_{i+j=d} a_{i,j} x^i y^j$  of maximal degree is a product of  $d$  linear different factors.

An important property of a Hamiltonian function  $H$  transversal to infinity is the fact that its atypical values are only the critical values of  $H$ . The transversal to infinity condition can be equivalently expressed as:

- i) The principal homogeneous part has an isolated critical point at the origin.
- ii) The partial derivatives  $\partial L/\partial x$  and  $\partial L/\partial y$  are mutually prime.
- iii)  $H$  has exactly  $\mu = (\deg(H) - 1)^2$  critical points in  $\mathbb{C}^2$  if counted with multiplicities.
- iv) Each level curve  $\{H = t\}$  intersects transversely the infinite line  $\mathbb{CP}_\infty^1 \subset \mathbb{CP}^2$  after projective compactification.

Now for a regular value  $t$  the generic fiber  $L_t = H^{-1}(t)$  is a Riemann surface of genus  $g$ , with removed points at infinity. The first homology group  $H_1(L_t, \mathbb{Z})$  is of dimension  $\mu$  where

$$\mu = \dim \frac{\mathbb{C}[x, y]}{(H_x, H_y)}$$

The first homology group  $H_1(L_t, \mathbb{Z})$  is the abelization of the homotopy group  $\pi_1(L_t, b)$ , with  $b \in L_t$ , that is

$$H_1(L_t, \mathbb{Z}) = \pi_1(L_t, b) / [\pi_1(L_t, b), \pi_1(L_t, b)],$$

where  $[\pi_1(L_t, b), \pi_1(L_t, b)]$  is the subgroup of  $\pi_1(L_t, b)$  generated by the commutators  $aba^{-1}b^{-1}$ ,  $a, b \in \pi_1(L_t, b)$ .

## 4.2 Vanishing cycles

Vanishing cycles are mainly used in singularity theory and the topology of algebraic varieties. Let us be given a polynomial  $H \in \mathbb{C}[x, y]$  of degree  $d$ . We consider  $H$  as a function from  $\mathbb{C}^2$  to  $\mathbb{C}$ .

**Definition 4.2.** Let  $\gamma(t)$  be a cycle in  $H_1(L_t, \mathbb{Z})$  represented by the sphere  $S^1$ . It is called a *vanishing cycle* (along the path  $u_i$  in  $\mathbb{C}$ ) in the critical value  $c_i$  of  $H$  if it degenerates in a critical point of the singular fiber.

The set

$$C = \{c_1, c_2, \dots, c_s\}$$

of critical values of  $H$  is finite and so each point in  $C$  is isolated in  $\mathbb{C}$ . Let  $D \subset \mathbb{C}$  be a disc containing the critical values  $C$  and  $b$  be a regular value of  $H$  in the boundary of  $D$ . Consider a system of  $s$  paths  $u_1, u_2, \dots, u_s$  starting from  $b$  and ending at  $c_1, c_2, \dots, c_s$  respectively and such that: each path  $u_i$  has no self intersection points, two distinct paths  $u_i$  and  $u_j$  meet only at their common origin  $u_i(0) = u_j(0) = b$  and  $\{u_1, u_2, \dots, u_s\}$  near  $b$  is the anticlockwise direction. This is called a *distinguished set of paths*, see Figure 4.1. The set of vanishing cycles along the paths  $u_i$ ,  $i = 1, 2, \dots, s$  is called a distinguished basis of vanishing cycles related to the critical values  $c_1, c_2, \dots, c_s$ . We denote them by  $\gamma_1(b), \gamma_2(b), \dots, \gamma_s(b) \in H_1(L_b, \mathbb{Z})$ .

**Example 4.1.** To carry an example in mind, take the polynomial  $f = y^2 - x^3 + 3x$  in two variables  $x$  and  $y$ , see Figure 1.3. We have  $C = \{-2, 2\}$ . For  $t$  a real number between 2 and  $-2$  the level surface of  $f$  in the real plane  $\mathbb{R}^2$  has two connected pieces which one of them is an oval and we can take it as  $\gamma_t$ . In this example as  $t$  moves from

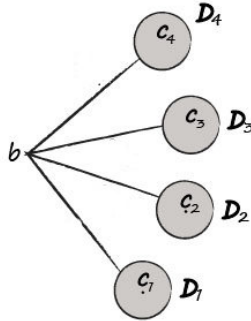


Figure 4.1: A distinguished set of paths

$-2$  to  $2$ ,  $\gamma_t$  is born from the critical point  $(-1, 0)$  of  $f$  and end up in the  $\alpha$ -shaped piece of the fiber  $f^{-1}(2) \cap \mathbb{R}^2$ . Another good example is

$$f = \frac{y^2}{2} + \frac{(x^2 - 1)^2}{4}$$

The set of critical values of  $f$  is  $C = \{0, \frac{1}{4}\}$  and we can distinguish three family of ovals.

### 4.3 Monodromy

Let  $\lambda$  be a path in  $\mathbb{C} \setminus C$  with the initial and end regular values in  $b_0 = \lambda(0)$  and  $b_1 = \lambda(1)$  of  $H$ . Consider the regular fibers  $L_{b_0} = H^{-1}(b_0)$  and  $L_{b_1} = H^{-1}(b_1)$ . The path  $\lambda$  can be lifted to a  $C^\infty$  map from  $L_{b_0}$  to  $L_{b_1}$  defining an isomorphism in the homology bundle, see Figure 4.2. Let  $D \subset \mathbb{C}$  be a disc containing the atypical values of  $H$  and  $t$  be a regular value of  $H$  in the boundary of  $D$ . Let  $\tilde{u}_i$  be a path in  $D \setminus C$ , in the complementary set of atypical values. It starts in  $t$ , goes along  $u_i$  until  $c_i$ , turns around  $c_i$  anticlockwise and returns back to  $t$  along  $u_i$ , we define the *monodromy* around  $c_i$  as the monodromy along  $\tilde{u}_i$ , i.e.

$$\mathbf{m}_i : H_1(L_t, \mathbb{Z}) \longrightarrow H_1(L_t, \mathbb{Z}) \quad (4.2)$$



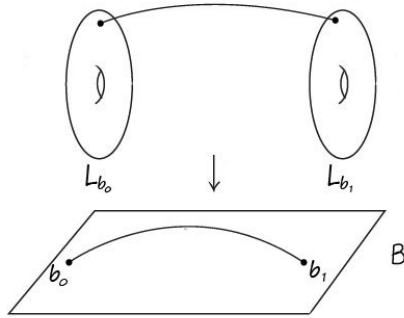


Figure 4.2: Monodromy along a path

The monodromy  $\mathbf{m}_i$  around the critical value  $c_i$  is given by the Picard-Lefschetz formula:

$$\mathbf{m}_i(\sigma) = \sigma - \sum_{j=1}^s (\sigma \circ \delta_j) \delta_j. \quad (4.3)$$

In this formula one considers non-degenerated critical points, where  $\delta_j$  are vanishing cycles of  $H_1(L_t, \mathbb{Z})$  at  $c_i$  and the notation  $(\sigma \circ \delta_j)$  represents the intersection number between the cycles  $\sigma$  and  $\delta_j$ , see Figure 4.3.

**Definition 4.3.** The *monodromy representation* associated to the function  $H$  is the group homomorphism

$$\pi_1(D \setminus C, t) \longrightarrow \mathcal{A}ut(H_1(L_t, \mathbb{Z})).$$

This homomorphism associates to the loop  $\gamma_i$  the monodromy operator  $\mathbf{m}_i$ . The image of the fundamental group  $\pi_1(D \setminus C, t)$  under the group homomorphism  $\mathcal{A}ut(H_1(L_t, \mathbb{Z}))$  is called the *Monodromy group* of  $H$ .

The following example is taken from [85].

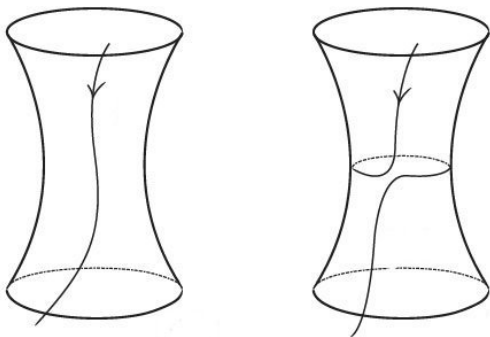


Figure 4.3: Picard Lefschetz Formula

**Example 4.2.** Consider the group  $SL(2, \mathbb{Z})$  defined by

$$SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

This group appears as the monodromy group of families of elliptic curves over  $\mathbb{C}$ . Let

$$E_t : y^2 - 4x^3 + t_2x + t_3 = 0,$$

$$t \in T := \mathbb{C}^2 \setminus \{(t_2, t_3) \in \mathbb{C} : \Delta = 0\},$$

where  $\Delta := 27t_3^2 - t_2^3$ . The elliptic curve  $E_t$  as a topological space is a torus minus a point and hence  $H_1(E_t, \mathbb{Z})$  is a free rank two  $\mathbb{Z}$ -module. We want to compute the monodromy representation

$$\pi_1(T, b) \rightarrow \text{Aut}(H_1(E_b, \mathbb{Z})),$$

where  $b$  is a fixed point in  $T$ . Fix the parameter  $t_2 \neq 0$  and let  $t_3$  varies. For  $\tilde{t}_3 = \pm \sqrt{\frac{t_2^3}{27}}$  of  $t_3$ , the curve  $E_t$  is singular. In  $E_b$  we can take two cycles  $\delta_1$  and  $\delta_2$  such that  $(\delta_1 \circ \delta_2) = 1$  and  $\delta_1$

(resp.  $\delta_2$ ) vanishes along a straight line connecting  $b_3$  to  $\tilde{t}_3$ . The corresponding anticlockwise monodromy around the critical value  $\tilde{t}_3$  can be computed using the Picard Lefschetz formula:

$$\delta_1 \mapsto \delta_1, \quad \delta_2 \mapsto \delta_2 + \delta_1 \text{ (resp. } \delta_1 \mapsto \delta_1 - \delta_2, \quad \delta_2 \mapsto \delta_2)$$

Therefore, the image of  $\pi_1(T, b)$  under the monodromy representation contains the following matrices in  $SL(2, \mathbb{Z})$ :

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

These matrices generates  $SL(2, \mathbb{Z})$ .

Figure 4.4, facilitated by Professor S. Rebollo shows the movement of cycles in the generic fiber under the action of monodromy.

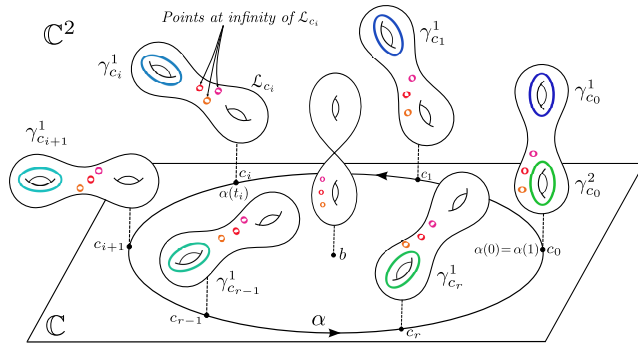


Figure 4.4: Monodromy of Hamiltonian  $H$

## 4.4 Properties of principal Poincaré-Pontryagin function

Consider a polynomial deformation defined by

$$dH - \varepsilon Q(x, y, \varepsilon)dx + \varepsilon, P(x, y, \varepsilon)dy = 0 \tag{4.4}$$

where  $P, Q$  are polynomials in  $x, y$  and analytic in  $\varepsilon$  a sufficiently small real parameter. Let  $C$  be the set of atypical values of  $H$ ,  $t_0 \notin C$ ,  $p_0 \in H^{-1}(t_0)$  and  $\Sigma \subset \mathbb{C}^2$  be a small complex disc centered at  $p_0$  and transversal to  $H^{-1}(t) \in \mathbb{C}^2$ . We will also suppose that the fibers  $H^{-1}(t)$  which intersect  $\Sigma$  are regular. Let  $\delta(t) \subset \{(x, y) : H(x, y) = t\}$  be a continuous family of cycles. For a closed loop

$$\ell_0 : [0, 1] \rightarrow H^{-1}(t_0); \quad \ell_0(0) = \ell_0(1) = p_0$$

we define the Poincaré first return map (holonomy) in the complex domain.

$$\mathcal{P}_{\ell_0} : \Sigma \rightarrow \Sigma.$$

There exists  $k \in \mathbb{N}_0$  such that

$$\mathcal{P}_{\ell_0}(t) = t + \varepsilon^k M_k(\ell_0, t) + \dots$$

The function  $M_k(\ell_0, t)$  is called in [29] *the generating function* associated to the deformation (4.4) and the loop  $\ell_0$ .

**Proposition 4.1. (Gavrilov and Iliev, [29])** *The number  $k$  and the generating function  $M_k(\ell_0, t)$  do not depend on  $\Sigma$ . They depend on the deformation (4.4) and on the free homotopy class of the loop  $\ell_0 \subset H^{-1}(t)$ . The generating function  $M_k(\ell_0, t)$  has an analytic continuation on the universal covering of  $\mathbb{C} \setminus C$ , where  $C$  is the set of atypical points of  $H$ .*

If  $\ell_0$  and  $\ell_1$  are two homotopic loops with the same initial point  $p_0$ , then  $\mathcal{P}_{\ell_0} = \mathcal{P}_{\ell_1}$ . In general, we write  $M_k(\ell_0, t) = M_k(t)$ . In [103, 104] the author uses principally the free homotopy condition to obtain a classification of the generating function  $M_k(t)$  associated to arbitrary polynomial perturbations of a Hamiltonian vector field  $H$  formed by a product of real function, see also [90].

**Theorem 4.1. (Gavrilov and Iliev, [29])** *The generating function  $M_k(t)$  satisfies a linear differential equation*

$$a_n(t)x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x = 0,$$

where  $n \leq \text{rank}H_1(H^{-1}(t), \mathbb{Z})^{\ell_0}$  and  $a_i(t)$  are suitable function on  $\mathbb{C} \setminus C$ .

Here  $H_1(H^{-1}(t), \mathbb{Z})^{\ell_0} \subset H_1(H^{-1}(t), \mathbb{Z})$  is obtained by the action of monodromy on  $\ell_0$ . The proof of Theorem 4.1 is obtained by the following considerations: The monodromy representation of  $M_k(t)$  and its moderate growth at singularities.

**Definition 4.4.** Let  $x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = 0$  be a linear differential equation with meromorphic coefficients in  $t$ . We say that the point  $t = t_0$  is singular if at least one of the coefficients  $a_i$  has no holomorphic continuation in  $t = t_0$ . We say that the point  $t = t_0$  is regular singular if  $(t - t_0)^i a_i(t)$  is holomorphic in  $t = t_0$ . We say that the differential equation is Fuchsian, if each singular point is a regular point.

Let  $\delta(t_0)$  be a loop in the first homology group  $H_1(H^{-1}(t_0), \mathbb{Z})$ . We denote by  $\delta(t)$  the monodromy  $\delta(t_0)$  to nearby fibers.

**Definition 4.5.** We define

$$\mathcal{P}_{\delta(t)} = \left\{ \int_{\delta(t)} \omega : \omega = p(x, y)dx + q(x, y)dy, p, q \in \mathbb{C}[x, y] \right\}. \quad (4.5)$$

$\mathcal{P}_{\delta(t)}$  is a  $\mathbb{C}[t]$ -module, with the multiplication

$$A(t) \int_{\delta(t)} \omega = \int_{\delta(t)} A(H)\omega.$$

We called  $\mathcal{P}_{\delta(t)}$  the Petrov module.

Let  $H \in \mathbb{C}[x, y]$  be a polynomial function,  $\delta(t)$  be a continuous family of ovals on the non-singular fibers  $H^{-1}(t)$  and let  $\omega$  be a polynomial 1-form in  $\mathbb{C}[x, y]$ . The Abelian integral  $I(t) = \int_{\delta(t)} \omega$  is a multivalued analytic function with a finite set of ramification points which depend only on  $H$ . This follows from the monodromy representation of  $H$ .

**Theorem 4.2. (Gavrilov [31])** Suppose that the representation of monodromy group associated to  $H$  is completely reducible, that is

$$H_1(H^{-1}(t), \mathbb{Q}) = \oplus_i V_i(t),$$

and each invariant space  $V_i(t)$  contains a vanishing cycle  $\delta_i(t)$ . Then the  $\mathbb{C}[t]$ -module  $\mathcal{P}_{\delta_i(t)}$  is free and its rank is equal to the dimension of  $V_{\delta_i(t)}$ .

**Example 4.3.** Let  $H$  be the polynomial function defined by

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + xy^2.$$

The polynomial  $H$  has four critical points. One elliptical point, on  $(0, 0)$ , three point of Morse type on  $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$  and  $(1, 0)$ . The critical values of  $H$  are 0 and  $1/6$ . We observe that the generic fiber is a torus minus three point at infinity and therefore

$$\text{rank}H_1(H^{-1}(t), \mathbb{Z}) = 4.$$

Let  $\gamma(t)$ ,  $\delta_1(t)$ ,  $\delta_2(t)$  and  $\delta_3(t)$  be the continuous families of vanishing cycles defined in the interval  $]0, 1/6[$  such that  $\gamma(t)$  vanishes at 0 and the cycles  $\delta_1(t)$ ,  $\delta_2(t)$  and  $\delta_3(t)$  vanish at  $1/6$ . Moreover, we can choose the orientation of the cycles such that  $(\gamma(t) \circ \delta_i(t)) = 1$ .

By Picard-Lefschetz formula, the vector space  $V_{\gamma(t)}$  is generate by  $\gamma(t)$  and  $\delta_1(t) + \delta_2(t) + \delta_3(t)$  and therefore the dimension of  $V_{\gamma(t)}$  is two. Moreover, we have

$$H_1(H^{-1}(t), \mathbb{Q}) = V_{\gamma(t)} \oplus V_{\delta_1 - \delta_2} \oplus V_{\delta_2 - \delta_3}$$

where  $V_{\delta_1 - \delta_2}$  is the one dimensional space generated by  $\{\delta_1 - \delta_2\}$ , etc. Over each subspace the monodromy group is irreducible and by Theorem 4.2 the  $\mathbb{C}[t]$ -module  $\mathcal{P}_{\gamma(t)}$  is two dimensional. Now, we consider the polynomial one forms  $\omega_1, \omega_2, \omega_3$  and  $\omega_4$  such that  $\omega_1 = ydx$ ,  $\omega_2 = y^2dx$ ,  $\omega_3 = y^3dx$  and  $\omega_4 = xydx$ , then we have

$$\int_{\gamma(t)} xydx = \int_{\gamma(t)} y^2dx = 0.$$

Therefore, we can conclude that the  $\mathbb{C}[t]$ -module  $\mathcal{P}_{\gamma(t)}$  is generated by the Abelian integrals  $I_1(t)$  and  $I_3(t)$ , where

$$I_1(t) = \int_{\gamma(t)} ydx ; I_3(t) = \int_{\gamma(t)} y^3dx.$$

□

**Definition 4.6.** A polynomial  $H \in \mathbb{C}[x, y]$ ,  $\text{deg}H = d$  is called *quasi-homogeneous* of weighted degree  $w\text{deg}(H) = d$  and of type  $\mathbf{w} = (w_x, w_y)$ , where  $w_x = \text{poids}(x)$  and  $w_y = \text{poids}(y)$  are integers, if

$$H(z^{w_x}x, z^{w_y}y) = z^d H(x, y) ; \forall z \in \mathbb{C}^*. \quad (4.6)$$

A polynomial  $H \in \mathbb{C}[x, y]$ , is called *semi quasi-homogeneous* of weighted degree  $wdeg(H) = d$  and of type  $\mathbf{w}$  if  $H$  can be written as

$$H(x, y) = \sum_{i=0}^d H_i(x, y),$$

where the polynomials  $H_i$  are quasi-homogeneous of weighted degree  $wdeg(H_i) = i$  and of type  $\mathbf{w}$ , and polynomials  $H_d(x, y)$  have an isolated critical point at the origin.

In [85] semi quasi-homogeneous polynomials are called tame polynomials, see also [84]. Let  $H \in \mathbb{C}[x, y]$  be a polynomial function,  $\Omega^1$  be the space of all polynomial one forms on  $\mathbb{C}[x, y]$  and  $\mathcal{B}$  the space of all one forms of type  $\omega = gdH + dR$ , with  $g$  and  $R$  polynomials on  $\mathbb{C}[x, y]$ . We define the quotient space

$$\mathcal{H}_{rel} = \frac{\Omega^1}{\mathcal{B}}, \quad (4.7)$$

The space  $\mathcal{H}_{rel}$  is called the Brieskorn or Petrov module of  $H$ . It is a module over the ring of polynomial function  $\mathbb{C}[t]$  and under the multiplication  $A(t) \cdot \omega = A(H)\omega$ ,  $A \in \mathbb{C}[t]$ .

**Theorem 4.3. (Gavrilov [31])**

1. If  $H \in \mathbb{C}[x, y]$  is a semi quasi-homogeneous polynomial of weighted degree  $wdeg(H) = d$  then the  $\mathbb{C}[t]$ -module  $\mathcal{H}_{rel}$  is free and finitely generated by  $\mu$  one forms  $\omega_1, \dots, \omega_\mu$ , where  $\mu = (d - w_x)(d - w_y)/w_x w_y$ . Each one form  $\omega_i$  satisfies the condition

$$d\omega_i = g_i dx \wedge dy,$$

where the set  $\{g_1, \dots, g_\mu\}$  is a monomial basis of the quotient  $\mathbb{C}[x, y]/\langle H_x, H_y \rangle$ .

2. If  $\omega$  is a polynomial one form then there exists polynomials  $P_k(t)$  of degree at most  $(wdeg(\omega) - wdeg(\omega_i))/wdeg(H)$  such that in  $\mathcal{H}_{rel}$  we have

$$\omega = \sum_{i=0}^{\mu} P_k(t)\omega_i.$$

For further generalization of this theorem in more variables and also algorithms for computing  $P_k(t)$  see [85, 84]. The polynomial  $H(x, y) = y^2 + P(x)$ , where  $P(x)$  is a polynomial of degree  $d \geq 2$  is semi quasi-homogeneous of weighted degree  $d$  and the type  $w_x = 1$ ,  $w_y = d/2$ , so the Milnor number of  $H$  is  $\mu = (d - 1)$  and therefore by Theorem 4.3 the set  $\{x^k y dx : k = 0, \dots, (d - 2)\}$  is a monomial basis of  $\mathcal{H}_{rel}$ .

For a polynomial one form  $\omega = A(x, y)dx + B(x, y)dy$ , the weighted degree  $wdeg(\omega)$  is

$$wdeg(\omega) = \max\{wdeg(A) + w_x, wdeg(B) + w_y\}.$$

As a consequence of Theorem 4.3 we have the following corollary:

**Corollary 4.1.** The Abelian integrals space form a  $\mathbb{C}[t]$ -module generated by the integral over one forms  $\omega_i$ , i.e.

$$\int_{\delta(t)} \omega = \sum_{i=1}^{\mu} P_i(t) \int_{\delta(t)} \omega_i,$$

where  $P_i \in \mathbb{C}[t]$  and  $deg\omega_i + degH \cdot degP_i \leq deg\omega$ .

## 4.5 Hamiltonian triangle

We consider a small polynomial perturbation of the Hamiltonian vector field with the Hamiltonian

$$H(x, y) = x[y^2 - (x - 3)^2]$$

having a center bounded by a triangle, and denote by  $\delta(t)$  the family of ovals defined by  $\{(x, y) \in \mathbb{R}^2 : H(x, y) = t\}$  with  $t \in (-4, 0)$ . We will denote by the same letters the corresponding continuous family of free homotopy classes of loops defined on the universal covering space of  $\mathbb{C} \setminus \{-4, 0\}$ .

The Hamiltonian triangle is the unique case belonging to the intersection of three strata of quadratic centers  $Q_3^H$ ,  $Q_3^{LV}$ , and  $Q_3^R$ . In [112] Zoladek studied the displacement map for small quadratic perturbations of the Hamiltonian triangle and proved that for small quadratic perturbations, the number of limit cycles produced by the



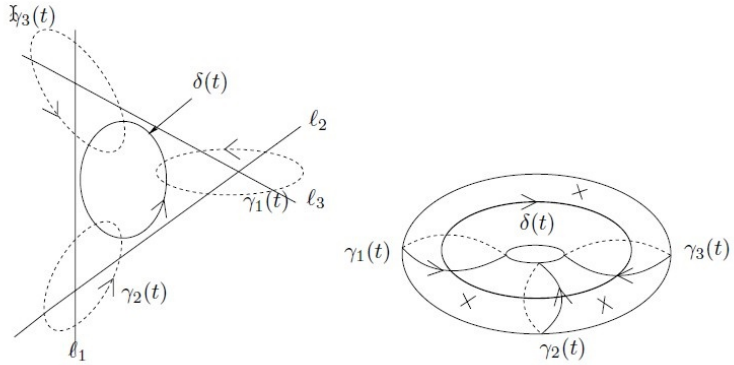


Figure 4.5: Cycles in the real and complex fiber

period annulus is equal to the number of zeros of a function  $J(t)$ , where  $J(t)$  is not an Abelian integral. This was also observed by Iliev in [48].

In [47], Iliev studies the quadratic perturbations of the Hamiltonian triangle. He determines a condition for the principal Poincaré-Pontryagin function under the hypothesis  $M_1(t) = M_2(t) = 0$ . More precisely, he shows that

$$tM_3(t) = \alpha(t)I_1(t) + (\beta(t) + t\gamma(t))I_0(t) + \delta_*(t)I_*(t),$$

where the constants  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$  and  $\delta_*(t)$  are polynomials in  $t$ . Indeed, the integrals  $I_1(t)$  and  $I_0(t)$  are Abelian integrals and

$$I_*(t) = \int_{\delta(t)} y(x-1) \ln x dx$$

is not Abelian integral. The asymptotic development of  $I_*(t)$  at  $t = 0$  contains a term in  $\ln^2(t)$ . An important and more general result for the Hamiltonian triangle is given in [103] and is describe as follows:

**Theorem 4.4. (Uribe, [103])** *The principal Poincaré-Pontryagin function  $M_k(t)$  associated with the family of ovals  $\delta(t)$  surrounding the center and with any polynomial perturbation of the symmetric*

*Hamiltonian triangle belongs to the  $\mathbb{C}[t, 1/t]$ -module generated by the integrals*

$$I_0(t) = \int_{\delta(t)} y dx, \quad I_2(t) = \int_{\delta(t)} x^2 y dx,$$

$$I^*(t) = \int_{\delta(t)} \ln x \cdot d \left( \ln \frac{y-x+3}{y+x-3} \right).$$

The integrals  $I_0(t)$  and  $I_2(t)$  are Abelian integrals, whereas  $I^*(t)$  is not an Abelian integral. Generally, in the space of polynomial perturbations of degree  $d$ ,  $d \geq 5$ , of the symmetric Hamiltonian triangle whose first Poincaré-Pontryagin function  $M_1(t)$  vanishes, the principal Poincaré-Pontryagin function  $M_2(t)$  is not an element of the  $\mathbb{C}[t, 1/t]$ -module generated by the integrals  $I_0(t)$  and  $I_2(t)$ . For the discussion of an arbitrary perturbation of the Hamiltonian triangle see [112], [72]. Finally, we would like to note that all the non-abelian integrals which appear in this context can be written as iterated integrals, see Chapter 6.

## 4.6 Product of linear function

Let  $f_i(x, y)$ ,  $i = 0, \dots, d$  be real linear functions in the variables  $x, y$ . Consider the polynomial function  $H$  given by

$$H(x, y) = f_0(x, y)f_1(x, y) \cdots f_d(x, y). \quad (4.8)$$

The zero level of the function  $H$  is formed by the union of the  $(d+1)$  lines,

$$\ell_k = \{(x, y) \in \mathbb{R}^2 \mid f_k(x, y) = 0\}, \quad k = 0, \dots, d. \quad (4.9)$$

Suppose that the lines  $\ell_k$  are in general position in  $\mathbb{R}^2$ . That is, they are distinct, non-parallel and no three of them have a common intersection point. Note that all the critical points of  $H$  lie on the real plane and are of Morse type. The total number of critical points of  $H$  is  $d^2$ . From this we have  $a_1 := \frac{d(d-1)}{2}$  critical points of center type and  $a_2 := \frac{d(d+1)}{2}$  critical points of saddle type. For a regular value  $t$ , the generic fiber  $L_t = H^{-1}(t)$  is an  $a_1$ -genus Riemann surface with

$(d+1)$  removed points at infinity. The first homology group  $H_1(L_t, \mathbb{Q})$  is of dimension  $\mu = a_1 + a_2$ .

Let  $\Delta_H = \{t_0, t_1, t_2, \dots, t_{a_1}\}$  be the set of critical values of  $H$ . The first critical value  $t_0 = 0$  corresponds to all the saddle points of  $H$ . For a generic choice of lines we have  $a_1$  distinct critical values  $t_i \neq 0$  corresponding to center points. We denote by  $\delta_i(t)$ ,  $i = 1, \dots, a_1$ , the continuous family of vanishing cycles at  $t_i \neq 0$  and  $\gamma_j(t)$ , with  $j = 1, \dots, a_2$ , the continuous family of vanishing cycles at the critical value  $t_0 = 0$ . The set  $\{\delta_1(t), \dots, \delta_{a_1}(t), \gamma_1(t), \dots, \gamma_{a_2}(t)\}$  is a distinguish basis of vanishing cycles of the first homology group  $H_1(L_t, \mathbb{Q})$ . We choose the orientation of the cycles  $\delta_i(t)$  and  $\gamma_j(t)$  so that the intersection number verify  $(\delta_i(t) \circ \gamma_j(t)) = -1$ . In the following, we will give a more convenient basis of  $H_1(L_t, \mathbb{Q})$ . Consider a vanishing cycle  $\delta(t)$  around a critical point of center type and the set

$$V_c = Orb_{\mathbf{m}}(\delta(t)), \quad (4.10)$$

where  $Orb_{\mathbf{m}}(\delta(t)) \subset H_1(L_t, \mathbb{Q})$  is obtained by the action of monodromy on  $\delta_t$ . Define the space  $V_\infty$  as follows

$$V_\infty = \{\delta'(t) \in H_1(L_t, \mathbb{Q}) \mid (\delta'(t) \circ \delta''(t)) = 0, \quad \forall \delta''(t) \in H_1(L_t, \mathbb{Q})\}. \quad (4.11)$$

Observe that the elements of  $V_\infty$  are invariant under the action of the monodromy  $\mathbf{m}_{t_i}$  for  $i = 0, 1, \dots, a_1$ . The elements of  $V_\infty$  are called cycles at infinity. For a regular value  $t$ , we consider the compactification  $\overline{L}_t$  of  $L_t$ . The surface  $\overline{L}_t$  is of genus  $g = a_1$  and we have the following result:

**Theorem 4.5. (Movasati, [80])** *The orbit of a vanishing cycle  $\delta(t)$  around a critical point of center type, by the action of the monodromy generates the first homology group  $H_1(\overline{L}_t, \mathbb{Q})$ .*

As an immediate consequence of Theorem 4.5 we have that

$$\dim V_c = 2a_1, \quad H_1(L_t, \mathbb{Q}) = V_c \oplus V_\infty.$$

Recall that each relatively compact component  $U_k$  of  $\mathbb{R}^2 \setminus H^{-1}(0)$  has one center point  $p_k$ . The variation map is obtained by  $Var_{t_k} \equiv \mathbf{m}_{t_k} - Id$ . Define  $\delta^k(t)$  the variation of the cycle  $\delta_k(t)$  by the monodromy  $\mathbf{m}_0$  around the critical value  $t_0$ . The cycle  $\delta^k(t)$  is equal to  $\sum_{i=1}^n \gamma_i(t)$ ,

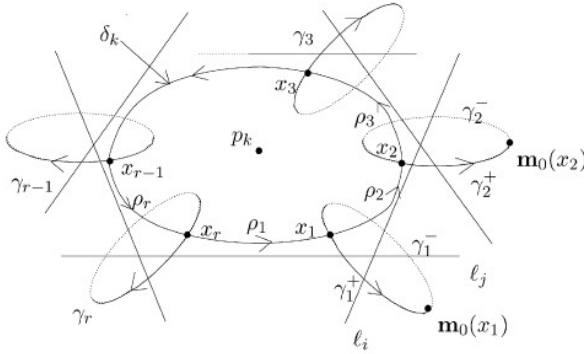


Figure 4.6: Closed loops in region  $U_k$

where  $\gamma_i(t)$  are vanishing in the vertices of  $U_k$  and  $n$  is the number the vertices of  $U_k$ .

We denote by  $\sigma_p^\infty$ ,  $p = 0, \dots, d$ , the cycle that turns once around the point at infinity  $p_i^\infty$  corresponding to the line  $l_i$ . The cycles  $\gamma_i(t)$  vanishing at saddle points are generated by the cycles  $\delta^i(t)$  and the cycles  $\sigma_p^\infty$ , see [80] for more details. The set of cycles  $\{\delta_i, \delta^i, \sigma_p^\infty : 1 \leq i \leq a_1 ; 1 \leq p \leq d\}$  is the convenient basis of  $H_1(L_t, \mathbb{Q})$ .

## 4.7 Relative cohomology for product of lines

**Definition 4.7.** For the polynomial function  $H$  and  $f_0, \dots, f_d$  given by (4.8), let

$$\eta_k := H \frac{df_k}{f_k}, \quad k = 0, \dots, d$$

and

$$\varphi_k(x, y) = \int_{a_k}^{(x,y)} \frac{1}{f_k} df_k = \log f_k ; \quad k = 0, \dots, d \quad (4.12)$$

where the points  $a_k$  and  $(x, y)$  are arbitrary points belonging to  $\mathbb{C}^2 \setminus \cup_{k=0}^d \ell_k$ , and the lines  $\ell_k$  are defined as in (4.9).

The points  $a_k$  are fixed, whereas the point  $(x, y)$  varies. Each function  $\varphi_k$  is a multivalued function in  $\mathbb{C}^2 \setminus \cup_{k=0}^d \ell_k$ , univalued in the universal covering of  $\mathbb{C}^2 \setminus \cup_{k=0}^d \ell_k$ . Let  $\delta(t)$  be a vanishing cycle that turns around a singular point of center type. The value of the function  $\varphi_k$  does not change along the cycle  $\delta(t)$ , because this cycle does not turn around the lines  $\ell_k$ ,  $k = 0, \dots, d$ . Hence

$$\int_{\delta(t)} d\varphi_k = 0, \quad k = 0, \dots, d. \quad (4.13)$$

We conclude that the  $\mathbb{C}$ -vector space generated by  $\eta_k = Hd\varphi_k$  is dual to  $V_\infty$ . Recall the definition of the Brieskorn module in (4.7) and Petrov module in (4.5). The  $\mathbb{C}[t]$ -module  $\mathcal{P}_{\delta(t)}$  is of rank  $\mu_1 = 2a_1$  and it is generated by the Abelian integrals

$$I_i(t) = \int_{\delta(t)} \omega_i, \quad i = 1, \dots, 2a_1,$$

where  $\omega_i$ 's together with  $\eta_i$ 's form a basis of the Brieskorn module.

**Definition 4.8.** Let  $\mathcal{H}$  be the ring defined by

$$\mathcal{H} = \mathbb{C}[x, y, H, 1/H, \varphi_1, \varphi_2, \dots, \varphi_d]. \quad (4.14)$$

We say that a polynomial 1-form  $\omega$  is *relatively exact in  $\mathcal{H}$* , if it can be written  $\omega = QdH + dR$ , with  $Q, R \in \mathcal{H}$ .

Let  $\tilde{\Omega}^1$  be the space of all 1-forms with coefficients in  $\mathcal{H}$  and  $\tilde{B}^1$  be the set of 1-forms of  $\tilde{\Omega}^1$  that are relatively. The first relative cohomology group with coefficients in  $\mathcal{H}$  is

$$\tilde{H}_{rel}^1(\tilde{\Omega}^1) = \frac{\tilde{\Omega}^1}{\tilde{B}^1}.$$

**Proposition 4.2.** *Let  $\omega$  be a polynomial 1-form in  $\mathbb{C}[x, y]$ . The form  $\omega$  verifies condition  $\int_{\delta(t)} \omega \equiv 0$ , if and only if  $\omega$  is relatively exact in  $\mathcal{H}$ .*

For a proof see [104, 80]. Now, if  $M_1(t) = 0$ , then we generalize Françoise's algorithm to the space  $\mathcal{H}$  in order to calculate the Poincaré-Pontryagin function of order two. Consider the functions  $\varphi_k$  as defined in (4.12) and denote by

$$\eta_{i,j}^* = \varphi_i d\varphi_j, \quad i, j = 1, \dots, d, \quad (4.15)$$

the 1-forms in the variables  $x, y$ . The integrals  $\int_{\delta(t)} \eta_{i,j}^*$ , with  $1 \leq i < j \leq d$ , appear naturally in our calculations. For the functions  $\varphi_k$ , given in (4.12) let

$$I_{i,j}^*(t) = \int_{\delta(t)} \varphi_i d\varphi_j, \quad 1 \leq i < j \leq d. \quad (4.16)$$

which are not Abelian integrals.

**Proposition 4.3. (Poincaré-Pontryagin function of order two)**

If  $M_1(t) \equiv 0$  then  $M_2(t)$  belongs to the  $\mathbb{C}[t, 1/t]$ -module generated by the Abelian integrals  $\{I_1(t), \dots, I_{2a_1}(t)\}$  and the integrals  $I_{i,j}^*(t)$ ,  $1 \leq i < j \leq d$ .

*Proof.* Let  $\omega$  be a polynomial 1-form with coefficients in  $\mathbb{C}[x, y]$ . Under the condition  $M_1(t) = 0$ ,  $\omega$  can be written as

$$\omega = \underbrace{\left( Q_1 - \sum_{k=1}^d \tilde{P}'_k(H) \varphi_k \right)}_Q dH + d \underbrace{\left( R_1 + \sum_{k=1}^d \tilde{P}_k(H) \varphi_k \right)}_R. \quad (4.17)$$

The Poincaré-Pontryagin function of order two is

$$M_2(t) = \int_{\delta(t)} Q dR. \quad (4.18)$$

From (4.17), we have that

$$Q dR = \left[ Q_1 - \sum_{k=1}^d \tilde{P}'_k(H) \varphi_k \right] d \left[ R_1 + \sum_{k=1}^d \tilde{P}_k(H) \varphi_k \right].$$

In order to simplify the notations, we write  $\alpha \sim \beta$ , for  $\int_{\delta(t)} \alpha \equiv \int_{\delta(t)} \beta$ .  
Now

$$\begin{aligned} QdR &\sim Q_1dR_1 + \sum_{k=1}^d (Q_1\tilde{P}_k(H) - R_1\tilde{P}'_k(H))d\varphi_k \\ &\quad + \sum_{1 \leq i < j \leq d} (\tilde{P}'_i(H)\tilde{P}_j(H) - \tilde{P}_i(H)\tilde{P}'_j(H))\varphi_id\varphi_j. \end{aligned}$$

It follows hence from (4.18) that

$$\begin{aligned} M_2(t) &= \int_{\delta(t)} Q_1dR_1 + \sum_{k=1}^d \int_{\delta(t)} (Q_1\tilde{P}_k(H) - R_1\tilde{P}'_k(H))d\varphi_k \\ &\quad + \sum_{1 \leq i < j \leq d} (\tilde{P}'_i(t)\tilde{P}_j(t) - \tilde{P}_i(t)\tilde{P}'_j(t)) \int_{\delta(t)} \varphi_id\varphi_j. \end{aligned} \tag{4.19}$$

As  $Q_1dR_1$  and  $Hd\varphi_k$  are polynomial 1-forms in  $\mathbb{C}[x, y]$ , we have that the integrals  $\int_{\delta(t)} Q_1dR_1$  and  $\int_{\delta(t)} (Q_1\tilde{P}_k(H) - R_1\tilde{P}'_k(H))d\varphi_k$  are Abelian. Finally, denoting by  $\tilde{P}_{i,j}^*(t)$  the polynomial

$$\tilde{P}_{i,j}^*(t) = \tilde{P}'_i(t)\tilde{P}_j(t) - \tilde{P}_i(t)\tilde{P}'_j(t), \tag{4.20}$$

it follows that the coefficient of each  $I_{i,j}^*(t)$  in (4.19) is  $\tilde{P}_{i,j}^*(t)$ . Hence, we see that generically, this coefficient is non-zero and hence in (4.19) the function  $M_2(t)$  can be written as

$$M_2(t) = \{\text{Abelian Integrals}\} + \sum_{1 \leq i < j \leq d} \tilde{P}_{i,j}^*(t)I_{i,j}^*(t). \tag{4.21}$$

□

The general situation is given in the following result

**Theorem 4.6. (Uribe, [104])** *The higher Poincaré-Pontryagin function  $M_k(t)$  associated to a family of ovals  $\delta(t)$  surrounding a critical point of center type and associated to a small polynomial perturbation of the Hamiltonian formed by a product of  $(d+1)$  lines in general position, belongs to the  $\mathbb{C}[t, 1/t]$ -module generated by Abelian integrals  $I_i(t)$ , with  $i = 1, \dots, 2a_1$  and special transcendental integrals of type  $I_{i,j}^*(t) = \int_{\delta(t)} \varphi_id\varphi_j$ , with  $1 \leq i < j \leq d$ .*

**Corollary 4.2.** *The principal Poincaré-Pontryagin function  $M_k(t)$  is an iterated integral of length two.*

For the definition of an iterated integral see Chapter 6.



# Chapter 5

## Center problem

In this chapter we consider the space of differential equations in  $\mathbb{C}^2$  which have at least one center singularity. It turns out that such a space is algebraic. The classification of all irreducible components of such an algebraic variety is known as the *center condition*. For further discussion of this topic in the context of holomorphic foliations see [65, 79].

### 5.1 Foliations with a center in $\mathbb{C}^2$

Let  $\mathbb{C}[x, y]_{\leq d}$  be the set of polynomials in two variables  $x, y$  with coefficients in  $\mathbb{C}$  and of degree at most  $d \in \mathbb{N}_0$ . The space of foliations

$$\mathcal{F} = \mathcal{F}(\omega), \omega \in \Omega_d^1 \tag{5.1}$$

where

$$\Omega_d^1 := \{P(x, y)dy - Q(x, y)dx : P, Q \in \mathbb{C}[x, y]_{\leq d}\}$$

is the projectivization of the vector space  $\Omega_d^1$  and it is denoted by  $\mathcal{F}(d)$ . The degree of  $\mathcal{F}$  is the maximum of  $\deg(P)$  and  $\deg(Q)$ . The points in the set  $\{(x, y) : P(x, y) = 0, Q(x, y) = 0\}$  are called the singularities of  $\mathcal{F}$ .

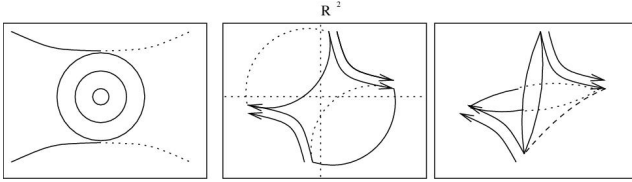


Figure 5.1: A cylinder in the complex plane

**Definition 5.1.** An isolated singularity  $p$  of  $\mathcal{F}$  is called reduced if  $(P_x Q_y - P_y Q_x)(p) \neq 0$ . A reduced singularity  $p$  is called a center singularity or simply a center, if there is a holomorphic coordinate system  $(\tilde{x}, \tilde{y})$  around  $p$  with  $\tilde{x}(p) = \tilde{y}(p) = 0$  and such that in this coordinate system, we have

$$\omega \wedge d(\tilde{x}^2 + \tilde{y}^2) = 0.$$

If  $0 \in \mathbb{C}^2$  is a center singularity of  $\mathcal{F}$ , then there exists a germ of holomorphic function  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  which has non-degenerate critical point at  $0 \in \mathbb{C}^2$ , and the leaves of  $\mathcal{F}$  near  $0$  are given by  $f = \text{const.}$ . The point  $0$  is also called a Morse singularity of  $f$ . Morse lemma in the complex case implies that there exists a local coordinate system  $(x, y)$  in  $(\mathbb{C}^2, 0)$  with  $x(0) = 0$ ,  $y(0) = 0$  and such that  $f(x, y) = \frac{1}{2}(x^2 + y^2)$ . Near the center the leaves of  $\mathcal{F}$  are homeomorphic to a cylinder, therefore each leaf has a nontrivial closed cycle. Note that the two curves  $x^2 + y^2 = 1$  and  $xy = 1$  are different in the real plane  $\mathbb{R}^2$  but isomorphic in the complex plane  $\mathbb{C}^2$ . Which pictures in Figure 5.1 is a correct intuition of these curves?

Let  $\mathcal{M}(d)$  be the closure of the subset of  $\mathcal{F}(d)$  containing  $\mathcal{F}(\omega)$ 's with at least one center. The problem of identifying irreducible components of  $\mathcal{M}(d)$  is known as *center condition*. The complete classification of the irreducible components of  $\mathcal{M}(2)$  is done by Dulac in [17], see also [12]. In this case any foliation in  $\mathcal{M}(2)$  has a Liouvillian first integral. Since this problem finds applications on the number of limit cycles in the context of real differential equations, this classification problem is very important. In what follows we give some examples of irreducible components of  $\mathcal{M}(d)$ .

We have learned the statement and proof of the following proposition from A. Lins Neto in [65]. This must go back to Poincaré and Dulac.

**Proposition 5.1.**  $\mathcal{M}(d)$  is an algebraic subset of  $\mathcal{F}(d)$ .

*Proof.* Let  $\mathcal{M}_0(d)$  be the set of all foliations in  $\mathcal{M}(d)$  with a center at the origin  $(0, 0) \in \mathbb{C}^2$  and with a local first integral of the type

$$f = xy + f_3 + f_4 + \cdots + f_n + h.o.t. \quad (5.2)$$

in a neighborhood of  $(0, 0)$ . Let us prove that  $\mathcal{M}_0(d)$  is an algebraic subset of  $\mathcal{F}(d)$ . Let  $\mathcal{F}(\omega) \in \mathcal{M}_0(d)$  and  $\omega = \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}$  be the homogeneous decomposition of  $\omega$ , then in a neighborhood around  $(0, 0)$  in  $\mathbb{C}^2$ , we have

$$\omega \wedge df = 0 \Rightarrow (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0.$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$\begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = k \cdot d(xy), \text{ } k \text{ is constant,} \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy), \\ \dots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy). \\ \dots \end{cases} \quad (5.3)$$

Dividing the 1-form  $\omega$  by  $k$ , we can assume that  $k = 1$ . Let  $\mathcal{P}_n$  denote the set of homogeneous polynomials of degree  $n$ . Define the operator

$$S_n : \mathcal{P}_n \rightarrow (\mathcal{P}_n dx \wedge dy),$$

$$S_n(g) = \omega_1 \wedge d(g).$$

We have

$$\begin{aligned} S_{i+j}(x^i y^j) &= d(xy) \wedge d(x^i y^j) = (xdy + ydx) \wedge (x^{i-1} y^j dx + j x^i y^{j-1} dy), \\ &= (j - i) x^i y^j dx \wedge dy. \end{aligned}$$

This implies that when  $n$  is odd  $S_n$  is bijective and so in (5.3),  $f_n$  is uniquely defined by the terms  $f_m, \omega_m$ 's  $m < n$ , and when  $n$  is even

$$Im(S_n) = A_n dx \wedge dy,$$

where  $A_n$  is the subspace generated by the monomials  $x^i y^j$ ,  $i \neq j$ . When  $n$  is even the existence of  $f_n$  implies that the coefficient of  $(xy)^{\frac{n}{2}}$  in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say  $P_n$ , with variables

$$\text{coefficients of } \omega_2 \dots \omega_{n-1}, f_2, \dots, f_{n-1}$$

is zero. The coefficients of  $f_i, i \leq n-1$  is recursively given as polynomials in coefficients of  $\omega_i, i \leq n-1$  and so the algebraic set

$$X : P_4 = 0 \ \& \ P_6 = 0 \ \& \ \dots \ \& \ P_n = 0 \ \dots$$

consists of all foliations  $\mathcal{F}$  in  $\mathcal{F}(d)$  which have a formal first integral of the type (5.2) at  $(0,0)$ . From a results of Mattei and Moussu in [73], it follows that  $\mathcal{F}$  has a holomorphic first integral of the type (5.2). This implies that  $\mathcal{M}_0(d)$  is algebraic. Note that by Hilbert nullstellensatz theorem, a finite number of  $P_i$ 's defines  $\mathcal{M}_0(d)$ . The set  $\mathcal{M}(d)$  is obtained by the action of the group of automorphisms of  $\mathbb{C}^2$  on  $\mathcal{M}_0(d)$ .  $\square$

## 5.2 Components of holomorphic foliations with a center

Let  $\mathcal{P}_{d+1}$  be the set of polynomials of maximum degree  $d+1$  in  $\mathbb{C}^2$  and  $f \in \mathcal{P}_{d+1}$ . The leaves of the foliation  $\mathcal{F}(df)$  are contained in the level surfaces of  $f$ . Let  $\mathcal{I}(d)$  be the set of  $\mathcal{F}(df)$  in  $\mathcal{F}(d)$ .

**Theorem 5.1.** (Ilyashenko [50])  $\mathcal{I}(d)$ ,  $d \geq 2$  is an irreducible component of  $\mathcal{M}(d)$ .

We can restate the above result as follows: Let  $\mathcal{F} \in \mathcal{I}(d)$ ,  $p$  one of the center singularities of  $\mathcal{F}$  and  $\mathcal{F}_t$  a holomorphic deformation of  $\mathcal{F}$  in  $\mathcal{F}(d)$  such that its unique singularity  $p_t$  near  $p$  is still a center.

**Theorem 5.2.** In the above situation, there exists an open dense subset  $U$  of  $\mathcal{I}(d)$ , such that for all  $\mathcal{F}(df) \in U$ , there exists polynomial  $f_t \in \mathcal{P}_{d+1}$  such that  $\mathcal{F}_t = \mathcal{F}(df_t)$ .

This theorem also says that the persistence of one center implies the persistence of all other centers. Let  $\mathcal{F}$  be a foliation in  $\mathbb{C}^2$  given by the polynomial 1-form

$$\omega(f, \lambda) = \omega(f_1, \dots, f_r, \lambda_1, \dots, \lambda_r) = f_1 \cdots f_r \sum_{i=1}^r \lambda_i \frac{df_i}{f_i} \quad (5.4)$$

where the  $f_i$ 's are irreducible polynomials in  $\mathbb{C}^2$  and  $d_i = \deg(f_i)$ .  $\mathcal{F}$  is called a logarithmic foliation and it has the multi-valued first integral  $f = f_1^{\lambda_1} \cdots f_r^{\lambda_r}$  in  $U = \mathbb{C}^2 \setminus (\cup_{i=1}^r \{f_i = 0\})$ . We can prove that generically, the degree of  $\mathcal{F}$  is  $d = \sum_{i=1}^r d_i - 1$ . Let  $\mathcal{L}(d_1, d_2, \dots, d_r)$  be the set of all logarithmic foliations of the above type.

**Theorem 5.3. (Movasati [78])** *The set  $\mathcal{L}(d_1, d_2, \dots, d_r)$  is an irreducible component of  $\mathcal{M}(d)$ , where  $d = \sum_{i=1}^r d_i - 1$ .*

The main tool in the proof of the above theorems is Picard-Lefschetz theory. For some other results for foliations in  $\mathbb{P}^2$  see [78]. The pull-back foliations have also center singularities and they form an irreducible component of  $\mathcal{M}(d)$ . This has been worked out by Y. Zare in his Ph.D. thesis, see [109].



# Chapter 6

## Iterated integrals

In this chapter we study iterated integrals and how they appear in differential equations. We define the corresponding Petrov/Brieskorn type modules, give a formula for the Gauss-Manin connection of iterated integrals and calculate the Melnikov functions for certain topological cycles in terms of iterated integrals. We collect the necessary machinery for dealing with iterated integrals. Our approach to iterated integrals is the homology type  $\mathbb{Z}$ -modules  $H_{1,r}(U, \mathbb{Z})$ ,  $r = 1, 2, \dots$  (see §6.2) and the construction of their duals in terms of differential forms (see §6.4). This approach, which is more convenient in holomorphic foliations, is not the classical approach in the literature and this is the main reason why we have reproduced some well-known materials in this section. This chapter was originally written as an appendix of the article [83]. However, due to its expository nature it was removed from there.

### 6.1 Definition

Let  $\bar{U}$  be compact Riemann surface,  $U$  be the complement of a finite non-empty set of points of  $\bar{U}$  and  $p_i \in U$ ,  $i = 0, 1$ . Let  $\Omega_U^\bullet$  be the set of meromorphic differential forms in  $\bar{U}$  with poles in  $\bar{U} \setminus U$  and

$$\Omega_U^{\bullet,r} := \mathbb{C} + \Omega_U^\bullet + \Omega_U^\bullet \Omega_U^\bullet + \cdots + \underbrace{\Omega_U^\bullet \Omega_U^\bullet \cdots \Omega_U^\bullet}_{r \text{ times}}.$$

For simplicity, in the above definition  $+$  denotes the direct sum and  $\Omega_U^\bullet, \Omega_U^\bullet$  denotes  $\Omega_U^\bullet \otimes_{\mathbb{C}} \Omega_U^\bullet$ . An element of  $\Omega_U^{\bullet, r}$  is called to be of length  $\leq r$ . By definition  $\Omega_U^{1, r} \subset \Omega_U^{\bullet, r}$  contains only differential 1-forms and in each homogeneous piece of an element of  $\Omega_U^{0, r} \subset \Omega_U^{\bullet, r}$  there exists exactly one differential 0-form. We have the differential map

$$d = d_U : \Omega_U^{0, \bullet} \rightarrow \Omega_U^{1, \bullet}$$

which is  $\mathbb{C}$ -linear and is given by the rules

$$d(g) = dg - g(p_1) + g(p_0) \quad (6.1)$$

$$d(g\omega_1\omega_2 \cdots \omega_r) = (dg)\omega_1\omega_2 \cdots \omega_r - (g\omega_1)\omega_2 \cdots \omega_r + g(p_0)\omega_1\omega_2 \cdots \omega_r$$

$$d(\omega_1 \cdots \omega_i g \omega_{i+1} \cdots \omega_r) =$$

$$\omega_1 \cdots \omega_i (dg)\omega_{i+1} \cdots \omega_r - \omega_1 \cdots \omega_i (g\omega_{i+1}) \cdots \omega_r + \omega_1 \cdots (\omega_i g)\omega_{i+1} \cdots \omega_r$$

$$d(\omega_1\omega_2 \cdots \omega_r g) = \omega_1\omega_2 \cdots \omega_r (dg) - g(p_1)\omega_1\omega_2 \cdots \omega_r + \omega_1\omega_2 \cdots (\omega_r g),$$

where  $1 \leq i \leq r-1$ . Let

$$B = \frac{\Omega_U^{1, \bullet}}{d\Omega_U^{0, \bullet}} \quad (6.2)$$

and

$$\mathbb{C} = B_0 \subset B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_r \subset \cdots \subset B$$

be the filtration given by the length:

$$B_r := \frac{\Omega_U^{1, \leq r}}{d\Omega_U^{0, \leq r}}.$$

The map  $\epsilon : B \rightarrow \mathbb{C}$  associates to each  $\omega$  its constant term in  $B_0 = \mathbb{C}$ . Take a basis  $\omega_1, \omega_2, \dots, \omega_m$  of the  $\mathbb{C}$ -vector space

$$H^1(U, \mathbb{C}) \cong H_{\text{dR}}^1(U) = \frac{\Omega_U^1}{d\Omega_U^0}.$$

Note that  $\bar{U} \setminus U$  is not empty. The  $\mathbb{C}$ -vector space  $B$  is freely generated by  $\omega_{i_1}\omega_{i_2} \cdots \omega_{i_k}$ ,  $1 \leq i_1, i_2, \dots, i_k \leq m$ ,  $k \in \mathbb{N}_0$ . The fact that these elements generate  $B$  follows from the definition of the differential  $d$



and various applications of the fact that every  $\omega \in \Omega_U^1$  can be written as a  $\mathbb{C}$ -linear combination of  $\omega_i$ 's plus some  $dg$ ,  $g \in \Omega_U^0$ . We obtain an isomorphism between  $B$  and the abstract associative ring generated by  $\omega_i$ 's. In this way  $B$  turns to be an associative, but noncommutative,  $\mathbb{C}$ -Algebra. Note that the  $\mathbb{C}$ -algebra structure of  $B$  does depend on the choice of the basis and  $p_0, p_1$ . However, the isomorphism of  $\mathbb{C}$ -vector spaces obtained in the quotient  $B_r/B_{r-1}$ ,  $r = 1, 2, \dots$  does not depend on the base  $p_0, p_1$ .

Let  $\delta : [0, 1] \rightarrow U$  be a path which connects  $p_0$  to  $p_1$  and  $\omega_i \in \Omega_U^1$ ,  $i = 1, 2, \dots, r$ . The *iterated integral* is defined by induction and according to the rule:

$$\int_{\delta} \omega_1 \omega_2 \cdots \omega_r = \int_{\delta} \omega_1 \left( \int_{\delta_x} \omega_2 \cdots \omega_r \right),$$

where for  $\delta(t_1) = x$  we have  $\delta_x := \delta|_{[0, t_1]}$ . By  $\mathbb{C}$ -linearity one extends the definition to  $\Omega_U^{1, \bullet}$  and it is easy to verify that an iterated integral of the elements in  $d\Omega_U^{0, \bullet}$  is zero ([42] Proposition 1.3) and hence  $\int_{\delta} \omega, \omega \in B$  is well-defined. It is homotopy functorial. This can be checked by induction on  $r$ . We will frequently use the equality

$$\int_{\delta} \omega_1 \omega_2 \cdots \omega_r = \int_{\delta} \omega_1 \cdots \omega_i \left( \int_{\delta_x} \omega_{i+1} \cdots \omega_r \right), \quad i = 1, 2, \dots, r-1.$$

## 6.2 Homotopy groups

From now on we take  $p := p_0 = p_1$  and let

$$G := \pi_1(U, p), \quad m = \text{number of generators of } G.$$

We denote by 1 the identity element of  $G$ . For  $\delta_1, \delta_2 \in G$  we denote by  $(\delta_1, \delta_2) = \delta_1 \delta_2 \delta_1^{-1} \delta_2^{-1}$  the commutator of  $\delta_1$  and  $\delta_2$  and for two sets  $A, B \subset G$  by  $(A, B)$  we mean the group generated by  $(a, b)$ ,  $a \in A, b \in B$ . Let

$$G_r := (G_{r-1}, G), \quad r = 1, 2, 3, \dots, \quad G_1 := G.$$

Each quotient

$$H_{1,r}(U, \mathbb{Z}) := G_r/G_{r+1}$$

is a free  $\mathbb{Z}$ -module of rank

$$M_m(r) := \frac{1}{r} \sum_{d|r} \mu(d) m^{\frac{r}{d}},$$

where  $\mu(d)$  is the möbius function:  $\mu(1) = 1$ ,  $\mu(p_1 p_2 \cdots p_s) = (-1)^s$  for distinct primes  $p_i$ 's, and  $\mu(n) = 0$  otherwise. Note that for  $r$  prime we have  $M_m(r) = \frac{m^r - m}{r}$ . A basis of  $H_{1,r}(U, \mathbb{Z})$  is given by basic commutators of weight  $r$  (see [41] Chapter 11).

There is another way to study  $G$  by finite rank  $\mathbb{Z}$ -modules mainly used in Hodge theory, see [42]. Let  $\mathbb{Z}[G]$  be the integral group ring of  $G$ ,  $J$  be the kernel of  $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ ,  $\sum_{i=1}^k a_i \alpha_i \mapsto \sum_{i=1}^k a_i$ ,  $a_i \in \mathbb{Z}, \alpha_i \in G$ . We have the canonical filtration of  $\mathbb{Z}[G]$  by subideals:

$$\cdots \subset J^3 \subset J^2 \subset J^1 = J \subset \mathbb{Z}[G].$$

Each quotient  $\mathbb{Z}[G]/J^r$  is a freely generate  $\mathbb{Z}$ -module of finite rank.

## 6.3 The properties of iterated integrals

In this section we list properties of iterated integrals in the context of the present text. The following four statements can be considered as the axioms of iterated integrals.

**I 1.** By definition the iterated integral is  $\mathbb{C}$ -linear (resp.  $\mathbb{Z}$ -linear) with respect to the elements of  $B$  (resp.  $\mathbb{Z}[G]$ ) and

$$\int_1 \omega := \epsilon(\omega), \omega \in B, \int_\alpha 1 = 1, \alpha \in G.$$

We use the convention  $\omega_1 \omega_2 \cdots \omega_r = 1$  for  $r = 0$ .

**I 2.** For  $\alpha, \beta \in G$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^r \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r$$

([42], Proposition 2.9).

**I 3.** For  $\alpha \in G$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$

$$\int_{\alpha^{-1}} \omega_1 \omega_2 \cdots \omega_r = (-1)^r \int_{\alpha} \omega_r \cdots \omega_1.$$

([42], Proposition 2.12).

**I 4.** For  $\alpha \in G$  and  $\omega_1, \omega_2, \dots, \omega_{r+s} \in \Omega_U^1$  we have the shuffle relations

$$\int_{\alpha} \omega_1 \cdots \omega_r \int_{\alpha} \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma} \int_{\alpha} \omega_{\sigma(1)} \omega_{\sigma(2)} \cdots \omega_{\sigma(r+s)}, \quad (6.3)$$

where  $\sigma$  runs through all shuffles of type  $(r, s)$  ([42], Lemma 2.11). Recall that a permutation  $\sigma$  of  $\{1, 2, 3, \dots, r+s\}$  is a shuffle of type  $(r, s)$  if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

Note that I1, I2 and I3 imply that every iterated integral can be written as a polynomial in  $\int_{\delta} \omega_1 \omega_2 \cdots \omega_r$ , where  $\delta$  runs through a set which generated  $G$  freely and  $\omega_i$  runs through a fixed basis of  $H_{\text{dR}}^1(U)$ . However by I4 this way of writing is not unique. By various applications of I4, we can get shuffle type formulas for the products of  $s \geq 2$  integrals. All the well-known properties of iterated integrals in the literature can be deduced from I1, I2, I3 and I4.

**I 5.** For  $\alpha, \beta \in J$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$ ,  $r \geq 1$

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=1}^{r-1} \int_{\alpha} \omega_1 \cdots \omega_i \int_{\beta} \omega_{i+1} \cdots \omega_r.$$

In particular,  $\int_{\alpha\beta} \omega_1 = 0$ . This statement follows from I1 and I2.

**I 6.** We have

$$\int_{J^s} B_r = 0, \text{ for } 0 \leq r < s.$$

This follows by induction on  $r$  from I5.

**I 7.** For  $\alpha_1, \alpha_2, \dots, \alpha_r \in G$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$

$$\int_{(\alpha_1-1)(\alpha_2-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_r = \prod_{i=1}^r \int_{\alpha_i} \omega_i.$$

This follows by induction on  $r$  from I5, I6 and I1.

We conclude that  $\int_{\alpha} \omega$ ,  $\omega \in B_r/B_{r-1}$ ,  $\alpha \in J^r/J^{r+1}$  is well-defined. Now we list some properties related to  $G_r$ 's.

**I 8.** For  $r < s$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$  we have

$$\int_{\beta_s} \omega_1 \omega_2 \cdots \omega_r = 0, \quad \beta_s = (\alpha_1, \alpha_2, \dots, \alpha_s) \text{ or its inverse,}$$

where  $(\alpha_1, \alpha_2, \dots, \alpha_r) = ((\cdots((\alpha_1, \alpha_2), \alpha_3) \cdots), \alpha_r)$ .

It is enough to prove the statement for  $\beta_s$ . For  $\beta_s^{-1}$  it follows from I2 applied on  $\beta_s \beta_s^{-1} = 1$ . The proof for  $\beta_s = (\beta_{s-1}, \alpha_s)$  is by induction on  $s$ . For  $s = 1$  it is trivially true. Suppose that the statement is true for  $s$  and let us prove it for  $s + 1$ . After various applications of I2 and the induction hypothesis we have

$$\int_{\beta_{s+1}} \omega_1 \omega_2 \cdots \omega_r = \int_{\beta_s} \omega_1 \omega_2 \cdots \omega_r + \int_{\beta_s^{-1}} \omega_1 \omega_2 \cdots \omega_r.$$

Now we apply I2 for  $\beta_s \beta_s^{-1} = 1$  and we conclude that the right hand side of the above equality is zero.

**I 9.** For  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_U^1$  we have

$$\begin{aligned} \int_{\alpha} \omega_1 \omega_2 \cdots \omega_r &= 0, \quad \alpha \in G_s, \quad r < s, \\ \int_{\alpha\beta} \omega_1 \cdots \omega_r &= \int_{\alpha} \omega_1 \cdots \omega_r + \int_{\beta} \omega_1 \cdots \omega_r, \quad \alpha, \beta \in G_r, \\ \int_{\alpha^{-1}} \omega_1 \cdots \omega_r &= - \int_{\alpha} \omega_1 \cdots \omega_r, \quad \alpha \in G_r, \\ \int_{\alpha} (\omega_1 \omega_2 \cdots \omega_r + (-1)^r \omega_r \cdots \omega_1) &= 0, \quad \alpha \in G_r. \end{aligned}$$

I9 implies that  $\int_{\alpha} \omega$ ,  $\alpha \in G_r/G_{r+1}$ ,  $\omega \in B_r/B_{r-1}$  is well-defined.

**I 10.** For  $\alpha \in G_r$  and  $\beta \in G_s$

$$\begin{aligned} \int_{(\alpha,\beta)} \omega_1 \omega_2 \cdots \omega_{r+s} &= \int_{\alpha} \omega_1 \cdots \omega_r \int_{\beta} \omega_{r+1} \cdots \omega_{r+s} \\ &\quad - \int_{\beta} \omega_1 \cdots \omega_s \int_{\alpha} \omega_{s+1} \cdots \omega_{r+s} \end{aligned}$$

In particular

$$\int_{(\alpha,\beta)} \omega_1 \omega_2 = \det \begin{pmatrix} \int_{\alpha} \omega_1 & \int_{\beta} \omega_1 \\ \int_{\alpha} \omega_2 & \int_{\beta} \omega_2 \end{pmatrix}, \alpha, \beta \in G, \omega_1, \omega_2 \in \Omega_U^1. \quad (6.4)$$

The above statement follows by several application of I2,I9, see also [33] Lemma 3.

**I 11.** For  $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_r, \beta_r \in G$  and  $\omega_1, \omega_2 \in \Omega_U^1$

$$\int_{\prod_{i=1}^s (\alpha_i, \beta_i)} \omega_1 \omega_2 = \sum_{i=1}^s \det \begin{pmatrix} \int_{\alpha_i} \omega_1 & \int_{\beta_i} \omega_1 \\ \int_{\alpha_i} \omega_2 & \int_{\beta_i} \omega_2 \end{pmatrix}.$$

The above statement follows by induction on  $s$ . The remarks after I7 and I9 suggest that there might be a relation between  $G_r/G_{r+1}$  and  $J^r/J^{r+1}$ . In fact the maps  $G_r/G_{r+1} \rightarrow J^r/J^{r+1}$  induced by  $x \mapsto x - 1$  are well-defined and gives us a morphism of Lie algebras over  $\mathbb{Z}$ :

$$\bigoplus_{r=1}^{\infty} G_r/G_{r+1} \rightarrow \bigoplus_{r=1}^{\infty} J^r/J^{r+1}.$$

For further information see [99].

**Proposition 6.1.** *Let  $\alpha$  and  $\beta$  be differential forms of length  $r$  and  $s$ , and  $a \in J^r$ ,  $b \in J^s$ . We have*

$$\int_{ab} \alpha \beta = \int_a \alpha \int_b \beta. \quad (6.5)$$

*In particular if  $r = s$  then*

$$\int_{[a,b]} [\alpha, \beta] = \det \begin{pmatrix} \int_a \alpha & \int_b \beta \\ \int_b \alpha & \int_a \beta \end{pmatrix}.$$

*Proof.* Since the equality (6.5) is  $\mathbb{Z}$ -linear in  $\alpha, \beta, a$  and  $b$ , we can assume that  $\alpha = \omega_1 \cdots \omega_r$  and  $\beta = \omega_{r+1} \cdots \omega_{r+s}$  and  $a = (a_1 - 1) \cdots (a_r - 1)$ ,  $b = (b_{r+1} - 1)(b_{r+s} - 1)$ ,  $b_i \in F$ . Now the first part of the proposition follows from the fact that: For  $\alpha_1, \alpha_2, \dots, \alpha_r \in G$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega$

$$\int_{(\alpha_1-1)(\alpha_2-1)\cdots(\alpha_r-1)} \omega_1 \cdots \omega_r = \prod_{i=1}^r \int_{\alpha_i} \omega_i.$$

This follows by induction on  $r$  from I5, I6 and I1. The second part is an immediate consequence of the first part.  $\square$

## 6.4 The dual of $H_{1,r}(U, \mathbb{C})$

Using the properties of iterated integrals it is easy to see that

$$\int_{\delta} \omega, \delta \in H_{1,r}(U, \mathbb{Z}), \omega \in B_r/B_{r-1}$$

is well-defined (see Section 6.3). Knowing the fact that

$$\dim_{\mathbb{C}}(B_r/B_{r-1}) = m^r \geq \text{rank}_{\mathbb{Z}}(G_r/G_{r-1}) = M_m(r)$$

we expect that

$$V_r := \{\omega \in B_r/B_{r-1} \mid \int_{H_{1,r}(U, \mathbb{Z})} \omega = 0\}$$

has non-zero dimension. In fact by shuffle formula (see Section 6.3), we know that in general  $V_r \neq 0$ . It has been recently proved in [35] that  $V_r$  is generated by the shuffle relations. By the extension of Atiyah-Hodge-Grothendieck theorem to iterated integrals, see [43] commentary after Theorem 13.5 and Corollary 7.3, we know that for every  $\delta \in H_{1,r}(U, \mathbb{Z})$  there is a  $\omega \in B_r/B_{r-1}$  such that  $\int_{\delta} \omega \neq 0$ . Therefore,

$$H_{\text{dR}}^{1,r}(U) := B_r/(B_{r-1} + V_r) \cong \check{H}_{1,r}(U, \mathbb{C}),$$

where  $\check{\phantom{x}}$  means dual and  $H_{1,r}(U, \mathbb{C}) = H_{1,r}(U, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ . One may be interested to find a basis of  $H_{\text{dR}}^{1,r}(U)$  similar to basic commutators,

see [41]. For instance one can construct  $M_m(r)$  elements of  $H_{\text{dR}}^{1,r}(U)$  in the following way: In the construction of basic commutators we replace the set which generates  $G$  freely with a basis of  $H_{\text{dR}}^1(U)$  and  $(\cdot, \cdot)$  with  $[\cdot, \cdot]$ . By definition  $[u, v] = uv - vu$  for  $u, v \in B$ . The basic commutators of weight  $r$  obtained in this way form a basis of  $H_{\text{dR}}^{1,r}(U)$ . This is easy to see for  $r = 1, 2$  and the complete proof is given in [35].

**Remark 6.1.** The authors of [87] have used another construction of  $H^{1,r}(U, \mathbb{Z})$  by means of iterated integrals over Feynman diagrams. Such a construction is associated to a basis of the freely generated group  $G$  and has the advantage that it does not require to perform a tensor product of the  $\mathbb{Z}$ -module  $H^{1,r}(U, \mathbb{Z})$  with  $\mathbb{C}$ .

## 6.5 Gauss-Manin connection of iterated integrals

In this section we consider a two dimensional complex manifold  $M$ , a one dimensional submanifold  $D$  of  $M$  (possibly not connected) and a regular proper holomorphic map  $f : M \rightarrow V$  such that  $f|_D$  is also regular, where  $V$  is some small open disk in  $\mathbb{C}$ . By Ehresmann's theorem  $f : (M, D) \rightarrow V$  is topologically trivial over  $V$ . We are going to work with iterated integrals in  $U_t := f^{-1}(t) \setminus D$ ,  $t \in V$ . In other words, the Riemann surface of the previous section depends on the parameter  $t$ . Instead of two points  $p_0, p_1$  we use two transversal section  $\Sigma_0, \Sigma_1$  to the fibers of  $f$  at points  $p_0, p_1 \in U_{t_0}$  for some  $t_0 \in V$ . We assume that  $\Sigma_i, i = 0, 1$  are parameterized by the the image  $t \in V$  of  $f : \Sigma_i \rightarrow V$ .

Let  $U := M \setminus D$ ,  $\Omega_U^1$  be the set of meromorphic differential 1-forms in  $M$  with poles along  $D$ ,  $\Omega_V^1$  be the set of holomorphic differential 1-forms in  $V$  and  $\Omega_{U/V}^1 = \frac{\Omega_U^1}{f^* \Omega_V^1}$  be the set of relative differentials. The set  $\Omega_{U/V}^1$  is a  $\mathcal{O}(V)$ -module in a canonical way, where  $\mathcal{O}(V)$  is the  $\mathbb{C}$ -algebra of holomorphic functions in  $V$ . We redefine the set  $B$  in (6.2) using

$$\Omega_{U/V}^{\bullet,r} = \mathcal{O}(V) + \Omega_{U/V}^{\bullet,1} + \Omega_{U/V}^{\bullet,2} \Omega_{U/V}^{\bullet,1} + \cdots + \underbrace{\Omega_{U/V}^{\bullet,1} \Omega_{U/V}^{\bullet,1} \cdots \Omega_{U/V}^{\bullet,1}}_{r \text{ times}}.$$

The differential  $d = d_{U/V}$  is  $\mathcal{O}(V)$ -linear and is defined by the equalities in (6.1). Here by  $f(p_i)$ ,  $i = 0, 1$  we mean  $f|_{\Sigma_i}$  as a function in  $t$  (one has to verify that  $d$  is well-defined).

Let  $\delta$  be a path in  $U_{t_0}$  which connects  $p_0$  to  $p_1$ . We denote by  $(M, \delta)$  a small neighborhood of  $\delta$  in  $M$  which can be homotopically contracted to  $\delta$ . By a holomorphic object (function, differential form etc.) along  $\delta$  we mean a holomorphic object defined in a universal covering of  $(M, \delta)$ . Therefore it can be viewed as a holomorphic object in a neighborhood of  $\delta$  in  $M$  which may be multi-valued in the self intersection points of  $\delta$ .

Let  $\omega$  be a holomorphic 1-form defined along the path  $\delta$ . Let  $x_0 \in \Sigma_0$  and  $\delta_{x, x_0}$  be a path which connects  $x_0$  to  $x \in M$  in  $f^{-1}(f(x_0))$  along the path  $\delta$ . For simplicity, we use  $\int_{x_0}^x \omega = \int_{\delta_{x, x_0}} \omega$  and consider it as a holomorphic function along  $\delta$ . The *Gelfand-Leray form*  $\frac{d\omega}{df}$  restricted to  $U_t$ 's is well-defined. For  $\omega \in \Omega_{U/V}^1$ , the map  $\omega \mapsto \frac{d\omega}{df}$  is also called the Gauss-Manin connection with respect to the parameter  $t$ . The reader is referred to [5, 79] for more details.

We denote by  $\tilde{\omega}$  (resp.  $\bar{\omega}$ ) the pullback of  $\omega|_{\Sigma_i}$  by the the holonomy map  $(M, \delta) \rightarrow \Sigma_i$  with  $i = 0$  (resp.  $i = 1$ ). The form  $\tilde{\omega}$  is of the form  $a(f)df$ ,  $a_2 \in \mathcal{O}(V)$  and so we define

$$\frac{\tilde{\omega}_1}{df} := a \in \mathcal{O}(V).$$

If there is no confusion we will also use  $\frac{\tilde{\omega}_1}{df}$  to denote  $a(f)$ . In a similar way we define  $\frac{\bar{\omega}}{df}$ .

**Proposition 6.2.** We have

$$d\left(\int_{x_0}^x \omega\right) = \left(\int_{x_0}^x \frac{d\omega}{df}\right)df + \omega - \tilde{\omega}. \quad (6.6)$$

This is [33] Lemma 1. For the convenience of the reader we prove it here.

*Proof.* First, we remark that if the equality (6.6) is true for  $\omega$  then it is also true for  $\omega + gdf$ , where  $g$  is a holomorphic function along  $\delta$ . By analytic continuation argument, it is enough to prove the proposition



in a small neighborhood of  $p_0$ . We take coordinates  $(z_1, z_2) : V_{p_0} \rightarrow (\mathbb{C}^2, 0)$  around  $p_0$  such that

$$p_0 = (0, 0), x = (z_1, z_2), x_0 = (0, z_2), f = z_2, \Sigma_0 = \{0\} \times (\mathbb{C}, 0).$$

Based on the first remark we can assume that  $\omega = a(z_1, z_2)dz_1$ ,  $a \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ . We have

$$\begin{aligned} d\left(\int_{x_0}^x \omega\right) &= d\left(\int_0^{z_1} a(\tilde{z}_1, z_2)d\tilde{z}_1\right) \\ &= a(z_1, z_2)dz_1 + \left(\int_0^{z_1} \frac{\partial a(\tilde{z}_1, z_2)}{\partial z_2}d\tilde{z}_1\right)dz_2 \\ &= \omega - \tilde{\omega} + \left(\int_{x_0}^x \frac{d\omega}{df}\right)df. \end{aligned}$$

Here  $\int_0^{z_1}$  is the integration on the straight line which connects 0 to  $z_1$ .  $\square$

Let  $\omega_i$ ,  $i = 1, 2, \dots, t$  be holomorphic differential 1-forms along  $\delta$ ,  $p_1 = \int_{x_0}^x \omega_1$  and  $\omega = \omega_2 \cdots \omega_r$ . We have

$$\int_{x_0}^{x_1} \omega \frac{d(\omega_2 p_1)}{df} = \int_{x_0}^{x_1} \omega \left( -\frac{\omega_2 \wedge \omega_1}{df} - \frac{\tilde{\omega}_1}{df} \omega_2 + \omega_2 \frac{d\omega_1}{df} + \frac{d\omega_2}{df} \omega_1 \right) \quad (6.7)$$

because

$$\begin{aligned} \int_{x_0}^{x_1} \omega \frac{d(\omega_2 p_1)}{df} &= \int_{x_0}^{x_1} \omega \left( \frac{dp_1 \wedge \omega_2 + p_1 d\omega_2}{df} \right) \\ &= \int_{x_0}^{x_1} \omega \left( -\frac{\omega_2 \wedge \omega_1}{df} - \frac{\tilde{\omega}_1}{df} \omega_2 \right. \\ &\quad \left. + \omega_2 \left( \int_{x_0}^x \frac{d\omega_1}{df} \right) + \frac{d\omega_2}{df} \left( \int_{x_0}^x \omega_1 \right) \right) \end{aligned}$$

which is the left hand side of (6.7). Now we define the Gauss-Manin connection:

$$\Omega_{U/V}^{1, \bullet} \rightarrow \Omega_{U/V}^{1, \bullet}, \omega \mapsto \omega'.$$

By definition it is a  $\mathbb{C}$ -linear map, it is zero on  $\Omega_U^{1,0}$  and for  $\omega \in \Omega_{U/V}^1$  we have:

$$\omega' = \frac{d\omega}{df} + \frac{\bar{\omega}}{df} - \frac{\tilde{\omega}}{df} \quad (6.8)$$

For  $r \geq 2$  and  $\omega_1, \omega_2, \dots, \omega_r \in \Omega_{U/V}^1$

$$(\omega_1 \omega_2 \cdots \omega_r)' := \quad (6.9)$$

$$\sum_{i=1}^r \omega_1 \omega_2 \cdots \omega_{i-1} \frac{d\omega_i}{df} \omega_{i+1} \cdots \omega_r - \sum_{i=1}^{r-1} \omega_1 \cdots \omega_{i-1} \frac{\omega_i \wedge \omega_{i+1}}{df} \omega_{i+2} \cdots \omega_r +$$

$$\frac{\bar{\omega}_1}{df} \omega_2 \cdots \omega_r - \omega_1 \cdots \omega_{r-1} \frac{\tilde{\omega}_r}{df}.$$

(For  $r = 1$  this is (6.8)). We have to show that this definition is well-defined and does not depend on the choice of  $\omega_i$  in its class in  $\Omega_{U/V}^1$ . Since (6.9) is linear in  $\omega_i$ , it is enough to prove that  $(\omega_1 \cdots \omega_r)' = 0$  if for some  $i$  we have  $\omega_i \in f^* \Omega_V^1$ . This can be easily checked using the facts  $\frac{d\omega_i}{df} = 0$ ,  $\frac{\omega_i \wedge \omega_j}{df} = \frac{\omega_i}{df} \omega_j$ .

Note that the definition (6.9) does depend on the choice of the transversal sections. The idea behind the definition (6.9) lies in the proof of the following proposition:

**Proposition 6.3.** *For continuous family of paths  $\delta_t$  connecting  $x_0 \in \Sigma_0$  to  $x_1 \in \Sigma_1$  in  $U_t$ ,  $t \in V$ , we have*

$$\frac{\partial}{\partial t} \int_{\delta_t} \omega = \int_{\delta_t} \omega', \quad \omega \in \Omega_{U'}^{1, \bullet} \quad (6.10)$$

*Proof.* Let  $\omega = \omega_1 \omega_2 \cdots \omega_r$ . For  $r = 0$  the equality (6.10) is true by definition. For  $r = 1$  it follows from Proposition 6.2. Let us assume that  $r \geq 2$ . Define

$$p_i(x) := \int_{x_0}^x \omega_i \cdots \omega_r = \int_{x_0}^x \omega_i p_{i+1}, \quad i = 1, 2, \dots, r, \quad p_{r+1} := 1.$$

Let  $P_i$ ,  $i = 1, 2, \dots, r$  be the restriction of  $p_i$  to  $\Sigma_1$ . We consider  $P_i$

as a function in  $t$ . We have:

$$\begin{aligned}
 \frac{\partial P_1}{\partial t} &\stackrel{(6.6)}{=} \left( \int_{x_0}^{x_1} \frac{d(\omega_1 p_2)}{df} \right) + \frac{\bar{\omega}_1 p_2}{df} - \frac{\widetilde{\omega_1 p_2}}{df} \\
 &= \left( \int_{x_0}^{x_1} \frac{d(\omega_1 p_2)}{df} \right) + \frac{\bar{\omega}_1}{df} P_2 \\
 &\stackrel{(6.7)}{=} \int_{x_0}^{x_1} \left( -\frac{\omega_1 \wedge \omega_2}{df} \omega_3 \cdots \omega_r - \frac{\widetilde{\omega_2 p_3}}{df} \omega_1 + \right. \\
 &\quad \left. \omega_1 \frac{d(\omega_2 p_3)}{df} + \frac{d\omega_1}{df} \omega_2 \omega_3 \cdots \omega_r + \frac{\bar{\omega}_1}{df} \omega_2 \cdots \omega_r \right) \\
 &\quad \vdots \\
 &= \int_{x_0}^{x_1} \omega'.
 \end{aligned}$$

In the  $(i-1)$ -th line,  $2 \leq i \leq r$ , we have used the fact that  $p_i|_{\Sigma_0} = 0$  and so  $\widetilde{\omega_{i-1} p_i} = 0$ . □

Similar to the previous section we define

$$V_r = \{ \omega \in B_r / B_{r-1} \mid \int_{H_{1,r}(U_t, \mathbb{Z})} \omega = 0, \forall t \in V \}$$

$$H_{\text{dR}}^{1,r}(U/V) = B_r / (B_{r-1} + V_r)$$

for the case  $\Sigma_0 = \Sigma_1$ . The Gauss-Manin connection does not necessarily maps  $d\Omega_{U/V}^{0,r}$ ,  $r \geq 2$  to itself (for instance check it for  $r = 2$ ) and so it may not induce a well-defined operator from  $B$  to itself. However, we have:

**Proposition 6.4.** *If  $\Sigma_0 = \Sigma_1$  then the Gauss-Manin connection induces a well-defined map*

$$H_{\text{dR}}^{1,r}(U/V) \rightarrow H_{\text{dR}}^{1,r}(U/V), \omega_1 \omega_2 \cdots \omega_r \rightarrow \sum_{i=1}^r \omega_1 \omega_2 \cdots \omega_{i-1} \frac{d\omega_i}{df} \omega_{i+1} \cdots \omega_r \tag{6.11}$$

which is independent of the choice of the transversal section  $\Sigma_0$ .

*Proof.* First note that the Gauss-Manin connection induces a well-defined map in  $B_r/B_{r-1} = (H_{\text{dR}}^1(U/V))^r$  even if it is not well-defined in  $B_r$ . By Proposition 6.3 it maps  $V_r$  to itself and so it induces a well-defined map in  $H_{\text{dR}}^{1,r}(U/V)$ . In the formula (6.9) the terms after the first sum have length less than  $r$  and so they are zero in  $H_{\text{dR}}^{1,r}(U/V)$ .  $\square$

## 6.6 Melnikov functions as iterated integrals

Recall the notations of the previous section. Let

$$\mathcal{F}_\epsilon : \omega_\epsilon = df + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots = 0, \quad \omega_i \in \Omega_U^1 \quad \epsilon \in (\mathbb{C}, 0) \quad (6.12)$$

be a holomorphic deformation of  $\mathcal{F} = \mathcal{F}_0$ . Let  $h_\epsilon(t) : \Sigma_0 \rightarrow \Sigma_1$  be the holonomy of  $\mathcal{F}_\epsilon$  along the path  $\delta$ . We write

$$h_\epsilon(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + \cdots, \quad i!.M_i(t) = \frac{\partial^i h_\epsilon}{\partial \epsilon^i} \Big|_{\epsilon=0}.$$

$M_i$  is called the  $i$ -th Melnikov function of the deformation along the path  $\delta$ . Let  $M_1 \equiv M_2 \equiv \cdots \equiv M_{k-1} \equiv 0$  and  $M_k \not\equiv 0$ . It is a well known fact that the multiplicity of  $M_k$  at  $t = 0$  is the number of fixed points of the holonomy  $h_\epsilon$  (as a function in  $t$ ).

**Proposition 6.5.** *If  $M_1 \equiv M_2 \equiv \cdots \equiv M_k \equiv 0$  then*

$$M_{k+1}(t) = - \int_{\delta_t} \left( \sum_{i=0}^k \omega_{k+1-i} p_i \right),$$

where  $p_i$  and  $g_i$  are holomorphic functions along  $\delta$  defined recursively by

$$p_i df + dg_i = - \sum_{j=0}^{i-1} \omega_{i-j} p_j, \quad i = 1, 2, \dots, k, \quad p_0 = 1. \quad (6.13)$$

Moreover, the restriction of  $p_i$  (resp.  $g_i$ ) to  $\Sigma_0$  and  $\Sigma_1$  coincide (as functions in  $t$ ).

This is just the reformulation of Theorem 3.3 in the context of holomorphic foliation in complex surfaces, see [101] Proposition 6 p. 73 or [79] Theorem 7.1. Now we want to write  $M_{k+1}$  as an iterated integral. This has been done in [33] Theorem 2, for the linear deformation of  $df$ . The idea of the proof is based on various usages of

$$p_i = \int_{x_0}^x \frac{d}{df} \left( \sum_{j=0}^{i-1} \omega_{i-j} p_j \right) \quad (6.14)$$

and the equality (6.7). Note that by Proposition 6.6

$$dp_i = \left( \int_{x_0}^x \frac{d^2}{df^2} \left( \sum_{j=0}^{i-1} \omega_{i-j} p_j \right) \right) df + \frac{d}{df} \left( \sum_{j=0}^{i-1} \omega_{i-j} p_j \right) - \frac{d}{df} \left( \widetilde{\sum_{j=0}^{i-1} \omega_{i-j} p_j} \right).$$

Since  $\frac{d}{df} \left( \sum_{j=0}^{i-1} \omega_{i-j} p_j \right)$  is defined modulo relatively zero differential 1-forms, the term  $\frac{d^2}{df^2} \left( \sum_{j=0}^{i-1} \omega_{i-j} p_j \right)$  is not uniquely defined even modulo relatively zero 1-forms. Note that we can add any holomorphic differential form  $\eta$  with  $\int_{\delta_t} \eta \equiv 0$  to  $\sum_{j=0}^{i-1} \omega_{i-j} p_j$  in the definition of  $p_i$  and so  $p_i$  and  $g_i$ 's are not uniquely defined.

For simplicity we define  $\omega^* = \frac{d\omega}{df}$  and define  $(\omega_1 \omega_2 \cdots \omega_r)^* = \sum_{i=1}^r \omega_1 \cdots \omega_{i-1} \omega_i^* \omega_{i+1} \cdots \omega_r$ . The first Melnikov functions are given by:

$$M_1(t) = - \int_{\delta_t} \omega_1.$$

$$M_2(t) = - \int_{\delta_t} \omega_2 + \omega_1 p_1 = - \int_{\delta_t} \omega_2 + \omega_1 \omega_1^*.$$

We have

$$p_2 = \int_{x_0}^x (\omega_2 + \omega_1 p_1)^* = \int_{x_0}^x \omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} - \omega_1 \frac{\tilde{\omega}_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**}$$

and so

$$\begin{aligned} M_3(t) &= - \int_{\delta_t} \omega_3 + \omega_2 p_1 + \omega_1 p_2 \\ &= - \int_{\delta_t} \omega_3 + \omega_2 \omega_1^* + \omega_1 \left( \omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} - \omega_1 \frac{\tilde{\omega}_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**} \right) \\ &= - \int_{\delta_t} \omega_3 + \omega_2 \omega_1^* + \omega_1 \left( \omega_2^* - \frac{\omega_1 \wedge \omega_1^*}{df} + \omega_1^* \omega_1^* + \omega_1 \omega_1^{**} \right). \end{aligned}$$

In the last equality we have used  $\int_{\delta_t} \omega_1 \omega_1 \equiv 0$ . In a similar way one calculates  $M_i$ 's as iterated integrals.

**Remark 6.2.** In the process of writing  $M_k(t)$  as an iterated integral, we do not use the fact that  $M_i(t) = 0, i < k$ . However, we have used them in the proof of Proposition 6.5. They may simplify the formula for  $M_k(t)$  as we have seen in  $M_3(t)$ .

**Proposition 6.6.** *If  $\Sigma_0 = \Sigma_1$  and  $\delta \in G_k$  then  $M_1(t) = M_2(t) = \dots = M_{k-1}(t) = 0$  and*

$$M_k(t) = \int_{\delta_t} \underbrace{\omega_1(\omega_1(\dots(\omega_1(\omega_1)'\dots))')}_{k-1 \text{ times}}. \tag{6.15}$$

*Proof.* For  $k = 1, 2$  we have already checked the equalities. In general the proof is as follows: For an arbitrary path  $\delta$  connecting  $p_0$  to  $p_1$  we claim that  $M_k(t)$  can be written as the iterated integral in (6.15) plus integrals, call it  $I_{k-1}$ , of differential forms of length strictly less than  $k$ . It is enough to prove that  $p_i, i \leq k - 1$  is given by

$$p_i = \int_{x_0}^x \underbrace{(\omega_1(\dots(\omega_1(\omega_1)^* \dots)^*)^*)^*}_{i \text{ times}} + I_{i-1}$$

because if this claim is true then

$$M_k(t) = - \int_{\delta_t} \omega_1 p_{k-1} + I_{k-1} = \int_{\delta_t} \underbrace{\omega_1(\omega_1(\dots(\omega_1(\omega_1)'\dots))')}_{k-1 \text{ times}} + I_{k-1}.$$

Our claim on  $p_i$ 's follows by various applications of (6.7) in the formula of (6.14). Note that if in (6.7)  $\omega_1$  is an arbitrary homogeneous element in  $\Omega_{ij}^{1,\bullet}$  of length  $k$  then we have

$$\int_{x_0}^{x_1} \omega(\omega_2 p_1)^* = \int_{x_0}^{x_1} \omega(\omega_2 \omega_1)^* + I_{k+r-2}.$$

□

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