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Intersection Theory on Moduli Spaces of Curves

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A Nadia

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Dois anos após a primeira versão deste manuscrito, ele vai ser publicado na série de Monografias do IMPA. Eu quero agradecer ao meu colega e amigo Eduardo Esteves *e ao* Prof. Paulo Sad *por me encorajar na publicação destas notas na coleção do IMPA.*

(Recife, 20 Julho 2000)

A Foreword (and more)

The Foreword...

The pages which follow were originally intended to provide a minimum of references for the series of introductory talks I delivered at the *Departamento de Matemática* of the *Universidade Federal de Pernambuco* at Recife, about *Intersection Theory over Moduli Spaces of Curves*, during the "Eschola de Verão", Janeiro 1998.

The purpose of these notes is not to substitute the several excellent books (such as, e.g. the very recent [45]) or research papers on the subject, available in the current literature, listed more or less in the references that do not pretend to be exhaustive. Rather, the aim is to focus on the most basic techniques one needs for computing classes in the Chow ring of M_g, the coarse moduli space of smooth projective curves of genus g, and $\overline{M}_g$, its Deligne-Mumford compactification.

Here is a list of problems (or, rather, exercises) that shall be studied in the notes.

1. How many elements, running in a pencil of lines of $\mathbb{P}^2$, are tangent to a plane conic?
2. How many reducible fibers may one find in a pencil of plane conics?
3. How many flexes does an irreducible plane curve of degree d have?
4. How many bitangents does an irreducible smooth plane quartic curve has?
5. How many Weierstrass points does a smooth curve of genus g have?
6. How many hyperelliptic fibers may one find in a general proper flat family of smooth curves?
7. How many fibers with a special Weierstrass point may be found on a flat proper family $\pi : \mathfrak{X} \longrightarrow S$ of smooth curves of genus g ?

8. How to compute divisors classes in $\overline{M}_g$ expressing loci of curves having special Weierstrass points?

9. How to find relations between the *tautological classes* of the moduli space M_g? [1]

The most important goal of these notes is to convince the reader that all the above exercises may be solved essentially by using the same embarassingly simple method, which always amounts, more or less, to be able to compute determinants.

Although there is no claim of originality about the contents of these notes (with the possible exception of Chapter 7), the presentation of the subject may offer some different points of view here and there. The core of the notes is certainly formed by the chapters where is developed the formalism of the *jets extension of relative bundles.* The extra material beside that (like the quick review of intersection theory or of the (sketch of) construction of the Deligne Mumford compactification of M_g via Kuranishi families) has been included in the attempt of trying to keep the lectures as self contained as possible.

For the detailed description of the topics here treated, we refer to the table of contents below as well as to the very first chapter, where one tries (not necessarily in a successful way) to get started in the smoothest way possible, by searching for the classical roots of current problems.

...and more.

What is going to follow are acknowledgments for all people and institutions that helped me to write this yet informal version of the present notes.

My first debt of gratitude is with Elizabeth Gasparim: by making possible my visit to Brasil, she forced me to give a deeper look at the foundations of the subject I am currently working on. These notes are the natural output of her kind confidence in me and her fine e-mail organization job of my trip. Moreover I got a great benefit from her detailed reading of the notes, which let me to correct several mistakes and to improve the shape of some english sentences.

For many enlighting discussions I have to express my gratitude to Prof. Israel Vainsencher. My gratitude is especially for warmly encouraging me to keep working on the subject of these notes and to communicate me his insight on many problems which I am still interested in.

[1] This is an exercise only for small values of g, such as $g = 3, 4, 5$ (Cf. Chap. 8).

Many thanks also to Joaquim Kock (for LaTeX advices and a careful reading of the notes, suggesting me several improvements) and to Prof. Anders Kock for explaining me some category theory and a couple of nice examples of fine moduli spaces.

Back to Italy. I have to express my gratitude to my colleague and office-mate Caterina Cumino. I shared a graduate course with her, at the University of Torino, on the same subject of these notes. She helped me quite a lot to improve the manuscript with many remarks here and there. Thanks are due to my collaborator F. Ponza for discussing the matter treated in the notes and for suggesting me several improvements, and to Tommaso de Fernex who immediately picked up, at a very first reading, two quite bad mistakes in the first pages of the notes. For sharing with me many mathematical discussions, I want also to thank my good friend Jorge Cordovez.

Two very bad mistakes in the substance of the notes were detected by Prof. Edoardo Ballico, who read them with special care. This is a wonderful opportunity for me to express to him my thanks for his mathematical support and, above all, for his precious friendship which I enjoy for eight years at least. Thank you very much for your support, Edoardo.

Very special thanks are due to the unknown Referee: his careful reading and his sharp criticisms helped me to substantially improve the shape of the notes.

I am sincerely grateful to the institution where I work, the "Dipartimento di Matematica del Politecnico di Torino" and to the people working there for allowing my stay abroad during the Recife Summer School. I thank my department for providing me the use of computers and printers, Mr. Dino Ricchiuti and Mrs. Rosa Rogano for helping me in doing some xeros copies of the manuscript.

Thanks a lot to my beloved mentor, Prof. S. Greco, for having accepted, on the behalf of the Algebraic Geometry group of my Department, to partially support my trip in Brasil. Without such a support it would not have been possible.

I want also to acknowledge many people who encouraged me in keep trying to do mathematical research or gave me moral support in several occasions. Among them I want to list Prof. G. Monegato, the Chairman of the *Dipartimento di Matematica del Politecnico di Torino*, my dear friends Prof. Aristide Sanini, Prof. Paolo Valabrega and Prof. Marco Codegone. For the same reason, many thanks are also due to Prof. Edoardo Vesentini.

Thanks also to my colleague Letizia Scuderi for her precious friendship and for the joy she communicated me during many chats.

Last, but absolutely not the least, I wish to express my special gratitude to Renza Cortini. Renza has certainly done a wonderful job in reading the first part of the preliminary form of this manuscript and in suggesting many improvements. However, the most important help I got from her was her constant and enthusiastic encouragement to pursue this project, especially in the bad moments when I felt tired and/or in lack of selfconfidence. Her friendly support has been the key that helped me to keep working on these *Recife Notes*. Grazie Renza.

For the several misprints, mistakes, omissions, misunderstanding and more, still left in this final form of the manuscript, I am, of course, the only responsible.

Questo libro è dedicato a Nadia, perché se non l'avessi trovata e poi smarrita, non sarebbe mai stato scritto.

Torino, 1 Marzo 1999

Contents

Chapter 1

Getting Started

1.1 Generalities and Notation

1.1.1 The projective space $\mathbb{P}^n_{\mathbb{C}} =: \mathbb{P}^n$

In these lectures we shall only deal with projective spaces defined over an algebraically closed field of characteristic 0 which, by the *Lefschetz principle* ([74], p. 164-165) shall be identified throughout with the field of the complex numbers $\mathbb{C}$.

For our purposes, it is often useful to think of $\mathbb{P}^n$ as a compact complex variety which admits the following *cellular decomposition*:

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \ldots \cup \mathbb{A}^1 \cup \mathbb{A}^0, \tag{1.1}$$

$\mathbb{A}^0$ being a point. Of course, $\mathbb{A}^n$ will denote the n-dimensional affine space, again thought of as a scheme over $\mathbb{C}$, i.e.

$$\mathbb{A}^n = Spec(\mathbb{C}[X_1, \ldots, X_n]),$$

or, in the language of schemes, the $\mathbb{C}$-valued points of the universal affine n-dimensional scheme $\mathbb{A}^n_{\mathbb{Z}} = Spec(\mathbb{Z}[X_1, \ldots, X_n])$, so that

$$\mathbb{A}^n_{\mathbb{C}} = \mathbb{A}^n_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} Spec(\mathbb{C}).$$

The above decomposition (1.1) recalls the well known classical fact that $\mathbb{P}^n$ may be seen as an affine n-dimensional space $\mathbb{A}^n$ with the addition of an extra hyperplane, called the *hyperplane at infinity*, which is itself a projective space of lower dimension.

A closed irreducible algebraic subvariety of $\mathbb{P}^n$ may be thought as the zero set locus of an homogeneous prime ideal in the ring $\mathbb{C}[X_0, \ldots, X_n]$. Moreover, recall that the *degree* of a closed irreducible subvariety V of $\mathbb{P}^n$ is defined to be the cardinality of the set of points of intersection between V and a general linear subspace of complementary dimension.

1.1.2 The intersection ring of $\mathbb{P}^n$

We want to learn how to attach a $\mathbb{Z}$-algebra to each n-dimensional projective space $\mathbb{P}^n$. We shall call such an algebra the *intersection ring* or the *Chow ring* of $\mathbb{P}^n$ and we shall denote it as $A^*(\mathbb{P}^n)$. As a $\mathbb{Z}$-module it is generated by all the closed irreducible algebraic subvarieties of $\mathbb{P}^n$. The name "intersection ring" has to do with the fact that such ring should reflect the intersection properties of subvarieties of $\mathbb{P}^n$. For example, we would like to say that, in $\mathbb{P}^2$, two lines intersect in a point, while three lines (in general position) do not intersect at all. In $\mathbb{P}^n$ we would like to say that two hyperplanes intersect in a codimension 2 linear subspace or that a hyperplane and a quadric hypersurface do intersect along a subvariety of codimension two of degree 2.

Here is the way to formalize such an intuitive idea. Start by setting, by definition:

$$A^*(\mathbb{P}^n) = \oplus_{i\in\mathbb{Z}} A^i(\mathbb{P}^n),$$

with $A^i(\mathbb{P}^n) = 0$ for each $i < 0$ and for each $i > n$ (still by definition). We are hence dealing with the finite direct sum:

$$A^*(\mathbb{P}^n) = A^0(\mathbb{P}^n) \oplus A^1(\mathbb{P}^n) \oplus \ldots \oplus A^n(\mathbb{P}^n), \tag{1.2}$$

where we must now declare what we mean by each degree of the above $\mathbb{Z}$-module. It is a positive integer. The codimension j of V is the *height* of the prime ideal in the ring $A(U)$ defining it on any affine open subset U of $\mathbb{P}^n$, and the dimension of V is $i = n - j$. We require that to each closed irreducible subvariety V of codimension j in $\mathbb{P}^n$ corresponds a generator $[V]$ of the module $A^j(\mathbb{P}^n)$, which shall be said to be the *Chow class* of V in $A^*(\mathbb{P}^n)$. Let H' be any hyperplane of $\mathbb{P}^n$. Abusing notation, the *Chow class* (the *class* for short) of H' in $A^1(\mathbb{P}^n)$ shall be denoted with the letter H, with no bracket surrounding it, instead of $[H']$. We define the Chow ring of $\mathbb{P}^n$ by agreeing that all the hyperplanes fall in the same class (so that H shall be said to be the class *of the* hyperplane) and that if V is an irreducible subvariety of codimension i in $\mathbb{P}^n$ of degree d, then $[V] = d \cdot H^i$.

The last relation implicitly defines the ring structure of $A^*(\mathbb{P}^n)$. In other words we are claiming that the ring $A^*(\mathbb{P}^n)$ is generated by H as a $\mathbb{Z}$-algebra, with the relation $H^{n+1} = 0$. This is the same as saying that:

$$A^*(\mathbb{P}^n) \cong \frac{\mathbb{Z}[H]}{(H^{n+1})}.$$

A cycle in $A^*(\mathbb{P}^n)$ is a formal finite $\mathbb{Z}$-linear combination:

$$[c] = \sum a_i[V_i],$$

where the V_i's are irreducible subvarieties of $\mathbb{P}^n$, of any codimension. If $[c] \in A^i(\mathbb{P}^n)$, then $[c]$ is said to be homogeneous of degree i. Of course, by what has been previously said, any cycle may be expressed as a polynomial in H of degree smaller than or equal to n, i.e.:

$$[c] = a_0[\mathbb{P}^n] + a_1 H + a_2 H^2 + \ldots + a_n H^n, \quad a_i \in \mathbb{Z}$$

where $[\mathbb{P}^n] = H^0$ is said to be the *fundamental class of* $\mathbb{P}^n$. One may set, as a definition, $H^n = [pt]$, said to be the *class of a point*.

Example 1.1 Let C_d and $C_{d'}$ be two irreducible curves in $\mathbb{P}^2$. Then, if L is the class of a line, we have:

$$[C_d] \cdot [C_{d'}] = dL \cdot (d'L) = dd'L^2 = dd'[pt],$$

in the ring $A^*(\mathbb{P}^2)$, expressing the well known *Bézout theorem* which says that two irreducible plane curves intersect in dd' points, counting multiplicities, where dd' is the product of the degrees.

Let $[c]$ be the Chow class in $A^*(\mathbb{P}^n)$. As we said, it may be uniquely written as a polynomial:

$$[c] = a_0 + a_1 H + \ldots + a_n H^n;$$

The *degree map* is a morphism of $\mathbb{Z}$-modules:

$$\int_{\mathbb{P}^n} : A^*(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

defined as:

$$[c] \mapsto \int_{\mathbb{P}^n} [c] = \int_{\mathbb{P}^n} a_0 + a_1 H + \ldots + a_n H^n = a_n.$$

In particular, $\int_{\mathbb{P}^n} H^r = 0$ unless $r = n$, in which case it is equal to 1. This means that the degree of the class of a point, $[pt]$ is 1. The reason for calling this the *degree* will be clear in a moment. Let $Pic(\mathbb{P}^n)$ be the *Picard group* of $\mathbb{P}^n$, i.e. the $\mathbb{Z}$-module of all the *Cartier divisors* of $\mathbb{P}^n$ modulo *linear equivalence*, or, alternatively, the group of all the isomorphism classes of line bundles $H^1(\mathbb{P}^n, O^*_{\mathbb{P}^n})$, the *analytic Picard group of* $\mathbb{P}^n$. We ask here to the reader to assume as a fact, which is well known, that $Pic(\mathbb{P}^n)$ is generated by the hyperplane bundle $O_{\mathbb{P}^n}(1)$, in the sense that any line bundle on $\mathbb{P}^n$ is isomorphic to the bundle $O_{\mathbb{P}^n}(1)^{\otimes N}$, for some $N \in \mathbb{Z}$.

Then there is a $\mathbb{Z}$-module (iso)-morphism:

$$c_1 : Pic(\mathbb{P}^n) \longrightarrow A^1(\mathbb{P}^n)$$

defined by:

$$c_1(O_{\mathbb{P}^n}(1)) = H$$

and extended by $\mathbb{Z}$-linearity. If L is any line bundle over $\mathbb{P}^n$, hence necessarily of the form $O_{\mathbb{P}^n}(m)$, then

$$c_1(O_{\mathbb{P}^n}(m)) = mH$$

is said to be the *first Chern class* of the bundle $O_{\mathbb{P}^n}(m)$.

For the informal discussion that shall follow below, let us suppose that X is a smooth closed subvariety of $\mathbb{P}^n$ and let $\iota : X \hookrightarrow \mathbb{P}^n$ be the inclusion morphism. Let us denote by $O_X(1)$ the *pull-back bundle* $\iota^* O_{\mathbb{P}^n}(1)$ and set, in a perfect formal way:

$$c_1(O_X(1)) = c_1(\iota^*(O_{\mathbb{P}^n}(1))) = \iota^*(c_1((O_{\mathbb{P}^n}(1))),$$

where $\iota^*(c_1(O_{\mathbb{P}^n}(1)))$ must be thought of as the class of the divisor on X which is supported on $X \cap H'$, where H' is a hyperplane not containing X. We have the following identity:

$$\int_X c_1(O_X(1)) = \int_X \iota^*(c_1(O_{\mathbb{P}^n}(1))) = \int_{\mathbb{P}^n} c_1(O_{\mathbb{P}^n}(1)) \cdot [X], \tag{1.3}$$

so that the last expression on the right hand side is non zero iff X is 1-dimensional (a curve!).

The formal rules quoted above shall be used in this section for an informal discussion about the motivation of the subject to be treated in the lectures. They shall be explained in a more detailed way in section 4.2.

1.2 Warming up: Two Exercises of High School Geometry.

1.2.1 I - How many lines in a pencil are tangent to a conic?

- A "numerical example".

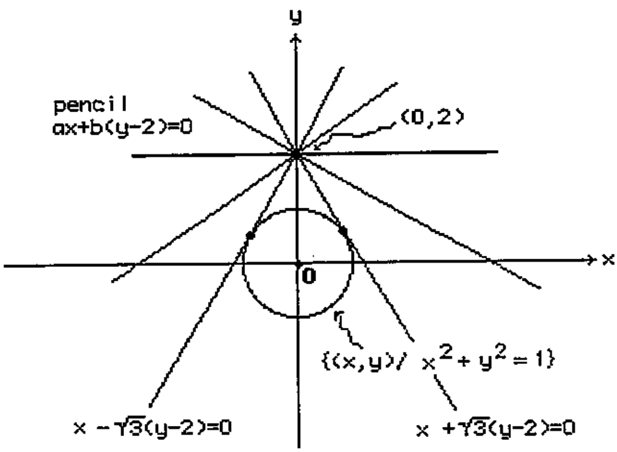
Let C be the curve in the affine space $\mathbb{A}^2$ defined by the points $P(x, y)$ whose coordinates satisfy the quadratic relation:

$$x^2 + y^2 = 1 \tag{1.4}$$

which is, as everybody knows, the equation of a circle (a conic!) in the plane with center in the origin. Let Q be the point of coordinates $(0, 2)$. Of course $Q \notin C$. The family of affine linear forms parametrized by the homogeneous coordinates $[a, b]$ of a projective line,

$$ax + b(y - 2) = 0, \tag{1.5}$$

represents a pencil of line through Q. As well known, by Bézout's theorem, the general line of the pencil intersects C at two distinct points. We wonder about lines that intersect C at two coincident points, corresponding to the tangent lines to C belonging to the pencil. This is an easy exercise.



Wlog [1], we may assume $a \neq 0$. Then equation 1.4 is equivalent to:

$$a^2x^2 + a^2y^2 = a^2,$$

but, by (1.5), $a^2x^2 = b^2(y-2)^2$. Hence the y coordinates of the intersection points of the line $L_{[a,b]}$ of the given pencil with the circle are the solutions of the quadratic equation:

$$b^2(y-2)^2 + a^2y^2 - a^2 = 0,$$

i.e.

$$(b^2 + a^2)y^2 - 4b^2y + 4b^2 - a^2 = 0.$$

The tangent lines of the pencil correspond to values of a, b where the discriminant:

$$\frac{\Delta}{4} = 4b^4 - (b^2 + a^2)(4b^2 - a^2) = a^2(a^2 - 3b^2)$$

vanishes. Since $a \neq 0$, this gives $b = \pm\sqrt{3}a/3$, so that there are exactly 2 tangent lines at C in the pencil. The tangent lines are exactly $L_{[\sqrt{3},1]}$ and $L_{[-\sqrt{3},1]}$. We hence solved a very easy enumerative problem.

[1] Without loss of generality!

- Never solve such a simple exercise as follows!

Now we shall try to solve the previous simple exercise with (just a little bit) more sophisticated techniques. The reason is not that we want to get the bad habit of solving easy exercises with difficult methods. Rather, the purpose of the speculations below is twofold: on one hand we want to show techniques which may be easily generalized to the more interesting (and definitely more difficult) situations which we are interested in this course while, on the other hand, it will be clear that the *enumerative geometry on the moduli spaces of curves* has very classical roots.

To begin with, we shall warn the reader that here some elementary results on the cohomology of the projective space and a minimum of prerequisites on line bundles over algebraic varieties shall be assumed.

Recall, first of all, that the homogeneous polynomial ring $\mathbb{C}[X_0, X_1, X_2]$ may be seen as the symmetric algebra of the 3-dimensional $\mathbb{C}$-vector space $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$. As a matter of fact, it is well known that $H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ is isomorphic to the space $(\mathbb{C}^{n+1})^\vee \cong \mathbb{C}_1[X_0, X_1, \ldots, X_n]$ of linear forms on $\mathbb{C}^{n+1}$. In our present situation we have:

$$\mathbb{C}[X_0, X_1, X_2] = \oplus_{n\geq 0} Sym^n(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))).$$

For the reader convenience, it is worth to remind that the homogeneous part $Sym^n(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$ is the same as $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(n))$, which is isomorphic to the vector space of the homogeneous linear forms of degree n in X_0, X_1, X_2, and where $O_{\mathbb{P}^2}(n) = O_{\mathbb{P}^2}(1)^{\otimes n}$.

Hence we shall identify an irreducible conic C in $\mathbb{P}^2$ with the closed zero subscheme $Z(F)$ of an irreducible $F \in H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2))$. The ideal sheaf of C is isomorphic to $O_{\mathbb{P}^2}(-2)$. Let $i : C \hookrightarrow \mathbb{P}^2$ be the inclusion morphism of C in $\mathbb{P}^2$. Now, as it should be known, $O_{\mathbb{P}^2}(1)$ may be thought of as a bundle over $\mathbb{P}^2$, that is the dual of the most natural line bundle living on it, the so called *tautological bundle* $\mathcal{T}_{\mathbb{P}^2}$ [2] Each non-zero global section of $O_{\mathbb{P}^2}(1)$ selects (via the zero-scheme of the linear form representing it) a unique line of $\mathbb{P}^2$. Hence, the pencil of lines passing through a given point of $\mathbb{P}^2$ may be thought of as a 2 dimensional linear subspace V of $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$. Let us denote by $O_C(1)$ the line bundle over C which is the pull-back $i^*O_{\mathbb{P}^2}(1)$ of $O_{\mathbb{P}^2}(1)$ via i. We contend that the space of the holomorphic global section of $O_C(1)$, $H^0(C, O_C(1))$, is

[2] The *tautological bundle* over $\mathbb{P}^n$ has the feature that the fiber over each point is the line of $\mathbb{C}^{n+1}$ represented by the point of $\mathbb{P}^n$ itself, whence the name.

isomorphic to $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$. This comes out by analyzing the exact sequence:

$$0 \longrightarrow O_{\mathbb{P}^2}(-2) \longrightarrow O_{\mathbb{P}^2} \longrightarrow O_C \longrightarrow 0 \tag{1.6}$$

that may be safely tensored by the sheaf $O_{\mathbb{P}^2}(1)$ (it is locally free!!!), getting:

$$0 \longrightarrow O_{\mathbb{P}^2}(-1) \longrightarrow O_{\mathbb{P}^2}(1) \longrightarrow O_C(1) \longrightarrow 0. \tag{1.7}$$

Writing down the *long exact cohomology* sequence:

$$\begin{aligned} 0 \longrightarrow \quad & H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(-1)) \quad \longrightarrow H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1)) \longrightarrow H^0(\mathbb{P}^2, O_C(1)) \longrightarrow \\ \longrightarrow \quad & H^1(\mathbb{P}^2, O_{\mathbb{P}^2}(-1)) \quad \longrightarrow \dots, \end{aligned}$$

using the fact that $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(-1)) = 0$, $H^1(\mathbb{P}^2, O_{\mathbb{P}^2}(-1)) = 0$ and that $H^0(\mathbb{P}^2, O_C(1)) \cong H^0(C, O_C(1))$, we get the desired isomorphism:

$$H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1)) \cong H^0(C, O_C(1)), \tag{1.8}$$

explicitly described by the map:

$$\sigma \mapsto \sigma_{|C}. \tag{1.9}$$

The geometrical meaning of the isomorphism (1.8) should be quite clear. An element of the left hand side vector space is a linear form $\sigma = a_0X_0 + a_1X_1 + a_2X_2$. Hence it represents, via its zero scheme $Z(\sigma)$, a line in $\mathbb{P}^2$. The restriction $\sigma_{|C}$ means that we simply limit ourselves to evaluate σ at points of C. Now $Z(\sigma_{|C})$ is a closed subscheme of C equal to the intersection of $Z(\sigma)$ with C. Let $\sharp(Z(\sigma_{|C}))$ be the lenght of the closed subscheme $Z(\sigma_{|C})$. Since C is smooth, $Z(\sigma_{|C})$ is a Cartier divisor and $\sharp(Z(\sigma_{|C}))$ must hence coincide with the degree $\int_C c_1(O_C(1))$ of the *first Chern class* of the bundle $O_C(1)$. One has:

$$c_1(O_C(1)) = c_1(i^*O_{\mathbb{P}^2}(1)) = i^*c_1(O_{\mathbb{P}^2}(1)) = L \cdot [C] = 2[pt], \tag{1.10}$$

where in (1.10) we used the intersection product of the ring $A^*(\mathbb{P}^2)$ (see 1.1.1), and where $[pt]$ is the class of a point. The total degree of the section σ is hence:

$$deg(\sigma) = \int_C 2[pt] = p_*(2[pt]) = 2,$$

where $p : C \longrightarrow Spec(\mathbb{C})$ is the structural morphism from C to $Spec(\mathbb{C})$. Hence we know that each section of $H^0(C, O_C(1))$ has degree 2 (indeed, we already knew that, because of our high school years). Hence *each line of the plane determines a pair of points on C.* Conversely, each pair of points on C, possibly coincident, identifies a unique line of $\mathbb{P}^2$, possibly tangent to C. This explains geometrically the isomorphism (1.8).

Because of the $1-1$ correspondence between $H^0(C, O_C(1))$ and $H^0(O_{\mathbb{P}^2}(1))$ and since the intersection points of a line with C may be viewed as the zero scheme of the section gotten by restricting the given line to C, it follows that the tangent lines correspond to sections σ of $H^0(C, O_C(1))$ such that $Z(\sigma)$ consists of two "coincidents" points. Put otherwise, we want that σ has a double zero on C. Even in this case we may proceed according to the taste. We may argue in an algebraic way, involving maximal ideals of regular local rings of points of C, or we may rely on our analytical intuition. We shall follow the latter because it may help to understand the underlying ideas of our future computations, keeping at a minimum the technical prerequisites. From now on, hence, we shall use the fact that *our conic C is an honest Riemann Surface.*

Getting back to our original problem, the situation is now as follows: we are given a pencil, $\mathcal{P}$, of lines in $\mathbb{P}^2$ passing through a point which does not belong to C. This pencil corresponds to a 2-dimensional $\mathbb{C}$-vector subspace of sections $V \subseteq H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$ which, via the isomorphism (1.8) may be identified with a 2-dimensional $\mathbb{C}$-vector subspace of $H^0(C, O_C(1))$, denoted, abusing notation, by the same letter V. The tangent lines of $\mathcal{P}$ correspond to the sections σ for which there exists $P \in C$ such that $ord_P\sigma \geq 2$.

By the latter sentence we mean the following. Let $z : U \subseteq C \longrightarrow \mathbb{C}$ be a local coordinate chart around P such that $z(P) = 0$, trivializing $O_C(1)$ over O_C. We shall often write (U, z) to denote such a chart. Then, in the neighbourhood U, one has:

$$\sigma_{|U} = s(z)\psi_U,$$

where $s \in O_C(U)$ (i.e. s is a holomorphic function on U) and ψ_U generates $H^0(U, O_C(1))$ freely over $O_C(U)$. As a holomorphic function on U, s admits a Taylor series expansion around $z = 0$, and its order of vanishing at $z = 0$ is, by definition, the order of vanishing of the section σ at P. Moreover σ vanishes twice at P if and only if $s(0) = 0$ and $s'(0) = \dfrac{ds}{dz}(0) = 0$. One may check, in fact, that such a definition does not depend neither on the local representative s nor on the local chart chosen. Let then $\sigma \in V$ be a non zero section vanishing at least twice at P. If $\{\sigma_1, \sigma_2\}$ is a $\mathbb{C}$-basis of V, then:

$$\sigma = a_1\sigma_1 + a_2\sigma_2,$$

for some a_1, a_2 not both zero. Let us set, in the given local chart (U, z), $\sigma_{i|_U} = s_i(z)\psi$, with $s_i \in O_C(U)$. Then:

$$\begin{aligned} s(0) &= 0 \Rightarrow \quad a_1 s_1(0) + a_2 s_2(0) = 0 \quad \textit{and} \\ s'(0) &= 0 \Rightarrow \quad a_1 s_1'(0) + a_2 s_2'(0) = 0 \end{aligned} \tag{1.11}$$

The system (1.11) has a non trivial solution if and only if the determinant:

$$\begin{vmatrix} s_1(z) & s_2(z) \\ s_1'(z) & s_2'(z) \end{vmatrix} \tag{1.12}$$

vanishes at $z = 0$. The determinant (1.12) is a holomorphic function on U, which should behave in a nice way when changing the local representation. In particular, we would like that its vanishing at a point in a local coordinate chart implied its vanishing at the same point in every local coordinate system. This is actually the fact. But then, such a determinant should be the local counterpart of a global object, which must describe intrinsically our situation.

Here is how things should work in *Heaven*. Let us consider the trivial family $p : C \longrightarrow Spec(\mathbb{C})$ together with the following map of vector bundles:

$$\begin{array}{ccc} C \times V & \xrightarrow{\;D\;} & J^1 O_C(1) \\ & {\scriptstyle pr_1}\searrow \quad \swarrow & \\ & C & \\ & \downarrow{\scriptstyle p} & \\ & Spec(\mathbb{C}) & \end{array} \tag{1.13}$$

We have not defined the vector bundle $J^1 O_C(1)$ yet, but we shall define it by describing the map D. To each pair $(P, \sigma) \in C \times V$ we associate $D(P, \sigma) = (D\sigma)(P)$. If (U, z) is a local holomorphic chart on C around P, $D\sigma_{|_U}$ is represented by the pair $(s(z), s'(z)) \in O_C(U)^{\oplus 2}$, where, as usual, the derivative is taken with respect to the local parameter z and $\sigma_{|_U} = s(z)\psi_U$. Suppose now that (U_α, z_α), (U_β, z_β) are both trivializing charts of $O_C(1)$ with respect to O_C, and such that $U_\alpha \cap U_\beta \neq \emptyset$. If $l_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ is the transition function with respect to this intersection, one has:

$$\sigma_{|_{U_\alpha \cap U_\beta}} = s_\alpha \psi_\alpha = s_\beta \psi_\beta,$$

with

$$\begin{aligned} s_\beta &= l_{\beta\alpha} s_\alpha \\ s_\beta' &= \frac{d}{dz_\beta}(s_\beta) = \frac{d}{dz_\beta}(l_{\beta\alpha} s_\alpha) = \frac{dz_\alpha}{dz_\beta} \cdot l_{\beta\alpha}' s_\alpha + \frac{dz_\alpha}{dz_\beta} l_{\beta\alpha} s_\alpha' \end{aligned} \tag{1.14}$$

We may organize the relations (1.14) in a matrix form, by setting:

$$(s_\beta, s'_\beta)^T = \begin{pmatrix} l_{\beta\alpha} & 0 \\ l'_{\beta\alpha}\frac{dz_\alpha}{dz_\beta} & l_{\beta\alpha}\frac{dz_\alpha}{dz_\beta} \end{pmatrix} \cdot (s_\alpha, s'_\alpha)^T. \tag{1.15}$$

Equation (1.15) so defines a $O_C(U_\alpha \cap U_\beta)$ valued matrix $M_{\alpha\beta}$ which, as the reader may easily check by reminding the celebrated *chain rule* of the elementary calculus, on triple overlapping $U_\alpha \cap U_\beta \cap U_\gamma$ satisfies:

$$M_{\alpha\beta} M_{\beta\gamma} = M_{\alpha\gamma},$$

i.e. $\{M_{\alpha\beta}\} \in \mathcal{Z}^1(\mathcal{U}, Gl_2(O_C))$, where $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of C trivializing $O_C(1)$. The *Čech cocycle* $\{M_{\alpha\beta}\}$ hence defines a rank 2-vector bundle, which is exactly the one we denoted by $J^1(O_C(1))$ in the diagram (1.13). We now claim that the points of C where we may find tangent lines of the pencil are the *degeneracy locus* of the map D. In fact we have just proved that a local representation of the map of vector bundles (1.13) is given by:

$$D_{|U} \equiv \begin{pmatrix} s_1 & s_2 \\ s'_1 & s'_2 \end{pmatrix}. \tag{1.16}$$

This map has maximal rank $\iff \begin{vmatrix} s_1 & s_2 \\ s'_1 & s'_2 \end{vmatrix} \neq 0$. Hence we are claiming that: *the locus of points of C such that there exists a non zero section $\sigma \in V$ vanishing twice at P, coincides with the locus where the vector bundle map D drops rank.*

Let us denote this locus by $Z(D)$: the notation is consistent with the fact that we are led to consider the zero locus of the holomorphic function occurring at the right hand side of (1.16) - a determinant.

Roughly speaking:

$$\begin{aligned} &\sharp\ \ \{\text{lines of } \mathcal{P} \text{ tangent to } C\} = \\ &\sharp\ \ \{\text{points of } C \text{ for which } \exists \sigma \in \mathrm{V} : \mathrm{ord}_{\mathrm{P}}\sigma \geq 2\} = \sharp(Z(D)), \end{aligned}$$

which rigorously may be rephrased as follows:

$$\sharp(Z(D)) = \int_C [Z(D)] = \int_C c_1(J^1 O_C(1) - V \otimes O_C) = \int_C c_1(J^1 O_C(1)). \tag{1.17}$$

Formula (1.17) may seem quite obscure. And it is. In fact we have used tools which have not been introduced yet. First of all we used the so-called *Porteous' formula* (see

Section 4.3), and then we used the fact, not explained yet, that, in our case, the Chern polynomial (see section 4.2) is a homomorphism between the Grothendieck group of locally free sheaves on C and its Chow ring $A^*(C)$. Hence, the second "integrand" of (1.17) has been computed as:

$$c_1(J^1O_C(1) - V \otimes O_C) = c_1(J^1O_C(1)) - c_1(V \otimes O_C) = c_1(J^1O_C(1)),$$

using the fact that the first Chern class vanishes on trivial vector bundles. By the way, for the time being, we shall content ourselves to evaluate the very right hand side of (1.17). Let us start by recalling that if E is a rank r vector bundle, then $c_1(E) = c_1(\wedge^r E)$. The bundle $\wedge^r(E)$, the top exterior power of E, is often called the *determinant bundle* associated to E. We are hence led to compute:

$$c_1(J^1O_C(1)) = c_1(\wedge^2(J^1O_C(1))).$$

But $\wedge^2(J^1O_C(1))$ is a line bundle whose transition functions are the determinant of the transition functions of the $M_{\alpha\beta}$. In other words, if $\{M_{\alpha\beta}\} \in \mathcal{Z}^1(\mathcal{U}, Gl_2(O_C))\}$, then $\{det(M_{\alpha\beta})\}$ is the defining cocycle of the line bundle $\wedge^2 J^1O_C(1)$. Hence, looking at (1.15), one has that, relatively to the covering $\{U_\alpha\}$, the transition functions of $\wedge^2 J^1O_C(1)$ are $\left\{U_\alpha, l^2_{\alpha\beta}\frac{dz_\alpha}{dz_\beta}\right\}$, i.e. they are the transition functions of the line bundle:

$$O_C(1)^{\otimes 2} \otimes K_C \cong O_C(1)^{\otimes 2} \otimes O_C(-1) = O_C(1).$$

Hence:

$$p_*c_1(O_C(1)) = \int_C c_1(O_C(1)) = 2,$$

as expected.

- Revisiting the numerical example in a numerical way.

Let us go back to the numerical example we started with. After some easy algebraic manipulations we got the explicit equations of the tangent lines to the circle $x^2+y^2 = 1$ belonging to the pencil $ax + b(y-2) = 0$. The equations of the tangent lines were:

$$\pm\sqrt{3}x + y - 2 = 0.$$

We want to show – not because we doubt it, but because we want to get familiar with maps of vector bundles – that we can get the result by computing explicitly the locus where the map D of (1.13) drops rank. We are working in the affine open subset $X_0 \neq 0$ of a projective plane with homogeneous coordinates $[X_0, X_1, X_2]$, where we set $x = X_1/X_0$ and $y = X_2/X_0$. Hence $x^2 + y^2 = 1$ is the affine piece of the conic $Z(X_0^2 - X_1^2 - X_2^2)$. In such an affine piece, a basis of the vector space V representing the pencil of lines through $Q = (0, 2)$ is given by:

$$\sigma_1 = x \quad \text{and} \quad \sigma_2 = y - 2.$$

We shall denote by the same expressions their image in $H^0(\mathcal{D}(X_0), \mathcal{O}_C(1))$ via the restriction isomorphism (1.9) ($\mathcal{D}(X_0)$ obviously means $\mathbb{P}^2 \setminus Z(X_0)$). The locus of points P of C where the map D of (1.13) drops rank have coordinates satisfying the following equation (see (1.16))

$$\begin{vmatrix} x & y-2 \\ x' & y' \end{vmatrix} = 0, \tag{1.18}$$

where the derivatives are taken with respect to some local parameter. Now, for all points P such that $y \neq 0$, we may choose x as a local coordinate, by virtue of the *implicit function theorem*. In fact, at such points, $\frac{\partial(x^2+y^2-1)}{\partial y} = 2y \neq 0$ and so y may be locally expressed as a function of x in a neighbourhood of each point $P_0(x_0, y_0) \in C$ such that $y_0 \neq 0$. Now, the only points on the circle C, having $y = 0$, are $(\pm 1, 0)$. But at these two points the tangent lines to the circle are $x = \pm 1$, which do not belong to the given pencil. This means that, for finding the searched points of tangency, we may safely use the parameter x itself. Hence, equation (1.18) can be simplified to:

$$\begin{vmatrix} x & y-2 \\ 1 & y' \end{vmatrix} = 0,$$

But since (x, y) are the coordinates of a point running on the given circle, by differentiating the expression $x^2 + y^2 = 1$, we get the relation:

$$x + yy' = 0,$$

i.e. $y' = -x/y$ (we are using now that $y \neq 0$). Hence we must solve the system of simultaneous equations:

$$\begin{cases} \begin{vmatrix} x & y-2 \\ 1 & -x/y \end{vmatrix} = 0, \\ x^2 + y^2 = 1 \end{cases} \tag{1.19}$$

The first equation, keeping the second one into account, yields $y = \frac{1}{2}$, hence $x = \pm\frac{\sqrt{3}}{2}$.

If one writes – as learned in the high schools – the equations of the lines joining the point $Q(0,2)$ and, respectively, $P_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $P_2\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$, one gets exactly the same we found at the beginning of this section.

1.2.2 II - How many reducible fibers are there in a pencil of plane conics?

Let $F \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ be an irreducible quadratic form. The associated zero scheme in $\mathbb{P}^2$ corresponds to an irreducible conic. The space of all the conics is parametrized by a $(\mathbb{P}^5)^\vee = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$. Let U be the set parametrizing the smooth conics. We claim that U is a Zariski principal affine subset of $(\mathbb{P}^5)^\vee$. In fact if:

$$C := Z(\sum_{i,=0}^{2} a_{ij}X_iX_j),$$

and considering $[a_{ij}]_{0\leq i\leq j\leq 2}$ as homogeneous coordinates of $(\mathbb{P}^5)^\vee$, it turns out – by high school geometry – that the reducible conics come out from the relation:

$$det(a_{ij}) = 0.$$

Hence $(\mathbb{P}^5)^\vee \setminus Z(det(a_{ij})) = \mathcal{D}(det(a_{ij})) = U$ is a principal affine open subset of $(\mathbb{P}^5)^\vee$. The boundary of U, ∂U, is hence the cubic hypersurface $det(a_{ij})$ in $(\mathbb{P}^5)^\vee$.

Of course U turns out to be an affine scheme over $Spec(\mathbb{C})$, but, being affine, it is not *complete* or, otherwise said, is not *proper* over $Spec(\mathbb{C})$. Analitically, this means that there exists at least one holomorphic function $\phi : D^* \longrightarrow U$, where D^* is the punctured disk $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$, that cannot be extended holomorphically at $z = 0$. For instance, consider the map:

$$\begin{cases} C \ : \ D^* \longrightarrow U \\ \qquad \lambda \longmapsto C_\lambda \end{cases}$$

where C_λ is the curve whose equation in $\mathbb{P}^2$ is $X_1X_2 + \lambda(X_0X_2 - X_1^2) = 0$. For each $\lambda \neq 0$, C_λ is an irreducible conic, but extending the map C in a holomorphic way would mean to fill the puncture with $C_0 = Z(X_0X_2)$, which is reducible. Hence we end out of U. Notice that because U is affine, we may conclude that any family of conics parametrized by some positive dimensional complete variety must have reducible fibers (the only complete subvarieties of the affine spaces are points!). As a matter of example we may hence try to solve the announced exercise.

We may try to compactify U, i.e. to find a proper scheme over $Spec(\mathbb{C})$ which contains U as a dense open subset. In this case it is very easy to guess what boundary component should we add to compactify U: it is the cubic hypersurface in $(\mathbb{P}^5)^\vee$ representing the reducible conics. Let us analyze then the following exercise: *How many reducible fibers are there in a pencil of plane conics whose general one is irreducible?* Such a pencil of conics is of the form

$$\lambda F_1 + \mu F_2 = 0,$$

and high schools methods provide us a simple answer: we look for homogeneous pairs $[\lambda, \mu]$ satisfying the equation:

$$\begin{vmatrix} \lambda a_{00} + \mu b_{00} & \lambda a_{01} + \mu b_{01} & \lambda a_{02} + \mu b_{02} \\ \lambda a_{10} + \mu b_{10} & \lambda a_{11} + \mu b_{11} & \lambda a_{12} + \mu b_{12} \\ \lambda a_{20} + \mu b_{20} & \lambda a_{21} + \mu b_{21} & \lambda a_{22} + \mu b_{22} \end{vmatrix} = 0 \tag{1.20}$$

where $F_1 = \sum a_{ij}X_iX_j$, $F_2 = \sum b_{ij}X_iX_j$, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. A quick inspection of (1.20) clearly shows that we are led to solve an homogeneous equation of third degree in $[\lambda, \mu]$, hence to find three reducible conics in the given pencil.

This last result may be interpreted geometrically as follows. The pencil of conics may be seen as a projective line in $(\mathbb{P}^5)^\vee \cong \mathbb{P}^5$. The reducible conics of such a pencil correspond to the intersections of this line with ∂U. To count such intersections we pass to the Chow ring of $\mathbb{P}^5$. A line is equivalent to H^4, H being the hyperplane class, while $[\partial U] = 3H$ (because ∂U is a hypersurface of degree 3). What we get is:

$$\sharp\{\text{reducible conics in the given pencil}\} = \int_{\mathbb{P}^5} 3H \cdot H^4 = \int_{\mathbb{P}^5} 3[pt] = 3.$$

Now we want to try to make the above heuristic reasoning a little bit more precise. In fact, our pencil of conics $\mathcal{P}$, may be seen as a proper flat family $\mathfrak{X} \longrightarrow \mathbb{P}^1$, where $\mathfrak{X}$ is of course a *surface fibered in conics.* I claim that there is a unique morphism $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^5$, such that the family $\mathfrak{X} \longrightarrow \mathbb{P}^1$ fits in the following *cartesian diagram:*

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^5 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \underset{\phi}{\longrightarrow} & \mathbb{P}^5 \end{array},$$

where $\mathcal{C}$ is the *universal conic* defined by:

$$\mathcal{C} = :\{((x_0 : x_1 : x_2), (a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22})) \in \mathbb{P}^2 \times \mathbb{P}^5 : \sum a_{ij}x_i x_j = 0\}.$$

Recall that saying that the above diagram is cartesian means that $\mathfrak{X} = \phi^*(\mathcal{C}) \cong \mathbb{P}^1 \times_{\mathbb{P}^5} (\mathcal{C})$. In other words the family $\mathfrak{X}$ is induced, via ϕ, from the *tautological family* of conics on $\mathbb{P}^5$, i.e. the family is gotten by parametrizing all the conics of $\mathbb{P}^2$ by their coefficients. The map ϕ is set-theoretically defined as:

$$\phi([a,b]) = \{\text{the point of } \mathbb{P}^5 \text{ corresponding to } \mathfrak{X}_{[a,b]}\}.$$

It is injective. In fact $\phi([a_1, b_1]) = \phi([a_2, b_2])$ implies that $[a_1, b_1]$ and $[a_2, b_2]$ correspond to the same conic, i.e. $[a_1, b_1] = [a_2, b_2]$. The map ϕ is a morphism in the category of schemes of finite type over $\mathbb{C}$ (i.e. in a down-to-the-earth terms, is a morphism of varieties). In fact it is expressed as:

$$[\lambda, \mu] \mapsto [\lambda a_{ij} + \mu b_{ij}],$$

and this is a polynomial map. Moreover ϕ is the unique map making the above diagram cartesian. For if ψ were another, by definition of pull-back family one should have:

$$\psi^*\mathcal{C}[a,b] = \mathcal{C}_{\psi([a,b])}.$$

Now, the fiber of $\mathcal{C}_{\psi([a,b])}$ sits over $\psi([a,b]) \in \mathbb{P}^5$ and, on the other hand, by the very definition of the tautological bundle, $\psi([a,b])$ should be the point of $\mathbb{P}^5$ representing the 1-dimensional $\mathbb{C}$-subspace of $\mathbb{C}^6$,

$$\mathcal{C}_{\psi([a,b])} = \psi^*\mathcal{C}_{[a,b]} = \mathfrak{X}_{[a,b]}.$$

Hence:

$$\psi([a,b]) = \{\text{the point of } \mathbb{P}^5 \text{ corresponding to } \mathfrak{X}_{[a,b]}\},$$

proving the uniqueness of ϕ. Therefore, as we already guessed, our family may be seen as a line sitting in $\mathbb{P}^5$. The Chow ring of $\mathbb{P}^5$ is $\mathbb{Z}[H]/(H^6)$, where H is the hyperplane class, so that the singular fiber of our pencil $\mathcal{P}$ correspond to the intersection of a line $\mathbb{P}^1 \sim H^4$ with a cubic hypersurface $S \sim 3H$. Hence, in the Chow ring of $\mathbb{P}^5$, one has:

$$[\mathbb{P}^1] \cdot [S] = 3H^5 = 3[pt],$$

i.e. the line representing the family intersects the boundary 3 times. We just showed that $\mathbb{P}^5$ is the *fine moduli space* of all the conics in the plane and that any family of conics may be seen as a way to embed the base into $\mathbb{P}^5$.

1.3 A more Difficult Problem

1.3.1 Flexes of smooth plane quartics

Let F be an irreducible element of $H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(4))$. Suppose also that the homogeneous ideal generated by $\left(F, \dfrac{\partial F}{\partial X_i}\right)_{i=0,1,2}$ is contained in the irrelevant ideal (X_0, X_1, X_2). Then, the zero scheme $Z(F)$ defines a smooth curve in $\mathbb{P}^2$ of degree 4 which, by the genus formula, may be thought also as a compact Riemann surface of genus 3. A point $P \in Z(F)$ is said to be a flex if and only if the tangent line L to $Z(F)$ at P is such that $L \cdot Z(F) = 3P + Q$, as a divisor on $Z(F)$, with Q possibly coincident to P. Of course we have a very good way to characterize the flexes on $C =: Z(F)$. In fact, arguing as in 1.2.1, there is an isomorphism:

$$H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1)) \longrightarrow H^0(C, O_C(1)),$$

sending $\sigma \in H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$ in $\sigma_{|C}$. Hence, $P \in C$ is a flex if and only if there exists $\omega \in H^0(C, O_C(1))$ such that $ord_P\omega \geq 3$. But let $(\omega_0, \omega_1, \omega_2)$ be a $\mathbb{C}$-basis of $O_C(1)$, such that $\omega = a_0\omega_0 + a_1\omega_1 + a_2\omega_2$. Then:

$$ord_P\omega \geq 3 \iff \begin{cases} a_0f_0(0) + a_1f_1(0) + a_2f_2(0) = 0 \\ a_0f_0'(0) + a_1f_1'(0) + a_2f_2'(0) = 0 \\ a_0f_0"(0) + a_1f_1"(0) + a_2f_2"(0) = 0 \end{cases}$$

where, in a local chart (U, z) around P, such that $z(P) = 0$ and trivializing $O_C(1)$ over O_C, we set:

$$\omega_{i|_U} = f_i(z)\psi_U,$$

and the derivatives of the $f_i \in O_C(U)$ are taken with respect to z. Then P is a flex if and only if the holomorphic function on U:

$$\begin{vmatrix} f_0 & f_1 & f_2 \\ f_0' & f_1' & f_2' \\ f_0" & f_1" & f_2" \end{vmatrix} \tag{1.21}$$

vanishes at $z = 0$. The equation (1.21) is often said to be the local *wronskian* associated to the holomorphic $\mathbb{C}$-basis $(\omega_0, \omega_1, \omega_2)$. By the way, as seen in 1.2.1, what comes out is that the holomorphic function (1.21) is the local representation of a map of vector bundles denoted by D^2 as in the following diagram:

$$\begin{array}{ccc} C \times H^0(C, O_C(1)) & \xrightarrow{D^2} & J^2 O_C(1) \\ & {\scriptstyle pr_1} \searrow \quad \swarrow & \\ & C & \\ & \downarrow{\scriptstyle p} & \\ & Spec(\mathbb{C}) & \end{array}$$

Again we shall apply *Porteous' formula*, not explained yet, getting:

$$\sharp\{flexes\} = \int_C c_1(J^2 O_C(1) - H^0(C, O_C(1)) \otimes O_C) = \int_C c_1(J^2 O_C(1)).$$

We shall use again the fact that $c_1(J^2(O_C(1))) = c_1(\wedge^3 J^2(O_C(1))) = c_1(O_C(6))$. Hence:

$$\sharp\{flexes\} = \int_C c_1(O_C(6)) = \int_C 6c_1(O_C(1)) = 6\int_C c_1(O_C(1)) = 24$$

a well known classical result (consequence, e.g., of the Plücker's formulas).

1.3.2 More geometry and less technique

Let us see, now, how to compute the number of flexes without using Porteous' formula but rather a little bit more of geometry. A smooth plane quartic, as said in 1.3.1, is a Riemann surface of genus 3. By the *theory of adjunction* [38], denoting by $\iota : C \hookrightarrow \mathbb{P}^2$ the *immersion morphism* of C in $\mathbb{P}^2$, we have that the pull back bundle $\iota^* O_{\mathbb{P}^2}(1) = O_C(1)$ is nothing but the *canonical bundle* K_C of the curve C. This is because the family of lines of $\mathbb{P}^2$ induce on the quartic a family of divisors of degree 4 (the intersection of the lines with the quartic C) parametrized by a $\mathbb{P}^2$. Hence the lines of $\mathbb{P}^2$ induces on C what is classically denoted as a g^2_4. But, as well known, there exists a unique g^{g-1}_{2g-2} on a curve of genus g and such a linear series coincides with the *canonical series.*

To further clarify the situation let us fix an open covering $\mathcal{U} = \{(U_\alpha, z_\alpha)\}$, where $z_\alpha : U_\alpha \longrightarrow \mathbb{C}$ is a local coordinate and each U_α trivializes K_C over O_C. Then we may assume that the generator ψ_{U_α} of $H^0(U_\alpha, K_C)$ over $H^0(U_\alpha, O_C)$ is nothing but the differential dz_α. Thus, if $\underline{\omega} = (\omega_0, \omega_1, \omega_2)$ is a $\mathbb{C}$-basis of $H^0(C, K_C)$, we may write, on each U_α:

$$\underline{\omega}_{|U_\alpha} = \mathrm{f}_\alpha(z_\alpha) \cdot dz_\alpha = (f_{\alpha,0}(z_\alpha)dz_\alpha, f_{\alpha,1}(z_\alpha)dz_\alpha, f_{\alpha,2}(z_\alpha)dz_\alpha),$$

so that the *local wronskian* relatively to a given U_α is:

$$W_\alpha(z_\alpha) = \mathbf{f}(z_\alpha) \wedge \mathbf{f}'(z_\alpha) \wedge \mathbf{f}"(z_\alpha) := \begin{vmatrix} f_{\alpha,0} & f_{\alpha,1} & f_{\alpha,2} \\ f'_{\alpha,0} & f'_{\alpha,1} & f'_{\alpha,2} \\ f"_{\alpha,0} & f"_{\alpha,1} & f"_{\alpha,2} \end{vmatrix}$$

As usual the derivatives occuring in (1.3.2) are taken with respect to the coordinate z_α.

Exercise 1.1 Show, by a straightforward use of the *chain rule* for the derivative of the composition of functions, that the local wronskians W_α and W_β are related by:

$$W_\alpha(z_\alpha) = W_\beta(z_\beta) \cdot \left(\frac{dz_\alpha}{dz_\beta}\right)^6 \tag{1.22}$$

The above exercise, once solved, tells us some nice things. First, that the vanishing of W_α at $P \in U_\alpha$ implies the vanishing of W_β at P for each U_β containing P. Moreover, for each pair (α, β),

$$\frac{dz_\alpha}{dz_\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

are the *transition functions of the canonical bundle.* It follows that the collection $\{(U_\alpha; W_\alpha)\}_{\alpha \in \mathcal{A}}$ ($W_\alpha \in O_C(U_\alpha)$) defines a holomorphic section of the sixth tensor power, $K_C^{\otimes 6}$, of the canonical bundle K_C. Such a section defines a *Cartier divisor* on C. Its degree, as seen already, corresponds to the degree of the first Chern class of the bundle $K_C^{\otimes 6}$, i.e.

$$\int_C c_1(K_C^{\otimes 6}) = 6 \int_C c_1(K_C) = 6 \cdot 4 = 24,$$

and the result is now achieved.

Exercise 1.2 Let C_d be a smooth plane curve of degree d. Using the *Clebsh formula* $d(d-3) = 2g - 2$ relating the genus to the degree of the curve, show that the number f of the flexes of C_d, counted according to multiplicities, is given by:

$$f = 3d(d-2)$$

Exercise 1.3 Weierstrass points on a curve of genus g

Let C be a smooth projective curve of genus $g \geq 2$ defined over $\mathbb{C}$ (a *Compact Riemann Surface*). Let $L \in Pic(C)$, that is L is (the isomorphism class of) a line bundle over C. Suppose that $h^0(C,L) = dim_{\mathbb{C}} H^0(C,L) > 0$ and that the *degree* of L is d (by this we mean that each holomorphic section has exactly d zeroes counted according to the multiplicities or, if we like, $\int_C c_1(L) = d$). Let $V \subseteq H^0(C,L)$ be an $r+1$ dimensional $\mathbb{C}$-subspace of $H^0(C,L)$. We say that the pair (L,V) is a g^r_d on C. Let P be a point of C. For each non negative integer n, set:

$$V - nP = \{\sigma \in V : ord_P\sigma \geq n\}. \tag{1.23}$$

We say that P is a *ramification point* of the $g^r_d = (L,V)$ iff:

$$dim_{\mathbb{C}}(V - (r+1)P) > 0.$$

1. Show that the total degree $wt(g^r_d)$ of the ramification points of the g^r_d is given by the *Brill-Segre formula*:
$$wt(g^r_d) = (r+1)d + (g-1)r(r+1).$$
The ramification points of the canonical series, i.e. of the unique g^{g-1}_{2g-2} living on C that coincides with $(K_C, H^0(C,K_C))$, are said to be the *Weierstrass points* of C.

2. The *total weight*, i.e. the total degree of the ramification divisor, is given by:
$$wt = \sum_{P\in C} wt(P) = (g-1)g(g+1).$$

3. Show that $P \in C$ is a Weierstrass point iff $h^0(C, K_C(-gP)) > 0$.

4. Show that $P \in C$ is a Weierstrass point iff there exists $n \leq g$ such that $h^0(C, O_C(nP)) \geq 2$.

The non negative integer $wt(P)$ is defined to be as the order of the wronskian section at the point $P \in C$. It is called the *Weierstrass weight* of $P \in C$.

1.3.3 Pencils of plane quartics versus pencils of curves of genus 3

Let us go back to Riemann surfaces of genus 3 such that K_C is *very ample*, which is a way of saying that the canonical system embeds C in $\mathbb{P}^2$ as a plane quartic. Once having carefully thought about the exercises and examples of the previous sections, we are led to conclude that the Weierstrass points of a plane quartic are nothing but the flexes. Let $P \in C$. We say that $P \in C$ is a hyperflex iff there exists a holomorphic differential vanishing at P with multiplicity (exactly) 4. In other words, there should exist $\omega \in H^0(C, K_C)$ such that $ord_P\omega = 4$. If C has been chosen in a sufficiently general way it would not be reasonable to expect that. In fact, such P must be described by the locus of points of C where the map:

$$\begin{array}{ccc} H^0(C,K)\otimes O_C & \xrightarrow{D^3} & J^3(K_C) \\ & {\scriptstyle pr_1}\searrow \qquad \swarrow & \\ & C & \\ & \downarrow{\scriptstyle p} & \\ & Spec(\mathbb{C}) & \end{array}$$

drops rank. As the reader should now be able to guess by himself, if (U, z) is a local chart of C trivializing K_C, the local representation of the map D^3 is given by:

$$D^3_{|U} = \begin{pmatrix} \mathbf{f} \\ \mathbf{f}' \\ \mathbf{f}'' \\ \mathbf{f}''' \end{pmatrix}$$

where $\underline{\omega}$ is a $\mathbb{C}$-basis of $H^0(C, K_C)$ and $\underline{\omega}_{|U} = \mathbf{f}dz$. Now, the fact that, as mentioned earlier, K_C is very ample implies that $rank(D^3) \geq 2$ at each $P \in C$. If $P \in C$ is a hyperflex such a rank has to be exactly 2 and by the *Kronecker rule* that means that at least two 3×3 determinants must vanish. The hyperflexes are cut out by two independent equations (in fact it is easy to construct a quartic having a flex which is not a hyperflex). Hence the hyperflexes have *expected codimension* 2. This is the reason why we *should not expect* to find a hyperflex on a sufficiently general quartic. Actually, this is a consequence of a theorem which says that the general curve of genus $g \geq 3$ has only *normal Weierstrass points.*

But then, we may ask ourselves the following question: let $\mathfrak{X} \longrightarrow \mathbb{P}^1$ be a pencil of plane quartics such that the general fiber is a smooth quartic with only ordinary f lexes: how many fibers of such a family are quartics having at least one hyperflex? This problem is in the same philosophical stream of the question suggested in section 1.2.2, but more difficult. The reason is, first of all, that in our pencil we should expect non-isomorphic fibers (in the holomorphic category) and even singular fibers. The latter, in particular, suggests that the *space* (whatever we mean by this word) of the smooth quartic may not be complete. Actually this is the case and we advise the reader to look at the paper [17] for a complete argument. Hence, over such singular fibers the map D^3 may give us some problem. May be not, but who knows? Moreover we cannot argue in a very uniformly way as we did for the pencil of conics. To look for fiber with hyperflexes we must use all the canonical bundles of all the fibers and hope to be able to glue them together to have something global (a line bundle? Yes, a line bundle) on the total space of our family. More generally we may ask a even more difficult question : let $\mathfrak{X}\longrightarrow\mathbb{P}^1$ be a proper flat family of curves of genus 3 such that the general fiber is a compact Riemann surface which is not *hyperelliptic*. What may we say about the fibers of such a family? Are there singular fibers? The answer is yes: but it is not easy to keep under control the singularities that may appear. Are there *hyperelliptic fibers*? The answer is yes, but how to count them? Clearly we may not expect to be able to embed all our family in the plane: one reason for this is the existence of the hyperelliptic fibers. Let us sketch briefly the way we may count the hyperelliptic fibers. First of all recall that a curve of genus g is hyperelliptic iff there exists a point P such that $h^0(C, K_C(-2P)) = g-1$. Then we may solve our problem in case there exists on the total space of our pencil $\pi : \mathfrak{X}\longrightarrow\mathbb{P}^1$ a line bundle K_π that restricts on each fiber to the canonical bundle of the fiber itself. Then we would like to write a vector bundle map similar to the one we already saw:

$$\begin{array}{ccc} \pi^*\pi_*K_\pi & \xrightarrow{D} & J^1K_\pi \\ & \searrow \quad \swarrow & \\ & \mathfrak{X} & \\ & \downarrow \pi & \\ & S & \end{array}$$

In this case we would be able, in some way to be explained in the next lectures, to compute a class in $A^1(\mathfrak{X})$, let us call it $[VH]$, and then we would push it down on $\mathbb{P}^1$ via π_* to compute its degree over $\mathbb{P}^1$, that is:

$$\int_{\mathbb{P}^1} \pi_*[VH].$$

This is exactly what we shall try to do in the sequel.

But notice: on the one hand the idea is very simple, being exactly the same we used to look for singular fibers in a pencil of conics or flexes on smooth quartics. But on the other hand no object written in the map (1.3.3) has been rigorously defined yet. For instance: what is the meaning K_π? On $\mathbb{P}^1$ we have reducible fibers and so we need to know what is the substitute of the canonical bundle for singular curves.

Secondly, what is the meaning of D, in this case, and what is the *jets extension* over a family of line bundles patched together, where pathologies arising from the singular fibers may very well happen?

Moreover: how to interpret the number of hyperelliptic fibers, or of the fiber with a hyperflex or of the singular fibers? We may suspect that there exists some *universal space* parametrizing holomorphic isomorphism classes of smooth curves of genus 3. Inside this space we may imagine "divisors" like the locus of the hyperelliptic curves or of the curves with a hyperflex and think about the family $\mathfrak{X}\longrightarrow\mathbb{P}^1$ as an image of $\mathbb{P}^1$ embedded in such a space. Then the number of special fibers may be thought of as the number of the intersections of such a $\mathbb{P}^1$ with the divisors of special curves.

The space we are looking for is denoted by M_3 and, as a set, it is the collection of all isomorphisms classes of curves of genus 3. Because of the singular fibers that may occur in a family parametrized by a positive dimensional variety, we certainly know that such a space cannot be complete, once that a reasonable scheme structure has been defined on it. We may thence try to compactify it: but what is the most reasonable compactification? Is it a good idea to simply add all the singular curves of arithmetic genus 3? Let us suppose that a reasonable compactification of M_3 has been achieved and let us denote it by $\overline{M}_3$. May we hope that such a space is a *fine moduli space* i.e. a space carrying a *universal family* of curves such that any other family may be seen as the pull-back of the universal one? We shall see in the sequel that the answer to this question is no. We shall look for a weaker solution, by seeing that there exists a space $\overline{M}_g$ which carries a structure of normal projective algebraic variety, which is a *coarse moduli space* for the so-called *stable curves* of genus g.

The purpose of the next lectures will be to learn to make some intersection theoretical computation on such a space and to convince ourselves that the basic ideas of the subjects have their roots in the classical geometry we studied in high school.

1.4 An appendix: The Quantum Cohomology Ring of $\mathbb{P}^2$.

Let $\mathbb{P}^r$ be the r-dimensional projective space. As for some other smooth algebraic varieties enjoying some good positivity properties for the anticanonical bundle (like *strict Del Pezzo rational surfaces*), see [11]), one may define an *intersection ring* of different kind, which has been basically introduced in mathematics by the physicists. We shall not want to be very general, here, but it is probably worth to axiomatically define the *quantum cohomology ring* of $\mathbb{P}^2$. To this purpose, let $A^*(\mathbb{P}^2)$ the $\mathbb{Z}$-module generated by the irreducible projective subvarieties of $\mathbb{P}^2$ (we are forgetting the *intersection ring* structure). We shall try to put an algebra structure on the $\mathbb{Q}[[T_0, T_1, T_2]]$-module:

$$\mathcal{A}(\mathbb{P}^2) = A^*(\mathbb{P}^2) \otimes_{\mathbb{Z}} \mathbb{Q}[[T_0, T_1, T_2]],$$

where $\mathbb{Q}[[T_0, T_1, T_2]]$ is the ring of formal power series in the indeterminates T_0, T_1, T_2. Define now a *misterious function* (basically invented by physicists):

$$f(y^0, y^1, y^2) = \frac{1}{2}\left(\left(y^0\right)^2 y^2 + \left(y^1\right)^2 y^0\right) + \sum_{d\geq 1} N_d \frac{\left(y^2\right)^{3d-1} e^{dy^1}}{(3d-1)!}, \tag{1.24}$$

where the N_d's are rational numbers. Strictly speaking the f defined in (1.24) is not a function. It is rather a formal power series in the indeterminates (y^0, y^1, y^2) and this is the meaning of the expression e^{dy_1} there occurring. Define now coefficients as follows:

$$\Phi_{ijk} = \frac{\partial f}{\partial y^i \partial y^j \partial y^k}.$$

Let (g_{ij}) be the matrix (Φ_{0ij}) that, as the reader may easily verify is exactly the 3×3 matrix:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Set $(g^{ij}) = (g_{ij})^{-1}$. Of course, the entries of the matrix (g^{ij}) are the same as the entries of the matrix (g^{ij}). However it is important to make the distinction because it is the way this construction may be generalized to other situations. Define coefficients:

$$\Phi_{ij}^k = g^{hk}\Phi_{hij}. \tag{1.25}$$

In the right hand side of (1.25) we are using the *Einstein convention* for the sums, i.e. we mean that the above expression is summed on the index h. We use the Φ_{ij}^k to define an algebra structure on the module $\mathcal{A}(\mathbb{P}^2)$, by setting:

$$T_i * T_j = \Phi_{ij}^k T_k.$$

We call $*$ the *quantum algebra product* of $\mathcal{A}(\mathbb{P}^2)$. We ask whether this product is associative, making $(\mathcal{A}, *)$ into a ring. As it may be easily checked, the associativity is equivalent to check that the relation:

$$(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2), \tag{1.26}$$

holds. We have hence the following:

Exercise 1.4 The product $*$ defined on $\mathcal{A}(\mathbb{P}^2)$ is associative if and only if the rational numbers N_d's satisfy the following recursive relations:

$$N_d = \sum_{\substack{d_A + d_B = d \\ d_A \geq 1, d_B \geq 1}} N_{d_A} N_{d_B} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A - 1} \right) \tag{1.27}$$

where N_1 may be chosen arbitrarily.

If the recursive relations above are satisfied, the pair $(\mathcal{A}(\mathbb{P}^2), *)$ is a ring, which is said to be the *Quantum Cohomology ring* of $\mathbb{P}^2$ or, also, the *Quantum Intersection ring* of $\mathbb{P}^2$. The following fundamental result is due to Kontsevich.

Theorem 1.1 *Define N_d as the* number of rational nodal plane curves of degree d passing through $3d-1$ points of $\mathbb{P}^2$ in general position. *Then the* quantum algebra product *defined by the* generating function *(1.24) is associative.*

The above theorem has a beautiful geometrical proof based on intersection theoretical properties of the *moduli space of stable maps* defined by Kontsevich and briefly discussed in Section 3.3. As a corollary one has that the number N_d of *rational nodal plane curves* of degree d passing through $3d-1$ points is prescribed by the recursive formula (1.27). Now the initial datum cannot be arbitrary: we must set $N_1 = 1$ (Euclid's theorem: *there is a unique line passing through two distinct points in the plane*), so that $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, known classically. The first out of reach was $N_5 = 87304$ which was first confirmed by I. Vainsencher ([75])

Chapter 2

Moduli Spaces

2.1 What is a Moduli Space?

2.1.1 A quick review of "Abstract non-sense"

We recall, following [55] (p. 53. and ff.), a few notions about categories, which will be useful to clarify the notion of *moduli spaces* associated to a moduli problem. A category $\mathcal{C}$ consists of a collection of *objects* $Obj(\mathcal{C})$, denoted by roman capital letters $A, B, \ldots$, such that to each pair $A, B \in Obj(\mathcal{C})$ [1] of objects of $\mathcal{C}$ is associated a (true) set, $Hom_{\mathcal{C}}(A, B)$, called the *set of $\mathcal{C}$-morphisms*, which satisfy the following axioms:

1. $Hom_{\mathcal{C}}(A,B) \cap Hom_{\mathcal{C}}(A',B') = \emptyset \quad unless \quad A = A' \quad \wedge \quad B = B'$, in which case $Hom_{\mathcal{C}}(A,B) = Hom_{\mathcal{C}}(A',B')$.

2. If $A, B, C \in Hom_{\mathcal{C}}(A, B)$, then there is a map of sets:

   $$Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \longrightarrow Hom_{\mathcal{C}}(A,C)$$

   sending $(f,g) \in Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C)$ to a morphism of $Hom_{\mathcal{C}}(A,C)$ denoted by $g \circ f$, which is *associative*. In other words, if $h \in Hom_{\mathcal{C}}(C,D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

3. For each object $A \in Obj(\mathcal{C})$, there is a distinguished morphism $id_A \in Hom_{\mathcal{C}}(A, A)$ characterized by the property that:

   $$f \circ id_A = f \quad \forall f \in Hom_{\mathcal{C}}(A,B),$$

[1] For our limited purposes, the most important thing in the quite misterious definition of a category is to recognize it when it happens to meet one somewhere.

and

$$id_B \circ f = f \quad \forall f \in Hom_C(A, B).$$

In particular, $Hom_C(A, A) \neq \emptyset$, for any $A \in Obj(C)$.

The collection of all the morphisms in C are the *arrows* of C. When one writes $f \in Ar(C)$ one means that $f \in Hom_C(A, B)$ for some $A, B \in Obj(C)$. $f \in Hom_C(A, B)$ is said to be a C-*isomorphism* iff there exists $g \in Hom_C(B, A)$ such that $f \circ g = id_B \in Hom_C(B, B)$ and $g \circ f = id_A \in Hom_C(A, A)$. $f \in Hom_C(A, A)$ is said to be a C-*endomorphism.* If an endomorphism is also an isomorphism, then it is said to be an *automorphism.*

Definition 2.1 *Let C be a category. One says that the category C has a* product *iff for each pair of objects $A, B \in Obj(C)$ there exists an object P together with two morphisms $p_A : P \longrightarrow A$ and $p_B : P \longrightarrow B$*

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{p_A} & & \searrow^{p_B} & \\ A & & & & B \end{array} \tag{2.1}$$

such that the following universal property is satisfied: for each $Q \in Obj(C)$ and morphisms $q_A : Q \longrightarrow A$ and $q_B : Q \longrightarrow B$ there exist a unique morphism $\pi : Q \longrightarrow P$ making the following diagram commutative.

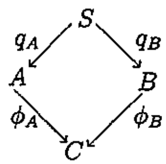
$$\begin{array}{ccccc} & & Q & & \\ & \swarrow^{q_A} & \downarrow^{\pi} & \searrow^{q_B} & \\ & & P & & \\ & \swarrow_{p_A} & & \searrow_{p_B} & \\ A & & & & B \end{array} \tag{2.2}$$

Of course, if a product exists in C it is unique up to a canonical isomorphism. Another important definition which will be used to define the notion of moduli space is:

Definition 2.2 *A category C is said to admit a* fiber product *over each of its objects, if and only if for each triple $A, B, C \in Obj(C)$ and each pair of morphisms $\phi_A : A \longrightarrow C$ and $\phi_B : B \longrightarrow C$ there exists an object T together with two morphisms $p_A : T \longrightarrow A$ and $p_B : T \longrightarrow B$, such that the following diagram:*

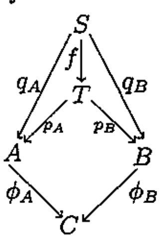
$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{p_A} & & \searrow^{p_B} & \\ A & & & & B \\ & \searrow_{\phi_A} & & \swarrow_{\phi_B} & \\ & & C & & \end{array} \tag{2.3}$$

commutes and such that the following universal property is satisfied: for each $S \in Obj(C)$ and each pair of morphisms $q_A : S \longrightarrow A$ and $q_B : S \longrightarrow B$ making commutative the diagram*:*



(2.4)

then there exists a unique morphism $f : S \longrightarrow T$ such that $q_A = p_A \circ f$ and $q_B = p_B \circ f.$, *i.e., the diagram:*



(2.5)

commutes.

We now come to analyze the notion of *functor* which, for refined people, is the way all the categories becomes themselves a category, called (Cat).

Definition 2.3 *Let C and C' be two categories. A* covariant functor $F : C \rightsquigarrow C'$ *is a law which associates to each $A \in Obj(C)$ an object $F(A) \in Obj(C')$ and such that to each $f \in Hom_C(A, B)$, associates a morphism $F(f) \in Hom_{C'}(F(A), F(B))$ such that:*

1. *For each $A \in Obj(C)$, $F(id_A) = id_{F(A)}$.*

2. *If $(f, g) \in Hom_C(A, B) \times Hom_C(B, C)$ then $F(g \circ f) = F(g) \circ F(f)$ in $Hom_{C'}(F(A), F(C))$. A* contravariant functor *is defined similarly, but by replacing 2 by:*

3. *If $(f, g) \in Hom_C(A, B) \times Hom_C(B, C)$ then $F(g \circ f) = F(f) \circ F(g)$ in $Hom_{C'}(F(C), F(A))$.*

In other words, a covariant functor is what educated people would define as a morphism in the category (Cat) of all the categories while a contravariant functor would be a morphism in the category $(Cat)^\circ$, the *opposite category* of (Cat). We shall not enter in details, here.

The last important notion we need is the following. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are, as usual, two categories. A functor $F : \mathcal{C} \rightsquigarrow \mathcal{C}'$ may be viewed as the object of a category, $Fun(\mathcal{C}, \mathcal{C}')$, of all the *functors from* $\mathcal{C}$ and $\mathcal{C}'$, whose morphisms are defined in the following way. If F, G are two functors, a *morphism* $\mathcal{H} : F \longrightarrow G$ is a rule that, to each $A \in \mathcal{C}$, associates a morphism $\mathcal{H}_A \in Hom_{\mathcal{C}'}(F(A), G(A))$:

$$\mathcal{H}_A : F(A) \longrightarrow G(A),$$

such that the diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\mathcal{H}_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\mathcal{H}_B]{} & G(B) \end{array} \tag{2.6}$$

The diagram (2.6) has been drawn in the case of a pair of covariant functors. If F and G are contravariant, one has simply to revert the arrows $F(f)$ and $G(f)$. The morphism $\mathcal{H} : F \longrightarrow G$ in the category $Fun(\mathcal{C}, \mathcal{C}')$ is said to be a *natural tranformation of functors*. Of course, we have a corresponding notion of isomorphism of functors. Two functors F, G are isomorphic if there are two natural tranformation $\mathcal{H}$ and $\mathcal{H}'$ such that $\mathcal{H} \circ \mathcal{H}' = id_{\mathcal{C}'}$ and $\mathcal{H}' \circ \mathcal{H} = id_{\mathcal{C}}$, where, if $\mathcal{C}$ is a category, $id_{\mathcal{C}}$ is the functor which sends each $A \in Obj(\mathcal{C})$ and each $f \in \mathcal{A}r(\mathcal{C})$ to itself.

2.1.2 Moduli Problems

The aim of such a subsection is to sketch the underlying idea of a *moduli problem*, without being too formal: it is our aim to be more precise in the examples we are interested in, to be treated below. We advise the interested reader to look at [23] and [67] for more details. By the way, the general setting to formulate a "Moduli Problem" is a sufficiently nice category $\mathcal{C}$ of *objects* to be classified in some way. Sufficiently nice means, for instance, that in such a category there are *fibered products* and *products*. We also want to be equipped with an equivalence relation $\sim$ between the objects of $\mathcal{C}$ and a suitable notion of a *family* parametrized by the object of some sub-category of the category of schemes (e.g., as we shall do in the following, the category of *schemes of finite type over Spec*$(\mathbb{C})$). Roughly speaking, a family of objects of $\mathcal{C}$ parametrized by some scheme S, should mean a collection of objects X_s, one for each $s \in S$, which are reasonably patched together according to the nature of the parameter space. In a sense, we look for a notion of *continuously varying family* of objects. The basic features we would ask for our *moduli problem* are:

1. A family parametrized by a *point* (in the category of schemes meaning the spectrum of a field) is the same as a single object of C.

2. The equivalence relation $\sim$ between objects of C may be extended to families: in other words, we are given of a notion of equivalence between families of objects of C which give us back the relation $\sim$ for families parametrized by a single point.

3. We may pull back families (this has to do with the fact that we required our category C to be equipped with *fibered products*). In other words, suppose that $\mathfrak{X}\longrightarrow S$ is a family and that $\phi : S'\longrightarrow S$ is a morphism in the category of schemes we are working in. Then there exists a family $\phi^*\mathfrak{X}\longrightarrow S'$, said to be the *induced family* parametrized by S', which makes *cartesian* the following diagram:

$$\begin{array}{ccc} \phi^*\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

i.e. $\phi^*\mathfrak{X} = S' \times_{S'} \mathfrak{X}$. The way of associating induced families to morphisms between parameter spaces has to be functorial. Put otherwise we would like that 1_S^* be the identity of $\mathfrak{X}$, for each $\mathfrak{X}\longrightarrow S$, and that $(\phi \circ \psi)^* = \psi^* \circ \phi^*$. This is the so called *universal property* of C.

4. The *pullback* of families has to be compatible with the relation $\sim$ in the sense that if $\mathfrak{X}\longrightarrow S$ and $\mathfrak{X}'\longrightarrow S$ are two equivalent families (parametrized by the same space S) then for any morphism $\phi : S'\longrightarrow S$ we have $\phi^*\mathfrak{X} \sim \phi^*\mathfrak{X}'$.

To solve the moduli problem means to equip $C/\sim$ with a structure which, if possible, makes it an object of the category where the parameter spaces have been taken.

2.1.3 Fine and coarse moduli spaces

Here we shall try to give a brief but possibly precise account on the basic definitions of a given moduli space. Such definitions shall not be given in the most general setting, but only in the one we are interested in. For details and generalizations see [16]. For the moment let us denote by $\mathcal{PS}$ be a category which shall be called the *category of the parameter spaces.* Moreover we are given of a moduli problem, i.e. of a pair $(C, \sim)$, where C is a *subcategory* of $\mathcal{PS}$ and $\sim$ is an equivalence relation between families $\mathfrak{X}\longrightarrow S$ of objects of C parametrized by S. A family $A\longrightarrow\{pt\}$ parametrized by a point $\{pt\}$ is, by definition, an object of $\mathcal{C}$ which, by abuse of notation, shall be denoted with the symbol A. This means, in particular, that $\sim$ induces an equivalence relations between the objects of $\mathcal{C}$, thought as families parametrized by a point. Let:

$$\mathcal{F} : \mathcal{PS} \longrightarrow (Sets),$$

be a functor which associate to each $S \in \mathcal{PS}$ the set $\mathcal{F}(S)$ of all the equivalence classes of families of objects of $\mathcal{C}$ parametrized by S. We have hence the following fundamental:

Definition 2.4 *The functor*

$$\mathcal{F} : \mathcal{PS} \longrightarrow (Sets),$$

is said to be representable in $\mathcal{PS}$ iff there exists an object $M \in Obj(\mathcal{S})$ *such that the functor* $\mathcal{F}$ *is isomorphic to the functor* $Hom_{\mathcal{PS}}(\cdot, M)$. *In such a case* M *is said to be a* fine moduli space *for the moduli problem* $(\mathcal{C}, \sim)$.

This means, in particular, that for each $S \in Obj(\mathcal{S})$ there exists a set bijection between:

$$\mathcal{F}(S) \cong Hom_{\mathcal{PS}}(S, M),$$

or, for further emphasis, that each family parametrized by S corresponds to one and only one morphism between S and M.

We want to show now that 2.4 is equivalent to the the following:

Definition 2.5 *A fine moduli space for the moduli problem* $(\mathcal{C}, \sim, \mathcal{PS})$ *is an object* $M \in Obj(\mathcal{PS})$ *together with a family* $\mathcal{U} \longrightarrow M$ *which is universal in the following sense. For each family* $\pi : \mathfrak{X} \longrightarrow S$ *there is a unique morphism* $f \in Hom_{\mathcal{PS}}(S, M)$ *such that* $\mathfrak{X} = S \times_M \mathcal{U} := f^*\mathcal{U}$.

To construct $\mathcal{U}$ from Definition 2.4, one chooses the unique family $\mathcal{U} \longrightarrow M$ associated to the identity morphism id_M of M. In this case, if $\mathfrak{X} \longrightarrow S$ is any family, there exists by definition a unique morphism $f : S \longrightarrow M$. Consider the cartesian diagram:

$$\begin{array}{ccc} S \times_M \mathcal{U} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ S & \underset{f}{\longrightarrow} & M \end{array}$$

then $S \times_M \mathcal{U} \longrightarrow S$ is a family parametrized by S which induces the morphism $f: S \longrightarrow M$. By the uniqueness of f, it follows that $S \times_M \mathcal{U} \longrightarrow S$ must coincide with $\mathfrak{X}$. Conversely, definition 2.5 clearly implies 2.4.

In general, given a moduli problem, the existence of a fine moduli space is not guaranteed. This happens when, for instance, as we shall see with the moduli space of curves, there is an object A with non trivial automorphisms, such that $Aut(A)$ acts non-trivially on $\mathcal{PS}$.

Example 2.1 [2] The existence of automorphisms for some objects of the category $\mathcal{C}$ does not prevent, in general, the existence of a *fine* solution of the moduli problem $(\mathcal{C}, \sim)$. In fact let $\mathcal{C}$ be the category of finite sets, and $\sim$ the relation:

$$A \sim B \iff \sharp(A) = \sharp(B).$$

Let $\mathcal{PS}$ be the category of all sets. Then the set $\mathbb{N}$ of natural numbers is a fine moduli space for the problem $(\mathcal{C}, \sim, \mathcal{PS})$. The *tautological family* $\mathcal{U} \longrightarrow \mathbb{N}$ (given by attaching the set $0, 1, \ldots, n-1$ over the integer $n \in \mathbb{N}$) is the universal family over $\mathbb{N}$ for the given moduli problem.. Surely, objects of $\mathcal{C}$ (finite sets) do admit automorphisms.

Definition 2.6 *A* coarse moduli space *for a moduli problem* $(\mathcal{C}, \sim, \mathcal{PS})$ *is an object* M *of* $\mathcal{PS}$*, for which there is a natural transformation of functors:*

$$\Psi_M : \mathcal{F} \longrightarrow Hom(\cdot, M),$$

such that:

1. $\Psi(\{pt\})$ *is bijective;*

2. *For any object* N *of* $\mathcal{PS}$ *and any natural transformation of functors*

$$\Psi_N : \mathcal{F} \longrightarrow Hom(\cdot, N),$$

there is a unique transformation of functors

$$\chi : Hom(\cdot, M) \longrightarrow Hom(\cdot, N),$$

[2] I am grateful to Prof. Anders Kock for pointing me out this example.

making commutative the following diagram of natural transformation of functors:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Psi_M} & Hom(\cdot, M) \\ & \searrow^{\Psi_N} & \downarrow \chi \\ & & Hom(\cdot, N) \end{array} \tag{2.7}$$

2.1.4 Examples of Moduli Spaces.

Now we turn our abstract picture seen in 2.1.3, into more concrete models by making some choices. First of all, the *category of the parameter spaces* will be the category $(Sch/\mathbb{C})$ whose objects are (not necessarily reduced) $\mathbb{C}$-schemes of finite type (i.e. of finite type over $Spec(\mathbb{C})$) and whose morphisms are morphisms of $\mathbb{C}$-schemes. Recall that a $\mathbb{C}$-scheme is of finite type if, locally in the Zariski topology, it is the *prime spectrum* of a finitely generated $\mathbb{C}$-algebra. Let us fix some sub-category $\mathcal{C}$ of $(Sch/\mathbb{C})$. The notion of family of objects of $\mathcal{C}$ is provided by the following:

Definition 2.7 *A family $\pi : \mathfrak{X} \longrightarrow S$ is a proper flat $\mathbb{C}$-morphism of $\mathbb{C}$-schemes such that, for each $s \in S$, the scheme theoretical fiber $Spec(\mathbf{k}(s)) \times_S \mathfrak{X} =: \mathfrak{X}_s$ is an object of $\mathcal{C}$.*

Notice that, because we are working over $\mathbb{C}$, $\mathbf{k}(s) \cong \mathbb{C}$ for each closed point $s \in S$: however, varying s, the isomorphism giving the identification with $\mathbb{C}$, in general, varies as well . Now, let $\pi_1 : \mathfrak{X}_1 \longrightarrow S$ and $\pi_2 : \mathfrak{X}_2 \longrightarrow S$ be two families. We say that these two families are *equivalent* if there exists an S-isomorphism between them. Explicitly, we ask that there is $F : \mathfrak{X}_1 \longrightarrow \mathfrak{X}_2$ such that the following diagram

$$\begin{array}{ccc} \mathfrak{X}_1 & \xrightarrow{F} & \mathfrak{X}_2 \\ & {}_{\pi_1}\searrow \quad \swarrow_{\pi_2} & \\ & S & \end{array}$$

commutes.

Notice that if $X_1 \longrightarrow Spec(\mathbb{C})$ and $X_2 \longrightarrow Spec(\mathbb{C})$ are two trivial families, then the notion of equivalence implies an equivalence between objects of $\mathcal{C}$, which says that two objects X_1, X_2 of $\mathcal{C}$ are equivalent if they are isomorphic as $\mathbb{C}$-schemes. The moduli problem we shall deal with will hence be $\{\mathcal{C}, \sim\}$. A fine moduli space for the moduli problem would be a $\mathbb{C}$-scheme M of finite type representing the functor:

$$\mathcal{F} : (Sch/\mathbb{C}) \rightsquigarrow (Sets),$$

defined as:

$$\begin{aligned}\mathcal{F}(S) \;=\; & \{\text{isomorphism classes of S} - \text{schemes } \mathfrak{X} \longrightarrow \text{S such that} \\ & \mathfrak{X}_s \text{ is an object of } \mathcal{C}\}.\end{aligned}$$

A *coarse moduli space* would be, instead, a $\mathbb{C}$-scheme for which there exists a natural transformation of functors:

$$\Psi_M : \mathcal{F} \longrightarrow Hom_{(Sch/\mathbb{C})}(\cdot, M),$$

such that:

$$\Psi_M(\{pt\}) := \mathcal{F}(\{pt\}) = \{\text{isomorphisms classes of schemes} \in \mathrm{Obj}(\mathcal{C})\} \cong \mathrm{M},$$

is a bijection and such that for each $\mathbb{C}$-scheme N and for each natural tranformation of functors $\Psi_N : \mathcal{F} \longrightarrow Hom_{(Sch/\mathbb{C})}(\cdot, N)$ there is a unique natural transformation

$$\chi : Hom_{(Sch/\mathbb{C})}(\cdot, M) \longrightarrow Hom_{(Sch/\mathbb{C})}(\cdot, N),$$

making commutative the diagram (2.7).

Example 2.2 Let us consider the functor:

$$\mathcal{F} : (Sch/\mathbb{C}) \rightsquigarrow (Sets),$$

defined as:

$$\begin{aligned}\mathcal{F}(S) = \{ \;& \text{isomorphisms classes of locally trivial families over S} \\ & \text{of } 1 - \text{dimensional vector } \mathbb{C} - \text{subspaces of } \mathbb{C}^{n+1}\}\end{aligned}$$

We claim that such a functor is represented by $\mathbb{P}^n$, i.e. that $\mathbb{P}^n$ is a a fine solution for a moduli problem. This works more or less as in 1.2.2. Let $\pi : \mathfrak{X} \longrightarrow S$ be a family where S is a scheme of finite type over $\mathbb{C}$. Over $\mathbb{P}^n$ we have the tautological family $(O_{\mathbb{P}^n}(1))^{\vee}$. Define:

$$\nu_S : S \longrightarrow \mathbb{P}^n,$$

as

$$\nu_S(s) = \{\text{the point of } \mathbb{P}^n \text{ parametrizing the line } \mathfrak{X}_s\}.$$

To check that ν_S is a morphism it is not restrictive to assume that S is affine, i.e. that $S = Spec(R)$, where R is some finitely generated $\mathbb{C}$-algebra which can be expressed as the quotient of a polynomial ring $\mathbb{C}[T_1, \ldots, T_m]$ by some ideal I. In other words:

$$R = \frac{\mathbb{C}[T_1, \ldots, T_m]}{I}.$$

Hence the family $\mathfrak{X}$ can be locally expressed by means of an algebraic family of 1-dimensional vector subspace of $\mathbb{C}^{n+1}$, i.e. as a family of lines:

$$\begin{cases} X_0 & = & [v_0(T_1, \ldots, T_m)]t \\ X_1 & = & [v_1(T_1, \ldots, T_m)]t \\ \vdots & \vdots & \vdots \\ X_n & = & [v_n(T_1, \ldots, T_m)]t \end{cases}$$

where the v_i's are polynomials in $\mathbb{C}[T_1, \ldots, T_m]$ and $[v_i]$ their classes in R. Hence, the map f may be locally written as:

$$([T_1], \ldots, [T_m]) \longmapsto [v_0(T_1, \ldots, T_m), \ldots, v_n(T_1, \ldots, T_m)],$$

where $[T_i]$ means the class of T_i modulo the ideal I, proving that ν_S is a morphism. It is then clear that $\mathfrak{X} \cong S \times_{\mathbb{P}^n} (O_{\mathbb{P}^n}(1))^\vee$ (prove it!) Of course ν_S is unique, and one argues exactly as in the example of the pencil of conics worked out in 1.2.2.

Exercise 2.1 Prove that the functor:

$$\mathcal{F} : (Sch/\mathbb{C}) \rightsquigarrow (Sets)$$

defined by:

$$\begin{aligned} \mathcal{F}(S) \; = \; & \{ \text{locally trivial families/S} \\ & \text{of } k-\text{dimensional vector } \mathbb{C}-\text{subspaces of } \mathbb{C}^n \} \end{aligned}$$

is representable by the smooth scheme $G(k, \mathbb{C}^n)$, the *grassmannian of k-planes* in $\mathbb{C}^n$.

Example 2.3 The Hilbert Scheme

If S is a scheme of finite type over $\mathbb{C}$ we may construct the fiber product $\mathbb{P}^r_S := \mathbb{P}^r \times_{Spec(\mathbb{C})} S$. As a matter of fact, what we are going to say hold universally over $Spec(\mathbb{Z})$, but we are not interested in being very general. Hence, the reason for which we wrote $\mathbb{P}^r_S := \mathbb{P}^r \times_{Spec(\mathbb{C})} S$, specifying $Spec(\mathbb{C})$ is to emphasize that we are considering $\mathbb{P}^r_S$ as a scheme over $Spec(\mathbb{C})$ and not, e.g. over $Spec(\mathbb{Z})$, or some other algebraically closed field with non zero characteristic. We want to stay as much adherent to the situations which, later on, we shall be interested in. We start now with a definition:

Definition 2.8 *A closed subscheme $\mathfrak{X}$ of $\mathbb{P}^r_S$ is a* flat family of closed subschemes of $\mathbb{P}^r$ *if and only if the projection $\pi : \mathfrak{X} \longrightarrow S$ induced by $pr_2 : \mathbb{P}^r_S \longrightarrow S$ is a flat morphism.*

It is a very well known fact (see e.g. [42]) that any flat family of closed subschemes of $\mathbb{P}^r$ parametrized by some S enjoys the property that the *Hilbert Polynomial* is fiberwise constant. Suppose that we are given of a polynomial of the form:

$$P(t) = \sum_{i=0}^{r} a_i \binom{T+r}{i} \in \mathbb{Q}[t]$$

where the a_i's are integers. Let us consider the contravariant functor:

$$\underline{Hilb}^r_{P(t)} = (Sch/\mathbb{C}) \rightsquigarrow (Sets),$$

where:

$$\underline{Hilb}^r_{P(t)}(S) = \{\text{flat } S-\text{families of closed subschemes of } \mathbb{P}^r \text{ having P(t) as Hilbert polynomial}\}$$

The important result proven by Grothendieck ([37]), in the most general situation, is that there exists a projective scheme $Hilb^r_{P(t)}$ that represents the functor $\underline{Hilb}^r_{P(t)}$. This in particular means, by recalling the definition of *fine moduli space* that there exists a universal family $\mathcal{U} \subseteq Hilb^r_{P(t)} \times \mathbb{P}^r$ together with a projection $\pi : \mathcal{U} \longrightarrow Hilb^r_{P(t)}$, such that each fiber is a closed subscheme of $\mathbb{P}^r$ having $P(t)$ as Hilbert polynomial, and such that for each flat family $\mathfrak{X} \longrightarrow S$ of closed subschemes of $\mathbb{P}^r$, having the same Hilbert polynomial, there exists a unique *classifying map* ν_S such that $\nu_S^* \mathcal{U} = \mathfrak{X}$.

Example 2.4 The Hilbert scheme parametrizing curves of a given genus and degree.

Let C be a projective curve in $\mathbb{P}^r$. Such a curve has a Hilbert polynomial which is given by $P_C(n) = \chi(O_C(n))$ (If $\mathcal{F}$ is a coherent sheaf on a noetherian scheme, then $\chi(\mathcal{F})$ is its *Euler Characteristic* (see [42])). Such a polynomial has degree 1 (which corresponds to the dimension of the scheme, in our case a curve). Hence it may be written as:

$$P_C(t) = dt + 1 - p_a.$$

It turns out that d is the *degree* [3], while p_a is said to be the *arithmetic genus* of the curve C. The curve C hence corresponds to a point of the Hilbert scheme $Hilb^r_{d,p_a}$ which universally parametrizes projective curves in $\mathbb{P}^r$ of degree d and arithmetic genus p_a.

Example 2.5 Any closed subscheme of $\mathbb{P}^n$ with Hilbert polynomial $\binom{t+k}{k}$ is a linear subspace of $\mathbb{P}^n$ of dimension k. It follows that:

$$Hilb^n_{\binom{t+k}{k}} = G(k+1, \mathbb{C}^{n+1}),$$

is the grassmannian of $k+1$ dimensional subspaces of $\mathbb{C}^{n+1}$.

[3] Such a *degree* is related to the *degree* defined in the introduction: the curve C defines a cycle in $\mathbb{P}^r$ which is equivalent to dH^{r-1} in the Chow ring $A^*(\mathbb{P}^r)$. If one intersect such a curve with a general hyperplane we get a cycle equivalent to dH^r, so that $\int_{\mathbb{P}^r} dH^r = d$.

Chapter 3

Moduli Spaces of Curves

3.1 An Informal Introduction

Fix an integer $g \geq 2$ and let us consider the category $\mathcal{C}(g)$ whose objects are all the complex smooth projective curves of genus g (*Compact Riemann Surfaces*) and the morphisms are the morphisms between curves (i.e. regular or holomorphic maps according to the algebraic or analytic language). Clearly on the objects of the category $\mathcal{C}(g)$, which in this case is a set, we may define an equivalence relation, by putting in the same class isomorphic curves (biholomorphic if we work in the category of the Riemann Surfaces). Let us denote by M_g the quotient, i.e. the *set* of all isomorphism classes of smooth curves of genus g. We would like to get the right to call such a set the *moduli space of the curves of genus g*. To do this we should provide the set M_g with some scheme or analytic space structure. Once such a structure has been provided one may ask the following question: is the functor

$$\mathcal{M}_g : (Sch/\mathbb{C}) \rightsquigarrow (Sets),$$

defined by:

$$\begin{aligned} \mathcal{M}_g(S) \; = \; & \{ \text{isomorphisms classes of flat proper families} \\ & \pi : \mathfrak{X} \longrightarrow S, \text{ such that } \mathfrak{X}_s \text{ is a smooth curve of genus } g \} \end{aligned}$$

representable? And, in case of positive answer, is M_g its representing scheme? As it will be shown later on, although in a sketchy way, the answer to this question is no, at least in the above formulation. The reason, as we shall see, is that there are plenty of curves having nontrivial automorphisms. As a matter of fact M_g turns out to be only a *coarse moduli space* for the above moduli functor. If M_g were a fine moduli space we would have a *universal curve* over M_g, by considering the tautological family $M_{g,1}$ of 1-*pointed smooth curves*. It would be defined as the set of all pairs (C, P) such that C is a smooth connected projective curve and $P \in C$. The projection $\pi : M_{g,1} \longrightarrow M_g$ would consist in *forgetting* the marking.(i.e. the fiber over $[C]$ would be the curve C itself!). However, if C has a non trivial automorphism σ, the point (C, P) and the point $(C, \sigma(P))$ must be identified, so that over such a $[C]$, the fiber would consist of C together with all its automorphic images. This is the reason, as we shall see, for which the morphism π is not solution of the universal moduli problem for projective smooth curves.

To go on, we may also observe that simple experiments suggest that M_g is not a complete scheme. Consider for instance the following proper flat family of curves of genus 3 parametrized by the punctured disc:

$$x^4 + y^4 + x^2 - y^2 + \lambda(x - y) = 0.$$

The fiber of such a family over $\lambda \neq 0$ is a smooth curve. For $\lambda = 0$ instead, we get the nodal curve:

$$x^4 + y^4 + x^2 - y^2 = 0$$

which has still *arithmetic genus* 3, but has also a node at the point $(0, 0)$. Hence we cannot fill, in a holomorphic way, the family with a smooth *plane* curve of genus 3. This is no yet sufficient to conclude that M_3 is not complete: one should check that it is not possible to replace the special fiber of the family with a smooth curve even after a finite base change (see [17] for details). This is actually the case and hence M_3 is not complete. We shall not enter in the details of this proof, since it goes beyond the scopes of these lectures.

It is natural to ask, then, if is there any natural way to compactify M_g. More precisely, one would like to find a *proper, separated* scheme of finite type over $\mathbb{C}$ (or a compact Hausdorff analytic space), containing M_g as an open dense subset. Such a space may be constructed in a purely algebraic way by using *geometric invariant theory* (see [64]) or, analytically, by using the analytic theory of deformation and patching together the so-called *Kuranishi families*. We shall sketch the construction of $\overline{M}_g$ using the latter method in the next section. By the way, we think it is worth to declare here what (isomorphism classes of) degenerate curves should one add to M_g to get its so-called *Deligne-Mumford* compactification.

Definition 3.1 *A* stable n-pointed curve *of genus g is a connected projective curve with at most nodes as singularities and having a finite automorphism group.*

It is well-known (for instance by using the theory of *Weierstrass points*, see e.g. [4], [36]) that each smooth curve of genus $g \geq 2$ has a finite automorphism group $Aut(C)$. The curves of genus 1 have a 1-dimensional group of automorphisms. Hence the 1-pointed curves of genus 1 are stable (because an automorphism must fix the marking). The smooth rational curves have a 3 dimensional group of automorphism ($PGL_2(\mathbb{C})$). It follows that a stable n-pointed curve of genus g is a reduced connected curve of arithmetic genus g such that each smooth rational component has at least 3 *special points*, where a point is said to be special either if it is a marked point or if it is a singular point (i.e. an intersection point with the other components). For instance, a stable curve of genus g is such that any smooth rational component must meet the rest of the curve in at least 3 points. This suffices to make finite the group of automorphisms. A stable n-pointed curve of genus g is the data of a proper flat family $\mathfrak{X} \longrightarrow S$ together with n disjoint sections $\sigma_1, \ldots, \sigma_n$, such that, for each $s \in S$, $(\mathfrak{X}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is a stable n-pointed curve of genus g. Clearly, if n is big enough, $\overline{M}_{g,n}$ is a fine moduli space.

3.1.1 A first example to warm up.

Let S be the set of all the configurations of 4 ordered distinct points on a projective line. We shall denote any one of such a configuration as $(\mathbb{P}^1, P_1, P_2, P_3, P_4)$, where $P_1, P_2, P_3, P_4 \in \mathbb{P}^1$, and we shall refer to it as a *smooth 4-pointed rational curve.* In the set of all smooth 4-pointed rational curves, we can define an equivalence relation $\sim$ by setting:

$$(\mathbb{P}^1, P_1, P_2, P_3, P_4) \sim (\mathbb{P}^1, Q_1, Q_2, Q_3, Q_4),$$

if and only if there exists an automorphism $\sigma \in Aut(\mathbb{P}^1) \cong PGL_2(\mathbb{C})$ such that $\sigma(P_i) = Q_i$. Let us denote by $M_{0,4}$ the set of equivalence classes of elements of S modulo the relation $\sim$. We claim that $M_{0,4}$ is a 1-dimensional affine variety. Indeed, since $PGL_2(\mathbb{C})$ is 3 dimensional, for each 4 pointed rational curve:

$$(\mathbb{P}^1, P_1, P_2, P_3, P_4)$$

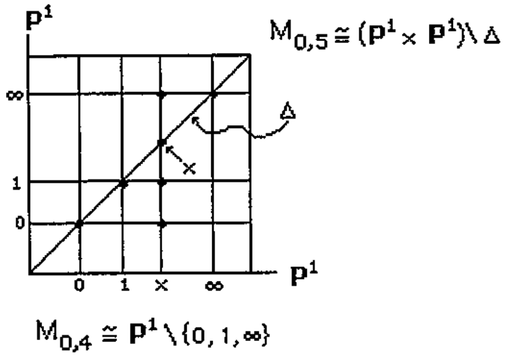
there exists $\sigma \in Aut(\mathbb{P}^1)$ such that $\sigma(P_1) = 0$, $\sigma(P_2) = 1$, $\sigma(P_3) = \infty$. Let $x = \sigma(P_4)$. Hence any element of $M_{0,4}$ has a unique representative of the form $(\mathbb{P}^1, 0, 1, \infty, x)$, where $x \notin \{0, 1, \infty\}$. Hence $M_{0,4}$ is parametrized by the coordinate x defined above, which runs in a $\mathbb{P}^1$ minus 3 points. If $[X_0, X_1]$ is a system of homogeneous coordinates such that $\infty = [0,1]$ and $0 = [1,0]$, it follows that $M_{0,4} = Spec\left(\mathbb{C}[X_0, X_1]_{(X_0X_1(X_0-X_1))}\right)$ or, which is the same, coincides with the affine principal open subset $D(X_0X_1(X_0 - X_1))$. On $M_{0,4}$ one may easily construct a *tautological family*, which may be denoted as $M_{0,5}$, the set of $\sim$-isomorphism classes of smooth 5-pointed rational curves. If $(\mathbb{P}^1, 0, 1, \infty, x, y) \in M_{0,5}$ we define:

$$\pi : M_{0,5} \longrightarrow M_{0,4},$$

by setting:

$$\pi((\mathbb{P}^1, 0, 1, \infty, x, y)) = (\mathbb{P}^1, 0, 1, \infty, x).$$

Of course afirming that such a projection is not defined for $x = y$ does not make sense since on $M_{0,5}$ marks are always distinct. What we did is simply to construct a surface which is fibered on $M_{0,4}$ whose fiber over each $x \in M_{0,4}$ is the smooth 4-pointed rational curve. The situation may be pictorially represented as follows:



where we have a total space $M_{0,4} \times M_{0,4}$ that projects onto $M_{0,4}$ off of the diagonal Δ. We have three distinguished sections that, by abuse of notation, shall be denoted as 0, 1 and ∞ and that cut out on any fiber over $x \notin \{0, 1, \infty\}$ the points $0, 1, \infty$ respectively. The point x on the fiber is, instead, cut out by the diagonal section. We claim that $M_{0,4}$ is a fine moduli space for the moduli problem $\{4-\text{pointed rational curves}, \sim\}$. In order to prove this, we begin by saying what we mean by a stable 4-pointed rational curve over S. It is the data $(\mathfrak{X} \longrightarrow S, \sigma_1, \ldots, \sigma_4)$, where $\mathfrak{X} \longrightarrow S$ is a proper flat family such that for each $s \in S$ $\mathfrak{X}_s$ is a rational curve, plus 4 disjoint sections $\sigma_1, \ldots, \sigma_4$, which for each $s \in S$ define the marking on $\mathfrak{X}_s$. We say that $(\pi_1 : \mathfrak{X}_1 \longrightarrow S, \sigma_1, \ldots, \sigma_4)$ and $(\pi_2 : \mathfrak{X}_2 \longrightarrow S, \tau_1, \ldots, \tau_4)$ are isomorphic if and only if there exists an S-isomorphism ϕ, such that the diagram:

$$\begin{array}{ccc} \mathfrak{X}_1 & \xrightarrow{\Phi} & \mathfrak{X}_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & S & \end{array}$$

commutes, and $\Phi \circ \sigma_i = \tau_i$. Let us consider the moduli functor:

$$\mathcal{M}_{0,4} : (Sch/\mathbb{C}) \longrightarrow (Sets),$$

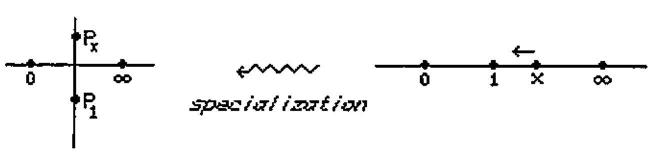
where:

$$\begin{aligned} \mathcal{M}_{0,4}(S) \;=\; & \{\text{isomorphism classes of } 4-\text{pointed rational} \\ & \text{curves parametrized by } S\} \end{aligned}$$

Then $\mathcal{M}_{0,4}$ is isomorphic to the functor $Hom_{(Sch/\mathbb{C})}(\cdot, M_{0,4})$. In fact, given any flat map $\mathfrak{X} \longrightarrow S$, consider the map:

$$\begin{aligned} S \;& \longrightarrow M_{0,4} \\ s \;& \longmapsto \{\textit{the rational curve } \mathfrak{X}_s \textit{ parametrized by } s \in S\} \end{aligned}$$

Arguing as in Ex. 2.2, one may easily see that f is a morphism. Moreover the fiber product coincides with $\mathfrak{X}$. In fact there is a map $\phi^* M_{0,5} \longrightarrow \mathfrak{X}$, and hence $\mathfrak{X}$ is canonically isomorphic to $\phi^* M_{0,5}$. However, $M_{0,4}$ is not compact or, if one thinks algebraically, it is not complete. For instance, consider the tautological family over $M_{0,4}$ itself. It is parametrized by the coordinate x, but there is no holomorphic way to fill the *punctures* $0, 1, \infty$ with 4-pointed stable smooth rational curves. Our hope is to be able to fill the "holes" of $M_{0,4}$ to get a bigger compact space $\overline{M}_{0,4}$. Of course, in the case we succeded in this attempt, such a $\overline{M}_{0,4}$ must be isomorphic, as a scheme, with the projective line $\mathbb{P}^1$. The idea, which shall be made precise below, is to fill the "holes" with 4-pointed stable *reducible* rational curves. In other words, imagine, as in the picture, that the point x is running toward 1.

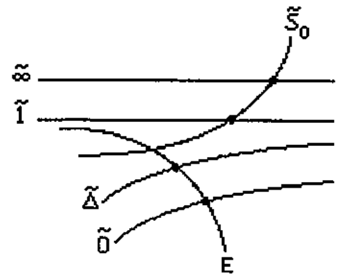


Then, when $x = 1$, the rational curve where x was walking on, splits in two components: on one component we still have the marked points 0 and ∞ and on the other components two new points, say P_x and P_1 corresponding respectively to x and 1.

This is the way to make precise the above intuitive picture. Consider $\mathcal{S} = \mathbb{P}^1 \times \mathbb{P}^1$ (the product is of course taken over $Spec(\mathbb{C})$). Then $\mathcal{S}$ projects onto $\mathbb{P}^1$ and we have 4 sections, three of which denoted, abusing notation, as $0, 1, \infty$, while the fourth is simply the diagonal Δ. Notice that if $x \notin \{0, 1, \infty\}$ the 4 sections give us back the configuration parametrized by x on the rational curve $\{x\} \times \mathbb{P}^1$. The problems arise at $x = 0, 1, \infty$, where the diagonal section meets each of the sections 0, 1 and ∞. Let us call $Q_0, Q_1, Q_\infty \in \mathbb{P}^1$ such intersection points and define:

$$\mathcal{U}_{0,4} = Bl_{Q_0,Q_1,Q_\infty}(\mathcal{S}),$$

the blow up of the surface $\mathcal{S}$ at Q_0, Q_1, Q_∞. Let us see what happens in correspondence of each of this points. For example, blowing up Q_0 we have in $\mathcal{U}_{0,4}$ the strict transforms $\tilde{0}$ and $\tilde{\Delta}$ of the sections 0 and Δ, respectively, and the strict transform $\tilde{\mathcal{S}}_0$ of $\{0\} \times \mathbb{P}^1$ (see the picture below).



Such stricts tranforms are mutually disjoint, but they are crossed by the exceptional divisor E_0 gotten by blowing up Q_0. Hence on $\tilde{S}_0$ we have two markings due to the intersection with the strict transforms of the sections 1 and ∞, while on E we have two markings, P_0 and P_x due to the intersection of E with the strict transform of the sections 0 and Δ respectively. Then the rational curve $E \cup \tilde{S}_0$ is the fiber of $\mathcal{U}_{0,4}$ over the point $0 \in \mathbb{P}^1$. Of course, $\mathcal{U}_{0,4}$ is isomorphic to $M_{0,5}$ off the points Q_0, Q_1, Q_∞. We have a natural projection:

$$\pi : \mathcal{U}_{0,4} \longrightarrow \overline{M}_{0,4} \cong \mathbb{P}^1,$$

which makes $\overline{M}_{0,4}$ a fine moduli space which parametrizes 4-pointed stable curves.

We incidentally remark that because of the isomorphism between $\overline{M}_{0,4}$ and $\mathbb{P}^1$, we may conclude that $\overline{M}_{0,4}$ is smooth and also complete. Notice, moreover, that all the points of $\overline{M}_{0,4}$ parametrize rational stable curves *with no automorphisms* other than the identity. This is a feature that is shared by all the fine moduli spaces. As soon as we have a parameter space parametrizing objects having non trivial automorphisms, we may get in trouble for having a fine moduli space, as we shall see in the next section.

We should end this example by observing that this seemingly innocuous moduli space plays a fundamental role in the algebraic geometric proof of the associativity of the *quantum cohomology ring* of the projective plane.

3.2 Construction of the Moduli Spaces of Stable Curves via Kuranishi Families

3.2.1 Some Deformation Theory.

In this subsection we shall give some rudiments of the theory of deformations restricting our attention to analytic spaces of dimension 1 (curves). From now on, let C be a stable curve of genus $g \geq 2$. A *deformation* of C parametrized by a pointed scheme (S, s_0) is a pair (ϕ, f) where $\phi : \mathfrak{X} \longrightarrow (S, s_0)$ is a proper flat morphism and

$$f : \phi^{-1}(s_0) =: \mathfrak{X}_{s_0} \longrightarrow C$$

is an isomorphism between C and the fiber of ϕ over s_0. Suppose that $(\phi_1, f_1) : \mathfrak{X}_1 \longrightarrow (S, s_0)$ and $(\phi_2, f_2) : \mathfrak{X}_2 \longrightarrow (S, s_0)$ are two deformations of C parametrized by the same pointed scheme (S, s_0). Two such deformations are said to be *equivalent* if and only if there exists an S-isomorphism F:

$$\begin{array}{ccc} \mathfrak{X}_1 & \xrightarrow{F} & \mathfrak{X}_2 \\ & {\scriptstyle\phi_1}\searrow \quad \swarrow{\scriptstyle\phi_2} & \\ & S & \end{array}$$

such that, if $f_1 : \phi_1^{-1}(s_0) \longrightarrow C$ and $f_2 : \phi_2^{-1}(s_0) \longrightarrow C$ are the two isomorphisms between the fiber over s_0 and C, then:

$$f_2 \circ F \circ f_1^{-1} = f_1 \circ F \circ f_2^{-1} = id_C.$$

For more emphasis, we should hence say that two deformations of C parametrized by the same (S, s_0) and such that $\phi_1 = \phi_2$ have to be considered *distinct* if the identification between C and the fibers over s_0 are different.

If $\mathfrak{X} \longrightarrow (S, s_0)$ is a deformation of C and (T, t_0) is any other pointed scheme together with a morphism $g : (T, t_0) \longrightarrow (S, s_0)$ (i.e. such that, $g(t_0) = s_0$), then $g^*\mathfrak{X} := T \times_T \mathfrak{X}$ is another deformation of C parametrized by (T, t_0), which is called the *induced deformation* on (T, t_0), or the *pull-back* of the deformation (ϕ, f) to (T, t_0).

For the remaining of this subsection we want to concentrate our attention to the case where C is smooth connected complex projective curve (compact Riemann surface). We will study the space of *infinitesimal deformations* of C.

Definition 3.2 *An* infinitesimal deformation *of C is a deformation parametrized by $S = Spec(\mathbb{C}[\epsilon])$.*

We recall that the ring $\mathbb{C}[\epsilon]$ is by definition the quotient ring $\mathbb{C}[X]/(X^2)$, so that $\epsilon = X+(X^2)$ and, consequently, $\epsilon^2 = 0$. Notice that $Spec(\mathbb{C}[\epsilon])$ is naturally a pointed scheme, since it contains only one (closed) point.

Let $\mathfrak{X}\longrightarrow Spec(\mathbb{C}[\epsilon]) = S$ be one such deformation. It will be convenient to think of C as a complex manifold of dimension 1, so that it may be viewed as the quotient manifold $\cup_{\alpha\in\mathcal{A}}(\{\alpha\}\times U_\alpha)/\sim$, where $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of C (which can naturally be chosen finite) such that each U_α is the domain of a local coordinate z_α and $\sim$ is the equivalence relation given by the glueing maps:

$$z_\alpha \sim z_\beta \iff z_\alpha = f_{\alpha\beta}(z_\beta).$$

$f_{\alpha\beta} : z_\beta(U_\alpha \cap U_\beta)\longrightarrow z_\alpha(U_\alpha\cap U_\beta)$, being bi-holomorphic functions satisfying the cocycle rule:

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma) = f_{\alpha\gamma}(z_\gamma).$$

Hence, the deformation $\mathfrak{X}\longrightarrow S$ of the curve C may be thought as a deformation of its transition functions. More precisely, the total space $\mathfrak{X}$ of the family may be covered by open sets of the form $U_\alpha \times S$ with coordinates (z_α, ϵ), glued by means of the transition functions:

$$(z_\alpha, \epsilon) = \tilde{f}_{\alpha\beta}(z_\beta, \epsilon) = (f_{\alpha\beta}(z_\beta) + \epsilon b_{\alpha\beta}(z_\beta), \epsilon).$$

Notice that the "first component" of the right hand side represents a first order deformation of the transition functions for C. Of course we must require that the functions $\tilde{f}_{\alpha\beta}$ satisfy a cocycle-like rule, i.e. that:

$$(z_\alpha, \epsilon) = \tilde{f}_{\alpha\beta}(\tilde{f}_{\beta\gamma}(z_\gamma, \epsilon)) = \tilde{f}_{\alpha\gamma}(z_\gamma, \epsilon).$$

Now:

$$\begin{aligned}
\tilde{f}_{\alpha\beta}(\tilde{f}_{\beta\gamma}(z_\gamma, \epsilon)) &= \tilde{f}_{\alpha\beta}\left(f_{\beta\gamma}(z_\gamma) + \epsilon b_{\beta\gamma}(z_\gamma), \epsilon\right) = \\
&= \left(f_{\alpha\beta}\left(f_{\beta\gamma}(z_\gamma) + \epsilon b_{\beta\gamma}(z_\gamma)\right) + \epsilon b_{\alpha\beta}\left(f_{\beta\gamma}(z_\gamma) + \epsilon b_{\beta\gamma}(z_\gamma)\right), \epsilon\right) = \\
&= \left(f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma) + \epsilon \frac{df_{\alpha\beta}}{dz_\beta} b_{\beta\gamma} + \epsilon b_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)), \epsilon\right)
\end{aligned}$$

so that, if we want that the last side of the above chain of equalities coincides with $\tilde{f}_{\alpha\gamma}$ we must require:

$$b_{\alpha\gamma} = b_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) + \frac{df_{\alpha\beta}}{dz_\beta} b_{\beta\gamma},$$

which means, by considering the natural basis $\frac{d}{dz_\alpha}$ of the holomorphic vector fields on U_α (for each $\alpha \in \mathcal{A}$), that:

$$b_{\alpha\gamma}\frac{d}{dz_\alpha} = b_{\alpha\beta}(z_\beta)\frac{d}{dz_\alpha} + b_{\beta\gamma}\frac{d}{dz_\beta}.$$

In other words, putting $\theta_{\alpha\beta} = b_{\alpha\beta}\frac{d}{dz_\alpha}$ we have that

$$\theta_{\alpha\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma},$$

i.e. we have $\{\theta_{\alpha\beta}\} \in Z^1(\mathcal{U}, \Theta_C)$ where Θ_C is the tangent sheaf of the curve C, i.e. the dual of the canonical bundle K_C. We have hence proven that an infinitesimal deformation induces a cocycle with values in the sheaf Θ_C. It is left as an exercise for the reader to check that if two infinitesimal deformation are equivalent, then the two cocycle gotten from the deformations are co-homologous. Hence the isomorphism classes of infinitesimal deformations of C map injectively into $H^1(C, \Theta_C)$. But any $t \in H^1(C, \Theta_C)$, is represented, on some open covering of C, by a Cĕch cocycle, which may be used to construct a deformation. Hence we have the bijection:

$$\{\text{isomorphism classes of infinitesimal deformations}\} \cong \mathrm{H}^1(\mathrm{C}, \Theta_\mathrm{C}).$$

Clearly, the above bijection induces on the infinitesimal deformations a structure of $\mathbb{C}$-vector space isomorphic to $H^1(C, \Theta_C)$. Now, by Serre duality and the Riemann-Roch formula:

$$h^1(C, \Theta_C) = h^0(C, 2K_C) = 1 - g + \int_C 2K_C = 3g - 3.$$

If $\phi : \mathfrak{X} \longrightarrow Spec(\mathbb{C}[\epsilon])$ is an infinitesimal deformation, then the associated class $[\theta_\phi] \in H^1(C, \Theta_C)$ is said to be the *Kodaira-Spencer class* associated to the deformation ϕ. Let us now consider any deformation of C, $\mathfrak{X} \longrightarrow (B, b_0)$, parametrized by a pointed scheme (B, b_0). Take any tangent vector to B at the point b_0. It is, by definition of the Zariski tangent space, a morphism $g : Spec(\mathbb{C}[\epsilon]) \longrightarrow B$, such that $g(\epsilon) = b_0$. The pull-back of the family $\mathfrak{X}$ to $Spec(\mathbb{C}[\epsilon])$ gives rise to an infinitesimal deformation $g^*\mathfrak{X} \longrightarrow Spec(\mathbb{C}[\epsilon])$. Hence we have a natural vector-space homomorphism:

$$\rho : T_{b_0}B \longrightarrow H^1(C, \Theta_C),$$

called the *Kodaira-Spencer homomorphism.*

Summing up, what we saw in this section is that a smooth curve of genus g may be "infinitesimally moved" in infinitely many ways, as many as the points of a $(3g-3)$-dimensional $\mathbb{C}$-vector space. The idea would be now to *integrate* such a *distribution of vector spaces* to get a (possibly open) $3g-3$ complex variety, parametrizing isomorphism classes of smooth curves in a *neighbourhood* of C. This actually may be done by using, e.g., the so-called *Schiffer variations* (see [5]) but we do not enter in this subject which is largely beyond the goals of these notes. Rather, we shall state, in the next subsection, the existence of some *local universal moduli spaces* around any stable curve C, called its *Kuranishi family.*

3.2.2 Kuranishi families for stable curves

Let C be a complex stable curve of (arithmetic) genus $g \geq 2$. Let $(\pi, p) : \mathfrak{X} \longrightarrow (B, b_0)$ be a deformation of C, $p : \pi^{-1}(b_0) \longrightarrow C$ being the isomorphism which identifies the distinguished fiber with C.

Definition 3.3 *We say that a deformation $(\pi, p) : \mathfrak{X} \longrightarrow (B, b_0)$ of C, parametrized by the pointed scheme (B, b_0), is a* Kuranishi family *for C if the deformation (π, p) satisfies the* local universal property *below. For each deformation $\phi : \mathcal{Y} \longrightarrow (S, s_0)$ of C, there exists a neighbourhood (in the Zariski or in the complex topology according to the category one is working in) U of s_0 and a unique morphism $f_U : (U, s_0) \longrightarrow (B, b_0)$ such that $\mathcal{Y}_{\phi^{-1}(U)} = f_U^* \mathfrak{X}$.*

One may prove the fundamental theorem of deformation theory that, along with the classical Geometric Invariant Theory by Mumford [64], is the corner stone for the construction of the moduli space of stable curves of genus g. It states the existence of a Kuranishi family around each stable curve C and gives a recipe to construct it. For the details, which are not in the main goals of these notes, we refer to [5]. The statement below is almost literally copied from [5]. We refer to that book for the proof.

Theorem 3.1 *Let $\nu \geq 3$ be an integer. Let $C \subset \mathbb{P}^r$ be a stable curve of genus g, embedded by the global sections of the sheaf $\omega_C^{\otimes \nu}$ (ω_C is the* dualizing sheaf *of C). Hence $r = (2\nu - 1)(g-1) - 1$ (by Riemann-Roch). Let $b_0 \in Hilb^r_{2\nu(g-1),g}$ be the corresponding point of the Hilbert scheme $Hilb^r_{2\nu(g-1),g}$ (Cf. ex. 2.3), and let $Aut(C) = G_{b_0} \subset PGL_{r+1}(\mathbb{C})$ be the stabilizer of b_0. Then there is a locally closed $(3g-3)$-dimensional smooth subscheme B of $Hilb^r_{2\nu(g-1),g}$ passing through b_0 such that the restriction to B of the universal family over $Hilb^r_{2\nu(g-1),g}$ is a Kuranishi family* *for all of its fibers and hence, in particular, a Kuranishi family for C. In addition one can choose B with the following properties:*

1. *B is affine;*
2. *B is G_{b_0}-invariant;*
3. *for every $b \in B$, the stabilizer G_b of b is contained in G_{b_0};*
4. *there is a G_{b_0}-invariant neighbourhood U of b_0 in B, for the analytic topology, such that $\{\gamma \in G : \gamma U \cap U \neq \emptyset\} = G_{b_0}$.*

3.2.3 Sketch of the construction of $\overline{M}_g$ via Kuranishi families.

The purpose of this subsection is to sketch the construction of the moduli space $\overline{M}_g$ by means of Kuranishi families. Of course there is no room to go into details, but at least we may achieve a quite important goal, namely to give the feeling that $\overline{M}_g$ is a quite complicated object. Moreover, using another result which shall be assumed without proof, we may explain why $\overline{M}_g$ cannot be a fine moduli space. Also, it will be clear that the obstruction arises because of the existence of curves with nontrivial automorphism group. Even in this subsection we shall mimic the exposition of [5].

Recall now that as far as we know $\overline{M}_g$ is only a set, namely the set of all *isomorphism classes of stable curves* of genus g ($g \geq 2$). Let C be a curve representing $[C] \in \overline{M}_g$. Then, by theorem 3.1, there exists a Kuranishi family $\pi : \mathfrak{X} \longrightarrow (B, b_0)$ for C (of course we have also the identification map $p : C \longrightarrow \pi^{-1}(x)$), where B may be taken as a locally closed $3g-3$ dimensional subscheme of the Hilbert scheme $H := Hilb^r_{(2\nu-1)(g-1)-1,g}$ of the curves of genus g embedded ν canonically in $\mathbb{P}^r$ (with $r = (g-1)(2\nu-1)-1$). Of course $G = PGL_{r+1}(\mathbb{C})$ acts on (B, b_0) in an obvious way: if $g \in G$, and $b \in B$, $g \cdot b = \{\text{the action of } g \text{ on } \pi^{-1}(b) \subset \mathbb{P}^r\}$. The *isotropy subgroup* (or the *stabilizer*), G_b, of each point $b \in B$ (i.e. the subgroup of G which acts trivially on b) is contained in G_{b_0} and we know by property 2 of Thm. 3.1, that B is G_{b_0} invariant. Notice that G_{b_0} may be identified with the group $Aut(C)$ of the automorphisms of C. This means that there is a natural map of sets:

$$\psi : B/G_{b_0} \longrightarrow \overline{M}_g,$$

defined as: $\psi(b) = [\pi^{-1}(b)]$. Of course it may well happen that given $[C_1] \in \overline{M}_g$, C_1 may have more than one preimage in B/G_{b_0}. But using Thm. 3.1, property 4, there exists a G_{b_0}-invariant open subset of b_0 (in the analytic topology) such that $\{\gamma \in G / \gamma U \cap U \neq \emptyset\} = G_{b_0}$. This means that the map:

$$\psi_U := \psi_{|U/G} : U/G_{b_0} \longrightarrow \overline{M}_g,$$

is injective. In fact, let $b_1, b_2 \in U$ such that $\psi(b_1) = \psi(b_2)$. Then $[\pi^{-1}(b_1)] = [\pi^{-1}(b_2)]$. This means that b_1 and b_2 belong to the same orbit of G, i.e., that there exists $\gamma \in G$ such that $b_2 = \gamma \cdot b_1$, so that $\gamma U \cap U \neq \emptyset$. But then, by virtue of Thm. 3.1,4, $\gamma \in G_{b_0}$, so that b_1 and b_2 give rise to the same equivalence class in U/G_{b_0}, proving the claimed injectivity.

This easy remark tells us something very important: now we know that the moduli space $\overline{M}_g$ locally looks like the quotient of a smooth variety by the action of a finite group. What we want to do now is to cover $\overline{M}_g$ by means of *charts* $(U_\alpha, \psi_a : U_\alpha \longrightarrow \overline{M}_g)$ and to check their compatibility in the category of the analytic spaces. We shall remark here that given the Kuranishi family $\mathfrak{X} \longrightarrow (B, b_0)$ around C, then B/G_{b_0} may have singularities; however, it is a normal affine variety. This follows from a not very difficult algebraic fact, quoted below for the reader's convenience, copied from [5], where we refer the reader for the proof.

Lemma 3.1 *Let Spec(A) be a normal affine variety acted on algebraically by a finite group Γ. Then the ring A^Γ of γ-invariant elements in A is an integrally closed finitely generated $\mathbb{C}$-algebra. Moreover, if X is the set of closed points of Spec(A), then the set of closed points of Spec(A^Γ) can be identified with X/Γ.*

Now, let us suppose that $(V, \psi_V : V/G_{c_0} \longrightarrow \overline{M}_g)$ is another *local chart* such that:

$$A_{UV} = \psi_U(U/G_{b_0}) \cap \psi_V(V/G_{c_0}) \neq \emptyset$$

Now $\psi_U^{-1}(A_{UV})$ and $\psi_V^{-1}(A_{UV})$ are open sets in U/G_{b_0} and in V/G_{c_0} respectively. In fact let b be a point, say, of $\psi_U^{-1}(A_{UV})$ and let $\tilde{b}$ be a preimage of b in U. Recall now that U is a Kuranishi family (see Thm. 3.1) for each of its points. The same as for the pre-image $\tilde{A}_U$ of $\psi_U^{-1}(A_{UV})$ in U with respect to the quotient induced by the action of G_{b_0}. Hence there exists a $G_{\tilde{b}}$ invariant neighbourhood $W \subseteq \tilde{A}_U$ of $\tilde{b}$, such that $W/G_{\tilde{b}}$ is an open set of A_{UV} containing b, by definition of the quotient topology. Of course here we have used the fact, stated in property 3 of Thm. 3.1, that $G_{\tilde{b}} \subseteq G_{\tilde{b}_0}$. Hence we now have to look at the transition functions:

$$T_{UV} = \psi_U^{-1} \circ \psi_V : \psi_V^{-1}(A_{UV}) \longrightarrow \psi_U^{-1}(A_{UV}),$$

by inspecting the commutative diagram:

$$\begin{array}{ccc} \tilde{A}_V & \xrightarrow{\gamma_{VU}} & \tilde{A}_U \\ \alpha \downarrow & & \downarrow \beta \\ \psi_V^{-1}(A_{UV}) & \xrightarrow{\psi_U^{-1}\circ\psi_V} & \psi_U^{-1}(A_{UV}) \\ & \searrow \quad \swarrow & \\ & \overline{M}_g & \end{array} \tag{3.1}$$

Now α and β are finite and holomorphic, and γ_{VU} is holomorphic. Hence $\psi_U^{-1} \circ \psi_V$ is certainly holomorphic off of the branch locus of α. But now, $\psi_V^{-1}(A_{UV}) \subseteq V/G_{c_0}$ which is normal, and hence it is normal, and its image through $\psi_U^{-1} \circ \psi_V$ lands in U/G_{b_0}, which wlog[1] may be thought of as a bounded analytic subset of $\mathbb{C}^N$ for some N. These are the needed hypotheses to apply the *Riemann extension theorem*, which ensures us that $\psi_U^{-1} \circ \psi_V$ may be extended holomorphically to all $\psi_V^{-1}(A_{UV})$.

Hence we have equipped $\overline{M}_g$ with a structure of a normal analytic space. The construction tells us that $\overline{M}_g$ cannot be a fine moduli space for the problem of classifying all the isomorphism classes of stable curves. This result is a consequence of a fact which is far from being trivial and whose proof can be read, e.g., in [5].

Lemma 3.2 *Let C be a stable curve and let $\pi : \mathfrak{X} \longrightarrow (B, b_0)$ a Kuranishi family for it. Suppose that $Aut(C)$ is non trivial. Then $Aut(C)$ acts non-trivially on B.*

In other words, if $g \in Aut(C)$ then there exists at least one $b \in B$ such that $g \cdot b \neq b$. Using lemma 3.2, we may prove the following theorem:

Theorem 3.2 *The analytic space $\overline{M}_g$ constructed above, parametrizing stable curves of genus $g \geq 2$ is not a fine moduli space.*

Proof.

Let C be a stable curve such that $Aut(C)$ is non trivial and assume, by contradiction, that $\overline{M}_g$ is a fine moduli space, so that there exists a universal family $\pi : \overline{\mathcal{C}}_g \longrightarrow \overline{M}_g$. Then, in particular, the universal family $\overline{\mathcal{C}}_g \longrightarrow \overline{M}_g$ would be a Kuranishi family for C. But this is impossible, because on one hand $Aut(C)$ should act non-trivially on $\overline{M}_g$ (lemma 3.2) while, on the other hand, $\overline{M}_g$ is already the set of stable curves modulo isomorphisms (and hence $Aut(C)$ should act trivially on it).[2]

QED

[1] see footnote 1, Sect. 1.2.1

[2] I am grateful to Joaquim Kock for suggesting me this argument which is much easier than the one shown in a previous form of these notes.

3.3 Appendix - An Example in the Trend: the Kontsevich Moduli Space of Stable Maps

3.3.1 Introduction

In this appendix we want to give another example of coarse moduli space, without proving that it is so (for details see [31]). This example is quite important in the actual trends of geometry, because it sets up a rigorous framework for some enumerative questions to be treated with techniques borrowed from *quantum cohomology*.

3.3.2 An informal discussion

Let $\mathbb{P}^2$ be the complex projective plane. Choose in $\mathbb{P}^2$ α points and β lines in *general position*, whatever it means, such that $\alpha+\beta=3d-1$. Look for all plane rational nodal irreducible curves of degree d passing through the assigned α points and tangent to the assigned β lines. It turns out that the set of such rational curves is non-empty and its cardinality is finite. The numbers $N_{\alpha,\beta}$ $(\alpha+\beta=3d-1)$ will be said the characteristic numbers of the plane rational curves of degree d. If $\beta=0$ (no tangency condition imposed) $N_{3d-1,0}$ will be simply denoted by N_d, and by the associativity law in quantum cohomology [51] it can be proven that the N_d's satisfy the recursive relations:

$$N_1=1,$$

and, for $d\geq 2$,

$$N_d=\sum_{\substack{d_A+d_B=d\\ d_A\geq 1, d_B\geq 1}} N_{d_A}N_{d_B}\left(d_A^2d_B^2\binom{3d-4}{3d_A-2}-d_A^3d_B\binom{3d-4}{3d_A-1}\right).$$

This recursive relation can be obtained via intersection theory on the Kontsevich moduli space of stable maps on rational pointed curves $\overline{M}_{0,n}(r,d)$. The aim of this appendix is to provide a friendly and informal introduction to the moduli space of stable maps of pointed rational curves in $\mathbb{P}^r$, just to give the reader the flavour of the new exciting directions which intersection theory is moving to.

3.3.3 Some elementary plane geometry

The feature of this subsection is very informal and so I do not want to start by axiomatically describing all the objects I need for the purpose of computing characteristic numbers. Rather, the aim is to start from the bottom, showing that elementary and concrete enumerative problems cry for the definition of the moduli space of stable maps. To achieve this goal, let us consider a very simple question: *how many plane cubics do pass through* 8 *points in general position* [3]? Everybody knows that there are infinitely many such cubics and, actually, that such curves are parametrized by a line in $\mathbb{P}^9$. Such a line is gotten by intersecting the 8 hyperplanes corresponding to cubics passing through any of the specified 8 points. The idea is now to intersect such a line with a hypersurface not containing the line to get a finite number of cubics which, beside passing through the 8 points, enjoy some other properties depending on the hypersurface we are intersecting with. For instance: choose a point P_9 in general position with respect to $P_1, \ldots, P_8$ and look for all the cubics passing through those 9 points. One finds exactly 1 cubic. We are in fact intersecting our line in $\mathbb{P}^9$ with a hypersurface (the hyperplane of plane cubics passing through P_9). But we may like to impose some non linear condition, e.g. "*how many rational cubics pass through* 8 *points in general position?*". It is not difficult to realize that the space of rational cubics is represented by a hypersurface in $\mathbb{P}^9$. To see why, it suffices to notice that, after all, a rational cubic is the image of a $\mathbb{P}^1$ in $\mathbb{P}^2$, under a morphism of degree 3, $\mu : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$, that can be expressed as:

$$\begin{aligned}[u, v] \mapsto \quad & [a_1u^3 + a_2u^2v + a_3uv^2 + a_4v^3, b_1u^3 + b_2u^2v + b_3uv^2 + b_4v^3, \\ & c_1u^3 + c_2u^2v + c_3uv^2 + c_4v^3].\end{aligned}$$

At least one coefficient must be $\neq 0$ and moding out by the automorphisms of $\mathbb{P}^1$, we see that our maps are parametrized by an eight dimensional space. Let us incidentally notice that since the general rational cubic is nodal, the general line in $\mathbb{P}^9$ will intersect the locus of rational cubics in its open set of irreducible nodal ones. As we have seen, we translated the problem of "counting" rational cubics in a plane into a problem of *counting maps* from $\mathbb{P}^1$ to $\mathbb{P}^2$ of degree 3. We may then restate our enumerative problem in another way: let $M_{0,8}$ be the moduli space of eight-pointed smooth rational curves, parametrizing the *configurations* of 8 distinct points in $\mathbb{P}^1$. Clearly, due to the 3-dimensional automorphism group of $\mathbb{P}^1$ acting on such configurations, one has

[3] Whatever that means.

$dim(M_{0,8}) = 5$. Fix $Q_1, \ldots, Q_8$ general points in $\mathbb{P}^2$ and look for all the maps:

$$\mu : (\mathbb{P}^1, P_1, \ldots, P_8) \longrightarrow \mathbb{P}^2,$$

such that $deg(\mu) = 3$ and $\mu(P_i) = Q_i$. This is a subset of the space $M_{0,8}(2,3)$ of all the maps from the eight labelled $\mathbb{P}^1$'s to $\mathbb{P}^2$ of degree 3. Here the degree has to be meant in the sense that the image is equivalent to three times the class of a line in the codimension 1 Chow group of $\mathbb{P}^2$. Notice that $dim(M_{0,8}(2,3) = 6+2+3+8-3 = 16$, which is exactly the dimension of the parameter space of 8 distinct points in the plane. If $Q_i = (x_i, y_i)$ (in an affine chart), then $(\mathbb{P}^1, P_1, \ldots, P_8; \mu)$ is such that $\mu(P_i) = Q_i$ if and only if $\mu(P_i)$ belongs to the lines $L_{x_i} : x_i = 0$ and $L_{y_i} : y_i = 0$, i.e. if and only if $P_i \in \mu^{-1} L_{x_i} \cap \mu^{-1} L_{y_i}$. Such a condition defines a 2-codimensional cycle which could be intersected with the other codimensional cycles defined by analogous conditions. In the full space $M_{0,8}(2,3)$ we are hence led to consider some very natural line bundles: roughly speaking, define maps

$$\nu_i : M_{0,8}(2,3) \longrightarrow \mathbb{P}^2,$$

as follows: if $\xi = (\mathbb{P}^1, P_1, \ldots, P_8, \mu) \in M_{0,8}(2,3)$, $\nu_i(\xi) = \mu(P_i)$. Define line bundles L_i in $M_{0,8}(2,3)$ as:

$$L_i = \nu_i^* O_{\mathbb{P}^2}(1)$$

and notice that $c_1(L_i)^2$ represents the two codimensional locus of stable maps such that $\mu(P_i) = Q_i$. The cup product:

$$c_1(L_1)^2 \cup \ldots \cup c_1(L_8)^2,$$

is a cohomology class in top codimension, which may be evaluated on the fundamental class of $M_{0,8}(2,3)$, hopefully giving the desired answer to the enumerative question we asked ourselves.

3.3.4 The moduli space of stable maps

The time has come to make precise the last part of the previous heuristic discussion. Let us consider the following data:

$$(\pi : C \longrightarrow S; P_1, \ldots, P_n; \mu),$$

where:

1. π is a projective flat morphism over an algebraic scheme S [4];

2. $P_1, \ldots, P_n$ are disjoint sections ($n \geq 0$);

3. Each geometric fiber C_s together with $P_1(s), \ldots, P_n(s)$ is an n-labelled *tree of projective lines*;

4. $\mu : C \longrightarrow \mathbb{P}^r_S$ is an S-morphism.

We say that $(C \longrightarrow Spec(\mathbb{C}), \{P_i\}; \mu)$ satisfying 1, 2, 3, 4 is a *Kontsevich stable map* from C to $\mathbb{P}^r$ if $\omega_{C/S}(P_1 + \ldots + P_n) \otimes \mu^* O_{\mathbb{P}^r}(3)$ is π-relatively ample. This means that if an irreducible component of $(C \longrightarrow Spec(\mathbb{C}), \{P_i\}; \mu)$ is mapped to a point of $\mathbb{P}^r$, then that component contains at least 3 special points.

Notice that a Kontsevich stable map to $\mathbb{P}^0$ is nothing but a Deligne-Mumford stable curve.

Let $\overline{\mathcal{M}}_{0,n}(r,d) : (Sch/\mathbb{C}) \rightsquigarrow (Sets)$ be defined as

$$\overline{\mathcal{M}}_{0,n}(r,d)(S) = \{\text{isomorphism classes of Kontsevich stable maps /S}\}$$

Then the following holds:

Theorem 3.3 *There exists a projective coarse moduli space $\overline{M}_{0,n}(r,d)$, i.e. a* natural transformation of functors*:*

$$\phi : \overline{\mathcal{M}}_{0,n}(r,d) \longrightarrow Hom_{Sch/\mathbb{C}}(-; \overline{M}_{0,n}(r,d)),$$

satisfying 1 and 2 below:

1. *$\phi(\{pt\}) : \overline{\mathcal{M}}_{0,n}(r,d)(Spec(\mathbb{C})) \longrightarrow Hom_{Sch/\mathbb{C}}(Spec(\mathbb{C}); \overline{M}_{0,n}(r,d))$ is a set bijection;*

2. *If $\psi : \overline{\mathcal{M}}_{0,n}(r,d) \longrightarrow Hom_{Sch/\mathbb{C}}(-; Z)$ is another natural transformation of functors (Z being an algebraic scheme), then there exists a unique scheme homomorphism:*

$$\gamma : \overline{M}_{0,n}(r,d) \longrightarrow Z,$$

making commutative the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(r,d) & \xrightarrow{\phi} & Hom_{Sch/\mathbb{C}}(\cdot, \overline{M}_{0,n}(r,d)) \\ & \searrow^{\psi} & \downarrow \tilde{\gamma} \\ & & Hom_{Sch/\mathbb{C}}(\cdot, Z) \end{array} \tag{3.2}$$

[4] A short way for saying of finite type over $\mathbb{C}$

where $\tilde{\gamma}$ is the obvious map induced by γ.

Proof.

See [31]

Notice now that we are given of a set of natural transformation of functors:

$$\theta_i : \overline{\mathcal{M}}_{0,n}(r,d) \longrightarrow Hom(-, \mathbb{P}^r),$$

defined as follows. If S is a scheme and $(C \longrightarrow S, \{P_i\}; \mu)$ is a stable map representing an element ξ of $\overline{\mathcal{M}}_{0,n}(r,d)(S)$, then $\theta_i(S)(\xi) = \mu \circ P_i$. It follows that there exists a unique map:

$$\nu_i : \overline{M}_{0,n}(r,d) \longrightarrow \mathbb{P}^r,$$

so that the marking induces on $\overline{M}_{0,n}(r,d)$ natural line bundles $L_i = \nu_i^* O_{\mathbb{P}^r}(1)$. It is very reasonable to expect that top codimensional intersection of the cohomology classes $c_1(L_i)$ should represent solution of enumerative questions. We shall continue to analyze such an example at the end of chapter 4, after reviewing some basic intersection theory.

Chapter 4

Intersection Theory: a Quick Review

4.1 Chow Rings

4.1.1 Introduction

In this section we shall list briefly some basic definitions and properties taken from one of the most fascinating subjects of Algebraic Geometry, named *intersection theory.* Needless to say there are very big treatises and very important papers on the subject, so that such a review is not intended to be a reference, but it is only an attempt to list some results which may be used later on, when we shall perform computation on the moduli space of curves, and in order to keep these notes self contained. Clearly there is no hope to be exhaustive. No details will be fully worked out and we shall not provide proofs that the reader may easily find in very good books such as [29], or expository paper such as [48] and [49]. On the contrary, we shall try , for the time being, to heavily appeal to the geometrical intuition of the reader.

4.1.2 Chow groups and Chow rings

We start from an n-dimensional ($n \geq 1$) ambient variety X. A variety will be, throughout this chapter, an irreducible reduced $\mathbb{C}$-scheme of finite type. For the time being *a priori* no smoothness assumption on the variety X is made. Let $K(X)$ denote the function field of X. For each affine open subset U of X, $K(X) = K(U)$ is the quotient field of the integral domain $O_X(U)$, where O_X is the structural sheaf of X.

Let now V be a *Weil divisor*, that is to say a 1-codimensional subvariety of X (i.e. an irreducible, reduced closed subscheme of codimension 1 of X) and suppose that $V \cap U \neq \emptyset$. Let $a \in O_X(U)$. Such an a induces an element in the local ring $O_{X,V}$, the stalk of the structural sheaf O_X at V, that shall be still denoted by a, abusing notation. Define:

$$ord_V(a) = \ell_{O_{X,V}}\left(\frac{O_{X,V}}{(a)}\right),$$

where ℓ denotes the *lenght* of a module. The positive integer $ord_V(a)$ is said to be the *order* of the element a along V. Let us suppose then that $r \in K(X)^*$ and let V be any 1-codimensional subvariety of X. If U is an affine open set of X such that $U \cap V \neq \emptyset$, then r admits a representation as a quotient of two elements in the domain $O_X(U)$, say $r = \dfrac{a}{b}$.

Definition 4.1 *The* order of a rational function $r = a/b$ along the subvariety V *is the integer* $ord_V(r)$ *defined as:*

$$ord_V(r) = ord_V(a) - ord_V(b).$$

By working a little bit on Definition 4.1 one may prove that such a function is well defined (e.g. it does not depend by the representative a, b of r) and that it defines a group homomorphism:

$$ord_V : K(X)^* \longrightarrow \mathbb{Z}.$$

Let now W be a k-dimensional subvariety of X (hence an irreducible scheme). To such a subvariety we associate a symbol, $[W]$, which shall be called a *prime k-dimensional cycle.* We may hence give the following:

Definition 4.2 *The* group of k-cycles *of* X, *denoted by* $Z_k(X)$ *is, by definition, the* $\mathbb{Z}$*-module freely generated by all the prime k-cycles.*

Hence, by definition, a cycle c is nothing but a formal finite $\mathbb{Z}$-linear combination in the symbols $[V]$, V running over the set of all k-dimensional subvarieties. We shall often write:

$$c = \sum_V n_V[V],$$

where $n_V \in \mathbb{Z}$.

To each rational function $r \in K(X)^*$ one may associate a $(n-1)$-cycle, $div(r)$, defined as:

$$div(r) = \sum_V ord_V(r)[V].$$

The above sum is meaningful because $\{[V] \in Z_{n-1}(X) : ord_V(r) \neq 0\}$ is a finite set (see [29], Appendix B.4.3.). Suppose now that $f : X \longrightarrow Y$ is a proper morphism of $\mathbb{C}$-schemes. Then there is an induced $\mathbb{Z}$-module homomorphism:

$$f_* : Z_k(X) \longrightarrow Z_k(Y),$$

defined as follows. If $[V]$ is a generator of $Z_k(X)$, then:

$$f_*[V] = \begin{cases} deg(f_{|V}) \cdot [f(V)] & \text{if } dim(V) = dim(f(V)) \\ 0 & \text{if } dim(f(V)) < dim(V) \end{cases},$$

where by $deg(f_{|V})$ we mean the degree of the algebraic field extension $[K(V) : K(f(V))]$. Moreover, if $f : X \longrightarrow Y$ is a flat morphism of relative dimension m, one may define a *pull-back* map f^* at the level of cycles by:

$$\begin{aligned} f^* \ : \ & Z_k(Y) \longrightarrow Z_{k+m}(X) \\ & [V] \mapsto f^*([V]) = [f^{-1}(V)]. \end{aligned}$$

We may now define the *Chow groups* of X as follows. Suppose that α is a k-cycle.

Definition 4.3 *The k-cycle α is said to be* rationally equivalent *to 0, and one writes $\alpha \sim 0$, if and only if there exists finitely many $(k+1)$-dimensional subvarieties $V_1, V_2, \dots, V_t$ and rational functions $r_i \in K(V_i)^*$ such that:*

$$\alpha = \sum_i div(r_i).$$

One says that two k-cycles α_1 and α_2 are rationally equivalent iff

$$\alpha_1 - \alpha_2 \sim 0,$$

where the "difference" in the left hand side is taken in $Z_k(X)$.

Let us denote, following [29], by $Rat_k(X)$ the set of all the k-cycles rationally equivalent to 0. As it may be easily checked, $Rat_k(X)$ is a $\mathbb{Z}$-submodule of $Z_k(X)$, so that $\alpha_1 \sim \alpha_2 \iff \alpha_1 - \alpha_2 \in Rat_k(X)$ is an equivalence relation. Taking the quotient, we set:

$$A_k(X) := \frac{Z_k(X)}{Rat_k(X)}.$$

The $\mathbb{Z}$-module $A_k(X)$ is said to be the *Chow group* of the k-cycles modulo rational equivalence. The graded $\mathbb{Z}$-module:

$$A_*(X) = \bigoplus_{i=0}^{dim(X)} A_i(X),$$

is said to be the *Chow group* of the variety X. Now, let $f : X \longrightarrow Y$ be a proper morphism. If $\alpha \in Rat_k(X)$, then $f_*(\alpha) \in Rat_k(Y)$, while if f is flat of relative dimension m and $\alpha \in Rat_k(Y)$, then $f^*\alpha \in Rat_{k+m}(X)$. This means that f_* and f^* define two homomorphisms of graded $\mathbb{Z}$-modules:

$$f_* : A_*(X) \longrightarrow A_*(Y) \quad \text{and} \tag{4.1}$$

$$f^* : A_*(Y) \longrightarrow A_*(X). \tag{4.2}$$

Such a fact is proven in [29], Ch. 1. The homomorphisms (4.1) and (4.2) shall be called respectively the *proper push-forward* (under the assumption that f is proper) and the *flat pull-back* (under the assumption that f is flat). From now on, if V is a k-dimensional subvariety, by the symbol $[V]$ we shall intend its Chow class, i.e. the k-cycle determined by it, *modulo rational equivalence.*

Let now W be a k-dimensional scheme. Let W_i be its irreducible components. Then the local rings O_{W,W_i} are *artinian local rings* [7]. Define the *multiplicity* as the O_{W,W_i}-lenght of the module O_{W,W_i} itself. Set:

$$m_i = \ell_{O_{W,W_i}}(O_{W,W_i}).$$

Definition 4.4 *The* fundamental class *of a k-dimensional subscheme is defined to be the Chow class:*

$$[W] = \sum_i m_i[W_i] \in A_k(X).$$

Notice that if W is irreducible and reduced then the fundamental class of W is simply the rational equivalence class of the cycle $[W]$.

Another very important definition, which we already used more than once, without setting it, is that of the *degree of a cycle.*

Definition 4.5 *Let X be a proper scheme of finite type over $Spec(\mathbf{K})$, $\mathbf{K}$ being any field. The* degree homomorphism*:*

$$\int_X : A_*(X) \longrightarrow \mathbb{Z},$$

is defined as:

$$\int_X \alpha = \begin{cases} \sum_P n_P[k(P):\mathrm{K}] & if \quad \alpha = n_P[P] \in A_0(X) \\ 0 & if \quad \alpha \in A_i(X), \quad i > 0. \end{cases}$$

In the above formula $[k(P):\mathrm{K}]$ denotes the degree of the algebraic extension of K by the field of the point P, $k(P) = m_P/m_P^2$, m_P being the maximal ideal of the local ring O_P. Clearly, if we take (as in most part of these notes) K to be algebraically closed, $[k(P):\mathrm{K}] = 1$, for all closed points.

If $f : X \longrightarrow Y$ is a morphism of schemes proper over $Spec(\mathrm{K})$, then:

$$\int_X \alpha = \int_Y f_*(\alpha).$$

4.1.3 The Chow Ring of a smooth variety.

In this section we shall make an additional hypothesis on our ambient variety X, namely that it is smooth. Set, by definition, $A^k(X) = A_{n-k}(X)$. Then, we may consider the Chow group $A_*(X)$ graded by the codimension instead of the dimension. As a notation set:

$$A^*(X) = \bigoplus_{i=0}^{n} A^k(X) \tag{4.3}$$

where, of course, $A^k(X)$ means the $\mathbb{Z}$-module freely generated by all the *prime cycles* of codimension k modulo rational equivalence. If X is a smooth variety, its Chow group has an extra nice *intersection product* which make $A^*(X)$ into a (graded) commutative ring, which is said to be the *Chow ring* of the variety X. From a formal point of view, if $\alpha \in A^i(X)$ and $\alpha' \in A^j(X)$, then $\alpha \cdot \alpha' \in A^{i+j}(X)$. Morally this is what should geometrically happen. Suppose that V and W are two subvarieties of X of respective codimension i and j. Suppose that they intersect *transversally*. To the last sentence one may give a precise meaning, but morally it means that, in an affine open set, their intersection is described by means of the zero-scheme of an ideal having $i + j$ independent generators, which may be extended to a regular system of parameters. With these hypothesis, let $V \cap W$ be the scheme theoretical intersection of V and W. Set

$$[V] \cdot [W] = [V \cap W],$$

where the right hand side means the fundamental class of the scheme $V \cap W$ in the Chow group $A^{i+j}(X)$. Of course it may well happen that V and W do not intersect transversally. In this case the intersection product may still be defined: to this purpose look at [29], p. 141, and it coincides with the usual one when V and W do intersect transversally. In some case one may avoid additional definitions.

For instance if X is a smooth surface, then one may compute the product of divisors $[V]\cdot[W]$, even if V and W do not cut transversally each other, essentially using only the notion of transversality. More precisely, if V and W do cut transversally, $[V]\cdot[W]$ is nothing but the number of points of intersections of V and W times the class of a point. If V and W are not transversal then, as explained in [42], p. 359, one can find divisors V_1, V_2, W_1, W_2 such that they are mutually transversal, $V \sim V_1 - V_2$, $W = W_1 - W_2$ ($\sim$ means linear equivelence and, hence, rational equivalence). One then define:

$$[V]\cdot[W] = ([V_1]-[V_2])\cdot([W_1]-[W_2]),$$

estending by $\mathbb{Z}$-linearity. The definition above does not depend on the choice of V_1, V_2, W_1, W_2, but only on the linear equivalence class of V and W. For more details see [42], p. 358–359.

The above discussion is quite heuristic and not much mathematical, but we do not need to be too much formal here. For future applications it is better to try to understand the geometrical meaning of the intersection product in the Chow ring.

The Chow ring is attached to each smooth variety in a functorial way, i.e. if $f : X \longrightarrow Y$ is a proper map and $g : X \longrightarrow Y$ is a map of relative dimension m one has two $\mathbb{Z}$-module homomorphisms:

$$f_* : A^*(X) \longrightarrow A^*(Y),$$

the *proper push-forward* and:

$$g^* : A^*(Y) \longrightarrow A^*(X),$$

the *pull-back.*

If X is not smooth and the map g is flat, then the pull-back of cycle modulo rational equivalence is still defined and is known as *flat pull-back.*

In these notes we shall frequently need the *projection formula:*

Proposition 4.1 The projection formula. *Let $f : X \longrightarrow Y$ be a proper morphism between smooth varieties. Then the* projection formula *holds:*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta, \tag{4.4}$$

for any $\alpha \in A^*(Y)$ and any $\beta \in A^*(X)$.

4.1.4 Chow Rings of varieties with nice singularities

If X is not smooth it is not possible in general to define an intersection product on $A^*(X)$. By the way there are some cases where if the singularities are not so bad there may be found a nice substitute for the intersection ring. However, there is a mild price to pay, that is that we need to extend the coefficient ring $\mathbb{Z}$ to the field $\mathbb{Q}$ of the rational numbers. This is the case in the situation we shall encounter in studying the intersection theory on the moduli space of curves, namely the one where X is *globally* the quotient of a smooth variety $\tilde{X}$ modulo the action of a finite group G. In other words $X = \tilde{X}/G$. The purpose is then the following. We are given of the Chow group of X:

$$A_*(X) = \bigoplus_{i=1}^{n} A_i(X), \tag{4.5}$$

and we shall define a *product*, called again *intersection product*, on the $\mathbb{Q}$-vector space:

$$A_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \left[\bigoplus_{i=1}^{n} A_i(X)\right] \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We proceed as follows. By construction, X is the quotient of $\tilde{X}$ by G. On $A_*(\tilde{X})$ we have the usual intersection product for smooth varieties. Notice now that the action of G on $\tilde{X}$ induces an action of G on the group $A_*(\tilde{X})$ itself. Indeed, if $g \in G$ and $L_g : \tilde{X} \longrightarrow \tilde{X}$ is the *left translation* by $g \in G$ on $\tilde{X}$, which is, indeed, an automorphism of $\tilde{X}$, then, for each prime cycle $[V]$ in $A_*(\tilde{X})$, we may define the *translated cycle* $(L_g)_*([V])$, where $(L_g)_*$ is nothing but the proper push-forward from $A_*(\tilde{X})$ into itself. Notice that, because L_g is an isomorphism, it follows that $deg(L_g) = 1$, so that we have $(L_g)_*([V]) = [L_g(V)]$. It is a trivial matter to prove that the group G acts on $A_*(\tilde{X})$ via the push-forward of the left translations. In particular it is meaningful to consider in $A_*(\tilde{X})$ the subgroup $A_*(\tilde{X})^G$ of the G-invariant cycles classes of $A_*(\tilde{X})$, but we shall not need that.

We are now in position to define a product structure on $A_*(X) \otimes \mathbb{Q}$, making it into a ring.

Let $p : \tilde{X} \longrightarrow X$ be the canonical projection of $\tilde{X}$ onto X and let α, β be two arbitrary cycles in $A_*(X)$. Set:

$$\alpha \cdot \beta := \frac{1}{\sharp(G)} \left(p_*[p^*(\alpha) \cdot p^*(\beta)]\right). \tag{4.6}$$

As one may see, in general, the right hand side does not land in $A_*(X)$ but in some rational number times a class in $A_*(X)$. This has to do with the fact that we are dividing out (but what else may we do? We are just using the definition of the proper push-forward) by the order of the finite group G. Of course, the product defined in (4.6) can be extended by $\mathbb{Q}$-linearity , so that it works for an arbitrary pair of linear combination of $\mathbb{Q}$-cycles.

It is easy, using the fact that $A_*(\tilde{X}) \cong A^*(\tilde{X})$ is a ring, to see that $A_*(X) \otimes \mathbb{Q}$ is a ring with the product defined by (4.6). We shall index it by codimension, writing $A^*(X)$ and omitting in the notation the fact that our coefficients are taken in $\mathbb{Q}$. It will be, by definition, the *intersection ring* of a variety which is globally the quotient of a smooth variety by the action of a finite group. One may check that if X can be viewed in two different ways as the quotient of a smooth variety modulo the action of a finite group, then the product defined on $A_*(X) \otimes \mathbb{Q}$ does not change.

Fortunately enough, as we shall see in the next chapter, M_g and $\tilde{M}_g$ are two nice examples of such varieties.

4.2 Chern Classes

Let X be a variety and let $L \in Pic(X)$, i.e. L is a line bundle on X or, equivalently, an invertible sheaf of O_X-modules. Let $\sigma \in H^0(X, L)$ be a holomorphic section of L. Let $Z(\sigma)$ be its zero scheme. Either $Z(\sigma)$ is empty, or it is a 1-codimensional subscheme said to be a *Cartier Divisor*. Its class in $A^1(X)$, $[Z(\sigma)]$, does not depend on the section chosen and it will be denoted by the symbol $c_1(L)$, said to be the *first Chern class* of the line bundle L (if $Z(\sigma) = \emptyset$, such a class is 0, which also means that the line bundle is trivial, possessing a never vanishing holomorphic section). Hence, $c_1(L)$ represents the *class of linear equivalence* of the Cartier divisor $Z(\sigma)$ defined by any $\sigma \in H^0(C, L)$. This procedure may be generalized. Let E be a holomorphic vector bundle of rank r over a n-dimensional scheme X. Let σ be a non-zero holomorphic section of E. Suppose that on an affine open subset of X, σ is represented by the r-tuple of holomorphic functions:

$$\sigma_{|U} \cong (s_1, \ldots, s_r).$$

The zero scheme of σ, $Z(\sigma)$, coincides, on the affine U, with the common zero-locus of the functions f_i's. Suppose that either $Z(\sigma) = \emptyset$ or the f_i's form a *regular sequence* in the ring $O_X(U)$. In this case we say that σ is a *regular section*. Hence, the locus is either empty or its *expected codimension* is r (we have r equations!). When the zero locus is not empty, saying that the section is regular in the above sense, means that the expected codimension coincide with the *actual codimension* of the locus $Z(\sigma)$.

Therefore, in this situation, $Z(\sigma)$ is a $(n-r)$-dimensional closed subscheme giving rise to a cycle $[Z(\sigma)]$, modulo rational equivalence, in $A^r(X)$. Such a class $[Z(\sigma)]$ is said to be the r-th Chern class, $c_r(E)$, of the bundle E, and it is defined to be O if $Z(\sigma)=\emptyset$. In general, to each vector bundle of rank r over a scheme X, it is possible to associate an element of $A^i(X)$ for each $0 \leq i \leq n = dim(X)$, $c_i(E)$, said to be the i-th Chern class of the bundle E. We list below the basic properties of the Chern classes of a vector bundle, whose geometrical meaning has been explained at least for the *top* one, i.e. for $c_r(E)$ when $rk(E)=r$.

Properties of Chern classes

1. $c_i(E) \in A^i(X)$;

2. If E is a trivial vector bundle, then $c_i(E)=0$, for each $i>0$.

3. $c_i(E)=0$ if $i > dim(X)$ (this, trivially, because $A^i(X)=0$ if $i \geq dim(X)$: there are no cycles of dimension less than 0!).

   The expression $c(E)=\sum_i c_i(E)$ is often said to be the *total Chern class*. However we shall use more often an analogous substitute, the *Chern polynomial*, defined to be:

4. $$c_t(E)=\sum_{i=0}^{n} c_i(E)t^i.$$

5. $c_0(E)=1=[X]$.

6. $c_i(E)=0$ for $i>rk(E)$.

   We have also some normalization properties, which reflects the geometrical meaning we recalled above for the top Chern classes. In particular:

7. If D is a Cartier divisor on X, and $O_X(D)$ is the invertible sheaf associated to D, then:
   $$c_t(E)=1+[D]t.$$
   i.e. $c_1(E)=[D]$.

8. *Naturality* For any morphism $g : Z \longrightarrow X$ and any holomorphic vector bundle on X, one has:
$$c_t(g^*E) = g^*c_t(E).$$

9. The Chern polynomial is a homomorphism between the Grothendieck group $K^0(X)$ of locally free sheaves on X and the Chow ring $A^*(X)$. In other words, if
$$0 \longrightarrow E' \longrightarrow E \longrightarrow E" \longrightarrow 0$$
is an exact sequence of vector bundles on X, then
$$c_t(E) = c_t(E')c_t(E").$$

10. Let $E^\vee$ be the dual bundle of E, then:
$$c_i(E^\vee) = (-1)^i c_i(E).$$

11. Suppose that E is a vector bundle of rank r over X and that its Chern polynomial admits a linear factorizations like:
$$c_t(E) = (1 + a_1 t) \cdot \ldots \cdot (1 + a_r t).$$
Then the coefficients $a_i \in A^1(X)$ are said to be the *Chern roots* of E. The Chern classes may hence be expressed as the elementary symmetric polynomials in the Chern roots. For instance, $c_1(E) = a_1 + \ldots + a_r$.

Exercise 4.1 Let $E = L \oplus L$, L being a line bundle over X. Write the Chern polynomial of E and describe its Chern roots.

12. **Chern character.** Let E be a vector bundle of rank r over X. Let $a_1, \ldots, a_r$ be its Chern roots. Then the (exponential) *Chern character* is defined to be:
$$ch(E) = \sum_i exp(a_i),$$
where of course $exp(a_i)$ has to be thought of as a formal power series. The first few terms of the Chern character are given by:
$$\begin{aligned} ch(E) &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \\ &+ \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots, \end{aligned} \tag{4.7}$$

where in (4.7) c_i is a shorter way to denote $c_i(E)$. The Chern character enjoys of a nice behaviour with respect to exact sequences of vector bundles and with respect to tensor products. If:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E" \longrightarrow 0,$$

is an exact sequence, then $ch(E) = ch(E') + ch(E")$, while if $E = E' \otimes E"$, one has that $ch(E) = ch(E')ch(E")$.

13. **Todd Class.**

If E and X are as above, the *Todd class* of E is defined to be:

$$td(E) = \prod_{i=1}^{r} Q(a_i), \tag{4.8}$$

where the a_i's are the *Chern roots* of E and $Q(x)$ is the generating function of the *Bernoulli numbers*:

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k}.$$

In the above formula, B_k stands for the k-th *Bernoulli number*. The first few terms of the Todd class of E are:

$$\begin{aligned} td(E) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \\ &+ \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots \end{aligned} \tag{4.9}$$

The *Todd class* $td(X)$ of a variety X is, by definition, the Todd class of its tangent sheaf.

If $0 \longrightarrow E' \longrightarrow E \longrightarrow E" \longrightarrow 0$ is an exact sequence of vector bundles, then one has:

$$td(E) = td(E') \cdot td(E").$$

14. **Projective Bundles.**

Let $\mathbb{P}(E)$ be the bundle $Proj(S(\mathcal{E}))$, where $S(\mathcal{E})$ is the symmetric algebra associated to the locally free sheaf $\mathcal{E}$ of the sections of E, and let $p : \mathbb{P}(E) \longrightarrow X$ be the structure map and:

$$0 \longrightarrow K \longrightarrow p^*\mathbb{E} \longrightarrow O_{\mathbb{P}(E)}(1) \longrightarrow 0,$$

be the *tautological exact sequence*. Set $\xi = c_1\left(O_{\mathbb{P}(E)}(1)\right)$. One then gets:

$$\xi^r - p^*c_1(E)\xi^{r-1} + \ldots + (-1)^r p^* c_r(E) = 0.$$

15. Let L be a line bundle and E any vector bundle. Then:

$$c_t(E \otimes L) = \sum_i \binom{r-k+i}{i} c_{k-i}(E) c_1(L)^i.$$

For instance, if $rk(E) = r$, one has:

$$c_1(E \otimes L) = c_1(E) + rc_1(L).$$

16. If $\wedge^r E$ is the determinant line bundle, one has that:

$$c_1(\wedge^r E) = c_1(E).$$

These are the only properties we shall need in the following. All what should we do, in the next sections, is to learn how to get familiar with the manipulation rules of the Chern classes of bundles on the moduli space of curves.

Example 4.1 Suppose that X is a variety which is globally the quotient of a smooth variety $\tilde{X}$ modulo a finite group acting by left translations on it. Let E be a vector bundle on X. Then we may still define the Chern classes as elements of $A^*(X)$, where $A^*(X)$ is the intersection ring of X defined in the subsection 4.1.4. In other words we may associate the Chern class $c_i(E) \in A^*(X)$ by setting:

$$c_i(E) = \frac{1}{\sharp(G)} p_* c_i(p^*E). \tag{4.10}$$

Notice that $c_i(p^*E)$ are the Chern classes defined above, in this section.

4.3 Porteous' Formula

The main purpose of this section is to introduce a fundamental tool for performing computations in several problems coming from *enumerative geometry*: the *Thom-Porteous' formula*, briefly said, in the following, *Porteous' formula*. We shall not try to sketch a proof of such a formula, which may be found, e.g., in [4], p. 86-ff. Rather, we prefer to explain how to use it and what it serves for. To this purpose, recall our informal discussions about the geometrical meaning of the top Chern class of a vector bundle $\mathbf{F}$ of rank n . Suppose that X is a $\mathbb{C}$-scheme of finite type and that σ is a regular holomorphic section (in the sense that locally it is expressed by a regular sequence of holomorphic functions) of a vector bundle $\mathbf{F}$ over X. Then we said that the class $[Z(\sigma)]$ in $A^n(X)$ of the zero scheme of the section σ is the same as $c_n(\mathbf{F})$, the top Chern class of the bundle $\mathbf{F}$. The section σ canonically induces a morphism between the trivial vector bundle O_X and the vector bundle $\mathbf{F}$:

$$\begin{array}{ccc} O_X & \xrightarrow{\sigma} & \mathbf{F} \\ & {\scriptstyle pr_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array} \tag{4.11}$$

Such a σ may be locally expressed (like when we thought of it as a section) by a $(1 \times n)$-matrix $(s_1, \ldots, s_n)$ of holomorphic functions. Generically on X, the rank of such a matrix is 1 (i.e. $min(1,n)$) and hence $Z(\sigma)$ is the locus of the points P of X such that $rank_P(\sigma) \leq 0 = min(1,n) - 1$. The *expected codimension* of such a degeneracy locus is n (because we are considering the simultaneous vanishing of all the 1×1 minors of a $1 \times n$ matrix). Since the section was assumed to be regular, the *expected codimension* of the locus $Z(\sigma)$ coincides with the *actual codimension* of it. Hence we may compute the class in $A^n(X)$ of the degeneracy locus of the map σ in (4.11) as:

$$\begin{aligned} [Z(\sigma)] &= c_n(\mathbf{F} - O_X) = \\ &= \text{the degree n part of } (c_t(\mathbf{F}))(c_t(O_X))^{-1} \end{aligned}$$

where the inverse is taken in the local ring of the formal power series in t. But O_X is the trivial bundle, so that $c_t(O_X) = 1$. And we hence get:

$$[Z(\sigma)] = c_n(\mathbf{F}),$$

as (of course) we already knew. To go further let us introduce a piece of notation: Let $A[[t,t^{-1}]]$ be the ring of *formal Laurent series* with coefficients in a commutative ring A (e.g. $\mathbb{Z}$, the rational field $\mathbb{Q}$, or even the Chow ring of a smooth variety) and let $P(t) \in A[[t,t^{-1}]]$, and:

$$P(t) = \sum_{-\infty}^{+\infty} a_p t^p,$$

Define:

$$\Delta_{p,q}(P(t)) = \begin{vmatrix} a_p & a_{p+1} & \cdots & a_{p+q-1} \\ a_{p-1} & a_p & \cdots & a_{p+q-2} \\ \vdots & \cdots & \ddots & \vdots \\ a_{p-q+1} & a_{p-q} & \cdots & a_p \end{vmatrix} \in A$$

It turns out that if one thinks σ as a map of vector bundles like in (4.11), $c_n(\mathbf{F})$ is nothing but:

$$\Delta_{n,1}(c_t(\mathbf{F} - O_X),$$

which is a particular case of the Porteous' formula, which deals with the following general situation, keeping the same underlying idea. Let

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\ \phi\ } & \mathbf{G} \\ & {\scriptstyle\pi_1}\searrow \quad \swarrow{\scriptstyle\pi_2} & \\ & X & \end{array} \tag{4.12}$$

be a map of vector bundles over a Cohen-Macaulay algebraic variety, such that $rank(\mathbf{F}) = m$ and $rank(\mathbf{G}) = n$. Let $r = min(m,n)$ and set, for each $k \leq r$:

$$Z_k(\phi) = \{P \in X : rk_P\phi \leq k\},$$

For $k = r-1$ we shall abbreviate $Z_{r-1}(\phi)$ by $Z(\phi)$, consistently with the meaning of Z for morphisms induced by sections of vector bundles. We then have:

Theorem 4.1 (Porteous' Formula) *If $Z_k(\phi)$ has the expected codimension $(m-k)(n-k)$ or it is empty, then the fundamental class of $Z_k(\phi)$ in $A_*(X)$ is given by the Porteous' formula below:*

$$[Z_k(\phi)] = \Delta_{n-k,m-k}(c_t(\mathbf{G} - \mathbf{F})).$$

We shall get more than one pretest to digest Porteous' formula by applying it.

4.4 The Kontsevich Moduli Space of Stable Maps (continued)

Notation as in section 3.3. As we claimed there, top codimensional intersection of the cohomology classes $c_1(L_i)$ should represent solution of enumerative questions. We would like to feel why. Suppose $n \geq 3$. Then, for any $\xi \in \overline{\mathcal{M}}_{0,n}(r,d)(S)$ there is a map $S \longrightarrow \overline{M}_{0,n}(r,d)$ (the DM compactification of stable pointed rational curves), hence a unique $\zeta : \overline{M}_{0,n}(r,d) \longrightarrow \overline{M}_{0,n}$.

Definition 4.6 *Let H denote the class of the hyperplane in $\mathbb{P}^r$. A* tree level system of Gromov-Witten classes *for $\mathbb{P}^r$ is the family of maps:*

$$I_{0,n}(r,d) : A_*(\mathbb{P}^r)^{\otimes n} \longrightarrow A_*(\overline{M}_{0,n}),$$

defined as:

$$I_{0,n}(r,d)(H^{\alpha_1} \otimes \ldots \otimes H^{\alpha_n}) = \zeta_*(c_1(L_1)^{\alpha_1} \cap \ldots \cap c_1(L_1)^{\alpha_n})$$

The reason for this terminology is because of Witten's paper [78] where τ classes are defined and because $I_{0,n}(r,d)$ satisfy the properties in [51].

By abuse of terminology and following Fulton [31], the *Gromov-Witten invariants* for $\overline{M}_{0,n}(r,d)$ are the numbers:

$$I_{0,n}(r,d)(\alpha_1,\ldots,\alpha_n) = \int_{\overline{M}_{0,n}(r,d)} (c_1(L_1)^{\alpha_1} \cup \ldots \cup c_1(L_1)^{\alpha_n})$$

i.e. the evaluation of the cohomology class gotten by the cup product of the $c_1(L_i)$'s against the fundamental class of $\overline{M}_{0,n}(r,d)$. For notational convenience, in the following we shall denote:

$$I_{0,n}(r,d)(\alpha_1,\ldots,\alpha_n) = < L_1^{\alpha_1} \ldots L_n^{\alpha_n}, \overline{M}_{0,n}(r,d) > .$$

Clearly, by the very definition of the degree of Chow (or cohomology) class, one has that $I_{0,n}(r,d)(\alpha_1,\ldots,\alpha_n) = 0$ if $\alpha_1 + \ldots + \alpha_n \neq rd + r + d + n - 3$. A relevant property of the Gromov-Witten invariants is:

$$< L_1^{\alpha_1} \ldots L_{n-1}^{\alpha_{n-1}} L_n ; \overline{M}_{0,n}(r,d) > = d \cdot < L_1^{\alpha_1} \ldots L_{n-1}^{\alpha_{n-1}}, \overline{M}_{0,n-1}(r,d) > . \quad (4.13)$$

It is worth, at this point, to settle a remarkable geometrical meaning of the Gromov-Witten invariants.

Theorem 4.2 *Set:*

$$N_d =: I_{0,3d-1}(2,d)\underbrace{(2,\dots,2)}_{3d-1\ times} =< L_1^2 \cdot \ldots \cdot L_{3d-1}^2; \overline{M}_{0,3d-1}(2,d) > \tag{4.14}$$

Then N_d is the number of irreducible nodal rational plane curves of degree d passing through $3d-1$ general points in $\mathbb{P}^2$.

Proof.

First we show that, from a set theoretical point of view, we get exactly the locus we are searching for. Fix $3d-1$ general points in $\mathbb{P}^2$, $Q_1, \dots, Q_{3d-1}$. The point Q_i is the intersection of two lines L_i and M_i. For $\xi = (C, P_1, \dots, P_{3d-1}, \mu)$,

$$\mu(P_i) = Q_i \iff \xi \in \nu_i^* L_i \cap \nu_i^* M_i,$$

i.e. if and only if:

$$\xi \in c_1(L_i)^2 \cap \overline{M}_{0,3d-1}(2,d)$$

Hence $c_1(L_1)^2 \cap \ldots \cap c_1(L_i)^2 \cap \overline{M}_{0,3d-1}(2,d)$ represents the locus of curves we were looking for. By *Bertini's theorem*, the scheme theoretic intersection cycle above is reduced and of dimension 0 (the general line of $\mathbb{P}^2$ is transverse to the map μ). As we noticed before, the general intersection cycle corresponds to an irreducible nodal rational plane curves of degree d passing through $3d-1$ general points in $\mathbb{P}^2$.

QED

By using an analogous proposition, we argue that:

$$< L_1^2 L_2^2 L_3^2 L_4^2, \overline{M}_{0,4}(3,1) >= 2,$$

which is the number of lines in $\mathbb{P}^3$ meeting four lines in general position. Notice that, if we were able to compute it, $I_{0,8}(2,\dots,2)$ would represent the solution of the enumerative problem about cubics stated in the subsection 3.3.3.

The *first reconstruction theorem* by Kontsevich and Manin, in such a setting, says that *there exists an explicit effective algorithm for calculating all the top intersection products $I(0,n)(\alpha_1,\dots,\alpha_n)$* ([51]).

Chapter 5

Jets Extensions of Relative Line Bundles on Stable Curves

5.1 The Relative Dualizing Sheaf.

Let $\pi : \mathfrak{X} \longrightarrow S$ be a noded curve over S, meaning that if some fiber of the map π is not a smooth curve it has only ordinary nodes as singularities. Each such family comes equipped with two important sheaves: the former is the *sheaf of the relative differentials* $\Omega^1_{\mathfrak{X}/S}$, and the latter is the *relative dualizing sheaf* ω_π. They are both coherent sheaves of $O_{\mathfrak{X}}$-modules which coincide if all the fibers of π are smooth curves. Moreover, as we shall see below, for families of stable curves, ω_π is an invertible sheaf.

From now on we shall make, for simplicity, the assumption that S is a smooth scheme. This assumption, which is not necessary for the validity of the properties of $\Omega^1_{\mathfrak{X}/S}$ and ω_π we are interested in, is not restrictive in our case, because in the sequel we shall be mainly concerned with families having a smooth connected base. Let now $s \in S$ be a point such that $\mathfrak{X}_s$, the geometric scheme-theoretical fiber of $\mathfrak{X}$, is a noded curve.

We say that $P \in \mathfrak{X}_s$ is a node if and only if the completion of O_P thought of as $\pi^* O_s$ algebra, is isomorphic to:

$$\mathbb{C}[[X, Y, T_1, \ldots, T_r]]/(XY - f(T_1, \ldots, T_r))$$

thought of as a $\mathbb{C}[[T_1, \ldots, T_r]]$-algebra and where

$$f(T_1, \ldots, T_r) \in \mathbb{C}[[T_1, \ldots, T_r]]$$

and that the map π is induced by the structural morphism:

$$\mathbb{C}[[T_1,\ldots,T_r]]\longrightarrow\mathbb{C}[[X,Y,T_1,\ldots,T_r]]/(XY-f(T_1,\ldots,T_r)).$$

In the complex analytic language such a definition means that a holomorphic map $\pi:\mathfrak{X}\longrightarrow S$ between complex analytic spaces has a node at P if and only if there is a complex analytic neighbourhood V of P and a complex analytic neighbourhood U of $\pi(P)$ such that the projection $\pi:V\longrightarrow U$ is given by $(X,Y,T_1,\ldots,T_r)\longrightarrow(T_1,\ldots,T_r)$ and U looks like the set of points $(X,Y,T_1,\ldots T_r)$ such that $XY-f(T_1,\ldots,T_r)=0$ for some holomorphic function $f\in O_{\mathfrak{X}}(V)$. The locus $f(T_1,\ldots,T_r)=0$ is a local equation of the so called *discriminant locus* or the *nodal locus* of the family. In the sequel, by definition, for a noded curve over S of genus g we shall mean a proper flat family $\pi:\mathfrak{X}\longrightarrow S$ such that each geometric fiber is either smooth or noded.

Now, by standard basic commutative algebra (see e.g. [18], [6], [62]), the module of sections on V of the sheaf of relative differentials $\Omega^1_{\mathfrak{X}}(V)$ is generated by dX, dY and dT_i over $\mathbb{C}[[X,Y,T_1,\ldots,T_m]]$, subject to the relation:

$$XdY+YdX-df=0.$$

By virtue of the well know exact sequence:

$$\pi^*\Omega^1_U\longrightarrow\Omega^1_V\longrightarrow\Omega^1_{V/U}\longrightarrow 0, \tag{5.1}$$

it follows that $\Omega^1_{\mathfrak{X}/S}(V)$ is generated as $\mathbb{C}[[X,Y,T_1,\ldots,T_m]]$-module by dX and dY subject to the relation:

$$XdY+YdX=0. \tag{5.2}$$

The relative dualizing sheaf ω_π, instead, is such that the module $\omega_\pi(V)$ is generated, over $\mathbb{C}[[X,Y,T_1,\ldots,T_m]]$ by $\frac{dX}{X}$ and $\frac{dY}{Y}$, subject to the relation:

$$\frac{dX}{X}+\frac{dY}{Y}=0,$$

which, after all, is a way of formally rephrasing (5.2). By the way, it turns out that the relative dualizing sheaf ω_π of the family π is an invertible $O_{\mathfrak{X}}$-module.

If $s\in S$ is such that $\mathfrak{X}_s$ is smooth, $\Omega^1_{\mathfrak{X}_s}$ and $\omega_{\mathfrak{X}_s}$, their restriction at $\mathfrak{X}_s$, coincide. In other words, if $\pi:\mathfrak{X}\longrightarrow S$ is a family of smooth curves, the dualizing sheaf is the same as the sheaf of the relative differentials.

Suppose that $C \longrightarrow Spec(\mathbb{C})$ is a trivial family and C a curve having only nodes as singularities. Let $N_1, \ldots, N_k$ be such nodes and let

$$n : \tilde{C} \longrightarrow C,$$

be the normalization of C and (P_i, Q_i) the preimages of N_i through n. Then the relative dualizing sheaf, in this case simply said the *dualizing sheaf* of the curve C, is the sheaf of regular differentials μ on $\tilde{C}$ with possible exceptions at the points $\{(P_i, Q_i)\}$, where μ may have at most simple poles such that:

$$Res_{P_i}(\mu) + Res_{Q_i}(\mu) = 0,$$

for each $1 \le i \le k$. The reason of the name *dualizing sheaf* ω_C for the curve C is that if $\mathcal{F}$ is any coherent sheaf on C, then there is an isomorphism:

$$Hom(H^1(C, \mathcal{F}), \mathbb{C}) \cong Hom_{\mathcal{O}_C}(\mathcal{F}, \omega_C)$$

If $\pi : \mathfrak{X} \longrightarrow S$ is any stable curve on S, by what has been said there is a natural map:

$$\mathcal{R} : \Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi. \tag{5.3}$$

On the restricted family $\pi_V : V \longrightarrow U$ as above, it is defined simply by sending the generators dX and dY of $\Omega^1_{\mathfrak{X}/S}(V)$ into, respectively, $\frac{dX}{X}$ and $\frac{dY}{Y}$. If $\zeta = \frac{dX}{X} = -\frac{dY}{Y}$ is a generator of ω_π around a node of a fiber of $\mathfrak{X}$, then the image of $\Omega^1_{\mathfrak{X}/S}$ via the map $\mathcal{R}$ is given by, in the above open set:

$$dX \mapsto X\zeta \quad \text{and} \quad dY \mapsto Y\zeta,$$

i.e., if $N \in \mathfrak{X}$ is a node we are looking at, $\Omega^1_{\mathfrak{X}/S,N} = (X, Y)\zeta$ where (X, Y) is the maximal ideal in the fiber $\pi^{-1}(\pi(N))$ corresponding to the node N. In other words the image of the map $\mathcal{R}$ is nothing but $\mathcal{I}_{[nodes]}\omega_\pi$, where $\mathcal{I}_{[nodes]}$ is the ideal sheaf corresponding to the closed subscheme of nodal points on fibers of the family π. Following [63], p. 101, we shall give the proof of the following remarkable fact, that we shall need in the sequel:

Theorem 5.1 *Let $\pi : \mathfrak{X} \longrightarrow S$ be a stable curve over a (germ of) smooth curve S and assume that $\mathfrak{X}$ is integral. Let $Sing(\mathfrak{X})$ be the closed subscheme of the singular points on fibers of π and let $\mathcal{I}_{sing}$ be its ideal sheaf. Then:*

i) $codim(Sing(\mathfrak{X})) = 2$

ii) the canonical homomorphism $\Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi$ *induces an isomorphism* $\Omega^1_{\mathfrak{X}/S} = \mathcal{I}_{[nodes]} \cdot \omega_\pi$.

Proof.

We are going to check that all the stalks of

$$\Omega^1_{\mathfrak{X}/S} \quad \text{and} \quad \Omega^1_{\mathfrak{X}/S} = \mathcal{I}_{[nodes]} \cdot \omega_\pi,$$

are isomorphic. They are clearly isomorphic off the singular locus $Sing(\mathfrak{X})$, by what has been previously said. Around a singular point N, $\mathfrak{X}$ has a local equation of the form $xy = t^n$, where (x, y, t) are generators of the maximal ideal corresponding to N and t is a local parameter on S. Locally, $\mathfrak{X}$ is singular only at the point $(0, 0, 0)$ corresponding to the fiber $t = 0$. It follows that $Sing(C)$ has codimension 2 in $\mathfrak{X}$ (a point in a surface). We already saw that locally around N:

$$\Omega^1_{\mathfrak{X}/S} = (O_{\mathfrak{X}}dx + O_{\mathfrak{X}}dy)/(xdy + ydx)O_{\mathfrak{X}},$$

while ω_π is invertible, generated by ζ which may be expressed as dx/x for $x \neq 0$ and $-dy/y$ for $y \neq 0$. We have the map:

$$\Omega^1_{\mathfrak{X}/S} \longrightarrow (x, y)\zeta = \mathcal{I}_{[nodes]} \cdot \omega_\pi.$$

Now $\alpha dX + \beta dY$ is sent, via the above map, to $(\alpha X - \beta Y)\zeta$. Such an image is 0 iff $\alpha X - \beta Y = 0$. Now, $\alpha, \beta \in O_{\mathfrak{X}}(U)$, and hence, since X, Y are local parameters around N, one may write $\alpha = Y\tilde{\alpha}$ and $\beta = X\tilde{\beta}$, so that:

$$\alpha\, dX + \beta\, dY = (Y\tilde{\alpha}dX + X\tilde{\beta}dY)$$

which is sent to:

$$XY(\tilde{\alpha} - \tilde{\beta})\zeta. \tag{5.4}$$

Now, since ζ generates ω_π, 5.4 gives:

$$\pi^*(t)(\tilde{\alpha} - \tilde{\beta}) = 0,$$

where $\pi^*(t)$ is the local parameter of the base S pulled back to $\mathfrak{X}$. But 5.1 is an equation in an open set of $\mathfrak{X}$ which is integral. Hence, because $\pi^*(t) \neq 0$, one has $\tilde{\alpha} = \tilde{\beta}$, giving $\alpha\, dX + \beta\, dY = 0$ as required.

QED

As a consequence we have that the following exact sequence must hold:

$$0 \longrightarrow \Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi \longrightarrow \omega_\pi \otimes O_{[nodes]} \longrightarrow 0, \tag{5.5}$$

as may be easily seen by considering the standard exact sequence:

$$0 \longrightarrow \mathcal{I}_{[nodes]} \longrightarrow O_{\mathfrak{X}} \longrightarrow O_{[nodes]} \longrightarrow 0,$$

tensoring it by the invertible sheaf ω_π and using theorem 5.1.

5.2 Jet Extensions of Relative Line Bundles

Let $\pi : \mathfrak{X} \longrightarrow S$ be, as usual, a stable curve over S. The sheaf of the relative differentials $\Omega^1_{\mathfrak{X}/S}$ comes equipped with a *universal derivation* ([6], p. 108):

$$d_{\mathfrak{X}/S} : O_{\mathfrak{X}} \longrightarrow \Omega^1_{\mathfrak{X}/S}. \tag{5.6}$$

Unless $\mathfrak{X}$ is a family of smooth curves, $\Omega^1_{\mathfrak{X}/S}$ is not in general invertible, but fortunately enough we are given of the natural map (5.3):

$$\mathcal{R} : \Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi,$$

having as a target the relative dualizing sheaf of the family ω_π, which is invertible. Let us denote, in the sequel, by d_π the composition of $\mathcal{R}$ with $d_{\mathfrak{X}/S}$. In other words:

$$d_\pi = \mathcal{R} \circ d_{\mathfrak{X}/S} : O_{\mathfrak{X}} \longrightarrow \omega_\pi$$

The map d_π will be called, in the following, by a slight abuse of terminology, the *exterior derivative along the fibers* of π (compare with [56], where in the case of families of smooth curves, an analytic description of d_π is provided). As it is well known, ω_π and d_π enjoy some nice functorial properties (see, e.g. [16], p. 77), which reflects in a nice behaviour of the map d_π by base change. More precisely, combining the properties of the dualizing sheaf under base change with the behaviour of the universal derivation described, e.g., in [6], p. 110, we have that if:

$$\begin{array}{ccc} \mathfrak{X}_T & \xrightarrow{p_2} & \mathfrak{X} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow[\phi]{} & S \end{array} \tag{5.7}$$

is a cartesian diagram, i.e. $\mathfrak{X}_T$ is the induced family over T, defined by $T \times_S \mathfrak{X}$, then

$$p_2^* \omega_\pi = \omega_{p_1}, \tag{5.8}$$

and

$$p_2^* d_\pi = d_{p_1}. \tag{5.9}$$

In particular, if $s \in S$ and $T = Spec(\mathbf{k}(s))$, then $p_2^* \omega_\pi = \omega_{p_1} = \omega_{\mathfrak{X}_s}$, the dualizing bundle of the fiber $\mathfrak{X}_s$ and $d_{p_2} \equiv d : O_{\mathfrak{X}_s} \longrightarrow \omega_{\mathfrak{X}_s}$. This in particular means that ω_π is a *dualizing bundle along the fibers*, i.e. its fiber at a point $P \in \mathfrak{X}$ is $\omega_{\mathfrak{X}_{\pi(P)}}$.

We want to use the previous remarks for defining a suitable notion of n-th *jets extension* of a relative line bundle $\mathcal{L}$ on $\mathfrak{X}/S$ along the fibers of π, where $\pi : \mathfrak{X} \longrightarrow S$ is a stable curve over S of genus $g \geq 2$. By a relative line bundle $\mathcal{L}$ on $\mathfrak{X}/S$ we shall intend a line bundle over $\mathfrak{X}$ modulo pull-backs of line bundles over S. Let $\mathcal{L}$ be a representative of such a line bundle and let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open affine covering of $\mathfrak{X}$ such that $\omega_\pi(U_\alpha)$ and $\mathcal{L}(U_\alpha)$ are generated by σ_α and ψ_α, respectively, over $O_{\mathfrak{X}}(U_\alpha)$.

If $\lambda \in H^0(\mathfrak{X}, \mathcal{L})$, we can then write $\lambda_{|U_\alpha} = \ell_\alpha \psi_\alpha$, for some $\ell_\alpha \in O_{\mathfrak{X}}(U_\alpha)$. The purpose is now to define the higher order derivatives of λ with respect to the generator σ_α (compare with [53] and [54]). We set $d_\alpha^{(0)}(\ell_\alpha) = \ell_\alpha$ and, recursively,

$$d_\pi(d_\alpha^{(n-1)}(\ell_\alpha)) = d_\alpha^{(n)}(\ell_\alpha)\sigma_\alpha. \tag{5.10}$$

It is now a standard patching game to show that the data relative to the open set U_α:

$$\left\{U_\alpha; \left(\ell_\alpha, d_\alpha \ell_\alpha, \ldots, d_\alpha^{(n)} \ell_\alpha\right)^T\right\},$$

is related to the data relative to the open set U_β:

$$\left\{U_\beta; \left(\ell_\beta, d_\beta \ell_\beta, \ldots, d_\beta^{(n)} \ell_\beta\right)^T\right\},$$

in the intersection $U_\alpha \cap U_\beta$ through the relation:

$$\left(\ell_\alpha, d_\alpha \ell_\alpha, \ldots, d_\alpha^{(n)} \ell_\alpha\right)^T = M_{\alpha\beta} \cdot \left(\ell_\beta, d_\beta \ell_\beta, \ldots, d_\beta^{(n)} \ell_\beta\right)^T. \tag{5.11}$$

where $M_{\alpha\beta}$ is a $(n+1) \times (n+1)$ matrix whose entries are regular function on $U_\alpha \cap U_\beta \subset \mathfrak{X}$. Moreover, it turns out that $\{M_{\alpha\beta}\} \in Z^1(\mathcal{U}, Gl_{n+1}(O_{\mathfrak{X}}))$, i.e.:

$$M_{\alpha\alpha} = id_{O_{\mathfrak{X}}(U_\alpha)^{\oplus(n+1)}} \quad and \quad M_{\alpha\beta} \cdot M_{\beta\gamma} = M_{\alpha\gamma}.$$

Hence, the collection $\{M_{\alpha\beta}\}$ so described, defines a rank $n+1$ vector bundle.

Definition 5.1 *The vector bundle of rank* $(n+1)$ *defined by the transition functions* $M_{\alpha\beta}$ *relatively to the covering* $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ *is said to be the* n-th jets extension along the fibers *of the relative line bundle* $\mathcal{L}$*, and, following [56], will be denoted by* $J^n_\pi\mathcal{L}$.

Example 5.1 For the reader's convenience it seems worth, here, to sketch the construction of $J^1\mathcal{L}$. The construction of $J^n\mathcal{L}$ may be gotten from this example by imitating an inductive procedure used by F. Ponza in [72]. Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open covering which trivializes simultaneously the representative $\mathcal{L}$ of a relative line bundle on $\mathfrak{X}/S$ and the relative dualizing sheaf ω_π. Let $\lambda \in H^0(\mathfrak{X}, \mathcal{L})$ be a global regular section of $\mathcal{L}$, and on $U_\alpha \cap U_\beta \neq \emptyset$ write: $\lambda_{|U_\alpha \cap U_\beta} = \ell_\alpha\psi_\alpha = \ell_\beta\psi_\beta$. By ψ_α and ψ_β we mean the restriction to $U_\alpha \cap U_\beta$ of the generators of $\mathcal{L}(U_\alpha)$ and $\mathcal{L}(U_\beta)$ over $O_{\mathfrak{X}}(U_\alpha)$ and $O_{\mathfrak{X}}(U_\beta)$ respectively. Similarly, we shall denote by σ_α and σ_β the restrictions to $U_\alpha \cap U_\beta$ of some generators of $\omega_\pi(U_\alpha)$ and $\omega_\pi(U_\beta)$ over $O_{\mathfrak{X}}(U_\alpha)$ and $O_{\mathfrak{X}}(U_\beta)$ respectively. Let $l_{\alpha\beta}$ and $k_{\alpha\beta}$ be the transition functions of $\mathcal{L}$ and ω_π, defined by:

$$\psi_\beta = l_{\alpha\beta}\psi_\alpha \quad \text{and} \quad \sigma_\beta = k_{\alpha\beta}\sigma_\alpha.$$

Set:

$$d_\pi(U_\alpha)(\ell_\alpha) = d_\alpha\ell_\alpha\sigma_\alpha.$$

One has then:

$$\ell_\alpha = l_{\alpha\beta}\ell_\beta, \tag{5.12}$$

while:

$$\begin{aligned} d_\alpha\ell_\alpha &= d_\alpha(l_{\alpha\beta}\ell_\beta) = k_{\alpha\beta}d_\beta(l_{\alpha\beta}\ell_\beta) = \\ &= k_{\alpha\beta}d_\beta(l_{\alpha\beta})\cdot\ell_\beta + k_{\alpha\beta}l_{\alpha\beta}d_\beta\ell_\beta. \end{aligned} \tag{5.13}$$

One may hence organize the transformation rules (5.12) and (5.13) in the following matrix equation:

$$\begin{pmatrix} \ell_\alpha \\ d_\alpha\ell_\alpha \end{pmatrix} = \begin{pmatrix} l_{\alpha\beta} & 0 \\ k_{\alpha\beta}d_\beta(l_{\alpha\beta}) & k_{\alpha\beta}l_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \ell_\beta \\ d_\beta\ell_\beta \end{pmatrix}. \tag{5.14}$$

Set:

$$M_{\alpha\beta} = \begin{pmatrix} l_{\alpha\beta} & 0 \\ k_{\alpha\beta}d_\beta(l_{\alpha\beta}) & k_{\alpha\beta}l_{\alpha\beta} \end{pmatrix} \tag{5.15}$$

Let us check that $\{M_{\alpha\beta}\} \in Z^1(\mathcal{U}, Gl_2(O_{\mathfrak{X}}))$. One has:

$$M_{\alpha\alpha} = \begin{pmatrix} l_{\alpha\alpha} & 0 \\ k_{\alpha\alpha}d_\alpha(l_{\alpha\alpha}) & k_{\alpha\alpha}l_{\alpha\alpha} \end{pmatrix}.$$

Now, $d_\alpha(\ell_{\alpha\alpha}) = d_\alpha(1_{O_{\mathfrak{X}}(U_\alpha)}) = 0$ while the other entries are the identity of the ring $O_{\mathfrak{X}}(U_\alpha)$. Hence:

$$M_{\alpha\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover

$$\begin{aligned} M_{\alpha\beta} \cdot M_{\beta\gamma} &= \begin{pmatrix} l_{\alpha\beta} & 0 \\ k_{\alpha\beta} d_\beta(l_{\alpha\beta}) & k_{\alpha\beta} l_{\alpha\beta} \end{pmatrix} \cdot \begin{pmatrix} l_{\beta\gamma} & 0 \\ k_{\beta\gamma} d_\gamma(l_{\beta\gamma}) & k_{\beta\gamma} l_{\beta\gamma} \end{pmatrix} = \\ &= \begin{pmatrix} l_{\alpha\beta} l_{\beta\gamma} & 0 \\ k_{\alpha\beta} l_{\beta\gamma} d_\beta(l_{\alpha\beta}) + k_{\alpha\beta} l_{\alpha\beta} k_{\beta\gamma} d_\gamma(l_{\beta\gamma}) & k_{\alpha\beta} l_{\alpha\beta} k_{\beta\gamma} l_{\beta\gamma} \end{pmatrix}. \end{aligned}$$

Now, by construction, $l_{\alpha\beta} l_{\beta\gamma} = l_{\alpha\gamma}$, while

$$k_{\alpha\beta} l_{\alpha\beta} k_{\beta\gamma} l_{\beta\gamma} = k_{\alpha\beta} k_{\beta\gamma} l_{\alpha\beta} l_{\beta\gamma} = k_{\alpha\gamma} l_{\alpha\gamma}.$$

For the entry located in the second row and in the first column, one has:

$$\begin{aligned} & k_{\alpha\beta} l_{\beta\gamma} d_\beta(l_{\alpha\beta}) + k_{\alpha\beta} l_{\alpha\beta} k_{\beta\gamma} d_\gamma(l_{\beta\gamma}) = \\ = \; & k_{\alpha\beta} l_{\beta\gamma} d_\beta(l_{\alpha\beta}) + k_{\alpha\gamma} d_\gamma(l_{\alpha\gamma}) - k_{\alpha\gamma} l_{\beta\gamma} k_{\alpha\gamma} d_\gamma(l_{\alpha\beta}) = \\ = \; & k_{\alpha\gamma} d_\gamma(l_{\alpha\gamma}), \end{aligned}$$

so that:

$$M_{\alpha\beta} \cdot M_{\beta\gamma} = \begin{pmatrix} l_{\beta\gamma} & 0 \\ k_{\beta\gamma} d_\gamma(l_{\beta\gamma}) & k_{\beta\gamma} l_{\beta\gamma} \end{pmatrix} = M_{\alpha\gamma}.$$

Hence $\{M_{\alpha\beta}\} \in Z^1(\mathcal{U}, GL_2(O_{\mathfrak{X}}))$ defines a vector bundle of rank 2 denoted by $J^1_\pi\mathcal{L}$. Let us check that such vector bundle is well defined on the class of $\mathcal{L}$ modulo pull-back of line bundles on the base. Let $\mathcal{M}$ be a line bundle on S, and let $\mathcal{L} \otimes \pi^*\mathcal{M}$ be another representative of the class which $\mathcal{L}$ belongs to. The transition functions of the line bundle $\mathcal{L} \otimes \pi^*\mathcal{M}$ are the product of the transition function $\{l_{\alpha\beta}\}$ of $\mathcal{L}$ and the pull-back $\pi^*(m_{\alpha\beta})$ of the transition functions of $\mathcal{M}$, which are fiberwise constant. Hence, $d_\pi(\pi^*(m_{\alpha\beta})) = 0$, so that the transition functions of the bundle $J^1(\mathcal{L} \otimes \pi^*\mathcal{M})$ are $M_{\alpha\beta} \cdot \pi^*(m_{\alpha\beta})$, i.e.:

$$J^1(\mathcal{L} \otimes \pi^*\mathcal{M}) = J^1(\mathcal{L}) \otimes \pi^*\mathcal{M}$$

which means that the first jet of a relative line bundle is defined as the first jet of a representative modulo the pull-back of line bundles on S.

By the above definitions, it turns out that the collection:

$$\{U_\alpha; (\ell_\alpha, d_\alpha(\ell_\alpha), \ldots, d_\alpha^{(n)}(\ell_\alpha))^T\}$$

defines a global holomorphic section, written $D^n\lambda$, of the bundle $J^n_\pi\mathcal{L}$. The construction outlined above is the relative version of the *jet bundles for Gorenstein curves* constructed in [33]. If $\pi : \mathfrak{X}\longrightarrow S$ is a family of smooth curves and $\mathcal{L} = \omega_\pi$ is the relative canonical sheaf, then $J^n_\pi\omega_\pi$ is exactly the n-th jet of the relative canonical bundle along the fibers defined by Lax in [56]. There, the construction was performed by using Patt's local coordinates ([69]) in the Teichmüller space.

To continue with, because of the property (5.9) of d_π, we also have, referring to the diagram (5.7),

$$p_2^*(D^k\tau) = D^k(p_2^*\tau), \tag{5.16}$$

and, hence:

$$p_2^*(J^k_\pi\mathcal{L}) = J^k_{p_1}(p_2^*\mathcal{L}),$$

where the D on the right hand side is precisely the D relatively to the induced family $p_1 : \mathfrak{X}_T\longrightarrow T$. In particular, if $T = Spec(\mathbf{k}(s))$, D is still defined as above and $J^k_{p_1}\mathcal{L}_s$ is the k-th jet of the bundle of $\mathcal{L}_{|\mathfrak{X}_s} = p_2^*\mathcal{L} = \mathcal{L}_s$.

Now we come to prove one (easy) technical result that establishes an exact sequence of vector bundles which shall be useful to compute Chern classes. It is nothing but formula (2.1.1) in [53], p. 138, rephrased in the language of jet bundles. However, we should say, the present framework is more general, since we shall prove it for bundles defined on a family of stable curves. Before stating such an important technical proposition, let us remark that if $M_{\alpha\beta}$ is the transition matrix of the bundle $J^n\mathcal{L}$, then such a matrix is upper triangular. This comes out because each k-th derivative of, say, ℓ_α, by passing to a trivialization (U_β, ψ_β), changes through a linear combination of all the $d_\beta^{(i)}(\ell_\beta)$'s, with $0 \leq i \leq k$. Moreover, the i-th diagonal entry of $M_{\alpha\beta}$ is exactly the transition function of the bundle $\mathcal{L}\otimes{\omega_\pi}^{\otimes i}$ (for $0 \leq i \leq n$), as the reader may check himself by doing an easy exercise. We may now prove the following fundamental:

Proposition 5.1 *For each line bundle $\mathcal{L}$ over $\mathfrak{X}/S$ and each $n \geq 1$, the following exact sequence holds:*

$$0\longrightarrow\mathcal{L}\otimes{\omega_\pi}^{\otimes n}\longrightarrow J^n_\pi\mathcal{L}\longrightarrow J^{n-1}_\pi\mathcal{L}\longrightarrow 0 \tag{5.17}$$

Proof.

Let $U_\alpha \in \mathcal{U}$ be an open set in $\mathfrak{X}$ belonging to a trivializing open covering. Let $(P, (u_{\alpha,0}, \ldots, u_{\alpha,n})^T)$ be a point of $J^n_\pi\mathcal{L}$ in the given trivialization. The data

$(u_{\alpha,0},\ldots,u_{\alpha,n})^T$ is a $(n+1)$-tuple of complex numbers: they do not need to come from the evaluation at P of a section $D^n\lambda$ of $J^n_\pi\mathcal{L}$. The only constraint is that if $(P,(u_{\beta,0},\ldots,u_{\beta,n})^T)$ is the representation of the same point of $J^n_\pi\mathcal{L}$ in the open trivializing set U_β, then these two sets of data must be related by the transition functions $\{M_{\alpha\beta}\}$ of the bundle $J^n_\pi(\mathcal{L})$. Define the map $p_{n-1}: J^n_\pi\mathcal{L}\longrightarrow J^{n-1}_\pi\mathcal{L}$ as follows:

$$(P,(u_{\alpha,0},\ldots,u_{\alpha,n})^T)\mapsto (P,(u_{\alpha,0},\ldots,u_{\alpha,n-2},u_{\alpha,n-1})^T).$$

This map is clearly surjective, and its kernel is formed by all the $(n+1)$-tuples $(P,(\underbrace{0,\ldots,0}_{(n-1)-times},v_{n-1,\alpha})^T)$ *belonging to* $J^n_\pi\mathcal{L}$. The latter sentence means that such n-tuple corresponds, in the trivialization (U_β,ψ_β) to the n-tuple:

$$(v_{0,\beta},\ldots,v_{n-1,\beta})^T = M_{\beta\alpha}\cdot(\underbrace{0,\ldots,0}_{n-times},v_{n-1,\alpha})^T,$$

which, because of the remarked structure of the matrix $M_{\alpha\beta}$, means that:

$$(P,(v_{0,\beta},\ldots,v_{n-1,\beta})^T) = (P;(\underbrace{0,\ldots,0}_{n-times},v_{n-1,\alpha}l_{\beta\alpha}\cdot(k_{\beta\alpha})^n)^T),$$

i.e. that $Ker(p_{n-1})$ may be identified with a line bundle isomorphic to $\mathcal{L}\otimes{\omega_\pi}^{\otimes n}$.

QED

5.3 Partial Jets

The title *partial jets* means that we want to learn to make *partial derivatives* of sections of line bundles on families which are fibered products of good families $\pi:\mathfrak{X}\longrightarrow S$. To be more precise, let $\pi:\mathfrak{X}\longrightarrow S$ be a stable curve over S and let ω_π be its dualizing sheaf. Let $\rho:\mathfrak{X}^n\longrightarrow S$ be the natural map from the n-fold fibered product:

$$\mathfrak{X}\times_S\ldots\times_S\mathfrak{X}$$

of $\mathfrak{X}$ over S. The notation $\mathfrak{X}^n$ is certainly abused. For each $1\le i\le n$ we have the *projections onto the i-th factor*:

$$p_i:\mathfrak{X}^n\longrightarrow\mathfrak{X},$$

so that $\rho = \pi \circ p_i$, for each $i \in \{1, \ldots, n\}$. Let $\Omega^1_{\mathfrak{X}^n/S}$ be the sheaf of relative differentials of $\mathfrak{X}^n \longrightarrow S$. By general theorems on the sheaf of the relative differentials, (see [18], pp. 392-393) it turns out that:

$$\Omega^1_{\mathfrak{X}^n/S} = p_1^*\Omega^1_{\mathfrak{X}/S} \oplus \ldots \oplus p_n^*\Omega^1_{\mathfrak{X}/S}. \tag{5.18}$$

The geometrical meaning of the above equality should be clear: it means that such a module is locally generated over $O_{\mathfrak{X}^n}$ by expressions like df_i, where f_i is the pull-back of a regular function on the i-th factor. Using the map:

$$\mathcal{R} : \Omega^1_{\mathfrak{X}/S} \longrightarrow \omega_\pi,$$

we have an induced map:

$$\tilde{\mathcal{R}} : \Omega^1_{\mathfrak{X}^n/S} \longrightarrow p_1^*\omega_\pi \oplus \ldots \oplus p_n^*\omega_\pi. \tag{5.19}$$

The target of the map $\tilde{\mathcal{R}}$ in (5.19) is a vector bundle of rank n and it is the *Whitney sum* of the line bundles $p_i^*\omega_\pi$. Pick $f \in O_{\mathfrak{X}^n}(U)$. Then $df \in \Omega^1_{\mathfrak{X}^n/S}(U)$. In a sense we want to take the component of df along the i-th factor and to call it the *partial derivative* along the i-th factor. It would be nice to denote such a partial derivative by $d_{p_i}(f)$, but this may create some confusion and we need to invent a new symbol for it, as it will be clear in a moment. In fact let us consider the cartesian diagram:

$$\begin{array}{ccc} \overbrace{\mathfrak{X}\times_S\ldots\times_S\mathfrak{X}}^{n-times} & \xrightarrow{\quad p_i \quad} & \mathfrak{X} \\ {\scriptstyle p_{1,\ldots,i-1,\hat{i},i+1,\ldots,n}}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\ \underbrace{\mathfrak{X}\times_S\ldots\times_S\mathfrak{X}}_{(n-1)-times} & \xrightarrow{\qquad\quad} & S \end{array} \tag{5.20}$$

Here $p_{1,\ldots,i-1,\hat{i},i+1,\ldots,n}$ means the projection onto all the factors but the i-th (the "hat" over the "i" means that "i" is omitted). The i-th partial derivative we are looking for is something living in $p_i^*\omega_\pi$. Now, there is a map $p_i^*d_\pi : O_{\mathfrak{X}^n} \longrightarrow p_i^*\omega_\pi$, which, by functoriality, is the derivative along the fiber of the map $p_{1,\ldots,i-1,\hat{i},i+1,\ldots,n}$. Hence, strictly speaking, the i-th *partial derivative* of the function f in our sense would be:

$$d_{p_{1,\ldots,i-1,\hat{i},i+1,\ldots,n}}(f) \in p_i^*\omega_\pi(U),$$

which is a clearly too long expression. Hence, from now on, we shall denote by the symbol ∂_i the derivation $d_{p_{1,\dots,i-1,\bar{i},i+1,\dots,n}}$, and to furtherly simplify notation, we shall set:

$$p_i^*\omega_\pi = \omega_i.$$

Moreover we shall set $K_i = c_1(\omega_i)$. If the morphism $\pi : \mathfrak{X} \longrightarrow S$ is smooth (i.e. if all the fibers are smooth), in $\mathfrak{X}^n$ we have also others Cartier divisors, corresponding to the diagonals: D_{ij} shall mean the *big* diagonal $P_i = P_j$. We shall be interested in their intersection properties stated and proven in the following:

Proposition 5.2 (Formularium)

Let K_i and D_{ij} as above. Then the following properties hold:

1. $D_{ij} \cdot D_{ik} = D_{ij} \cdot D_{jk}$.
2. $K_i D_{ij} = K_j D_{ij}$.
3. $D_{ij}^2 = -K_i D_{ij} = -K_j D_{ij}$

Proof.

The proof of 1 is as follows: on an analytic neighbourhood of $\mathfrak{X}^k$, with coordinates $(X_i)_{1\leq i\leq k}$, one has that $D_{ij} = Z(X_i - X_j)$, while $D_{jk} = Z(X_j - X_k)$. Since $X_i - X_j$ and $X_j - X_k$ are linearly independent linear form, it turns out that D_{ij} and D_{jk} do intersect transversally each other. Hence the classes $D_{ij}D_{jk}$ and $D_{ij}D_{ik}$ are the classes of the scheme theoretical intersections $D_{ij} \cap D_{jk}$ and $D_{ij} \cap D_{ik}$, which locally are defined by $Z(X_i - X_j, X_i - X_k)$ and $Z(X_i - X_j, X_j - X_k)$, and hence equal. It follows that even their classes in $A^*(\mathfrak{X}^n)$ are equal, so that:

$$D_{ij}D_{jk} = D_{ij}D_{ik}.$$

as desired.

As for 2, let us denote by ι the natural embedding:

$$\iota : D_{ij} \hookrightarrow \mathfrak{X}^n.$$

We hence have:

$$\begin{aligned} K_i \cdot D_{ij} &= \iota^*(K_i) = \iota^*(p_i^* c_1(\omega_\pi)) = (p_i \circ \iota)^* c_1(\omega_\pi) = \\ &= (p_j \circ \iota)^* c_1(\omega_\pi) = \iota^* p_j^* c_1(\omega_\pi) = K_j \cdot D_{ij}. \end{aligned}$$

It remains to prove 3. For such a purpose, we shall use the fact that the self intersection of a divisor coincides with the first Chern class of its normal sheaf. Hence we use the exact sequence defining the normal sheaf:

$$0 \longrightarrow \mathcal{T}_{D_{ij}} \longrightarrow \iota^* \mathcal{T}_{\mathfrak{X}^n} \longrightarrow \mathcal{N}_{D_{ij}/\mathfrak{X}^n} \longrightarrow 0$$

We have

$$c_1(\mathcal{N}_{D_i/\mathfrak{X}^n}) = c_1(\iota^* \mathcal{T}_{\mathfrak{X}^n}) - c_1(\mathcal{T}_{D_{ij}}),$$

i.e.:

$$D_{ij}^2 = -c_1(\wedge^{max} T^{\vee}_{\mathfrak{X}^n}) \cdot D_{ij} + K_{D_{ij}}.$$

But $c_1(\wedge^{max} T^{\vee}_{\mathfrak{X}^n}) D_{ij} = K_{\mathfrak{X}^n} D_{ij} \cong (K_1 + \ldots + K_n) D_{ij}$, while $K_{D_{ij}}$ is equal to $K_i D_{ij} + K_\ell D_{ij}$ with $\ell \neq i$ and $\ell \neq j$ (We passed from the product of n-curves to the product of $n-1$-curves. Hence:

$$D_{ij}^2 = -K_i D_{ij} - K_j D_{ij} - \sum_{\ell \neq i,j} K_i D_{ij} + K_i D_{ij} + \sum_{\ell \neq i,j} K_i D_{ij} = -K_i D_{ij},$$

as required.

QED

The above results are functorial, in the sense that they are independent on the base of the family. For instance consider a single curve C of genus g. Then it may be seen as a trivial family:

$$C \longrightarrow Spec(\mathbb{C}),$$

In this case C^2 is simply the two-fold product of the curve by itself. Let Δ be the diagonal and $K_i = c_1(p_i^* K_C)$ $(i = 1, 2)$.

Exercise 5.1 Prove that $\int_{C\times C} \Delta^2 = 2 - 2g$ and that $\int_{C\times C} K_i \Delta = 2g - 2$. May you feel the underlying geometrical idea of the latter equality?

Let us go back to the jets bundles, and let $\mathcal{L}$ be a relative line bundle over $\mathfrak{X}^n/S$. The bundle $J_i^n \mathcal{L}$ is the bundle gotten in the following way. Take a section $\lambda \in H^0(\mathfrak{X}^n, \mathcal{L})$. Here we are making the simplifying assumption that $\mathcal{L}$ has non-zero non constant global sections, which is more than enough for our purposes. Take a trivializing open set $U \subset \mathfrak{X}^n$. Then:

$$\lambda = \ell \psi,$$

where $\ell \in H^0(U, 0_{\mathfrak{X}^n})$. The bundle $J_i^n(\mathcal{L})$ is hence the bundle whose transition functions are prescribed by the transformation rules of the set of data:

$$(U; \ell, \partial_i \ell, \dots, \partial_i^n \ell).$$

Obviously we have the exact sequence:

$$0 \longrightarrow \mathcal{L} \otimes \omega_i^{\otimes n} \longrightarrow J_i^n \mathcal{L} \longrightarrow J_i^{n-1} \mathcal{L} \longrightarrow 0. \qquad (5.21)$$

The proof works out exactly as the one for the exact sequence (5.17).

Chapter 6

Intersection Theory on the Moduli Spaces of Curves

6.1 Introduction

6.1.1 The Chow rings of M_g and $\overline{M}_g$

Let $g \geq 2$. Studying the intersection theory of $\overline{M}_g$ means primarily to understand what should one mean by the Chow ring of $\overline{M}_g$. In fact, to begin with, $\overline{M}_g$ is a singular variety. Hence, if on one hand we have no trouble in defining the Chow groups, on the other hand we should try to make precise what should be a reasonable intersection product in the Chow group $A_*(\overline{M}_g)$. But even if one doesn't aim to be particularly sharp, one sees that problems arise already in the divisor theory. For instance: what is $Pic(\overline{M}_g)$ and what are its generators? The answer to this question has been provided during several years of studies by several authors. And once one has understood what the generators are, how to express classes of natural defined loci in terms of such generators?

The first big step toward the foundations of the intersection theory on the moduli space of curves has been walked by Mumford in his pionieristic paper [65]. At that time it was of course clear that $\overline{M}_g$ was a singular variety, due to curves possessing non trivial automorphisms. However the difference between M_g and $\overline{M}_g$ (its *Deligne Mumford compactification*) was quite relevant. In fact M_g is a singular variety, but fortunately enough, it is globally the quotient of a smooth variety modulo the action of a finite group. The smooth variety which does the job is the so called space M_g^n, the *fine moduli space* parametrizing smooth projective complex curve with a *level n-*

structure. Roughly speaking, a *level n structure* is a choice of an explicit isomorphism between the group $(\mathbb{Z}/n\mathbb{Z})^2$ and the subgroup $Jac(C)[n]$ of the Jacobian $Jac(C)$, parametrizing n-torsion points[1]. We are not interested to enter in details here. For our own purposes it will be sufficient to know that we may define the Chow ring of M_g, $A^*(M_g)$ exactly as explained in 4.1.4.
However, for $\overline{M}_g$, it was not known, at the time of [65], if it was *globally* the quotient of a smooth variety by the action of a finite group. To circumvent this difficulty, Mumford used the fact, that is clear from the construction of the moduli space via Kuranishi families, that $\overline{M}_g$ is at least locally the quotient of a smooth complex variety by the action of a finite group acting on it. Moreover he notices that, globally, $\overline{M}_g$ is the quotient of a Cohen-Macaulay variety by the action of a finite group. In such a way, using the theory of the *operational Chow rings* developed by Fulton and MacPherson (see [29] and [30]) Mumford succeeds in defining a Chow ring structure on the Chow group $A^*(\overline{M}_g)$, whose classes are called Q-classes. In the Q-Chow ring of $\overline{M}_g$ the rational coefficients are used. Of course Mumford knew that life was much easier for M_g, and in fact in the introduction of [65] he claims that if $\overline{M}_g$ "were globally the quotient of a smooth variety by a finite group, it would be easy to define a product in $A_*(\overline{M}_g) \otimes \mathbb{Q}$."

Fortunately, in recent years, Looijenga ([59], see also [70]) was able to prove that actually $\overline{M}_g$ turns out to be globally the quotient of a smooth variety by the action of a finite group, confirming the best possible hopes. We will not describe here the work of Looijenga, but his result is enough to allow us to know what is the framework which gives a rigorous meaning to the computations to be performed in the following of these notes. As a conclusion of this brief introduction, we may hence set the following fundamental:

Definition 6.1 *The Chow ring of $\overline{M}_g$ is the ring whose support is the $\mathbb{Q}$-vector space $A_*(\overline{M}_g) \otimes \mathbb{Q}$ and whose product is the one induced by the subring of the G-invariants of any of its smooth global covering $\tilde{\overline{M}}_g$ such that $\overline{M}_g \cong \tilde{\overline{M}}_g/G$, where G is a finite group acting on $\tilde{\overline{M}}_g$. The Chow ring of M_g, $A^*(M_g)$ is defined similarly.*

6.1.2 Basic Classes in $A^*(\overline{M}_g)$.

From now on, and for all the rest of these notes, by a *stable curve over S* we shall mean a flat proper family of stable curves of genus g parametrized by a smooth scheme of

[1] A n-torsion point of the Jacobian of a curve C is a line bundle $L \in Pic^0(C)$ such that $L^{\otimes n} = O_C$.

finite type over $Spec(\mathbb{C})$. Let $\pi : \mathfrak{X} \longrightarrow S$ be one such. Let ω_π be the relative canonical sheaf of the family . One can then define some classes in the Chow ring, $A^*(S)$, of S:

$$\kappa_i = \pi_* \left[c_1(\omega_\pi)^{i+1}\right], \tag{6.1}$$

the so-called *κ-classes*, and

$$\lambda_1 = \lambda = c_1(\pi_*\omega_\pi) = c_1(\bigwedge^g \pi_*\omega_\pi) \quad \text{and} \quad \lambda_i = c_i(\pi_*\omega_\pi), \tag{6.2}$$

the so-called *λ-classes*.

As for definitions (6.1), one has that (see [13]) $\kappa_0 = (2g-2)[S]$, where $[S]$ is the fundamental class of S (the identity of the ring $A^*(S)$). This comes out because if $\pi : \mathfrak{X} \longrightarrow S$ is a good family of stable curves, the restriction of π to the zero scheme of any holomorphic section of the dualizing sheaf ω_π, is generically finite of degree $2g-2$. Hence, by applying the definition of the push-forward one has:

$$\kappa_0 = \pi_*(c_1(\omega_\pi)) = (2g-2)[S].$$

As for definitions (6.2), instead, we remark that $\pi_*\omega_\pi$ is a locally free rank g sheaf of O_S-modules, often denoted in the literature as $\mathbb{E}$ and called the *Hodge bundle* relative to π. Let $\overline{M}_g$ and $\overline{M}_{g,1}$ be respectively the coarse moduli space of DM-stable curves and of the DM-pointed stable curves of genus g. Let now $\pi : \overline{M}_{g,1} \longrightarrow \overline{M}_g$ be the natural morphism that "forgets" the *marking*. As already remarked, due to curves with non trivial automorphisms, $\overline{M}_g$ is not smooth for $g \geq 2$. However, via non abelian level n structures (see [59]), $\overline{M}_g$ can be globally seen as the quotient of a smooth variety $\tilde{\overline{M}}_g$ under a finite group G acting faithfully on it.

This allows, following [29], p. 142, to define an intersection product on $A_*(\overline{M}_g) \otimes \mathbb{Q} = \left[\oplus A_i(\overline{M}_g)\right] \otimes \mathbb{Q}$ which makes $A_*(\overline{M}_g) \otimes \mathbb{Q}$ itself into a ring. This will be, by definition, the Chow ring of $\overline{M}_g$. In the following it will be denoted as $A^*(\overline{M}_g)$, with contravariant notation. Moreover there is a proper flat family of curves $\tilde{\pi}$: $\tilde{\overline{M}}_{g,1} \longrightarrow \tilde{\overline{M}}_g$, with G acting faithfully on $\tilde{\overline{M}}_{g,1}$ compatibly with the action of G on $\tilde{\overline{M}}_g$ and the morphism π, such that $\tilde{\overline{M}}_{g,1}/G \cong \overline{M}_{g,1}$. Hence, following Mumford ([65], p. 298), the morphism

$$\pi : \overline{M}_{g,1} \longrightarrow \overline{M}_g,$$

comes equipped with a Q-line bundle ω_π, represented by the relative canonical bundle $\omega_{\tilde{\pi}}$, which shall be briefly said to be the *relative dualizing sheaf* of π, with no further mention. Similarly, $\mathbb{E} = \pi_* \omega_\pi$ is a Q-bundle of rank g over $\overline{M}_g$, represented by $\tilde{\pi}_* \omega_{\tilde{\pi}}$ over $\tilde{\overline{M}}_g$. The κ-classes and the λ classes will be defined as in (6.2) and (6.1), where the Chern classes are taken in the sense explained in Example 4.1. The λ-classes and the κ-classes are said to be the *tautological classes* of $A^*(\overline{M}_g)$ (Cf. [65]). It is worth of mentioning that:

$$A^1(\overline{M}_g) = Pic(\overline{M}_g) \otimes \mathbb{Q} \cong Pic(\overline{\mathcal{M}}_g) \otimes \mathbb{Q},$$

where the last right hand side means the Picard group (with $\mathbb{Q}$ coefficients) of the *moduli functor* $\overline{\mathcal{M}}_g$ ([44], p. 50), which is briefly described below.

6.1.3 Some known facts on $Pic(\overline{M}_g) \otimes \mathbb{Q}$

Let $\overline{M}_g$ be the Deligne-Mumford compactification of the moduli space M_g of the smooth projective curves of genus g, defined over the field $\mathbb{C}$. For studying intersection theory on $\overline{M}_g$, one of the first goals would be to describe the Picard group of line bundles on it. Unfortunately, $\overline{M}_g$ is a singular space and $Pic(\overline{M}_g)$ is not known in general, and is very hard to describe. The way to overcome this difficulty is to study a better behaving object, the so called *Picard group of the moduli functor* $\overline{\mathcal{M}}_g$, $Pic(\overline{\mathcal{M}}_g)$, which is going to be explained. The underlying idea of $Pic(\overline{\mathcal{M}}_g)$ is to consider line bundles on families of curves all at once. More precisely, a line bundle on the moduli functor of stable curves is a line bundle $L(\pi)$ on the base S of every stable curve $\pi : \mathfrak{X} \longrightarrow S$ over S, enjoying the following property: if

$$\begin{array}{ccccc} & X_1 & \longrightarrow & X_2 & \\ \pi_1 & \downarrow & & \downarrow & \pi_2 \\ & S_1 & \longrightarrow & S_2 & \\ & & f & & \end{array}$$

is a morphism of families with *cartesian* square (i.e. $\mathfrak{X}_1 = S_1 \times_{S_2} \mathfrak{X}_2$), then there is an isomorphism between $L(\pi_1)$ and $f^*L(\pi_2)$. The isomorphisms should be compatible in a obvious sense. Namely, if both the squares of the diagram:

$$\begin{array}{ccccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 & \longrightarrow & \mathfrak{X}_3 \\ \pi_1 \downarrow & & \pi_2 \downarrow & & \pi_3 \downarrow \\ S_1 & \xrightarrow[f_1]{} & S_2 & \xrightarrow[f_2]{} & S_3 \end{array} \tag{6.3}$$

are cartesian, then:

$$L(\pi_1) \cong f_1^* L(\pi_2) \cong f_1^* f_2^* L(\pi_3) \cong (f_2 \circ f_1)^* L(\pi_3).$$

Two lines bundles L_1 and L_2 on the moduli functor are isomorphic iff for any family $\pi : \mathfrak{X} \longrightarrow S$, there is an isomorphism between $L_1(\pi)$ and $L_2(\pi)$ which respects the above compatibility requirements.

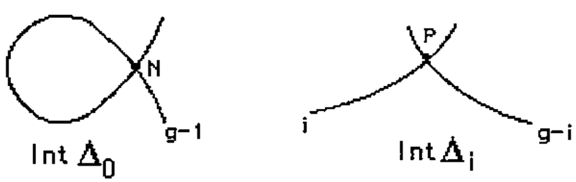
As the reader can easily check, the tensor product is also well defined and it is compatible with the relation of isomorphism, so that we can attach an abelian group $Pic(\overline{\mathcal{M}}_g)$ to the moduli functor $\overline{\mathcal{M}}_g$, the *Picard group of the moduli functor.* Analogously one can define $Pic(\mathcal{M}_g)$: one simply considers families of smooth curves of genus g instead of families of stable curves. Here is a list of known results:

1. A theorem by Harer implies that $Pic(M_g) \otimes \mathbb{Q}$ is 1-dimensional for $g \geq 3$. From this, $Pic(\overline{M}_g) \otimes \mathbb{Q}$ is $(h+2)$-dimensional, $h = [g/2]$, generated by λ (see below) and the boundary components of $\overline{M}_g$. The boundary of $\overline{M}_g$, $\overline{M}_g - M_g$ is the union $\cup_{i=0}^{h} \Delta_i$, where:

   $\Delta_0 =$ {closure of the locus of irreducible curves having one node}

   and

   $\Delta_i =$ {closure of the locus of the connected curves having two irreducible components, one of genus i and the other of genus $g-i$}.



   Pictorially, $Int\,\Delta_0$ and $Int\,\Delta_i$ are represented in the pictures above, where the integer near each irreducible component is the geometric genus of the component itself.

2. Let $\mathbf{H}_g$ be the locally closed subscheme of a suitable Hilbert scheme parametrizing stable curves in a fixed projective space $\mathbb{P}^{\nu-1}$. For instance one can construct such $\mathbf{H}_g$ by means of the tri-canonical embedding as in [16], with $\nu = 5g-5$. Over $\mathbf{H}_g$, lives a universal family $p : Z_g \longrightarrow \mathbf{H}_g$. The group $\mathbb{P}Gl(\nu)$ acts on $\mathbf{H}_g$ in the obvious way, so that it makes sense to consider the group $Pic(\mathbf{H}_g)^{\mathbb{P}Gl(\nu)}$ of isomorphisms classes of $\mathbb{P}Gl(\nu)$-invariant line bundles on $\mathbf{H}_g$. In [63] it is shown that $Pic(\overline{\mathcal{M}}_g) \cong Pic(\mathbf{H}_g)^{\mathbb{P}Gl(\nu)}$ and $Pic(\overline{M}_g)$ is a subgroup of these of

finite index. Moreover, Mumford shows that $Pic(\overline{\mathcal{M}}_g)$ is torsion free, so that $Pic(\overline{M}_g)$ and $Pic(\mathcal{M}_g)$ are related by the identification:

$$Pic(\mathcal{M}_g) \otimes \mathbb{Q} \cong Pic(M_g) \otimes \mathbb{Q} \cong Pic(\mathbf{H}_g)^{\mathbb{P}Gl(\nu)} \otimes \mathbb{Q}$$

i.e. all the Picard groups are lattices in the same $\mathbb{Q}$-vector space $Pic(\cdot) \otimes \mathbb{Q}$.

3. By 1 and 2 above, it follows that $Pic(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}\{\lambda, \Delta_0, \cdots, \Delta_h\}$, where with Δ_i we have denoted the divisor classes that they determine as well.

4. More results:
   If $g \geq 3$ (which will be supposed from now on) we have:
   $Pic(\mathcal{M}_g) = \mathbb{Z} \cdot \lambda$ (λ to be defined below)
   $Pic(\overline{\mathcal{M}}_g) = \mathbb{Z} \cdot \{\lambda, \delta_0, \ldots, \delta_h\}$
   The two preceeding results are due to Arbarello and Cornalba ([2]). To continue with, let us define λ on the moduli functor $\overline{\mathcal{M}}_g$. If $\pi : \mathfrak{X} \longrightarrow S$ is any stable curve over S and if ω_π is its relative dualizing sheaf, then $\pi_*\omega_\pi$ is a rank g vector bundle over S such that

$$\pi_*\omega_\pi \otimes \mathbf{k}(s) \cong H^0(\mathfrak{X}_s, \omega_{\mathfrak{X}_s}),$$

   $\mathfrak{X}_s$ being $\mathfrak{X} \times_S Spec(\mathbf{k}(s))$ (the scheme theoretical fiber over s) and ω_{C_s} the dualizing sheaf of the fiber. A line bundle on the moduli functor is hence given by the assignment $\lambda \mapsto \lambda_\pi$, where $\lambda_\pi = \wedge^g(\pi_*\omega_\pi) \in Pic(S)$.

We are going to define now, the classes δ_i's related to the divisors Δ_i in $\overline{M}_g$. Let us fix a family $\pi : \mathfrak{X} \longrightarrow S$ with $dim(S) = 1$ and S smooth.

1. Assume that the general fiber does not contain a singular point of type i ($0 \leq i \leq h$). Locally around each singular point of the special fiber, the family is given as $xy = \gamma(s)$, where δ_i is the divisor given by $\prod_{type\, i} \gamma = 0$.

2. Assume now that the general fiber does contain singular points of type i. We want sections of singular points (of type i): after a finite base change, we can obtain this: $\Sigma_1, \ldots, \Sigma_k$ sections of singular points of type i, and $P_1, \ldots, P_l$ isolated singular points of type i. Then one partially normalizes the family along the Σ_k, obtaining a family of (not necessarily connected) nodal curves, plus sections S_k, T_k of smooth points (in such a way that identifying S_k and T_k we obtain Σ_k). Then one has:

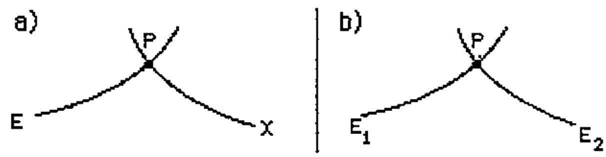
$$\delta_i = \Sigma_k \left(c_1(N_{S_k}) + c_1(N_{T_k}) + \sum_l m_l[\pi(p_l)] \right) \tag{6.4}$$

m_l being the multiplicity of p_l: locally, p_l is given by $xy = t^{n_l}$, where t is a local parameter on the base.

In $Pic(\overline{M}_g) \otimes \mathbb{Q}$ the relation between δ_i and Δ_i is given by the formula:

$$\delta_i = \frac{[\Delta_i]}{\sharp(aut(\eta_{\Delta_i}))} = [\Delta_i]_Q,$$

the so-called "Q-class", where by η_{Δ_i} we mean the generic point of Δ_i.
For $g \geq 3$ and $i \neq 1$ one has, e.g. $\delta_i = [\Delta_i]$ and $\delta_1 = 1/2[\Delta_1]$ while for $g = 2$ one has $\delta_0 = 1/2[\Delta_0]$ and $\delta_1 = 1/4[\Delta_1]$. In fact, for general curves of type i, with $g \geq 4$ and $i \geq 2$, there are no automorphisms. If $i = 2$ this is still true, provided that the intersection point P is not a Weierstrass point for the component of genus i. If $i = 1$, and $g \geq 3$, all the curves of Δ_1



have an automorphism of order 2 which is given by the map which sends $E \cup_P X$ onto $\iota(E) \cup_P X$, where E is an elliptic tail, $\iota : E \longrightarrow E$ the unique involution of E which fixes the intersection point P and X is a component of genus $g-1$.

If $g = 2$, arguing as above, we see that if $C_0 = E_1 \cup_P E_2$ then the two involutions of E_1 and E_2 fixing P give rise to an automorphism group of C_0 of order 4, i.e. $\{(\iota_1, 1), (\iota_1, \iota_2), (1, \iota_2), (1, 1)\}$.

Another interesting class in $Pic(\overline{\mathcal{M}}_g)$ is $\kappa_1 = \pi_*(c_1(\omega_\pi)^2)$ computed on each good family (i.e. proper and flat) $\pi : \mathfrak{X} \longrightarrow S$. There is an important relation linking κ_1, λ and $\delta = \sum_i \delta_i$.

6.1.4 The relation $\kappa_1 = 12\lambda - \delta$.

In the computations of the examples worked out in the following of these notes, we shall use the following important theorem, which is interesting in its own right.

Theorem 6.1 *The following relation holds in $A^1(\overline{M}_g)$:*

$$\kappa_1 = 12\lambda - \delta \quad \text{where} \quad \delta = \sum_{i=0}^{h} \delta_i. \tag{6.5}$$

The rest of this section shall be devoted to prove this theorem. The main tool to achieve such a result is the *Grothendieck-Riemann-Roch formula* for families of curves (GRR in the following), whose statement shall be briefly recalled below in the particular case we are interested in.

Suppose then that $\pi : X \longrightarrow S$ is any flat proper family of stable curves of genus g, where S is a regular scheme of dimension 1. After normalizations and blow-ups we can suppose X is a smooth surface. For such a situation, the GRR formula says that:

$$ch_t(\pi_! \omega_\pi) = \pi_*[(ch(\omega_\pi)) \cdot (td_t(\mathcal{T}_{X/S}))], \tag{6.6}$$

where $\mathcal{T}_{X/S}$ is the relative tangent sheaf of the morphism $\pi : X \longrightarrow S$.

Proof of the formula (6.5).

One applies the GRR formula, by computing both sides of (1.3). Due to the fact that the fibers of π are 1-dimensional, $\pi_! \omega_\pi$ is given by:

$$\pi_! \omega_\pi = R^0 \pi_* \omega_\pi - R^1 \pi_* \omega_\pi,$$

where the sum is taken in the Grothendieck group of the coherent sheaves on S. Since ω_π is flat and $h^0(X_s, \omega_{\pi|X_s}) = g$ and $h^1(X_s, \omega_{\pi|X_s}) = h^0(X_s, O_{X_s}) = 1$ are constant, $R^0\pi_*\omega_\pi$ and $-R^1\pi_*\omega_\pi$ are locally free of rank g and 1 respectively. Moreover, by properties of duality, $R^1\pi_*\omega_\pi \cong O_S$. We have then:

$$ch_t(R^0\pi_*\omega_\pi) = ch_t(\pi_*\omega_\pi) = g + c_1(\pi_*\omega_\pi)t + \ldots$$

and

$$ch_t(R^1\pi_*\omega_\pi) = 1.$$

The right hand side of (1.4) is:

$$ch_t(\omega_\pi) = 1 + c_1(\omega_\pi)t + \frac{1}{2}c_1(\omega_\pi)^2t^2 + \ldots,$$

and there are no higher order Chern classes involved, because ω_π is a line bundle. For the polynomial Todd class (whose definition follows obviously from the one of the total Todd class, see 13 in Sect. 4) $td_t(\mathcal{T}_{X/S})$, recall first that $c_i(\mathcal{T}_{X/S}) = (-1)^i c_i(\Omega^1_{X/S})$.

Let us suppose now that in our family $\pi : \mathfrak{X} \longrightarrow S$, the singular fibers occur in codimension 1. Then, as shown in [63], $\Omega^1_{X/S} \cong \omega_\pi \otimes \mathcal{I}_{[nodes]}$ ($\mathcal{I}_{[nodes]}$ is the ideal sheaf in X of the zero-dimensional closed subscheme of the nodes), from which it follows easily (cfr. [13]) that $c_1(\Omega^1_{X/S}) = c_1(\omega_\pi)$ and $c_2(\Omega^1_{X/S}) = [nodes]$. Hence:

$$td_t(\mathcal{T}_{X/S}) = 1 - \frac{1}{2}c_1(\omega_\pi)t + \frac{1}{12}(c_1(\omega_\pi)^2 + [nodes]t^2), \tag{6.7}$$

where $[nodes]$ denotes the class of the nodes.

Plugging all of these data in (1.2), we get the equality:

$$\begin{aligned} & g - 1 + c_1(\pi_*\omega_\pi)t + \ldots \\ = \; & \pi_*\left[(1 + c_1(\omega_\pi)t + \frac{1}{2}c_1(\omega_\pi)^2t^2 + \ldots)(1 - \frac{1}{2}c_1(\omega_\pi)t + \frac{1}{12}(c_1(\omega_\pi)^2 + \delta t^2)\right]. \end{aligned}$$

Multiplying and equating terms in equal codimension (notice that applying π_* the codimension of the pushed down classes "decreases" with respect to S), one gets:

$$\pi_*(c_1(\omega_\pi)) = 2g - 2 \tag{6.8}$$

together with the formula we wanted to prove:

$$\kappa_1 = \pi_*(c_1(\omega_\pi)^2) = 12c_1(\pi_*\omega_\pi) - \pi_*[nodes] = 12\lambda - \delta,$$

with the obvious meaning of the symbols.

QED

6.2 Review of some basic facts on the Brill-Nöther Matrix

6.2.1 A Riemann-Roch formula for effective divisors on curves.

In this section we shall review some elementary classical geometry of curves which, nevertheless, seems to be very important for stating and solving some enumerative problems in the moduli spaces of curves. To begin with, let D be an effective divisor of degree d on a smooth complex curve of genus $g \geq 2$. Let us write:

$$D = d_1P_1 + \ldots + d_kP_k, \tag{6.9}$$

with $\sum_i d_i = d$. Let $O_C(D)$ be the line bundle associated to the Cartier divisor induced by D. Then, the dimension of the $\mathbb{C}$-vector space of global holomorphic sections of $O_C(D)$ is prescribed by the *Riemann-Roch theorem* (i.e. Riemann-Roch formula + Serre-Duality):

$$h^0(O_C(D)) = 1 - g + d + h^0(C, K_C(-D)), \tag{6.10}$$

By $H^0(C, K_C(-D)) \subseteq H^0(C, K_C)$ we mean the $\mathbb{C}$-vector subspace of the holomorphic differentials vanishing at $P_i \in Supp(D)$ with multiplicity prescribed by D itself. Thanks to the work we made in section 5.2, we can make precise such a condition. Let $\omega \in H^0(C, K_C)$. We say that ω *vanishes at P_i with multiplicity at least d_i* iff $D^{i-1}\omega \in H^0(C, J^{i-1}K_C)$ vanishes at P_i. Let now $\omega = (\omega_1, \ldots, \omega_g)$ be a $\mathbb{C}$-basis of $H^0(C, K_C)$. If ω is any non zero section of $K_C(-D)$, then there is a non-zero g-tuple $(a_1, \ldots, a_g) \in \mathbb{C}^g$ such that:

$$\omega = a_1\omega_1 + \ldots + a_g\omega_g =: \underline{\omega} \cdot A$$

where we set $A = (a_1, \ldots, a_g)^T$. Hence the following equations must be satisfied:

$$\begin{cases} D^{d_1-1}\underline{\omega}(P_1) \cdot A = 0, \\ D^{d_2-1}\underline{\omega}(P_2) \cdot A = 0, \\ \vdots \\ D^{d_k-1}\underline{\omega}(P_k) \cdot A = 0 \end{cases} \tag{6.11}$$

The system (6.11) admits a local representation as follows. Pick k local (analytic) charts (U_j, z_j) trivializing K_C such that $P_j \in U_j$ and $z_j(P_j) = 0$. Then, we have local representations of the ω_i in each open subset $U = U_j$ of the form:

$$\omega_{i|U_j} = f_{ij}(z_j)dz_j,$$

By the very definitions of the jets bundles, the system (6.11) may be translated in the open set $U =: U_1 \times \ldots \times U_k \subseteq C \times \ldots \times C$ as:

$$\mathcal{BN}_U(D) \cdot (a_1, \ldots, a_g)^T = 0 \tag{6.12}$$

where we set:

$$\mathcal{BN}_U(D) = \begin{pmatrix} f_1(P_1) & \dots & f_g(P_1) \\ f_1'(P_1) & \dots & f_g'(P_1) \\ & \vdots & \\ f_1^{(d_1-1)}(P_1) & \dots & f_g^{(d_1-1)}(P_1) \\ \vdots & \ddots & \vdots \\ f_1(P_k) & \dots & f_g(P_k) \\ & \vdots & \\ f_1^{(d_k-1)}(P_1) & \dots & f_g^{(d_k-1)}(P_k) \end{pmatrix}. \tag{6.13}$$

Obviously, the notation $f_i^{(h)}(P_j)$ means that we are considering the h-th derivative of the local holomorphic function f_i with respect to the local parameter z_j, $\frac{d^h f_i}{dz_j^h}$, evaluated at $z_j = 0$. The system (6.12) has a non trivial solution if and only if $rank(\mathcal{BN}_U(D)) < g$.

Exercise 6.1 Prove that the rank of the matrix $\mathcal{BN}_U(D)$ depends only on the divisor D. In particular it does not depend nor on the choice of the basis $\underline{\omega}$ of holomorphic differentials neither on the local trivializing charts (U_j, z_j) chosen around each P_i $(1 \leq j \leq k)$.

By the above exercise, in equation (6.13), we may hence skip in the notation the dependence on the open set U and on the local coordinates z_i's. We may so speak of the *Brill-Nöther matrix* associated to the divisor D. We shall simply write $\mathcal{BN}(\mathcal{D})$ to denote the expression:

$$\underline{\omega}(P_1) \wedge D\underline{\omega}(P_1) \wedge \ldots \wedge D^{d_1-1}\underline{\omega}(P_1) \wedge \ldots \wedge \underline{\omega}(P_k) \wedge D\underline{\omega}(P_k) \wedge \ldots \wedge D^{d_k-1}\underline{\omega}(P_k)$$

where the *wedge* serves to recall us that we are looking at the rank of the matrix (6.13), and the use of the basis $\underline{\omega}$ is to remind us that such a rank does not depend on the local coordinates (indeed it does not depend on the basis $\underline{\omega}$ too).

Example 6.1 Let C be a curve of genus g and consider the divisors $D_1 = 2P + Q + R$, $D_2 = 3P + 2Q$. Then:

$$\begin{aligned} \mathcal{BN}(D_1) &= \underline{\omega}(P) \wedge D\underline{\omega}(P) \wedge \underline{\omega}(Q) \wedge \underline{\omega}(R), \\ \mathcal{BN}(D_2) &= \underline{\omega}(P) \wedge D\underline{\omega}(P) \wedge D^2\underline{\omega}(P) \wedge \underline{\omega}(Q) \wedge D\underline{\omega}(Q). \end{aligned}$$

Exercise 6.2 This is a revisiting of the exercise 1.3. A Weierstrass point on a curve C of genus $g \geq 2$ is a point P such that $h^0(C, K_C(-gP)) > 0$. What does it mean in terms of the Brill-Nöther matrix? Once one has fixed a basis $\underline{\omega}$ of $H^0(C, K_C)$, the determinant of the matrix $\mathcal{BN}(gP)$ is said to be the *wronskian section* associated to $\underline{\omega}$. Shows that there are only finitely many Weierstrass points, and count them.

By the above considerations it turns out that the dimension of the vector space $H^0(C, K_C(-D))$ of all holomorphic differentials on C vanishing at $P_i \in Supp(D)$ with multiplicity at least the one prescribed by D itself is $g - rank(\mathcal{BN}(D))$. Hence formula (6.10) can be rewritten as:

$$h^0(\mathcal{O}_C(D)) = d + 1 - rank(\mathcal{BN}(D)).$$

6.2.2 Pointed jets bundles on curves.

In this section we want to deal with some useful tools sometimes called in the literature *pointed jets bundle* and used quite often for solving enumerative problems. The idea consists, inspired by what have been seen in Section 5.2, in trying to interpret the Brill-Nöther matrix as the local representation of a map of vector bundles on some n-fold cartesian product of a curve by itself.

Let C^k be the k-fold product of C by itself, i.e. the k-th fold fibered product of C over $Spec(\mathbb{C})$. Let $p_i : C^k \longrightarrow C$ be the projection onto the i-th factor. Let $p_i^* K_C$ be the pull-back of K_C via p_i. Let $D = d_1 P_1 + \ldots + d_k P_k$ be a divisor and consider the following map of vector bundles over C^k:

$$\begin{array}{ccc} C^k \times H^0(C, K_C) & \xrightarrow{D^{d_1-1} \oplus \ldots \oplus D^{d_k-1}} & J^{d_1-1}(p_1^* K_C) \oplus \ldots \oplus J^{d_k-1}(p_k^* K_C) \\ & {\scriptstyle pr_1} \searrow \quad \swarrow & \\ & \overbrace{C \times \ldots \times C}^{k\ times} & \\ & \downarrow {\scriptstyle p} & \\ & Spec(\mathbb{C}) & \end{array} \qquad (6.14)$$

defined by:

$$((Q_1, \ldots, Q_k), \omega) \mapsto (D^{d_1-1}\omega(Q_1), \ldots, D^{d_k-1}\omega(Q_k)).$$

For sake of brevity we shall adopt the following notational convention: $\underline{d}$ will denote a multi-index $\underline{d} = (d_1, \ldots, d_k)$ such that $|\underline{d}| = d$. Moreover we shall set:

$$D^{\underline{d}} =: D^{d_1-1} \oplus \ldots \oplus D^{d_k-1}.$$

Clearly, if $D = d_1P_1 + \ldots + d_kP_k$ is such that $h^0(C, K_C(-D)) > 0$, then $(P_1, \ldots, P_k) \in \underbrace{C \times \ldots \times C}_{k\ times}$ is such that:

$$rk(D^{d_1-1} \oplus \ldots \oplus D^{d_k-1}) < g,$$

or, which is the same, $rk(\mathcal{BN}(D) < g$. If D is general, then $rk(\mathcal{BN}(D) = min(d, g)$. The *degeneracy locus* of the map $D^{d_1-1} \oplus \ldots \oplus D^{d_k-1}$ is the locus of points $(Q_1, \ldots, Q_k) \in C^k$ such that $rk(\mathcal{BN}(D) < min(d, g)$. A point of C^k, $(Q_1, \ldots, Q_k)$ which lies in the degeneracy locus of the map $D^{\underline{d}}$ will correspond to a *special divisor* in the sense that it imposes to the canonical divisors less conditions than it should do. However the degeneracy locus of the map $D^{\underline{d}}$ contains the diagonals $D_{ij} = \{(Q_1, \ldots, Q_k) \in C^k : Q_i = Q_j\}$ as fixed components. In fact, if $P_i = P_j$, one has that the rank of $\mathcal{BN}(D)$ is certainly less than $min(d, g)$. In fact one has at least 2 equal rows in $\mathcal{BN}(D)$. The purpose now is to figure out the multiplicities of the diagonals. Let us start with an example.

Example 6.2 Let $d < g$. Consider the points $(P, Q) \in C \times C$ which are in the degeneracy locus of the map:

$$D^{d-2} \oplus D^0 : C^2 \times H^0(C, K_C) \longrightarrow J^{d-2} p_1^* K_C \oplus p_2^* K_C,$$

Then, by writing down the local representation, it turns out that this is equivalent to consider:

$$\{(P, Q) : \underline{\omega}(P) \wedge \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge \underline{\omega}(Q) = 0\}$$

where by the vanishing of the above formal *exterior form* we mean the locus of (P, Q) such that $rk(\mathcal{BN}((d-1)P + Q)) < d$. We want to compute the multiplicity of the diagonal D_{12} in such a degeneracy locus. Pick $P \in C$ which is not a Weierstrass point. Then:

$$\underline{\omega}(P) \wedge D\underline{\omega} \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge \underline{\omega},$$

vanishes at P. By derivating the above expression as a section of a subspace of $H^0(C, K_C)$, we see that:

$$\underline{\omega}(P) \wedge \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge D^i\underline{\omega},$$

vanishes at P for all $0 \le i \le d-2$, because for i in that range:

$$\underline{\omega}(P) \wedge \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge D^i\underline{\omega}(P) = 0,$$

because two rows at least are equal. Hence $\underline{\omega}(P) \wedge \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge \underline{\omega}$ vanishes at P with multiplicity at least $d-1$. Moreover:

$$\underline{\omega}(P) \wedge \ldots D^{d-2}\underline{\omega}(P) \wedge D^{d-1}\underline{\omega}(P) = 0 \iff P \text{ is a Weierstrass point.}$$

Hence, for a general P, the form:

$$\underline{\omega}(P) \wedge \ldots \wedge D^{d-2}\underline{\omega}(P) \wedge \underline{\omega},$$

vanishes at P with multiplicity exactly $d-1$, so that, as a matter of fact, the map

$$D^{d-2} \oplus D^0$$

induces an evaluation map from the bundle $C^2 \times H^0(C, K_C)$ to a bundle $\mathcal{F}_{d-1,1}$ which has the same Chern polynomial of the bundle $J^{d-2}p_1^*K_C \oplus p_2^*K_C(-(d-1)D_{12})$ (or, in other words, it is an extension of the bundle $J^{d-2}p_1^*K_C$ by means of $p_2^*K_C(-(d-1)D_{12})$).

Example 6.3 We modify the above example a little bit. We now look for all the points $(P, Q) \in C \times C$ such that $(d-2)P + 2Q$ imposes less than d conditions on the canonical divisor. This means that it should live in the degeneracy locus of the map:

$$D^{d-3} \oplus D : C^2 \times H^0(C, K_C) \longrightarrow J^{d-3}p_1^*K_C \oplus J^1 p_2^*K_C.$$

As above, by fixing P, this means that:

$$\underline{\omega}(P) \wedge \ldots \wedge D^{d-3}\underline{\omega}(P) \wedge D\underline{\omega},$$

must vanish at Q. But the above expression surely vanishes at $Q = P$. If $\underline{\omega} = \underline{f} \cdot dz$ in a neighbourhood of P, the above expressions may be locally represented as a $2 \times d$ matrix of functions:

$$(\underline{u}(z) \wedge \underline{u}'(z)),$$

where $\underline{u}(z)$ is a local representation of:

$$\underline{\omega}(P) \wedge \ldots D^{d-3}\underline{\omega}(P) \wedge \underline{\omega}$$

that by example 6.2 vanishes at P with multiplicity $d-2$, so that $D^{d-3} \oplus D$ induces a map which lands to a bundle $\mathcal{F}_{d-2,2}$ which has the same Chern polynomial of the bundle:

$$J^{d-3}p_1^*K_C \oplus J^1 p_2^*K_C(-(d-2)D_{12})$$

The matter of the two previous examples may be easily generalized. In fact, for any d, if $deg(D) = d$, the support of the divisors $d_1P_1 + \ldots + d_kP_k$ that imposes on K_C less than $min(d, g)$ conditions, must be searched in the degeneracy locus of the map of vector bundles:

$$\begin{array}{ccc} C^k \times H^0(C, K_C) & \xrightarrow{D^{d_1-1}\oplus\ldots\oplus D^{d_k-1}} & \mathcal{F}_{d_1,\ldots,d_k} \\ & {\scriptstyle pr_1}\searrow \quad \overbrace{C \times \ldots \times C}^{k\ times} \quad \swarrow & \\ & \downarrow {\scriptstyle p} & \\ & Spec(\mathbb{C}) & \end{array} \tag{6.15}$$

where $\mathcal{F}_{d_1,\ldots,d_k}$ is defined as follows. Let $\mathcal{D}$ be the closed subscheme of C^{k+1} whose corresponding cicle in $A^1(C^k)$ is given by:

$$[\mathcal{D}] = d_1D_{1,k+1} + \ldots + d_kD_{k,k+1},$$

and let $\mu : C^{k+1} = C^k \times C \longrightarrow C^k$ be the projection onto the first k factors. Then:

$$\mathcal{F}_{d_1,\ldots,d_k} = \mu_*\left(O_{\mathcal{D}} \otimes p_{k+1}^* K_C\right).$$

This is a vector bundle on C^k whose fiber at $(P_1, \ldots, P_k)$ is exactly $H^0\left(C^k, K_C/K_C(-d_1P_1 - \ldots - d_kP_k)\right)$.

Exercise 6.3 Check that the following exact sequences hold:

$$0 \longrightarrow J^{d-k-1}\left(p_k^*K_C\left(-\sum_{1\leq i<k} d_iD_{ik})\right)\right) \longrightarrow \mathcal{F}_{d_1,\ldots,d_k} \longrightarrow p_k^*\mathcal{F}_{d_1,\ldots,d_{k-1}} \longrightarrow 0.$$

The above exercise proves that the Chern polynomial of $\mathcal{F}_{d_1,\ldots,d_k}$ coincides with the Chern polynomial of the direct sum bundle:

$$J^{d_1-1}\left(p_1^*K_C \oplus J^{d_2-1}p_2^*K_C(-d_1D_{12}) \oplus \ldots \oplus J^{d_k-1}\left(p_k^*K_C(-\sum_{1\leq i<k} d_iD_{ik}\right)\right). \tag{6.16}$$

We shall often denote the bundle $\mathcal{F}_{d_1,\ldots,d_k}$ as:

$$\mathcal{E}\left(J^{d_1-1}\left(p_1^*K_C\right) \oplus J^{d_2-1}p_2^*K_C(-d_1D_{12}) \oplus \ldots \oplus J^{d_k-1}\left(p_k^*K_C(-\sum_{1\le i<k} d_iD_{ik}\right)\right), \tag{6.17}$$

for the following reason: if $k=2$,

$$\mathcal{F}_{d_1,d_2} = \mathcal{E}\left(J^{d_1-1}K_C \oplus J^{d_2-1}\left(K_C(-d_1D_{12})\right)\right)$$

is the *extension* of the bundle $J^{d_1-1}K_C$ by the kernel bundle $J^{d_2-1}\left(K_C(-d_1D_{12})\right)$. In other words, the notation (6.17) is to emphasize the fact that the $\mathcal{F}_{d_1,\ldots,d_k}$ are (in general non-split) extensions of bundles $\mathcal{F}_{d_1,\ldots,d_h}$, with $h \le k$ and the fundamental blocks appearing in the direct sum on the right hand side of (6.17).

Before going on we give a couple of examples to get familiar with the above formula.

Example 6.4 Let C be a curve of genus 5. Consider all $(P,Q,R) \in C^3$ such that:

$$4P + 3Q + 2R,$$

move in a canonical divisor. Then (P,Q,R) must lie in the degeneracy locus of the map $D^3 \oplus D^2 \oplus D$ of vector bundles:

$$O_{C^3} \otimes H^0(C,K_C) \longrightarrow \mathcal{E}\left(J^3p_1^*K_C \oplus J^2p_2^*K_C(-4D_{12}) \oplus J^1p_3^*K_C(-4D_{13}-3D_{23})\right).$$

The bundle $\mathcal{F}_{\underbrace{1,\ldots,1}_{d\ times}}$ is denoted in the literature (see [65], [26]) by $\mathcal{F}_d$. Consider the natural surjection:

$$\mathcal{F}_d \longrightarrow p_{1,2,\ldots,d-1}^*\mathcal{F}_{d-1} \longrightarrow 0$$

Hence, by recalling the meaning of formula (6.17) one has the exact sequence:

$$0 \longrightarrow p_d^*K_C(-\Delta_d) \longrightarrow \mathcal{F}_d \longrightarrow p_{1,2,\ldots,d-1}^*\mathcal{F}_{d-1} \longrightarrow 0, \tag{6.18}$$

having set, following [26],

$$\Delta_d = D_{1,d} + \ldots + D_{d-1,d}.$$

Using the multiplicativity property of Chern classes, its hence a simple computational exercise to show that:

Proposition 6.1 *The Chern polynomial of the bundle $\mathcal{F}_d$ is given by:*

$$c_t(\mathcal{F}_d) = (1+K_1t)(1+(K_2-D_{12})t)\cdot \quad (6.19)$$

$$\cdot(1+(K_3-D_{13}-D_{23})t)\cdot\ldots\cdot \quad (6.20)$$

$$\cdot(1+(K_d-\sum_{i=1}^{d-1}D_{id})t). \quad (6.21)$$

In particular

$$c_1(\mathcal{F}_d) = K_1+K_2+\ldots+K_d-\left(\sum_{i=1}^{d-1}D_{id}\right), \quad (6.22)$$

6.3 Some Easy Classical Enumerative Problems

The purpose of this section is to provide some easy examples of application of the above techniques.

Example 6.5 The total weight of the Weierstrass points on a curve of genus $g\geq 2$. Let C be a curve of genus $g\geq 2$ and C^g its g-fold fiber product over $Spec(\mathbb{C})$. D_{ij} are the big diagonals and $K_i = c_1(p_i^*K_C)$. We shall use the following relations (see Proposition 5.2).

$$D_{ij}^2 = -K_iD_{ij} = -K_jD_{ij},$$

and

$$\int_{C^g} K_iD_{12}\cdot\ldots\cdot D_{d-1,d} = \int_C K_C = 2g-2.$$

On C^g define the divisor $Z(\underbrace{\omega\wedge\ldots\wedge\omega}_{g\ times})$, i.e. the support is given by the points in the degeneracy locus of the map:

$$D^0\oplus D^0\oplus\ldots\oplus D^0 : C^g\times H^0(C,K_C)\longrightarrow\mathcal{F}_g,$$

Then, by applying *Porteous' formula*, we find:

$$[Z(\underbrace{\omega\wedge\ldots\wedge\omega}_{g\ times})] = c_1(\mathcal{F}_g).$$

By virtue of formula (6.22), the right hand side is

$$c_1(\mathcal{F}_g) = K_1+\ldots+K_g-\sum_{1\leq i<j<g}D_{ij};$$

and this is the class $[Z(\underbrace{\underline{\omega}\wedge\ldots\wedge\underline{\omega}}_{g\ times})]$ in the Chow group $A^1(C^g)$. If we intersect with the cycle $D_{12}\cdot\ldots\cdot D_{g-1,g}$, we should get the canonical divisors of the form gP. One finds:

$$\int_{C^g}\left(K_1+K_2+\ldots+K_g-\sum_{1\leq i<j<g} D_{ij}\right) D_{12}\cdot\ldots\cdot D_{d_1,d}=$$
$$=2g(g-1)+\frac{g(g-1)}{2}\cdot 2(g-1)=g(g-1)(g+1).$$

Exercise 6.4 Example 6.5 may be also treated as follows. The map:

$$D^0\oplus D^0\oplus\ldots\oplus D^0 : C^g\times H^0(C,K_C)\longrightarrow\mathcal{F}_g$$

induces a map between the top exterior powers of the two bundles above. In other words we have:

$$\bigwedge^g(D^0\oplus D^0\oplus\ldots\oplus D^0) : O_{C^g}\longrightarrow\bigwedge^g\mathcal{F}_g,$$

hence a section $\bigwedge^g(D^0\oplus D^0\oplus\ldots\oplus D^0)\in H^0(C^g,\bigwedge^g\mathcal{F}_g)$.

i) Prove that the line bundle $\bigwedge^g\mathcal{F}_g$ is isomorphic to the bundle

$$\bigotimes_{i=1}^g p_i^*K_C\otimes O_{C^g}\left(-\sum_{i=2}^d\Delta_i\right),$$

and that:

$$\bigwedge^g(D^0\oplus D^0\oplus\ldots\oplus D^0)\cong\underbrace{\underline{\omega}\wedge\ldots\wedge\underline{\omega}}_{g\ times}.$$

ii) Use i) to get the expression of the class $[Z(\underbrace{\underline{\omega}\wedge\ldots\wedge\underline{\omega}}_{g\ times})]$

The exercise explains why we used the notation $[Z(\underbrace{\underline{\omega}\wedge\ldots\wedge\underline{\omega}}_{g\ times})]$ in example 6.5.

Example 6.6 Weierstrass points again. Fix $a, b > 0$ and $a + b = g$. Let

$$Z(\underline{\omega} \wedge \ldots \wedge \ldots D^{(a-1)}\underline{\omega} \wedge \underline{\omega} \wedge \ldots \wedge D^{(b-1)}\underline{\omega})$$

be the divisor in $C \times C$ defined by all the points (P, Q) such that

$$h^0(K_C - aP - bQ) > 0.$$

Its class is given by applying *Porteous' formula* to the map:

$$D^{(a-1)} \oplus D^{(b-1)} : O_{C\times C} \otimes H^0(C, K_C) \longrightarrow \mathcal{E}\left(J^{a-1}(p_1^*K_C) \oplus J^{b-1}(p_2^*K_C(-aD_{12}))\right). \quad (6.23)$$

Hence we get:

$$[Z(\underline{\omega} \wedge \ldots \wedge \ldots D^{(a-1)}\underline{\omega} \wedge \underline{\omega} \wedge \ldots \wedge D^{(b-1)}\underline{\omega})] = c_1(J^{a-1}(p_1^*K_C) \oplus J^{b-1}(p_2^*K_C(-aD_{12})).$$

i.e.:

$$[Z(\underline{\omega} \wedge \ldots \wedge \ldots D^{(a-1)}\underline{\omega} \wedge \underline{\omega} \wedge \ldots \wedge D^{(b-1)}\underline{\omega})] = c_1(J^{a-1}(p_1^*K_C)) + c_1(J^{b-1}(p_2^*K_C(-aD_{12})).$$

Now we use the exact sequence (5.17). For instance we have:

$$0 \longrightarrow (p_2^*K_C)^{\otimes b}(-aD_{12}) \longrightarrow J^{b-1}(p_2^*K_C(-aD_{12})) \longrightarrow J^{b-2}(p_2^*K_C(-aD_{12})) \longrightarrow 0,$$

so that:

$$c_1(J^{b-1}(p_2^*K_C(-aD_{12})) = c_1((p_2^*K_C)^{\otimes b}(-aD_{12})) + c_1(J^{b-2}(p_2^*K_C(-aD_{12}))).$$

By repeatingly applying the exact sequence, at last, one finds:

$$c_1(J^{a-1}(p_1^*K_C)) = \frac{a(a+1)}{2}K_1,$$

and

$$c_1(J^{b-1}(p_2^*K_C(-aD_{12}))) = \frac{b(b+1)}{2}K_2 - abD_{12}$$

Hence:

$$[Z(\underline{\omega} \wedge \ldots \wedge \ldots D^{(a-1)} \wedge \underline{\omega} \wedge \ldots \wedge D^{(b-1)}\underline{\omega})] = \frac{a(a+1)}{2}K_1 + \frac{b(b+1)}{2}K_2 - abD_{12}$$

in $A^1(C \times C)$. Of course, intersecting with the divisor D_{12} and taking the degree, one should obtain the *total weight of the Weierstrass points.* In fact:

$$[Z(\underline{\omega}\wedge\ldots\wedge\ldots D^{(a-1)}\underline{\omega}\wedge\underline{\omega}\wedge\ldots\wedge D^{(b-1)}\underline{\omega})]D_{12}=$$
$$=\left(\frac{a(a+1)}{2}K_1+\frac{b(b+1)}{2}K_2-abD_{12}\right)D_{12}=$$
$$=(\frac{a(a+1)}{2}+\frac{b(b+1)}{2}+ab)K_1D_{12}.$$

i.e.

$$\int_{C\times C}[Z(\underline{\omega}\wedge\ldots\wedge D^{(a-1)}\underline{\omega}\wedge\underline{\omega}\wedge\ldots\wedge D^{(b-1)}\underline{\omega})]D_{12}=$$
$$\left(\frac{a(a+1)}{2}+\frac{b(b+1)}{2}+ab\right)\int_{C\times C}K_1D_{12}=$$
$$=(g-1)(a^2+b^2+2ab+a+b)=(g-1)(g^2+g)=(g-1)g(g+1).$$

Exercise 6.5 The other way to solve the above enumerative problem is to consider the top exterior product of the map (6.23), namely:

$$\bigwedge^g(D^{a-1}\oplus D^{b-1}):O_{C\times C}\longrightarrow\bigwedge^g\mathcal{E}\left((J^{a-1}(p_1^*K_C)\oplus J^{b-1}(p_2^*K_C(-aD_{12}))\right),$$

which may hence identified with a holomorphic section of the line bundle

$$\bigwedge^g(J^{a-1}(p_1^*K_C))\oplus J^{b-1}(p_2^*K_C(-aD_{12})).$$

i) What is the line bundle $\bigwedge^g(J^{a-1}(p_1^*K_C)\oplus J^{b-1}(p_2^*K_C(-aD_{12})))$ isomorphic to?

ii) Describe explicitly the section $\bigwedge^g(D^{a-1}\oplus D^{b-1})$. It should be represented by a $g\times g$ matrix which, in a sense, looks like a wronskian.

Example 6.7 The number of the bitangents on a plane quartic Of course this number is 28. It is a very classical number that may be easily gotten by passing to the *dual curve* and repeatedly using the *Plücker formulas.*

Exercise 6.6 Compute the number of bitangents on a smooth quartic by using the Plücker formulas and the *dual curve.*

Here, we shall use instead our *vector-bundles' maps machinery* to make some practise. We are given of a smooth plane quartic C in $\mathbb{P}^2$. A *bitangent* is a line L such that $L\cdot C = 2P+2Q$. Since the canonical series on a plane quartic is cut out by the linear series of the lines of $\mathbb{P}^2$, it turns out that the number of bitangents corresponds to the number of unordered pairs (P,Q) such that $2P+2Q$ is a canonical divisor. The map we should look at, is:

$$D\oplus D : C\times C\times H^0(C,K_C)\longrightarrow \mathcal{E}\left(J^1p_1^*K_C\oplus J^1p_2^*K_C(-2D_{12})\right),$$

The expected codimension of the locus is 2 and it coincides with the actual codimension (for evident geometrical reasons). We may hence apply the Porteous' formula, getting:

$$\sharp(\textit{bitangents}) = \frac{1}{2}\int_{C\times C} c_2\left(J^1p_1^*K_C\oplus J^1p_2^*K_C(-2D_{12})\right)$$

By the two exact sequences:

$$0\longrightarrow p_1^*K_C^{\otimes 2}\longrightarrow J^1p_1^*K_C\longrightarrow p_1^*K_C\longrightarrow 0$$

and:

$$0\longrightarrow p_2^*K_C^{\otimes 2}(-2D_{12})\longrightarrow J^1p_2^*K_C(-2D_{12})\longrightarrow p_2^*K_C(-2D_{12})\longrightarrow 0$$

one gets:

$$c_t(J^1p_1^*K_C) = (1+K_1t)(1+2K_1t) = 1+3K_1t+2K_1^2,$$

and

$$\begin{aligned} & c_t(J^1p_2^*K_C(-2D_{12})) = (1+(K_2-2D_{12})t)(1+(2K_2-2D_{12})t) = \\ = \;& 1+(3K_2-4D_{12})t+(2K_2^2-6K_2D_{12}+4D_{12}^2)t^2 \end{aligned}$$

Hence:

$$\begin{aligned} & c_2(J^1p_1^*K_C\oplus J^1p_2^*K_C(-2D_{12}) = \\ = \;& 9K_1K_2-12K_1D_{12}+2K_1^2+2K_2^2-6K_2D_{12}+4D_{12}^2. \end{aligned}$$

Now $K_1^2 = K_2^2 = 0$ (because is the self-intersection of a fiber on a surface - $C\times C$ - mapped on a curve - C - thought as first or second factor. Moreover

$$\int_{C\times C} K_1D_{12} = \int_{C\times C} K_2D_{12} = -\int_{C\times C} D_{12}^2 = 4$$

and $\int_{C\times C} K_1K_2 = 16$.

The solution of our enumerative problem is then:

$$\begin{aligned}\sharp(bitangents) &= \frac{1}{2}\int_{C\times C} c_2(J^1p_1^*K_C\oplus J^1p_2^*K_C(-2D_{12}))\\ &= \frac{1}{2}(9\times 16-12\cdot 4-6\cdot 4-4\cdot 4)=28.\end{aligned}$$

6.4 Working with Families of Curves

The nice thing of the speculations we did above about these pointed jets bundles, is that one may extend them for families of curves. Let $\pi:\mathfrak{X}\longrightarrow S$ be a smooth curve of genus g over S. In this case, since all the fibers are smooth, the relative dualizing sheaf ω_π coincides with the sheaf of the relative differentials $\Omega^1_{\mathfrak{X}/S}$, which may hence also called the *relative canonical bundle*. By the way, for sake of uniformity, we shall denote such a bundle with the same symbol ω_π. Let us consider the k-fold fibered product of $\mathfrak{X}$ over S.

$$\begin{array}{c}\overbrace{\mathfrak{X}\times_S\ldots\times_S\mathfrak{X}}^{k-times}\\ \rho\Big\downarrow\\ S\end{array}\tag{6.24}$$

Then, associated to a multi-index $\underline{d}=(d_1,\ldots,d_k)$ of lenght d, there is a map of vector bundles over $\mathfrak{X}^k_S:=\mathfrak{X}^k=\times^k_{i=1S}\mathfrak{X}$ defined as:

$$\begin{array}{ccc}D^{d_1-1}\oplus\ldots\oplus D^{d_k-1}:\rho^*\mathbb{E} & \longrightarrow & \mathcal{F}_{d_1,\ldots,d_k}\\ & \searrow\ \mathfrak{X}^k\ \swarrow & \\ & \Big\downarrow\rho & \\ & S & \end{array}\tag{6.25}$$

where we are setting now:

$$\begin{aligned}&\mathcal{F}_{d_1,\ldots,d_k}=\\ &=\mathcal{E}\left(J^{d_1-1}_{p_1}(p_1^*\omega_\pi)\oplus\ldots\oplus J^{d_k-1}_{p_k}(p_k^*\omega_\pi(-d_1D_{1k}-d_2D_{2k}-\ldots-d_{k-1}D_{k-1,k}))\right)\end{aligned}$$

The particular case corresponding to $\underline{d}=\underbrace{(1,\ldots,1)}_{d\ times}$ gives rise to the bundle $\mathcal{F}_d$ (see [65], [26]), together with its standard exact sequence:

$$0\longrightarrow p_d^*\omega_\pi(-\Delta_d)\longrightarrow\mathcal{F}_d\longrightarrow p^*_{1,2,\ldots,d-1}\mathcal{F}_{d-1}\longrightarrow 0.\tag{6.26}$$

6.4.1 Generalities on the classes of $A^*(M_g)$

Recall the definition of $A^*(M_g)$. It is nothing but the $\mathbb{Q}$-vectorspace $[\oplus A_i(M_g)] \otimes \mathbb{Q}$ equipped with the product induced by the ring $A^*(\tilde{M}_g)^G$ of any smooth branched covering $\tilde{M}_g \longrightarrow M_g$. Consider the *forgetful map*:

$$\pi : M_{g,1} \longrightarrow M_g.$$

It is equipped with an invertible Q-sheaf, ω_π, which is represented by $\omega_{\tilde{\pi}}$, the relative dualizing sheaf of the map $\tilde{\pi} : \tilde{M}_{g,1} \longrightarrow \tilde{M}_g$. We define:

$$\kappa_i = \pi_* c_1(\omega_\pi)^{i+1},$$

and

$$\lambda_i = c_i(\mathbb{E}),$$

where $\mathbb{E}$ is a locally free Q-sheaf of rank g called the *Hodge Bundle* over M_g. In the sequel the following result by Mumford, ([65], p. 307), here assumed without proof, shall be used.

Proposition 6.2

$$c_t(\mathbb{E}^\vee) = \frac{1}{c_t(\mathbb{E})} = (1 - \lambda_1 t + \lambda_2 t^2 - \ldots + (-1)^g \lambda_g t^g)$$

The Chern classes are taken in the following sense:

$$c_i(\omega_\pi) =: \frac{1}{\sharp(G)} (\phi_{g,1})_* (c_i(\omega_{\tilde{\pi}})))$$

where $\phi_{g,1} : \overline{\tilde{M}_{g,1}} \longrightarrow \overline{M_{g,1}}$ is a global smooth covering of $\overline{M_{g,1}}$.

6.4.2 The hyperelliptic locus

The purpose of this section is to convince ourselves that the fundamental class of the hyperelliptic locus in the Chow group $A^{g-2}(M_g)$ may be easily computed by means of the tools developed above. Recall that a smooth curve is hyperelliptic if and only if there is a point $P \in C$ such that $h^0(C, K_C(-2P)) = g - 1$. This means that the Brill-Nöther matrix associated to the divisor $2P$, $\mathcal{BN}(2P) = \underline{\omega}(P) \wedge D\underline{\omega}(P)$ has rank 1.

The equation $\underline{\omega}(P) \wedge D\underline{\omega}(P) = 0$ means the common zero locus of the $g-1$ minors of the matrix $\mathcal{BN}(2P)$. The *expected codimension* of the hyperelliptic locus is therefore $g-1$ in $M_{g,1}$. But we also know that the hyperelliptic locus has codimension $g-2$ in M_g: we are hence in condition to apply the Porteous' formula to the map of vector bundles:

$$\pi^*\mathbb{E} \longrightarrow J^1\omega_\pi .$$

It turns out that:

$$[H_g] = \frac{1}{2g+2}\pi_*[Z(D)] = \pi_*\Delta_{1,g-1}(c_t(J^1\omega_\pi - \pi^*\mathbb{E}))$$

Now, using Proposition 6.2, one has:

$$c_t(\pi^*\mathbb{E}) = 1 + \pi^*\lambda t + \pi^*\lambda_2 t^2 + \pi^*\lambda_{g-1} t^{g-1}$$

So that:

$$\frac{1}{c_t(\pi^*\mathbb{E})} = 1 - \pi^*\lambda_1 t + \pi^*\lambda_2 t^2 - \ldots + (-1)^{g-2}\pi^*\lambda_{g-2} t^{g-2}$$

The Chern polynomial $c_t(J^1\omega_\pi - \pi^*\mathbb{E})$ is hence given by:

$$c_t(J^1\omega_\pi - \pi^*\mathbb{E}) = (1 + 3Kt + 2K^2t^2)\pi^*(1 - \pi^*\lambda_1 t + \pi^*\lambda_2 t^2 - \ldots + (-1)^{g-2}\pi^*\lambda_{g-2} t^{g-2})$$

Porteous' formula gives us:

$$[H_g] = \frac{1}{2g+2}\pi_*[Z(D)] = \frac{1}{2g+2}\pi_* \begin{vmatrix} c_1 & c_2 & \ldots & c_{g-1} \\ 1 & c_1 & \ldots & c_{g-2} \\ \vdots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & c_1 \end{vmatrix} (J^1\omega_\pi - \pi^*\mathbb{E}).$$

After some combinatorics, the above formula yields ([65], p. 314):

$$\begin{aligned} [H_g] &= \frac{1}{2g+2}\{(2^g-1)\kappa_{g-2} - (2^{g-1}-1)\lambda_1\kappa_{g-3} + \ldots + \\ &+ (-1)^{(g-3)} \cdot 7 \cdot \lambda_{g-3} \cdot \kappa_2 + \ldots + (-1)^{g-2} \cdot (6g-6)\lambda_{g-2}\}. \end{aligned} \tag{6.27}$$

Exercise 6.7 Check the above formula, by direct computation, for $g = 3$ and $g = 4$.

There is another way to perform the computation of the fundamental class of the hyperelliptic locus in M_g. For the purpose, let us set for simplicity, $\mathcal{C}_g = M_{g,1}$. Firstly, we learned how to characterize the triples $\mathcal{P} = (C; P_1, P_2) \in \mathcal{C}_g^2$ such that $P_1 + P_2$ moves in a g_2^1 on C, which is the same as claiming that C is hyperelliptic . It is matter of studying the map $D^0 \oplus D^0 : \rho^*\mathbb{E} \longrightarrow \mathcal{F}_2$. Notice that at $\mathcal{P} \in \mathcal{C}_g^2$, if $(\omega_1, \ldots, \omega_g)$ is a basis of the holomorphic differential on C, and if $\omega_i = a_i(u)du$ around P_1 and $\omega_i = b_i(v)dv$ around P_2, then:

$$D^0 \oplus D^0{}_{\mathcal{P}} \cong A =: \begin{pmatrix} a_1(0) & \ldots & a_g(0) \\ b_1(0) & \ldots & b_g(0) \end{pmatrix}, \tag{6.28}$$

so that $P_1 + P_2$ moves in a g_2^1 if and only if the rank of the matrix A is less or equal than 1. Using the well known fact, here assumed without proof, that as $(C; P_1, P_2)$ moves in $\mathcal{C}_g^2$, then the matrix A varies analitically, it follows that the hyperelliptic locus can be described determinantally so that its *expected codimension* in M_g is $g - 2$. Since we know for other geometrically reasons that H_g actually has such a codimension, we may apply Porteous' formula 4.1 to the map $D^0 \oplus D^0 : \rho^*\mathbb{E} \longrightarrow \mathcal{F}_2$ to compute the class of $[H_g]$. One gets:

$$\Delta_{1,g-1} c_t(\mathcal{F}_2 - \mathbb{E})$$

In this case, recall that:

$$c_t(\mathcal{F}_2) = (1 + K_1 t)(1 + (K_2 - D_{1,2})t).$$

In the next section we shall show, on a very nice example, how one may get relations between the tautological classes by computing the fundamental class of some locus using different ways.

6.4.3 Computing the class $[H_3]$ in two different ways

We shall work out the computation of the class of $[H_3]$ in $A^1(M_3)$ in two different ways, providing all the details [2] . Firstly:

$$\Delta_{1,2}(c_t(\mathcal{F}_2 - \mathbb{E}) = \begin{vmatrix} c_1 & c_2 \\ 1 & c_1 \end{vmatrix} (\mathcal{F}_2 - \mathbb{E}).$$

Now

$$c_1(\mathcal{F}_2 - \mathbb{E}) = (K_1 + K_2 - D_{1,2} - \lambda_1),$$

[2] I learned the beautiful idea to compute the class of H_3 in two different ways from Carel Faber, during his Levico's Lectures on "Intersection theory on Moduli spaces of Curves (1995)".

while:

$$c_2(\mathcal{F}_2 - \mathbb{E}) = K_1(K_2 - D_{1,2}) - (K_1 + K_2 - D_{1,2})\lambda_1 + \lambda_2.$$

Hence:

$$\begin{aligned}\Delta_{1,2}(c_t(\mathcal{F}_2 - \mathbb{E})) &= (K_1 + K_2 - D_{1,2} - \lambda_1)^2 - K_1(K_2 - D_{1,2}) + \\ &+ (K_1 + K_2 - D_{1,2})\lambda_1 - \lambda_2.\end{aligned}$$

Notice that $\Delta_{1,2}(c_t(\mathcal{F}_2 - \mathbb{E})$ describes the class of a codimension 2 locus in $\mathcal{C}_3 \times_{M_3} \mathcal{C}_3$. One knows that $dim\, \mathcal{C}_3^2 = 8$. Hence, if $p : \mathcal{C}_3^2 \longrightarrow M_3$ is the structural morphism from $\mathcal{C}_3^2$ to M_3, then $[H_3] = p_* \Delta_{1,2}(c_t(\mathcal{F}_2 - \mathbb{E}))$ has codimension 1 in M_3. Let us now compute $[H_3]$ in two different ways.

1^{st} method

First we cut the class $\tilde{H} = \Delta_{1,2}(c_t(\mathcal{F}_2 - \mathbb{E}))$ with $D_{1,2}$. That means to find the locus of points (P, P) such that P is a Weierstrass point on a hyperelliptic curve. Then we shall push down via p_*, where $p : \mathfrak{X} \times_S \mathfrak{X} \longrightarrow S$. We have:

$$\begin{aligned}&p_*(D_{1,2} \cdot \tilde{H}) = p_*(D_{1,2}(K_1 + K_2 - D_{1,2} - \lambda_1)^2) + \\ &-p_*(D_{1,2}K_1(K_2 - D_{1,2})) + \\ &-p_*(D_{1,2} - (K_1 + K_2 - D_{1,2})\lambda_1) + p_*(D_{1,2} \cdot \lambda_2).\end{aligned} \tag{6.29}$$

Keeping into account the relations, described in 5.2,

$$D_{1,2}^2 = -K_1 D_{1,2} = -K_2 D_{1,2},$$

we shall proceed in computing separately each summand of (6.29):

$$\begin{aligned}\boxed{1}\quad & p_*(D_{1,2}(K_1 + K_2 - D_{1,2} - \lambda_1)^2) = \\ = \ & p_*(D_{1,2} \cdot K_1 + D_{1,2} \cdot (K_2 - D_{1,2}^2 - D_{1,2} \cdot \lambda_1)(K_1 + K_2 - D_{1,2} - \lambda_1)) = \\ = \ & p_*((3K_1 - \lambda_1) \cdot D_{1,2} \cdot (K_1 + K_2 - D_{1,2} - \lambda_1) D_{1,2} \cdot (K_1 + K_2 - D_{1,2} - \lambda_1)) = \\ = \ & p_*(D_{1,2}(3K_1 - \lambda_1)^2) = (p_1)_*((3K_1 - \lambda_1)^2) = \\ = \ & (p_1)_*(9K_1^2 - 6K_1\lambda_1 + \lambda_1^2) = 9\kappa_1 - 6\kappa_0\lambda_1.\end{aligned}$$

$$\boxed{2}\quad -p_*(D_{1,2}K_1(K_2-D_{1,2}) = -p_*(2\,D_{1,2}K_1^2) = -2(p_1)_*(K^2) = -2\kappa_1.$$
$$\boxed{3}\quad p_*(D_{1,2}\lambda_1(3K_1)) = 3\kappa_0\lambda_1.$$
$$\boxed{4}\quad -p_*(D_{1,2}\lambda_2) = -(p_1)_*\lambda_2 = 0$$

The last piece of information we need to achieve the result is that $\kappa_0 = (2\cdot 3-2)[M_3] = 4[M_3] \cong 4$. Hence, by summing:

$$\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}$$

one finally gets:

$$8[H_3] = 9\kappa_1 - 6\cdot 4\lambda - 2\kappa_1 + 3\cdot 4\lambda_1 = 7\kappa_1 - 12\lambda = 72\lambda$$

the last equality coming from the relation gotten in Sect. 1, $\kappa_1 = 12\lambda$. The factor 8 multiplying $[H_3]$ is due to the fact that the locus is counted with multiplicity [8], because the morphism $(\pi_1)_*: D_{1,2}\cdot\tilde{H}\longrightarrow[H_3]$ is finite of degree 8 (the fiber consisting in the 8 Weierstrass points on each hyperelliptic curve of genus 3). We hence conclude that:

$$[H_3] = 9\lambda.$$

2^{nd} method.

In the locus $\tilde{H}$ described above, we can require that the 1^{st} point lies in a fixed canonical divisor. This correspond to intersecting $\tilde{H}$ with K_1. What one gets is the class of $4[H_3]$ (the factor 4 being due to the degree of the canonical divisor). One has then:

$$\begin{aligned}
4[H_3] &= p_*(K_1\cdot\tilde{H}) = p_*\left[K_1\cdot[(K_1+K_2-D_{1,2}-\lambda_1)^2+\right.\\
&- K_1(K_2-D_{1,2}) + (K_1+K_2-D_{1,2})\lambda_1 - \lambda_2]] =\\
&= p_*(K_1^2K_2+K_1K_2^2) + p_*\left[K_1(K_1^2-K_1D_{1,2}-2K_2D_{1,2}+D_{1,2}^2+\right.\\
&- \left.K_1\lambda_1 - K_2\lambda_1 + D_{1,2}\lambda_1 + \lambda_1^2 - \lambda_2)\right]. \qquad (6.30)
\end{aligned}$$

We shall compute separately the two summands of the right hand side of the last equality.

$$\begin{aligned} \boxed{1} \quad & p_*(K_1^2K_2 + K_1K_2^2) = p_*(K_1^2K_2) + p_*(K_1K_2^2) = \\ = \quad & \pi_* \circ p_{1*}(p_1^*K \cdot p_1^*K \cdot p_2^*K) + \pi_* \circ p_{2*}(p_1^*K \cdot p_2^*K \cdot p_2^*K), \end{aligned}$$

and, by repeatingly applying the projection formula, one gets:

$$\begin{aligned} p_*(K_1^2K_2 + K_1K_2^2) &= \pi_*(K^2 \cdot p_{1*}(p_2{}^*K)) + \pi_*(K^2 \cdot p_{2*}(p_1{}^*K)) = \\ &= \pi_*(4K^2) + \pi_*(4K^2) = 8\kappa_1, \end{aligned}$$

where we used the fact that

$$p_{1*}(p_2{}^*K) = p_{2*}(p_1{}^*K) = 4[\mathcal{C}_3],$$

$[\mathcal{C}_3]$ being the fundamental class of $\mathcal{C}_3$. As for the second summand of the r.h.s. of the last equality of (6.4), one has, by applying the projection formula and using the relations:

$$D_{1,2}^2 = -K_1 \cdot D_{1,2} = -K_2 \cdot D_{1,2}$$

$$\begin{aligned} \boxed{2} \quad & p_* \Big[K_1(K_1^2 - K_1D_{1,2} - 2K_2D_{1,2} + D_{1,2}^2 + \\ & -K_1\lambda_1 - K_2\lambda_1 + D_{1,2}\lambda_1 + \lambda_1^2 - \lambda_2)\Big] = \\ = \quad & \pi_* \Big[K \cdot p_{1*}(K_1^2 - 4K_1D_{1,2} - K_1\lambda_1 - K_2\lambda_1 \qquad (6.31) \\ + \quad & D_{1,2}\lambda_1 + \lambda_1^2 - \lambda_2)\Big]. \qquad (6.32) \end{aligned}$$

Now, in the r.h.s. of (6.32) one has:

$$p_{1*}(K_1^2) = p_{1*}(K_1\lambda_1) = p_{1*}(hb_1^2) = p_{1*}(\lambda_2) = 0$$

(for example, $p_{1*}(K_1^2) = p_{1*}(p_1{}^*Kp_1{}^*K) = K \cdot p_{1*}p_1^*(K) = 0$). On the other hand:

$$p_{1*}(-4K_1D_{1,2}) = -4K \cdot [\mathcal{C}_3];$$

$$p_{1*}(-K_2\lambda_1) = -4[\mathcal{C}_3]\pi^*\lambda$$

and

$$p_{1*}(D_{1,2}\lambda_1) = \lambda \cdot [S].$$

Plugging all these data in the r.h.s. of (6.32), one has:

$$r.h.s.\ of\ (6.32) = \pi_*[K \cdot (-4K - 4\pi^*\lambda + \pi^*\lambda] = -4\kappa_1 - 12\lambda$$

Hence:

$$4[H_3] = p_*(K_1 \cdot \tilde{H}) = \boxed{1} + \boxed{2} = 4\kappa_1 - 12\lambda.$$

***

Patching together the results gotten with the first method and the second one, i.e.:

$$\frac{4\kappa_1 - 12\lambda}{4} = \frac{7\kappa_1 - 12\lambda}{8},$$

one gets $[H_3] = 9\lambda$ and the relation $\kappa_1 = 12\lambda$ which we already knew by the general result coming from the Grothendieck Riemann-Roch formula (see Sect. 2).

For $g = 4$: do the same for H_4 in codimension 2. We find 2 expressions for $[H_4]$ involving κ_1^2, κ_2, λ_1 and λ_2. This time we use relations (derived by Mumford) expressing λ_i (on M_g) in the κ_i. The formula by Mumford is:

$$\sum_{i=0}^{\infty} \lambda_i t^i = exp\left(\sum_{i=0}^{\infty} \frac{b_{i+1}}{i(i+1)} \kappa_i t^i\right), \tag{6.33}$$

where t is a formal variable and b_i are the Bernouilli numbers with signs. Recall that $b_{odd} = 0$. The first few values of the b_i's are: $b_2 = 1/6$, $b_4 = -1/30$. Plugging these numbers in the above formula one finds, e.g.,

$$\lambda_1 = \frac{\kappa_1}{12} \quad \lambda_2 = \frac{\kappa_1^2}{288}.$$

We have 2 expressions, 1 relation: $\kappa_1^2 = 32\kappa_2/3$, and $[H_4]_Q = (\text{nonzero})\kappa_1^2$.

A corollary of this fact is quite remarkable: suppose $S \in M_4$ is a projective surface (it is not known whether S exists!). Then: $S \cap H_4 \neq \emptyset$, since κ_1 is ample. Moreover the intersection is finite.

6.4.4 Some divisor classes in M_g

This subsection is devoted to compute classes of naturally geometrical defined loci in M_g as a way to practise with Porteous' formula. The reader will realize that it is matter which may left as an exercise.

We want to deal with the loci in M_g ($g \geq 3$) set-theoretically described as:

$$\mathbb{D}_{g-1} = \{[C] \in M_g : C \text{ has a point } P \text{ such that } h^0(C, O_C((g-1)P)) > 1\},$$

introduced in [13],

$$E(1) = \{[C] \in M_g : C \text{ has a point } P \text{ such that } h^0(C, O_C((g+1)P)) > 2\}$$

introduced in [13] and studied in [12], and:

$$wt(2) = \{[C] \in M_g / C \text{ has a Weierstrass point } P \text{ of weight at least } 2\}$$

introduced in [72] and studied in [72], [35] and [34].

Our aim is to endow the above sets with a scheme structure and then to compute their Chow class in the group $A^1(M_g)$. To this purpose it is useful to define the loci:

$$\mathcal{V}\mathbb{D}_{g-1} = \{[(C,P)] \in M_{g,1} : h^0(C, O_C((g-1)P)) > 1\},$$

$$\mathcal{V}E(1) = \{[(C,P)] \in M_{g,1} : h^0(C, O_C((g+1)P)) > 2\},$$

and

$$\mathcal{V}wt(2) = \{[(C,P)] \in M_{g,1} / P \text{ is a Weierstrass point of } C / wt(P) \geq 2\}$$

where $wt(P)$ is the *Weierstrass weight* of the point P.

Notice that if (C,P) belongs to $\mathcal{V}\mathbb{D}_{g-1}$, by the Riemann-Roch formula it follows that $h^0(C, K_C(g-1)P) \geq 2)$, i.e. that:

$$rk(\mathcal{BN}((g-1)P)) \leq g-2.$$

Similarly (C,P) belongs to $\mathcal{V}E(1)$ if and only if:

$$rk(\mathcal{BN}((g+1)P)) \leq g-1.$$

The idea is, clearly, to pick in $M_{g,1}$ all the pairs (C,P) fulfilling the above conditions by globalizing the above description. The way to do that is to consider the following two maps of vector bundles:

$$a) \quad \begin{array}{ccc} \pi^*\mathbb{E} & \xrightarrow{D^{g-2}} & J_\pi^{g-2}\omega_\pi \\ & \searrow \quad \swarrow & \\ & \mathfrak{X} & \\ & \downarrow \pi & \\ & S & \end{array} \qquad b) \quad \begin{array}{ccc} \pi^*\mathbb{E} & \xrightarrow{D^{g}} & J_\pi^{g}\omega_\pi \\ & \searrow \quad \swarrow & \\ & \mathfrak{X} & \\ & \downarrow \pi & \\ & S & \end{array} \qquad (6.34)$$

so that to $\mathcal{V}\mathbb{D}_{g-1}$ and to $\mathcal{V}E(1)$ we may put a scheme structure by setting, for each good family $\pi : \mathfrak{X} \longrightarrow S$:

$$\mathcal{V}\mathbb{D}_{g-1}(S) = Z(D^{g-2}), \tag{6.35}$$

referring to the map (6.34) a), and:

$$\mathcal{V}E(1)(S) = Z(D^{g}), \tag{6.36}$$

referring to the map (6.34) b). For each general family $\pi : \mathfrak{X} \longrightarrow S$ of smooth curves over S, we may hence define on the base S the 2 closed subschemes:

$$\mathbb{D}_{g-1}(S) = \pi(\mathcal{V}\mathbb{D}_{g-1}(S)),$$

and

$$E(1)(S) = \pi(\mathcal{V}E(1)(S)).$$

By looking at the local representations of the map D^{g-2} and D^{g}, it turns out that the *expected codimension* of the schemes $Z(D^{g-2})$ and $Z(D^{g})$ is 2. $Z(D^{g-2})$ and $Z(D^{g})$ are loci of special Weierstrass points, and since the general curve of genus g has only normal Weierstrass points, it turns out that the expected codimension coincides with the *actual codimension.*

We may hence apply the Porteous' formula for computing the fundamental classes of $\mathbb{D}_{g-1}$ and $E(1)$ in the Picard group of the moduli functor $Pic(\mathcal{M}_g)\otimes\mathbb{Q}$ of the smooth projective curves of genus $g \geq 3$. We may hence solve the following two exercises here recalled as propositions:

Proposition 6.3 *The class of $\mathbb{D}_{g-1}$ and $E(1)$ in $Pic(\mathcal{M}_g)\otimes\mathbb{Q}$ are given by:*

$$[\mathbb{D}_{g-1}] = \frac{1}{2}g^2(g-1)(3g-1)\lambda_\pi. \tag{6.37}$$

and

$$[E(1)] = \frac{1}{2}(g+1)(g+2)(3g^2+3g+2)\lambda_\pi \tag{6.38}$$

Proof.

We may assume to work on a family parametrized by a smooth curve S. One has:

$$[\mathbb{D}_{g-1}] = \pi_*\Delta_{1,2}(c_t(J^{g-2}\omega_\pi - \pi^*\mathbb{E})),$$

Now:

$$c_t(J^{g-2}\omega_\pi - \pi^*\mathbb{E}) = (1 + c_1(J^{g-2}\omega_\pi)t + c_2(J^{g-2}\omega_\pi)t^2)(1 - \pi^*\lambda t),$$

so that:

$$c_1(J^{g-2}\omega_\pi - \pi^*\mathbb{E}) = c_1(J^{g-2}\omega_\pi) - \pi^*\lambda$$

and

$$c_2(J^{g-2}\omega_\pi - \pi^*\mathbb{E}) = c_2(J^{g-2}\omega_\pi) - c_1(J^{g-2}\omega_\pi)\pi^*\lambda$$

Applying the fundamental exact sequence (5.17) one may easily get:

$$c_1(J^k\omega_\pi) = \frac{1}{2}(k+1)(k+2)c_1(\omega_\pi), \tag{6.39}$$

and

$$c_2(J^k\omega_\pi) = \frac{1}{24}k(k+1)(k+2)(3k+5)c_1(\omega_\pi)^2. \tag{6.40}$$

Hence:

$$c_1(J^{g-2}\omega_\pi - \pi^*\mathbb{E}) = \frac{1}{2}g(g-1)c_1(\omega_\pi) - \pi^*\lambda$$

while:

$$c_2(J^{g-2}\omega_\pi - \pi^*\mathbb{E}) = \frac{1}{24}g(g-2)(g-1)(3g-1)c_1(\omega_\pi)^2 - \frac{1}{2}g(g-1)c_1(\omega_\pi)\pi^*\lambda.$$

We have all the needing data to plug in Porteous' formula:

$$\begin{aligned}
&[\mathbb{D}_{g-1}] = \\
= \;& \pi_* \begin{vmatrix} \frac{1}{2}g(g-1)c_1(\omega_\pi) - \pi^*\lambda & \frac{1}{24}g(g-2)(g-1)(3g-1)c_1(\omega_\pi)^2 - \frac{1}{2}g(g-1)c_1(\omega_\pi)\pi^*\lambda \\ 1 & \frac{1}{2}g(g-1)c_1(\omega_\pi) - \pi^*\lambda \end{vmatrix} = \\
= \;& \pi_*\left(\frac{1}{4}g^2(g-1)^2c_1(\omega_\pi)^2 - \frac{1}{2}g(g-1)c_1(\omega_\pi)\pi^*\lambda + \right. \\
- \;& \left. \frac{1}{24}g(g-2)(g-1)(3g-1)c_1(\omega_\pi)^2\right) = \\
= \;& \left(\frac{1}{8}g^4 - \frac{1}{12}g^3 - \frac{1}{8}g^2 + \frac{1}{12}g\right)\pi_*(c_1(\omega_\pi)^2) - \frac{1}{2}g(g-1)\pi_*(c_1(\omega_\pi)\pi^*\lambda).
\end{aligned} \tag{6.41}$$

As a matter of the first summand of (6.41) we recall the definition of the tautological class $\kappa_1 = \pi_*(c_1(\omega_\pi)^2)$. As for the second summand, we apply the *projection formula* (4.4):

$$\pi_*(c_1(\omega_\pi)\pi^*\lambda) = \pi_*c_1(\omega_\pi)\cdot\lambda = (2g-2)\lambda.$$

We hence got, from (6.41):

$$[\mathbb{D}_{g-1}] = \left(\frac{1}{8}g^4 - \frac{1}{12}g^3 - \frac{1}{8}g^2 + \frac{1}{12}g\right)\kappa_1 - g(g-1)^2\lambda.$$

Using the relation $\kappa_1 = 12\lambda$ one finally gets:

$$[\mathbb{D}_{g-1}] = \frac{1}{2}g^2(g-1)(3g-1)\lambda_\pi.$$

as required.

For the class $E(1)$ we argue exactly in the same way (skipping some details in the easy computations):

$$[E(1)] = \pi_*(\Delta_{2,1}(c_t(J^g\omega_\pi - \pi^*\mathbb{E})) = \pi_*[c_2(J^g\omega_\pi - \pi^*\mathbb{E})].$$

Now:

$$c_2(J^g\omega_\pi - \pi^*\mathbb{E}) = c_2(J^g\omega_\pi) - c_1(J^g\omega_\pi)\pi^*\lambda.$$

Applying the formulas (6.39) and (6.40) one gets:

$$[E(1)] = \pi_*[\frac{1}{24}g(g+1)(g+2)(3g+5)c_1(\omega_\pi)^2 - \frac{1}{2}(g+1)(g+2)c_1(\omega_\pi)\pi^*\lambda]$$

Performing all the computations, using the relation $\kappa_1 = 12\lambda$ one finally gets:

$$[E(1)] = \frac{1}{2}(g+1)(g+2)(3g^2+3g+2)\lambda.$$

and the proof is now complete.

QED

We are still left to deal with the divisor $wt(2)$ of curves possessing a Weierstrass point with weight greater than or equal to 2. Such a locus has codimension 1 in the moduli space M_g (the reason being always the same: the general curve of genus g has only normal Weierstrass points).

The quickest way to define a scheme structure on such a set is to consider, for each smooth curve over S, $\pi : \mathfrak{X} \longrightarrow S$, (whose general fiber is not hyperelliptic), the following map (already studied in [12], [53] and [54]):

$$
\begin{array}{ccc}
\pi^*\mathbb{E} & \xrightarrow{D^{g-1}} & J^{g-1}_{\pi}\omega_{\pi} \\
& \searrow \quad \swarrow & \\
& \mathfrak{X} & \\
& \downarrow \pi & \\
& S &
\end{array}
\tag{6.42}
$$

The closed subscheme $Z(D^{g-1})$ represents in $\mathfrak{X}$ the locus of all the Weierstrass points on fibers of π.

Exercise 6.8 Compute the class of $Z(D^{g-1})$ in $A^1(\mathfrak{X})$.

To study the locus $wt(2)$ we shall take the top exterior product of the map (6.42), getting:

$$
\begin{array}{ccc}
\wedge^g\pi^*\mathbb{E} & \xrightarrow{\wedge^g D^{g-1}} & \wedge^g J^{g-1}_{\pi}\omega_{\pi} \\
& \searrow \quad \swarrow & \\
& \mathfrak{X} & \\
& \downarrow \pi & \\
& S &
\end{array}
$$

The map $\mathbb{W}_\pi = \wedge^g D^{g-1}$ is said to be the *relative wronskian*. It may be identified with a section of the line bundle $\wedge^g(J^{g-1}\omega_\pi) \otimes (\wedge^g\pi^*\mathbb{E})^\vee$.

Exercise 6.9

1. Show that the bundle $\wedge^g(J^{g-1}\omega_\pi) \otimes (\wedge^g\pi^*\mathbb{E})^\vee$ is isomorphic to the bundle ${\omega_\pi}^{\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee$.
2. If $C \longrightarrow Spec(\mathbb{C})$ is a trivial family (a single curve), show that $\mathbb{W}_\pi$ is the usual wronskian section of the bundle $\omega_C^{\otimes\frac{g(g+1)}{2}}$, where ω_C is the canonical bundle.

From the above exercise, it turns out that the locus of the Weierstrass points on fibers of $\pi : \mathfrak{X} \longrightarrow S$ are described by the scheme $Z(\mathbb{W}_\pi)$ where $\mathbb{W}_\pi \in H^0(\mathfrak{X}, {\omega_\pi}^{\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee)$.

Exercise 6.10 Show that the locus of the Weierstrass points on fibers of π having weight at least 2 is scheme theoretically described by $Z(D\mathbb{W}_\pi)$, where

$$
D\mathbb{W}_\pi \in H^0\left(\mathfrak{X}, J^1_\pi\left({\omega_\pi}^{\otimes\frac{g(g+1)}{2}}\right) \otimes (\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee\right).
$$

By using exercise 6.10, we may hence define, for $g \geq 4$:

$$wt(2) = \pi(Z(D\mathbb{W}_\pi)).$$

Because the general curve of genus g has only normal Weierstrass points, $D\mathbb{W}_\pi$ is a regular section of the rank 2 vector bundle

$$J^1_\pi\left(\omega_\pi{}^{\otimes\frac{g(g+1)}{2}}\right) \otimes (\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee.$$

We may hence prove the following theorem, due to Ponza ([72],[35]):

Theorem 6.2 *The class of $wt(2)$ in $Pic(\mathcal{M}_g) \otimes \mathbb{Q}$ is given by:*

$$[wt(2)] = (3g^4 + 4g^3 + 9g^2 + 6g + 2)\lambda.$$

Proof.

To compute the class of $wt(2)$, $[wt(2)]$ in $A^1(M_g)$, it suffices to compute it on the base of a 1-parameter (proper, flat) family $\pi : \mathfrak{X} \longrightarrow S$ of smooth curves of genus g. One has just to push down on $A^1(S)$, via π, the top Chern class of the rank 2 vector bundle $J^1_\pi\left(\omega_\pi{}^{\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee\right)$. We have, hence, using standard properties of Chern classes:

$$\begin{aligned} \pi_*\left[Z(D\mathbb{W}_\pi)\right] &= \pi_*c_2\left[J^1_\pi\omega_\pi{}^{\otimes\frac{g(g+1)}{2}} \otimes \left((\pi^* \bigwedge^g \pi_*\omega_\pi)^\vee\right)\right] = \\ &= \pi_*c_2(J^1_\pi\omega_\pi{}^{\otimes\frac{g(g+1)}{2}} - \pi^* \bigwedge^g \pi_*\omega_\pi), \end{aligned}$$

where the "difference" is taken in the Grothendieck group of coherent sheaves on $\mathfrak{X}/S$. One has:

$$\begin{aligned} & [wt(2)] = \pi_*c_2(J^1_\pi\omega_\pi{}^{\otimes\frac{g(g+1)}{2}} - \pi^* \bigwedge^g \pi_*\omega_\pi) = \\ = \; & \pi_*\left[c_2(J^1_\pi\omega_\pi{}^{\otimes\frac{g(g+1)}{2}}) - c_1(J^1_\pi\omega_\pi{}^{\otimes\frac{g(g+1)}{2}})\right]\pi^*\lambda = \\ = \; & \pi_*\left\{\left[\frac{g(g+1)}{2} + 1\right]\frac{g(g+1)}{2}c_1(\omega_\pi)^2 - [g(g+1)+1]\pi^*\lambda c_1(\omega_\pi)\right\}, \end{aligned}$$

where the last equality is gotten by applying the fundamental exact sequence to the bundle $\omega_\pi^{\otimes\frac{g(g+1)}{2}}$ for $n = 2$ and $n = 1$, and taking the Chern classes. Now, by using the push-pull formula and simplifying all the expressions involved:

$$[wt(2)] = \frac{1}{4}g(g^3 + 2g^2 + 3g + 2)\kappa_1 - 2(g^3 - 1)\lambda. \tag{6.43}$$

Using the relation $\kappa_1 = 12\lambda$ one gets the desired result.

QED

As the reader may easily guess there is some relations between the divisors $\mathbb{D}_{g-1}$, $E(1)$ and $wt(2)$, clarified at the level of divisors classes in [72].

By direct computation it may in fact proven the following:

Proposition 6.4 *For $g \geq 4$*

$$[wt(2)] = [\mathbb{D}_{g-1}] + [E(1)].$$

If $g = 3$ then $Z(D)$ is the locus of hyperelliptic points, and the above formula must be modified as:

$$\pi_*[Z(D\mathbb{W}_\pi)] = 16[H_3] + [E(1)].$$

where $[H_3]$ is the class of the hyperelliptic locus.

Actually, for $g \geq 4$, in [34] it is proven that $wt(2)$ is the scheme theoretical union of $E(1)$ and $\mathbb{D}_{g-1}$. This turns out to be useful for computational purposes, see [34].

Exercise 6.11 ([35], Proposition 4.9) Let $\pi : \mathfrak{X} \longrightarrow S$ be a general smooth curve of genus $g \geq 4$, such that $dim(S) \geq 2$. For $g \geq 4$ it has been proven in [35] that:

$$wt(3) = \pi(Z(D^2\mathbb{W}_\pi)),$$

has the expected codimension in S. Compute the class of $wt(3)$ in $A^2(M_g)$.

(Solution:

$$\begin{aligned}[wt(3)] &= \frac{1}{8}g(g+1)(g^2+g+2)(g^2+g+4)\,\kappa_2 + \\ &- \frac{1}{4}[2g(g+1)(g^2+g+3) + (g^2+g+2)(g^2+g+4)]\,\kappa_1\lambda.\end{aligned}$$

6.4.5 Concluding remarks

What we have basically done, until now, is to compute the classes of some natural loci of the moduli space in the Chow group $A^1(M_g)$. With the class of $wt(3)$, see exercise 6.11, we also got a Chow class in $A^2(M_g)$ for $g \geq 4$. A natural problem arise. How to compute the classes of such loci in the Deligne-Mumford compactification of M_g? As a matter of the divisors classes $\mathbb{D}_{g-1}$ and $E(1)$ this problem has been solved by Diaz in [13] and [12], respectively. To this purpose they heavily use the theory of the *compactified Hurwitz schemes* developed by Harris and Mumford ([44], see also [13] p. 22 and [12] p. 329). We give a brief account of such a theory in Sect. 7.1. The proof by Diaz and Cukierman are very long and they probably would need a full course to study them in all the details. The results they find are:

$$\begin{aligned}[\overline{\mathbb{D}_{g-1}}] &= \frac{1}{2}g^2(g-1)(3g-1)\lambda - \frac{1}{6}g(g-1)^2(g+1)\delta_0 + \\ &- \frac{1}{2}g(g^2+g-4)\sum_{1\leq i\leq [g/2]} i(g-i)\delta_i. \end{aligned} \tag{6.44}$$

and

$$\begin{aligned}[\overline{E(1)}] &= \frac{1}{2}(g+1)(g+2)(3g^2+3g+2)\lambda - \frac{1}{6}g(g+1)^2(g+2)\delta_0 + \\ &- \frac{1}{2}(g+1)(g+2)^2\sum_{1\leq i\leq [g/2]} i(g-i)\delta_i. \end{aligned} \tag{6.45}$$

However, we think it is important to see at least one example of explicit computation of a class in $A^*(\overline{M}_g)$. Fortunately enough, it is very cheap to compute the class of the closure of $wt(2)$ in $\overline{M}_g$. This has been shown in [34], by using the absolutely standard techniques we learned in this chapter. Hence, next chapter will be devoted to explain this example providing all the details which in a research paper must be referred to the existing literature. The interesting thing is that, as the reader may easily check (after computing it) the class of the closure of $wt(2)$ in $\overline{M}_g$, is the sum of the expressions found by Diaz (6.44) and Cukierman (6.45). In a sense, while it seems to be hard to get their results, the sum of them is not.

Chapter 7

The Divisor Class of Special Weierstrass Points

7.1 A Review about the Hurwitz Schemes.

Let S be, as usual, an algebraic scheme over $\mathbb{C}$. By a $2g+n$ pointed *smooth* rational curve we mean the data (compare with the appendix 3.3) $(D \longrightarrow S, P_1, \ldots, P_{2g+n})$ where $D \longrightarrow S$ is a flat proper family of rational curves together with an ordered $(2g+n)$-tuple of disjoint sections, $P_1, \ldots, P_{2g+n}$. Consider the functor:

$$\mathcal{H}_n : (Sch/\mathbb{C}) \rightsquigarrow (Sets), \tag{7.1}$$

associating to each scheme S an isomorphism classes of maps:

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & D \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & S & \end{array} \tag{7.2}$$

such that for each $s \in S$, $\mathfrak{X}_s$ is a curve of genus g which is a branched n-fold covering of $D_s \cong \mathbb{P}^1$ with branch points $P_1(s), \ldots, P_{2g+n}(s)$ and totally ramified at $P_1(s)$ (notice that such requirements agree with the *Hurwitz formula* for computing the degree of the ramification divisor in $\mathfrak{X}_s$). Two such families, say $(\mathfrak{X}, D, S, P_1, \ldots, P_{2g+n})$ and $(\mathfrak{X}', D', S, Q_1, \ldots, Q_{2g+n})$ are said to be isomorphic if and only if there is an S-isomorphism between D and D', sending the sections P_i's into the sections Q_i's compatibly with an S-isomorphism between $\mathfrak{X}$ and $\mathfrak{X}'$ such that the ramification points of $\mathfrak{X}\longrightarrow D$ are sent to the ramification points of $\mathfrak{X}\longrightarrow D'$. It may be shown that the functor (7.1) is coarsely represented by a $\mathbb{C}$-scheme H_n, said to be the *Hurwitz scheme*, whose points are isomorphism classes of n-fold covering of a curve of genus g over $\mathbb{P}^1$ branched at $2g+n$ ordered points with a total ramification point. The scheme H_n is not complete, for the simple reason that the moduli space of smooth curves M_g is not. The way to compactify it has been elaborated in [44]. As a matter of fact it is possible to get a natural compactification of H_n, $\overline{H}_n$, by adding the so-called *admissible coverings.* We state formally the definition for the reader's convenience.

Definition 7.1 *Let S be a $\mathbb{C}$-scheme of finite type. A n-fold* admissible covering *over S is the set of data described below:*

i) A stable $2g+n$-pointed curve $(D\longrightarrow S, P_1, P_2, \ldots, P_{2g+n})$.

ii) A proper flat family $\pi : \mathfrak{X}\longrightarrow S$ such that each scheme theoretical fiber is a reduced connected curve *with at most ordinary double points*[1].

iii) A morphism $f : \mathfrak{X}\longrightarrow D$ which factors $\pi : \mathfrak{X}\longrightarrow S$, which is almost everywhere étale. More precisely it is simply branched along sections Q_i $(2 \le i \le 2g+n)$ and totally branched along a section $Q_1 : S\longrightarrow\mathfrak{X}$ over the sections $P_i : S\longrightarrow D$. Moreover:

iv) for each $s \in S$ and any $N \in \mathfrak{X}_s$ projecting onto a node M of D_s, $\mathfrak{X}_s$ has an ordinary double point. In a neighbourhood of N the family may be described as follows:

$$\mathfrak{X}_{|U} : xy = a, \quad a \in \hat{O}_{\mathfrak{X},s}, \quad \text{and } x, y \text{ generate } \hat{m}_{\mathfrak{X}_s,N}$$
$$D : uv = a^{\nu}, \quad \text{and } u, v \text{ generate } \hat{m}_{D_s,M}$$
$$\pi : u = x^{\nu}, v = y^{\nu} \text{ for some } \nu.$$

Exercise 7.1 Let P be a point on a curve C of genus g and let $n \le g$. One says that n is the *first Weierstrass non gap* at P if and only if $h^0(C, O_C(n-1)P) = 1$ and $h^0(C, O_C(nP)) = 2$. In such a case P is said to be a *Weierstrass point* having n as a first non gap.

[1] Warning: we are not requiring that the fibers of π are stable curves. They may well have smooth rational components intersecting the rest of the curve in only one point.

i) Prove that if P is a WP having n as first non gap then (C, P) defines a point of H_n. In other words: there exists a n-fold branched covering $f : C \longrightarrow \mathbb{P}^1$ having P as a total ramification point.

ii) Let $\mathbb{D}_n$ be the closure in M_g of all the points $[C]$ such that C has a Weierstrass point whose first non gap is n (see [14]). Can you guess the codimension of $\mathbb{D}_n$ inside M_g?

7.2 Computations in the Space of Stable Curves

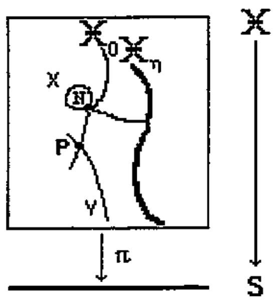
7.2.1 A degeneration problem

The purpose of this subsection is to prove the following theorem:

Theorem 7.1 *Let $g \geq 3$ and let $C_0 = X \cup_P Y$ such that:*

1. *X is a rational nodal irreducible curve. Let N be its non separating node.*
2. *Y is a connected smooth curve of genus $g-1$ intersecting X transversally at the point P which is not a Weierstrass point for Y.*

Then N is not a limit of a Weierstrass point of weight at least 2 on nearby smooth curves.



Some remarks are in order. At first, notice that a Weierstrass point of weight at least 2 on a curve C is either of type $g-1$, i.e. its first non gap is $g-1$, or of type $g+1$, i.e. there exists a non-zero holomorphic differential σ vanishing at P with multiplicity at least $g+1$ (or, in other words $(\sigma) \geq (g+1)P$). The former case has been studied by Diaz, by using the theory of admissible covers, introduced by Harris and Mumford in [44] (Cf. Section 7.1).

Theorem 7.2 (Diaz, see [13]) *Suppose that $C_0 = X \cup_P Y$ and that X is a rational nodal curve intersecting transversally an irreducible smooth curve Y of genus $g-1$ at a point P. Assume that P is not a Weierstrass point for Y. Then the node N (see picture) cannot be a limit of a Weierstrass point of type $g-1$.*

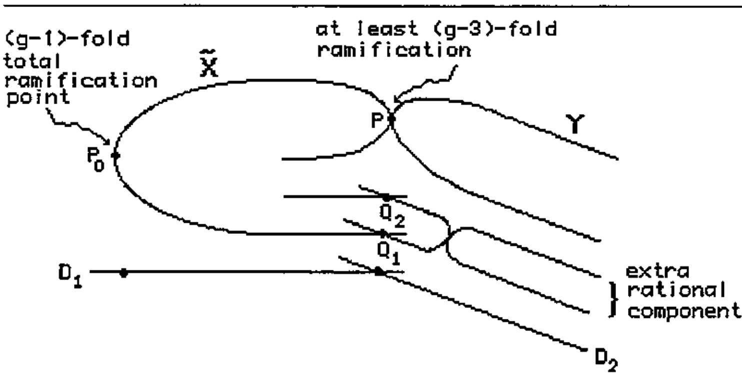
Proof.

Suppose, by contradiction, that there exists a 1-parameter family of smooth curves of genus g, $\mathfrak{X} \longrightarrow Spec(\mathbb{C}[[T]])$, such that the generic fiber is geometrically smooth and possesses a Weierstrass point P_η of type $g-1$. This means that the generic fiber may be exhibited as a $(g-1)$-sheeted branched covering of $\mathbb{P}^1$ with a total ramification point P_η. By the theory of the *compactification of the Hurwitz schemes* ([44], p. 56 and ff., [13] p. 22-23), such a covering must degenerate to an *admissible covering*, that we may assume having a base with two rational components, and a total ramification point P_0 on the component $\tilde{X}$, the normalization of X. Let us denote the reducible rational base as $D = D_1 \cup D_2$. First of all we claim that no both components $\tilde{X}$ and Y may cover D_2. In fact, if it were so, we should add some rational components that, after a contraction, should give back the curve C_0 with the node N and the node P, the intersection of X and Y. In fact, any rational curve connecting P_X and P_Y (the preimage of the point P of the partial normalization of C_0 at P) must be part of a $(g-1)$-sheeted covering of D_2. The other sheets would necessarily connect points of X and Y other than P_X and P_Y. But then, blowing down the resulting curve, we would have extra singularities. This is not possible.

The second more serious possibility is the one corresponding to the situation in which $\tilde{X}$ covers D_1 and Y cover D_2. We may construct an admissible covering as follows. Let Q_1 and Q_2 be the points of $\tilde{X}$ that when identified give rise to the node N. Then observe that there exists a function $f_{\tilde{X}} : \tilde{X} \longrightarrow \mathbb{P}^1$ such that the principal divisor defined by $f_{\tilde{X}}$, $(f_{\tilde{X}})$, is given by:

$$(f_{\tilde{X}}) = (g-1)P_0 - Q_1 - Q_2 - (g-3)P.$$

Now, to get the identification of Q_1 and Q_2, we must connect these two points by means of a rational curve which covers at least $2:1$ the $\mathbb{P}^1$ labelled by D_2. Hence, the curve Y must cover $\mathbb{P}^1$ with at most $g-3$ sheets and must totally ramify at P, as in the picture below.



But this contradicts the generality of the curve $X \cup Y$, since Y would be a curve with a (very) special Weierstrass point. QED

Once established the above result by Diaz, we are left to prove that the non separating node of a stable curve like in Fig. 3.1 cannot be limit of a WP of type $g+1$. We shall get such a result by relying on a lemma on families of curves degenerating to a cuspidal curve which seems to be interesting in its own. From now on the exposition will be adherent to the content of section 3 of ([34]).

Lemma 7.1 *Suppose that $\pi : \mathfrak{X} \longrightarrow S$ is a flat proper family of curves of arithmetic genus $g \geq 3$ parametrized by some smooth scheme of finite type over Spec($\mathbb{C}$). Suppose that the general curve of the family is smooth and that the special fiber $\mathfrak{X}_0$ is a curve having a cusp at a point P_0. Let $n : \tilde{\mathfrak{X}}_0 \longrightarrow \mathfrak{X}_0$ be the normalization of $\tilde{\mathfrak{X}}_0$ and let $Q = n^{-1}(P_0)$. If there is a section of WP's having weight at least 2 degenerating to the cusp P_0, then Q is a Weierstrass point for $\tilde{\mathfrak{X}}_0$.*

Proof.

Let ω_π be the relative dualizing sheaf of the family. Then, by hypothesis, there is a section σ_η such that $(\sigma_\eta) \geq (g+1)P_\eta$. This section extends to a section σ on all the family, such that $(\sigma_0) \geq (g+1)P_0$. The induced σ_0 is a section of the dualizing sheaf of the curve $\mathfrak{X}_0$. Let $n^*\sigma_0$ be the pull-back of σ_0 to $\tilde{\mathfrak{X}}_0$. It is a section of the sheaf $n^*\omega_{C_0}$, which is isomorphic, by adjunction theory, to $K_{\tilde{\mathfrak{X}}_0}(2Q)$, $K_{\tilde{\mathfrak{X}}_0}$ being the canonical sheaf of $\tilde{\mathfrak{X}}_0$. Hence σ_0 induces a section $\tilde{\sigma_0} \in H^0(\tilde{\mathfrak{X}}_0, K_{\tilde{\mathfrak{X}}_0})$ such that $(\tilde{\sigma}_0) \geq (g-1)Q$, which is the same as claiming that Q is a WP for $\tilde{\mathfrak{X}}_0$.

QED

Lemma 7.2 *Suppose that $C_0 = X \cup_P Y$ and that X is a rational nodal curve intersecting transversally an irreducible smooth curve Y of genus $g-1$ at a point P_0. Assume that P_0 is not a Weierstrass point for Y. Then the node N (see Fig. 3.1) is not a limit of a WP of type $g+1$.*

Proof.

Suppose that there is a family $\pi : \mathfrak{X} \longrightarrow S$ parametrized by $Spec(\mathbb{C}[[T]])$, $\mathfrak{X}$ a smooth surface, such that $\mathfrak{X}_\eta$ is geometrically smooth, and that there is a WP of type $g+1$, P_η, such that $P_0 \in \overline{\{P_\eta\}}$. We can assume, up to replacing the special fiber by an equivalent semistable model, that P_η is $\mathbb{C}((T))$-rational. Now we play with our family as follows. Let us consider the sheaf $\omega_\pi(-2Y)$. One has $\omega_\pi(-2Y)_{|\mathfrak{X}_\eta} = \omega_{\pi|\mathfrak{X}_\eta}$.
Therefore:

$$\pi_*[\omega_\pi(-2Y)] \otimes \mathbb{C}(0) \cong \pi_*[\omega_\pi(-2Y)] \otimes \mathbb{C} \cong H^0(X \cup Y, \omega_\pi(-2Y)_{|X\cup Y}).$$

We now claim that:

$$H^0(X \cup Y, \omega_\pi(-2Y)_{|X\cup Y}) \cong H^0(Y, \omega_Y(2P)).$$

In fact $h^0(X \cup Y, \omega_\pi(-2Y)_{|X\cup Y}) \geq g$. Moreover the inclusion $Y \hookrightarrow X \cup Y$ induces a natural restriction map:

$$\rho : H^0(X \cup Y, \omega_\pi(-2Y)_{|X\cup Y}) \longrightarrow H^0(Y, \omega_Y(2P)), \tag{7.3}$$

defined by $\sigma \mapsto \sigma_{|Y}$. The map ρ is injective. In fact, suppose that:

$$\sigma \in H^0(X \cup Y, \omega_\pi(-2Y)_{|X\cup Y}),$$

identically vanishes on Y, i.e.:

$$\sigma_{|Y} = 0.$$

Hence $\sigma(P) = 0$, and since $deg(\sigma_{|X}) = 0$, it follows that $\sigma_{|X} = 0$, i.e. that $\sigma = 0$. This proves injectivity. By dimension reasons, (7.3) is actually an isomorphism.

Now the sheaf $\pi_*\omega_\pi(-2Y)$ embeds the family $\pi : \mathfrak{X} \longrightarrow S$ in $\mathbb{P}(\pi_*\omega_\pi(-2Y))$, i.e. we have the following diagram:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\phi_{\pi_*(\omega_\pi(-2Y))}} & \mathbb{P}(\pi_*(\omega_\pi(-2Y))) \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{7.4}$$

The generic fiber is a geometrically smooth curve of genus g while the special fiber is a cuspidal curve having a cusp in P_0 with the rational nodal component of C_0 contracted in P_0 by the map $\phi_{\pi_*(\omega_\pi(-2Y))}$. In fact such a map has degree 0 when restricted to X. The generic fiber has a WP P_η of type $g+1$ degenerating onto the cusp P_0 (because it degenerated onto X in the initial family and X has been contracted in the cusp). But then P_0 would be a Weierstrass point by Lemma 7.1, contradicting the hypothesis. Hence the node of X is not a limit of a Weierstrass point of type $g+1$.

QED

Patching together Theorem 7.2 and Lemma 7.2 we have proven Theorem 7.1

We are now interested to an intersection theoretical consequence of 7.1.

Let us consider in $\overline{M}_g$ a family lying entirely in the divisor Δ_1, whose general point corresponds to a stable curve consisting of an elliptic curve E intersecting transversally a smooth curve X of genus $g-1$ at a point P which is not a WP for X. The one parameter family one wants to construct is gotten by fixing the curve of genus $g-1$ and the point of intersection on it and varying the j-invariant of the elliptic curve (for a rigorous and detailed construction of such a family see [HM], p. 83). It is a family of curves parametrized by the j-line (hence the parameter space is complete). According to Diaz [13], we denote it by $\mathcal{F}_2$. We claim that:

$$\mathcal{F}_2 \cdot [\overline{wt(2)}] = 0, \tag{7.5}$$

in $A^1(\overline{M}_g)$. In fact if $E \cup_P X$ is a stable curve where E is elliptic, with X a general curve of genus $g-1$, we know, by a rough dimension count, that $X \cup Y$ is not in the closure of $wt(2)$ in $\overline{M}_g$. Otherwise one may find a family of curves $\pi : \mathfrak{X} \longrightarrow S$ with $\mathfrak{X}_\eta$ smooth and a WP $P_\eta \in \mathfrak{X}_\eta$ degenerating to $E \cup_P X$. The only possibility would be that P_η degenerated to the node N of $E \cup_P X$, when E is rational nodal. But Theorem 7.1 says us that such a curve cannot be in the closure of the locus of curves having a WP of weight at least 2. Hence:

$$[\overline{wt(2)}] \cdot \mathcal{F}_2 = 0. \tag{7.6}$$

Example 7.1 We want to show the rigorous construction of the family $\mathcal{F}_2$. This construction[2] will turn out to be useful later on, in order to determine some intersection theoretical numbers. Let $F_1, F_2 \in H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(3))$ two irreducible forms of degree 3. Their zero scheme correspond to two smooth elliptic curves. We shall make a pencil out of them by considering the family of cubic forms:

$$F_{[\lambda,\mu]} = \lambda F_1 + \mu F_2.$$

Let $C_i = Z(F_i)$ and $C_{[\lambda,\mu]} = Z(F_{[\lambda,\mu]})$. Consider the rational map:

$$\phi : \mathbb{P}^2 \cdots \to \mathbb{P}^1, \tag{7.7}$$

sending a general point P of $\mathbb{P}^2$ in the unique $[\lambda, \mu]$ parametrizing the unique cubic of the pencil passing through P. Clearly, such a map is not defined in the 9 intersection points $\{P_1, \ldots, P_9\} \in C_1 \cap C_2$. Let $\epsilon : S \longrightarrow \mathbb{P}^2$ be the *blow-up* of $\mathbb{P}^2$ in the 9 points $P_1, \ldots, P_9$. Let $E_1, \ldots, E_9$ be the nine *exceptional divisors* corresponding to $P_1, \ldots, P_9$. The composition of the map ϕ with ϵ yields a family:

$$\tilde{\phi} : S \longrightarrow \mathbb{P}^1,$$

[2] I learned such a construction of the family $\mathcal{F}_2$, from C. Faber during his Levico's Lectures on intersection theory over moduli space of curves.

of elliptic curves together with 9 sections $E_1, \ldots, E_9$. Pick one of them, say E_1. Notice that the fiber of $\tilde{\phi}$ over a point $[\lambda, \mu]$ of $\mathbb{P}^1$ is the strict transform in S of the cubic $Z(\lambda F_1 + \mu F_2)$, whose class will be denoted by $\tilde{F}$. Of course, recalling that ([9], p. 17):

$$Pic(S) = Pic(\mathbb{P}^2) \oplus \mathbb{Z}[E_1] \oplus \ldots \oplus \mathbb{Z}[E_9],$$

it turns out that $\tilde{F} = 3L$, L being the first Chern class of $O_{\mathbb{P}^2}(1)$.

Let now C be a general curve of genus $g-1$ and let $Q \in C$ be a general point of it. Construct the surface $\mathcal{T} = \mathbb{P}^1 \times C$ and denote by S_1 the section $\mathbb{P}^1 \times \{Q\}$. Construct a surface $\mathfrak{X}$ by glueing $\mathcal{T}$ and S along the sections S_1 and E_1 respectively. What we get is a family $\mathfrak{X} \longrightarrow \mathbb{P}^1$ of stable curves whose general fiber is an elliptic curve glued to a fixed curve C of genus $g-1$ at a point $Q \in C$, in such a way that the intersection is transversal.

7.2.2 A useful theorem by Cukierman

The purpose of this section is to prove the following important theorem proven by Cukierman, which is especially important for our purposes.

Theorem 7.3 *Let $\pi : \mathfrak{X} \longrightarrow S$ be a flat proper family of stable curves, of genus $g \geq 3$, with $\mathfrak{X}$ a smooth surface, $S = Spec(\mathbb{C}[[T]])$, smooth generic fiber and special fiber:*

$$\mathfrak{X}_0 = X \cup_P Y,$$

a general member of Δ_i (so, in particular, P is not a Weierstrass point neither for X nor for Y). Then one has:

$$[Z(\mathbb{W}_\pi)] = \left[\overline{Z(\mathbb{W}_{\pi|_{\mathfrak{X}-\mathfrak{X}_0}})}\right] + \binom{g_Y+1}{2}[X] + \binom{g_X+1}{2}[Y]. \tag{7.8}$$

For proving this fundamental theorem we need few steps. We recall, for the reader convenience, an algebraic result. If $\phi : M \longrightarrow N$ is a morphism of A-modules, with M a *torsion module* and N a free A-module (and hence torsion-free), then ϕ is the zero morphism. In fact, if ϕ were not the zero morphism, there would exists a $m \neq 0_M$ such that $\phi(m) \neq 0$. But since M is a torsion module, there exists at least a $0 \neq a \in A$ such that $a \cdot m = 0$, i.e. $\phi(a \cdot m) = a\phi(m) = 0$. Contradiction. We shall need this (easy) algebraic result in the following:

Proposition 7.1 *Let ω_π be the relative dualizing sheaf of the family π. In the hypotheses specified above, consider the restriction map:*

$$\rho_k : \pi_*(\omega_\pi(-kX)) \longrightarrow \pi_*(\omega_\pi(-kX)_{|Y}) = H^0(Y, \omega_Y(-k+1)P).$$

Then ρ_k is onto for all $k \geq 0$.

Proof.

Let us consider the exact sequence of sheaves of $O_{\mathfrak{X}}$-modules:

$$0 \longrightarrow O_{\mathfrak{X}}(-Y) \longrightarrow O_{\mathfrak{X}} \longrightarrow O_Y \longrightarrow 0,$$

and tensor it by $\omega_\pi(-kY)$. Because $\omega_\pi(-kX)$ is an invertible sheaf of $O_{\mathfrak{X}}$-modules we may safely tensor the above exact sequence by it, to get the new exact sequence:

$$0 \longrightarrow \omega_\pi(-kX - Y) \longrightarrow \omega_\pi(-kX) \longrightarrow \omega_\pi(-kX)_{|Y} \longrightarrow 0. \tag{7.9}$$

We now apply to the exact sequence (7.9) the functor π_*, passing to the long exact cohomology sequence:

$$\begin{aligned} &0 \longrightarrow \pi_*\omega_\pi(-kX - Y) \longrightarrow \pi_*(\omega_\pi(-kX)) \longrightarrow \pi_*\omega_\pi(-kX)_{|Y} - \delta \longrightarrow \\ &\longrightarrow R^1\pi_*(\omega_\pi(-kX - Y)) \longrightarrow R^1\pi_*(\omega_\pi(-kX)) \longrightarrow 0 \end{aligned} \tag{7.10}$$

The reason of the last zero is clear: it is matter of a first cohomology group of a sheaf supported at a point. We contend that the homomorphism δ is zero. In fact the sheaf $\pi_*(\omega_\pi(-kX)_{|Y})$ is torsion (the stalk at the generic point is 0), and we are left to prove that the sheaf $R^1\pi_*[\omega_\pi(-kX - Y)]$ is free.

It is sufficient to show, by the *theorem of Grauert* ([42], Corollary 12.9), that the dimension of the $\mathbf{k}(b)$-vector-spaces:

$$h(b) = h^1(\mathfrak{X}_b, \omega_\pi(-kX - Y)_{|X_b}) = dim_{\mathbf{k}(b)} R^1\pi_*[\omega_\pi(-kX - Y)] \otimes \mathbf{k}(b),$$

is constant, for all $b \in S$. But $S = Spec(\mathbb{C}[[T]])$ has only 2 points: the *generic point* η and the *closed point* 0, the latter corresponding to the maximal ideal (T). Hence we have to compute $h(b)$ only for these two points. First of all we have:

$$h(\eta) = h^1(\mathfrak{X}_\eta, \omega_\pi(-kX - Y)_{|X_\eta}). \tag{7.11}$$

But $[\mathfrak{X}_\eta] \cdot [-kX - Y] = 0$ (notice: $[-kX - Y]$ is a Cartier divisor, so that the above intersection makes sense) because they are the classes of two fibers of the map $\pi : \mathfrak{X} \longrightarrow S$ (see [9], Prop. I.8, p. 8). Hence, by the *adjunction formula* (see [42], p. 361), it follows that (7.11) may be written as:

$$h(\eta) = h^1(\mathfrak{X}_\eta, \omega_{\pi|\mathcal{X}_\eta}) = h^0(\mathfrak{X}_\eta, O_{\mathcal{X}_\eta}) = 1.$$

It remains to check the dimension at the special fiber. One has:

$$h(0) = h^1(\mathfrak{X}_0, \omega_\pi(-kX - Y)_{|X_0}), \tag{7.12}$$

Now, for the same reasons recalled above, we have $[-kX - Y][\mathfrak{X}_0] = 0$ and, by adjunction formula, we get again, from (7.12):

$$h(0) = h^1(\mathfrak{X}_0, \omega_{\pi|X_0}) = 1$$

where the last equality comes from well known properties of the relative dualizing sheaf (e.g. the Riemann-Roch theorem for stable curve, see e.g. [8], p. 677). So we have proven that $R^1\pi_*[\omega_\pi(-kX - Y)]$ is a free sheaf, so that δ is the zero homomorphism. By the exact sequence (7.10), it follows that the map ρ_k is onto as stated.

QED

The next step consists in finding a useful O_S-basis of $\pi_*\omega_\pi$ in order to compute explicitly the zero-scheme associated to the relative wronskian $\mathbb{W}_\pi$ of the family. Here is how we may construct a convenient basis, following [12]. Consider the commutative diagram:

$$\begin{array}{ccc} \pi_*(\omega_\pi(-kX)) & \xhookrightarrow{\iota_k} & \pi_*\omega_\pi \\ {\scriptstyle\rho_k}\downarrow & & \downarrow{\scriptstyle\rho_0} \\ H^0(Y, \omega_Y((-k+1)P)) & \xhookrightarrow[\iota_k]{} & H^0(Y, \omega_Y(P)) \end{array} \tag{7.13}$$

and choose $\overline{\alpha}_k \in H^0(Y, \omega_Y(-k+1)P)$ such that $\alpha_k(P) \neq 0$, for each $1 \leq k \leq g_Y$. Let $\{\alpha_k\}$ be a system of preimages of $\overline{\alpha}_k$ via ρ_k. Then $\iota_k(\overline{\alpha}_k) \in H^0(Y, \omega_Y(P))$ is a basis of $H^0(Y, \omega_Y(P))$. Since, by assumption, P is not a Weierstrass point for Y it follows that such a basis has $1, 2, \ldots, g_Y$ as *vanishing sequence* at P. Such a basis may be lifted to $\pi_*\omega_\pi$ via the surjection ρ_0, giving $(\omega_1, \ldots, \omega_{g_Y})$. The idea consists in extending such a system of sections to a basis of all of $\pi_*\omega_\pi$. To achieve this goal, choose an arbitrary basis $\{\overline{\omega}_{g_Y+1},\ldots,\overline{\omega}_g\}$ of $H^0(X,\omega_X(P))$. Let $\omega_i\in\pi_*\omega_\pi$ such that $\rho_0(\omega_i)=\overline{\omega}_i$ for $g_Y+1\leq i\leq g$. Because of the isomorphism:

$$\pi_*\omega_\pi\otimes\mathbf{k}(0)=H^0(\mathfrak{X}_0,\omega_{\mathfrak{X}_0})=H^0(X,\omega_X(P))\oplus H^0(Y,\omega_Y(P)),$$

Since $(\omega_1,\ldots,\omega_g)$ so constructed projects onto a basis of $\pi_*\omega_\pi\otimes\mathbf{k}(0)$, it follows, by *Nakayama's lemma* that $(\omega_1,\ldots,\omega_g)$ is actually a basis of $\pi_*\omega_\pi$. This is the basis we want to use to make wronskians computations. We are now in position to provide the

Proof of Theorem 7.3.,

which consists in showing that, using the O_S-basis for $\pi_*\omega_\pi$ constructed above, the order of vanishing of the wronskian $\mathbb{W}_\pi$ along X is

$$1+2+\ldots+g_Y.$$

Repeating the same construction for the component Y, one gets, similarly, that the order of vanishing of the Wronskian $\mathbb{W}_\pi$ along Y is $1+2+\ldots+g_X$. This would prove the theorem, since the order of vanishing of the wronskian does not depend on the particular basis chosen.

Let $U\in\mathfrak{X}$ be an open Zariski subset of $\mathfrak{X}$, trivializing ω_π over $O_{\mathfrak{X}}$, such that $U\cap Y=\emptyset$, and $U\cap X\neq\emptyset$. Suppose that ω_0 is a generator of $H^0(U,\omega_\pi)$ over $O_{\mathfrak{X}}(U)$. On such an open set, X is described by the equation $T=0$, where T is the local parameter of $Spec(\mathbb{C}[[T]])$. Hence, by the very construction of the basis $(\omega_1,\ldots,\omega_g)$ above, it follows that:

$$\omega_{k|_U}=t^k\phi_k\omega_0,\quad 1\leq k\leq g_Y \tag{7.14}$$
$$\omega_{k|_U}=\phi_k\omega_0,\quad g_Y+1\leq k\leq g \tag{7.15}$$

for some $\phi_k\in O_{\mathfrak{X}}(U)$. Then we have that:

$$\begin{aligned}\mathbb{W}_{\pi|_U} &= W(t\phi_1,t^2\phi_2,\ldots,t^{g_Y}\phi_{g_Y},\phi_{g_Y+1},\ldots,\phi_g)=\\ &= t^{\binom{g_Y+1}{2}}\cdot W(\phi_1,\phi_2,\ldots,\phi_g)\end{aligned}$$

To be done, we only need to prove that $W(\phi_1,\phi_2,\ldots,\phi_g)$ does not vanish along X. But for each $1\leq k\leq g_Y$, we have that $\phi_k\omega_0$ is exactly the restriction to U of the element $\alpha_k\in H^0(X,\omega_\pi(-kY)_{|_Y})$ and $\alpha_{k|_X}=\overline{\alpha}_k\in H^0(X,\omega_X(P))$ does not vanish at $P\in Y$. For $g_Y+1\leq k\leq g$, instead, $\phi_k\omega_0=\omega_{k|_U}$ and $\omega_{k|_X}=\overline{\omega}_k\in H^0(X,\omega_X(P))$.

Now, considering the natural inclusion:

$$H^0(X, \omega_X(k+1)P) \subseteq H^0(X, \omega_X(g_Y+1)P),$$

it follows that:

$$(\alpha_{1|_X}, \dots, \alpha_{g_Y|_X}, \overline{\omega}_{g_Y+1}, \dots, \overline{\omega}_g),$$

is a basis of $H^0(Y, \omega_Y(g_Y+1)P)$. Hence, its wronskian,

$$W(\phi_1, \dots, \phi_g),$$

does not vanish at $T = 0$.

Arguing in the same way for the component Y, one finally gets equality (7.8).

QED

7.2.3 The closure of $wt(2)$ in $\overline{M}_g$

Let now $\pi : \mathfrak{X} \longrightarrow S$ be a family of stable curves such that the general curve is smooth and non hyperelliptic. Let Δ be the locus of S corresponding to the singular fibers. Over the base $S' = S \setminus \{\Delta\}$ we have a family of smooth curves whose total space is $\mathfrak{X} \setminus \pi^{-1}(\Delta)$. For each stable curve over S, $\pi : \mathfrak{X} \longrightarrow S$, set, by definition, $\overline{wt(2)}(S) = \overline{wt(2)(S')}$.

Let us write $[\overline{wt(2)}]$ as:

$$[\overline{wt(2)}] = a\lambda - b_0\delta_0 - b_1\delta_1 - \ldots - b_{[\frac{g}{2}]}\delta_{[\frac{g}{2}]}, \tag{7.16}$$

the equality holding in $A^1(M_g)$ or, which is the same, in the Picard group of the moduli functor $Pic(\mathcal{M}_g) \otimes \mathbb{Q}$. Thinking our problem in the moduli functor turns out to be more useful. In fact, equality (7.16) thought in the moduli functor, means that it may be evaluated on test families. For instance, if one wanted to compute the coefficient a, it would be sufficient to consider a 1-parameter family $\pi : \mathfrak{X} \longrightarrow S$ of *smooth* curves. In this case, $deg_S(\delta_i) = 0$ for each $0 \leq i \leq [g/2]$, because the family has not singular fibers! Hence, on a family $\pi : \mathfrak{X} \longrightarrow S$ of smooth curves, the equality (7.16) would translate as:

$$[\overline{wt(2)}]_S = [\overline{wt(2)}(S)] = a\lambda_\pi.$$

As it should be clear, our aim is to explicitly determine $a, b_0, b_1, \ldots, b_{[g/2]}$. The value of the coefficient a has been already computed in section 6.4.4 (Theorem 6.2) and it is given by:

$$a = 3g^4 + 4g^3 + 9g^2 + 6g + 2.$$

It remains to evaluate $b_0, \ldots, b_{[g/2]}$. The strategy consists in computing first all the coefficients $b_1, b_2, \ldots$ up to $b_{[g/2]}$ (i.e. not b_0). To do this we use the previous remark and evaluate expression (7.16) on germs of 1-parameter families of stable curves with smooth generic fiber and a (only one) reducible special fiber of type i. In this way we kill all the contributions coming from the coefficients of δ_j's, for $j \neq i$. At the very hand we shall fill the resulting numbers in the formula (7.16).

To compute the coefficients b_i ($1 \leq i \leq [g/2]$) one at a time, according to the suggested strategy, we shall heavily rely on the theorem 7.3 by Cukierman. We shall briefly recall it here, for the reader's convenience. Let $\pi : \mathfrak{X} \longrightarrow S$ be a stable curve over $S = Spec(\mathbb{C}[[T]])$. Suppose that the geometric generic fiber of the family, $\mathfrak{X}_{\overline{\eta}}$, is smooth and non hyperelliptic and that $\mathfrak{X}_0$, the special fiber, is a stable curve of genus g that is the union of an irreducible smooth curve X of genus i which intersects transversally at a point P an irreducible smooth curve Y of genus $g-i$. Assume that P is not a Weierstrass point neither for X nor for Y. Then, by theorem 7.3 we have that:

$$Z(\mathbb{W}_\pi) = \overline{Z(\mathbb{W}_{\pi|_{\mathfrak{X}-\mathfrak{X}_0}})} + \alpha X + \beta Y,$$

where, for notational convenience, we set:

$$\alpha = \binom{g-i+1}{2} \quad \text{and} \quad \beta = \binom{i+1}{2}$$

having set $g_X = i$ and $g_Y = g - i$. Notice that $\mathbb{W}_\pi$ is a section of the bundle ${\omega_\pi}^{\otimes\otimes\frac{g(g+1)}{2}} \otimes (\pi^* \bigwedge^g \pi_* \omega_\pi)^\vee$ (and here we are using the fact that we are able to define the Weierstrass divisor for the family of stable curve. Also we are going to use the fact that we are able to take jets of such a bundle). Then $\mathbb{W}_\pi$ induces a section, denoted in the same way by abuse of notation, of the bundle:

$${\omega_\pi}^{\otimes\frac{g(g+1)}{2}} \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee.$$

The aim, now, is to compute $\pi_*([Z(D\mathbb{W}_\pi)])$, where:

$$D\mathbb{W} \in H^0\left(\mathfrak{X}, J^1_\pi\left({\omega_\pi}^{\otimes\frac{g(g+1)}{2}}\right) \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee\right).$$

Clearly, because $\mathbb{W}_\pi$ does not vanish on the special fiber, the same holds for the section $D\mathbb{W}_\pi$ (in fact, $D\mathbb{W}_\pi$ locally looks like a pair (w, w') where w is a local equation for $\mathbb{W}_\pi$ and w' is its derivative (in the sense of Section 5.2) along the fibers. We use now the exact sequence 5.17 for $n = 1$ and for $\mathcal{L} = \omega_\pi$.

$$0 \longrightarrow \omega_\pi^{\otimes\left(\frac{g(g+1)}{2}+1\right)}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee \longrightarrow$$
$$\longrightarrow J^1_\pi\left(\omega_\pi^{\otimes\frac{g(g+1)}{2}}\right)(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee \longrightarrow$$
$$\longrightarrow \omega_\pi^{\otimes\frac{g(g+1)}{2}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee \longrightarrow 0$$

so that:

$$\begin{aligned}
& c_2(J^1_\pi\left(\omega_\pi^{\frac{g(g+1)}{2}} \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee\right)) = \\
= & \left[\left(\frac{g(g+1)}{2} + 1\right) c_1(\omega_\pi) - \alpha X - \beta Y - \pi^*\lambda\right] \cdot \\
& \cdot \left[\frac{g(g+1)}{2} c_1(\omega_\pi) - \alpha X - \beta Y - \pi^*\lambda\right] = \\
= & \frac{g(g+1)}{2}\left(\frac{g(g+1)}{2} + 1\right) c_1(\omega_\pi)^2 + \\
- & (\alpha X + \beta Y)[g(g+1)+1]c_1(\omega_\pi) + 2(\alpha X + \beta Y)\pi^*\lambda + \\
- & [g(g+1)+1]c_1(\omega_\pi)\pi^*\lambda + (\alpha X + \beta Y)^2 + (\pi^*\lambda)^2.
\end{aligned}$$

By pushing down the above equality via π_* one has:

$$\begin{aligned}
& \pi_*\left[c_2(J^1_\pi\left(\omega_\pi^{\frac{g(g+1)}{2}} \otimes O_{\mathfrak{X}}(-\alpha X - \beta Y) \otimes \pi^* \bigwedge^g \mathbb{E}^\vee\right))\right] = \\
& = \frac{g(g+1)}{2}\left(\frac{g(g+1)}{2} + 1\right) \kappa_1 + \\
& -(g(g+1)+1)[\alpha(2i-1) + \beta(2(g-i)+1)]\delta_i + \\
& -2(g-1)[g(g+1)+1]\lambda - \alpha^2\delta_i - \beta^2\delta_i + 2\alpha\beta\delta_i = \\
& = \left[\frac{g^2(g+1)^2}{4} + \frac{g(g+1)}{2}\right]\kappa_1 +
\end{aligned}$$

$$-\{[g(g+1)+1](\alpha(2i-1)+\beta(2(g-i)-1))+$$
$$+\alpha^2+\beta^2-2\alpha\beta\}\delta_i-2(g-1)[g(g+1)+1]\lambda=$$
$$=[3g^2(g+1)^2+6g(g+1)-2g(g-1)(2g+1)-2(g-1)]\lambda+$$
$$-\{[g(g+1)](\alpha(2i-1)+\beta(2(g-i)-1))+$$
$$+\alpha^2+\beta^2-2\alpha\beta+\left(\frac{g^2(g+1)^2}{4}+\frac{g(g+1)}{2}\right)\right\}\delta_i.$$

Performing all the computations, after replacing α and β, respectively, by their values $\binom{g-i+1}{2}$ and $\binom{i+1}{2}$, and by using the fundamental relation proven in section 6.1.4:

$$\kappa_1 = 12\lambda - \delta_i,$$

one finally gets:

$$[\overline{wt(2)(S)}] = (3g^4+4g^3+9g^2+6g+2)\lambda - (g^3+3g^2+2g+2)i(g-i)\delta_i. \quad (7.17)$$

We have hence proven that:

Proposition 7.2 *In $Pic(\overline{\mathcal{M}}_g)\otimes\mathbb{Q}$ the following equality holds:*

$$[\overline{wt(2)}] = (3g^4+4g^3+9g^2+6g+2)\lambda - b_0\delta_0 - (g^3+3g^2+2g+2)\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i \quad (7.18)$$

It remains only to compute the coefficient b_0. To do this, we shall need the intersection theoretical consequence (7.5) of theorem 7.1. We rewrite it here:

$$[\overline{wt(2)}]\cdot\mathcal{F}_2 = 0. \qquad (7.5)$$

Now, as shown by a slightly different argument by Harris and Mumford, in [44], pp. 83-84, one has the following:

Lemma 7.3

$$\mathcal{F}_2\cdot\lambda = 1, \quad \mathcal{F}_2\cdot\delta_0 = 12, \quad \mathcal{F}_2\cdot\delta_1 = -1 \quad \text{and} \quad \mathcal{F}_2\cdot\delta_j = 0, \quad \text{for} \quad j>1.$$

Proof.

We want to evaluate the degrees of the divisors λ and δ_i on the base of the family $\mathcal{F}_2$, $\mathfrak{X} \longrightarrow \mathbb{P}^1$ constructed in the example 7.1. The reader is referred to such an example for what concerns the notation here used. First of all we notice that the degree of δ_i for $i \geq 2$ is zero. In fact there is obviously no fiber of $\mathcal{F}_2$ having a node of type $i \geq 2$. For computing $deg(\delta_1)$ we use the recipe explained in Section 6.1.3, formula (6.4). First we normalize $\mathfrak{X}$ along the nodal section, getting back the families $\mathcal{T}$ and $\mathcal{S}$ together with the sections S_1 and E_1. Then we apply formula (6.4), so that:

$$\int_{\mathbb{P}^1} \delta_1 = \int_{\mathbb{P}^1} \left(c_1(N_{S_1/T}) + c_1(N_{E_1/S}) \right) = 0 + E_1^2 = -1.$$

As a matter of $deg(\lambda)$ recall that:

$$\lambda = c_1(\pi_* \omega_\pi).$$

But $\pi_* \omega_\pi = (H^0(C, K_C) \otimes O_C) \oplus \tilde{\phi}_* \omega_{\tilde{\phi}}$, so that, actually:

$$\lambda = c_1(\tilde{\phi}_* \omega_{\tilde{\phi}}).$$

We contend that $\int_{\mathbb{P}^1} \lambda_{\tilde{\phi}} = 1$. We shall get such a result by using the fact that:

$$\kappa_{1\tilde{\phi}} = 12\lambda_{\tilde{\phi}} - \delta_{0\tilde{\phi}},$$

and by proving that $\int_{\mathbb{P}^1} \kappa_{1\tilde{\phi}} = 0$ and that $\int_{\mathbb{P}^1} \delta_{0\tilde{\phi}} = 12$. For simplicity we shall skip the $\tilde{\phi}$ from the notation. First of all $deg(\delta_0) = 12$, because there are 12 rational cubics in the pencil $\lambda F_1 + \mu F_2 = 0$. As for κ_1, notice that:

$$\omega_{\tilde{\phi}} = K_{\tilde{S}} - \tilde{\phi}^* O_{\mathbb{P}^1}(-2).$$

Let us denote by $\tilde{F}$ the class of the fiber. We have:

$$c_1(\omega_{\tilde{\phi}}) = -3L + \Sigma + 2\tilde{F} = -\tilde{F} + \Sigma + 2\tilde{F} = \tilde{F} + \Sigma = \epsilon^* F,$$

where we set $\Sigma = E_1 + \ldots + E_9$ and $\epsilon^* F = \tilde{F} + \Sigma$ is the total transform of the fiber of the rational map $\phi : \mathbb{P}^2 \cdots \rightarrow \mathbb{P}^1$ (7.7), so that $c_1(\omega_{\tilde{\phi}})^2 = (\epsilon^* F \cdot \epsilon^* F) = F \cdot F = 0$ and therefore $\kappa_1 = 0$. This concludes the proof.

QED

By the above computations it turns out that the relation:

$$a - 12b_0 + b_1 = 0,$$

must hold. Since we already know the values of a and b_1 in the expression of $\overline{wt(2)}$, we are also able to compute b_0. One gets:

$$\begin{aligned} b_0 &= \frac{1}{12}(a+b_1) = \frac{1}{12}\Big(3g^4+4g^3+9g^2+6g+2+ \\ &+ (g^3+3g^2+2g+2)(g-1)\Big) = \\ &= \frac{1}{12}(4g^4+6g^3+8g^2+6g) = \frac{1}{6}g(g+1)(2g^2+g+3), \end{aligned}$$

We have hence proven the following:

Theorem 7.4 *Let $g \geq 3$. In the Chow group $A^1(\overline{M}_g)$, the Q-class $[\overline{wt(2)}]$ of the locus $wt(2)$ is given by:*

$$\begin{aligned} [\overline{wt(2)}] &= (3g^4+4g^3+9g^2+6g+2)\lambda - \frac{1}{6}g(g+1)(2g^2+g+3)\delta_0 + \\ & -(g^3+3g^2+2g+2)\sum_{i=1}^{[\frac{g}{2}]} i(g-i)\delta_i. \end{aligned} \tag{7.19}$$

QED

Chapter 8

The Tautological Ring of M_g

8.1 Generalities on $R^*(M_g)$

Let $g \geq 2$. The *tautological ring* of M_g is by definition the subring of $A^*(M_g)$ generated over $\mathbb{Q}$ by the "tautological classes" κ_i defined in Section 6.4.1. $\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1})$, where $\pi : C_g \longrightarrow M_g$ is the *"universal" curve.*

Here are some known results about $R^*(M_g)$:

1. $\lambda_i \in R^*(M_g)$;

2. The classes of many "geometrically defined" subvarieties of M_g lie in $R^*(M_g)$.

3. Mumford shows that $\kappa_1, \ldots, \kappa_{g-2}$ generate the ring.

4. There are some relations. For instance κ_{g-2} is not needed as generator for $g \geq 4$ (for proving it, the idea is to find 2 formulas for $[H_g]$). For instance, if $g = 4$ one finds $\kappa_1^2 = 32\kappa_2/3$. We shall get such a relation in an alternative way in Section 8.4.2. Notice that for $\forall i \geq 1$, one has $\kappa_i[H_g] = 0$ (e.g. $\kappa_1[H_g] = 0$, by pushing forward the class of hyperelliptic locus). One gets in this way a relation in codimension $g - 2 + i$.

5. For small genus is known that $R^*(M_g) = A^*(M_g)$. It is known for $g \leq 5$ in characteristic 0 ($g = 2$ being proven by Mumford, $g = 3, 4$ by Faber, [24]–[25], and $g = 5$ by Izadi, [47]). However it does not seem possible that the Chow ring of M_g coincides with the tautological ring for g sufficiently large. For a discussion of this, see [26].

8.2 C. Faber's Conjectural Description of the Tautological Ring

What is going to follow is a conjectural description of the tautological ring $R^*(M_g)$ due to Carel Faber ([26]). We advise the reader to have a look to the exciting description contained in [45].

1. $R^*(M_g)$ is *Gorenstein with socle in degree* $g-2$, i.e.:
   (a) $R^j(M_g) = 0$ for $j \geq g-2$,
   (b) $R^{g-2}(M_g) = \mathbb{Q}$ modulo the choice of an isomorphism.
   (c) There is a perfect pairing
   $$R^i(M_g) \times R^{g-2-i}(M_g) \longrightarrow R^{g-2}(M_g) \cong \mathbb{Q}.$$
2. $\kappa_1, \kappa_2, \ldots, \kappa_{[g/3]}$ generate the ring with no relations in codim $\leq [g/3]$.
3. The formula for the *projection on the socle* holds. To explain what does this mean, one needs two ingredients.
   (a) Set, by definition:
   $$< \tau_{d_1+1}\tau_{d_2+1}\ldots\tau_{d_k+1} >:= \frac{(2g-3+k)!(2g-1)!!}{(2g-1)!\prod_{j=1}^{k}(2d_j+1)!!}\kappa_{g-2},$$
   for $\sum_{j=1}^{k} d_j = g-2$ $(d_j \geq 0)$.
   (b) A second definition for the symbol above:
   $$< \tau_{d_1+1}\tau_{d_2+1}\ldots\tau_{d_k+1} >= \sum_{\sigma\in S_k} \kappa_\sigma,$$
   where σ is written as a product of distinct cycles acting on $\{1,\ldots,k\}$ (including the 1-cycle): if $\sigma = \alpha_1\alpha_2\ldots\alpha_n$, then:
   $$\kappa_\sigma = \kappa_{|\alpha_1|}\kappa_{|\alpha_2|}\ldots\kappa_{|\alpha_n|}.$$
   and for a cycle α we define:
   $$|\alpha| := \sum_{p\in\alpha} d_p.$$

The relations between the two definitions of the same symbol is inspired by a formula on $\overline{M}_{g,n}$ taken from [78]:

$$\sum_{\sigma \in S_k} =< \tau_{d_1+1}\tau_{d_2+1}\dots\tau_{d_k+1} >= \int_{\overline{M}_{g,n}} \psi_1^{d_1+1}\psi_2^{d_2+1}\dots\psi_k^{d_k+1}.$$

for $\sum_{j=1}^{k} d_j = 3g-3$. In the above formula the ψ_i are the so-called *gravitational descendants*, defined as follows. Consider the forgetful map $\pi_0 : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$ gotten forgetting the first marking, labelled by 0. Then the above maps come equipped with n sections $P_1, \dots, P_n$ (repeating the marking and stabilizing). if ω_{π_0} is the relative dualizing sheaf associated to π_0, then $P_i^*\omega_{\pi_0} = \psi_i$. See [78] for more details.

Example 8.1 For $g = 4$ one has:

$$< \tau_2\tau_2 >= \frac{7!7!!}{7!3!!3!!}\kappa_2 = \frac{7\cdot 5\cdot 3\cdot 1}{3\cdot 1\cdot 3\cdot 1}\kappa_2 = \frac{35}{3}\kappa_2.$$

On the other hand, using Witten formulas as in [Wi], one has:

$$< \tau_2\tau_2 >= \kappa_1^2 + \kappa_2,$$

so that we get the relation:

$$\kappa_1^2 = \frac{32}{3}\kappa_2.$$

In other words we have the equations:

$$< \tau_i\tau_j >= \kappa_{i-1}\kappa_{j-1} + \kappa_{g-2} = \frac{(2g-1)!!}{(2i-1)!!(2j-1)!!}\kappa_{g-2} \sim \binom{2g-1}{2i-1}\kappa_{g-2}.$$

For $g = 5$ we have:

$$< \tau_2\tau_3 >= \frac{9!!}{5!!3!!}\kappa_3 = 21\kappa_3 = \kappa_1\kappa_2 + \kappa_3.$$

We also have:

$$< \tau_2\tau_2\tau_2 >= 350\kappa_3 = \kappa_1\kappa_1\kappa_1 + 3\kappa_1\kappa_2 + 2\kappa_3,$$

and $\kappa_1^3 = 288\kappa_3$ together with $\kappa_1\kappa_2 = 20\kappa_3$. So, we get the relation :

$$\kappa_1(20\kappa_1^2 - 288\kappa_2) = 0,$$

yielding the already known inequality:

$$\kappa_1^2 = \frac{72}{5}\kappa_2.$$

8.3 Evidences for the Faber's Conjecture

1. Results by Harer ([41]) imply indeed that $\kappa_1, \ldots, \kappa_{[g/3]}$ have no relations in codimension $\leq g/3$.

2. Looijenga, ([60]) proved that $R^j(M_g) = 0$ for $j > g-2$ and that $dim R^{g-2}(M_g) \leq 1$.

   Incidentally, notice that Looijenga's result imply Diaz's Theorem (see [15]): if $Z \in M_g$ is a projective subvariety, then $dim Z \leq g-2$. Moreover this hold now true in every characteristic. In fact $\kappa_1^{g-1} = 0$ on M_g and κ_1 is ample on $\overline{M}_g$. If it existed a complete subvariety Z of M_g of dimension $g-1$, then $[Z] \cdot \kappa_1^{g-1} \neq 0$, and then it should intersect the boundary of $\overline{M}_g$, which contradicts the assumption on the completeness of Z in M_g.

3. Soon after, see [26], C. Faber proved that actually the dimension of $R^{g-2}(M_g)$ is 1. In fact, by the Looijenga result $R^{g-2}(M_g)$ is generated by the class of the hyperelliptic locus. So the result comes out as a consequence of the following:

   Theorem 8.1 *(Faber, [26]) The class $\kappa_{g-2} \neq 0$ in $A^{g-2}(M_g)$.*
   Hence $dim_{\mathbb{Q}} R^{g-2}(M_g) = 1$

   In particular the class of $[H_g]$ in $A^{g-2}(M_g)$ is not zero. See [26] for a sketch of the proof of Theorem 8.1

4. Because $\kappa_{g-2} \neq 0$ in $A^{g-2}(M_g)$, Faber checked that his conjecture is actually a theorem for all $g \leq 15$. The reason for that is that he is able to express, by explicit calculations, the tautological ring $R^*(M_g)$ as the quotient of $\mathbb{Q}[\kappa_1, \ldots, \kappa_{g-2}]$ by a certain ideal of relations to be described in the examples below. Such a quotient ring turns out to be Gorenstein with socle in degree $g-2$, and since $R^{g-2}(M_g)$ is non zero by Theorem 8.1, the surjection is in fact an isomorphism.

In the examples we shall treat below in genus 3 and 4, as exercises to practise with our jets bundles techniques, we shall check that - for such genera - in fact the following conjectural expression of the ideal of relations in $R^*(M_g)$, hold[1]:

[1]The statement of this beautiful conjecture is almost literally copied from [26].

Conjecture Let $\pi : \mathcal{C}_g \longrightarrow M_g$ be the "universal curve" over M_g, and let $\mathcal{C}^k$ be the k-fold fibered product of $\mathcal{C}_g$ over M_g. Let I_g be the ideal of relations in the polynomial ring $\mathbb{Q}[\kappa_1, \dots, \kappa_{g-2}]$ generated by the relations of the form:

$$\rho_*(M \cdot c_j(\mathcal{F}_{2g-1} - \rho^*\mathbb{E})),$$

with $j \geq g$ and M is a monomial in the $K_i = c_i(p_i^*\omega_\pi)$ and D_{ij} and $\rho : \mathcal{C}_g^{2g-1} \longrightarrow M_g$ is the structural map, forgetting all the points. Then the quotient ring $\mathbb{Q}[\kappa_1, \dots, \kappa_{g-2}]/I_g$ is Gorenstein with socle in degree $g-2$; hence it is isomorphic to the tautological ring $R^*(M_g)$.

The beautiful underlying idea about the conjectural description of the ideal of relations in $R^*(M_g)$ consists in noticing that computing the Chow class of the empty set may sometime give non trivial relations between the tautological classes. We shall apply concretely such an idea (due to Faber), by using the fact that there is no canonical divisor having degree bigger than $2g-2$!!!

8.4 Explicit Computation of the Tautological ring for $g = 3$ and $g = 4$

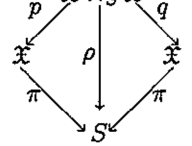
As a matter of example of the techniques involved in the previous section we want to compute explicitly the tautological rings of M_3 and M_4. We use the theorem by Looijenga (already known by Faber, see [24], [25]) to write:

$$R^*(M_3) = \mathbb{Q}[\kappa_1]/I_3 \quad R^*(M_4) = \mathbb{Q}[\kappa_1, \kappa_2]/I_4. \tag{8.1}$$

so that our purpose is to find the ideal of the relations I_3 and I_4. We want to do this directly, without using the formulae by Mumford and the Grothendieck Riemann Roch formula.

8.4.1 The relation $\kappa_1 = 12\lambda$ in genus 3

The situation is as usual:



On $\mathcal{X} \times_S \mathcal{X}$ lives the bundle $\rho^*\mathbb{E}$ and the bundle $J^2K_1 \oplus J^1(K_2 - 3\Delta)$ which has rank 5. The locus of points in $\mathcal{X} \times_S \mathcal{X}$ for which the natural evaluation map (Cf. formula (6.17)):

$$\rho^*\mathbb{E} \longrightarrow \mathcal{E}\left(J^2K_1 \oplus J^1(K_2 - 3\Delta)\right),$$

has rank strictly smaller than 3 (i.e. ≤ 2) is empty.

In fact, such a locus should correspond to the set of curves of genus 3 possessing a pair of points (P, Q), such that, e.g. $3P + 2Q$ is a canonical divisor. Hence:

$$\rho_*[c_3(J^2K_1 \oplus J^1(K_2 - 3\Delta) - \rho^*\mathbb{E})] = 0.$$

Now:

$$\begin{aligned} &c_3(J^2K_1 \oplus J^1(K_2 - 3\Delta) - \rho^*\mathbb{E}) = \\ &[(1 + 6K_1t + 11K_1^2t^2 + 6K_1t^3) \cdot (1 + (3K_2 - 6\Delta)t + \\ &+(2K_2^2 - 9\Delta K_2 + 9\Delta^2)t^2)(1 - \rho^*\lambda t)]_3, \end{aligned} \tag{8.2}$$

i.e.

$$\begin{aligned} &c_3(J^2K_1 \oplus J^1(K_2 - 3\Delta) - \rho^*\mathbb{E}) = 6K_1^3 + 11K_1^2(3K_2 - 6\Delta) + \\ &+6K_1(2K_2^2 - 9\Delta K_2 + 9\Delta^2) - (11K_1^2 + 2K_2^2 - 9\Delta K_2 + 9\Delta^2 + \\ &+6K_1(3K_2 - 6\Delta))\rho^*\lambda = 0. \end{aligned}$$

Pushing down the above expression via ρ one gets:

$$\begin{aligned} 0 &= \rho_*c_3(J^2K_1 \oplus J^1(K_2 - 3\Delta) - \rho^*\mathbb{E}) = 11 \cdot 12\kappa_1 - 66\kappa_1 + 12 \cdot 4\kappa_1 + \\ &- 54\kappa_1 - 54\kappa_1 - (-36 - 36 + 18 \cdot 16 - 36 \cdot 4)\lambda = 6\kappa_1 - 72\lambda, \end{aligned}$$

i.e.

$$\kappa_1 = 12\lambda,$$

as well known.

As for $R^*(M_4)$, for not getting lost with too heavy computation, we shall assume the relation $\kappa_1 = 12\lambda$ e we shall instead concentrate on the relations in codimension 2.

8.4.2 Relations in $R^2(M_4)$

We know that the locus in $\mathfrak{X} \times_S \mathfrak{X}$ (S being a surface) of points (P, Q) such that $aP + bQ$ is a canonical divisor is empty if $a + b = 7 = 2 \cdot 4 - 1$. The equations of such loci in the case (a, b) equal respectively to $(5, 2)$ and $(4, 3)$ are given by:

$$c_4(J^4(p^*K) \oplus J^1(q^*K - 5\Delta) - \rho^*\mathbb{E}) \quad and \tag{8.3}$$
$$c_4(J^3(p^*K) \oplus J^2(q^*K - 4\Delta) - \rho^*\mathbb{E}) \tag{8.4}$$

respectively. Let us set $K_1 = p^*K$ and $K_2 = q^*K$. Hence the relations we seek for are:

$$\rho_*(c_4(J^4(K_1) \oplus J^1(K_2 - 5\Delta) - \rho^*\mathbb{E}) = 0 \tag{8.5}$$
$$\rho_*(c_4(J^3(K_1) \oplus J^2(K_2 - 4\Delta) - \rho^*\mathbb{E}) = 0. \tag{8.6}$$

They are the relations we want if they are independent as it will turn out to be. We compute (8.5) first. We have [2]:

$$\begin{aligned}
& c_4(J^4K_1 \oplus J^1(K_2 - 5\Delta) - \rho^*\mathbb{E}) = \\
= \ & \frac{1}{4!}\frac{d^4}{dt^4}\{(1 + K_1t)(1 + 2K_2t)(1 + 3K_1t)(1 + 4K_1t)(1 + 5K_1t)\cdot \\
& \cdot(1 + (K_2 - 5\Delta)t)(1 + (2K_2 - 5\Delta)t)((1 - \rho^*\lambda t + \rho^*\lambda_2t^2)\Big\}_{t=0} = \\
= \ & 274K_1^4 + 675K_1^3K_2 + 170K_1^2K_2^2 + \\
- \ & 2250K_1^3\Delta - 1275K_1^2K_2\Delta + +2125K_1^2\Delta^2 + \\
+ \ & (-225K_1^3 - 255K_1^2K_2 + 850K_1^2\Delta + 225K_1K_2\Delta - 375K_1\Delta^2)\rho^*\lambda + \\
+ \ & (85K_1^2 + 45K_1K_2 + 2K_2^2 - 150K_1\Delta - 15K_2\Delta + 25\Delta^2)\rho^*\lambda_2.
\end{aligned}$$

Pushing down both sides via ρ_* one gets (recalling the basic formularium of Prop. 5.2):

$$0 = \rho_*c_4(J^4K_1 \oplus J^1(K_2 - 5\Delta) - \rho^*\mathbb{E}) = 1600\kappa_2 + 170\kappa_1^2 - 260\kappa_1\lambda_1 + 480\lambda_2. \tag{8.7}$$

We compute now (8.6). One has, at first:

$$c_4(J^3K_1 \oplus J^2(K_2 - 4\Delta) - \rho^*\mathbb{E}) =$$

[2] We shall not consider λ_3 because it vanishes on a smooth surface.

$$
\begin{aligned}
= & \ \frac{1}{4!}\frac{d^4}{dt^4}\{(1+K_1t)(1+2K_2t)(1+3K_1t)(1+4K_1t)\cdot \\
& \cdot(1+(K_2-4\Delta)t)(1+(2K_2-4\Delta)t)(1+(3K_2-4\Delta)t)\cdot \\
& (1-\rho^*\lambda t+\rho^*\lambda_2t^2)\big\}_{t=0} = \\
= & \ 24K_1^2+300K_1^3K_2+385K_1^2K_2^2+60K_1K_2^3+ \\
- & \ 600K_1^3\Delta-1680K_1^2K_2\Delta+ \\
- & \ 440K_1K_2^2\Delta+1680K_1^2\Delta^2+960K_1K_2\Delta^2-640K_1\Delta^3+ \\
+ & \ (-50K_1^3-210K_1^2K_2-110K_1K_2^2-6K_2^3+420K_1^2\Delta+ \\
+ & \ 480K_1K_2\Delta+44K_2^2\Delta-480K_1\Delta^2-96K_2\Delta^2+64\Delta^3)\rho^*\lambda+ \\
+ & \ (35K_1^2+60K_1K_2+11K_2^2-120K_1\Delta-48K_2\Delta+48\Delta^2)\rho^*\lambda_2.
\end{aligned}
$$

Pushing down both sides via ρ_* one gets:

$$0=\rho_*c_4(J^3K_1\oplus J^2(K_2-4\Delta)-\rho^*\mathbb{E})=-3840\kappa_2+385\kappa_1^2-336\kappa_1\lambda+864\lambda_2. \quad (8.8)$$

We put together (8.7) and (8.8) using the relation $\kappa_1=12\lambda$. One has:

$$\begin{cases} \frac{445}{3}\kappa_1^2 + 480\lambda_2 = 1600\kappa_2 \\ 357\kappa_1^2 + 864\lambda_2 = 3840\kappa_2 \end{cases} \quad (8.9)$$

with

$$\begin{vmatrix} \frac{445}{3} & 480 \\ 357 & 864 \end{vmatrix} = -43200 \neq 0 \quad (!)$$

Solving the linear system (8.9) with respect to κ_1^2 and λ_2 one gets:

$$\begin{cases} \kappa_1^2 = \frac{32}{3}\kappa_2 \\ \lambda_2 = \frac{1}{27}\kappa_2 \end{cases} \quad (8.10)$$

Moreover, λ_2 can be also expressed as

$$\lambda_2=\frac{1}{27}\kappa_2=\frac{1}{27}\cdot\frac{3}{32}\kappa_1^2=\frac{1}{288}\kappa_1^2,$$

as well known by [65].

Hence we may conclude that:

$$R^*(M_3) = \mathbb{Q}[\kappa_1]/(\kappa_1^2)$$

and that:

$$R^*(M_4) = \mathbb{Q}[\kappa_1]/(\kappa_1^3)$$

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