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REGULAR TREES AND THEIR AUTOMORPHISMS

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Regular Trees and their Automorphisms

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Introduction

The subject matter of these notes concerns automorphisms of 1-rooted infinite regular trees. In writing them, I have tried to explore and develop ideas surrounding the construction of certain Burnside groups acting on trees.

Trees which are 1-rooted grow in a variety of theories such as that of automata and of groups. They are subgraphs (half) of infinite homogeneous trees which have been the subject of intensive study for their connection with number theory and the theory of 3-manifolds; see [27], [28].

The necessity for studying the structure of the automorphism groups of trees and their subgroups is clearly recognized in R. Lyndon's survey lecture of problems in Combinatorial Group Theory (see, [19]).

Various classes of groups have faithful representations as 1-rooted tree automorphisms. For any prime p, groups which are finitely generated and residually "finite p-groups" act faithfully on the p-adic 1-rooted tree. The Burnside p-groups constructed by Golod in 1964 are of this kind. By a classical theorem of Schur ([34], page 57) a finitely generated periodic subgroup of any finite dimensional linear group is necessarily finite. Therefore, the Golod groups are nonlinear.

Automorphisms of 1-rooted trees have a natural interpretation as automata. The set of automorphisms of a given 1-rooted tree which are representable as finite automata form the subgroup of *finite state automorphisms.* This subgroup is of special interest to us. In 1972, Aleshin [1] constructed

[1]Support from CNPq and FAPDF is acknowledged

the first infinite Burnside subgroup generated by two finite state automorphisms of the binary tree. Since then other constructions of Burnside groups which afford such representations were given by Suchanskii [33], Grigorchuk [13], Gupta-Sidki [15]. Due to their multiple origins, these groups have been named after their different authors individually or collectively.

The recursive definitions of these examples of Burnside groups were of such simplicity that it became possible to visualize many properties of "residually finite p-groups" having infinite exponent. However, we are still very far from clearly understanding the infinite exponent phenomenon. On the other hand, residually finite groups with finite exponent are known nowadays to be locally finite, thanks to Zelmanov's celebrated solution of the Restricted Burnside problem [36].

General techniques developed in Gupta-Sidki [16] allow the extension of finitely generated periodic groups by adequate automorphisms of a tree, possibly of infinite valence, in such a manner that the new groups continue being periodic and finitely generated, yet acquiring a richer subgroup structure. This work had its continuation in Fournelle-Dixon [4].

Finite state Burnside groups were studied in more depth in [17], [26], [30], [31]. It was shown in [30] that a 3-group of Gupta-Sidki cannot be finitely presented. The same result was also shown for Grigorchuk's 2-group by Lysonok [20].

These Burnside groups also served as important counter-examples to a conjecture of J. Milnor concerning growth functions of groups. Grigorchuk proved that they possess sub-exponential growth [14]. Furthermore he proved these groups to be amenable. A description of this material and a comment on its connections with other areas can be found in Chapter 12 of Ol'shanskii's [22].

In joint work with A.Brunner [2], [3], we have undertaken a study of the group of finite state automorphisms. Considering that this group contains non-linear Burnside groups, the degree to which it differs from linear groups needs to be clarified. Toward this end, we have produced in the second cited work a faithful representation of the linear group $GL(n, \mathbb{Z})$ into the group of finite state automorphisms of a tree naturally associated with the module $\mathbb{Z}^n$. Since the free group F_m of rank m is a subgroup of $GL(2, \mathbb{Z})$, it follows that F_m too has a representation as a group of finite state automorphisms. In view of the theorem of Formanek-Processi [11] that $Aut(F_m)$ is nonlinear for $m \geq 3$, it would be interesting to decide for which $m \geq 2$, the group $Aut(F_m)$

has a faithful representation on some n-ary tree, and, if it does, whether it has a faithful representation as a finite state group of automorphisms.

The Golod groups were, in reality, born out of finitely generated non-nilpotent nil algebras which Golod had constructed as an answer to the Kurosh problem (see, [5], [6]). In this respect, the existence of corresponding algebras for finite state Burnside groups of automorphisms has been raised and remains unanswered. The answer may depend upon deeper properties of nil algebras in finite characteristic and of their corresponding Golod groups (see, [9]).

¿From a different view point, knowing that the total group of automorphisms of a 1-rooted tree is an infinite iteration of group extensions, what about the existence of algebras having similarly iterated structure? We constructed in [32] an invariant ideal within the group algebra of the group of automorphisms of the tree such that the quotient of the algebra by this ideal has infinitely iterated structure, and where the original group is isomorphically embedded. The quotient algebra allowed us to prove the existence of a linear faithful irreducible infinite dimensional representation of one of the Gupta-Sidki 3-groups, in characteristic 3. This result marks a deep contrast between finitely generated infinite p-groups which are residually finite, and finite p-groups. In this respect, Passman and Temple studied in [24] the number of non-equivalent representations of a given degree of p-groups such as ours, over non-denumerable algebraically closed fields having characteristic the same prime p. They showed that if the group admits a faithful irreducible representation then it admits an infinite number of non-equivalent faithful irreducible representations. The new algebras with recursive structure raise a host of interesting questions concerning their special properties.

My thanks go to the participants of the Algebra Seminar, my colleagues Noraí Rocco, Pavel Schumyatski, and students Ana Cristina Vieira, Claus Halkjaer, Frederico Cid, for comments which contributed to an improved exposition. I owe Ana Cristina special thanks for taking notes during my talks.

Content

1. Trees and their automorphisms

1.1 Rooted trees

In order to give a formal definition of our trees, we start with a non-empty set Y , called *alphabet*, and we let $M = M(Y)$ be the set of all finite sequences from Y . Then M is the free monoid generated by Y where the operation is indicated by . , and the neutral element is the empty set ϕ. The length of an element u of M is denoted by $|u|$. The monoid M admits the following ordering:

$$v \leq u \text{ if and only if } u \text{ is a prefix of } v.$$

The tree $\mathcal{T} = \mathcal{T}(Y)$ is none but the graph of $(M, \leq)$, and it is a metric space with metric induced from the length function $| u |$. Given two vertices u and v with largest common prefix w, then the distance between u and v is defined by

$$d(u, v) = | u | + | v | -2 | w | .$$

The tree satisfies two important properties.
(i) Given $u \in M$, the set of its descendents $u.M = \{u.v \mid v \in M\}$ form a subtree isomorphic to the original tree $\mathcal{T}$.. The canonical isomorphism is given by deleting the prefix u.
(ii) The tree $\mathcal{T}$ is the direct limit of the set of finite subtrees $\mathcal{T}_k(M) = \{v \in M : |v| \leq k\}$, defined for $k \geq 0$.

We note that our tree is a subtree of a homogeneous tree obtained from $\mathcal{T}(Y)$ by adding to the latter another copy $\mathcal{T}(Y)'$ and by connecting ϕ with ϕ' with an edge.

The first interesting tree occurs when Y has two elements, say 0 and 1. Then we have the *binary tree* [Figure 1] which we denote by $\mathcal{T}_2$. It is a subtree of the *ternary* homogeneous tree [Figure 2].

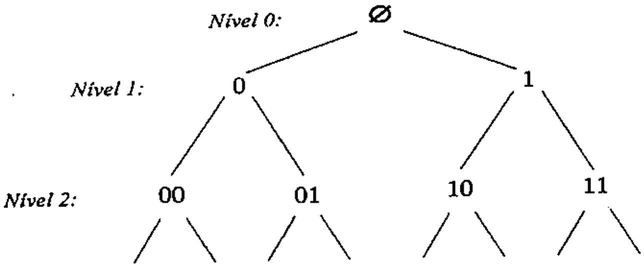

Figure 1: 1-rooted binary tree

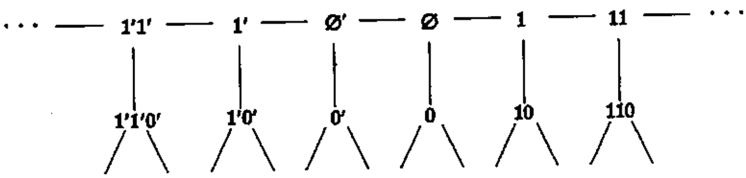

Figure 2: Homogeneous ternary tree

1.2. The group of tree automorphisms

An automorphism of a 1-rooted regular tree is an isometry of its metric space. In other words, it is a bijection on the vertices, which preserves the length function. As we will see below the description of an element α of the group of automorphisms $\mathcal{A} = Aut(\mathcal{T})$ of the tree is made by specifying an infinite sequence of permutations of Y.

Given a permutation σ of Y , with action on the left , it may be extended (rigidly) to an automorphism of the tree $\mathcal{T}$ in the following simple manner :

$$(y.u)\sigma = (y)\sigma.u, \forall y \in Y,\ \forall u \in M.$$

This gives us an immersion of the group $P(Y)$ of permutations of the set Y into the group of automorphisms of the tree $\mathcal{A}$. .

An automorphism $\alpha \in \mathcal{A}$ induces a permutation $\sigma_\phi(\alpha)$ on the set Y, which we identify with its extension as explained above. Therefore the automorphism α factors as $\alpha = \alpha' \sigma_\phi(\alpha)$ where α' fixes Y pointwise. Furthermore, α' itself induces for each y in Y an automorphism $\alpha'(y)$ of the subtree whose vertices form the set $y.M$. On using the canonical isomorphism between this subtree and the tree $\mathcal{T}$, we may consider α' as function from Y into $\mathcal{A}$; in notational form, $\alpha' \in \mathcal{F}(Y, \mathcal{A})$. Thus, the group $\mathcal{A}$ factors as a semi-direct product

$$\mathcal{A} = \mathcal{F}(Y, \mathcal{A}) >\!\lhd P(Y).$$

It is convenient to denote α by $\alpha(\phi)$ and $\alpha'(y)$ by $\alpha(y)$. In order to describe $\alpha(y)$, we follow the same procedure used in the case of α. On repeating successively this procedure, we obtain the set $\sum(\alpha) = \{\sigma_u(\alpha) \mid u \in M\}$ of permutations of Y which describes faithfully the automorphism α. We also obtain the set of automorphisms of the tree $Q(\alpha) = \{\alpha_u \mid u \in M\}$ which we call the *set of states of* α.

It follows that for each α we have a corresponding function

$$\begin{array}{rccc} [\alpha]: & M & \to & P(Y) \\ & u & \mapsto & \sigma_u(\alpha) \end{array}$$

and [ ] is a bijection between $\mathcal{A}$ and the set of functions $\mathcal{F}(M, P(Y))$. It then follows that $\mid \mathcal{A} \mid = \mid P(Y) \mid^{|M|}$.

The above representation of a tree automorphism α is in the form of an infinite product of automorphisms. In general, given a sequence of automorphisms $\{\alpha_i : i \geq 0\}$, their products $\beta = ..\alpha_i..\alpha_2\alpha_1$, $\gamma = \alpha_1\alpha_2..\alpha_i..$ are well-defined automorphisms, provided for any level k,there exists a natural number m_k such that only finitely many of the factors α_i have nontrivial action for $i \leq m_k$,. If $u \in M$ is of length k, then $(u)\beta = (u)\alpha_{m_k}..\alpha_2\alpha_1$, and $(u)\gamma = (u)\alpha_1\alpha_2..\alpha_{m_k}$. Thus we may define the following closure operation: given H a subset of $\mathcal{A}$, let H^* be the set of definable infinite products of elements from H. It follows from the observation about the infinite products β, and γ that if H is a subgroup then H^* is also a subgroup.

We recall (see, [21] page 74) that a subgroup H of a group G is said to be *verbal* provided it satisfies a set of group equations

$$\{a_1x_1a_2x_2..a_sx_sa_{s+1} = e, b_1y_1b_2y_2...b_tu_tb_{t+1} = e, ..\}$$

with variables

$$x_1, x_2, .., x_s, y_1, y_2, .., y_t, ..$$

and constants (i.e, elements of G)

$$a_1, a_2, .., a_{s+1}, b_1, b_2, .., b_{t+1}, ...$$

Proposition 1 *If H is a verbal subgroup of $\mathcal{A}$, then H^* is also verbal .*

Proof. To show that $\beta_1, \beta_2, ..\beta_s$ satisfies one of the group equations , say $\omega_{1=}a_1x_1a_2x_2..a_sx_sa_{s+1} = e$, it suffices to prove this fact modulo $\mathcal{A}_k$, the group stabilizer of the k-th level of the tree, for all $k \geq 1$. This is so because modulo $\mathcal{A}_k$, the β_i's are congruent to elements from H , and so $a_1\beta_1a_2\beta_2..a_s\beta_sa_{s+1} \in \mathcal{A}_k$ for all k, and therefore it is the identity element.

Let Y be a finite set, and α an automorphism of the tree $\mathcal{T}(Y)$. Suppose α induces permutations of order n_k on the k-th level vertices of the tree. Define $\alpha_k = \alpha^{n_k}$. Then infinite products of powers of these α_k's are well-defined.

Let m be the exponent of the group $P(Y)$. Then α^m fixes pointwise the set Y , and clearly, α^{m^k} fixes all $u \in M$ of length k. We conclude that if α has finite order n, then n is a divisor of m^k for some $k \geq 0$.

Suppose $Y = \{0,1\}$. Then torsion automorphisms have orders which are powers of 2. Now, for any automorphism α , the set $\{\alpha, \alpha^2, .., \alpha^{2^k}, ..\}$ allows infinite products. Indeed, for $\xi = \sum j_k.2^k$, $(j_k = 0, 1)$, any element of the ring of dyadic integers $\mathbb{Z}_2$, we may define $\alpha^\xi = ..\alpha^{j_2 4}.\alpha^{j_1 2}.\alpha^{j_0}$. Thus, the closure of α is

$$< \alpha >^* = \{\alpha^\xi : \xi \in \mathbb{Z}_2\}.$$

The following exponentiation rules hold:

$$\alpha^\xi.\alpha^\eta = \alpha^{(\xi+\eta)}, (\alpha^\xi)^\eta = \alpha^{\xi.\eta}, \forall \alpha \in \mathcal{A}, \ \forall \xi, \eta \in \mathbb{Z}_2;$$

in other words, $\mathcal{A}$ is a $\mathbb{Z}_2$-group.

Also, we note that $\langle\alpha\rangle^*$ is isomorphic to $\langle\alpha\rangle$ if α has finite order, and to $\mathbb{Z}_2$ if it has infinite order.

1.3. Automata

A Mealy automaton is a Turing machine defined by a sextuple $(Q, L, \Gamma, f, l, q_0)$, where Q is the set of states, L is the input alphabet, Γ is the output alphabet,

$f : Q \times L \to Q$ is the state transition function, $l : Q \times L \to \Gamma$ is the output function, and q_0 is the initial state; see [10].

An automorphism α of the tree $\mathcal{T}$ can be interpreted in a simple manner as a Mealy automaton. Its input and output alphabet are the same set Y, its set of states is $Q(\alpha)$, the transition function is described by

$$y : \alpha(u) \to \alpha(u.z)$$

and the output function by

$$\alpha(u) : y \to z,$$

where z is the image of y under the permutation $\sigma_u(\alpha)$.

If the alphabet Y is finite and the automorphism α is such that its set of states $Q(\alpha)$ is also finite, then we say α is a finite state automorphism. We denote the set of finite state automorphisms by $F(Y)$.

We will see below some examples of automorphisms of $\mathcal{T}_2$ and their interpretation as automata. The symbol σ will be reserved for the transposition $(0, 1)$ extended to an automorphism of the tree in the manner already indicated. Each automorphism α is represented as $\alpha = (\alpha_0, \alpha_{1)}.\sigma^i$ for some $i = 0, 1$; clearly, $\sigma_\phi(\alpha) = \sigma^i$.

Example 1. Let $\alpha = (\alpha, \sigma)$. Then, the square of α is $\alpha^2 = (\alpha^2, e)$. As this automorphism fixes all the vertices of the tree, it is the identity. Easily, the set of states of α is $Q(\alpha) = \{\alpha, \sigma\}$.

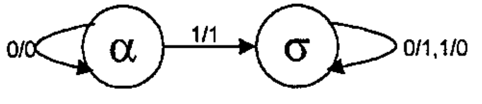

Figure 3: Automaton 1

Example 2. Let $\tau = (e, \tau)\sigma$. Then, $\sigma_\phi(\tau) = \sigma$, $\sigma_0(\tau) = e$, $\sigma_1(\tau) = \tau$. We note that $\tau^2 = (\tau, \tau)$ and therefore

$$\tau^{2n} = (\tau^n, \tau^n), \quad \tau^{2n+1} = (\tau^{n+1}, \tau^n)\sigma$$

Then τ has infinite order and the set of states of τ is $Q(\tau) = \{\tau, e\}$.

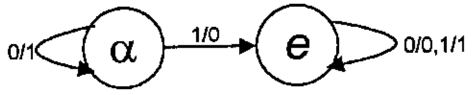

Figure 4: Automaton 2

Example 3. Let $\alpha = (\alpha, \alpha^2)\sigma$. Then, $\alpha^2 = (\alpha, \alpha)$, and

$$\alpha^{2n} = (\alpha^n, \alpha^n), \quad \alpha^{2n+1} = (\alpha^{n+1}, \alpha^{n+2})\sigma$$

Therefore α has infinite order and the set of its states $Q(\alpha) = \{\alpha^n \mid n \geq 1\}$ is also infinite.

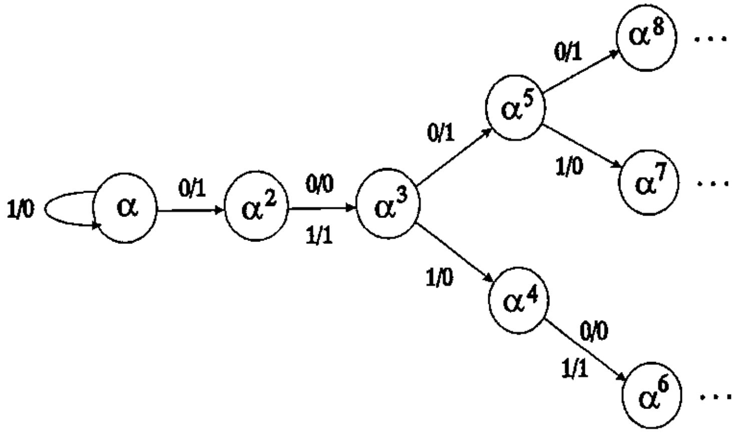

Figure 5: Automaton 3

Example 4. Let $\alpha = (\alpha^{-1}, \alpha^2)\sigma$. Then, $\alpha^{-1} = (\alpha^{-2}, \alpha)\sigma$, $\alpha^2 = (\alpha, \alpha)$ and $\alpha^{-2} = (\alpha^{-1}, \alpha^{-1})$. Therefore,

$$\alpha^{2n} = (\alpha^n, \alpha^n), \quad \alpha^{2n+1} = (\alpha^{n-1}, \alpha^{n+2})\sigma$$

and α has infinite order. The states of α is $Q(\alpha) = \{\alpha^n \mid n = -1, -2, 1, 2\}$.

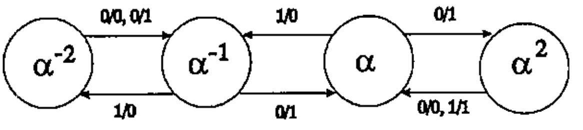

Figure 6: Automaton 4

1.4. The group of finite state automorphisms

For any automorphisms α, β, it is easily verified that their states satisfy

$$Q(\alpha^{-1}) = Q(\alpha)^{-1}, Q(\alpha\beta) \subseteq Q(\alpha)Q(\beta).$$

Therefore,

Lemma 2 *The set of finite state automorphisms $F(Y)$ is a subgroup of $\mathcal{A}$.*

We will make quick remarks on the enumeration of finite state automorphisms of the binary tree.

If $\mid Q(\alpha) \mid = 1$, then there are only two possibilities for α :

$$\alpha = e \text{ ou } \alpha = (\alpha, \alpha)\,\sigma.$$

Let $\mid Q(\alpha) \mid = 2$. As every state is reached from the initial state α, the graph of the automaton is connected. On calculating the number of possibilities for the directed edges and for their labelings, we find that there are 2^7 such automorphisms.

In general, by a result of Harrison [8], the number of automata with n states is asymptotic to $(2n)^{2n}/(2n!)$.

Another type of consideration concerns the relationship between the group operation and the number of states. For an automorphism $\alpha \in \mathcal{A}$with a finite number of states n,we know that α^{-1}also has $n$states. However, as far as $\alpha^2$is concerned, we know that it has at most n^2states. But this number may be much smaller. Consider for example $\tau = (e, \tau)\sigma$.Then, $Q(\tau) = \{\tau, e\}, Q(\tau^2) = \{\tau, \tau^2, e\}$, $Q(\tau^3) = \{\tau^3, \tau^2, \tau, e\}$,and in general, $| Q(\tau^m) | \leq m + 1$, for all $m \geq 1$.

We formulate below a typical problem about the growth of states.

Problem 1. Let H be a subgroup of $\mathcal{A}$ generated by $\alpha_1, ..., \alpha_k$. Then an element $\omega \in H$ is expressible as a word $\omega = \alpha_{i_1}^{m_1} \alpha_{i_2}^{m_2} ... \alpha_{i_s}^{m_s}$ of length $| \omega | = m_1 + m_2 + ... + m_s$; let this be a word of minimum length. Suppose that the growth of states of ω is linear in terms of the length of ω;that is, there exists a constant c such that $| Q(\omega) | \leq c. | \omega |$, $\forall \omega \in H$. What is the group theoretic impact of this condition on H?

2. Cosets

2.1. The coset tree

Given a group $H$and a chain of its subgroups:

$$H = H_0 > H_1 > ... > H_i > H_{i+1} > ...$$

such that $\bigcap H_i = \{1\}$, we consider the partition of each of the subgroups in this chain as cosets of the subsequent subgroup

$$H = \bigcup H_1 x_{j,1}, H_1 = \bigcup H_2 x_{j,2}$$

and for $i \geq 1$,

$$H_i = \bigcup H_{i+1} x_{j,i+1}.$$

Thus we produce the cosets

$$H_s x_{j_s,s} x_{j_{(s-1)},s-1} ... x_{j_1,1}$$

which will be the vertices of a tree $\mathcal{T}$ where the incidence relation is defined by set theoretic inclusion . The group H acts on this tree simply by translation

$$h : H_i w \mapsto H_i wh$$

and this action is faithful.

The set of vertices fixed by $h \in H$ is a subtree of $\mathcal{T}$ which can be irregular. In case H_i is a normal subgroup of H and h fixes some coset $H_i w$ then h fixes all the cosets of H_i in H, and thus fixes all the vertices of the tree down to and including the i-th level .

We note that the coset tree obtained from the chain of subgroups is regular provided $[H_i : H_{i+1}]$ is constant. If the set of consecutive indices is bounded by some number m, then we can embed the coset tree in an m-ary tree, and extend the action of H to the possibly larger tree by having it fix pointwise the extra subtrees that it may have.

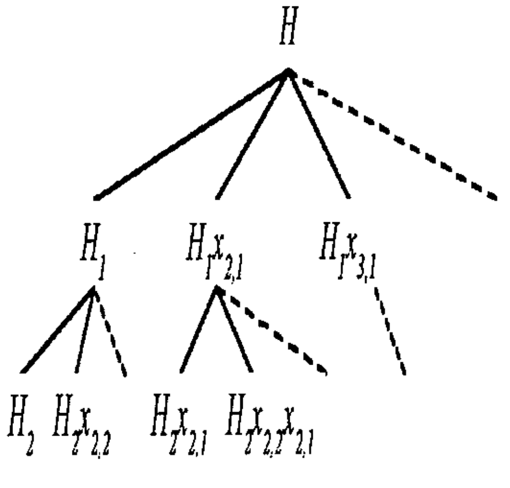

Figure 7: The coset tree

Example 5. The group $G = \sum\{C_p \mid p \text{ :prime number }\}$does not have a chain of subgroups $\{H_i\}$ satisfying both conditions $\bigcap H_i = \{e\}$,$\exists c$ such that $\forall i$,$[H_i : H_{i+1}]< c$.

2.2. Residually finite groups

A group $H$is said to be *residually finite* (see [21], page 116) provided for each nontrivial element $h \in H$, there exists a normal subgroup $N_h \trianglelefteq H$ such that $h \notin N_h$ and $[H : N_h] < \infty$. This implies the existence in H of a chain of normal subgroups $\{H_i\}$, such that $[H : H_i] < \infty$ and $\bigcap H_i = \{1\}$. If the group H is generated by an enumerable set then there exists a enumerable chain $\{H_i\}$ with the required properties. However, as we have already commented, the existence of a chain where the subsequent indices are constant, or even bounded, cannot be guaranteed.

On the other hand, given a prime number p and an enumerable group Hwhich is also residually a finite p-group, in the sense that H/N_h is a finite p-group, then it has an enumerable chain of normal subgroups $\{H_i\}$ whose successive quotients H_i/H_{i+1} are finite p-groups. We may refine this chain to a chain $\{K_j\}$ of subnormal subgroups of H, which starts with $H = K_0$, such that $[K_j : K_{j+1}] = p$, and whose intersection is the identity. The coset tree corresponding to this chain is isomorphic to the p-adic tree $\mathcal{T}(Y)$ for $Y = \{0, 1, ..., p\}$ which we denote by $\mathcal{T}_p$.

It is a well-know that a free group of finite rank is residually a finite $p-$group, for any prime number p. We conclude easily from this

Theorem 3 *Every free group of finite rank is isomorphic to a subgroup of $Aut(\mathcal{T}_p)$, for any prime number p.*

The linear group $\Gamma = GL(n, \mathbb{Z})$ is residually finite. The usual argument proceeds as follows. The the ring epimorphism $\vartheta_k : \mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$, for $k \geq 1$ induces a group epimorphism $\phi_k : GL(n, \mathbb{Z}) \to GL(n, \mathbb{Z}/p^k\mathbb{Z})$ whose kernel is the congruence subgroup N_k formed by linear transformations of the from $I + p^k L$ where L is any $n \times n$ integral linear transformation. Therefore we have :

(i) $\Gamma/N_1 \simeq GL(n, \mathbb{Z}/p\mathbb{Z})$ is a finite group,
(ii) for $i \geq 1, N_i/N_{i+1}$ is an elementary abelian p-group of rank n^2,
(iii) $\cap N_i = \{e\}$.

Actually, the chain of congruence subgroups satisfy the stronger condition:

(ii)$'$ for $1 \leq i \leq j$, N_i/N_j is a p-group generated by n^2 elements.

If we replace in the above $\mathbb{Z}$ by any residually finite ring R (for example, any finitely generated subring of $\mathbb{C}$) then $GL(n, R)$ is a residually finite group.

Going back to $GL(n, \mathbb{Z})$, since N_1 is a finitely generated and residually a finite p-group, it has a faithful representation on the p-adic tree $\mathcal{T}_p$. In order to represent Γ itself on a 1-rooted regular tree, we consider the problem of the extension of a tree automorphism group by a finite group. To this effect we recall the result of Kaloujnine-Krasner [23].

Theorem 4 *Let G be a group having a normal subgroup N with quotient group $H = G/N$. Then G is faithfully embedded in the wreath product of N by H, $W = \mathcal{F}(H, N).H$, in such manner that N is mapped into $\mathcal{F}(H, N)$, and some set of coset representatives of N in G cover H.*

The proof of the theorem amounts simply to codifying a factor set of N in G as a 'vector' in $\mathcal{F}(H, N)$.

Corollary 5 *Let G be a group having a normal subgroup N such that $H = G/N$ is finite of order m. Suppose N is faithfully represented as a group of automorphisms of an n-ary 1-rooted tree $\mathcal{T}_n$. Then this representation can be extended to a faithful representation of G on a q-tree where $q = max\{m, n\}$.*

Proof. Extend the tree by taking m copies of the tree $\mathcal{T}_n$ and attach them to a root vertex. Have N act on the first copy as before and have it fix point-wise the other copies. Also let H permute the n copies of $\mathcal{T}_n$ by the regular representation. Then the group generated by these representations of N and H is isomorphic to W, the wreath product of N by H. Now easily, the tree can be embedded in a q-tree, where $q = max(m, n)$ and the action of W can be extended to the this q-tree. Therefore by the previous theorem, G is faithfully represented on the q-tree.

Corollary 6 *The linear group $GL(n, \mathbb{Z})$ is faithfully representable on a q-tree for some natural number q.*

Of course, the above representation is no way explicit. However, we have given an explicit representation in [3].

A theorem of Lubotsky is quite relevant to the question concerning which subgroups of the group of tree automorphisms are linear. His result is modeled on the chain of congruence subgroups in the linear group. A group G is said to satisfy the p-congruence condition with bound $c \in \mathbb{N}$ provided it has a descending series of normal subgroups $\{N_i\}$ such that: G/N_1 is a finite group, N_i/N_{i+1} is a finite p-group, $\cap N_i = \{e\}$, and for $1 \leq i \leq j$, N_i/N_j is generated by c elements.

Theorem 7 *[18] Let G be a finitely generated group. Then G is isomorphic to a subgroup of $GL(n, \mathbb{C})$ iff G satisfies the p-congruence condition for some bound c.*

It is time we look at a concrete coset tree and a calculation of one of its automorphisms.

Example 6. The $(+1)$ function

Let $H = \mathbb{Z}$ be the additive group of the integers, and define for every non-negative integer k, the subgroup $H_k = 2^k\mathbb{Z}$. Then we have a chain of subgroups of H and the coset tree is the binary tree with set of vertices $\{2^k\mathbb{Z} + i \mid 0 \leq i \leq 2^k - 1, 0 \leq k\}$.

The $(+1)$ function on the integers, $\tau : i \to i+1$, induces an automorphism of the binary tree

with the following action

$$\tau : 2^k\mathbb{Z} + i \to \begin{cases} 2^k\mathbb{Z} + i + 1, \text{ se } i < 2^k - 1 \\ \\ 2^k\mathbb{Z}, \text{ se } i = 2^k - 1 \end{cases}$$

As τ interchanges the two vertices of the first level in the tree, it has the representation $\tau = (\tau_0, \tau_1)\sigma$. Also, since $2^k\mathbb{Z} + 2i \overset{\sigma}{\longleftrightarrow} 2^k\mathbb{Z} + 2i + 1$, then

$$\tau\sigma : 2^k\mathbb{Z} + 2i \overset{\tau}{\longrightarrow} 2^k\mathbb{Z} + 2i + 1 \overset{\sigma}{\longrightarrow} 2^k\mathbb{Z} + 2.$$

Therefore $\tau\sigma$ acts as the identity on the subtree headed by $2\mathbb{Z}$; in other words, $\tau_0 = e$ and $\tau\sigma = (e, \tau_1)$. Now we look at the action of $\tau\sigma$ on the subtree headed by $2\mathbb{Z}+1$:

$$\tau\sigma : \begin{cases} 2^k\mathbb{Z}+2i-1 \xrightarrow{\tau} \mathbb{Z}+2i \xrightarrow{\sigma} 2^k\mathbb{Z}+2i+1 \\ 2^k\mathbb{Z}+2^k-1 \xrightarrow{\tau} 2^k\mathbb{Z} \xrightarrow{\sigma} 2^k\mathbb{Z}+1 \end{cases}$$

In order to determine τ_1, we use the isomorphism between the subtree headed by $2\mathbb{Z}+1$and $\mathbb{Z}$

$$\varphi : 2^k\mathbb{Z}+2i+1 \longrightarrow 2^{k-1}\mathbb{Z}+i$$

Therefore

$$\tau_1 : \begin{cases} 2^{k-1}\mathbb{Z}+i-1 \longrightarrow 2^{k-1}\mathbb{Z}+i \\ 2^{k-1}\mathbb{Z}+2^{k-1}-1 \longrightarrow 2^{k-1}\mathbb{Z} \end{cases}$$

and this is precisely the definition of τ. Therefore we have reached the recursive definition $\tau = (e, \tau)\sigma$.

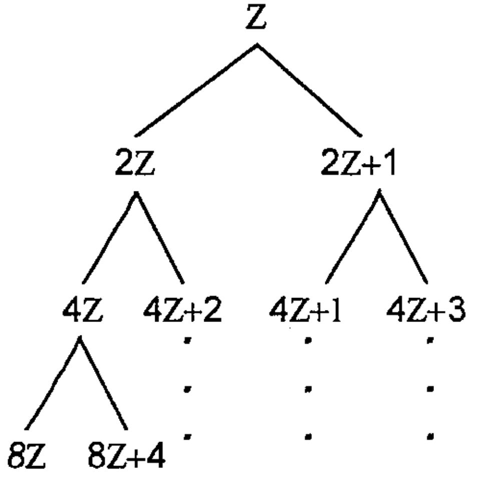

Figure 8: The binary coset tree of $\mathbb{Z}$

3. Some subgroups of automorphisms

One of the consequences of the iterative structure of the group of automorphisms $\mathcal{A}$ is the abundance of subgroups which are direct products. Indeed, $\mathcal{A}$ contains the direct product of enumerably many copies of itself as a subgroup. This is easily illustrated on the binary tree: we create isomorphic copies of $\mathcal{A}$ on the subtrees headed by the labels $u = 00..01$. As these copies commute , and as only finitely many of them have nontrivial actions on any given level, their direct product is well-defined.

Indecomposable subgroups with iterated structure can be obtained by using the following construction. First let us introduce some notation. Given a set Y and a group H, define the groups

$$\mathcal{F}_0(Y,H) = H, \mathcal{F}_1(Y,H) = \mathcal{F}(Y,H),$$

and for $i \geq 1$,

$$\mathcal{F}_i(Y,H) = \mathcal{F}(Y, \mathcal{F}_{i-1}(Y,H)).$$

It is clear that the i-th group in this definition is isomorphic to a direct product of $|Y|^i$ copies of H.

Now suppose that H is a subgroup of $P(Y)$. We construct from H two subgroups of automorphisms of the tree $\mathcal{T}(Y)$. The first, $H^\#$ is the group generated by $\mathcal{F}_i(Y,H)$ for $i \geq 0$. The second, $\widetilde{H}$ is the closure of $H^\#$ under infinite products; its elements are represented as infinite products

$$\alpha = ...f_k...f_0,$$

where $f_i \in \mathcal{F}_i(Y,H)$, for $i \geq 0$. Consider the direct product of groups

$$\mathcal{X}(H) = .. \times \mathcal{F}_k(Y,H) \times .. \times \mathcal{F}_1(Y,H) \times \mathcal{F}_0(Y,H),$$

then the map

$$\alpha \to (.., (f_k...f_0), .., (f_1 f_0), f_0)$$

is an embedding of $\widetilde{H}$ into $\mathcal{X}(H)$. Note that the following decompositions hold

$$H^\# = \mathcal{F}(Y, H^\#) >\!\!\lhd H, \widetilde{H} = \mathcal{F}(Y, \widetilde{H}) >\!\!\lhd H.$$

3.1. The base group

In the above construction, if $H = P(Y)$, then $H^{\#}$ is called the *base group* and is denoted by $G(Y)$. In this case, $\widetilde{H}$ is the whole group of automorphisms $\mathcal{A}$.

An element α of $G(Y)$ has finite description, in the sense that $\alpha_u = e$, except for finitely many indices u. Clearly, when Y is finite, $G(Y)$ is a subgroup of the group of finite state automorphisms $F(Y)$.

The decompositions for both $\mathcal{A}$ and $G(Y)$ can be further developed as

$$\mathcal{A} = \mathcal{F}_k(Y, \mathcal{A}) >\!\!\lhd G_{0,k-1}(Y)$$

$$G(Y) = \mathcal{F}_k(Y, G(Y) >\!\!\lhd G_{0,k-1}(Y)$$

where by definition

$$G_{0,k-1}(Y) =< \mathcal{F}_{k-1}(Y, P(Y)), ..., P(Y) >$$

This last group is isomorphic to the group of automorphisms of the subtree formed by the vertices u with length at most k . Also, $G(Y)$ is the union of its subgroups $G_{0,k}(Y)$ for $k \geq 0$.

Since $\mathcal{F}_i(Y, \mathcal{A})$ is a direct product of copies of $\mathcal{A}$, on fixing $y_0 \in Y$, we may define for $i \geq 0$ the following groups:

$$P_0(Y) = P(Y),$$

$$and\ for\ i \geq 0,$$

$$P_i(Y) = \{f \in \mathcal{F}_i(Y, \mathcal{A}) \mid f(y_0) \in P_{i-1}(Y),\ f(y) = e,\ \forall y \neq y_0\}.$$

Lemma 8 : *Suppose Y is finite. Then the group $G(Y)$ is locally finite and is generated by $P_i(Y)$, for $i \geq 0$.*

Proof. Since Y is finite, the group $G_{0,k}(Y)$ is also finite and is generated by its subgroups $P_i(Y)$, for $0 \leq i \leq k$. Now the assertion follows from the fact that $G(Y)$ is the union of $G_{0,k}(Y)$, for all $k \geq 0$.

The group $G_{0,k}(Y)$ is of *type* $P(Y)$, in the sense that it contains subnormal subgroups where the successive quotients are isomorphic to $P(Y)$. Thus $\mathcal{A}$ itself is residually a $P(Y)$-group. We also observe that is $\mathcal{A}$ the *inverse limit* of the groups $G_{0,k}(Y)$. This means that any automorphism α is a limit of the sequence formed by its images, $\mathcal{F}_k(Y,\mathcal{A})\ \alpha$, for $k \geq 0$. Since the quotient group $\mathcal{A}/\mathcal{F}_k(Y,\mathcal{A}) \cong G_{0,k-1}(Y)$, the above sequence corresponds to one formed by elements from the groups $P_k(Y)$.

Let p be a prime number, $Y = \{0,1,2,..,p-1\}$ and let σ be the permutation the cycle $(0,1,2,..,p-1)$. If we choose the subgroup of $P(Y)$ to be $H = \langle\sigma\rangle$, then $H^{\#}$ is a locally finite p-group generated by elements defined inductively by

$$\sigma_0 = \sigma, \sigma_1 = (e,e,..,e,\sigma_0),..,\sigma_{i+1} = (e,e,..,e,\sigma_i),...,$$

and it is a (restricted) infinitely iterated wreath product $(...C_p)wrC_p)wrC_p$ of cyclic p-groups. As is well known, every finite p-group is isomorphic to a subgroup of a wreath product of C_p iterated a finite number of times, therefore it follows that $\langle\sigma\rangle^{\#}$ contains a copy of every finite p-group.

In the case of the binary tree, the group $P(Y)$ is generated by σ, $P_i(Y)$ is generated by σ_i and $G(Y) = \langle\sigma\rangle^{\#}$.

3.2. Functionally recursive automorphisms

We saw that the automorphism of the binary tree $\alpha = (\alpha, \alpha^2).\sigma$ has an infinite number of states, inspite of its simple description. In order to explain the process by which α_u is obtained from α, we need to introduce the notion of a *functionally recursive* set of automorphisms. Let $\Psi = \{\alpha, \beta, \gamma, ...\}$ be a finite set of elements from $\mathcal{A}$ where

$$\alpha = \alpha'\sigma_\phi(\alpha), \beta = \beta'\sigma_\phi(\beta), \gamma = \gamma'\sigma_\phi(\gamma), ...$$

We call $\{\sigma_\phi(\alpha), \sigma_\phi(\beta), \sigma_\phi(\gamma), ...\}$ the *initial data* for Ψ. Now we define a set of distinct symbols $\{a,b,c,...\}$ corresponding to α, β, γ, .. respectively, and use them as free generators of the free group $\mathcal{L}$. Also we let $\widetilde{\mathcal{L}} = \mathcal{L} * G(Y)$ be the free product of $\mathcal{L}$ with the base group $G(Y)$.

The set Ψ is fuctionally recursive provided there exist words A_y, B_y, C_y, ... $\in \widetilde{\mathcal{L}}$, $\forall y \in Y$, such that α_y is obtained from A_y by substituting the symbols $a, b, c, ...$ by their values $\alpha, \beta, \gamma, ...$ In the same manner, β_y is obtained from B_y, etc.

Having defined a funcionally recursive set of automorphisms, an automorphism α is functionally recursive provided it belongs to some set of functionally recursive automorphisms Ψ. Let $R(Y)$ denote the set of functionally recursive automorphisms.

In the case of the automorphism $\alpha = (\alpha, \alpha^2)\sigma$, we may take Ψ to be simply the unitary set $\{\alpha\}$. Here the free group is generated by a and the words are $A_0 = a$, $A_1 = a^2$.

Theorem 9 *(i) The set of functionally recursive automorphisms $R(Y)$ is a subgroup of $\mathcal{A}$.*
(ii) The subgroup of finite state automorphisms $F(Y)$ is a proper subgroup of $R(Y)$.
(iii) If Y is finite then $R(Y)$ is enumerable.

Proof. (i) (a) Easily, $e \in R(Y)$.

(b) Let α be a functionally recursive automorphism. Then $\alpha \in \Psi$ for some functionally recursive set. Therefore $\alpha^{-1} \in \Psi^{-1}$ which is also functionally recursive. This is so, since $(\alpha^{-1})_y = (\alpha_z)^{-1}$ where $\sigma(z) = y$, and α_z is obtained by substituting elements of Ψ in the word A_z. We see that α_y^{-1} can be similarly obtained from Ψ^{-1} using the inverse of A_z.

(c) Let Ψ and Φ be functionally recursive sets, and let $\alpha \in \Psi$ e $\beta \in \Phi$. Then $\alpha\beta = \gamma$ is an element of $\Psi\Phi$. On using the product of adequate words, it can be shown easily that $\Psi\Phi$ is funcionaly recursive.

(ii) Let α be a finite state automorphism. Then $\alpha \in \Psi = Q(\alpha)$ which is funcionally recursive, for we may simply use the generators a_u of the free group as the words in which to substitute; naturally, we need to maintain the identification $a_u = a_v$ whenever $\alpha_u = \alpha_v$. The example $\alpha = (\alpha, \alpha^2)\sigma$ proves the proper incusion of $F(Y)$ in $R(Y)$.

(iii) Let Y have finite cardinality n. Given a natural number k we can enumerate the funcionally recursive sets with k elements, by enumerating all the $(n+1).k$-tuples $(A_y, B_y, ..; i_{\phi(\alpha)}, i_{\phi(\beta)}, ..)$ where $A_y, B_y.., \in \widetilde{\mathcal{L}}$, and where $i_{\phi(\alpha)}, i_{\phi(\beta)}, .., \in Y$. Therefore, the group $R(Y)$ is enumerable.

Problem 2. Describe group theoretic differences between $F(Y)$ and $R(Y)$.

4. Conjugacy classes and Centralizers

4.1 Conjugates of automorphisms

We start with two examples of automorphisms of the binary tree, and consider their conjugacy classes.

(i) Let $\alpha = (\alpha_0,\alpha_1).\sigma$. Then on conjugating α by $\beta = (\alpha_0, e)$ we get $\gamma = \alpha^\beta = (e,\gamma_1).\sigma$,where $\gamma_1 = \alpha_1\alpha_0$. Observe that $\gamma^2 = (\gamma_1, \gamma_1)$. Thus , if $o(\alpha) = 2$, then $\gamma_1 = e$, and $\gamma = \sigma$ follow.

(ii) Let $\alpha = (\alpha, \sigma)$. Then, $o(\alpha) = 2$, and α cannot be conjugated, in the same manner as in the first example, to an element of $G(Y)$.

Let Y be finite, and let us fix the canonical representatives of the conjugacy classes of $P(Y)$. We will describe how to produce inductively representatives of the conjugacy classes of automorphisms of the tree. The following two basic situations will explain the procedure.

(i) Suppose that on some tree, we have an automorphism represented by $\alpha = (\alpha_0,\alpha_1,.., \alpha_k).\sigma$, where σ permutes the entries transitively. Then, we may assume, by using conjugation from $P(Y)$, that σ corresponds to the cycle $(0,1,..,k)$. Then on conjugating by

$\beta = (\alpha_0, e, (\alpha_1)^{-1},(\alpha_1\alpha_2)^{-1},(\alpha_1..\alpha_{k-1})^{-1})$ we obtain, as in part (i) above,

$$\begin{aligned}
\alpha^\beta &= ((\alpha_0)^{-1}, e, \alpha_1, \alpha_1\alpha_2,.., \alpha_1..\alpha_{k-1}).(\alpha_0,\alpha_1,..,\alpha_k)\sigma. \\
&\quad .(\alpha_0, e, (\alpha_1)^{-1}, (\alpha_1\alpha_2)^{-1},..,(\alpha_1..\alpha_{k-1})^{-1}) \\
&= (e, \alpha_1, \alpha_1.\alpha_2,.., \alpha_1..\alpha_{k-1}.\alpha_k).(e, (\alpha_1)^{-1}, (\alpha_1\alpha_2)^{-1},..,(\alpha_1..\alpha_{k-1})^{-1})\sigma \\
&= (e, e,.., e, \alpha_1..\alpha_{k-1}\alpha_k.\alpha_0)\sigma.
\end{aligned}$$

(ii) If on some tree an automorphism has the form $\alpha = (e, e,.., e, \alpha_k)\sigma$, then in order to effect the conjugation of α_k by some β, we simply conjugate α by $\gamma = \beta^{(1)} = (\beta,\beta,..,\beta)$ producing the desired effect $\alpha^\gamma = (e,e,..,e, \alpha_k^\beta)\ \sigma$.

With these observations it becomes possible to prove

Theorem 10 *Let $\alpha \in \mathcal{A}$, and write $\alpha = ...f_k..f_1f_0$ with $f_i \in \mathcal{F}_i(Y,P(Y))$, for $i \geq 0$. Then the representative of the conjugacy class of α is $\gamma =$*

$...g_k..g_1g_0$, where for all $k \geq 0$, the γ-section $g_k..g_1g_0$ is the representative of the conjugacy class of the α-section $f_k..f_1f_0$ in $G_{0,k}(Y)$.

In the next result, we illustrate how orbit structures of tree automorphisms may determine their conjugacy classes.

Theorem 11 *Let γ be an automorphism of the binary tree.*
(i) Suppose γ has 2^{k-1} orbits on the k-th level for all $k \geq 1$. Then γ is conjugate to σ.
(ii) Suppose that γ induces a transitive permutation on the k-th level for all $k \geq 1$. Then γ is conjugate to the " $(+1)$ function" $\tau = (e, \tau)\sigma$.

Proof. (i) First we note that γ does not fix any vertex at any level $k \geq 1$. For, if γ fixes some vertex u, it will then fix all the vertices on the path connecting u to the root ϕ. Therefore every orbit on the k-th level has size two for all $k \geq 1$. Now, $\gamma = (\gamma_0, \gamma_1)\sigma$ and we may assume by conjugation that $\gamma_0 = e$. If ρ is some orbit of γ_1 at the k-th level, then $\rho \cup \rho^\sigma$ is an orbit of γ on that level of size $2 \mid \rho \mid$. Therefore, $\mid \rho \mid = 1$, and $\gamma_1 = e$.

(ii) Clearly, $\gamma = (\gamma_0, \gamma_1)\sigma$ which when conjugated by $\beta = (\gamma_0, e)$ becomes $\gamma^\beta = \gamma' = (e, \gamma_1\gamma_0)\sigma$. Now $\gamma_1' = \gamma_1\gamma_0$ is again an automorphism that induces a transitive permutation on every level of the tree, and so $\gamma_1' = (\gamma_{10}', \gamma_{11}')\sigma$, and we may conjugate γ^β by $\delta = (\delta_0 , \delta_0)$ where $\delta_0 = (\gamma_{10}', e)$ to obtain

$$\gamma^{\beta\delta} = (e, (e, \gamma_{11}'\gamma_{10}')\ \sigma)\ \sigma.$$

We produce in this manner an infinite sequence of conjugators β,δ,.., whose product is well defined. In the end, γ is conjugated to an automorphism α of the tree, where $\alpha_u = (e, \alpha_{1u})\sigma$, if the index $u = \phi$ or 11..1, and $\alpha_u = e$ for other indices. The definition of α clearly coincides with that of τ.

It is possible to write down explicitly the conjugacy classes of the base group $G(Y)$ when $Y = \{0, 1\}$. We recall that $G_{0,k}(Y) =< \sigma_k, ...\sigma_1, \sigma_0 >$, where $\sigma_0 = \sigma$ and $\sigma_i = (e, \sigma_{i-1})$, for $i \geq 1$, and that $G(Y) = \bigcup G_{0,k}(Y)$.

We start off with the first classes $C_{-1} = \{e\}$ and $C_0 = \{\sigma\}$. Next we define $\widetilde{C}_0 = C_{-1} \cup C_0$, a set of representatives of the conjugacy classes of $G_{0,1}(Y)$, and order it by $e < \sigma$. Let $C_1 = \{(e, \sigma), (\sigma, \sigma), (e, \sigma)\sigma\}$ and define $\widetilde{C}_1 = \widetilde{C}_0 \cup C_1$, the set consisting of the representatives of the classes of

$G_{0,2}(Y)$, and ordered it by $e < \sigma < (e,\sigma) < (\sigma,\sigma) < (e,\sigma)\sigma$. Having defined by induction the set of representatives $\widetilde{C}_k$ of $G_{0,k+1}(Y)$, we define

$$C_{k+1} = \{(\zeta_0,\zeta_1) \mid \zeta_0 \in \widetilde{C}_k, \zeta_1 \in C_k, \zeta_0 \leq \zeta_1\} \cup (\{e\} \times C_k)\sigma$$

and define the set of representatives of the classes of $G_{0,k+2}(Y)$ to be $\widetilde{C}_{k+1} = \widetilde{C}_k \cup C_{k+1}$. We extend the ordering in the obvious way. With this, the process of enumaration of a set of representatives of the conjugacy classes of $G(Y)$ is completely described.

4.2 Centralizers of automorphisms of the binary tree

A straightforward analysis shows that the centralizer of $\alpha \in \mathcal{A} = Aut(\mathcal{T}_2)$, which we denote by $C(\alpha)$, can be described as follows.

(1) If $\alpha = (\alpha_0, \alpha_1)$, then there are two possibilities:

(1.1) if α_0 and α_1 are non-conjugate, then $C(\alpha) = C(\alpha_0) \times C(\alpha_1)$;

(1.2) if $\alpha_1 = \alpha_0^{\beta}$, for some tree automorphism β, then

$$C(\alpha) = [C(\alpha_0) \times C(\alpha_0^{\beta})] < (\beta, \beta^{-1})\sigma > .$$

(2) If $\alpha = (\alpha_0, \alpha_1)\sigma$, then $C(\alpha) = \{(\beta, \beta^{\alpha_0}) \mid \beta \in C(\alpha_0\alpha_1)\} \cdot < \alpha > .$

Proposition 12 *Let τ be the $(+1)$ function . Then, $C(\tau) = \langle\tau\rangle^*$. Furthermore, if $\alpha \in \mathcal{A}$ is such that $C(\alpha) = \langle\alpha\rangle^*$ then α is conjugate to τ .*

Proof. Recall that $\tau = (e,\tau)\sigma$, and that $\tau^2 = (\tau,\tau)$. Let $\beta = (\beta_0,\beta_1)\sigma^i \in C(\tau)$. We multiply β by τ^i and thus assume $\beta = (\beta_0,\beta_1)$. Now, $\beta^\tau = (\beta_1^\tau, \beta_0) \in C(\tau)$ holds if and only if $\beta_0 = \beta_1$, $\beta_0 \in C(\tau)$. On multiplying (β_0,β_0) by τ^{2j} for some $j = 0,1$, we assume that $\beta_0 = (\beta_{00}, \beta_{01})$. The argument can repeated to reach $\beta = \tau^{\ \xi}$ for some ξ, a dyadic integer.

Suppose that $\alpha = (\alpha_0,\alpha_1)\sigma^i$ is such that $C(\alpha) = \langle\alpha\rangle^*$. If $i = 0$, then $(\alpha_0, e) \in C(\alpha)$, and so there exists $\xi \in \mathbb{Z}$ such that $\alpha^\xi = (\alpha_0^\xi, \alpha_1^\xi) = (\alpha_0, e)$ from which we conclude that α_1 has finite order; similarly, α_0 has finite order. Therefore α has finite order and this easily leads to a contradiction. Therefore $i = 1$, and we may assume that $\alpha = (e,\alpha_1)\sigma$. Hence, $C(\alpha) = \{(\beta,\beta) \mid \beta \in C(\alpha_1)\} \cdot < \alpha >$. Now we observe from $C(\alpha) = \langle\alpha\rangle^*$, that $C(\alpha_1) = \langle\alpha_1\rangle^*$ and the argument may be repeated to reach the desired conclusion.

4.3 The affine group of the dyadic integers

Let α be an automorphism of the binary tree, and let $\xi = \sum j_i.2^i$ be a dyadic unit; that is $j_0 = 1$.We will show that the exponentiation $\alpha \to \alpha^\xi$ can be effected by conjugation.

The automorphism α is one of two types (α_0, α_1), $(\alpha_0, \alpha_1)\sigma$. Accordingly,

$$\alpha^\xi = (\alpha_0^\xi, \alpha_1^\xi) \text{ or } ((\alpha_0\alpha_1)^{(\xi-1)/2}\alpha_0,\ (\alpha_1\alpha_0)^{(\xi-1)/2}\alpha_1)\sigma.$$

We want to find λ an automorphism of the tree such that $\alpha^\lambda = \alpha^\xi$. For both possible types of α we will choose $\lambda = (\lambda_0\ ,\lambda_1)$, and observe that

$$\alpha^\lambda = (\alpha_0^{\lambda_0},\ \alpha_1^{\lambda_1}), \text{ or } (\lambda_0^{-1}\alpha_0\lambda_1, \lambda_1^{-1}\alpha_1\lambda_0)\sigma.$$

If α is of the first type, then the conditions are $\alpha_0^{\lambda_0} = \alpha_0^\xi$, $\alpha_1^{\lambda_1} = \alpha_1^\xi$. So the problem is repeated at the next lower level in the tree. If α is of the second type then the conditions are

$$\lambda_0^{-1}\alpha_0\lambda_1 = (\alpha_0\alpha_1)^{(\xi-1)/2}\alpha_0,\ \lambda_1^{-1}\alpha_1\lambda_0 = (\alpha_1\alpha_0)^{(\xi-1)/2}\alpha_1.$$

These conditions are equivalent to

$$\begin{aligned} \lambda_1 &= \alpha_0^{-1}\lambda_0(\alpha_0\alpha_1)^{(\xi-1)/2}\alpha_0 \\ (\alpha_0\alpha_1)^{\lambda_0} &= (\alpha_0\alpha_1)^\xi. \end{aligned}$$

Again, the second equation repeats our problem at the next lower level of the tree, and we choose $\lambda_0 = (\lambda_{00}\ ,\lambda_{01})$. By an iteration of this process we arrive at an inductive definition of the conjugator λ. To illustrate, consider the $(+1)$ function defined by $\tau = (e, \tau)\sigma$. Then, $\lambda = (\lambda, \lambda\tau^{(\xi-1)/2})$ conjugates τ to τ^ξ .

Let us go back to the general case and indicate λ by $\lambda(\alpha;\xi)$. It follows from the above definitions that

$$\begin{aligned} \lambda(\alpha;1) &= e, \\ \lambda(\alpha;\xi)\ .\lambda(\alpha;\eta) &= \lambda(\alpha;\xi.\eta), \forall \xi,\eta \in U(\mathbb{Z}_2). \end{aligned}$$

Therefore, $\lambda(\alpha;\xi) = \lambda(\alpha;\eta)$ if and only if $o(\alpha)$ divides $\xi\eta^{-1} - 1$.

Let $\Lambda(\alpha) = \{\lambda(\alpha;\xi) : \xi \in U(\mathbb{Z}_2)\}$. Then,

$$\begin{aligned}\Lambda(\alpha) &\simeq U(\mathbb{Z}_2), \text{ if } \alpha \text{ has infinite order} \\ &\simeq U(\mathbb{Z}/2^k\mathbb{Z}), \text{ if } o(\alpha) = 2^k.\end{aligned}$$

Actually, when α has infinite order, then $U(\mathbb{Z}_2) \simeq Aut(\langle\alpha\rangle^*)$, and when $o(\alpha) = 2^k$, then $U(\mathbb{Z}/2^k\mathbb{Z}) \simeq Aut(\langle\alpha\rangle)$. When α is of infinite order, the subgroup $\langle\alpha\rangle^*\Lambda(\alpha)$ is isomorphic to the affine group of $\mathbb{Z}_2$. Whatever the order of α , we have shown

Theorem 13 *Let α be an automorphism of the binary tree. Then the holomorph group of $\langle\alpha\rangle^*$, is isomorphic to the subgroup of tree automorphisms $\langle\alpha\rangle^*\Lambda(\alpha)$.*

4.4. The automorphism group of the base group

As we commented earlier, the base group G of the binary tree is a locally finite 2-group generated by $\{\sigma_i : i = 0, 1, ..\}$, where by definition, $\sigma_0 = \sigma$, and $\sigma_i = (e, \sigma_{i-1})$ for all $i \geq 1$. The stabilizers of the k-th level vertices of the tree $G_k = G \cap \mathcal{A}_k$ are normal subgroups of G, and $\mathcal{A} / \mathcal{A}_k \simeq G/G_k \simeq G_{0,k-1}$, the finite 2-group generated by $\{\sigma_i : 0 \leq i \leq k-1\}$. Indeed, the group $\mathcal{A}$ is a pro-2 completion of its base group G. This intimate relationship between $\mathcal{A}$ and G is also revealed at the algebraic level. The material of this section is based upon our work with A.Brunner in [2].

We will exhibit some automorphisms of $G(Y)$ and for this purpose we define for a given $\alpha \in \mathcal{A}$, the sequence of elements

$$\alpha^{(0)} = \alpha, \alpha^{(1)} = (\alpha, \alpha), \alpha^{(i)} = (\alpha^{(i-1)}, \alpha^{(i-1)}), i \geq 2.$$

As $G(Y)$ is transitive on the the k-th level vertices for all k, it follows that its centralizer in $\mathcal{A}$ is trivial.
Now consider the subgroup of $G(Y)$, $D = < \sigma, \sigma^{(1)}, ..., \sigma^{(i)}, ... >$. Then D is an enumerable elementary abelian subgroup, and for any $k \geq 0$ only finitely many of the generators have non-trivial action on the k-th level of the binary tree. Therefore, the set of infinite products $...\sigma^{(i)j_i}...\sigma^{(1)j_1}\sigma^{j_0}$ where $j_k \in \{0, 1\}$, is the closure D^* which is a non-enumerable elementary abelian

2–group. It is easy to show that D^* normalizes $G(Y)$, and therefore induces a non-enumerable group of automorphisms on $G(Y)$. Clearly, $G(Y)D^* \leq \mathcal{A}$ is a locally finite group.
As for the normalizer subgroup of $G(Y)$ in $F(Y)$, we can construct for any $\delta \in D$ the finite state automorphism $w(\delta) \in D^*$, defined by $w(\delta) = w^{(1)} .\delta$. This set of $w(\delta)$'s form a subgroup D_f^* of $F(Y)$, and the base group $G(Y)$ is a proper subgroup of $G(Y).D_f^*$.

To reach a description of the automorphism group of the base group, first we study the normal subgroup stucture of G. We show that the normal subgroups are controlled by the stabilizers $G_k{=}G{\cap}\mathcal{A}_k$ of the k-th level vertices of the tree.

Theorem 14 *Let G be the base group of the binary tree and N be a nontrivial subgroup of G. Then, there exists $k \geq 1$ such that N contains the derived subgroup of G_k, and thus the quotient group G/N is an abelian by finite 2-group.*

With this result, and given the types of centralizers of elements of $\mathcal{A}$ seen in Section 5.2, it becomes possible to handle the isomorphism problem for the centralizer subgroups of the base group G and also obtain control on the automorphisms of G.

Theorem 15 *(1) Suppose s is a conjugacy class representative of an involution in the base group G such that $C_G(s) \simeq C_G(\sigma_i)$ for some i. Then, $s = \sigma_i$. (2) An automorphism of G maps each σ_i into a conjugate ${\sigma_i}^{g_i}$ for some $g_i \in G$. (3) The group $Aut(G)$ is isomorphic to the normalizer subgroup $N_{\mathcal{A}}(G)$ of G in $\mathcal{A}$. (4) The group $Aut(G)$ contains a copy of $\mathcal{A}$.*

5. Periodic subgroups of tree automorphisms

5.1. Maximal 2-subgroups of automorphisms of the binary tree

We investigate the maximal 2-subgroups of the automorphism group $\mathcal{A}$ of the binary tree. As a starting point, we show that $\mathcal{A}$ has at least two conjugacy classes of maximal 2-subgroups. Therefore classical Sylow theory fails here, thus answering a question raised on page 10 of R. Lyndon's [19].

Proposition 16 *Let $\alpha = (\alpha, \sigma) \in \mathcal{A}$. Then a maximal 2-subgroup that contains α is is not conjugate to any maximal 2-subgroup that contains σ.*

Proof. We claim that $o(\alpha\ \sigma^u)$ is infinite for any $u \in \mathcal{A}$. As we may consider u modulo the centralizer of σ, it can be assumed that $u = (e, u_1)$. Thus, $\alpha\ \sigma^u = (\alpha\ u_1\ ,\ \sigma\ u_1^{-1})\ \sigma$, and

$$(\alpha\sigma^u)^2 = (\alpha\sigma^v, \sigma.u_1^{-1}\alpha u_1)$$

where $v = u_1^{-1}$. Therefore, $o(\alpha\ \sigma^u) \geq 2.o(\alpha\ \sigma^v)$. We note that the conjugator u of σ has changed to v inside the bracket and now the argument may be repeated to obtain $o(\alpha\ \sigma^u)$ is infinite.

Let N be a 2-subgroup of $\mathcal{A}$. Let K be the point-wise stabilizer of $0, 1$, within N, and let K_0, K_1, be the projections of K on the first and second coordinates, respectively.

(i) If $N = K$, then $N \leq K_0 \times K_1$, is a 2-subgroup of A. If K_1 is a conjugate of K_0, then N is conjugate to a subgroup of the larger 2-group $L = (K_0 \times K_0).\langle\sigma\rangle$.

(ii) If $N \neq K$, then N contains an element $a = (a_0, a_1)\sigma$. On conjugating N by (a_0, e), we may assume $a_0 = e$. Since $a = (e, a_1)\sigma$, it follows that $a^2 = (a_1, a_1)$, and $a_1 \in K_0 \cap K_1$. Now since a normalizes K, we have that $K_0 = K_1$. Therefore, N is embeddable in the 2-subgroup $L = (K_0 \times K_0).\langle\sigma\rangle$.

The above analysis proves

Proposition 17 *Let N be a subgroup of $\mathcal{A}$. Then, N is a maximal 2-subgroup of $\mathcal{A}$ if and only if (i) $N = H \times K$, where H and K are non-conjugate maximal 2-subgroups of $\mathcal{A}$, or (ii) N is conjugate to $L = (K \times K).\langle\sigma\rangle$, where K is a maximal 2-subgroup of $\mathcal{A}$.*

In particular, we obtain,

Corollary 18 *A maximal 2-subgroup of $\mathcal{A}$ is infinitely generated.*

There is much that can be asked about these maximal subgoups. For example,

Problem 3. Let N be a maximal 2-subgroup of $\mathcal{A}$. Can N be enumerable? Can N be locally finite?

Let N be a maximal 2-subgroup of $\mathcal{A}$ and suppose $\sigma \notin N$. Define the following sequence of subgroups of $\mathcal{A}$: , $N_{(0)} = N$, $N_{(i)} = (N_{(i-1)} \times N_{(i-1)}).\langle\sigma\rangle$ for all $i \geq 1$. Then by the above result, these are maximal 2-subgroups of $\mathcal{A}$. Clearly, since $N_{(1)}$ contains σ, it is not conjugate to $N_{(0)}$ .By a direct argument we can show that no two of the set $\{N_{(i)}\}$ are conjugate.

We can use direct products of subgroups to produce more maximal 2-subgroups. Let $\{j_i\}$ be an infinite sequence of non-negative integers , define the subgroup $L(j) = ((...) \times N_{j_1}) \times N_{j_0}$, and let $tL(j)$ be the torsion subgroup of L. Then $tL(j)$ is a maximal 2-subgroup of $\mathcal{A}$. It can be shown that different sequences j, k produce non-conjugate $tL(j)$, $tL(k)$. This proves

Theorem 19 *The set of conjugacy classes of maximal 2-subgroups of $\mathcal{A}$ is non-denumerable.*

5.2 Burnside groups

First we sketch some simple instances of finite state Burnside groups taken from classes constructed by different authors.

(1) Alëshin, 1972 [1] A Burnside 2-group generated by two automorphisms of the binary tree: $a = (\sigma, e)$, $b = (c, (e, (e, c)))$, where c is defined by $c = (\sigma, (\sigma, (e, c)))$.

(2) Grigorchuck, 1980 [13] A Burnside 2-group H generated by three automorphisms of the binary tree , σ, $u = (e, v), v = (\sigma, w)$, where $w = (\sigma, u)$.We note that these automorphisms satisfy $\sigma^2 = u^2 = v^2 = w^2 = e$, $uv = vu = w$. We also note that the states of u form the set $Q(u) = \{u, e, v, \sigma, w\}$.

To prove that H is infinite, first we observe that $u^\sigma = (v, e), v^\sigma = (w, \sigma)$ and the pointwise stabilizer of $Y = \{0,1\}$ in H is $H_1 =< u, v, u^\sigma, v^\sigma >$. Next, we observe that the map $\pi_1 : H_1 \to H$ defined as the projection on the first coordinate is an epimorphism, and so the infiniteness of H follows very easily.

(3) Gupta-Sidki, 1983 [15]. We consider the ternary tree with $Y = \{0,1,2\}$ and on it the automorphism σ that is the rigid extension of the permutation $(0,1,2)$. Define the automorphism $\gamma = (\gamma, \sigma, \sigma^{-1})$. We verify easily that $o(\gamma) = 3$,and that the states of γ form the set $Q(\gamma) = \{\gamma, \sigma, \sigma^{-1}\}$. The group $\mathcal{G}$ generated by σ, γ is a Burnside 3-group.

The proof that $\mathcal{G}$ is infinite follows the same argument used in the case of Grigorchuk's group. We observe that $\gamma^\sigma = (\sigma^{-1}, \gamma, \sigma), \gamma^{\sigma^{-1}} = (\sigma, \sigma^{-1}, \gamma)$, and that the group normal closure of γ is $< \gamma >^{\mathcal{G}} =< \gamma, \gamma^\sigma, \gamma^{\sigma^{-1}} >= \mathcal{G} \cap \mathcal{A}_1 = \mathcal{G}_1$, the point-wise stabilizer of $Y = \{0,1,2\}$. Clearly, the projection map on the first coordinate $\pi_1 : \mathcal{G}_1 \to \mathcal{G}$ defines an epimorphism and so $\mathcal{G}$ is infinite. To prove that every element has order a power of 3 we start with $(\gamma\sigma)^3 = (\gamma, \gamma, \gamma^\sigma)$ and conclude that $o(\gamma\sigma) = 3^2$. It is worthwhile to make the following observation: if $\mathcal{G}$ is seen as a group generated by $\gamma, \gamma^\sigma, \gamma^{\sigma^2}, \sigma$, then in the previous calculation, $(\gamma\sigma)^3 = \gamma\gamma^{\sigma^{-1}}\gamma^\sigma$ has length three, while its three entries in $(\gamma, \gamma, \gamma^\sigma)$ have length one. It is precisely this property that allows to us to proceed by induction on the length of the elements and prove that our group is a 3–group, as well as showing that every proper quotient of it is finite (that is, just infinite).

As we already commented, Golod's p-groups have faithful representations on a p-adic tree.

Problem 4. Decide whether the Golod groups can be realized as groups of finite state automorphisms.

Let us go back to the 3-group $\mathcal{G}$ and exhibit more of its properties. An easy calculation verifies the following important commutator relation

$$c = [\gamma^{\sigma^{-1}}\gamma, \gamma\gamma^\sigma] = ([\gamma^{-1}, \sigma^{-1}], e, e).$$

Since $\mathcal{G}_1$ projects onto $\mathcal{G}$, on conjugating the above element by $\mathcal{G}_1$,we are able to produce in the first coordinate the group normal closure of $[\gamma^{-1}, \sigma^{-1}]$.

Therefore the group normal closure of c in $\mathcal{G}$ is $N = \mathcal{G}' \times \mathcal{G}' \times \mathcal{G}'$. It is easy to show that $\mathcal{G}/N$ is isomorphic to the wreath product $C_3 \wr C_3$.

We put together some of the properties of the group $\mathcal{G}$ in

Theorem 20 *The group $\mathcal{G}$ satisfies the following properties:*

(i) it is a 2-generated infinite 3-group and is residually finite ;

(ii) any proper quotient of it is finite (that is, it is just-infinite) ;

(iii) it does not satisfy the ACC condition for subnormal subgroups ;

(iv) the only element g of G such that $C_{\mathcal{G}}(g)$ has finite index in $\mathcal{G}$ is $g = e$;

(v) it contains a direct sum of an infinite number of copies of its derived subgroup $\mathcal{G}'$;

(vi) it contains a subgroup isomorphic to the restricted infinitely iterated wreath product of cyclic groups of order 3, and in particular, it contains a copy of every finite 3-group.

One of the many problems concerning Burnside p-subgroups of the group of automorphisms of a p-adic tree is

Problem 5. Let H be a finitely generated infinite p-subgroup of the automorphism group of the p-adic tree. Does H contain an infinite abelian group?

Another question is about the embedding of a Burnside p-group, such as $\mathcal{G}$, in a maximal p-subgroup of the automorphism group of the tree. Is it possible to describe " constructively" such an embedding ?

We will show that $\mathcal{G}$ and the locally finite 3-group $\langle\sigma\rangle^{\#}$ generate a larger 3-group. Define the sequence of subgroups of $\mathcal{A}$: : $\mathcal{G}_{(0)} = \mathcal{G}$, and $\mathcal{G}_{(i+1)} = \mathcal{F}(Y,\mathcal{G}_{(i)})\langle\sigma\rangle$ for $i \geq 0$. The group $\mathcal{G}_{(1)} = \mathcal{F}(Y,\mathcal{G})\langle\sigma\rangle$ is generated by $\{(\gamma, e, e), \sigma_1 = (e, e, \sigma), \sigma\}$,and therefore it is generated by $\{\gamma, \sigma_1, \sigma\}$. Similarly, $\mathcal{G}_{(i)}$ is generated by $\{\gamma, \sigma_i , .., \sigma_1, \sigma\}$. Hence, for all $i \geq 0$, $\mathcal{G}_{(i)}$ is a proper subgroup of $\mathcal{G}_{(i+1)}$. Easily, $\mathcal{G}_\infty = \cup\, \mathcal{G}_{(i)}$ is a periodic 3-group generated by γ and $\langle\sigma\rangle^{\#}$, and the identity $\mathcal{G}_\infty = \mathcal{G}_\infty \wr \langle\sigma\rangle$ holds.

The above constructions of Burnside groups have been considerably generalized in [16]. Given a group H, let $Y = \underline{H} = \{\underline{e}, \underline{h}_2, ..\}$ and define the

H-tree $\mathcal{T}(Y)$. We extend the regular representation $h : \underline{h'} \mapsto \underline{h'h}$ to the whole tree in the usual manner. We call a subset S of the tree vertices a *connecting set* provided every vertex is comparable to an element of S. Let $Y' = Y \backslash \{e\}$, and define the set of vertices

$$S = Y' \cup \underline{e}Y' \cup .. \cup (\underline{e})^i Y' \cup ....$$

Then this connecting set is minimal and will be our *special connecting set.* Given a minimal connecting set S, a function $\gamma : S \to H$ is called a *decorating function* of the tree. Such a function determines an automorphism of the tree which will be denoted by the same symbol γ.

Let H be a non-trivial finitely generated periodic group and S be the special connecting set. Then a decorating function $\gamma : S \to H$ is called *periodicity preserving* if it satisfies the following conditions:

(i) $\gamma(y) = \gamma((\underline{e})^i y)$, for all, $i \geq 0$, $y \in Y'$,
(ii) H is generated by $\gamma(Y')$,
(iii) $\{\underline{h} : \gamma(\underline{h}) \neq e\}$ is finite,
(iv) $\{\gamma((\underline{h})^i) : i \neq 0\}$ is a commutative set for all $h \neq e$,
(v) $\Pi\{\gamma((\underline{h})^i) : i = 1, .., o(h) - 1\} = e$, for all $h \in H \backslash \{e\}$.

It is not difficult to see that the only finitely generated periodic groups which do not admit a periodicity preserving function are generalized dihedral- we mean by that, $H = B\ \langle t \rangle$, B abelian, $b^t = b^{-1}$ for all $b \in B$.

We are now ready to state the following result on periodicity.

Theorem 21 *Let H be a non-trivial finitely generated periodic group and suppose it admits a periodicity preserving function γ. Let $\mathcal{E}$ be the subgroup of the automorphism group of the tree generated by H and γ . Then $\mathcal{E}$ is periodic and its elements involve the same primes as the orders of elements of H. Furthermore, $\mathcal{E}$ contains an infinite series of subnormal subgroups such that the successive quotients are isomorphic to H.*

The methods of the above paper were employed by M.Dixon and T.Fournelle(see, [4]) in the more general setting of *wreath powers* introduced by P.Hall [7], to prove the following result.

Theorem 22 *Let n be a natural number with $n \geq 4$, and let π denote the set of primes dividing n. Then there is a 2-generator infinite periodic π-subgroup E_n whose generators are elements of order n. The group E_n is hypoabelian, residually finite, has trivial center and contains an isomorphic copy of every countable, residually finite, locally soluble FC-π-group. Moreover, if n divides m then the groups E_n and E_m can be constructed so that $E_n \leq E_m$.*

6. Presentation of tree automorphisms

The question we address here is how to describe in terms of generators and relations the interdependence between an automorphism α , its states $Q(\alpha) = \{\alpha_u \mid u \in M\}$ and its set of permutations $\Sigma(\alpha) = \{\sigma_u(\alpha) \mid u \in M\}$. Let K be the group generated by $Q(\alpha)$ and C the group generated by $\Sigma(\alpha)$. We interpret the question as being about the presentation of the group $H = \langle K, C \rangle$.

The following two ideas can be employed toward obtaining a presentation for H. First, given groups U, V, and a homomorphism $\varphi : U \to V$, then this last can be extended naturally to a homomorphism

$$\varphi' : \mathcal{F}(Y, U) \to \mathcal{F}(Y, V)$$

and then extended further to a homomorphism

$$\varphi'' : \mathcal{F}(Y, U) >\!\lhd P(Y) \to \mathcal{F}(Y, V) >\!\lhd P(Y)$$

by having it map $P(Y)$ identically onto itself.

Second, let U be a group and C a group of permutations of the set Y, contained as a subgroup of U. Denote U by U_0, and define inductively, $U_{i+1} = \mathcal{F}(Y, U_i) >\!\lhd P(Y)$, for $i \geq 0$. Now suppose there exists a homomorphism $\varphi_0 : U_0 \to U_1$ such that φ_0 maps C identically into $P(Y)$. Then by the first idea, φ_0 induces a homomorphism

$$\varphi_1 : U_1 (= \mathcal{F}(Y, U_0) >\!\lhd P(Y)) \to U_2 \ (= \mathcal{F}(Y, U_1) >\!\lhd P(Y)).$$

and inductively produce the corresponding homomorphisms $\varphi_i : U_i \to U_{i+1}$, $i \geq 0$. Thus, we obtain a *direct mapping family* $\{U_i, \varphi_i\}$, and therefore a *direct limit* group $\widehat{U}$ which is determined by the map φ_0.

One type of group which admits homomorphisms such as φ_0 is $U = W * C$, the free product of a free group W and a subgroup C of $P(Y)$. Clearly, any function from a free generating set of W into $\mathcal{F}(Y,U) > \triangleleft P(Y)$ can be extended to a homomorphism, and this in turn can be extended further to U by mapping C identically into $P(Y)$.

Let us consider the example of the automorphism $\gamma = (\gamma, \sigma, \sigma^{-1})$ defined on the 3-tree. Then its set of states is $Q(\gamma) = \{\gamma, \sigma, \sigma^{-1}\}$ and $\Sigma(\gamma) = \{e, \sigma, \sigma^2\}$. Let W be a cyclic group of order three generated by g, let C be the cyclic group generated by σ and $U = W * C$. Define φ_0 as the extension of the map $g \to (g, \sigma, \sigma^{-1})$, $\sigma \to \sigma$. Then the image of g by φ_1 is $((g, \sigma, \sigma^{-1}), \sigma, \sigma^{-1})$. Let R_i be the kernel of φ_i for $i \geq 0$, and $R_\infty = \cup R_i$. Then the direct limit group $\widehat{U}$ is isomorphic to $\overline{U} = U/R_\infty$, and $\mathcal{G}$ is a homomorphic image of $\overline{U}$.

To illustrate the representation of elements in $\overline{U}$, we note

$$R_\infty g = R_\infty(g, \sigma, \sigma^{-1}) = R_\infty((g, \sigma, \sigma^{-1}), \sigma, \sigma^{-1}).$$

It is easy to see that

$$R_\infty (g\sigma)^3 = R_\infty(g, g, g^\sigma), \; R_\infty(g\sigma)^9 = R_\infty.$$

As was done in the case of our group $\mathcal{G}$, here a length function can be defined in a similar manner and it can be shown that $\overline{U}$ is a 3−group and is just infinite. This establishes that $\overline{U}$ is isomorphic to $\mathcal{G}$. Basically, this was the procedure used to obtain a concrete presentation for $\mathcal{G} = < \gamma, \sigma >$ and to show that the group is not finitely presentable (see, [30]).

We state the following

Problem 6. : Let K be a group of finite state automorphisms of a p-adic tree. If K is a finitely presentable p-group, is K then finite?

This problem recalls the more general question:

Does there exist a finitely presented infinite periodic group? (see:[35], page 643; [19], Problem 10b.)

7. The group algebra of the group of tree automorphisms

Given a group G and a field k, questions about the linear representations of G over k are intimately related to the study of the group algebra $k[G]$ and its ideals, such as the Jacobson Radical $J(k[G])$ and the Augmentation Ideal $\omega(k[G])$. It is an elementary fact that if G is a p-group and k has characteristic p, then the only finite dimensional irreducible representation of G over k is the trivial one. If we make the further assumption that the representation is faithful then G is reduced to the trivial group $\{e\}$. In the case of infinite p-groups, non-trivial irreducible representations in the same characteristic have to be infinite dimensional, but it is not clear if they exist, and if they do, how to obtain them. These considerations fall under the general theme of *semiprimitivity of group algebras.* We recommend D.Passman's [25] for an extensive and up-to-date survey of this topic.

Since the automorphism group $\mathcal{A}$ of a 1-rooted tree $\mathcal{T}(Y)$ has recursive mutiplicative structure, the group algebra $k[\ \mathcal{A}\ ]$ has a quotient that is also recursive in its *additive structure.* This is shown in our forthcoming [32]. We review below the contents of this paper.

The idea of the construction is based on the following fact.

Proposition 23 *Let C and D be groups. Consider their direct product group $H = C \times D$ and the direct sum of their algebras $R = k[C] \oplus k[D]$. Then*

(i) the embedding of H into R defined by $\varphi : (c,d) \to c+d$ extends naturally to a k-algebra homomorphism $\varphi : k[C \times D] \to k[C] \oplus k[D]$;

(ii) $image(\varphi) = (\ \omega(k[C]) + k.1_C\ ) \oplus (\ \omega(k[D]) + k.1_D)$;

(iii) $ker(\varphi) = \omega(k[C]).\omega(k[D])$.

The homomorphism φ above which we call a *summation thinning* process, is employed to obtain the intended quotient algebra of $k[\mathcal{A}]$.

Since $\mathcal{A} = \mathcal{F}(Y, \mathcal{A}) > \lhd P(Y)$, its group algebra is a *crossed product* algebra

$$k[\mathcal{A}] = k[\mathcal{F}(Y, \mathcal{A})] > \lhd P(Y).$$

Furthermore , since $\mathcal{A}_1 = \mathcal{F}(Y, \mathcal{A})$, we have the inclusion map:

$$v_1 : \mathcal{A} \to \mathcal{F}(Y, k[\mathcal{A}])$$

which extends canonically to a $k-$algebra homomorphism denoted by

$$\upsilon_1 : k[\mathcal{A}] \to \mathcal{F}(Y, k[\mathcal{A}]) >\!\!\lhd P(Y);$$

this is the first step of summation thinning. We note that

$$\ker(\upsilon_1) \cap (A - 1) = (0).$$

We continue the process as follows. The homomorphism υ_1 induces a homomorphism of $k-$algebras

$$\upsilon_2 \ : \ \mathcal{F}(Y, k[\mathcal{A}]) >\!\!\lhd P(Y) \to \mathcal{F}(Y, k[\mathcal{A}]\upsilon_1) >\!\!\lhd P(Y)$$

and call the composition $\widetilde{\upsilon}_2 = \upsilon_1\upsilon_2$ the second stage of the process. Proceeding inductively for $i \geq 3$, the i-the stage homomorphism is $\widetilde{\upsilon}_i = \widetilde{\upsilon}_{i-1}\upsilon_i$,where

$$\upsilon_i : \mathcal{F}(Y, k[\mathcal{A}]\widetilde{\upsilon}_{i-2}) >\!\!\lhd P(Y) \to \mathcal{F}(Y, k[\mathcal{A}]\widetilde{\upsilon}_{i-1}) >\!\!\lhd P(Y).$$

Let $T_i = \ker(\widetilde{\upsilon}_i),\ i \geq 1$. Then $\{T_i\}_{i\geq 1}$ is an ascending chain of ideals of $k[\mathcal{A}]$ such that $(1 + T_i) \cap \mathcal{A} = \{1\}$, $\forall i$. Denote by T the union of the ideals T_i for $i \geq 1$. Then the thinned algebra is

$$\overline{k[\mathcal{A}]} = \frac{k[\mathcal{A}]}{T}.$$

Proposition 24 *(1) The quotient algebra $\overline{k[\mathcal{A}]}$ is a subalgebra of $\mathcal{F}(Y, \overline{k[\mathcal{A}]}) >\!\!\lhd P(Y)$, and they are equal when Y is finite.*

(2) The group $\mathcal{A}$ and the algebra $k[P(Y)]$ are both embedded in $\overline{k[\mathcal{A}]}$.

We will apply the above construction to the Burnside 3-group $\mathcal{G} = \langle \gamma, \sigma \rangle$ where $\gamma = (\gamma, \sigma, \sigma^{-1})$.

Let k be a field of characteristic 3, and let $R = k[\mathcal{G}]$. We consider the image $\mathcal{P}$ of R in $\overline{k[\mathcal{A}]}$; that is,

$$\mathcal{P} = \frac{R + T}{T} \cong \frac{R}{R \cap T}$$

We observe that $\mathcal{P}$ is a proper quotient of R. This follows from the fact that $\mathcal{G}$ contains direct product of groups, such as $\mathcal{G}' \times \mathcal{G}' \times \mathcal{G}'$.

Two elements in the augmentation ideal $\omega(\mathcal{P})$ play an important role in the study of $\mathcal{P}$. These are

$$\gamma^\flat = 1 + \gamma + \gamma^2,\ \sigma^\flat = 1 + \sigma + \sigma^2.$$

Using the fact that we can effect addition coordinate-wise in $\mathcal{P}$, we obtain that

$$\gamma^\flat = (1,1,1) + (\gamma, \sigma, \sigma^{-1}) + (\gamma^2, \sigma^2, \sigma^{-2}) = (1+\gamma+\gamma^2, 1+\sigma+\sigma^2, 1+\sigma+\sigma^2)$$

and therefore we have

Lemma 25 $\gamma^\flat = (\gamma^\flat, \sigma^\flat, \sigma^\flat)$ *holds in the algebra* $\mathcal{P}$.

The formulas below allow us to simplify calculations which involve $\sigma^\flat$.

Lemma 26 *(i)* $\sigma^\flat(u_0, u_1, u_2) = (u_0, u_1, u_2) + (u_1, u_2, u_0)\sigma + (u_2, u_0, u_1)\sigma^2$;

(ii) $\sigma^\flat(u_0, u_1, u_2)\sigma^\flat = (u_0 + u_1 + u_2, u_0 + u_1 + u_2, u_0 + u_1 + u_2)\sigma^\flat$.

With the use of these lemmas it is a direct argument to show that

Proposition 27 *The element* $\gamma^\flat\sigma^\flat$ *of* $\mathcal{P}$ *is transcendental over* k. *That is,* $k[\gamma^\flat\sigma^\flat]$ *is a polynomial algebra .*

The next proposition confirms that the ring $\mathcal{P}$ imitates $\mathcal{G}$ in that is has tree-like behavior.

Proposition 28 *(i) Let* $\mathcal{I}$ *be a 2-sided ideal in* $\mathcal{P}$ *and let* $u = (u_0, u_1, u_2)\sigma^\flat \in \mathcal{I}$ *. Then there for all* $c \in \mathcal{G}'$*, and for* $i = 0, 1, 2$,

$$((c-1)u_i(c-1)^2, 0, 0) \in \mathcal{I}\ .$$

(ii) Let $\mathcal{I}$ be a nontrivial 2-sided ideal in $\mathcal{P}$, then there exists $c \in \mathcal{G}'$, $c \neq 1$ such that $\mathcal{I} \supseteq N - 1$, where N is the normal closure of $< c >$ in $\mathcal{G}$.

Since $\mathcal{G}$ is just-infinite, it is immediate that the algebra $\mathcal{P}$ is also just-infinite, in the sense that all its proper quotients are finite dimensional algebras. A further consequence is

Corollary 29 *(i) $\mathcal{P}$ is a prime ring.*

(ii) $J(\mathcal{P}) = (0)$ or $J(\mathcal{P}) = \omega(\mathcal{P})$.

It is well-known that elements of the Jacobson Radical $J(\mathcal{P})$ are quasi-invertible. Therefore, if we can exhibit some element of $\omega(\mathcal{P})$ which is not quasi-invertible then we would reach $J(\mathcal{P}) = (0)$.

Proposition 30 *The element $\eta = 1 + \gamma\sigma^b$ is not invertible in $\mathcal{P}$.*

At the end of this sequence of steps we reach

Theorem 31 *Let k be a field of characteristic 3, $\mathcal{G}$ be the Burnside 3-group $\mathcal{G}$ and $\mathcal{P}$ be the image of $k[\mathcal{G}]$ in the thinned algebra of $k[\mathcal{A}\ ]$. Then, the algebra $\mathcal{P}$ is just-infinite, contains transcendental elements, and is primitive.*

An immediate consequence of this theorem is

Corollary 32 *The Burnside 3-group $\mathcal{G}$ admits an irreducible and faithful representation over any field of k of characteristic 3.*

Passman and Temple studied in [24] the function $F_G(n)$ which counts the number of non-equivalent irreducible representations of $k[G]$ of degree $\leq n$, for groups G which contain a normal subgroup H of finite index in G, such that $H \simeq G \times G \times .. \times G$. This is the case of the commutator subgroup $\mathcal{G}'$ of the Burnside 3-group $\mathcal{G}$. Based on some fine estimates of this function and using the above corollary, they proved that $\mathcal{G}$ admits an infinite number of non-equivalent faithful irreducible representations over a field k of characteristic 3 , which is algebraically closed and non-denumerable.

Problem 7. Decide whether $k[\mathcal{G}]$ itself is primitive.

In comparison with the Golod groups, it would be very interesting to answer the

Problem 9. Decide whether in the case of $\mathcal{G}$ there exists an ideal I contained in $\omega k[\mathcal{G}]$ such that $\omega k[\mathcal{G}]/I$ is nil and $I \cap (1 - \mathcal{G}) = (0)$.

Indeed, one of the outstanding conjectures in this area is:

> *Let G be a finitely generated p-group which is residually finite, and let $char(k) = p > 0$. Then, $J(k[G]) = \omega(k[G])$ if and only if G is finite.*

For the 3-group $\mathcal{G}$, on considering the thinning ideal T of $\mathcal{A}$ and $T(\mathcal{G}) = T \cap \omega k[\mathcal{G}]$, we observe the following proper inclusions

$$J(k[G]) \subset T(\mathcal{G}) \subset \omega(k[G]).$$

The inequality $J(k[\mathcal{H}]) \neq \omega(k[\mathcal{H}])$ was shown for the Grigorchuk 2-group $\mathcal{H}$ (see,[12]). In the case of a Golod group G, constructed from a nil ring R, it was shown by Siderov [29] that $R \neq \omega(k[G])$ holds under some natural conditions on the parameters which define the ring R.

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