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SPECTRAL GEOMETRY: DIRECT AND INVERSE PROBLEMS

by

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(with an Appendix by G. BESSON)

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I S B N

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To Rachel, Philippe, Izabel

INTRODUCTION

The purpose of these notes is to describe some aspects of direct problems in spectral geometry.

Eigenvalue problems were motivated by questions in mathematical physics. In these notes, we deal with eigenvalue problems for the Laplace-Beltrami operator on a compact Riemannian manifold. To such a manifold (M,g), we can associate a sequence of non-negative real numbers $\{\lambda_i\}_{i\geq 1}$, the eigenvalues of the Laplace-Beltrami operator Δ^g acting on $C^\infty(M)$. One can think of a Riemannian manifold as a musical instrument together with the musician who plays it. In this picture, the eigenvalues of the Laplace operator correspond to the harmonics of the instrument; they may depend on the music player, i.e. on the Riemannian metric: think of a kettledrum, or better of a Brazilian "cuíca".

Spectral geometry aims at describing the relationships between the musical instrument and the sounds it is capable of sending out.

The problems which arise in spectral geometry are of two kinds: direct problems and inverse problems. In a direct problem, we want information on the sounds produced by the instrument, in terms of its geometry. For example, we know that the bigger the tension of the parchment head of a kettledrum, the higher the pitch. In an inverse problem, we investigate what geometric information on the instrument can be recovered from the sounds it sends out.

Both types of problems are relevant to deep questions arising in mathematical physics (for example in elasticity theory, in plasma physics, in spectroscopy...).

This book could be divided into three parts: Chapters I to III; Chapters IV to VI and Appendix A; Chapter VII and Appendices B and C.

In Chapter I, we give some very simple-minded motivations from mathematical physics. Our purpose is not to derive mathematical models for some physical phenomena, but rather to show how some mathematical objects which will be introduced later on, arise naturally from physical principles. For further reading, we suggest [C-H] and [TL].

Chapter II is devoted to Riemannian geometry. We introduce the basic notions (geodesics, curvature,...) and we state, mainly without proofs, the basic results. In order to understand Chapter VI, the reader must have in mind the comparison theorems which involve the curvature of a Riemannian manifold. For further reading, we suggest [B-C], [CO], [C-E], [M-S] and [SI].

In Chapter III, we introduce the Laplace-Beltrami operator, and we describe the eigenvalue problems we will deal with in this book. An important part of this chapter is devoted to the variational characterizations of the eigenvalues. This is very important for later purposes. Although this material can be considered as classical ([KO], [R-S] or [C-H]), we have tried to describe it at length. The last paragraph of Chapter III contains general considerations on direct and inverse problems, and some answers to such problems as an illustration of the variational characterizations of the eigenvalues.

Chapters IV to VI <u>form the core</u> of this book. They contain results related to isoperimetric inequalities and to an important topic in Riemannian geometry, namely the interactions between local geometry (curvature estimates) and global geometry (topology...).

Many of the results we present in these chapters are new and are not yet available in print. These results were obtained in collaboration with G. Besson and S. Gallot (see [B-B-G1 to 3], [B-G]).

In Chapter IV, we introduce isoperimetric methods on compact Riemannian manifolds without boundary. The general setup described in § B, as well as the proof of J. Cheeger's lower bound for the first non-zero eigenvalue of a closed Riemannian manifold, are new. They arose from the above mentioned papers, and from brainstorming sessions with G. Besson and S. Gallot.

In Chapter V, we introduce the heat equation and then go directly to the main <u>tool</u> in this book: the isoperimetric inequality for the heat kernel. The ideas we develop here are those of [B-G]; our presentation differs however from that of [B-G] and is more in the spirit of Chapter IV.

Chapter VI is devoted to some applications of isoperimetric inequalities to Riemannian geometry. We use the ideas of [B-G], and the isoperimetric inequality obtained in [B-B-G1], to give bounds on topological invariants. The underlying method is the analytic method introduced by S. Bochner in the early 1940's, to obtain vanishing theorems. This method was improved by P. Li (1980) to give estimating theorems for Betti numbers, and later by S. Gallot (1981) to give estimating theorems in a more general framework. Both used isoperimetric estimates for Sobolev constants. In Chapter VI, we introduce a new idea (that of using Kato's inequality on heat kernels) which is due to M. Gromov, and came to life with the isoperimetric inequalities on the heat kernel given in [B-G]. It is important to read this chapter keeping in mind the compactness theorems of M. Gromov. These theorems are briefly described in the last paragraph of Chapter VI (see [SI] for a review).

These chapters are completed by an Appendix written by G. Besson.

In Appendix A, G. Besson shows how one can think of symmetrization procedures as relationships between Riemannian Geometry/Spectral Geometry on the one hand, and Operator Theory in a Hilbert space on the other hand; he views Kato's inequality (Chapter VI), and the symmetrization à la Faber-Krahn (Chapters IV and V), as particular cases of a unique general theorem. This interpretation is important because it distinguishes geometric techniques (isoperimetric inequalities) from analytic techniques (quadratic forms and operator theory); it also separates technicalities from fundamental ideas.

I am very grateful to G. Besson for writing this Appendix.

Spectral geometry has witnessed much research activity since the late 1960's. In Chapter VII, we very briefly sketch some of the important recent developments (in particular, see the very end of Chapter VII for more references).

Appendix B is a bibliography which I compiled in collaboration with M. Berger. I would like to thank M. Berger for allowing me to include it here. This bibliography is reproduced from the printed original; I thank the publisher Kaigai Publications (Japan) who left us the copyright. This bibliography is referred to as [B-B] in the text. It is divided into several chapters dealing with the different aspects of spectral geometry. Although the title refers to 1982, we revised the bibliography in September 1983. In Appendix C, I have added some new references.

This book was originally written both as a support for, and as a complement to lectures delivered at the 15º Colóquio Brasileiro de Matemática, July 1985. Although I have tried to give many complete proofs, I deliberately put emphasis on ideas rather than on technicalities. In a sense this book is an invitation to spectral geometry, rather than a course on spectral geometry. The original notes were first published by IMPA, in the series "Colóquio Brasileiro de Matemática". This new edition differs very little from the original one, as far as the mathematics are concerned: in order to avoid delay, I have only corrected some mistakes in the original text. In an attempt to make these notes more useful, I have added Appendix C (as a complement to [B-B]) and two indexes.

I thank the organizing Committee of the 15º Colóquio Brasileiro de Matemática for the opportunity to give a course on spectral geometry, and IMPA for its hospitality.

It is a pleasure for me to thank M.F. Cordel and P. Strazzanti who typed the first version of these notes, as well as Rogério Dias Trindade who typed the present text, for their care and competence.

I profited very much from regular brainstorming sessions with G. Besson and S. Gallot over the last three years. This book is an outgrowth of our collaboration. I owe them very much.

This book is dedicated to Marcel Berger in acknowledgement of his teachings.

Rio de Janeiro, April 1986.

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APPENDIX A by G. BESSON

ON SYMMETRIZATION

CHAPTER I

MOTIVATIONS AND THE PHYSICAL POINT OF VIEW

1. The purpose of this chapter is to introduce some basic concepts which arise naturally from problems in mathematical physics. Our presentation might appear childish...; we do not aim at establishing good mathematical models for some elasticity problems. We only want to show how the notions of energy integral, variational methods, boundary conditions, wave equation, separation of variables, eigenvalue problems... arise naturally from problems in mathematical physics, and how they are related to other fields in mathematics (partial differential equations, spectral theory, Riemannian geometry).

A. AN ELEMENTARY EXAMPLE

2. Let us consider a homogeneous elastic string S whose position at rest is represented by the line segment $[0,L]$ in the plane. The string being elastic, the tension forces are tangential to the string. The string being homogeneous, the linear density ρ and the tension μ of the string are constant along the string.

The first problem we shall deal with is that of the equilibrium position of the string S submitted to an external force which acts in the plane, transversally to the string, with intensity $f(x)$.

We represent the equilibrium position of the string by a function $u: [0,L] \to \mathbb{R}$, the amplitude of the deflection of the string, therefore assuming that the points of the string can only move transversally.

The potential energy of the string consists of two terms: the energy $E_t(u)$ which arises from the tension μ, and the external energy $E_e(u)$ which arises from the force applied to the string. The energy E_t equals the tension times the increase of length of the string; the external energy is the work of the force f. We have

$$(3) \qquad \begin{cases} E_t(u) = \mu[\int_0^L (1+(u_x')^2)^{1/2}dx - L]; \\ E_e(u) = \int_0^L f(x)u(x)dx. \end{cases}$$

We shall now make the assumption that the deflection of the string is "very small" in the sense that we can replace $(1+(u_x')^2)^{1/2}$ by $\frac{1}{2}(u_x')^2$. The potential energy of the string can then be replaced by

$$(4) \qquad E(u) = \frac{\mu}{2}\int_0^L (u_x')^2dx + \int_0^L f(x)u(x)dx.$$

In order to find u, we apply the <u>principle of least potential energy</u> which says that a stable equilibrium u is a local minimum of the energy E, which implies that

$$(5) \qquad \frac{d}{d\epsilon}E(u+\epsilon v)\bigg|_{\epsilon=0} = 0,$$

where $u + \epsilon v$ represents a position of the string close to the equilibrium u.

If we plug condition (5) into (4), we find

$$(6) \qquad \mu\int_0^L u_x'v_x'dx + \int_0^L f(x)v(x)dx = 0.$$

We can of course take local variations v, i.e. variations with compact support in $]0,L[$. Taking such a variation and integrating by parts, we find that for all v in $C_o^\infty(]0,L[)$,

$$\int_0^L [-\mu u''_{xx} + f(x)]v(x)dx = 0 \quad \text{and hence}$$

(7) $$\mu \frac{d^2u}{dx^2}(x) = f(x) \quad \text{in} \quad]0,L[.$$

8. <u>Remark</u>. We have implicitely made the assumption that u is twice differentiable, in order to be able to write (7). We shall show how one can make weaker assumptions later on (nº 43).

9. Let us now take the function v in $C^\infty([0,L])$. Equation (6) becomes, after integration by parts,

$$\mu u'_x v|_0^L + \int_0^L (f(x)-\mu u''_{xx})\, v(x)dx = 0.$$

Taking (7) into account, we then have

(10) $$u'_x(L)v(L) - u'_x(0)v(0) = 0.$$

The fact that one can take one v or another depends on the physical problem at hand. If we do not impose any condition on v, we deduce from (10) that u must satisfy the <u>natural boundary condition</u> (Neumann boundary condition)

(10N) $$u'_x(0) = 0 \quad \text{and} \quad u'_x(L) = 0.$$

If we assume that the string is fixed at both ends (think of a violin or a piano string), we must impose that the deflection of the string is 0 at $x = 0$ and $x = L$. This means that both u and v must satisfy the boundary condition (Dirichlet boundary condition)

(10D) $u(0) = 0$ and $u(L) = 0$.

In that case, (10) is void. The boundary condition (10N) corresponds to a free string, for which all deflections are allowed or admissible. The boundary condition (10D) corresponds to a string which is fixed at both ends. We then impose that the deflections satisfy $u(0) = 0$ and $u(L) = 0$. It is physically very intuitive that such conditions must be imposed to determine the equilibrium position of the problem under consideration.

11. Summary. In order to determine the equilibrium of a string submitted to a transversal external force f, we can

(i) either seek the local extrema of the energy

$$E(u) = \frac{\mu}{2}\int_0^L (u'_x)^2 dx + \int_0^L f(x)u(x)dx,$$

when u varies in a space of admissible functions, corresponding to the physical problem under consideration;

(ii) or solve the equation

$$\mu \frac{d^2u}{dx^2}(x) = f(x) \quad \text{in} \quad]0,L[,$$

where some boundary conditions are imposed to u at $x = 0$ and $x = L$, depending on the problem which is considered.

Examples:

Dirichlet problem (string fixed at both ends):

- Admissible functions: $u \in C^2([0,L])$ (see nº 8) such that $u(0) = u(L) = 0$ ($u+\epsilon v$ must also be admissible),
- Boundary conditions: $u(0) = 0$ and $u(L) = 0$;

Neumann problem (free string)

- Admissible functions: $u \in C^2([0,L])$ (see nº 8),
- Boundary conditions: $u'(0) = 0$ and $u'(L) = 0$ (imposed by the least potential energy principle).

12. Let us now consider the problem of the vibrating string, i.e. let us determine the laws of motion of an elastic string. We denote by $u: \mathbb{R} \times [0,L] \to \mathbb{R}$ the deflection of the string which is assumed to be transverse and small (in the sense used to derive (4)). The function f considered above may also depend on the time parameter t. We then have to consider the kinetic energy of the string, namely

$$E_k(u) = \int_0^L \frac{1}{2}\,\rho(u_t')^2(t,x)dx. \tag{13}$$

Let t_1 and t_2 be two instants of time. Hamilton's principle states that the motion $u(t,x)$ of the string between the instants of time t_1 and t_2 should minimize the expression

$$J(u) = \int_{t_1}^{t_2}\int_0^L \{\frac{1}{2}\rho(\frac{\partial u}{\partial t}(t,x))^2 - \frac{1}{2}\,\mu(\frac{\partial u}{\partial x})^2(t,x) - f(t,x)u(t,x)\}dtdx,$$

among all admissible motions close to u, taking the same values as u at $t = t_1$, and $t = t_2$ i.e.

$$\frac{d}{d\varepsilon}J(u+\varepsilon v)\Big|_{\varepsilon=0} = 0, \tag{14}$$

for all admissible functions v such that $v(t_1,x) = 0$ and $v(t_2,x) = 0$, for all x in $[0,L]$.

The adjective admissible refers to functions describing the physical problem under consideration as above (see nº 9 to 11).

Applying Hamilton's principle with $v \in C^\infty(\mathbb{R} \times [0,L])$ satisfying $v(t_1,x) = 0$, $v(t_2,x) = 0$, for all x, and integrating by parts, we deduce from (14) that

$$\int_{t_1}^{t_2}\int_0^L \{\rho \frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x)\}\ v(t,x)dt\ dx$$

$$+ \int_{t_1}^{t_2} \mu \frac{\partial u}{\partial x}(t,x)v(t,x)dt\ \Big|_0^L = 0.$$

The choice of v being arbitrary we conclude that

(15) $$\rho \frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x) = 0 \quad \text{in} \quad \mathbb{R} \times]0,L[,$$

(16) $$\frac{\partial u}{\partial x}(t,x)v(t,x)\Big|_0^L = 0 \quad \text{for all admissible } v, \text{ and all } t.$$

In the case of a string with free ends (i.e. no condition on u and v), Equation (16) gives (Neumann conditions)

(16N) $$\frac{\partial u}{\partial x}(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,L) = 0 \quad \text{for all } t.$$

In the case of a string with fixed ends, we must impose $u(t,0) = u(t,L) = 0$ and $v(t,0) = v(t,L) = 0$ for all t. Equation (16) is then always satisfied, and we only write the condition that u is admissible (Dirichlet conditions)

(16D) $$u(t,0) = 0 \quad \text{and} \quad u(t,L) = 0 \quad \text{for all } t.$$

Equation (15) is called the one-dimensional <u>wave equation</u> (the space variable x being one-dimensional).

17. <u>Remark</u>. In order to be able to determine the motion $u(t,x)$ of the string, we need Equation (15), boundary conditions e.g. (16D) or (16N) and <u>initial conditions</u>; these initial conditions already appeared in the statement of Hamilton's principle; we also consider the <u>Cauchy data</u> $u(t_o,x) = u_o(x)$ and $u'_t(t_o,x) = u_1(x)$, $0 \le x \le L$,

which describe the string at time t_o.

18. <u>Summary</u>. In order to determine the motion of a vibrating string submitted to a transversal external force f, we can

(i) either seek the extrema of the integral

$$J(u) = \int_{t_1}^{t_2} \int_0^L \{\frac{1}{2} \rho (\frac{\partial u}{\partial t})^2 - \frac{1}{2} \mu (\frac{\partial u}{\partial x})^2 - fu\} dt dx,$$

when u varies in a space of <u>admissible functions</u> corresponding to the physical problem under consideration;

(ii) or solve the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial x^2} - f = 0 \quad \text{in} \quad \mathbb{R} \times]0,L[,$$

with initial conditions $\left\{ \begin{array}{l} u(t_o,x)=u_o(x) \\ \\ u'_t(t_o,x)=u_1(x) \end{array} \right\}$, x in $[0,L]$,

and boundary conditions at $x = 0$ and $x = L$. (e.g. Dirichlet or Neumann conditions described in nº 11).

We can reduce the problem of the equilibrium to the present one by making all functions independent of the time variable t.

19. <u>Some Comments</u>

(a) Equations and boundary conditions: the transversal force acting on the string could also be related to the deflection $u(t,x)$ e.g. this could be an elastic force proportional to $u(t,x)$; we could also assume that there is a force acting on the ends of the string e.g. the ends could be elastically attached instead of fixed. Such conditions can give rise to other contributions to the energy of the string, thus modifying both the equation and the boundary conditions.

(b) Considering elastic bars instead of strings, we would arrive at the following situation

$$(20)\quad \begin{cases} \cdot\ J(u) = \displaystyle\int_{t_1}^{t_2}\int_0^L \{\tfrac{1}{2}\rho(x)(\tfrac{\partial u}{\partial t})^2 - \tfrac{1}{2}\mu(x)(\tfrac{\partial u}{\partial x})^2 - fu\}dt\,dx, \\ \cdot\ \rho(x)\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}(\mu(x)\dfrac{\partial u}{\partial x}) - f = 0, \end{cases}$$

where $\rho(x)$ is the linear density of the bar and where $\mu(x)$ describes the elasticity of the bar (both functions are positive). We assume these functions to depend on the space variable x, but not on the time variable.

References

[C-H] Chap. IV § 10, p. 242 ff,

[FO] Chap. 5 p. 130 ff (p. 168 for the elastic bar),

[WR] Chap. I, for both the elastic string and the elastic bar.

B. THE METHOD OF SEPARATION OF VARIABLES

21. In order to study the wave equation which appears in (20), it is convenient to look at a simpler problem, namely the case $f \equiv 0$, and to seek solutions $u(t,x)$ of the form $F(t)G(x)$ (i.e. to separate variables, a method which goes back to the 18th century). Equation (20) becomes

$$(22)\qquad \rho(x)G(x)\frac{d^2F}{dt^2}(t) - F(t)\frac{d}{dx}(\mu(x)\frac{dG}{dx}(x)) = 0,$$

which is easily seen to split into two equations

$$
(23) \quad \begin{cases} \text{(i)} \quad \dfrac{d}{dx}(\mu(x)\dfrac{dG}{dx}(x)) + \lambda\rho(x)G(x) = 0, \quad x \in]0,L[, \\ \\ \text{(ii)} \quad \dfrac{d^2F}{dt^2}(t) + \lambda F(t) = 0, \quad t \in \mathbb{R}, \end{cases}
$$

for some constant λ.

If we now recall that $u(t,x)$ must be an admissible function, e.g. that it satisfies one of the boundary conditions (16D) or (16N), we have to impose boundary conditions on G, e.g.

$$
(24) \quad \begin{cases} G(0) = G(L) = 0 \quad \text{(Dirichlet conditions)}, \\ \text{or} \\ G'(0) = G'(L) = 0 \quad \text{(Neumann conditions)}. \end{cases}
$$

Let us for example consider the Dirichlet boundary conditions. We are led to the <u>Sturm-Liouville problem</u>

$$
(25) \quad \begin{cases} (\mu(x)G'(x))' + \lambda\rho(x)G(x) = 0, \\ \\ G(0) = G(L) = 0. \end{cases}
$$

It can be shown ([SR] Chap. IV or [C-H]) that the λ's for which (25) has a non-trivial solution form an infinite sequence $\lambda_1 < \lambda_2 < \ldots \uparrow +\infty$ of positive real numbers going to infinity (these numbers are called the <u>eigenvalues</u> of the Sturm-Liouville Problem (25)). To the eigenvalue λ_n of Problem (25) corresponds a one-dimensional space of <u>eigenfunctions</u>.

We can choose an eigenfunction G_n corresponding to λ_n, normalized by $\int_0^L \rho(x)G_n^2(x)dx = 1$. The basic fact is that a given function $f(x)$ can, under certain mild conditions, be represented by an infinite series in the G_n's; $f(x) = \sum_{n=1}^{\infty} a_n G_n(x)$.

The case of Fourier-sine series is a particular instance of this fact ($\mu=1$, $\rho=1$). Let us make some formal computations. The functions which appear in Equation (20) can be written as infinite series in the G_n's (as far as the x variable is concerned); we thus have (summations from 1 to ∞)

$$f(t,x) = \Sigma\, a_n(t)G_n(x);$$
$$u(t,x) = \Sigma\, b_n(t)G_n(x);$$
$$u(t_o,x) = u_o(x) = \Sigma\, c_nG_n(x);$$
$$u'_t(t_o,x) = u_1(x) = \Sigma\, d_nG_n(x).$$

At least at the formal level, plugging these series into Equation (20), we obtain

$$(26) \qquad \left\{\begin{array}{l} b''_n(t) + \lambda_n b_n(t) = a_n(t) \\ b_n(t_o) = c_n \\ b'_n(t_o) = d_n \end{array}\right\}, \quad n \text{ in } \mathbb{N}.$$

Since it is easy to solve (26), we have an expression of $u(t,x)$ in terms of series representing $f(t,x)$, $u(t_o,x)$ and $u'_t(t_o,x)$.

These formal calculations explain why it is so important to determine the eigenvalues of the Sturm-Liouville Problem (25). In these notes, we shall deal with generalizations of the situation we have just described.

For more details on Sturm-Liouville problems and their eigenfunctions expansions, we refer to [C-H], [SR] Chap. IV and [FO] (for the case of Fourier series), or [D-M].

C. GENERALIZATIONS

27. Let us now consider a vibrating homogeneous membrane, whose position at rest is represented by a bounded, regular domain Ω in $\mathbb{R}^2$.

We are again interested in transverse vibrations of the membrane (i.e. normal to the plane $\mathbb{R}^2$). We denote by $u(t,x)$, $(t,x) \in \mathbb{R} \times \Omega$, the amplitude of such a vibration.

In order to make things simpler, we shall assume that no external force acts on the membrane, and that the membrane is either fixed on its boundary $\partial\Omega$ (this is a drum) or free. The corresponding <u>admissible functions</u> in the sense of nº 10-11 are in $C^2(\Omega)$, and assumed to vanish on the boundary $\partial\Omega$ when the membrane is fixed (no condition when the membrane is free). We denote the density of the membrane by ρ, and its tension by μ. The kinetic energy of the membrane is given by

$$E_k(u) = \frac{1}{2}\rho \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2(t,x)dx, \tag{28.1}$$

and the potential energy is given by μ times the increase of area of the membrane, i.e.

$$\mu \int_\Omega [(1+|\nabla_x u|^2(t,x))^{1/2} - 1]\, dx,$$

which we shall approximate (under the assumption that the vibration is "small", compare with nº 3-4) by

$$E_p(u) = \frac{1}{2}\mu \int_\Omega |\nabla u|^2(t,x)dx, \tag{28.2}$$

where ∇u is the gradient of u in the x-variable, i.e. (in Cartesian coordinates)

$$\nabla u(t,x) = \left(\frac{\partial u}{\partial x_1}(t,x), \frac{\partial u}{\partial x_2}(t,x)\right), \quad \text{if} \quad x = (x_1,x_2),$$

and where $|x|^2 = x_1^2 + x_2^2$, for $x \in \mathbb{R}^2$.

In order to derive the laws of motion of the membrane, we again use Hamilton's principle.

We define

$$(29) \qquad J(u) = \frac{1}{2} \int_{t_1}^{t_2} \int_{\Omega} [\rho(\frac{\partial u}{\partial t})^2(t,x) - \mu|\nabla u|^2(t,x)]dt\ dx,$$

and we seek admissible functions u, such that for all admissible functions v, with $v(t_1,x) = 0$, $v(t_2,x) = 0$ for all x, we have

$$(30) \qquad \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = 0.$$

If we plug (29) into (30), we obtain

$$\int_{t_1}^{t_2} \int_{\Omega} [\rho \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} - \mu(\nabla u|\nabla v)]dt\ dx = 0,$$

where $(.|.)$ is the scalar product in $\mathbb{R}^2$.

If we apply integration by parts in the t-variable, and Green's formula in the x-variable (n being the inner unit normal to $\partial\Omega$), we obtain

$$(31) \qquad \int_{t_1}^{t_2} \int_{\Omega} (\rho \frac{\partial^2 u}{\partial t^2} + \mu\Delta u)v\ dt\ dx - \int_{t_1}^{t_2} \int_{\partial\Omega} \mu v(\nabla u|n)dt\ d\sigma = 0$$

where $\Delta u = -(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1^2})$ (<u>Note our sign convention</u>), and

$d\sigma$ = arc length on $\partial\Omega$.

If we take v with compact support in the x-variable (inside Ω), we deduce that u must satisfy the two-dimensional wave equation

$$(32) \qquad \rho \frac{\partial^2 u}{\partial t^2}(t,x) + \mu\Delta u(t,x) = 0 \quad \text{in} \quad \mathbb{R}\times\Omega.$$

If we deal with a fixed membrane, we have to impose the condition that u and v vanish on $\partial\Omega$, so that the second term in (31) is always 0; we then have the <u>boundary condition</u> (Dirichlet condition)

(33D) $\qquad u(t,x) = 0$ for all $(t,x) \in \mathbb{R} \times \partial\Omega$.

If we deal with a free membrane, we can take a v in $C^\infty(\mathbb{R}\times\Omega)$ so that (31) and (32) imply that u must satisfy the <u>natural boundary condition</u> (Neumann condition)

(33N) $\qquad (\nabla u|n) = 0$ on $\mathbb{R} \times \partial\Omega$

(We shall write $\frac{\partial u}{\partial n} = (\nabla u|n)$).

For more details, we refer the reader to [C-H] Chap. IV §10 p. 242 ff and [PY] p.7.

As for the vibrating string, we can now apply the method of separation of variables, and we have to deal with the following problem

$$(34) \qquad \begin{cases} \Delta U(x) = \lambda U(x) \quad \text{in} \quad \Omega, \\ U(x) = 0 \quad \text{on} \quad \partial\Omega \quad (\text{for (33D)}), \\ \text{or} \\ \frac{\partial U}{\partial n}(x) = 0 \quad \text{on} \quad \partial\Omega \ (\text{for (33N)}). \end{cases}$$

Problem (34) is far more difficult than its one-dimensional analogue (25). As was shown by H. Poincaré at the end of the $19^{\underline{th}}$ century, Problem (34) admits a non-trivial solution for values of λ which form an infinite sequence of non-negative numbers, which goes to infinity, $(0\leq)\lambda_1 \leq \lambda_2 \leq \ldots$. Given an <u>eigenvalue</u> λ_n of (34), the vector space formed by the solutions of Equation (34) with $\lambda=\lambda_n$ is <u>finite dimensional</u> (its elements are called <u>eigenfunctions</u> associated with λ_n, and its dimension the <u>multiplicity</u> of λ_n).

In these notes we shall be interested in problems similar to Problem (34) with Ω (a domain in) a differentiable manifold, and

Δ an operator which will generalize the ordinary Laplacian in $\mathbb{R}^2$.

We shall not go into any further details now. The reader interested in Problem (34) may read the appropriate chapters of [TS].

35. Let us now indicate a generalization of the above situation. In certain problems of elasticity, dealing with non-homogeneous media, one has to consider an expression of the potential energy of the following form

$$(36) \qquad E_p(u) = \int_\Omega Q(x,\nabla u)\, dx,$$

where $Q(x,.)$ is a positive definite quadratic form on $\mathbb{R}^2$ whose coefficients are functions of the space variable (in some mechanical problems $Q(x,.)$ describes the tensor of constraints). If we plug the expression (36) into (29), and if we apply Hamilton's principle, we obtain an equation similar to (32) with an operator Δ_Q which generalizes Δ. We shall meet such expressions later, $Q(x,\nabla u)$ will then be associated with some <u>Riemannian metric</u> on the manifold Ω.

D. <u>OTHER POINTS OF VIEW</u>

Let us now look back at what we did in paragraphs A to C. The energy or Dirichlet integral

$$(37) \quad \begin{cases} \text{(i)} \quad E(u) = \int_0^L (u_x')^2 dx & \text{(in the case of the vibrating string),} \\ \text{or} & \\ \text{(ii)} \quad E(u) = \int_\Omega |\nabla u|^2 dx & \text{(in the case of the vibrating membrane),} \end{cases}$$

plays a prominent role. To this energy integral, the <u>variational approach</u> associates the Laplacian

$$
(38) \quad \begin{cases} \text{(i)} \quad \Delta = -\dfrac{d^2}{dx^2} \quad \text{or} \quad -\dfrac{d}{dx}(\mu(x)\dfrac{d}{dx}) & \text{(one-dimensional case),} \\[2ex] \text{(ii)} \quad \Delta = -(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}) & \text{(two-dimensional case).} \end{cases}
$$

(<u>We use the minus sign for convenience; we shall keep this convention throughout this text</u>).

Having in mind the eigenvalue Problems (25) and (34), and recalling Lagrange's multipliers method, we also introduce the <u>Rayleigh</u> (-<u>Ritz</u>) <u>quotient</u>

$$
(39) \quad \begin{cases} \text{(i)} \quad R(u) = \displaystyle\int_0^L (u_x')^2 dx \Big/ \int_0^L u^2 dx & \text{(one-dimensional case),} \\[2ex] \text{(ii)} \quad R(u) = \displaystyle\int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega u^2 dx & \text{(two-dimensional case),} \end{cases}
$$

where u is not identically zero. Indeed, if we write

$\frac{d}{d\epsilon} R(u+\epsilon v)\Big|_{\epsilon=0} = 0$, we find, say with (39ii),

$$
(40) \quad \int_\Omega \langle \nabla u, \nabla v \rangle \, dx = R(u) \int_\Omega uv \, dx.
$$

Assume that (40) holds for all functions v in $C_0^\infty(\Omega)$, and let $R(u) = \lambda$. Integrating by parts gives

$$
\Delta u = \lambda u \quad \text{in} \quad \Omega \quad (\Delta \text{ as in (38ii)!}).
$$

<u>SUMMARY</u> - THE MAIN CHARACTERS OF THIS PLAY ARE

(41)
- the energy or Dirichlet integral $\int_\Omega |\nabla u|^2 dx$,
- the (positive) Laplacian Δ (positive refers to the sign convention made above nº (38) and (47ii) below),
- the Rayleigh (-Ritz) quotient $R(u) = \int_\Omega |\nabla u|^2 dx \,/\, \int_\Omega u^2 dx$.

42. In the preceding paragraphs, we have shown how the partial differential equations governing the vibrations of an elastic string or membrane can be deduced from Hamilton's principle, once we know the expression of the energy. These partial differential equations involve the Laplacian Δ. The method of separation of variables led us to some eigenvalue problems for the Laplacian. These eigenvalue problems are related to the extrema of the Rayleigh quotient $R(u)$ (or equivalently to the extrema of the energy $\int_\Omega |\nabla u|^2 dx$, under the constraint $\int_\Omega u^2 dx = 1$).

We shall now explain how these considerations are related to other points of view or formulations.

43. Let us first deal with the <u>point of view of partial differential equations</u> (P.D.E). Let Ω denote an elastic membrane which is fixed along $\partial\Omega$ and submitted to a transversal force f. In order to find the equilibrium position of the membrane, we have to look for local extrema of the energy

$$E(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega f(x)u(x)dx.$$

This leads us to the boundary value problem (compare with §A; the admissible functions are required to vanish on $\partial\Omega$)

(44) $$\begin{cases} \Delta u(x) + f(x) = 0 \quad \text{in} \quad \Omega, \quad \text{with the boundary condition} \\ u(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Multiplying (44) by any function v in $C_0^\infty(\Omega)$ we obtain, after integration by parts,

$$(45) \quad \begin{cases} \text{(i)} \quad \int_\Omega \langle \nabla u, \nabla v \rangle dx + \int_\Omega f(x)v(x)dx = 0 \\ \text{(ii)} \quad \int_\Omega u\Delta v \, dx + \int_\Omega f(x)v(x)dx = 0 \end{cases}$$

If u is twice differentiable and satisfies (44), we say that u is a <u>classical solution</u> of Equation (44). If u is in $L^1_{loc}(\Omega)$ and satisfies Equation (45ii), we say that u is a solution of Equation (44) in the <u>sense of distributions</u>. If u and $|\nabla u|$ are in $L^2(\Omega)$ and if u satisfies (45i), we say that u is a <u>weak solution</u> of Equation (44). We do not want to go into technical details here, for precise definitions and results see [TS], [G-T] or [SW].

It turns out that it is much easier to prove the existence of a weak solution than that of a classical solution. Once the existence of a weak solution is proved (e.g. by using Hilbert space methods and appropriate Sobolev spaces), one has to prove that the weak solution is indeed a classical solution: one has to prove interior regularity in Ω, and regularity up to $\partial\Omega$. Note that the bilinear form $\int_\Omega \langle \nabla u, \nabla v \rangle dx$ in Equation (45i) is just the bilinear form associated with the quadratic form giving the energy, $\int_\Omega |\nabla u|^2 dx$.

46. We have seen (nº 21-26) that it is very important to solve the eigenvalue problem $\Delta u = \lambda u$ in Ω, with some appropriate boundary condition, e.g. the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The Laplacian Δ is a linear (partial differential) operator. We could view it as a linear operator from $C^2(\Omega)$ into $C^0(\Omega)$ but this is not so good if we want to consider the eigenvalues of Δ. We

could also consider Δ as a linear operator from $C^\infty(\Omega)$ into $C^\infty(\Omega)$. It turns out that this is not an appropriate choice because the C^∞-topology is too complicated and because Equation (45i) is so much related to the L^2-inner product in Ω, $(u,v) \to \int_\Omega u(x)v(x)dx = (u|v)$ (if we deal valued functions or $(u,v) \to \int_\Omega u(x)\bar{v}(x)dx$, if we deal with complex valued functions). It turns out that the good choice is to view Δ as an unbounded linear operator on $L^2(\Omega)$, with domain $C_o^\infty(\Omega)$; this means that we consider Δ as a linear operator from the dense linear subspace $C_o^\infty(\Omega)$ of $L^2(\Omega)$ into $L^2(\Omega)$. Spectral theory was devised to deal with such operators. The Laplacian has the following properties

(47i) $\forall\, u,v \in C_o^\infty(\Omega)\ (\Delta u|v) = (u|\Delta v)$,

we say that Δ is a symmetric operator;

(47ii) $\forall\, u \in C_o^\infty(\Omega)\ (\Delta u|u) = \int_\Omega |\nabla u|^2 \geq 0$,

we say that Δ is a positive operator.

It follows from a theorem of Friedrichs ([R-S] Vol. II) that Δ can be extended to an unbounded self-adjoint operator Δ_e in $L^2(\Omega)$. The manner in which this extension is made depends on the boundary conditions which are imposed on $\partial\Omega$.

In the finite dimensional case, there is a very strong relationship between self-adjoint operators and quadratic forms. This can be generalized to more complicated situations. For example, in order to study the Laplacian Δ one can study the quadratic form given by the energy integral $u \to \int_\Omega |\nabla u|^2 dx$ (see Formula 47ii).

Let us deal with an example. If we want to study the eigenvalue problem

$$(48)\qquad \begin{cases} \Delta u = \lambda u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

we look at the Rayleigh quotient $R(u) = \dfrac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2\, dx}$.

In order to study $R(u)$, we have to determine what are the admissible functions. For u in $C^\infty(\Omega)$, we define $\|u\|_1^2 = \int_\Omega u^2(x)dx + \int_\Omega |\nabla u(x)|^2 dx$. We call $H^1(\Omega)$ (resp. $H_o^1(\Omega)$) the completion of $C^\infty(\Omega)$ (resp. $C_o^\infty(\Omega)$) for the norm $\|.\|_1$. Let $\|.\|_o$, $\|u\|_o^2 = \int_\Omega u^2(x)dx$, be the L^2-norm. Since $\|u\|_o \le \|u\|_1$ for $u \in C^\infty(\Omega)$, we have

$$H_o^1(\Omega) \subset H^1(\Omega) \subset L^2(\Omega).$$

The admissible functions for the Dirichlet problem are the functions in $H_o^1(\Omega)$ (we would take $H^1(\Omega)$ for the Neumann problem). Since Ω is a compact set, the inclusion $H_o^1(\Omega) \subset L^2(\Omega)$ is <u>compact</u> (this is the case for $H^1(\Omega) \subset L^2(\Omega)$, under some regularity conditions on $\partial\Omega$).

From this it follows that

$$\lambda_1(\Omega) = \inf\{R(u) \mid u \neq 0,\ u \in H_o^1(\Omega)\}$$

is achieved on a subspace $E_1(\Omega)$ of $H_o^1(\Omega)$; $E_1(\Omega)$ is characterized by the property

$$u \in E_1(\Omega) \Leftrightarrow \forall\ v \in H_o^1(\Omega),\ \int_\Omega \langle \nabla u, \nabla u\rangle dx = \lambda_1(\Omega) \int_\Omega uv\, dx.$$

In order to find the other eigenvalues, one has to consider the orthogonal $H^{(1)}$ of $E_1(\Omega)$ in $L^2(\Omega)$. One then defines $\lambda_2(\Omega) = \inf\{R(u) \mid u \neq 0\ u \in H_o^1(\Omega) \cap H^{(1)}\}$... See nº III. 18 ff.

This manner of dealing with the eigenvalue problem (48) is very close to the underlying physical properties.

For example, consider two membranes Ω_1, Ω_2 with the same physical properties and such that $\Omega_1 \subset \Omega_2$. We then have $H_o^1(\Omega_1) \subset \subset H_o^1(\Omega_2)$, and we conclude that $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$: the smaller drum has a higher fundamental tone.

As was already alluded to before, a Riemannian metric g on Ω may account for some physical properties (stress,...). Assume that one is given Ω with two Riemannian metrics g_1 and g_2 such that for any tangent vector U, $g_1(U,U) \leq g_2(U,U)$. Then, for any u in $C_o^\infty(\Omega)$, one has $R(u;g_1) \geq R(u;g_2)$ (recall that R involves the dual metric) and hence $\lambda_1(\Omega,g_1) \geq \lambda_1(\Omega,g_2)$.

We shall see in Chapter III nº 26 that there are variational characterizations of eigenvalues which are very similar to the one above for $\lambda_1(\Omega)$ (it is good to keep the finite dimensional case in mind). These characterizations are very important because they are very close to the original physical problems through the Rayleigh quotient.

It will be important in the sequel to keep in mind the physical motivations we described in this chapter.

Further references for Chapter I: Partial differential equations: [B-J-S],[ES],[GN],[GT],[PY],[TL],[TS],[WR],[WS]; Spectral theory: [AN],[R-S],[SW] and [BZ] for functional analysis; other possible references for motivations and results: [BE],[C-H],[CL],[TL].

CHAPTER II

TOPICS FROM RIEMANNIAN GEOMETRY

A. GENERALITIES

The purpose of this chapter is to introduce the basic objects we shall deal with in the core of these notes (Chapters IV to VI): Riemannian manifolds, curvatures, the covariant derivative...

The reader interested in Riemannian geometry itself is referred to [B-G-M] Chapters I and II, [C-E] or [CO], for more details and proofs.

All manifolds we shall consider will be C^∞ connected manifolds (unless otherwise stated).

1. A Riemannian manifold (M,g) is a manifold M equipped with a Riemannian metric g: for any point x in M, g_x is a scalar product on the tangent space T_xM which depends C^∞ on x (this can be checked in a local coordinate system).

2. Examples

(a) $(\mathbb{R}^n, \text{can})$: the space $\mathbb{R}^n$ equipped with the usual Euclidean structure is a Riemannian manifold; we can also consider $(\mathbb{R}^n, g_A)$, where A is a C^∞ map from $\mathbb{R}^n$ to the space $S_+(n)$ of positive definite symmetric $n \times n$ matrices on $\mathbb{R}^n$ and $g_A(x,y) = (Ax|y)$ for any vectors x,y in $\mathbb{R}^n$ (here $(.|.)$ denotes the usual Euclidean structure). We can also restrict our attention to a smooth bounded domain D in $\mathbb{R}^n$; in that case g_A could represent a strain tensor inside the body D.

We call (H^n, can) the Riemannian manifold (B, g_H) where B is the open ball centered at 0, with radius 2 in $\mathbb{R}^n$; for U, V tangent to B at x, g_H is defined by

$$g_H(U,V) = \left(1 - \frac{|x|^2}{4}\right)^{-2} (U|V)$$

where $|x|$ is the Euclidean norm of the vector x in $\mathbb{R}^n$.

This Riemannian manifold is called the n-dimensional <u>hyperbolic space</u>.

(b) Let $f: M \to \mathbb{R}^N$ be an imbedding of a manifold M into $\mathbb{R}^N$. The <u>induced metric</u> g on M is defined as the pull-back by f of the canonical metric on $\mathbb{R}^N$; for any vectors U and V in T_xM, we define $g_x(U,V) = (f_*U|f_*V)$ the scalar product in $\mathbb{R}^N$ of the images of U and V by the tangent map f_* to f. A very important instance of such a Riemannian submanifold of $\mathbb{R}^N$ is the <u>canonical sphere</u> (S^n, can), where S^n is the unit sphere in $\mathbb{R}^{n+1}$ with induced Riemannian metric, $S^n = \{x \in \mathbb{R}^{n+1} \mid (x|x) = 1\}$ (For example S^1 in $\mathbb{R}^2$).

(c) The <u>Riemannian product</u> $(M \times N, g \times h)$ of two Riemannian manifolds (M,g) and (N,h) is defined in such a way that Phythagoras theorem be true: if (U,V) (resp. (U',V')) are tangent vectors at (x,y) in $M \times N$, then

$$(g \times h)((U,V),(U',V')) = g(U,U') + h(V,V').$$

For example, the n-torus (T^n, can) is the product of n copies of (S^1, can) (see Example (b) and Example 4(d)).

3. <u>An isometry</u> $f: (M,g) \to (N,h)$ between two Riemannian manifolds is a diffeomorphism f between M and N such that $f^*h = g$, i.e. for any x in M and U in T_xM, $h_{f(x)}(f_*U, f_*U) = g_x(U,U)$.

4. Examples (continued)

(d) Let (M,g) be a Riemannian manifold and let G be a discrete group of isometries of (M,g), such that the quotient space $M/G = N$ is a manifold.

It is then clear that one can define a Riemannian manifold $(N,h) = (M/G, g/G)$. For example (T^n, can) defined in 2(c) is isometric to $(\mathbb{R}^n/\mathbb{Z}^n, \text{can}/\mathbb{Z}^n)$. Other tori can be defined as follows: let G be a lattice in $\mathbb{R}^n$ i.e. $G = e_1\mathbb{Z}+\ldots+e_n\mathbb{Z}$, where $[e_1,\ldots,e_n]$ is a basis of $\mathbb{R}^n$. We can define the torus (T^n_G, can) as $(\mathbb{R}^n/G, \text{can}/G)$. The tori (T^n, can) and (T^n_G, can) are not necessarily isometric (they are howewer always diffeomorphic).

Another instance of such a situation is the canonical Riemannian metric on the projective space $\mathbb{R}P^n$. We can view $\mathbb{R}P^n$ as the quotient of the sphere S^n by the antipodal map which sends the point x in S^n to $-x$. We can then write $\mathbb{R}P^n = S^n/\{1,\sigma\}$. We define the Riemannian manifold $(\mathbb{R}P^n, \text{can})$ as $(S^n/\{1,\sigma\}, \text{can}/\{1,\sigma\})$ because σ is an isometry of (S^n, can) (it is induced by the symmetry about 0 in $\mathbb{R}^{n+1}$).

From the definition of the quotient Riemannian manifold $(M/G, g/G)$ it follows that one can also define a natural Riemannian metric on a covering space M over a Riemannian manifold (N,h).

For more details see [B-G-M] Chap. I, [CO] Chap. 1.

5. A Riemannian invariant is a function F defined on the space of Riemannian metrics on a manifold M, which is invariant under isometries. This means that F is in fact a function on the space of Riemannian structures on M, i.e. on the quotient space of the space of Riemannian metrics by the group of diffeomorphisms. For example, we do not necessarily want to view the Riemannian manifold (S^n, can) as the unit sphere in $\mathbb{R}^{n+1}$ with induced metric; any other

isometric representation can serve. Any positive definite quadratic form on $\mathbb{R}^n$ with constant coefficients gives rise to the same Riemannian structure on $\mathbb{R}^n$: $(\mathbb{R}^n, \text{can})$.

6. Scaling. Given a Riemannian manifold (M,g), one has a whole family of Riemannian manifolds (M,g_a) which are obtained from (M,g) by multiplying the Riemannian metric g by the positive constant a : $g_a = ag$. A Riemannian invariant $F(g)$ may have a weight r, i.e. satisfy $F(ag) = a^r F(g)$. Since dilating the metric is very often a trivial operation, we shall be mainly interested in Riemannian invariants with weight 0. This will appear in a crucial way later on.

7. On a Riemannian manifold one can define the length of a curve $c: [0,1] \to M$ by

$$\ell(c) = \int_0^1 g(\dot{c}(t),\dot{c}(t))^{1/2} dt,$$

where $\dot{c}(t)$ is the velocity vector of the curve.

One can now define the Riemannian distance $d(x,y)$ or $\overline{xy}$ between two points x and y of (M,g) as the infimum of the lengths of the curves in M going from x to y.

Caution. Let us consider (S^2,can) in $\mathbb{R}^3$. It shall be clear later on that the Riemannian distance between two antipodal points is π; we have to consider curves lying on S^2, not curves going through the ball. For this reason, the Riemannian distance on a submanifold of $\mathbb{R}^n$ is also referred to as the intrinsic distance (vs extrinsic distance).

8. Properties

(i) d is a distance (in the sense of metric spaces) and this distance defines the same topology on M as the one given with the differentiable manifold structure;

(ii) A classical theorem of H. Hopf and W. Rinow ([CO] Chap. 7 or [B-C] § 8.2) states that if the metric space (M,d) is complete, then any two points x,y in M can be joined by a curve (called shortest path) whose length is exactly $d(x,y)$.

9. Variational arguments show that a shortest path is carried by a geodesic. Geodesics are curves which satisfy a certain second order (non-linear) differential equation on M, see nº 41. Given x in M and U in T_xM there exists a unique geodesic c_U starting from x with velocity vector U at x. An assertion in the Hopf-Rinow theorem states that $c_U(t)$ is defined for all values of t if and only if (M,d) is a complete metric space. In that case one says that (M,g) is a complete Riemannian manifold.

From now on, all Riemannian manifolds are assumed to be complete (unless otherwise stated).

Geodesics are always parametrized proportionally to arc-length and are locally length minimizing (for ε small enough, c is a shortest path between the points $c(t)$ and $c(t+\varepsilon)$).

The geodesics of (S^2,can) are the great circles. A shortest path between two points x,y of S^2 is the piece of a great circle through x and y with smallest length. Antipodal points are joined by infinitely many shortest paths. Any two points of the sphere can be joined by at least two geodesics one of them of shortest length (two arcs of a great circle passing through x and y).

10. The diameter $\mathrm{Diam}(M,g)$ of the Riemannian manifold (M,g) is defined by

$$\mathrm{Diam}(M,g) = \sup\{d(x,y) : x,y \text{ in } M\}.$$

It is finite if and only if M is compact. (As indicated above (M,g) is already assumed to be complete).

11. Let $\{x_1,\ldots,x_n\}$ be local coordinates near a point p in M. In these coordinates, the metric g can be represented by the matrix (g_{ij}), $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. The measure $[\mathrm{Det}((g_{ij}))]^{1/2}dx_1\ldots dx_n$ does not depend on the choice of a local coordinate system (use the theorem on change of variables in an integral). It defines the canonical Riemannian measure which we shall denote by v_g (or sometimes simply by dx) ([B-G-M] Chap. II.A). Given a continuous function f on M, we shall write $\int_M f(x)dv_g(x)$, $\int_M f dv_g$ or simply $\int_M f$ for the integral of the function f on M.

12. Properties Let (M,g) be a Riemannian manifold. Then

(i) $v_{ag} = a^{n/2}v_g$;

(ii) $\mathrm{Diam}(M,ag) = a^{1/2}\,\mathrm{Diam}(M,g)$

if $\dim M = n$ $(a > 0)$; see nº 6.

13. Given a point x in a complete Riemannian manifold (M,g), define the exponential map at the point x, $\exp_x\colon T_xM \to M$ as follows.

Given a vector U in T_xM we define $\exp_x(U)$ as the point $c_U(1)$ on the geodesic c_U issued from x with initial velocity vector U.

For nº 7-13, see [B-G-M] Chap. II.C, [CO] Chap. 3 and 7, or [C-E].

B. CURVATURES: The geometric point of view

As we shall see in a minute, there are several notions of curvature. These Riemannian invariants are very difficult to grasp and we will meet them under various circumstances. We first give definitions of a geometric flavor. (See [B-G-M] Chap. II.D and E, [CO] Chap. 4 and 8 or [C-E], [B-C]).

14. Sectional curvature. Let x be a point in (M,g) and let P be a (2-dimensional) plane in T_xM. We call $C_P(r)$ the image, under the exponential map $\exp_x$, of the circle centered at 0, with radius r, in P. This is a curve in (M,g), whose length we call $\ell_P(r)$ (if (M,g) is not complete the map $\exp_x$ might not be defined on the whole of T_xM, but it is always defined on a small ball centered at 0 in T_xM).

It turns out that one can prove the following estimate

$$(15) \qquad \ell_P(r) = 2\pi r\left(1 - \frac{r^2}{6}\sigma(P) + O(r^3)\right)$$

as r goes to zero ([B-G-M] Chap. II.E.III).

The number $\sigma(P)$ which appears in (15) is called the sectional curvature of the 2-plane P at x. This defines a function on the Grassmannian $G_{m,2}(T_xM)$ (the set of 2-planes in T_xM) and, when x varies, a function on $G_{m,2}(M)$ the Grassmannian bundle over M (see [NN]). In dimension 2, this is only a function on (M,g). When (M,g) is a surface in $\mathbb{R}^3$, with induced metric, the sectional curvature coincides with the Gaussian curvature of the surface (product of theprincipal curvatures), see [HF]. When $\dim M$ is bigger than 2, this is a much more complicated object.

16. Comments. The fact that $\ell_P(r) \sim 2\pi r$ as r goes to zero, means that a Riemannian manifold looks like Euclidean space in the

small. The fact that there is no second order term in (15) comes from the fact that in a "good" coordinate system centered at p in $M\{x_1,\dots,x_n\}$, (namely the one given by $\exp_p$) one has $g_{ij}(0) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x_k}(0) = 0$.

Local calculations show that the curvature involves second order derivatives of the metric.

17. Examples.

(a) As is easily seen (see Fig. 1), in the case of the canonical 2-sphere (or more generally n-sphere) we have

$$\ell_p(r) = 2\pi \sin r = 2\pi r\left(1- \frac{r^2}{6} + 0(r^3)\right)$$

which shows that $\sigma \equiv 1$ on (S^n, can);

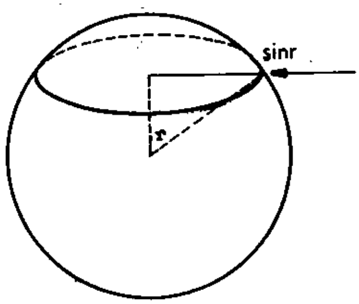

Fig. 1

(b) It is even easier to see that $\sigma \equiv 0$ on $(\mathbb{R}^n, \text{can}), (T^n_G, \text{can}/G)$;

(c) Exercise. Consider the Riemannian manifold (H^n, can) given in Example 3(a). Show that the geodesics issued from 0 are the rays issued from 0 (Hint: use the differential equation and the fact that the image of a geodesic by an isometry again is a geodesic). Compute the length of the curve $t \to (tr, 0)$ where $r < 2$. Show that for any 2-plane P in T_oH^n, $\ell_p(r) = 2\pi \operatorname{sh} r$. Conclude that $\sigma(P) = -1$.

In fact one can show that given any two points x, y in (H^n, can) there exists an isometry f of (H^n, can) such that

$f(x) = y$. It follows that $\sigma \equiv -1$ for (H^n, can).

18. Remarks.

By adjusting the definitions of (S^n, can) and (H^n, can) (scaling) one can easily construct the simply connected Riemannian manifolds $(\$^n_k, can)$ whose sectional curvature is constant and equal to k (any real number). They are called space forms. If $k > 0$ $(\$^n_k, can)$ is homothetic to (S^n, can); if $k = 0$ $(\$^n_o, can)$ is just $(\mathbb{R}^n, can)$; if $k < 0$ $(\$^n_k, can)$ is homothetic to (H^n, can) ([CO] Chap. 8).

The sectional curvature is a very strong invariant. Any Riemannian manifold (M, g) whose sectional curvature is constant equal to k, is locally isometric to $(\$^n_k, can)$: given any point x in (M, g), there exists a neighborhood of x which is isometric to a neighborhood of a point in $(\$^n_k, can)$. This local property is in fact global when M is simply-connected: a simply-connected complete Riemannian manifold (M, g) with constant sectional curvature equal to k is isometric to $(\$^n_k, can)$ (These results are known as E. Cartan's theorems, [B-G-M] Chap. II.E.III).

19. An Example.

We denote by $\mathbb{C}P^n$ the complex projective space (complex lines in $\mathbb{C}^{n+1}$) i.e. $\mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^*$.

We can identify $\mathbb{C}^{n+1}$ and $\mathbb{R}^{2n+2}$, so that the unit sphere S^{2n+1} is the set $\{(z_o, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_o|^2 + \ldots + |z_n|^2 = 1\}$. The circle S^1 acts on S^{2n+1} by $e^{it}.(z_o, \ldots, z_n) = (z_o e^{it}, \ldots, z_n e^{it})$. It is easy to see that $\mathbb{C}P^n$ is diffeomorphic to S^{2n+1}/S^1. Here we view $\mathbb{C}P^n$ as a real manifold. Let $p: S^{2n+1} \to \mathbb{C}P^n$ the projection map. Given x in S^{2n+1}, we let H_x denote the space $\mathbb{C}^{n+1}$-orthogonal to the complex line $\mathbb{C}x$ of $\mathbb{C}^{n+1}$ ($\dim_{\mathbb{C}} H_x = n$). The tangent map to p defines an isomorphism p^T from H_x onto $T_{p(x)}\mathbb{C}P^n$. We define a metric g on $\mathbb{C}P^n$ as follows. Given any vector X in H_x we let $g(p^T(X),p^T(X)) = |X|^2$ (norm in $\mathbb{C}^{n+1}$).

This metric g turns the map p into a <u>Riemannian submersion</u> $p: (S^{2n+1},\text{can}) \to (\mathbb{C}P^n,g)$ with fibers S^1. This means that p is a (differentiable) submersion and that the tangent map T_xp is an isometry from the orthogonal to the tangent space to the fiber at x onto the tangent space to $\mathbb{C}P^n$ at $p(x)$. The geodesics are easily seen to be the images under p of the great circles of S^{2n+1} which are orthogonal to the fibers of p. One can show that the sectional curvature of a 2-plane P in $T\mathbb{C}P^n$ lies between 1 and 4 ([BS] Chap. 3).

20. The sectional curvature measures how geodesics diverge from one another (this should be at least intuitive from the very definition of sectional curvature: see nº (15)).

Let us make this statement more precise. Let us consider a point x in the manifold (M,g), and two geodesics $c_1(t)$, $c_2(t)$ such that

$$(21)\qquad \begin{cases} \text{(i)} & c_1(0) = c_2(0) = x; \\ \text{(ii)} & c_1'(0) \text{ and } c_2'(0) \text{ form an angle } A; \\ \text{(iii)} & c_1(t),\ c_2(t) \text{ are parametrized by arc-length.} \end{cases}$$

22. Since geodesics are locally length minimizing, it follows that for t small enough, the Riemannian distance $d(x,c_i(t))$ is just equal to t. Let us consider the <u>geodesic triangle</u> $\{x,c_1(t_1),c_2(t_2)\}$ whose sides are the minimizing geodesics between the vertices (t_1,t_2 are assumed to be small). Let us call $T(t_1,t_2)$ the length of the side from $c_1(t_1)$ to $c_2(t_2)$.

Let us now call $\{\bar{x},\bar{c}_1(t_1),\bar{c}_2(t_2)\}$ a geodesic triangle on $(\$^n_k,\text{can})$ where $\bar{x}$ is in $(\$^n_k,\text{can})$ and where $\bar{c}_i(t)$ are geodesics satisfying assumptions analogous to (21). Let us call $\bar{T}_k(t_1,t_2)$ the length of the side from $\bar{c}_1(t_1)$ to $\bar{c}_2(t_2)$ (again t_1,t_2 are assumed to be small enough). See Fig. 2.

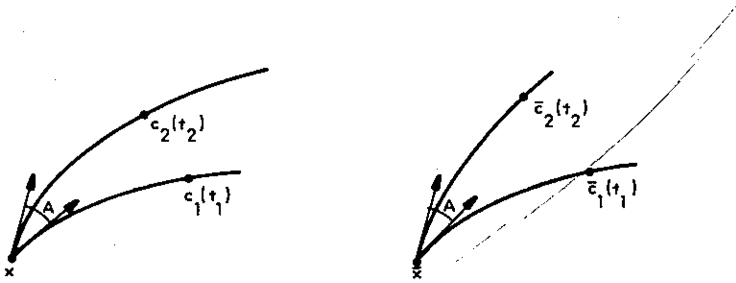

Fig. 2

23. <u>Definition</u>

We say that the sectional curvature $\sigma = \sigma(M,g;.)$ of (M,g) is <u>bigger than</u> k if for any 2-plane P in TM (i.e. any point x in M, any 2-plane P_x in T_xM), $\sigma(P) > k$ (one defines $\geq$, $<$, $\leq$ similarly).

We then have the following <u>comparison theorem</u>

24. <u>Theorem</u> (i) (Rauch)

<u>Let</u> (M,g) <u>be a Riemannian manifold whose sectional curvature satisfies</u> $\sigma \geq k$ (<u>resp</u>. $\sigma \leq k$) <u>then, with the notations of</u> nº 22, <u>we have</u>

$$T(t_1,t_2) \leq \bar{T}_k(t_1,t_2) \quad (\text{resp. } T(t_1,t_2) \geq \bar{T}_k(t_1,t_2)),$$

for all geodesic triangles constructed an in nº 22, for t_1, t_2 small enough.

(ii) (Rauch-Alexandrov-Toponogov)

If $\sigma \leq k$ then $T(t_1,t_2) \geq \bar{T}_k(t_1,t_2)$ holds for all geodesic triangles constructed as in nº 22 (whatever the size of t_1,t_2).

This theorem is difficult; see [C-E] Chap. 2 for more precise statements and proofs, see also [SI] for a survey.

25. Ricci curvature. Given a point x in the Riemannian manifold (M,g), it is easy to show that $\exp_x$ is a local diffeomorphism from a neighborhood of 0 in T_xM onto a neighborhood of x in M. The pulled-back measure $\exp_x^*(v_g)$ has a density with respect to the Lebesgue measure in T_xM. Using polar coordinates in T_xM, $(t,u) \in \mathbb{R}_+^{\bullet} \times S^{n-1}$, we can write $\exp_x^*(v_g) = \theta_x(t,u)dt\, du$, at least for t small enough. More precisely, let $\phi_x: \mathbb{R}_+^{\bullet} \times S^{n-1} \to M$ be defined by $\phi_x(t,u) = \exp_x(tu)$, where S^{n-1} is the unit sphere in (T_xM,g_x). Then $\phi_x^* v_g = \theta_x(t,u)dt\, du$. The following expansion holds

$$(26) \qquad \theta_x(t,u) = t^{n-1}\{1 - \frac{t^2}{6} r_x(u,u) + O(t^3)\} \qquad (n=\dim M),$$

as t goes to zero ([B-G-M] Chap. II.E.III).

Here $r_x(u,u)$ is a quadratic form in u, whose associated symmetric bilinear form is called the Ricci curvature at x (we shall often forget the index x in θ_x, r_x). The function $\theta_x(t,u)$ is the density of the Riemannian measure viewed through the map ϕ_x. See also [BS] Chap. 6 p. 154.

27. Comments

The fact that $\theta_x(t,u) \sim t^{n-1}$ as t goes to zero means that the Riemannian measure is asymptotically Euclidean. The Ricci curvature measures how the Riemannian measure differs from the Euclidean Lebesgue measure, at least infinitesimally. Let u be a unit vector in T_xM and let $\{u,e_2,\dots,e_n\}$ be an orthonormal basis of T_xM. Let $P_i = [u,e_i]$ denote the 2-plane spanned by the vectors u and e_i in T_xM. We then have

$$r_x(u,u) = \sum_{i=2}^{n} \sigma(P_i). \tag{28}$$

It follows from Formula (28) that the Ricci curvature of (S^n_k,can) satisfies $r(u,u) = (n-1)k$, for any unit tangent vector u or equivalently $r = (n-1)k$ can.

Formula (28) also shows that an assumption on the Ricci curvature is weaker than an assumption on the sectional curvature.

Let us mention the following important theorem which relates the Ricci curvature and the diameter of the manifold (M,g).

29. <u>Theorem</u> (Myers)

<u>Let</u> (M,g) <u>be a complete Riemannian manifold whose Ricci curvature satisfies</u> $r(u,u) \geq (n-1)k > 0$ <u>for any unit tangent vector</u> u.

<u>Then the diameter of</u> (M,g) <u>satisfies</u>

$$\mathrm{Diam}(M,g) \leq \pi/\sqrt{k},$$

<u>and hence</u> (M,g) <u>is compact. Furthermore the fundamental group</u> $\pi_1(M)$ <u>is finite</u>. ([CO] Chap. 9, [C-E] Chap. 1).

The following <u>comparison theorem</u> will be of utmost importance in the sequel.

30. <u>Theorem</u>

<u>Let</u> (M,g) <u>be a</u> n-<u>dimensional Riemannian manifold whose</u>

Ricci curvature r satisfies $r(u,u) \geq (n-1)k$ for any unit tangent vector u, k any real number. Let $B_k^n(t)$ be any geodesic ball with radius t in $(\$_k^n, can)$ (all such balls are isometric). Then

(i) (Bishop) For any point x in M and t in $\mathbb{R}_+$, $Vol(B(x,t)) \leq Vol(B_k^n(t))$ ($B(x,t)$ is the geodesic ball with radius t and center x in (M,g));

(ii) (Gromov) The function

$$t \to Vol(B(x,t))/Vol(B_k^n(t))$$

is non-increasing. ([B-C] or [GV1]).

31. Caution. Reverse inequalities when $r(u,u) \leq (n-1)k$ DO NOT HOLD (exept when $n = 2$ or $n = 3$).

32. Curvature versus scaling. Let (M,g) be a Riemannian manifold and let $c: [0,1] \to M$ be a curve. If the length of c in (M,g) is equal to L, then the length of c in (M,ag) (where $a > 0$) is $\sqrt{a}L$. If we denote by $\sigma(M,g)$ the sectional curvature of the Riemannian manifold (M,g), it follows from (15) that $\sigma(M,ag) = a^{-1}\sigma(M,g)$ $(a > 0)$; the sectional curvature is Riemannian invariant of weight -1 (see nº 6). The products $Diam(M,g)^2\sigma(M,g)$, $Vol(M,g)^{2/n}\sigma(M,g)$, are therefore Riemannian invariants of weight 0.

Let $r(M,g) = r$ (resp. $r_{min}(M,g)$) denote the Ricci curvature (resp. $\inf\{\frac{r(u,u)}{g(u,u)};\ u \in TM,\ u \neq 0\}$) of the Riemannian manifold (M,g).

33. Exercise. Show that $r(M,g)$ is a Riemannian invariant of weight 0, and that $r_{min}(M,g)$ is a Riemannian invariant of weight -1 (Hint: use formula (28) and the fact that $r(M,g)$ is a bilinear form on TM).

C. THE COVARIANT DERIVATIVE

34. Given a manifold M and a function $f: M \to \mathbb{R}$, one can define the differential of f, df as follows. Let p be a point in M and U a tangent vector in T_pM. Let $\{x_1, \ldots, x_n\}$ be a local coordinate system centered at p. The function f can then be viewed as a function $f(x_1, \ldots, x_n)$. We let

$$df_p(U) = \sum_{i=1}^{n} \frac{\delta f}{\delta x_i}(0)\, u_i,$$

if $U = \sum_{i=1}^{n} u_i \frac{\delta}{\delta x_i}$. It is easy to prove that df is invariantly defined on TM. A straightforward computation shows that $\left(\frac{\delta^2 f}{\delta x_i \delta x_j}(0)\right)$ $1 \le i,j \le n$, does not define an invariant object on M, unless $df_p = 0$.

One of the main features of Riemannian geometry is that to a Riemannian metric g on a manifold M, is naturally attached an intrinsic notion of derivation.

Let $\mathfrak{A}(M)$ denote the vector space of C^∞ vector-fields on M.

35. <u>Theorem and Definition</u>.

<u>Let</u> (M,g) <u>be a Riemannian manifold. There is a unique map</u>

$$D: \mathfrak{A}(M) \times \mathfrak{A}(M) \to \mathfrak{A}(M)$$

$$(X,Y) \to D_X Y,$$

<u>with the following properties (for any</u> X,Y,Z <u>in</u> $\mathfrak{A}(M)$ <u>and</u> f <u>in</u> $C^\infty(M)$)

(i) $X.g(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z)$;

(ii) $D_X Y - D_Y X = [X,Y]$;

(iii) D is $\mathbb{R}$-bilinear;

(iv) $D_{(fX)}Y = f\, D_X Y$;

(v) $D_X(fY) = (X.f)Y + f\, D_X Y$.

<u>This map</u> D <u>is called the</u> Levi-Civita <u>connection of the Riemannian manifold</u> (M,g). ([B-G-M], II.B. I. 4, p. 24).

36. <u>Exercise</u>. Using Property (iv) above, show that $(D_X Y)(x)$ depends only on the value X_x of the vector-field X at x.

<u>Caution</u>: the same property does not hold for Y, see nº 39(3).

37. <u>Metrics and Connections on tensor products</u>. Let $\{e_i\}_1^n$ be an orthonormal basis of T_xM. Let $\{e_i^*\}_1^n$ be the dual basis in T_x^*M, the dual space of T_xM. We extend g_x to a scalar product g_x^* on T_x^*M such that the basis $\{e_i^*\}_1^n$ be orthonormal.

<u>Exercise</u>. Show that the matrix (g_{ij}^*) of g^* in a local coordinate system $\{x_1,\ldots,x_n\}$ is $(g_{ij})^{-1}$, the inverse of the matrix (g_{ij}) of the metric g.

We extend g_x to a scalar product on $\otimes^p T_xM \otimes^q T_x^*M$ by taking an orthonormal basis $\{e_i\}_1^n$ of T_xM and by requiring that the natural basis of $\otimes^p T_xM \otimes^q T_x^*M$ deduced from $\{e_i\}_1^n$ be orthonormal.

We can also extend the Levi-Civita connection on tensors. For this purpose we require that Leibnitz rule be true, e.g. if U,V,X are sections of TM and if W is a section of T^*M, we let (using the same symbol D for the extension of the connection)

$$(38)\quad \begin{cases} \text{(i)} & D_X(U\otimes V) = (D_XU)\otimes V + U\otimes(D_XV), \\ \text{(ii)} & X.(W(U)) = (D_XW)(U) + W(D_XU). \end{cases}$$

We also define $D_X f$, for f in $C^\infty(M)$ as $X.f$. This extension of the Levi-Civita connection satisfies properties similar to those of Theorem 35.

39. Exercises.

The "musical" isomorphisms $TM \underset{\#}{\overset{\flat}{\rightleftarrows}} T^*M$ are defined as follows: for u in TM and f in T^*M, we let

$$g(u, f^\#) = f(u), \quad \text{and}$$
$$u^\flat = g(u, .).$$

(1) Show that for all X, Y in $\mathfrak{X}(M)$,

$$D_X(Y^\flat) = (D_X Y)^\flat.$$

(2) Show that for all X in $\mathfrak{X}(M)$,

$$D_X g = 0 \quad (\text{view } g \text{ as a section of } \otimes^2 T^*M).$$

(3) Let p be a point in (M,g) and let (U,F) be a chart centered at p, i.e. U is an open set in M, containing p and $F\colon U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism, with $F(p) = 0$. Let $\{x_1,\ldots,x_n\}$ be local coordinates on U. Let X, Y be vector-fields on M, whose expressions in (U,F) are $\sum_{i=1}^n X_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$, $\sum_{i=1}^n Y_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$. Show that there exist C^∞ functions on U, Γ^i_{jk}, such that $D_X Y$ is represented by the vector-field

$$\sum_{i=1}^n \{ \sum_{j=1}^n X_j(x_1,\ldots,x_n)[\frac{\partial Y_i}{\partial x_j}(x_1,\ldots,x_n) + \sum_{k=1}^n \Gamma^i_{jk}(x_1,\ldots,x_n)Y_k(x_1,\ldots,x_n)]\} \frac{\partial}{\partial x_i}.$$

Compare with Exercise 36. The coefficients Γ^i_{jk} are called the Christoffel symbols of the metric g.

(4) Let p in M, and u in T_pM. Let Y_1, Y_2 be vector-fields on M in a neighborhood of p. Let $c:]-a,a[\to M$ be a C^∞-curve such that $c(0) = p$ and $c'(0) = u$. Assume that for all t in $]-a,a[$, $Y_1(c(t)) = Y_2(c(t))$. Show that $D_uY_1 = D_uY_2$.

(5) Find the Levi-Civita connection on $(\mathbb{R}^n, \text{can})$ (Hint: compare it with the usual derivation of a vector-field).

(6) Let D be the Levi-Civita connection on (S^n, can), and let $\tilde{D}$ be the Levi-Civita connection on $(\mathbb{R}^{n+1}, \text{can})$. Let X, Y be vector fields on S^n. Show that they can be extended to local vector fields on $\mathbb{R}^{n+1}$, $\tilde{X}, \tilde{Y}$, and that for any x in S^n, $(D_XY)(x)$ is the orthogonal projection of $(\tilde{D}_{\tilde{X}}\tilde{Y})(x)$ on T_xS^n (Hint: use Theorem 35 and Exercise 39(4)).

40. Let $c:]-a,a[\to M$ be a C^∞-curve in M. A <u>vector-field along</u> c is a C^∞ map $X:]-a,a[\to TM$ such that $X(t)$ is in $T_{c(t)}M$ for all t in $]-a,a[$. It follows from Exercise 39(4) that one can define $D_{c'(t)}X$. We say that the <u>vector-field</u> X <u>is parallel</u> along the curve c if $D_{c'(t)}X = 0$ for all t.

In a local coordinate system $\{x_1,\dots,x_n\}$ near the point $c(0)$, we can write $c(t)$ as the curve $(A_1(t),\dots,A_n(t))$ in $\mathbb{R}^n$, and $c'(t)$ as the vector $(A_1'(t),\dots,A_n'(t))$ in $\mathbb{R}^n$. Let $\{X_i(t)\}_1^n$ be the coordinates of $X(t)$ in this coordinate system. According to Exercises 39(3) and (4) the condition $D_{c'(t)}X = 0$ can be written as

$$\sum_{j=1}^n A_j'(t) \left[\frac{\partial \tilde{X}_i}{\partial x_j}(A_1(t),\dots,A_n(t)) + \sum_{k=1}^n \Gamma_{jk}^i \tilde{X}_k(A_1(t),\dots,A_n(t))\right] = 0$$

or

$$\text{(41)} \qquad \frac{d}{dt}X_i(t) + \sum_{j,k=1}^n A_j'(t)\Gamma_{jk}^i X_k(t) = 0, \qquad 1 \le i \le n$$

(where $\tilde{X}$ is an extension of X in a neighborhood of $c(0)$ and $\tilde{X}_k(A_1(t),\dots,A_n(t)) \equiv X_k(t)$).

Equation (41) is a system of ordinary differential equations on $]-a,a[$. Given a vector u in $T_{c(0)}M$, one can therefore find a vector field X along c, such that $D_{c'(t)}X = 0$. A <u>geodesic</u> is a curve $c(t)$, whose tangent vector $c'(t)$ is parallel along $c(t)$, $D_{c'(t)}c'(t) = 0$.

42. <u>Remark</u>. The map $u \to X(t)$ where X is the parallel vector-field along c, such that $X(0) = u$, is called the <u>parallel translation</u> along c. This very important notion can be generalized to tensors on M.

<u>Exercise</u>. Let $c(t)$ be a curve in $(\mathbb{R}^n, can)$ and let u be a vector at $c(0)$. Find the parallel vector-field $X(t)$ along c, such that $X(0) = u$.

For § C see [B-G-M] Chap. II. B, [CO] Chap. 2, or [C-E].

D. <u>CURVATURES</u>: <u>The analytic point of view</u>.

43. The <u>curvature tensor</u> R of the Riemannian manifold (M,g) is defined as follows. Given X,Y in $\mathfrak{X}(M)$ one defines the map $R(X,Y)$ from $\mathfrak{X}(M)$ to $\mathfrak{X}(M)$ by $R(X,Y) = [D_X,D_Y] - D_{[X,Y]}$, i.e. for any U in $\mathfrak{X}(M)$,

$$R(X,Y)U = D_X(D_YU) - D_Y(D_XU) - D_{[X,Y]}U.$$

<u>Caution</u>. The sign of R may differ from one book to another.

44. <u>Properties</u>.

(i) <u>For any</u> X,Y <u>in</u> $\mathfrak{X}(M)$, $R(X,Y) + R(Y,X) = 0$;

(ii) <u>The map from</u> $\mathfrak{X}(M)^3$ <u>to</u> $\mathfrak{X}(M)$, <u>which associates</u> $R(X,Y)U$ <u>to</u> (X,Y,U) <u>is in fact a map from</u> $(T_xM)^3$ <u>to</u> T_xM <u>i.e.</u>

$(R(X,Y)U(x)$ depends only on the values of the vector fields X_x, Y_x, U_x at x (we say that it is a tensor).

Proof. Use Theorem 35 to show that for any $f,g,h,$ in $C^\infty(M)$ and any X,Y,U in $\mathfrak{A}(M)$, $R(fX,gY)(hU) = fgh\, R(X,Y)U$ (i.e. that R is $C^\infty(M)$ -3-linear) ■

45. Definitions. (Caution with sign conventions)

Let us define the tensor $R(X,Y;U,V)$ by

$$R(X,Y;U,V) = g(R(X,Y)V,U). \quad \text{Then}$$

(i) If P is a 2-plane spanned by $\{X,Y\}$, we define the sectional curvature σ of (M,g) on the 2-plane P by $\sigma(P) = R(X,Y;X,Y)/g(X\wedge Y, X\wedge Y)$, i.e. $\sigma(P) = R(e,f;e,f)$ if $\{e,f\}$ is an orthonormal basis of the 2-plane P;

(ii) The Ricci curvature of (M,g) is defined by

$$r(X,X) = \sum_{i=1}^{n} R(X,e_i;X,e_i),$$

for any vector X in T_xM, where $\{e_i\}_{1,\dots,n}$ is any orthonormal basis in T_xM;

(iii) The scalar curvature u of the Riemannian manifold (M,g) at x is defined by

$$u(x) = \sum_{i,j=1}^{n} R(e_i,e_j;e_i,e_j) = \sum_{j=1}^{n} r(e_j,e_j),$$

where $\{e_i\}$ is any orthonormal basis of T_xM. The scalar curvature is a function on (M,g).

46. Claim. These definitions coincide with those given in nº 14 and 25.

47. Notations.

In order to make things explicit without statements, we will often use the following obvious notations Sect, Sect(M,g); Ricci, Ricci(M,g); Scal, Scal(M,g). (see [B-G-M] Chap. II, [CO] Chap. 4).

48. Let f be a C^∞ function defined on the Riemannian manifold (M,g). Let X,Y be two vector-fields on M. We denote by $Ddf(X;Y)$ the one-form $D_X(df)$ evaluated on the vector-field Y. The $\mathbb{R}$-bilinear map Ddf is called the Hessian of f and denoted by Hess f (with respect to the Riemannian metric g). According to nº (38) we have.

$$\text{Hess } f(X,Y) = X.(df(Y)) - df(D_X Y). \tag{49}$$

50. Proposition.

The Hessian of a C^∞ function f, Hess f, is a symmetric two tensor i.e.

(i) $(\text{Hess } f(X,Y))(x)$ depends only on X_x and Y_x;

(ii) $\text{Hess } f(X,Y) = \text{Hess } f(Y,X)$.

Proof. Use the fact that being a tensor is equivalent to $C^\infty(M)$-linearity and Theorem 35. ■

This proposition answers the question which was raised in nº 34, and generalizes for 2nd order derivatives the well-known Schwarz theorem on functions of several variables. An important fact in Riemannian geometry is that Schwarz theorem no longer holds for higher order derivatives.

Let f be a C^∞ function on M and let X,Y,Z be three vector fields on M. The following lemma holds.

51. <u>Lemma</u>.

$$(D_X(D_Y df))(Z) - (D_Y(D_X df))(Z) - (D_{[X,Y]}df)(Z) = -R(X,Y;df^{\#},Z).$$

<u>Proof</u>. Write (using nº (38) the first term in the left-hand side as

$$\begin{aligned}(D_X(D_Y df))(Z) &= X.((D_Y df)(Z)) - (D_Y df)(D_X Z)\\ &= X.[Y.(df(Z)) - df(D_Y Z)] - Y.(df(D_X Z)) + df(D_Y(D_X Z)),\end{aligned}$$

and a similar expression for the second term. This gives $([D_X,D_Y]df)(Z) = [X,Y].(df(Z)) - df([D_X,D_Y]Z)$. Then use the definition of the curvature tensor to conclude that

$$\begin{aligned}([D_X,D_Y]df)(Z) - (D_{[X,Y]}df)(Z) &= -df(R(X,Y)Z)\\ &= -R(X,Y;df^{\#},Z),\end{aligned}$$

using nº 45 and nº 39. ■

52. Take an orthonormal basis $\{e_i\}_1^n$ at x in M. Using nº 40, one can extend $\{e_i\}_1^n$ to a local orthonormal frame $\{X_i\}_1^n$ such that $X_i(x) = e_i$ and $(D_{X_i}X_j)(x) = 0$; from Theorem 35(ii), we deduce that $[X_i,X_j](x) = 0$ and from Exercise 36, we conclude that $(D_{[X_i,X_j]}df)(x) = 0$. Finally, we deduce from Lemma 51 that

$$([D_{X_i},D_{X_j}]df)(X_k) = D^3f(X_i;X_j;X_k) - D^3f(X_j;X_i;X_k) = -R(X_i,X_j;df^{\#},X_k)$$

(the second equality is a notation), which shows that Schwarz theorem does not hold for derivatives of order 3, unless $R=0$. We can view the <u>curvature</u> as <u>an obstruction to commuting derivatives</u>.

53. The vector-field $df^{\#}$, dual to the 1-form df, which appears in Lemma 51 is called the <u>gradient</u> of f; it depends on the Riemannian metric g whereas df does not.

Further references for Chapter II: Introduction to Riemannian geometry: [CL] Chap. III, [MR], [SI]; Riemannian geometry: [B-C], [B-G-M], [C-E], [CO], [KG], [SK].

CHAPTER III

THE LAPLACIAN AND RELATED TOPICS

ALL RIEMANNIAN MANIFOLDS ARE ASSUMED TO BE SMOOTH, CONNECTED AND COMPLETE

Unless otherwise stated, vector-fields, forms, functions... will also be assumed smooth.

This chapter is mainly devoted to the Laplace-Beltrami operator (or Laplacian) acting on C^∞-functions on a Riemannian manifold (M,g).

A. STARRING: The Laplacian and the Rayleigh quotient.

1. Let (M,g) be a Riemannian manifold, with Levi-Civita connection D. Given a smooth vector-field X on M, one defines the divergence of X with respect to the Riemannian metric g, as the function $\mathrm{Div}_g X$ (or simply $\mathrm{Div}\, X$) defined by

$$(2) \qquad (\mathrm{Div}_g X)(x) = \mathrm{Trace}\, \{u \to D_u X\}\, (x),$$

where the trace of the endomorphism $u \to D_u X$ is taken in $T_x M$. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M$ $(n = \dim M)$, one can also write

$$(3) \qquad (\mathrm{Div}_g X)(x) = \sum_{i=1}^{n} g(D_i X, e_i).$$

Note. When $\{e_1,\dots,e_n\}$ is an orthonormal basis we use the notation D_i instead of D_{e_i}.

4. The Laplace-Beltrami operator (or Laplacian) acting on C^∞ functions is defined by the formula

$$\Delta^g f = -\mathrm{Div}_g(df^\#)$$

where $df^\#$ is the gradient of f (see nº II.53). We shall also write Δ instead of Δ^g (Notice our sign convention: compare with III.11(c)).

The following proposition gives useful formulas for the Laplacian.

5. Proposition. Let (M,g) be an n-dimensional Riemannian manifold, D its Levi-Civita connection and Δ its Laplacian. Let $\{e_1,\dots,e_n\}$ be a local orthonormal frame near the point p in M. Let f be a C^∞ function on M. The following formulas hold

(i) $\Delta f(p) = -\mathrm{Trace\ Hess}\ f(p)$,

the trace of the bilinear form Hess f(p) in T_pM or equivalently

$$\Delta f(p) = -\sum_{i=1}^{n} Ddf(e_i(p);e_i(p))$$

(see nº II.48);

(ii) $\Delta f(p) = -\sum_{i=1}^{n} \{e_i.(e_i.f) - (D_i e_i).f\}(p)$;

(iii) Let $\{x_1,\dots,x_n\}$ be a local coordinate system centered at p. Let $g_{ij}(x) = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $v = \mathrm{Det}(g_{ij})^{1/2}$. We denote by (g^{ij}) the inverse matrix $(g_{ij})^{-1}$. The local expression of the Laplacian is

$$(\Delta f)(x_1,\dots,x_n) = -[v^{-1} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(g^{ij} v \frac{\partial f}{\partial x_j})](x_1,\dots,x_n);$$

(iv) Let $\{c_1(t),\dots,c_n(t)\}$ denote geodesics such that $c_i(0) = p$ and $c_i'(0) = e_i(p)$.

The function Δf can be calculated at p by the following formula

$$(\Delta f)(p) = -\sum_{i=1}^{n} \left.\frac{d^2}{dt^2}\right|_{t=0} (f\circ c_i)(t);$$

(v) Let p be a fixed point in M and let h be a C^∞ function on $\mathbb{R}_+^{\bullet}$. For $r = d(p,x)$ small enough we assume that the function $x \to h(d(p,x)) = f(x)$ is C^∞ (for $x \neq p$, near p). We can write $x = \exp_p(ru)$ (for r in $\mathbb{R}_+^{\bullet}$ and u in the unit sphere S^{n-1} of T_pM) and $\exp_p^*(v_g) = \theta(r,u)drdu$ (see II.25).

The following formula holds (near p)

$$(\Delta f)(x) = -h''(d(p,x)) - \frac{\theta'(r,u)}{\theta(r,u)} h'(d(p,x)),$$

where h' and h'' are derivatives of h and $\theta'(r,u) = \frac{\partial\theta}{\partial r}(r,u)$.

Proof. The assertions (i) and (ii) follow from the definition of Δ and from the definition of Hess f, see Exercise III. 39(1) and nº II. 48.

(iii) Let $V = v dx_1 \dots dx_n$ be the local volume form which represents the measure v_g in the local coordinate system. A classical result ([B-G-M] Chap. II.G) states that given a vector field X, the Lie derivative $\mathcal{L}_X V$ is just $\mathrm{Div}_g(X)V$ or equivalently, since V is an n-form, $d(i_X V) = \mathrm{Div}_g(X)V$. Writing X as $\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ in the local coordinates, one can deduce that the local expression for $\mathrm{Div}_g X$ is

(6) $$\mathrm{Div}_g X = v^{-1} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(vX_i).$$

The local expression for df is $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$, so that using the duality between TM and T^*M induced by the metric g, we have

$$df^{\#} = \sum_{i,j=1}^{n} (g^{ij} \frac{\partial f}{\partial x_j}) \frac{\partial}{\partial x_i}.$$

The assertion (iii) follows from these computations.

(iv) The local frame $\{e_1,\dots,e_n\}$ can be obtained from $\{e_1(p),\dots,e_n(p)\}$ by parallel translation along the geodesics issued from p. In particular (see nº II.41) we can choose e_i such that $e_i(c_i(t)) = c_i'(t)$. It follows from this choice of $\{e_i\}$ that

(i) $(e_i\cdot(e_i\cdot f))(p) = \frac{d^2}{dt^2} f\circ c_i(t)\Big|_{t=0}$, and

(ii) $(D_{e_i} e_i)(p) = 0$. It suffices to apply Assertion (ii) and the definition of Ddf (Notice that the final result is independent of the choice of the local orthonormal frame $\{e_i\}$).

(v) See [B-G-M] Chap. II.G or [CL]. ■

7. <u>Comments</u>. The definition of Δ^g given in nº 4 shows that Δ^g is invariantly defined on (M,g) and that Δ^g is a Riemannian invariant. Proposition 5(iii) shows that Δ is a 2nd order linear differential operator whose leading terms are $-\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. The function $\xi \to g^*(\xi,\xi)$, whose expression in local coordinates is $\sum_{i,j=1}^{n} g^{ij} \xi_i \xi_j$ is well defined on T^*M and is called the <u>principal symbol</u> of the operator Δ. For x in M

the principal symbol maps T_x^*M into $\mathbb{R}_+$ by $\xi \to g_x^*(\xi,\xi)$. It follows that Δ is elliptic (see [G-T] and [NN] Chap. 3); this fact will be very important in the sequel. Proposition 5(v) shows that the Laplacian Δ is strongly related to the Ricci curvature through $\theta(r,u)$ (see nº II.(26)). We shall use this property later on. Some of the formulas in Proposition 5 can be deduced from the fact that Δ is the Friedrichs extension of the quadratic form $u \to \int |\nabla u|^2$ ([R-S] Vol. II).

8. Let (M,g) be a Riemannian manifold with boundary. The boundary ∂M of M is a Riemannian manifold with the induced metric $g|\partial M$. We use the following notations

a) $\overset{\circ}{M} = M \setminus \partial M$ (the interior of M);

b) a_g the Riemannian measure on $(\partial M, g|\partial M)$;

c) ν the unit normal vector-field on ∂M, pointing inward.

FROM NOW ON ALL RIEMANNIAN MANIFOLDS WILL BE ASSUMED TO BE COMPACT unless otherwise stated.

The following theorems are standard (see [LG] p.204)

9. Divergence Theorem. Let X be a C^∞ vector-field on M. Then

$$\int_M (\mathrm{Div}_g X)(x)\,dv_g(x) = -\int_{\partial M} g(X,\nu)(x)\,da_g(x) = -\int_{\partial M} \langle X,\nu\rangle(x)\,da_g(x).$$

10. Green's Theorem. Let f,h be C^∞ functions on M. Then

(i) $$\int_M \{h(x)\Delta f(x) - g(\nabla h,\nabla f)(x)\}dv_g(x) = \int_{\partial M} h(x)(\nu.f)(x)da_g(x)$$

(ii) $$\int_M \{h(x)\Delta f(x) - f(x)\Delta h(x)\}dv_g(x) =$$

$$= \int_{\partial M} \{h(x)(\nu.f)(x) - f(x)(\nu.h)(x)\}da_g(x)$$

where $\nabla f = df^{\#}$ is the gradient of f with respect to the Riemannian metric g on M (see nº II.53).

11. Remarks.

(a) In the sequel, we will simply write, e.g. for (ii)

$$\int_M (h\Delta f - f\Delta h)dv_g = \int_{\partial M} \{h(\nu.f)-f(\nu.h)\}da_g.$$

(b) Both theorems are true under more general assumptions: M could be non-compact, provided that the integrations be in fact performed on compact sets (e.g. f, and X with compact supports...); one can also weaken the regularity assumptions on X, h, f (e.g. Theorem 9 works for X a C^1-vector-field...), or on ∂M (∂M might only be piece-wise smooth).

(c) In order to make things clear let us insist that our Laplacian is written $-f''$ on $\mathbb{R}$ and that our normal ν points inward.

Before we go any further with the study of the Laplacian, let us introduce some basic objects (compare with Chapter I nº43ff).

12. We denote by $L^2(M,v_g)$ or simply $L^2(M)$, the space of measurable functions f on M such that $\int_M |f(x)|^2 dv_g < +\infty$. This

space is a Hilbert space with inner product $(f|h)_o = \int_M fh\,dv_g$, and norm $\|f\|_o = (f|f)_o^{1/2}$ (we shall mainly deal with real-valued functions; when dealing with complex-valued functions we shall use $(f|h)_o = \int_M f\bar{h}\,dv_g$ as inner product).

We denote by $C_o^\infty(M)$ the set of C^∞ functions on M, with compact support in $\mathring{M}$.

We define a norm on $C^\infty(M)$ by

$$\|f\|_1 = \{\int_M |f(x)|^2 dv_g(x) + \int_M |df|^2(x) dv_g(x)\}^{1/2}$$

where $|df|^2(x)$ is the square of the norm of the 1-form $df(x)$ in T_x^*M i.e. $|df|^2(x) = g_x^*(df(x),df(x)) = g_x(\nabla f(x),\nabla f(x))$.

We shall now use the notation $\langle .|.\rangle_x$ or $\langle .|.\rangle$ for the natural scalar products on tensor products above the point x in M (see nº II.37) and $\langle\langle .|.\rangle\rangle$ for the integrated inner product. For example

$$\langle df|df\rangle_x = g_x^*(df(x),df(x)), \quad \text{and}$$

$$\langle\langle df|df\rangle\rangle = \int_M \langle df|df\rangle_x \, dv_g(x).$$

The norm $\|.\|_1$ is associated with the inner product

$$(f|g)_1 = (f|g)_o + \langle\langle df|dg\rangle\rangle$$

on $C^\infty(M)$.

13. Let us recall that $C^\infty(M)$ and $C_o^\infty(M)$ are dense in $L^2(M,v_g)$ for the norm $\|.\|_o$. The closure of $C^\infty(M)$ (resp. $C_o^\infty(M)$) in $L^2(M,v_g)$ for the norm $\|.\|_1$ will be denoted by $H^1(M,g)$ (resp. $H_o^1(M,g)$) or

simply $H^1(M)$ (resp. $H^1_o(M)$). The following inclusions are continuous (with the natural norms)

$$H^1_o(M) \subset H^1(M) \subset L^2(M).$$

These spaces are called Sobolev spaces (see [G-T]).

Let us point out that whereas $L^2(M,v_g)$ only depends on the measure v_g (M being a fixed manifold), $H^1(M,g)$ and $H^1_o(M,g)$ depend on the Riemannian metric itself since $\langle\langle df|dg\rangle\rangle$ does.[(1)]

The elements in $H^1(M)$ are L^2-functions on M, with first derivatives (in the sense of distributions) in $L^2(M)$ (see [G-T] p. 142). If $\partial M = \emptyset$ then $H^1_o(M) \equiv H^1(M)$.

The following theorem is standard ([G-T] p. 160 or [NN] Chap. 3).

14. **Theorem.** *Let (M,g) be a smooth compact Riemannian manifold with boundary (possibly empty). The inclusion maps*

(i) $(H^1_o(M,g), \|.\|_1) \to (L^2(M,v_g), \|.\|_o)$, *and*

(ii) $(H^1(M,g), \|.\|_1) \to (L^2(M,v_g), \|.\|_o)$

are compact (or completely continuous): the image in L^2 of a bounded set in H^1 or H^1_o is relatively compact.

15. **Remark.** The theorem remains true under weaker assumptions on the regularity of ∂M; however the second assertion might be false if ∂M is too irregular.

---

(1) Here, we consider H^1 or H^1_o equipped with the norm $\|.\|_1$, and not only the spaces of functions, which do not depend on g because M is compact.

B. <u>EIGENVALUE PROBLEMS ON RIEMANNIAN MANIFOLDS I</u>.

In these notes we shall be interested in the following eigenvalue problems:

(16C) (M,g) is a compact Riemannian manifold without boundary,

$\Delta u = \lambda u$ (<u>closed eigenvalue problem</u>);

(16D) (M,g) is a compact Riemannian manifold with boundary,

$$\begin{cases} \Delta u = \lambda u & \text{in } \mathring{M}, \\ u = 0 & \text{on } \partial M, \end{cases}$$ (<u>Dirichlet eigenvalue problem</u>);

(16N) (M,g) is a compact Riemannian manifold with boundary,

$$\begin{cases} \Delta u = \lambda u & \text{in } \mathring{M}, \\ \nu . u = 0 & \text{on } \partial M, \end{cases}$$ (<u>Neumann eigenvalue problem</u>),

where ν is the unit normal vector-field on ∂M, pointing inward (see nº 8).

This means that given the compact Riemannian manifold (M,g), we look for all numbers λ for which there exists a nontrivial solution u in $C^\infty(M)$ of the (boundary value) Problem (16C), (16D) or (16N).

Notice that if u is a nontrivial solution of one of the Problems (16), then the corresponding number λ must be a nonnegative real number (apply Green's Theorem 10(i) with $h = f = u$).

17. The numbers λ for which (16*) has a nontrivial solution u in $C^\infty(M)$ are called the <u>eigenvalues</u> of problem (*), * = C,D or N. The corresponding functions u, called the <u>eigenfunctions</u> of problem (*) associated with the eigenvalue λ, form a vectorspace whose dimension is called the <u>multiplicity</u> of the eigenvalue λ.

18. Theorem. *Let* (M,g) *be a compact Riemannian manifold and let* $(*)$ *be one of the eigenvalue problems* (C),(D) *or* (N) *of* nº 16.

(i) *The set of eigenvalues of problem* $(*)$ *consists of an* infinite sequence $(0\le)\ \bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 < \dots \uparrow +\infty$;

(ii) *Each eigenvalue* $\bar{\lambda}_i$ *has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are* $L^2(M,v_g)$ *- orthogonal*;

(iii) *The direct sum of the eigenspaces* $E(\bar{\lambda}_i)$ $i = 1,2,\dots,$ *is dense in* $L^2(M,v_g)$ *for the* L^2*-norm topology and dense in* $C^k(M)$ *for the uniform* C^k*-topology*, $k = 0,1,2,\dots$.

19. Notations. From now on, we will list the eigenvalues of Problem $(*)$ as $(0\le)\ \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \uparrow +\infty$

with each eigenvalue repeated a number of times equal to its multiplicity. If necessary we will write

$$\lambda_i,\ \lambda_i(M,g,*),\ \lambda_i(M,g) \text{ or } \lambda_i(*)$$

to point out the dependence on the manifold (M,g), the eigenvalue problem $*$ which is considered, or both the manifold and the eigenvalue problem.

To the sequence $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ formed by the eigenvalues of Problem $(*)$, one can associate an orthonormal family $\phi_1, \phi_2, \dots$ of eigenfunctions such that ϕ_i satisfies the eigenvalue Problem (16*) with $\lambda = \lambda_i$, $* = C,D,N$.

The third assertion in Theorem 18 shows that the sequence $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis of $L^2(M,v_g)$. For any f in $L^2(M)$, one can write

$$f = \sum_{i=1}^{\infty} (f|\phi_i)_o \phi_i \quad \text{in } L^2\text{-sense, and} \quad \|f\|_o^2 = \sum_{i=1}^{\infty} (f|\phi_i)_o^2 .$$

Let us sketch two possible proofs for Theorem 18.

20. Let D_*, $* = C,D,N$, denote the following subspaces of $C^\infty(M)$, which are dense in $L^2(M)$

$$D_C = C^\infty(M)$$

$$D_D = \{f \text{ in } C^\infty(M) \mid f = 0 \text{ on } \partial M\}$$

$$D_N = \{f \text{ in } C^\infty(M) \mid \nu.f = 0 \text{ on } \partial M\}.$$

In order to study the eigenvalue Problem (16*), one is led to view the Laplacian Δ as an unbounded operator in $L^2(M)$, with domain D_*. It follows from Green's Theorem 10 that Δ is

<u>Symmetric</u> i.e. for any f,h in D_*,

$$(f|\Delta h)_o = (\Delta f|h)_o;$$

<u>Positive</u> i.e. for any f in D_*,

$$(\Delta f|f)_o = \langle\langle \nabla f|\nabla f\rangle\rangle \geq 0.$$

A classical theorem in spectral theory ([R-S] Chap. X or [TR1] Part 3 and [TR2] Section 3) states that (D_*,Δ) has a unique extension $(\mathcal{E}_*,\Delta_*)$ as an unbounded self-adjoint operator in $L^2(M,v_g)$. The vector-space $\mathcal{E}_*$ is contained in the subset of functions in $L^2(M)$, whose derivatives (in the sense of distributions) up to order 2 are in $L^2(M)$. The operator Δ_* is then Δ viewed on $\mathcal{E}_*$ as a differential operator acting on distributions. It must be pointed out that $(\mathcal{E}_D,\Delta_D)$ and $(\mathcal{E}_N,\Delta_N)$ are quite different operators: they contain both the Laplacian Δ as a differential operator acting on distributions and the boundary conditions (Dirichlet or Neumann). The positivy of Δ implies that $(\mathcal{E}_*,\Delta_*)$ is a positive self-adjoint operator which in turn implies that the <u>spectrum</u> of $(\mathcal{E}_*,\Delta_*)$ is contained in $\mathbb{R}_+$. The compactness and regularity assumptions on

M imply that the inclusion $\mathcal{E}_* \to L^2(M)$ is compact. It follows that for $\lambda \notin \mathbb{R}_+$, the resolvent $(\Delta_* - \lambda)^{-1}$ is a compact operator in $L^2(M)$. Theorem 18 follows from the classical results on the spectral theory of compact operators and from the fact that the Laplacian Δ is an elliptic differential operator (for more details see [TR1] and [TR2] or [AN], [R-S] Vol. II).

21. Instead of looking at the Laplacian Δ, one can consider the Dirichlet integral or the Rayleigh (-Ritz) quotient: for u in $C^\infty(M)$ we define

$$E(u) = \int_M |du|^2 v_g \qquad \text{(Dirichlet or energy integral);}$$

$$R(u) = \int_M |du|^2 v_g \Big/ \int_M u^2 v_g \qquad \text{(Rayleigh quotient),}$$

where $\int_M u^2 v_g \neq 0$. For motivations see Chap. I, where we pointed out the two points of view: operator vs quadratic form.

Both $E(u)$ and $R(u)$ are defined on $H^1_*(M)$ (see nº 13).

In order to prove Theorem 18, one considers the extrema of $R(u)$ on $H^1_*(M)$ or equivalently on $C^\infty_*(M)$, where

$$H^1_C(M) = H^1(M), \quad C^\infty_C(M) = C^\infty(M);$$

$$H^1_D(M) = H^1_o(M), \quad C^\infty_D(M) = C^\infty_o(M);$$

$$H^1_N(M) = H^1(M), \quad C^\infty_N(M) = C^\infty(M);$$

denote the sets of admissible functions (see Chapter I, nº I.9-11) respectively for the Closed, Dirichlet or Neumann Eigenvalue problems.

Let $\mu_1^* = \inf\{R(u) : u \in H^1_*(M), \int_M u^2 \neq 0\}$.

This infimum exists because $R(u)$ is non-negative for all u. Let $\{u_n\}_1^\infty$ be a sequence in $H_*^1(M)$, normalized by $\int_M u_n^2 = 1$, such that $\mu_1^* \le R(u_n) \le \mu_1^* + \frac{1}{n}$.

From the definition of $R(u)$ we deduce that, for all n, $\|u_n\|_1^2 \le \mu_1^* + 2$. The sequence $\{u_n\}$ being a bounded sequence in the Hilbert space $H_*^1(M)$, we can find a weakly convergent subsequence $\{u_{1,n}\}$ in $H_*^1(M)$ with weak limit v in $H_*^1(M)$. This subsequence being bounded in $H_*^1(M)$, its image $\{u_{1,n}\}$ in $L^2(M)$ is relatively compact and hence one can find a subsequence $\{v_n = u_{2,n}\}$ which converges weakly to v in $H_*^1(M)$ and <u>strongly</u> to an element u in $L^2(M)$. Since $\|v_n\|_o = 1$, we have $\|u\|_o = 1$. Since strong convergence implies weak convergence, $\{v_n\}$ converges weakly to u in $L^2(M)$. The inclusion $H_*^1(M) \to L^2(M)$ being continuous, the $H_*^1(M)$ weak convergence of $\{v_n\}$ to v implies the $L^2(M)$-weak convergence of $\{v_n\}$ to v (viewed as an element of $L^2(M)$) and hence $u = v$.

Finally we have proved that $\{v_n\}$ converges $H_*^1(M)$-weakly and $L^2(M)$-strongly to an element v in $H_*^1(M)$ such that $\|v\|_o = 1$.

From Cauchy-Schwarz inequality, we deduce that for any f in $H_*^1(M)$

$$(v_n|f)_1^2 \le \|v_n\|_1^2 \, \|f\|_1^2 \le (\mu_1^* + 1 + \tfrac{1}{n})\|f\|_1^2.$$

It follows that

$$(v|f)_1^2 \le (\mu_1^*+1)\|f\|_1^2, \quad \text{and taking} \quad f = v, \ R(v) \le \mu_1^*.$$

Since $R(v) \ge \mu_1^*$, by definition of μ_1^*, we conclude that the infimum μ_1^* of $R(u)$ on H_*^1 is achieved.

Let E_1 be the set of all elements v in H_*^1 such that $v = 0$ or $v \ne 0$ and $R(v) = \mu_1^*$. Let $v \in E_1$. For any u in

$H_*^1(M)$ and t small enough in $\mathbb{R}$, we have $R(v+tu) \geq R(v) = \mu_1^*$. Writing that the derivative at $t = 0$ of the function $t \to R(v+tu)$ is zero, we have the following characterization of E_1

(22) $\quad v \in E_1 \Leftrightarrow \quad \forall\, u \in H_*^1, \quad (u|v)_1 = (\mu_1^*+1)(u|v)_o.$

From this characterization one can conclude that E_1 is a vector space. From the fact that the $\|.\|_1$-norm and $\|.\|_o$-norm are proportional on E_1, we conclude (Theorem 14) that the unit-$\|.\|_o$-ball of E_1 is compact and hence that E_1 is finite dimensional (for the elementary functional analysis we used above see [BZ]).

23. Summary. The infimum

$$\mu_1^* = \inf\{R(u) \mid u \in H_*^1(M), u \neq 0\}$$

is achieved on a finite dimensional subspace E_1 of $H_*^1(M)$ which is characterized by (22).

Given u in $H_*^1(M)$, we also denote by ∇u the gradient of u in the sense of distributions (in a local coordinate system $\nabla u = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij} \frac{\partial u}{\partial x_j}\right) \frac{\partial}{\partial x_i}$ where $\frac{\partial u}{\partial x_j}$ are derivatives in the sense of distributions).

The fact that u belongs to $H_*^1(M)$ means that $|\nabla u|$ belongs to $L^2(M)$. Formula (22) can be written as follows: for any u in E_1 and any f in $H_*^1(M)$,

$$\int_M \langle \nabla u | \nabla f \rangle v_g = \mu_1^* \int_M u f\, v_g$$

which we can state as (see I.43-45)

"any element u in E_1 is a weak solution of the eigenvalue problem (16*)"

(The boundary conditions are taken into account through $H_*^1(M)$). The classical regularity theory of weak solutions of elliptic problems ([G-T] or [TR2]) shows that E_1 is in fact contained in $C^\infty(M)$. Green's Theorem 10 finally shows that u is in fact a classical solution of the eigenvalue Problem (16*): if u is in $H^1_D \cap C^\infty$, then $u = 0$ on ∂M and hence u satisfies (16D); if u is in $H^1_N \cap C^\infty$, then $\Delta u = \lambda u$ in $\mathring{M}$ (take f in $C^\infty_o(M)$) and $\int_{\partial M} (\nu.u)f\, da_g = 0$ which implies that $\nu.u = 0$ on ∂M (take f in $C^\infty(M)$: compare with nº I.9 ff).

24. So far we have proved the existence of the first eigenvalue and its finite dimensional eigenspace. Let us denote by L_1 (resp. H_1) the subspace of $L^2(M)$ (resp. $H^1_*(M)$) which is orthogonal to E_1. Formula (22) shows that $H_1 \subset L_1$. These spaces are closed in $L^2(M)$ and $H^1_*(M)$ respectively and the inclusion $H_1 \subset L_1$ is compact.

We now define

$$\mu_2^* = \inf\{R(u) \mid u \in H_1,\ u \neq 0\}.$$

Following the same arguments as those used above, we can prove that μ_2^* is indeed achieved on a finite dimensional subspace E_2 of H_1 which is characterized by

$$u \in E_2 \Leftrightarrow \text{for any } v \in H_1,\ (u|v)_1 = (\mu_2^*+1)(u|v)_o.$$

Noticing that the right-hand side equality holds trivially for v in E_1, we deduce that E_2 is characterized by a formula analogous to (22) (change μ_1^* to μ_2^* in (22)). It is also clear that $\mu_2^* > \mu_1^*$. We can construct an increasing sequence of non-negative real numbers

$$\mu_1^* < \mu_2^* < \dots$$

and a sequence of associated finite dimensional subspaces of $H^1_*(M)$ which are mutually orthogonal (in $H^1_*(M)$ and in $L^2_*(M)$)

$$E_1, E_2, \ldots$$

Due to elliptic regularity theory, the functions in the E_i's are C^∞ and satisfy the eigenvalue Problem (16*). Notice that these sequences are infinite because $H^1_*(M)$ is infinite dimensional and the E_i's are finite dimensional.

The sequence $\{u^*_i\}$ either increases to infinity or is bounded. If it were bounded by some number, we would have an infinite sequence $\{\phi_i\}$ of orthonormal functions in $L^2(M)$ (take L^2-orthonormal bases in the E_j's) satisfying $R(\phi_i) \leq u$ and hence $\|\phi_i\|_1 \leq u+1$, in $H^1_*(M)$.

This is not possible because the inclusion $H^1(M) \to L^2(M)$ is compact.

Let E denote the closure in $H^1_*(M)$ of the vector-space spanned by the vectors in the E_j's. Assume $E \neq H^1_*(M)$. We can then find a function u in $H^1_*(M)$, orthogonal to all the E_j's in $H^1_*(M)$ or equivalently in $L^2(M)$ (because of (22)). It follows that $R(u) \geq u^*_i$ and $(u|u)_1 \geq (u^*_i+1)\ (u|u)_o$ for all i, which is impossible because u^*_i tends to infinity. It follows that $\oplus E_i$ is dense in $H^1_*(M)$ and hence in $L^2(M)$. The other assertions in Theorem 18 follow from elliptic regularity results.

For more details see [SW] Chap. III.

25. Given f in $H^1_*(M)$, one can write

$$R(f) = \sum_{i=1}^{\infty} \lambda_i a_i^2 \Big/ \sum_{i=1}^{\infty} a_i^2$$

where $a_i = (f|\phi_i)_o$, $f \neq 0$ (see nº 19).

This expression of $R(f)$ justifies a posteriori the second proof we sketched for Theorem 18; it also proves the following characterization of the eigenvalues and eigenfunctions of Problem (16*).

26. Variational Characterization I. (Notations as in nº 19)

The k^{th} eigenvalue (with multiplicities) λ_k is characterized by

$$\lambda_k = \inf\{R(u) \mid u \neq 0,\ u\ L^2\text{-orthogonal to } \phi_1, \ldots, \phi_{k-1}\}$$

where u is taken in $H_*^1(M)$ or in $C_*^\infty(M)$. Furthermore, if u in $H_*^1(M)$ is L^2-orthogonal to $\phi_1, \ldots, \phi_{k-1}$, and $R(u) = \lambda_k$ then u is an eigenfunction of Problem (16*) associated with the eigenvalue λ_k.

Let us consider $\lambda_1 = \inf\{R(u) \mid u$ in $C_*^\infty(M),\ u \neq 0\}$. If we know enough functions u on which we can calculate $R(u)$, then we know an upperbound for λ_1. This will turn out to be very important in the future. However, if we want upper bounds on λ_2 instead of λ_1, we have to know the eigenfunction ϕ_1 and take u L^2-orthogonal to ϕ_1. Things are even more complicated with λ_k, $k \geq 3$. The following characterizations deal with these difficulties.

27. Variational Characterization II

The following variational characterization holds

$$\lambda_k = \sup_{M_{k-1}} \inf\{R(u) \mid u\ L^2\text{-orthogonal to } M_{k-1},\ u \neq 0\},$$

where M_{k-1} runs through $(k-1)$-dimensional subspaces of $H_*^1(M)$ or $C_*^\infty(M)$.

Proof. Let $\Lambda(M_{k-1}) = \inf\{R(u) \mid u\ L^2\text{-orthogonal to } M_{k-1},\ u \neq 0\}$. Take $M^o_{k-1} = [\phi_1,\dots,\phi_{k-1}]$ the vector-space spanned by the eigengunctions $\phi_1,\dots,\phi_{k-1}$. Then $\Lambda(M^o_{k-1}) = \lambda_k$ according to the first variational characterization. This implies that

$$\sup_{M_{k-1}} \Lambda(M_{k-1}) \geq \lambda_k .$$

It is easy to show (by an argument on dimensions) that given a subspace M_{k-1}, one can find an element v in $M^o_k = [\phi_1,\dots,\phi_k]$ such that $v \neq 0$ and v L^2-orthogonal to M_{k-1}. For such a v one has $R(v) \leq \lambda_k$ and hence $\Lambda(M_{k-1}) \leq \lambda_k$. This implies that $\sup_{M_{k-1}} \Lambda(M_{k-1}) \leq \lambda_k$ ∎

28. *Variational Characterization III*. (Notations as in nº 19)

The k^{th} *eigenvalue (with multiplicities)* λ_k *is characterized by*

$$\lambda_k = \inf_{L_k} \sup\{R(u) \mid u \text{ in } L_k,\ u \neq 0\}$$

where L_k *runs through* k-*dimensional subspaces of* $H^1_*(M)$ *or* $C^\infty_*(M)$.

Proof. Taking $L_k = [\phi_1,\dots,\phi_k]$ the vector-space spanned by the eigenfunctions $\phi_1,\dots,\phi_k$, we find that

$$\Lambda_k = \inf_{L_k} \sup\{R(u) \mid u \text{ in } L_k,\ u \neq 0\} \text{ satisfies } \Lambda_k \leq \lambda_k .$$

Let L_k be a k-dimensional subspace in $H^1_*(M)$ or $C^\infty_*(M)$. Then there exists an element u in L_k such that u is orthogonal to $\phi_1,\dots,\phi_{k-1}$. It follows from nº 25 that $R(u) = \sum_{j=k}^{\infty} \lambda_j a_j^2 / \sum_{j=k}^{\infty} a_j^2$ where $a_j = (u|\phi_j)_o$ and hence that $R(u) \geq \lambda_k$. We then deduce that $\Lambda_k \geq \lambda_k$ ∎

For other characterizations see [BE] Chap. III.

The following proposition turns out to be useful when one wants to determine explicitely the eigenvalues and eigenfunctions of one of the Problems (16*).

29. Proposition

Let (M,g) be a Riemannian manifold. Let $\{V_i\}_{i=1}^{\infty}$ be a sequence of non-trivial subspaces of $D_*(M)$ (see nº 20) with the following properties

(i) For all $i \geq 1$, there exists a real number μ_i such that, for all f in V_i, $\Delta f = \mu_i f$;

(ii) The sum $\sum_{i=1}^{\infty} V_i$ (finite linear combinations of elements in the V_i's) is dense in $L^2(M,v_g)$ for $\|.\|_o$.

Then the sequence $\{\mu_i\}$ is the sequence of eigenvalues of Problem (16*), up to increasing rearrangement, and the V_i's are the associated eigenspaces.

Proof. Exercise 31(2).

One can give an analogous statement at the level of Rayleigh quotient.

30. Proposition

Let (M,g) be a Riemannian manifold and let $\{V_i\}_{i=1}^{\infty}$ be a sequence of non-trivial subspaces of $H_*^1(M)$ with the following properties

(i) For all $i \geq 1$ there exists a real number μ_i such that for all u in V_i and all v in $H_*^1(M)$

$$(u|v)_1 = (\mu_i+1)(u|v)_o,$$

(ii) $\sum_{i=1}^{\infty} V_i$ is dense in $L^2(M)$ for $\|.\|_o$.

Then the sequence $\{\mu_i\}$ is the sequence of eigenvalues of the Rayleigh quotient in $H^1_*(M)$, up to increasing rearrangement, and the V_i's are the associated eigenspaces.

Proof. Take (μ^*_i, E_i) as in the proof of Theorem 18.

For u in V_i and v in E_j we can write

$$(u|v)_1 = (\mu_i+1)(u|v)_o, \quad \text{and (formula (22))}$$

$$(u|v)_1 = (\mu^*_j+1)(u|v)_o.$$

We then conclude that either $(u|v)_o = 0$ or $\mu_i = \mu^*_j$. This fact and Assertion (ii) show that the sequences $\{\mu_i\}$ and $\{\mu^*_i\}$ are equal. The characterization of the E_i's (Formula (22)) shows that (up to an increasing rearrangement on the μ_i's) $V_i \subset E_i$. Assertion (ii) shows that $V_i = E_i$, because V_i is orthogonal to E_j, $j \neq i$. ∎

31. Exercises

(a) Prove Proposition 29;

(b) Let (M,g) (resp. (N,h)) be a Riemannian manifold without boundary, with eigenvalues $\bar{\lambda}^M_i$ (resp. $\bar{\lambda}^N_i$) and eigenspaces E^M_i (resp. E^N_i).

Find the eigenvalues and eigenspaces of the product Riemannian manifold $(M \times N, \ g \times h)$;

(c) Let $(N,h) \xrightarrow{p} (M,g)$ be a finite (Riemannian) covering i.e. $N \xrightarrow{p} M$ is a finite covering of manifolds without boundaries and $p^*g = h$.

Describe the eigenfunctions of (M,g) in terms of the eigenfunctions of (N,h) (see [B-G-M] Chap. III Prop. A.II.5 p.145);

(d) Let Ω be a smooth bounded domain in $(\mathbb{R}^n, \text{can})$.

Show that for all i (notations in nº 19)

$$\lambda_i(\Omega,N) \leq \lambda_i(\Omega,D)$$

(Hint: use nº 26-28);

(e) Let $\Omega_1 \subset \Omega_2$ be two smooth bounded domains in $(\mathbb{R}^n, \text{can})$. Show that for all i (Notations in nº 19)

$$\lambda_i(\Omega_1,D) \geq \lambda_i(\Omega_2,D)$$

(Hint: use nº 26-28);

(f) Let $\Omega_{a,b}$ be a rectangle with sides a and b in $(\mathbb{R}^2, \text{can})$. Find the eigenvalues and eigenfunctions of Problems (16D) and (16N) in $\Omega_{a,b}$ (Hint: use separation of variables and Proposition 29). When are all the eigenvalues Problem (16D) or Problem (16N) in $\Omega_{a,b}$ simple?

(g) Give upper and lower bounds for $\lambda_1(B(0,1),D)$ where $B(0,1)$ is the unit ball in $(\mathbb{R}^n, \text{can})$ (Hint: use nº 26-28 and generalize Exercise (f));

C. EIGENVALUE PROBLEMS ON RIEMANNIAN MANIFOLDS II.

32. Much research effort has been devoted to eigenvalue problems since the 18th century. These problems arise from (linear) mathematical models for questions in mathematical physics: acoustics, elasticity, plasma physics, spectroscopy, wave guides... These problems can be roughly divided into two types: direct problems and inverse problems.

In <u>a direct problem</u>, one seeks information about the eigenvalues and the eigenfunctions of the Laplacian Δ^g in terms of geometrical data. It turns out that it is usually not possible to determine explicitely the eigenvalues or the eigenfunctions (except when symmetries allow to reduce the original problem to one-dimensional eigenvalue problems: see Exercise 30(f) and [CL] Chap.II). Important progress have been made in the field of high-speed computers which allow reliable numerical computations of eigenvalues. Although these numerical methods are now used extensively, they do not discard theoretical investigations on the eigenvalues and eigenfunctions (see [K-S]).

A very important theoretical problem consists in finding bounds on the eigenvalues. Due to the variational characterizations of eigenvalues (nº 26-28) it is easier to obtain upper bounds than lower bounds. It turns out that lower bounds are more interesting from both the mathematical and physical points of view. For example, lower bounds determine safety limits of some mechanical systems in order to avoid buckling: rods, plates, beams.

The most powerful methods which have been developped in order to obtain bounds on the eigenvalues are the <u>isoperimetric methods</u>. These methods owe very much to the works of G. Polya and G. Szegö in the 40s (see [P-S] or [PE]); we will study them in Chapters IV-V of these notes.

In <u>an inverse problem</u>, one assumes that one or several eigenvalues of the Laplacian Δ^g are known and one seeks information on the metric g: curvature, form (i.e. topology) of the manifold.... Let us quote Sir A. Schuster (1882) who created the word <u>spectroscopy</u>: "to find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certains cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction" (quoted in [G-S] Introduction p.8).

We will not deal with inverse problems in these notes but for a brief survey (Chap. VII).

However, we shall be interested in an <u>inverse geometric problem</u>; one important question in Riemannian geometry is to determine the <u>global influence</u> on the manifold of <u>(local) estimates</u> on the curvature. Theorem II.29 (Myers theorem) gives a partial answer to this question in dimension n; the Gauss-Bonnet theorem also gives a partial answer in the two-dimensional case ([HF] Theorem III p.113). Chapter VI is devoted to an analytic approach to the above question. In Chapter V-VI, we shall show how local estimates on the curvature (after scaling the metric appropriately) imply bounds on geometric invariants such as the Betti numbers of the manifold.

Let us now give two examples, one example of an inverse problem (H. Weyl's asymptotic formula) and one example of a direct problem (S.Y. Cheng's upper bounds on the eigenvalues).

33. Let Ω be a smooth bounded domain in $(\mathbb{R}^n, \text{can})$ i.e. with the usual Euclidean structure and Laplacian. We consider the Dirichlet eigenvalue problem (16D) in Ω

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us consider a grid with size a in $\mathbb{R}^n$ i.e. the pattern of cubes in $\mathbb{R}^n$ made by a lattice $(a\mathbb{Z})^n$ centered at an interior point in Ω: see Figure 3.

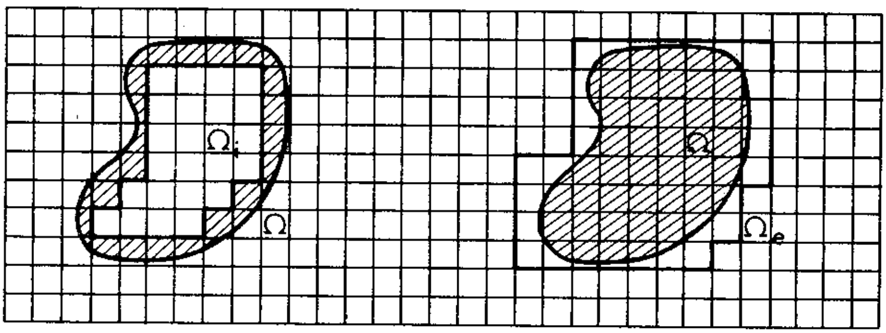

Fig.3

Let us call $\mathcal{C}_i$ (resp. $\mathcal{C}_e$) the collection of all cubes of the grid which are contained in Ω (resp. which contain a point lying in Ω).

Let $\mathcal{E}_i$ (resp. $\mathcal{E}_e$) be the collection of all eigenvalues (with multiplicities) of all the cubes in $\mathcal{C}_i$ (resp. $\mathcal{C}_e$) with Dirichlet (resp. Neumann) boundary condition: an eigenvalue which appears for two different cubes should be counted twice. Arrange the sets $\mathcal{E}_i$ and $\mathcal{E}_e$ in increasing sequences $\{\mu_j^{(i)}\}_{j=1}^{\infty}$ and $\{\mu_j^{(e)}\}_{j=1}^{\infty}$.

Denote by $\{\lambda_j\}_{j=1}^{\infty}$ the increasing sequence of the eigenvalues of the Dirichlet problem in Ω.

34. Proposition. For $j \geq 1$ the following inequalities hold

$$\mu_j^{(e)} \leq \lambda_j \leq \mu_j^{(i)}.$$

Proof. Denote the generic cube of the grid by C. Let

$$L_i = \oplus_{C \in \mathcal{C}_i} L^2(C), \quad L_e = \oplus_{C \in \mathcal{C}_e} L^2(C)$$

$$H_i = \oplus_{C \in \mathcal{C}_i} H_o^1(C), \quad H_e = \oplus_{C \in \mathcal{C}_e} H^1(C)$$

We can view the (open) cubes as disjoint manifolds. We then have two manifolds $M_i = \coprod_{C \in \mathcal{C}_i} C$, $M_e = \coprod_{C \in \mathcal{C}_e} C$

(disjoint unions).

In that case, we also have

$$L_i = L^2(M_i), \; H_i = H_o^1(M_i), \; L_e = L^2(M_e), \; H_e = H^1(M_e).$$

Proposition 31 shows that $\{\mu_j^{(i)}\}$ (resp. $\{\mu_j^{(e)}\}$) is the sequence of the eigenvalues of the Rayleigh quotient on $H_o^1(M_i)$ (resp. $H^1(M_e)$). In particular, we have the following variational characterization

$$\mu_k^{(i)} = \inf_{L_k} \sup\{R(u) \mid u \text{ in } L_k, \; u \neq 0\}$$ (resp. $\mu_k^{(e)}$), where L_k runs through the k-dimensional subspaces of H_i (resp. H_e).

Let $\Omega_i = \bigcup_{C \in \mathcal{C}_i} \bar{C}$, $\Omega_e = \bigcup_{C \in \mathcal{C}_e} \bar{C}$. It is clear that $\Omega_i \subset \Omega \subset \Omega_e$.

Since $H_i \subset H_o^1(\Omega_i) \subset H_o^1(\Omega)$ we have

$$\mu_k^{(i)} \geq \lambda_k(\Omega_i, \text{Dirichlet}) \geq \lambda_k(\Omega, \text{Dirichlet}).$$

Since $H_o^1(\Omega) \subset H^1(\Omega_e) \subset H_e$ we have

$$\lambda_k = \lambda_k(\Omega, \text{Dirichlet}) \geq \lambda_k(\Omega_e, \text{Neumann}) \geq \mu_k^{(e)}.$$

Finally, we obtain $\mu_k^{(e)} \leq \lambda_k \leq \mu_k^{(i)}$ ∎

35. Let $N(\lambda) = \text{Card}\{j \mid \lambda_j \leq \lambda\}$,

$$N^{(e)}(\lambda) = \text{Card}\{j \mid \mu_j^{(e)} \leq \lambda\},$$

$$N^{(e)}(\lambda) = \text{Card}\{j \mid \mu_j^{(i)} \leq \lambda\}.$$

<u>Lemma</u>. <u>When</u> λ <u>goes to infinity the following limits hold</u>

$$\lim \lambda^{-n/2} N^{(e)}(\lambda) = c(n) \text{ Vol}(\Omega_e)$$

$$\lim \lambda^{-n/2} N^{(i)}(\lambda) = c(n) \text{ Vol}(\Omega_i)$$

<u>where</u> $c(n) = (2\pi)^{-n}$.Vol unit ball in $\mathbb{R}^n$.

<u>Proof</u>. The eigenvalues of a cube with side a are of the form

$$\frac{\pi^2}{a^2} \sum_{\ell=1}^{n} k_\ell^2 \quad \text{where}$$

$$k_\ell \in \mathbb{N} \text{ (Neumann)} \quad \text{or} \quad k_\ell \in \mathbb{N} \setminus \{0\} \quad \text{(Dirichlet)}$$

Hint: use separation of variables and Theorem 30.

Counting the eigenvalues less than λ amounts to counting the points with positive integer coordinates inside the ball of radius $\frac{a}{\pi}\sqrt{\lambda}$. As λ goes to infin..ty the equivalent for the number

of such points is $(2\pi)^{-n}$(Vol unit ball in $\mathbb{R}^n$) $a^n \lambda^{n/2}$ (2^{-n} appears because we only consider points with positive coordinates). We can interpret a^n as the volume of a generic cube in the grid.
The Lemma follows from the definitions of $\mu_k^{(i)}$ and $\mu_k^{(e)}$ ∎

36. <u>Theorem</u> (Weyl's asymptotic formula)

<u>Let</u> Ω <u>be a smooth bounded set in</u> $(\mathbb{R}^n, \text{can})$.

<u>Let</u> $\{\lambda_k\}_{k=1}^{\infty}$ <u>denote the sequence of eigenvalues of the Dirichlet eigenvalue problem for the Laplacian in</u> Ω. <u>Let</u> $N(\lambda) = \text{Card}\{j \mid \lambda_j \leq \lambda\}$. <u>The asymptotic behavior of</u> $N(\lambda)$ <u>is given by</u>

$N(\lambda) \sim c(n)\text{Vol}(\Omega)\lambda^{n/2}$ <u>as</u> $\lambda \to \infty$ ($c(n) = (2\pi)^{-n}$Vol unit ball in $\mathbb{R}^n$).

<u>Proof</u>. Take a grid as above and notice that one can take $\text{Vol}(\Omega_i)$ and $\text{Vol}(\Omega_e)$ very close to $\text{Vol}(\Omega)$ by taking a small.
Compare with [C-H] Chap. VI.4 and Chap. VII.14 [R-S], §XIII.15 ∎

37. <u>Remarks</u>

(i) Weyl's formula also holds for any of the eigenvalue Problems (16): see nº VII.11;

(ii) Weyl's formula shows that if we know all the eigenvalues of Ω, say for the Dirichlet eigenvalue problem, then we know the dimension n of Ω and its volume $\text{Vol}(\Omega)$. This is an example of an answer to an <u>inverse problem</u>: the knowledge of the Dirichlet-spectrum of Ω gives both the dimension and the volume of Ω.

38. <u>Corollary</u>. <u>For</u> Ω <u>as in</u> Theorem 36 <u>we have</u>

$$\lambda_j(\Omega) \underset{j\to\infty}{\sim} c(n)^{-2/n}\left(\frac{j}{\text{Vol}(\Omega)}\right)^{2/n}.$$

Our next result is an example of an answer to a direct problem.

39. Theorem. Let (M,g) be an n-dimensional Riemannian manifold without boundary, whose Ricci curvature is bounded from below by $(n-1)k$ (k in $\mathbb{R}$):

$$\text{Ricci}(M,g) \geq (n-1)kg.$$

Let r be less than the injectivity radius of (M,g).

For any x in M, the following inequality holds

$$\lambda_1(B(x,r),\text{Dirichlet}) \leq \lambda_1(k,r)$$

where $\lambda_1(k,r)$ is the first eigenvalue for the Dirichlet problem in a geodesic ball of radius r in the space $(\mathbb{S}^n_k,\text{can})$ with constant curvature k (see nº II.18).

40. Comments.

(1) The injectivity radius $\text{Inj}(M,g)$ is the largest r such that for all x in M, $\exp_x$ is an embedding on the open ball of radius r in T_mM. When M is compact this number is (strictly) positive ([C-E] Chap. 5).

(2) It follows from Theorem 18 and Green's formula (Theorem 10) that for a Riemannian manifold with boundary (N,h), the first eigenvalue $\lambda_1(N,h,D)$ of the Dirichlet eigenvalue problem is (strictly) positive. For the Closed or Neumann eigenvalue problems the first eigenvalue is 0.

(3) All the geodesic balls of radius r in $(\mathbb{S}^n_k,\text{can})$ are isometric so that the definition of $\lambda_1(k,r)$ in the theorem makes sense.

(4) In fact, Theorem 39 also holds for radii larger than $\mathrm{Inj}(M,g)$: see S.Y. Cheng, [CG].

Proof. It can be shown ([CL], Chap. II.5) that the first eigenfunction of the Dirichlet eigenvalue problem in the ball $B(p,r)$ in $(\$^n_k, \mathrm{can})$ can be written as $\phi_1 = \varphi(d_k(p,.))$ where φ is a positive function and $d_k(p,.)$ is the Riemannian distance function to p in $(\$^n_k, \mathrm{can})$.

Let $\bar{\theta}(r,u) = \bar{\theta}(r)$ be the volume density in $(\$^n_k, \mathrm{can})$ viewed through $\exp_p$ (see nº II.25). The function φ satisfies

$$\varphi''(s) + \frac{\bar{\theta}'(r)}{\bar{\theta}(r)}\varphi'(s) + \lambda_1(k,r)\varphi(s) = 0;$$

$$\varphi(r) = 0;$$

$$\varphi \ C^\infty.$$

From this equation it follows that φ is decreasing.

For x in M, let $f(y) = \varphi(d(x,y))$ where $y \in B(x,r)$ and $d(x,y)$ is the Riemannian distance in (M,g) (this procedure is called transplantation: see [BE], [P-S]).

From the first variational characterization of eigenvalues (nº 26) we can write

$$\lambda_1(B(x,r)) \le \int_{B(x,r)} |df|^2 \Big/ \int_{B(x,r)} f^2 \quad \text{because } f$$

vanishes on the boundary of $B(x,r)$. Let $\theta(s,u)$ denote the volume density in (M,g) viewed through $\exp_x$ (nº II.25).

Pulling back the above integrals to T_xM we obtain

$$\int_{B(x,r)} |df|^2 = \int_{S^{n-1}} \int_0^r (\varphi'(s))^2 \theta(s,u)\,ds\,du,$$

$$\int_{B(x,r)} f^2 = \int_{S^{n-1}} \int_0^r \varphi^2(s)\theta(s,u)ds\ du.$$

Integration by parts gives

$$\int_{B(x,r)} |df|^2 = \int_{S^{n-1}} \int_0^r \varphi(s)\{-\varphi''(s) - \frac{\theta'}{\theta}(s,u)\varphi'(s)\}\theta(s,u)ds\ du.$$

41. Lemma. With the above notations and under the assumption Ricci $(M,g) \geq (n-1)kg$, we have

$$\frac{\partial}{\partial s}\{\theta(s,u)/\bar{\theta}(s)\} \leq 0$$

for s smaller than $\mathrm{Inj}(M,g)$.

For a proof see [B-C] Chap. 11.10; this Lemma is the key-point in the proof of the Bishop-Gromov comparison theorem (nºII.30).

Now since $\varphi'(s) < 0$ we conclude that

$$\int_B |df|^2 \leq \int_{S^{n-1}} \int_0^r \varphi(s)[-\varphi''(s) - \frac{\bar{\theta}'}{\bar{\theta}}(s)\varphi'(s)\}\theta(s,u)ds\ du =$$

$$= \lambda_1(k,r) \int_{S^{n-1}} \int_0^r \varphi^2(s)\theta(s,u)ds\ du \quad \blacksquare$$

42. Corollary. Let (M,g) be an n-dimensional Riemannian manifold without boundary, whose Ricci curvature is bounded from below by $(n-1)k$ i.e. $\mathrm{Ricci}(M,g) \geq (n-1)kg$. Let D denote the diameter of (M,g) and let $\{\lambda_i(M)\}_{i=1}^{\infty}$ denote the eigenvalues of the closed eigenvalue problem on (M,g) counted with multiplicities.

The following inequalities holds

$$\lambda_m(M) \leq \lambda_1(k, D/2(m-1)), \text{ for } m \geq 2 \quad (\text{recall that } \lambda_1(M)=0),$$

where $\lambda_1(k,r)$ denotes the first eigenvalue for the Dirichlet eigenvalue problem in a ball of radius r in the space $(\mathbb{S}^n_k, \mathrm{can})$.

Proof. Take x and y in (M,g) such that $d(x,y) = D$ and consider a shortest path (see II.8) from x to y. One can find $x_1,\ldots,x_m$ on this path such that the open balls $B(x_i, D/2(m-1))$ are pair-wise disjoint.

Consider the vector-space in $H^1(M)$ spanned by the m functions $f_1,\ldots,f_m$, where

$$f_i = \begin{cases} 1^{\text{st}} \text{ eigenfunction (for Dirichlet) in } B(x_i, D/2(m-1)), \\ 0 \quad \text{outside } B(x_i, D/2(m-1)). \end{cases}$$

This subspace has dimension m. By Theorem 39, for any u in this subspace $R(u) \leq \lambda_1(k, D/2(m-1))$.

Corollary 42 then follows from the third variational characterization of eigenvalues (nº 28) ∎

43. Comments.

Corollary 42 says that the eigenvalues of (M,g) can be estimated from above in terms of a lower bound on the Ricci curvature and an upper bound on the diameter. The estimate given in Corollary 42 is not satisfactory for several reasons.

For k and n fixed, we have $\lim\limits_{r\to 0_+} r^2\lambda_1(k,r) = c(k,n)$ (this follows from the fact that a Riemannian manifold is asymptotically Euclidean). It then follows from Corollary 42 that $\lim\limits_{m\infty} D^2 m^{-2}\lambda_m(M) \leq c_1(k,n)$. Recall that Weyl's asymptotic formula reads $\lim\limits_{m\infty} m^{-2/n} \operatorname{Vol}(M)^{2/n}\lambda_m(M) = c(n)$, and notice that both $\operatorname{Diam}^2(M,g)\lambda_m(M,g)$ and $\operatorname{Vol}(M,g)^{2/n}\lambda_m(M,g)$ are Riemannian invariants of weight 0 (see nº II.6).

It is easy to explain why Corollary 42 does not give an estimate compatible with Weyl's asymptotic formula. We took a shortest path from x to y, so that we acted as if (M,g) were one-dimensional and we therefore found $m^{-2/1}$ instead of $m^{-2/n}$ and its "1-dimensional volume" $\mathrm{Diam}(M) = D$ instead of $\mathrm{Vol}(M,g)$, its n-dimensional volume.

In order to get closer to Weyl's formula, we have to fill the Riemannian manifold with balls. The following argument is due to M. Gromov ([GV1]). For a given $\varepsilon > 0$, let $\{x_i\}_{i=1}^{N(\varepsilon)}$ denote a maximal set of points in M such that the balls $B(x_i,\varepsilon)$ are pair-wise disjoint and $\bigcup_{i=1}^{N} B(x_i,2\varepsilon) = M$. Let $b_k(r)$ denote the volume of a geodesic ball with radius r in $\$_k^n$.

Applying Bishop's comparison Theorem (II.30(i)), we can write

$$\mathrm{Vol}(M,g) \le \sum_{i=1}^{N(\varepsilon)} \mathrm{Vol}\, B(x_i,2\varepsilon) \le N(\varepsilon)\, b_k(2\varepsilon).$$

The 3rd variational characterization of eigenvalues (nº 28) and Theorem 39 give

(44) $$\lambda_{N(\varepsilon)}(M,g) \le \lambda_1(k,\varepsilon).$$

For k and n fixed, we have

$$\lim_{\varepsilon\to 0} \varepsilon^2\lambda_1(k,\varepsilon) = c_1(k,n)$$

$$\lim_{\varepsilon\to 0} b_k(2\varepsilon)\varepsilon^{-n} = c_2(k,n)$$

$$\lim_{\varepsilon\to 0} N(\varepsilon)\varepsilon^n \ge \frac{\mathrm{Vol}(M,g)}{c_2(k,n)}$$

and finally we obtain the estimate

$$\lim_{m\infty} \lambda_m(M,g)\, \mathrm{Vol}(M,g)^{2/n} m^{-2/n} \le c_3(k,n)$$

which is very close to Weyl's estimate.

In fact, (44) shows that one can estimate the eigenvalues $\lambda_m(M,g)$ from above in terms of a lower bound on the Ricci curvature and the volume.

In [GV1], M. Gromov also gave lower bounds for the eigenvalues $\lambda_m(M,g)$, in terms of Ricci and Diameter.

More precisely, let $r_{min} = \inf\{Ricci(u,u) \mid u \in UM\}$. Assume that $r_{min}Diam^2(M,g) \geq (n-1)\alpha$. Then there exists a constant $C(\alpha,n)$ such that

$$(45) \qquad \lambda_m(M,g)\ Diam(M,g)^2 \geq C(\alpha,n)m^{2/n}$$

for all $m \geq 2$.

We shall prove such an estimate in Chapter VI (our constant will be better than Gromov's and our method quite different). In view of Weyl's asymptotic formula, one can ask whether one can replace $Diam(M,g)^2$ by $Vol(M,g)^{2/n}$ in (45), as we did above for the upper bounds. In V.32, we will give a counter-example showing that (45) is best possible qualitatively: a general lower bound on $\lambda_m(M,g)$ must depend on a lower bound on the Ricci curvature and an upper bound on the diameter.

46. <u>Remarks</u>.

(1) Theorem 39, Corollary 42 and (44) give partial answers to a <u>direct problem</u> (find information on the eigenvalues in terms of geometric data).

(2) As the variational characterizations show, to a stronger stress correspond larger eigenvalues of given rank. This should be kept in mind together with our motivations from mathematical physics in Chapter I.

Further references for Chapter III: [AN], [B-J-S], [B-G-M], [BN], [BZ], [C-H], [CL], [ES], [WS].

CHAPTER IV

ISOPERIMETRIC METHODS

In this chapter, we give the basic ideas concerning the isoperimetric methods together with some direct applications.

A. MOTIVATIONS: THE FABER-KRAHN INEQUALITY.

Given Ω, a smooth bounded domain in $\mathbb{R}^n$, we denote by $\lambda_1(\Omega)$ the first eigenvalue for the Dirichlet eigenvalue problem in Ω; we denote by Ω^* the Euclidean ball centered at 0 in $\mathbb{R}^n$, whose volume is equal to $\mathrm{Vol}(\Omega)$. The following inequality holds

(1) $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ (Faber-Krahn's inequality).

This inequality was stated in dimension 2 by Lord Rayleigh in his treatise on the theory of sound ([RH], Section 210; this is still a very stimulating reading). The proof of inequality (1) was given independently by C. Faber and E. Krahn in the 1920s.

2. Sketch of the proof of the Faber-Krahn inequality.

3. Lemma. Let f be any eigenfunction associated with the first eigenvalue λ_1 of the Dirichlet problem in Ω. Then f is (strictly) positive or (strictly) negative in the open set Ω.

Proof. Since $f \in H_0^1(\Omega)$, we have $|f| \in H_0^1(\Omega)$ and $R(f)=R(|f|)=\lambda_1$ ([G-T] §7.4). It follows that $|f|$ also is an eigenfunction associated with λ_1 (III.26) and hence $|f| \in C^2(\Omega) \cap C^0(\bar{\Omega})$ by elliptic regularity theory ([G-T] Chap. 8). Now, $\Delta(|f|) = \lambda_1 |f| \geq 0$ (recall our sign convention on Δ) and it follows that $\inf_{\bar{\Omega}} |f|$ is achieved only on $\partial\Omega$, unless $f \equiv 0$ (maximum principle, [G-T] Theorem 3.5 p.34). Finally, we have $|f| > 0$ in Ω and hence $f > 0$ or $f < 0$ ■

Another reference for the maximum principle is M. Spivak [SK], Vol. V, Addendem 2 to Chapter 10 p. 181ff. In order to prove Lemma 3, one can also use a local argument (Taylor's expansion near a point where f vanishes in Ω) and the unique continuation property ([AZ]) ■

4. <u>Corollary</u>. <u>The first eigenvalue</u> λ_1 <u>for the Dirichlet problem in</u> Ω <u>is simple</u>.

<u>Proof</u>. Assume not. Take f_1, f_2 two orthogonal eigenfunctions. They can also be assumed to be positive in Ω by Lemma 3, and hence $\int_\Omega f_1 f_2 > 0$, a contradiction ■

Let f be the first eigenfunction in Ω associated with $\lambda_1(\Omega)$.

The main idea in the proof of inequality (1) is the <u>idea of symmetrization</u>.

We consider the sets $\Omega_t = \{x \in \Omega \mid f(x) > t\}$ and we symmetrize them by considering the Euclidean balls Ω_t^* in $\mathbb{R}^n$, with center 0, satisfying $\text{Vol}(\Omega_t) = \text{Vol}(\Omega_t^*)$.

Equivalently, we symmetrize the graph F of f above Ω into a set F^* which is invariant under rotations about the axis D^* (Fig. 4).

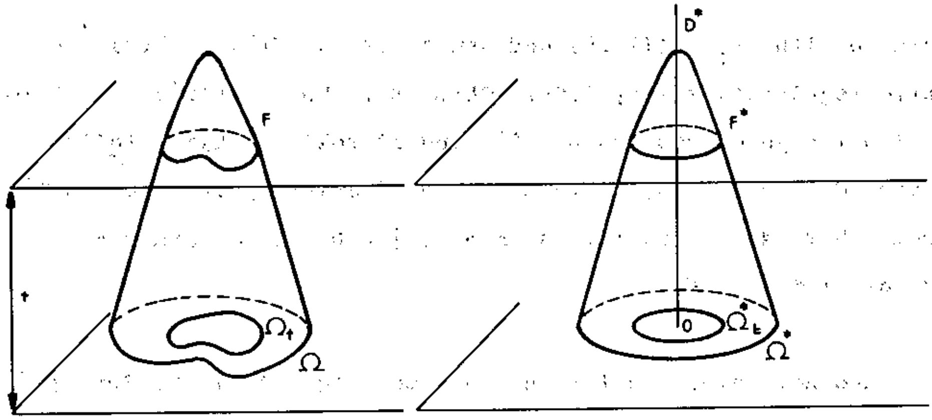

Fig. 4

We define a function f^* on Ω^* by the following properties:

(i) the graph of f^* is F^*, or equivalently,

(ii) $\begin{cases} f^* \text{ is a radial decreasing function} \\ \text{and} \\ f^* \text{ takes the value } t \text{ on } \partial\Omega_t^*. \end{cases}$

This procedure is called the <u>symmetrization of the function</u> f. (we do not wish to go into formal definitions now: for more details see [BE] Chap. II, [MO] Chap. I, Appendix A or [KL]).

In order to estimate the Rayleigh quotient $R(f)$ of the function f, we introduce new coordinates on Ω, by considering the level hypersurfaces and the lines of gradient of f. The following formula is known as the <u>co-area formula</u> (see [CL] Chap. IV, [BE] Lemma 2.5 p. 53, [B-M] Appendix A, [FR]).

5. Lemma.

For any continuous function h on Ω one has

$$\int_\Omega h(x)dx = \int_0^{\sup f} \left(\int_{G(t)} h|df|^{-1} da_t\right)dt,$$

where da_t is the volume element of the Riemannian metric induced by $\mathbb{R}^n$ on the hypersurface $G(t) = f^{-1}(t)$ (this makes sense for t in the set $\mathfrak{R}_f$ of regular values of f; the complement of $\mathfrak{R}_f$ has measure zero by Sard's theorem).

If we now take $h = |df|^2$ and if we apply the co-area formula, we obtain $(m=\sup f)$

$$\int_\Omega |df|^2(x)dx = \int_0^m \left(\int_{G(t)} |df| da_t\right)dt.$$

Applying Cauchy-Schwarz inequality, we find

$$\int_{G(t)} |df| da_t \geq \left(\int_{G(t)} da_t\right)^2 \Big/ \int_{G(t)} |df|^{-1} da_t$$

(for t in $\mathfrak{R}_f$).

Now $\int_{G(t)} da_t$ is just the $(n-1)$-dimensional volume of $G(t) = f^{-1}(t)$, hence, by the classical isoperimetric inequality in $\mathbb{R}^n$ (see nº 8 below), $\left(\int_{G(t)} da_t\right)^2 \geq \mathrm{Vol}(\partial\Omega_t^*)^2$.

It also follows from the co-area formula that

$$-\int_{G(t)} |df|^{-1} da_t = \frac{d}{dt}\mathrm{Vol}(\Omega_t) = \frac{d}{dt}\mathrm{Vol}(\Omega_t^*).$$

If we now apply the same construction to the radial function f^*, (see [CL] Chap. IV or [B-M] Appendix B) we have

$$\int_{G^*(t)} |df^*| da_t^* = \left(\int_{G^*(t)} da_t^*\right)^2 \Big/ \int_{G^*(t)} |df^*|^{-1} da_t^*$$

($|df^*|$ is constant on $G^*(t) = f^{*-1}(t)$) and hence,

$\int_{G(t)} |df| da_t \geq \int_{G^*(t)} |df^*| da_t^*$. Integrating in t, this gives

$$\int_\Omega |df|^2(x)dx \geq \int_{\Omega^*} |df^*|^2(x)dx.$$

It follows easily from the co-area formula that

$$\int_\Omega f^2(x)dx = \int_{\Omega^*} f^{*2}(x)dx.$$

Finally we have proved that $R(f;\Omega) \geq R(f^*;\Omega^*)$, from which it follows that $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ (First variational characterization of the eigenvalues: III.26).

6. Remarks

(i) One way of dealing with the difficulties arising from the co-area formula is to approximate functions in $H_o^1(\Omega)$ by "nice" Morse functions (this argument was introduced by Th. Aubin: see [B-M] Lemma 10 bis p. 519);

(ii) One can also use a more general form of the co-area formula: see [TI], [MO], [KL], [FR];

(iii) The Faber-Krahn inequality can be generalized to other situations: see [CL] Chap. IV, [B-M] and nº 22 below.

7. Comments.

The main ideas in the proof are the <u>principle of symmetrization</u> and the use of the <u>classical isoperimetric inequality</u> (both Cauchy-

Schwarz inequality and the co-area formula are technical details which are easily generalized to other situations). The classical isoperimetric inequality in $\mathbb{R}^n$ states that among all domains in $\mathbb{R}^n$, with given n-dimensional volume, the Euclidean balls have least boundary (n-1)-dimensional volume. Because dilations act on $\mathbb{R}^n$, this inequality can be written as follows: for any bounded domain Ω in $\mathbb{R}^n$,

$$(8) \qquad \mathrm{Vol}_{n-1}(\partial\Omega) \geq C^*(n)\ \mathrm{Vol}_n(\Omega)^{(n-1)/n},$$

where $C^*(n) = \mathrm{Vol}_{n-1}(S^{n-1})/\mathrm{Vol}_n(B^n)^{(n-1)/n}$,

$$B^n = \{x \text{ in } \mathbb{R}^n \mid |x| = 1\}, \quad S^{n-1} = \partial B^n.$$

(For a more general statement, for example when $\partial\Omega$ is very irregular, see [FR] p. 278).

Inequality (8) explains our choice of the symmetrization procedure, so that the principle of symmetrization and the isoperimetric inequality amount to the same idea. This idea can be generalized to analogous situations on the sphere (S^n,can) or on the hyperbolic space (H^n,can): among domains of $(\$^n_k,\mathrm{can})$ with given volume, the geodesic balls have least boundary volume (see [ON] for a survey on isoperimetric inequalities).

The point is that the model spaces $(\$^n_k,\mathrm{can})$ have many isometries and nice geodesic balls. We cannot expect anything like that on a generic Riemannian manifold. We will now explain how the symmetrization procedure can be extented to the general case.

B. ISOPERIMETRIC INEQUALITIES AND SYMMETRIZATION

Although some of the ideas we will deal with can be generalized to other situations, we will from now on assume that

ALL RIEMANNIAN MANIFOLDS ARE COMPACT, CONNECTED, WITHOUT BOUNDARY.

9. An <u>isoperimetric inequality</u> on a Riemannian manifold (M,g) (compact, without boundary) is an estimate from below of the volume of the boundary $\partial\Omega$ of a domain Ω in M, in terms of $Vol(\Omega)$. If we take Ω to be the complement of a small ball in M, we see that it is more realistic to consider $Vol(\Omega)/Vol(M)$, the relative volume of Ω in M, instead of $Vol(\Omega)$.

We define the <u>isoperimetric function</u> of (M,g) as

$$h(\beta) = h(M,g;\beta) = \inf_{\Omega}\left\{\frac{Vol(\partial\Omega)}{Vol(M)} \;\middle|\; \Omega \subset M,\ Vol(\Omega) = \beta Vol(M)\right\},$$ for β in $[0,1]$ (it should be clear that $Vol(\Omega)$ is an n-dimensional volume, $n = \dim M$, and that $Vol(\partial\Omega)$ is an $(n-1)$-dimensional volume), and Ω smooth domains in M.

An <u>isoperimetric estimator</u> on (M,g) is a function H from $[0,1]$ to $\mathbb{R}_+$ such that $h(\beta) \geq H(\beta)$ for all β in $[0,1]$.

10. <u>Example</u>

Let us consider (S^2,can). The volume of a geodesic ball of radius r in (S^2,can) is $2\pi(1-\cos r)$, and the volume of the corresponding geodesic sphere of radius r is $2\pi \sin r$. It follows from the isoperimetric inequality on $(\$^n_k,can)$ (see end of §A) that $h(S^2,can;\beta) = \sqrt{\beta(1-\beta)}$.

11. Proposition

The function $h(M,g;\beta)$ has the following properties (n=dim M)

(i) $h(\beta) \geq 0$;

(ii) $h(\beta) = h(1-\beta)$;

(iii) $h(\beta) \sim C^*(n)\mathrm{Vol}(M)^{-1/n}\,\beta^{(n-1)/n}$, when β is close to 0 (for $C^*(n)$ see formula (8));

(iv) $h(M,g;.)$ is continuous on $[0,1]$ and has right and left derivatives at each β in $]0,1[$ and is differentiable in $]0,1[$ except on a denumerable set.

Proof. (i) and (ii) are clear;

(iii) is called the asymptotic isoperimetric inequality; it says that for domains with small volume, the isoperimetric inequality looks very much like the classical isoperimetric inequality in $\mathbb{R}^n$: see [B-M] Appendix C;

(iv) is much more delicate: see [B-B-G2] ∎

12. Philosophy

Isoperimetric function vs. Isoperimetric inequality.

If we want to use isoperimetric methods on a given Riemannian manifold (M,g), the best thing we can hope for is to know the isoperimetric function $h(M,g;\beta)$ itself. In general this is not the case and we have to replace $h(\beta)$ by some minorizing function $H(\beta)$.

Now if we want to use isoperimetric methods on a class of Riemannian manifolds, we have to choose an isoperimetric inequality which is valid for any manifold in the given class.

An example of such a class of Riemannian manifolds is

$$\mathfrak{M}_{n,k,D} = \{(M,g) \mid \dim M = n,\ \mathrm{Ricci}(M,g) \geq (n-1)kg,\ \mathrm{Diam}(M,g) \leq D\}.$$

We then consider isoperimetric inequalities of the form $h(M,g;\beta) \geq H(n,k,D;\beta)$, for any (M,g) in $\mathfrak{M}_{n,k,D}$. We will give examples of such situations in Chapters V and VI.

In the sequel we will study isoperimetric methods on Riemannian manifolds with a given isoperimetric estimator $H(\beta)$ and with the above philosophy in mind.

13. Symmetrization

Let (M,g) be a Riemannian manifold equipped with an isoperimetric estimator $H(\beta)$. Because a generic Riemannian manifold does not have symmetries, we cannot compare a domain in M with a geodesic ball in M. Keeping in mind the symmetrization procedure which we used in the proof of the Faber-Krahn inequality, we will instead construct a model space with nice balls having isoperimetric properties related to $H(\beta)$.

For this purpose we consider $S^{n-1} \times]0,L[$ with Riemannian metric $g^* = a^2(s)d\theta^2 + ds^2$, where θ is in S^{n-1}, s in $]0,L[$, and $d\theta^2$ is the canonical Riemannian metric on (S^{n-1},can). We also assume that $a(0) = a(L) = 0$. We call (M^*,g^*) this Riemannian manifold (it is not necessarily complete; we also use M^* for $S^{n-1} \times]0,L[\cup \{N,S\}$, where the north and south poles N and S are the points corresponding to $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{L\}$; for M^* to be smooth one needs that $a'(0) = 1$ and $a'(L) = -1$; (M^*,g^*) can be viewed as a manifold with revolution symmetry).

We denote the volume of (M^*,g^*) by V^*. We call $A(s)$ the relative volume of the ball $B(N,s)$ with center N and radius s (i.e. $B(N,s) = \{N\} \cup S^{n-1} \times]0,s[$). We then have

(14) $$A(s) = V^{*-1}\, Vol(S^{n-1}) \int_0^s a^{n-1}(t)dt.$$

(see Fig. 5).

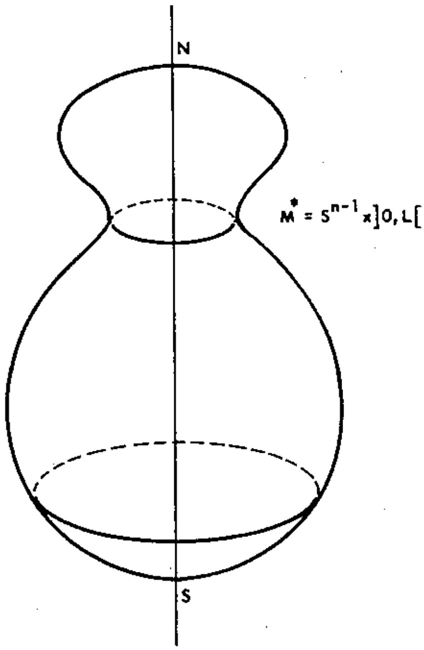

Fig.5

15. In order to have a nice symmetrization procedure on the manifold $(M,g;H(\beta))$, we want to construct a model (M^*,g^*) such that the balls $B(N,s)$ have H as isoperimetric function i.e. such that

$V^{*-1}\, Vol(\partial B(N,s)) = H(A(s))$, (recall that $A(s)$ is the relative volume of $B(N,s)$).

This can also be written as

(16) $A'(s) = H(A(s))$, $s \in]0,L[$, because $Vol(\partial B(N,s)) = Vol(S^{n-1})a^{n-1}(s)$.

Given a Riemannian manifold (M,g), with isoperimetric estimator H, we can construct (M^*,g^*) as follows. We determine $A(s)$ by the differential equation (16) and the initial condition $A(0) = 0$ (we could also use another condition: see nº 22 below).

This determines $A(s)$ by the equality

(17) $$s = \int_0^{A(s)} \frac{du}{H(u)} ,$$

from which we deduce the value of L

(18) $$L = \int_0^1 \frac{du}{H(u)} .$$

19. <u>Remark</u>

It is clear that the isoperimetric function h satisfies $h(0) = h(1) = 0$. This implies that the isoperimetric estimator H must satisfy $H(0) = H(1) = 0$. It follows that Equations (17) and (18) only make sense if the integrals converge. Notice that in view of Proposition 11 (iii), the integral $\int_0^1 \frac{du}{h(u)}$ converges. For this reason, we will usually assume that $H(\beta) \sim C\beta^{\alpha}$ when β is close to 0 with $1 > \alpha \geq \frac{n-1}{n}$ and a similar assumption near $\beta = 1$ (the second inequality comes from Proposition 11 (iii) and the assumption that $h(\beta) \geq H(\beta)$ for all β). We shall also give an example with $\alpha = 1$ (see nº 22 below).

20. So far, given $(M,g;H(\beta))$, we have determined $A(s)$ and L. In order to determine (M^*,g^*), we still have a degree of freedom, namely the choice of V^*. This will turn out to be convenient later (nº 21 and Chap. V) but the choice of V^* is in fact <u>irrelevant</u> for the following reasons. Let $\tilde{a}(s) = V^{*-1/(n-1)}a(s)$. This function is determined by $A(s)$ because

$$A'(s) = V^{*-1}\mathrm{Vol}(S^{n-1})a^{n-1}(s) = \mathrm{Vol}(S^{n-1})\,\tilde{a}^{n-1}(s).$$

Now let f be a function on M^* which depends only on the variable s. The Rayleigh quotient $R(f)$ of f on (M^*,g^*) is given by

$$R(f) = \int_{S^{n-1}}\int_0^L \dot{f}^2(s)a^{n-1}(s)dvds \Big/ \int_{S^{n-1}}\int_0^L f^2(s)a^{n-1}(s)dvds,$$

where dv is the Riemannian measure on (S^{n-1},can). It follows that

$$R(f) = \int_0^L \dot{f}^2(s)\,\tilde{a}^{n-1}(s)ds \Big/ \int_0^L f^2(s)\tilde{a}^{n-1}(s)ds$$

does not depend on V^*: this is a Rayleigh quotient in one dimension, with measure $\tilde{a}^{n-1}(s)ds$. In the sequel we will compare the Rayleigh quotient of a function on (M,g) to the Rayleigh quotient of a radial function on (M^*,g^*) (i.e. depending only on the s-variable) so that we will be able to ignore V^*.

On the other hand, the radial part of the Laplacian on (M^*,g^*) is given by (see nº III.5(v))

$$-\frac{\partial^2}{\partial s^2} - (n-1)\frac{a'(s)}{a(s)}\frac{\partial}{\partial s}$$

(because $\theta(s,u) = a^{n-1}(s)$ in the local chart $\exp_N$).
This operator does not depend on a choice of V^*.

As a matter of fact, it turns out that our manifold with revolution symmetry (M^*,g^*) is just a convenient way of visualizing a <u>one-dimensional model</u>.

21. <u>Example</u>

Let (M,g) be a Riemannian surface with isoperimetric estimator $H(\beta) = \sqrt{\beta(1-\beta)}$. Formula (17) gives

$$s = \int_0^{A(s)} [u(1-u)]^{-1/2} du, \quad \text{i.e.} \quad A(s) = \sin^2 \frac{s}{2} \text{ and } L=\pi.$$

It then follows that $a(s) = \frac{V^*}{4\pi} \sin s$, and that

$$(M^*, g^*) = (S^1 \times]0,\pi[,\ g^* = a^2(s)d\theta^2 + ds^2).$$

A pleasant choice of V^* is $V^* = 4\pi$, in which case (M^*, g^*) is just (S^2, can), whose isoperimetric function is $h(S^2, can; \beta) = \sqrt{\beta(1-\beta)}$ (see Example 10). Another choice of V^* gives a "cigar" with two conic points and constant curvature 1.

22. <u>Application</u>: Cheeger's isoperimetric inequality.

In 1970, J. Cheeger introduced the following isoperimetric constant, known as <u>Cheeger's isoperimetric constant</u>. For a closed Riemannian manifold (M,g) we define

$$h_C = h_C(M,g) = \inf\{Vol(\partial\Omega)/Vol(\Omega) \mid \Omega \subset M,\ 2Vol(\Omega) \leq Vol(M)\},$$

Ω smooth domains in M. It follows that

$$h(\beta) = h(M,g;\beta) \geq h_C \min(\beta, 1-\beta),$$

so that we can choose $H(\beta) = h_C \min(\beta, 1-\beta)$. Notice that $H(\beta) = H(1-\beta)$, $\int_0^1 \frac{du}{H(u)}$ diverges at 0 and 1 (see Remark 19). Taking into account the symmetry of $H(\beta)$, we construct the model space (M^*, g^*) as $S^{n-1} \times]-\infty\infty[$ with $g^* = a^2(s)d\theta^2 + ds^2$, and we solve the differential equation (16) with the initial condition $A(0) = 1/2$, which gives

$$s = \int_{1/2}^{A(s)} \frac{du}{H(u)}. \tag{23}$$

Using the symmetry of H, we find that for <u>positive</u> s, $A(s) + A(-s) = 1$ and hence $a(s) = a(-s)$. Finally (23) gives

For $s \geq 0$,

$$(24) \quad \begin{cases} A(s) = 1 - \frac{1}{2}\exp(-h_C s) \\ \\ a(s) = (V^* \mathrm{Vol}(S^{n-1})^{-1} h_C/2)^{1/(n-1)} \exp(-h_C s/(n-1)) \end{cases}$$

so that (M^*, g^*) is made of two "cusps" glued together in a symmetric manner (see Fig. 6 in dimension 2).

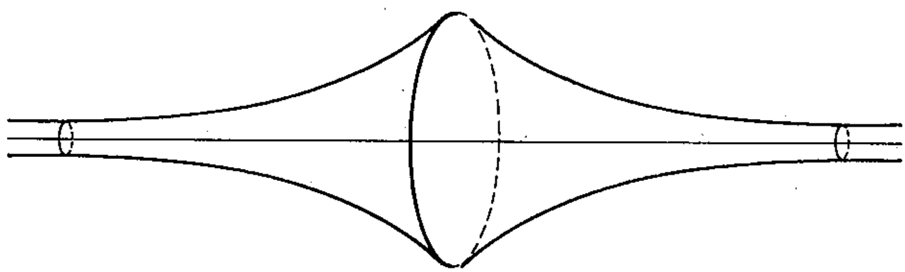

Fig. 6

The manifold (M^*, g^*) is complete, noncompact and has finite volume. Using the symmetry of M^*, we can view N as "the point at ∞ : $S^{n-1} \times \{\infty\}$".

25. Cheeger's isoperimetric constant was introduced in order to give a lower bound for $\lambda_2(M, g;\ \text{closed})$ (see nº III.19). The first

eigenvalue $\lambda_1(M,g;\text{ Closed})$ is always 0, the corresponding eigenspace corresponding to constant functions. Cheeger proved that

$$\lambda_2(M,g;\text{ Closed}) \geq h_C^2/4. \tag{26}$$

An eigenfunction f associated with $\lambda_2(M)$ being orthogonal to the constants must change sign in M. We can therefore find a connected component Ω of $M\backslash f^{-1}(0)$, such that $2\text{Vol}(\Omega) \leq \text{Vol}(M)$. We can also assume that f is positive in Ω. It turns out that $f|\Omega$ is a first eigenfunction associated with the first eigenvalue $\lambda_1(\Omega,g;\text{ Dirichlet})$ of the Dirichlet eigenvalue problem in Ω, and that $\lambda_1(\Omega,g;D) = \lambda_2(M,g;C)$ (think of Green's formula, Theorem III.10 or see [B-M] Appendix D).

We will now estimate $\lambda_1(\Omega,g;D)$ from below in terms of Cheeger's isoperimetric constant h_C. For this purpose we use the model (M^*,g^*) of Fig. 6 above, with $V^* = \text{Vol}(M)$ and we mimic the proof of the Faber-Krahn inequality (§A).

Let $\Omega_t = \{x \text{ in } \Omega \mid f(x) \geq t\}$ and let $\Omega_t^* = S^{n-1}\times[r(t),\infty[\subset M^*$ be such that $\text{Vol}(\Omega_t) = \text{Vol}(\Omega_t^*)$. Define a function $\phi\colon [r(0),\infty[\to \mathbb{R}_+$ by $\phi(r(t)) = t$. The function ϕ increases from 0 to $\sup f = m$ in $[r(0),\infty[$. For (θ,s) in $\Omega^* = \Omega_o^*$, let $f^*(\theta,s) = \phi(s)$. The co-area formula gives (see § A)

$$\int_\Omega |df|^2 dv_g = \int_0^m \left(\int_{G(t)} |df|\, da_t\right) dt,$$

where $G(t) = (f|\Omega)^{-1}(t)$. By Cauchy-Schwarz inequality, we can write

$$\int_{G(t)} |df|\, da_t \geq \left(\int_{G(t)} da_t\right)^2 \Big/ \int_{G(t)} |df|^{-1} da_t .$$

Since $2\text{ Vol }\Omega_t \leq \text{Vol}(M)$, we have

$$\text{Vol}(\partial\Omega_t) \geq h_C \text{Vol}(\Omega_t) = h_C \text{Vol}(\Omega_t^*) = \text{Vol}(\partial\Omega_t^*),$$

by definition of (M^*, g^*). This can be written as

$$\mathrm{Vol}(G(t)) \geq \mathrm{Vol}(G^*(t))$$

where $G^*(t) = f^{*-1}(t)$. Since f^* only depends on the s-variable, we can write

$$\int_{G(t)} |df|\, da_t \geq \int_{G^*(t)} |df^*|\, da_t^*$$

(we again used the fact that $\mathrm{Vol}\ \Omega_t = \mathrm{Vol}\ \Omega_t^*$: see § A).
As in § A we conclude that

$$(27) \qquad \int_\Omega |df|^2 dv_g \Big/ \int_\Omega f^2 dv_g \geq \int_{\Omega^*} |df^*|^2 dv_{g^*} \Big/ \int_{\Omega^*} f^{*2}\, dv_{g^*}.$$

The right-hand side of Inequality (27) can be written as

$$\int_{r(0)}^{\infty} \dot{\phi}^2(s) \exp(-h_C s) ds \Big/ \int_{r(0)}^{\infty} \phi^2(s) \exp(-h_C s) ds = R_1(\phi),$$

because $g^* = a^2(s) d\theta^2 + ds^2$ (note that $r(0) \geq 0$ and that $|df^*|$ is the norm of df^* on M^* for the dual metric g^{*-1}). It follows that

$$\lambda_2(M, g; \mathrm{Closed}) = \int_\Omega |df|^2 dv_g \Big/ \int_\Omega f^2 dv_g \geq \inf\{R_1(\phi)\} = \Lambda,$$

where the infimum in the right-hand side is taken over all functions ϕ such that

(i) ϕ and $\dot{\phi}$ are in $L^2(\mathbb{R}_+, \exp(-h_C s) ds)$,

(ii) $\phi(0) = 0$

($\dot{\phi}$ the derivative of ϕ is the sense of distributions).

It is easy to see that

$$\Lambda = \inf\{R_1(\phi) \mid \phi \text{ in } C_o^\infty(\mathbb{R}_+^\bullet)\},$$

and that

$$\Lambda = \inf_{n>0} \inf\{R_1(\phi) \mid \phi \text{ in } C_o^\infty(]0,n[)\}.$$

Using the first variational characterization of the eigenvalues nº III.26), it follows that

$$\Lambda_n = \inf\{R_1(\phi) \mid \phi \text{ in } C_o^\infty(]0,n[)\}$$

is the first eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} \phi''(s) - h_C\phi'(s) + \lambda\phi(s) = 0 \\ \phi(0) = \phi(n) = 0 \end{cases}$$

which can be solved explicitly showing that $\Lambda_n \geq h_C^2/4$. Finally we can conclude that $\Lambda \geq h_C^2/4$, which proves Cheeger's estimate (26).

28. Remarks.

(i) Cheeger's original proof is shorter than the above one. Although it uses the same technical details as that of the Faber-Krahn inequality, it is quite different. We found it interesting to show that Cheeger's inequality can be reduced to an inequality à la Faber-Krahn, with an appropriate model space (M^*, g^*);

(ii) One can also consider the surface with boundary $S^1 \times [0,\infty[$, with the above metric g^*. This manifold is not compact but is complete with finite volume. One can still consider the Laplacian Δ as an unbounded operator on $L^2(M^*,g^*)$ with Dirichlet boundary condition. The number $h_C^2/4$ then appears as the lower bound of the spectrum of the Friedrichs extension of Δ (compare with [CL]

Chap. IV.3) (here Δ has a continuous spectrum see [R-S]);

(iii) Cheeger's estimate (26) would be void of sense if we did not know that $h_C > 0$. In fact one can prove the following estimate (see [GA1]).

29. Theorem. Let (M,g) be an n-dimensional compact Riemannian manifold without boundary. Define r_{min} by

$r_{min} = \inf\{Ricci(M,g)(u,u) \mid u \text{ in } UM\}$, where UM is the unit tangent bundle to M.

Assume that $r_{min}.Diam(M,g)^2 \geq \varepsilon(n-1)k^2$, $\varepsilon \in \{-1,0,1\}$, $k \in \mathbb{R}_+^{\bullet}$.

Then

$$Diam(M,g).h_C(M,g) \geq K(k), \quad \text{where}$$

$$K(k) = \begin{cases} k\left[\displaystyle\int_0^{k/2} (\cos t)^{n-1}dt\right]^{-1} & \text{if} \quad \varepsilon = 1 \\ 2 & \text{if} \quad \varepsilon = 0 \\ k\left[\displaystyle\int_0^{k/2} (\mathrm{ch}\, t)^{n-1}dt\right]^{-1} & \text{if} \quad \varepsilon = -1 \end{cases} .$$

30. Remarks.

(i) Notice that both $r_{min}.Diam(M,g)^2$ and $Diam(M,g).h_C(M,g)$ are Riemannian invariants of weight 0 (see nº II.6);

(ii) Theorem 29 shows that $h_C(M,g)$ is uniformly bounded from below on the class $\mathfrak{M}_{n,k,D}$ given in nº 12;

(iii) In Chapter VI we will give an estimate on $\lambda_2(M,g;C)$ which is sharper than Cheeger's and we will generalize this estimate to $\lambda_i(M,g;C)$, $i \geq 2$.

31. Comments.

(i) Let (M,g) be a Riemannian manifold (always assumed to be compact without boundary) equipped with an isoperimetric estimator $H(\beta)$. If $\int_0^1 \frac{du}{H(u)}$ converges, the manifold M^* is compact (possibly with two conic points) and we can easily mimic the proof of the Faber-Krahn inequality to show the following assertion.

Let Ω be a domain in M and let Ω^* be the ball $B(N,r)$ in M^* such that $\mathrm{Vol}(\Omega)/\mathrm{Vol}(M) = \mathrm{Vol}(\Omega^*)/\mathrm{Vol}(M^*)$. Then

(32) $$\lambda_1(\Omega,g;D) \geq \lambda_1(\Omega^*,g^*;D).$$

Notice that in order to find $\lambda_1(\Omega^*,g^*;D)$, one only has to solve a one-dimensional eigenvalue problem (indeed the first eigenfunction is radial: compare with nº 20).

As was already pointed out in Remark 28 (iii), the estimate (32) is void if we do not know $H(\beta)$ i.e. if we cannot give lower bounds for $h(\beta)$ in terms of geometric data. So again the main difficulty is to find a good isoperimetric inequality. This fact will turn out to be even more important in Chapter V: see nº V § C;

(ii) One can also investigate isoperimetric inequalities on a manifold with boundary. In the case of a domain Ω in a manifold without boundary M, we have used the isoperimetric inequality in M to obtain results on the Dirichlet eigenvalue problem in Ω. One can also consider isoperimetric constants adapted to the Dirichlet boundary condition. For example one can define Cheeger's iso-perimetric constant

(33) $$h_C(\Omega,g;\mathrm{Dirichlet}) = \inf\{\mathrm{Vol}(\partial\omega)/\mathrm{Vol}(\omega) \mid \omega \subset \mathring{\Omega}\}$$

for the Dirichlet boundary conditions on $\partial\Omega$. If we want to deal with the Neumann problem, we have to allow subdomains ω such that $\partial\omega \cap \partial\Omega \neq \phi$; see [BR] p. 29. It turns out that the isoperimetric constants adapted to the Neumann boundary conditions are much more difficult do deal with than the other ones. In fact, estimates on $\lambda_1(\Omega,g;\text{Neumann})$ involve the geometry of $(\Omega,\partial\Omega)$ is a very strong way. We shall not deal with these problems here: see [ME1] for more details.

For further reading on Cheeger's constant h_C we recommend [BR].

<u>Further references for Chapter IV</u>: [BE], [CL] Chap. IV, [PE], [P-S], [ON].

CHAPTER V

ISOPERIMETRIC METHODS AND THE HEAT EQUATION

ALL RIEMANNIAN MANIFOLDS ARE ASSUMED TO BE

COMPACT, CONNECTED, WITHOUT BOUNDARY

In Chapter I, we used the wave equation and separation of variables to motivate eigenvalue problems; we could have used the heat equation as well.

In the present chapter, we give direct results concerning the heat kernel of a Riemannian manifold. They will be useful in Chapter VI.

A. THE HEAT EQUATION.

Let (M,g) be a compact Riemannian manifold without boundary. To determine the heat flow $u(t,x)$ on (M,g) is to find a function $u(t,x)$, solution of the heat equation:

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t}(t,x) + \Delta_x u(t,x) = f(t,x), \text{ for } (t,x) \text{ in } \mathbb{R}_+^{\bullet} \times M, \\ u(0,x) = f_o(x), \text{ for } x \text{ in } M, \end{cases}$$

where f_o and f are given functions (e.g. C^∞ functions).

An easy way to solve this problem is to introduce the notion of fundamental solution of the heat equation (or heat kernel) on (M,g). The heat kernel is a function k on $\mathbb{R}_+^{\bullet} \times M \times M$ which satisfies the following properties

$$(2)\quad \begin{cases} \text{(i)} & k(t,x,y) \text{ is continuous on } \mathbb{R}_+^{\bullet} \times M \times M,\ C^1 \text{ in the } t\text{-variable and } C^2 \text{ in the } x\text{-variable;} \\ \text{(ii)} & (\frac{\partial}{\partial t} + \Delta_x)k(t,x,y) = 0 \text{ for all } (t,x,y) \text{ in } \mathbb{R}_+^{\bullet} \times M \times M; \\ \text{(iii)} & \lim_{t\to 0} k(t,x,y) = \delta_x(y), \text{ the Dirac measure at } x. \end{cases}$$

Property (2.iii) means that for any h in $C^\infty(M)$, we have

$$\lim_{t\to 0_+} \int_M k(t,x,y)h(y)dv_g(y) = h(x),$$

and is usually written as $k(0,x,y) = \delta_x(y)$.

At least at the formal level, the solution $u(t,x)$ of (1) is given by the following formula (known as Duhamel's formula)

$$u(t,x) = \int_M k(t,x,y)f_o(y)dv_g(y) + \int_0^t (\int_M k(t-s,x,y)f(s,y)dv_g(y))ds.$$

For the following theorem we refer to [CL] Chap. VI, [B-G-M] Chap. III.E., or [GY].

3. Theorem. *Let* (M,g) *be an* n-*dimensional compact Riemannian manifold without boundary, with eigenvalues (counted with multiplicities)* $\{\lambda_i\}_{i\geq 1}$ *and associated orthonormal real eigenfunctions* $\{\phi_i\}_{i\geq 1}$. *There exists a unique heat kernel* $k(t,x,y)$ *on* (M,g). *This is a* C^∞ *function on* $\mathbb{R}_+^{\bullet} \times M \times M$ *which satisfies* $k(t,x,y) = k(t,y,x)$ *for all* (t,x,y) *in* $\mathbb{R}_+^{\bullet} \times M \times M$. *Furthermore,* $k(t,x,y)$ *can be expressed as*

$$k(t,x,y) = \sum_{j=1}^{\infty} \exp(-\lambda_j t)\phi_j(x)\phi_j(y),$$

where the series in the right-hand side converges in the C^k-*topology* on any subset of the form $[a,\infty[\times M\times M$, $a>0$, for any k.

For example, this theorem justifies Duhamel's formula. Theorem 3 also justifies the following equalities

$$(4)\qquad \begin{cases} \text{(i)} & k(t,x,x) = \sum_{j=1}^{\infty} \exp(-\lambda_j t)\phi_j^2(x) \\ \text{(ii)} & Z(t) = \sum_{j=1}^{\infty} \exp(-\lambda_j t) = \int_M k(t,x,x)dv_g(x). \end{cases}$$

The function $Z(t)$ (the trace of the heat kernel on (M,g)) is called the partition function of (M,g). We shall also use the notation $Z(M,g;t)$ to stress the dependence of $Z(t)$ on (M,g).

5. Exercise. Prove that giving the sequence of eigenvalues (with multiplicities) $\{\lambda_i\}_{i\geq 1}$ of (M,g) is equivalent to giving the partition function $Z(t)$ of (M,g).

6. Examples and Exercises.

(i) Although $(\mathbb{R}^n, can)$ is not a compact Riemannian manifold, it has a heat kernel: $(4\pi t)^{-n/2}\exp(-\|x-y\|^2/4t)$; however, we cannot take its trace as we did for $k(t,x,y)$ in (4);

(ii) Let Γ be a lattice in $\mathbb{R}^n$ and let Γ^* be the dual lattice: $\Gamma^* = \{\gamma^*\in\mathbb{R}^n \mid \text{for all } x\in\Gamma,\ \langle x|\gamma^*\rangle\in\mathbb{Z}\}$. The eigenvalues of the torus $T_\Gamma = (\mathbb{R}^n/\Gamma, can/\Gamma)$ are the numbers $4\pi^2\|\gamma^*\|^2$, with associated orthonormal complex eigenfunctions $\exp(2i\pi\langle x|\gamma^*\rangle)\mathrm{Vol}(T_\Gamma)^{-1/2}$, $\gamma^*\in\Gamma^*$. The heat kernel of T_Γ is given by

$$k(t,x,y) = (4\pi t)^{-n/2}\sum_{\gamma\in\Gamma}\exp(-\|x-y-\gamma\|^2/4t).$$

In particular, Formula (4ii) can be written as

$$(4\pi t)^{-n/2}\mathrm{Vol}(T_\Gamma) \sum_{\gamma\in\Gamma} \exp(-\|\gamma\|^2/4t) = \sum_{\gamma\in\Gamma^*} \exp(-4\pi^2\|\gamma^*\|^2 t)$$

(this formula is known as Poisson summation formula).

It follows from the Poisson summation formula that $Z(T_\Gamma,\mathrm{can};t) \sim (4\pi t)^{-n/2}\mathrm{Vol}(T_\Gamma)$ when t goes to 0_+ ($n=\dim T_\Gamma$). In fact the following property holds.

7. Property. For any n-dimensional Riemannian manifold (M,g)

$$Z(M,g;t) \sim (4\pi t)^{-n/2}\mathrm{Vol}(M,g) \quad \text{when } t \to 0_+.$$

For more details see Chap. VII or [CL] Chap. VI, [B-G-M] Chap. III.E.

The purpose of this chapter is to give an isoperimetric inequality for the heat kernel.

B. ISOPERIMETRIC INEQUALITY FOR THE HEAT KERNEL, I.

8. Let (M,g) be an n-dimensional Riemannian manifold (compact, without boundary) and assume that one is given an isoperimetric estimator H on (M,g). As in Chapter IV, we construct a model space (M^*,g^*) which is associated with H (and with a choice of V^*). We assume that H satisfies

$$H(\beta) \sim C\beta^\alpha,\ 1 > \alpha \geq \frac{n-1}{n},$$

for β close to 0 and a similar property when β is close to 1 (see nº IV.19). This implies that (M^*,g^*) is a "nice Riemannian manifold" possibly with two conic points (the north and south poles);

recall that although we see a manifold with revolution symmetry, the mathematics only see a one-dimensional model: see nº IV.20.

Our main theorem is the following (see [B-G] § 2, [B-B-G1] §III).

9. Theorem. Under the above assumptions (nº 8), let $k(t,x,y)$ denote the heat kernel of the Riemannian manifold (M,g) and let $k_*(t,N,N)$ denote the heat kernel of the Riemannian manifold (M^*,g^*) evaluated at (t,N,N), where N is the north pole of (M^*,g^*). The following inequalities hold

$$Z(M,g;t) \leq \mathrm{Vol}(M,g) \sup_x k(t,x,x) \leq \mathrm{Vol}(M^*,g^*) k_*(t,N,N).$$

10. The main ideas in the proof of Theorem 9 are as follows (compare with [BE] Chap. IV §3, [M-T])

(a) We consider Problem (1) with $f \equiv 0$ and $f_o > 0$ on (M,g), and we compare the solution $u(t,x)$ to the solution of a symmetrized problem on (M^*,g^*);

(b) We apply a symmetrization procedure similar to the one described in Chapter IV, but we take the relative volume as new parameter;

(c) We then let f_o tend to a Dirac measure on (M,g), so that we obtain a comparison theorem for $k(t,x,y)$.

11. Since we are mainly interested in geometry in these notes, we only give a rough sketch of the proof of Theorem 9. For full analytic details see [BE] IV.3, [B-G], [M-T].

We divide the proof into several steps.

12. Step 1. Let f be a C^∞ positive function on M. We define

$$D(r) = \{x \in M \mid f(x) > r\} \quad \text{and}$$

$$a(r) = \mathrm{Vol}(D(r))/\mathrm{Vol}(M).$$

We now define a function $\bar{f}$ by

$$\bar{f}(s) = \inf\{r \mid a(r) < s\}.$$

The function $a(r)$ is non-increasing, and varies from $a(0) = 1$ to 0, when r increases from 0 to sup f. If a were strictly decreasing and continuous, $\bar{f}$ would be the inverse function of a (see [TI] Chap. I). Since f is C^∞, it follows from Sard's theorem that a is C^∞ on an open set whose complement has measure zero (use the co-area formula nº IV.5). We also have $\bar{f}(a(r)) = r$ for all regular values r. See Fig. 7

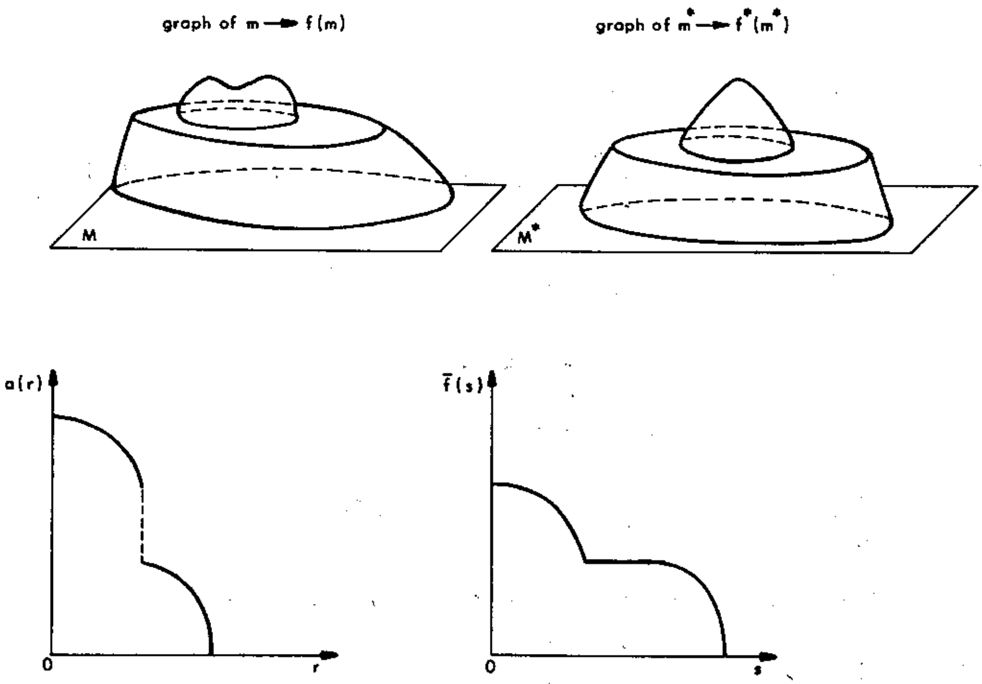

Fig. 7

Let $E(s) = D(\bar{f}(s))$, $G(s) = \partial E(s)$ and

$$F(s) = \int_{E(s)} f(x)dv_g(x).$$

It follows from the co-area formula (nº IV.5) that

$$Vol(M)a'(r) = - \int_{\partial D(r)} |df|^{-1} da_r,$$

when r is a regular value of f. We deduce that

$$(13) \qquad F(s) = Vol(M) \int_0^s \bar{f}(u)du.$$

Since the sets $D(r)$ are level sets of f, we deduce from Green's formula (Theorem III.10) that

$$\int_{E(s)} \Delta f(x)dv_g(x) = \int_{G(s)} \frac{\partial f}{\partial \nu} da_s = \int_{G(s)} |df| da_s.$$

Applying Cauchy-Schwarz inequality to the righ-hand side of the above equality, we can write

$$(14) \quad \int_{E(s)} \Delta f dv_g \geq (Vol G(s))^2 / \int_{G(s)} |df|^{-1} da_s = -(Vol G(s))^2 / Vol(M) a'(\bar{f}(s)).$$

It follows from the definitions of a, $\bar{f}$ and H, that

$$Vol(G(s)) \geq Vol(M)H(s).$$

From (13) we deduce that

$$\frac{d^2F}{ds^2}(s) = Vol(M) \frac{df}{ds}(s) = Vol(M)/a'(\bar{f}(s)).$$

From these relations and (14), we deduce that

$$(15) \qquad \int_{E(s)} \Delta f \, dv_g \geq -H^2(s) \frac{d^2F}{ds^2}(s).$$

Let $f^\wedge$ be a radial C^∞ function on (M^*,g^*). With the obvious notations and taking into account the fact that $f^\wedge$ is radial, and the very definition of M^* we have

(16) $$\int_{E^\wedge(s)} \Delta^* f^\wedge \, dv_{g^*} = -H^2(s) \frac{d^2F^\wedge}{ds^2}(s)$$

(where Δ^* is the Laplacian on (M^*,g^*)).

17. <u>Step 2</u>. Let f_o be a positive C^∞ function on M. Let $u(t,x)$ be the solution of Problem (1) (with $f \equiv 0$). It follows from the maximum principle for the heat equation ([P-W]) that $u(t,x)$ is positive.

Fixing t, we can apply the first step to $u(t,.)$. We define

$$a_t(r) = \mathrm{Vol}\{x \in M \mid u(t,x) > r\}/\mathrm{Vol}(M),$$

$$\bar{u}_t(s) = \inf\{r \mid a_t(r) < s\},$$

$$E(t,s) = \{x \in M \mid u(t,x) > \bar{u}_t(s)\},$$

$$F(t,s) = \int_{E(t,s)} u(t,x) dv_g(x).$$

We deduce from the first step that

$$\int_{E(t,s)} \Delta u(t,x) dv_g(x) \geq -H^2(s) \frac{\partial^2 F}{\partial s^2}(t,s).$$

18. <u>Lemma</u>. ([BE] Lemma 4.23 p. 212)

$$\int_{E(t,s)} \frac{\partial u}{\partial t}(t,x) dv_g(x) = \frac{\partial F}{\partial t}(t,s).$$

Finally, we conclude from Lemma 18 and the preceding inequality (recall that $u(t,x)$ solves the heat equation) that

(19) $$\frac{\partial F}{\partial t}(t,s) - H^2(s)\,\frac{\partial^2 F}{\partial s^2}(t,s) \le 0.$$

In particular, by letting f_o tend to the Dirac measure δ_y at y in M we conclude that we can also take $u(t,x)=k(t,x,y)$ (y fixed).

20. Let $f_o^{\wedge}$ be a C^∞ radial decreasing function in (M^*,g^*). We consider the solution $u^{\wedge}(t,x)$ of

$$(\frac{\partial}{\partial t} + \Delta^*)u^{\wedge}(t,x) = 0,$$

$$u^{\wedge}(0,x) = f_o^{\wedge}(x),$$

on (M^*,g^*). We conclude that $u^{\wedge}(t,x)$ is also a C^∞-radial decreasing function on (M^*,g^*) ([BE] Prop. 4.8 p.214).

If we apply to $u^{\wedge}$ what we did before for u and if we take the first step into account, we conclude (using obvious notations) that

(21) $$\frac{\partial F^{\wedge}}{\partial t}(t,s) - H^2(s)\,\frac{\partial^2 F^{\wedge}}{\partial s^2}(t,s) = 0.$$

Now we choose a sequence of radial decreasing function $f_{o,n}^{\wedge}$ converging to δ_N, the Dirac measure at N in (M^*,g^*). It follows that (21) also holds for $k_*(t,N,.) = u^{\wedge}(t,.)$ where k_* is the heat kernel on (M^*,g^*).

22. <u>Step 3</u>. Using $u(t,x) = k(t,x,y)$, y fixed in M, and $u^{\wedge}(t,x) = k_*(t,N,x)$ we define $h(t,s)$ by

$$h(t,s) = F(t,s) - F^{\wedge}(t,s).$$

This function satisfies the following properties.

23. Properties.

(i) $\frac{\partial h}{\partial t}(t,s) - H^2(s)\,\frac{\partial^2 h}{\partial s^2}(t,s) \le 0$ for (t,s) in $\mathbb{R}_+^{\bullet} \times [0,1]$;

(ii) $h(t,0) = 0$ for all $t > 0$ (we integrate functions on a set with volume equal to 0);

(iii) $\lim_{t\to 0_+} h(t,s) = 0$ for all $s \in [0,1]$ (because k and k_* are heat kernels);

(iv) $h(t,1) = 0$ for all $t > 0$ ($\frac{\partial h}{\partial t}(t,1) = 0$ for all t, because $\int_M \Delta f \, dv_g = 0$ for all C^∞ f and $h(0,1) = 0$).

From these properties and the maximum principle (see [P-W]) applied to h, we conclude that $h(t,s) \le 0$ or

$$(24)\quad \begin{cases} \text{For all } (t,s) \text{ in } \mathbb{R}_+^{\bullet} \times [0,1],\ F(t,s) \le F^{\wedge}(t,s), \text{ or equivalently,} \\ \text{For all } (t,s)\ \mathrm{Vol}(M) \int_0^s \bar{u}_t(r)dr \le \mathrm{Vol}(M^*) \int_0^s \bar{u}_t^{\wedge}(r)dr. \end{cases}$$

It follows from the convexity of $t \to t^2$ and from the second mean value theorem ([BE] p. 173-174), that

$$(25)\quad \text{For all } t > 0,\ \mathrm{Vol}(M)^2 \int_0^1 \bar{u}_t^2(r)dr \le \mathrm{Vol}(M^*)^2 \int_0^1 \bar{u}_t^{\wedge 2}(r)dr.$$

$$\text{Now,}\quad \mathrm{Vol}(M) \int_0^1 \bar{u}_t^2(r)dr = \int_M k^2(t,x,y)dv_g(y) = k(2t,x,x),$$

where the second equality follows from the semi-group property of the heat kernel k on M (e.g. use Theorem 3) and similarly $\mathrm{Vol}(M^*) \int_0^1 \bar{u}_t^{\wedge 2}(r)dr = k_*(2t,N,N)$. Finally we have proved that

$$\mathrm{Vol}(M)k(2t,x,x) \le \mathrm{Vol}(M^*)k_*(2t,N,N),$$

From which Theorem 9 easily follows ■

C. ISOPERIMETRIC INEQUALITY FOR THE HEAT KERNEL, II.

As we already mentionned in relation with Cheeger's estimate in Chapter IV nº 28 (iii), Theorem 9 is only interesting if we have a "good" isoperimetric estimator $H(\beta)$ on (M,g). As we also pointed out in nº IV.12, it is not always possible to use the isoperimetric function $h(\beta)$ itself, although its properties (Proposition IV.11) allow us to construct a model (M_h^*, g_h^*) with a "good" heat kernel (see nº 29 below).

The following theorem (see [B-B-G1]) gives a nice isoperimetric inequality for heat kernel comparisons (for another Theorem see [B-G] p. XV.17).

26. Theorem. Let (M,g) be an n-dimensional compact Riemannian manifold without boundary. We define

$$r_{min}(M) = \inf\{\mathrm{Ricci}(M,g)(u,u) \mid u \text{ unit tangent vector to } M\},$$

$$d(M) = \mathrm{Diam}(M,g).$$

If (M,g) satisfies $r_{min}(M)d(M)^2 \ge \epsilon(n-1)\alpha^2$ for $\epsilon \in \{-1,0,1\}$ and $\alpha \in \mathbb{R}_+^*$, there exists a positive number $a(n,\epsilon,\alpha)$ such that for all β in $[0,1]$,

$$h(M,g;\beta) \ge h(S^n(R), \mathrm{can};\beta)$$

where $S^n(R)$ is the sphere of radius $R = d(M)/a(n,\epsilon,\alpha)$, in $\mathbb{R}^{n+1}$ with induced metric.

The number $a(n,\varepsilon,\alpha)$ is defined by

$$a(n,\varepsilon,\alpha) = \begin{cases} \alpha\omega_n^{1/n}\left(2\int_0^{\alpha/2}\cos^{n-1}(t)dt\right)^{-1/n} & \text{if } \varepsilon = 1; \\ (1 + n\omega_n)^{1/n} - 1 & \text{if } \varepsilon = 0; \\ \alpha c(\alpha) & \text{if } \varepsilon = -1, \end{cases}$$

where $\omega_n = \mathrm{Vol}(S^n)/\mathrm{Vol}(S^{n-1})$, and where $c(\alpha)$ is the unique positive root of the equation

$$x\int_0^{\alpha}(\mathrm{ch}t + x\,\mathrm{sh}t)^{n-1}dt = \omega_n.$$

The proof of this theorem is rather difficult, we refer to [B-B-Gl],

27. Remarks.

(a) When $\varepsilon = 1$, Myers' Theorem (nº II.29) implies that $\alpha \le \pi$;

(b) Myers' theorem also shows that the estimate when $\varepsilon = 1$ improves Gromov's isoperimetric inequality on manifolds with positive Ricci curvature (see [GV1]); Theorem 26 generalizes Gromov's theorem to all manifolds, whatever the sign of their Ricci curvature;

(c) In the case $\varepsilon = -1$, we can also replace $a(n,-1,\alpha)$ by the following lower bound for $\alpha c(\alpha)$ (see [B-B-Gl]) $\alpha c(\alpha) \geq$ $\geq \alpha \min\{C(\alpha), C(\alpha)^{1/n}\}$, where

$$C(\alpha) = (n-1)\omega_n/[\exp((n-1)\alpha) - 1];$$

(d) If we now take $H(M,g;\beta) = h(S^n(R),can;\beta)$ as isoperimetric estimator on (M,g), we notice that $H(\beta) = H(1-\beta)$ and that $H(\beta) \sim C\beta^{(n-1)/n}$ when β tends to 0 (for some constant C). The model space (M^*,g^*) associated with $H(\beta)$ is then $(S^n(R),can)$.

Taking into account the behaviour of $k_M(t,x,y)$ under scaling (e.g. use Theorem 3) and the fact that (S^n, can) is 2-point homogeneous, which implies that $k_{S^n}(t,x,x)$ is independent of x, we deduce from Theorem 9 and Theorem 26 the following (see [B-B-G1] §III).

28. <u>Theorem</u>. <u>Under the assumptions of Theorem</u> 26, <u>we have</u>

$$Z(M,g;t) \leq \mathrm{Vol}(M,g) \sup_M k_M(t,x,x) \leq Z(S^n, can; t/R^2)$$

(<u>where</u> $R = d(M)/a(n,\epsilon,\alpha)$).

29. <u>Remarks</u>. (of a philosophical flavor) In Theorem 9, we gave a comparison theorem using the heat kernel k_* on (M^*, g^*). As the proof of Theorem 9 shows, we only used the function $K_*(t,x) = k_*(t,N,x)$ on (M^*, g^*). The function $K_*(t,.)$ is a radial function with respect to N, so that we only use the radial part of the heat kernel, associated with the radial part of the Laplacian (see nº IV.20)

$$\Delta_r^* = -\{\frac{\partial^2}{\partial r^2} + (n-1)\frac{a'(r)}{a(r)}\frac{\partial}{\partial r}\},$$

where $(M^*, g^*) = (S^{n-1} \times]0,L[,\ a^2(r)d\theta^2 + dr^2)$.

Assume that $H(\beta) \sim C\beta^{\alpha}$, $1 > \alpha \geq (n-1)/n$, when β tends to 0 (for some constant C). It is then easy to check (by making an appropriate choice for V^*) that

$$a(r) \sim r \quad \text{if} \quad \alpha = (n-1)/n, \quad \text{and}$$

$$a(r) \sim r^{\gamma},\ \gamma > 1 \quad \text{if} \quad 1 > \alpha > (n-1)/n,$$

when r tends to 0.

Now recall that the radial part of the Laplacian in $(\mathbb{R}^n, can)$ is

$\Delta_r^n = -\{\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r}\}$ (this non-standard notation will only be used in these few lines).

We conclude that for r close to 0, Δ_r^* looks like Δ_r^n when $\alpha = \frac{n-1}{n}$ and looks like Δ_r^p, $p > n$ ($p \in \mathbb{R}$), when $1 > \alpha > \frac{n-1}{n}$. This means that our comparison function $k_*(t,N,N)$ will look like an "n-dimensional" heat kernel if we choose $\alpha = \frac{n-1}{n}$, and like a "p-dimensional" heat kernel ($p > n$) when $1 > \alpha > \frac{n-1}{n}$. In particular, its behaviour when t tends to 0_+ will be in $t^{-n/2}$ (resp. $t^{-p/2}$) when $\alpha = \frac{n-1}{n}$ (resp. $1 > \alpha > \frac{n-1}{n}$).

Because of Proposition IV.11, we see that it is much better to take an isoperimetric estimator H such that $\alpha = \frac{n-1}{n}$; this is the case if we use Theorem 26.

Another interpretation can be made in terms of the behaviour of the function $A(s)$ (see nº IV.14) when s goes to 0. If $A(s) \sim C\, s^m$ when s goes to 0, the volume of a small geodesic ball in (M^*,g^*) is of the order of s^m and hence M^* has "isoperimetric" dimension m (*) (recall that we are near a conic point). It is clear that it is better to compare (M,g) to an m-dimensional manifold, with $m = \dim M$. This means again that we have to take $\alpha = \frac{n-1}{n}$.

The case $\alpha = 1$ is even worst because (M^*,g^*) is no longer compact (see nº IV.22).

Let us also mention here that the isoperimetric function $h(\beta)$ has exactly the required properties which allow us to define the heat kernel for Δ_r^* on the associated model space (see Proposition IV.11 or [B-B-G2]).

* This is a non-standard notion which we use here for convenience.

D. APPLICATIONS

In this paragraph we give some direct applications of Theorem 28; for further results we refer to [B-G] §3 or [B-B-G1] § III.

30. Let (M,g) be a compact Riemannian manifold without boundary such that (<u>notations as in</u> nº 26-29)

$$n = \dim M;$$

$$r_{min}(M)d(M)^2 \geq \epsilon(n-1)\alpha^2.$$

Let $\{\lambda_i\}_{i\geq 1}$ be the sequence of eigenvalues (counted with multiplicities) of (M,g) [Notice that we count the eigenvalues from $i = 1,\ldots,$ some authors begin with $i = 0$ e.g. [B-G-M]].

31. <u>Theorem</u>. <u>Let</u> (M,g) <u>be as above. Then</u>

(i) $\lambda_2(M,g) \geq a^2(n,\epsilon,\alpha)\, n\, d(M)^{-2}$;

(ii) <u>there exists a number</u> $C(n,\epsilon,\alpha)$ <u>such that for</u> $i \geq 2$,

$$\lambda_i(M,g) \geq C(n,\epsilon,\alpha)\, i^{2/n}\, d(M)^{-2}.$$

<u>Proof</u>. Using Theorem 28, we can write

$$(*)\ Z(M,g;t) \leq Z(S^n,\text{can};a^2(n,\epsilon,\alpha)t\, d(M)^{-2}).$$

For any Riemannian manifold (compact, connected, without boundary), we have

$$Z(M,g;t) = 1 + \sum_{j=2}^{\infty} \exp(-\lambda_j(M,g)t)$$

because $\lambda_1(M,g) = 0$ has multiplicity 1.

Assertion (i) follows from (*) by substracting 1 to both sides, by taking Log and by letting t tend to infinity.

It follows from Property 7 that there exists constants $C(n)$ and $D(n)$ such that

$$(**) \text{ for all } t > 0,\ Z(S^n, can; t) \leq C(n) t^{-n/2} + D(n).$$

Let $N(\lambda) = \mathrm{Card}\{j \mid \lambda_j(M,g) \leq \lambda\}$. We can then write

$$k \leq N(\lambda_k) \leq e \sum_{\lambda_j \leq \lambda_k} \exp(-\lambda_j/\lambda_k) \leq e Z(M,g;1/\lambda_k)$$

and, using Theorem 28 and (**),

$$k \leq e\, C(n)(d(M)/a(n,\varepsilon,\alpha))^n \lambda_k^{n/2} + e\, D(n)$$

Assertion (ii) follows ■

32. <u>Remarks</u>.

(i) A theorem of Lichnerowicz states that if $r_{min}(M) \geq (n-1)$ then $\lambda_2(M,g) \geq n = \lambda_2(S^n, can)$. Recall that Myers' theorem implies that $d(M) \leq \pi$. The expression of $a(n,\varepsilon,\alpha)$ in Theorem 26 together with Theorem 31(i) give

$$r_{min}(M) \geq n-1 \Rightarrow \lambda_2(M,g) \geq$$

$$\geq n \left\{ \int_0^{\pi/2} \cos^{n-1}(t)dt \Big/ \int_0^{d(M)/2} \cos^{n-1}(t)dt \right\}^{2/n} > n$$

when $d(M) < \pi$. In fact, one can show that $\lambda_2(M,g) = n$ implies that (M,g) is isometric to (S^n, can); this is Obata's theorem: see [B-G-M] Chap. III.D and compare with [CG] and [CL] Chap. III.4. In fact Assertion (i) in Theorem 31 can also be proved by an argument à la Faber-Krahn (see nº IV.31); the Lichnerowicz-Obata theorem also follows from this method: see [B-M];

(ii) For $r_{min}(M) \geq 0$, Theorem 31(i) and Theorem 26 give

$\lambda_2(M,g) \geq 8/d(M)^2$ when $\dim M = 2$; on the other hand,

$\lambda_2(S^1(R) \times S^1) \sim \pi^2/d^2(S^1(r) \times S^1)$ when r goes to infinity;

(iii) Weyl's estimate gives

$\lambda_k(M,g) \sim C(n)k^{2/n}\,\mathrm{Vol}(M,g)^{-2/n}$ when k goes to infinity.

It turns out that one cannot substitute $\mathrm{Vol}(M,g)^{-2/n}$ to $d(M)^{-2}$ in Theorem 31(i), as the following example shows. Consider the Riemannian manifolds

$$M_a = S^{n-1}(a^{1/(n-1)}) \times S^1(1/a)$$

with the product metric; they satisfy $\mathrm{Vol}(M_a) = \mathrm{Vol}(S^{n-1}) \times \mathrm{Vol}(S^1) = \mathrm{Vol}(M_1)$, and $\mathrm{Diam}(M_a)$ goes to infinity when a goes to zero. The number $N_a(\lambda)$ of eigenvalues of M_a less than λ satisfies $N_a(\lambda) \geq 2\,\mathrm{Card}\{p \in \mathbb{N}^{\bullet} \mid a^2p^2 \leq \lambda\}$. This shows that for fixed $\lambda, N_a(\lambda)$ goes to infinity when a goes to zero. In particular, this implies that $\lambda_k(M_a)\mathrm{Vol}(M_a)^{2/n}$ goes to zero with a. However it follows from Theorem 31(ii) that $\lambda_k(M_a)d(M_a)^2$ is bounded from below when a goes to zero.

For a counter-example involving the lower bound on Ricci see [GA2], I.1.2.

At least qualitatively, the estimate in Theorem 31(ii) is best possible (it was obtained by Gromov in [GV1] with a worst constant).

33. Remark. Since $k_M(t,x,x) = \sum_{j=1}^{\infty} \exp(-\lambda_j t)\phi_j^2(x)$, Theorem 28 also gives bounds on the L^∞-norm of the eigenfunctions ϕ_j of Δ on (M,g).

The next chapter is devoted to <u>inverse geometric results</u>. We will use Theorem 28 in a crucial way.

Further references for Chapter V: For other comparison theorems on the heat kernel see [CL] or the references in [B-B]. For the heat kernel itself, see [DK], [GY].

CHAPTER VI

GEOMETRIC APPLICATIONS OF ISOPERIMETRIC METHODS

ALL RIEMANNIAN MANIFOLDS ARE COMPACT,
CONNECTED WITHOUT BOUNDARY

In this chapter, we will give some partial answers to the following geometric inverse problem.

1. Problem. To what extent do local estimates on the curvature of a Riemannian manifold (M,g) inforce global restrictions on the manifold?

A. INTRODUCTION.

In order to explain the meaning of Problem 1, let us give an appropriate formulation of the Gauss-Bonnet theorem ([HF] Part II, Chap. III).

2. Theorem. Let (M,g) be any compact Riemannian surface, whose curvature K is bounded from below by the real number k. Then

(i) $$\chi(M) = \frac{1}{2\pi}\int_M K(M)dv_g \geq \frac{k}{2\pi}\mathrm{Vol}(M,g);$$

(ii) $$b_1(M) \leq 2 - \frac{k}{2\pi}\mathrm{Vol}(M,g).$$

Here $\chi(M)$ denotes the Euler characteristic of M and $b_1(M) = 2 - \chi(M)$ the first Betti number of M (these are topological invariants which do not depend on the choice of a Riemannian metric g on M); $Vol(M,g)$ is the 2-dimensional volume of (M,g).

3. Corollary. The number of differentiable surfaces which admit a Riemannian metric whose curvature is bounded from below by k, and whose volume is bounded from above by V (k in $\mathbb{R}$, V in $\mathbb{R}_+$) is finite.

4. Comments.

(i) Let us first point out that the product $K(M)Vol(M)$ is a Riemannian invariant with weight 0 in dimension 2;

(ii) By scaling the metric, it is always possible to bound any curvature of an n-dimensional manifold by 1 in absolute value, so that we cannot expect any general theorem answering Problem 1, without scaling. In order to scale the metric, we can use a Riemannian invariant, e.g. the volume or the diameter. In the Gauss-Bonnet theorem, the metric is scaled by giving an upper bound on the volume. In general, we will have to use the diameter; the following example shows that fixing the volume is a very weak condition. Take any manifold (N,g). By an appropriate choice of R, the Riemannian manifold $(N \times S^1(R), g \times can_R) = (M,g_R)$ has volume one. However the topology of M may be very complicated;

(iii) In dimension bigger than 2, we have several notions of curvature. We will always try to use the weakest possible notion. In general we will try to use the Ricci curvature (the scalar curvature is very often too weak an invariant). This is the case if we want bounds on the eigenvalues of the Laplacian Δ^g of (M,g): see nº V.31. In other situations, we will have to make assumptions on the <u>sectional curvature</u> (see nº 24 (ii)).

Finally, we reduce Problem 1 to the following.

5. *Problem*. Give global bounds on M (e.g. on topological invariants) in terms of (Diam(M,g) and Ricci(M,g)) or (Diam(M,g) and Sect(M,g)).

6. *Examples*.

(i) Myers' theorem (see nº II.29) says that if $\text{Ricci}(M,g) > 0$, then $\pi_1(M)$, the fundamental group of M, is finite. This is a partial answer to Problem 5. Notice that no scaling is required here. In fact, a consequence of Myers' theorem is that $\text{Diam}(M,g) \leq \pi/k$, if $\text{Ricci}(M,g) \geq (n-1)k^2 > 0$. However, taking Diam(M,g) into account gives sharper results (see nº V.32 and [B-B-G1] Corollary 17), so that in some sense scaling is also necessary here;

(ii) In the 1940's, S. Bochner proved the following results

$$\text{Ricci}(M,g) > 0 \Rightarrow b_1(M) = 0 \quad \text{(1st Betti number)},$$

$$\text{Ricci}(M,g) \geq 0 \Rightarrow b_1(M) \leq \dim(M).$$

These results were obtained by an analytic method which we now describe (notice that Myers' theorem is proved by geometric methods).

B. THE ANALYTIC APPROACH, I.

7. Let (M,g) be an n-dimensional Riemannian manifold (compact, connected, without boundary). We denote by $\Lambda^p T^* M$, $0 \leq p \leq n$, the p^{th} exterior product of T^*M and by $E^p(M)$ the C^∞ sections of

$\wedge^p T^*M$, i.e. the exterior forms of degree p on M. The exterior differential d is a first order differential operator from $E^p(M)$ to $E^{p+1}(M)$. This operator only depends on the differentiable structure.

We now define an operator $\delta: E^{p+1}(M) \to E^p(M)$ by

(8) $$\langle\langle \alpha | \delta\beta \rangle\rangle = \langle\langle d\alpha | \beta \rangle\rangle,$$

for all α in $E^p(M)$ and β in $E^{p+1}(M)$: the metric g on TM induces a metric on each $\wedge^p T_x^*M$ which we denote by $\langle . | . \rangle_x$; we define δ by

$$\int_M \langle \alpha | \delta\beta \rangle_x dv_g(x) = \int_M \langle d\alpha | \beta \rangle_x \, dv_g(x)$$

(note that in the first integral we have the scalar product of two p-forms, and in the second integral the scalar product of two (p+1)-forms). We say that δ is the <u>formal adjoint</u> of d (note that δ depends on the Riemannian metric g).

9. <u>Properties</u>.

(i) <u>If</u> $f \in C^\infty(M) = E^\circ(M)$, <u>then</u> $\delta f = 0$;

(ii) <u>If</u> $\alpha \in E^1(M)$, <u>then</u> $\delta\alpha = -\operatorname{Div}_g(\alpha^\#)$;

(iii) <u>Let</u> $\{e_1,\ldots,e_n\}$ <u>be a local orthonormal frame in</u> M. <u>For</u> α <u>in</u> $E^p(M)$ <u>we note</u>

$\alpha(i_1,\ldots,i_p)$ <u>for</u> $\alpha(e_{i_1},\ldots,e_{i_p})$ <u>where</u> $i_1,\ldots,i_p \in \{1,\ldots,n\}$.

<u>Then</u>

$$d\alpha(i_1,\ldots,i_{p+1}) = \sum_{k=1}^{p+1} (-1)^{k+1} (D_{i_k}\alpha)(i_1,\ldots,\hat{i}_k,\ldots,i_{p+1}),$$

$$\delta\alpha(i_2,\ldots,i_p) = -\sum_{k=1}^{n} (D_k\alpha)(k,i_2,\ldots,i_p)$$

(recall that we note D_k for D_{e_k}, where D is the Levi-Civita connection of (M,g));

(iv) If M is oriented and if $*$ denotes the Hodge operator on M, $*: E^p(M) \to E^{n-p}(M)$, then

$\delta: E^p(M) \to E^{p-1}(M)$ satisfies

$\delta = (-1)^{n(p+1)} * d *$.

Proof.

(i) is dual to the fact that $d\alpha = 0$ for all α in $E^n(M)$;

(ii) follows from the definitions of δ and Div_g (see also [B-G-M] Chap. II.G);

(iii) the formulae for d and δ follow from the definitions of d, δ and of the Levi-Civita connection D of (M,g);

For (iv) see [WA] Chap. 6. [The formula $\langle\alpha|\beta\rangle_x dv_g(x) = \beta\wedge *\alpha$, defines $*$ uniquely] ■

10. Definition. One defines the Laplace-Beltrami operator (or Laplacian) on p-forms by $\Delta = \delta d + d\delta$ (this operator is also called Hodge-de Rham Laplacian).

For further details on δ, Δ see [WA] Chap. 6 and [CL] Appendix, or [LZ] and [RM].

The classical Hodge- de Rham theory ([WA] Chap. 5 and 6) states that

(11) $b_p(M) = \dim \mathrm{Harm}^p(M)$;

the p^{th} Betti number of the manifold M is equal to the dimension of the space of harmonic p-forms $(\mathrm{Harm}^p(M) = \{\alpha \in E^p(M) \mid \Delta\alpha=0\})$.

Note that $b_p(M)$ is a topological invariant, while $\mathrm{Harm}^p(M)$ depends on the Riemannian metric.

In order to prove Bochner's results, and to introduce the analytic method, we need the following.

12. __Lemma__. __Let__ (M,g) __be a Riemannian manifold and let__ α __be a__ 1-__form on__ M. __The following formulae hold__

(i) $\Delta\alpha = D^*D\alpha + \mathrm{Ricci}(\alpha^\#,.)$;

(ii) $\langle\Delta\alpha|\alpha\rangle = \frac{1}{2}\Delta(\langle\alpha|\alpha\rangle) + |D\alpha|^2 + \mathrm{Ricci}(\alpha^\#,\alpha^\#)$.

In Formula (i), the Laplacian is the Laplacian acting on 1-forms, D the Riemannian connection on 1-forms, and D^* its adjoint; for a tensor field β, D^* is given by $D^*\beta = -\mathrm{Trace}\, D\beta$ (contraction of the first two indices or $D^*\beta(.) = -\sum_k D_k\beta(k,.)$, in a local orthonormal frame (notations as in nº 9). Equivalently, we have

$$D^*D\alpha = -\sum_{i=1}^{n}\{D_i(D_i\alpha) - D_{D_i i}\alpha\},$$

in a local orthonormal frame (notations as in nº 9).

In formula (ii), the left hand side is the point-wise scalar product of two 1-forms; $\Delta(\langle\alpha|\alpha\rangle)$ is the Laplacian of the function $\langle\alpha|\alpha\rangle$ and $|D\alpha|$ is the norm of the 2-tensor $D\alpha$.

13. __Definitions__. The operator $\bar{\Delta} = D^*D$ is called the __rough Laplacian__ (here on 1-forms). Formulae (i) and (ii) are called __Weitzenböck formulae__.

Proof of Lemma 12. We use the notations of nº 9; we denote by $\{e_i\}$ a local orthonormal frame near x; we can always assume that we have $(D_{e_i} e_i)(x) = 0$, at the point x (see III.40).

Claim 1. For β a section of $\otimes^p T^*M$, we have

$$D^*\beta = -\text{Trace } D\beta = -\sum_k D_k\beta(k,.).$$

Let γ be a section of $\otimes^{p-1}T^*M$. We consider the 1-form $\omega = \sum_I \beta(.,I)\gamma(I)$, where I is a multi-index of length $(p-1)$. Now $\delta\omega$ is a function on M which is given at x by (see nº 9 (iii))

$$\delta\omega = -\sum_{k=1}^{n} e_k \cdot (\sum_I \beta(k,I)\gamma(I)).$$ An easy computation gives

$$\delta\omega(x) = -\langle D^\wedge\beta|\gamma\rangle_x + \langle\beta|D\gamma\rangle_x$$

where $D^\wedge\beta = -\text{Trace } D\beta$. Since this is valid for all x in M, and since $\int_M \delta\omega \, dv_g = 0$ (Divergence Theorem III.9), we have $\langle\langle D^\wedge\beta|\gamma\rangle\rangle = \langle\langle\beta|D\gamma\rangle\rangle$, which shows that $D^\wedge = D^*$ (see nº III.9 and [B-G-M] Chap. II.GII).

Claim 2. For any 1-form α,

$$D^*D\alpha = -\sum_{i=1}^{n} \{D_i(D_i\alpha) - D_{D_i i}\alpha\}.$$

Using the first claim we have for all k, $D^*D\alpha(e_k) = -\sum_j D_j\beta(j,k)$, where $\beta = D\alpha$, i.e.

$$D^*D\alpha(e_k) = -\sum_j e_j \cdot \beta(j,k) + \sum_j[\beta(D_j j,k) + \beta(j,D_j k)].$$

Now $\beta(j,k) = D\alpha(j;k) = (D_j\alpha)(k)$, so that

$$D^*D\alpha(e_k) = -\sum_j D_j((D_j\alpha))(k) - \sum_j (D_j\alpha)(D_jk)$$

$$+ \sum_j (D_{D_jj}\alpha)(k) + \sum_j (D_j\alpha)(D_jk)$$

$$= -\sum_j \{D_j(D_j\alpha) - D_{D_jj}\alpha\}(k).$$

Note that at the point x we can write

$$D^*D\alpha_x = -\sum_{i=1}^{n} D_i(D_i\alpha)_x \quad \text{since} \quad D_i i(x) = 0.$$

<u>Proof of formula (i)</u>

We use the formula

$$d\alpha(i,j) = (D_i\alpha)(j) - (D_j\alpha)(i), \quad \text{see 9(iii)}$$

(this formula easily follows from the definitions of d and D:

$$d\alpha(X,Y) = X.\alpha(Y) - Y.\alpha(X) - \alpha([X,Y])).$$

This gives, at the point x $(D_i j(x) = 0)$,

$$\delta d\alpha(i)_x = -\sum_{k=1}^{n} e_k.(d\alpha(e_k,e_i))_x$$

$$= -\sum_{k=1}^{n} D_k(D_k\alpha)(i)_x + \sum_{k=1}^{n} e_k.((D_i\alpha)(k))_x.$$

We can also write

$$d\delta\alpha(i)_x = e_i.(\delta\alpha) = -\sum_{k=1}^{n} e_i.(D_k\alpha(k)).$$

Finally we can write

$$\Delta\alpha(i)_x = \bar{\Delta}\alpha(i)_x + \sum_{k=1}^{n}\{e_k.((D_i\alpha)(k))_x - e_i.((D_k\alpha)(k))_x\}.$$

Since $(D_j\alpha)^{\#} = D_j(\alpha^{\#})$, the second term in the right-hand side can be written as

$$\langle (D_kD_i - D_iD_k)\alpha^{\#}, e_k\rangle_x = \langle R(e_k,e_i)\alpha^{\#} + D_{[e_k,e_i]}\alpha^{\#}, e_k\rangle_x,$$

in view of nº II.43. Now we have $[e_k,e_i](x) = 0$ (see nº II.35) so that

$$\Delta\alpha(i)_x = \bar{\Delta}\alpha(i)_x + \sum_{k=1}^{n} \langle R(e_k,e_i)\alpha^{\#}, e_k\rangle_x$$

$$= \bar{\Delta}\alpha(i)_x + \sum_{k=1}^{n} R(e_k,e_i;e_k,\alpha^{\#})_x \quad \text{(see nº II.45)}$$

$$= \bar{\Delta}\alpha(i)_x + \mathrm{Ricci}_x(\alpha^{\#},e_i) \quad \text{(see nº II.45).}$$

<u>Proof of formula (ii)</u>

In order to prove (ii), it suffices to prove

$$\langle \bar{\Delta}\alpha|\alpha\rangle = \frac{1}{2}\,\Delta(\langle\alpha|\alpha\rangle) + |D\alpha|^2.$$

We can write

$$e_k.\langle\alpha|\alpha\rangle = 2\langle D_k\alpha|\alpha\rangle, \quad \text{and}$$

$$\Delta(\langle\alpha|\alpha\rangle)_x = -\sum_k e_k.(e_k(\langle\alpha|\alpha\rangle))_x$$

$$= -2\sum_k\langle D_k\alpha|D_k\alpha\rangle_x - 2\sum_k\langle D_k(D_k\alpha)|\alpha\rangle_x$$

($D_k i = 0$ at x). Finally we have

$$\Delta(\langle\alpha|\alpha\rangle)_x = -2|D\alpha|_x^2 + 2\langle\Delta\alpha|\alpha\rangle_x \quad ■$$

14. <u>Definition</u>. A 1-form α is <u>parallel</u> if $D\alpha = 0$.

15. **Exercises.**

(i) Show that the point-wise norm $|\alpha|_x$ of a parallel 1-form α is constant;

(ii) Show that the vector-space of parallel 1-forms has dimension less than or equal to $n = \dim M$ [Hint: Take a curve $c(t)$ in M and let $\{e_i\}$ be a parallel orthonormal frame along c (see nº II.40). Show that if $\alpha_{c(t)} = \Sigma\, \alpha_i(t) e_i^\flat(t)$, then the $\alpha_i(t)$ are constant functions].

16. **Proof of Bochner's results.**

Integrating formula 12(ii) on M, we obtain

$$\langle\langle \Delta\alpha | \alpha \rangle\rangle = \int_M |D\alpha|^2 \, dv_g + \int_M \mathrm{Ricci}(\alpha^\#, \alpha^\#) dv_g,$$

because $\int_M \Delta f \, dv_g = 0$ for any C^∞ function f.

In view of the Hodge- de Rham theory (nº (11)), we take α to be a harmonic 1-form. Finally we obtain

(17) Any harmonic 1-form α satisfies

$$\int_M |D\alpha|^2 dv_g + \int_M \mathrm{Ricci}(\alpha^\#, \alpha^\#) dv_g = 0.$$

For x in M, we define

$$(18) \quad \begin{cases} \rho(x) = \inf\{\mathrm{Ricci}(u,u) \,|\, u \text{ unit vector in } T_x M\}, \text{ and} \\ r_{min} = \inf\{\rho(x) \,|\, x \text{ in } M\}. \end{cases}$$

We have the inequalities

(19) $$\int_M \mathrm{Ricci}(\alpha^\#,\alpha^\#)dv_g \geq \int_M \rho(x)\langle\alpha|\alpha\rangle_x dv_g \geq r_{min} \langle\langle\alpha|\alpha\rangle\rangle.$$

20. Proposition. With the above notations, we have the following results.

(i) If $\rho(x) \geq 0$, and if there exists an x_o such that $\rho(x_o) > 0$, then $b_1(M) = 0$;

(ii) If $r_{min} = 0$, then $b_1(M) \leq n = \dim M$.

Proof. Assumption (i) implies that $\int_M |D\alpha|^2 \, dv_g < 0$ for any non-zero harmonic 1-form: this is impossible, hence $b_1(M) = 0$.

Assumption (ii) implies that $\int |D\alpha|^2 dv_g \leq 0$ and hence that $D\alpha = 0$. Any harmonic 1-form is parallel and hence $b_1(M) \leq n$, by Exercise 15(ii) ■

Notice that Assertion (i) in the Proposition is sharper than Bochner's result as stated in nº 6 (ii).

21. Generalizations.

The above situation can be generalized as follows. We consider a fiber bundle E over the Riemannian manifold (M,g). We assume that E is equipped with a Riemannian metric (we say that E is a Riemannian fibre bundle) i.e. with a scalar product $\langle .|.\rangle_x$ in the fibers E_x of E, depending C^∞ on x. We also assume that E is equipped with a connection $\bar{D}$, which is compatible with the scalar product i.e.

$$\bar{D}: \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(E)$$

is an $\mathbb{R}$-linear map which satisfies the following properties

(i) $\quad X.\langle u|v\rangle = \langle \bar{D}_X u|v\rangle + \langle u|\bar{D}_X v\rangle$,

for all X in $\mathfrak{A}(M)$ and u,v in $C^\infty(E)$;

(ii) $\quad \bar{D}_{fX}u = f\,\bar{D}_X u$,

for all f in $C^\infty(M)$, X in $\mathfrak{A}(M)$ and u in $C^\infty(E)$;

(iii) $\quad \bar{D}_X(fu) = (X.f)u + f\,\bar{D}_X u$,

for all f in $C^\infty(M)$, X in $\mathfrak{A}(M)$ and u in $C^\infty(E)$.

Finally, we assume that there is a natural Laplacian $\tilde{\Delta}$ acting on $C^\infty(E)$ (i.e. a 2^{nd} order linear partial differential operator, with properties similar to those of the Laplace-Beltrami operator) and that $\tilde{\Delta}$ satisfies the following Weitzenböck formula

(22) $$\tilde{\Delta}s = \bar{\Delta}s + \mathfrak{R}s,$$

where $\bar{\Delta}s = -\sum_{i=1}^{n}\{\bar{D}_{e_i}(\bar{D}_{e_i}s) - \bar{D}_{D_{e_i}e_i}s\}$ is the rough Laplacian, $\{e_i\}$ local orthonormal frame in M, and where $\mathfrak{R}$ is a symmetric endomorphism of the bundle E, $\mathfrak{R}_x : E_x \to E_x$ is an endomorphism of E_x which satisfies $\langle \mathfrak{R}_x u|v\rangle_x = \langle u|\mathfrak{R}_x v\rangle_x$ for all u,v in E_x.

As above we define

(23) $$\begin{cases} \mathfrak{R}(x) = \inf\{\langle \mathfrak{R}_x u|u\rangle_x \mid u \text{ in } E_x,\ \langle u|u\rangle_x = 1\}, \\ \mathfrak{R}_{min} = \inf\{\mathfrak{R}(x) \mid x \text{ in } M\}. \end{cases}$$

24. Examples.

(i) $E = T^*M$, $\bar{D}$ is the Levi-Civita connection, $\tilde{\Delta}$ is the Laplacian on 1-forms, $\mathfrak{R}s = \mathrm{Ricci}(s^\#,.)$; in that case,

$$\mathfrak{R}_{min} = r_{min} \quad \text{(see nº 7.20);}$$

(ii) $E = \Lambda^p T^* M$, $\bar{D}$ is the Levi-civita connection, $\tilde{\Delta}$ is the Laplacian on p-forms, $\mathfrak{R}s$ can be expressed in terms of the curvature tensor of (M,g); in that case $\mathfrak{R}_{min}$ can be computed in terms of upper and lower bounds on $\mathrm{Sect}(M,g)$ (see [LZ] p. 3, [G-M] p. 264 and [B-G] p. XV. 8);

(iii) $E = S(M)$, the bundle of spinors, $\bar{D}$ is the Levi-Civita connection, $\tilde{\Delta}$ is the Dirac operator, $\mathfrak{R} = \frac{u}{4}$ (u is the scalar curvature of (M,g): see nº III.45);

(iv) Other examples include the moduli space of Einstein metrics ([B-G] § 3), Jacobi fields for harmonic maps or minimal immersions ([E-L] or [UA]).

25. It follows from the Weitzenböck formula (22) that

$$\int_M \langle \tilde{\Delta} s | s \rangle dv_g = \int_M |\bar{D} s|^2 dv_g + \int_M \langle \mathfrak{R} s | s \rangle dv_g .$$

The following Proposition is a direct consequence of the above formula (same methods as in the proof of Proposition 20).

26. <u>Proposition</u>. <u>Under the above assumptions, let</u> $\delta(E)$ <u>denote the dimension of the space of harmonic sections of</u> E,

$$\delta(E) = \dim\{s \in C^\infty(E) \mid \tilde{\Delta} s = 0\}$$

(<u>this dimension is finite because</u> $\tilde{\Delta}$ <u>is elliptic</u>).

(i) <u>If</u> $\mathfrak{R}(x) \geq 0$, <u>and if there exists an</u> x_o <u>in</u> M <u>such that</u> $\mathfrak{R}(x_o) > 0$, <u>then</u> $\delta(E) = 0$.

(ii) <u>If</u> $\Re_{min} = 0$ <u>then</u> $\delta(E) \le \ell = \operatorname{rank}(E)$.

Such results are called vanishing theorems. We will now deal with the following problem.

27. <u>Problem</u>. Give upper bounds on $\delta(E)$, in terms of estimates on $\Re(x)$ or $\Re_{min}$ and on the curvature of (M,g).

Proposition 26 gives a partial answer to Problem 27 when $\Re_{min} \geq 0$. In the following paragraph, we investigate the case $\Re_{min} < 0$.

C. THE ANALYTIC APPROACH, II.

28. Assume that $\Re_{min} = -k^2$, $k \in \mathbb{R}_+^{\bullet}$, and let s be a harmonic section of E, $\tilde{\Delta}s = 0$. It follows from nº 25 that

$$\int_M |\bar{D}s|^2 v_g \le k^2 \int_M |s|^2 v_g. \tag{29}$$

It follows from the very definition of $\bar{\Delta}$ and from the min-max principle (nº III.26), that (29) implies

$$\delta(E) \le \text{number of eigenvalues of } \bar{\Delta} \text{ less than } k^2.$$

30. <u>Note</u>. The operators $\tilde{\Delta}$ and $\bar{\Delta}$ are non-negative, symmetric, 2^{nd} order elliptic linear partial differential operators on $C^{\infty}(E)$, so that the spectrum of $\tilde{\Delta}$ (resp. $\bar{\Delta}$) consists of a sequence of non-negative eigenvalues, with finite multiplicities

$$(0\le)\ \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots \uparrow +\infty \quad (\text{resp. } (0\le)\ \bar{\lambda}_1 \le \bar{\lambda}_2 \le \dots \uparrow +\infty).$$

In fact, the above inequality does not say much because we do not know the eigenvalues of $\bar{\Delta}$. Since $|s|$ is a function on M, we can try to obtain an inequality on scalar functions on M (recall that we have bounds on the eigenvalues of Δ acting on $C^\infty(M)$ by Chapter V). For this purpose, we use the following lemma known as Kato's inequality (see [H-S-U]).

31. Lemma. *For any* s *in* $C^\infty(E)$, *the pointwise norm of* $s, |s|$, *is in* $H^1(M,g)$ *and we have the following inequality (in the sense of distributions)*

$$|d|s|| \le |\bar{D}s|.$$

Proof. In the sense of distributions, we can write, for any f in $C^\infty(M)$

$$\int_M (D_X|s|) f\, dv_g = -\int_M |s| \mathcal{L}_X(f\, dv_g)$$

$$= -\lim_{r\to 0} \int_M (|s|^2+r^2)^{1/2} \mathcal{L}_X(f\, dv_g)$$

$$= \lim_{r\to 0} \int_M \langle \bar{D}_X s|s\rangle (|s|^2+r^2)^{-1/2} f\, dv_g,$$

so that

$$d|s|(x) = \begin{cases} 0, & \text{if } s(x) = 0 \\ \langle \bar{D}_{\cdot} s|s\rangle_x / |s|_x, & \text{if } s(x) \ne 0, \end{cases}$$

and hence $d|s|$ is in $H^1(M)$ and satisfies

$$|d|s||_x \le |\bar{D}s|_x \quad \blacksquare$$

Note that $|d|s||$ is the norm of the 1-form $d|s|$ and that $|\bar{D}s|$ is the norm of the element $|\bar{D}s|$ in $T^*M\otimes E$.

In we apply Lemma 31 to (29), we find

$$(32) \qquad \int_M |df|^2 dv_g \leq k^2 \int_M f^2 \, dv_g, \quad \text{with } f = |s|.$$

From the min-max principle (nº III.26), we conclude that ... $k^2 \geq 0$, because $\lambda_1(M,g)$, closed) = 0 ... this is not very interesting! An interesting estimate would be $k^2 \geq \lambda_2(M,g)$, because we know how to bound $\lambda_2(M,g)$ from below by Cheeger's estimate (see nº IV. 26-29). In order to obtain such an estimate, we need to write (32) with a function f such that $\int_M f = 0$ (see nº III. 26).

Define $h(x)$ by $h(x) = |s|_x - \int_M |s|_x \, dv_g \, / \, \mathrm{Vol}(M)$.

It is clear that $\int_M h = 0$. In order to substitute h to f in (32), we use the following general lemma due to Daniel Meyer.

33. Lemma ([ME2]). <u>Let</u> E <u>be a Riemannian vector bundle over the Riemannian manifold</u> (M,g). <u>Let</u> F <u>be a finite dimensional subspace of</u> $L^2(E,v_g) = \{s \mid \int_M |s|_x^2 \, dv_g(x) < \infty\}$, <u>such that</u>

$$\dim F = N > \ell = \mathrm{rank}(E).$$

<u>Then there exists an element</u> s_o <u>in</u> F <u>such that</u>

$$\mathrm{Vol}(M)^{-1/2} \int_M |s_o|_x dv_g \leq C(\ell,N)\left(\int_M |s_o|_x^2 \, dv_g\right)^{1/2},$$

<u>where</u> $C(\ell,N)$ <u>is a universal function of</u> (ℓ,N) <u>which satisfies:</u>

$C(\ell,N)$ is strictly decreasing in N, $C(\ell,\ell) = 1$, $C(\ell,N)$ goes to 0 when N goes to infinity.

In order to apply Lemma 33, we take $F = \{s \in C^\infty(E) | \tilde{\Delta} s=0\}$. Then $N = \dim F = \delta(E)$. We assume that $N > \ell = rk(E)$. Let s_o be the section given by Lemma 33, and denote $|s_o|$ by f and $f - \int_M f\, dv_g / Vol(M)$ by h. We can then write

$$(1 - C^2(\ell,N)) \int_M f^2 \leq \int_M h^2,$$

so that (32) gives

$$\int_M |dh|^2 dv_g \leq k^2(1-C^2(\ell,N))^{-1} \int_M h^2\, dv_g;$$

this last inequality implies that

(34) $\qquad \lambda_2(M,g);\text{closed}) \leq k^2(1-C^2(\ell,N))^{-1}.$

We can now prove the following generalization of Bochner's results.

35. Theorem. Let E be a Riemannian vector bundle over the Riemannian manifold (M,g) (compact without boundary) as in nº 21. Assume that

$$r_{min}(M,g)d(M,g)^2 \geq \epsilon(n-1)a^2,\ \epsilon \in \{-1,0,1\},\ a \in \mathbb{R}_+^\cdot \text{ and } n = \dim M$$

(see nº V. 26; the interesting case here is $\epsilon = -1$).

Then there exists a positive number $b = b(n,\epsilon,a)$ such that $\mathfrak{R}_{min} d(M,g)^2 \geq -b$ implies $\delta(E) \leq \ell$.

Proof. By Cheeger's inequality (nº IV. 26-29), there exists a constant $c = c(n,\varepsilon,a)$ such that

$\lambda_2(M,g;\text{closed}) \geq c/d(M,g)^2$. If $\delta(E) \geq \ell+1$ we can write, in view of (34) and Lemma 33 (see nº 28),

$$c(n,\varepsilon,a)(1-C^2(\ell,\ell+1)) \leq |\mathfrak{R}_{min}|\ d(M,g)^2.$$

This proves the theorem ■

36. <u>Example</u>. Take $E = T^*M$, $\delta(E) = b_1(M)$. We obtain

$$r_{min}(M)d^2(M) \geq -b(n,-1,1) \Rightarrow b_1(M) \leq n = \dim M,$$

which extends Bochner's result to the case in which the curvature of (M,g) is allowed to take negative values.

37. <u>Comments</u>.

(i) Lemma 33 is quite general. It also applies to manifolds with boundary. See [ME2] for more applications;

(ii) Theorem 35 does not yet answer Problem 27. We could imagine to use (32) with enough functions f in order to apply the variational characterization of eigenvalues (nº III, 28), and the estimates of Chapter V nº 31. The map $s \to |s|$ maps $C^\infty(E)$ to a <u>cone</u> in $H^1(M)$, so that it is not clear at all that one can apply the above idea (remember that if s is a parallel section, then $|s|$ is a constant; in the case of a trivial bundle, the parallel sections form a vector space of dimension $\ell = rk(E)$, whose image by the application $s \to |s|$ is $\mathbb{R}_+$).

(iii) The first improvements of Bochner's result (i.e. when the curvature is allowed to take negative values) were given by P. Li (1980) for Betti numbers; they were then generalized by S. Gallot (1981). Both used Sobolev inequalities with Sobolev constants estimated in terms of isoperimetric inequalities. In 1980, M.Gromov gave bounds on the Betti numbers using geometric methods. He also pointed out that one should be able to use the heat equation and Kato's inequality. However he did not have the isoperimetric inequality for the heat kernel (see nº V. 28) and could therefore not go any further with this idea.

In the next paragraph we describe how heat kernel methods give partial answers to Problem 27.

D. THE ANALYTIC APPROACH, III.

The idea is very simple. First of all notice that $\delta(E)$ is the multiplicity of 0 as eigenvalue of $\tilde{\Delta}$,

(38) $\delta(E) = \dim \operatorname{Ker} \tilde{\Delta}$.

It can be shown that the operator $\tilde{\Delta}$ (resp. $\bar{\Delta}$) has a heat kernel (see Chap. V § A), and that the trace of this heat kernel can be written as

$$\tilde{Z}(t) = \sum_{j=1}^{\infty} \exp(-\tilde{\lambda}_j t)$$

(resp. $\bar{Z}(t) = \sum_{j=1}^{\infty} \exp(-\bar{\lambda}_j t)$) (see nº 30),

where the series converges for $t > 0$.

Now recall that

$$\int_M \langle \tilde{\Delta} s | s \rangle \, dv_g \geq \int_M \langle \bar{\Delta} s | s \rangle \, dv_g + \mathfrak{R}_{min} \int_M \langle s | s \rangle \, dv_g ,$$

so that the variational characterization of the eigenvalues (nº III.28) gives, for all $j \geq 1$, $\tilde{\lambda}_j \geq \bar{\lambda}_j + \Re_{min}$, from which we can deduce

(39) $\tilde{Z}(t) \leq \exp(-t\Re_{min})\bar{Z}(t)$, and hence

(40) $\delta(E) = \dim \operatorname{Ker} \tilde{\Delta} \leq \tilde{Z}(t) \leq \exp(-t\Re_{min})\bar{Z}(t)$, for all $t > 0$.

Finally we notice that

$$\operatorname{Ker} \bar{\Delta} = \{s \in C^\infty(E) \mid \bar{\Delta}s = 0\} \subset \{s \in C^\infty(E) | \bar{D}s = 0\}$$

so that

(41) $\dim \operatorname{Ker} \bar{\Delta} \leq \ell = \operatorname{rank}(E)$

(compare with Exercise 15 (ii)).

Consequence. Proposition 26 is an easy consequence of (39) - (41):

If $\Re_{min} > 0$, $\lim_{t\infty} \tilde{Z}(t) = 0$, so that $\delta(E) = 0$;

If $\Re_{min} = 0$, $\lim_{t\infty} \tilde{Z}(t) \leq \lim_{t\infty} \bar{Z}(t) \leq \ell$, so that $\delta(E) \leq \ell$.

We now use the following theorem (see [H-S-U]), which generalizes Kato's inequality (Lemma 31).

42. Theorem. Let E be a Riemannian vector bundle of rank ℓ over the Riemannian manifold (M,g) (see nº 21). Then

$$\bar{Z}(t) \leq \ell\, Z(M,g;t), \text{ for all } t > 0.$$

43. Remark. Notice that equality holds in Theorem 42 when $E = M \times \mathbb{R}^\ell$.

Proof of Theorem 42. (for the results on operator theory we use here, see [KO], in particular Chap. 9).

Let ε be a positive number. One can write (see Lemma 12)

$\frac{1}{2}\,\Delta(\langle s|s\rangle + \varepsilon^2) = \frac{1}{2}\,\Delta(\langle s|s\rangle) = \langle\bar{\Delta}s|s\rangle - |\bar{D}s|^2$. On the other hand,

$\frac{1}{2}\,\Delta(|s|_\varepsilon^2) = |s|_\varepsilon\,\Delta(|s|_\varepsilon) - |d|s_\varepsilon||^2$, where $|s|_\varepsilon^2 = \langle s|s\rangle + \varepsilon^2$,

so that

$$|s|_\varepsilon.\Delta(|s|_\varepsilon) = \langle\bar{\Delta}s|s\rangle + |d|s|_\varepsilon|^2 - |\bar{D}s|^2$$

from which we deduce (see Lemma 31),

$$|s|_\varepsilon.\Delta(|s|_\varepsilon) \le \langle\bar{\Delta}s|s\rangle.$$

Passing to the limit when ε tends to 0, this shows that

$$(*) \qquad |s|.\Delta(|s|) \le \langle\bar{\Delta}s|s\rangle,$$

where $\Delta(|s|)$ is understood in the sense of distributions, in particular this implies that $\Delta(|s|)$ is a measure (this extends Kato's inequality, Lemma 31).

For $\lambda \in \mathbb{R}_+^*$, we deduce from (*) that

$$|s|.(\Delta+\lambda\,\mathrm{Id})(|s|) \le \langle(\bar{\Delta}+\lambda\,\mathrm{Id})s|s\rangle.$$

Let $S = (\bar{\Delta}+\lambda\,\mathrm{Id})s$ so that, by Cauchy-Schwarz,

$$|(\bar{\Delta}+\lambda\,\mathrm{Id})^{-1}S|.(\Delta+\lambda\,\mathrm{Id})|(\bar{\Delta}+\lambda\,\mathrm{Id})^{-1}S| \le |S|\,|(\bar{\Delta}+\lambda\,\mathrm{Id})^{-1}S|$$

i.e. $$(\Delta+\lambda\,\mathrm{Id})|(\bar{\Delta}+\lambda\,\mathrm{Id})^{-1}S| \le |S|.$$

Recall that $\exp(-t\Delta)$, the heat operator, preserves positivity. For $\lambda > 0$, we write, in the sense of operators,

$$(\Delta+\lambda \mathrm{Id})^{-1} = \int_0^\infty e^{-t\lambda} e^{-t\Delta} dt,$$

so that $(\Delta+\lambda \mathrm{Id})^{-1}$ also preserves positivity. We conclude that for all $\lambda > 0$ and $n \in \mathbb{N}$,

$$|(\bar{\Delta}+\lambda \mathrm{Id})^{-n} S| \leq (\Delta+\lambda \mathrm{Id})^{-n}(|S|) \text{ and then}$$

$$\langle(\bar{\Delta}+\lambda \mathrm{Id})^{-n} S|T\rangle \leq |T|(\Delta+\lambda \mathrm{Id})^{-n}(|S|).$$

Recalling that $e^{-t\Delta} = \lim_{t\infty} (1+\frac{t}{n}\Delta)^{-n}$, we conclude that for all S,T in $C^\infty(E)$,

$$\langle \exp(-t\bar{\Delta}).S|T\rangle \leq |T| \exp(-t\Delta).(|S|)$$

and finally, we conclude that the norm $|\bar{k}(t,x,x)|$, of the endomorphism $\bar{k}(t,x,x)$ of the Euclidean space E_x, satisfies

$|\bar{k}(t,x,x)| \leq k_M(t,x,x)$, for any $x \in M$ ($\bar{k}(t,x,x)$ is the heat kernel of $\bar{\Delta}$).

We then deduce that $\bar{Z}(t) \leq \ell\, Z(t)$ ■ (Compare with Appendix A).

Finally the above results prove the following

44. Theorem. With the notations of nº 21, we have

$$\delta(E) \leq \ell \inf_{t>0} \exp(-t\Re_{min}) Z(M,g;t).$$

Theorem 44 together with Theorem V. 28 give partial answers to Problem 27.

45. Summary.

Let (M,g) *be a compact Riemannian manifold without boundary, such that*

$$r_{min}(M,g)d(M,g)^2 \geq (n-1)\varepsilon\alpha^2, \ \varepsilon \in \{-1,0,1\}, \ \alpha \in \mathbb{R}_+^{*} \ \text{and } n=\dim M$$

(*see* nº V. 26).

Let $E \to M$ *be a Riemannian vector bundle of rank* ℓ, *equipped with a compatible connection* $\bar{D}$, *and a Laplacian* $\tilde{\Delta}$ *which satisfies the Weitzenböck formula*

$$\tilde{\Delta}s = \bar{\Delta}s + \Re s .$$

Let $\Re_{min} = \inf\{\langle \Re s|s\rangle \mid s \in E, \ \langle s|s\rangle = 1\}$.

Then there exists a positive number $a(n,\varepsilon,\alpha)$ (*see* nº V.26) *such that*

$$\delta(E) = \dim \operatorname{Ker} \tilde{\Delta} \leq \ell \inf_{t>0} F(t),$$

where $F(t) = \exp(-\Re_{min}d^2(M,g)t).Z(S^n,can;a^2(n,\varepsilon,\alpha)t)$.

In particular, there exists a positive number $b(n,\varepsilon,\alpha)$ *such that* $\Re_{min}d^2(M,g) \geq -b(n,\varepsilon,\alpha)$ *implies* $\delta(E) \leq \ell$.

Note that since $Z(S^n,can;t)$ (see [CL] Chap. II. 4) and $a^2(n,\varepsilon,\alpha)$ (see nº V. 26) are easily computable, the above estimate for $\delta(E)$ can be made very explicit, with intermediate values of t, i.e. with t neither close to 0 nor very large. For explicit numerical computations, see [B-G] § 5.

46. Let H_γ be a surface with genus γ and constant curvature -1. The Gauss-Bonnet theorem ([HF] Part II Chap. III) implies that the volume of H_γ is $4\pi(\gamma-1)$. Let M_γ denote the Riemannian

product of H_γ with a flat torus T^{n-2} with volume $1/4\pi(\gamma-1)$. For M_γ, we have $r_{min}(M_\gamma) = -1$, $Vol(M_\gamma) = 1$. However $r_{min}(M_\gamma)d^2(M_\gamma)$ and $b_1(M_\gamma)$ tend to infinity with γ. This example shows that the above result (nº 45) is qualitatively best possible.

For more technical details, examples and counter-examples, we refer to [B-G] § 3; see also [GA2].

47. Remarks. In fact, Theorem V. 28 and Theorem 42 give the following estimates

(i) $Vol(M,g)k_M(t,x,x) \le Z(S^n, can; a^2(n,\varepsilon,\alpha)t\, d(M)^{-2})$,

(ii) $\tilde{k}_E(t,x,x) \le \ell \exp(-t\mathfrak{R}_{min})\, k_M(t,x,x)$,

so that we also get bounds on the L^∞-norms of the eigenfunctions of Δ on $C^\infty(M)$ or of the eigensections of $\tilde{\Delta}$ on $C^\infty(E)$.

E. UNDERLYING PHILOSOPHY.

48. In nº IV. 12, we introduced the following class of Riemannian manifolds $\mathfrak{M}_{n,k,D} = \{(M,g) \mid \dim M = n,\ Ricci(M,g) \geq (n-1)kg,\ Diam(M,g) \le D\}$.

In nº V. 26, we stated that $h(M,g;\beta)$ is bounded from below by a uniform function $H(\beta) = H(n,k,D;\beta)$ on $\mathfrak{M}_{n,k,D}$; in nº V. 28 we proved that $Z(M,g;t)$ is bounded from above by a uniform function $Z(t) = Z(n,k,D;t)$ on $\mathfrak{M}_{n,k,D}$. From the second estimate, we deduced, in this chapter, that $b_1(M)$ is uniformly bounded in the class of manifolds M which admit a metric g such that (M,g) is in $\mathfrak{M}_{n,k,D}$. It is very important to visualize these results in the following picture.

49. Given two subspaces X and Y of a given metric space Z, we denote by $d_H^Z(X,Y)$ the infimum of the positive numbers r such that X (resp. Y) is contained in the r - neighborhood of Y (resp. X). This is the <u>Hausdorff distance</u> of X and Y in Z.

We now define the Hausdorff distance $d_H(X,Y)$ between two metric spaces X and Y as the infimum of the number $d_H^Z(i(X),j(Y))$, for all isometric embeddings $i: X \to Z$, $j: Y \to Z$ in the some metric space Z, as Z varies.

In [GV2], M. Gromov proves the following fundamental theorems

50. <u>Theorem</u>. (precompactness theorem: [GV2], Chap. 5). <u>The space</u> $\mathfrak{M}_{n,k,D}$ <u>is precompact for the Hausdorff distance</u> d_H <u>between Riemannian manifolds (compact, without boundary)</u>.

Let $\mathfrak{M}_{n,K,D,V}$ denote the class of all Riemannian manifolds (M,g) such that $\dim M = n$, $|\mathrm{Sect}(M,g)| \leq K$, $\mathrm{Diam}(M,g) \leq D$ and $\mathrm{Vol}(M,g) \geq V$.

Let $\mathfrak{D}_{n,K,D,V}$ denote the class of all differentiable manifolds M such that there exists a metric g on M with $(M,g) \in \mathfrak{M}_{n,K,D,V}$.

51. <u>Theorem</u> (compactness theorem: [GV2], Chap. 8)

(i) (Cheeger) <u>The set</u> $\mathfrak{D}_{n,K,D,V}$ <u>is finite</u>;

(ii) (Gromov) <u>The set</u> $\mathfrak{M}_{n,K,D,V}$ <u>is compact for the Hausdorff distance</u> d_H <u>and the map</u> $(M,g) \to M$ from $\mathfrak{M}_{n,K,D,V}$ <u>to</u> $\mathfrak{D}_{n,K,D,V}$ <u>is locally constant</u>.

It follows from Theorem 51 that $b_I(M)$, $b_p(M)$, and more generally any topological invariant is bounded on $\mathfrak{M}_{n,K,D,V}$. Indeed, such an invariant does not depend on the Riemannian metric, and

$\mathfrak{D}_{n,K,D,V}$ is finite(1), hence there are only finitely many homeomorphism classes of manifolds in $\mathfrak{D}_{n,K,D,V}$. Unfortunately, given (n,K,D,V), we do not know in general the elements in $\mathfrak{D}_{n,K,D,V}$ so that we do not know any explicit bound on the topological invariants of the elements in $\mathfrak{M}_{n,K,D,V}$, at least from Theorem 51.

However, it follows from nº 45 that we can give explicit bounds on

(i) $b_1(M)$ in terms of $\dim(M)$, a lower bound on $\mathrm{Ricci}(M,g)$ and an upper bound on $\mathrm{Diam}(M,g)$;

(ii) $b_p(M)$, $2 \le p \le n-2$, in terms of $\dim(M)$, an upper bound on $|\mathrm{Sect}(M,g)|$ and an upper bound on $\mathrm{Diam}(M,g)$.

In view of (i) we could conjecture that any reasonable geometric invariant is bounded on $\mathfrak{M}_{n,k,D}$. Theorem 50 would prove this conjecture if these geometric invariants were continuous for the Hausdorff distance. This is not so in general, as the following examples show.

52. Counter-Examples. Let us consider $b_2(M)$, the second Betti number.

(i) $(M_\varepsilon,g) = (T^1,\mathrm{can}) \times (T^{n-1},\varepsilon\mathrm{can})$ converges to (T^1,can) for the Hausdorff distance. However, $b_2(M_\varepsilon) = \binom{n}{2}$ and $b_2(T^1) = 0$.

(ii) Consider the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$, whose fiber is S^1. We can multiply the metric in the fiber by ε so that we obtain a sequence (S^{2n+1},g_ε) of manifolds whose 2^{nd} Betti number is 0. This sequence converges for the Hausdorff distance to $(\mathbb{C}P^n,\mathrm{can})$,

(1) The number of elements in $\mathfrak{D}_{n,K,D,V}$ was recently bounded from above by S. Peters.

whose 2^{nd} Betti number is non-zero.

One can also show that $\lambda_2(M,g;\text{closed})$ is not continuous for d_H (see [TSG]).

53. Pre-compactness revisited.

Our Theorem V. 28 gives a precompactness result for $\mathfrak{M}_{n,k,D}$ which might have some relationship with Gromov's precompactness theorem (Theorem 50): see [B-B-G3].

We denote by ℓ^2 the Hilbert space of sequences $\{a_i\}_{i\geq 1}$ such that $\Sigma|a_i|^2 < \infty$.

We denote by h^1 the Hilbert space of sequences $\{a_i\}_{i\geq 1}$ such that $\Sigma|a_i|^2(1+i^{2/n}) < \infty$ (n fixed in $\mathbb{N}$).

It is a classical result that the inclusion $h^1 \to \ell^2$ is compact.

Now, given a Riemannian manifold (M,g), we define a map ψ from (M,g) to ℓ^2 as follows

$$\psi(x) = \{\text{Vol}(M,g)^{1/2} \exp(-\lambda_j)\phi_j(x)\}_{j\geq 1},$$

where $\{\lambda_j = \lambda_j(M,g;\text{closed})\}_{j\geq 1}$ is the spectrum of the Laplacian on $C^\infty(M)$, and where $\{\phi_j\}_{j\geq 1}$ is an associated orthonormal family of eigenfunctions.

If (M,g) is in $\mathfrak{M}_{n,k,D}$, we can write (use nº V. 28)

(i) $\|\psi(x)\|^2_{\ell^2} = \text{Vol}(M,g)k_M(2,x,x) \leq Z(S^n,\text{can};A(n,k,D))$, and

$$\|\psi(x)\|^2_{h^1} = \text{Vol}(M,g) \sum_{j=1}^{\infty} (1+j^{2/n})\exp(-2\lambda_j)\phi_j^2(x).$$

Since $\lambda_j \geq B(n,k,D)j^{2/n}$ in view of Theorem V. 31, we conclude that

(ii) $\|\psi(x)\|^2_{h^1} \leq \mathrm{Vol}(M,g)C(n,k,D) \sum_{j=1}^{\infty} (1+\lambda_j)\exp(-2\lambda_j)\phi_j^2(x).$

Using nº V. 28 again and a summation by parts, we conclude that

(iii) $\|\psi(x)\|^2_{h^1} \leq E(n,k,D).$

Here $A(n,k,D)$, $B(n,k,D)$, $C(n,k,D)$, $E(n,k,D)$ are universal functions of n,k,D.

From (iii) we conclude that the image of the set $\mathfrak{M}_{n,k,D}$ by the application ψ is bounded in h^1, and hence relatively compact in ℓ^2. This is the announced precompactness result.

<u>Further references for Chapter VI</u>: For § B to D see also [BD1]. For § E, see [GV2] and [M-S], [SI].

CHAPTER VII

A BRIEF SURVEY OF SOME RECENT DEVELOPMENTS IN SPECTRAL GEOMETRY

ALL RIEMANNIAN MANIFOLDS ARE COMPACT,
CONNECTED, WITHOUT BOUNDARY

(Unless otherwise stated)

In Chapter III.C, we divided the problems concerning the relationship between the eigenvalues of the Laplacian and the geometry of a Riemannian manifold (spectral geometry) into two categories: direct problems and inverse problems. Both types of problems are relevant to mathematical physics.

In Chapters IV and V, we dealt with direct problems and more precisely with isoperimetric methods applied to direct problems (e.g. lower bounds on the eigenvalues). These chapters do not give an exhaustive survey of known results on direct problems. For more details we refer to [CL+BD2], [PE] and [ON]

In the present chapter, we give a brief overview of inverse problems. As we neither plan nor wish to give a thorough survey, we refer to [B-B] for references (see in particular the list of basic references given page 156 or Appendix B, p. 210).

In Chapter I, we motivated the study of eigenvalue problems by applying the method of separation of variables to the wave equation. We could have applied the same method to the heat equation. Now it turns out that this is the analysis of the heat and wave equations which leads to inverse results in spectral geometry.

A. THE HEAT EQUATION AND APPLICATIONS.

1. In Chapter V § A, we introduced the heat kernel $k(t,x,y)$ of a Riemannian manifold (M,g).

If we denote by $\{\lambda_i\}_{i\geq 1}$ the eigenvalues of the Laplacian Δ^g acting on $C^\infty(M)$ and by $\{\phi_i\}_{i\geq 1}$ an associated family of orthonormal real eigenfunctions, we can write (see nº V. 4)

$$\text{(i)} \quad k(t,x,x) = \sum_{j=1}^{\infty} \exp(-\lambda_j t)\phi_j^2(x),$$

$$\text{(ii)} \quad \int_M k(t,x,x)dv_g(x) = Z(t) = \sum_{j=1}^{\infty} \exp(-\lambda_j t).$$

Recall that $Z(t)$ determines $\{\lambda_i\}_{i\geq 1}$ (nº V. 5).

The following theorem (known as Minakshisundaram - Pleijel asymptotic expansion) has been used extensively to investigate inverse problems.

2. Theorem. Let (M,g) be an n-dimensional Riemannian manifold. The following asymptotic expansions hold when t goes to 0_+

$$\text{(i)} \quad k(t,x,x) \sim (4\pi t)^{-n/2} \sum_{m=0}^{\infty} u_m(x)t^m;$$

$$\text{(ii)} \quad Z(t) \sim (4\pi t)^{-n/2} \sum_{m=0}^{\infty} a_m t^m;$$

(these are asymptotic expansions; the series in the right hand sides do not converge in general).

The functions $u_m(x)$ are C^∞ functions on M which can be expressed as universal polynomials in the components of the curvature tensor and its covariant derivatives.

In particular,

$$a_o = \mathrm{Vol}(M,g)$$

$$a_1 = \frac{1}{6}\int_M u(x)dv_g(x)$$ (u is the scalar curvature of M: II.45).

3. Remarks. It is in fact very difficult to give explicit formulae for the functions u_m: see [B-G-M] Chap. III. E and [GY] or [BD].

In the sequel, we denote the sequence $\{\lambda_i\}_{i\geq 1}$ of the eigenvalues (with multiplicities) of the operator Δ^g acting on $C^\infty(M)$ by Spec(M,g) (the spectrum of (M,g)).

4. Some Consequences.

Assume we know the spectrum Spec(M,g). It follows from Theorem 2 that we then know

(i) the dimension of M,

(ii) the volume of (M,g),

(iii) the integral $\int_M u(x)\, dv_g(x)$ of the scalar curvature of (M,g), and hence, in dimension 2, the Euler-characteristic $\chi(M)$ of M, by the Gauss-Bonnet theorem.

5. Definition. We say that two Riemannian manifolds (M,g) and (N,h) are isospectral if Spec(M,g) = Spec(N,h).

One of the important questions in spectral geometry was formulated by M. Kac in the 1960's.

6. <u>Question</u>. "Can one hear the shape of a drum", or are two isospectral Riemannian manifolds isometric?

7. <u>Some positive answers to Question 6</u>.

(i) 2-dimensional flat tori are characterized by their spectra <u>among flat tori</u> ([B-G-M] Chap. III. B);

(ii) Let (M,g) be one of the following Riemannian surfaces: (S^2, can), $(\mathbb{R}P^2, can)$, (T^2_Γ, can). Then (M,g) is characterized by its spectrum ([B-G-M] Chap. III. E);

(iii) (S^n, can) and $(\mathbb{R}P^n, can)$ are characterized by their spectra for $n \leq 4$ ([B-G-M] Chap. III. E).

For further results see [B-B] Chap. 6.

8. <u>Some negative answers to Question 6</u>.

The first counter-example to Question 6 was given by J.Milnor in 1964. We can summarize the negative answers to Question 6 as follows

(i) There exist isospectral 16-dimensional flat tori which are not isometric (Milnor 1964); however, they are diffeomorphic;

(ii) There exist isospectral 5-dimensional lens spaces which are neither isometric nor homeomorphic (Ikeda 1980); (curvature+1);

(iii) There exist isospectral Riemann surfaces (with curvature-1) which are not isometric (they are homeomorphic by 4(iii)); there exist isospectral 3-dimensional manifolds with curvature-1, which are neither isometric nor homeomorphic (Vignéras 1980); recent examples were given by Sunada (1984) and Buser (1985);

(iv) There exists a <u>one-parameter family</u> of 5-dimensional Riemannian manifolds, such that any two elements of the family are isospectral but not isometric (C. Gordon - E. Wilson, 1983, J. Diff. Geom. 19 (1984)).

For further results see [B-B] Chap. 6 or references in § C.

9. Comments. The examples described by Milnor, Ikeda and Vignéras arise from number theoretic considerations; those of Gordon-Wilson from group theoretic considerations. It can be shown (Wolpert 1979) that there are finitely many non-isometric flat tori (resp. Riemann surfaces) with a given spectrum. However, no upper bound on the number of such tori (resp. Riemann surfaces) is known (except for 3-dimensional tori, J.P. Berry 1981).

In 1982, H. Urakawa gave the first examples of non-congruent domains in $\mathbb{R}^4$ with same Dirichlet and Neumann spectra. Examples of domains in $\mathbb{R}^3$ with the same property were recently given by P. Buser (1985). It would be interesting to have other counter-examples of such domains in $\mathbb{R}^n$ and specialy in $\mathbb{R}^2$.

More generally, it would be interesting to have examples of non-isometric, isospectral manifolds (possibly with boundary), which are not locally isometric.

We conclude this paragraph with two ramarks.

10. Remarks.

(i) The heat equation is a diffusion equation and is very much related to Brownian motion. Some results in spectral geometry are easily interpreted or proved in terms of Brownian motion and probability theory: see [CL] Chap. IX, [R-S] and [B-B] Chap. 12; probabilistic methods might turn out to be very powerful, for example to investigate the heat kernel with Dirichlet boundary condition in a domain with very irregular boundary.

(ii) A consequence of Theorem 2 (ii) is Weyl's asymptotic formula (see III. 36)

(11) $$N(\lambda) = \mathrm{Card}\{j \mid \lambda_j \le \lambda\} = C(n)\ \mathrm{Vol}(M,g)\lambda^{n/2} + o(\lambda^{n/2}),$$

which follows from the asymptotic formula for $Z(t)$ by applying Karamata's Tauberian theorem. However, one cannot give a sharp estimate for $N(\lambda) - C(n)\ \mathrm{Vol}(M,g)\lambda^{n/2}$ with this method. One has to use wave equation techniques.

B. THE WAVE EQUATION AND APPLICATIONS.

The fundamental solution of the wave equation (or wave kernel) on (M,g) is the distribution $E(t,x,y)$, $(t,x,y) \in \mathbb{R} \times M \times M$, which satisfies

(12) $$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} + \Delta_y\right) E(t,x,y) = 0; \\ E(0,x,y) = \delta_x(y); \\ \dfrac{\partial E}{\partial t}(0,x,y) = 0. \end{cases}$$

In the sense of spectral theory, this is the kernel of the operator $\cos(t\sqrt{\Delta})$.

For example the wave kernel of $(\mathbb{R}^2, \mathrm{can})$ is

$E(t,x,y) = C_2|t|\ (|x-y|^2 - t^2)_-^{-3/2}$, for some constant C_2,

where

$$x_-^a = \begin{cases} 0 & \text{if } x \ge 0, \\ |x|^a & \text{if } x < 0. \end{cases}$$

The wave-kernel of (T^2_Γ, can) is given by

$$E(t,x,y) = C_2|t| \sum_{\gamma\in\Gamma} (\|x-y-\gamma\|^2 - t^2)_-^{-3/2}. \tag{13}$$

The wave equation techniques were introduced in the late 1960's by L. Hörmander to study the function $N(\lambda)$ (see nº 11). We now summarize the main results concerning the estimates on $N(\lambda)$.

14. Results on $N(\lambda)$. Let (M,g) be a Riemannian manifold without boundary then

(i) $N(\lambda) = C(n)\mathrm{Vol}(M,g)\lambda^{n/2} + O(\lambda^{(n-1)/2})$ (Avakumovic 1956, Hörmander 1968) and this estimate is best possible, as the example of (S^n, can) shows;

(ii) It was then observed by J. Duistermaat and V. Guillemin (1975) that the nature of $R(\lambda) = N(\lambda) - C(n)\mathrm{Vol}(M,g)\lambda^{n/2}$ is very much related to the geodesic flow of (M,g). Roughly speaking, $R(\lambda)$ is of the order of $\lambda^{(n-1)/2}$ if and only if the geodesic flow of (M,g) is periodic i.e. all geodesics are closed with same period. This is exactly why (S^n, can) appears in (i). All the geodesics of (S^n, can) are periodic with period 2π. Again roughly speaking, if the geodesic flow is not periodic then $R(\lambda) = o(\lambda^{(n-1)/2})$;

(iii) In some cases, the estimate for $R(\lambda)$ can be improved. In the case of flat tori, one has $R(\lambda) = O(\lambda^{(n-2)/2+1/(n+1)})$. This estimate is not best possible, and the true nature of $R(\lambda)$ is not known; to investigate $R(\lambda)$ in that special case is a very difficult problem related to number theory. For manifolds with negative curvature one can prove that $R(\lambda) = O(\lambda^{(n-1)/2}/\mathrm{Log}\lambda)$ (Bérard, Randol 1976).

15. The case of manifolds with boundary is <u>much more difficult</u>. It can be shown on certain examples that the counting function $N_D(\lambda)$ for the Dirichlet eigenvalue problem in the manifold with boundary (M,g) satisfies

$$(16) \quad N_D(\lambda) = C(n)\mathrm{Vol}(M,g)\lambda^{n/2} - C'(n)\mathrm{Vol}(\partial M,g)\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}).$$

This estimate is know as <u>Weyl's conjecture</u>.

Estimate (16) turns out to be much more difficult to prove than 14(i). In fact Hömander's estimate 14(i) was proved for manifolds with boundary only a few years ago (Pham The Lai, R.Seeley 1980). Counter-examples to Weyl's conjecture were given by D.Gromes (1967) and Bérard-Besson (1980): they again involve manifolds with boundary, whose geodesic flow (allowing reflections at the boundary, as in geometric optics) is periodic. Weyl's conjecture was <u>settled</u>, undergeneral assumptions, by Melrose (1980), Ivrii (1981) and Petkov (1985).

17. The wave kernel is very much related to the geodesic flow of (M,g). In particular, wave equation techniques clarified the relationship between the spectrum of a Riemannian manifold and the lengths of the closed geodesics on the manifolds. Results in this direction were obtained by Y. Colin de Verdière (1973) and Duistermaat-Guillemin (1975). We explain this relationship for a flat torus T^2_Γ.

The Poisson summation formula (Example V. 6 (ii))

$$(4\pi t)^{-n/2}\mathrm{Vol}(T_\Gamma) \sum_{\gamma\in\Gamma} \exp(-\|\gamma\|^2/4t) = \sum_{\gamma^*\in\Gamma^*} \exp(-4\pi^2\|\gamma^*\|^2 t)$$

shows that the spectrum of T_Γ, $\{4\pi^2\|\gamma^*\|^2, \gamma^* \in \Gamma^*\}$ determines the lengths of the closed geodesics of T_Γ, $\{\|\gamma\|^2, \gamma \in \Gamma\}$, the so-called length-spectrum. This relation can also be seen as follows. Using nº (13) and the fact that $E(t,x,y)$ is the kernel of $\cos(t\sqrt{\Delta})$ we can write, at least at the formal level, for T_Γ^2

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\phi_j^2(x) = E_\Gamma(t,x,x) = C_2|t| \sum_{\gamma\in\Gamma} (\|\gamma\|^2-t^2)_-^{-3/2}.$$

In fact, one can show that $\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})$ is a tempered distribution whose singular support (points away from which the distribution is C^∞) is contained in the set of lengths of closed geodesics (and their opposites). Again we see that the spectrum determines the set of lengths of the closed geodesics. This phenomenon can be understood by thinking of ripples propagating on a cylindrical lake: see Fig. 8 (think of a cylinder as a rectangle with two sides identified).

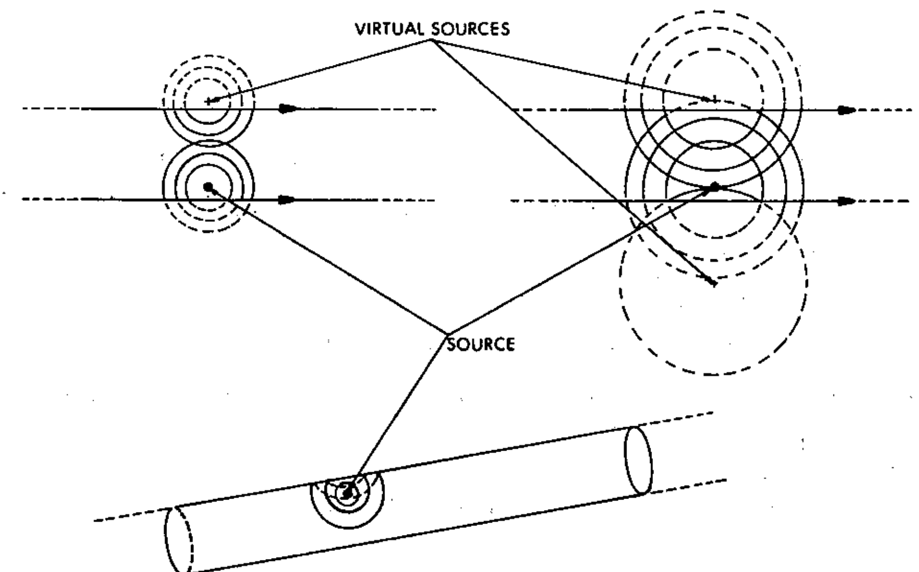

Fig. 8

Similar results can be established on manifolds with boundary, relating the spectra of the manifold, with either Dirichlet or Neumann boundary conditions, to billiard trajectories.

For more references on the wave equation see [B-B] Chap. E and § C. Let us end this paragraph by pointing out that wave equation techniques belong to the realm of symplectic geometry rather than to that of Riemannian geometry.

C. FINAL COMMENTS.

Many problems arise in spectral geometry, both direct and inverse problems, both on manifolds with or without boundary. Leafing through the "Leitfaden" of [B-B], the reader will discover some of these problems. We only hope that these notes will arouse the interest of the readers and will lead them to solve some of these problems.

Some further references

Heat equation: [BD], [CL+BD2], [DK]

Wave equation: [GL1], [GL2], [G-S]

Partial differential equations and geometry: [YU]

Open problems: [YU]

For the estimates concerning the counting functions $N(\lambda)$ (see (11)), we also refer the reader to: L. Hörmander, the analysis of partial differential operators, Vol.III-IV, Springer Grundlehren 1985.

General references: see [B-B] and the following review papers or books: [BE], [B-G-M], [BN], [CL+BD2], [GL1,2], [G-S], [GY], [PE], [P-S], [ON].

A P P E N D I X   A

ON SYMMETRIZATION

by

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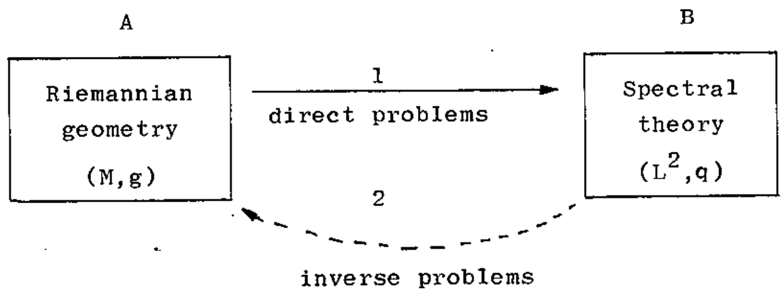
I - INTRODUCTION

The spectral theory of Riemannian manifolds is a typical example of an interaction between two different aspects of mathematics: Riemannian geometry and operator theory in Hilbert spaces.

For geometers the aim is, of course, to obtain geometric informations using the well-known methods of Hilbert spaces analysis. This transfer can be summarized in the following picture:



The link between box A and box B is done by associating to the metric space constituted by a Riemannian manifold (M,g) another metric space built on the Hilbert space $L^2(M)$ endowed with a quadratic form q in the sense of spectral theory (i.e. not necessarily everywhere defined) which we shall call the energy. In the case of a smooth compact connected Riemannian manifold, this quadratic form generates (by a standard process: the so-called Friedrichs' extension) a self-adjoint operator which has a compact resolvent. This is one of the best-known situation occuring in box B.

Once this link is established (may be the right words in this situation would be functors and categories, but such a formal approach is not our goal here) a lot of problems arise immediately. In Chapter III.C, the problems which are related to arrow 1 were called direct problems and those which come from interpretation of box B results, inverse problems. Let us now give some examples:

<u>Question 1</u>: Is arrow 1 injective?

This is the famous question asked by M. Kac [see Appendix B for references]: "Can one hear the shape of a drum?". It is known that the answer is no (in a general situation); however the following question has not yet been answered;

<u>Question 2</u>: Are two isospectral manifolds locally isometric?

Another interesting problem when dealing with the Laplacian (for example) would be to separate precisely PDE theoretic results from Hilbert spaces algebra ones, or in other words, to understand more accuratly the arrow 1.

This appendix illustrates an interesting principle which can be summarized as follows:

Once a problem in box A is translated into a problem in box B, it is very often useful to study thoroughly the latter problem before going back to the former one. This gives rise to more elementary and simple proofs.

The aim of the following text is to try to gather in a general picture a number of inequalities involving the heat semigroups. We wish to prove that such inequalities are (algebraic) consequences of inequalities on the quadratic forms associated with the operators under considerations, and are obtained in a manner which is very similar to the proof of Kato's inequality ([H-S-U1]: References p.190).

In the geometric applications it will then be clear that these criteria rely on isoperimetric inequalities. The former are in some sense, a Hilbert space translation of the latter. This relationship being established the desired results on diffusion processes are then easily proved.

Section II consists in recalling the Beurling-Deny criterion. We shall then give the corresponding criterion in our case, in box B, forgetting the geometrical meaning (Section IV). For this purpose we give a formal definition of the notion of symmetrization. The applications (Section V, VI) then consist in a verification of the conditions established in Section IV (geometric symmetrization decreases energies) in the particular cases under considerations: Fiber bundles, Schwarz symmetrization. Recall that a symmetrization generally yields a comparison between a generic space and a more symmetric one, in the sense that it has more isometries. Certain results are then incorporated in this framework such as the paper [B-G] which has been a guide for this text.

The reader will easily see that this appendix deals neither with the most general case nor with the most formal one, and that extensions and modifications are possible.

Finally, it must be noted that the idea of the formal approach of the inequalities appearing in [B-G] has been suggested to the author by the article [H-S-U1]. Our method is just a slight modification of the one presented in [H-S-U1]. In a forthcoming paper we shall develop a similar approach to the notion of transplantation (which is in some sense dual to that of symmetrization).

The results which we present in this Appendix are not yet polished, as they pertain to current research. A hopefully more satisfactory version of this Appendix will appear elsewhere.

References for this Appendix are given p. 190.

II - THE ABSOLUTE VALUE AS A SYMMETRIZATION PROCEDURE AND THE BEURLING-DENY CRITERION

This section is our first contact with symmetrization, in its weakest form. However all the ideas (which are very simple) will appear here.

Let (M,μ) be a σ-finite measure space. We will deal with the Hilbert space $L^2(M,d\mu)$. It is then natural to give the

1. Definition:

Let A be a bounded operator on L^2, it is called positivity preserving if Af is positive whenever f is positive (see [R-S]IV, p. 201).

We can now give sufficient conditions for a self-adjoint operator to be positivity preserving.

2. Proposition:

Let H be a self-adjoint operator, bounded from below by $E = \inf\{\mathrm{Spec}(H)\}$. Then e^{-tH} is positivity preserving for all $t > 0$ if and only if $(H-\lambda)^{-1}$ is positivity preserving for all $\lambda < E$.

Proof. Use the formulae

$$(H-\lambda)^{-1} = \int_0^\infty e^{\lambda t}\, e^{-tH} dt \qquad (\lambda < E)$$

$$e^{-tH} = \lim_{n\to+\infty} \left(1+\frac{tH}{n}\right)^{-n} \qquad (t > 0). \quad ■$$

In the following we will only consider self-adjoint operators and real valued functions (for the sake of simplicity only).

There is a very simple criterion for a positive self-adjoint operator to generate a positivity preserving semi-group.

3. Theorem (Beurling-Deny criterion)

Let $H \geq 0$ be a self-adjoint operator on $L^2(M, d\mu)$ and let q be the associated quadratic form with domain $\mathcal{D}(q)$. The following assertions are equivalent:

a) e^{-tH} is positivity preserving for all $t > 0$,

b) if $u \in \mathcal{D}(q)$ then $|u| \in \mathcal{D}(q)$ and $q(|u|) \leq q(u)$.

Remark.

As in Proposition 2, it is sufficient to have H bounded from below by some constant $-c$, i.e.

$$\langle H\varphi|\varphi\rangle \geq -c|\varphi|^2, \quad \text{for all } \varphi \text{ in the domain of } H.$$

Indeed, in that case $H + c\mathrm{Id}$ is non-negative, and the quadratic form associated to this operator is,

$$q(\varphi) + c|\varphi|^2.$$

The proof is very easy and it can be found in [R-S]IV,p.210.

In 1973, T. Kato [KÓ] proved a simple but very useful inequality, the so-called "Kato's inequality" (nº VI.31 and proof of VI.42). Later B. Simon [SN] gave an interpretation of this inequality in terms of positivity preserving semigroup. More precisely,

4. Definition:

With the notations of the above theorem, we say that H obeys Kato's inequality if and only if:

i) $u \in \mathfrak{D}(q)$ implies $|u| \in \mathfrak{D}(q)$

ii) for $u \in \mathfrak{D}(H)$ and $f \in \mathfrak{D}(q)$ with $f \geq 0$

$$\langle f | H(|u|)\rangle \leq \langle (\text{sign } u) f | Hu\rangle$$

where $\text{sign}(u)$ is defined to be

$$(\text{sign } u)(x) = 0 \quad \text{if} \quad u(x) = 0$$

$$= \frac{u(x)}{|u(x)|} \quad \text{otherwise}.$$

The link with positivity is given by the following

5. Theorem [SN]

A self-adjoint operator H which is bounded from below satisfies Kato's inequality if and only if e^{-tH} is positivity preserving for all t.

Proof. If Kato's inequality holds, taking $f = |u|$ we get, for $u \in \mathfrak{D}(H)$

$$q(|u|) \leq q(u).$$

A limiting argument and the use of Beurling-Deny criterion give the result.

Conversely if e^{-tH} is positivity preserving, then for any u and any $f \geq 0$

$$\langle (\text{sign } u) f | e^{-tH} u\rangle \leq \langle f | e^{-tH} |u|\rangle$$

and equality holds at $t = 0$.

If $u \in \mathcal{D}(H)$, taking $f = |u|$ and differentiating at $t = 0$ gives

$$0 \leq q(|u|) \leq q(u) \text{ and hence,}$$

$|u| \in \mathcal{D}(q)$. A limiting argument shows that the same is true when $u \in \mathcal{D}(q)$. If $u \in \mathcal{D}(H)$ and $f \in \mathcal{D}(q)$, differentiating again gives the desired inequality. ∎

6. Interpretation:

Let us assume that the self-adjoint operator H is bounded from below and satisfies the Beurling-Deny criterion. Then the absolute value can be seen of as a mapping from $L^2(M, d\mu)$ to itself which decreases the quadratic form associated to H (and thus which preserves its domain). Assume that $H \geq -c\,\mathrm{Id}$.

The consequences are the positivity preserving properties of $(H+\lambda)^{-1}$ (for $\lambda > c$) and e^{-tH} (for $t > 0$), and Kato's inequality.

If $(H+\lambda)^{-1}$ (resp. e^{-tH}) is an integral operator with kernel $R(\lambda;.,.)$ (resp. $K(t;.,.)$), this leads to the positivity of the function on $M \times M$, $R(\lambda;.,.)$, for $\lambda > c$ (resp. $K(t;.,.)$ for $t > 0$).

So this property of the absolute value allows us to compare the operator to 0, the trivial operator (positivity). The question which arises now is:

Is it possible to compare different operators, even acting on different Hilbert spaces?

III - SYMMETRIZATION

This section is devoted to a formal approach of symmetrization. It aims at giving a criterion analogous to that of Beurling-Deny for semigroup domination. It is clearly inspired by the paper [H-S-U1].

Unless otherwise specified we shall deal with self-adjoint operators. This is not really necessary.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. We assume $\mathcal{K}$ to be real.

A nonempty cone $\mathcal{K}^+$ of $\mathcal{K}$ is a subset such that:

i) $\mathcal{K}^+ + \mathcal{K}^+ \subseteq \mathcal{K}^+$,

ii) $a\mathcal{K}^+ \subseteq \mathcal{K}^+$ for all $a \geq 0$,

we assume furthermore that

iii) $\langle \mathcal{K}^+ | \mathcal{K}^+ \rangle \geq 0$.

The cone will be said to be <u>self-dual</u> if the following condition holds

iv) $\langle g | \mathcal{K}^+ \rangle \geq 0$ implies $g \in \mathcal{K}^+$.

In this situation, we have the

7. <u>Definition</u>:

<u>A map</u> S <u>from</u> $\mathcal{H}$ <u>to</u> $\mathcal{K}^+$ <u>is called a symmetrization if</u>

1) <u>for all</u> (f,f') <u>in</u> $\mathcal{H}$, $|\langle f|f'\rangle| \leq \langle S(f)|S(f')\rangle$, <u>and equality if</u> $f = f'$;

2) <u>Let</u> g <u>be any element in</u> $\mathcal{K}^+$, <u>then for any</u> $f_1 \in \mathcal{H}$ <u>there exists an</u> $f_2 \in \mathcal{H}$ <u>such that</u>

$$g = S(f_2) \quad \underline{\text{and}}$$

$$\langle f_1|f_2\rangle = \langle S(f_1)|S(f_2)\rangle = \langle S(f_1)|g\rangle$$

(<u>in this case</u> f_1, f_2 <u>are said to be</u> g-<u>paired</u>).

We then have the

8. <u>Proposition</u>:

<u>A symmetrization is a Lipschitz map</u>.

<u>Proof</u>. $|S(f)-S(g)|^2 = |S(f)|^2 + |S(g)|^2 - 2\langle S(f)|S(g)\rangle$

but $|S(f)|^2 = |f|^2$ and

$$\langle f|g\rangle \leq |\langle f|g\rangle| \leq \langle S(f)|S(g)\rangle,$$

then

$$|S(f)-S(g)|^2 \leq |f-g|^2. \blacksquare$$

9. <u>Remark</u>.

This property allows us to define the symmetrization on a dense subset of $\mathcal{H}$ only.

Another property which will be important in the sequel is given by the following:

10. <u>Proposition</u>:

<u>Let</u> $\mathcal{H}, \mathcal{K}$ <u>and</u> $\mathcal{L}$ <u>be three Hilbert spaces</u>, $\mathcal{K}$ <u>and</u> $\mathcal{L}$ <u>being real</u>, S (<u>resp</u>. T) <u>a symmetrization map from</u> $\mathcal{H}$ <u>to</u> $\mathcal{K}^+$ (<u>resp</u>. <u>from</u> $\mathcal{K}$ <u>to</u> $\mathcal{L}^+$); <u>if</u> T <u>has the property that whenever</u> $f_1 \in \mathcal{K}^+$ <u>and</u> $g \in \mathcal{L}^+$, <u>the element</u> $f_2 \in \mathcal{K}$ <u>such that</u> (f_1, f_2) <u>are</u> g-<u>paired</u> <u>can be chosen in</u> $\mathcal{K}^+$ <u>then</u> $T \circ S$ <u>is a symmetrization</u>.

<u>Proof</u>. Clear.

The following proposition will lead to a definition of the domination relation.

11. Proposition:

Let A (resp. B) be a bounded operator on $\mathcal{H}$ (resp. $\mathcal{K}$). The following inequalities are equivalent (for all $f_1, f_2 \in \mathcal{H}$ and $g \in \mathcal{K}^+$):

i) $\langle S(Af_1)|g\rangle \leq \langle B(S(f_1))|g\rangle$;

ii) $\Re e\langle Af_1|f_2\rangle \leq \langle B(S(f_1))|S(f_2)\rangle$;

iii) $|\langle Af_1|f_2\rangle| \leq \langle B(S(f_1))|S(f_2)\rangle$;

if futhermore $\mathcal{K}^+$ is a self-dual cone, we can add

iv) $S(Af_1) \leq B(S(f_1))$, i.e. $B(S(f_1)) - S(Af_1) \in \mathcal{K}^+$.

Proof of the Proposition:

(iii) $\Rightarrow$ (ii) trivial

(ii) $\Rightarrow$ (i)

Choose f_2 such that (Af_1, f_2) are g-paired, then

$$\langle S(Af_1)|g\rangle = \langle Af_1|f_2\rangle = \Re e\langle Af_1|f_2\rangle \leq \langle B(S(f_1))|g\rangle.$$

(i) $\Rightarrow$ (iii) clear.

Finally it is clear that (iv) implies (i), (ii) and (iii), and that the self-dual property of $\mathcal{K}^+$ allows the converse to be true. ■

12. Definition:

If A and B satisfy one of these inequalities we will say that B dominates A.

13. Remark.

The fact that the cone is self-dual allows to pass from integral inequalities to pointwise ones when working on spaces of functions. This will be important in the applications and explains the differences between inequalities obtained from various types of symmetrizations.

The following lemma will be important in the sequel.

14. Lemma:

1) If B_i dominates A_i then $|\alpha_1|B_1 + |\alpha_2|B_2$ dominates $\alpha_1 A_1 + \alpha_2 A_2$ $(\alpha_1, \alpha_2 \in \mathbb{C})$.

2) If B_i dominates A_i and if B_i and A_i converge respectively to B and A (weakly or strongly) then B dominates A.

3) If B_i dominates A_i $(i=1,2)$ and if B_1 preserves the cone $\mathcal{K}^+$ then $B_1 \circ B_2$ dominates $A_1 \circ A_2$. If furthermore $\mathcal{K}^+$ is self-dual then the preservation of $\mathcal{K}^+$ by B_1 is a consequence of the domination relation.

Proof. The only non trivial point is 3). For $f \in \mathcal{H}$, $g \in \mathcal{K}^+$ we have

$$\langle S(A_1 \circ A_2(f)) | g \rangle \leq \langle B_1(S(A_2 f)) | g \rangle \leq \langle S(A_2 f) | B_1 g \rangle$$

(recall that the operators are assumed to be self-adjoint). Then

$$\langle S(A_1 \circ A_2(f)) | g \rangle \leq \langle B_2(S(f)) | B_1 g \rangle = \langle (B_1 \circ B_2)(S(f)) | g \rangle,$$

because $B_1 g$ belongs to $\mathcal{K}^+$.

If $\mathcal{K}^+$ is self-dual then the relation

$$S(A_1 f) \leq B_1(S(f))$$

implies that for all $g \in \mathcal{K}^+$, choosing $f \in \mathcal{H}$ such that $S(f) = g$, we have

$$\langle B_1 g | h \rangle \geq \langle S(A_1 f) | h \rangle \geq 0 \quad \text{for all} \quad h \in \mathcal{K}^+.$$

Thus

$$B_1 g \in \mathcal{K}^+. \quad \blacksquare$$

15. Corollary:

Let H (resp. K) be a self-adjoint operator on $\mathcal{H}$ (resp. $\mathcal{K}$) both bounded from below by $-c$ and let $P_t = \exp(-tH)$ and $T_t = \exp(-t K)$. The following propositions are equivalent:

i) T_t dominates P_t for all $t > 0$;

ii) $(\lambda+K)^{-1}$ dominates $(\lambda+H)^{-1}$ for all $\lambda > c$;

iii) $(\lambda+K)^{-n}T_t$ dominates $(\lambda+H)^{-n}P_t$ for all $\lambda > c$, $t > 0$ and $n \in \mathbb{N}$.

Proof. Use the formulae

$$(\lambda+H)^{-1} = \int_0^{+\infty} e^{-t\lambda} P_t \, dt \quad (\lambda > c),$$

$$P_t = \lim_{n\to+\infty} (1+ \frac{t}{n} H)^{-n} = \lim_{n\to+\infty} [(\frac{n}{t})^n(\frac{n}{t} + H)^{-n}] \quad \blacksquare$$

IV - INTERPRETATION IN TERMS OF QUADRATIC FORMS: A CRITERION FOR SEMIGROUP DOMINATION

We can now give a necessary and sufficient condition in terms of quadratic forms for a symmetrization to give rise to semigroup domination. It will be a generalization of Beurling-Deny's criteria.

The main theorem of this text is:

16. Theorem:

Let H and K be self-adjoint operators, both bounded from below by $-c$, on $\mathcal{H}$ and $\mathcal{K}$ respectively; q_H, q_K the associated quadratic forms (which will be considered as bilinear forms as well). Let P_t and T_t be the semigroups generated by H and K. If $\mathcal{D}_0$ is a core for H, and if we assume that $(\lambda+K)^{-1}$ preserves $\mathcal{K}^+$ for all $\lambda > c$, then the following conditions are equivalent:

a) Semigroup domination

T_t dominates P_t for all $t > 0$;

b) Resolvent domination

$(\lambda+K)^{-1}$ dominates $(\lambda+H)^{-1}$ for all $\lambda > c$;

c) Kato's inequality

(K_1) $u \in \mathcal{D}(q_H)$ implies $S(u) \in \mathcal{D}(q_K)$

$$(K_2)\quad |q_H(f_1,f_2)| = |\langle Hf_1|f_2\rangle| \geq \Re e\langle Hf_1|f_2\rangle \geq$$

$$\geq \langle S(f_1)|KS(f_2)\rangle = q_K(S(f_1),S(f_2)),$$

for all $f_1 \in \mathcal{D}_0$, f_2 such that $S(f_2) \in \mathcal{D}(K)$ and (f_1,f_2) are $S(f_2)$-paired.

Proof.

i) The equivalence of a) and b) is clear from the previous section.

ii) a) implies c). The hypothesis a) implies the inequality

$$\langle (\frac{1-e^{-tH}}{t}) f | f \rangle \geq \langle (\frac{1-e^{-tK}}{t}) S(f) | S(f) \rangle,$$

for $f \in \mathfrak{D}_0$. Letting t go to zero yields,

$$+\infty > q_H(f) = |H^{1/2} f|^2 \geq |K^{1/2} S(f)|^2 = q_K(S(f)) \geq 0,$$

which implies

$$S(f) \in \mathfrak{D}(q_K).$$

A similar argument gives inequality (K_2).

Recall that a core for H is a subset $\mathfrak{D}_o$ of $\mathfrak{D}(H)$ such that the graph of $H|_{\mathfrak{D}_o}$ is dense in the graph of H.

iii) c) implies b).

Let us assume that $f_1 \in (\lambda+H)(\mathfrak{D}_o)$ and $g \in K^+$, then

$$\langle S((H+\lambda)^{-1} f_1) | g \rangle = \langle S(h_1) | g \rangle,$$

with $h_1 = (H+\lambda)^{-1} f_1$.

Since $h_1 \in \mathfrak{D}_o$, and by assumption $(\lambda+K)^{-1} g \in K^+$, we can write

$(\lambda+K)^{-1} g = S(f_2)$ with h_1 and f_2, $S(f_2)$-paired.

(Notice that $S(f_2) \in \mathfrak{D}(K)$). Then (by (K2))

$$\langle S(h_1)|g\rangle = \langle S(h_1)|(\lambda+K)S(f_2)\rangle \leq \mathcal{R}e\langle (H+\lambda)h_1|f_2\rangle = \mathcal{R}e\langle f_1|f_2\rangle \leq$$

$$\leq |\langle f_1|f_2\rangle| \leq \langle S(f_1)|S(f_2)\rangle = \langle S(f_1)|(\lambda+K)^{-1} g\rangle.$$

Then we have

$$\langle S((H+\lambda)^{-1}f_1)|g\rangle \le \langle(\lambda+K)^{-1}S(f_1)|g\rangle,$$

which is the desired inequality. A limiting argument allows us to conclude. ■

Let $Eg(H)$ (resp. $Eg(K)$) be the ground state energy of the self-adjoint operator H (resp. K), i.e. the infimum of the spectrum of H (resp. K), the following consequence is immediate.

17. Corollary:

If one of the conditions in Theorem 16 is verified then,

$$Eg(H) \geq Eg(K).$$

Proof. Use the min-max principle. ■

18. Interpretation:

The theorem can be summarized in:

the semigroup P_t is dominated by T_t if and only if S does not increase the energy integral.

19. Remarks.

i) We assumed the operators to be self-adjoint. Clearly, this class can be enlarged. As an example, in [H-S-U1], the operators are just assumed to be maximally accretive.

ii) If e^{-tH_i} is dominated by e^{-tK_i} $(i=1,2)$ and if H_1+H_2 and K_1+K_2 are in the class of operator under consideration then e^{-tH} is dominated by e^{-tK}, H and K being the closures of H_1+H_2 and K_1+K_2 respectively. This is easily proved by applying Trotter product Formula.

iii) As a conclusion, let us summarize the ideas of the three preceding sections in the following comparative chart:

<u>Beurling-Deny-Simon Criterion</u>	<u>Generalized Criterion</u>
H self-adjoint operator in $L^2(M,d\mu)$ bounded from below by $-c$. q_H associated quadratic form.	H self-adjoint operator on $\mathcal{H}$, $H \geq -c$; q_H. K self-adjoint operator on $\mathcal{K}$, $K \geq -c$; q_K. S symmetrization from $\mathcal{H}$ to $\mathcal{K}$ with positive cone $\mathcal{K}^+$ preserved by $(\lambda+K)^{-1}$ for all $\lambda > c$. $\mathcal{D}_o$ a core for H.

The following assertions are equivalent

i) e^{-tH} positivity preserving for all $t > 0$;	i) e^{-tK} dominates e^{-tH} for all $t > 0$;
ii) $(H+\lambda)^{-1}$ positivity preserving for all $\lambda > c$;	ii) $(\lambda+K)^{-1}$ dominates $(\lambda+H)^{-1}$ for all $\lambda > c$;
iii) $u \in \mathcal{D}(q_H) \Rightarrow \lvert u\rvert \in \mathcal{D}(q_H)$ and $q_H(\lvert u\rvert) \leq q_H(u)$;	iii) Kato's inequality (K_1) $u \in \mathcal{D}(q_H) \Rightarrow S(u) \in \mathcal{D}(q_K)$
iv) $u \in \mathcal{D}(q_H) \Rightarrow \lvert u\rvert \in \mathcal{D}(q_H)$ and $\forall u \in \mathcal{D}(H)$ $\forall f \in \mathcal{D}(q_H)$, $f \geq 0$ $\langle f \mid H(\lvert u\rvert)\rangle \leq \langle \mathrm{sign}(u)f \mid H(u)\rangle$	(K_2) for all $f_1 \in \mathcal{D}_o$ and for all f_2 such that $S(f_2) \in \mathcal{D}(K)$ and (f_1,f_2) are $S(f_2)$-paired, $\mathcal{R}e\langle Hf_1 \mid f_2\rangle \geq q_K(S(f_1),S(f_2))$.

V - APPLICATION 1: KATO'S INEQUALITY ON FIBER BUNDLES

Kato's inequality is the subject of the article [H-S-U1]. For an application of this inequality, see [H-S-U] or Chap. VI. §III. Let us briefly summarize the situation.

Let M be a compact Riemannian manifold and $E \xrightarrow{\pi} M$ a hermitian vector bundle, i.e. a vector bundle such that each fiber is equipped with a hermitian structure varying smoothly in the base point.

Let D be a connection on the sections of E, compatible with the hermitian product.

The Riemannian metric on M gives rise to the Laplace-Beltrami operator Δ.

The connection D together with the Riemannian structure on M allow to define a Laplacian type operator on the space $L^2(M;E)$ of L^2-sections of the bundle E, called the <u>rough Laplacian</u> $\bar{\Delta}$, and which is a non-negative self-adjoint operator (see VI.13).

Finally, for a section u in $L^2(M;E)$, we define the function $|u|$ in $L^2(M;\mathbb{R})$ by the relation,

$$|u|(m) = |u(m)| \quad \text{for all} \quad m \in M.$$

The norm in the right hand side is taken in the fiber E_m.

We then have the

20. <u>Theorem</u> ([H-S-U1])

<u>With the above notations</u>

$$|e^{-t\bar{\Delta}}(u)| \leq e^{-t\Delta}(|u|).$$

Proof. The proof is done by showing that the map

$$S\colon L^2(M;E) \to L^2(M,\mathbb{R})$$

$$u \to |u|$$

is a symmetrization which decreases energies. Recall that the quadratic form defining $\bar{\Delta}$ is

$$\bar{q}(u) = \int_M |Du|^2.$$

i) Define, for $\varepsilon > 0$ and s a smooth section of the bundle

$$|s|_\varepsilon = (|s|^2 + \varepsilon^2)^{1/2}$$

which is then a smooth function on the manifold M. An easy computation gives

$$\Delta|s|_\varepsilon = \frac{(\bar{\Delta}s,s)}{|s|_\varepsilon} - \left(\frac{|Ds|^2}{|s|_\varepsilon} - \frac{|(Ds,s)|^2}{|s|_\varepsilon^3}\right).$$

Here Ds is considered as a one-form with values in the bundle and (Ds,s) a real one-form, and $(.,.)$ is the scalar product in each fiber of the bundle. By the Cauchy-Schwarz inequality

$$\frac{|(Ds,s)|^2}{|s|_\varepsilon^3} \le \frac{|Ds|^2|s|^2}{|s|_\varepsilon^3} \le \frac{|Ds|^2}{|s|_\varepsilon} .$$

Therefore,

$$\Delta|s|_\varepsilon \le (\bar{\Delta}s, \frac{s}{|s|_\varepsilon}) .$$

ii) Let $\mathcal{S}_o$ be the space of C^∞ sections. It is a core for $\bar{\Delta}$. Let s_1 be in $\mathcal{S}_o$ and g a non-negative function on M in the domain of Δ. Define

$$s_2(m) = \begin{cases} g(m)\,\dfrac{s_1}{|s_1|}(m) & \text{if} \quad s_1(m) \neq 0 \\ \\ g(m)e(m) & \text{if} \quad s_1(m) = 0 \end{cases}$$

where $e(m)$ is any fixed measurable section of E over M such that

$$|e(m)| = 1 \quad \text{for all} \quad m.$$

(such sections exist since away from a set of measure zero the manifold is contractible).

Then clearly s_2 and s_1 are g-paired.

Furthermore,

$$\langle |s_1|_\varepsilon \big| \Delta g\rangle = \langle \Delta |s_1|_\varepsilon \big| g\rangle = \int_M (\Delta |s_1|_\varepsilon) g \leq$$

$$\leq \int_M (\bar{\Delta} s_1, g\,\frac{s_1}{|s_1|_\varepsilon}) = \langle \bar{\Delta} s_1 \Big| g\,\frac{s_1}{|s_1|_\varepsilon}\rangle$$

since g is non-negative and in the domain of Δ.

Letting ε go to zero yields

$$\langle \bar{\Delta} s_1 | s_2\rangle \geq \langle |s_1| \Big| \Delta |s_1|\rangle$$

then the result follows by Theorem 16. ■

Let us point out that in that case the cone is self-dual since it is the set of non-negative functions of $L^2(M;\mathbb{R})$ and thus allows to get a strong domination inequality in Theorem 20.

VI - APPLICATION 2: SCHWARZ SYMMETRIZATION

This is the key section of this appendix. We aim at giving an alternative proof of Theorems V.9 and V.28, using the formal approach of symmetrization. For the sake of simplicity, we will give the construction in the simpler case of bounded open subsets with smooth boundary in $\mathbb{R}^2$. It generalizes steadily to all the other situations in which Schwarz symmetrization can be used. Some of them are described at the end of this section.

For symmetrization, we have used the basic reference [H-L-P] pages 260 to 299.

The Geometric Symmetrization.

From now on the domain under consideration will be connected, bounded and with smooth boundary in $\mathbb{R}^2$.

The symmetrization is a map which associates to each such domain a more symmetric one. The Schwarz symmetrization (which is the one we consider here) associates to Ω the ball Ω^* of $\mathbb{R}^2$ with center 0 and the same area as Ω.

Notice that we consider concentric balls in order to have a totally ordered family of balls as target space for the symmetrization.

The main feature of this operation is given by the

21. Theorem

With the above notations if L (resp. L^*) is the length of the boundary of Ω (resp. Ω^*), then

$$L \geq L^*.$$

22. Remarks.

i) This is the classical isoperimetric inequality in dimension 2. For a review on the different proofs, see [B-Z] and [ON].

ii) We will see later on that this inequality implies all the inequalities which will appear in this section.

iii) Here the smoothness of the boundary is not essential.

23. Some elementary properties

Let A and B be two bounded measurable sets then:

i) $\mathrm{Vol}([A\cap B]^*) = \mathrm{Vol}(A\cap B) \leq \mathrm{Vol}(A^*\cap B^*) = \mathrm{Min}\{\mathrm{Vol}(A),\mathrm{Vol}(B)\}$.

ii) $\mathrm{Vol}([A\cup B]^*) = \mathrm{Vol}(A\cup B) \geq \mathrm{Vol}(A^*\cup B^*) = \mathrm{Max}\{\mathrm{Vol}(A),\mathrm{Vol}(B)\}$.

iii) The symmetrized sets of a finite sequence of sets (A_k) can always be arranged into a decrasing sequence. So, this map is often called "decreasing rearrangement" (see [H-L-P]).

Symmetrization of Functions.

Let A be a measurable set of finite volume then χ_A, the characteristic function of A, is measurable and integrable. We define,

$$S(\chi_A) = \chi_{A^*}.$$

The finite sums of characteristic functions of measurable sets is dense in the space of integrable functions on a bounded domain. Unfortunately the map S cannot be extended as a linear map indeed

$$S(\chi_A+\chi_B) \neq (S(\chi_A) + S(\chi_B))$$

in general.

However, if $A \subset B$ or $B \subset A$, we can define $S(\chi_A+\chi_B)$ by

(*) $\quad S(\chi_A+\chi_B) = S(\chi_A) + S(\chi_B)$.

Let Ω be a bounded domain in $\mathbb{R}^2$ and f be a non-negative integrable function on Ω. We will write f as a sum (integral) of characteristic functions of an increasing family of bounded measurable sets.

Define

$$D_t = \{x \in \Omega : f(x) \geq t\}$$

the set D_t is measurable of finite volume.

24. Lemma:

With the above notations $f = \int \chi_{D_t} dt$.

Proof. Define the functions

$$F: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$$

by

$$F(x,t) = \chi_{D_t}(x)$$

and

$$\phi: \Omega \times \mathbb{R}_+ \to \mathbb{R} \quad \text{by} \quad \phi(x,t) = f(x)-t .$$

Then

$$F(x,t) = \chi_E(x,t) \quad \text{where} \quad E = \phi^{-1}(\{y \geq 0\}).$$

The function ϕ is measurable and so is E, thus F is a measurable and non-negative function.

Then by Fubini's Theorem (see [RN] pg. 140) both functions

$$t \to \int_\Omega \chi_{D_t}(x)dx, \quad x \to \int_{\mathbb{R}_+} \chi_{D_t}(x)dt = \int_0^{f(x)} 1\, dt = f(x)$$

are measurable and

$$\int_{\mathbb{R}_+} \mathrm{Vol}(D_t)dt = \int_{\mathbb{R}_+} (\int_\Omega \chi_{D_t}(x)dx)dt = \int_\Omega (\int_{\mathbb{R}_+} \chi_{D_t}(x)dt)dx = \int f < \infty$$

The family D_t is a decreasing family of bounded measurable sets; taking (*) into account we define

$$S(f) = S(\int \chi_{D_t} dt) = \int S(\chi_{D_t})dt = \int \chi_{D_t^*}\, dt.$$

Now it remains to verify that if $f \in L^1(\Omega)$, i.e. is a class of functions defined up to measure zero sets, so is $S(f)$.

More precisely, if g is in the class of f (defines the same L^2-function) then f and g differ on a set N of measure zero. Then if

$$E_t = \{x | g(x) \geq t\}$$

$$(E_t \backslash D_t) \cup (D_t \backslash E_t) \subset N$$

and has measure zero. Therefore

$$\mathrm{Vol}(E_t) = \mathrm{Vol}(D_t)$$

and

$$E_t^* = D_t^* \quad \text{for all} \quad t$$

which implies

$$S(f) = S(g). \blacksquare$$

For another approach of the symmetrization of functions the reader is referred to Chapter IV, §A, and Chapter V, nº 12.

If f is any integrable function (not necessarily non-negative), we define:

$$S(f) = S(|f|).$$

The next theorem shows why S deserves its name.

25. Theorem:

With above notations the map S is a symmetrization in the sense of Section III.

Proof. The domain Ω being bounded, any function f in $L^2(\Omega,\mathbb{R})$ is integrable, so S is defined on $L^2(\Omega,\mathbb{R})$.

The symmetrized function $S(f)$ is radially symmetric on Ω^*, i.e. it depends only on the distance to the origin, and is non-increasing. So, the target Hilbert space is the space $L^2([0,R],rdr)$ (or equivalently $L_o^2(\Omega^*,dx)$, the set of functions in $L^2(\Omega^*,dx)$ which are radially symmetric) where R is the radius of Ω^*.

The cone $\mathcal{K}^+$ is the cone constituted by the non-increasing functions (i.e. functions which are in L^2 and whose derivative in the distribution sense is a non-positive measure). It is clearly not self-dual.

Let us then verify the conditions which appear in the definition of a symmetrization.

i) If f and g are non-negative function in $L^2(\Omega;\mathbb{R})$

$$f = \int \chi_{D_t} dt \qquad g = \int \chi_{E_s} ds$$

then

$$fg = \int \chi_{D_t}\chi_{E_s} dtds = \int \chi_{D_t \cap E_s} dtds$$

so

$$\int_\Omega fg = \int_{\mathbb{R}^2} \mathrm{Vol}(D_t \cap E_s) dtds \ ;$$

similarly

$$\int_\Omega S(f)S(g) = \int_{\mathbb{R}^2} \mathrm{Vol}(D_t^* \cap E_s^*) dtds$$

and from the property 23.i)

$$\langle f|g\rangle \leq \langle S(f)|S(g)\rangle .$$

26. <u>Remark</u>.

If the reader is not satisfied with the formula proved in Lemma 26 and used in the following calculations, he can use the following property of non-negative measurable functions f on a σ-finite measured space (see [RN] page 15): there exists an increasing sequence of simple functions s_n

$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq f$$

which converges to f at any point. Recall that a simple function is a finite linear combination of characteristic functions of measurable sets; in the case at hand, the sets are in the family $\{D_t\}_{t\in\mathbb{R}}$. Then by Lebesgue's dominated convergence theorem and the uniform continuity of the symmetrization on $L^2(\Omega;\mathbb{R})$ we can avoid using the integral representation.

ii) Now if f is a non-negative real valued function on Ω and F a non-decreasing function on $\mathbb{R}_+$, we can write

$$F\circ f = \int \chi_{D_t} dF$$

the integral being a Stieljès integral (F is of bounded variation). Then using the positivity of the derivative of F or the method given in Remark 26, one can easily prove that

$$S(F\circ f) = \int S(\chi_{D_t}) dF = \int \chi_{D_t^*} dF = F\circ(S(f)).$$

Then let f be a function in $L^2(\Omega;\mathbb{R})$ and g in $\mathcal{K}^+$, g is in $L^2([0,R],xdx)$ and is non-increasing. For each $t \in \mathbb{R}_+$, the set $D_t^* = (\{x|f(x)| > t\})^*$ is a ball of radius $r(t)$, $0 \leq r(t) \leq R$. The function r is non-increasing. For the sake of simplicity (and this will be sufficient for the sequel) we assume that r is a homeomorphism onto $[0,R]$.

If we define the function h to be $g\circ r$, h is a non-decreasing function and by the previous formula

$$S(h\circ|f|) = h\circ S(|f|) = g.$$

The last equality being achieved because r is assumed to be a homeomorphism. Indeed in that case

$$S(|f|)(r_o) = r^{-1}(r_o)$$

$$h\circ S(|f|)(r_o) = g(r_o).$$

The functions $h\circ|f|$ and $|f|$ having the same level sets,

$$\int_\Omega |f|.h\circ|f| = \int_{\Omega^*} S(f)g.$$

If we now define

$$\mathrm{sign}(f) = \begin{cases} \dfrac{f(x)}{|f(x)|} & \text{if} \quad f(x) \neq 0 \\ 1 & \text{if} \quad f(x) = 0 \end{cases}$$

$$\bar{f} = (\text{sign } f)h\circ|f|$$

we have the equalities,

$$S(\bar{f}) = g$$

$$\langle f|\bar{f}\rangle = \int_{\Omega} f\bar{f} = \int_{\Omega^*} S(f)g = \langle S(f)|g\rangle$$

which is the pairing condition for the particular case where r is a homeomorphism. ■

27. Remarks.

1) The verification of the pairing condition for a general situation (r not necessarily a homeomorphism) being not necessary we limit ourselves to the above case (this will be clear in the next theorem).

2) The reader can easily verify that this definition of symmetrization coincides with the usual one (see [BE] pp. 47) when the functions under considerations are sufficiently regular.

We can now formulate the main theorem of this section. Let Δ be the Laplace operator on $L^2(\Omega;\mathbb{R})$ (resp. Δ^* on $L^2(\Omega^*;R)$) with Dirichlet boundary condition. Recall that it is associated to the quadratic form

$$q(u) = \int_{\Omega} |du|^2$$

with domain

$$\mathcal{D}(q) = H^1_o = \{u \in L^2(\Omega;\mathbb{R}) \,/\, \int_{\Omega} |du|^2 < +\infty,\ u|_{\partial\Omega} \equiv 0\}.$$

Notice that if u is a function in $L^2(\Omega;\mathbb{R})$ which satisfies $\int_{\Omega}|du|^2 < +\infty$ then $u|_{\partial\Omega}$ is a function in $H^{1/2}(\partial\Omega)$ and so at least in $L^2(\partial\Omega)$ (see [C-P] pg. 101). The equality $u|_{\partial\Omega} \equiv 0$ has to be understood in this sense. We have,

28. Theorem

With the above notations the semigroup $e^{-t\Delta^*}$ dominates the semigroup $e^{-t\Delta}$ for all $t > 0$.

Step 1.

The theorem will be proved if we show that for any $f_1 \in L^2(\Omega)$ and $g \in K^+$ we have the inequality

$$(*) \qquad \langle S[(\Delta+\lambda)^{-1}f_1]|g\rangle \leq \langle(\Delta^*+\lambda)^{-1}S(f_1)|g\rangle \quad \text{for all} \quad \lambda > 0$$

(See Theorem 16).

The operators $(\Delta+\lambda)^{-1}$ and $(\Delta^*+\lambda)^{-1}$ are positivity preserving (this is a well known fact which can be proved by using the Beurling-Deny criterion). Thus we have,

$$|(\Delta+\lambda)^{-1}f_1| \leq (\Delta+\lambda)^{-1}|f_1|$$

(inequality between functions)

and $|f_1|$ being non-negative,

$$0 \leq (\Delta+\lambda)^{-1}|f_1|.$$

Using the facts that

$$S[(\Delta+\lambda)^{-1}f_1] = S[|(\Delta+\lambda)^{-1}f_1|]$$

and

$$S(u) \leq S(v) \quad \text{whenever} \quad 0 \leq u \leq v,$$

we see that it is sufficient to prove the inequality (*) for f_1 a non-negative function.

Step 2.

The operators $(\Delta+\lambda)^{-1}$, $(\Delta^*+\lambda)^{-1}$ and S being continuous (for the L^2 norms; see proposition 8) it suffices to prove (*) for g in a dense subset of $\mathcal{K}^+$ and f_1 in a dense subset in $\mathcal{H}^+$ (the set of non-negative functions of $L^2(\Omega;\mathbb{R})$).

The proof of Theorem 16 relies on the inequality

$$(**) \qquad \langle \Delta h_1 | h_2 \rangle \geq \langle S(h_1) | \Delta^* S(h_2) \rangle$$

for $h_1 = (\Delta+\lambda)^{-1} f_1$, $S(h_2) = (\Delta^*+\lambda)^{-1} g$ and (h_1, h_2) $S(h_2)$-paired. If f_1 is a non-negative function then h_1 is also non-negative, the operator $(\Delta+\lambda)^{-1}$ being positivity preserving.

It is then clear that it is sufficient to prove (**) for $S(h_2)$ in a core for Δ^* and h_1 in a dense subset of $\mathcal{D}(\Delta) \cap \mathcal{H}^+$ (for the $\mathcal{D}(\Delta)$ topology).

Step 3.

We choose for $S(h_2) = u$ a smooth function in $\bar{\Omega}^*$ vanishing on the boundary, radially symmetric and non-increasing in the radial variable.

Then, we have seen that

$$h_2 = g \circ r \circ h_1 \quad \text{where} \quad r(t) = \text{radius of } D_t^*$$

whenever r is continuous. In fact, we will choose h_1 to be smooth non-negative and such that r is piecewise smooth and absolutely continuous. Let us assume for a while that this can be done, and define

$$\bar{g} = g \circ r$$

$$h_2 = \bar{g} \circ h_1$$

and $\bar{g}$ is non-decreasing. Thus

$$\langle \Delta h_1 | h_2 \rangle = \int_\Omega (dh_1 | dh_2) = \int_\Omega (\bar{g}' \circ h_1) |dh_1|^2 .$$

Let us define a non-decreasing function on $[0,s]$ (where $s = \sup h_1$), k by

$$\bar{g}' = (k')^2$$

then

$$\langle \Delta h_1 | h_2 \rangle = \int_\Omega (k' \circ h_1)^2 |dh_1|^2 = \int_\Omega |dw|^2$$

with $w = k \circ h_1$.

Now it is a well known fact that symmetrization decreases the Dirichlet integral (see Chap. V, §A or [BE] p. 55)

$$\int_\Omega |dw|^2 \geq \int_\Omega |dS(w)|^2$$

but $S(w) = k \circ S(h_1)$ and by the same process we get

$$\int_\Omega |dw|^2 \geq \langle S(h_1) | \Delta^* S(h_2) \rangle$$

and the theorem is proved.

<u>Step 4</u>.

1) It remains to choose nice functions for h_1. Recall that $f_1 = (\Delta + \lambda) h_1$ must be in a dense subset in $\mathcal{H}^+$. For example we can assume that f_1 is a smooth non-negative function with compact support in the interior of Ω. Then h_1 is smooth up to the boundary ($h_1 \in C^\infty(\bar{\Omega})$), vanishes on $\partial\Omega$ and is positive in the interior of Ω. Thus if $\frac{\partial}{\partial \nu}$ is the derivative in the direction of the inward normal, then

$$\frac{\partial h_1}{\partial \nu} \geq 0.$$

Now we can perturb h_1 in $C^\infty(\bar{\Omega})$ in such a way that the new function, let us say u, has the following properties

$$\frac{\partial u}{\partial n} > 0 \quad \text{on} \quad \partial\Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

$$u > 0 \quad \text{in the interior of} \quad \Omega.$$

2) Let $\Omega_s = \{x \in \Omega \setminus \operatorname{dist}(x,\partial\Omega) \geq s\}$. For sufficiently small s this set has a smooth boundary. Rescaling u if necessary we can assume, for the sake of simplicity, that $\frac{\partial u}{\partial s} \geq 1$ on $\Omega\setminus\Omega_{s_o}$ for sufficiently small s_o where $s(x) = \operatorname{dist}(x,\partial\Omega)$.

Furthermore, the function u being in $C^\infty(\bar{\Omega})$ is the restriction on $\bar{\Omega}$ of a smooth, compactly supported, function in $\mathbb{R}^2$.

3) Applying Milnor's theorem (see [MR], page 37) to this extended function, we can approximate uniformly u by a smooth function with non-degenerate critical points on $\mathbb{R}^2$ and in the C^k topology on $\bar{\Omega}$, (for k chosen arbitrarily and ≥ 3). Let v be such a function

$$v \in C^\infty(\bar{\Omega})$$

v has non-degenerate critical points

$$\|u-v\|_{C^k(\bar{\Omega})} \leq \eta/10 \quad \text{(small positive number)}$$

if η is small enough

$\frac{\partial v}{\partial s} \geq \frac{1}{2}$ on $\Omega\setminus\Omega_{s_o}$ (and so has no critical points in this set).

From the properties of u it is clear that the set

$$\{x \backslash u(x) = \eta\} = \Gamma_\eta$$

is a smooth curve for η small which converges to $\partial\Omega$ as η goes to zero and which is almost parallel to $\partial\Omega$.

Taking η small, this curve is in $\Omega\backslash\Omega_{s_o}$. Thus

$$u(x) = \eta \quad \text{on} \quad \Gamma_\eta \quad \text{implies} \quad v(x) \geq \frac{9\eta}{10} \quad \text{on} \quad \Gamma_\eta$$

$$u(x) = 0 \quad \text{on} \quad \partial\Omega \quad \text{implies} \quad v(x) \leq \frac{\eta}{10} \quad \text{on} \quad \Gamma_\eta.$$

Then the set $\{x \backslash v(x) = \frac{\eta}{5}\} = \gamma_\eta$ is a smooth curve close to $\partial\Omega$.

It now suffices to construct a diffeomorphism ψ_η from the interior of γ_η onto Ω such that:

$$\begin{cases} \psi_\eta \text{ is the identity in } \Omega_{s_o} \\ \psi_\eta \text{ sends diffeomorphically } \gamma_\eta \text{ onto } \partial\Omega \\ \psi_\eta \text{ is close to the identity in the } C^k \text{ topology.} \end{cases}$$

Then the function

$$w(x) = v[\psi_\eta^{-1}(x)] - \eta/5 \quad \text{for} \quad x \in \Omega$$

is a smooth function, with finitely many critical points in the interior of Ω, vanishing on $\partial\Omega$ and arbitrary close to h_1 in the $\mathcal{D}(\Delta)$-topology.

4) We can then work with such functions, which clearly have the property that the associated function $r(t)$ is absolutely continuous (the only possible points at which r is not smooth are the critical values of h_1).

Finally we have to verify that $(\Delta^*+\lambda)^{-1}$ preserves $\mathcal{K}^+$. Recall that if $g \in \mathcal{K}^+$, it is radially symmetric. Because the rotations about the origin which are isometries of $\mathbb{R}^2$, commute with Δ^*, $f = (\Delta^*+\lambda)^{-1}g$ is radially symmetric.

The function f verifies (by definition)

$$(\Delta^*+\lambda)f \geq 0$$

thus by applying the maximum principle we see that it is non-increasing in the radial variable. ∎

29. <u>Remarks</u>.

i) The technical details have not been completely written here, they will appear elsewhere. We just wanted to show that although the criterion given in theorem 28 is not so beautiful as Beurling-Deny's one, it can be improved in some particular cases; it is sufficient to prove that the energy integral is not increased by symmetrization.

ii) This proof steadily generalizes to higher dimension.

iii) It also gives an alternative proof of Theorem 9 and 28 of Chapter V. In fact, it is much simpler technically in the case of a compact manifold without boundary. Let (M,g) be a n-dimensional compact connected Riemannian manifold without boundary such that

$$\mathrm{Ricci}(g) \geq (n-1)g$$

and define the number β by,

$$\beta = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S^n)} \leq 1$$

then the geometric symmetrization associates to each measurable set D on M a ball centered at the north pole of S^n of volume $\frac{1}{\beta}\,\mathrm{Vol}(D)$. Gromov's isoperimetric inequality ([GV]) asserts that this map decreases the volume of ∂D (when it is smooth) up to the factor β. This yields a symmetrization on functions

$$f = \int \chi_{D_t} dt \qquad S(f) = \int \chi_{D_t^*} dt ;$$

one then has to prove

$$\beta^{-1} \int_M |dw|^2 \geq \int_{S^n} |dS(w)|^2$$

for nice w which is easy because we deduce from Milnor's theorem that the functions with non-degenerate critical points are dense in $C^k(M)$ for all $k \in \mathbb{N}$.

iv) The theorems on forms which appear in [B-G] are obtained by composing symmetrizations.

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APPENDIX B

A GUIDE TO THE LITERATURE

"Le Spectre d'une Variété Riemannienne en 1982"

by

Pierre Bérard and Marcel Berger

This bibliography was first published in "Spectra of Riemannian Manifolds" Kaigai Publications, Tokyo 1983, p. 139-194.

Spectra of Riemannian Manifolds
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LE SPECTRE D'UNE VARIÉTÉ RIEMANNIENNE EN 1982

PIERRE H. BÉRARD & MARCEL BERGER

Table des Matières

Introduction

Depuis 1970, date à laquelle a été publié "LE SPECTRE D'UNE VARIÉTÉ RIEMANNIENNE", en abrégé, BGM (M. Berger, P. Gauduchon, et E. Mazet, Lecture Notes in Mathematics, n° 194, Springer) l'étude du spectre a connu une grande effervescence. Il nous a paru utile de rassembler une *bibliographie classée et assez complète* (mais bien sûr difficilement exhaustive) pour compléter le BGM.

Suivant en cela BGM, mais aussi pour des raisons de temps, d'espace et d'incompétence, nous avons fait cette bibliographie avec un A PRIORI: quand nous disons SPECTRE, nous sous-entendons *"le spectre du Laplacien d'une variété riemannienne compacte sans bord"*.

Il ne nous a cependant pas paru raisonnable de nous limiter à ce seul sujet; c'est pourquois nous donnons aussi des éléments de bibliographie concernant le spectre de l'opérateur de Laplace-Beltrami agissant sur les p-formes (essentiellement regroupés au paragraphe 3.2) et la théorie spectrale des variétés non compactes (voir chapitre 10). Compte tenu de l'importance "physique" des variétés à bord, et aussi des développements assez spectaculaires dont la théorie a été l'objet ces dernières années, nous donnons un certain nombre de références concernant le *"cas à bord"*. Ces références sont ventilées dans les différents chapitres. Notons que, dans les chapitres 3 à 8, et 10, ces références sont regroupées à la fin de chaque paragraphe, précédées de la mention explicite *"cas à bord"*.

Peut-être convient-il de noter que le cas des variété non compactes, comme celui des variétés à bord, sort du cadre strict de la géométrie riemannienne (problèmes de théorie spectrale dans l'un, de géométrie symplectique dans l'autre).

Le lecteur trouvera dans le *"Mode d'emploi"* ci-après des informations plus détaillées qui l'aideront, du moins nous l'espérons, à utiliser fructueusement cette bibliographie.

Plusieurs collègues ont bien voulu nous signaler des erreurs, des omissions ou de nouvelles références; nous les prions de bien vouloir accepter ici nos remerciements.

Nous remercions Mesdames Cordel et Strazzanti qui ont assuré la frappe du manuscrit.

Mode d'Emploi

"Le spectre en 1982" est divisé en douze chapitres. Les chapitres 1 et 2 sont essentiellement consacrés aux résultats préliminaires. Le chapitre 9 traite des VARIÉTÉS SPÉCIALES et le chapitre 10 des VARIÉTÉS NON COMPACTES. Notons que les références concernant certains sujets *pointus* (et souvent encore peu explorés) ont été regroupées, sous différentes rubriques au chapitre 12. Nous renvoyons le lecteur au *"Leitfaden"* pour plus de détails concernant le contenu des différents chapitres.

Quelques commentaires complémentaires:

"cas à bord": dans les chapitres 3 à 8 et 10, cette mention précède, à la fin de chaque paragraphe, les références concernant le spectre des variétés à bord.

"Tableau des interactions fortes": toutes les références bibliographiques n'ont pas été inscrites dans chacun des paragraphes où elles devraient l'être. Ce tableau est destiné à compenser cet inconvénient.

"Ouvrages de base": nous avons choisi (choix personnel, donc sujet à caution) un certain nombre de références: livres, cours, articles de synthèse, pour aider le lecteur à se faire une idée générale d'un chapitre précis, avant d'aborder la jungle de la bibliographie spécialisée. Pour être plus repérables, ces références sont données en MAJUSCULES (exemples: CLARK [1], GUILLEMIN [2]). Une liste spécifique de ces ouvrages est donnée après le *"Tableau des interactions fortes"* (chaque référence est suivie des numéros des paragraphes auxquels elle se rapporte).

"Actes de colloques": il est parfois intéressant de connaître le développement historique d'un sujet. C'est pourquoi nous avons regroupé, en une liste séparée, les actes des colloques où sont publiés certains des articles cités en référence (ceci par ordre chronologique).

"Preprints": pour permettre au lecteur de localiser (ou de se procurer) plus facilement les articles encore sous forme de prétirage, nous avons essayé de donner un lieu d'émission et une date (en général celle à laquelle nous avons eu le prétirage en main pour la première fois; cette date, on s'en doute, est assez relative et on ne peut lui donner de valeur absolue).

"Validité": cette bibliographie a été arrêtée à l'état de nos fiches en septembre 81. La composition du texte ayant été retardée, cela nous a amenés à compléter notre texte un an plus tard, soit en décembre 82. Nous osons espérer que le lecteur nous pardonnera de ne pas avoir respecté l'ordre lexicographique dans la classification par sujets.

Dans le "Leitfaden" qui suit, nous donnons quelques indications complémentaires sur la manière dont nous avons ventilé les références suivant les chapitres.

Terminons en indiquant trois références qui peuvent être utilisées par le lecteur parallèlement à cette bibliographie:

Simon-Wissner [1]: article de synthèse sur une partie des chapitres qui constituent cette bibliographie;

Yau [3]: le lecteur y trouvera une liste des applications des équations aux dérivées partielles à la géométrie et en particulier au spectre;

Yau [4]: liste de problèmes ouverts en géométrie, contient une section propre au spectre (certains problèmes sont peut-être déjà résolus).

Leitfaden

Chapitre 1: *"Préliminaires à l'étude du spectre"*

Ce chapitre est surtout destiné aux non spécialistes. Nous y donnons quelques références (personnelles) sur les connaissances requises pour aborder la littérature spécialisée.

Chapitre 2: *"Motivations; Equations de la physique mathématique"*

L'intérêt porté au spectre du Laplacien nous vient sans doute de la physique. Les références concernant les rapports avec la physique sont données au paragraphe 2.1. Dans le paragraphe 2.2. nous donnons des références relatives à l'étude a priori des équations de la physique mathématique utiles dans l'étude du spectre.

Chapitre 3: *"Exemples de spectres"*

Comme le montrent les références données dans le paragraphe 3.1., les variétés dont le spectre est donné par des formules explicites sont rares.

On a cependant une bonne description du spectre de certaines variétés spéciales (groupes de Lie, espaces symétriques, . . .), parfois par le biais de solutions explicites pour l'équation de la chaleur ou des ondes. Nous donnons ces références dans le paragraphe 3.3.

Nous avons choisi de rassembler toutes les références relatives au spectre de l'opérateur de Laplace-Beltrami agissant sur les formes différentielles dans le paragraphe 3.2. (en particulier, elles ne sont pas ventilées systématiquement dans les différents paragraphes). C'est une conséquence de l'A PRIORI cité dans l'introduction.

Dans ce chapitre, les références au *"cas à bord"* sont systématiquement regroupées en fin de chaque paragraphe.

Chapitre 4: *"Asymptotiques"* et Chapitre 5: *"Spectre et géométrie"*

De très nombreux résultats sur le spectre ont été obtenus par l'intermédiaire du comportement asymptotique de certaines fonctions du spectre.

L'équation fonctionnelle de l'exponentielle a permis, par le biais de l'asymptotique du noyau de la chaleur (à la Minakshisundaram-Pleijel) d'étudier certains invariants spectraux: paragraphes 4.1. et 5.1. respectivement.

L'étude de la propagation des ondes, liée aux géodésiques (en fait aux trajectoires d'un hamiltonien) a permis d'établir les rapports existant entre le spectre et le spectre des longueurs, que sont les formules de Poisson: paragraphes 4.2. et 5.2. (à comparer aussi avec les formules de traces de Selberg: paragraphe 9.1.).

Ces études ont permis de mieux cerner le comportement asymptotique des valeurs propres: paragraphe 4.3.

Les paragraphes 5.4. et 5.5. sont consacrés aux références relatives à des sujets connexes.

La fonction zeta associée aux valeurs propres joue aussi un rôle important, le paragraphe 5.3. lui est consacré.

Dans ces deux chapitres, les références au *"cas à bord"* sont regroupées à la fin de chaque paragraphe.

Chapitre 6: *"Isospectralité"*

Ce sujet, presque intouché en 1970, a connu d'importants développements récents. Le paragraphe 6.1. est consacré aux résultats positifs (souvent très liés au paragraphe 5.1.): zoologie des variétés caractérisées par leur spectre, et à certains théorèmes généraux.

Le paragraphe 6.2. est consacré à la faune des variétés isospectrales non isométriques.

Ici encore, le *"cas à bord"* fait l'objet d'attentions particulières.

Chapitre 7: *"Perturbations et généricité"*

Outre les références relatives aux propriétés génériques du spectre (paragraphe 7.2.) ce chapitre contient des références sur le comportement du spectre sous divers types de perturbations (paragraphe 7.1.).

Les références pour le *"cas à bord"* sont données à la fin de chaque paragraphe.

Chapitre 8: *"Equations aux dérivées partielles, applications"*

Le fait que le spectre étudié soit celui d'un opérateur différentiel impose des conditions locales ou globales sur les fonctions propres et par conséquent sur le spectre lui-même. De même le fait que le Laplacien soit très lié à la structure riemannienne conduit à des utilisations spécifiques des fonctions propres (immersions isométriques ...). Des références à ces divers aspects de l'étude du spectre sont données dans ce chapitre.

Les références pour le *"cas à bord"* sont données à la fin de chaque paragraphe.

Chapitre 9: *"Variétés spéciales"*

Le paragraphe 9.1. est consacré au cas très particulier des variétés de courbure -1: techniques et résultats sont propres à la géométrie hyperbolique (comme par exemple la formule des traces de Selberg) mais la comparaison avec le "cas général" n'en est pas moins intéressante.

Le paragraphe 9.2. contient les références qui traitent du spectre d'autres variétés particulières.

Chapitre 10: *"Cas non compact"*

Selon l'A PRIORI indiqué dans l'introduction, nous avons regroupé les références relatives à l'étude du spectre des variétés non compactes en un seul chapitre, sans les ventiler en différents paragraphes. Le *"cas à bord"* fait quand même l'objet d'un traitement séparé.

Chapitre 11: *"Etude individuelle des valeurs propres"*

La première valeur propre (non triviale) joue un rôle particulier (comme en physique), le paragraphe 11.1. lui est consacré.

Les autres valeurs propres se contentent du seul paragraphe 11.2.

Le paragraphe 11.3. est consacré aux questions connexes: inégalités isopérimétriques et inégalités de Sobolev. Ces questions sont liées à l'étude des valeurs propres. Ce texte *n'étant pas* une bibliographie spécifique sur ce sujet, et compte tenu des excellentes références Payne [1], Osserman [2, 3] et Bandle [1, 3] nous ne donnons que certaines références antérieures à ces trois articles et bien sûr celles d'articles plus récents (sans doute en avons nous oubliées!).

Chapitre 12: *"Last but not least"*

Dans ce chapitre nous avons regroupé, sous diverses rubriques, les références des articles qui traitent d'aspects particuliers du spectre. Ce sont souvent des domaines peu explorés, au moins actuellement, ou des domaines connexes au spectre. Aussi, il convient de ne pas considérer ce chapitre comme mineur.

Le Spectre en 1982

1. PRÉLIMINAIRES À L'ÉTUDE DU SPECTRE
   (Théorie spectrale abstraite; théorie spectrale des opérateurs différentiels; équations aux dérivées partielles; matériel riemannien)
2. MOTIVATIONS: ÉQUATIONS DE LA PHYSIQUE MATHÉMATIQUE
   - 2.1. Motivations; Physique et modèles mathématiques;
   - 2.2. Résultats généraux sur les équations étudiées: problèmes de Dirichlet et de Neumann pour le Laplacien; équation de la chaleur, équation des ondes; fonctions de Green;
3. EXEMPLES DE SPECTRES
   - 3.1. Exemples numériques explicites de spectres ou de valeurs propres du Laplacien sur les fonctions;
   - 3.2. Le Laplacien sur les formes;
   - 3.3. Spectre du Laplacien sur les fonctions et variétés spéciales (groupes de Lie, espaces symétriques, quotients, submersions, . . .); expressions "explicites" pour les noyaux de la chaleur et des ondes;
4. ASYMPTOTIQUES
   - 4.1. Développements asymptotiques à la MINAKSHISUNDARAM-PLEIJEL;
   - 4.2. Formules de POISSON et équation des ondes;
   - 4.3. Asymptotique des valeurs propres;
5. SPECTRE ET GÉOMÉTRIE
   - 5.1. Spectres et invariants locaux et globaux;
   - 5.2. Spectre des longueurs; spectre et longueurs des géodésiques périodiques;
   - 5.3. Fonctions zeta; invariant êta;
   - 5.4. Quasimodes; fonctions propres concentrées près d'une géodésique périodique;
   - 5.5. Spectre du Laplacien plus potentiel;
6. ISOSPECTRALITÉ
   - 6.1. Résultats positifs et théorèmes généraux;
   - 6.2. Contre-exemples;

1. PRÉLIMINAIRES A L'ETUDE DU SPECTRE

Généralités
Cime 1973:3, CLARK [1], FRIEDLAND [2], GARABEDIAN [1], Gelfand [1], Gelfand-Yaglom [1], GOULAOUIC [1]
Théorie spectrale abstraite
Glazman [1], REED-SIMON [1] (vol II)
Théorie spectrale des opérateurs différentiels
BROWDER [1], Protter [2]
Equations aux dérivées partielles
Gilbarg-Trudinger [1], Petrovsky [1], PROTTER [3]
Matériel Riemannien

BERGER-GAUDUCHON-MAZET [1], Besse [1]

2. MOTIVATIONS; ÉQUATIONS DE LA PHYSIQUE MATHÉMATIQUE

Généralités

CLARK [1], Courant-Hilbert [1] (vol 1), Gelfand-Yaglom [1], Kac [1, 2], Morse-Feshbach [1], VENKOV [1]

2.1. Mativations; Physique et modèles mathématiques

Balian-Bloch [1], Petrovsky [1], PROTTER [3]

2.2. Résultats généraux sur les équations étudiées: problèmes de Dirichlet et de Neumann pour le Laplacien; équation de la chaleur, équation des ondes; fonctions de Green

Atiyah-Bott-Patodi [1], Aubin [1], Benabdallah [1], Cheeger-Yau [1], Cheng-Li-Yau [1, 2], Colin de Verdière [3, 7, 8], Colin de Verdière-Frisch [1], Dodziuk [2, 3], Fegan [1, 3], Frisch [1], Greiner [1, 2], GUILLEMIN-STERNBERG [1], Hall-Stedry [1], Hess-Schrader-Uhlenbock [1], Hörmander [2], Ivrii [1, 2], Kannai [1], Keller [1],Keller-Rubinow [1], Lascar [1], Malliavin [1], Meyer [1], Minakshisundaram-Pleijel [1], Mneimne [1], Molchanov [1], Rauch [1], Reilly [5], Seeley [1, 4, 5, 6], Smale [1], Urakawa [4], Weinstein [5], Zucker [1], Arnal [1], Cheng-Li [1], Clements [1], Danet [1], Eichhorn [5], Günther [5], Har'el [2], Kalnins-Miller [1], Oersted [1], Rinke-Wunsch [1], Varopoulos [1 à 4]

3. EXEMPLES DE SPECTRES

Généralités

BERGER [1], Courant-Hilbert [1], Morse-Feshbach [1], Paquet [1]

3.1. Exemples numériques explicites de spectres ou de valeurs propres du Laplacien sur les fonctions

Buser [8], Friedland-Hayman [1], Urakawa [9]

"*cas à bord*": Bérard [4, 6], Bérard-Besson [2], Nooney [1], Pinsky [3], Polya [1], Urakawa [6]

3.2. Le Laplacien sur les formes

Asada [1, 2], Donnelly [19], Eichhorn [3, 7], Fegan [2, 3], Ikeda-Taniguchi [1], Iwasaki-Katase [1], Kuwabara [3], Levy-Bruhl [2, 3], Millman [1], Tachibana-Yamaguchi [1], Tanno [2, 8], Tsagas [1], Tsagas-Kochinos [1], Wolpert [5], Dodziuk [5, 6, 7, 8, 9, 10], Tsukamoto [1]

3.3. Spectre du Laplacien sur les fonctions et variétés spéciales (groupes de Lie, espaces symétriques, quotients, submersions, ...); expressions "explicites" pour les noyaux de la chaleur et des ondes

Bedford-Suwa [1], Beers-Millman [1], Benabdallah [1], Bérard Bergery- Bourguignon [1, 2], Besson [1], Cheeger-Taylor [1], Chen-Vanhecke [1], S. S. Chen [1], Gallot-Meyer [1], Huber [3, 4, 5], Ikeda [1, 3], Ikeda-Yamamoto [1], Y. Mutō [1, 2], Sakai [2], Strese [1, 4], Sunada [1], Tandai-Sumitomo [1], Taniguchi [1], Tanno [4], Tsagas [2], Yamaguchi [1], Berezin [1], Furutani [1], |Marbes [1, 2], Tsukada [3], Urakawa [9]

4. ASYMPTOTIQUES

Généralités

Balian-Bloch [1, 2, 3], BÉRARD [3], BERGER [1], CLARK [1], Colin de Verdière [3, 8], Duistermaat-Guillemin [1], Duistermaat-Kolk-Varadarajan [1], Gangolli [1], Guillemin [1, 3, 4], GUILLEMIN [2], GUILLEMIN-STERNBERG [1]

"cas à bord": Balian-Bloch [1, 2, 3], CLARK [1], Seeley [4, 5, 6]

4.1. Développements asymptotiques à la MINAKSHISUNDARAM-PLEIJEL

Bérard [1, 2], Cahn-Gilkey-Wolf [1], Chavel-Feldman [4], Cheeger [1], COLIN DE VERDIERE [1, 6], Dlubek-Friedrich [1], Dodziuk [3], Fegan [1], Greiner [1, 2], Hess-Schrader-Uhlenbock [1], Kannai [1], Miatello [1], Minakshisundaram-Pleijel [1], Mneimne [1], L. Smith [1], Wallach [1], Atiyah [1], Bott [1], Sunada [5]

"cas à bord": Hasegawa [1], L. Smith [1], Atiyah [1], Bott [1]

4.2. Formules de POISSON et équation des ondes

Bérard [2], Besse [1], Chazarain [1, 3, 4, 5], CHAZARAIN [2], Colin de Verdière [5, 7], Kolk [1]

"gas à bord": Bardos-Guillot-Ralston [1, 2], Harthong [1], Kurylev [1]

4.3. Asymptotique des valeurs propres

Bérard [5], Boutet de Monvel [1, 4], Boutet de Monvel-Grisvard [1], Boutet de Monvel-Guillemin [1], Chachère [1], Clerc [1, 2], Colin de Verdière [4], Fleckinger-Pellé [1], Frisch [2, 3], GOULAOUIC [1], Grubb [1, 2], Haitov [1], HEJHAL [1], Hejhal [2], Helffer-Robert [1], Helton [1], Hörmander [1, 2, 3], Kolk [1, 2], Lieb [3], Meyer [1], Randol [2, 3], Taylor [1], Vasil'ev [1], VENKOV [1], Weinstein [1, 2, 3, 4], Weyl [1], Widom [1, 2, 3], Asurov [1]

"cas à bord": Arnold J. M. [1], Babich [1], Babich-Levitan [1], Bérard [4, 6], Bérard-Besson [4], Brüning [1], Ivrii

[1, 2, 3, 4, 5], Keller-Rubinov [1], Kurylev [1], Majda-Ralston [1, 2], Melrose [1, 2], Pham The Lai [1], Pinsky [3], Polya [1], Seeley [2, 3], Boimatov-Kostjucenko [1], Carleman [1], Lazutkin-Terman [1], Tamura [1 à 4]

5. SPECTRE ET GÉOMÉTRIE

Généralités

BÉRARD [3], BERGER [1, 3], Fischer [1], GUILLEMIN-STERNBERG [1], Kac [1], Singer [1]

"cas à bord": Kac [1], Fischer [1]

5.1. Spectres et invariants locaux et globaux

Atiyah-Bott-Patodi [1], Benko et al [1], Bérard [1], Brooks [1, 2, 3], Brüning [3], Brüning-Heintze [1], Cahn-Gilkey-Wolf [1], Cheeger [1], COLIN DE VERDIÈRE [6], Dodziuk [3], Dodziuk-Patodi [1], Donnelly [1, 2, 3, 4, 6, 12, 13, 14, 15], Donnelly-Patodi [1], Gallot-Meyer [1], Gilkey [1, 2, 4, 6à23], GILKEY [3], Gilkey-Sacks [1], Greiner [1, 2], Günther-Schimming [1], Har'el [1], Hasegawa [2], Ii [1], Levy-Bruhl [1], Mc Kean-Singer [2], Müller [1], Patodi [1, 2, 3], Olszak [1], Perrone [1], Pinsky [4], Ray [1], Ray-Singer [1, 2], Sakai [1], Sunada [2], Tanno [1], Urakawa [5], Véron [1],

"cas à bord": Gilkey [5], Hasegawa [1], Mc Kean-Singer [1], L. Smith [1], Kennedy [1], Schimming [1], Schimming-Teumer [1]

5.2. Spectre des longueurs; Spectre et longueurs des géodésiques périodiques

Balian-Bloch [1 à 4], Bérard [2], Bérard Bergery [1], Besse [1], Boutet De Monvel [1], Boutet de Monvel-Guillemin [1], Buser [10, 11], Chachère [1], CHAZARAIN [2], Chazarain [1, 3], Colin de Verdière [1, 3, 7, 8], De George [1], Donnelly [9], Duistermaat-Guillemin [1], Frisch [1], Gangolli [2], Guillemin [1, 3, 4], GUILLEMIN [2], Guillemin-Weinstein [1], Helton [1], Kudla-Millson [1], Müller [2], Randol [1, 3, 4, 6], Weinstein [1, 2, 5], Wolpert [3],

"cas à bord": Balian-Bloch [1 à 4], Guillemin-Melrose [1, 2], Harthong [1], Marvizi-Melrose [1], Millson [1]

5.3. Fonctions zeta; invariant êta

Atiyah-Bott-Patodi [1], Atiyah-Patodi-Singer [1], Cahn [1], Cahn-Wolf [1], Dlubek-Friedrich [1], Donnelly [5, 7, 10, 12], Gangolli [3], Gilkey [11], Randol [5], Seeley [1], VENKOV [1], S. Tanaka [1], Wodzicki [1], Atiyah-Donnelly-Singer [1], Gilkey-Smith [1], Millson [1]

5.4. Quasimodes; fonctions propres concentrées près d'une géodé-

sique périodique
Arnold [1], Colin de Verdière [2], Guillemin-Weinstein [1], Pyshkina [1], Ralston [1, 2], Weinstein [5],
"cas à bord": Babich [1], Babich-Lazutkin [1], Keller-Rubinow [1], Lazutkin [1, 2, 3], Babich-Ulin [1] Lazutkin-Terman [2]

5.5. Spectre du Laplacien plus potentiel
Barthel-Kümritz [1], Colin de Verdière [3, 4, 5, 8, 9], Flaschka [1], Fleckinger-Pellé [1], Guillopé [1, 2], Lax-Phillips [1, 2], Majda-Ralston [1], Prosser [2], Weinstein [3, 4], Widom [1, 2, 3], Fegan [4], Fleckinger [1], Moser [3]
"cas à bord": Balian-Bloch [3], Chung-Li [1], Guillemin [5, 6], Guillemin-Uribe [1], Li-Yau [3], Voros [1, 2]

6. ISOSPECTRALITÉ

Généralités—A titre d'exemple (mais un peu en dehors du sujet) FRIEDLAND [4, 5], KITAOKA [1], Calogero [1, 2], Carison [1], Carroll [1], Carroll-Gilbert [1], Carroll-Santosa [1] Chudnovsky-Chudnovsky [1], Levitan [1]

6.1. Résultats positifs et théorèmes généraux
Bérard [1], Berry [1], Buser [10, 11], Donnelly [1, 2, 3], Fischer [1], Flaschka [1], Global Analysis [1], Guillemin-Kazhdan [1, 2], Hochstadt [1], Gilkey [4, 7, 9], Krein [1], Kuwabara [1, 2, 4], Mc KEAN [3] p. 122, Mc Kean-Van Moerbeke [1], Moser [1, 2], Prosser [1, 2], Randol [6], Sakai [1], Sunada [1], Symes [1], M. Tanaka [1, 2], Tanno [1, 6, 7], Wolpert [1, 2, 3], Zalcman [1], Guillemin [5, 6], Marvizi-Melrose [1, 2]
"cas à bord": Borg [1], Guillemin-Melrose [1, 2], Kac [1], Levinson [1], Mallows-Clark [1], Waechter [1], Gelgand-Levitan [1]

6.2. Contre-exemples
Berger-Gauduchon-Mazet [1], Ejiri [1], Ikeda [2, 4, 5], Vignéras [1, 2]
"cas à bord": Hersch [4], Urakawa [8]

7. PERTURBATIONS ET GÉNÉRICITÉ

Généralités
Albert [1 à 4]; Bando-Urakawa [1], Bleecker-Wilson [1], Krein [1], Uhlenbeck [1, 2], Urakawa [6, 7]
"cas à bord": Driscoll [1]

7.1. Perturbations du spectre, du Laplacien, formules de variation à la Hadamard
Aomoto [1], Donnelly [16], Fujiwara [1, 2], GARABEDIAN

[1], Lobo Hidalgo-Sanchez Palencia [1], Rauch [2], Rauch-Taylor [1], Svendsen [1], Tanikawa [2], Wolpert [4], Weber [1]

"cas à bord": Chavel-Feldman [3, 4], Fujiwara et al. [1], Ozawa [1 à 15], Swanson [1], Fujiwara [3], I'lin [1], Maz'ja et al. [1 à 3], Shimakura [1, 2, 3], Vanninathan [1]

7.2. Résultats sur la généricité

Arnold [1], Millman [1]

"cas à bord": Tanikawa [1]

8. ÉQUATIONS AUX DÉRIVÉES PARTIELLES: APPLICATIONS.

Généralités

Besson [1], Borell [1], Cheng [1, 2], Cheng-Yau [1], Gallot [1], Huber [6], Tanno [3], Uchiyama [1], Yau [2]

"cas à bord": Brascamp-Lieb [1, 2, 3], Hersch [1]

8.1. Etude locale et applications

Albert [1, à 4], Goldberg-Ishihara [1], H. Muto [2]

"cas à bord": Nooney [1]

8.2. Etude globale et applications

Aubin [2], Brüning [2], Gallot [6], Kobayashi [1], Kobayashi-Takeuchi [1], Li [5], Müller Pfeiffer-Stande [1], Nagano [1], Payne [2], Serrin [1], Takahashi [1]

"cas à bord": Bérard-Meyer [1, 2], Biollay [1], Brüning-Gromes [1], Meyer [1], Peetre [1], Pleijel [1]

9. VARIÉTÉS SPÉCIALES

Généralités

Donnelly [13, 16], Duistermaat-Kolk-Varadarajan [1], Fegan [3], Randol [8], Wolpert [5]

9.1. Cas Hyperbolique: courbure-1, formules de traces de Selberg

Bérard Bergery [1], Buser [1 à 11, 14], BUSER [9], Buzzanca [1], Donnelly [19], Good [1], GUILLEMIN [2], HEJHAL [1], Hejhal [2, 3], Huber [1 à 6], Jenni [1, 2], Kolk [2], Kudla-Millson [1], Lax-Phillips [1, 2, 3], Mc Kean [2], Müller [2, 3], Patterson [1], Randol [1 à 6], Sunada [3], S. Tanaka [1], VENKOV [1], Vignéras [1, 2, 3], Wolpert [1, 3], Ehrenpreis [1], Elstrodt [1, 2], Günther [1, 5], Zograf [1]

9.2. Autres variétés spéciales:

Submersions Riemanniennes

Bérard Bergery [1], Bérard Bergery-Bourguignon [1, 2], Goldberg-Ishihara [1], Y. Mutō [, 2]

Espaces lenticulaires

Ikeda [2], Ikeda-Yamamoto [1], Sakai [2], Tanaka [1, 2]
Variétés et tores plats
Berry [1], Kuwabara [2], Sunada [1], Tsukada [2], Wolpert [2]
Espaces Riemanniens symétriques de rang 1
Bonami-Clerc [1], Bourguignon [1], Cahn-Wolf [1], Gangolli [3], Guillemin [1, 4], Hasegawa [2], Ikeda-Taniguchi [1], Iwasaki-Katase [1], Levy Bruhl [2, 3], H. Muto [1, 2], R. T. Smith [1], Tandai-Sumitomo [1], Tanno [5, 6, 7], Widom [2], Günther [2, 3]
Groupes de Lie et quotients discrets
Beers-Millman [1], Cahn [1], Cahn-Gilkey-Wolf [1], S. S. Chen [1], Clerc [1, 2], Donnely [13], Fegan [1, 2], Urakawa [2, 4], Wallach [1], Berezin [1], Greiner [3], Jerison [1, 2], Nachman [1], Rothschild-Wolf [1]
Autres variétés spéciales
Bedford-Suwa [1], Benabdallah [1], Bérard [1], Chachere [1], Chen-Vanhecke [1], Clerc [3], Colin de Verdière [4], De George [1], Donnelly [1, 2, 3, 17, 21], Donnelly-Li [2], Duistermaat-Kolk-Varadarajan [1], Eichhorn [1], Frisch [1], Gangolli [1, 2], Gilkey [4, 8, 9, 15], Gilkey-Sachs [1], Helgason [1, 2, 3], Ikeda [1, 3], Kashiwara et al. [1], Kuwabara [3], Li [3, 4], Miatello [1], Mneimne [1], Müller [2], Muto-Urakawa [1], Olszak [1, 2], Oshima-Sekiguchi [1], Patodi [3], Sekiguchi [1], Simon [1, 3], Strese [1 à 4], Tandai-Sumitomo [1], Taniguchi [1], Tsagas [2 à 4], Tsukada [1, 4], Urakawa [3, 5, 7], Widom [3], Wolpert [4], Yamaguchi [1], Yang-Yau [1], Bleecker [1], Hano [1], Toimer [1], Yamaguchi [2]

10. LE CAS NON COMPACT

Baider [1], Buser [7], Cheng-Li-Yau [2], Colin de Verdière [5, 9, 10], Donnelly [11, 16, 17, 18, 20, 21], Donnelly-Li [1], Eichhorn [1, 2, 3, 6, 8, 9], Good [1], Guillopé [1], Helffer-Robert [1, 2, 3, 4], Hörmander [3], Jørgensen [3], Mc Kean [1, 2], Müller [2], Randol [8], Sekiguchi [1], Xavier [1], Brooks [1 à 5], Melrose [3]

Asakura [1], Bardos-Guillot-Ralston [1], Jørgensen [1, 2], Majda-Ralston [2], Cantor-Brill [1], Cheeger-Gromov-Taylor [1], Combes-Ghez [1], Dodziuk [4, 6, 7, 8, 9], Elstrodt [1, 2], Elstrodt-Roelke [1], Friedlander [1], Gasimov-Levitan [1], Gehtman [1], Strichartz [1], Vol'pert [1], Voros [1, 2]

11. ÉTUDE INDIVIDUELLE DES VALEURS PROPRES

Généralités

BANDLE [1, 2, 3], Berger [2], Biollay [1], Chavel-Feldman [3], Cheng [1 à 4], Donnelly-Li [3], FRIEDLAND [2], GALLOT [4], Gallot-Meyer [1], GARABEDIAN [1] Garabedian-Schiffer [1], Gromov [1], Hersch [2, 3], Li [1 à 4], Mc Kean [1], OSSERMAN [2], Osserman [3, 4], PAYNE [1], POLYAS-ZEGÖ [1], PROTTER [3], Reid [1], Schoen-Wolpert-Yau [1], Simon [1, 2], Yang-Yau [1], Yau [1], Friedland-Novosad [1], Reilly [6]

11.1. Estimées sur le λ_1 et applications

Aomoto [1], Asada [1, 2], Aubin [1, 3], Barbosa-do Carmo [1, 2], Barthel-Kümritz [1], Bérard-Besson [1], Bérard-Meyer [1, 2], Bérard Bergery-Bourguignon [1], Berger [4], Besson [2], Bleecker-Weiner [1], Borell [1], Bourguignon [1], Brascamp-Lieb [1, 2, 3], Buser [2 à 8, 12, 14], BUSER [9], do Carmo [1], Chavel [1], Chavel-Feldman [1], Cheeger [3, 5], Chen [1], Croke [1, 2, 5], Debiard-Gaveau-Mazet [1], Friedland [1, 3], Friedland-Hayman [1], Friedrich [1, 2], Fujiwara [1], Gage [1], Gallot [3, 8], Gallot-Meyer [1], Hersch [4], Hoffman [2], Huber [1, 2], Komorowski [1, 2], Li-Treibergs [1], Li-Yau [2], Li-Zhong [1], Marcellini [1], Matsuzawa-Tanno [1], Mazet [1], H. Muto [1, 2, 3], H. Muto-Urakawa [1], Y. Mutō [3], Nehari [1], Obata [1], Osserman [1], Ozawa [6, 8], de Paris [1], Payne-Rayner [1], Philippin [1, 2], Pinsky [4], Protter [1], Randol [1], Reilly [2, 3, 4], Schoen-Wolpert-Yau [2], Sperb [1], Sperner [1], Tachibana-Yamaguchi [1], Tanno [5], M. Taylor [2], Trudinger [1], Tsukada [2], Uchiyama [1], Urakawa [1, 2, 3], Kasue [3], Lichnerowicz [1], Meyer [1], Schoen [1], Sulanke [1], Vignéras [3], Watanabe [1]

11.2. Estimées faisant intervenir les λ_k, $k \geqq 2$ et applications

Berger [2, 8], Cheng-Li-Yau [1], Gromov [1], Hile-Protter [1], Huber [4], Li-Yau [1], Polya [1], Simon [4, 5], Bareket [1], Cheng-Li [1], Hersch [5], Donnelly-Li [2, 3], Li-Yau [3], Urakawa [10]

11.3 Inégalités isopérimétriques, inégalités de Sobolev, applications

Aubin [2, 4], Barbosa-do Carmo [3], Benko et al. [1], Berger [7, 8], Berger-Kazdan [1], Buser [4, 12], BUSER [9], Chavel-Feldman [2, 5, 6], Cheeger [3], Croke [1], Gallot [2, 5, 7, 9], Gromov [1], Hoffman [1], Ilias [1, 2], Kohler-Jobin [1 à 5], Li-Yau [1], Lieb [1, 2], Peetre [1], Schmidt [1], Cheeger-Gromov-Taylor [1], Chiti [1], Hersch-Monkiewicz [1], I'lin-Moiseev [1], Li [6], Pansu [1, 2], Talenti [1]

12. LAST BUT NOT LEAST

Lignes et surfaces nodales

Brüning [2, 4], Brüning-Gromes [1], Cheng [1, 4], Meyer [1], Payne [2], Pleijel [1], Bérard-Meyer [1],

Spectre et actions de groupes

Brüning [3], Brüning-Heintze [1], Donnelly [6, 8, 10, 14], Donnelly-Patodi [1], Gilkey [15, 16], Helgason [1, 2, 3], Höppner [1], Huber [5], R. T. Smith [1], Yen [1], Shafii-Dehabad [1]

Approximations et Triangulations

Dodziuk [1], Dodziuk-Patodi [1], Komorowski [3], Patodi [4]

Calculs numériques

Bassotti Rizza [1], Chachère [1], Forsythe [1]

Variétés avec singularités

Cheeger [4], Cheeger-Taylor [1], Kalka-Menikoff [1]

Convergence des séries de fonctions propres

Alimov et al. [1], Bérard [5], Bonami-Clerc [1], Boutet de Monvel [2], Clerc [1, 2], Hörmander [1], Kenig-Thomas [1], Randol [7], Smale [1], Taylor [1], Meaney [1], Rothschild-Wolf [1]

Opérateurs autres que le Laplacien (opérateurs elliptiques généraux dont opérateur de Dirac; opérateur de Schrödinger...)

J. M. Arnold [1], Balian-Bloch [3], Berthier [1], Boutet de Monvel [3], Boutet de Monvel-Grisvard [1], Boutet de Monvel-Guillemin [1], Buzzanca [1], Chazarain [2, 3], Colin de Verdière [3, 5, 8], Dlubek-Friedrich [1], Flaschka [1], Fleckinger Pellé [1], Friedrich [1], Geller [1], Gilkey [14], Grubb [1, 2], Guillopé [1], Hall-Streedry [1], Kolk [1], Lieb [1, 2, 3], Mc Kean-Van Moerbeke [1], Pyshkina [1], Seeley [1], Strese [2, 3], Sunada [2], Vasil'ev [1], Baxley [1], Bezjaev [1], Charbonnel [1], Cheng [1], Chico [1], Friedrich [2], Kalf [1], Sulanke [1],

Invariant êta

Atiyah-Patodi-Singer [1], Donnelly [5, 7, 10, 12], Gilkey [11], Gilkey-Smith [2], Atiyah-Donnelly-Singer [1]

Torsion analytique

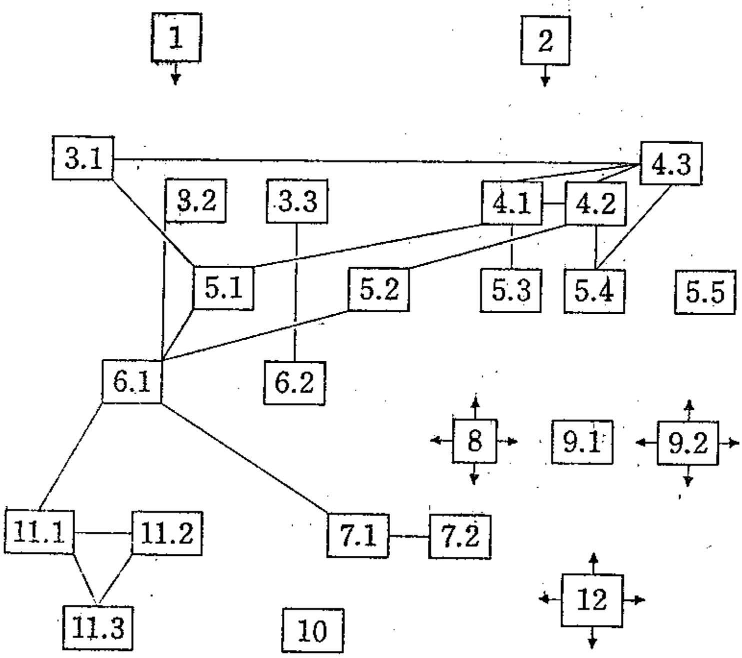
Cheeger [1], Donnelly [12], Müller [1], Ray [1], Ray-Singer [1, 2], Schwarz [1], Urakawa [5]

Inégalités type isopérimétriques autres que §11.3

Berger [5, 6, 7], Berger-Kazdan [1], Kasue [2, 4, 5], Kohler-Jobin [6, 7], Lions [1], Pach [1], Parks [1]

Géométrie intégrale et problèmes spectraux

Tableau des Interactions Fortes



Liste Ouvrages de Base

Tous les paragraphes
BERGER-GAUDUCHON-MAZET [1]
SIMON-WISSNER [1, 2]
YAU [3, 4]

Autres
BANDLE [1, 3] (§11)
BÉRARD [3] (§4, 5)
BERGER [1, 3] (§3, 4, 5)
BROWDER [1] (§1)
BUSER [9] (§9.1, 11.1, 11.3)
CHAZARAIN [2] (§4.2, 5.2)
CLARK [1] (§1, 2, 4)
COLIN DE VERDIÈRE [6] (§4.1, 5.1)
FRIEDLAND [2] (§1, 11)
GALLOT [4] (§11)
GARABEDIAN [1] (§1, 7.1, 11)
GILKEY [3] (§5.1)
GOULAOUIC [1] (§1, 4.3)
GUILLEMIN [2] (§4, 5.2, 9.1)
GUILLEMIN-STERNBERG [1] (§2.2, 4.5)
HEJHAL [1] (§4.3, 9.1)
Mc KEAN [3] (§6.1)
OSSERMAN [2] (§11)
PAYNE [1] (§11)
POLYA-SZEGO [1] (§11)
PROTTER [3] (§1, 2.1, 11)
REED-SIMON [1] (§1)
VENKOV [1] (§2, 4.3, 5.3, 9.1)

Liste Chronologique des Colloques Cités en Références

1973 CIME 1973; Proceedings of Symposia n° 27 A.M.S.
1977 Partial differential equations and geometry Stochastic Differential Equations and Applications
Minimal submanifolds including geodesics
1978 Global Analysis; Pseudo-differential Operators with Applications; Probabilistic analysis and related topics
1979 Geometry of the Laplace operator; Non linear problems in Geometry
1980 Geometry and analysis; Free Boundary Problems I & II; Geometry Symposium;

1981 Séminaire Franco-Japonais; Contribution to Analysis and Geometry; Global Differential Geometry and Global Analysis; Nonlinear Partial Differential Equations and their Applications; Seminar on Harmonic Analysis; Spectral Theory of Differential Operators;

1982 Differential Geometry; Differential Geometric Methods in Mathmatical Physics; Seminar on Differential Geometry;

Références

Albert, J. H.

1. Topology of the nodal and critical sets for eigenfunctions of elliptic operators, Thèse M.I.T. 1971.
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A P P E N D I X C

A COMPLEMENT TO APPENDIX B

APPENDIX C

Complement to Appendix B

The following (obviously not exhaustive) list of references is an addendum to the bibliography [B-B] given in Appendix B. We do not intend to give an up to date 1986 - version of [B-B], but we think that it might be useful to point out some contributions to spectral geometry which appeared in the last four years. Some papers are major contributions; some others are less important, but give interesting developments to the subject.

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INDEX OF NOTATIONS

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