MONOGRAFIAS DE MATEMÁTICA N.º 34

NOTES ON PARTIAL DIFFERENTIAL EQUATIONS

F. JAVIER THAYER

INSTITUTO DE MATEMÁTICA PURA E APLICADA

RIO DE JANEIRO
1980

DEPARTAMENTO DE INFORMAÇÃO CIENTÍFICA (DIC)

MONOGRAFIAS DE MATEMÁTICA

1) Alberto Azevedo & Renzo Piccinini - Introdução à Teoria dos Grupos
2) Nathan M. Santos - Vetores e Matrizes (esgotado)
3) Manfredo P. Carmo - Introdução à Geometria Diferencial Global
4) Jacob Palis Jr. - Sistemas Dinâmicos
5) João Pitombeira de Carvalho - Introdução à Álgebra Linear (esgotado)
6) Pedro J. Fernandez - Introdução à Teoria das Probabilidades
7) R.C. Robinson - Lectures on Hamiltonian Systems
8) Manfredo P. do Carmo - Notas de Geometria Riemanniana
9) Chain S. Hönig - Análise Funcional e o Problema de Sturn-Liouville
10) Welington de Melo - Estabilidade Estrutural em Variedades de Dimensão 2
11) Jaime Lesmes - Teoria das Distribuições e Equações Diferenciais
12) Clóvis Vilanova - Elementos da Teoria dos Grupos e da Teoria dos Anéis
13) Jean Claude Douai - Cohomologie des Groupes
14) H. Blaine Lawson Jr. - Lectures on Minimal Submanifolds, Vol.1
15) Elon L. Lima - Variedades Diferenciáveis
16) Pedro Mendes - Teoremas de Ω-estabilidade e Estabilidade Estrutural em Variedades Abertas
17) Herbert Amann - Lectures on Some Fixed Point Theorems
18) Exercícios de Matemática - IMPA
19) Djairo G. de Figueiredo - Números Irracionais e Transcendentes
20) C.E. Zeeman - Uma Introdução Informal à Topologia das Superfícies
21) Manfredo P. do Carmo - Notas de um curso de Grupos de Lie
22) A. Prestel - Lectures on Formally Real Fields
23) Aron Simis - Introdução à Álgebra
24) Jaime Lesmes - Seminário de Análise Funcional
25) Fred Brauer - Some Stability and Perturbation Problem for Differential and Integral Equations
26) Lucio Rodriguez - Geometria das Subvariedades
27) Mario Miranda - Frontiere Minime
28) Fernando Cardoso - Resolubilidade Local de Equações Diferenciais Parciais
29) Eberhard Becker - Hereditarily-Pythagorean Fields and Orderings of Higher Level
30) Hyman Bass - Projective Modules and Symmetric Algebras
31) J. Neyman - Probabilidade Freqüentista e Estatística Freqüentista
32) Freddy Dumortier - Singularities of Vector Fields
33) T.M. Viswanathan - Introdução à Álgebra e Aritmética
34) F. Javier Thayer - Notes On Partial Differential Equations

INTRODUCTION

These are notes for a course on PDE's which I gave at IMPA in the spring semester 1978. They suppose the reader is already familiar with functional analysis and in the part on evolution equations with spectral theory. I also suppose the reader has had some contact with PDE's before. This accounts for some of the major omissions such as the maximum principle.

The notes are divided into four sections. The first section covers the Cauchy Kowaleskva theorem using the abstract approach of Oscianikov-Nirenberg. The second section is a rapid introduction to the theory of distributions. The third section constitutes the core of the notes; We cover the elliptic theory using Hilbert space methods. We give in particular detailed proofs of the regularity theorems for boundary value problems. The final section covers evolution equations.

A Nora

CONTENTS

SECTION 3. ELLIPTIC EQUATIONS

Pág.

SECTION 1

THE CAUCHY KOWALESKVA THEOREM

Ordinary and Partial Differential Equations.

There is a formal similarity between ordinary differential equations and certain partial differential equations (PDE's) which single out a certain variable with the appelative implicit or explicit of time. The well known equations

$$\partial_t^2 u = \partial_x^2 u \qquad \text{[Wave]}$$

$$\partial_t u = \partial_x^2 u \qquad \text{[Heat]}$$

$$\partial_t u = -i\partial_x^2 u \qquad \text{[Schrödinger]}$$

$$\partial_t u = -u\partial_x u - \partial_x^3 u \qquad \text{[Korteweg - de Vries]}$$

all have the formal structure of an ODE with unknown funtion φ taking values in a function space:

$$\varphi'(t) = f(\varphi(t))$$

If we pursue this idea, that is view a PDE as a jazzed up ODE we will not get very far because the operator f in all the above equations is discontinous in any natural topology one can think of. There is a sense however in which derivation is a continous operator in the uniform topology of holomorphic functions on an open set. This is the content of Cauchy's inequalities which we now briefly review.

1.1 Analytic Functions

Let $W \subseteq \mathbb{C}^n$ be an open set; $f: W \to E$ where E is a Banach space. f is an analytic function iff f has a convergent power series expansion $f(z) = \{\Sigma\, c_\alpha(z-z_o)^\alpha : \alpha \in \mathbb{N}^n\}$ around each $z_o \in W$.

We will assume familiarity with Banach space valued analytic functions; This theory is not much different from the scalar valued theory, and in fact follows from it via the Hahn Banach theorem. We will not dwell on this here; however we do recall the following

1.1.1 <u>Let</u> f <u>be an analytic function</u> $W \to E$; <u>Let</u> $z_o = (z_o^1,\ldots,z_o^n) \in W$ <u>and let</u> $r > 0$ <u>be such that if</u> $|y-z_o^i| < r$ then $(z_o^1,\ldots,z_o^{i-1},y,z_o^{i+1},\ldots,z_o^n) \in W$. Then

$$|\partial_{e_i} f(z_o)| \leq 1/r \sup\{|f(z_o^1,\ldots,z_o^{i-1},y,z_o^{i+1},\ldots,z_o^n)| : |y-z_o^i| = r\}.$$

<u>Proof</u>. By the Hahn Banach theorem it suffies to consider scalar valued f, and since only one variabl (i.e.y) is involved this reduces to a well known theorem about ordinary holomorphic functions.

This is a particular case of Cauchy's inequalities.

1.2 The Cauchy Kowaleskva Theorem

In general differentiation is not continuous; However the sense of Cauchy's inequalities is that in some sense for complex analytic functions they are continuous: Specifically if U,V are open sets $U \subseteq V$ and $d(U,\complement V) > r$ then ∂_{e_i} is a continuous linear operator from the Banach space of bounded holomorphic functions on V to the Banach space of bounded holomorphic functions on U. [Both with the supremum norm]. The norm of this operator furthermore is $\leq 1/r$. In order to apply an iteration process we must have an infinite sequence of decreasing open sets U_m such that ∂_{e_i} is a continuous operator $H_b(U_m) \to H_b(U_n)$, $n > m$. The notion that we will use to realize this idea is that of a scale of Banach spaces. This is a very general concept which we illustrate first with complex analytic functions.

Let $A \subseteq \mathbb{C}^n$ be a compact set, $U_s = \{x \in \mathbb{C}^n : d(x,A) < (1-s)r\}$. Let $E_s = H_b(U_s)$, that is bounded holomorphic functions on U_s with the uniform norm. If $s' > s$, then ∂_{e_i} is a continuous linear map $E_s \to E_{s'}$, of norm $1/r(s'-s)$ as follows readily by Cauchy's inequalities. This family $\{E_s\}$ of vector spaces, can be considered as an increasing family of subspaces of a fixed vector space. In fact, restriction is a contractive injective linear map $E_s \to E_{s'}$ for $s' > s$.

This example motivates the formal definition in the following section.

1.3 <u>The Abstract Cauchy Kowaleskva Theorem</u>

We begin with the basic concept of a <u>scale of Banach spaces</u>:

1.3.1 Let E be a vector space, E_s a family of subspaces of E parametrized by $s \in [a,b[$ such that:

(i) E_s carries the structure of a Banach space.

(ii) $E_s \subseteq E_{s'}$ whenever $s' \geq s$ as vector spaces, and the inclusion is contrative.

(iii) $E = \cup E_s$.

We now extend the existence theorems of ODE's valid for Banach spaces to the context of scales of Banach spaces. This will permit us to handle certain PDE's which formally are like ODE's.

1.3.2 _Let_ $I \subseteq \mathbb{R}$ _be some interval,_ $V \subseteq E$ _a set such that_ $V \cap E_s$ _is open for all_ s, $f: I \times V \to E$ _a function satisfying_

(1) $f: I \times (V \cap E_s) \to E_{s'}$ _is continuous whenever_ $s' > s \geq a$.

(2) _There is some_ $C > 0$ _such that_

$$\|f(t,x) - f(t,y)\|_{s'} \leq [c/s'-s] \|x-y\|_s$$

whenever $s' > s$ _and_ $x,y \subseteq V \cap E_s$.

The theorem on existence is as follows:

1.3.3 If $t_o \in I$, $x_o \in V \cap E_a$, _then for each_ $s_o \in]a,b[$ _there is a function_ g _defined on some interval around_ t_o, _taking values in_ $V \cap E_{s_o}$, _is_ C^1 _in the norm_ of E_{s_o} _and satisfies_

(a) $g(t_o) = x_o$

(b) $g'(t) = f(t,g(t))$

Before proving the theorem we observe the following:

Conditions (1), (2) above are invariant under linear rescaling of the parameter s. Thus if $F_s = E_{\alpha s+\beta}$ for $s \in [a',b'[$ then (1), (2) are still valid. We may thus assume $[a,b[= [0,1[$.

Obviously we may take $t_o = 0$ in the above theorem.

If $s_o > s_1 > a$ then $t \mapsto f(t,x_o)$ is continuous $I \to E_{s_1}$ as follows by (1). By rescaling $[s_1,b[$ so $s_1 \to 0$, $b \to 1$ we may assume in addition $t \to f(t,x_o)$ is continuous $I \to W_o$, $s_o > 0$.

We first prove the theorem when $V = E$. The method of proof is by succesive approximations. We define by induction a sequence of functions g_i: $I \to E$ as follows

$$g_o(t) = x_o$$
$$g_{n+1}(t) = x_o + \int_o^t f(\lambda,g_n(\lambda))d\lambda \qquad (.1)$$

We must check this makes sense. In fact:

If for all $s \in]0,1[$ g_n maps I into E_s continuously then the integral (.1) makes sense as a (Riemann) vector valued integral for any E_s with $s > 0$ and in fact the value of the integral does not depend on s. Furthermore g_{n+1} so defined maps I into E_s for all s.

To verify the first assertion suppose $s > s' > 0$. Then by hypothesis $g_n: I \to E_{s'}$ is continuous, $f: I \times E_{s'} \to E_s$ is continuous by (1) and so $\lambda \mapsto f(\lambda, g_n(\lambda))$ is continuous $I \to E_s$. If $s, s'' > 0$ and $s'' > s$ then $E_s \subseteq E_{s''}$ is continuous and

$$[\int_0^t f(\lambda, g_n(\lambda) d\lambda]_{E_s} = [\int_0^t f(\lambda, g_n(\lambda)) d\lambda]_{E_{s''}}$$

The second assertion is immediate: $\lambda \longmapsto f(\lambda, g_n(\lambda))$ is continuous $I \to E_s$ for any s, so as a function of t, the integral (.1) is continuous.

Next we prove there is an interval I_s around 0 such that $\{g_n \mid I_s\}$ is a uniform Cauchy sequence $I_s \to E_s$. In fact there is an $M > 0$ such that for all $s > 0$ and all t

$$\|g_{n+1}(t) - g_n(t)\|_s \leq M[Ce|t|/s]^n$$

Here $e = \exp 1$. We prove this by induction on n. Let $M = \int_I \|f(\lambda, x_o)\|_o d\lambda$. This makes sense as $\lambda \longmapsto f(\lambda, x_o)$ is continuous $I \to E_o$. Now if $t \in I$

$$\|g_1(t) - g_o(t)\|_s \leq \|g_1(t) - g_o(t)\|_o \leq \int_I \|f(\lambda, x_o)\|_o d\lambda = M$$

In general, let $s' > s > 0$. Then

$$\|g_{n+1}(t) - g_n(t)\|_{s'} = \|\int_o^t [f(\lambda, g_n(\lambda)) - f(\lambda, g_{n-1}(\lambda))] d\lambda\|_{s'} \leq$$

$$\leq \int_o^t \|f(\lambda, g_n(\lambda)) - f(\lambda, g_{n-1}(\lambda))\|_{s'} d\lambda \leq$$

$$\leq C/(s'-s) \int_o^t M[Ce|\lambda|/s]^{n-1} d\lambda \leq$$

$$\leq C/(s'-s)\, M[Ce/s]^{n-1}|t|^n/n \qquad (.2)$$

Take $s = s'-s'/n$. Then (.2) is $\leq$

$$\leq [C/s'] n M[Ce/s']^{n-1} [n/n-1]^{n-1} |t|^n/n$$

$$\leq M[C/s']^n e^{n-1} [n/n-1]^{n-1} |t|^n$$

$$\leq M[Ce/s']^n |t|^n$$

since $e \geq (n/n-1)^{n-1}$.

Thus if $t \in I_s =]-s/Ce, s/Ce[\cap I, \{g_n(t)\}$ is a Cauchy sequence and in fact $\{g_n\}$ is Cauchy on any compact subinterval of I_s. Thus if $[-a,a] \subseteq I_s$ there is a continuous function $g: [-a,a] \to E_s$ such that $g = \lim g_n$ uniformly on $[-a,a]$. If $s < s'$, then

$$\|f(\lambda,g_n(\lambda)) - f(\lambda,g(\lambda))\|_{s'} \leq C/s'-s \cdot \|g_n(\lambda) - g(\lambda))\|_s$$

so $f(\lambda,g_n(\lambda)) \to f(\lambda,g(\lambda))$ uniformly (in the nom of $E_{s'}$) and

$$\int_0^t f(\lambda,g_n(\lambda))d\lambda \to \int_0^t f(\lambda,g(\lambda))d\lambda$$

in $E_{s'}$ for any $t \in [-a,a]$. Thus

$$g(t) = \lim_n g_n(t) = \lim_n [x_o + \int_0^t f(\lambda,g_n(\lambda))d\lambda]$$

$$= x_o + \int_0^t f(\lambda,g(\lambda))d\lambda$$

where all the limits are in $E_{s'}$. As $\lambda \mapsto f(\lambda,g(\lambda))$ is continuous $[-a,a] \to E_{s'}$ it follows g is C^1 as a function $[-a,a] \to E_{s'}$ and $g'(t) = f(t,g(t))$ by the fundamental theorem of calculus. We can obviously arrange matters so $s' = s_o$ and this proves the theorem for the case $V = E$

1.3.4 In the general case, we observe there is no loss of generality in assuming $E_1=E$ is a Banach space and

(1)-(2) are satisfied for $1 \geq s' > s \geq 0$. [e.g.,, let $s_o < s_1 < 1$ and consider the spaces E_s, $0 \leq s \leq s_1$ and rescale $[0,s_1]$]. Now let $V_o = \{x: \|x-x_o\|_1 < r\} \subseteq V$ and let $\varphi: \mathbb{R} \to [0,1]$ be a Lipschitz function supported in $]-r,r[$ which is 1 on $[-r/2, r/2]$. Define $\lambda(x) = \varphi(\|x-x_o\|)x + (1 - \varphi(\|x-x_o\|_1))x_o$. λ maps E into V_o and is the identity on $\{x: \|x-x_o\|_1 \leq r/2\}$. To see this observe that if $x \notin V_o$ then $\lambda(x) = x_o \in V_o$. If $x \in V_o$ then as $0 \leq \varphi(\|x-x_o\|_1) \leq 1$ and V_o is convex $\lambda(x) \in V_o$. If $\|x-x_o\|_1 \leq r/2$, $\varphi(\|x-x_o\|_1) = 1$ so $\lambda(x) = x$.

Next set $h(t,x) = f(t,\lambda(x))$. h satisfies (1) and (2): First λ maps $E_s \to E_s$ continuously: Note $x \to \varphi(\|x-x_o\|_1)$ is continuous $E_s \to \mathbb{R}$ so λ is the sum of continuous functions $E_s \to E_s$. Since h is the composition

$$I \times E_s \xrightarrow{id \times \lambda} I \times E_s \xrightarrow{f} I \times E_{s'}, \quad (s' > s)$$

it is continuous. This proves (1). To show (2) observe

$$\|h(t,x) - h(t,y)\|_{s'} \leq C/(s'-s) \|\lambda(x) - \lambda(y)\|_s .$$

If both $x,y \notin V_o$ then $\lambda(x) = \lambda(y) = x_o$. In estimating $\|\lambda(x) - \lambda(y)\|_s$ we may therefore assume say $y \in V_o$. In this case;

$$\|\lambda(x) - \lambda(y)\|_s = \|\varphi(\|x-x_o\|_1)(x-x_o) - \varphi(\|y-x_o\|_1)(y-x_o)\|_s =$$

$$= \|\varphi(\|x-x_o\|_1)(x-x_o) - \varphi(\|x-x_o\|_1)(y-x_o) + \varphi(\|x-x_o\|_1)(y-x_o) -$$

$$- \varphi(\|y-x_o\|_1)(y-x_o)\|_s = \varphi(\|x-x_o\|_1) \|x-y\|_s + |\varphi(\|x-x_o\|_1) -$$

$$- \varphi(\|y-x_o\|_1)| \|y-x_o\|_s \leq \|x-y\|_s + K|\|x-x_o\|_1 - \|y-x_o\|_1| r \leq$$

$$\leq \|x-y\|_s + rK\|x-y\|_1 \leq K_1\|x-y\|_s .$$

Applying the previous result to h we see that for any $s_o > 0$ there is a norm C^1 function $g: I_o \to E_{s_o}$ satisfying $g(0) = x_o$ and $g'(t) = h(t,g(t))$. For t in some small interval $I_1 \subseteq I_o$ $\|g(t)-x_o\|_1 < r/2$ and so $g'(t) = h(t,g(t)) =$ $= f(t,g(t))$. ∎

1.4 Uniqueness of the Solution

The solution to the initial value problem is also unique:

1.4.1 *If* g, g_1 *are norm* C^1 *functions* $J \to V \cap E_{s_o}$ *such that* (a), (b) *of* 1.3.3 *hold for* g, g_1 *then* $g = g_1$ *on* J. *Here* J *is any interval* $\subseteq I$.

Proof. By connectivity it suffices to show $\{t: g(t) = g_1(t)\}$ is an open set. To prove this in turn it suffices to show that if $g(t_1) = g_1(t_1)$ then g, g_1 coincide in some interval around t_1. Evidently we may take $t_1 = t_o = 0$, and so we are reduced to showing g, g_1 agree on an interval around 0.

Rescale $[s_o, b[$ so $s_o \to 0, b \to 1$. We then prove that there is an $M > 0$ such that for all $s' > 0$ and all $n \in \mathbb{N}$:

$$\|g(t) - g_1(t)\|_{s'} \le M[C_1 e|t|/s']^n \qquad (.1)$$

The proof of (.1) is by induction. Let $M = \sup\{\|g(t)-g_1(t)\|_o : t \in J\}$. $M < \infty$ by continuity of $g, g_1: J \to E_o$.

In general if $s'' > s'$ then $\lambda \to f(\lambda, g(\lambda))$, $\lambda \mapsto f(\lambda, g_1(\lambda))$ are continuous functions $J \to E_{s''}$. By (b) and the fundamental theorem of Calculus

$$g(t) = x_o + \int_o^t f(\lambda, g(\lambda))d\lambda; \quad g_1(t) = x_o + \int_o^t f(\lambda, g_1(\lambda))d\lambda$$

Thus

$$\|g(t) - g_1(t)\|_{s''} \le \int_o^t \|f(\lambda, g\ (\lambda)) - f(\lambda, g_1(\lambda))\|_{s''} d\lambda$$

$$\le \int_o^t C_1/(s''-s')\cdot M[C_1 e|\lambda|/s']^{n-1} d\lambda$$

$$\le C_1/(s''-s')\cdot M[C_1 e/s']^{n-1}|t|^n/n \qquad (.2)$$

Taking as in the proof of existence the particular choice $s' = s'' - s''/n$ we have (.2) is $\le M[C_1 e/s'']^n |t|^n$.

From (.1) we deduce that for all $s' > 0$ and all t such that $[C_1 e|t|/s'] < 1$, $\|g(t) - g_1(t)\|_{s'} = 0$. Thus $g(t) = g_1(t)$ for $|t|$ small. ■

1.5 Extension to Holomorphic Equations

1.5.1 Let E, F be Banach spaces, $U \subseteq E$ an open set. A function $f: U \to F$ is analytic iff for any open set $W \subseteq \mathbb{C}$ and every analytic function $\varphi: W \to U$, the composition $f\circ\varphi: W \to F$ is analytic.

We may extend the preceding theorem to the complex domain.

1.5.2 Let $D = \{z \in \mathbb{C}: |z-z_o| < r\} \subseteq \mathbb{C}$, $\{E_s\}$, $s \in [a,b[$ a scale of complex Banach spaces, $V \subseteq E$ a set such that $V \cap E_s$ is open for all s and $f: D\times V \to E$ a function such that

(1)' f maps $D\times(V \cap E_s)$ analytically into $E_{s'}$, whenever $s' > s \ge a$.

(2) <u>There is a constant</u> $C > 0$ <u>such that</u>

$$\|f(z,x) - f(z,y)\|_{s'} \le [C/s'-s]\cdot\|x-y\|_s$$

<u>whenever</u> $s' > s \ge a$ <u>and</u> $x,y \in V \cap E_s$.

<u>If</u> $x_o \in V \cap E_a$ <u>then for each</u> $s_o \in]a,b[$ <u>there is function</u> g <u>defined on some disc</u> $D_1 = \{z \in \mathbb{C}: |z-z_o| < r_1\}$ <u>taking values in</u> $V \cap E_{s_o}$, <u>is analytic in the norm of</u> E_{s_o} <u>and satisfies</u>

(a) $g(z_o) = x_o$

(b) $g'(z) = f(z,g(z))$.

<u>The function is unique</u>.

<u>Proof</u>. The proof is the same as the previous theorem; Merely interpret $\int_o^z h(\lambda)d\lambda$ for a function h to be the line integral along the straight line segment $[0,z]$. As a function of z, this integral is analytic whenever h is. The remainder of the proof is carried out almost verbatim. ∎

Actually the same idea permits a Banach scale version of the Frobenius integrability theorem.

1.6 <u>Higher Order Equations</u>

Let E_s, $s \in [a,b[$ be a scale of Banach spaces; Then we may consider $E_s^m = E_s \times \ldots \times E_s$ (m times), $s \in [a,b[$ as a scale of Banach spaces. Now suppose $V \subseteq E^m$ is such that $V \cap E_s^m$ is open for all s and $f: I \times V \to E$ $[I \subseteq \mathbb{R}$ an interval$]$ a function satisfying

(1)' $f: I \times (U \cap E_s^m) \to E_{s'}$ is continuous whenever $s' > s \geq a$

(2)' There is some constant $C > 0$ such that

$$\|f(t,x_o,x_1,\ldots,x_{n-1}) - f(t,y_o,\ldots,y_{n-1}\|_{s'} \leq$$
$$\leq C/s'-s \operatorname*{Max}_i\{\|x_i-y_i\|_s\}.$$

We then have the following:

1.6.1 If $t_o \in I$, $(x_o,x_1,\ldots,x_{n-1}) \in V \cap E_a^m$, then for each $s_o \in]a,b[$ there is an interval I_{s_o} and a unique function $g: I_{s_o} \to E_{s_o}$ which is C^1 in the norm of E_{s_o} and satisfies

(a) $g(t_o) = x_o,\ldots,g^{(n-1)}(t_o) = x_{n-1}$

(b) $g^{(n)}(t) = f(t,g(t), g'(t),\ldots,g^{(n-1)}(t))$.

<u>Proof</u>. To prove this, let $F: I\times V \to E^m$ be the function

$$F(t,x_o,x_1,\ldots,x_{n-1}) = (x_1,x_1,\ldots,x_{n-1}, f(t,x_o,x_1,\ldots,x_{n-1})).$$

Then F satisfies conditions (1), (2) of the abstract Cauchy Kowaleskva theorem. This is trivial. Thus there exists an interval I_{s_o} and $G: I_{s_o} \to E_{s_o}$, such that

(a) $G(t_o) = (x_o,\ldots,x_{n-1})$

(b) $G'(t) = F(t,G(t))$

1.7 Cauchy Kowaleskva Theorem

To apply the preceding abstract theory, let $A \subseteq C^n$ be a compact set, X a finite dimensional Banach space, E_s the space of X valued bounded holomorphic functions defined on the open set $U_s = \{x \in C^n: d(x,A) < (1-s)r\}$ with the uniform norm.

The following is a substitute for continuity of ∂_{e_i}:

1.7.1 If $f \in E_s$ then for $s' > s$

$$\sup\{|\partial_{e_i} f(x)| : x \in U_{s'}\} \leq 1/r(s'-s) \|f\|_s$$

<u>Proof</u>. For if $x \in U_{s'}$ then $B(x,(s'-s)r) \subseteq U_s$ so that by Cauchy's inequalities

$$|\partial_{e_i} f(x)| \leq 1/r(s'-s) \sup\{|f(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)| : |y-x_i| \leq |s'-s|r\}$$

$$\leq 1/r(s'-s) \|f\|_s \quad \blacksquare$$

Note that any bound on higher order derivatives would involve higher powers of $1/(s'-s)$. This imposes a serious limitation on this general Cauchy Kowaleskva existence theorem as we shall see.

To complete the data for the abstract theory, let $E = \cup E_s$. Observe that we may regard $E_s \subseteq E_{s'}$ whenever $s' \geq s$. In fact restriction is an injective map $E_s \to E_{s'}$. Elements of E are thus germs of analytic functions at A.

Let us assume $P: I \times E \to E$ is a local operator, that is there is a function $G: I \times U_0 \times X^{(n+1)} \to X$ such that

$$P(t,u)(x) = G(t,x,u(x), \partial_{e_1} u(x),\ldots,\partial_{e_n} u(x))$$

1.7.2 It is very easy to translate conditions 1.3.2 on P which guarantee the initial value problem $g'(t) = P(t,g(t))$ is solvable to conditions on G. In fact: _Suppose_

(a) $G\colon I\times U_o \times X^{(n+1)} \to X$ _is continuous_

(b) _For fixed_ t, $G(t,\ ,)$ _is analytic_ $U_o \times X^{(n+1)} \to X$

(c) G _is Lipschitzian:_ _If_ $(t,x) \in I \times U_o$

$$\|G(t,x,\xi_1,\ldots,\xi_{n+1}) - G(t,x,\xi_1',\ldots,\xi_{n+1}')\| \le K \sup_{1\le i\le n+1} \|\xi_i - \xi_i'\| .$$

Then if $u_o \in E_o$ _and_ $1 > s > 0$ _there is an interval_ I, _depending on_ s _and_ a norm C^1 _function_ $\varphi\colon I_1 \to E_s$ _with the properties_

(i) $\varphi(0) = u_o$

(ii) $\varphi'(t)(x) = G(t,x,u(x),\ldots,\partial_{e_n} u(x))$

If in addition

(d) G _is analytic in all its variables, where_ I _is now some disc in_ $\mathbb{C}$, _there is an analytic function_ $\varphi\colon I_1 \to E_s$ _satisfying_ (i), (ii).

Proof. To begin with (a), (b) imply that P actually maps $I \times E_s$ into $E_{s'}$, for any $s' > s$ and that it does so continuously: If $u \in E_s$ then all the functions $u, \partial_{e_1} u,\ldots, \partial_{e_1} u$ are analytic on U_s and by Cauchy's inequalites are also bounded on $U_{s'}$, for $s' > s$. From this and the fact $G(t,\ )$ is continuous $\overline{U}_{s'} \times X^{(n+1)} \to X$ follows that $x \mapsto G(t,x,u(x),\ldots,$ $\partial_{e_i} u(x),\ldots)$ is bounded on $U_{s'}$. This map is obviously analytic as it is the composition of analytic maps. To show P maps $I \times E_s$ into $E_{s'}$ continuously let $u_\lambda \to u$ uniformly on U_s, $t_\lambda \to t$. By Cauchy's inequalities $\partial_{e_i} u_\lambda \to \partial_{e_i} u$ uniformly on $U_{s'}$. By uniform continuity of G on relatively compact subsets of $I \times U_o \times X^{(n+1)}$ it follows $G(t_\lambda,x,\ldots,\partial_{e_i} u_\lambda(x),\ldots) \to$ $\to G(t,x,\ldots,\partial_{e_i} u(x),\ldots)$ uniformly on $U_{s'}$.

By (c) and Cauchy's inequalities:

$$\|P(t,u) - P(t,u')\|_{s'} \leq K \max_{1\leq i\leq n} \{\|u-u'\|_{s'}, \|\partial_{e_i}(u-u')\|_{s'}\} \leq$$

$$\leq K\, 1/r(s'-s)\cdot\|u-u'\|_s$$

so P is Lipschitzian.

If (d) holds we show $P\colon I \times E_s \to E_{s'}$ is analytic. This means that for any analytic function $\varphi = (\varphi_1,\varphi_2)\colon W \to I \times E_s$ defined on an open set $W \subseteq \mathbb{C}$, the composition $z \to P(\varphi_1(z),\varphi_2(z))$ is analytic $W \to E_{s'}$. By Morera's theorem in complex variables it suffices to prove:

$$\int_\gamma P(\varphi_1(z), \varphi_2(z))dz = 0$$

for any closed curve γ. Now for all $x \in U_s$

$$\left[\int_\gamma P(\varphi_1(z),\varphi_2(z))dz\right](x) = \int_\gamma P(\varphi_1(z), \varphi_2(z))(x)dz =$$

$$= \int_\gamma G(\varphi_1(z),x,\varphi_2(z)(x),\partial_{e_1}\varphi_2(z)(x),\ldots,\partial_{e_n}\varphi_2(z)(x))dz = 0.$$

We thus have the following theorem for first order equations:

1.7.3 Let $A \subseteq \mathbb{C}^n$ be a compact set, X a finite dimensional complex space $G: I \times U_o \times X^{(n+1)} \to X$ a function satisfying conditions (a), (b), (c) above where $U_s = \{x \in \mathbb{C}^n: d(x,A) < (1-s)r\}$. Given u_o a bounded analytic function on U_o there is for each $1 > s > 0$ an interval $I_1 \subseteq I$ containing 0 and a function $\varphi: I_1 \times U_s \to X$ such that:

(1) For fixed t, $z \to \varphi(t,z)$ is bounded analytic

(2) $t \to \varphi(t,\ )$ is uniformly C^1

(3) φ satisfies the partial differential equation

$$\partial_t \varphi(t,z) = G(t,z,\varphi(t,z), \partial_{e_1}\varphi(t,z),\dots,\partial_{e_n}\varphi(t,z))$$

for all $(t,z) \in I_1 \times U_s$

(4) φ satisfies the initial condition

$$\varphi(0,z) = u_o(z)$$

for all $z \in U_s$.

If (d) holds, we have the following: There is a function $\varphi: I_1 \times U_s \to X$ where I_1 is some disc around 0, such that

(1) φ is bounded analytic

(2) φ satisfies the partial differential equation

$$\partial_t \varphi(t,z) = G(t,z,\varphi(t,z), \partial_{e_1}\varphi(t,z),\dots,\partial_{e_n}\varphi(t,z))$$

(3) φ _satisfies_ $\varphi(0,z) = u_o(z)$ _for all_ $z \in U_s$

This is true under weaker hypotheses on G:

1.7.4 _Suppose_ $G\colon I \times U_o \times \prod_{i=0}^{n} V_i \to X$ _is an analytic function where_ $V_i \subseteq X$ _are open sets_. _Let_ u_o _be an analytic function defined on_ U_o _and such that_ $u_o(x) \in V_o$, $\partial_{e_i} u_o(x) \in V_i$ _for all_ i _and all_ $x \in U_o$. _Then if_ $1 > s > 0$ _there is a bounded analytic function_ $\varphi\colon I_1 \times U_s \to X$ _satisfying_ (1), (2) _and_ (3) of 1.7.3 _above_.

Proof. Aside from the fact that the domain of G is allowed to be smaller the only novelty in the above statement is that the Lipschitz condition is not explicit. To prove the statement observe $\overline{U}_s \subseteq U_o$ is a compact set; Thus for a compact disc $I_o \ni 0$, the set

$$I_o \times \overline{U}_s \times u_o(\overline{U}_s) \times \partial_{e_1} u_o(\overline{U}_s) \times \dots \times \partial_{e_n}(\overline{U}_s)$$

is compact. Let $J \times A \times \prod_{i=0}^{n} B_i \subseteq I \times U_o \times \prod_{i=0}^{n} V_i$ be a compact neighborhood of this set. G and all its derivatives are bounded on $J \times A \times \prod_{i=0}^{n} B_i$ and so G is Lipschitzian (in all its variables) there.

1.8 Higher Order Systems

The previous existence theory for first order equations can be transformed into an existence theorem for higher order equations in much the same way as is done ordinary differential equations.

Consider a differential equation of order m

$$\partial_t^m u(t,x) = P(t,x,\dots,\partial_t^{\alpha_1} \partial_x^\alpha u(t,x),\dots) \tag{.1}$$

where the derivatives which enter on the right are of order $\leq m$ i.e. $\alpha_1 + |\alpha| \leq m$ for every tem $\partial_t^{\alpha_1} \partial_x^\alpha u$ and derivatives in t have order $< m$ i.e. $\alpha_1 < m$ for every such term. Let us ignore momentarily the initial conditions. Now introduce new variables so as to obtain an equation of order m-1 which satisfies the same condition on the orders of the derivatives in t. We do this as follows

$$u_o = u, \quad u_j = \partial_{e_j} u \quad (1 \leq j \leq n) \quad \text{and} \quad u_{n+1} = \partial_t u$$

The left hand side of (.1) becomes $\partial_t^{m-1} u_{n+1}(x)$ whereas the right hand side is a function of derivatives of u, u_j, u_{n+1}. Obviously there is no unique way of carrying this out and many of these ways will be of no use to us. To be specific rewrite the typical term $\partial_t^{\alpha_1} \partial^\alpha u(t,x)$ as:

$\partial_t^{\alpha_1 - 1} \partial_x^\alpha u_{n+1}$ if $\alpha_1 \neq 0$.

$\partial_x^{\alpha - e_j} u_j$ if $\alpha_1 = 0$ and $\alpha_j \neq 0$ [say j is the smallest index for which $\alpha_j \neq 0$].

It is clear that the order of the resulting equation is $\leq$ $\leq$ m-1 and all derivatives in t have order $<$ m-1. To complete the reduction we have the supplementary equations

$$\partial_t^{m-1} u_o = \partial_t^{m-2} \partial_t u = \partial_t^{m-2} u_{n+1}$$

$$\partial_t^{m-1} u_j = \partial_t^{m-1} \partial_{e_j} u = \partial_t^{m-2} \partial_{e_j} u_{n+1}$$

The resulting equation in the vector $[u_o,\ldots,u_{n+1}]$ is obviously of the type (.1).

Eventually we reduce this to a first order equation to which the Cauchy Kowaleskva theorem directly applies.

As far as side conditions go, we adjoin to (.1) the initial values

$$\partial_t^k u(0,x) = v_k(x) \quad 0 \leq k \leq m-1$$

These get transformed into the initial values:

$$\partial_t^k u_o(0,x) = v_k(x)$$

$$\partial_t^k u_j(0,x) = \partial_{x_j} \partial_t^k u(0,x) = \partial_{x_j} . v^k(x) \qquad 1 \leq j \leq n$$

$$\partial_t^k u_{n+1}(0,x) = \partial_t^{k+1} u(0,x) = v_k(x) .$$

The range of k is $0,\ldots,m-2$. When we reach a first order equation the range of k is the single value 0, which is what it should be if Cauchy Kowaleskva to be applied.

1.9 Cauchy Kowaleskva Real Analytic Case

If f is an analytic function defined on an open set $W \subseteq \mathbb{R}^n$ with values in a complex Banach space E, then f extends to a holomorphic function $\tilde{f}: W' \to E$ where $W = W' \cap \mathbb{R}^n$ and $W' \subseteq \mathbb{C}^n$ is an open set.

1.9.1 Thus let E be a real finite dimensional vector space; Let us consider the partial differential equation for E valued functions:

$$\partial_t^m u(t,x) = P(t,x,\dots,\partial_t^{\alpha_1}\partial_x^{\alpha} u(t,x),\dots)$$

where P is an analytic function

$$P\colon I \times U \times \prod V_{(\alpha_1,\alpha)} \to E$$

where $I \subseteq \mathbb{R}$ is an interval, $U \subseteq \mathbb{R}^n$ an open set and $V_{(\alpha_1,\alpha)} \subseteq E$ for $\alpha_1 + |\alpha| \leq m$ and $\alpha_1 < m$ are open

Given analytic functions $v_i\colon U \to E$ $0 \leq i \leq m-1$ such that $\partial_x^{\alpha} v_i \in V_{(i,\alpha)}$ for $i \leq m-1$ and $i + |\alpha| \leq m$ and $x_o \in U$ there is an analytic function $\varphi\colon I_1 \times U_1 \to E$ defined on a neighborhood of $(0,x_o)$ satisfying the initial conditions

$$\partial_t^i \varphi(0,x) = v_i(x) \quad 0 \leq i \leq m-1$$

for $x \in U_1$ and the partial differential equation

$$\partial_t^m u(t,x) = P(t,x,\dots,\partial_t^{\alpha_1} \partial_x^{\alpha} u(t,x),\dots,)$$

Proof. To prove this extend P to a holomorphic function in a neighborhood of $(0,x_o,v_o(x_o),\dots,\partial_x^{\alpha} v_i(x_o),\dots,)$ with values in the complexification of E. Then apply the complex analytic Cauchy Kowaleskva theorem and restrict the solution to the real domain.

SECTION 2

RUDIMENTS OF DISTRIBUTION THEORY

2.1 Review of Basic Facts

2.1.1 A topological vector space is a vector space E with a topology such that the maps $(x,y) \longmapsto x+y$, $(\lambda,x) \longmapsto \lambda\cdot x$ are continuous. We will consider exclusively topologies defined by a family $\{p_\lambda\}_{\lambda\in\Lambda}$ of function $E \to \mathbb{R}^+$ (called seminorms) satisfying the conditions

(a) $p_\lambda(x+y) \leq p_\lambda(x) + p_\lambda(y)$

(b) $p_\lambda(r\cdot x) = |r|p_\lambda(x)$ $\quad r \in K$ $(= \mathbb{R}$ or $\mathbb{C})$

(c) If $x \neq 0$ there is a $\lambda \in \Lambda$ s.t. $p_\lambda(x) \neq 0$

The associated topology is defined as follows: $V \subseteq E$ is open iff for all $x_o \in V$ there is an $r > 0$ and a finite set $\Lambda_o \subseteq \Lambda$ such that $p_\lambda(x-x_o) < r$ for $\lambda \in \Lambda_o$ implies $x \in V$.

Convergence of a net $x_i \to x$ means that $p_\lambda(x_i-x) \to 0$ for every seminorm p_λ, $\lambda \in \Lambda$. It follows from this that the operations $(x,y) \longmapsto x+y$, $(\lambda,x) \longmapsto \lambda\cdot x$ are continuous, so that E is a topological vector space. Condition (c) furthemore implies the topology is Hausdorff.

We will use repeatedly the following characterization of continuity for linear maps between spaces E,F equipped with seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$, $\{q_\omega\}_{\omega\in\Omega}$.

2.1.2 <u>Suppose</u> $T: E \to F$ <u>is a linear map</u>. T <u>is continuous iff for every</u> $\omega \in \Omega$ <u>there are</u> $\lambda_1,\ldots,\lambda_n \in \Lambda$ <u>and</u> $C > 0$ <u>such that for</u> all $x \in E$

$$q_\omega(Tx) \le C\ \mathrm{Max}\{p_{\lambda_i}(x): \quad i = 1,\ldots,n\} \tag{.1}$$

<u>Proof</u>. If T is continuous, it is continuous at 0 and hence there is an open neighborhood V of 0 in E which is mapped by T into the open neighborhood $\{y: q_\omega(y) < 1\}$ of 0 in F. By the definition of open set this implies there are $\lambda_1,\ldots,\lambda_n \in \Lambda$ and $r > 0$ such that $p_{\lambda_i}(x) < r$ for all λ_i implies $q_\omega(Tx) < 1$. If $x \in E$ then $r\ \mathrm{Max}\{p_{\lambda_i}(x)\}^{-1}x \in V$ if $\max\{p_{\lambda_i}(x)\} \ne 0$ or else $tx \in V$ for all $t > 0$. In the first instance

$$q(Tx) < r^{-1}\ \mathrm{Max}_i\{p_{\lambda_i}(x)\}$$

whereas in the second

$$q(Ttx) = tq(Tx) < 1$$

for all $t > 0$. Then $q(Tx) = 0 < r^{-1}\mathrm{Max}\{p_{\lambda_i}(x)\}$ in this case also.

If condition (.1) is satisfied it is easy to see T is continuous at 0 and this implies by linearity, continuity at very point $x_0 \in E$.

For more background the reader should consult the texts [7], [9].

We will assume the reader is familiar with calculus on manifolds. Our notation is standard: $C^\infty(M)$ is the space of C^∞ functions on M, $C_o^\infty(M)$ the functions in $C^\infty(M)$ with compact support. If U is an open subset of $\mathbb{R}^n$, Then $C_b^\infty(U)$ is the space of C^∞ functions f on U st for every multiindex $\alpha \in \mathbb{N}^n$, $D^\alpha f$ is a bounded function on U. Note that we use the standard multiindex notation and the convention:

$$D_{e_i} f = i^{-1} \partial_{e_i} f$$

2.1.3 Let M be a C^∞ manifold. A partition of unity on M is a family $\{\rho_\lambda\}$ of functions in $C^\infty(M)$, such that Supp ρ_λ is locally finite, $\rho_\lambda \geq 0$ and $\Sigma\rho_\lambda = 1$.

The fundamental result on partitions of unit is as follows:

2.1.4 Let M be a σ-compact [or more generally, paracompact] C^∞ manifold $\{V_i\}$ a cover of M by open sets. Then there exists a partition of unity $\{\rho_i\}$ of functions in $C_o^\infty(M)$ subordinate to the cover $\{V_i\}$. In other words for each ρ_j there is a V_i such that Supp $\rho_j \subset V_i$.

For a proof, consult [4] p. 85.

As a corollary

2.1.5 Let M be a σ-compact C^∞ manifold, $\{V_i\}$ a cover of M. Then there exists a partition of unity $\{\omega_i\}$ of functions in $C^\infty(M)$ such that Supp $\omega_i \subseteq \overline{V}_i$

Proof. Let $\{\rho_j\}$ be a partition of unity subordinate to $\{V_i\}$. Partition the functions $\{\rho_i\}$ into families $\{\rho_j^i\}$ such that Supp $\rho_j^i \subseteq V_i$. Let $\omega_i = \sum_j \rho_j^i$. Then evidently, Supp $\omega_i \subseteq V_i$

and ω_i is C^∞. □

2.2 Formal Adjoints

Let $W \subseteq \mathbb{R}^n$ be an open set. The space $C_o^\infty(W)$ of infinitely differentiable functions with compact support is a prehilbert space with the inner product $\langle \phi, \psi \rangle = \int \phi(x) \overline{\psi(x)}\, d\mu(x)$.

2.2.1 <u>If</u> $P = \Sigma\, c_\alpha D^\alpha$ <u>is a differential operator with coefficients in</u> $C^\infty(W)$ <u>there is a unique operator</u> $P^o: C_o^\infty(W) \to C_o^\infty(W)$ <u>with the property:</u>

$$\langle P\phi, \psi \rangle = \langle \phi, P^o \psi \rangle$$

<u>for all</u> $\phi, \psi \in C_o^\infty(W)$. P^o <u>is also a differential operator of the same order as</u> P.

<u>Proof</u>. Observe $\langle D^\alpha \phi, \psi \rangle = \langle \phi, D^\alpha \psi \rangle$ for $\phi, \psi \in C_o^\infty(W)$ and all $\alpha \in \mathbb{N}^n$. To see this note

$$0 = \int D_{e_i}(\phi \cdot \psi)\, d\mu = \int [D_{e_i} \phi\, \overline{\psi} - \phi D_{e_i} \psi]\, d\mu = 0$$

and apply induction. Thus

$$\langle \Sigma\, c_\alpha D^\alpha \phi, \psi \rangle = \langle \phi, \Sigma\, D^\alpha \overline{c}_\alpha \psi \rangle$$

Let $P^o \psi = \Sigma\, D^\alpha(\overline{c}_\alpha \psi)$. It is casy to see P^o is a differential operator [Apply for example Leibniz' product formula]. To show uniqueness, if Q is another operator satisfying the above condition then $\langle \phi, P^o \psi - Q\psi \rangle = 0$ for all $\psi, \phi \in C_o^\infty$ and so $P^o = Q$. □

2.2.2 P^o is called the <u>formal adjoint</u> of P. The adjective "formal" distinguishes it from the adjoint given by spectral theory. This is an operator P^* acting on $D \subset L^2(w)$ consisting of $u \in L^2$ s.t. $\phi \mapsto \langle P\phi, u\rangle$ is a bounded linear functional on $C_o^\infty(W)$; P^*u is the element which represents this linear functional.

Obviously P^* is an extension of P^o. One of the major problems of the theory is to find suitable extensions of P to which techniques of spectral theory are applicable. Now if Hilbert space methods should be applicable to any space of generalized functions it seems reasonable to require that any $u \in L^2(w)$ should be a generalized function.

Next let τ be any locally convex topology on $C_o^\infty(W)$ coavser than the $\sigma(C_o^\infty(W), L^2(W))$ topology and such that all differential operators are continuous in this topology. Let $C_o^\infty(W)'_\tau$ be the space of conjugate linear τ continuous functionals on $C_o^\infty(W)$. This is a complex vector space and includes the linear functionals $\Lambda_\phi: \psi \mapsto \langle\phi, \psi\rangle$ for $\phi \in L^2(W)$. If $u \in C_o^\infty(W)'_\tau$ we define Pu by

$$Pu(\phi) = u(P^o\phi)$$

for any differential operator P. Pu is also in $C_o^\infty(W)'_\tau$ and evidently extends P acting on functions [we identify $u \in L^2$ to the linear functional Λ_u].

Among the various topologies τ which satisfy the above conditions we will only consider two, which we now describe. We observe however that the topologies τ have only marginal interest for us; Our main concern is with the dual space $C_o^\infty(W)'_\tau$.

2.3 Distributions

2.3.1 Let $W \subseteq \mathbb{R}^n$ be an open set, $K \subseteq W$ a compact set. Then $C_k^\infty(W)$ is the vector space $\{\phi \in C_o^\infty(W): \text{Supp } \phi \subseteq K\}$. $C_k^\infty(W)$ has the seminorms

$$p_{m,k}(\phi) = \text{Sup}\{|D^\alpha \phi(x)|: x \in K, |\alpha| \le m\}$$

Note $C_k^\infty(W) \neq \{0\}$ iff $\overset{o}{K} \neq \phi$. Also if $K \subseteq K'$ then $C_k^\infty(W) \subseteq C_{k'}^\infty(W)$ and the inclusion map is continuous. Finally $C_o^\infty(W) = \cup\{C_k^\infty(W): K \subseteq W \text{ compact}\}$

2.3.2 Definition. A distribution on W is a conjugate linear map $u: C_o^\infty(W) \to \mathbb{C}$ such that for each compact $K \subseteq W$ $u \restriction C_k^\infty(W)$ is continuous. We say $u \in C_o^\infty(W)'$.

In other words a conjugate linear functional u is a distribution iff for every compact set $K \subseteq W_1$ there existe $m \in \mathbb{N}$ and $C > 0$ s.t. $|u(\phi)| \le Cp_{m,k}(\phi)$ for all $\phi \in C_k^\infty(W)$.

2.3.3 There exist non trivial distributions. In fact we explicitly exhibit large classes of distributions.

If $p \in [1,\infty]$, $L^p_{loc}(W)$ is the space of measurable functions $f: W \to \mathbb{C}$ s.t. $f \cdot \chi_K \in L^p(W)$ for every compact set $K \subseteq W$. Define

$$\Lambda_f(\phi) = \int f(x)\, \overline{\phi(x)}\, d\mu(x)$$

Then $\Lambda_f \in C_o^\infty(W)'$.

First Λ_f is well defined. If Supp $\subseteq K$, K compact then

$$|\Lambda_f(\phi)| = |\int_K f(x)\,\overline{\phi(x)}\,d\mu(x)| \le \|f\cdot\chi_K\|_p \cdot \|\phi\|_q$$

as follows from Hölders inequality, where $p^{-1} + q^{-1} = 1$.
Evidently

$$\|\phi\|_q \le \mu(K)^{1/q} \operatorname{Sup}\{|\phi(x)|: x \in K\}$$

so u is a distribution

Suppose λ is a countably additive measure on the Borel subsets of W, such that $\lambda(K) < \infty$ for $K \subseteq W$ compact. Then

$$\Lambda_\lambda(\phi) = \int_W \overline{\phi(x)}\,d\lambda(x)$$

is a distribution on W. This is obvious.

We will generally not distinguish between a function f and the distribution Λ_f, using the same symbol f for both.

It is an immediate consequence of the above, that if $\phi \neq 0$ in $C_o^\infty(W)$ then there is a distribution $u \in C_o^\infty(W)$, such that $u(\phi) \neq 0$.

From this it follow distributions separate functions in $C_o^\infty(W)$.

2.3.4 Next we show <u>the formula</u> $Pu(\phi) = u(P^o\phi)$ <u>defines a distribution for any differential operator</u> P. By an inductive argument the proof reduces to showing this for $P = D_{e_i}$ and P a multiplication operator $c \in C^\infty(W)$. Suppose $K \subseteq W$ is compact. Then there is an $m \in \mathbb{N}$ and $C > 0$ such that $|u(\phi)| \le Cp_{m,k}(\phi)$ for all $\phi \in C^\infty(W)$. Thus

$$|D_{e_i}u(\phi)| = |u(D_{e_i}\phi)| \le C\cdot \mathrm{Sup}\{|D^{\alpha}(D_{e_i}\phi(x)|: |\alpha| \le m, x \in K\}$$
$$\le Cp_{m+1,k}(\phi)$$

$$|cu(\phi)| = |u(\bar{c}\phi)| \le C\cdot \ \mathrm{Sup}\{|D^{\alpha}(\bar{c}\phi(x)|: |\alpha| \le m, x \in K\} \le$$
$$\le C\ \mathrm{Sup}\{|\sum_{\beta} \{d_{\alpha\beta}(x)\ D^{\beta}\phi(x): |\beta| \le |\alpha|: |\alpha| \le M,\ \ x \in K\}$$

where $d_{\alpha\beta}$ are C^{∞} functions. [Apply Leibniz' rule]. This is $\le C_1\ \mathrm{Sup}\{|D^{\beta}\ \phi(x)|: |\beta| \le m,\ x \in K\} = C_1p_{m,k}(\phi)$.

Both linear functionals $D_{e_i}u$, cu are therefore distributions.

2.4 Differential operators on Manifolds

We define the concept of differential operator on a manifold. As usual we reduce everything to coordinate charts. In order to do this we must show invariance under change of coordinates.

2.4.1 Thus let $V \subseteq \mathbb{R}^n$ be an open set $P: C_o^{\infty}(V) \to C_o^{\infty}(V)$ a linear operator. We show that the property of being a differential operator is unchanged under coordinate transformations. This means that if $T: V \to V'$ is a diffeomorphism $(T^{-1})^* PT^*(\varphi) = P(\varphi\circ T)\circ T^{-1}$ is a differential operator on V'.

Proof. To prove this it is enough to consider the special cases $P = c$, $c \in C^{\infty}(V)$ and $P = D_{e_i}$. The first case is clear. Now

$$D_{e_i}(\varphi\circ T)(T^{-1}x) = \Sigma\, D_{e_k}\varphi(x)\cdot \partial_{e_i}T_k(T^{-1}x) \qquad (.1)$$

which is evidently a differential operator with C^∞ coefficients. It is also easy to deduce from (.1) that the order of P on V is invariant.

2.4.2 Now let M be a C^∞ manifold, $P:C_o^\infty(M)\to C_o^\infty(M)$ a $\mathbb{C}$ linear operator. P is a differential operator of order $\le m$ iff for every coordinate chart V, $P(C_o^\infty(V)) \subseteq C_o^\infty(V)$ and $P|C_o^\infty(V)$ is a differential operator of order $\le m$. We define the order of P to be $\mathrm{Sup}\{\text{order } P|C_o^\infty(V):\ V \text{ a coordinate chart}\}$.

We associate to P another invariant, the symbol of P. Intuitively this is the polynomial of highest degree in P. If P is a differential operator on $V \subseteq \mathbb{R}^n$ and $P = \Sigma\, c_\alpha D^\alpha$ then

$$\mathrm{Symb}\, P(x,\xi) = \sum_{|\alpha|=m} c_\alpha(x)\, \xi^\alpha \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n .$$

In general the symbol of a differential operator is a function on the cotangent bundle. We will discuss this in more generality later.

2.5 Distributions on Manifold

2.5.1 Let M be a C^∞ manifold. Distributions on M are defined in exactly the same way as for open sets in $\mathbb{R}^n$. First we define the topology on $C_K^\infty(M)$ for $K \subseteq M$ compact. This is defined by the family of seminorms

$$P_{Q,K}(\phi) = \mathrm{Sup}\{|Q(\phi)(x)|,\ x \in K\}$$

where Q is a differential operator on M. One can show that $C_k^\infty(M)$ with this topology is a Fréchet space. We will not use this however.

2.5.2 A distribution on M is a conjugate linear functional $u\colon C_0^\infty(M) \to \mathbb{C}$ such that $u \mid C_k^\infty(M)$ is continuous for every compact $K \subseteq M$. This means that for every compact $K \subseteq M$ and every linear differential operator Q there is a $C > 0$ such that for all $\phi \in C_k^\infty(M)$

$$|u(\phi)| \leq C p_{Q,K}(\phi) \tag{.1}$$

Actually <u>it suffices to consider compact sets</u> K <u>contained in coordinate charts</u> V. To see this, let $\{\omega_\alpha\}$ be a C^∞ partition of unity subordinate to an atlas $\{V_\alpha\}$. Given $K \subseteq M$ compact let $K_\alpha = \operatorname{Supp} \omega_\alpha \cap K$, which is contained in some V_β. Now

$$\begin{aligned} p_{Q,K}(\phi) &= \operatorname{Sup}\{|Q(\phi)|(x)\colon x \in K\} \\ &\leq \sum_\alpha \{|\omega_\alpha Q(\phi)|(x)\colon x \in K_\alpha\} = \\ &= \sum_\alpha p_{\omega_\alpha Q, K_a}(\phi) \end{aligned}$$

where the sum is over a finite set of indices α, since the sets $\{\operatorname{Supp} \omega_\alpha\}$ are locally finite. Thus an estimate of the type (.1) for compact sets contained in charts implies a similar estimate for all compact sets. Thus

2.5.3 <u>A conjugate linear functional</u> u on $C_0^\infty(M)$ <u>is a distribution iff for every compact set</u> K <u>contained in</u>

a coordinate neighborhood $u \mid C_k^\infty(M)$ is continuous.

Alternatively

2.5.4 u is a distribution iff $u \mid C_o^\infty(W)$ is a distribution for every coordinate chart W.

This reduces everything down to charts.

On a C^∞ manifold without additional structure there is no natural inner product on the vector space $C_o^\infty(M)$. Consequently there is no way of associating a distribution to a function. To this one must have a measure λ on M. We now consider this.

2.6 Formal Adjoints on Manifolds

2.6.1 Let M be a C^∞ manifold with a distinguished measure λ which in a coordinate chart V has a non vanishing C^∞ Radon Nikodym derivative w.r. to the Lebesgue measure $d\mu = dx_1, \ldots, dx_r$ induced by the coordinates $(x_1, \ldots, x_n)$ of the chart V. In other words

$$\int_V f(\omega)\, d\lambda(\omega) = \int_V f(x)\, \rho(x)\, d\mu(x)$$

for any measurable function f, with $\rho \in C^\infty(V)$ and $\rho(x) > 0$ every where. We say the pair (M, λ) is a manifold with C^∞ density.

The space $C_o^\infty(M)$ is a prehilbert space with inner product

$$\langle \phi, \psi \rangle = \int \phi\, \overline{\psi}\, d\lambda\ .$$

2.6.2 Let $P\colon C_o^\infty(M) \to C_o^\infty(M)$ be a differential operator. Then there is a unique operator $P^o\colon C_o^\infty(M) \to C_o^\infty(M)$ which _satisties_

$$\langle P\phi, \psi\rangle = \langle \phi, P^{o}\psi\rangle$$

for all $\phi, \psi \in C_o^\infty(M)$. P^o _is also a differential operator_.

Proof. The uniqueness is as in the case of $W \subseteq \mathbb{R}^n$ a simple consequence of the non degeneracy of the inner product $\langle\ ,\ \rangle$. To show its existence, let $\{V_i\}$ be an atlas for M, $\{\omega_i\}$ a partition of unity subordinate to $\{V_i\}$. It suffices to show each operator $\omega_i P$ has a formal adjoint Q_i for then

$$\langle P\phi, \psi\rangle = \Sigma_i \langle \omega_i P\phi, \psi\rangle = \Sigma_i \langle \phi, Q_i\psi\rangle$$

where the above sum is actually finite. It is easy to see $\Sigma\, Q_i$ is a differential operator.

Now as $\mathrm{Supp}\ \omega_i$ is contained in some chart we use local coordinates and write

$$\int \omega_i P\phi \overline{\psi}\, d\lambda = \int \omega_i(x)\ P\ \phi(x)\ \overline{\psi}(x)\ \rho(x)\ d\mu(x)$$

for some non vanishing C^∞ function ρ. Now $R = \rho\,\omega_i P$ a differential operator on an open set in $\mathbb{R}^n$, so has a formal adjoint in the sense defined previously. Thus

$$\int \omega_i(x)\ P\ \phi(x)\ \overline{\psi}(x)\ \rho(x)\ d\mu(x) = \int \phi(x)[\overline{R^o\psi}](x)\ d\mu(x) =$$

$$= \int \phi(x)\ \rho^{-1}(x)[\overline{R^o\ \psi}](x)\ \rho(x)\ d\mu(x) = \int \phi(x)\ \overline{S\ \psi}(x)\ d\lambda(x)$$

for some differential operator S.

2.6.3 We may thus extend the action of linear differential operators on $C_o^\infty(M)$ to $C_o^\infty(M)'$ in the same way as before. If $u \in C^\infty(M)'$

$$Pu(\phi) = u(P^o\ \phi)\ .$$

It is clear that if $\phi \in C_o^\infty(M)$, then $P\Lambda_\phi = \Lambda_{P\phi}$ where Λ_ϕ is the distribution $\psi \to \langle \phi, \psi \rangle$.

2.7 Integration on Semiriemannian Manifolds

Any semiriemannian manifold carries a canonical C^∞ density. To see this we first review the change of variables formula. If $\Omega, \Omega' \subseteq \mathbb{R}^n$ are open sets $T: \Omega' \to \Omega$ a diffeomorphism then

$$\int_{\Omega'} f(Tx)dx = \int_\Omega f(y)\ |\det JT^{-1}(y)|dy$$

for any $f \in C_o^\infty(\Omega)$. In this formula JT is the Jacobian matrix

$$\begin{bmatrix} \partial_{e_1} T_1 & \cdots & \partial_{e_n} T_1 \\ \partial_{e_1} T_n & \cdots & \partial_{e_n} T_n \end{bmatrix}$$

Clearly then if M is a C^∞ manifold there is no natural way of defining the integral of a function by integrating in local coordinate systems

Suppose now M is a semiriemannian manifold and V is a coordinate chart on M. We may suppose V is an open subset of $\mathbb{R}^n$. Now for $f \in C^\infty(V)$ consider the integral

$$\int_V f(x)\ |\det g(x)|^{1/2} dx$$

where $\{g_{ij}\}$ are the components of the metric tensor expressed in the coordinates of V. If $T: V' \to V$ is any diffeomorphism then the metric tensor pulls back to V' and its expression there is:

$$g'_{ij}(x) = g'(e_i,e_j)(x) = g(JT(e_i),\ JT(e_j))(Tx) =$$

$$= \Sigma_k\ \Sigma_\ell\ \partial_{e_i} T_k(x)\ \partial_{e_j} T_\ell(x)\ g_{k\ell}(Tx)$$

so that

$$g'(x) = (JT)(x)\ g(Tx)(JT)^t(x)$$

and hence $|\det g'(x)|^{1/2} = |\det JT(x)|\ |\det g(T(x)|^{1/2}$.

Thus

$$\int_{V'} f(Tx)\ |\det g'(x)|^{1/2}\ dx =$$

$$= \int_{V'} f(Tx)\ |\det JT(x)|\ |\det g(Tx))|^{1/2}\ dx$$

$$= \int_V f(g)\ |\det JT(T^{-1}\ y)|\ |\det g(y)|^{1/2}\ |\det JT^{-1}(y)|dy =$$

$$= \int_V f(y)\ |\det g(y)|^{1/2}\ dy\ .$$

The integral is therefore invariant under arbitrary C^∞ coordinate changes. Using a C^∞ partition of unity it is now clear we may define in an invariant way the integral of a function in C_o^∞.

2.8 The Schwartz Space

2.8.1 The space $\mathcal{S}(\mathbb{R}^n)$ consists of C^∞ functions φ for which

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty \quad \text{for all } \alpha,\ \beta.$$

$\mathcal{S}(\mathbb{R}^n)$ is obviously a seminormed space with the seminorms

$$p_{\alpha,\beta}(\phi) = \operatorname{Sup}|x^\alpha D^\beta \varphi(x)|$$

As $\mathcal{S}(\mathbb{R}^n)$ is defined by a countable family of seminorms it is metrisable. In fact

2.8.2 $\mathcal{S}(\mathbb{R}^n)$ <u>is a Fréchet space which contains</u> $C_o^\infty(\mathbb{R}^n)$ <u>as a dense subspace.</u>

<u>Proof</u>. Suppose $\{\phi_i\}$ is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$. For every $\alpha,\beta \in \mathbb{N}^n$, the sequence of continuous functions $x^\alpha D^\beta \varphi_i(x)$ converges uniformly to a continuous and bounded function $\varphi_{\alpha,\beta}$. Now

$$\varphi_{\alpha,\beta}(x) = \lim x^\alpha D^\beta \varphi_i(x) = x^\alpha D^\beta \lim \varphi_i(x) = x^\alpha D^\beta \varphi_{o,o}(x)$$

so that $\varphi_{o,o}$ is C^∞ and $x^\alpha D^\beta \varphi_{o,o} \in \mathcal{S}(\mathbb{R}^n)$. Evidently $x^\alpha D^\beta \varphi_i(x) \to x^\alpha D^\beta \varphi_{o,o}(x)$ uniformly, so $\varphi_i \to \varphi$ in the topology of $\mathcal{S}(\mathbb{R}^n)$.

2.8.3 Obviously $C_o^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\psi \in C_o^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ on $\{x \in \mathbb{R}^n: |x| \leq 1\}$. Then if

$$\varphi_r(x) = \varphi(x)\ \psi(rx)$$

$\varphi_r \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ as $r \to 0$. To see this observe

$$x^\alpha D^\beta[\varphi(x) - \varphi_r(x)] = x^\alpha \sum_{\gamma\leq\beta} C_{\gamma,\beta}\, D^{\beta-\gamma}\varphi(x)\cdot r^{|\gamma|}\, D^\gamma(1-\psi)(rx).$$

All terms in the sum are bounded. If $\gamma \neq 0$, furthermore $r^{|\gamma|} \to 0$ as $r \to 0$. As $r \to 0$ therefore all the terms corresponding to $\gamma \neq 0$ go to zero uniformly in the above sum. When $\gamma = 0$ $(1-\psi)(rx)=0$ as soon as $r \leq |x|^{-1}$. Consequently, as $c_{0,\beta}\ D^{\beta}\varphi$ vanishes at ∞, the term corresponding to $\gamma = 0$ also goes to 0 uniformly. □

By the density of $C_0^\infty(\mathbb{R}^n)$ in $\mathcal{S}(\mathbb{R}^n)$ it follows that any continuous linear functional on $C_0^\infty(\mathbb{R}^n)$ [with the topology induced from $\mathcal{S}(\mathbb{R}^n)$] extends uniquely to a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$.

2.8.4 Definition. <u>A tempered distribution</u> is a conjugate linear functional $u: C_0^\infty(\mathbb{R}^n) \to \mathbb{C}$ continuous in the topology induced from $\mathcal{S}(\mathbb{R}^n)$.

A tempered distribution is a distribution in the previous sense. The converse is false.

To see this observe u is a tempered distribution iff there is a $C > 0$ and an $m > 0$ such that

$$|u(\phi)| \leq C\ \mathrm{Sup}\{|x^\alpha D^\beta \phi(x)|:\ x \in \mathbb{R}^n,\ |\alpha|,\ |\beta| \leq m\}\ .$$

Thus if $K \subseteq \mathbb{R}^n$ is compact and $\phi \in C_k^\infty(\mathbb{R}^n)$

$$|u(\phi)| \leq C\ \mathrm{Sup}\{|x^\alpha|: x \in K,\ |\alpha| \leq m\}\ \mathrm{Sup}\{|D^\beta\phi(x)|: x \in K,\ |\beta| \leq m\}$$
$$= C_1\ p_{m,k}(\phi).$$

u is thus a distribution.

The function $f(x) = e^x$ is locally integrable so Λ_f is a distribution, but is not tempered.

2.8.5 Examples. Suppose f is a measurable function on $\mathbb{R}^n$ such that $(1 + |x|^2)^{-N} f(x) \in L^p(\mathbb{R}^n)$ for $N > 0$ and some $1 < p < \infty$. Then $\Lambda_f(\phi) = \int_{\mathbb{R}^n} f(x)\, \overline{\phi(x)}\, d\mu(x)$ is a tempered distribution. For

$$|\Lambda_f(\phi)| \leq$$

$$\leq \left[\int_{\mathbb{R}^n} |(1+|x|^2)^{-N} f(x)|^p \, d\mu(x)\right]^{1/p} \left[\int_{\mathbb{R}^n} |(1+|x|^2)^N \phi(x)|^q d\mu(x)\right]^{1/q}$$

$$\leq C \int_{\mathbb{R}^n} (1+|x|^2)^N C_1 p(\phi)(1+|x|^2)^{-M} d\mu(x) \leq$$

$$\leq CC_1\, p(\phi) \int_{\mathbb{R}^n} (|1+|x|^2|^{N-M} d\mu(x) \leq C_2 p(\phi)$$

where p is a continuous seminorm in $\mathcal{S}(\mathbb{R}^n)$, and M is chosen so that the last integral is finite.

If $f \in L^1(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$ then Λ_f is also a tempered distribution. The proof is actually simpler in these cases.

Distributions $u \in C^\infty(\mathbb{R}^n)'$ with compact support are tempered. For if supp $u = K$, then $\chi \cdot u = u$ for some $\chi \in C_0^\infty(\mathbb{R}^n)$. There are $m \in \mathbb{N}$ and $C > 0$ such that for $K_1 = \text{Supp}\ \chi$

$$|u(\phi)| \leq C\, p_{m,K_1}(\phi) \quad \text{for } \phi \in C_{K_1}^\infty(\mathbb{R}^n).$$

In particular for any $\phi \in C_0^\infty(\mathbb{R}^n)$

$$|u(\phi)| = |u(\chi\phi)| \leq C\, \text{Sup}\{|D^\alpha \chi \cdot \phi(x)\colon |\alpha| \leq m, x \in K_1\}$$

$$\leq C_1\, \text{Sup}\{|D^\alpha \phi(x)|\colon |\alpha| \leq m,\ x \in K_1\}$$

$$\leq C_1\, \text{Sup}\{|D^\alpha \phi(x)|\colon |\alpha| \leq m,\ x \in \mathbb{R}^n\}$$

which is a continuous seminorm in $\mathcal{S}(\mathbb{R}^n)$

2.9 The Fourier Transform

From now on we assume Lebesgue measure μ on $\mathbb{R}^n$ is normalized so that $\mu([0,1]^n) = (2\pi)^{-n/2}$. This simplifies the formulae.

2.9.1 If $f \in L^1_\mu(\mathbb{R}^n)$ we define the Fourier transform $\mathfrak{F}f$ of f as follows:

$$\hat{f}(x) = (\mathfrak{F}f)(x) = \int_{\mathbb{R}^n} f(\theta) \exp - i\langle x,\theta\rangle \, d\mu(\theta)$$

Obviously this is well defined since $|f(\theta) \exp - i\langle x,\theta\rangle| = |f(\theta)| \in L^1$.

2.9.2 For $f \in L^1$, $\mathfrak{F}f$ is a continuous function and $|\mathfrak{F}f(x)| \leq \|f\|_1$.

Proof. The fact $\mathfrak{F}f$ is continuous follows from the Lebesgue dominated convergence theorem. □

2.9.3 If $f, x_i f \in L^1$, then $\partial_{e_i} \mathfrak{F}f$ exists and

$$\partial_{e_i} \mathfrak{F}f(x) = \mathfrak{F}_\theta[i^{-1} \theta_i f] .$$

Proof. The hypotheses enable us to differentiate under the integral sign and obtain the above result. This is in virtue of the following:

2.9.4 Lemma. Let (X,λ) be a measure space, $I \subseteq \mathbb{R}$ an interval, $g: I \times X \to \mathbb{R}$ a function such that: $x \mapsto g(t,x) \in L^1_\lambda(X)$ for each $t \in I$, $\partial_t g(t,x)$ exists for all $(t,x) \in I \times X$ and $|\partial_t g(t,x)| \leq \phi(x)$ for all $(t,x) \in I \times X$

where $\phi \in L^1_\lambda$. Then the function f

$$f(t) = \int_X g(t,x)\ d\lambda(x)$$

is differentiable and

$$f'(t) = \int_X \partial_t g(t,x)\ d\lambda(x)$$

This follows by the mean value theorem and dominated convergence. □

In other words $D_{e_i}\ \mathcal{F}f = -\mathcal{F}_\theta[\theta_i f(\theta)]$.

2.9.5 <u>If $\phi \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore</u>

(a) $D^\alpha\ \mathcal{F}\ \phi = (-1)^\alpha\ \mathcal{F}_\theta[\theta^\alpha\ \phi(\theta)]$

(b) $x^\beta[\mathcal{F}\phi](x) = \mathcal{F}(D^\beta\ \phi)(x)$

<u>Finally the map</u> $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ <u>is continuous</u>

<u>Proof</u>. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ then $x^\alpha\ \phi(x) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ for all α.

Thus $\mathcal{F}\phi$ is infinitely differentiable and (a) is satisfied. To show (b) note

$$x^\beta \int \phi(\theta)\ \exp - i\langle x,\theta\rangle\ d\mu(\theta) = \int \phi(\theta)\ D^\beta_\theta[\exp - \langle x,\theta\rangle]\ d\mu(\theta)$$

$$= \int D^\beta\ \phi(\theta)\ \exp - i\langle x,\theta\rangle\ d\mu(\theta) = \mathcal{F}(D^\beta\ \phi)(x)$$

We have used the integration by parts formula

$$\int D^\alpha\ \phi.\psi\ du = \int \phi . D^\alpha\psi du$$

valid for functions $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in C^\infty_b(\mathbb{R}^n)$. This proves statement (b). (a) & (b) together with the fact that the

Fourier transform of an L^1 function is bounded imply $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^n)$.

To show $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, note for M large

$$|x^\alpha D^\beta \mathcal{F}\phi(x)| \le \int |D^\alpha \theta^\beta \phi(\theta)| \, d\mu(\theta) \le$$

$$\le \operatorname{Sup}\{(1+|\theta|^2)^M |D^\alpha \theta^\beta \phi(\theta)|: \theta \in \mathbb{R}^n\}: \int (1+|\theta|^2)^{-M} d\mu(\theta) \le$$

$$\le Cp(\phi).$$

where p is a continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$. Thus for every seminorm $p_{\alpha,\beta}$ there is a continuous seminorm p in $\mathcal{S}(\mathbb{R}^n)$ such that $p_{\alpha,\beta}(\mathcal{F}\phi) \le p(\phi)$. $\mathcal{F}$ is therefore continuous. $\square$

2.9.6 Example. The function $\phi_n(x) = \exp(-1/2|x|^2)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}\phi_n = \phi_n$. It is elementary that $\phi_n \in \mathcal{S}(\mathbb{R}^n)$; Next observe $\phi_n(x_1,\ldots,x_n) = \phi_1(x_1)\ldots\phi_1(x_n)$ and so $\hat{\phi}_n(x_1,\ldots,x_n) = \hat{\phi}_1(x_1)\ldots \hat{\phi}_1(x_n)$. Thus it suffices to consider the case $n = 1$. In this case ϕ_1 satisfies the ordinary differential equation $\phi_1'(x) + x\phi_1(x) = 0$. From this and the formulas (a) and (b) above it follows $\hat{\phi}_1$ satisfies the same differential equation. As $\phi_1(0) = 1$ and

$$\hat{\phi}_1(0) = \int \exp(-1/2x^2) \, du(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp(-1/2\, x^2)dx = 1$$

it follows $\phi_1 = \hat{\phi}_1$

2.10 The Fourier Inversion Formula

2.10.1 If $\phi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\phi(x) = \int \hat{\phi}(\theta) \exp i\langle x,\theta\rangle \, d\mu(\theta) = \mathcal{F}^2\phi(-x)$$

Proof. To prove this fundamental formula, observe first that for $f,g \in L^1$

$$\int f(x)\, \hat{g}(x)\, d\mu(x) = \int \hat{f}(x)\, g(x)\, d\mu(x)$$

To see this apply Fubini's theorem. Thus if $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\int \phi(\lambda^{-1}x)\, \mathcal{F}\psi(x)\, d\mu(x) = \int [\mathcal{F}_\theta\, \phi(\lambda^{-1}\theta)](x)\, \psi(x)\, d\mu(x) =$$

$$= \int \lambda^n\, \mathcal{F}\phi(\lambda x)\cdot\psi(x)\, d\mu(x) = \int \mathcal{F}\phi(x)\, \psi(\lambda^{-1}x)\, d\mu(x) \quad .$$

Now letting $\lambda \to \infty$, $\phi(\lambda^{-1}x) \to \phi(0)$, $\psi(\lambda^{-1}x) \to \psi(0)$ for all x. Furthermore the family of functions $\phi(\lambda^{-1}x), \psi(\lambda^{-1}x)$ is uniformly bounded, so that by dominated convergence it follows

$$\phi(0) \int \mathcal{F}\psi(x)\, d\mu(x) = \psi(0) \int \mathcal{F}\phi(x)\, d\mu(x)$$

for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Taking $\psi(x) = \exp\{-1/2|x|^2)$, we have

$$\phi(0) = \int \mathcal{F}\phi(x)\, d\mu(x) \; .$$

Thus

$$\int \exp i\langle x,\theta\rangle\, \mathcal{F}\phi(\theta)\, d\mu(\theta) = \int (\mathcal{F}_y\, \phi(y+x)](\theta)\, d\mu(\theta) = \phi(x) \; .$$

as asserted. □

As a corollary to this it follows $\mathcal{F}^4 = \mathrm{id}$ and in particular $\mathcal{F}$ is a homeomorphism with continuous inverse $\mathcal{F}^3$.

We now extend the Fourier transform to all tempered distributions in much the same way as differential operators were extended to distributions. Note that for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}\phi, \psi \rangle = \int \mathcal{F}\phi(x) \overline{\psi(x)}\, d\mu(x) = \int \phi(x) \overline{\mathcal{F}\psi(x)}\, d\mu(x)$$
$$= \int \phi(x) \overline{\mathcal{F}\psi(-x)}\, d\ (x) = \langle \phi, J\mathcal{F}\psi \rangle \qquad (.1)$$

where $J\psi(x) = \psi(-x)$. Thus $J\mathcal{F}$ is the adjoint of $\mathcal{F}$.

2.10.2 If $u \in \mathcal{S}(\mathbb{R}^n)'$ then define $\mathcal{F}u$ by the formula

$$\mathcal{F}u(\psi) = u(J\mathcal{F}\psi) .$$

It is clear from this definition that $\mathcal{F}$ on tempered distributions extends $\mathcal{F}$ acting on $\mathcal{S}(\mathbb{R}^n)$, that is $\mathcal{F}\Lambda_\phi = \Lambda_{\mathcal{F}\phi}$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. This is in fact true for $\phi \in L^1(\mathbb{R}^n)$ as (.1) is valid for $\phi \in L^1(\mathbb{R}^n)$.

The Fourier inversion formula $\phi = J\mathcal{F}^2\phi$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$ extends immediately to tempered distributions:

2.10.3 _If_ $u \in \mathcal{S}(\mathbb{R}^n)'$ _then_ $u = J\mathcal{F}^2u$. _In this formula, the operator_ $J: \mathcal{S}(\mathbb{R}^n)' \to \mathcal{S}(\mathbb{R}^n)'$ _is defined by_ $Ju(\phi) = u(J\phi)$

Proof. To see this note $J^2 = \text{id}$ and $J\mathcal{F} = \mathcal{F}J$ on $\mathcal{S}(\mathbb{R}^n)$. Thus

$$J\mathcal{F}^2u(\phi) = u((J\mathcal{F})^2\, J\phi) = u(J\mathcal{F}^2\phi) = u(\phi)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. □

Finally as it is sometimes convenient to have a symbol for the inverse Fourier transform, we set $\check{u} = J(\hat{u}) = J\mathcal{F}u$.

2.11 The Plancherel Theorem

By the formula (.1) of the proceding section and the Fourier inversion formula we have for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}\phi, \mathcal{F}\psi \rangle = \langle \phi, J\mathcal{F}^2 \psi \rangle = \langle \phi, \psi \rangle$$

The map $\mathcal{F}$ is thus isometric on the subspace $\mathcal{S}(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$. $\mathcal{F}$ thus extends to an isometry U on $L^2(\mathbb{R}^n)$. U is in fact unitary as its image contains $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$. The content of the Plancherel theorem is that U is in fact the restriction to $L^2(\mathbb{R}^n)$ of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)'$.

To see this suppose $u \in L^2(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$Uu(\phi) = \langle Uu, \phi \rangle = \langle Uu, U^2 J\phi \rangle = \langle u, J\mathcal{F}\phi \rangle$$

$$= u(J\mathcal{F}\phi) = \mathcal{F}u(\phi)$$

proving the assertion. Thus

2.11.1 <u>The Fourier transform</u> $\mathcal{F}$ <u>is a bijective isometric map</u> $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

2.12 Fourier Transforms of Products and Convolutions

2.12.1 <u>Suppose $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then</u>

(a) $(\varphi \cdot \psi)^\wedge = \varphi^\wedge * \psi^\wedge$

(b) $(\varphi * \psi)^\wedge = \varphi^\wedge \cdot \psi^\wedge$

Proof. For by Fubini:

$$(\varphi * \psi)^{\wedge}(x) = \int \exp - i\langle x,y\rangle \left[\int \varphi(y-z)\ \psi(z)dz\right]dy =$$

$$= \iint \left[\varphi(y-z)\ \psi(z) \exp - i\langle x,y\rangle\ dzdy = \right.$$

$$= \iint \left[\varphi(y-z)\ \psi(z) \exp - i\langle x,y-z\rangle \exp - i\langle x,z\rangle dzdy\right.$$

$$= \int \left[\psi(z) \exp - i\langle x,z\rangle \int \varphi(y-z) \exp - i\langle x,y-z\rangle dy\right] dz$$

$$= \int \left[\psi(z) \exp - i\langle x,z\rangle \int \varphi(y) \exp - i\langle x,y\rangle dy\right] dz =$$

$$= \hat{\psi}(x)\ \hat{\varphi}(x).$$

This proves (b). (a) Follows almost immediately:

$$(\hat{\varphi} * \hat{\psi})^{\wedge} = \varphi^{\wedge\wedge} \cdot \psi^{\wedge\wedge} = J\varphi \cdot J\psi$$

so $J(\varphi^{\wedge} * \psi^{\wedge}) = J(\varphi \cdot \psi)^{\wedge}$, and thus $\varphi^{\wedge} * \psi^{\wedge} = (\varphi \cdot \psi)^{\wedge}$. □

2.13 The Sobolev Spaces

2.13.1 Let $m \in \mathbb{Z}^+$, $\Omega \subseteq \mathbb{R}^n$ an open set. $H^m(\Omega)$ is the space of $u \in C^\infty(\Omega)'$ such that $D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq m$.

$H^m(\Omega)$ is an inner product space with the following inner product.

$$\langle u,v\rangle_m = \Sigma_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v\rangle$$

2.13.2 $H^m(\Omega)$ for $m \in \mathbb{Z}^+$ is a Hilbert space

Proof. Suppose $\{u_n\}$ is Cauchy in $H^m(\Omega)$. For each α, $|\alpha| \leq m$ $u_n^\alpha = D^\alpha u_n$ is a Cauchy sequence in $L^2(\Omega)$ and hence there is a $u^\alpha \in L^2(\Omega)$ such that $u_n^\alpha \to u^\alpha$ in $L^2(\Omega)$. Now for all $\phi \in C_0^\infty$

$$\langle u_n^\alpha,\phi\rangle_o = \langle u_n, D^\alpha\phi\rangle_o \longrightarrow \langle u^o, D^\alpha\phi\rangle_o = \langle D^\alpha u^o,\phi\rangle_o$$

so that $u^\alpha = D^\alpha u^o$. Thus $u^o \in H^m(\Omega)$ and $u_n \to u^o$ in $H^m(\Omega)$. □

2.13.3 Next we define for $m > 0$ the space $H^{-m}(\Omega)$. This is the space of $u \in C^\infty(\Omega)'$ such that the functional $\phi \longmapsto \langle u,\phi\rangle$ is continuous on $C_o^\infty(\Omega)$ in the topology of H^m. In other words $u \in H^{-m}(\Omega)$ iff there is a $C > 0$ such that for all $\phi \in C_o^\infty(\Omega)$

$$|\langle u,\phi\rangle| \le C\|\phi\|_m$$

<u>$H^{-m}(\Omega)$ is in a natural way a Hilbert space</u>. To see this, let $H_o^m(\Omega)$ be the closure of $C_o^\infty(\Omega)$ in $H^m(\Omega)$. We then have:

There is a unique map $R: H^{-m}(\Omega) \to H_o^m(\Omega)$ such that for all $\phi \in C_o^\infty$,

$$\langle Ru,\phi\rangle_m = \langle u,\phi\rangle_o \qquad (.1)$$

If $u \in H^{-m}(\Omega)$ then the Riesz representation theorem implies the existence of a unique $Ru \in H_o^m$ satisfying (.1). Next, if $w \in H_o^m$ then $\langle u,\phi\rangle = \langle w,\phi\rangle_m$ defines an element $u \in C_o^\infty(\Omega)'$. In fact $|\langle u,\phi\rangle| \le \|w\|_m \|\phi\|_m$, so $u \in H^{-m}(\Omega)$. Furthermore $\langle Ru, \phi\rangle_m = \langle u,\phi\rangle = \langle w,\phi\rangle_m$ for all $\phi \in C^\infty(\Omega)$ so that $Ru = w$. □

We thus give H^{-m} the Hilbert space structure it inherits via the mapping R. In this manner we have a sequence of Hilbert spaces

$$H^r(\Omega) \subseteq \ldots \subseteq H^1(\Omega) \subseteq H^o(\Omega) \subseteq H^{-1}(\Omega) \subseteq \ldots \subseteq H^{-r}(\Omega)$$

It is obvious the inclusions $H^r \subseteq H^s$ for $r \geq s$ are continuous. The verification of these facts is trivial.

The H^m spaces, $m \in \mathbb{Z}$ have natural restriction properties, viz:

2.13.4 <u>Suppose</u> $\Omega, \Omega_1 \subseteq \mathbb{R}^n$ <u>are open sets and</u> $\Omega_1 \subseteq \Omega$. <u>Then</u> $u \mapsto u|\Omega_1$ <u>maps</u> $H^m(\Omega)$ <u>continuously into</u> $H^m(\Omega_1)$ <u>for all</u> $m \in \mathbb{Z}$.

<u>Proof</u>. If $m \geq 0$ this is trivial: Essentially, because the restriction of an L^2 function is in L^2. For $-m, m > 0$

$$|\langle u|\Omega_1, \phi\rangle_o| = |\langle u, \phi\rangle_o| \leq \|u\|_{-m} \|\phi\|_m$$

Thus $\|u|\Omega_1\|_{-m} \leq \|u\|_{-m}$. □

This raises the question whether $H^m(\mathbb{R}^n) \to H^m(\Omega)$ is surjective. An open set Ω for which this occurs is said to have the <u>m-extension property</u>. We will return to this later on.

2.14 Differential Operators and the Sobolev Spaces

2.14.1 <u>For any</u> $m \in \mathbb{Z}$ $C_b^\infty(\Omega)$ <u>acts on</u> $H^m(\Omega)$ <u>boundedly. In fact if</u> $|||\, a \,|||_r$ <u>is the norm</u> $|||\, a \,|||_r = \operatorname{Max}\{\|D^\alpha a\|_\infty : |\alpha| \leq r\}$ for $a \in C_b^\infty(\Omega)$, <u>then there is a</u> $C > 0$ <u>such that</u>

$$\|a \cdot u\|_m \leq C |||\, a \,|||_{|m|} \|u\|_m$$

<u>Proof</u>. To see this suppose first $m \geq 0$. Then

$$\|a.u\|_m^2 = \sum_{|\alpha|\le m} \|D^\alpha a.u\|_o^2 = \sum_\alpha \|\sum_{\beta\le\alpha} C_{\beta,\alpha} D^\beta a D^{\alpha-\beta} u\|_o^2 \le$$

$$\le C \sum_\alpha \sum_{\beta\le\alpha} C_{\beta,\alpha} ||| a |||_m^2 \|D^{\alpha-\beta}u\|_o^2 \le$$

$$\le C_1 ||| a |||_m^2 \Sigma_{|\gamma|\le m} \|D^\gamma u\|_o^2$$

$$\le C_1 ||| a |||_m^2 \|u\|_m^2 .$$

For $-m$, we have for any $\phi \in C_o^\infty(\Omega)$

$$|\langle au,\phi\rangle_o| = |\langle u,\bar{a}\phi\rangle_o| \le \|u\|_{-m} \|\bar{a}\phi\|_m$$

$$\le C_1 ||| a |||_m \|\phi\|_m \|u\|_{-m}$$

so that $\|au\|_{-m} \le C_1 ||| a |||_m \|u\|_{-m}$. □

Next we observe how derivatives behave:

2.14.2 <u>For any</u> $m \in \mathbb{Z}$ <u>and any</u> $\alpha \in \mathbb{N}^n$, D^α <u>is a continuous linear map</u> $H^m(\Omega) \to H^{m-|\alpha|}(\Omega)$.

<u>Proof</u>. It suffices to prove this for $|\alpha| = 1$. This is almost vacuous If $r \ge 0$

$$\|D_{e_i} u\|_r^2 = \sum_{|\alpha|\le r} \|D_{e_i} D^\alpha u\|_o^2 \le \sum_{|\gamma|\le r+1} \|D^\gamma u\|_o^2 = \|u\|_{r+1} .$$

Thus $D_{e_i} : H^{r+1}(\Omega) \to H^r(\Omega)$ is continuous for $r \ge 0$. In general the result follows by duality: If $m = -r \le 0$, then for $u \in H^m(\Omega)$

$$|\langle D_{e_i} u,\phi\rangle_o| = |\langle u,D_{e_i}\phi\rangle_o| \le \|u\|_m \|D_{e_i}\phi\|_r \le \|u\|_m \|\phi\|_{r+1}$$

so that $D_{e_i} u \in H^{-r-1}(\Omega) = H^{m-1}(\Omega)$ and

$$\|D_{e_i} u\|_{m-1} \leq \|u\|_m \quad \square$$

As a Corollary,

2.14.3 <u>If</u> L <u>is an operator of order</u> m <u>with coefficients in</u> $C_b^\infty(\Omega)$, <u>then</u> L <u>maps</u> $H^r(\Omega)$ <u>continuously into</u> $H^{r-m}(\Omega)$.

2.14.4 This result may be stated without recurring to the negative spaces $H^{-m}(\Omega)$. In fact, let $m = r+s$ with $r,s \geq 0$. Then there is a $C > 0$ such that for $u \in H^r(\Omega)$, $\phi \in C_o^\infty(\Omega)$

$$|\langle Lu, \phi\rangle| \leq C\|u\|_r \|\phi\|_s$$

For this means exactly $\|Lu\|_s \leq C\|u\|_r$.

2.15 <u>H^m Spaces under Coordinate Change</u>

2.15.1 Suppose Ω, Ω' are open sets in $\mathbb{R}^n$, $T: \Omega \to \Omega'$ a diffeomorphism. If $u \in C_o^\infty(\Omega)'$, then we define $T^* u \in C_o^\infty(\Omega')'$ by the formula

$$T^* u(\phi) = u(\phi T) .$$

Assume first $u = \Lambda_f$, for $f \in L^2_{loc}(\Omega)$. Thus for $\phi \in C^\infty(\Omega)$:

$$u(\phi) = \int \overline{\phi(x)}\, f(x)dx$$

Consequently

$$(T^* u)(\phi) = u(\phi T) = \int_\Omega \overline{\phi(Tx)}\, f(x)dx = \int_\Omega \overline{\phi(Tx)}\, fT^{-1}(Tx)dx =$$

$$= \int_{\Omega'} \overline{\phi(y)}\, fT^{-1}(y)\, |\det JT^{-1}(y)|dy$$

T^*u is thus at least formally $\Lambda_{f'}$, for $f'(y) = = fT^{-1}(y) \mid \det JT^{-1}(y)|$. In fact

$$\int (|fT^{-1}(y)| \; |\det JT^{-1}(y)|)^2 \, dy =$$

$$= \int |f(x)|^2 \; |\det JT^{-1}(Tx)|^2 \; |\det JT(x)| dx =$$

$$= \int |f(x)^2 \; |\det JT^{-1}(Tx)| dx$$

If $|\det JT^{-1}(y)| \le M$ then $f' \in L^2_{loc}(\Omega')$. Furthermore, if $u \in H^o(\Omega)$ then $f' \in L^2(\Omega')$ and thus $T^*u \in H^o(\Omega')$. It is also clear $\|T^*u\|_o \le M\|u\|_o$. In summary:

2.15.2 <u>Suppose</u> $T: \Omega \to \Omega'$ <u>is a diffeomorphism such that</u> $|\det JT^{-1}(y)|$ <u>is uniformly bounded on</u> Ω. <u>Then</u> T^* <u>maps</u> $H^o(\Omega')$ <u>into</u> $H^o(\Omega')$ boundedly.

We next consider invariance under diffeomorphisms of the distribution spaces $H^m(\Omega)$. First note that for any $u \in C_o^\infty(\Omega)'$

$$[D_{e_i} T^*u](\phi) = T^*u(D_{e_i} \phi) = u[(D_{e_i} \phi)T]$$

By the chain rule

$$D_{e_i}(\phi T)(x) = \sum_{j=1}^{n} D_{e_j} \phi(Tx) \; \partial_{e_i} T_j(x)$$

where $T(x) = (T_1(x),\ldots,T_n(x))$. Thus

$$D_{e_i} \phi(Tx) = \sum_{j=1}^{n} A_{ij}(x) \; D_{e_j}(\phi T)(x)$$

where $A(x) = [JT(x)]^{t^{-1}}$. From this we deduce

$$[D_{e_i}\ T^*u](\phi) = \sum_{j=1}^{n} [A_{ij}u](D_{e_j}(\phi T))$$
$$= \sum_{j=1}^{n} D_{e_j}[A_{ij}u](\phi T)$$
$$= \sum_{j=1}^{n} T^*(D_{e_j}[A_{ij}u])(\phi)$$

With this formula it is easy to prove the following

2.15.3 <u>Suppose</u> Ω, Ω' <u>are open sets in</u> $\mathbb{R}^n$, $T: \Omega \to \Omega'$ <u>a diffeomorphism such that the components</u> $T_i (1 \le i \le n)$ <u>of</u> T <u>and</u> $(T^{-1})_i$ $(1 \le i \le n)$ <u>of</u> T^{-1} <u>are in</u> C_b^∞. <u>Then</u> T^* <u>is a bijective linear homeomorphism</u> $H^m(\Omega) \to H^m(\Omega')$, <u>for all</u> $m \ge 0$.

<u>Proof</u>. Since T^* is functorial in T it evidently suffices to prove T^* maps $H^m(\Omega)$ continuously into $H^m(\Omega')$. Now $JT(x)^{-1} = JT^{-1}(Tx)$ so that the functions A_{ij} are in C_b^∞. [To see this note

$$A_{ij}(x) = (JT^{-1}(Tx))_{ji} = \partial_{e_i}(T^{-1})_j(Tx)$$

and by hypothesis $(T^{-1})_j \in C_b^\infty(\Omega')$. As the T_i are also in C_b^∞ the chain rule implies the A_{ij} are in $C_b^\infty(\Omega)$].

The proof is now by induction. Obviously T^* maps $H^o(\Omega) \to H^o(\Omega')$. If T^* maps $H^m(\Omega)$ into $H^m(\Omega')$ continuously, then for $u \in H^{m+1}(\Omega)$, $T^*u \in H^m(\Omega')$ and $\|T^*u\|_m \le C\|u\|_m \le C\|u\|_{m+1}$. Also $A_{ij}u \in H^{m+1}(\Omega)$, so $D_{e_j} A_{ij}u \in H^m(\Omega)$ and thus by the induction hypothesis $T^*(D_{e_j}[A_{ij}u]) \in H^m(\Omega')$. Therefore $D_{e_i}T^*u \in H^m(\Omega)$. In addition

$$\|D_{e_i} T^* u\|_m \leq C \sum_j \|D_{e_j}[A_{ij}u]\|_m$$

$$\leq C' \sum_j \|A_{ij}u\|_{m+1} \leq C'' \|u\|_{m+1} .$$

In conclusion $T^*u \in H^{m+1}(\Omega)$ and

$$\|T^*u\|_{m+1} \leq \sum_i \|D_{e_i} T^*u\|_m + \|T^*u\|_m \leq C''' \|u\|_{m+1} .$$

One can show that if Ω is open then the definition of $H^m(\Omega)$ does depend on the coordinates used on Ω, that is on the particular imbedding $\Omega \hookrightarrow \mathbb{R}^n$. However the preceding result shows the dependence is fairly mild in the sense that if the coordinates are changed in "bounded" way then H^m does not change.

2.15.4 In particular, suppose $K \subseteq \Omega$ is compact and let $H^m_k(\Omega)$ consist of distributions $u \in H^m(\Omega)$ such that Supp $u \subseteq K$. We have the following:

2.15.5 <u>If</u> Ω, Ω' <u>are open sets in</u> $\mathbb{R}^n$ <u>and</u> $T: \Omega \to \Omega'$ <u>is any diffeomorphism then</u> T <u>maps</u> $H^m_K(\Omega)$ <u>bicontinuously onto</u> $H^m_{T(K)}(\Omega)$ <u>for any compact</u> $K \subseteq \Omega$

2.16 <u>H^m Spaces for Manifolds</u>

Let M be a compact C^∞ manifold. We define the spaces H^m, $m \in \mathbb{Z}$ essentially by localizing to coordinate neighborhoods.

Thus let $\{V_i\}$, $1 \leq i \leq n$ be an atlas for M and $\{\rho_i\}$, $1 \leq i \leq n$ a partition of unity on M such that $K_i = \text{Supp } \rho_i \subseteq V_i$

is a compact set. Note that if $u \in C_0^\infty(M)'$ then $\mathrm{Supp}\ \rho_i u \subseteq K_i$.

2.16.1 $H^m(M)$ is defined to consist of those $u \in C_0^\infty(M)'$ such that $\rho_i u \in H^m_{K_i}(V_i)$. [or equivalently, $\rho_i u \in H^m(V_i)$].

Obviously the definition is independent of the coordinates used on V_i. <u>It is also independent of the atlas $\{V_i\}$ and the partition</u> $\{\rho_i\}$.

To see this let $\{V_i\}$, $\{W_j\}$ $1 \le i \le n$, $1 \le j \le m$ be atlasses for M; $\{\rho_i\}$, $\{\tau_j\}$ corresponding partitions of unity. If $\rho_i u \mid V_i \in H^m(V_i)$ then $\tau_j\ \rho_i u \mid V_i \cap W_j \in H^m(V_i \cap W_j)$ so that $\tau_j\ \rho_i u \mid W_i \in H^m(W_j)$ for all i,j. Therefore $\tau_j u \mid W_i =$ $= \Sigma\ \tau_j\ \rho_i u \mid W_i \in H^m(W_i)$.

There is no natural Hilbert space structure on $H^m(M)$ although it carries a Hilbertizable locally convex structure. In fact if $u \in H^m(M)$ set $\|u\|_m^2 = \sum_{i=1}^{n} \|\rho_i u\|_m^2$. The topology is independent of $\{V_i\}$, $\{\rho_i\}$. To see this suppose $u_s \to 0$ in the norm $(\Sigma\|\rho_i u\|_m^2)^{1/2}$: Then $\Sigma \|\tau_j\ \rho_i u_s\|_m^2 \to 0$ and so $\tau_j u_s =$ $= \Sigma_i\ \tau_j\ \rho_i u_s \to 0$ in the $H^m(W_j)$ norm. Thus $(\Sigma \|\tau_j u_s\|^2)^{1/2} \to 0$. By symmetry if this last term $\to 0$ then so does $(\Sigma \|\rho_i u_s\|^2)^{1/2}$.□

2.16.2 We can reduce most questions about H^m spaces on manifolds to questions on open sets in $\mathbb{R}^n$ by the following device: Let $\{V_i\}_{1 \le i \le N}$ be an open cover for M, $\{\rho_i\}$ a partition of unity in C^∞ such that $\mathrm{Supp}\ \rho_i \subseteq V_i$. Now consider the maps

$$C^\infty(M)' \xrightarrow{i} \prod_{i=1}^{N} C^\infty(V_i)' \xrightarrow{h} C^\infty(M)'$$

given by $i : u \to (\rho_1 u, \ldots, \rho_n u)$, $h : (u_1, \ldots, u_n) \to \Sigma\, u_i$. Evidently $h \circ i = \mathrm{id}_{C^\infty(M)'}$ and for $m \in \mathbb{Z}$ we have

$$H^m(M) \to \prod_{r=1}^{N} H^m(V_k) \to H^m(M)$$

are continuous maps. This is immediate from the definitions.

Furthermore, if L is a differential operator with C^∞ coefficients on M and $L_i(v) = L(v)$ for $v \in C_o^\infty(M)$ supported in V_i, then L_i is an operator on V_i and

$$\begin{array}{ccc} \prod_{k=1}^{N} C^\infty(V_k)' & \xrightarrow{\;h\;} & C^\infty(M)' \\ \Big\downarrow \pi L_k & & \Big\downarrow L \\ \prod_{k=1}^{N} C^\infty(V_k)' & \xrightarrow{\;h\;} & C^\infty(M)' \end{array}$$

is commutative. With this observation, it is easy to prove.

2.16.3 L <u>is a continuous operator</u> $H^p(M) \to H^{p-m}(M)$ <u>for any</u> p, <u>where</u> $m = \deg L$.

<u>Proof</u>. Suppose $u \in H^p(M)$. Then

$$\|Lu\|_{p-m} = \|L(h \circ i(u))\|_{p-m} = \|h \circ \prod_{k=1}^{N} L_k \circ i(u)\|_{p-m}$$

$$\leq C \|\prod_{k=1}^{N} L_k \circ i(u)\|_{p-m} = C\|(L\rho_1 u, L\rho_2\, u, \ldots, L\rho_N u)\|_{p-m}$$

$$\leq C_1 \Big[\sum_{k=1}^{N} \|\rho_k\, u\|_m^2 \Big]^{1/2} = \|u\|_m \, .$$

2.17 The Spaces $H^m(\mathbb{R}^n)$

In case $\Omega = \mathbb{R}^n$ the Sobolev spaces $H^m(\mathbb{R}^n)$, $m \in \mathbb{Z}$ have elegant interpretations in terms of the Fourier transform:

2.17.1 $u \in H^m(\mathbb{R}^n)$ *iff* u *is tempered and the function* $(1 + |\xi|^2)^{m/2}\, \hat{u}(\xi)$ *is in* $L^2(\mathbb{R}^n)$.

Proof. Suppose $m \geq 0$. If $u \in H^m(\mathbb{R}^n)$, then $u \in L^2(\mathbb{R}^n)$ so u is tempered. Now $(D^\alpha u)^\wedge(\xi) = \xi^\alpha\, \hat{u}(\xi)$ and the Fourier transform carries L^2 onto L^2 so $\xi^\alpha\, \hat{u}(\xi)$ is in L^2 for $|\alpha| \leq m$. Thus $(1 + |\xi|^2)^{m/2}\, \hat{u}(\xi)$ is in L^2. Conversely if u is tempered and $(1+|\xi|^2)^{m/2}\, \hat{u}(\xi)$ is in L^2, then $\xi^\alpha\, \hat{u}(\xi)$ is in L^2 for $|\alpha| \leq m$ and thus by the Plancherel theorem $D^\alpha u \in L^2$ for $|\alpha| \leq m$.

Before proving the result for $m < 0$, let us make some observations on the norm of H^m. Note that

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha u\rangle_0 = \sum_{|\alpha| \leq m} \langle \xi^\alpha \hat{u}, \xi^\alpha \hat{u}\rangle_0$$

$$= \sum_{|\alpha| \leq n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\, \xi^{2\alpha}\, d\xi$$

$$= \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \Big(\sum_{|\alpha| \leq m} \xi^{2\alpha} \Big) d\xi$$

Thus it is clear this norm is equivalent to the norm

$$\|u\|_m^2 = \int |\hat{u}(\xi)|^2\, (1 + |\xi|^2)^m\, d\xi$$

For technical purposes this norm is often more convenient than the equivalent norm $\|\ \ \|_m$. We will use these norms interchangeably.

We now prove the result for $m < 0$. If $u \in H^m(\mathbb{R}^n)$ then the inequality $|\langle u,\phi\rangle_0| \le C\|u\|_m \|\phi\|_{-m}$ shows u is tempered. On the other hand $\phi \to \langle u,\check{\phi}\rangle$ is a continuous linear functional on $L^2((1 + |\xi|^2)^{-m} d\xi)$ so by the Riesz representation theorem, there is a unique $w \in L^2((1 + |\xi|^2)^{-m} d\xi)$ such that for $\phi \in s(\mathbb{R}^n)$

$$\langle u,\phi^{\vee}\rangle = \int W(\xi)\, \overline{\phi(\xi)}\, (1+|\xi|^2)^{-m}\, d\xi$$

This means exactly that $\hat{u}$ is the function $w(\xi)(1+|\xi|^2)^{-m}$ and so $(1 + |\xi|^2)^{m/2}\, \hat{u}(\xi) = w(\xi)(1 + |\xi|^2)^{-m/2}$ is in $L^2(\mathbb{R}^n)$.

Conversely, if u is tempered and $(1 + |\xi|^2)^{m/2}\hat{u}(\xi) \in L^2$, then

$$|\langle u,\phi\rangle_0| = |\langle \hat{u},\hat{\phi}\rangle_0| \le$$

$$\le \left[\int |\hat{u}(\xi)|^2\, (1 + |\xi|^2)^m\, d\xi\right] \cdot \left[\int |\hat{\phi}(\xi)|^2\, (1 + |\xi|^2)^{-m}\, d\xi\right]$$

$$= C\|\phi\|_{-m}$$

The Spaces $H^m(\mathbb{R}^n)$ [cont'd]

One of the fundamental reasons for studying the spaces $H^m(\mathbb{R}^n)$ is the following proposition:

2.17.2 The operator $(1-\Delta)$ is an isometry $H^m(\mathbb{R}^n) \to H^{m-2}(\mathbb{R}^n)$ for all values of m. [Δ is the Laplacian $-\sum_{i=1}^{n} D^2_{e_i} = \sum_{i=1}^{n} \partial^2_{e_i}$.] .

<u>Proof</u>. In fact Fourier transformation transforms the statement of the proposition into the following:

Multiplication by the function $(1 + |\xi|^2)$ transforms the space $L^2(1 + |\xi|^2)^m \, d\xi)$ isometrically into $L^2(1 + |\xi|^2)^{m-2} \, d\xi)$. However this is easy to prove. □

2.17.3 Another relevant feature about the family of spaces $\{H^m(\mathbb{R}^n)\}_{m\in\mathbb{R}}$ is that there is a unique continuous pairing

$$H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

which extends the pairing $(u,\phi) \longmapsto \langle u,\phi\rangle_0$

$$H^m(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

2.18 <u>Sobolev's Lemma</u>

If $u \in L^1(\mathbb{R}^n)$ then $\hat{u} \in C(\mathbb{R}^n)$. More generally if $(1 + |x|)^p \, u(x) \in L^1$ then $\hat{u}$ has continuous derivatives of order $\leq p$. By the Fourier inversion formula this provides a criterion for determining when a tempered distribution u is of the form Λ_f for $f \in C^p$.

2.18.1 <u>If</u> $u \in \mathcal{S}'(\mathbb{R}^n)$ <u>is such that</u> $(1 + |x|)^p \, \hat{u}(x) \in L^1$, <u>then there is an</u> $f \in C^p(\mathbb{R}^n)$ <u>such that</u> $u = \Lambda_f$. <u>Furthemore</u> if $|\alpha| \leq p$ <u>and</u> $x \in \mathbb{R}^n$ $|D^\alpha f(x)| \leq \int (1 + |x|)^p |u(x)| dx$.

<u>Proof.</u> The proof is immediate from the formula $u = \check{\hat{u}}$. □

In terms of the Hilbert spaces $H^p(\mathbb{R}^n)$ we can give a criterion as follows:

2.18.2 <u>Suppose</u> $u \in H^{m+p}(\mathbb{R}^n)$ <u>with</u> $m > n/2$. <u>Then</u> $u = \Lambda_f$ <u>for</u> $f \in C^p(\mathbb{R}^n)$. <u>In addition there is a</u> $C > 0$ <u>such that for</u> $|\alpha| \leq p$

$$|D^\alpha f(x)| \leq C\|u\|_{m+p} \text{ [uniformly in } x]$$

<u>so that</u> $H^{m+p}(\mathbb{R}^n)$ <u>imbeds continuously into</u> $C^p(\mathbb{R}^n)$.

<u>Proof</u>. We reduce this to the result 2.18.1. First consider the case $p = 0$.

If $u \in H^m(\mathbb{R}^n)$ and $v = \hat{u}$ we have by Cauchy Schwartz:

$$\int |v(x)|dx \leq$$

$$\leq \left\{\int_{|x|\leq c} |v(x)|^2 (1 + |x|^2)^m dx\right\}^{1/2} \left\{\int_{|x|\leq c} (1 + |x|^2)^{-m} dx\right\}^{1/2}$$

$$+ \left\{\int_{|x|>c} |v(x)|^2 (1 + |x|^2)^m dx\right\}^{1/2} \left\{\int_{|x|>c} (1 + |x|^2)^{-m} dx\right\}^{1/2}$$

The first summand is finite no matter what m is by the assumption that $u \in H^m(\mathbb{R}^n)$.

The second summand will be finite if

$$\infty > \int_{|x|>c} (|1 + |x|^2)^{-m} dx = \int_{r>c} (1 + r^2)^{-m} r^{n-1} dr d\Omega .$$

This quantity is $\leq$

$$\int_{r>c} r^{-2m+n-1} dr d\Omega = \int_{S^{n-1}} d\Omega \int_c^\infty r^{-2m+n-1} dr = Ar^{-2m+n}\Big]$$

This will be finite if $-2m+n < 0$, that is $m > n/2$.

Evidently both summands are $\leq C\|u\|_m$ for some constant C. By the previous result $|u(x)| \leq C\|u\|_m$

In the general case, note that if $u \in H^{m+p}(\mathbb{R}^n)$, then $w(x) = (1 + |x|^2)^{p/2}\,\hat{u}(x)$ is integrable (same proof as before) and an estimate $\int |w(x)|dx < C\|u\|_{m+p}$ holds. Thus by the previous result $u = \Lambda_f$ with $f \in C^p(\mathbb{R}^n)$ and with

$$|D^\alpha f(x)| \leq C_1\|u\|_{m+p}$$

for $|\alpha| \leq p$ $x \in \mathbb{R}^n$. □

2.19 Rellichs Lemma

2.19.1 Suppose $s > t > 0$, $\Omega \subseteq \mathbb{R}^n$ a Borel set of finite measure, and $K^s \subseteq H^s(\mathbb{R}^n)$ a closed subspace whose elements, as functions in $L^2(\Omega)$ vanish off Ω. Then the inclusion $H^s \to H^t(\mathbb{R}^n)$ is compact.

Proof. Notice $K^s \subseteq L^2(\Omega) \subseteq L^1(\Omega) \subseteq L^1(\mathbb{R}^n)$ where we identify a function which is zero off Ω with one on Ω. The inclusions are also all continuous. This follows either from the definitions and from the fact Ω is finite and Cauchy Schwartz. In particular $u \to \hat{u}(x)$ are continuous functions a $L^2(\Omega)$, for all $x \in \mathbb{R}^n$.

Let $\{u_i\}$ be a sequence in $[K^s]_1$. By weak compactness it has a subsequence $\{w_i\}$ which converges weakly in $[K^s]_1$ (and hence in $L^2(\Omega)$), to a $w \in K^s$. In particular $\hat{w}_i(x) \to \hat{w}(x)$ for all x. We have, in fact $w_i \to w$ in the norm of $H^t(\mathbb{R}^n)$. For ease of notation assume $w = 0$, as this involves no loss in generality. Then

$$\int |\hat{w}_i(x)|^2(1 + |x|^2)^t dx \le \int_{|x|\le c} |\hat{w}_i(x)|^2(1 + |x|^2)^t\, dx \; + $$
$$+ \int_{|x|>c} |\hat{w}_i(x)|^2\, (1 + |x|^2)^s\, (1 + |x|^2)^{t-s}\, dx$$

If χ is sufficiently large, ie. $> c(\epsilon)$ then $(1 + |x|^2)^{t-s} < \epsilon$. The second term is thus dominated by ϵ. $[\|w_i\|_s \le 1]$. Now $\hat{w}_i \to 0$ pointwise, so that by dominated convergence the first term $\to 0$ regard less of c. Thus for all $\epsilon > 0$

$$\lim\, \sup \|w_i\|_t^2 \le \epsilon$$

and thus $w_i \to 0$ in $H^t(\mathbb{R}^n)$. □

A Stronger Form of Rellich's Lemma

From the Sobolev inequalities it follows that for $m > n/2$ and $|\alpha| \le p$ there is a constant $C > 0$ such that $|D^{\alpha}u(x)| \le$ $\le C\|u\|_{m+p}$ for all $x \in \mathbb{R}^n$. The following is a strong form of Rellich's lemma:

2.19.2 <u>Suppose</u> $\Omega \subseteq \mathbb{R}^n$ <u>is Borel of finite measure</u>, $m > n/2$ $K \subseteq H^{p+m}(\mathbb{R}^n)$ <u>a closed subspace whose elements vanish of</u> Ω. <u>Then the inclusion</u> $K \hookrightarrow H^p(\mathbb{R}^n)$ <u>is a Hilbert Schmidt map</u>. <u>Here</u> $p \ge 0$.

<u>Proof</u>. To prove this, let $\{e_i\}$ be any orthonormal basis for the space K. By Sobolev's theorem there exists for each $x \in \Omega$ and $|\alpha| \leq p$ a vector $f_{\alpha,x} \in K$ such that $D^\alpha u(x) = \langle u, f_{\alpha,x}\rangle_{m+p}$. It is also clear that $\|f_{\alpha,x}\|_{m+p} \leq C$ for all $x \in \Omega$, $|\alpha| \leq p$. Thus if $x \in \Omega$, $|\alpha| \leq p$

$$\Sigma_i \ |D^\alpha \ e_i(x)|^2 = \Sigma_i \ |\langle e_i, f_{\alpha,x}\rangle|^2 = \|f_{\alpha,x}\|^2_{m+p} \leq C^2$$

By Lebesgue dominated convergence and the fact $\mu(\Omega) < \infty$

$$\infty > \int_\Omega \left\{\Sigma_{\alpha,i} |D^\alpha \ e_i(x)|^2\right\} dx = \Sigma_{\alpha,i} \int_\Omega |D^\alpha e_i(x)|^2 \ dx = $$

$$= \Sigma_i \ \|e_i\|^2_p. \quad \square$$

SECTION 3

ELLIPTIC EQUATIONS

3.0 The Dirichlet Problem

Intuitively a function belongs to $H_0^m(\Omega)$ iff all its derivatives of order $\le m-1$ vanish on $\partial\Omega$. We will justify this subsequently. Presently we show how this leads to a functional analytic treatment of the Dirichlet boundary value problem. This problem (roughly) consists in the following: Let L be an elliptic operator (for example $-\Delta$) of order 2m on an open set $\Omega \subseteq \mathbb{R}^n$. Let $f \in L^2(\Omega)$ and $\{g_\alpha\}$, $|\alpha| \le m-1$ be functions on $\partial\Omega$. Find $u \in C^{2m}(\Omega)$ so that $Lu = f$ and $D^\alpha u \mid \partial\Omega = g_\alpha$ [That is, $D^\alpha u$ extends continously to $\overline{\Omega}$ so that its restriction to $\partial\Omega$ is g_α].

Evidently, the functions g_α cannot be arbitrary, because relations between derivatives of $D^\alpha u$ will imply relations between the derivatives of g_α. Let us suppose there is a $g \in C^{2m}(\overline{\Omega})$ so that $D^\alpha g \mid \partial\Omega = g_\alpha$, for $|\alpha| \le m-1$. The original problem then becomes

$$Lu = f, \quad D^\alpha(u-g) = 0 \text{ on } \partial\Omega, \text{ for } |\alpha| \le m-1$$

Arguing heuristilly this becomes

$$Lu = f, \quad u-g \in H_0^m(\Omega)$$

Obviously if we put $w = u-g$ then the above problem becomes

$$Lw = f-Lg, \quad w \in H_0^m(\Omega) .$$

We will solve the Dirichlet problem in this way. The regularity theorems we will prove will show that the solution obtained in this way is in fact a "classical" solution to the Dirichlet problem.

3.1 The Spaces $H_o^m(\Omega)$

In the study of the Dirichlet problem via Hilbert space methods, the spaces $H_o^m(\Omega)$, $m \geq 0$ play a crucial role. Recall that $H_o^m(\Omega)$ is the closure of $C_o^\infty(\Omega)$ in $H^m(\Omega)$

3.1.1 $H_o^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)$. In fact for any $s \in \mathbb{R}$, $C_o^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof. $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. To see this, note $\mathcal{S}(\mathbb{R}^n)^\wedge = \mathcal{S}(\mathbb{R}^n) \supseteq C_o^\infty(\Omega)$ so $\mathcal{S}(\mathbb{R}^n)^\wedge$ is dense in $L^2((1+|\xi|^2)^s d\xi)$ and $\wedge$ is an isometry $H^s(\mathbb{R}^n) \to L^2((|1 + |\xi|^2)^s\, d\xi)$. Furthemore $C_o^\infty(\mathbb{R}^n)^- = \mathcal{S}(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^n)$ which is stronger than the H^s topology.

3.1.2 If $u \in H^m(\Omega)$ has compact support in Ω, then $u \in H_o^n(\Omega)$

Proof. Let $K = \text{supp } u$. Now u may be extended to a distribution $\tilde{u}$ on $\mathbb{R}^n$ so that $\tilde{u} \mid \mathbb{R}^n - K = 0$. [Let $\rho \in C_o^\infty(\mathbb{R}^n)$ be s.t. $\rho \mid K = 1$ and Supp $\subseteq \Omega$. Then $\tilde{u}(\phi) = u(\rho\phi)$ extends u and $\tilde{u}$ is 0 on $\mathbb{R}^n - K$]. Thus there is a sequence $\{\phi_i\}$ in $C_o^\infty(\mathbb{R}^n)$ s.t. $\phi_i \to \tilde{u}$ on $H^m(\mathbb{R}^n)$. Thus $\rho\phi_i \to \rho\tilde{u}$ on $H^m(\mathbb{R}^n)$, and so $\rho\phi_i|\Omega \to \rho\tilde{u} \mid \Omega = u$ in H^m. □

3.1.3 It is convenient to view $H_o^m(\Omega)$ as a subspace of $H^m(\mathbb{R}^n)$. In fact, <u>let</u> $U: C_o^\infty(\Omega) \to C_o^\infty(\mathbb{R}^n)$ <u>be the map s.t. for</u> $\phi \in C_o^\infty(\Omega)$ $U\phi \mid \Omega = \phi$, $U\phi \mid \mathbb{R}^n - \Omega = 0$. U <u>is an isometric map in the respective</u> H^m <u>norms. Therefore</u> U <u>extends in a unique way to an isometry</u> $V: H_o^m(\Omega) \to H^m(\mathbb{R}^n)$.

<u>Proof</u>. Note that $\langle u,\phi\rangle_o = \langle Vu, U\phi\rangle_o$ for $u \in H_o^m(\Omega)$, $\phi \in C_o^\infty(\Omega)$.

Thus $Vu \mid \Omega = u$: For this equality is satisfied for u, $\phi \in C_o^\infty(\Omega)$ and both sides are continuous in u with the topology of H_o^m.

3.1.4 One of the notable features of the space $H_o^m(\Omega)$ is the pairing

$$\langle\ ,\ \rangle_o: H^{-m} \times H_o^m \to \mathbb{C}$$

which extends the canonical pairing $\langle\ ,\ \rangle_o: H^{-m} \times C_o^\infty \to \mathbb{C}$. This exists in virtue of the definition of H^{-m} [distributions u s.t. $|\langle u,\phi\rangle_o| \le C\|\phi\|_m$ for $\phi \in C_o^\infty(\Omega)$] and the fact $C_o^\infty(\Omega)$ is dense in $H_o^m(\Omega)$. Obviously, if $u \in H^{-m}$, $v \in H_o^m$

$$|\langle u,v\rangle_o| \le \|u\|_{-m}\, \|v\|_m \quad .$$

We also have have the following consequence of Rellich's lemma (2.19.1 and 2.19.2) and 3.1.3.

3.1.5 <u>Suppose</u> $\Omega \subseteq \mathbb{R}^n$ <u>is an open set of finite measure</u>, $m_1 > m > 0$. <u>Then the inclusion</u> $H_o^{m_1}(\Omega) \to H_o^m(\Omega)$ <u>is a compact mapping</u>. <u>If furthermore</u> $m_1 - m > n/2$ <u>then the inclusion is a Hilbert Schmidt map</u>.

<u>Proof</u>. That $H_0^{m_1}(\Omega) \subseteq H_0^m(\Omega)$ is clear. To show the mapping is compact consider the commutative diagram

$$\begin{array}{ccc} H_0^{m_1}(\Omega) & \xrightarrow{V_1} & H^{m_1}(\mathbb{R}^n) \\ {\scriptstyle i(\Omega)}\downarrow & & \downarrow{\scriptstyle i(\mathbb{R}^n)} \\ H_0^m(\Omega) & \xrightarrow{V} & H^m(\mathbb{R}^n) \end{array}$$

V_1, V are isometries and $i(\mathbb{R}^n)$ is compact on the subspace $V_1(H_0^{m_1}(\Omega))$. Thus $i(\Omega)$ is compact. Similarly if $m_1-m > n/2$ we deduce $i(\Omega)$ is Hilbert Schmidt. □

3.2 <u>Interpretation of</u> $H_0^m(\Omega)$

3.2.1 <u>Let</u> $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$ $\{(x,x') \in \mathbb{R}^{n-1} \times \mathbb{R}: x' > 0\}$; <u>Suppose</u> u <u>is a function with continuous bounded derivatives of order</u> $\leq m$ <u>which vanishes outside a bounded set and such that if</u> $|\alpha| \leq m-1$ <u>and</u> $x \in \mathbb{R}^{n-1}$, $D^\alpha u(x,x') \to 0$ <u>as</u> $x' \to 0$. <u>Then</u> $u \in H_0^m(\Omega)$.

<u>Proof</u>. For $\varepsilon \geq 0$ let $u_\varepsilon(x,x') = u(x,x'-\varepsilon)$ if $x' > \varepsilon$ while $u_\varepsilon(x,x') = 0$ if $x' \leq \varepsilon$. u_ε has bounded continuous derivatives of order $\leq m-1$ and L^2 derivatives of order $= m$. In fact the distributional derivative $D^\alpha u_\varepsilon$ for $|\alpha| \leq m$ is the function w_α given by $w_\alpha(x,x') = D^\alpha u_\varepsilon(x,x')$ if $x' > \varepsilon$ and $w_\alpha(x,x') = 0$ if $x' \leq \varepsilon$. This follows from the definitions. Therefore $u_\varepsilon \in H^m(\Omega)$ for $\varepsilon \geq 0$. In addition if $\varepsilon > 0$, u_ε has compact support in Ω so $u_\varepsilon \in H_0^m(\Omega)$. Finally as $\varepsilon \to 0$,

$D^\alpha u_\varepsilon \to D^\alpha u$ pointwise; The family $\{D^\alpha u_\varepsilon\}$, $0 < \varepsilon < 1$ is uniformly bounded and vanishes outside a bounded set for $|\alpha| \le m$. Therefore by dominated convergence, $D^\alpha u_\varepsilon \to D^\alpha u$ in L^2 for $|\alpha| \le m$ and thus $u_\varepsilon \to u$ is $H^m(\Omega)$. In conclusion $u \in H_o^m(\Omega)$.

<u>Interpretation of $H_o^n(\Omega)$.(Cont'd).</u>

3.2.2 <u>Let</u> $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+ = \{(x,x') \subset \mathbb{R}^{n-1} \times \mathbb{R}: \; x' > 0\}$. <u>Then if</u> $u \in H_o^m(\Omega)$ <u>and</u> $|\alpha| \le m-1$ <u>are such that</u> $D^\alpha u$ <u>extends to a continuous function</u> w_a <u>on</u> $\overline{\Omega}$, <u>then</u> $w_\alpha \mid \partial\Omega = 0$.

We will first prove the following:

3.2.3 <u>Suppose</u> u <u>is a continuous</u> $\overline{\Omega}$ <u>such that as a distribution on</u> Ω, u <u>and</u> $D_{e_n} u \in L^2$ <u>and there is a sequence</u> $\phi_k \in C_o^\infty(\Omega)$ <u>such that</u> $\phi_k \to u$ <u>and</u> $D_{e_n}\phi_k \to D_{e_n} u$ in L^2. <u>Then</u> $u \mid \partial\Omega = 0$.

<u>Proof</u>. To prove this observe that for $\phi \in C_o^\infty(\Omega)$:

$$\phi(x,x') = \int_o^{x'} \partial_{e_n}\phi(x,s)ds$$

so that by the Cauchy Schwartz inequality:

$$|\phi(x,x')|^2 \le x' \int_o^{x'} |D_{e_n} \phi(x,s)|^2 ds$$

Therefore for any measurable set $E \subseteq \mathbb{R}^{n-1}$

$$\int_o^{x''} \int_E \left|\phi(x,x')\right|^2 dxdx' \le \int_o^{x''} \left[\int_E x' \int_o^{x'} |D_{e_n}\phi(x,s)|^2 \, ds \, dx\right] dx' \le$$

$$\le 1/2 \; x''^2 \int_o^{x''} \int_E |D_{e_n}\phi(x,s)|^2 \, dx \, ds \, .$$

Substituing $\phi = \phi_k$ and letting $k \to \infty$ we obtain

$$x''^{-1} \int_0^{x''} \int_E |u(x,x')|^2 \, dx dx' \le x''/2 \int_0^{x''} \int_E |D_{e_n} u(x,s)|^2 \, dx \, ds \quad .$$

Therefore if E is a bounded set so that

$$x' \longmapsto \int_E |u(x,x')|^2 \, dx \qquad x' \in \mathbb{R}^+ \cup \{0\}$$

is continuous, then letting $x'' \to 0$ in the above inequality we obtain

$$\int_E |u(x,0)|^2 \, dx = 0$$

So that $u(x,0) = 0$.

To prove the main proposition, observe that under the hypothesis $D^\alpha u$ and $D_{e_n} D^\alpha u \in L^2$; Furthermore there is a sequence $\phi_k \in C^\infty(\Omega)$ such that $D^\alpha \phi_k \to D^\alpha u$ and $D_{e_n} D^\alpha \phi_k \to D_{e_n} D^\alpha u$ in L^2. By the previous result therefore $w_\alpha \mid \partial\Omega = 0$.

The above proposition is true for more general open sets Ω. We will prove this for a class of open sets which have a smooth boundary.

3.3 Smooth Open Sets

We now deal with some technicalities concerning the boundary of an open set $\Omega \subseteq \mathbb{R}^n$.

3.3.1 An open set Ω has a smooth boundary (or is smooth) iff the following is satisfied for any $x_0 \in \partial\Omega$: There is a neighborhood V of x_0 in $\mathbb{R}^n$ and a diffeomorphism $T: V \cap \Omega \to B_1 = \{x \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : |x| < 1\}$ such that

1) The components $T_i (1 \le i \le n)$ of T and $(T^{-1})_i$ $(1 \le i \le n)$ of T^{-1} are functions in C_b^∞.

2) By the previous condition T is Lipschitzian and therefore uniformly continuous. From this follows T has a unique extension to a continuous map $T_1 : \overline{V \cap \Omega} \to \overline{B}_1$. We require T map $\overline{V} \cap \partial\Omega$ onto $\{x \in \overline{B}_1 : x_n = 0\}$.

3) This unique extension maps x_o onto 0.

The importance of condition 1) lies in the fact that T will the transport the spaces $H^r(V \cap \Omega)$, $H_o^r(V \cap \Omega)$ onto $H^r(B_1)$, $H_o^r(B_1)$ resp.

$H_o^m(\Omega)$ for smooth open sets

3.3.2 Let $\Omega \subseteq \mathbb{R}^n$ be an open set with a smooth boundary. Then if $u \in H_o^m(\Omega)$ and $|\alpha| \le m-1$ are such that $D^\alpha u$ extends to a continuous function w_α on $\overline{\Omega}$ then $w_\alpha \mid \partial\Omega = 0$.

Proof. Suppose $x_o \in \partial\Omega$. Let V be a neighborhood of x_o as in the previous definition, T a diffeomorphism $V \cap \Omega \to B_1$ satisfying 1)–3). Let $\rho \in C_o^\infty(\mathbb{R}^n)$ be s.t. Supp $\rho \subseteq V$ and ρ is identically 1 in a vicinity of x_o. Then the the distribution $u_1 = \rho|\Omega \cdot u \in H_o^m(\Omega)$ and Supp $u_1 \subseteq V \cap \Omega$. Evidently $u_1 | V \cap \Omega \in H_o^m(V \cap \Omega)$. Now T^* maps $H_o^m(V \cap \Omega)$ onto $H_o^m(B_1)$ so $T^*(u_1 | V \cap \Omega) \in H_o^m(B_1)$. Applying 3.3.1 (in the case Ω is a half space) one concludes the desired result. To see this explicitly, extend $T^*(u_1 | V \cap \Omega)$ to a distribution v on $\mathbb{R}^{n-1} \times \mathbb{R}^+$ in such a way that $v = 0$ outside the support of

$T^*(u_1 | V \cap \Omega)$. v is in $H_0^m(\mathbb{R}^{n-1} \times \mathbb{R}^+)$ and if $|\alpha| \leq m-1$ is such that $D^\alpha u$ extends continuously to $\overline{\Omega}$ the same is true of $D^\alpha v$. On the boundary this must be zero, and therefore on x_0 this extension is 0. Thus $D^\alpha u_1(x) \to 0$ as $x \to x_0$. But $D^\alpha u(x) = D^\alpha u_1(x)$ near x_0. This proves it.

One can prove a converse: If u vanishes along with its derivatives of order α, $|\alpha| \leq m-1$ on $\partial\Omega$ then $u \in H_0^m(\Omega)$. We will not need this so will leave it as an exercise.

3.4 Lax Milgram Lemma

3.4.1 Suppose H is a Hilbert space, B a bounded sesquilinear form on H. Then there is a unique bounded linear operator $T \in \mathcal{B}(H)$ such that $B(\xi,\eta) = \langle T\xi,\eta\rangle$. Furthermore if B is bounded away from 0 in the sense that there is a $C > 0$ such that $|B(\xi,\xi)| \geq C\|\xi\|^2$ for all $\xi \in H$ then T is invertible.

Proof. Given B, the existence of T is immediate from the Riesz representation theorem. If $|B(\xi,\xi)| \geq C\|\xi\|^2$ then $|\langle T\xi,\xi\rangle| \geq C\|\xi\|^2$, so obviously $\|T\xi\| \geq C\|\xi\|$ for all $\xi \in H$. From this it follows T is injective and the image of T is closed. To show T is surjective, suppose $\eta \perp \mathrm{Ran}\, T$. Then in particular $\langle T\eta,\eta\rangle = 0$ which implies $\|\eta\| = 0$. Therefore $\mathrm{Ran}\, T = H$. □

We will use this theorem in the following form:

3.4.2 <u>Suppose</u> L <u>is a differential operator of order</u> $2m$ <u>with coefficients in</u> $C_b^\infty(\Omega)$. <u>If there is a constant</u> $C > 0$ <u>such that for all</u> $\phi \in C_o^\infty(\Omega)$

$$|\langle L\phi,\phi\rangle_o| \geq C\|\phi\|_m \qquad (.1)$$

<u>Then there is a bijective bicontinuous linear map</u> $S\colon H_o^m(\Omega) \to$ $\to H_o^m(\Omega)$ <u>such that</u> $\langle Lu,\phi\rangle_o = \langle Su,\phi\rangle_m$ <u>for all</u> $u \in H_o^m(\Omega)$, $\phi \in C_o^\infty(\Omega)$:

<u>Proof</u>. To prove this note

$$B(u,v) = \langle Lu,v\rangle_o$$

is a bounded sesquilinear form $H_o^m(\Omega) \times H_o^m(\Omega) \to \mathbb{C}$. As $C_o^\infty(\Omega)$ is dense in $H_o^m(\Omega)$ (.1) extends to all $\phi \in H_o^m(\Omega)$. The result then follows by the abstract Lax Milgram Lemma [3.4.1] above.

It is evident the same theorem is true for compact C^∞ manifolds with a density.

3.4.3 These apparently trivial considerations are central to the Hilbert space approach to elliptic equations. In fact Gardings inequality states the following: If L is a strictly elliptic operator [we will define this subsequently] then there are constants C_1, C_2 such that

$$\mathrm{Re}\langle[L + C_1]\phi,\phi\rangle_o \geq C_2 \|\phi\|_m \qquad (.2)$$

for $\phi \in C_o^\infty(\Omega)$. In other words $L_1 = L + C_1$ satisfies (.1).

If L satisfies (.1) then L is a bijective map $H_o^m(\Omega) \to H^{-m}(\Omega)$. If $f \in H^{-m}(\Omega)$, let $Rf \in H_o^m(\Omega)$ be such that for all $\phi \in C_o^\infty(\Omega)$.

$$\langle f,\phi\rangle_o = \langle Rf,\phi\rangle_m$$

On the other hand there is an operator $S: H_o^m(\Omega) \to H_o^m(\Omega)$ such that $\langle Lu,\phi\rangle_o = \langle Su,\phi\rangle_m$, for al $\phi \in C_o^\infty$. S is bijective. Now $Lu = f$ iff for all $\phi \in C_o^\infty(\Omega)$

$$\langle Lu,\phi\rangle_o = \langle f,\phi\rangle_o$$

i.e. Iff $\langle Su,\phi\rangle_m = \langle Rf,\phi\rangle_m$ for all $\phi \in C_o^\infty(\Omega)$. As C_o^∞ is dense in H_o^m this is equivalent to $Su = Rf$ or $u = S^{-1} Rf$. whence L is bijective.

The solution of $Lu = f$ is a distribution solution: Even if $f \in C_b^\infty$ there is nothing in the above proof which implies u is even continuous. There are however strong regularity theorems: Our main objective is to prove that if $u \in H_o^m(\Omega)$ and $Lu \in H^p(\Omega)$ then $u \in H^{p+\deg(L)}(M)$

3.5 Approximation of Operators

The Fourier transform essentially diagonalizes partial differential operators with constant coefficient: If $P(D)$ is a constant coefficient operator then for $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\mathcal{F}^{-1}P(D)\,\mathcal{F}u = P(x)u$$

Now for many reasons constant coefficient operators are insufficient for any comprehensive theory. For example if the theory is to apply to operators on manifolds, then of necessity we must consider ones with variable coefficients as there is no coordinate invariant meaning one can attach to the adjective "constant coefficient". The problem that arises in dealing with general operators $P(x,D)$ via the Fourier transform is that multiplication operators get transformed into convolution operators which from our standpoint are unpleasant to handle.

One way out of this difficulty is to suitably approximate each variable coefficient operator L by an operator L_1 which almost has constant coefficients. This is made precise in the following.

3.5.1 *Let* L *be a differential operator of order* m *with* $C_b^\infty(\Omega)$ *coefficients,* $L = \sum_{|\alpha|\le m} a_\alpha D^\alpha$. *Suppose* $\{V_i\}_{1\le i\le n}$ *is an open cover of* Ω, $\{\omega_i\}$ *a* C_b^∞ *partition of unity such that Supp* $\omega_i \subseteq \overline{V}_i$ *and* $|a_\alpha(x) - a_\alpha(x')| \le M$ *for all* $|\alpha| = m$, *and* x, x' *in the same* V_i. *Let* $x_i \in V_i$ *be arbitrary and*

$$L_1 u = \Sigma_i \sum_{|\alpha|=m} a_\alpha(x_i)\, D^\alpha(\omega_i u)$$

Then if $0 \le p \le m$ *there is an operator* L_2 *of order* $\le m-1$ *with coefficients in* $C_b^\infty(\Omega)$ *such that for all* $u \in H^p$

$$\|\{L-L_1-L_2\}\|_{p-m} \le M\ C(m)\|u\|_p \qquad (.1)$$

Proof. To prove this we may write

$$L_1 u = \Sigma_i \{\sum_{|\alpha|=m} a_\alpha(x_i)\, D^{\alpha-\gamma(\alpha)}[\omega_i D^{\gamma(\alpha)} u] + L_3 u = (P + L_3)u$$

where $|\gamma(\alpha)| = p$ and $\alpha \ge \gamma(\alpha)$. Also order $(L_3) \le m-1$.

Similarly,

$$Lu = \Sigma_i \{ \sum_{|\alpha|=m} D^{\alpha-\gamma(\alpha)} [a_\alpha \ \omega_i \ D^\gamma u] + L_4 u = (Q + L_4)u$$

where order $(L_4) \le m-1$.

$$\|\{P-Q\}u\|_{p-m} = \|\Sigma_i \sum_{|\alpha|=m} D^{\alpha-\gamma(\alpha)}\{a_\alpha - a_\alpha(x_i)\} \ \omega_i \ D^{\gamma(\alpha)} u\|_{p-m} \le$$

$$\le \sum_{|\alpha|=m} \|\Sigma_i \{a_\alpha - a_\alpha(x_i)\} \ \omega_i \ D^{\gamma(\alpha)} \ u\|_o \le$$

$$\le \sum_{|\alpha|=m} M\|D^{\gamma(\alpha)} \ u\|_o \le M \ \ C(m)\|u\|_p$$

where $C(m)$ is the number of multindices α such that $|\alpha| = m$. Now $(L-L_1-(L_4-L_3)) = Q-P$ and $L_2 = L_4-L_3$ has C_b^∞ coefficients and order $\le m-1$. □

Notice that in the hypothesis of the preceding proposition we assume the partition of unity $\{\omega_i\}$ is in C_b^∞. This is an extra assumption as partitions of unity are not generally in C_p^∞. [They are bounded, obviously, but their derivatives may be unbounded].

3.5.2 Let $f \in C(\Omega)$. Then $\omega(f) = \inf\{r > 0$: There is a bounded open set $\Omega_1 \subseteq \Omega$ s.t. the oscillation of f on $\Omega-\Omega_1$ is $\le r\}$.

3.5.3 <u>Suppose</u> $f \in C_b^\infty(\Omega)$ <u>and</u> $\omega(f) < r$. <u>Then there is a finite open cover</u> $\{V_i\}$ <u>of</u> Ω <u>and a partition of unity</u> $\{\omega_i\}$ <u>in</u> C_b^∞ <u>such that</u> 1. <u>Oscillation of</u> f <u>on</u> $V_i \le r$; 2. Supp $\omega_i \subseteq V_i$.

<u>Proof</u>. Let $\Omega_1 \subseteq \Omega$ be as in the definition. f is uniformly continuous and so extends to a uniformly continuous function on $\bar{\Omega}$. There are open sets $\{W_i\}_{i\le m}$ in $\mathbb{R}^n$ such that

$UW_i \supseteq \overline{\Omega}$, and oscillation of f on $W_i \cap \overline{\Omega}$ is $\leq r$ [compactness of $\overline{\Omega}_1$] and $\rho_i \in C_o^\infty(\mathbb{R}^n)$ such that $\operatorname{Supp} \rho_i \subseteq W_i$, $\rho_i \geq 0$, $\Sigma \rho_i \leq 1$ on $\overline{\Omega}$ with equality on $\overline{\Omega}_1$. Evidently $\rho_i | \Omega \in C_b^\infty(\Omega)$. Furthermore $\eta = 1 - \Sigma \rho_i | \Omega \in C_b^\infty$, $\eta \geq 0$ and η is 0 on $\overline{\Omega}_1$. Therefore $\operatorname{Supp} \eta \subseteq (\Omega - \overline{\Omega}_1)^-$.

From this we deduce $\{\rho_i | \Omega\}_{i \leq m}$, η is a partition of 1 on Ω, $\operatorname{Supp} \rho_i | \Omega \subseteq W_i \cap \Omega$, $\operatorname{supp} \eta \subseteq (\Omega - \overline{\Omega}_1)^-$ and the oscillation of f on each of the open sets $W_i \cap \Omega$, $\Omega - \overline{\Omega}_1$ is $\leq r$. □

3.6 Order Structure for Operators

3.6.1 Let L be an operator of order m with coefficients in $C_b^\infty(\Omega)$ and such that

(1) $\omega(a_\alpha) < d$ for $|\alpha| = m$

(2) $\operatorname{Symb}(L)(x,\xi) \geq 0$ for all $(x,\xi) \in \Omega \times \mathbb{R}^n$.

Then given $r,s \geq 0$ such that $m = r+s$, there is an operator P of order $\leq m-1$ with C_b^∞ coefficients and such that for $u \in C_o^\infty(\Omega)$

$$\operatorname{Re}\langle (L-P)u, u\rangle_o \geq -d\, C(m)\, \|u\|_r\, \|u\|_s$$

Proof. Let $\{V_i\}$ be a cover of Ω on which the oscillation of all a_α for $|\alpha| = m$ is $\leq d$; $\{\omega_i\}$ a partition of unity in $C_b^\infty(\Omega)$ such that $\operatorname{Supp} \omega_i \subseteq \overline{V}_i$. We may arrange matters so that $\rho_i = \sqrt{\omega_i}$ is also in C_b^∞. Finally let L_1, L_2 be operators as in proposition 3.5.1. Thus

$$L_1 u = \sum_i \sum_\alpha a_\alpha(x_i)\, D^\alpha(\omega_i u) \quad [x_i \in V_i \quad \text{arbitrary}]$$

L_2 has C_b^∞ coefficients is of order $\le m-1$ and

$$|\langle (L-L_1-L_2)u,u\rangle_o| \le d\, C(m)\, \|u\|_r \|u\|_s$$

for all $u \in C_o^\infty(\Omega)$.

Now

$$L_1 u = \sum_i \sum_{|\alpha|=m} a_\alpha(x_i)\, \rho_i\, D^\alpha(\rho_i u) + L_5 u = (R + L_5)u$$

where order $(L_5) \le m-1$. Next

$$\langle Ru,u\rangle_o = \sum_i \langle \sum_{|\alpha|=m} a_\alpha(x_i)\, \rho_i\, D^\alpha(\rho_i u), u\rangle_o$$

$$= \sum_i \langle \sum_{|\alpha|=m} a_\alpha(x_i)\, D^\alpha(\rho_i u),\ \rho_i u\rangle_o$$

This quantity is ≥ 0. We prove each summand is ≥ 0. The simplest way of doing this is by the Fourier transform. Note first that $\rho_i u$ is of compact support in Ω, so we may view it as a function in $C_o^\infty(\mathbb{R}^n)$. Thus for each i

$$\langle \sum_{|\alpha|=m} a_\alpha(x_i)\, D^\alpha(\rho_i u), \rho_i u\rangle_o = \langle \sum_{|\alpha|=m} a_\alpha(x_i)\ [illegible]\ \widehat{(\rho_i u)}, \widehat{\rho_i u}\rangle_o =$$

$$= \langle \sum_{|\alpha|=m} a_\alpha(x_i)\, \xi^\alpha\, \widehat{\rho_i u},\ \widehat{\rho_i u}\rangle_o \ge 0 .$$

Here ξ^α is the operator multiplication by ξ^α. Now by 3.5.1 if $M = L - L_1 - L_2 = L - R - L_5 - L_2$, for all $u \in C_o^\infty(\Omega)$

$$|\langle Mu,u\rangle_o| \le d\, C(m)\, \|u\|_r\, \|u\|_s$$

Thus for all $u \in C_o^\infty(\Omega)$

$$\mathrm{Re}\langle (L - L_2 - L_5)u,u\rangle_o \geq \mathrm{Re}\langle Mu,u\rangle_o \geq - dC(m)\ \|u\|_r\ \|u\|_s\ .$$

Evidently L_2, L_5 have C_b^∞ coefficients, so taking $P = L_2 + L_5$ completes the proof. □

3.7 Gårdings Inequality

3.7.1 Let L be a differential operator of order $2m$ with coefficients in $C_b^\infty(\Omega)$. Suppose there is a constant $c > 0$ such that $\omega(a_\alpha) < c/C(2m)$ for all $|\alpha| = 2m$ and

$$\mathrm{Symb}\ L\ (x,\xi) \geq c|\xi|^{2m}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^n$. Then there are c_1, $c_2 > 0$ such that

$$\mathrm{Re}\langle Lu,u\rangle \geq c_1\|u\|_m^2 - c_2\|u\|_o^2$$

for all $u \in C_o^\infty(\Omega)$

Proof. Let S be the constant coefficient operator

$$S = c|D|^{2m} = c(\Sigma\ D_{e_i}^2\ )^m = c \sum_{|\alpha|=m} C_\alpha\ D^{2\alpha}$$

Since $\mathrm{Symb}(L-S)(x,\xi) = \mathrm{Symb}\ L(x,\xi)-c|\xi|^{2m} \geq 0$ by 3.6.1 there is an operator P of order $2m-1$ with C_b^∞ coefficients such that

$$\mathrm{Re}\langle (L-S-P)u,u\rangle_o \geq -d\|u\|_m^2$$

where $C(2m)\cdot\mathrm{Max}\ \omega(a_\alpha) < d < c$. Now if $u \in C_o^\infty(\Omega)$:

$$|\langle Pu,u\rangle_o| \leq C\|u\|_m\ \|u\|_{m-1}$$

$$|\langle Su,u\rangle_o| = c|\Sigma\langle C_\alpha\ D^{2\alpha}\ u,u\rangle_o| = c|\Sigma\ C_\alpha\langle D^\alpha u,D^\alpha u\rangle_o| \geq$$

$$\geq c|\sum_{|\alpha|=m} \|D^\alpha u\|_o^2| = c|\|u\|_m^2 - \|u\|_{m-1}^2|$$

Thus for all $u \in C_o^\infty(\Omega)$

$$\mathrm{Re}\langle Lu,u\rangle_o \geq -d\|u\|_m^2 - C\|u\|_m \|u\|_{m-1} + c|\|u\|_m^2 - \|u\|_{m-1}^2| .$$

We also have the following inequalities

$$\|u\|_m \|u\|_{m-1} \leq \varepsilon\|u\|_m^2 + \varepsilon^{-1}\|u\|_{m-1}^2$$

$$\|u\|_{m-1}^2 \leq \varepsilon\|u\|_m^2 + C(\varepsilon)\ \|u\|_o^2 \qquad (.1)$$

The latter inequality we prove below. Granting this we obtain

$$\begin{aligned}\mathrm{Re}\langle Lu,u\rangle_o &\geq -d\|u\|_m^2 - C(\varepsilon\|u\|_m^2 + \varepsilon^{-1}\|u\|_{m-1}^2) + c(\|u\|_m^2 - \|u\|_{m-1}^2) \geq \\ &\geq (-d-C\varepsilon+c)\|u\|_m^2 - (C\varepsilon^{-1} + c)\|u\|_{m-1}^2 \geq \\ &\geq (-d-C\varepsilon+c)\|u\|_m^2 - (C\varepsilon^{-1} + c)(\varepsilon_1\|u\|_m^2 + C(\varepsilon_1)\|u\|_o^2) \\ &\geq (-d-C\varepsilon+c - (C\varepsilon^{-1}+c)\varepsilon_1)\|u\|_m^2 - (C\varepsilon^{-1}+c)C(\varepsilon_1)\|u\|_o^2 .\end{aligned}$$

To complete the proof determine ε_1, ε, so the coefficient of $\|u\|_m^2$ is ≥ 0. □

3.7.2 Definition. An operator L of order $2m$ with $C_b^\infty(\Omega)$ coefficients such that there is a $c > 0$ such that for all $(x,\xi) \in \Omega \times \mathbb{R}^n$

$$\mathrm{Symb}\ L(x,\xi) \geq c|\xi|^{2m}$$

and $\omega(a_\alpha) = 0$ for all $|\alpha| = 2m$ is said to be strictly elliptic.

If this condition is satisfied locally then we say L has a positive symbol or is positive. Evidently this means $L(x,\xi)>0$ for all $(x,\xi) \in \Omega \times \mathbb{R}^n$.

It remains to prove the inequality (.1):

3.7.3 Lemma. Suppose $r > s$ so that $H^r(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n)$. Then for $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that for $u \in H^r(\mathbb{R}^n)$

$$\|u\|_s^2 \le \epsilon\|u\|_r^2 + C(\epsilon)\ \|u\|_o^2$$

Proof. Let $v = \hat{u}$. Then

$$\|u\|_s^2 = \int(1 + |\xi|^2)^s\ |v(\xi)|^2\ d\xi =$$

$$= \int_{|\xi|\le R}(1+|\xi|^2)^s\ |v(\xi)|^2 d\xi + \int_{|\xi|>R}(1+|\xi|^2)^s |v(\xi)|^2 d\xi$$

$$\le \int_{|\xi|\le R} |v(\xi)|^2 d\xi \cdot \max_{|\xi|\le R}(1+|\xi|^2)^s +$$

$$+ \int_{|\xi|>R}(1+|\xi|^2)^{s-r}\ |v(\xi)|^2(|1+|\xi|^2)^r d\xi$$

$$\le \|v\|_o^2\ (1+R^2)^s + \int|v(\xi)|^2(|1+|\xi|^2)^r d\xi \cdot (1+R^2)^{s-r}$$

$$= \|u\|_o^2(1+R^2)^s + \|u\|_r^2\ (1+R^2)^{s-r}\ .$$

As $(1+R^2)^{s-r} \to 0$ as $R \to \infty$ this proves the result. [And in fact gives an estimate for $C(\epsilon)$]. □

3.8 Elliptic Regularity

3.8.1 Let L be a strictly elliptic operator of order $2m$ on $\mathbb{R}^n$. Then if $p \in \mathbb{Z}$ there are $d_1, d_2 > 0$ such that for all $u \in H^p(\mathbb{R}^n)$

$$d_1\|u\|_p \le \|(L + \lambda)u\|_{p-2m} \qquad (.1)$$

if $\lambda \ge d_2$.

Proof. We apply Gårdings inequality to the strictly elliptic operators $L(1-\Delta)^s$, $(1-\Delta)^s L$ of order $2m+2s$. Thus there are constants c_1, c_2 (depending on s) such that for all $u \in H^{s+m}(\mathbb{R}^n)$

$$\mathrm{Re}\langle(1-\Delta)^s\, Lu,u\rangle_o \geq c_1\|u\|^2_{s+m} - c_2\|u\|^2_o$$

$$\mathrm{Re}\langle L(1-\Delta)^s\, u,u\rangle_o \geq c_1\|u\|^2_{s+m} - c_2\|u\|^2_o$$

(1) Suppose $p-m = s \geq 0$. Then if $\lambda \geq 0$

$$\|(L+\lambda)u\|_{s-m}\ \|u\|_{s+m} = \|(1-\Delta)^s(L+\lambda)u\|_{-s-m}\ \|u\|_{s+m} \geq$$

$$\geq \mathrm{Re}\langle(1-\Delta)^s\ (L+\lambda)u,u\rangle_o \geq c_1\|u\|^2_{s+m} - c_2\|u\|^2_o + \lambda\langle(1-\Delta)^s\ u,u\rangle_o \geq$$

$$\geq c_1\|u\|^2_{s+m} + (\lambda-c_2)\|u\|^2_o$$

This will be $\geq c_1\|u\|^2_{s+m}$ whenever $\lambda \geq c_2$. We thus obtain

$$\|(L+\lambda)u\|_{s-m} \geq c_1\|u\|_{s+m}$$

whenever $\lambda \geq c_2$, 0. Note $s+m = p$, $s-m = p-2m$

(2) Suppose $m-p = s \geq 0$. Then if $\lambda \geq 0$

$$\|(L+\lambda)u\|_{-s-m}\ \|u\|_{-s+m} = \|(L+\lambda)u\|_{-s-m}\ \|(1-\Delta)^{-s}\ u\|_{s+m} \geq$$

$$\geq \mathrm{Re}\langle(L+\lambda)u,\ (1-\Delta)^{-s}\ u\rangle_o$$

$$= \mathrm{Re}\langle L(1-\Delta)^s(1-\Delta)^{-s}\ u + \lambda u,\ (1-\Delta)^{-s}\ u\rangle_o$$

$$\geq c_1\|(1-\Delta)^{-s}u\|^2_{s+m} - c_2\|(1-\Delta)^{-s}\ u\|^2_o + \lambda\langle u,(1-\Delta)^{-s}u\rangle_o$$

$$\geq c_1\|u\|^2_{-s+m} - c_2\|u\|^2_{-2s} + \lambda\|u\|^2_{-s} \geq c_1\|u\|^2_{-s+m}$$

whenever $\lambda \geq c_2$. We thus obtain whenever $\lambda \geq 0, c_2$

$$\|(L+\lambda)u\|_{-s-m} \geq c_1\|u\|_{-s+m}$$

Note $-s+m = p$, $-s-m = p-2m$. □

As a corollary, under the same assumptions on L:

3.8.2 <u>For</u> $p \in \mathbb{Z}$, <u>there is a</u> $c(p) \geq 0$ <u>such that</u> $(L+\lambda)$ <u>is a bijective map</u> $H^p(\mathbb{R}^n) \to H^{p-2m}(\mathbb{R}^n)$ <u>whenever</u> $\lambda \geq c(p)$.

<u>Proof</u>. If $\lambda \geq d_2(p)$ then (.1) implies $L+\lambda$ is injective and has a closed image in $H^{p-2m}(\mathbb{R}^n)$. One must thus show that for λ large $(L+\lambda)\, H^p(\mathbb{R}^n)$ is dense in $H^{p-2m}(\mathbb{R}^n)$. If not there is a $v \neq 0$, $v \in H^{2m-p}(\mathbb{R}^n)$ such that for all $u \in H^p(\mathbb{R}^n)$:

$$\langle (L+\lambda)u, v\rangle_0 = 0$$

Thus $\langle u, (L^0+\lambda)v\rangle_0 = 0$ for all $u \in H^p(\mathbb{R}^n)$, where L^0 is the formal adjoint of L. This implies $(L^0+\lambda)v = 0$. However L^0 is strictly elliptic, and thus for λ large $L^0 + \lambda$ is injective. Thus for such λ, $v = 0$. For such λ, therefore $L + \lambda$ is bijective. □

3.8.3 <u>Let</u> L <u>be a strictly elliptic operator of order</u> $2m$ <u>on</u> $\mathbb{R}^n$. <u>If</u> $u \in H^{-\infty}(\mathbb{R}^n)$ <u>and</u> $Lu \in H^p(\mathbb{R}^n)$ <u>with</u> $p \in \mathbb{Z}$, <u>the</u> $u \in H^{p+2m}(\mathbb{R}^n)$.

<u>Proof</u>. For each $q \in \mathbb{Z}$, let $c(q)$ be such that $L+\lambda$ is a bijection $H^q(\mathbb{R}^n) \to H^{q-2m}(\mathbb{R}^n)$ for $\lambda \geq c(q)$. Now let $u, f = Lu \in H^s(\mathbb{R}^n)$. Suppose $\lambda \geq \max\,(c(s),\ c(s+2m))$. Then as $L+\lambda$ is surjective $H^{s+2m} \to H^s$, there is a $w \in H^{s+2m}(\mathbb{R}^n)$ such

that

$$(L+\lambda)w = f + \lambda u = (L+\lambda)u \ .$$

On the other hand $w,u \in H^s(\mathbb{R}^n)$ and $L+\lambda$ is injective there. Thus $w = u$ and $u \in H^{s+2m}(\mathbb{R}^n)$. If $p \geq s+2m$ so that $f \in H^p \subseteq H^{s+2m}(\mathbb{R}^n)$ we may repeat the argument. Eventually we arive at $u \in H^r(\mathbb{R}^n)$ with $r \geq p$. Then $u \in H^p(\mathbb{R}^n)$ so repeating the argument once more it follows $u \in H^{p+2m}(\mathbb{R}^n)$.

3.9 Regularity of Solutions in the Interior

The most convenient way of dealing with interior regularity of solutions from our viewpoint is by the distribution spaces $H^p_{loc}(\Omega)$: $u \in H^p_{loc}(\Omega)$ iff any $x \in \Omega$ has a neighborhood V such that $u \mid V$ is the restriction of a $\tilde{u} \in H^p(\mathbb{R}^n)$

3.9.1 Suppose L *is a differential operator of order* m *with* $C^\infty(\Omega)$ *coefficients and with a positive symbol. Then if* $u \in C_o^\infty(\Omega)'$ *is such that* $Lu = f \in H^p_{loc}(\Omega)$ *then* $u \in H^{p+m}_{loc}(\Omega)$.

Proof. Let $x \in \Omega$, $K_2 \subseteq \overset{o}{K}_1 \subseteq K_1 \subseteq \overset{o}{K}_o \subseteq K_o \subseteq \Omega$ compact neighborhoods of x, so that $f \mid \overset{o}{K}_o = g \mid \overset{o}{K}_o$ with $g \in H^p(\mathbb{R}^n)$ and $\rho, \phi \in C_o^\infty(\mathbb{R}^n)$ such that $\rho | K_1 = 1$, $\operatorname{Supp} \rho \subseteq K_o$ and ϕ is 1 on some neighborhood of x while $\operatorname{Supp} \phi \subseteq K_2$. Let L_1 be the constant coefficient operator obtained from L by evaluating the coefficients of L at x. Now $(1-\rho)\, L_1 + \rho L$ is an operator whose coefficients are constant outside a compact set: Furthemore if K_o is sufficiently small it is strictly elliptic.

Now

$$(1-\rho)\ L_1\phi w + \rho L\phi w = \rho\phi Lw + \rho Pw = \phi Lw + \rho Pw$$

where order $P \leq m-1$. Thus as distributions on $\mathbb{R}^n$

$$(1-\rho)\ L_1\phi u + \rho L\phi u = \phi f + \rho Pu\ .$$

We prove that if $u \in H^r_{loc}(\Omega)$ for $r \leq p+m-1$ then $\phi u \in H^{r+1}(\mathbb{R}^n)$: Evidently $\rho Pu \in H^{r-m+1}(\mathbb{R}^n)$; On the other hand $\phi f \in H^p(\mathbb{R}^n)$ by hypothesis. Therefore $\phi f + \rho Pu \in H^{r-m+1}(\mathbb{R}^n)$. By the global regularity theorem of the previous section it follows that

$$\phi u \in H^{r-m+1+m}(\mathbb{R}^n) = H^{r+1}(\mathbb{R}^n)\ .$$

As ϕ is 1 near x, and x is arbitrary $u \in H^{r+1}_{loc}(\Omega)$. From this it follows by induction that $u \in H^{p+m}_{loc}(\Omega)$.

If $u \in C^\infty_o(\Omega)'$, then for any relatively compact $\Omega' \subseteq \Omega$, $u \mid \Omega' \in H^r_{loc}(\Omega')$ for some r. To prove this we need the easy half of the Paley Wiener theorem: If v is a distribution of compact support on $\mathbb{R}^n$, then $\mathfrak{F}v$ is a function of at most polynomial growth. In particular any distribution v of compact support belongs to some Sobolev space $H^s(\mathbb{R}^n)$. Thus if $f \in C^\infty_o(\Omega)$ then $fu \mid \Omega'$ is obviously the restriction of a distribution of compact support on $\mathbb{R}^n$. Therefore by relative compactness of Ω', $u \mid \Omega' \in H^r_{loc}(\Omega')$ for some r.

Applying the result for the case $u \in H^{-\infty}(\Omega')$, we obtain $u \in H^{p+m}_{loc}(\Omega')$. As $\Omega' \subseteq \Omega$ is an arbitrary relatively compact set $u \in H^{p+m}_{loc}(\Omega)$.

3.10 Elliptic Equations on Compact Manifolds

At this point we pause in our proofs of the regularity theorems for elliptic equations. In fact for compact manifolds without boundary the theorem we have already proved is sufficient to prove a global regularity theorem. We will also show that symmetric elliptic operators on a compact manifold with a density (and of course without a boundary) have an orthonormal basis of C^∞ eigenvectors.

Probably the most significant example of an elliptic operator is the haplace Beltrami operator on a Riemannian manifold. The ordinary Laplacian Δ is an example of this. We begin this section by defining the Laplace Beltrami operator Δ for an arbitrary Riemannian manifold M. For generalities on Riemannian geometry we refer to [4]

3.11 The Laplace Beltrami Operator

We consider a Riemannian manifold M. Then M has an associated invariant connexion ∇. If X is a vector field, then the general covariant derivative of X is the tensor

$$\Delta X(\omega, Y) = \omega(\nabla_Y X)$$

(This obviously is $C^\infty(M)$ linear in ω and Y so is in fact a tensor). The contraction of this tensor is a function called the divergence of X, div X. In local coordinates $(x_1,\ldots,x_n)$ the contraction of ΔX with $X = \Sigma X_i e_i$ is calculated as follows:

$$\begin{aligned} \operatorname{div} X &= \Sigma_i \, dx_i \, \nabla_{e_i} X = \Sigma \, dx_i \, \nabla_{e_i}(\Sigma X_j \, e_j) = \\ &= \Sigma_{i,j} \, dx_i(\partial_{e_i} X_j \, e_j + X_j \, \nabla_{e_i} e_j) = \\ &= \Sigma_i \, \partial_{e_i} X_i + \sum_{i,j,k} dx_i \, X_j \, \Gamma^k_{ji} \, e_k \\ &= \Sigma_i \, \partial_{e_i} X_i + \sum_{i,j} X_j \, \Gamma^i_{ji} \end{aligned}$$

The <u>Laplace Beltrami operator</u> (on functions) is defined by

$$\Delta f = \operatorname{div} \operatorname{grad} f$$

In local coordinates grad f is the vector field $\Sigma X_i e_i$ with the $X_i = \Sigma_j (g^{-1})_{ij} \, \partial_{e_j} f$. Therefore the principal part of Δ is the operator $\sum_{i,j} (g^{-1})_{ij} \, \partial_{e_i} \partial_{e_j}$.

From this follows that $-\Delta$ <u>is an elliptic operator</u>. For we have just shown that on any coordinate chart its symbol is the function $S(x,\xi) = \Sigma \; (g^{-1})_{ij} \, \xi_i \, \xi_j$.

3.12 <u>Elliptic Equations on Compact Manifolds</u>

3.12.1 <u>Let</u> M <u>be a compact</u> C^∞ <u>manifold with density</u>, L <u>an elliptic differential operator of order</u> 2m <u>on</u> M. <u>Then there are constants</u> $c_1, c_2 > 0$ <u>such that</u>

$$\operatorname{Re}\langle Lu,u\rangle_o \geq c_1 \, \|u\|_m^2 - c_2 \|u\|_o^2 \qquad (.1)$$

<u>for all</u> $u \in H^m(M)$.

Proof. As $C^\infty(M)$ is dense in $H^m(M)$ it suffices to prove this for $u \in C^\infty$. Let $\{V_i\}_{1\le i\le N}$ be a finite atlas for M, $\{\rho_i\}_{1\le i\le N}$ a partition of unity such that Supp $\rho_j \subseteq V_j$. Assume also $\omega_i = \sqrt{\rho_i} \in C^\infty$. Now $P_i = L\omega_i - \omega_i L$ is an operator of order $2m-1$ with coefficients supported in V_i. If $u \in C^\infty(M)$, then

$$\langle Lu,u\rangle_o = \sum_{i=1}^{N} \langle \omega_i^2 Lu,u\rangle_o =$$

$$= \sum_{i=1}^{N} \langle L\omega_i,\ \omega_i u\rangle_o - \sum_{i=1}^{N} \langle P_i u,\ \omega_i u\rangle_o \tag{.2}$$

By Gårdings inequality for open sets in $\mathbb{R}^n$ we have that there exist constants c_1, c_2 such that

$$\mathrm{Re} \sum_{i=1}^{N} \langle L\omega_i u,\ \omega_i u\rangle_o \ge \sum_{i=1}^{N} [c_1\|\omega_i u\|_m^2 - c_2\|\omega_i u\|_o^2]$$

On the other hand the $\|\ \|_p$ nom on $H^p(M)$ for all p is equivalent to the norm $[\sum_{i=1}^{N}\|\omega_i u\|_p^2]^{1/2}$. Thus we have constants c_3, c_4 such that

$$\mathrm{Re} \sum_{i=1}^{N} \langle L\omega_i u, \omega_i u\rangle_o \ge c_3\|u\|_m^2 - c_4\|u\|_o^2$$

On the other hand the term $|\sum_{i=1}^{N} \langle P_i u,\ \omega_i u\rangle_o|$ may be estimated as follows

$$|\sum_{i=1}^{N} \langle P_i u,\ \omega_i u\rangle_o| \le c_5 \sum_{i=1}^{N} \|P_i u\|_{-m}\ \|\omega_i u\|_m \le$$

$$\le c_6 \sum_{i=1}^{N} \|u\|_{m-1}^2 \quad \|\omega_i u\|_m \le c_6 \sum_{i=1}^{N} [\epsilon^{-1}\|u\|_{m-1}^2 + \epsilon\|\omega_i\ u\|_m^2] \le$$

$$\le c_7[\epsilon^{-1}\|u\|_{m-1}^2 + \epsilon\|u\|_m^2] .$$

Thus by equation (.2)

$$\mathrm{Re}\langle Lu,u\rangle_o \geq c_3\|u\|_m^2 - c_4\|u\|_o^2 - c_7[\epsilon^{-1}\|u\|_{m-1}^2 + \epsilon\|u\|_m^2]$$

We complete the proof of (.1) exactly as in the proof of Gårding's inequality 3.6.1.

3.12.2 By Gardings inequality for manifolds, it follows that it L is an elliptic operator on the compact manifold M with density $L+c$ is a bijective map $H^m(M) \to H^{-m}(M)$ for $c \geq c_o$. To prove this apply the Lax-Milgram Lemma to the sesquilinear form $(u,v) \longmapsto \langle Lu,v\rangle_o$ defined on $H^m(M) \times H^m(M)$.

We now consider in detail the inverse $T_c = (L+c)^{-1}$, $c \geq c_o$. Now if $v \in H^r(M)$ and $(L+c)u = v$ elliptic regularity implies $u \in H^{2m+r}_{loc}(M)$; As M is compact however, $H^p_{loc}(M) = H^p(M)$ for any value of p. In particular:

3.12.3 <u>If</u> $c \geq c_o$, <u>and</u> $r \geq -m$ <u>then</u> T_c <u>maps</u> $H^r(M)$ <u>onto</u> $H^{r+2m}(M)$.

3.12.4 Now it is natural to consider the mapping $T_c|H^o(M)$ as an operator on $H^o(M)$. It is evdently a bounded operator, and in fact some power of it is Hilbert Schmidt. To prove this note that for $s > \dim M/2$ the inclusion map $H^s(M) \to H^o(M)$ is Hilbert Schmidt. Now $(T_c)^p$ maps maps $H^o(M)$ into $H^{2mp}(M)$ so that considered as a map $H^o(M) \to H^o(M)$, $(T_c)^p$ <u>will be Hilbert Schmidt if</u> $2mp > \dim M/2$.

3.13 Eigenfunction Expansions

Suppose M is a compact C^∞ manifold with density and L an elliptic differential operator which is formally self adjoint. We have seen that for c large, $L' = (L+c) \mid H^{2m}(M)$ has an inverse $T: H^o(M) \to H^{2m}(M)$.

3.13.1 T <u>is a compact self adjoint operator on</u> $H^o(M)$.

<u>Proof</u>. First we show that $\langle L'u,v\rangle_o = \langle u,L'v\rangle_o$ for all $u,v \in H^{2m}(M)$. Clearly the two sides of the equation are will defined and are continuous $H^{2m} \times H^{2m} \to \mathbb{C}$. The equation holds for $u,v \in C^\infty(M)$ by hypothesis and thus extends by continuity. If $u_1,v_1 \in H^o(M)$ therefore

$$\langle Tu_1,v_1\rangle_o = \langle Tu_1, L'Tv_1\rangle_o = \langle L'Tu_1, Tv_1\rangle_o = \langle u_1, Tv_1\rangle_o \quad \square$$

By the structure theory for compact self adjoint operators there is an orthonormal basis $\{e_i\}$ of $L^2(M)$ and real numbers λ_i such that $Te_i = \lambda_i e_i$. The numbers $\{\lambda_i\}$ form a set with at most one cluster 0. Thus $e_i = \lambda_i L'e_i$ or $e_i = \lambda_i(L+c)e_i$, so that $\lambda_i Le_i = (1-c\lambda_i)e_i$. Now $\lambda_i \neq 0$, for otherwise $0 = e_i$. Hence

$$Le_i = \lambda_i^{-1}(1-c\lambda_i)e_i$$

Thus L is diagonalized by $\{e_i\}$ with corresponding eigenvalues $\lambda_i^{-1} - c$. This sequence is discrete and converges to $\pm\infty$. In fact we show it must converge to $+\infty$. To see this note

$$\langle Tu,u\rangle_o = \langle Tu, L'Tu\rangle_o \geq c_1\|Tu\|_m^2 > 0$$

for all $u \in H^o(M)$ [as L' is symmetric the quantity $\langle Tu, L'Tu\rangle_o$ is automatically real so Gårding's inequality applies with no Re in front].This implies T is a non-negative operator and so its eigenvalues λ_i are ≥ 0.

3.13.2 By elliptic regularity one can show <u>all the distribution eigenvectors of</u> L <u>are actually in</u> $C^\infty(M)$. To see this suppose $u \in C^\infty(M)'$ is such that $Lu = \lambda u$. As M is compact $u \in H^r_{loc}(M)$ for some $r \in \mathbb{R}$. By elliptic regularity it follows immediately $u \in H^{r+2mk}_{loc}(M)$ for all $k \in \mathbb{N}$ and thus $u \in C^\infty(M)$ as asserted.

In particular any eigenvector of L is one of T, and the map $\lambda \longmapsto \lambda^{-1}-c$ is a bijection between eigenvalues of T and those of L.

3.13.3 As T^r is Hilbert-Schmidt for $r > \dim M/4m$, it follows that if $\{\lambda_i\}$ is the sequence of eigenvalues of T, counted with multiplicity,

$$\Sigma\, \lambda_i^{2r} < \infty$$

Now $\lambda_i = (\mu_i + c)^{-1}$ where μ_i is the eigenvalue sequence of L; Furthermore $\mu_i \geq c$ for all but finitely many i, so that for such i $2^{-1}\mu_i^{-1} \leq (\mu_i + c)^{-1}$. It follows therefore that

$$\sum_{\mu_i \neq 0} \mu_i^{-2r} < \infty$$

We have proven this only for integral r, but it can be shown the same result holds true for any real r.

3.14 Flat Tori

The simplest examples of compact Riemannian manifolds are flat tori: Let $\Gamma \subseteq \mathbb{R}^n$ be a lattice, that is a discrete subgroup such that $\mathbb{R}^n/\Gamma$ is compact. The space $\mathbb{R}^n/\Gamma$ with the quotient metric is called a flat torus.

It is not difficult to prove that tori $\mathbb{R}^n/\Gamma$, $\mathbb{R}^n/\Gamma'$ are isometric iff there is an isometry $V: \mathbb{R}^n \to \mathbb{R}^n$ which carries Γ onto Γ'.

3.14.1 The map $\pi_*: C^\infty(\mathbb{R}^n/\Gamma) \longrightarrow C^\infty(\mathbb{R}^n)$, which is composition with the canonical map $\mathbb{R}^n \to \mathbb{R}^n/\Gamma$ commutes with the Laplacians i.e. $\Delta\pi_* f = \pi_*\Delta' f$ where Δ, Δ' denote the Laplacions on $\mathbb{R}^n$, $\mathbb{R}^n/\Gamma$ resp. From this it is immediate that the eigenvectors of Δ' are quotientes of eigenvector of Δ.

3.14.2 <u>The eigenvectors of</u> Δ' <u>are the quotients</u> $\tilde{f}_x$ <u>of the functions</u> $f_x = \exp 2\pi i\langle x,y\rangle$ <u>with</u> $x \in \Gamma^* = \{y \in \mathbb{R}^n . \langle y,w\rangle \in \mathbb{Z}$ <u>for all</u> $w \in \Gamma\}$. <u>All these functions are linearly independent</u>.

<u>Proof</u>. Observe that $\Delta f_x = -4\pi|x|^2 . f_x$. Now f_x passes to the quotient iff $\exp 2\pi i\langle x,y\rangle = \exp 2\pi i\langle x,y'\rangle$ for $y-y' \in \Gamma$ i.e. $\exp 2\pi i\langle x,y\rangle = 0$ for all $y \in \Gamma$. Equivalently, $\langle x,y\rangle \in \mathbb{Z}$ for all $y \in \Gamma$. This means exactly that $x \in \Gamma^*$.

Next the functions $\{\tilde{f}_x\}$, $x \in \Gamma^*$ are linearly independent. This is a trivial consequence of the fact that the functions $\{f_x\}$, $x \in \mathbb{R}^n$ are linearly independent.

To show that the $\tilde{f}_x$ are all the eigenvectors of Δ' it is enough to show $\{\tilde{f}_x\}$ is total in $L^2(\mathbb{R}^n/\Gamma)$. This will follow if the $\{\tilde{f}_x\}$ are closed under multiplication, involution and separate points. In fact by the Stone Weierstrauss theorem the $\{\tilde{f}_x\}$ will be total in $C(\mathbb{R}^n/\Gamma)$. However $\tilde{f}_{x+y} = \tilde{f}_x \tilde{f}_y$, $\tilde{f}_{-x}=\overline{\tilde{f}_x}$ and if $\tilde{f}_x(z+\Gamma) = \tilde{f}_x(z'+\Gamma)$ for all $x \in \Gamma^*$, then $\exp 2\pi i\langle(z-z'),x\rangle = 0$ for all $x \in \Gamma^*$. Therefore $z-z' \in \Gamma^{**} =$ $= \Gamma$. □

As a Corollary:

3.14.3 <u>The eigenvalues of</u> Δ' <u>are the numbers</u> $4\pi^2|x|^2$ <u>with</u> $x \in \Gamma^*$.

3.15 <u>The Fredholm Alternative</u>

3.15.1 <u>Let</u> Ω <u>be an open set in</u> $\mathbb{R}^n$ <u>of finite Lebesgue measure or a compact</u> C^∞ <u>manifold with density, L a strictly elliptic operator on</u> Ω [<u>or simply elliptic in the case of a manifold] and</u> L^o <u>its formal adjoint. Let</u> $2m =$ order L. If $f \in H^{-m}(\Omega)$ <u>then</u> $Lu = f$ <u>has a solution</u> $u \in H_o^m(\Omega)$ iff. $f \in [\ker L^o]^\perp$, <u>that is iff</u>.

$$\langle f,u\rangle_o = 0 \text{ for all } u \in H_o^m \text{ s.t. } L^o u = 0 \qquad (.1)$$

<u>Furthermore</u>

$$\text{Codim Ran } L = \dim\ker L = \text{Codim Ran } L^o = \dim\ker L^o < \infty \qquad (.2)$$

<u>Codim is codimension in</u> $H^{-m}(\Omega)$. <u>If these numbers are zero then</u>

<u>the solution of</u> $Lu = f$ <u>is unique in</u> $H_o^m(\Omega)$.

<u>Proof</u>. By Gårding's inequality and the Lax-Milgram Lemma applied to L and L^o there is a c such that $L+c$, L^o+c are invertible operators $H_o^m(\Omega) \to H^{-m}(\Omega)$. Their inverses are bounded operators $S, S_1: H^{-m}(\Omega) \to H_o^m(\Omega)$.

Next note the equation $Lu = f$ for $u \in H_o^m(\Omega)$, $f \in H^{-m}(\Omega)$ is equivalent to the equation $[1-cS]u = Sf$ with $f,u \in H^{-m}(\Omega)$. To see this, note that $Lu = f$, $u \in H_o^m(\Omega)$, $f \in H^{-m}(\Omega)$ is equivalent to $(L+c)u = f+cu$ (with $u \in H_o^m(\Omega)$, $f \in H^{-m}(\Omega)$) iff $u = S(f+cu)$ (with $u,f \in H^{-m}(\Omega)$) iff $[1-cS]u = Sf$ with $f,u \in H^{-m}(\Omega)$.

Similarly $L^o u = f$ with $u \in H_o^m(\Omega)$, $f \in H^{-m}(\Omega)$ is equivalent to $[1-cS_1]u = S_1 f$ with $f \in H^{-m}(\Omega)$, $u \in H_o^m(\Omega)$.

Now let us consider S as an operator $H^{-m} \to H^{-m}$. S is compact for it is the composition of the operator $S \cdot H^{-m} \to H_o^m$ with the inclusion $H_o^m(\Omega) \to H^{-m}(\Omega)$ which is compact.

On the other hand let us view S_1 as an operator $H_o^m \to H_o^m$. It thus is the composition of the inclusion $H_o^m(\Omega) \to H^{-m}(\Omega)$ with the map $S_1: H^{-m}(\Omega) \to H_o^m(\Omega)$. S_1 is also compact.

The relevant thing about S, S_1 is that $S = S_1^*$ if we identity H^{-m} with the dual of H_o^m via the pairing $\langle\ ,\ \rangle_o$. To see this, suppose $g \in H^{-m}$, $v \in H_o^m$. Then

$$\langle g, S_1 v\rangle_o = \langle (L+c)\, Sg,\ S_1 v\rangle_o = \langle Sg,\ (L^o+c) S_1 v\rangle_o = \langle Sg, v\rangle_o$$

We now apply the Riesz theory of compact operators. To begin with $Lu = f$ has a solution $u \in H_o^m(\Omega)$ iff. $Sf \in \mathrm{Ran}(1-cS)$. This is equivalent to $Sf \in [\ker(1-cS_1)]^{\perp}$ i.e. $\langle Sf,v\rangle_o = 0$ for all $v \in H_o^m(\Omega)$ s.t. $(1-cS_1)v = 0$. This is equivalent to $\langle f,S_1v\rangle_o = 0$ for all $v \in H_o^m(\Omega)$ s.t. $(1-cS_1)v = 0$. This means $\langle f,v\rangle_o = 0$ for all $v \in H_o^m(\Omega)$ s.t. $L^o v = 0$. This proves (.1). To prove (.2), notice (.1) implies

$$\mathrm{Codim\ Ran}\ L = \dim\ker L^o .$$

On the other hand the Riesz theory gives

$$\dim\ker(1-cS_1) = \mathrm{Codim\ Ran}(1-cS_1) \quad [\text{in } H_o^m(\Omega)]$$

and this implies $\dim\ker L^o = \mathrm{Codim\ Ran}\ L^o$.

3.16 The m-extension property

It is intuitively obvious that for an open set Ω the m-extension property is a property of the boundary. In fact:

3.16.1 <u>Suppose</u> $\Omega \subseteq \mathbb{R}^n$ <u>has a compact boundary. If every</u> $x \in \partial\Omega$ <u>has a neighborhood</u> U <u>in</u> $\mathbb{R}^n$ <u>such that</u> $U \cap \Omega$ <u>has the m-extension property, then</u> Ω <u>has this property</u>.

<u>Proof</u>. Let V_{x_i} be an open neighborhood of $x_i \in \mathbb{R}^n (i=1,\ldots,M)$ such that $V_{x_i} \cap \Omega$ has the m-extension property and $\bigcup_{i=1}^{M} V_{x_i} \supseteq \partial\Omega$. Let $\rho_i \in C_o^\infty(\mathbb{R}^n)$, $i = 1,\ldots,M$ be such that $\mathrm{Supp}\,\rho_i \subseteq V_{x_i}$, $0 \le \rho_i \le 1$ and $\Sigma\, \rho_i = 1$ in a compact neighborhood neighborhood of $\partial\Omega$ in Ω. Thus

$$1 = (1-\Sigma\,\rho_i) + \Sigma\,\rho_i = \phi + \Sigma\,\rho_i$$

with supp $\phi \cap W = \phi$, where W is a neighborhood of $\partial\Omega$. If $u \in H^m(\Omega)$, then $\rho_i u \in H^m(\Omega)$, $\phi u \in H^m(\Omega)$. By the m-extension property for $\Omega \cap V_i$, there is a distribution $v_i \in H^m(\Omega)$ which extends $u \mid \Omega \cap V_i$. Thus $\rho_i u = \rho_i v_i \mid \Omega$; Next ϕu can be extended to a distribution w on $\mathbb{R}^n$ by setting it equal to 0 on $\mathbb{R}^n - \Omega$; $w \in H^m(\mathbb{R}^n)$. Therefore

$$u = \phi u + \Sigma\rho_i u = w|\Omega + \Sigma\rho_i v_i| \ \Omega = (w + \Sigma\rho_i v_i)|\Omega. \quad \square$$

For open sets Ω with $\partial\Omega$ sufficiently regular the m-extension property is a consequence of the m-extension property for a half-space. We now consider this; The proof is highly technical and our argument is a modification of one due to Lions [cf [5]].

3.17 The m-extension Property for a Half-space

We now investigate the m-extension property for the half space $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$. In the following discussion $s \in \mathbb{N}$ is fixed as is a sequence $\alpha_1, \ldots, \alpha_s$ of positive real numbers.

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$, let $\mathcal{L}(\lambda)$ be the operator

$$[\mathcal{L}(\lambda)\phi](x', x_n) = -\sum_{k=1}^{s} \lambda_k \phi(x', -\alpha_k x_n) + \phi(x', x_n)$$

for $(x', x_n) \in \Omega$. We consider $\mathcal{L}(\lambda)$ as a mapping $C_0^\infty(\mathbb{R}^n) \to C^\infty(\Omega)$. We have the following basic proposition:

3.17.1 1. <u>For any</u> $p \geq 0$, $\mathcal{L}(\lambda)$ <u>maps</u> $C_0^\infty(\mathbb{R}^n)$ <u>into</u> $H^p(\Omega)$. <u>Furthermore if</u> $C_0^\infty(\mathbb{R}^n)$ <u>has the topology induced from</u> $H^p(\mathbb{R}^n)$, $\mathcal{L}(\lambda)$ <u>is continuous</u>.

2. <u>If in addition the equations</u>

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^r = 1 \qquad 0 \le r \le p-1 \tag{.1}$$

<u>hold, then</u> $\mathcal{L}(\lambda)$ <u>maps</u> $C_o^\infty(\mathbb{R}^n)$ <u>into</u> $H_o^p(\Omega)$.

3. <u>If</u> λ_m <u>is the sequence</u> $\{\lambda_k(-\alpha_k)^{-m}\}_{1 \le k \le s}$ <u>we have</u>

$$\mathcal{L}(\lambda)\, D^{(\beta',\beta_n)} = D^{(\beta',\beta_n)}\, \mathcal{L}(\lambda_{\beta_n})$$

<u>Proof</u>. The first part of the proposition is easy to see as $\mathcal{L}$ is the composition of the restriction map $\phi \to \phi|\Omega$ with one that is obviously continuous. To verify the second statement, observe that if (.1) is satisfied then for $\beta_n \le p-1$

$$D_{e_n}^{\beta_n} \mathcal{L}(\lambda)\ \phi(x',0) = -\Sigma\, \lambda_k(-\alpha_k)^{\beta_n} D^{\beta_n} \phi(x',0) + D_{e_n}^{\beta_n} \phi(x',0) =$$

$$= -\Sigma\ \lambda_k(-\alpha_k)^{\beta_n} + 1]\ D_{e_n}^{\beta_n} \phi(x',0) = 0$$

so $\mathcal{L}(\lambda)\phi \in H_o^p(\Omega)$, by [3.2.1].

The final assertion is a straightforward computation:

Note that if $\beta_n = 0$ then

$$\mathcal{L}(\lambda)\ D^{(\beta',0)} = D^{(\beta',0)}\ \mathcal{L}(\lambda)$$

Thus consider

$$[\mathcal{L}(\lambda)\ D_{e_n}^m\ \phi](x',x_n) = -\sum_{k=1}^{s} \lambda_k [D_{e_n}^m \phi](x',-\alpha_k x_n) + D_{e_n}^m\ \phi(x',x_n) =$$

$$= -\sum_{k=1}^{s} \lambda_k(-\alpha_k)^{-m}\ D_{e_n}^m [\phi(x',-\alpha_k x_n)] + D_{e_n}^m\ \phi(x',x_n) =$$

$$= D_{e_n}^m [-\Sigma\, \lambda_k(-\alpha_k)^{-m}\ \phi(x',-\alpha_k x_n)] + \phi(x',x_n)] = [D_{e_n}^m \mathcal{L}(\lambda_m)\phi](x',x_n)$$

This proves the proposition.

We can now define the extension operators $\epsilon(\lambda)$. In fact if λ satisfies (.1) then $\mathcal{L}(\lambda): H^p(\mathbb{R}^n) \to H^p_o(\Omega)$ is continuous; We define $\epsilon(\lambda): H^{-p}(\Omega) \to H^{-p}(\mathbb{R}^n)$ to be the adjoint map. Thus $\langle \epsilon(\lambda)u, \phi\rangle = \langle u, \mathcal{L}(\lambda)\phi\rangle$. We have

$$|\langle \epsilon(\lambda)u, \phi\rangle| = |\langle u, \mathcal{L}(\lambda)\phi\rangle| \leq \|u\|_m \; \|\mathcal{L}(\lambda)\phi\|_m$$

$$\leq C\|u\|_{-m} \; \|\phi\|_m$$

$0 \leq m \leq p$ from which follows $\epsilon(\lambda)u \in H^{-m}$ if $u \in H^{-m}$ and $\epsilon(\lambda): H^{-m}(\Omega) \to H^{-m}(\mathbb{R}^n)$ is continuous.

We regard $\epsilon(\lambda)$ as a mapping defined on $H^{-p}(\Omega)$ for the largest value of p satisfying (.1). Obviously $\text{dom}\ \epsilon(\lambda) \supseteq H^o(\Omega)$ for any value of λ. $\epsilon(\lambda)$ are in fact extension operators: If $u \in \text{dom}\ \epsilon(\lambda)$, then $\epsilon(\lambda)u|\Omega = u$. To see this note that if $\text{Supp}\ \phi \subseteq \Omega$, then $\mathcal{L}(\lambda)\phi = \phi|\Omega$. For such ϕ therefore, $\langle \epsilon(\lambda)u, \phi\rangle = \langle u, \phi|\Omega\rangle$ proving the result.

Next we see how $\epsilon(\lambda)$ intertwines with derivatives.

Suppose $u \in H^o(\Omega) \subseteq \text{dom}\ \epsilon(\lambda)$, and λ_{β_n} satisfies (.1) for $0 \leq r \leq |\beta'| + \beta_n - 1$. Then

$$\epsilon(\lambda_{\beta_n})\, D^{(\beta',\beta_n)} u = D^{(\beta',\beta_n)}\, \epsilon(\lambda)u \qquad (.2)$$

Notice $\text{dom}\ \epsilon(\lambda_{\beta_n}) \supseteq H^{-[|\beta|+\beta_n']}(\Omega)$ so that as $u \in H^o$, $D^{(\beta',\beta_n)}u \in \text{dom}\ \epsilon(\lambda_{\beta_n})$. Next suppose $\phi \in C^\infty(\mathbb{R}^n)$. Then:

$$\langle \epsilon(\lambda_{\beta_n})\, D^{(\beta',\beta_n)} u,\phi\rangle = \langle D^{(\beta',\beta_n)} u, \mathcal{L}(\lambda_{\beta_n})\phi\rangle$$

$$= \langle u, D^{(\beta',\beta_n)} \mathcal{L}(\lambda_{\beta_n})\phi\rangle \qquad (.3)$$

[This is the case because $u \in H^o(\Omega)$ and $\mathcal{L}(\lambda_{\beta_n})\phi \in H^{|\beta'|+\beta_n}(\Omega)$].

Now (.3) equals:

$$\langle u, \mathcal{L}(\lambda)\, D^{(\beta',\beta_n)} \phi\rangle = \langle \epsilon(\lambda)u, D^{(\beta',\beta_n)} \phi\rangle =$$

$$= \langle D^{(\beta',\beta_n)} \epsilon(\lambda)u,\phi\rangle$$

This proves (.2)

Notice that the assumption λ_{β_n} satisfy (.1) for $0 \le r < |\beta'| + \beta_n$ means exactly

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^{-\beta_n} (-\alpha_k)^r = 1 \quad 0 \le r < |\beta'| + \beta_n$$

In other words

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^{r-\beta_n} = 1 \qquad 0 \le r < |\beta'| + \beta_n \qquad (.4)$$

3.17.2 <u>Suppose</u> $\lambda = (\lambda_1,\ldots,\lambda_s)$ <u>satisfies</u>

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^q = 1, \quad -p \le q < p, \quad p > 0 \qquad (.5)$$

<u>Then</u> $\epsilon(\lambda)$ <u>maps</u> $H^m(\Omega)$ <u>continuously into</u> $H^m(\mathbb{R}^n)$ <u>for</u> $-p \le m \le$ $\le p$.

<u>Proof</u>. This has already been established for $0 \ge m \ge -p$.

Now if $|\beta'|+\beta_n \le m \le p$, then we claim λ_{β_n} satisfies condition

(.4) of paragraph 3.17.1. To see this, note $r-\beta_n$ satisfies $-p \le -\beta_n \le r-\beta_n < |\beta'| < p$, whenever $0 \le r < |\beta'| + \beta_n$. Thus we may deduce that for all $u \in H^o(\Omega)$

$$\epsilon(\lambda_{\beta_n})\, D^{(\beta',\beta_n)} u = D^{(\beta',\beta_n)}\, \epsilon(\lambda) u$$

In particular, if $u \in H^m(\Omega)$ then $D^{(\beta',\beta_n)} u \in H^o(\Omega)$ and so $\epsilon(\lambda_{\beta_n})\, D^{(\beta',\beta_n)} u \in H^o(\mathbb{R}^n)$. As this is true of all $|(\beta',\beta_n)| \le m$ $\epsilon(\lambda)u \in H^m(\mathbb{R}^n)$. Furthermore

$$\|D^{(\beta',\beta_n)}\, \epsilon(\lambda)u\|_o \le C\|D^{(\beta',\beta_n)} u\|_o \le C\|u\|_m$$

so $u \to \epsilon(\lambda)u$ is continuous $H^m(\Omega) \to H^m(\mathbb{R}^n)$, for $m \le p$. □

3.17.3 It is a corollary of the proof that under the above assumptions if $u \in H^o(\Omega)$ and $|(\beta',\beta_n)| \le p$ then

$$\epsilon(\lambda_{\beta_n})\, D^{(\beta',\beta_n)} u = D^{(\beta',\beta_n)} \epsilon(\lambda) u \,.$$

It remains to show that there exist $s, \{\alpha_k\}_{1 \le k \le s}$ and $\{\lambda_k\} = \lambda$ satisfying the above equalities. However, let $s = 2p$. It is well known [and easy to check] the determinant

$$\det \begin{bmatrix} 1 & (-1) & (-1)^{s-1} \\ 1 & (-2) & (-2)^{s-1} \\ 1 & (-s) & (-s)^{s-1} \end{bmatrix} \ne 0$$

Letting $\alpha_k = k$ we have that (.5) above has a solution (λ_k).

3.17.4 As a corollary, Ω <u>has the m-extension property, for all</u> m.

The m-extension property for a half space has many important consequences, among them being an extension of the Sobolev inequalities to open sets with a compact and smooth boundary.

3.17.5 <u>Let</u> $\Omega \subseteq \mathbb{R}^n$ <u>be an open set with a compact smooth boundary</u>. <u>Suppose</u> $u \in H^{m+p}(\Omega)$ <u>with</u> $m > n/2$. <u>Then</u> $u = \Lambda_f$ <u>for</u> $f \in C^p(\bar{\Omega})$. <u>In addition there is a</u> $C > 0$ s.t. <u>for all</u> $|\alpha| \leq p$

$$|D^\alpha f(x)| \leq C\|u\|_{m+p} \text{ [uniformly in } x].$$

The proof is a straightforward application of partitions of unity, the invariance of the spaces involved under C_b^∞ diffeomorphisms and the m-extension property.

3.18 Regularity at the Boundary

We apply the results of the preceding section to prove the following technical result: Still $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$.

3.18.1 <u>Suppose</u> $u \in H^0(\Omega)$, $D_{e_i} u \in H^r(\Omega)$ <u>for</u> $1 \leq i \leq n-1$ <u>and</u> $D_{e_n}^m u \in H^{r-m+1}(\Omega)$. <u>Then</u> $u \in H^{r+1}(\Omega)$.

<u>Proof</u>. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be such that

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^q = 1 , \quad -(m+r+1) \leq q < (m+r+1)$$

$$\sum_{k=1}^{s} \lambda_k(-\alpha_k)^{-m} (-\alpha_k)^q = 1 \quad -(p+m) \leq q \leq p+m$$

where $p = |r-m+1|$. Then $\varepsilon(\lambda)$ maps $H^q(\Omega)$ continuously into $H^q(\mathbb{R}^n)$, q s.t. $-r \le q \le r$, while $\varepsilon(\lambda_m)$ maps $H^{r-m+1}(\Omega)$ continuously into $H^{r-m+1}(\mathbb{R}^n)$.

By assumption and 3.17.3

$$D_{e_i}\ \varepsilon(\lambda)u = \varepsilon(\lambda)\ D_{e_i}\ u \qquad 1 \le i \le n-1$$

and
$$D^m_{e_n}\ \varepsilon(\lambda)u = \varepsilon(\lambda_m)\ D^m_{e_n}\ u$$

so $D^m_{e_n}\ \varepsilon(\lambda)u \in H^{r-m+1}(\mathbb{R}^n)$, $D_{e_i}\ \varepsilon(\lambda)u \in H^r(\Omega)$. If we manage to show $\varepsilon(\lambda)u \in H^{r+1}(\Omega)$ then as $u = \varepsilon(\lambda)u|\Omega \in H^{r+1}(\Omega)$ the result will follow.

This is a general result which we state as a lemma.

3.18.2 Suppose $w \in H^{-\infty}(\mathbb{R}^n)$ is such that $D_{e_i} w \in H^r(\mathbb{R}^n)$ $1 \le i \le n-1$ and $D^m_{e_i}\ w \in H^{r-m+1}(\mathbb{R}^n)$. Then $w \in H^{r+1}(\mathbb{R}^n)$.

Proof. Applying the Fourier transform, we reduce this to showing: If $v \in L^2_{loc}(\mathbb{R}^n)$ satisfies the conditions

$$\int |x_i\ v(x)|^2\ (1+|x|^2)^r\ dx < \infty \qquad 1 \le i \le n-1$$

$$\int |x^m_n\ v(x)|^2\ (1+|x|^2)^{r-m+1}\ dx < \infty$$

Then

$$\int |v(x)|^2\ (1+|x|^2)^{r+1}\ dx < \infty\ .$$

Proof. To prove this we state the following inequality: For some $C > 0$

$$|x_n|^2 \le C(|x_n|^{2m}(1+|x|^2)^{-m+1} + \sum_{i=1}^{n-1} |x_i|^2) .$$

Thus

$$\int |v(x)|^2 |x|^2 (1+|x|^2)^r dx < \infty$$

and therefore:

$$\int |v(x)|^2 (1+|x|^2)^{r+1} dx \le \int_{|x|\ge 1} |v(x)|^2 (2|x|^2)(1+|x|^2)^r dx +$$

$$+ \int_{|x|\le 1} 2^{r+1} |v(x)|^2 dx < \infty .$$

3.18.3 <u>Suppose</u> Ω <u>is the half space</u> $\{(x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}: x_n > 0\}$. <u>Let</u> L <u>be a differential operator of order</u> m <u>with coefficients in</u> $C_b^\infty(\Omega)$ <u>and such that the coefficient a of</u> $D_{e_n}^m$ <u>satisfies</u> $\inf\{|a(x)|: x \in \Omega\} > 0$. <u>Then for any integer</u> $p \ge 0$, $u \in H^p(\Omega)$, $D_{e_i} u \in H^p(\Omega)$, $1 \le i \le n-1$ <u>and</u> $Lu \in H^{p-m+1}(\Omega)$ <u>imply</u> $u \in H^{p+1}(\Omega)$. $u \in H^{p+1}(\Omega)$.

<u>Proof</u>. $L = aD_{e_n}^m + L_1 + L_2$ where

$$L_1 = \Sigma\{|\alpha| = m, \ \alpha_n < m: a_\alpha D^\alpha\}$$

$$L_2 = \Sigma\{|\alpha| < m: a_\alpha D^\alpha\}$$

To show $u \in H^{p+1}(\Omega)$ it suffices to show $D_{e_n}^m u \in H^{p-m+1}(\Omega)$ and as $D_{e_n}^m = a^{-1}(L-L_1-L_2)$ this will follow from $L_1u, L_2u \in H^{p-m+1}(\Omega)$. Now each term of L_1 is of the form $a_\alpha D^{\alpha'} D_{e_i}$ with $1 \le i \le n-1$, $|\alpha'| \le m-1$. Thus as $D_{e_i} u \in H^p(\Omega)$

for $1 \le i \le n-1$ evidently $L_1 u \in H^{p-m+1}(\Omega)$. On the other hand $\deg L_2 \le m-1$ and $u \in H^p(\Omega)$ by hypothesis so $L_2 u \in H^{p-m+1}(\Omega)$. This proves the assertion. □

Note that if L is a strictly elliptic operator then it satisfies the above condition on a: For by strict ellipticity there is a $c > 0$. s.t.

$$a(x)\, \xi_n^{2m_1} \ge c\xi_n^{2m_1} \quad [2m_1 = \text{order } L]$$

for all $x \in \Omega$.

3.18.4 <u>Suppose</u> L <u>is a differential operator of order</u> $m = 2m_1$ <u>with coefficients in</u> $C_b^\infty(\Omega)$ <u>such that Gårding's inequality is valid and</u> $\inf\{|a(x)|: x \in \Omega\} > 0$. [<u>a is as before the coefficient of</u> $D_{e_n}^m$]. If $u \in H^p(\Omega) \cap H_o^{m_1}(\Omega)$ <u>and</u> $Lu \in H^{p-m+1}(\Omega)$ <u>then</u> $u \in H^{p+1}(\Omega) \cap H_o^{m_1}(\Omega)$.

<u>Proof</u>. To begin with the result is vacuous unless $p \ge m_1$. We then prove this by induction starting with $p = m_1$. By the previous proposition we have to show $D_{e_i} u \in H^p(\Omega)$ $1 \le i \le n-1$ [By hypothesis $Lu \in H^{p-m+1}(\Omega)$ and $u \in H^p(\Omega)$].

We require the following lemma which we prove in the next section.

3.18.5 Suppose $u \in H_o^{m_1}(\Omega)$ and $Lu \in H^{-m_1+1}(\Omega)$. Then (Assuming L is a differential operator as above) $D_{e_i} u \in H_o^{m_1}(\Omega)$ for all $1 \le i \le n-1$.

It is easily seen this is stronger than the induction starter corresponding to $p = m_1$. Granting 3.18.5 it remains only to prove the induction step. Thus assume the statement true for $p-1 \geq m_1$. Now $LD_{e_i} u = D_{e_i} Lu + L_1 u$ with order $L_1 \leq m-1$. As $u \in H^p(\Omega)$ by hypothesis, it follows $L_1 u \in$ $\in H^{p-m+1}(\Omega)$ and as $Lu \in H^{p-m+1}(\Omega)$, $D_{e_i} Lu \in H^{p-m}(\Omega)$. Thus $LD_{e_i} u \in H^{p-m}(\Omega)$. By the above lemma $D_{e_i} u \in H_o^{m_1}(\Omega)$ for $1 \leq i \leq n-1$ while $D_{e_i} u \in H^{p-1}(\Omega)$. Apllying the induction hypothesis to $D_{e_i} u$ we conclude $D_{e_i} u \in H^p(\Omega)$ for $1 \leq i \leq n-1$. But this is what we saw was necessary to establish $u \in H^{p+1}(\Omega)$.

As a corollary:

3.18.6 Let L be a differential operator of order $2m_1$ as above. If $u \in H_o^{m_1}(\Omega)$ and $Lu \in H^p(\Omega)$ then $u \in H_o^{m_1} \cap H^{p+2m_1}$.

It remains only to prove the lemma. To do so requires some results on difference quotients which we now develop.

3.19 Difference Quotients

Let $\Omega = \mathbb{R}^{n-1} \times \overset{+}{\mathbb{R}} \subseteq \mathbb{R}$. If $x \in \mathbb{R}^{n-1} \times \{0\}$ then define the operator $R_x\colon C_o^\infty(\Omega) \to C_o^\infty(\Omega)$ by $(R_x\phi)(y) = \phi(y-x)$. R_x has an extension $C_o^\infty(\Omega)' \to C_o^\infty(\Omega)'$ given by

$$\langle R_x u, \phi\rangle = \langle u, R_{-x}\phi\rangle$$

3.19.1 <u>If</u> $1 \le j \le n-1$ <u>and</u> $h \in \mathbb{R}$ <u>define the operator</u> Δ_{h,e_j} <u>as follows</u>:

$$\Delta_{h,e_j} u = (ih)^{-1} \{R_{he_j} u - u\}$$

<u>Suppose</u> $u, v \in L^2(\Omega)$ <u>and</u> $\Delta_{h,e_j} u \to v$ <u>as</u> $h \to 0$ <u>in the weak topology of</u> $L^2(\Omega)$. <u>Then</u> $v = D_{e_j} u$.

<u>Proof</u>. Assume $\Delta_{h,e_j} u \to v$ weakly. In particular for all $\phi \in C_o^\infty(\Omega)$

$$\langle \Delta_{h,e_j} u, \phi \rangle_o = (ih)^{-1} \langle R_{he_j} u - u), \phi \rangle_o$$

$$= (ih)^{-1} \langle u, (R_{-he_j} \phi - \phi) \rangle_o \to \langle v, \phi \rangle_o$$

as $h \to 0$. On the other hand as $h \to 0$

$$-(ih)^{-1} (R_{-he_j} \phi - \phi) \to D_{e_j} \phi$$

in the sense of $C_o^\infty(\Omega)$. It follows that as $h \to 0$.

$$-\langle u, (ih)^{-1} (R_{-he_j} \phi - \phi) \rangle_o \to \langle u, De_j \, \phi \rangle$$

so that $\langle D_{e_j} u, \phi \rangle = \langle u, De_j \, \phi \rangle = \langle v, \phi \rangle$ for all $\phi \in C_o^\infty(\Omega)$.

We will also require the following inequality

3.19.2 <u>If</u> $p \in \mathbb{Z}^+$ <u>and</u> $u \in H^{-p}(\Omega)$ <u>then</u>

$$\|\Delta_{h,e_j} u\|_{-p-1} \le \|u\|_{-p} \tag{.1}$$

Proof. To see this note

$$|\langle \Delta_{h,e_j} u, \phi \rangle_o| = |\langle u, -\Delta_{-h,e_j}\phi\rangle_o| \le \|u\|_{-p} \|\Delta_{-h,e_j}\phi\|_p$$

Now to estimate $\|\Delta_{-h,e_j}\phi\|_p$ we may use the Fourier transform

$$\|\Delta_{-h,e_j}\phi\|_p^2 = \int |(e^{ih\langle e_j,\theta\rangle} - 1)h^{-1}\,\hat{\phi}(\theta)|^2(1+|\theta|^2)^p d\theta \le$$

$$\le \int_{\mathbb{R}^n} |\theta_j \hat{\phi}(\theta)|^2(1+|\theta|^2)^p d\theta \le \int_{\mathbb{R}^n} |\hat{\phi}(\theta)|^2(1+|\theta|^2)^{p+1} d\theta = \|\phi\|_{p+1}^2$$

Thus $|\langle \Delta_{h,e_j} u, \phi\rangle_o| \le \|u\|_{-p}\,\|\phi\|_{p+1}$ which proves (.1). □

3.19.3 We now prove the Lemma 3.18.5 conserving the notations used there.

We suppose only that L satisfies Gårdings inequality and that it has $C_b^\infty(\Omega)$ coefficients. If $1 \le i \le n-1$ and $h \in \mathbb{R}$, then

$$L\Delta_{h,e_i} = \Delta_{h,e_i} L + L_{i,h} \qquad (.2)$$

$L_{i,h}$ is an operator of order m. Furthermore the coefficients $a_{\alpha,i,h}$ of $L_{i,h}$ have the following property: For each r the derivatives of order r of the $a_{\alpha,i,h}$ are uniformly bounded in h.

From this follows that there is a $C > 0$ independent of i and h such that for all $v \in H^{m_1}(\Omega)$

$$\|L_{i,h} v\|_{-m_1} \le C\|v\|_m \,.$$

Thus by (.2) and 3.19.2

$$\|L\Delta_{h,e_i}u\|_{-m_1} \le \|\Delta_{h,e_i}Lu\|_{-m_1} + \|L_{i,h}u\|_{-m_1} \le$$

$$\le \|Lu\|_{-m_1+1} + C\|u\|_{m_1}$$

so $\|L\Delta_{h,e_i}u\|_{-m_1} \le C_1$ uniformly in $h \in \mathbb{R}$. In particular

$$|\langle L\Delta_{h,e_i}u,\ \Delta_{h,e_i}u\rangle_o| \le C_1\|\Delta_{h,e_i}u\|_{m_1}$$

[which makes sense as $\Delta_{h,e_i}u \in H_o^{m_1}(\Omega)$]. By Gårding's inequality

$$|\langle L\Delta_{h,e_i}u,\ \Delta_{h,e_i}u\rangle_o| \ge c_1\|\Delta_{h,e_i}u\|_{m_1}^2 - c_2\|\Delta_{h,e_i}u\|_o^2$$

for appropriate constants $c_1, c_2 > 0$. Hence for all $h \in \mathbb{R}$.

$$c_1\|\Delta_{h,e_i}u\|_{m_1}^2 \le c_2\|\Delta_{h,e_i}u\|_o^2 + C_1\|\Delta_{h;e_i}\|_{m_1} .$$

i.e. $c_1\|\Delta_{h,e_i}u\|_{m_1} \le C_1 + c_2\|\Delta_{h,e_i}u\|_o \le C_1 + c_2\|u\|_1$.

[That $\|\Delta_{h,e_i}u\|_o \le \|u\|_1$ follows also from the proof of 3.19.2]. In particular the family $\{\Delta_{h,e_i}u\colon h \in \mathbb{R}\}$ is norm bounded in $H_o^{m_1}(\Omega)$. By weak compactness there is a $h_k \to 0$ such that $\Delta_{h_k,e_i}u$ converges weakly in $H_o^{m_1}(\Omega)$ to some $v \in H_o^{m_1}(\Omega)$. As the inclusion $H_o^{m_1}(\Omega) \to L^2(\Omega)$ is continuous it follows

$$\Delta_{h_k,e_i}u \to v$$

weakly in $L^2(\Delta)$ and by 3.19.1 $v = D_{e_i}u \in H_o^{m_1}(\Omega)$.

3.20 Global Regularity

3.20.1 Suppose $\Omega \subseteq \mathbb{R}^n$ is a smooth open set with compact boundary, and L is a strictly elliptic operator of order $2m$ on Ω. Then if $u \in H_o^m(\Omega)$ is such that $Lu \in H^p(\Omega)$, $u \in H^{p+2m}(\Omega)$.

Proof. The proof is a direct application of the result for $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$. It proceeds in a series of bootstrap steps analogous to those used in proving interior regularity. Evidently it suffices to prove that if $Lu = f$ with $f \in H^p(\Omega)$ and $u \in H^r(\Omega) \cap H_o^m(\Omega)$ with $r < p+2m$ then $u \in H^{r+1}(\Omega)$. We first prove that any $x_o \in \partial\Omega$ has a neighborhood U in $\mathbb{R}^n$ such that $u \quad \Omega \in H^{r+1}(U \cap \Omega)$. By the smoothness assumption there is a neighborhood V of x_o in $\mathbb{R}^n$ and a diffeomorphism $T: V \cap \Omega \to B_1 = \{x \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : |x| < 1\}$, s.t.:

1) The components of T and T^{-1} are functions in C_b^∞.

2) The unique extension of T to $\overline{V \cap \Omega}$ (which exists in virtue of 1) which guarantees T is Lipschitzian and hence uniformly continuous) maps $\overline{V} \cap \partial\Omega$ onto $\{x \in \overline{B}_1 : x_n = 0\}$.

3) This unique extension maps x_o onto 0.

There is a function $\rho \in C_b^\infty(B_1)$ which is 1 on $B_{1/3}$ and 0 on $B_1 - B_{2/3}$ [Generally, $B_r = \{x \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : |x| < r\}$]. Let $\tilde{\rho} = \rho \circ T$, $W_r = T^{-1}(B_r)$. Then $\tilde{\rho} \in C_b^\infty(V \cap \Omega)$ and clearly $\operatorname{supp} \rho \subseteq W_{3/4}$. $\tilde{\rho}$ has a canonical extension $\tilde{\rho}_1$ to Ω by setting $\tilde{\rho}_1(x) = 0$ if $x \in \Omega - V$. Now $\tilde{\rho}_1 \in C_b^\infty(\Omega)$. To show this it suffices to show $\tilde{\rho}_1$ is 0 in a neighborhood of any point $y \subset \Omega \cap \partial V$. As T extends to a homeomorphism $\overline{V \cap \Omega} \to \overline{B}_1$ it follows there is a neighborhood W of y in $\mathbb{R}^n$ such that $W \cap \overline{V \cap \Omega} \cap \overline{W}_{3/4} = \phi$, i.e. $\overline{W} \cap W_{3/4} = \phi$. Thus $\rho | W \cap \Omega = 0$.

This also proves $\operatorname{Supp} \tilde{\rho}_1 \cap \subseteq V \cap \Omega$.

Now $\tilde{\rho}_1 u \mid V \cap \Omega \in H_o^m(V \cap \Omega)$. Furthermore

$$L\, \tilde{\rho}_1 u = \tilde{\rho}_1 Lu + Pu = g$$

where P is an operator with coefficients in C_b^∞ and order $\leq$ $\leq 2m-1$. Also $Pu \in H^{r-2m+1}(\Omega)$ and $\tilde{\rho}_1 Lu \in H^{r-2m+1}(\Omega)$, so that $g|V \cap \Omega \in H^{r-2m+1}(V \cap \Omega)$. Since T satisfies condition 2) above the induced map on distributions preserves the Sobolev spaces H^r, H_o^r. Whence we may consider, by transport of structure

$$L\, \tilde{\rho}_1 u|V \cap \Omega = g|V \cap \Omega$$

as an equation on B_1, with $\tilde{\rho}_1 u|V \cap \Omega \in H_o^m(V \cap \Omega) \cap H^r(V \cap \Omega)$ and $g|V \cap \Omega \in H^{r-2m+1}(V \cap \Omega)$.

Applying the regularity theorem for a half space we may conclude $\tilde{\rho}_1 u|V \cap \Omega \in H^{r+1}(V \cap \Omega)$. To see this explicitly, let $u_1 = \tilde{\rho}_1 u$ considered as a distribution on B_1. Evidently, $\operatorname{Supp} u_1 \subseteq W_{3/4}$. Let $\varphi \in C_b^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^+)$ with $\operatorname{Supp} \varphi \subseteq B_1$ and such that φ is identically 1 on $W_{3/4}$. If L_1 is any strictly elliptic constant coefficient operator then $(1-\varphi)L_1+\varphi L$ is a strictly elliptic operator on $\mathbb{R}^{n-1} \times \mathbb{R}^+$. In addition regarding u_1 as a distribution on $\mathbb{R}^{n-1} \times \mathbb{R}^+$

$$[(1-\varphi)L_1 + \varphi L](u_1) = \varphi L u_1 = L u_1 \in H^{r-2m+1}(\mathbb{R}^{n-1} \times \mathbb{R}^+)$$

Whence $u_1 \in H^{r+1}(\mathbb{R}^{n-1} \times \mathbb{R}^+)$.

Thus $\tilde{\rho}_1 u|V \cap \Omega \in H^{r+1}(V \cap \Omega)$, and as $\tilde{\rho}_1$ is identically 1 on $W_{1/3}$, then $u|W_{1/3} \in H^{r+1}(W_{1/3})$. This completely proves the assertion that any $x_o \in \partial\Omega$ has a neigborhood U such that $u \mid U \cap \Omega$ is in $H^{r+1}(U \cap \Omega)$.

The next step in the proof is to show that if $\Omega' \subseteq \Omega$ is open and $\bar{\Omega}' \cap \partial\Omega = \phi$, then $u|\Omega' \in H^{r+1}(\Omega')$. This is proved along the same lines: An easy compactness argument shows there is a $\rho \in C_b^\infty(\Omega)$ which is 1 on Ω' and vanishes in a neighborhood of $\partial\Omega$. Then $\rho u \in H_o^m(\Omega') \cap H^r(\Omega')$ and

$$L\rho u = \rho Lu + Pu \in H^{r-2m+1}(\Omega)$$

order $P \leq 2m-1$. Consider as before an operator $(1-\varphi)L_1 + \varphi L$, in this case on $\mathbb{R}^n$, where L_1 is strictly elliptic, $\varphi \in C_b^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \Omega$ and $\varphi = 1$ on $\operatorname{supp}\rho$. By global regularity for elliptic operators on $\mathbb{R}^n$ it follows $\rho u \in H^{r+1}(\mathbb{R}^n)$. As $\rho = 1$ on Ω' $u|\Omega' \in H^{r+1}(\Omega')$.

To complete the proof that $u \in H^{r+1}(\Omega)$, use a partition of unity: Specifically, let $\{V_i\}$ be a finite family of open sets in $\mathbb{R}^n$ such that $\cup V_i \supseteq \partial\Omega$ and $u|V_i \cap \Omega \in H^{r+1}(V_i \cap \Omega)$. Let ψ be a function on $C_b^\infty(\Omega)$ supported in $\cup V_i$ and which is 1 on a neighborhood of $\partial\Omega$. By the first step one has $\psi u \in H^{r+1}$ $\in H^{r+1}(\Omega)$, whereas by the second $(1-\psi)u \in H^{r+1}(\Omega)$. Therefore $u \in H^{r+1}(\Omega)$. This completely proves the theorem

3.21 Green's Functions for the Dirichlet Problem

3.21.1 Let $\Omega \subseteq \mathbb{R}^n$ be an open set of finite Lebesgue measure, L a strictly elliptic operator of order $2m$ with coefficients in C_b^∞. Then there is a constante c_o such that

$L+c \mid H_o^m(\Omega)$ has a unique continuous inverse $T_c: H^{-m}(\Omega) \to H_o^m(\Omega)$ for $c \geq c_o$.

We have already shown this; Recall the proof is a simple application of the Lax Milgram Lemma and Gårding's inequality.

Suppose now Ω is a smooth bounded open set; If $u \in H_o^m$ and $Lu \in H^p$ then $u \in H^{2m+p}$. From this follows immediately that T_c maps H^p into $H^{2m+p} \cap H_o^m$. Thus $(T_c)^r$ maps H^p into $H^{2mr+p} \cap H_o^m$. We then conclude in exactly the same way as for compact manifolds without boundary:

3.21.2 If Ω is a smooth bounded open set and $2mr > n/2$ then $(T_c)^r$ for $c \geq c_o$ is a Hilbert Schmidt operator $H^o(\Omega) \to H^o(\Omega)$. In particular there is a function $k \in L^2(\Omega \times \Omega)$ such that for $u \in H^o(\Omega)$

$$(T_c)^r\, u(x) = \int_\Omega k(x,y)\, u(y)dy$$

3.21.3 As a corollary if $4m > n$ then T_c is an integral operator with an L^2 kernel. This kernel is called the Green's function of the boundary value problem.

3.22 Coercivity

We have seen that if L is elliptic of order $2m$, $\Omega \subseteq \mathbb{R}^n$ a smooth open set, then for $c > 0$ large $L+c$ is a bijective map $H_o^m(\Omega) \cap H^{2m}(\Omega) \to H^o(\Omega)$; More generally $L+c$ is a bijective map $H_o^m(\Omega) \cap H^{p+2m}(\Omega) \to H^p(\Omega)$. Furthermore $L+c$ is continuous between these two spaces: $H_o^m(\Omega) \cap H^{p+2m}(\Omega)$ is here considered with the topology induced from $H^{p+2m}(\Omega)$. $H_o^m(\Omega) \cap H^{p+2m}(\Omega)$ is closed in $H^{p+2m}(\Omega)$ and is therefore a Banach space. We may apply the open mapping theorem to conclude $L+c$ is a homeomorphism $H_o^m(\Omega) \cap H^{p+2m}(\Omega) \to H^p(\Omega)$. In particular there is a constant c_1 such that for $u \in H_o^m \cap H^{p+2m}$

$$\|u\|_{p+2m} \le c_1\|(L+c)u\|_p$$

$$\le c_1\|Lu\|_p + c_2\|u\|$$

Now $\|u\|_p \le \epsilon\|u\|_{p+2m} + C(\epsilon)\|u\|_o$, so we can deduce from this that for some constant $C > 0$,

3.22.1 $$\|u\|_{p+2m} \le C(\|Lu\|_p + \|u\|_o)_p$$

This type of inequality is usually refered to as a <u>coercive estimate</u>: $\|Lu\|_p$, $\|u\|_o$ small "coerce" $\|u\|_{p+2m}$ to be small.

SECTION 4

EQUATIONS OF EVOLUTION

4.1 Self Adjoint Extensions

Suppose L is a formally self adjoint differential operator on a manifold with density or on an open set $\Omega \subseteq \mathbb{R}^n$. As an operator on $L^2(\Omega)$ L is symmetric; To determine the self adjoint extensions of L we first determine $\mathrm{Ran}(i+L)^{\perp}$; The self adjoint extensions will then correspond to the partial isometries

$$V: \mathrm{Ran}(i+L)^{\perp} \to \mathrm{Ran}(-i+L)^{\perp}$$

Now $f \in \mathrm{Ran}(i+L)^{\perp}$ iff. for all $\varphi \in C_0^{\infty}(\Omega)$

$$0 = \langle f,(i+L)\varphi\rangle = \langle(-i+L)f,\varphi\rangle$$

i.e. $f \in L^2(\Omega)$ and $(-i+L)f = 0$, in the sense of distributions. As $(-i+L)$ is elliptic, this implies $f \in C^{\infty}(\Omega)$. Similarly $\mathrm{Ran}(-i+L)^{\perp}$ consists $f \in C^{\infty}(\Omega) \cap L^2(\Omega)$ such that $(i+L)f = 0$ in the sense of distributions. It is now very easy to prove:

4.1.1 Suppose Ω is a compact manifold with density. Then any formally self adjoint operator has exactly one self adjoint extension.

Proof. To prove this we show $\mathrm{Ran}(\pm i+L)^{\perp} = \{0\}$. Now if $f \in C^{\infty}(\Omega)$ is such that $Lf = if$ then $i\langle f,f\rangle = \langle Lf,f\rangle =$

$= \langle f, Lf\rangle = -i\langle f,f\rangle$ so that $\langle f,f\rangle = 0$ and hence $f = 0$. imilarly, $f \in C^\infty(\Omega)$ and $Lf = -if$ imply $f = 0$. □

4.2 The Dirichlet Extension

4.2.1 Suppose L is a formally self adjoint strictly operator of order $2m$. Then the distribution extension of L restricted to the space $V = \{u \in H_0^m(\Omega): Lu \in H^0(\Omega)\}$ is self adjoint

Proof. We know there is a $c > 0$ and a bounded $S: H^0(\Omega) \to H_0^m(\Omega)$ such that $Sf = u$ iff $(L+c)u = f$, $u \in H_0^m(\Omega)$. $(L+c)|V$ is evidently S^{-1}. If S is a bounded self adjoint operator, then $(L+c)$ will be self adjoint and so L itself will be selfadjoint. There is a unique sesquilinear form $B: H_0^m \times H_0^m \to \mathbb{C}$ such that

$$\langle (L+c)u, \phi\rangle = B(u,\phi)$$

By continuity, $B(u,u) \geq 0$ for all $u \in H_0^m$. Similarly one can show

$$B(Sf,v) = \langle f,v\rangle_0$$

for $f \in H^0$ and $v \in H^m$. Thus $\langle f, Sf\rangle_0 = B(Sf,Sf)$ is real valued for all $f \in H^0$, and so S is self adjoint. □

By a similar argument we may show:

4.2.2 If L is a before and formally positive, then the distribution extension of L restricted to V is positive in the operator sense.

Proof. One must show $(L+c)|V \geq c$. In terms of S this means $S^{-1} \geq c$ or $1 \geq cS$. Now by the fact L is positive on C_o^∞ and continuity it follows that $B(v,v) \geq c\|v\|_o^2$ for $v \in H_o^m$. Thus for all $f \in H^o$

$$\langle f, Sf\rangle_o = B(Sf,Sf) \geq c\|Sf\|_o^2 = c\langle S^*Sf,f\rangle_o$$

Whence $S \geq cS^2$ from which follows $1 \geq cS$, by the fact S is injective. □

4.2.3 In case $\Omega \subseteq \mathbb{R}^n$ is an open set with $\partial\Omega$ smooth and compact, the subspace V above is precisely the space $H^{2m}(\Omega) \cap H_o^m(\Omega)$. This follows immediately from the theorem on elliptic regularity. For this reason we call the distribution extension of L restricted to V the Dirichlet extension of the strictly elliptic operator L.

In the following sections we will consider strictly elliptic operators L under "standard" regularity assumptions on the coefficients of L and on $\partial\Omega$. These assumptions are exactly the above hypotheses which insure the validity of the global elliptic regularity theorem.

4.3 The Cauchy Problem for Linear Ordinary Differential Equations

The following theorem is an immediate consequence of the general existence theory for ODE's.

4.3.1 Let E be a Banach space, $t \mapsto A_i(t)$ $0 \leq i < s$ continuous functions from an interval I into $B(E)$.

Then if $t_o \in I$ and $x_1, \ldots, x_s \in E$ there is one and only one solution to the equation

$$\phi^{(s)}(t) + \sum_{i<s} A_i(t)\, \phi^{(i)}(t) = 0$$

and

$$\phi^{(i)}(t_o) = x_{i+1}$$

for $0 \leq i < s$.

Even if we can regard evolution equations as ordinary differential equations with values in Banach spaces, this theorem is of little value since the operators $A_i(t)$ are unbounded. Using spectral theory however, we can prove a similar theorem when the $A_i(t)$ are simultaneously diagonalisable operators. We now consider this although we restrict ourselves to a special case for simplicity of notation.

4.4 Solutions by Eigenfunction Expansions

Let A be a self adjoint operator on a Hilbert space $H, \{\beta_i\}_{0 \leq i < s}$ complex C^1 functions on the interval $[a,b]$. In this section we consider the differential equation

$$\phi^{(s)}(t) + \sum_{0<i<s} \beta_i(t)\, \phi^{(i)}(t) = \beta_o(t)\, A\phi(t) \qquad (.1)$$

We consider this equation in the obvious sense: ϕ is a norm C^s function $I \to \text{dom}\, A$ which satisfies (.1)

4.4.1 Applying the spectral theorem to A, we may assume $H = \int^{\oplus} H(\lambda)\, d\mu(\lambda)$ where $H(\lambda)$ is a μ-measurable family of Hilbert spaces and A is multiplication by a measurable real valued function g on X. The domain of g consists of $\xi = \int^{\oplus} \xi(\lambda)\, d\mu(\lambda) \in H$ such that $\int^{\oplus} g(\lambda)\, \xi(\lambda)\, d\mu(\lambda) \in H$.

This simplifies theoretically the solution of (.1). Let us dispose first of uniquness.

4.4.2 Suppose ϕ, ϕ_1 are solutions of (.1) such that for some $\phi^{(i)}(t_o) = \phi_1^{(i)}(t_o)$ for $0 \le i < s$. Then $\phi = \phi_1$, on I.

Proof. Let $P_n = \chi_{[-n,n]}(A)$. P_n is a projection, $P_n \le P_{n+1}$ and $\lim P_n = 1$. Now $P_n A$ is a bounded operator and if ϕ satisfies (.1), $P_n\phi$ also satisfies

$$(P_n\phi)^{(s)}(t) + \sum_{0<i<s} \beta_i(t)(P_n\phi)^{(i)}(t) = \beta_o(t)AP_n\phi(t) \qquad (.2)$$

This is an ordinary differential equation of order n, with continuous coefficients. As $P_n\phi$, is also a solution of (.2) and $(P_n\phi)^{(i)}(t_o) = (P_n\phi_1)^{(i)}(t_o)$ we conclude that $P_n\phi(t) = P_n\phi_1(t)$ for all t. Therefore $\phi_1 = \phi$.

To solve (.1) we proceed formally; Using the spectral representation of A, (.1) becomes

$$\partial_t^s\, \phi(t,\omega) + \sum_{0<i<s} \beta_i(t)\, \partial_t^i\, \phi(t,\omega) = \beta_o(t)\, A\phi(t) \qquad (.3)$$

For a fixed ω this is a homogeneous differential equation (with unknown function taking values in $H(\omega)$) but with scalar coefficients. Let $\{\omega_i(t,c)\}_{1\le i<s}$ be a basic system of solutions of the scalar ODE

$$v^{(s)}(t) + \sum_{0<i<s} \beta_i(t)\, v^{(i)}(t) = \beta_o(t)\, cv(t) \qquad (.4)$$

In other words the functions $\{w_i(t,c)\}$ each solve (.4) and take the initial values $w_i^{(k)}(a,c) = \delta_{i\ k+1}$; $1 \le i \le s$, $0 \le k \le s-1$. By the continuous dependence on parameters $\partial_t^k w_i(t,c)$ is a continuous function $I \times \mathbb{R} \to \mathbb{C}$ for $k \le s$. In particular $1 \to w_i(t,c)$ is in $C^s([a,b])$.

Let us write following formal solution to (.3)

$$\phi(t,\omega) = \sum_{i=1}^{s} w_i(t,g(\omega))\ \chi_i(\omega) \tag{.5}$$

where $x_1,\ldots,x_s$ are μ-measurable families of vectors. It remains to see when (.5) actually expresses a solution to (.1). To do this we need the following Hilbert spaces version of differentiating under the integral sign

4.4.3 Suppose $x(t) = \int^{\oplus} x(t,\omega)\ d\mu(\omega) \in H$ and $\rho \in L^2_\mu(X)$ are such that for each $\omega \in X$, $t \longmapsto x(t,\omega)$ is in $C^m(I)$, $|\partial_t^k x(t,\omega)| \le \rho(\omega)$ a.e. for all t, and $k \le m$. Then $I \to H$ is in C^m (normwise) and

$$x^{(k)}(t) = \int^{\oplus} \partial_t^k x(t,\omega)\ d\mu(\omega)$$

Proof. By induction it suffices to prove this for $m = 1$. In this case it is a consequence of the mean value theorem and the Lebesgue dominated convergence theorem.

Therefore to show the ϕ given by (.5) is actually a solution to (.1) we must verify (a) that for each ω as a function of t it satisfies (.3) and (b) the functions $\partial_t^k \phi(t,\omega)$ are dominated by an L^2 function ρ independent of

t, (c) $g(\omega)\ \phi(t,\omega) \in H$.

The condition (a) is obviously satisfied as the functions w_i, $1 \le i \le s$ as functions of t satisfy (.4).

To satisfy conditions (b) and (c) consider the function

$$f_i(c) = \text{Sup}\{|c\, w_i(t,c)|;\ \ |\partial_t^k\ w_i(t,c)|:\ \ 0\le k\le s,\ \ t \in [a,b]\} \qquad (.6)$$

f_i being the supremum of a family of continuous function (continuous dependence on parameters) is itself upper semicontinuous and in particular Borel. We then have the following:

If $x_1,\ldots,x_n$ are such that $f_i(g(\omega))x_i(\omega)$ is square integrable then ϕ satisfies conditions (b) and (c).

For $k \le s$

$$\|\partial_t^k\ \phi(t,\omega)\| = \|\sum_i x_i(\omega)\ \partial_t^k\ w_i(t,g(\omega))\| \le \sum_i \|x_i(\omega)\|\ f_i(g(\omega)) = \rho(\omega)$$

where $\rho(\omega)$ is an L^2 function. Similarly

$$\|g(\omega)\ \phi(t,\omega)\| = \|\sum_i g(\omega)\ w_i(t,g(\omega))\ x_i(\omega)\| \le$$

$$\le \Sigma\ \|x_i(\omega)\|\ f_i(g(\omega)) = \rho(\omega)$$

so that the family $g(\omega)\ \phi(t,\omega)$ is square integrable.

This essentially proves:

4.4.4 Suppose A is a self adjoint on H. Then the H valued differential equation

$$\phi^{(s)}(t) + \sum_{0<i<s} \beta_i(t)\ \phi^{(i)}(t) = \beta_o(t)\ A\phi$$

<u>with initial conditions</u> $\phi^{(i)}(a) = x_{i+1}$, $0 \le i \le s-1$ <u>has at most one solution</u> ϕ: I → H. <u>If in addition</u> $x_i \in \text{dom } f_i(A)$ <u>where the</u> f_i <u>are given</u> by (.6) <u>then the solution exists</u>.

For to say $f_i(g,(\omega))\, x_i(\omega)$ is square integrable means precisely that $x_i \in \text{dom } f_i(A)$. Furthermore

$$\partial_t^k \phi(a,\omega) = \Sigma\, \partial_t^k\, w_i(a,g(\omega))\, x_i(\omega) = \Sigma\, \delta_{i\ k+1}\, x_i(\omega) = x_{k+1}(\omega)$$

so that $\phi^{(k)}(a) = x_{k+1}$.

4.5 Evolution Equations

We now apply the preceding theory to formally self adjoint strictly elliptic operators L. These operators are, under certain regularity assumptions on the domain Ω and the coefficients of L actually self adjoint when considered on the domain $H^{2m}(\Omega) \cap H_o^m(\Omega)$, where deg L = 2m (cf. 4.2.3).

This leads to the following theorem

4.5.1 <u>Let</u> $\{\beta_i\}$ <u>be</u> C' <u>complex functions</u>, $\{f_i\}$, $1 \le i \le s$ <u>as defined in</u> (.6) <u>of paragraph</u> 4.4.3, <u>and</u> $u_i \in \text{dom } f_i(L)$. <u>Then the Cauchy problem</u>

$$\begin{aligned} &\phi^{(s)}(t) + \sum_{0<i<s} \beta_i(t)\, \phi^{(i)}(t) = \beta_o(t)\, L\phi(t) \\ &\phi^{(i)}(a) = u_{i+1} \end{aligned} \qquad (.1)$$

<u>has a unique solution</u> ϕ: $[a,b] \to L^2(\Omega)$

It is worth emphasizing that the above solution ϕ is differentiable as a mapping into $L^2(\Omega)$. This obviously does not mean that ϕ is every where differentiable as a function of t;

4.6 Regularity of Solutions

As is customary we will analyze regularity of solutions using the higher Sobolev spaces. L is a strictly elliptic operator on Ω with the standing assumptions [4.2.3]

4.6.1 *Let* $a \leq a_1 \leq b_1 \leq b$, *and define the function*:

$$k_i(c) = \operatorname{Sup}\{|c|^j \, |w_i(t,c)|: \; 1 \leq j \leq r, \; t \in [a_1,b_1]\}.$$

If $u_i \in \operatorname{dom} f_i(L)$ *for* $1 \leq i \leq s$ *and* $u_i \in \operatorname{dom} k_i(L)$ *then the solution to the Cauchy problem* (.1) *of* 4.5.1 *satisfies*

$$\phi(t) \in H^{2mr}(\Omega) \; .$$

for all $t \in [a_1,b_1]$.

Proof. To prove this consider the spectral representation of L with the domain $H^{2m}(\Omega) \cap H_0^m(\Omega)$.

$$H^0(\Omega) = \int^{\oplus} H(\lambda)d\mu(\lambda)$$

and L is multiplication by a real valued measurable function g. In the spectral representation the solution to the Cauchy problem is

$$\phi(t,\omega) = \Sigma \, w_i(t,g(\omega))u_i(\omega)$$

Thus for $j \leq r$ and $t \in [a_1,b_1]$

$$\|g(\omega)^j \, \phi(t,\omega)\| \leq \Sigma |g(\omega)|^j \, |w_i(t,g(\omega))| \; \|u_i(\omega)\|$$

$$\leq \Sigma \, k_i(g(\omega)) \; \|u_i(\omega)\| = \gamma(\omega)$$

The hypothesis $u_i \in \text{dom } k_i(L)$ means precisely $\gamma \in L^2$. By hypothesis therefore $g(\omega)^j \phi(t,\omega)$ is a function in $\text{dom } L = H^{2m} \cap H^m_o$, for $j \le r-1$. In particular if $j \le r-2$, $g^{j+1}\phi(t) = Lg^j\phi(t) \in H^{2m}$ Applying elliptic regularity [which we can do as $g^j\phi(t) \in H^m_o$] we deduce that for all $j \le r-2$, $g^j\phi(t) \in H^{4m}$.

Evidently this argument can be repeated and thus we conclude that $\phi(t) \in H^{2mr}$ for all $t \in [a_1,b_1]$.

The preceding result says nothing about regularity of time dependence. We now consider this; Let us assume for simplicity that the functions $w_i(t,c)$ are jointly C^∞ (which will be the case if the β_i are C^∞);

4.6.2 <u>Let</u> r,r' <u>be integers</u> > 0 <u>and consider the function</u>

$h_i = h_i(r,r')$ $h_i(c) = \text{Sup}\{|c|^j \, |\partial_t^k w_i(t,c)|: 0 \le j \le r,$ $0 \le k \le r', \ t \in [a_1,b_1]\}$. <u>If</u> $u_i \in \text{dom } h_i(L) \cap \text{dom } f_i(L)$ <u>for</u> $1 \le i \le s$ <u>then the Cauchy problem</u> 4.5.1. <u>has a (unique) solution</u> ϕ <u>which in the interval</u> $[a_1,b_1]$ <u>is in</u> $C^{r'}([a,b], H^{2mr}(\Omega))$.

<u>Remark</u>. The condition $u_i \in \text{dom } h_i(L)$ implies $u_i \in \text{dom } k_i(L)$ so $\phi(t) \in H^{2mr}(\Omega)$ for $t \in [a_1,b_1]$.

<u>Proof</u>. Consider the space K of $x = \int^{\oplus} x(\omega)\, d\mu(\omega) \in H^o$ such that $g^j x \in H^o$ for $j \le r$. For $j \le r-1$ it is clear that $x \in K$ implies $g^j x \in H^{2m} \cap H^m_o$ [= domain of g]. By induction it follows easily that $x \in K$ implies $x \in H^{2mr} \cap H^m_o$. [This is the same fact we used in the previous proposition]. Furthermore by coercivity of L

$$\|x\|_{2mr} \leq C\{\|g\cdot x\|_{2m(r-1)} + \|x\|_o\} \leq C_1\{\|g^r x\|_o + \|x\|_o\} .$$

It thus suffices to show that $\phi(t) \in K$ for $t \in [a_1,b_1]$ and that ϕ is differentiable as a function into K with the norm norm $x \to \|g^r x\|_o + \|x\|_o$. This we prove using the Hilbert space version of differentiation under the integral sign. The solution ϕ has the representation

$$\phi(t,\omega) = \Sigma\, w_i(t,g(\omega))u_i(\omega)$$

Now

$$|g(\omega)^j \, \partial_t^k \phi(t,\omega)| \leq \Sigma|g(\omega)|^j \, |\partial_t^k \, w_i(t,g(\omega))| \, \|u_i(\omega)\| \leq$$

$$\leq \Sigma\, h_i(g(\omega)) \, \|u_i(\omega)\| = \gamma(\omega)$$

whenever $j \leq r$, $k \leq r'$, and $t \in [a_1,b_1]$. By the assumption $u_i \in \text{dom}\, h_i(L)$ $\gamma \in L^2$. Therefore $\phi(t)$ is in $C^{r'}$ as a function from $[a_1,b_1]$ into the Hilbert space $\int^{\oplus} H(\lambda)\, dg(\lambda)^j \mu(\lambda)$ for $j \leq r$. This implies ϕ is $C^{r'}$ as a function into K with the norm $\|g^r x\|_o + \|x\|_o$, priving our result.

We can specialize the preceding results on existence and regularity when the data are C_o^∞.

4.6.3 <u>Suppose the basic solutions</u> $\{w_i(t,c)\}$ <u>satisfy the following growth conditions</u>. <u>For each</u> $k \in \mathbb{N}$ <u>there are constants</u> M_k, N_k <u>such that</u> $|\partial_t^k \, w_i(t,c)| \leq M_k(|c|^{N_k}+1)$ <u>for</u> $1 \leq i \leq s$ <u>and</u> $c \in Sp(L)$. <u>If</u> $u_i \in C_o^\infty(\Omega)$ <u>then the Cauchy problem</u> (.1) <u>of</u> 4.5.1 <u>has a unique solution</u> $\phi(t)$ <u>such that</u> $\phi(t) \in H^\infty(\Omega)$ <u>and</u> ϕ <u>is in</u> C^∞ <u>as a function</u> $[a,b] \to H^p(\Omega)$ <u>for every</u> $p > 0$.

Proof. The hypotheses imply that $u_i \in H^{2m}(\Omega) \cap H_o^m(\Omega)$ and $L^j u_i \in H^{2m}(\Omega) \cap H_o^m(\Omega)$ for all j. In the spectral representation of L therefore $g^j u_i \in H^o(\Omega)$ for all $j \in \mathbb{N}$. Now for $c \in Sp(L)$

$$f_i(c) = \mathrm{Sup}\{|cw_i(t,c)|; |\partial_t^k w_i(t,c)|: 0 \le k \le s, \quad t \in [a,b]\} \le$$
$$\le M(|c|^N + 1)$$

for appropriate M,N. Thus $f_i(g(\omega)) \le M(|g(\omega)|^N + 1)$ and so $\omega \longmapsto f_i(g(\omega))\, u_i(\omega)$ is square integrable. It thus follows that $u_i \in \mathrm{dom}\, f_i(L)$. Similarly we can show that no matter what r,r' are $u_i \in \mathrm{dom}\, h_i(L)$. This proves the result.

In particular applying the Sobolev imbedding theorem we obtain under the above assumptions:

$\phi(t) \in C^\infty(\overline{\Omega})$ and ϕ is in C^∞ as a function $[a,b] \to C_b^p(\overline{\Omega})$ for every $p > 0$. All derivatives of order $\le m-1$ of $\phi(t)$ are zero on $\partial\Omega$.

4.7 The Heat Equation

4.7.1 Let L be a strictly elliptic formally self adjoint and formally positive operator. The heat equation is the PDE

$$\phi'(t) = -L\phi(t)$$
$$\phi(o) = u_1 \in H^{2m}(\Omega) \qquad (.1)$$

Under the standard assumptions on L and Ω there is a unique mapping $\phi: [0,\infty[\to H^{2m} \cap H^m_o$, C^1 as a mapping $[0,\infty[\to L^2(\Omega)$ and satisfying the conditions (.1).

Furthermore for any $t > 0$, $\phi(t) \in H^\infty(\Omega)$ and as a function $]0,\infty[\to H^\infty(\Omega)$, ϕ is C^∞.

Proof. To prove this we apply the preceding theory. Note that the corresponding ordinary differential equation is $v'(t) = -cv(t)$ which has a basic solution $w(t,c) = e^{-ct}$. Thus

$$f(c) = \text{Sup}\{|c|e^{-ct}: t \geq 0\} \leq c$$

provided $c \geq 0$. As L is positive it following that $u \in \text{dom } L$ implies $u \in \text{dom } f(L)$.

The regularity follows from the following observation: Let $0 < a < b$ and

$$\begin{aligned}\rho(c) &= \text{Sup}\{|c|^j\ |\partial_t^k\ w(t,c)|: \ j \leq r, \ k \leq r', \ t \in [a,b]\} \\ &= \text{Sup}\{|c|^j\ |c|^k\ e^{-ct}: j \leq r, \ k \leq r', \ t \in [a,b]\} \\ &= (1+|c|)^{r+r'}\ e^{-ca}\end{aligned}$$

for $c \geq 0$. Furthermore ρ is bounded on $[0,\infty[$. Thus the operators $k(L)$, $h(L)$ (no matter what r,r' are) are bounded and so $u \in \text{dom } k(L)$ $u \in \text{dom } h(L)$ follows immediately.

4.8 The Wave Equation

4.8.1 Let L be strictly elliptic formally positive differental Operator.

$$\phi''(t) = -L\phi(t)$$
$$\phi(0) = u_1 \qquad (.1)$$
$$\phi'(0) = u_2$$

<u>Under the standard assumptions</u> (.1) <u>has a unique solution (with the Dirichlet boundary conditions</u> $\phi(t) \in H^{2m} \cap H^m_o$) <u>if</u> $u_1 \in \text{dom } L \;\; u_2 \in \text{dom } L^{1/2}$.

<u>Proof</u>. The ordinary differential equation corresponding to (.1) is $v''(t) = -cv(t)$ which has basic solutions

$$w_1(t,c) = \text{Cos } c^{1/2}t$$
$$w_2(t,c) = c^{-1/2} \text{ Sin } c^{1/2}t \quad \text{if } c \neq 0$$
$$= t \quad \text{if } c = 0$$

Thus by differentiating we obtain the following inequality for $c \geq 0$:

$$f_1(c) \leq \text{Sup}\{|c|, 1, |c|^{1/2}, |c|\} = \text{Max}(1,c)$$
$$f_2(c) \leq \text{Sup}\{|c|^{1/2}, 1, 1, |c|^{1/2} = \text{Max}(1,c^{1/2})$$

If $u_1 \in \text{dom } L$, then $u_1 \in \text{dom } f_1(L)$ and similarly if $u_2 \in \text{dom } L^{1/2}$ $u_2 \in \text{dom } f_2(L)$. This proves the existence of the solution.

Note that in the spectral representation of L the solution has the form

$$\phi(t,\omega) = \text{Cos } g(\omega)^{1/2}t \; u_1(\omega) + g(\omega)^{-1/2} \text{ Sin } g(\omega)^{1/2}t \cdot u_2(\omega)$$

for ω, when $g(\omega) \neq 0$.

Inhomogeneous Equations

We now treat the abstract inhomogeneous wave equation

$$\phi''(t) = - L\phi(t) + f(t) \qquad (.1)$$

where L is a positive self adjoint operator on H and f(t) an H valued function Again we consider the spectral representation of L

$$L = \int^{\oplus} g(\lambda)\, d\mu(\lambda), \quad g(\lambda) \in \mathbb{R}^+$$

The uniqueness of solutions is treated in the same way as for the homogeneous equation. The solution is determined by the value of ϕ and ϕ' at a point.

Next we consider existence. In the spectral representation (.1) becomes

$$\partial_t^2\, \phi(t,w) = -g(\omega)\, \phi(t,\omega) + f(t,\omega)$$

We can give a formal solution to this by the variation of parameters formula

$$\phi(t,w) = \int_0^t f(\theta,\omega) \det\begin{bmatrix} \mathrm{Cos}\; g(\omega)^{1/2}\theta & \mathrm{Sin}\; g(\omega)^{1/2}\theta \\ \mathrm{Cos}\; g(\omega)^{1/2}t & \mathrm{Sin}\; g(\omega)^{1/2}t \end{bmatrix} g(\omega)^{-1/2}\, d\theta$$

$$= g(\omega)^{-1/2}\, \mathrm{Sin}\; g(\omega)^{1/2}t \int_0^t f(\theta,\omega)\, \mathrm{Cos}\; g(\omega)^{1/2}\theta\, d\theta$$

$$- g(\omega)^{-1/2}\, \mathrm{Cos}\; g(\omega)^{1/2}t \int_0^t f(\theta,\omega)\, \mathrm{Sin}\; g(\omega)^{1/2}\theta\, d\theta$$

provided $g(\omega) \neq 0$; If $g(\omega) = 0$ the formula is

$$\phi(t,\omega) = \int_0^t f(\theta,w)(t-\theta)\, d\theta$$

This will be a solution in the abstract sense if the functions $\partial_t^k \phi(t,\omega)$ $k = 0,1,2$ are dominated by an L^2 function ρ independent of t and if $\omega \to g(\omega)\, \phi(t,\omega) \in H$. Now if f is continuous in the variable θ and $g(\omega) \neq 0$:

$$\partial_t \phi(t,\omega) = \int_0^t [\mathrm{Cos}\, g(\omega)^{1/2}(t-\theta)]\, f(\theta,\omega)\, d\theta$$

$$\partial_t^2 \phi(t,\omega) = \int_0^t -g(\omega)^{1/2}\, [\mathrm{Sin}\, g(\omega)^{1/2}(t-\theta)]\, f(\theta,\omega) d\theta + f(t,\omega)$$

We thus obtain the following estimates

$$|\partial_t\, \phi(t,\omega)| \leq t\, \mathrm{Sup}\{|f(\theta,\omega)|: 0 \leq \theta \leq t\}$$

$$|\partial_t^2\, \phi(t,\omega)| \leq t\, |g(\omega)|^{1/2}\, \mathrm{Sup}\{|f(\theta,\omega)|: 0 \leq \theta \leq t\} + |f(t,\omega)|$$

besides the estimate

$$|\phi(t,\omega)| \leq t\, \mathrm{Sup}\{|f(\theta,\omega)|: 0 \leq \theta \leq t\} .$$

[These are valid whether or not $g(\omega) \neq 0$] .

Thus:

Suppose f is continuous in θ for almost all ω and

$$\psi(\omega) = \mathrm{Sup}\{|f(\theta,\omega)|: 0 \leq \theta \leq t\}$$

$$\psi_1(\omega) = \mathrm{Sup}\{|g(\omega)||f(\theta,\omega)|: 0 \leq \theta \leq t\}$$

are in L^2. Then the formal solution to the inhomogeneous problem is a true solution.

4.9 The Schrödinger Equation

4.9.1 This is the equation $\phi'(t) = -iL\phi(t)$ together with the initial condition $\phi(0) = u_1$. Here L is a formally symmetric strictly elliptic operator.

If $u_1 \in H^{2m} \cap H_o^m$ then the Cauchy problem for the Schrödinger equation has a unique solution.

Proof. The corresponding differential equation $v'(t) = -icv(t)$ has the basic solution e^{-ict}. Thus $f(c) = = \mathrm{Sup}\{|c||e^{-ict}|: t \geq 0\} = |c|$ so $u \in \mathrm{dom}\ L$ implies $u \in \mathrm{dom}\ f(L)$. The result then follows by the general theory.

Notice that the solution ϕ can be extended to negative times t also, as is possible for the wave equation

4.10 Equations with Constant Coefficients

We have obtained general theorems of existence and uniqueness for evolution equations and even a formula for the solution in the so called spectral representation of L. In the case L is a constant coefficient operator the spectral representation is particularly simple and of course is implemented by the Fourir transform. Also we can obtain the solution itself by Fourier transforming the solution in the spectral representation. This we do now.

First we evaluate expressions of the form $\mathfrak{F}_\xi^{-1}[a(\xi)[\mathfrak{F}_\theta u(\theta)](\xi)](x)$ where a is a bounded measurable function and $u \in \mathcal{S}'(\mathbb{R}^n)$. Thus

$$\Phi(x) = \int \exp i\langle x,\xi\rangle\, a(\xi)[\int \exp -i\langle \xi,\theta\rangle\, u(\theta)d\theta]d\xi =$$

$$= \int a(\xi)[\int \exp i\langle \xi, x-\theta\rangle\, u(\theta)d\theta]d\xi =$$

$$= \int a(\xi)[\overline{\int \exp -i\langle -\xi,\theta\rangle\, \overline{u(\theta+x)d\theta}}]d\xi =$$

$$= \int a(\xi)\overline{[\mathcal{F}_\theta \overline{u(\theta+x)}]}\,(-\xi)\, d\xi$$

$$= \Lambda_a[J\mathcal{F}_\theta \overline{u(\theta+x)}] = \mathcal{F}\Lambda_a[R_{-x}\overline{u}]$$

$$= R_x\ \mathcal{F}\Lambda_a(\overline{u})$$

With this we may calculate explicit formulas for the so solutions to the heat, the wave and the Schrödinger equations on $\mathbb{R}^n$.

4.10.1 The Heat Equation

$$\partial_t u = \Delta u, \quad u(0,x) = u_1(x) \tag{.1}$$

has solution in the Fourier transform representation

$$\phi(t,\xi) = \exp[-t|\xi|^2]\ \hat{u}_1(\xi)$$

To find the solution itself it suffices to find $\mathcal{F}_\xi[\exp -t|\xi|^2]$. However

$$\mathcal{F}_\xi[\exp -t|\xi|^2](\theta) = (2t)^{-n/2}\ \mathcal{F}_\xi[\exp -|\xi|^2/2]\ ((2t)^{-1/2}\ \theta) =$$

$$=(2t)^{-n/2}\ \exp(-|\theta|^2/4t)$$

The solution to the Cauchy problem (.1) is thus

$$u(t,x) = (2t)^{-n/2} \int \exp(-|\theta-x|^2/4t)\, u_1(\theta)d\theta$$

4.10.2 The Schrödinger equation

$$\partial_t u = -i\Delta u, \quad u(0,x) = u_1(x)$$

is treated similarly. In the Fourier representation (.2) has the solution

$$\phi(t,\xi) = \exp[-it|\xi|^2]\, \hat{u}_1(\xi)$$

Now to find $\mathfrak{F}_\xi[\exp-it|\xi|^2]$ is some what more complicated. The answer is $\mathfrak{F}_\xi[\exp-it|\xi|^2](\theta) = (2t)^{-n/2}\, a^n \exp i|\theta|^2/4t$ $a = \exp - i\pi/4$. Granting this we can write the solution to (.2) as

$$u(t,x) = (2t)^{-n/2}\, a^n \int \exp[i|\theta-x|^2/4t]\, u_1(\theta)d\theta$$

APPENDIX

SPECTRAL THEORY OF UNBOUNDED OPERATORS ON A HILBERT SPACE

A.1 Symmetric Operators

A.1.1 Let H be a Hilbert space, A an operator with dom $A \subseteq H$. A is *symmetric* iff $\langle Ax,y\rangle = \langle x,Ay\rangle$ for all $x,y \in$ dom A.

A.1.2 *Let* $J \in B(H \oplus H)$ *be given by* $J(x,y) = (-y,x)$. *Then an operator* A *is symmetric iff* $G(A) \perp JG(A)$ [$G(A)$ *is the graph of* A] .

Proof. For $G(A) \perp JG(A)$ iff for all $(x,y) \in$ Dom A

$$\langle (x,Ax),\ J(y,Ay)\rangle = \langle (x,Ax),\ (-Ay,y)\rangle =$$

$$= -\langle x,Ay\rangle + \langle Ax,y\rangle = 0$$

A.1.3 Given an operator A, we define its adjoint A^* in the following way: Its domain $D(A^*) = \{y \in H: y \to \langle Ay,x\rangle$ is a bounded linear functional$\}$ and A^*y is an element in H such that $\langle Ay,x\rangle = \langle y,A^*,x\rangle$ for all $y \in$ Dom (A). In order for this to define a function it is necessary and sufficient A be densely defined and this will always be an explicit assumption on A whenever A^* is refered to.

A.1.4 *If* A *is a densely defined operator on* H, *then*

$$G(A^*) = (JG(A))^{\perp}$$

Proof. For $(x,y) \in G(A^*)$ iff $\langle Az,x\rangle = \langle z,y\rangle$ for all $z \in Dom(A)$ iff $\langle(-Az,z), (x,y)\rangle = 0$ for all $z \in dom(A)$ iff $\langle J(z,Az), (x,y) = 0$ for all $z \in Dom(A)$ iff $(x,y) \in JG(A)^{\perp}$

A.1.5 An operator A is self adjoint iff $A = A^*$. In particular A must be densely defined.

A.2 The Cayley Transform

Let H be a Hilbert space. J is the transformation on $H \oplus H$ given by $(x,y) \mapsto (-y,x)$ and Φ by $(x,y) \to (ix+y,-ix+y)$.

A partial isometry on H is an isometric operator U whose domain and range are subspaces of H. In particular any isometry $H \to H$ is a partial isometry.

A.2.1 Lemma 1. Let $G \subseteq H \oplus H$ be such that $G \perp J(G)$. Then $\Phi(G)$ is the graph of a partial isometry on H.

2. Suppose U is a partial isometry. Then the set $\Phi^{-1}(G(U)) = G'$ satisfies $G' \perp J(G')$ Φ^{-1} is given by

$$\Phi^{-1}(x,y) = (1/2i\,(x-y),\ 1/2(x+y))$$

Proof. 1. $\Phi(G)$ is the graph of a partial isometry means that for all $x,y \in G$, $\|ix+y\|^2 = \|^2 = \|-ix+y\|^2$. Expanding, this means

$$2(i\langle x,y\rangle - i\langle y,x\rangle) = 2(-i\langle x,y\rangle + i\langle y,x\rangle)$$

i.e. $\langle x,y\rangle = \langle y,x\rangle$ for $(x,y) \in G$. This is $(x,y) \perp J(x,y)$ for all $(x,y) \in G$. This is actually weaker than our hypothesis

2. Let $(x,y); (x',y') \in G$. Now

$$\langle(1/2i(x-y),\ 1/2(x+y)),\ J(1/2i(x'-y'),\ 1/2(x'+y'))\rangle =$$

$$= \langle(1/2i(x-y),\ 1/2(x+y)),\ (-1/2(x'+y'),\ 1/2i(x'-y'))\rangle =$$

$$= -1/4i\langle(x-y),\ (x'+y')\rangle - 1/4i\langle(x+y),\ (x'-y')\rangle\ .$$

This quantity will be zero iff we have for $(x,y), (x',y') \in G$

$$-\ \langle x-y,\ x'+y'\rangle = \langle(x+y),(x'-y')\rangle$$

This comes down to $\langle x,x'\rangle = \langle y,y'\rangle$, that is $\langle x,x'\rangle = \langle Ux,Ux'\rangle$ for $x,x' \in Dom(U)$. However this is just the definition of partial isometry. □

A.2.2 This establishes a bijection between isometries (with not necessarily closed domains) and certain subsets of $H \oplus H$. The subset $\Phi^{-1}(G(U))$ will be the graph of a function iff for all $(x,y) \in G(U)$ $x-y = 0$ implies $x+y = 0$. i.e. iff $x=Ux$ implies $x = Ux$ implies $x = Ux = 0$. Thus:

A.2.3 $\Phi^{-1}(G(U))$ _is the graph of a function iff_ $\ker(1-U)=\{0\}$.

Furthermore the domain of $\Phi^{-1}(G(U))$ is evidently $Im(1-U)$. It is also clear that U is closed iff $\Phi^{-1}(G(U))$ is closed. The mapping Φ^{-1} preserves inclusion; this means W is an extension of U iff $\Phi^{-1}(G(W))$ is an extension of $\Phi^{-1}(G(U))$.

A.2.4 _If_ $\Phi^{-1}(G(U))$ _is the graph of a densely defined operator_ A, _then_ A _is self adjoint iff_ U _is unitary_.

<u>Proof.</u> The mapping $\sqrt{2}\,\Phi^{-1} \in B(H \oplus H)$ is a unitary operator; For

$$\|\Phi^{-1}(x,y)\|^2 = 1/4\|x-y\|^2 + 1/4\|x+y\|^2$$

$$= 1/2\ (\|x\|^2 + \|y\|^2) = 1/2\|(x,y)\|^2$$

The operator A is self adjoint iff $G(A) = G(A^*)$. In other words iff $\Phi^{-1}(G(U)) = (J\Phi^{-1}\,G(U))^{\perp}$. As J and Φ preserve orthogonality, it follows that A is self adjoint iff $G(U)^{\perp} = \Phi J\Phi^{-1}(G(U))$. Now we always have:

$$\Phi J\Phi^{-1}(x,y) = \Phi J(1/2i(x-y),\ 1/2(x+y)) =$$

$$= \Phi(-1/2(x+y),\ 1/2i(x-y)) = (i(-1/2(x+y)) + 1/2i(x-y))\ ,$$

$$-i(-1/2i(x-y)) = (-ix,iy) = i(-x,y).$$

Thus suppose A is self adjoint. We prove U is a unitary; To show this, it suffices to show $\mathrm{Dom}(U) = \mathrm{Ran}(U) = H$. A is closed so $G(U)$ is closed. Thus it suffices to show $\mathrm{Dom}\ U$, $\mathrm{Ran}\ U$ are dense. Suppose $x' \in \mathrm{Dom}(U)^{\perp}$. Thus $(x',0) \perp (x,Ux)$ for all $x \in \mathrm{dom}(U)$ or in other words, $(x',0) \perp G(U)$. Therefore $(x',0) = i(-x,y)$ for some $(x,y) \in G(U)$ and so $y = 0$. It follows $x = 0$ and hence $x' = 0$. Similarly we show $\mathrm{Ran}\ U = H$.

Suppose U is unitary. Then $(x',y') \in G(U)^{\perp}$ iff for all $x \in H$

$$\langle x',x\rangle + \langle y',Ux\rangle = 0\ .$$

This means $\langle x'+U^*y', x\rangle = 0$ for all $x \in H$ or equivalently $x'+U^*y' = 0$. This is turn is equivalent to $y' = -Ux'$. It is easy to show this is equivalent to $(x',y') \in \Phi J \Phi^{-1}(G(U))$.

This proves the following:

A.2.5 $U \longmapsto \Phi^{-1}(G(U))$ is a bijective order preserving map from the set of partial isometries $U \in B(H)$ s.t. $1-U$ is injective into the set of graphs of symmetric [not necessarily densely defined] operators. U is closed iff $\Phi^{-1}(G(U))$ is closed; U is unitary iff $\Phi^{-1}(GU))$ is the graph of a self adjoint operator.

A.2.6 The mapping which assigns to a symmetric operator A the partial isometry $\Phi(G(A))$ is called the Cayley transform. In terms of operators (rather than graphs) the Cayley transform of A is the operator $U = (A-i)(A+i)^{-1}$ whose domain is the space $\mathrm{Ran}(A+i)$. The inverse of the Cayley transform assoicates to a partial isometry U such that $1-U$ is injective an operator $A = i(1+U)(1-U)^{-1}$ whose domain is $\mathrm{Ran}(1-U)$.

In general A is a symmetric operator. It will have a dense domain iff $\mathrm{Ran}(1-U)$ is dense. This condition on the other hand implies $1-U$ injective.

To see this, suppose $x-Ux = 0$. Then for all $y \in \mathrm{dom}\, U$

$$0 = \langle x-Ux, y-Uy\rangle = \langle x,y\rangle - \langle x,Uy\rangle - \langle Ux,y\rangle + \langle Ux,Uy\rangle = \langle x,y-Uy\rangle$$

As $\mathrm{Ran}(1-U)$ is dense, it follows $x = 0$.

A.2.7 This gives us criteria for determining when a symmetric operator T has self adjoint extensions. If $\Phi(T) = U$ is the Cayley transform of T then the symmetric extensions of T are in bijective correspondence with partial isometries which extend U. There is a unitary which extends U iff the initial and final projections E_1, E_2 of U have equal codimensions n_1, n_2. In particular T has a self adjoint extension iff. The numbers n_1, n_2 are equal.

A.2.8 These numbers n_1, n_2 are called the deficiency induces of T.

Observe that the domain of U is $\mathrm{Ran}(i+T)$ and its codomain is $\mathrm{Ran}(-i+T)$. Thus the integers n_i are given by

$$n_1 = \dim\{y: \langle ix+Tx, y\rangle = 0 \text{ for all } x \in \mathrm{Dom}(T)\}$$

$$n_2 = \dim\{y: \langle -ix+Tx, y\rangle = o \text{ for all } x \in \mathrm{Dom}(T)\}$$

Furthermore if both $i+T, -i+T$ are surjective (resp. have dense image) then T is self adjoint (resp. has a unique self adjoint extension).

A.2.9 In case T dooes have a unique self adjoint extension we say T is essentially self adjoint.

A.3 Structure of Self adjoint Operators

If T is a self adjoint operator and U its Cayley transform, then

$$T = i(1+U)(1-U)^{-1}$$

in the sense that dom $T = \mathrm{Im}(1-U)$ and on this domain T is given by the above formula. This makes sense because $1-U$ is injective.

As U is unitary the complete spectral theorem is applicable. In particular U is unitarily equivalent to a multiplication operator. More precisely there is a measure space (X,μ) and u-measurable partition $X = \cup X_n$ and μ-measurable function $f:X \to \mathbb{T}$ such that U is unitarily equivalent to the operator m_f acting on $H = \bigoplus_{n=1}^{\omega} L^2_\mu(X_n, H_n)$ by: If $\psi = \{\psi_n\} \in H$ then $m_f\psi = \{m_{f_n} \psi_n\}$ where for $x \in X_n$.

$$[m_{f_n} \psi_n](x) = f_n(x)\ \psi_n(x)$$

Now $g(x) = i(1+f(x))(1-f(x))^{-1}$ is a real valued measurable function. It is clear that on the domain of T, T is just multiplication by g. Further the domain of T is the image of $1-U$: $\psi \in \mathrm{dom}(T)$ iff. $x \to (1-f(x))^{-1}\psi(x)$ is in H. As $f(x) = (g(x) - i)(g(x) + i)^{-1}$, $\psi \in \mathrm{dom}(T)$ iff the function

$$[1-(g(x)-i)(g(x)+i)^{-1}]^{-1}\ \psi(x) = [2i(g(x)+i)^{-1}]^{-1}\ \psi(x)$$
$$= (2i)^{-1}(g(x)+i))\ \psi(x)$$

is in H. But this is so iff $x \to g(x)\ \psi(x)$ is in H.

This proves the complete spectral theorem for self adjoint operators.

A.3.1 Let T be a self adjoint operator on a separable Hilbert space. Then T is unitarily equivalent to an operator

$\mathfrak{m}_g$ for a real valued measurable function g. $\mathfrak{m}_g$ acts on $H = \bigoplus_{n=1}^{\omega} L^2_\mu(X_n, H_n)$ where X_n, μ are as above. The domain of $\mathfrak{m}_g$ consists of $\psi = \{\psi_n\}$ such that $\{\varphi_n'\} \in H$ where $\varphi_n'(x) = g(x)\,\psi(x)$ if $x \in X_n$. In more direct but imprecise language dom T consists of $\psi \in H$ such that $\mathfrak{m}_g\,\psi \in H$.

REFERENCES

[1] Agmon, Shmuel, "Lectures on Elliptic Boundary Value Problems", Van Nostrand, New York ; 1965.

[2] Folland, Gerald; "Introduction to Partial Differential Equations", Princeton University Press, Princeton ; 1976

[3] Friedman, Avner; "Partial Differential Equations", Holt, New York ; 1969.

[4] Hicks, Noel; "Notes on Differential Geometry", Van Nostrand, New York , 1971

[5] Lions, Jean, "Lectures on Elliptic Partial Differential Equations". Tata Institute of Fundamental Research, Bombay.

[6] Nirenberg, Louis; An Abstract Form of the Non linear Cauchy Kowalewski Theorem, J. Differential Geometry 6 (1972) 561-576.

[7] Rudin, Walter; "Functional Analysis," McGraw Hill, New York.

[8] Treves, François; Basic Linear Partial Differential Equations. Academic Press, New York ; 1975.

[9] Yosida, Kosaku; "Functional Analysis", Springer Verlag, Berlin, 1965.