MONOGRAFIAS DE MATEMÁTICA N.° 30

PROJECTIVE MODULES

AND

SYMMETRIC ALGEBRAS

by

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PREFACE

These notes are based on a course of lectures given at IMPA in the summer of 1977. The central theme is the solution, by Quillen and Suslin, of the so called Serre Conjecture, affirming that projective A-modules are free when A is a polynomial ring over a field. Quillen's techniques have been further exploited to yield structure theorems for algebras which are locally isomorphic to polynomial algebras. These applications are presented in the last portion of the notes.

T.-Y. Lam has prepared a splendid exposition of work on the Serre Conjecture, to appear shortly in the Springer Lecture Notes. Lam generously made an early draft of his manuscript available to me. The reader will easily discern my extensive indebtedness to Lam's exposition. It is a pleasure here to express my gratitude to him. The principal novelty in these lectures is the treatment of symmetric algebras, a topic not pursued by Lam.

These otes owe their existence to the generous and painstaking efforts of my colleague and friend, T.M. Viswanathan, to whom I am deeply grateful. I wish also to thank Wilson Góes for the excellent job of typing. Finally I am pleased to thank the staff of IMPA for the kind hospitality I received in Rio.

Hyman Bass
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CONTENTS

In the first five sections of these notes R will denote a possibly non-commutative ring with an identity element 1_R, while A will denote a commutative ring with an identity element. In sections 6 and 7, we work over a commutative ground ring K and we explicitly state it, when the algebras and rings involved are not commutative. All modules and ring homomorphisms are unitary. $\mathcal{P}(R)$ will denote the category of finitely generated projective R-modules.

1. Serre's Problem

(1.1) Unimodular rows.

Consider an $n \times n$ matrix

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

over the ring A. Then α is an invertible matrix if and only if its determinant $\det \alpha$ is an invertible element of A. Since $\det \alpha = a_1 a_1' + a_2 a_2' + \ldots + a_n a_n'$, where the a'-s are appropriate minors of the matrix, we see that α invertible implies that the ideal $Aa_1 + Aa_2 + \ldots + Aa_n$ is the whole ring A. It is natural to ask whether the converse holds. We may view the first row of the matrix α as an n-tuple $a = (a_1, a_2, \ldots, a_n) \in A^n$ and the condition $Aa_1 + Aa_2 + \ldots + Aa_n = A$ is equivalent to saying that there exist elements $b_1, b_2, \ldots, b_n$ in A such that $b_1 a_1 + b_2 a_2 + \ldots + b_n a_n = 1$. In this case, we call a a unimodular row; so our question can be stated as:

Unimodular row problem: Given a unimodular row $a \in A^n$, is a the first row of an invertible matrix $\alpha \in GL_n(A)$?

It is clear that if $n=1$, the answer to this question is affirmative; if $n=2$, the answer continues to be affirmative: For, if $a = (a_1, a_2)$ is a unimodular row, there exist elements b_1, b_2 in A such that $b_1a_1 + b_2a_2 = 1$. Clearly then a is the first row of the invertible matrix $\alpha = \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$. The following example shows that the answer may be negative, when $n=3$.

Example 1 - Let $A = \dfrac{\mathbb{R}[X,Y,Z]}{(X^2+Y^2+Z^2-1)} = \mathbb{R}[x,y,z]$, where $\mathbb{R}[X,Y,Z]$ is the polynomial ring in three variables X, Y, Z over the field $\mathbb{R}$ of real numbers. The ideal $(X^2+Y^2+Z^2-1)$ defines the sphere S^2 in $\mathbb{R}^3$ as an algebraic set. We view A as the ring of polynomial functions on S^2. The row $a = (x,y,z)$ is clearly unimodular in A^3, since $x^2+y^2+z^2 = 1$. We claim that a can not be the first row of any 3×3 invertible matrix

$$\alpha = \begin{pmatrix} x & y & z \\ f & g & h \\ * & * & * \end{pmatrix}$$

with $f, g, h, *, \ldots$ in A. If $t \in S^2$, we consider $v(t) = (x(t), y(t), z(t))$ and $\varphi(t) = (f(t), g(t), h(t))$. Since $v(t)$ is merely given by the coordinates of t, we may view it as the unit normal vector to S^2 at t. The invertibility of the matrix α

implies that the vectors determined by the rows of $\alpha(t)$ can be taken as coordinate axes at t in $\mathbb{R}^3$. Thus $\varphi(t)$ is not normal to S^2 at t. If $\tau(t)$ denotes the projection of the vector $\varphi(t)$ on the tangent plane to S^2 at t, then $\tau(t) \neq 0$. This means that the function $t \mapsto \tau(t)$ is a continuous non-vanishing tangent vector field on S^2. But by a well known theorem in topology, this is impossible*. Thus the invertible matrix α can not exist.

The example above shows that the answer to the unimodular row problem depends very much on the ring A.

We now want to reformulate the problem algebraically: Let $a = (a_1, a_2, \ldots, a_r) \in A^r$. If $e_1, e_2, \ldots, e_r$ is the canonical basis of the free A-module A^r, then any A-linear map f of A^r into A is completely determined by its values at the basis elements $e_1, e_2, \ldots, e_r$. Since $a = \sum_{i=1}^{r} a_i e_i$, we notice that $f(a) = \sum_{i=1}^{r} a_i f(e_i)$. It is now easy to see that a is unimodular if and only if, there exists an A-linear map f of A^r into A such that $f(a) = 1$.

Let $O(a) = \{f(a): f: A^r \to A$ being an A-linear map$\}$. It is easily seen that $O(a)$ is an ideal of A, called the <u>order ideal</u> of a. Then a is unimodular in A^r if and only if $O(a) = A$. This is an intrinsic description of unimodularity, not depending on the coordinate system. If $a \in A^r$ is unimodular and $f: A^r \to A$ is an A-linear map with $f(a) = 1$, then A being a free A-module, and f being an epimorphism, we have a splitting

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* No purely algebraic proof of this example seems to be known. See Theorem (16.5) in "Lectures on Algebraic Topology" by M.J. Greenberg.

$A^r = \ker f \oplus Aa$, as A-module; here Aa is a free A-module with a as a basis element. Conversely, if $A^r = P \oplus Aa$ as A-module with Aa free having a as a basic element, then a is a unimodular element of A^r: For define $f\colon A^r \to A$ by $f|P = 0$ and $f(a) = 1$ and extend f to an A-linear map.

To sum up, we have the following result:

Proposition 1 - Let $a = (a_1, a_2, \ldots, a_r) \in A^r$. Then the following conditions are equivalent:

i) a is unimodular in A^r.

ii) There exists an A-linear map $f\colon A^r \to A$ such that $f(a) = 1$.

iii) The order ideal $O(a) = A$.

iv) The A-submodule Aa is a free direct summand of A^r, having a as a basis element.

Recall that P is a projective A-module, if there exists an A-module Q such that $P \oplus Q$ is a free A-module F. If P is finitely generated, we can choose F to be finitely generated, say $F = A^n$ for some $n \geq 1$. In this case the complement direct summand Q is finitely generated. If a is unimodular in A^r, we get a direct sum $A^r = P \oplus Aa$, with Aa free. Thus $A^r \cong P \oplus A$. It is natural to ask whether the projective module P is isomorphic to the free module A^{r-1}. It turns out that this is the module theoretic formulation of the unimodular row problem:

Proposition 2 - Let $a \in A^r$ be a unimodular element with $a = (a_1, a_2, \ldots, a_r)$. Write $A^r = P \oplus Aa$. Then the following conditions are equivalent:

i) P <u>is isomorphic to the free module</u> A^{r-1}.

ii) <u>The unimodular row</u> $(a_1, a_2, \ldots, a_r)$ <u>is the first row of a matrix</u> $\alpha \in GL_r(A)$.

iii) <u>The element</u> a <u>can be extended to a basis of</u> A^r.

<u>Proof</u>: (i) ⇔ (iii), because $P \cong A^r/Aa$.

(ii) ⇔ (iii), because the rows of invertible matrices correspond to bases of A^r.

We are thus led to consider the following "cancellations property"

$(C)_r$ $$P \oplus A \cong A^{r+1} \Rightarrow P \cong A^r,$$

which we have just seen is equivalent to completability of unimodular rows of length $(r+1)$. More generally we can consider the property

$(C)_{r,s}$ $$P \oplus A^s \cong A^{r+s} \Rightarrow P \cong A^r.$$

Note that, by cancelling one copy of A at a time in $(C)_{r,s}$, we conclude that: $(C)_r$ for all $r \geq$ some r_o implies $(C)_{r,s}$ for all $r \geq r_o$ and all $s \geq 0$. Modules P as in $(C)_{r,s}$ are said to be <u>stably free</u> (<u>of rank</u> r).

<u>Remark</u>: The above considerations also hold for rings R which may be non-commutative; the reader is referred to Bass [1] and Swan [7]. To avoid, pathologies one usually assumes that R has the "invariant basis property" (IBP) i.e. that $R^n \cong R^m \Rightarrow n = m$. This property is very mild. For example fields obviously have it; and any R that admits a homomorphism into a ring B with IBP

also has IBP. In fact $R^n \cong R^m \Rightarrow B^n = B \otimes_R R^n \cong B \otimes_R R^m = B^m$, whence $n=m$. It follows that any commutative ring A has IBP.

(1.2) When are projective modules free?

Let $A = K[t_1, t_2, \ldots, t_n]$ be a polynomial ring in n variables $t_1, t_2, \ldots, t_n$ over a field K. Serre's Problem was the following: Are the finitely generated projective A-modules free? Notice that an affirmative answer to this question would settle positively the unimodular row problem and all its equivalent formulations, for the ring $A = K[t_1, t_2, \ldots, t_n]$. Serre's problem has a long and interesting history, since 1955 when it first appeared in FAC. If the number of variables n is 0 or 1, the affirmative answer is trivially proved. The case $n=2$ was settled in the affirmative by Seshadri in 1958. In the same year in dealing with the Generalized Riemann-Roch Theorem, Grothendieck and Serre showed that the projective A-modules are stably free. In the early sixties, two stability theorems were proved for projective modules of large rank over arbitrary commutative rings ([B2]'). We now recall their statements.

Let A be a commutative ring, and write Spec (A) for the set of prime ideals of A, with the Zariski topology, in which closed sets are of the form $V\mathfrak{A} = \{\mathfrak{P} : \mathfrak{P} \supset \mathfrak{A}, \mathfrak{P} \text{ prime}\}$ for any subset $\mathfrak{A}$ of A. We write dim A = dim Spec(A) which is called the Krull dimension of A; for example, if A is Noetherian, then $\dim A[t] = \dim A + 1$.

We write Max (A) for the set of maximal ideals of A, viewed as a subspace of Spec (A).

Let P be a finitely generated projective A-module.

If $\mathfrak{p} \in \text{Spec}(A)$, then the localized module $P_{\mathfrak{p}}$ is free over the local ring $A_{\mathfrak{p}}$ ((2.3) Theorem 1).
Let $r(\mathfrak{p}) = r_P(\mathfrak{p})$ denote its rank. Then $r\colon \text{Spec } A \to \mathbb{Z}$ is a locally constant function, which we call the rank of P. These ranks are always constant if and only if $\text{Spec}(A)$ is connected, which is equivalent to A containing no idempotents other than 0 and 1. This is the case for example when A is an integral domain.

Theorem 1 - Let A be a commutative Noetherian ring, and let P be a projective module of rank $>$ dim max(A). Then

1) (Serre) $P \cong P' \oplus A$ for some P'.
2) (Cancellation) $P \oplus A^s \cong Q \oplus A^s \Rightarrow P \cong Q$.

It must be remarked that stably free modules of rank 1 over any commutative ring are free, permitting cancellation. This can be seen using the determinant of a projective module. Let P be a projective module of constant rank r. Then the exterior power $\Lambda^s P$ is a projective module of rank $\binom{r}{s}$, as we see by localizing. In particular, $\det(P) = \Lambda^r P$ is a projective module of rank 1; note that $\det(P) = P$ if $r=1$. Moreover for any module P', there is a natural isomorphism of graded anti-commutative algebras $\Lambda(P \oplus P') \cong \Lambda(P) \otimes_A \Lambda(P')$. It follows that if P' is projective of constant rank r', then $\det(P \oplus P') \cong \det(P) \otimes_A \det(P')$. Suppose $P = P_1 \oplus P_2 \oplus \ldots \oplus P_r$ where each P_i has rank 1, then we see that $\det P \cong P_1 \otimes \ldots \otimes P_r$. Now we can show that stably free modules of rank 1 are free.

Proposition 1 - If a projective module P is stably free of rank r, then $\det P \cong A$. In particular, if $r = 1$ then $P \cong A$.

Proof: In fact suppose that $P \oplus A^s \cong A^{r+s}$. Then $A = \det(A^{r+s}) \cong \det(P) \otimes \det(A^s) \cong \det(P) \otimes A \cong \det(P)$.

Further progress on Serre's problem waited for about ten years. In 1974, Murthy and Towber affirmed Serre's conjecture for $n=3$ with K algebraically closed. Then Roitman improved upon Theorem 1 by showing that P is free, if rank $P = n$. Then Suslin showed that P is free for $\operatorname{rk} P \geq \frac{n}{2} + 1$. There followed results of Suslin and Vašerštein, confirming the conjecture for $n \leq 5$. Finally, Quillen at MIT and Suslin in Leningrad independently obtained affirmative solution to Serre's conjecture for all n. Their methods are substantially different. For a historical note on Serre's problem, the reader is referred to [B4], [BS], and [E2].

2. The Local Theory

Recall that R denotes a possibly non-commutative ring. We prove in this section that all $P \in \mathcal{P}(R)$ are free, when R is local. R^* denotes the group of invertible elements of R.

(2.1) The Jacobson Radical and Nakayama's Lemma.

Definition 1 - A left (or right) R-module S is a simple R-module if $S \neq 0$ and if S is the only non-zero submodule of S.

If S is a simple R-module, then every non-zero element x of S generates S; that is, $Rx = S$. The mapping $f: R \to S$ defined by $a \mapsto ax$ is R-linear, whose kernel is a maximal left ideal of R. We first define the Jacobson radical of a left R-module M.

Definition 2 - Let M be any left R-module. Then the Jacobson radical of $M = \bigcap \ker f$, f varying over all R-homorphisms $M \to S$, S being any simple R-module. We shall simply refer to the radical of M, denoted by rad(M).

Lemma 1 - *The radical of M is the intersection of all maximal (proper) submodules of M.*

Proof: If $N \subset M$ is maximal, then $\frac{M}{N}$ is simple. If $f: M \to S$ with S simple, then either $f = 0$ and $\text{Ker}(f) = M$ or $f(M) = S$ and $\text{Ker}(f)$ is maximal. The lemma results from these observations.

Remark: If $f: M \to N$ is any R-homomorphism of M into a left module N, then $f(\text{rad } M) \subseteq \text{rad } N$, since every R-homomorphism $g: N \to S$ of N into a simple module S, gives by composition $g \circ f: M \to S$ and $g \circ f(\text{rad } M) = 0$.

Definition 3 - The left Jacobson radical of the ring R is the radical of the left R-module R and will be denoted by rad R. The right Jacobson radical of R is similarly defined.

We shall show that the two radicals are equal; hence we simply speak of the radical of R. This is done by obtaining an intrinsic description of the radical of R.

Proposition 1 - In the ring R, the following sets are equal:

1) The left Jacobson radical of R.
2) The intersection of the maximal left ideals of R.
3) The intersection of the annihilators of all simple left modules S (i.e., the primitive ideals of R).
4) The set of all elements a in R for which $1+Ra \subseteq R^*$.
5) The set of all elements a in R for which $1+aR \subseteq R^*$.
6) The right Jacobson radical of R.

Proof: The equality of the sets 1) and 2) is given by Lemma 1.

The sets 1) and 3) are equal: If S is any simple module, pick $x \neq 0$ in S. Then $f: R \to S$ defined by $f(1) = x$ is an R-homomorphism onto S; $\ker f = \mathrm{Ann}(x)$, annihilator of the element x. Clearly, $\mathrm{Ann}\, S \subseteq \mathrm{Ann}(x) = \ker f$; hence $\bigcap_S \mathrm{Ann}\, S \subseteq$ $\subseteq \mathrm{rad}\, R$. Also $\mathrm{rad}\, R \subseteq \bigcap_{x \in S} \mathrm{Ann}(x) = \mathrm{Ann}\, S$, whence $\mathrm{rad}\, R \subseteq$ $\subseteq \bigcap_S \mathrm{Ann}\, S$. The sets 1) and 4) are equal. We first claim that, if $a \in \mathrm{rad}\, R$, then $1-a \in R^*$. For this, if $R(1-a) \neq R$, then, $R(1-a)$ will be contained in a maximal left ideal L. By 2), $a \in \mathrm{rad}\, R \subseteq L$ and so both a, and $1-a$ belong to L; that is $1 \in L$, a contradiction. Thus $R(1-a) = R$ and $(1-a)$ has a left inverse. So there exists $u \in R$ such that $u(1-a) = 1$. This means $u = 1+ua$; but $-ua \in \mathrm{rad}\, R$ and the same argument applied to $-ua$ implies that $u = 1-(-ua)$ has a left inverse v with $vu = 1$. Hence $(1-a)u = vu(1-a)u = v \cdot 1 \cdot u = vu = 1$. Thus $1-a$ is invertible. To show the equality of 1) and 4), we

see that the argument above shows that $1+Ra \subseteq R^*$, whenever $a \in \text{rad } R$. Conversely, suppose $1+Ra \subseteq R^*$. If $a \notin \text{rad } R$, it follows by 2) that there would exist a maximal left ideal L, with $a \notin L$. So $Ra+L = R$, which means $1 = ba+u$ with $b \in R$ and $u \in L$. But $u = 1-ba \in 1+Ra \subseteq R^*$. This contradicts the fact that $u \in L$. Thus a has to belong to rad R.
The equality of the sets 5) and 6) is just the right-handed version of the equality of 1) and 4).
Finally 1) and 6) are equal: For this, we notice that the set 3) is a two-sided ideal of R. Hence rad R is a two sided ideal. If $a \in \text{Rad } R$, then $aR \subseteq \text{rad } R$ and so $1+aR \subseteq R^*$. Hence a blongs to the set 5), which is equal to the right Jacobson radical $\text{rad}' R$. Thus $\text{rad } R \subseteq \text{rad}' R$ and by symmetry $\text{rad}' R \subseteq$ $\subseteq \text{rad } R$. Therefore, the left and the right radicals are equal.

Corollary 1 - *The radical of R is a two-sided ideal of R.*

Proposition 2 (Nakayama's Lemma) - *Let J be a two ideal contained in the radical of R. Then the following hold:*

First form: *If M is a finitely generated R-module, and $JM = M$, then $M = (0)$.*

Second form: *If N is a submodule of an R-module M such that $\frac{M}{N}$ is finitely generated, and if $M = N+JM$, then $M = N$.*

Third form: *Let $f: N \to M$ be an R-homomorphism of R-modules such that coker $f = \frac{M}{f(N)}$ is finitely generated. Suppose the induced map $\bar{f}: \frac{N}{JN} \to \frac{M}{JM}$ is surjective. Then f is surjective.*

Proof: 1) Let $x_1, x_2, \ldots, x_n$ generate M with n minimal. We claim that $n=0$. If $n \geq 1$, then $x_1 \in JM$; say, $x_1 = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, with $a_i \in J$. Then $(1-a_1)x_1 = a_2x_2 + \ldots + a_nx_n$. But $(1-a_1)$ is invertible, by Proposition 1. Hence $x_1 \in Rx_2 + \ldots + Rx_n$, giving $M = Rx_2 + \ldots + Rx_n$. This contradicts the minimality of n.

2) The second form is proved by applying 1) to the finitely generated module $\frac{M}{N}$.

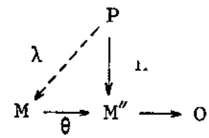
3) The third statement follows, if we apply 2) to the module M and the submodule $f(N)$; the surjectivity of $\bar{f}$ is precisely the statement: $M = f(N) + JM$.

(2.2) Projective modules.

Recall that the functor $\mathrm{Hom}_R(P,-)$ is a left exact functor from the category of left R-modules and R-homomorphisms to the category of abelian groups and group homomorphisms. This means that, whenever $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of R-modules and R-homomorphisms, $0 \to \mathrm{Hom}(P,M') \xrightarrow[f_*]{} \mathrm{Hom}(P,M) \xrightarrow[g_*]{} \mathrm{Hom}(P,M'')$ is an exact sequence of abelian groups; here if $\lambda: P \to M'$, then $f_*(\lambda) = f\circ\lambda$. The functor $\mathrm{Hom}(P,-)$, may not be exact. It is easily checked that projective modules as defined in (1.1) are precisely those modules P for which $\mathrm{Hom}(P,\cdot)$ is exact. To see this, just observe that exactness of $\mathrm{Hom}(P,\cdot)$ is equivalent to the following property of P: Given R-modules M and M'', an epimorphism $\theta: M \to M''$, and a homomorphism $h: P \to M''$, there exists a homomorphism $\lambda: P \to M$ such that $\theta\circ\lambda = h$ or equivalently if the following diagram of

R-modules and R-homomorphisms can be completed to be commutative:

(*)



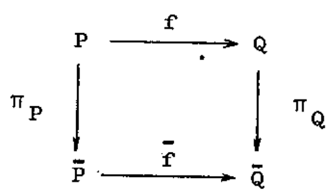
If P is a free module or a direct summand of a free module, then P clearly has the desired property. The converse also holds, since we always have an epimorphism $F \to P \to 0$ with F free, and we can take $h = 1_P$ above. These results are summed up in the following:

Proposition 1 - For an R-module P the following conditions are equivalent:

1) P is projective; that is, a direct summand of a free module.
2) Any diagram (*) as above can be completed.
3) Any surjective R-homomorphism $M \xrightarrow{\pi} P \to 0$ splits; that is π admits a section $s: P \to M$ such that $\pi \circ s = 1_P$.
4) The functor $\mathrm{Hom}(P,-)$ is exact.

(2.3) Reduction modulo the radical and projective modules.

Let J be a two-sided ideal, contained in rad R. If M is an R-module, then $\bar{M} = \frac{M}{JM} \simeq \frac{R}{J} \underset{R}{\otimes} M$ is an $\frac{R}{J}$ - module. We consider commutative diagrams of R-modules:



Remarks:

i) If $P \in \mathcal{P}(R)$, then every R-homomorphism $\lambda: \bar{P} \to \bar{Q}$ comes from some $f: P \to Q$; that is, $\lambda = \bar{f}$, for some f. We say that λ admits a lifting $f: P \to Q$.

ii) Suppose Q is finitely generated. If $\bar{f}$ is surjective, then f is surjective. This follows from the third form of Nakayama's Lemma.

iii) If $Q \in \mathcal{P}(R)$, and $\bar{f}$ is surjective, then f is surjective and moreover, f splits.

Proposition 1 - *Suppose* $P, Q \in \mathcal{P}(R)$ *and* $\bar{f}: \bar{P} \to \bar{Q}$ *is an* $\bar{R}$*-isomorphism. Then* $\bar{f}$ *admits a lifting* $f: P \to Q$, *and any such* f *is an R-isomorphism.*

Proof: By Remarks above, we know that there exists a lifting f of $\bar{f}$ and that f is a split surjection. Write $P = \ker f \oplus P_o$. Now $\overline{\ker f} = \ker \bar{f} = (0)$, since $\bar{f}$ is injective. Since $\ker f$ is a direct summand of the finitely generated module P, and $\overline{\ker f} = 0$, Nakayama's Lemma implies that $\ker f = 0$. Thus f is an isomorphism.

The following corollaries are now immediate:

Corollary 1 - *Let* $P, Q \in \mathcal{P}(R)$. *If* $\bar{P}$ *and* $\bar{Q}$ *are isomorphic as* $\bar{R}$*-modules, then* P *and* Q *are R-isomorphic.*

Corollary 2 - *If* $P \in \mathcal{P}(R)$, *then the canonical map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_{\bar{R}}(\bar{P})$ *is a surjection. In particular taking* P *to be free, we see that the canonical map* $GL_n(R) \to GL_n(\bar{R})$ *is surjective.*

Corollary 3 - Let $P \in \mathcal{P}(R)$ and $e_1, e_2, \ldots, e_n$ in P. If $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ is an $\bar{R}$-basis of $\bar{P}$, then P is free with basis $\{e_1, e_2, \ldots, e_n\}$.

Recall that R is a local ring if $\frac{R}{\text{rad } R}$ is a division ring. We can now state the principal result of this section.

Theorem 1 - Every finitely generated projective module over a local ring is free.

Proof: Let R be local and $P \in \mathcal{P}(R)$. Then $\bar{R} = \frac{R}{\text{rad } R}$ is a division ring; hence $\bar{P}$ is free as an $\bar{R}$-module. Then by Corollary 3 P is R-free.

3. Localization and Flat Base Change

The material of this section is mostly quite standard. We assemble it here mainly for reference, and to fix some notational conventions. The informed reader is advised to proceed directly to §4.

A. Localization.

(3.1) Review.

Let A be a commutative ring and S a multiplicative set in A; i.e., $1 \in S$ and $s_1, s_2 \in S$ implies that the product $s_1 s_2 \in S$. If M is an A-module, we introduce a relation in $M \times S$ as follows: $(x,s) \sim (y,t)$ if $\exists\, u \in S$ such that $u(tx-sy) = 0$. This is an equivalence relation; the equivalence class of (x,s) will be denoted by $\frac{x}{s}$; M_S will denote the set of all equivalence classes. Sometimes we shall also write $S^{-1}M$ or $M[S^{-1}]$ for M_S. Then A_S obtained from the ring A is again a ring, if we add and multiply the "fractions" in a natural way, M_S is isomorphic to $A_S \otimes_A M$ as A_S-modules.

We have a canonical map $\theta: M \to M_S$ given by $\theta(x) = \frac{x}{1}$; $\ker \theta = \{x \in M: \exists\, t \in S \ni tx = 0\}$.

There are two special cases of multiplicative sets of great interest: 1) If $s \in A$, let $S = \{s^n: n=0,1,2,\ldots\}$. In this case, we write M_s instead of M_S. 2) If $\mathfrak{P}$ is a prime ideal of A, then $S = A\backslash\mathfrak{P}$ is a multiplicative set. We write $M_{\mathfrak{P}}$ instead of M_S and call $M_{\mathfrak{P}}$ the <u>localization</u> of M with respect to the prime ideal $\mathfrak{P}$. $A_{\mathfrak{P}}$ is a local ring with radical $\mathfrak{P}A_{\mathfrak{P}}$.

<u>Localization is an exact functor</u>: If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then $0 \to M'_S \to M_S \to M''_S \to 0$ is an exact sequence of A_S-modules. Notice that this is equivalent to saying that $A_S \otimes -$ is an exact functor.

One important result we use is <u>the permutability of residue class ring and localization</u>: If S is any multiplicative set, and $\mathfrak{A}$ any ideal of A then

$\frac{A_S}{\mathfrak{A}A_S} \cong \left(\frac{A}{\mathfrak{A}}\right)_{S'}$, where S' denotes the image of S in $\frac{A}{\mathfrak{A}}$.

Finally we must remark that the above considerations hold for any ring R, if S is a central multiplicative set; that is, if $S \subseteq$ center of R. Hence we shall freely talk of R_S in this case.

We now define the dimension or Krull dimension dim A of a commutative ring A: A chain of prime ideals of A of length n is a strictly descending sequence $P_o \supset P_1 \supset \ldots \supset P_n$ of $(n+1)$ prime ideals. We define dim A to be the supremum of the lengths of all chains of prime ideals of A. If $A \neq (0)$, dim $A \geq 0$ or ∞.

B. Flat Base Change.

(3.2) Extended modules.

Let $\varphi: R \to R'$ be a ring homomorphism. We say that an R'-module E is extended, if $E \cong R' \overset{\varphi}{\underset{R}{\otimes}} M$ for some R-module M. In general, E does not determine M up to isomorphism. However, if there is a retraction $\psi: R' \to R$ such that $\psi \circ \varphi = 1_R$, then M is determined upto isomorphism. Indeed, $M \simeq R \underset{R}{\otimes} M \simeq R \overset{\psi}{\underset{R'}{\otimes}} (R' \overset{\varphi}{\underset{R}{\otimes}} M) \cong R \underset{R'}{\otimes} E$. Notice that free R'-modules are extended from R. If P is a projective R-module, then the extended module $R' \underset{R}{\otimes} P$ is a projective R'-module.

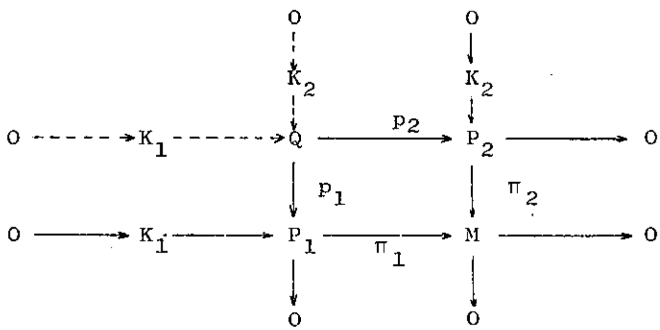
Suppose $A \to B$ is a homomorphism of commutative rings, and $\varphi: R \to R'$ a homomorphism of A-algebras. If the R'-module E is extended from R, then the $B \underset{A}{\otimes} R'$-module $B \underset{A}{\otimes} E$ is extended from $B \underset{A}{\otimes} R$ (via $B \otimes \varphi$). Indeed if $E \simeq R' \underset{R}{\otimes} M$ then $B \underset{A}{\otimes} E \simeq (B \underset{A}{\otimes} R') \underset{B \otimes R}{\otimes} (B \otimes M)$, by commutativity of the base change $A \to B$ with tensor products.

(3.3) Schanuel's Lemma; faithful flatness.

Schanuel's Lemma - Let R be any ring and $0 \to K_i \to P_i \xrightarrow{\pi_i} M \to 0$ be two exact sequences of R-modules, with P_i projective $(i=1,2)$. Then $K_1 \oplus P_2 \simeq K_2 \oplus P_1$.

Proof: Consider the direct sum $P_1 \oplus P_2$ and the submodule

$Q = \{(x_1,x_2) \in P_1 \oplus P_2 : \pi_1 x_1 = \pi_2 x_2\}$. Consider the following diagram:



The bottom row and the extreme right column are the given exact sequences; p_1 and p_2 are the natural projection maps. The kernel of $Q \xrightarrow{p_1} P_1$ is precisely $\{0\} \times K_2$ and similarly $\ker Q \xrightarrow{p_2} P_2$ is K_1. Moreover, the projections p_1 and p_2 are onto. These observations turn the top row and the left hand column to be exact sequences. Since P_1 and P_2 are projective, the last two sequences are split; that is, $P_1 \oplus K_2 \cong Q \cong P_2 \oplus K_1$.

Corollary 1 - Let K_1 and K_2 be as above. Then K_1 is projective if and only if K_2 is projetive.

Let R, R' be rings and F and additive functor from the category R-mod of R-modules and R-homomorphisms to R'-mod; that is F preserves direct sums.

To the sequence (ε): $M' \xrightarrow{f} M \xrightarrow{g} M''$ of R-modules, there corresponds a sequence $(F\varepsilon)$: $FM' \xrightarrow{Ff} FM \xrightarrow{Fg} FM''$ of R'-modules. We say that the functor F is exact, if the sequence $(F\varepsilon)$ is exact, whenever (ε) is exact. We say that F is *faithfally exact* if F is exact and $FM = 0$ implies $M = 0$ for R-modules M.

Proposition 1 - *The following conditions are equivalent for a functor* F

1) F *is faithfully exact*.
2) F *is exact and* $FS \neq 0$ *for all simple modules* S.
3) *For every sequence* (ε) *of* R-*modules*, (ε) *is exact if and only if* $(F\varepsilon)$ *is exact*.

Proof: It is obvious that 1) ⇒ 2). We prove 2) ⇒ 1). We must show that for any R-module M, $M \neq 0 \Rightarrow FM \neq 0$. We pick a finitely generated submodule N of M such that $N \neq 0$. Since N is finitely generated there exist quotient modules S of N which are simple. We have the following diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M \\ & & \downarrow & & \\ & & S & & \\ & & \downarrow & & \\ & & 0 & & \end{array} \qquad\qquad \begin{array}{ccccc} 0 & \longrightarrow & FN & \longrightarrow & FM \\ & & \downarrow & & \\ & & FS & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

Since F is exact, the rows and the columns are exact. Moreover $FS \neq 0$ implies that $FN \neq 0$ and hence $FM \neq 0$.

1) ⇒ 3): We need only show that $(F\varepsilon)$ exact implies that (ε) is exact. Notice that F exact implies that F preserves kernels, images, cokernels etc.; similarly, F faithfully exact implies that the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R'}(FM,FN)$ is injective:

For $Ff = 0$ implies $0 = \mathrm{Im}(Ff) = F(\mathrm{Im}(f)) \Rightarrow \mathrm{Im}(f) = 0 \Rightarrow f = 0$.

Let (ϵ): $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence, such that $(F\epsilon)$: $FM' \xrightarrow{Ff} FM \xrightarrow{Fg} FM''$ is exact. From $0 = Fg \circ Ff = F(g \circ f)$, we get $g \circ f = 0$. Hence $\mathrm{im}\, f \subseteq \ker g$. We have $F(\frac{\ker g}{\mathrm{im}\, f}) = \frac{\ker Fg}{\mathrm{im}\, Ff} = 0$, whence $\mathrm{im}\, f = \ker g$. Thus (ϵ) is an exact sequence.

3) $\Rightarrow$ 1): F is trivially exact. Suppose $FM = 0$ for some R-module M. From the exact sequence $0 \to FM \to 0$, we see that $0 \to M \to 0$ is exact. Hence done.

A right R-module M <u>is flat</u> (<u>faithfully flat</u>) if the functor $M \otimes_R \cdot$ is exact (faithfully exact).

<u>Examples</u>

(1) R itself is a faithfully flat R-module.

(2) A direct sum $\oplus M_i$ is flat if and only if each summand M_i is flat.

(3) A direct limit $\varinjlim M_i$ of flat modules is flat; this is because $\varinjlim$ is exact and commutes with $\otimes$; that is, $\otimes$ preserves direct limits.

(4) Let s be an element in the center of R. Then $R_s = \varinjlim (R \xrightarrow{s} R \xrightarrow{s} R \to \dots)$ and so R_s is a flat module.

(5) If S is a central multiplicative set, then $R_S = \varinjlim_{s \in S} R_s$, where S is ordered by divisibility; so R_S is flat.

(6) Let A be a commutative Noetherian ring, and J an ideal of A. Let $\hat{A} = \varprojlim \frac{A}{J^n}$ be the completion of A with respect

to the J-adic topology. Then $\hat{A}$ is flat. $\hat{A}$ is faithfully flat if and only if $J \subseteq \mathrm{rad}(A)$.

(3.4) Finite presentability; descent.

Let $\varphi: A \to A'$ be a homomorphism of commutative rings. If M is an A-module, we write $M' = A' \otimes_A M$. Let R be a possibly non-commutative A-algebra. Then $R' = A' \otimes_A R$ is an A'-algebra. If M is an R-module, then M' is an R'-module, since $M' = A' \otimes_A M = A' \otimes_A R \otimes_R M = R' \otimes_R M$. Let $P, M, \ldots$ be R-modules. Then the natural map $\varphi_P: [\mathrm{Hom}_R(P,M)]' \to \mathrm{Hom}_{R'}(P',M')$ is an A'-homomorphism for each P.

Recall that an R-module M is said to be finitely presented if there exists an exact sequence $R^m \to R^n \to M \to 0$ for some natural numbers m and n. It is clear that finitely generated projective modules are finitely presented.

Remarks:

1) If $P = R$, φ_R is an isomorphism. Hence φ_F is an isomorphism for every finitely generated free R-module F. Since both Hom and $\otimes$ are additive functors, it follows that φ_P is an isomorphism for every $P \in \mathcal{P}(R)$.

2) If A' is flat over A, then φ_P is an isomorphism for every finitely presented R-module P.

Proof: Fix a module M. Since P is finitely presented, choose an exact sequence $R^n \longrightarrow R^m \longrightarrow P \longrightarrow 0$. Since $\mathrm{Hom}(\cdot, M)$ is a right exact contravariant functor and A' is flat, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to [\mathrm{Hom}_R(P,M)]' & \to & [\mathrm{Hom}_R(R^m,M)]' & \to & [\mathrm{Hom}_R(R^n,M)]' \\
\downarrow \varphi_P & & \wr\!\!\downarrow \varphi_{R^m} & & \wr\!\!\downarrow \varphi_{R^n} \\
0 \to \mathrm{Hom}_{R'}(P',M') & \to & \mathrm{Hom}_{R'}(R'^m,M') & \to & \mathrm{Hom}_{R'}(R'^n,M')
\end{array}$$

By 1) above, the two extreme right maps are isomorphisms. Hence, by the Five Lemma, φ_P is an isomorphism.

Proposition 1 (Descent properties) - *Let* $A \to A'$ *be a homomorphism of commutative rings such that* A' *is a faithfully flat* A-*module*. *Let* R *be an* A-*algebra*, P *an* R-*module*, *and* $P' = A' \otimes_A P$, *a module over* $R' = A' \otimes_A R$. *Then*

1) P *is finitely generated* $\Leftrightarrow$ P' *is finitely generated*.
2) P *is finitely presented* $\Leftrightarrow$ P' *is finitely presented*.
3) $P \in \mathcal{P}(R) \Leftrightarrow P' \in \mathcal{P}(R')$.

Proof: We need only to prove $\Leftarrow$.

1) Suppose $P' = A' \otimes_A P$ is finitely generated as an R'-module. Then there exists a finitely generated R-submodule $Q \subseteq P$ such that $Q' = P'$; that is, $(P/Q)' = 0$. Faithful flatness implies that $P/Q = 0$. So $P = Q$ and P is finitely generated.

2) Suppose P' is finitely presented. Let $0 \to K \to R^n \to P \to 0$ be an exact sequence. Then $0 \to K' \to R'^n \to P' \to 0$ is an exact sequence. By (3.3) Schanuel's Lemma, K' is finitely generated, since P' is finitely presented. By (1) then K is finitely generated.

3) Suppose P' is finitely generated and projective. Then by (2), P is finitely presented. By (3.4), Remark 2, $\varphi_P: [\mathrm{Hom}_R(P,\cdot)]' \to \mathrm{Hom}_{R'}(P',\cdot)$ is an isomorphism. Since P' is projective, $\mathrm{Hom}_{R'}(P',\cdot)$ is an exact functor. Hence so is $\mathrm{Hom}_R(P,\cdot)$ by faithful flatness of A'. Thus P is projective.

Proposition 2 - Let S be a central multiplicative set in a ring R, ordered by divisibility. If M is finitely presented, the canonical map

$$\varinjlim_{s\in S} \mathrm{Hom}_{R_s}(M_s,N_s) \to \mathrm{Hom}_{R_S}(M_S,N_S)$$

is bijective. If N is also finitely presented, the canonical map

$$\varinjlim_{s\in S} \mathrm{Isom}_{R_s}(M_s,N_s) \to \mathrm{Isom}_{R_S}(M_S,N_S)$$

is bijective, where "Isom" denotes the set of isomorphisms.

Proof: By Remark 2 above, $\mathrm{Hom}_{R_S}(M_S,N_S) \cong \mathrm{Hom}_R(M,N)_S \cong$ $\cong \varinjlim_{s\in S} (\mathrm{Hom}_R(M,N))_s \cong \varinjlim_{s\in S} \mathrm{Hom}_{R_s}(M_s,N_s)$. The second assertion follows from this, once we show that if $u: M_s \to N_s$ and if $u_S: M_S \to N_S$ is an isomorphism, then $u_t: M_{st} \to N_{st}$ is an isomorphism for some t. Since N is also assumed finitely presented, we may after "enlarging" s to some ss' if necessary, assume there is a homomorphism $v: N_s \to M_s$ such that $v_S = u_S^{-1}$. Then $(1_{M_s} - vu)_S = 0$ and $(1_{N_s} - u\circ v)_S = 0$; so these equations hold already with s replaced by some $t \in S$, whence the claim.

C. Affine Patching.

(3.5) Affine Patching.

In any category C, a commutative square

$$\begin{array}{ccc} M & \xrightarrow{p_1} & M_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ M_2 & \xrightarrow[\alpha_2]{} & M' \end{array}$$

is said to be cartesian if it has the following universal property: For every commutative square

$$(*) \qquad \begin{array}{ccc} X & \xrightarrow{q_1} & M_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ M_2 & \xrightarrow[\alpha_2]{} & M' \end{array}$$

in C, there exists a unique morphism $f: X \to M$ such that $p_i \circ f = q_i$, $i=1,2$. A cartesian square is sometimes also called a pullback diagram, and we call M "the" fibre product of M_1 and M_2 over M'. When $C = R$ - mod, the category of R-modules, we can construct M as

$$M = \{(m_1,m_2) \in M_1 \times M_2 : \alpha_1(m_1) = \alpha_2(m_2)\}$$

and take p_1, p_2 to be coordinate projections. Hence the cartesian property of the square (*) can be expressed as an exact sequence:

$$0 \longrightarrow M \xrightarrow{\binom{p_1}{p_2}} M_1 \oplus M_2 \xrightarrow{(\alpha_1,\ -\alpha_2)} M' .$$

We also notice that p_1 is a monomorphism if and only if α_2 is.

Lemma 1 - Let A be a commutative ring and S_i, $i=0,1$ pairwise comaximal multiplicative sets; this means the following condition is satisfied:

$$\forall\ (s_o,s_1) \in S_o \times S_1, \qquad As_o + As_1 = A.$$

Write $S = S_oS_1 = \{s_os_1 \mid s_o \in S_o,\ s_1 \in S_1\}$. Let M be an A-module. Then the following square $\square_A$ of localizations with the natural maps is cartesian:

$$\square_A \qquad \begin{array}{ccc} M & \longrightarrow & M_{S_1} \\ \downarrow & & \downarrow \\ M_{S_o} & \longrightarrow & M_S \end{array}$$

Proof: If $B = A_{S_o} \times A_{S_1}$, then B is a faithfully flat A-algebra

B is A-flat, because each one of the factors is. We must show that the functor $B \otimes_A \cdot$ is faithfully exact. By (3.3) Proposition 1, item 2) it is enough to show that for every maximal ideal $\mathfrak{m}$ of A, $B \otimes_A \frac{A}{\mathfrak{m}} \neq (0)$. By comaximality $\mathfrak{m} \cap S_o = \emptyset$ or $\mathfrak{m} \cap S_1 = \emptyset$ and so either $\mathfrak{m} A_{S_o} \neq A_{S_o}$ or $\mathfrak{m}A_{S_1} \neq A_{S_1}$. Thus $B \otimes \frac{A}{\mathfrak{m}} =$ $= \frac{A_{S_o}}{\mathfrak{m}A_{S_o}} \oplus \frac{A_{S_1}}{\mathfrak{m}A_{S_1}} \neq (0)$. The square is cartesian if and only if the sequence

$$(\ ) \qquad 0 \longrightarrow M \longrightarrow M_{S_o} \oplus M_{S_1} \longrightarrow M_S$$

is exact and by faithful flatness, this is so if and only if the sequence obtained by tensoring with B is exact; this in turn holds if and only if the two sequences obtained by tensoring with A_{S_o} and A_{S_1} are exact. In other words, we have to prove that the squares $\square_{A_{S_o}}$ and $\square_{A_{S_1}}$ are cartesian. Thus we may assume, say, that $S_o \subseteq A^*$, the set of units of A, so that $S = S_1$. It is obvious that the square

$$\begin{array}{ccc} M & \longrightarrow & M_S \\ \| & & \| \\ M & \longrightarrow & M_S \end{array}$$

is cartesian.

Remark: The proof shows that the sequence ( ) is exact, even with a zero added on the right.

The above lemma shows that the module M can be reconstructed as the fibre product of the two localizations M_{S_o} and M_{S_1}. In fact, we can go one more step.

Lemma 2 - Let M_i be two A_{S_i}-modules, $i=0,1$ and let there be given an A_S-isomorphism $\alpha: M_{0S} \to M_{1S}$. Define the A-module M by the cartesian square below. Then the natural maps $M_{S_i} \to M_i$ are A_{S_i}-isomorphisms for $i = 0,1$.

Proof: We are given the cartesian square

$$\begin{array}{ccccc} M & & \longrightarrow & & M_1 \\ \downarrow & & & & \downarrow \\ M_o & \longrightarrow & M_{oS} & \xrightarrow[\alpha]{\sim} & M_{1S} \end{array}$$

As in the previous lemma, we can make base change $A \to B = A_{S_o} \times A_{S_1}$ and reduce to the case, where $S_o = \{1\}$, say. In this case, the cartesian square diagram reads:

$$\begin{array}{ccccc} M & & \longrightarrow & & M_1 \\ \downarrow & & & & \downarrow \\ M & \longrightarrow & M_S & \xrightarrow[\alpha]{\sim} & M_1 \end{array}$$

The assertion is now obvious.

It is convenient to think of the content of the above lemma as a statement of equivalence between categories:

Proposition 1 - Let A, A_{S_o}, A_{S_1} and S as in Lemma 1. The category of A-modules is equivalent to the category of triples (M_o, M_1, α), where M_o and M_1 are A_{S_o} and A_{S_1} modules respectively and $\alpha: M_{oS} \to M_{1S}$ is an A_S-isomorphism. If M corresponds to (M_o, M_1, α) then

1) M is finitely generated if and only if M_o and M_1 are.
2) M is finitely presented if and only if M_o and M_1 are.
3) $M \in \mathcal{P}(A)$ if and only if $M_i \in \mathcal{P}(A_{S_i})$, $i = 0,1$.

4. Serre's Conjecture

A. The Main Theorems

In this section, we shall state without proof the main theorems that yield a proof of Serre's conjecture: Throughout $t, t_1, t_2, \ldots, t$ etc. will denote indeterminates.

(4.1) Local Horrocks' Theorem (Algebraic Form) - Let $A[t]$ be a polynomial ring in t over a local ring A and P a finitely generated projective module over $A[t]$. If there exists a monic polynomial f in $A[t]$ such that the localization P_f is free as an $A[t]_f$-module, then P is already free over $A[t]$.

It was in the following geometric form that the above theorem was originally proved by Horrocks in 1964. In this geometric form, it gives a criterion for a vector bundle over the affine line $\mathbb{A}^1_A$ to be trivial, A being a commutative ring. As we do not use this geometric form, we do not explain the terms involved.

Local Horrocks' Theorem (Geometric form) - Let $A[t]$ be a polynomial ring in t over a local ring A and let P be a finitely generated projective $A[t]$-module. Write Spec $A[t]$ = $= \mathbb{A}^1_A$ and let $\tilde{P}$ be the locally free sheaf corresponding to P. If $\tilde{P}$ extends to a locally free sheaf on the projective line $\mathbb{P}^1_A$, then P is a free $A[t]$-module.

Serre's conjecture is solved by an affine version of Horrocks' Theorem, where the local ring A is replaced by any commutative ring A. This affine version, sometimes called the Affine Horrocks' Theorem is made possible by a localization theorem due to Quillen:

(4.2):
Quillen's Localization Theorem - Let A be a commutative ring and P a finitely presented $A[t]$-module. If for every $\mathfrak{P} \in$ Spec A, the localized module $P_{\mathfrak{P}}$ over the polynomial ring $A_{\mathfrak{P}}[t]$ is extended from $A_{\mathfrak{P}}$, then P is extended from A.

In other words, Quillen's Localization Theorem says that the problem of extending is local with respect to the base ring A.

(4.3) Affine Horrocks' Theorem (First form), (Quillen-Suslin)

Let A be a commutative ring and P a finitely generated projective $A[t]$-module. Suppose there exists a monic polynomial $f \in A[t]$ such that the $A[t]_f$-module P_f is extended from a projective module over A (*), then P itself is extended from A.

Proof: By Quillen's Localization Theorem, it is enough to show that for every $\mathfrak{P} \in \operatorname{Spec} A$, the localized module $P_{\mathfrak{P}}$ is extended from the local ring $A_{\mathfrak{P}}$. Fix $\mathfrak{P} \in \operatorname{Spec} A$, $P_{\mathfrak{P}}$ is a finitely generated projective $A_{\mathfrak{P}}[t]$-module. Clearly, f can be identified with a monic polynomial in $A_{\mathfrak{P}}[t]$. The rings $A_{\mathfrak{P}}[t]_f$ and $(A[t]_f)_{\mathfrak{P}}$ are isomorphic and so are the modules $(P_{\mathfrak{P}})_f$ and $(P_f)_{\mathfrak{P}}$. By hypothesis P_f is extended from a projective module Q over A; that is, $P_f \simeq A[t]_f \otimes_A Q$ over $A[t]_f$ and so $(P_{\mathfrak{P}})_f \simeq$ $\simeq (A_{\mathfrak{P}}[t]_f) \otimes_{A_{\mathfrak{P}}} Q_{\mathfrak{P}}$ as $(A_{\mathfrak{P}}[t]_f)$ modules. Since $A_{\mathfrak{P}}$ is a local ring, the projective module $Q_{\mathfrak{P}}$ is free by (2.3) Theorem 1. Thus $(P_{\mathfrak{P}})_f$ is free over $(A_{\mathfrak{P}}[t]_f)$, with $A_{\mathfrak{P}}$ local. By Local Horrocks' Theorem then, the projective module $P_{\mathfrak{P}}$ is $A_{\mathfrak{P}}[t]$-free and so extended from $A_{\mathfrak{P}}$. This proves the result.

Let A be a commutative ring and T the set of monic polynomials of $A[t]$. We denote by $A(t)$ the localization $A[t]_T$.

Affine Horrocks' Theorem (Second form) - Let P be a finitely generated projective $A[t]$-module, and let C be any intermediate ring between $A[t]$ and $A(t)$. If $C \otimes_{A[t]} P$ is extended from A, then P is already extended from A.

(*) The hypothesis that the base module indicated be projective is redundant; see Remark 1 below.

Proof: If we take $C = A[t]_f$, we get the first form from the second. We shall deduce the second form from the first. If $C \otimes_{A[t]} P$ is extended from A, then so is $A(t) \otimes_{A[t]} P$. Hence it is enough to prove the result for $C = A(t)$. Assume then that $A(t) \otimes_{A[t]} P$ is extended from an A-module Q. By Remark 1 below, Q is projective. By (3.4) Proposition 2, there exists a monic polynomial $f \in T$ such that $P_f \simeq Q[t]_f$ as $A[t]_f$-modules; that is, P_f is extended from A. By the first form, we conclude that P is extended from A.

Remark 1. If Q is an A-module and the extended module $A(t) \otimes_A Q$ is in $\mathcal{P}(A(t))$, then $Q \in \mathcal{P}(A)$. This follows from the fact that $A(t)$ is a faithfully flat A-algebra (vide (5.7) Proposition 2).

The above forms can be slightly sharpened as follows:

Affine Horrocks' Theorem (Strong form) - *Let A be a commutative ring, $P \in \mathcal{P}(A[t])$, and $P_o \in \mathcal{P}(A)$. Suppose the extended $A(t)$-modules $A(t) \otimes_{A[t]} P$ and $A(t) \otimes_A P_o$ are isomorphic, then* $P \cong P_o[t] = A[t] \otimes_A P_o$.

Proof: The strong form is clearly a consequence of the second form above and the following supplement.

We use the following notation: If M is an A-module, $M[t] = A[t] \otimes_A M$, and $M(t) = A(t) \otimes_A M$.

Supplement. *Let P_o and $Q_o \in \mathcal{P}(A)$ and suppose the extended modules $P_o(t)$ and $Q_o(t)$ are $A(t)$-isomorphic. Then* $P_o \cong Q_o$.

Proof: Write $s = t^{-1}$, and consider the affine patching diagram (see (3.5)):

$$\begin{array}{ccc} A[s] & \longrightarrow & A[s]_s = A[t,t^{-1}] \\ \downarrow & & \downarrow \\ B=A[s]_{1+sA[s]} & \longrightarrow & A[s]_{s(1+sA[s])} = A(t) \end{array}$$

For the last equality, see (5.9) Proposition 3.
The comaximality conditions on the multiplicative sets are clearly satisfied.

Define $V \in \mathcal{P}(A[s])$ by the cartesian square:

$$\begin{array}{ccccc} V & & \longrightarrow & & P_o[t,t^{-1}] \\ \downarrow & & & & \downarrow \\ B \otimes_A Q_o = Q_o[s]_{(1+sA[s])} & \longrightarrow & Q_o(t) & \xrightarrow{\alpha} & P_o(t) \end{array}$$

where α is a given isomorphism of $Q_o(t)$ and $P_o(t)$. By (3.5) Lemma 2, $V_s \cong P_o[t,t^{-1}]$ and $V_{1+sA[s]} \cong Q_o[s]_{1+sA[s]}$. Also, $A(t) \underset{A[s]}{\otimes} V \in \mathcal{P}(A(t))$ and so by Affine Horrocks' Theorem V is extended from an A-module $V_o \in \mathcal{P}(A)$. Moreover,

$$V_o \cong \frac{V}{sV} \cong \frac{A[s]}{sA[s]} \underset{A[s]}{\otimes} V \cong \frac{B}{sB} \underset{A[s]}{\otimes} V \cong (B \underset{A}{\otimes} Q_o)/s\,(B \underset{A}{\otimes} Q_o) \cong$$

$$\cong \frac{Q_o[s]}{sQ_o[s]} \cong Q_o$$ (on putting $s=0$). But since V is extended,

$$V_o \simeq \frac{V}{(s-1)V} \simeq \frac{A[s]}{(s-1)A[s]} \underset{A[s]}{\otimes} V \simeq \frac{A[s]_s}{(s-1)A[s]_s} \underset{A[s]}{\otimes} V =$$

$$= \frac{A[t,t^{-1}]}{(s-1)A[t,t^{-1}]} \underset{A[s]}{\otimes} V \simeq \frac{P_o[t,t^{-1}]}{(t^{-1}-1)P_o[t,t^{-1}]} \simeq P_o$$ (on putting $t=1$).

Thus P_o and Q_o are both isomorphic to V_o.

B. The proof of Serre's conjecture.

(4.4) Quillen Classes; conjecture $(B\text{-}Q_d)$.

Before we take up the proof of Serre's conjecture, a few remarks on extended modules are in order. Suppose E is an $A[t]$-module that is extended from A, then clearly $E \underset{A[t]}{\cong} E_o[t]$, where $E_o \underset{A}{\cong} \frac{E}{tE}$. By using the retraction $A[t] \to A$ given by $t \mapsto a$ for $a \in A$, we see that $E_o \underset{A}{\cong} E_a = \frac{E}{(t-a)E}$. More generally, these remarks hold for any ring R, provided we take $a \in R$ to be a central element. We also see that if $E \in \mathcal{P}(R[t])$, then $E_a \in \mathcal{P}(R)$ for all central elements a.

Following Lam's formulation, we shall call a class QC of commutative rings a Quillen class if the following conditions are satisfied:

Q_1: $A \in QC \Rightarrow$ the localization $A_{\mathfrak{P}} \in QC$, for every prime $\mathfrak{P} \in \operatorname{Spec} A$.

Q_2: $A \in QC \Rightarrow A(t) \in QC$. (Vide §5).

Q_3: $A \in QC$, A local $\Rightarrow$ all finitely generated projective $A[t]$-modules are free.

Theorem 1 - Let QC be a Quillen class of commutative rings. If $A \in QC$ and $n \geq 0$, then every finitely generated projective module P over the polynomial ring $A[t_1, t_2, \ldots, t_n]$ is extended from A.

Proof: The result follows from the Affine Horrocks' Theorem by induction on n. If $n=0$, there is nothing to prove.

If $n=1$, let P be a finitely generated projective $A[t]$-module. For every prime $\mathfrak{P} \in \operatorname{Spec} A$, the localization $P_{\mathfrak{P}} \in \mathcal{P}(A_{\mathfrak{P}}[t])$ and so by conditions Q_1 and Q_3, $P_{\mathfrak{P}}$ is free over $A_{\mathfrak{P}}[t]$ and so is extended from a (free) module over $A_{\mathfrak{P}}$. Hence by Quillen's Localization Theorem P is extended from A. Assume now that $n \geq 2$ and that the result is true for $n-1$, for all rings belonging to the given QC. Let there be given a finitely generated projective $A[t_1,t_2,\ldots,t_n]$-module P. Put $t = t_1$ and $B = A[t_2,\ldots,t_n]$. Hence $A[t_1,t_2,\ldots,t_n] = B[t]$. If we can show that P is extended from a finitely generated projective module Q' over B, then we will be done, on invoking the induction hypothesis over Q.

By Affine Horrocks' Theorem (Second form), it is enough to show that the projective module $C \otimes_{B[t]} P$ is extended from a projective module over B, for some intermediate ring C such that $B[t] \subseteq C \subseteq B(t)$. We choose $C = A(t)[t_2,t_3,\ldots,t_n]$ and in fact show that $C \otimes_{B[t]} P$ is even extended from a projective module over A and so *a fortiori* from B.

Write $P_o = \dfrac{P}{(t_1,t_2,\ldots,t_n)P} \in \mathcal{P}(A)$ and $P_1 = \dfrac{P}{(t_2,t_3,\ldots,t_n)P} \in$ $\in \mathcal{P}(A[t])$. By the case $n=1$, which we have already proved, P_1 is extended from A and by remark above, $P_1 \simeq P_o[t]$. Since $A(t) \in QC$, the induction hypothesis for the case $n-1$ implies that $C \otimes_{B[t]} P$, belonging to $\mathcal{P}(A(t)[t_2,\ldots,t_n])$ is extended from $A(t)$; that is, $C \otimes_{B[t]} P \simeq Q[t_2,\ldots,t_n]$ for some $Q \in \mathcal{P}(A(t))$. But $C \cong A(t) \otimes_{A[t]} B[t]$ as $A[t]$-algebras and so $C \otimes_{B[t]} P \simeq A(t) \otimes_{A[t]} P$ as C-modules. From the remark at the beginning of (4.4), we conclude that

$$Q \simeq_{A(t)} \frac{A(t) \otimes_{A[t]} P}{(t_2,\ldots,t_n)(A(t) \otimes_{A[t]} P)} \simeq A(t) \otimes_{A[t]} \frac{P}{(t_2,\ldots,t_n)P} \simeq$$

$\simeq A(t) \otimes_{A[t]} P_1 \simeq A(t) \otimes_{A[t]} P_o[t]$. Thus $A(t) \otimes_{A[t]} P \simeq$ $\simeq P_o(t)[t_2,\ldots,t_n]$ as $A(t)[t_2,t_3,\ldots,t_n]$-modules. This shows that $A(t) \otimes_{A[t]} P$ is extended from the projective A-module P_o to $C = A(t)[t_2,\ldots,t_n]$. By induction, the theorem is true for all n.

Serre's conjecture is now an easy corollary.

<u>Corollary 1</u> - <u>If</u> F <u>is a field, then all finitely generated projective modules over the polynomial ring</u> $F[t_1,t_2,\ldots,t_n]$ $(n \geq 0)$ <u>are free</u>.

<u>Proof</u>: All we have to do is to show that the class of fields is a Quillen class: Condition Q_1 is trivial; Q_2 is also trivial since $F(t)$ is nothing but the field of rational functions over F in one indeterminate. For Q_3, just observe that F is already local and that $F[t]$ is a principal ideal domain. From Theorem 1, we conclude that every finitely generated projective $F[t_1,t_2,\ldots,t_n]$-module P is extended from F and so is free, being the extension of a free module over F.

<u>Corollary 2</u> - <u>Let</u> A <u>be a principal ideal domain. Then every</u> $P \in \wp(A[t_1,t_2,\ldots,t_n])$ <u>is free</u>.

<u>Proof</u>: Since every $Q \in \wp(A)$ is free, all we have to do is to show that the class of PID-s is a Quillen class. Q_1 is trivial. Q_2 follows from the fact that $A(t)$ is a UFD (unique

factorization domain) and that $\dim A(t) = \dim A = 1$ ((5.9) Proposition 1). Q_3 follows from Affine Horrocks' Theorem (Second form with $C = A(t)$), if we observe that $A(t)$ is a PID and so every $P \in \mathcal{P}(A(t))$ is free and extended from A.

Recall that the Noetherian domain A is a Dedekind domain if the localization $A_{\mathfrak{p}}$ is a principal ideal domain for each $\mathfrak{p} \in \operatorname{Spec} A$. This is equivalent to saying that the domain A is Noetherian, integrally closed, and is of Krull dimension 0 or 1.

Corollary 3 - *Let* A *be a Dedekind domain. Then every* $P \in \mathcal{P}(A[t_1,t_2,\ldots,t_n])$ *is extended from* A.

Proof: Once again we must verify axioms Q_1, Q_2 and Q_3 for the class of Dedekind domains. Q_1 follows from the definition above. Q_2 holds because $A(t) = A[t]_T$ is Noetherian, integrally closed, and $\dim A(t) = \dim A \leq 1$ ((5.9) Proposition 1). Q_3 holds, since A local and Dedekind implies that A is a PID and Corollary 2 guarantees that in this case $P \in \mathcal{P}(A[t])$ is free.

We now indicate some of the open questions that arise in this situation. Recall that a local ring A is regular if A is Noetherian and the maximal ideal of A can be generated by d elements, where $d = \operatorname{Kr\,dim}(A) < \infty$. A commutative ring A is called regular if A is Noetherian and $A_{\mathfrak{M}}$ is a regular local ring for all $\mathfrak{M} \in \operatorname{Max}(A)$. Then A_S is regular for any multiplicative set S, and the polynomial ring $A[t]$ is also regular; hence $A(t)$ is regular as well.

Conjecture. If A is regular, then every $P \in \mathcal{P}(A[t_1,t_2,\ldots,t_n])$

is extended from A.

For each integer $d \geq 0$ let Reg_d denote the class of regular rings of dimension $\leq d$. The conjecture above would follow for all $A \in \mathrm{Reg}_d$ if one knew that Reg_d is a Quillen class. Property Q_1 follows from the remarks above, as does Q_2, since $\dim A(t) = \dim A$. Thus the conjecture above for all $A \in \mathrm{Reg}_d$ is equivalent to:

<u>Conjecture</u> $(B\text{-}Q_d)$. If A is regular local of dimension $\leq d$ then all $P \in \mathcal{P}(A[t])$ are free.

The conjecture follows for $d=0$ (respectively $d=1$) from Corollary 1 (resp. Corollary 2) above. If $d=2$, the affirmative answer is given by a theorem due to Horrocks and Murthy (see (5.12) Theorem 1). At the time of giving these lectures, the case $d \geq 3$ is still open. The conjecture is also proved, when A is the formal power series ring in d variables over a field (see (5.13) Corollary 1). The conjecture is also valid, when A is the ring of convergent power series in d variables over a field with a non-trivial absolute value ((5.13) Remark 1, p.81).

5. Local Horrocks' Theorem

We now take up the proof of Horrocks' Theorem for local rings. We shall present two proofs. The first one by Swan uses the so called Towber presentation of an $R[t]$-module $P \in \mathcal{P}(R[t])$.

A. The Towber Presentation.

(5.1) Characteristic sequence of an endomorphism.

As usual let R be a possibly non-commutative ring and M a left R-module. Given an R-endomorphism α of M, there is a natural way in which we can associate an $R[t]$-module structure M_α to the pair (M,α): M_α is the R-module M with t action given by $t \cdot m = \alpha(m)$, for all $m \in M$. Thus, for every polynomial $p(t) \in R[t]$, we have $p(t)\cdot m = p(\alpha)(m)$ for all $m \in M$. The R-module structure of M is obtained from the $R[t]$-module structure of M_α by restriction of the scalars to R.

Given the pair (M,α), we consider $M[t] = R[t] \otimes_A M$; $M[t]$ is an $R[t]$-module obtained by base change. We define an R-linear map $\varphi: M[t] \to M_\alpha$ by $\varphi(\sum_i t^i \otimes m_i) = \sum_i \alpha^i(m_i)$. It is easily checked that φ is an $R[t]$-homomorphism. On the other hand, the endomorphism α defines an endomorphism $1_{R[t]} \otimes \alpha$ by base change. We denote this extended endomorphism also by α; thus $\alpha(\sum_i t^i \otimes m_i) = \sum_i t^i \otimes \alpha(m_i)$. In this set up, we have the following characteristic sequence associated to the endomorphism α of M.

Proposition 1 - The sequence of $R[t]$-module homorphisms

$$0 \longrightarrow M[t] \xrightarrow{t-\alpha} M[t] \xrightarrow{\varphi} M_\alpha \longrightarrow 0$$

is exact.

Proof: The map $t-\alpha$ in the statement is multiplication by t minus the extended endomorphism α. First, notice that $\varphi\circ(t-\alpha) = 0$, since $\varphi\circ(t-\alpha)(t^i \otimes m_i) = \varphi(t^{i+1}\otimes m_i - t^i\otimes\alpha(m_i)) = \alpha^{i+1}(m_i) - \alpha^i\circ\alpha(m_i) = 0$. Clearly φ is surjective. Also the map $t-\alpha$ is injective: To see this, observe that $M[t]$ as an R-module is a direct sum $\bigoplus_{i\geq 0} t^i\otimes M$; multiplication by t increases the "degree", preserving "leading coefficients". Finally, we show that $\ker\varphi \subseteq \mathrm{Im}(t-\alpha)$: Suppose $x = \sum_{i\geq 0} t^i\otimes m_i \in \ker\varphi$; i.e., $\sum_{i\geq 0} \alpha^i(m_i) = 0$, then $x = x-0 = \sum_{i\geq 0} (t^i\otimes m_i) - 1\otimes \sum_{i\geq 0} \alpha^i(m_i) =$
$= \sum_{i\geq 0} (t^i\otimes m_i - 1\otimes\alpha^i m_i) = \sum_{i\geq 1} (t^i-\alpha^i)(1\otimes m_i) =$
$= (t-\alpha)(\sum_{i\geq 1} (t^{i-1} + t^{i-2}\alpha + \dots + \alpha^{i-1})(1\otimes m_i)) \in \mathrm{Im}(t-\alpha).$

(5.2) The Towber presentation.

Theorem 1 (Towber Presentation) - Let R be any ring. Given $P \in \mathcal{P}(R[t])$ and $F_o \in \mathcal{P}(R)$, let $F = F_o[t]$. Suppose f is a monic polynomial in the center of $R[t]$ such that $P_f \cong F_f$ over $R[t]_f$. Then, there exist $M,N \in \mathcal{P}(R)$, and linear maps $u,v \in \mathrm{Hom}_R(M,N)$ and split exact sequences:

$$0 \longrightarrow M \xrightarrow{v} N \longrightarrow F_o \longrightarrow 0, \qquad \text{over } R$$

$$0 \longrightarrow M[t] \xrightarrow{u+vt} N[t] \longrightarrow P \longrightarrow 0, \qquad \text{over } R[t].$$

Proof: The maps u and v in the second sequence are the extended homomorphisms which restrict to u and v

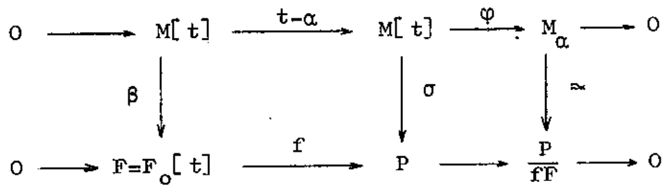
respectively on M. By (3.4) Remark 2, there exists an $R[t]$-homomorphism $\lambda: P \to F$ such that $\lambda_f: P_f \to F_f$ is an $R[t]_f$-isomorphism. Since f is monic, it is a non-zero divisor on free modules and so also on P. Thus the natural maps $P \to P_f$ and $F \to F_f$ are inclusions. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{\lambda} & F & \longrightarrow & F/P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & P_f & \xrightarrow[\lambda_f]{\simeq} & F_f & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

we see that λ is injective and that $(F/P)_f = (0)$. Since F/P is finitely generated, this implies that $f^nF \subseteq P$, for some integer $n \geq 0$. Since f^n is still monic, and $\frac{\lambda}{1}$ is an isomorphism of $P_{f^n} \simeq F_{f^n}$, we may replace f by f^n and assume that $fF \subseteq P \subseteq F$.

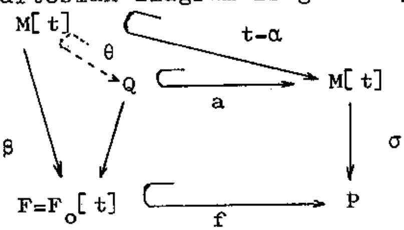
Let $d = \deg f$ and put $F_d = F_o + tF_o + \dots + t^{d-1} F_o$. Then $F = F_o[t] = \bigoplus_{i \geq 0} t^i F_o = F_d \oplus fF$, by the Euclidean division algorithm, and this is a direct sum as R-modules. Since $P \subset F$, we may write $P = M \oplus fF$, where $M = F_d \cap P$, and $fF \subseteq P$. The R-module M is isomorphic to P/fF. The t-action on the $R[t]$-module P/fF induces an endomorphism $\alpha \in \mathrm{Hom}_R(M,M)$; if $x \in M$, then $tx = \alpha(x) + f\beta(x)$, with $\alpha(x) \in M$ and $\beta(x) \in F$. In this way, we get an R-homomorphism $\beta: M \to F$, and in fact $\beta: M \to F_o$, by degree considerations.

Consider the following diagram of $R[t]$-modules:



The rows are exact, the top row being the characteristic sequence of the endomorphism α. The map β is just the extended homomorphism of $\beta: M \to F_o$; the map σ is the unique $R[t]$-linear map, such that $\sigma(x) = x$ for all $x \in M$. It is easily verified that the diagram is commutative.

We claim that the left hand square is cartesian: For suppose the associated cartesian diagram is given by



Then f injective implies that $Q \xrightarrow{a} M[t]$ is injetive. By definition of a cartesian diagram, we have a map $\theta: M[t] \to Q$ such that $t-\alpha = a \circ \theta$. Clearly θ is injective. Using (5.1) Proposition 1, we have

$$(t-\alpha)\, M[t] \subseteq \text{Im } a \subseteq M[t] = M \oplus (t-\alpha)\, M[t]$$

as R-modules; so $\text{Im } a = (M \cap \text{Im } a) \oplus (t-\alpha)\, M[t]$. Now $M \cap \text{Im } a = (0)$, since $x \in M \cap \text{Im } a$ implies that $x = \sigma x \in$ $\in fF \cap M = (0)$. Therefore, $\text{Im } a = (t-\alpha)\, M[t]$; i.e. $a\theta M[t] = (t-\alpha)M[t]$, which means θ is onto and hence an isomorphism. We may then take $Q = M[t]$. Thus the left hand square is indeed cartesian.

As remarked in (3.5), the statement of a Cartesian square can be written as an exact sequence. Combining this with $P = M \oplus fF$, we get an exact sequence:

$$0 \longrightarrow M[t] \xrightarrow{\binom{t-\alpha}{\beta}} \begin{matrix} M[t] \\ \oplus \\ F_o[t] \end{matrix} \xrightarrow{(\sigma,-f)} P \longrightarrow 0.$$

If we write $N = M \oplus F_o$, $u = \binom{-\alpha}{\beta}: M \to N$ and $v = \binom{1_M}{0}: M \to N$, we get the following exact sequences:

$$0 \longrightarrow M \xrightarrow{v} N \longrightarrow F_o \longrightarrow 0$$

$$0 \longrightarrow M[t] \xrightarrow{u+vt} N[t] \longrightarrow P \longrightarrow 0.$$

This proves half of the theorem.

It remains to show that M and N are finitely generated projective R-modules. Since $N = M \oplus F_o$, clearly it is enough to show that $M \in \mathcal{P}(R)$. Now $P \in \mathcal{P}(R[t])$ and $R[t]$ is R-free so that P is R-projective; hence so is M, since $P = M \oplus fF$ as R-modules. Now $M_\alpha = \frac{P}{fF}$ is a finitely generated $\frac{R[t]}{R[t]f}$ -module and f-monic implies that $\frac{R[t]}{R[t]f}$ is a finitely generated R-module. Hence $M_\alpha = M$ is a finitely generated R-module.

The Towber presentation has an application to the functor K_o of K-theory. It seems appropriate to present it here. For the definition of K_o, the reader is referred to (7.4).

<u>Corollary 1</u> - <u>Let</u> $R, t, P,$ <u>and</u> $F = F_o[t]$ <u>be as in the statement of the theorem.</u> <u>Then in</u> $K_o(R[t])$, <u>we have</u> $[P] = [F]$ <u>and so</u> $[P]$ <u>belongs to</u> $\text{Im}\ K_o(R) \to K_o(R[t])$.

<u>Proof</u>: From the split exact sequences of the Towber presentation, we get $[P] + [M[t]] = [N[t]] = [F_o[t] \oplus M[t]] = [F_o[t]] + [M[t]]$, whence $[P] = [F_o[t]]$.

B. <u>Swan's proof</u>.

(5.3) <u>Swan's proof</u>.

<u>Theorem 1</u> (Horrocks) - <u>Let</u> R <u>be a local ring, and</u> P <u>a finitely generated projective</u> $R[t]$-<u>module</u>. <u>If there exists a monic polynomial</u> f <u>in the center of</u> $R[t]$ <u>such that</u> P_f <u>is</u> $R[t]_f$-<u>free, then</u> P <u>is already</u> $R[t]$-<u>free</u>.

<u>Proof</u>: The proof uses an induction argument on the Tower presentation of P. Let F_o be a finitely generated free module over R such that $F_o[t]_f$ is isomorphic to P_f. Choose a Towber presentation as in (5.2):

$$0 \longrightarrow M \xrightarrow{\ v\ } N \longrightarrow F_o \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow M[t] \xrightarrow{\ u+vt\ } N[t] \longrightarrow P \longrightarrow 0\ , \qquad \text{where}$$

$N = M \oplus F_o$, $v = \binom{1}{0}$, $u+vt = \binom{t-\alpha}{\beta}$ and $u = \binom{-\alpha}{\beta}$. If $\mathfrak{M}$ is the radical of the local ring R, then $\bar{R} = R/\mathfrak{M}$ is a division ring. M and F_o are free R-modules. We induct on $r = \text{rank of } M$.

If $r=0$, then $N = F_o$ and so $P \simeq F_o[t]$ is free.

So assume that $r \geq 1$ and that $Q \in \mathcal{P}(R(t))$ is free, whenever there is a Towber presentation of Q, in which the corresponding

"M" is of rank $\leq r-1$.

Denoting passage modulo the radical $\mathfrak{M}$ by bar, our first claim is that $\bar{\beta}: \bar{M} \to \bar{F}_o$ is not zero: otherwise, from the split exact sequence

$$0 \longrightarrow \bar{M}[t] \xrightarrow{\binom{t-\bar{\alpha}}{0}} \bar{M}[t] \oplus \bar{F}_o[t] \longrightarrow \bar{P} \longrightarrow 0,$$

we get $\bar{P} \simeq \dfrac{\bar{M}[t]\oplus\bar{F}_o[t]}{(t-\bar{\alpha})\bar{M}[t]\oplus 0} \cong \bar{M}_{\bar{\alpha}} \oplus \bar{F}_o[t]$, since

$0 \longrightarrow \bar{M}[t] \xrightarrow{t-\bar{\alpha}} \bar{M}[t] \longrightarrow \bar{M}_{\bar{\alpha}} \longrightarrow 0$ is an exact sequence. Now $\bar{P}$ is a finitely generated projective $\bar{R}[t]$-module and so free. Hence $\bar{M}_{\bar{\alpha}}$ is a free $\bar{R}[t]$-module. It is a finitely generated $\bar{R}$-module. Hence $\bar{M}_{\bar{\alpha}} = (0)$; that is, $\bar{M} = (0)$ and by Nakayama's Lemma $M = (0)$, a contradiction. Thus $\bar{\beta} \neq 0$. Hence there exists $x \in M\backslash\mathfrak{M}M$ such that $y = \beta(x) \in F_o\backslash\mathfrak{M}F_o$. In particular $y \neq 0$ and so by(2.3), Corollary 3, $F_o = Ry \oplus F'_o$. Since $x \notin \mathfrak{M}M$ we appeal once again to (2.3) Corollary 3, to see that Rx generates a free submodule of M. We can complete $\{x\}$ to a basis of M, in a suitable way such that $M = Rx \oplus M'$, and $\beta M' \subseteq F'_o$.

Now, consider the following diagram of exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \ker(C\to P) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'[t] & \xrightarrow{\binom{t-\alpha/M'}{\beta/M'}} & M[t]\oplus F'_o[t] & \xrightarrow{\varphi} & \text{Coker} = C & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M[t] & \xrightarrow{\binom{t-\alpha}{\beta}} & M[t]\oplus F_o[t] & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Rx[t] & \xrightarrow{\simeq} & Ry[t] & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Using the Snake Lemma (p.26 [1]) or the 3x3 Lemma (p.49 [4a]), one sees readily that $C \cong P$.

We want to use the top row for a new Towber presentation of C. For this, we need an endomorphism α' of M' and a corresponding map β'. We define α' as the composition map $M' \xrightarrow{\alpha} M = \begin{matrix} M' \\ \oplus \\ Rx \end{matrix} \xrightarrow{\binom{1}{0}} M'$, this defines a map $\beta'': M' \to Rx$ so that $\alpha/M' = \binom{\alpha'}{\beta''}$. Let $\beta': M' \to Rx \oplus F'_0$ be given by the sum of $\binom{\beta''}{\beta/M'}$. If we observe that $M[t] \oplus F'_0[t] = M'[t] \oplus (Rx \oplus F'_0)[t]$ $= M'[t] \oplus F_1[t]$ say, then the top row reads

$$0 \longrightarrow M'[t] \xrightarrow{\binom{t-\alpha'}{\beta'}} M'[t] \oplus F_1[t] \xrightarrow{\varphi'} C \longrightarrow 0.$$

If we write $u_1 = \binom{1}{0}$ and $v_1 = \binom{-\alpha'}{\beta'}$, then we get a Tower presentation of $C \in \mathcal{P}(R[t])$

$$0 \longrightarrow M' \xrightarrow{v_1} N' = M' \oplus F_1 \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow M'[t] \xrightarrow{u_1 t + v_1} N'[t] \longrightarrow C \longrightarrow 0,$$

with rank of $M' = r-1$. The induction hypothesis implies that C is $R[t]$-free. Since C and P are isomorphic as $R[t]$-modules, it sollows that P is free.

(5.4) Lindel's matrix version.

We now give another poof of Local Horrocks' Theorem using matrices. This proof due to Lindel again uses the Towber presentation (5.2). The hypothesis in (5.3) stands and we want to show that $P \in \mathcal{P}(R[t])$ is free. Using the Towber presentation (5.2)

$$0 \longrightarrow M[t] \xrightarrow{u+vt} M[t] \oplus F_o[t] \longrightarrow P \longrightarrow 0,$$

we see that P is the cokernel of the $R[t]$-homomorphism $u+vt = = \binom{t-\alpha}{\beta}$, where $\alpha: M \to M$ and $\beta: M \to F_o$ are R-homomorphisms. Since R is local, these last R-modules are free. Thus if r and s are the ranks of M and F_o respectively, the R-linear maps α and β can be represented by $r\times r$ and $s\times r$ matrices $\alpha = (a_{ij})$ and $\beta = (b_{ij})$ respectively over R. Since the extended $R[t]$-homomprphisms α and β are represented by the same matrices over $R[t]$, the matrix of $u+vt$ over $R[t]$ is given by the $(r+s)\times r$ matrix

$$u + tv = \begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1r} \\ -a_{21} & t-a_{22} & \cdots & -a_{2r} \\ \vdots & & \ddots & \\ -a_{r1} & -a_{r2} & & -a_{rr} \\ \hline b_{11} & b_{12} & & b_{1r} \\ & \cdots & & \\ bs_1 & b_{s2} & & b_{sr} \end{pmatrix}$$

If by means of elementary operations, we can take $u+tv$ over $R[t]$ to a matrix of the form $\left(\frac{I_r}{0}\right)$, where I_r is the identity matrix, then $\operatorname{coker}(u+tv) \simeq \operatorname{coker}\left(\frac{I_r}{0}\right) \simeq \frac{M[t]\oplus F_o[t]}{M[t]} \simeq F_o[t]$ and so P will be free. Accordingly, we proceed by induction on r to show that this can be achieved.

The case $r=0$ is trivial. So assume $r \geq 1$. We assume as induction hypothesis that a matrix of the form $\binom{tI-\alpha'}{\beta'}$

can be taken by elementary operations to the form $\binom{I_m}{0}$, whenever rank $\alpha' = m \le r-1$, and where α' and β' are matrices of constants over R. Given $\binom{t-\alpha}{\beta}$ with rank $\alpha = r$, we pass modulo $\mathfrak{M}$ as in Swan's proof to see that $\bar{\beta} = (\bar{b}_{ij}) \neq 0$. Hence some $b_{ij} \notin \mathfrak{M}$. By permuting the rows and clolumns of β, we may assume that $b_{11} \in R^*$ is a unit. The required reduction is indicated below:

$$u + tv = \begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1r} \\ -a_{21} & t-a_{22} & \cdots & -a_{2r} \\ & \cdot & \cdot & \cdot \\ -ar_1 & -a_{r2} & \cdots & -a_{rr} \\ \hline b_{11} & b_{12} & \cdots & b_{1r} \\ & \cdot & \cdot & \cdot \\ bs_1 & bs_2 & \cdots & bs_r \end{pmatrix} \qquad \begin{matrix} a_{ij} \in R \\ b_{ij} \in R \end{matrix}$$

elem. row operations using $(r+1)^{th}$ row $\longrightarrow$

$$\begin{pmatrix} 1 & A_2 & \cdots & A_r \\ 0 & t-a'_{22} & \cdots & -a'_{2r} \\ & \cdot & \cdot & \cdot \\ 0 & -a'_{r2} & \cdots & -a'_{rr} \\ \hline b_{11} & b_{12} & \cdots & b_{1r} \\ 0 & b'_{22} & \cdots & b'_{2r} \\ & \cdot & \cdot & \cdot \\ 0 & b'_{s2} & \cdots & b'_{sr} \end{pmatrix} \qquad \begin{matrix} A_i \in R + Rt \\ a'_{ij} \in R \\ b_{1\lambda} \in R \\ b'_{ij} \in R \end{matrix}$$

elem row operation using 1^{st} row on $(r+1)^{th}$ row $\longrightarrow$

$$\left(\begin{array}{cccc} 1 & A_2 & \cdots & A_r \\ 0 & t-a'_{22} & \cdots & -a'_{2r} \\ 0 & -a'_{r2} & \cdots & -a'_{rr} \\ \hline 0 & B_2 & \cdots & B_r \\ 0 & b'_{22} & \cdots & b'_{2r} \\ & \cdot\ \cdot\ \cdot & & \\ 0 & b'_{s2} & \cdots & b'_{sr} \end{array}\right) \qquad \begin{array}{l} A_i \in R + Rt \\ a'_{ij} \in R \\ B_i \in R + Rt \\ b'_{ij} \in R \end{array}$$

$(r-1)$ row operations one at a time on the $(r+1)^{th}$ row using $2^{nd}, 3^{rd}, \ldots,$ r^{th} rows respectively

$$\left(\begin{array}{cccc} 1 & A_2 & \cdots & A_r \\ 0 & t-a'_{22} & \cdots & -a'_{2r} \\ & \cdot\ \cdot\ \cdot & & \\ 0 & -a'_{r2} & \cdots & -a'_{rr} \\ \hline 0 & b'_2 & \cdots & b'_r \\ 0 & b'_{22} & \cdots & b'_{2r} \\ & \cdot\ \cdot\ \cdot & & \\ 0 & b'_{s2} & \cdots & b'_{sr} \end{array}\right) \qquad \begin{array}{l} A_i \in R + Rt \\ a'_{ij} \in R \\ b'_i \in R \\ b'_{ij} \in R \end{array}$$

$(r-1)$ column operations using the 1^{st} column $\longrightarrow$

$$\left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & t-a'_{22} & \cdots & -a'_{2r} \\ & \cdot\ \cdot\ \cdot & & \\ 0 & -a'_{r2} & \cdots & -a'_{rr} \\ \hline 0 & b'_2 & \cdots & b'_r \\ & \cdot\ \cdot\ \cdot & & \\ 0 & b'_{s2} & \cdots & b'_{sr} \end{array}\right)$$

which is a matrix of the form

$$\left(\begin{array}{c|c} 1 & 0 \ \dots \ 0 \\ \hline 0 & \\ \vdots & tI_{r-1} - \alpha' \\ 0 & \\ \hline 0 & \beta' \end{array}\right)$$

with α', β' matrices over R. The induction hypothesis implies that the matrix $\begin{pmatrix} t-\alpha' \\ \beta' \end{pmatrix}$ can be taken by elementary operations to $\left(\frac{I_{r-1}}{0}\right)$. Hence the same is true of $u+vt$, and we are done.

C. Elementary Matrices.

We wish to present a second proof of Horrocks' Theorem due to Paul Roberts. We will be presenting an axiomatized version whose formulation is due to T.-Y. Lam. With this in view, we prepare some preliminary ground.

(5.5) The group $E_n(A)$.

Let R be a ring and $M_n(R)$ the ring of $n \times n$ matrices over R. As usual e_{ij} is the matrix having 1 as its (i,j) th entry and 0 elsewhere. We recall that the matrices e_{ij} form a (standard) basis for the free R-module $M_n(R)$. We also recall the following rules, governing their multiplication:

$$e_{ij}\, e_{k\ell} = \begin{cases} e_{i\ell} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

In particular, if $i \neq j$, then $e_{ij}^2 = 0$. If $a \in R$, consider formally the exponential function $e_{ij}^a = 1 + a\, e_{ij}$ (the higher terms in the exponential series are 0). e_{ij}^{a} is an invertible matrix, the inverse being e_{ij}^{-a}. Left multiplication of a matrix A by e_{ij}^{a} corresponds to an elementary row operation on A: that of replacing the i^{th} row of A by i^{th} row plus a times the j^{th} row. Right multiplication by e_{ij}^{a} corresponds similarly to an elementary column operation on A.

As usual $GL_n(R)$ is the group of $n \times n$ invertible matrices over R. The mapping $a \mapsto e_{ij}^{a}$ is a group homomorphism of the additive group $(R,+)$ of R into the multiplicative group $GL_n(R)$. The subgroup of $GL_n(R)$ generated by all e_{ij}^{a} with $i \neq j$ and $a \in R$ is denoted by $E_n(R)$ and is called the <u>elementary subgroup</u> of $GL_n(R)$. Notice $E_1(R) = (1)$. The matrices e_{ij}^{a} are called <u>elementary matrices</u>.

If $\theta: R \to R'$ is a ring homomorphism, then the correspondence $e_{ij}^{a} \mapsto e_{ij}^{a'}$ induces a group homomorphism of $E_n(R) \to E_n(R')$.

<u>Lemma 1</u> - <u>If</u> $\theta: R \to R'$ <u>is a surjective ring homomorphism then the induced homomorphism</u> $E_n(R) \to E_n(R')$ <u>is also surjective</u>.

In other words, an elementary matrix over R' can be lifted to one over R, if θ is surjective. Notice that such a lifting of matrices in $GL_n(R')$ to matrices in $GL_n(R)$ is not in general possible: For example consider $n = 1$ and $\theta: Z \to Z/(5)$.

(5.6) <u>Action on unimodular elements</u>.

We shall identify $GL_{n-1}(R)$ with a subgroup of $GL_n(R)$ as follows: To $\sigma \in GL_{n-1}(R)$, there corresponds

$\left(\begin{array}{c|c}\sigma & 0\\ 0 & 1\end{array}\right) \in GL_n(R)$. For the next lemma, we see that the definition of unimodular elements in A^n made in Section 1 for commutative rings A also makes sense for elements of R^n over any ring R.

Lemma 1 - Let $n \geq 2$. Suppose $E_n(R)$ acts transitively on unimodular elements of R^n. Then $GL_n(R) = E_n(R)GL_{n-1}(R)$.

Proof: Let $\alpha \in GL_n(R)$. Then $\alpha^{-1}\alpha = 1$. Hence columns of α are unimodular. From the hypothesis, we get a matrix $\epsilon_1 \in E_n(R)$ such that the last column of $\epsilon_1\alpha$ is the transpose of $(0,0,\dots,0,1)$. Write $\epsilon_1\alpha = \left(\begin{array}{c|c}\beta & 0\\ \hline \gamma & 1\end{array}\right)$ with $\beta \in GL_{n-1}(R)$, as is easily verified. We write this as

$$\epsilon_1\alpha = \left(\begin{array}{c|c} I & 0\\ \gamma\beta^{-1} & 1\end{array}\right)\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} = \epsilon_2\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix},$$

where

$$\epsilon_2 = \left(\begin{array}{ccc|c} & I & & 0\\ a_1 & a_2 \dots & a_{n-1} & 1\end{array}\right) \text{ say. Then}$$

$\epsilon_2 = (1 + a_1e_{n1})(1 + a_2e_{n2}) \dots (1 + a_{n-1}\, e_{n,n-1}) \in E_n(R)$.

Thus $\alpha = \epsilon_1^{-1}\,\epsilon_2\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} \in E_n(R)\; GL_{n-1}(R)$. This proves the lemma.

Corollary 1 - If for all $n \geq 2$, $E_n(R)$ acts transitively on unimodular elements of R^n, then $GL_n(R) =$ $= E_n(R)GL_1(R)$. In particular, if A is a commutative ring, then the inclusion $E_n(A) \subset SL_n(A)$ is an equality for all $n \geq 2$.

Proof: We have only to prove the last statement of the corollary. If $n \geq 2$ and $\alpha \in SL_n(A)$, we can write

$$\alpha = \epsilon\begin{pmatrix} u_1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots\; 1\end{pmatrix}$$

with $\varepsilon \in E_n(A)$ and $u_1 \in A^*$. Taking determinants, we get $1 = \det \alpha = \det \varepsilon \cdot u_1$, whence $u_1 = 1$. Hence $\alpha = \varepsilon \in E_n(A)$.

<u>Proposition 1</u> - <u>Let</u> A <u>be an Euclidean domain.</u> <u>Then</u> $E_n(A)$ <u>acts transitively on unimodular elements of</u> A^n <u>for all</u> $n \geq 2$.

<u>Proof</u>: If $a = (a_1, a_2, \ldots, a_n)^T$ is a unimodular column (T denoting transpose), we pick $\varepsilon \in E_n(R)$ so that one of the non-zero entries, say $b_i = d$ of $\varepsilon a = (b_1, b_2, \ldots, b_n)$ is such that $\varphi(b_i)$ is the least possible non-zero integer, φ being the Euclidean function. We claim that all the other b_j are multiples of d; if $b_j = q_j d + r_j$, with $0 < r_j < d$, then $e_{ji}^{\pm q_j} \varepsilon a$ will have r_j as its j^{th} entry contradicting the choice of ε. Hence all the other b_j are multiplies of d. A series of elementary row operations will take $(b_1, b_2, \ldots, b_n)^T$ to $(0, 0, \ldots, d, 0, 0)^T$ with d in the i^{th} entry. By using (1.1) Proposition 1, item ii) or directly, one sees that a unimodular element goes into an unimodular element under an invertible matrix. Hence d is a unit. Because of this and $n \geq 2$, we can send $(0, 0, \ldots, d, \ldots, 0)^T$ to $(1, 0, 0, \ldots, d, \ldots, 0)^T$ and then to $(1, 0, 0, \ldots, 0)^T$ by elementary row operations. We have thus proved that there exists an elementary matrix ε' taking the unimodular column $a = (a_1, a_2, \ldots, a_n)^T$ to $(1, 0, 0, \ldots, 0)^T$. The conclusion of the proposition is now immediate.

<u>Corollary 2</u> - If A is an Euclidean domain, then $SL_n(A) = E_n(A)$; hence $SL_n(A)$ is generated by elementary matrices.

<u>Proof</u>: The result follows from Corollary 1 and Proposition 1.

Lemma 2 - *Let* R *be a ring satisfying the following condition:*

(*) *If* $a \in R$ *and* L *is a left ideal of* R *such that* $Ra+L = R$, *then there exists* b *in* L *such that* $a+b$ *is invertible.*

Then $E_n(R)$ *acts transitively on unimodular elements of* R^n *for all* $n \geq 2$.

Proof: Let $a = (a_1, a_2, \ldots, a_n)^T$ be a unimodular column. Clearly it is enough to show that a can be taken by elementary operations to $(1,0,\ldots,0)^T$.

Case 1. Suppose a_1 is invertible in R. Then

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longrightarrow \begin{pmatrix} a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In the general case, take $L = Ra_2 + \ldots + Ra_n$. The unimodularity condition implies that $Ra_1+L = R$. By condition (*), there exists b in L such that $a_1+b \in R^*$. Now

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longrightarrow \begin{pmatrix} a_1+b \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

by Case 1.

Lemma 3 - *Let* R *be a semilocal ring; i.e.* $R/\mathrm{rad}(R)$ *is a semisimple (Artin) ring. Then* R *satisfies the condition* (*) *of Lemma 2. In particular, this holds when* R *is local.*

Proof: In these notes, the applications concern the case, when R is local. Hence it is instructive to give a simple proof in this case; if $a \in R$ and L is a left ideal such that $Ra + L = R$, then a and L are not both contained in rad R. In either case, we can find $\ell \in L$, possibly zero such that $a + \ell \notin$ rad R. Passing modulo rad R and using (2.1) Proposition 1, we see that $a + \ell$ is a unit of R.

Assume now that R is semilocal and that $Ra + L = R$. Denoting by bar passage modulo rad R, we see that $\overline{Ra} + \overline{L} = \overline{R}$, with $\bar{R}$ semisimple. If $\bar{a} + \bar{\ell}$ is a unit of $\bar{R}$, with $\ell \in L$, then $a + \ell$ is a unit of R by (2.1), Proposition 1. Hence it is enough to prove the lemma, when R is a semisimple ring. Since every R-module is projective, these exists a left ideal $M \subseteq L$ such that $L = (Ra \cap L) \oplus M$. Hence $R = Ra \oplus M$. Again, the map $R \to Ra$ given by $r \overset{\hat{a}}{\longmapsto} ra$ is onto the projective module Ra, whence it splits. Denoting by K, the kernel, we get an exact sequence:

$$0 \longrightarrow K \underset{f}{\rightleftarrows} R \xrightarrow{\hat{a}} Ra \longrightarrow 0.$$

Let $f: R \to K$ be a splitting. We note that K is R-isomorphic to M. If we denote such an isomorphism by g, we get $\ell = g(f(1)) \in M \subseteq L$. Now, the composition j of the isomorphisms $R \xrightarrow{(\hat{a},f)} Ra \oplus K \xrightarrow{(1,g)} Ra \oplus M \xrightarrow{=} R$ sends 1 to $a + \ell$. If h denotes the inverse of this composition, then $1 = h \circ j(1) = h(a+\ell) = (a+\ell)\, h(1)$. Hence $a + \ell$ is a right unit, so also a unit, since A is Artinian.

Corollary 3 - If A is a commutative semilocal ring, then

$$E_n(A) = SL_n(A) \text{ for all } n \geq 2.$$

Proof: Using Lemmas 2 and 3, the result follows from the proof of Corollary 1.

Proposition 2 - Let R' be a semilocal ring or an Euclidean domain. Let $R \to R'$ be a surjective ring homomorphism such that $R^* \to R'^*$ is surjective. Then for every $n \geq 1$, the induced map $GL_n(R) \to GL_n(R')$ is surjective.

Proof: Using Lemmas 1, 2, 3, and Proposition 1, we see that the hypotheses in Corollary 1 are satisfied for the ring R'. Hence the conclusion of that corollary holds. The proposition now follows, if we observe that $GL_1(R') = R'^*$, via (5.5) Lemma 1.

D. The Ring A(t).

We recall that $A(t) = A[t]_T$ is the ring of fractions of the polynomial ring $A[t]$ with respect to the multiplicative set T of monic polynomials in $A[t]$; here, as always, A denotes a commutative ring. The ring $A(t)$ plays a important role in the solution of Serre's problem. It first seems to occur explicitly in the 1965 paper of Claborn.

(5.7) First properties.

Suppose $A \xrightarrow{\varphi} A'$ is a ring homomorphism. If we denote by T' the multiplicative set of monic polynomials in A', we get the following commutative diagram:

$$\begin{array}{ccc} A(t) & \xrightarrow{\varphi(t)} & A'(t) = A'[t]_{T'} \\ & \searrow & \Big\uparrow j \\ & & A' \underset{A}{\otimes} A(t) = A'[t]_{\varphi(T)} \end{array}$$

The map j is an inclusion. In special cases j may be an equality as happens for example, when A' is integral over A. We shall need only the following special case of this.

Proposition 1 - *If* φ *is a surjective ring homomorphism then* $\varphi(t)$ *is surjective and* j *is an equality. In particular, if* $\mathfrak{A}$ *is an ideal of* A, *then* $\frac{A}{\mathfrak{A}}(t) \simeq \frac{A(t)}{\mathfrak{A}A(t)}$.

Proof: The first statement is obvious, since $T' = \varphi(T)$. For the second statement, we apply the functor $A(t) \underset{A}{\otimes} \cdot$ to the right exact sequence

$$\mathfrak{A} \longrightarrow A \longrightarrow \frac{A}{\mathfrak{A}} \longrightarrow 0.$$

Proposition 2 - *The ring* $A(t)$ *is a faithfully flat A-algebra.*

Proof: The polynomial ring $A[t]$ is flat over A and $A(t)$ is flat over $A[t]$. We observe that $A(t) \underset{A}{\otimes} \cdot \simeq A(t) \underset{A[t]}{\otimes} A[t] \underset{A}{\otimes} \cdot$ Hence $A(t) \underset{A}{\otimes} \cdot$ is an exact functor, and so $A(t)$ is flat over A. To show that $A(t)$ is faithfully flat, it is enough to show by (3.3), Proposition 1, item 2) that $A(t) \underset{A}{\otimes} \frac{A}{\mathfrak{M}} \neq 0$ for every maximal ideal $\mathfrak{M}$ of A. By Proposition 1, this last object is $\frac{A}{\mathfrak{M}}(t)$, which is a field and not zero.

(5.8) Units of $A(t)$. We denote by G the multiplicative group generated by T in $A(t)$.

Lemma 1 - $A(t)^* = A^*\cdot G$, if A is an integral domain.

Proof: We only have to show that $A(t)^* \subseteq A^*\cdot G$. Suppose $\frac{f(t)}{g(t)} \in A(t)^*$ with g monic. Then there exist $f_1, g_1 \in$ $\in A[t]$ with g_1 monic such that $\frac{f}{g}\frac{f_1}{g_1} = 1$. Hence $ff_1 = gg_1$ is a monic polynomial. Since A is an integral domain, this implies that the leading coeficient of f is a unit u and $f = uf'$ with f' monic in $A[t]$. We then have $\frac{f}{g} = u\frac{f'}{g} \in A^*\cdot G$.

Proposition 1 - If A' is an integral domain, and $A \xrightarrow{\varphi} A'$ a surjective ring homomorphism inducing a surjective group homomorphism $A^* \to A'^*$, then the induced map $A(t)^* \to$ $\to A'(t)^*$ is surjective.

Proof: If G' denotes the multiplicative group generated by T' in $A'(t)$, then it is enough to observe that φ induces a surjective group homomorphism $G \to G'$ and apply Lemma 1 to the integral domain A'.

Remark: If we take $A = \mathbb{Z}$ and $A' = \frac{\mathbb{Z}}{6\mathbb{Z}}$ and consider the natural map $A \to A'$, then the induced map $A(t)^* \to A'(t)^*$ is not surjective: For, the element $\bar{3}t - \bar{2}$ is a unit of $A'(t)$ as $(\bar{3}t-\bar{2})(\bar{2}t-\bar{3}) = -t$; but $\bar{3}t - \bar{2}$ does not lift to a unit of $A(t)$, even though $A^* \to A'^*$ is surjective.

(5.9) $A(t)$ and $A\{[t^{-1}]\}$.

Proposition 1 - Let A be a commutative Noetherian ring of dimension d. Then 1) $\dim A[t] = \dim A+1$, and 2) $\dim A(t) = \dim A$.

Proof: 1) is well-known. For 2): If $\mathfrak{P}$ is a prime ideal of A, then $\mathfrak{P}A(t)$ is prime in $A(t)$. Hence $\dim A(t) \geq d$. If $d = \infty$, then equality clearly holds. So assume $d < \infty$. Since $A(t)$ is a localization of $A[t]$, it is enough if we look at the contracted primes of $A[t]$. Let $\wp \in \operatorname{Spec} A[t]$, and $\mathfrak{P} = \wp \cap A$. So $\operatorname{ht} \mathfrak{P} \leq d$ and $\operatorname{ht} \wp \leq \operatorname{ht} \mathfrak{P} + 1$. If $\operatorname{ht} \mathfrak{P} < d$, then $\operatorname{ht} \wp \leq d$. If $\operatorname{ht} \mathfrak{P} = d$, then $\mathfrak{P}$ is maximal in A and so $\frac{A(t)}{\mathfrak{P}A(t)} \cong \frac{A}{\mathfrak{P}}(t)$, a field, showing that $\mathfrak{P}A(t)$ is maximal in $A(t)$. Thus if $\wp$ is a contracted prime of $A[t]$, then $\wp = \mathfrak{P} A[t]$. So in all cases, contracted primes of $A[t]$ have height $\leq d$. So $\dim A(t) = d$.

Proposition 2 - Let $s \in A$ and consider the multiplicative set $M = 1 + sA$ of A and the ring of fractions $A_M = A_{1+sA}$. Then the following hold:

1) $\frac{s}{1} \in \operatorname{rad} A_M$

2) $\frac{A_M}{sA_M} \cong \frac{A}{sA}$

3) If $\frac{A}{sA}$ is local, so also is A_M.

Proof: 1) By (2.1) Proposition 1, it is enough to show that $1 - \frac{s}{1}\frac{a}{t} \in A_M^*$ for all a in A and $t \in M$. Fix a in A, and $t = 1 + sb$ in $M = 1 + sA$, with b in A. Therefore, $1 + \frac{sa}{t} = \frac{1}{t}(t+sa) = \frac{1}{t}(1+(b+a)s) = \frac{1+sc}{1+sb}$ say. This last element is a unit of A_M.

2) This follows from (3.1) permutability..., on observing that the image of $M = 1+sA$ under the natural map $A \to \frac{A}{sA}$ is $\{1\}$.

3) Clearly, this follows from 1) and 2).

In Proposition 2, take $A = A[t]$ and $s = t$; we obtain the ring we shall denote $A\{[t]\} = A[t]_{(1+tA[t])}$. One can roughly think of $A\{[t]\}$ as a rational power series ring. We conclude from Proposition 2 that $t \in \text{rad } A\{[t]\}$ and that $\frac{A\{[t]\}}{tA\{[t]\}} \cong A$. If A is local with $\text{rad } A = \mathfrak{M}$, then $A\{[t]\}$ is local with radical equal to $(\mathfrak{M},t)$. In what follows, ∂f denotes the degree of a polynomial f.

<u>Proposition 3</u> - <u>Let</u> $s = t^{-1}$ <u>and</u> $B = A\{[s]\}$. <u>Then the following hold:</u>

1) $B = \{\frac{f(t)}{g(t)} \in A(t)$: g <u>monic and</u> $\partial f \leq \partial g\}$.

2) $B_s = A(t) = A[t] + B$

3) $A[t] \cap B = A$.

<u>Proof</u>: 1) Let $\frac{f(t)}{g(t)} \in A(t)$ with g monic and $\partial f \leq \partial g = n$.

Dividing by t^n, we can write $g = t^n g_1(t^{-1}) = t^n g_1(s)$, where $g_1(s)$ is a polynomial in s with constant term 1; i.e., $g_1(s) \in 1 + sA[s]$. Since $\partial f \leq n$, we can write $f(t) = t^n f_1(s)$. Hence $\frac{f(t)}{g(t)} = \frac{f_1(s)}{g_1(s)} \in A[s]_{1+sA[s]} = B$. This proves that the right set of the equality in 1) is contained in B. For the other inclusion, take $\frac{f_1(s)}{g_1(s)} \in B$, with $g_1 \in 1 + sA[s]$. Choose $n \geq \partial f_1, \partial g_1$ (degree in s). Since $B \subseteq A(T)$, we write $\frac{f_1(s)}{g_1(s)} = \frac{t^n f_1}{t^n g_1} = \frac{f(t)}{g(t)}$ with g monic and $\partial f \leq \partial g$. This proves the equality in 1).

2) Since $B \subseteq A(t)$ and $A[t] \subseteq B_s$ the equality $B_s = A(t)$ follows from $A(t) = A[t] + B$. To prove this last equality, we employ the division algorithm:

If $\frac{f(t)}{g(t)} \in A(t)$ with g monic of degree n, then there exist polynomials $q(t)$ and $r(t)$ such that $f(t) = q(t)g(t) + r(t)$, where $\partial r < n$. Now $\frac{f}{g} = q(t) + \frac{r(t)}{g(t)}$ with $\frac{r(t)}{g(t)} \in B$. So $A(t) = A[t] + B$.

3) Let $h = \frac{f}{g} \in A[t] \cap B$, with $h \in A[t]$, $f,g \in A[t]$, g monic and $\partial f \leq \partial g$. From $hg = f$, we get $\partial f = \partial h + \partial g$, since g is monic. Hence $\partial h = 0$ and $h \in A$.

E. Robert's proof of Horrocks' Theorem.

We present an axiomatized version of Robert's proof of Horrocks' Theorem, whose formulation is due to T.-Y. Lam. In this section $(A,\mathfrak{M})$ will denote a commutative local ring with rad $A = \mathfrak{M}$. Bar will denote passage modulo $\mathfrak{M}$.

(5.10) Statement and proof.

Theorem 1 - Let R be a possibly non-commutative A-algebra over the commutative local ring $(A,\mathfrak{M})$, let T be a central multiplicative set of non-zero divisors of R, and let n be an integer ≥ 1. Assume that the following conditions hold:

1) The natural map $GL_n(R_T) \to GL_n(\bar{R}_T)$ is surjective.

2) $\forall f \in T$, $\frac{R}{fR}$ is a finite A-algebra; i.e., a finitely generated A-module.

3) There exists a sub A-algebra B of R_T such that $\mathfrak{M}B \subseteq \text{rad } B$ and $R_T = R+B$.

Let P be a finitely generated R-module such that $P \xrightarrow{f} P$ is injective, for all $f \in T$. Further, suppose that $\bar{P} \simeq \bar{R}^n$, and $P_T \simeq R_T^n$. Under these conditions $P \simeq R^n$.

Before we take up the proof of Theorem 1, we want to use it to derive Local Horrocks' Theorem for commutative rings.

Corollary 1 - Let A be a commutative local ring, and $P \in \mathcal{P}(A[t])$. If $A(t) \underset{A[t]}{\otimes} P$ is $A(t)$-free, then P is already $A[t]$-free.

Proof: In Theorem 1, we put $R = A[t]$, T = the set of monic polynomials in t, $B = A\{[t^{-1}]\}$. Then $R_T = A(t)$ and $\bar{R}_T = \frac{A(t)}{\mathfrak{M}A(t)} \cong \frac{A}{\mathfrak{M}}(t) = \bar{A}(t)$, which is a field. The hypothesis on the multiplicative set T is clearly satisfied. We now check the three conditions of Theorem 1. For condition 1), we see that $GL_n(\bar{A}(t)) = D_n(\bar{A}(t))\, E_n(\bar{A}(t))$, where $D_n(\bar{A}(t))$ is the group of invertible diagonal matrices.

Combining (5.5) Lemma 1 and (5.7), Proposition 1, we see that $E_n(A(t)) \to E_n(\bar{A}(t))$ is surjective. Since $\bar{A}$ is local, the surjectivity of $A(t)^* \to \bar{A}(t)^*$ is guaranteed by (5.8), Proposition 1. This shows that invertible diagonal matrices over $\bar{A}(t)$ can be lifted to invertible matrices over $A(t)$. Thus condition 1) is satisfied. It is easily seen that condition 2) holds. The validity of condition 3) is the content of (5.9). Proposition 3, and the remark preceding that proposition. Assume now that $P \in \mathcal{P}(A[t])$ and let $P(t) \cong A(t)^n$. It follows that $\bar{P} \cong \bar{A}[t]^n$, since $\bar{A}[t]$ is a PID. By Theorem 1, $P \cong A[t]^n$ and so is $A[t]$-free.

<u>Proof of Theorem 1</u>: The hypotheses on the multiplicative set guarantee the inclusions in the following diagrams:

$$\begin{array}{ccc} R & \hookrightarrow & R_T \\ \downarrow & & \downarrow \\ \bar{R} & \hookrightarrow & \overline{R_T} \end{array} \qquad \begin{array}{c} y_i \\ \downarrow \\ \bar{y}_i \end{array} \qquad \begin{array}{ccc} P & \overset{\alpha}{\hookrightarrow} & P_T \\ \downarrow & & \downarrow \\ \bar{P} & \xrightarrow{\bar{\alpha}} & \overline{P_T} \end{array} \qquad \begin{array}{c} x_i \\ \downarrow \\ \bar{x}_i \end{array}$$

Pick elements $y_1, y_2, \ldots, y_n \in P$ such that $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n\}$ is an $\bar{R}$-basis of $\bar{P}$. Pick an R_T-basis $\{x_1, x_2, \ldots, x_n\}$ of P_T. In this way, we get two $\overline{R_T}$ bases $\{\bar{\alpha}\bar{y}_1, \bar{\alpha}\bar{y}_2, \ldots, \bar{\alpha}\bar{y}_n\}$, and $\{\bar{x}_1, \ldots, \bar{x}_n\}$ of $\overline{P_T}$. Hence there exists a matrix $\bar{\beta} \in GL_n(\overline{R_T}) = \mathrm{Aut}_{\overline{R_T}}(\overline{P_T})$ such that $\bar{\beta}\,\bar{x}_i = \bar{\alpha}\,\bar{y}_i$, for all $i=1,2,\ldots,n$. By hypothesis, we can lift $\bar{\beta}$ to $\beta \in GL_n(R_T) = \mathrm{Aut}_{R_T}(P_T)$, so that $\overline{\beta x_i} = \bar{\alpha}\bar{y}_i$ for all i. Thus replacing x_i by βx_i, we may assume that $\bar{x}_i = \bar{\alpha}\bar{y}_i$ for all i.

We have $P_T = \sum_{i=1}^{n} R_T\, x_i = \Sigma\, Rx_i + \Sigma\, Bx_i$ (by condition 3)) $= P' + Q$ say. Then $P' = \Sigma\, Rx_i$ and $Q = \Sigma\, Bx_i$ are free R- and B-modules respectively. We notice that $\bar{P}$ and $\overline{P'}$ have the same image in $\bar{P}_T$. We claim that $P_T = P+Q$: We already have $\overline{P_T} = \mathrm{Im}\ \overline{P'} + \mathrm{Im}\ \bar{Q} = \mathrm{Im}\ \bar{P} + \mathrm{Im}\ \bar{Q}$. Hence by the third form of Nakayama's Lemma, it is enough to show that $C = \frac{P_T}{P+Q}$ is a finitely generated A-module. Since $P_T = P' + Q$, P' maps onto C, and so C is a quotient of $\frac{P'}{P' \cap P}$. Now, P' is a finitely generated R-submodule of P_T, and so, there exists $f \in T$ such that $fP' \subseteq P$. This implies that C is a quotient of $\frac{P'}{fP'} \simeq (\frac{R}{fR})^n$, since P' is R-free. But by condition 2) $\frac{R}{fR}$ is a finitely generated A-module, whence so is C.

In P_T, we have $x_i - y_i \in \mathfrak{M}P_T = \mathfrak{M}P + \mathfrak{M}Q$, say $x_i - y_i = -x_i'' + y_i''$, with $-x_i'' \in \mathfrak{M}P$, and $y_i'' \in \mathfrak{M}Q$. Thus $x_i' = x_i + x_i''$ equals $y_i' = y_i + y_i''$, and $\{\overline{y_1'}, \overline{y_2'}, \ldots, \overline{y_n'}\}$ is an $\bar{R}$-basis of $\overline{P_T}$. Similarly, $\{x_1', x_2', \ldots, x_n'\}$ is an R_T-basis of P_T: To see this, observe that the x_i form a B-basis of Q, and $\mathfrak{M}B \subseteq \text{rad } B$, and $\overline{x_i'} = \bar{x}_i$ in $\bar{Q}$. By (2.3), Corollary 3), the x_i form a B-basis of Q. Since $B \supsetneq R_T$, it follows that the x_i form an R_T-basis of P_T. Hence we can start all over, replacing the x_i by the x_i', and the y_i by the y_i'. Moreover, after the replacement, we have $x_i = y_i$ for all i. So $x_i \in P$ and $P' = \Sigma\, Rx_i \subseteq P$. Since $\overline{x_i'} = \bar{x}_i$, we also have $\overline{P'} = \bar{P}$. Thus $\mathfrak{M}P + P' = P$, and so for the R-module $D = \frac{P}{P'}$, we have $\bar{D} = (0)$. Since P is a finitely generated R-module, so is D. Since $P'_T = P_T$ we see, as above, that $fP \subseteq P'$ for some $f \in T$, so $fD = 0$, and D is finitely generated over R/fR, hence over A by condition 2) of the theorem. By Nakayama's Lemma, we conclude that $D = (0)$; i.e., $P = P'$ whence $P \cong \bigoplus_{i=1}^{n} Rx_i$, a free R-module.

F. <u>Regular Local Rings</u>.

(5.11) <u>Special PID's</u>.

Recall the conjecture from (4.4) pp.36:

<u>Conjecture</u> $(B\text{-}Q_d)$: <u>Let</u> A <u>be a regular local ring of dimension</u> $\leq$ d. <u>Then every</u> $P \in \mathcal{P}(A[t])$ <u>is free</u>.

In this section, we establish the validity of the conjecture for d=2. The result is due to Horrocks in the geometric case, and to Murthy in the general case. The Murthy-Horrocks Theorem relies on facts for a special class of PID-s, the so called special PID-s. The method gives yet another proof that $P \in \mathcal{P}(B[t])$ is free, when B is a PID (Corollary 1).

Definition 1 - We say that an integral domain A is an n-special PID, if A is a PID and $SL_n(A) = E_n(A)$. We say that A is a special PID, if it is an n-special PID for all $n \geq 2$. We say that an element $p \in A$ is n-special (or special), if $\frac{A}{pA}$ is an n-special (or special) PID. Notice that an n-special elment $p \in A$ is prime in the sense that Ap is a prime ideal, i.e., if $a, b \in A$ and $p|ab$, then $p|a$ or $p|b$.

Lemma 1 - Let $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_k$ be invertible primes of the integral domain A. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are ideals of A and $r_1, r_2, \ldots, r_k$ positive integers such that $\mathfrak{P}_1^{r_1} \mathfrak{P}_2^{r_2} \ldots \mathfrak{P}_k^{r_k} = \mathfrak{A}\mathfrak{B}$. Then $\mathfrak{A} = \mathfrak{P}_1^{s_1} \mathfrak{P}_2^{s_2} \ldots \mathfrak{P}_k^{s_k}$ with $0 \leq s_i \leq r_i$, $i=1,2,\ldots,k$.

Proof: By induction on the sum $n = \sum_{i=1}^{k} r_i$. If $n=1$, then $\mathfrak{A} = \mathfrak{P}_1$ or A. So assume that $n > 1$ and that the lemma holds for $\sum r_i < n$. Since $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{P}_1$, we have $\mathfrak{A} \subseteq \mathfrak{P}_1$ or $\mathfrak{B} \subseteq \mathfrak{P}_1$. If $\mathfrak{A} \subseteq \mathfrak{P}_1$ then $\mathfrak{A} = \mathfrak{P}_1\mathfrak{A}'$, whence $\mathfrak{P}_1^{r_1-1} \mathfrak{P}_2^{r_2} \ldots \mathfrak{P}_k^{r_k} = \mathfrak{A}'\mathfrak{B}$. Hence $\mathfrak{A}'$ is a product of these primes; so is $\mathfrak{A} = \mathfrak{P}_1\mathfrak{A}'$. If $\mathfrak{B} \subseteq \mathfrak{P}_1$, then $\mathfrak{B} = \mathfrak{P}_1\mathfrak{B}'$ and so $\mathfrak{P}_1^{r_1-1} \mathfrak{P}_2^{r_2} \ldots \mathfrak{P}_k^{r_k} = \mathfrak{A}\mathfrak{B}'$. Hence $\mathfrak{A}$ is a product of the required form.

Theorem 1 - Let A be an integral domain and S a multiplicative set generated by n-special elements. Let $P \in \mathcal{P}(A)$. If $P_S \simeq A_S^n$, then $P \simeq A^n$.

Proof: Choose elements $x_1, x_2, \ldots, x_n \in P$ such that $\{x_1, x_2, \ldots, x_n\}$ is an A_S-basis of P_S. Put $L = \sum_{i=1}^{n} Ax_i$. We have $L \cong A^n$, and $L_S = P_S$. Consider the exterior powers $\Lambda^n P$, and $\Lambda^n L$, and let $\det(P,L) = \operatorname{Ann}(\frac{\Lambda^n P}{\Lambda^n L})$, which is an ideal of A. Locally, that is at each maximal ideal $\mathfrak{M}$ of A, this ideal $\det(P,L)$ is generated by the determinant of a matrix and so is principal. Hence by Kaplansky [3] Theorem 62, $\det(P,L)$ is an invertible ideal of A. Since $L_S = P_S$, we must have $S \cap \det(P,L) \neq \phi$. Let $s \in S \cap \det(P,L)$, where $s = p_1^{r_1} \cdots p_k^{r_k}$ for some primes $p_i \in S$. Since $\det(P,L)$ is invertible, there exists an ideal $\mathfrak{A}$ such that $As = \det(P,L) \cdot \mathfrak{A}$. Since the elements p_i are prime, it follows from Lemma 1 that $\det(P,L) = At$ for some $t = p_1^{u_1} p_2^{u_2} \cdots p_k^{u_k}$ with $u_i \leq r_i$, $i = 1, 2, \ldots, k$.

We induct on the number $\ell = \Sigma u_i$ of prime factors of t. If $\ell = 0$, $t = 1$ and so $L = P$. So assume $\ell \geq 1$, and pick an n-special divisor p of t. Denoting by bar reduction modulo p, we have the following diagram

$$\begin{array}{ccc} L & \xhookrightarrow{\ \varphi\ } & P \\ \downarrow & & \downarrow \\ \bar{L} & \xrightarrow{\ \bar{\varphi}\ } & \bar{P} \end{array}$$

φ is injective, while $\bar{\varphi}$ is not, since it is given locally by a matrix of determinant 0. Since $\bar{A}$ is a PID, $\bar{P}$ is free and so is the submodule $\bar{\varphi}\bar{L}$. We have a splitting $\bar{L} = \bar{\varphi}\bar{L} \oplus \ker \bar{\varphi}$. We can choose $\bar{\sigma} \in SL_n(\bar{A}) = E_n(\bar{A})$ so that $\bar{\sigma}\bar{x}_1 \in \operatorname{Ker} \bar{\varphi}$. Since $E_n(A) \to E_n(\bar{A})$ is surjective, we can lift $\bar{\sigma}$ to $\sigma \in SL_n(A)$. If we write $y_i = \sigma x_i$, then $\{y_1, y_2, \ldots, y_n\}$ is a new basis of L.

From $\bar{\varphi}\bar{y}_1 = \bar{0}$, we get $y_1 \in pP \cap L$. If $y_1 = pz_1$, with $z_1 \in P$, then $L \subseteq L' = Az_1 \oplus Ay_2 \oplus \ldots \oplus Ay_n \subseteq P$ and $\det(P,L) = \det(P,L')\det(L',L)$. Hence $\det(P,L') = A\frac{t}{p}$, which has fewer than ℓ factors. By induction P is free.

Corollary 1 (Seshadri) - *Let B be a PID and $P \in \wp(B[t])$. Then P is free.*

Proof: Let $A = B[t]$, and S the multiplicative set generated by all primes of B. If $p \in S$ is a prime of B, then $\frac{A}{pA} = \frac{B}{pB}[t]$ is an Euclidean domain of the form $K[t]$, with K field. Hence p is a special prime. We have $A_S = B_S[t] = F[t]$, F being a field. Since P_S is free, we deduce that P is free.

It is worthwhile to mention a generalization of this argument (see [1], Ch. IV, Th. (6.1)).

Definition 2 - Let A be a domain. Call a *prime ideal* $\mathfrak{P}$ n-*special* if $\mathfrak{P}$ is invertible and $A/\mathfrak{P}$ is an n-special PID.

Theorem 2 - *Let S be a multiplicative set of invertible ideals generated by n-special primes. We have $A_S = \bigcup_{\mathfrak{A}\in S} \mathfrak{A}^{-1}$. Suppose $P \in \wp(A)$ and that $P_S \cong P_1 \oplus P_2 \oplus \ldots \oplus P_n$, with P_i of rank 1 for all $i = 1,2,\ldots,n$. Then*

1) *There exist rank 1 projective A-modules L_i such that $L_{iS} = P_i$, $i=1,2,\ldots,n$.*
2) *For any $L_1,\ldots,L_n$ as in 1), there exists an invertible ideal $\mathfrak{A}$ in S such that $P = \mathfrak{A}L_1 \oplus L_2 \oplus \ldots \oplus L_n$.*

Corollary 2 - *Let A be a Dedekind domain. Then every*

$P \in \mathcal{P}(A[t])$ <u>is extended from</u> A.

(5.12) <u>The Murthy-Horrocks' Theorem</u> $(B\text{-}Q_2)$.

We now proceed to give some examples of special PID-s. First some useful formulas, involving elementary matrices.

Let R be a ring and let $u \in R^*$. Define

$$w_{ij}(u) = e_{ij}^{u}\, e_{ji}^{-u^{-1}}\, e_{ij}^{u}$$

$$h_{ij}(u) = w_{ij}(u)\, w_{ij}(-1);$$

for example, in $E_2(R)$, we have

$$w_{12}(u) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}$$

$$h_{12}(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

If $u,v \in R^*$, then

$$w_{12}(u)\, w_{12}(v) = \begin{pmatrix} -uv^{-1} & 0 \\ 0 & -u^{-1}v \end{pmatrix},$$

and

$$h_{12}(u) = w_{12}(u)\, w_{12}(-1).$$

We have

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix} \in \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} E_2(R)$$

If $n \geq 3$ and $u_1, u_2, \ldots, u_n \in R^*$ then

$$\operatorname{diag}(u_1,u_2,\ldots,u_n) = \operatorname{diag}(u_1u_2\ldots u_n,1,1,\ldots,1)\, h_{12}(u_n^{-1}\, u_{n-1}^{-1} \ldots u_2^{-1})\, h_{23}(u_n^{-1}\, u_{n-1}^{-1} \ldots u_3^{-1}) \ldots h_{n-1,n}(u_n^{-1}) \in \operatorname{diag}(u_1u_2\ldots u_n,1,1,\ldots,1)E_n(R)$$

Summing up, we have the following important result.

Whitehead Lemma - *Let R be a ring and let $u_1,u_2,\ldots,u_n \in R^*$ with $n \geq 2$. Then the diagonal matrix $\operatorname{diag}(u_1,u_2,\ldots,u_n)$ is congruent modulo $E_n(R)$ to $\operatorname{diag}(u_1u_2\ldots u_n, 1,1,\ldots,1)$.*

Remark: Let $D_n(R)$ denote the group of diagonal matrices in $GL_n(R)$ and let $n \geq 2$. Then $D_n(R)$ normalizes $E_n(R)$: For, if $\delta = \operatorname{diag}(d_1,d_2,\ldots,d_n)$, then $\delta\, e_{ij}^{a}\, \delta^{-1} = e_{ij}^{d_i a d_j^{-1}}$. If $R = A$ is commutative, then $D_n(A) \cap SL_n(A) \subseteq E_n(A)$. This is an immediate consequence of Whitehead Lemma.

Proposition 1 - *Let A be an integral domain and s an n-special element of A. Then $SL_n(A_s) = SL_n(A)\, E_n(A_s)$.*

Proof: On taking determinants, one sees with the aid of the remark above that it suffices to show that $SL_n(A_s) \subseteq SL_n(A)\cdot D_n(A_s)\cdot E_n(A_s)$. Choose $\alpha \in SL_n(A_s)$. Multiplying by a suitable power s^N, we may assume that α has entries in A; so $\alpha \in M_n(A)$, and $\det \alpha = s^m$, say. We claim that if $\alpha \in M_n(A)$ and $\det(\alpha) = s^m$, then $\alpha \in SL_n(A)\, D_n(A_s)\, E_n(A_s)$. For this, we induct on m. If $m = 0$, then $\alpha \in SL_n(A)$. Suppose $m > 0$, write $\bar{A} = \frac{A}{sA}$. Now $\bar{\alpha} \in M_n(\bar{A})$, and $\det(\bar{\alpha}) = 0$. Since

$\bar{A}$ is an n-special PID, by elementary divisor theory, there exist $\bar{\epsilon}_1, \bar{\epsilon}_2 \in SL_n(\bar{A}) = E_n(\bar{A})$ such that $\bar{\epsilon}_1 \bar{\alpha} \bar{\epsilon}_2$ is diagonal with last entry 0. We lift $\bar{\epsilon}_i$ to ϵ_i in $E_n(A)$, by (5.5) Lemma 1. If $\beta = \epsilon_1 \alpha \epsilon_2$, then the entries of β in the last column are in As. We let $\delta = \mathrm{diag}(1,1,\ldots,s)$ and consider $\gamma = \beta\delta^{-1} \in M_n(A)$; $\det \gamma = \det \beta \det \delta^{-1} = \det \alpha \det \delta^{-1} = s^{m-1}$. By induction, $\gamma = \sigma\delta'\epsilon'$ for some $\sigma \in SL_n(A)$, $\delta' \in D_n(A_s)$, $\epsilon' \in E_n(A_s)$. Hence $\alpha = \epsilon_1^{-1} \beta \epsilon_2^{-1} = \epsilon_1^{-1} \gamma \delta \epsilon_2^{-1} = (\epsilon_1^{-1}\sigma)(\delta')(\epsilon'\delta\epsilon_2^{-1})$, where the first matrix in the parenthesis is in $SL_n(A)$, and the second in $D_n(A_s)$ and the third in $E_n(A_s)$. This proves the proposition.

Corollary 1 - If A is an n-special PID, then so is A_S for every multiplicative set S of A.

Proof: If $\alpha \in SL_n(A_S)$ then $\alpha \in SL_n(A_s)$ for some $s \in S$. We can assume $s = p_1 p_2 \ldots p_k$ with each p_i a prime element of A. Since A/Ap_i is a field, each p_i is special. Arguing by induction on k, we further reduce to the case $k=1$, so s is itself speical. Now the corollary follows from Proposition 1.

Corollary 2 - Let $(A,\mathfrak{M})$ be a regular local ring of dimension 2, and $s \in \mathfrak{M}\setminus\mathfrak{M}^2$. Then A_s is a special PID.

Proof: A is a UFD and so is A_s. Also $\dim(A_s) \leq 1$. Thus A_s is a PID. By (5.6) Corollay 3, $SL_n(A) = E_n(A)$. The quotient ring $\frac{A}{sA}$ is also a regular local ring of dimension 1, and so is a discrete valuation ring. Once again by (5.6), Corollary 3 we conclude that $\frac{A}{sA}$ is a special PID. Thus s is a special element. The corollary now follows from Proposition 1.

Finally, we have the following result

Proposition 2 - If A is an n-special PID, then so is $A(t)$.

The proof of the proposition depends upon the following two lemmas. Let B be a commutative ring, and $\mathfrak{A}$ an ideal of B. Then the natural homomorphism $B \to \frac{B}{\mathfrak{A}}$ induces a group homomorphism $GL_n(B) \to GL_n(B/\mathfrak{A})$, whose kernel is denoted by $GL_n(B,\mathfrak{A})$; the corresponding kernel in case of $SL_n(\cdot)$ is denoted by $SL_n(B,\mathfrak{A}) = SL_n(B) \cap GL_n(B,\mathfrak{A})$.

Lemma 1 - Suppose J is an ideal of the commutative ring B such that $J \subseteq \text{rad } B$. Then $SL_n(B,J) \subseteq E_n(B)$ for all $n \geq 1$.

Proof: We start with $\alpha = (a_{ij}) \in GL_n(B,J)$. We claim that there exist matrices ϵ, ϵ' in $E_n(B)$, and a diagonal matrix $\delta = \text{diag}(d,1,1,\ldots,1)$ such that $\alpha = \epsilon\delta\epsilon'$. If this can be done, we would be finished, since $\alpha \in SL_n(B)$ would imply that $1 = \det \alpha = d$, so that $\alpha = \epsilon\epsilon' \in E_n(B)$.

By definition of $GL_n(B,J)$ $a_{ii} \equiv 1 \pmod J$ and $a_{ij} \equiv 0 \pmod J$, if $i \neq j$. From (2.1), Proposition 1, we see that $a_{11} \in B^*$. The proof is by induction on n. If $n = 1$, α is already diagonal with $a_{11} \in B^*$. So assume $n \geq 2$. If we left multiply α by the elementary matrix $\epsilon_1 = \prod_{i=2}^{n} e_{i1}^{-a_{i1}a_{11}^{-1}}$, we get

$$\alpha' = \left(\begin{array}{c|c} a_{11} & * \\ \hline 0 & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right)$$

Similarly right multiplying α' by a suitable elementary matrix ϵ_2, we get

$$\alpha'' = \left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & \beta \end{array}\right),$$

where $\beta \in GL_{n-1}(B,J)$. By induction, these exist elementary matrices ϵ_3, ϵ_4 such that $\epsilon_3\,\beta\,\epsilon_4 = \delta'$ is a diagonal matrix. If we write $\delta'' = \mathrm{diag}(a_{11},1,1,\ldots,1)$, then $\alpha = \epsilon_1^{-1}\,\delta''\,\beta\,\epsilon_2^{-1} = \epsilon_1^{-1}\,\epsilon_3\,\delta''\delta'\epsilon_4\,\epsilon_2^{-1} = \bar{\epsilon}\bar{\delta}\bar{\epsilon}'$, with $\bar{\epsilon},\bar{\epsilon}' \in E_n(B)$ and $\bar{\delta}$ a diagonal matrix in $GL_n(B,J)$. By an observation at the beginning of this proof, the diagonal entries of $\bar{\delta}$ are in B^*. If we write $\bar{\delta} = \mathrm{diag}(u_1,u_2,\ldots,u_n)$ with $u_i \in B^*$, then by Whitehead Lemma, $\bar{\delta} = \mathrm{diag}(d,1,\ldots,1).\epsilon_5$, for some $d \in B^*$ and some $\epsilon_5 \in E_n(B)$. Thus $\alpha = \epsilon\delta\epsilon'$, with $\delta = \mathrm{diag}(d,1,\ldots,1)$, and $\epsilon,\epsilon' \in E_n(B)$. The claim is now established.

<u>Lemma 2</u> - <u>Let</u> B <u>be a commutative ring and</u> $J \subseteq \mathrm{rad}\,B$ <u>an ideal.</u> <u>If</u> $SL_n(\frac{B}{J}) = E_n(\frac{B}{J})$, <u>then</u> $SL_n(B) = E_n(B)$.

<u>Proof</u>: Let $\alpha \in SL_n(B)$, and $\bar{\alpha}$ its image under $SL_n(B) \xrightarrow{\theta} SL_n(\frac{B}{J})$.

By hypothesis $\bar{\alpha} = \bar{\epsilon}$ for some $\bar{\epsilon} \in E_n(\frac{B}{J})$. By (5.5) Lemma 1, we can lift $\bar{\epsilon}$ to ϵ in $E_n(B)$. Hence $\delta = \epsilon^{-1}\alpha \in \ker\theta = SL_n(B,J) \subseteq E_n(B)$, by Lemma 1. Thus $\alpha = \epsilon\delta \in E_n(B)$, whence

$SL_n(B) = E_n(B)$.

We can now finish the proof of Proposition 2. Consider the ring $B = A\{[s]\}$, with $s = t^{-1}$. By (5.9), Proposition 3, we have $B_s = A(t)$. By Proposition 1, it suffices to show that $SL_n(B) = E_n(B)$, and that $\frac{B}{sB}$ is an n-special PID. Again by (5.9) Proposition 2, we have $J = sB \subseteq \mathrm{rad}(B)$ and $\frac{B}{sB} \simeq A$, an n-special PID by hypothesis; so $SL_n(\frac{B}{J}) = E_n(\frac{B}{J})$. By Lemma 2, we conclude that $SL_n(B) = E_n(B)$.

We are now ready for the principal result of this section:

<u>Theorem 1</u> (Murthy-Horrocks) - <u>Let</u> $(A,\mathfrak{M})$ <u>be a regular local ring of dimension</u> 2. <u>Then every</u> $P \in \mathcal{P}(A[t])$ <u>is free</u>.

<u>Proof</u>: By Local Horrocks' Theorem, it is enough to show that $Q = A(t) \underset{A[t]}{\otimes} P$ is free. Let $s \in \mathfrak{M}\backslash\mathfrak{M}^2$. Now s is a special element of $A(t)$: For $\frac{A(t)}{sA(t)} \simeq \frac{A}{sA}(t)$; but $\frac{A}{sA}$ is a regular local ring of dimension 1 and so is a discrete valuation ring. From (5.6), Corollary 3 and (5.12), Proposition 2, it follows that $\frac{A}{sA}(t)$ is a special PID. By (5.11), Theorem 1, it is enough, if we show that Q_s is $A(t)_s$-free. But $A(t)_s =$ $= A_s(t)$ and Q_s is extended from P_s which is an $A_s[t]$-module. Now A_s is a PID and so by (4.4) Corollary 2, P_s is free. Hence the extended module Q_s is free.

G. Formal Power Series Rings over Fields.

(5.13) Mohan Kumar's Theorem (B-Q_d) for power series).

In this section, we deal with the (B-Q_d) conjecture, when the base ring $A = k[[X_1,X_2,\ldots,X_d]]$ is the power series ring in d-indeterminates over a field k. The result will be deduced from the following more general result:

Theorem 1 (Mohan Kumar) - *Let $A = k[[X_1,\ldots,X_d]]$ be the power series ring over a field k, and K the field of fractions of A. Let B be any commutative k-algebra. Let $P \in \mathcal{P}(A \otimes_k B)$ and $Q \in \mathcal{P}(B)$. If $K \otimes_A P$ and $K \otimes_k Q$ are isomorphic as $K \otimes_k B$-modules, then $P \simeq A \otimes_k Q$.*

Corollary 1 - *Let A be as above. Then every $P \in \mathcal{P}(A[t_1,t_2,\ldots,t_n])$ is free.*

Proof: Take $B = k[t_1,t_2,\ldots,t_n]$ and Q to be a free B-module of suitable rank, and apply (4.4) Corollary 1.

Before we take up the proof of Theorem 1, we need some basic facts about formal power series. Recall that if $f \in k[[X]]$, we write $f = \sum_{i=0}^{\infty} a_i X^i$, with $a_i \in k$. If $f \neq 0$, there exists a first non-vanishing coefffficient a_r, and we write $\mathrm{Ord}_X(f) = r$. If $f = 0$, $\mathrm{Ord}_X(f) = \infty$. We also recall that the degree of the zero polynomial is $-\infty$.

Definition 1 - Let $f(X_1,X_2,\ldots,X_d) \in A$. We say that f is *regular of order* $m < \infty$ in X_d, if $\mathrm{Ord}_{X_d} f(0,0,\ldots,0,X_d) = m$. In other words, a regular element in X_d has a term aX_d^m, $a \neq 0$, but no term bX_d^i with $b \neq 0$, $0 \leq i < m$.

We write $A' = k[[X_1,X_2,\dots,X_{d-1}]]$.

Proposition 1 - Let $f \in A$ be regular of ord m in X_d. Then given $g \in A$, there exist unique elements q in A and r in the polynomial ring $A'[X_d]$ such that $g = qf + r$, where $\deg_{X_d} r < m$.

Proof: We induct on the number of variables d. If $d = 0$, the result is trivially valid.

If $d = 1$, then A is a discrete valuation ring, and every non-zero element g can be written as $g = \epsilon X_1^n$, with ϵ a unit, and $n \geq 0$. We proceed by induction when $d \geq 2$. We show that the coefficients of q and r can be inductively determined; write $f = \Sigma\, f_i\, X_1^i$, $g = \Sigma\, g_i\, X_1^i$, $q = \Sigma\, q_i\, X_1^i$, and $r = \Sigma\, r_i\, X_1^i$, where $f_i, g_i, q_i \in B = k[[X_2,X_3,\dots,X_d]]$, and $r_i \in B'[X_d]$, the ring B' being $k[[X_2,\dots,X_{d-1}]]$. We want $g = qf+r$, with $r = 0$ or $\deg_{X_d} r < m$. Comparing the coefficients of X_1^i we have the following equations:

$$g_o = q_o f_o + r_o$$

$$g_1 = q_1 f_o + q_o f_1 + r_1$$

$$\cdots\cdots\cdots\cdots$$

$$g_i = q_i f_o + q_{i-1} f_1 + \dots + q_o f_i + r_i$$

$$\cdots\cdots \quad \text{etc.}$$

We notice that $f_o(0,0,\dots,0,X_d) = f(0,0,\dots,0,X_d)$ and so f_o in B is regular of ord m in X_d. Since g_o is also in B and B has fewer than d variables, the induction hypothesis guarantees the unique existence of $q_o \in B$ and $r_o \in B'[X_d]$

such that $\deg_{X_d} r_o < m$. We repeat the process with $g_1 - q_o f_1 \in B$ to get q_1 and r_1 as desired and uniquely.
Proceeding in this way at the i^{th} stage, we apply the division algorithm to $g_i - (q_o f_i + q_1 f_{i-1} + \dots + q_{i-1} f_1)$ to find q_i and r_i. Thus all the coefficients q_i and r_i are determinable uniquely. At each stage we have $\deg_{X_d} r_i < m$ so that $\deg_{X_d} r < m$, where $r = \sum_i r_i X_1^i$.

Corollary 2 - *Let* A, f, *and* A' *as above. Then* $\frac{A}{Af}$ *is an* A'*-free module with basis the image* of $\{1, X_d, X_d^2, \dots, X_d^{m-1}\}$.

Definition 2 - Let $(B, \mathfrak{M})$ be a commutative local ring. We say that a polynomial w in $B[X]$ is a *Weierstrass polynomial* of (degree m) if $w = X^m + a_{m-1} X^{m-1} + \dots + a_o$, with $a_i \in \mathfrak{M}$ for $i = 0, 1, \dots, m-1$.

Theorem 2 (Weierstrass Preparation Theorem) - *If* $f \in A$ *is regular of order* m *in* X_d, *then there exist a unit* $q \in A^*$ *and a Weierstrass polynomial* w *in* $A'[X_d]$ *of degree* m *in* X_d *such that* $qf = w$; *in other words* f *and* w *generate the same ideal in* A.

Proof: By Proposition 1, there exist unique elements $q \in A$ and $r \in A'[X_d]$ such that $X_d^m = qf + r$, with $\deg_{X_d} r < m$. If $m = 0$, then $r = 0$, and if $m \geq 1$, then $r(0,0,\dots,0) = 0$ and so $r \in \mathfrak{M}$, the maximal ideal of A. If $r = a_{m-1} X_d^{m-1} + \dots + a_o \in A'[X_d]$, then we have

$$(\dagger) \qquad X_d^m = q(0,0,\dots,X_d) f(0,\dots,0,X_d) + r(0,0,\dots,X_d).$$

Here $f(0,\dots,0,X_d)$ is of the form $a X_d^m$ + higher terms, with $a \neq 0$ and so $q(0,0,\dots,X_d)f(0,0,\dots,X_d)$ has only terms of $\deg \geq m$. This means for $i = 0,1,2,\dots,m-1$, $a_i(0,0,\dots,0) = 0$ and so $a_i \in \mathfrak{M}'$ of A'. Hence $X_d^m - r = w$ is a Weierstrass polynomial of degree m in X_d over A'. Comparing the coefficient of X_d^m on both sides of (†), we see that $q(0,0,\dots,X_d)$ has a non-zero constant term, i.e., $q(0,0,\dots,0)\neq 0$ and so q is a unit of A^*. If $w = X_d^m + r'$ with $\deg_{X_d} r' < m$ the uniqueness of w follows from Proposition 1 via the equation $X_d^m = qf - r'$.

<u>Lemma 1</u> - <u>Let</u> $f(X_1,X_2,\dots,X_d) \neq 0$ <u>in</u> A. <u>There exists a change of variables</u> $X_i \mapsto Y_i$, <u>with</u> $Y_d = X_d$ <u>such that</u> $f(Y_1,Y_2,\dots,Y_d)$ <u>is regular in</u> X_d.

<u>Proof</u>: We first give a proof which works when k is an infinite field. Put $Y_i = X_i + a_i X_d$ for $i = 1,2,\dots,d-1$ and $Y_d = X_d$. Write $g(X_1,X_2,\dots,X_d) = f(Y_1,\dots,Y_d)$. Then $g(0,0,\dots,X_d) = f(a_1X_d, a_2X_d,\dots,X_d)$; write $f = \sum_i f_i$, with f_i homogeneous of degree i. We have $g(0,0,\dots,X_d) =$ $= \sum_i f_i(a_1,a_2,\dots,a_{d-1},1)X_d^i$. Since $f \neq 0$, there exists $f_j \neq 0$. Since k is infinite, we can choose $a_1,a_2,\dots,a_{d-1}$ such that $f_j(a_1,a_2,\dots,a_{d-1},1) \neq 0$. Hence the change of variables can be effected such that $f(Y_1,\dots,Y_d)$ is regular in X_d.

<u>2nd proof</u>: If $f = \sum a_{s_1,s_2,\dots,s_d} X_1^{s_1} \dots X_d^{s_d}$, we write $s = (s_1,s_2,\dots,s_d)$ and $X^s = X_1^{s_1} X_2^{s_2} \dots X_d^{s_d}$. We write f itself as $f = \sum a_s X^s$, s running through all d-tuples.

We consider the lexicographic ordering on these d-tuples: $s = (s_1, s_2, \ldots, s_d) < s' = (s'_1, s'_2, \ldots, s'_d)$ if $s_1 < s'_1$ or $s_1 = s'_1$ and $s_2 < s'_2$, etc. Let support $f = \{s: a_s \neq 0\}$. Let α be minimal in the lexicographic ordering of supp f. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ choose $r >$ all α_i. We now make the following change of variables: $Y_i = X_i + X_d^{r^{d-i}}$, $i = 1,2,\ldots,d-1$ and $Y_d = X_d$. Write as before $g(X_1, X_2, \ldots, X_d) = f(Y_1, \ldots, Y_d)$; therefore $g(0,0,\ldots,X_d) = f(X_d^{r^{d-1}}, X_d^{r^{d-2}}, \ldots, X_d^r, X_d)$. If we write $f = \Sigma\, a_s X^s$, then $g(0,0,\ldots,X_d) = \Sigma\, a_s\, X_d^{\lambda_r(s)}$ where $\lambda_r(s) = s_d + s_{d-1}\, r + \ldots + s_1 r^{d-1}$. We claim that if $f \neq 0$, the above choice of r guarantees that $g(0,0,\ldots,X_d) \neq 0$. In fact the term $X_d^{\lambda_r(\alpha)}$ can not get cancelled with the other terms. This follows from the following two observations:

(i) $\lambda_r(\alpha) < r^d$, and

(ii) for every s in the supp f, $\alpha < s \Rightarrow \lambda_r(\alpha) < \lambda_r(s)$.

To see (i), we have $\lambda_r(\alpha) = \alpha_d + \alpha_{d-1}\, r + \ldots + \alpha_1\, r^{d-1} \leq (r-1)\,(1 + r + \ldots + r^{d-1}) = r^d - 1$. To see (ii), first suppose $\alpha_1 < s_1$; then $\lambda_r(s) \geq r^{d-1}\, s_1 \geq r^{d-1}(\alpha_1 + 1)$, while $\lambda_r(\alpha) = \alpha_d + \alpha_{d-1}\, r + \ldots + \alpha_2\, r^{d-2} + \alpha_1\, r^{d-1} < r^{d-1} + \alpha_1\, r^{d-1} = r^{d-1}\,(1+\alpha_1)$. Thus $\lambda_r(\alpha) < \lambda_r(s)$. If $\alpha_1 = s_1$, we settle the question in a similar way. Finite induction then establishes (ii).

The following theorem provides a means of descent from power series rings to polynomial rings; the element w should be thought of as a Weierstrass polynomial.

Descent Theorem - Let $T_o \hookrightarrow T$ be commutative rings, $w \in T_o$ not a zero-divisor in T such that

$$(*) \qquad \frac{T_o}{wT_o} \simeq \frac{T}{wT}.$$

Let P be a T-module, on which w is not a zero-divisor, and W a T_{ow}-module such that $P_w \cong T \otimes_{T_o} W$. Then there exists a T_o-module V such that (i) $P \cong T \otimes_{T_o} V$ and (ii) $V_w \cong W$. Moreover, if T is faithfully flat over T_o, then $V \in \mathcal{P}(T)$ implies that $V_o \in \mathcal{P}(T_o)$.

Proof: From the hypothesis on w, we get $P \hookrightarrow P_w \xrightarrow[f]{\simeq} T \otimes_{T_o} W$.

Form the exact sequence of T_o-modules:

$$(**) \qquad 0 \longrightarrow P \xrightarrow{f} T \otimes_{T_o} W \xrightarrow{g} C \longrightarrow 0.$$

Localizing we get $C_w = 0$. Since $(*)$ implies that $T = T_o + wT = T_o + w^2T$, etc., we can identify $T \otimes_{T_o} C$ with C and g with $1_T \otimes_{T_o} g_o$, where $g_o\colon W \to C \to 0$. Now form the exact sequences,

$$0 \longrightarrow V \longrightarrow W \xrightarrow{g_o} C \longrightarrow 0$$

and

$$\begin{array}{ccccccccc} \mathrm{Tor}_1^{T_o}(T,C) & \longrightarrow & T \otimes_{T_o} V & \longrightarrow & T \otimes W & \xrightarrow{g} & T \otimes C & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ 0 & \longrightarrow & P & \longrightarrow & T \otimes W & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

and compare with $(**)$.

If we show $\mathrm{Tor}_1^{T_o}(T,C) = 0$, then we can conclude that $T \otimes_{T_o} V = \ker g = P$ and moreover $V_w \cong W$, since $C_w = 0$.

To prove $\mathrm{Tor}_1^{T_o}(T,C) = 0$: From the exact sequence of

T_o-modules:

$$0 \longrightarrow T_o \longrightarrow T \longrightarrow \frac{T}{T_o} \longrightarrow 0$$

we get for all $i \geq 1$

$$0 = \mathrm{Tor}_i^{T_o}(T_o, C) \longrightarrow \mathrm{Tor}_i^{T_o}(T, C) \longrightarrow \mathrm{Tor}_i^{T_o}(\frac{T}{T_o}, C).$$

Assume that C is finitely generated as a T_o-module. Since $C_w = 0$, we can find an integer $r \geq 1$ such that $w^r C = 0$. Hence w^r annihilates $\mathrm{Tor}_i^{T_o}(\cdot, C)$. On the other hand, the condition (*) implies that multiplication by w is an automorphism of $\frac{T}{T_o}$, and so also of $\mathrm{Tor}_i(\frac{T}{T_o}, \cdot)$. These two observations together enable us conclude that $\mathrm{Tor}_i(\frac{T}{T_o}, C) = 0$, if C is finitely generated. In the general case, it is enough to observe that C is the direct limit of its finitely generated T_o-submodules, so that $\mathrm{Tor}_i(\frac{T}{T_o}, C) = 0$, always.

Let $(C, \mathfrak{M})$ be a commutative local ring, t an indeterminate, and $S = 1 + tC[t]$. We have $C[t]_S = C[[t]] = C[t]_{(\mathfrak{M},t)}$, by remark preceding (5.9) Proposition 3.

<u>Lemma 2</u> - <u>Let</u> $w \in C[t]$ <u>be a Weierstrass polynomial of degree</u> m. <u>Then</u> $C[t]s + C[t]w = C[t]$, <u>for all</u> $s \in S$.

<u>Proof</u>: Let $B = \frac{C[t]}{C[t]s + C[t]w}$. B is a finitely generated C-module, since w is monic, and $\frac{C[t]}{C[t]w}$ is. Hence by Nakayama's Lemma, it is enough to show that $\frac{B}{\mathfrak{M}B} = (0)$. But $t^m = 0$ in $\frac{B}{\mathfrak{M}B}$. Thus it suffices to show that $\frac{B}{\mathfrak{M}B + tB} = (0)$, since $B = \mathfrak{M}B + tB$ would imply $B = \mathfrak{M}B + tB \subseteq \mathfrak{M}B + t^2B \subseteq \ldots \subseteq \mathfrak{M}B + t^mB = \mathfrak{M}B$. Now $\frac{B}{\mathfrak{M}B + tB} = \frac{C[t]}{\mathfrak{M}C[t] + tC[t] + sC[t]}$, if we use

if we use the natural mapping from $C[t] \to B$ and lift $\mathfrak{M}B + tB$. If we now use the specialization $t \mapsto 0$, we see that $\frac{B}{\mathfrak{M}B+tB} \simeq$ $\simeq \frac{C}{\mathfrak{M}+Cs(0)} = (0)$, since $s(0) = 1$.

Proposition 2 - Let $(C,\mathfrak{M})$ be a commutative local ring, and w a Weierstrass polynomial of degree m. Let $A = C[[t]]$, and B any commutative C-algebra. Write $T = A \otimes_C B$. Suppose there be given $P \in \wp(T)$, and $Q \in \wp(B)$ such that $P_w \simeq A_w \otimes_C Q$, then $P \simeq A \otimes_C Q$.

Proof: To start with we have the following diagram:

$$(\Delta)\qquad \begin{array}{ccccccc} C & \longrightarrow & C[t] & \longrightarrow & C[t]_S & \xrightarrow{\text{faithfully flat}} & A = C[[t]] \\ & & \downarrow & & & & \downarrow \\ & & C[t]_w & \longrightarrow & C[t]_{wS} & \longrightarrow & A_w \end{array}$$

From this we form $(\Delta) \otimes_C B$:

$$(\Delta) \otimes_C B \qquad \begin{array}{ccccccc} B & \longrightarrow & B[t] & \longrightarrow & B[t]_S & \xrightarrow{\text{faithfully flat}} & A \otimes_C B \\ & & \downarrow & \text{I} & \downarrow & \text{II} & \downarrow \\ & & B[t]_w & \longrightarrow & B[t]_{wS} & \longrightarrow & A_w \otimes_C B \end{array}$$

Putting $T_o = B[t]_S$ and $T = A \otimes_C B$, we can rewrite square II:

$$\begin{array}{ccc} T_o & \xrightarrow{\text{faithfully flat}} & T \\ \downarrow & & \downarrow \\ T_{ow} & \longrightarrow & T_w \end{array}$$

Since w is a Weierstrass polynomial, we get $\frac{T_o}{wT_o} \simeq \frac{T}{wT}$. Let

$W = Q[t]_{wS} \in \mathcal{P}(T_{ow})$. We have $P_w \simeq A_w \otimes_C Q$. We can thus apply the Descent Theorem, which implies that there exists $V \in \mathcal{P}(T_o)$ such that (i) $P \simeq T \otimes_{T_o} V$, and (ii) $V_w \simeq W$.

Lemma 2 guarantees that square I is an affine patching square, since the comaximality property there is preserved under base extensions of C. Hence we can define $U \in \mathcal{P}(B[t])$ by the Cartesian square

$$\begin{array}{ccccc} U & & \longrightarrow & & V \\ \downarrow & & & & \downarrow \\ Q[t]_w & \longrightarrow & Q[t]_{wS} & \xrightarrow{\simeq} & V_w \end{array}$$

so that (a) $U_w \simeq Q[t]_w$, and (b) $U_S \simeq V$. We conclude from (a) by the strong form of Affine Horrocks Theorem that $U \simeq Q[t]$. Thus, we have from (i) and (b) that $P \simeq T \otimes_{T_o} V \simeq T \otimes_{T_o} Q[t]_S \simeq$ $\simeq T \otimes_{T_o} (T_o \otimes_B Q) \simeq A \otimes_C Q$, which is what we wanted to prove.

Theorem 1 can now be deduced easily.

<u>Proof of Theorem 1</u>: We are given that $K \otimes_A P$ and $K \otimes_k Q$ are isomorphic as $K \otimes_k B$ modules. By (3.4) Proposition 2, there exists $w \neq 0$ in A such that $P_w \cong A_w \otimes_k Q$ as $A_w \otimes_k B$ modules. After a change of variables (Lemma 1) and multiplication by a unit (Theorem 2) we may assume that w is a Weierstrass polynomial in $X = X_d$ with coefficients in $A' = k[[X_1, X_2, \ldots, X_{d-1}]]$. Write $D = A' \otimes_k B$, an A'-algebra, and $Q' = A' \otimes_k Q \in \mathcal{P}(D)$. We have $P_w \simeq A_w \otimes_{A'} Q'$. We can apply Proposition 2 with $C = A'$, and $Q = Q'$. We conclude that

$P \simeq A \otimes_{A'} Q' \simeq A \otimes_k Q$, getting what we wanted.

Remark 1: Let k be a field with a non-trivial absolute value and let $A = k\{\{X_1, X_2, \ldots, X_d\}\}$ be the ring of convergent power series over k (See Artin [O]). Then the proof of Theorem 1 may be modified to show that $P \in \mathcal{P}(A[t_1, t_2, \ldots, t_n])$ is free.

6. Quillen's Localization Theorems

We shall present a somewhat axiomatized version of the theorem from which we can deduce a number of further important applications, notably the following: If a finitely presented algebra is locally a polynomial algebra, then it is the symmetric algebra of a finitely generated projective module. This material is taken from [BCW].

All rings and algebras are commutative, unless indicated otherwises.

A. Quillen Induction.

(6.1) Formulation.

We fix a commutative ring K. Roughly speaking, the localization theorems we present say that two objects A and B over K which are locally isomorphic (i.e., $A_{\mathfrak{M}} \cong B_{\mathfrak{M}}$ over $K_{\mathfrak{M}}$ for all maximal ideals $\mathfrak{M}$ of K) are isomorphic over K. The following proposition formulates a useful argument called Quillen Induction.

Let Loc(K) denote the category of K-algebras of the form $L = K_S$, where S is a multiplicative set in K.

Proposition 1 - Let $P(L)$ be a proposition about K-algebras $L \in Loc(K)$. In order that $P(L)$ hold for all $L \in Loc(K)$, and in particular for $L = K$, it suffices that the following conditions be satisfied:

1) Local validity: $P(K_{\mathfrak{M}})$ holds for all maximal ideals $\mathfrak{M}$ of K.
2) Specialization: If $L, L' \in Loc(K)$, and if there is a K-algebra homomorphism $L \to L'$, then $P(L)$ implies $P(L')$.
3) Finiteness: If S is a multiplicative set in K, then $P(K_S)$ implies $P(K_s)$ for some $s \in S$.
4) Sheaf condition: If $L \in Loc(K)$ and if s_o, s_1 in L generate the unit ideal in L, then $P(L_{s_o})$ and $P(L_{s_1})$ together imply $P(L)$.

Proof: Let $I = \{s \in K: P(K_s) \text{ holds}\}$. By specialization, it suffices to show that $1 \in I$. By local validity and finiteness, it follows that for a given maximal ideal $\mathfrak{M}$ of K, there exists $s \notin \mathfrak{M}$ for which K_s holds. Hence I is contained in no maximal ideal of K, and so we will be done, if we show that I is an ideal. Let $t_o, t_1 \in I$ and let $t \in Kt_o + Kt_1$. We want to prove that $P(K_t)$ holds. Write $L = K_t$ and s_i = image of t_i in L, $i = 0,1$. Then we have $L = Ls_o + Ls_1$. Moreover $L_{s_i} \cong K_{t_i t}$, $i = 0,1$ and so the L_{s_i} are localizations of K_{t_i}. Since $t_i \in I$, it follows by specialization that $P(L_{s_i})$ hold for $i = 0,1$. The sheaf condition now implies that $P(L)$ holds.

(6.2) Strategy of applications.

Without being specific we shall present a typical application of the argument. We shall assume that A, B, etc., are some K-linear structures belonging to some "K-linear category" $C(K)$; for example, they may be K-modules, K-algebras, $K[T]$-modules, etc. We want a criterion permitting us to pass from local isomorphisms to isomorphisms in $C(K)$:

Local criterion for isomorphisms: If $A_{\mathfrak{M}} \cong B_{\mathfrak{M}}$ in $C(K_{\mathfrak{M}})$ for all $\mathfrak{M} \in \mathrm{Max}(K)$, then $A \cong B$ in $C(K)$.

We now outline our approach to this result. This will help to motivate the subsequent development of this section.

Apply Quillen induction to the proposition about K-algebras $L \in \mathrm{Loc}(K)$: $P(L)$: $L \otimes_K A \simeq L \otimes_K B$ in $C(L)$; i.e., $P(K_S)$: $A_S \simeq B_S$ in $C(K_S)$. We must check the four conditions of (6.1) Proposition 1. Local validity is given as hypothesis. Specialization is obvious, since base change is a functor.

For finiteness, it suffices to show that

$$\mathrm{Isom}_{C(K_S)}(A_S,B_S) = \varinjlim_{s\in S} \mathrm{Isom}_{C(K_s)}(A_s,B_s) \qquad (*)$$

where the directed system on the right side is given by divisibility: $K_s \to K_{st}$ for $s,t \in S$. We claim that if A and B are finitely presented objects, then $*$ is valid, guaranteeing finiteness.

In fact we will show the following:

(i) If A is finitely presented, then

$$\underline{FP(A)}: \ \forall\, C \in C(K), \quad \mathrm{Hom}_{C(K_S)}(A_S,C_S)=\varinjlim_{s\in S} \mathrm{Hom}_{C(K_s)}(A_s,C_s).$$

(ii) FP(A) and FP(B) together imply (*).

To prove (i), suppose A is finitely presented in $C(K)$. Then A is presented by a finite set $a_1, a_2, \ldots, a_n$ of generators, and a finite set of defining relations $f_1(a) = 0$, $f_2(a) = 0, \ldots, f_m(a) = 0$, where $f_i(X) = f_i(X_1, X_2, \ldots, X_n)$ and $a = (a_1, a_2, \ldots, a_n)$. Here the $f_i(X)$ are some K-multilinear expressions for K-linear structures of the type under consideration. So if $C \in C(K)$, we have a canonical identification

$$H(C) = \mathrm{Hom}_{C(K)}(A, C) = \{c = (c_1, c_2, \ldots, c_n) \in C^n : \ f_i(c) = 0, \quad i = 1, 2, \ldots, m\}.$$

Since base change will preserve such presentations, we obtain an analogous description of $\mathrm{Hom}_{C(K_S)}(A_S, C_S)$ $(= \mathrm{Hom}_{(C(K)}(A, C_S))$ as

$$H(C_S) = \{c = (c_1, c_2, \ldots, c_n) \in C_S^n : \ f_i(c) = 0, \quad i = 1, 2, \ldots, n\}.$$

Moreover this identfication is functorial in $L = K_S$. Now $C_S = \varinjlim_{s \in S} C_s$ and since n is finte $C_S^n = \lim_{s \in S} C_s^n$. From this (i) follows, since $c \in H(C_S) \Rightarrow \exists\ s \in S$ and $c_1 \in C_s^n$ such that $c_{1S} = c$. Then $f_i(c) = 0 \Rightarrow f_i(c_1)_S = 0$ so that $\exists\ t \in S \ni$ $\ni f_i(c_1)_t = 0 = f_i(c_{1t})$, $i = 1, 2, \ldots, m$. Hence $H(C_S) = \varinjlim_{s \in S} H(C_s)$.

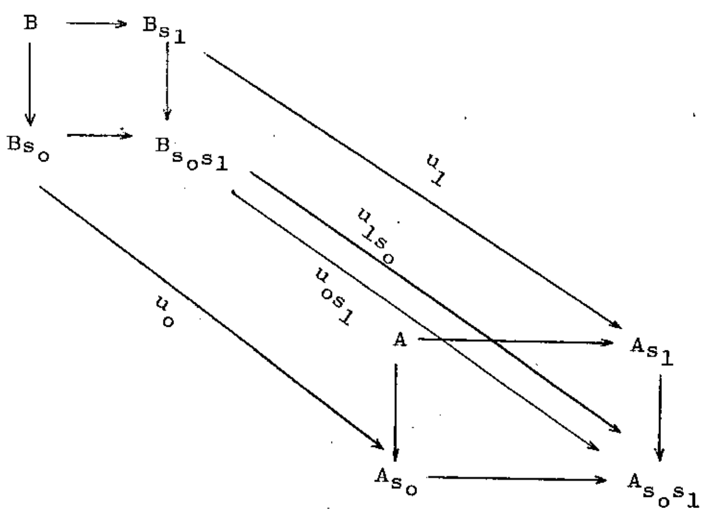
To prove (ii): Assume that <u>FP(A)</u> and <u>FP(B)</u> hold, and let $u: A_S \to B_S$ be an isomorphism. From FP(A), we have $u = u_{1S}$ for a lifting $u_1: A_{s_1} \to B_{s_1}$ for some $s_1 \in S$. Similarly from <u>FP(B)</u>, we get $u^{-1} = u_{2S}$ for some $u_2: B_{s_2} \to A_{s_2}$, with $s_2 \in S$. Write $v = u_{2s_1} \circ u_{1s_2}: A_{s_1 s_2} \to A_{s_1 s_2}$; then $v_S = 1_{A_S}$. We conclude from <u>FP(A)</u> again that there exists $t \in S$ for which $v_t = 1_{A_{s_1 s_2 t}}$. Similarly, if we write $v' = u_{1s_2} \circ u_{2s_1}$: $B_{s_1 s_2} \to B_{s_1 s_2}$, we have $v'_S = 1_{B_S}$, whence $v'_{t'} = 1_{B_{s_1 s_2 t'}}$ for some $t' \in S$. Then $u_{1s_2tt'}: A_{s_1s_2tt'} \to B_{s_1s_2tt'}$ is a $K_{s_1s_2tt'}$-isomorphism with inverse $u_{2s_1tt'}$. Thus (ii) holds.

In practice thus, we can deduce the finiteness condition by the "finite presentability" of the objects in $C(K)$.

Finally, we turn to the sheaf condition. To simplify notation assume $L = K$ and that $K_{s_o} + K_{s_1} = K$. We are given isomorphisms $u_i: B_{s_i} \to A_{s_i}$ in $C(K_{s_i})$, $i = 0,1$. We want to prove that A and B are isomorphic in $C(K)$. We have an affine patching square:

$$\begin{array}{ccc} K & \longrightarrow & K_{s_1} \\ \downarrow & & \downarrow \\ K_{s_o} & \longrightarrow & K_{s_os_1} \end{array}$$

Hence by affine patching, using the fibre product, we think of objects in $C(K)$ as a pair of objects in $C(K_{s_o})$ and $C(K_{s_1})$ respectively, together with an isomorphism in $C(K_{s_os_1})$ of the last two objects localized. We have a diagram as below:



We see that if $u_{os_1} = u_{1s_o}$, then by the remark above, the fibre product gives the sought for isomorphism $u: B \to A$ such that $u_{s_i} = u_i$, $i = 0,1$, and we are done.

Of course, there is no _a priori_ reason why we should have

$$(*) \qquad u_{os_1} = u_{1s_o}.$$

Next, we ask if we can replace u_o and u_1 by isomorphisms which satisfy (*). The freedom of choice we have is to replace u_i by $u_i' = u_i \circ v_i$, where the v_i are automorphisms of B_{s_i}, $i = 0,1$. Then the relation $u'_{os_1} = u'_{1s_o}$ is equivalent to

$$(**) \qquad v_{1s_o} v_{os_1}^{-1} = u_{1s_o}^{-1} u_{os_1} = u \in \mathrm{Aut}(B_{s_o s_1}).$$

Let us introduce the following notation: For any commutative K-algebra L, let $G(L) = \mathrm{Aut}_{C(L)}(L \underset{K}{\otimes} B)$. Then we have $u \in G(K_{s_o s_1})$. The relation (**) means that we wish to have $u \in G(K_{s_1})_{s_o} \cdot G(K_{s_o})_{s_1}$. If we can ahieve this, then we would succeed, since then we can appeal to (6.1) Proposition 1.

In general however, this is unattainable. However in the setting of Quillen's Localization Theorem as well as others, there is a naturally defined subgroup $G_o(L)$ of $G(L)$ to which we can make u belong, and we will be able to show that

$$(***) \qquad G_o(K_{s_o s_1}) = G_o(K_{s_1})_{s_o} \cdot G_o(K_{s_o})_{s_1}.$$

Thus in what follows, we aim for the local criterion for isomorphism in $C(K)$, imposing some finite presentability in the category, and choosing G_o adequately to guarantee (***).

The strength of Quillen Induction is in the sheaf condition, which enables us to pass from local isomorphisms to global ones, and as such it should be thought of as gluing of isomorphisms.

B. Axiom Q and Scalar Operations on Group Functors.

(6.3) The formula $G_o(L_{s_o s_1}) = G_o(L_{s_o})_{s_1} \cdot G_o(L_{s_1})_{s_o}$.

We now look for formulas of the type (***) indicated in (6.2) and want to interpret G_o. First some notation:

G will denote a functor from the category $C(K)$ of commutative K-algebras to groups; a K-algebra homomorphism $f: L \to L'$ yields a group homomorphism $G(f): G(L) \to G(L')$. In special cases, we use a more suggestive notation. Localization: If $f: L \to L_S$ with S a multiplicative set of L, and $G(f)$: $G(L) \to G(L_S)$, we write u_S for the image of $u \in G(L)$ under $G(f)$; we also write $G(L)_S (\subseteq G(L_S))$ for the image of $G(L)$ under $G(f)$. Similarly, if $S = \{1, s, s^2, \ldots\}$, we write u_s etc. If $\mathfrak{P} \in \operatorname{Spec} K$, we write $u_{\mathfrak{P}}$ etc.

Polynomials: Let T be an indeterminate and $f: L[T] \to L'$. Write $f(T) = t \in L'$. Here L' is an L-algebra. If $u \in G(L[T])$, we write $u(T) = u$, and $G(f): u(T) \longmapsto u(t)$.

Example (i): If $s \in L$, and $f: L[T] \to L[T]$ is defined by

$f|L = 1_L$ and $f(T) = sT$. Then for $u \in G(L[T])$, $G(f)(u) = u(sT)$.

Example (ii): Let $f: L[T] \to L$ be defined by $f|L = 1_L$ and

$f(T) = 0$. We write $u(0) = G(f)(u)$, and

$G(TL[T]) = \{u(T) \in G(L[T]) \mid u(0) = 1\}$. We have an exact sequence

$$1 \to G(TL[T]) \to G(L[T]) \xrightarrow{G(f)} G(L).$$

With these notations, we make two definitions. Examples will be given in the next two subsections.

Definition 1 - Let G be a group functor from commutative K-algebras to groups. We say that G satisfies axiom Q, if given a commutative K-algebra L, an element s of L and an element $u(T)$ of $G(TL_s[T])$, there is an integer $r \geq 0$ and an element $v(T)$ in $G(TL[T])$ such that $u(s^r T) = v(T)_s$.

Definition 2 - Let G be a group functor as above. A scalar operation on G consists of an action $L \times G(L) \to G(L)$ for each commutative K-algebra L, denoted $(s,u) \longmapsto {}^s u$, satisfying the following:

$$^1u = u, \quad ^s(^tu) = {}^{st}u, \quad ^s{uv} = {}^su \cdot {}^sv$$

for $s,t \in L$, $u,v \in G(L)$. Further these actions are to be natural, in the sense that if $f\colon L \to L'$ is a K-algebra homomorphism and if the corresponding map $G(f)\colon G(L) \to G(L')$ sends $u \in G(L)$ to $u' \in G(L')$, then it sends ${}^su$ to ${}^{f(s)}u'$ for $s \in L$.

The action of L on $G(L)$ amounts to a multiplicative monoid homomorphism $L \to \mathrm{End}\,(G(L))$. In particular $u \longmapsto {}^0u$ is an idempotent endomorphism of $G(L)$. If we denote the image of this endomorphism by ${}^0G(L)$ and the kernel by $G_0(L)$, then $G_0(L) = \{u \in G(L)\colon\ {}^0u = 1\}$. Thus $G(L)$ is the semi-direct product $G_o(L) \rtimes {}^oG(L)$, and this decomposition is functorial in L.

The relevance of these definitions is brought out by the following result:

Theorem 1 - *Let G be a functor from commutative K-algebras to groups. Assume that G satisfies axiom Q and that G admits a scalar operation. For any commutative K-algebra L, let $G_o(L) = \{u \in G(L): {}^ou = 1\}$. Suppose $s_o, s_1 \in L$ and $L = Ls_o + Ls_1$. Then we have*

$$G_o(L_{s_os_1}) = G_o(L_{s_o})_{s_1} \cdot G_o(L_{s_1})_{s_o}.$$

The proof of the theorem depends on the following lemma.

Lemma 1 - *Let G be as in Theorem 1. Let L be a K-algebra, $s \in L$, $u \in G(L_s)$. Then there exists an integer $r \geq 0$ such that if $a,b \in L$ and $a \equiv b \pmod{Ls^r}$, then $({}^bu)({}^au)^{-1} = v_s$ for some $v \in G_o(L)$.*

Proof: Let Y, T be indeterminates. We identify $G(L_s)$ with a subgroup of $G(L_s[Y,T])$. Put $w = w(Y,T) = ({}^{(Y+T)}u)({}^Yu)^{-1} \in$ $\in G(L_s[Y,T])$. Clearly ${}^ow = 1$. For, ${}^ow = ({}^{o(Y+T)}u) \cdot {}^o(({}^Yu)^{-1}) =$ $= {}^ou\ {}^o(({}^Yu)^{-1}) = {}^o({}^Yu({}^Yu)^{-1}) = {}^o1 = 1$. Also $w(Y,0) = 1$, so $w \in G(TL_s[Y,T])$. By axiom Q conclude that there exists $r \geq 0$ and $v(Y,T) \in G(TL[Y,T])$ such that $v(Y,t)_s = w(Y,s^rT)$. Since ${}^ow = 1$, we can replace v by $({}^ov)^{-1} \cdot v$ without affecting the above conditions and so we may further arrange that ${}^ov = 1$. Now suppose $a,b \in L$ and that $a \equiv b \pmod{Ls^r}$. Write $b = a + s^rt$, with $t \in L$. Then $v(a,t) \in G_o(L)$, and $v(a,t)_s =$

$= w(a,s^r t) = {}^{(a+s^r t)}u\,({}^a u)^{-1} = {}^b u({}^a u)^{-1} = {}^b u({}^a u)^{-1}$, as we wanted.

Proof of Theorem 1 (Quillen): Given $u \in G_o(L_{s_o s_1})$, we apply Lemma 1 to the localizations $L_{s_i} \to L_{s_o s_1} = (L_{s_i})_{s_{1-i}}$, $i = 0,1$. As a result, we can choose r large enough to work for both $i = 0$ and $i = 1$. For such an r, whenever $x,y \in L_{s_i}$ satisfy $x \equiv y \pmod{L_{s_i} \cdot s_{1-i}^r}$, then there exist $v_i \in G_o(L_{s_i})$ such that $(v_i)_{s_{1-i}} = ({}^y u)({}^x u)^{-1}$.

Now we are given that $Ls_o + Ls_1 = L$, say $1 = a+b$ with $a \in Ls_o^r$ and $b \in Ls_1^r$. We write $u = [{}^1 u({}^a u)^{-1}][{}^a u({}^o u)^{-1}]$, since ${}^1 u = u$ and ${}^o u = 1$. We see that if we take $y=1$, $x=a$ and $i = 0$ above, then $y-x = b \in Ls_1^r$; so $x \equiv y \pmod{L_{s_o} s_1^r}$ and there exists $v_o \in G_o(L_{s_o})$ such that $(v_o)_{s_1} = {}^1 u({}^a u)^{-1}$. Similarly by taking $y=a$, $x=0$ and $i=1$ we get $y-x = a \in Ls_o^r$ so that $x \equiv y \pmod{L_{s_1} s_o^r}$. In this case we get $v_1 \in G_o(L_{s_1})$ such that $(v_1)_{s_o} = {}^a u({}^o u)^{-1}$. Thus

$$u = (v_o)_{s_1} \cdot (v_1)_{s_o} \in G_o(L_{s_o})_{s_1} \cdot G_o(L_{s_1})_{s_o}.$$

C. Scalar Operations on Polynomial Extensions.

We begin with examples of scalar operations of interest to us. FROM NOW ON WE MAKE THE CONVENTION THAT THE SYMBOL $\cdot\otimes\cdot$ INDICATES $\otimes$ OVER THE BASE RING K.

(6.4) The functor $G'(A) = G(A[T])$.

Example 1 - Let G be any functor from commutative K-algebras to groups, and let T be an indeterminate. Define a new functor G' by $G'(L) = G(L[T])$.

If $u = u(T) \in G'(L)$ and if $s \in L$, we can define ${}^s u = u(sT)$. It is easily checked that this defines a scalar operation on G'. The map $u \mapsto {}^o u$ is the retraction $u(T) \mapsto u(0)$ from $G(L[T])$ onto $G(L) = {}^o G'(L)$ with kernel $G'_o(L) = G(TL[T])$.

We next verify that the functor G' satisfies axiom Q, if G does. Let there be given an element s of L and an element $u(Y)$ of $G'(YL_s[Y])$, Y being an indeterminate. Now $G'(YL_s[Y]) = G(YL_s[Y,T])$. But G satisfies axiom Q. So there is an $r \geq 0$ and a $v(Y) \in G(YL[Y,T])$ such that $u(s^r Y) = v(Y)_s$; that is, $v(Y) \in G'(YL[Y])$ such that $u(s^r Y) = v(Y)_s$, which shows that G' indeed satisfies axiom Q.

Theorem 1 - Let G be a functor from commutative K-algebras to groups, satisfying axiom Q. Let L be a commutative K-algebra, and $s_o, s_1 \in L$ such that $Ls_o + Ls_1 = L$. Then $G(TL_{s_o s_1}[T]) = G(TL_{s_o}[T])_{s_1} \cdot G(TL_{s_1}[T])_{s_o}$.

Proof: Let $G'(L) = G(L[T])$. We verified above that G' satisfies axiom Q and that it admits a scalar operation. Moreover $G'_o(L) = G(TL[T])$. The result thereupon follows from (6.3) Theorem 1.

Example 2 - We can generalize Example 1 as follows. Let $H = H_o \oplus H_1 \oplus \ldots$ be any commutative graded K-algebra with $H_o = K$. We put $\bar{H} = H_1 \oplus H_2 \oplus \ldots$ We have an exact sequence $0 \longrightarrow \bar{H} \longrightarrow H \underset{\epsilon}{\rightleftarrows} K \longrightarrow 0$, ϵ being the retraction. If G is any functor as before from commutative K-algebras to groups, we define a new functor G' by $G'(L) = G(L \otimes H)$. If $s \in L$, let $\epsilon_s: L\otimes H \to L\otimes H$ be the graded L-algebra endomorphism defined by $\epsilon_s(x) = s^n x$, if $x \in L\otimes H_n$. The following properties are easily verified:

(i) ϵ_1 = Identity

(ii) $\epsilon_s \cdot \epsilon_t = \epsilon_{st}$ for $s,t \in L$

(iii) If $f: L \to L'$ is a K-algebra homomorphism sending s to s', then the following diagram commutes:

$$\begin{array}{ccc} L\otimes H & \xrightarrow{f\otimes 1_H} & L'\otimes H \\ \Big\downarrow{\scriptstyle \epsilon_s} & & \Big\downarrow{\scriptstyle \epsilon'_s} \\ L\otimes H & \xrightarrow[f\otimes 1_H]{} & L'\otimes H \end{array}$$

Thus G' admits the scalar operation defined by ${}^s u = G(\epsilon_s)(u)$ for $s \in L$ and $u \in G'(L)$, since $G(\epsilon_s): G'(L) \to G'(L)$. The map $u \longmapsto {}^o u$ is the retraction $G(L\otimes H) \xrightarrow{G(1_L\otimes\epsilon)} G(L)$ with kernel $G'_o(L) = G(L\otimes\bar{H})$. If we put $H = L[T]$, we get Example 1.

We again verify that the functor G' satisfies axiom Q, if G does. If we are given an element s of L and an element $u(T)$ of $G'(TL_s[T])$, we note that $G'(L_s[T]) =$ $= G(L_s[T]\otimes H) = G((L[T]\otimes H)_{s\otimes 1}$ and that $G'(TL_s[T]) = G(T\, L_s[T]\otimes H)$ $= G(T(L[T]\otimes H)_{s\otimes 1})$. Since G satisfies axiom Q, there is an $r \geq 0$ and a $v(T) \in G(T(L[T]\otimes H))$ such that $v(T)_{s\otimes 1} =$ $= u((s^r\otimes 1)T)$. Hence G' satisfies axiom Q.

This leads to the generalization of Theorem 1.

Theorem 2 - Let G *be a functor from commutative* K-*algebras to* groups satisfying axiom Q. Let $H = H_o \oplus H_1 \oplus \cdots$ be a graded K-algebra with $H_o = K$, and let $\epsilon: H \to K$ be the retraction with kernel $\bar{H} = H_1 \oplus H_2 \oplus \cdots$. For any commutative K-algebra L, put $G(L\otimes\bar{H}) = \mathrm{Ker}(G(L\otimes H) \xrightarrow{G(1_L\otimes\epsilon)} G(L))$. If $s_o, s_1 \in L$ are such that $Ls_o + Ls_1 = L$, then $G(L_{s_o s_1}\otimes\bar{H}) = G(L_{s_o}\otimes\bar{H})_{s_1}\cdot G(L_{s_1}\otimes\bar{H})_{s_o}$.

Proof: Write $G'(L) = G(L\otimes H)$. We have just seen that G' satisfies axiom Q and that it admits a scalar operation. Also $G'_o(L) = G(L\otimes\bar{H})$. The result follows from (6.3) Theorem 1, if we use the commutativity of tensor product formation and localization.

D. Scalar Operations on Filtration preserving Homomorphisms of Graded Algebras.

(6.5) Definition.

We consider graded NOT NECESSARILY COMMUTATIVE K-algebras $A = A_o \oplus A_1 \oplus \cdots$. We equip A with the descending filtration defined by $A_{(n)} = A_n \oplus A_{n+1} \oplus \cdots$, $n = 0,1,2,\ldots$. Let $B = B_o \oplus B_1 \oplus \cdots$ be another such graded K-algebra, and $u: A \to B$ a K-algebra homomorphism preserving filtrations; that is, $u(A_{(n)}) \subseteq B_{(n)}$ for all $n = 0,1,\ldots$. As a K-linear map u can be decomposed into homogeneous components $u = u_o+u_1+u_2+\cdots$, where $u_p: A \to B$ is homogeneous of degree p for all p: $u_p(A_n) \subseteq B_{n+p}$ for all n, and where for a given $a \in A$, $u_p(a) = 0$ for all but finitely many p. The fact that u is a ring homomorphism is expressed by the conditions: $u_o(1) = 1$ and

(*) $$u_n(ab) = \sum_{p+q=n} u_p(a)u_q(b),$$

for $a,b \in A$ and all $n \geq 0$. Also, it suffices to know (*) for homogeneous elements of A.

Now, if $s \in K$, define ${}^{s}u\colon A \to B$ by ${}^{s}u = u_o + su_1 + s^2u_2 + \dots$; that is, $({}^{s}u)_n = s^n u_n$. We have ${}^{s}u(1) = 1$, and $({}^{s}u)_n(ab) = s^n \sum_{p+q=n} u_p(a)u_q(b) = \sum_{p+q=n} ({}^{s}u)_p(a)({}^{s}u)_q(b)$, for all n and $a,b \in A$. Hence ${}^{s}u$ is again a homomorphism of filtered algebras from A to B.

We want to speak of scalar operations admitted by a suitable functor G. For this we see that ${}^{1}u = u$ and that ${}^{s}({}^{t}u) = {}^{st}u$, for $s,t \in K$. Moreover ${}^{o}u = u_o$. Suppose $v\colon B \to C$ is a filtration preserving algebra homomorphism, then $(v \circ u)_n = \sum_{p+q=n} v_p u_q$, so $({}^{s}(v \circ u))_n = s^n (v \circ u)_n = \sum_{p+q=n} s^n v_p \circ u_q = \sum_{p+q=n} ({}^{s}v)_p \circ ({}^{s}u)_q = ({}^{s}v \circ {}^{s}u)_n$. Hence ${}^{s}(v \circ u) = ({}^{s}v) \circ ({}^{s}u)$. It follows from this that if u is an isomorphism and if u^{-1} is also filtration preserving, then the same is true of ${}^{s}u$.

Let L be a K-algebra. Then $L\otimes A$ and $L\otimes B$ are graded L-algebras. Thus the scalars $s \in L$ operate as above on the filtration preserving L-algebra homomorphisms $u\colon L\otimes A \to L\otimes B$. If $f\colon L \to L'$ is a K-algebra homomorphism, then one sees easily that for $s \in L$, the following diagram is commutative:

$$\begin{array}{ccc} L\otimes A & \xrightarrow{\quad {}^{s}u \quad} & L\otimes B \\ {\scriptstyle f\otimes 1_A}\downarrow & & \downarrow{\scriptstyle f\otimes 1_B} \\ L'\otimes A & \xrightarrow[\ {}^{f(s)}(1_{L'} \otimes_L u)\ ]{} & L'\otimes B \end{array}$$

Now for a fixed graded algebra A as above, let $G^A(L)$ = Filtered L-algebra autmorphisms of $L\otimes A$, for any commutative K-algebra L. Then G^A is a functor from commutative K-algebras to groups. The discussion above shows that the maps $u \to {}^s u$ ($s \in L$, $u \in G^A(L)$) define a scalar operation on the functor G^A. Notice that passing from u to ${}^o u = u_o = gr(u)$ is passing to the associated graded homomorphism induced by u, providing the retraction $G^A(L) \to {}^oG^A(L)$, where ${}^oG^A(L)$ is the group of automorphisms of the <u>graded</u> L-algebra $L \otimes A$. If we denote by $G_o^A(L)$ the kernel of this retraction, then $G_o^A(L) = \{u \in G^A(L): gr(u) = 1_{L\otimes A}\}$. In this way $G^A(L)$ is the semidirect product $G^A(L) = G_o^A(L) \rtimes {}^oG^A(L)$.

It is a fact that the functor G^A <u>satisfies axiom</u> Q, <u>when</u> A <u>is a finitely presented</u> K-<u>algebra</u>. We refer the reader to [BCW] for the proof.

(6.6) <u>Axiom Q for G^A</u>.

Recall that a <u>finitely presented</u> K-<u>algebra</u> A is a not necessarily commutative K-algebra of the form $\dfrac{K\{X_1,X_2,\dots,X_n\}}{(f_1(X),\dots,f_n(X))}$, where $R = K\{X_1,X_2,\dots,X_n\}$ is the free K-algebra on non-commuting indeterminates and $(f_1(X),\dots,f_n(X))$ is a finitely generated ideal of R generated by some finite set $f_1(X) = f_1(X_1,X_2,\dots,X_n)$, $f_2(X),\dots,f_n(X)$. Hence we can write $A = K[x_1,x_2,\dots,x_n]$, x_i denoting the class of X_i. Thus $x = (x_1,\dots,x_n) \in A^n$ is a sequence of elements generating A as K-algebra. If B is a K-algebra, and $u: A \to B$ a K-algebra homomorphism, then u is determined by $u(x) = (u(x_1),u(x_2),\dots,u(x_n)) \in B^n$. We thus obtain a bijection $u \mapsto u(x)$ from $\mathrm{Hom}_{K\text{-alg}}(A,B)$ to $H(A,B) = \{y \in B^n: f_j(y) = 0,\ j=1,2,\dots,n\}$.

Let S be a multiplicative set in K. Finite presentability of A implies by (6.2) the validity of

FP(A): ∀ K-algebras C,

$$\mathrm{Hom}_{K_S}(A_S, C_S) = \varinjlim_{s \in S} \mathrm{Hom}_{K_s}(A_s, C_s),$$

and if both A and B are finitely presented algebras, then

$$\mathrm{Isom}_{K_S}(A_S, B_S) = \varinjlim_{s \in S} \mathrm{Isom}_{K_s}(A_s, B_s).$$

Proposition 1 - *Let A be a possibly non-commutative finitely presented K-algebra. For any commutative K-algebra L, let $G^A(L) = \mathrm{Aut}_{L\text{-alg}}(L \otimes A)$, i.e., the group of L-algebra automorphisms of $L \otimes A$. Then the functor G^A attaching to each L the group $G^A(L)$ satisfies axiom Q.*

Proof: Clearly it is enough to treat the case when $L = K$. In this case $L \otimes A = A$. Let there be given an element s of K and an element $u(T)$ of $G^A(TA_s[T])$. This means $u(0)$ is the identity automorphism of A_s. Now $\mathrm{Hom}_{K_s[T]\text{-alg}}(A_s[T], A_s[T]) = \mathrm{Hom}_{K\text{-alg}}(A, A_s[T]) = H(A, A_s[T])$, by the remark and notation above. Hence we can identify $u(T)$ with the element $y(T) = u(T)(x)$ in $H(A, A_s[T]) \subseteq A_s[T]^n$, where $x = (x_1, \ldots, x_n)$ is a sequence of generators of A as a K-algebra; notice that x generates A_s as a K_s-algebra, and so the condition $u(0) = 1_{A_s}$ implies that modulo $TA_s[T]$ we get the identity map. Hence $y(T) = x + Ty_1(T)$, for some $y_1(T) \in A_s[T]^n$.

We can clear denominators by choosing r_1 large enough so that $s^{r_1} y_1(s^{r_1}T) = w_1(T)_s$ for some $w_1(T) \in A[T]^n$. If we put $w(T) = x + Tw_1(T)$, then $w(T)_s = y(s^{r_1}T)$. We have $f_j(w(T))_s = 0$, which means there is an $r_2 \geq 0$ such that

$s^{r_2} f_j(w(T)) = 0$. Expanding formally, $f_j(w(T)) = f_j(x+Tw_1(T)) = = f_j(x) + Tf'_j(w(T)) = 0 + Tf'_j(w(T))$. On substitution, this gives $s^{r_2} Tf'_j(w(T)) = 0$. Similarly we get $s^{r_2} Tf'_j(w(s^{r_2}T)) = 0$. Moreover we can choose one r_2 to work for all $j = 1,\ldots,n$. If we now replace $w(T)$ by $w(s^{r_2}T)$, we get $f_j(w(T)) = 0$, for all j; i.e., $w(T) \in H(A,A[T])$. With this replacement, we also have $w(T)_s = y(s^{r_1+r_2}T) = y(s^{r_3}T)$, where $r_3 = r_1+r_2$. Analogously, if we work with the inverse automorphism $u(T)^{-1}$, we get a $y'(T) = u(T)^{-1}(x) \in H(A,A_s[T])$, a $w'(T) = x+Tw'_1(T) \in \in H(A,A[T])$ and an r_4 such that $w'(T)_s) = y'(s^{r_4}T)$. Replacing T suitably, we can further arrange $r_3 = r_4$. The endomorphisms $w(T)$ and $w'(T)$ of $A[T]$ have composites corresponding to elements of $H(A,A[T])$. We shall denote these composites in $H(A,A[T])$ by $w(T)\circ w'(T)$ and $w'(T)\circ w(T)$.

Since $w(0) = w'(0) = x$, we can write $w(T)\circ w'(T) = = x + Tz(T)$ and $w'(T)\circ w(T) = x + Tz'(T)$. On localizing to $A_s[T]$, $w(T)_s$ and $w'(T)_s$ correspond to inverse automorphisms, which in turn correspond to x. Hence $(z(T)_s = (z'(T))_s = 0$ from which, we can get an $m \geq 0$ such that $s^m z(s^m T) = 0 = = s^m z'(s^m T)$. Hence $w(s^m T)\circ w'(s^m T) = x = w'(s^m T)\circ w(s^m T)$. This means that $w(s^m T)$ defines an automrphism $v(T)$ of $A[T]$. Clearly $v(0) = 1_A$ and $v(T)_s = w(s^m T)_s = y(s^{r_3+m}T) = y(s^r T) = = u(s^r T)$, with $r = r_3+m$. This proves Proposition 1.

(6.7) <u>Axiom</u> Q <u>for</u> GL_p.

We will devote the rest of the section to verification of Axiom Q for some important functors.

If E is any ring (not necessarily commutative), we denote by E^* its group of units. If J is a two-sided ideal, we put

$$(1+J)^* = \mathrm{Ker}(E^* \to (\tfrac{E}{J})^*).$$

<u>Proposition 1</u> (Quillen) - <u>Let</u> E <u>be a ring (not necessarily commutative),</u> s <u>an element in the center of</u> E, <u>and</u> T <u>an indeterminate. Given</u> $u(T) \in (1+TE_s[T])^*$, <u>there is an</u> $r \geq 0$ <u>and a</u> $v(T)$ <u>in</u> $(1+TE[T])^*$ <u>such that</u> $u(s^rT) = v(T)_s$.

<u>Proof</u>: Write $u(T) = 1+Tu_1(T)$ and $u(T)^{-1} = 1+Tu_1'(T)$. For r_1 sufficiently large, the elements $s^{r_1}\, u_1(s^{r_1}T) = w_1(T)_s$ and $s^{r_1}\, u_1'(s^{r_1}T) = w_1'(T)_s$ for some $w_1, w_1' \in E[T]$. Put $w(T) = 1 + Tw_1(T)$ and $w'(T) = 1 + Tw_1'(T)$; we then have $w(T)_s = u(s^{r_1}T)$ and $w'(T)_s = u(s^{r_1}T)^{-1}$. Thus $w(T)w'(T) = 1+TX(T)$ and $w'(T)w(T) = 1 + TX'(T)$, with $X(T)_s = 0 = X'(T)_s$. Hence there exists $r_2 \geq 0$ such that $s^{r_2}X(s^{r_2}T) = 0$ and $s^{r_2}X'(s^{r_2}T) = 0$. Put $v(T) = w(s^{r_2}T)$ and $v'(T) = w'(s^{r_2}T)$. Then $v(T)_s = w(s^{r_2}T)_s = u(s^{r_1+r_2}T)$ and $v(T)v'(T) = 1 + s^{r_2}TX(s^{r_2}T) = 1$ and $v'(T)v(T) = 1 + s^{r_2}T\, X'(s^{r_2}T) = 1$. Thus $v(T)$ is a unit and we are done.

<u>Corollary 1</u> - <u>Let</u> E <u>be a</u> K-<u>algebra (not necessarily commutative). Let</u> G <u>be the functor attaching to each commutative</u> K-<u>algebra</u> L <u>the group</u> $G(L) = (L \otimes_K E)^*$ <u>of units of</u> $L \otimes_K E$. <u>Then</u> G <u>satisfies axiom</u> Q.

<u>Proof</u>: The result is immediate from Proposition 1.

<u>Corollary 2</u> - <u>Let</u> P <u>be a finitely presented</u> K-<u>module and let</u> $GL_P(L) = \mathrm{Aut}_{L\text{-mod}}(L\otimes P)$, <u>for each commutative</u>

K-algebra L. Then the functor GL_P satisfies axiom Q.

Proof: Let $E = End_A(M)$. The natural homomorphism $L\otimes E \to$
$\to End_{L\otimes A}(L\otimes M)$ is an isomorphism, when L is flat over K, since P is finitely presented. Hence on flat K-algebras L, $GL_P(L)$ coincides with $(L\otimes E)^*$ of Corollary 1. This shows that axiom Q holds, when $L = K$. The general case follows by base change, replacing K, P by L, $L\otimes P$ respectively.

F. Localization Theorems for Finitely Presented Algebras.

(6.8) Localization for K[T]-algebras.

Theorem 1 - Let $H = H_o \oplus H_1 \oplus \dots$ be a commutative graded K-algebra with $H_o = K$. Put $\bar{H} = H_1 \oplus H_2 \oplus \dots$, and let A be a finitely presented H-algebra. Write ${}^oA = \frac{A}{\bar{H}A}$ and let $B = H \otimes_K {}^oA$. If $A_{\mathfrak{M}} \cong B_{\mathfrak{M}}$ as $H_{\mathfrak{M}}$-algebras for all $\mathfrak{M} \in Max(K)$, then $A \cong B$.

Proof: Apply Quillen Induction ((6.1) Proposition 1) to the proposition: $P(L)$: $L\otimes A \cong L\otimes B$ as $L\otimes H$ algebras, for $L \in Loc(K)$. Local validity is just the hypothesis. Specialization is obvious. The finiteness condition is guaranteeed by the finite presentability of A and B (see (6.2) and (6.6)). It remains to verify the sheaf conditon. Modulo notation, we may assume $L = K$. Suppose $s_o, s_1 \in K$ such that $Ks_o + Ks_1 = K$. We want to show that the validity of $P(K_{s_o})$ and $P(K_{s_1})$ imply the validity of $P(K)$.

We want to apply (6.4) Theorem 2. We take for G the functor $G^{{}^oA}$ of (6.6) Proposition 1: $G^{{}^oA}(L) = Aut_{L\text{-}alg}(L\otimes {}^oA)$, for each commutative K-algebra L. It is clear that ${}^{o}A$ is a finitely presented K-algebra, since ${}^{o}A \cong A \otimes_{H} K$. From (6.6) Proposition 1, the functor $G^{{}^{o}A}$ satisfies Axiom Q. As in (6.4) Example 2, we define $G'(L) = G^{{}^{o}A}(L\otimes H)$, for any commutative K-algebra L; then the functor G' also satisfies axiom Q and admits a natural scalar operation. Specifically, $G^{{}^{o}A}(L) =$ $= \mathrm{Aut}_{L\text{-alg}}(L\otimes {}^{o}A)$ and $G'(L) = \mathrm{Aut}_{L\otimes H\text{-alg}}(L \otimes H \otimes {}^{o}A) =$ $= \mathrm{Aut}_{L\otimes H\text{-alg}}(L\otimes B)$. In view of this, we write $G'(L) = G^{B}(L)$. If $u \in G^{B}(L)$, then ${}^{o}u$ is the canonical map obtained by passage modulo $\bar{H}$. Hence $G_{o}^{B}(L) = \{u \in G^{B}(L): {}^{o}u = 1\}$ is the kernel of reduction modulo $\bar{H}$.

We are given H_{s_i}-algebra isomorphisms $u_i: B_{s_i} \to A_{s_i}$, i=0,1. Passing modulo $\bar{H}_{s_i}$, we get

$$ {}^{o}u_i = K_{s_i} \otimes_{H_{s_i}} u_i : \frac{B_{s_i}}{\bar{H}_{s_i} B_{s_i}} \longrightarrow \frac{A_{s_i}}{\bar{H}_{s_i} A_{s_i}} . $$

These two last objects are ${}^{o}A_{s_i}$. Since $A_{s_i} \to {}^{o}A_{s_i}$ is a retraction of ${}^{o}A_{s_i} \to A_{s_i}$, we replace u_i by $u_i(H_{s_i} \otimes_{K_{s_i}} {}^{o}u_i)^{-1}$, and arrange ${}^{o}u_i = 1_{{}^{o}A_{s_i}}$, i = 0,1.

Now we try to make our gluing of isomorphisms. We have the following diagram:

$$\begin{array}{ccccc} B_{s_o} & \longrightarrow & B_{s_o s_1} & \longleftarrow & B_{s_1} \\ \downarrow u_o & & u_{os_1}\downarrow \;\; \downarrow u_{1s_o} & & \downarrow u_1 \\ A_{s_o} & \longrightarrow & A_{s_o s_1} & \longleftarrow & A_{s_1} \end{array}$$

$u = u_{os_1}^{-1} u_{1s_o} \in \mathrm{Aut}_{H_{s_os_1}\text{-alg}}(B_{s_os_1}) = G^B(K_{s_os_1}) = G^{{}^oA}(H_{s_os_1})$.

Modulo $\bar{H}$, we have $1_{K_{s_os_1}} \underset{H_{s_os_1}}{\otimes} u = 1_{{}^oA_{s_os_1}}$. Thus $u \in G_o^B(K_{s_os_1})$. we can write

$$u = u_{os_1}^{-1} u_{1s_o} = v_{os_1} v_{1s_o}^{-1}$$

for some $v_i \in G_o^B(K_{s_i})$, $i = 0,1$ ((6.4) Theorem 2).

Replacing u_i by $u_i' = u_i \circ v_i$, we get $u'^{-1}_{os_1} u'_{1s_o} = v_{os_1}^{-1} u_{os_1}^{-1} u_{1s_o} v_{1s_o} = 1$ from (*). By the affine patching argument in (6.2) we conclude that there exists a H-algebra homomorphism $w: B \to A$ such that $w_{s_i} = u_i'$, $i = 0,1$. Thus w is the sought for isomorphism implying the validity of $P(K)$.

We want to deduce Quillen's Localization Theorem as a Corollary to Theorem 1. For this we need a passage from algebras to modules:

<u>Lemma 1</u> - <u>Let</u> M <u>and</u> N <u>be</u> K-<u>modules</u>. <u>Suppose the symmetric algebras</u> S(M) <u>and</u> S(N) <u>are isomorphic as</u> K-<u>algebras</u>, <u>then</u> M <u>and</u> N <u>are isomorphic as</u> K-<u>modules</u>.

<u>Proof</u>: Let $s: S(M) \to S(N)$ be an isomorphism. For $x \in M$, write $s(x) = v(x) - t(x)$, with $t(x) \in K$ and $v(x)$ in the augmentation ideal $S_+(N)$. Let $\bar{t}$ be the automorphism of $S(M)$ defined by $\bar{t}(x) = x + t(x)$ for $x \in M$. Put $w = s \circ \bar{t}$. Then $w(x) = v(x)$ for $x \in M$. So w is an isomorphism of augmented K-algebras. We thus have K-module isomorphisms

$$M \cong \frac{S_+(M)}{(S_+(M))^2} \cong \frac{S_+(N)}{(S_+(N))^2} \cong N.$$

Corollary 1 - *Let* $H = H_o \oplus H_1 \oplus \ldots$ *be a commutative graded K-algebra with* $H_o = K$. *Put* $\bar{H} = H_1 \oplus H_2 \oplus \ldots$. *Let* P *be a finitely presented H-module,* ${}^oP = \frac{P}{\bar{H}P}$, *and* $Q = H \otimes {}^oP$. *Suppose* $P_{\mathfrak{M}} \cong Q_{\mathfrak{M}}$ *as* $H_{\mathfrak{M}}$*-modules for all* $\mathfrak{M} \in \mathrm{Max}(K)$, *then* $P \cong Q$ *as H-modules.*

Proof: In Theorem 1, we take $A = S_K(P)$, the symmetric algebra of P over K. Then A is a finitely presented H-algebra, since P is a finitely presented H-module. We have $B = H \otimes {}^oA = H \otimes \frac{A}{\bar{H}A}$. The module homomorphism $P \to \frac{P}{\bar{H}P} = {}^oP$ gives an algebra homomorphism $S_K(P) \to S_K({}^oP)$ whose kernel is generated by $\bar{H}P$; i.e., the ideal $S_K(\bar{H}P) = \bar{H} \otimes S_K(P)$. Hence $S_K({}^oP) = {}^oA$. Also $B = H \underset{K}{\otimes} {}^oA = H \underset{K}{\otimes} S_K({}^oP) = S_K(H \otimes {}^oP) = S_K(Q)$. Also for $\mathfrak{M} \in \mathrm{Max}(K)$, $A_{\mathfrak{M}} = K_{\mathfrak{M}} \otimes A = S_K(P_{\mathfrak{M}})$ and $B_{\mathfrak{M}} = S_K(Q_{\mathfrak{M}})$. Thus the hypothesis implies $A_{\mathfrak{M}} \cong B_{\mathfrak{M}}$ as $H_{\mathfrak{M}}$-algebras. From Theorem 1, we conclude that $A \cong B$ as H-algebras. By Lemma 1, $P \cong Q$ as H-modules.

Corollary 2 (Quillen's Localization Theorem) - *Let* $K[T]$ *be the polynomial ring in one variable* T *over a commutative ring* K, *and* P *a finitely presented* $K[T]$*-module. Put* $P_o = \frac{P}{TP}$ *and* $Q = P_o[T]$. *If* $P_{\mathfrak{M}}$ *and* $Q_{\mathfrak{M}}$ *are isomorphic* $K_{\mathfrak{M}}[T]$*-modules for all* $\mathfrak{M} \in \mathrm{Max}(K)$, *then* $P \cong Q$ *as K-modules.*

Proof: In Corollary 1, take $H = K[T]$ with the natural grading.

7. Symmetric and Invertible Algebras

A. The Automorphism Group of the Symmetric Algebra.

(7.1) $GA_P(K) = GA'_P(K) \cdot GL_P(K) \cdot \overline{P^*}$.

Let P be a K-module and let $B = S_K(P)$ be the symmetric algebra of P over K. We have a grading $B = B_o \oplus B_1 \oplus \ldots$, with $B_o = K$ and $B_1 = P$. An algebra homomorphism $u: B \to C$ where C is any commutative algebra is determined by its restriction to P, i.e., by $u|P: P \to C$. On the other hand any K-linear map $v: P \to C$ can be extended to an algebra homomorphism $u: S_K(P) \to C$. We write $\bar{B} = B_1 \oplus B_2 \oplus \ldots$ which is the augmentation ideal of B. We have a descending filtration on B given by $\bar{B}^n$.

Let $GA_P(K) = \mathrm{Aut}_{K\text{-}alg}\,(S_K(P))$ be the group of K-algebra automorphisms of $S_K(P)$. Three subgroups of this group are of interest to us:

(i) $GL_P(K) = \{u \in GA_P(K): u(P) \subseteq P\}$; these automorphisms are graded algebra automorphisms of B.

(ii) $GA_P^o(K) = \{u \in GA_P(K): u(P) \subseteq \bar{B}\}$; these are automorphisms preserving augmentation; they also preserve the descending filtration on B defined by $\bar{B}^n$.

Given $u \in GA_P^o(K)$ the associated graded map $gr(u) \in GL_P(K)$ is extended from the automorphism $P \xrightarrow{u} \bar{B} \to \bar{B}/\bar{B}^2 = P$ of P. Then $u \to gr(u)$ is a retraction of $GA_P^o(K)$ onto $GL_P(K)$. Denoting its kernel $GA'_P(K)$, we have the split exact sequence

$$1 \to GA'_p(K) \hookrightarrow GA^o_p(K) \overset{gr}{\rightleftarrows} GL_p(K) \to 1.$$

$u \in GA'_p(K)$ if and only if $u(x) = x+y$ with $y \in B_2 \oplus B_3 \oplus \ldots$ for all $x \in P$. We have a semidirect product $GA^o_p(K) = GA'_p(K) \rtimes GL_p(K)$. The map gr is sometimes called the Jacobian at 0.

(iii) Let $Af_p(K) = \{u \in GA_p(K) : u(P) \subseteq K \oplus P\}$; these are the automorphisms preserving the ascending filtration:
$B_o \subseteq (B_o \oplus B_1) \subseteq \ldots \subseteq (B_o \oplus B_1 \oplus \ldots \oplus B_n), \ldots$. The notation Af is suggestive of the affine group, where the maps consist of a K-linear map and a translation. Again grade defines a map

$$Af_p(K) \xrightarrow{gr} GL_p(K) \to 1$$

by $u \longmapsto gr(u)$. To find the corresponding kernel, we see that if $t\colon P \to K$ is a linear map, then the map $\bar{t}\colon P \to K \oplus P$ defined by $\bar{t}(x) = x+t(x)$ belongs to $\ker(Af_K(P) \to GL_p(K))$. Conversely, if u is in the kernel, then $u(x) = x+t_u(x)$, with $t_u(x) \in K$ for all $x \in P$. Hence $u = \bar{t}_u$. Thus if we write
$P^* = \{t \in GA_p(K) : t(P) \subseteq K$, then we have an exact sequence

$$1 \to P^* \to Af_p(K) \xrightarrow{gr} GL_p(K) \to 1.$$
$$t \longrightarrow \bar{t}$$

If we denote the image of P^* by $\overline{P^*}$, we have a semidirect product $Af_p(K) = \overline{P^*} \rtimes GL_p(K)$.

<u>Proposition 1</u> - $GA_p(K) = GA'_p(K) \cdot GL_p(K) \cdot \overline{P^*} = GA^o_p(K) \cdot \overline{P^*} =$
$= GA'_p(K) \cdot Af_p(K)$.

<u>Proof</u>: Let $u \in GA_p(K)$. Then for $x \in P$, we have $u(x) = t(x) +$
$+ v(x)$, with $t(x) \in K$ and $v(x) \in \bar{B}$. The map t

defined thus is in P^* and so we may consider $\bar{t} \in Af_p(K)$. Write $u_1 = u_o(\overline{-t})$. Then for $x \in P$, $u_1(x) = u((\overline{-t})(x)) = u(x-t(x)) = u(x) - t(x)$, since $t(x) \in K$. Thus $u_1(x) = v(x) \in \bar{B}$ i.e., $u_1 \in GA_p^o(K)$. Notice that the inverse of $(\overline{-t})$ is $\bar{t}$, since for $x \in P$, $(\overline{-t})\circ\bar{t}(x) = (\overline{-t})(x+t(x)) = (\overline{-t})(x) + t(x) = x-t(x) + t(x) = x$. Hence $(\overline{-t})\circ\bar{t} = 1$, and similarly $\bar{t}\circ(\overline{-t}) = 1$. From $u_1 = u\circ(\overline{-t})$, we get $u = u_1\circ\bar{t} \in GA_p^o(K)\cdot\bar{P}^*$. Hence $GA_p(K) = GA_p^o(K)\cdot\overline{P^*} = GA_p'(K)\cdot GL_p(K)\cdot\overline{P^*}$, from (ii) above. The third equality follows from (iii) above.

For any commutative algebra L, put $GA_p(L) = GA_{L\otimes P}(L) = \mathrm{Aut}_{L\text{-alg}}\, S_L(L\otimes P) = \mathrm{Aut}_{L\text{-alg}}\, L \otimes_K S_K(P)$.

We obtain a decomposition as in Proposition 1.

Proposition 2 - *Let L be a commutative K-algebra. Then*

$$GA_p(L) = GA_p'(L)\cdot GL_p(L)\cdot\overline{(L\otimes P)^*} = GA_p^o(L)\cdot\overline{(L\otimes P)^*} = GA_p'(L)\cdot Af_p(L).$$

Theorem 1 - *Suppose P is a finitely presented K-module, and L a commutative K-algebra. If $s_o, s_1 \in L$ such that $Ls_o + Ls_1 = L$, then*

$$GA_p(L_{s_os_1}) = GA_p'(L_{s_o})_{s_1}\cdot Af_p(L_{s_os_1})\cdot GA_p(L_{s_1})_{s_o}.$$

We first abbreviate our notation and write $GA(L) = GA_p(L)$, $G(L) = GA_p^o(L)$ and $G_o(L) = GA_p'(L)$. (See proof of Theorem 1 for the choice of the notation.) Similarly, we write $H(L) = Af_p(L)$, $H_o(L) = \overline{(L\otimes P)^*}$. With these notations, we have $GA(L) = G(L)\cdot H_o(L) = G_o(L)\cdot H(L)$. We shall first prove some lemmas.

<u>Lemma 1 - The functor</u> G <u>satisfies axiom</u> Q.

<u>Proof</u>: Given $s \in L$, $u \in G(TL_s[T])$, we want to show that there exists $r \geq 0$ and $v \in G(TL[T])$ such that $v(T)_s = u(s^r T)$. By (6.6) Proposition 1, the functor GA satisfies axiom Q, since $S_L(L \otimes P)$ is a finitely presented L-algebra. Since $G(L) \subseteq GA(L)$, we get an $r_1 \geq 0$ and $v_1(T) \in GA(TL[T])$ such that $v_1(T)_s = u(s^{r_1}T)$. By Proposition 2, we can write $v_1(T) = v_G(T) \cdot v_{H_o}(T)$, with $v_G(T) \in G(TL[T])$ and $v_{H_o}(T) \in H_o(TL[T])$. Localizing we get $v_1(T)_s = v_G(T)_s \cdot v_{H_o}(T)_s$. But $v_1(T)_s = u(s^{r_1}T) \in G(TL_s[T])$; hence $v_{H_o}(T)_s \in G(TL_s[T]) \cap H_o(TL_s[T])$. We conclude that $v_{H_o}(T)_s = 1$. Now $v_{H_o}(T) \in H_o(TL[T])$ and so $v_{H_o}(T)_s = 1$ implies that there exists $r_2 \geq 0$ such that $v_{H_o}(s^{r_2}T) = 1$. Take $r = r_1 + r_2$ and $v(T) = v_1(s^{r_2}T) = v_G(s^{r_2}T) \in G(TL[T])$. Now $v(T)_s = v_1(s^{r_2}T)_s = u(s^r T)$, and we are done.

Since $GA_P^o(L)$ is the group of filtration preserving automorphisms of B, we know from (6.5) that the functor G admits scalar operations. Hence if $u \in G(L)$ and $a \in L$ then ${}^a u \in G(L)$.

<u>Lemma 2 - Let</u> L <u>be a commutative</u> K-<u>algebra</u>, $u \in G_o(L_{s_o s_1})$ <u>and</u> $w \in H(L_{s_o s_1})$. <u>Then there exists</u> $r \geq 0$ <u>such that whenever</u> $a \in Ls_o^r$, <u>we have</u> $w^{-1}({}^a u)w = v_{1 s_o}$ <u>for some</u> $v_1 \in GA(L_{s_1})$.

<u>Proof</u>: Let $L' = L_{s_1}$ and $s' = s_o$. Hence $L'_{s'} = L_{s_1 s_o}$. We have $w^{-1}\,{}^o u\, w \in GA(L'_{s'})$ which means that $w^{-1}\,{}^T u\, w \in GA(TL'_{s'}[T])$. Axiom Q for GA implies that there exists $r \geq 0$ and $v(T) \in$

$\in GA(TL'[T])$ such that $v(T)_{s_o} = w^{-1}\,{}^{s_o^r T}u\, w$. Now if $a \in Ls_o^r$ put $a = s_o^r t$. Then $w^{-1}\,{}^{a}u\, w = w^{-1}\,{}^{s_o^r t}u\, w = v(t)_{s_o}$ with $v(t) \in GA(L') = GA(L_{s_1})$. If $v_1 = v(t)$, we are done.

Proof of Theorem 1: As remarked in the beginning, $G(L)$ is the group of filtration preserving automorphisms of the L-algebra $L\otimes S_K(P)$. Hence by (6.5), the functor G admits scalar operations. Moreover $G_o(L)$ is precisely $GA_P'(L)$. By Lemma 1, G satisfies Axiom Q. Hence the conditions of (6.3) Lemma 1 are satisfied. In that lemma, we take $L = L_{s_o}$, $s = s_1$. Accordingly, we get an integer $r_1 \geq 0$ satisfying the conclusion of that lemma.

We want to prove that $GA(L_{s_o s_1}) = G_o(L_{s_o})_{s_1}\cdot H(L_{s_o s_1})\cdot GA(L_{s_1})_{s_o}$. Let $g \in GA(L_{s_o s_1})$. By Proposition 2, we can write $g = uw$ with $u \in G_o(L_{s_o s_1})$ and $w \in H(L_{s_o s_1})$.

For this choice of u and w, we get an $r_2 \geq 0$ as in Lemma 2. Choose $r = \max(r_1, r_2)$. From $Ls_o + Ls_1 = 1$, we get $Ls_o^r + Ls_1^r = 1$. Hence there exists $x \in Ls_o^r$ and $y \in Ls_1^r$ such that $1 = x+y$.

We write $g = uw = [{}^{1}u({}^{x}u)^{-1}]\,[w][w^{-1}\,{}^{x}u\, w]$. In (6.3) Lemma 1, take $a=1$, $b=x$; we see that $1-x \in L_{s_o}s_1^r$ (after taking images in L_{s_o}); so ${}^{1}u({}^{x}u)^{-1} = v_{o s_1}$ for some $v_o \in G_o(L_{s_o})$. Now, if we take in Lemma 2, $a=x$, we see that $x \in Ls_o^r \subseteq Ls_o^{r_2}$, and so $w^{-1}\,{}^{a}u\, w = v_{1 s_o}$ for some $v_1 \in GA(L_{s_1})$. Also $w \in H(L_{s_o s_1})$. Thus $g = v_{o s_1}\cdot w\cdot v_{1 s_o} \in G_o(L_{s_o})_{s_1}\cdot H(L_{s_o s_1})\cdot GA(L_{s_1})_{s_o}$, which proves the result.

B. Locally Polynomial Algebras are Symmetric.

(7.2) The proof

We now come to one of the principal results of these lectures:

Theorem 1 - Let A be a finitely presented K-algebra. Suppose for every $\mathfrak{M} \in \mathrm{Max}(K)$, $A_{\mathfrak{M}} \cong S_{K_{\mathfrak{M}}}(M)$ for some finitely presented $K_{\mathfrak{M}}$-module M depending on $\mathfrak{M}$, then there exists a finitely presented K-module P, unique upto isomorphism, such that $A \cong S_K(P)$.

Proof: We apply Quillen Induction to the following proposition defined for every $L \in \mathrm{Loc}(K)$:

P(L): The L-algebra $L \otimes A$ is isomorphic to the symmetric algebra $S(M)$ of some finitely presented L-module M.

We must verify the four conditions of (6.1) Proposition 1. Local validity is just the hypothesis, and specialization is obvious. To check finiteness: If S is a multiplicative set in K, and if $A_S \cong S_{K_S}(M)$ for some finitely presented K_S-module M, then we will have an exact sequence $K_S^p \to K_S^q \to M \to 0$. If $K^p \to K^q \to N \to 0$, then $N_S \cong M$ and N is finitely presented. If now $B = S_K(N)$, then $B_S = K_S \otimes B = S_{K_S}(N_S) = A_S$. Since both A and B are finitely presented, we conclude by remarks at the beginning of (6.6) that there exists $s \in S$ such that $A_s \cong B_s$. Now $B_s = S_{K_s}(N_s)$.

We now proceed to verify the sheaf condition. After change of notation, we may assume $L = K$. We are given that

$Ks_o + Ks_1 = K$ with $s_o, s_1 \in K$, and we must prove that $P(K_{s_o})$ and $P(K_{s_1})$ together imply $P(K)$. Under these assumptions, we are given finitely presented K_{s_i}-modules M_i and K_{s_i}-algebra isomorphisms from A_{s_i} onto $S_{K_{s_i}}(M_i)$, $i=0,1$. From this we get that the $K_{s_os_1}$-algebras, $S_{K_{s_os_1}}(M_{os_1})$ and $S_{K_{s_os_1}}(M_{1s_o})$ are isomorphic. From (6.8) Lemma 1, we conclude that the $K_{s_os_1}$-modules M_{os_1} and M_{1s_o} are isomorphic. We now use affine patching to get a K-module P such that $P_{s_i} \cong M_i$, $i=0,1$:

$$\begin{array}{ccccc} P & & \dashrightarrow & & M_1 \\ \downarrow & & & & \downarrow \\ M_o & \dashrightarrow & M_{os_1} & \xrightarrow{\cong} & M_{1s_o} \end{array}$$

Write $B = S_K(P)$; P is finitely presented by (3.5) Proposition 1. Then there exist isomorphisms $u_i: B_{s_i} \to A_{s_i}$, $i = 0,1$. We form the usual diagram:

$$\begin{array}{ccccc} B_{s_o} & \longrightarrow & B_{s_os_1} & \longleftarrow & B_{s_1} \\ {\scriptstyle u_o}\downarrow & & {\scriptstyle u_{os_1}}\downarrow \;\downarrow{\scriptstyle u_{1s_o}} & & \downarrow{\scriptstyle u_1} \\ A_{s_o} & \longrightarrow & A_{s_os_1} & \longleftarrow & A_{s_1} \end{array}$$

Now $u = u_{os_1}^{-1} \circ u_{1s_o} \in \mathrm{Aut}_{K_{s_os_1}}(B_{s_os_1}) = GA_p(K_{s_os_1})$. From (7.1) Theorem 1, we can write $u = v_{os_1} w\, v_{1s_o}^{-1}$ with $v_o \in GA'_p(L_{s_o})$, $w \in Af_p(K_{s_os_1})$ and $v_1 \in GA_p(K_{s_1})$. If we put $u'_i = u_i \circ v_i$, then $u'_i \in GA_p(K_{s_i})$, $i=0,1$ and $(u'_{os_1})^{-1} u'_{1s_o} = v_{os_1}^{-1} u_{os_1}^{-1} u_{1s_o} v_{1s_o} =$ $= w \in Af_p(K_{s_os_1})$, which is the group of automorphisms of

The example above justifies the following definition.

Definition 1 - Let C be a commutative augmented K-algebra and $\bar{C}$ the augmentation ideal. We have an exact sequence $0 \to \bar{C} \subset C \xrightarrow{\epsilon_C} K \to 0$. A tensor decomposition of C is a pair (α,β) of endomorphisms of the augmented K-algebra C such that

$$(*) \qquad \alpha^2 = \alpha, \quad \beta^2 = \beta, \quad \alpha\beta = \epsilon_C = \beta\alpha$$

where we consider ϵ_C as an endomorphism of C, and such that the homomorphism $\Gamma: \alpha C \otimes \beta C \to C$ induced by the inclusions of αC and βC is an isomorphism. Thus, to say that a K-algebra A is a polynomial tensor factor is to say that $A \cong \alpha C$ for some tensor decomposition (α,β) of some $C = K^{[n]}$. We then call $(C;\alpha,\beta)$ a fulfillment of A.

We notice that $A = \alpha C$ and $B = \beta C$ are augmented algebras, and that $C = K \oplus \bar{C}$. Also $C = A \otimes B = (K \oplus \bar{A}) \otimes (K \oplus \bar{B}) = K \oplus \bar{A} \oplus \bar{B} \oplus (\bar{A} \otimes \bar{B})$. If $K \to K'$ is a base change, and $C' = K' \otimes_K C$, $\alpha' = K' \otimes \alpha$, $\beta' = K' \otimes \beta$, then (α',β') is a tensor decomposition of the augmented K'-algebra C' with $\alpha' C' \cong K' \otimes \alpha C$ and $\beta' C' \cong$ $\cong K' \otimes \beta C$.

Proposition 1 - *Let C be a finitely presented K-algebra, S a multiplicative set in K, and (α,β) a tensor decomposition of the K_S-algebra C_S. Then there exist $s \in S$ and a tensor decomposition (α',β') of C_s such that $\alpha'_S = \alpha$ and $\beta'_S = \beta$.*

Proof: By finite presentability of C, there exists $s_1 \in S$, and $\alpha_1, \beta_1 \in \mathrm{End}_{K_{s_1}\text{-alg}}(C_{s_1})$ such that $\alpha_{1S} = \alpha$ and $\beta_{1S} = \beta$. The desired equations $\alpha_1^2 = \alpha_1$, $\beta_1^2 = \beta_1$, $\alpha_1\beta_1 = \epsilon_{C_{s_1}} = \beta_1\alpha_1$ hold after localizing at S, hence also at s_2 for some $s_2 \in S$. Hence replacing s_1 by $s_1 s_2$ we can assume that the above equations hold at s_1. Consider $\Gamma: \alpha_1 C_{s_1} \otimes \beta_1 C_{s_1} \to C_{s_1}$ induced by the inclusions.

Localizing at S, Γ_S becomes an isomorphism. Hence there exists $t_1 \in S$ such that Γ_{t_1} is an isomorphism of $K_{s_1 t_1}$-algebras, since the algebras in question are finitely presented. If we now replace s_1 by $s_1 t_1$ we are done.

Corollary 1 - Let S be a multiplicative set in K, and let A be a polynomial factor over K_S of some polynomial K_S-algebra. Then there exists $s \in S$ and a polynomial factor B over K_s such that $B_S \cong A$.

Proof: If $A \underset{K_S}{\otimes} A'$ is a polynomial algebra $K_S^{[n]}$, we take $C = K^{[n]}$ in Proposition 1. The result is now immediate.

Theorem 1 - *Let A be a finitely presented K-algebra. If $A_{\mathfrak{M}}$ is a polynomial tensor factor over $K_{\mathfrak{M}}$ for every $\mathfrak{M} \in \mathrm{Max}(K)$, then A is a polynomial tensor factor over K.*

Proof: Once again we apply Quillen Induction to the following proposition:

$P(L)$: If $L \in \mathrm{Loc}(K)$, then $L \underset{K}{\otimes} A$ is a polynomial factor over L of some polynomial L-algebra.

We must once again verify the four conditions of (6.1) Proposition 1.

3) $J(S(P))$ is canonically isomorphic to P.

Proof: 1) and 3) are obvious. For 2), we notice that if

$A \cong K \oplus \bar{A}$ and $B \cong K \oplus \bar{B}$ as K-modules, then $A \otimes B \cong K \oplus \bar{A} \oplus \bar{B} \oplus$ $\oplus \bar{A} \otimes \bar{B}$. Hence the augmentation ideal $\overline{A \otimes B}$ is isomorphic to $\bar{A} \otimes K \oplus$ $\oplus K \otimes \bar{B} \oplus \bar{A} \otimes \bar{B}$. Thus $J(A \otimes B) \cong \dfrac{\bar{A} \otimes K \oplus K \otimes \bar{B} \oplus \bar{A} \otimes \bar{B}}{\bar{A}^2 \otimes K \oplus K \otimes \bar{B}^2 \oplus \bar{A} \otimes \bar{B}} \cong \dfrac{\bar{A} \otimes K \oplus K \otimes \bar{B}}{\bar{A}^2 \otimes K \oplus K \otimes \bar{B}^2}$ by the Second Isomorphism Theorem, whence $J(A \otimes B) \cong \dfrac{\bar{A}}{\bar{A}^2} \oplus \dfrac{\bar{B}}{\bar{B}^2} = J(A) \oplus J(B)$.

Recall that the K-modules M and N are stably isomorphic if there exists a positive integer n such that $M \oplus K^n \cong$ $\cong N \oplus K^n$. If A and B are commutative augmented K-algebras, we say that A and B are stably isomorphic if there exists a positive integer n such that $A \otimes K^{[n]}$ and $B \otimes K^{[n]}$ are isomorphic as augmented K-algebras. The following is a Corollary to (7.3) Theorem 1.

Corollary 1 - Let A and B be polynomial tensor factors over K. Suppose that JA and JB are stably isomorphic as K-modules, and that for all $\mathfrak{M} \in \mathrm{Max}(K)$, $A_{\mathfrak{M}}$ and $B_{\mathfrak{M}}$ are stably isomorphic as $K_{\mathfrak{M}}$-algebras, then A and B are stably isomorphic as K-algebras.

Proof: It is given that $JA \oplus K^{n_1} \cong JB \oplus K^{n_1}$ for some $n_1 \geq 0$.

Also for each $\mathfrak{M} \in \mathrm{Max}(K)$, there is a positive integer $n_{\mathfrak{M}}$ such that $A_{\mathfrak{M}}$ such that $A_{\mathfrak{M}} \otimes_{K_{\mathfrak{M}}} K_{\mathfrak{M}}^{[n_{\mathfrak{M}}]} \cong B_{\mathfrak{M}} \otimes_{K_{\mathfrak{M}}} K_{\mathfrak{M}}^{[n_{\mathfrak{M}}]}$. Since A and B are finitely presented K-algebras, the above isomorphisms lift to isomorphisms over K_s for some $s \notin \mathfrak{M}$, (s depending on $\mathfrak{M}$) giving $A_s \otimes_{K_s} K_s^{[n_{\mathfrak{M}}]} \cong B_s \otimes_{K_s} K_s^{[n_{\mathfrak{M}}]}$. We can take $n_s = n_{\mathfrak{M}}$ for all $\mathfrak{M}$ which do not contain s. Since $\mathrm{Max}(K)$ is compact, the integers

n_s are bounded. Choose an integer $n \geq n_1$ and all the $n_{\mathfrak{M}}$. If we write $A' = A \otimes K^{[n]}$ and $B' = B \otimes K^{[n]}$, then A' and B' are isomorphic over $K_{\mathfrak{M}}$ for all maximal ideals $\mathfrak{M}$ of K. Moreover $JA' \cong JB'$, since $n \geq n_1$. Choose a K-algebra B'' so that $B' \otimes B'' \cong K^{[m]}$ for some $m \geq 0$. Write $C = A' \otimes B''$. Then $C_{\mathfrak{M}} = $ $= A'_{\mathfrak{M}} \otimes_{K_{\mathfrak{M}}} B''_{\mathfrak{M}} \cong B'_{\mathfrak{M}} \otimes_{K_{\mathfrak{M}}} B''_{\mathfrak{M}} \cong K_{\mathfrak{M}}^{(m)}$. Hence C is locally a polynomial algebra. By (7.2) Corollary 1, $C \cong S(P)$ for some $P \in \mathcal{P}(K)$. We have $P \cong JS(P) \cong JA' \oplus JB'' \cong JB' \oplus JB'' \cong J(B' \otimes B'') \cong K^m$. Hence $C \cong S(K^m) \cong K^{[m]}$; that is, $A' \otimes B'' \otimes B' \cong B' \otimes K^{[m]}$. This gives $A' \otimes K^{[m]} \cong B' \otimes K^{[m]}$, from where $A \otimes K^{[n+m]} \cong B \otimes K^{[n+m]}$, proving the corollary.

We proceed now to formulate the above result in terms of the Grothendieck groups of K-theory. Recall that if $\mathcal{C}$ is a category equipped with a coherently associative and commutative product $\perp$, then the Grothendieck group of $\mathcal{C}$ is an abelian group $K_o(\mathcal{C})$ defined as follows:
Consider the free abelian group F generated by [A] where A runs over isomorphism classes of objects of $\mathcal{C}$ and the subgroup T of F generated by elements of the form $[A \perp B) - [A] - [B]$ for $A,B \in \mathcal{C}$. Then $K_o(\mathcal{C}) = \frac{F}{T}$. We observe that any map f from the objects of $\mathcal{C}$ into an abelian group G factors through ob $\mathcal{C} \xrightarrow{[\ ]} K_o(\mathcal{C})$, provided f satisfies the following:
1) If $A \cong B$ in $\mathcal{C}$ then $f(A) = f(B)$; 2) $f(A \perp B) = f(A)+f(B)$, for $A,B \in \mathcal{C}$.

If $\mathcal{C} = \mathcal{P}(R)$ is the category of finitely generated projective modules over a ring R, then the direct sum operation $\oplus$ defines a product in $\mathcal{C}$. In this case it is customary to denote $K_o(\mathfrak{a})$ by $K_o(R)$.

If $\mathfrak{a} = \mathfrak{a}(K)$ is the category of polynomial tensor factors over a commutative ring K, then $\otimes$ defines a product in $\mathfrak{a}$. The corresponding Grothendieck group will be denoted by $KA_o(K)$.

Proposition - 1) Let $P,Q \in \mathcal{P}(R)$. Then $[P] = [Q]$ in $K_o(R)$ if and only if P and Q are stably isomorphic in $\mathcal{P}(R)$.

2) Let $A,B \in \mathfrak{a}(K)$. Then $[A] = [B]$ in $KA_o(K)$ if and only if A and B are stably isomorphic in $\mathfrak{a}(K)$.

Proof: We shall only prove 2); the proof of 1) is similar.

Since $[A] = [B]$ in $KA_o(K) = \frac{F}{T}$ there exist suitable algebras C_i, D_i, C'_j, $D'_j \in \mathfrak{a}(K)$ such that $[A]-[B] =$
$= \Sigma([C_i \otimes D_i] - [C_i] - [D_i]) - \Sigma([C'_j \otimes D'_j] - [C'_j] - [D'_j])$.
Transposing all negative terms to the opposite side of the equation, and observing that F is free abelian, we get

$$A \otimes ((\underset{i}{\otimes} C_i) \otimes (\underset{i}{\otimes} D_i) \otimes (\underset{j}{\otimes} C'_j \otimes D'_j)) \cong$$
$$B \otimes ((\underset{i}{\otimes} C_i \otimes D_i) \otimes (\underset{j}{\otimes} C'_j) \otimes (\otimes D'_j)).$$

The algebras inside the long parentheses are clearly isomorphic. Hence, we may write $A \otimes E \cong B \otimes E$. Now if F is a K-algebra such that $E \otimes F \cong K^{[n]}$, then $A \otimes K^{[n]} \cong B \otimes K^{[n]}$. So A and B are stably isomorphic.

Since the functors $S: \mathcal{P}(K) \to \mathfrak{a}(K)$ and $J: \mathfrak{a}(K) \to \mathcal{P}(K)$ respect isomorphisms and products (Lemma 1), we have induced homomorphisms $S: K_o(K) \to KA_o(K)$ and $J: KA_o(K) \to K_o(K)$ such that the composition map $J \circ S = 1_{K_o(K)}$. Hence we have a splitting $KA_o(K) \cong K_o(K) \oplus KA'_o(K)$, say.

Confession. We do not know a single commutative ring $K \neq 0$ for which $KA'_o(K)$ is known to be zero or non-zero.

It seems very likely that these groups are non-trivial in general, though it may be conjectured that they vanish when K is a field.

With these notations, we can restate Corollary 1 as follows:

Corollary 2 - The canonical homomorphism

$$KA'_o(K) \to \prod_{\mathfrak{m} \in Max(K)} KA'_o(K_{\mathfrak{m}})$$

is injective.

(7.5) Some classical open problems.

Write $GA_n(K) = Aut_{K\text{-}alg}(K^{[n]})$. This is sometimes called the integral Cremona group. As in (7.1) we have a decomposition

$$GA_n(K) = GA_n^o(K) \cdot \bar{K}^n$$

where $\bar{K}^n$ denotes the group of translations, and

$$GA_n^o(K) = GA'_n(K) \rtimes GL_n(K).$$

If $K^{[n]} = K[t_1, \ldots, t_n]$ an endomorphism of $K^{[n]}$ can be identified with the image $f = (f_1, \ldots, f_n)$ of $t = (t_1, \ldots, t_n)$. This is the nonlinear analogue of the matrix representation of a linear map. We call f a (non linear) transvection if, for some i, $f_j = t_j$ for $j \neq i$ and $f_i = t_i + h$ where h depends only on $t_1, \ldots, \hat{t}_i, \ldots, t_n$. Such an f is clearly invertible. Let

$EA_n(K)$ denote the group generated by all such transvections.

<u>The tame generation problem.</u> Assume that K is a field. Is

$$GA_n^o(K) = \langle GL_n(K), EA_n(K) \rangle ?$$

This is currently known to be so only for $n = 2$. Letting $n \to \infty$ we have the <u>stable form of the problem:</u> Is

$$GA^o(K) = \langle GL(K), EA(K) \rangle ?$$

If $f = (f_1, \ldots, f_n)$ is an endomorphism of $K^{[n]}$ as above its <u>Jacobian</u> is

$$J(f) = \left(\frac{\partial f_i}{\partial t_j}\right)_{1 \le i,j \le n} \in M_n(K^{[n]}).$$

If f is invertible then so also is the matrix $J(f)(t)$ for all $t \in K^n$, so that $\det(J(f)) \in K^{[n]}$ is a polynomial with only invertible values in any K-algebra, hence it is a unit u of K. Replacing say f_1 by $u^{-1}f_1$ we can make $u = 1$. We can ask, conversely, whether the condition

(J1) $\det(J(f)) = 1$

implies that f is invertible. There are easy counterexamples in characteristic $p > 0$.

<u>Jacobian 1 Problem.</u> If K is a field of characteristic zero, does condition (J1) imply that f is invertible?

This is unknown even for $n = 2$. See Vitůshkin,..., Manifolds, Tokyo, for a good discussion of this problem.

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