LECTURES ON MINIMAL SUBMANIFOLDS

VOLUME 1

H. BLAINE LAWSON, JR.

MONOGRAFIAS DE MATEMÁTICA

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TO CAROLYN

PREFACE

These notes are the result of a course taught at IMPA in the fall of 1970. They have been designed to serve as an introduction for advanced students to the study of minimal submanifolds of riemannian spaces. The subject is enourmous and abounds with excellent literature, so there is no attempt here to be complete. I have tried rather to give an over-view of the area, that is, to present the major historical developments in a detailed and unified way. In each subarea an indication is given of the present state of affairs, with sufficient referen-ces that the reader may pursue in depth any topic of special interest to him.

The notes begin with a discussion of the general concept of minimal submanifolds (variational formulas, etc.) and the formulation of a number of examples which give focus and color to the subsequent study.

The remainder of this volume is concerned with two-dimensional minimal surfaces in euclidean space. This branch of the subject has enjoyed a long history in mathematics. Its close relationship with complex function theory has made it possible to prove profound results which have a certain resistance to generalization. The discussion here centers principally on the classical Plateau problem and on the global geometry of complete minimal surfaces, in particular, the Bernstein-Osserman Theorem. In the last chapter we give a detailed exposition of Calabi's theory of moduli for isometric minimal surfaces.

The second volume will contain an exposition of aspects of geometric measure theory with applications to the general Plateau problem and the codimension-one Bernstein conjecture. It will also contain a study of the global geometry of minimal varieties in compact manifolds, in particular, in symmetric spaces.

The background necessary for reading these notes is not excessive. The reader is presumed to have a working relationship with the basic concepts of fiemannian geometry and complex function theory. The special topics of central importance (e.g., certain aspects of Kähler geometry) are introduced in detail in the text.

The chapters in this volume, with the exception of the first, are mutually independent. Readers are neither required nor particularly encouraged to take up the topics in the order presented.

I would like to express may thanks to both Manfredo do Carmo and Robert Osserman for reading the manuscript and making a number of valuable comments. I also want to thank Manfredo for arranging my extremely pleasant visit to IMPA. I gratefully acknowledge the support of the Organization of American Sates and the Sloan Foundation during the preparation of this volume.

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CHAPTER I

A PRELIMINARY DISCUSSION

The purpose of this chapter is to develop some motivation for the study of minimal submanifolds and to give the reader some insight into the geometry involved. We shall begin with a proof that submanifolds which are stationary with respect to volume are characterized by a local geometric invariant, namely, the mean curvature vector. We shall then develop in detail a number of important examples which will illuminate all of our subsequent discussion.

§1. Notations and conventions

Let M be a C^∞ riemannian manifold where at each point $p \in M$ the metric in the tangent space $T_p(M)$ at p is denoted by the bracket $\langle \cdot, \cdot \rangle$. Let $\mathfrak{X}_M$ denote the space of C^∞ vector fields on M. Then by a connection on M we mean a rule which assigns to each $X \in \mathfrak{X}_M$ a linear map $\nabla_X : \mathfrak{X}_M \to \mathfrak{X}_M$ such that for all $X, Y, Z \in \mathfrak{X}_M$ and all $f, g \in C^\infty(M)$ we have

$$(1) \quad \nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$$

(2) $\nabla_X(fY) = (X \cdot f)Y + f\, \nabla_X Y$.

We recall that there exists on M a unique connection, called the riemannian connection, which satisfies the further conditions:

(3) $X \cdot \langle Y, Z \rangle = \langle \nabla_X Y,\ Z \rangle + \langle Y,\ \nabla_X Z \rangle$

(4) $\nabla_X Y - \nabla_Y X = [X, Y]$.

As a basic reference for connections defined as above we cite either Helgason [1] or Hicks [1]. We shall assume without proof the elementary properties established in these sources and note two of them here for future reference.

FACT 1 - Let $X, Y \in \mathfrak{X}_M$ and let $p \in M$. Then the value of the field $\nabla_X Y$ at p depends only on the vector $X_p \in T_p(M)$ and on the values of Y along any smooth curve γ such that $\gamma(0) = p$ and $\frac{d\gamma}{dt}(0) = X_p$. That is, if $\tilde{X} \in \mathfrak{X}_M$ satisfies $\tilde{X}_p = X_p$, then $(\nabla_{\tilde{X}} Y)_p = (\nabla_X Y)_p$; and if $\tilde{Y} \in \mathfrak{X}_M$ satisfies $\tilde{Y}|\gamma = Y|\gamma$, then $(\nabla_X \tilde{Y})_p = (\nabla_X Y)_p$. Hence, one may speak of $\nabla_X Y$ where Y is defined only along a submanifold and X is tangent to that submanifold.

FACT 2 - For each $X \in \mathfrak{X}_M$, the map ∇_X can be uniquely extended to the algebra of all tensor fields on

M as a derivation which commutes with contractions. For example, let ω be a 1-form on M, i.e., a $C^\infty(M)$-linear map $\mathfrak{X}_M \to C^\infty(M)$. Then $\nabla_X\omega$ is the 1-form given by

$$(\nabla_X\omega)(Y) = X\cdot\omega(Y) - \omega(\nabla_X Y)$$

for all $Y \in \mathfrak{X}_M$.

By a <u>submanifold</u> of M we shall mean simply an immersion $\psi: N \to M$ where N is some differentiable manifold. It will always be assumed that N carries the induced riemannian metric. By the implicit function theorem we know that any point $p \in N$ has a neighborhood $\mathfrak{U}$ such that $\psi|\mathfrak{U}$ is a topological embedding. Thus, when we are working on local questions we will often, for convenience, assume that our submanifold is embedded and suppress all mention of the immersion ψ.

§2. The first variational formula

Let $\bar{M}$ be a riemannian $\bar{m}$-manifold and $M \subset \bar{M}$ a topologically embedded submanifold of dimension m. Denote the metric on $\bar{M}$ by $\langle\cdot,\cdot\rangle$ and the associated riemannian connection by $\bar{\nabla}$. For any $p \in M \subset \bar{M}$ we have an orthogonal splitting

$$T_p(\bar{M}) = T_p(M) \oplus N_p(M)$$

into the tangent and normal spaces of M at p respectively. With respect to this splitting we decompose any vector $X \in T_p(\overline{M})$ as

$$X = (X)^T + (X)^N$$

The (unique) riemannian connection ∇ of M can then be given as follows. Denote by $\mathfrak{X}_p$ the set of tangent vector fields of M each of which is defined in some neighborhood of p on M. Then for $X, Y \in \mathfrak{X}_p$,

$$\nabla_X Y = (\overline{\nabla}_X Y)^T$$

(From Fact 1 above, the term on the right is a well defined element of $\mathfrak{X}_p$.) In an analogous manner we can define a local normal vector field at p by

$$B_{X,Y} = (\overline{\nabla}_X Y)^N$$

Recall that the covariant derivative $(\overline{\nabla}_X Y)_p$ depends only on X_p and not the choice of $X \in \mathfrak{X}_p$ extending it. However, we note that $(\overline{\nabla}_X Y)^N = (\overline{\nabla}_Y X + [X,Y])^N = (\overline{\nabla}_Y X)^N$, and thus

$$B_{X,Y} = B_{Y,X}$$

Therefore, $B_{X,Y}$ at p depends only on the vectors X_p, Y_p and not on the choice of local fields X,Y. This is to say that B represents a C^∞-section of the bundle $T^*(M) \otimes T^*(M) \otimes N(M)$ called the <u>second fundamental form</u> of the submanifold M. At each point p, B_p represents a symmetric bilinear map of $T_p(M)$ into $N_p(M)$. Thus, we can define

$$K_p = \text{trace}(B_p)$$

for each p. K is a smooth field of normal vectors on M called the mean curvature vector field. Locally, if $\mathcal{E}_1, \ldots, \mathcal{E}_n \in \mathfrak{X}_p$ are pointwise orthonormal fields, then

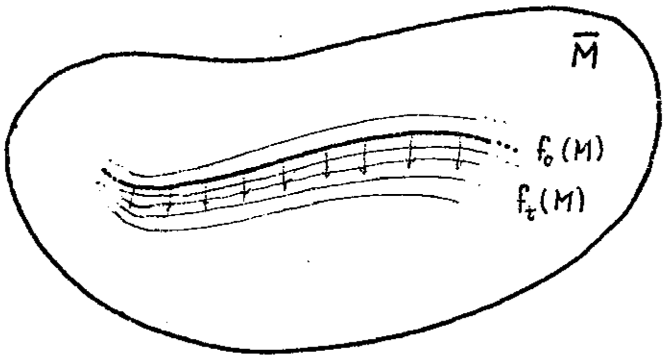
$$K_p = \sum_{k=1}^{m} (\bar{\nabla}_{\mathcal{E}_k} \mathcal{E}_k)^N$$

in a neighborhood of p.

NOTE - This definition can be immediately carried over to a general submanifold $\psi: M \to \bar{M}$ by using the fact that ψ is locally an embedding as mentioned above.

Observe that the field K is an invariant of the local geometry of the pair $(M, \bar{M})$, that is, of the way the submanifold M is situated locally in $\bar{M}$. If $F: \bar{M} \to \bar{M}$ is an isometry such that $F(M) = M$, then $F_*K = K$. This invariant is the simplest and most important one, and we shall now proceed to give an interpretation of it.

Roughly speaking, the mean curvature vector field is "the gradient of the area function on the space of immersions, $\mathcal{J}(M, \bar{M})$, of M into $\bar{M}$." For example, suppose M is compact. Then $\mathcal{J}(M, \bar{M})$ can be naturally given the structure of an (infinite-dimensional) C^∞-manifold where at $f \in \mathcal{J}(M, \bar{M})$ the tangent space corresponds to the set of $T(\bar{M})$-valued vector fields along f. More explicitly, a smooth curve f_t in $\mathcal{I}(M,\bar{M})$ with $f_0 = f$ corresponds to a smooth variation of f, and the tangent vector $\left.\frac{df}{dt}\right|_{t=o}$ naturally corresponds to the variational vector field.



The inner product in this tangent space is given by integrating the pointwise inner product of the variational vector fields in $\bar{M}$ over $f_0(M)$ (i.e., over M in the metric induced by f_0). Now, for $f \in \mathcal{I}(M,\bar{M})$ let $A(f)$ = the volume of M in the metric induced by f. Then for each $f \in \mathcal{I}(M,\bar{M})$ the claim is that $-(\nabla A)_f$ corresponds to the mean curvature vector field of the immersion. This means that by deforming f along the field K we decrease the area of M most rapidly. Moreover, if we deform f along another field E, then the infinitesimal rate of change of A at f is given by the inner product of E with $-K$ (e.g. $(E\cdot A)_f = -\langle E,K\rangle_f = \langle E,\nabla A\rangle_f$).

The precceding discussion was intended for reasons of motivation and will not be pursued in detail. Rather we proceed as follows. Let $\bar{M}$ be a riemannian manifold and let $f: M \to \bar{M}$ be an immersion where M is a compact oriented manifold with boundary ∂M (possibly $= \emptyset$).

DEFINITION - By a <u>smooth variation</u> of f we mean a C^∞-mapping $F: I \times M \to \bar{M}$, where $I = (-1,1)$, such that:

(a) Each map $f_t = F(t,\cdot): M \to \bar{M}$ is an immersion.

(b) $f_0 = f$.

(c) $f_t | \partial M = f | \partial M$ for all $t \in I$.

Let $\partial/\partial t$ denote the canonical vector field along the I factor in $I \times M$ and set $E = F_* \frac{\partial}{\partial t}\Big|_{t=0}$. E is considered as a section of $T(M) \oplus N(M)$. Finally, let $A(t)$ be the volume of M at time t, i.e., let dV_t be the volume element of the metric induced by f_t and set

$$A(t) = \int_M dV_t$$

Then we have

THEOREM 1 (The first variational formula):

$$\frac{dA}{dt}\Big|_{t=0} = - \int_M \langle K,E \rangle \, dV_0 \quad .$$

<u>Proof</u>: We first note that

$$\frac{dA}{dt} = \frac{d}{dt}\int_M dV_t = \int_M \frac{d}{dt}\, dV_t \ .$$

This follows immediately from the definition of the integral and standard theorems of calculus. We shall prove the theorem by demonstrating that

$$(1.1) \qquad \frac{d}{dt}\, dV_t\Big|_{t=0} = -\langle K,E\rangle\, dV_o = d\Omega$$

where Ω is an $(m-1)$-form on M such that $\Omega|\partial M = 0$. In fact, let ω be the 1-form on M given by

$$(1.2) \qquad \omega(X) = \langle E,X\rangle$$

for tangent vector fields X on M. Then

$$\Omega \overset{\text{def.}}{=} *\omega$$

and since $E|\partial M = 0$, $\Omega|\partial M = 0$.

Let $p \in M$ and choose $\varepsilon_1,\ldots,\varepsilon_m \in \mathfrak{X}_p$ such that:

(1) $\varepsilon_1,\ldots,\varepsilon_m$ are pointwise orthonormal in the metric induced by f_o.

(2) $(\nabla_{\varepsilon_i}\varepsilon_j)_p \equiv (\overline{\nabla}_{f_{o*}\varepsilon_i} f_{o*}\,\varepsilon_j)^T_{f(p)} = 0$ for all i,j.

(This last can be arranged by parallel translation of an orthonormal basis of $T_p(M)$ along the geodesic rays emanating from p.) Let $\omega_1,\ldots,\omega_m$ be the local 1-forms dual to $\varepsilon_1,\ldots,\varepsilon_n$. Then the metric induced by f_t can be written

$$ds_t^2 = \sum_{i,j=1}^{m} g_{ij}(t)\ \omega_i \otimes \omega_j$$

where

$$g_{ij}(t) = \langle f_{t*}\mathcal{E}_i, f_{t*}\mathcal{E}_j\rangle \ .$$

Thus,

$$dV_t = \sqrt{g(t)}\ \omega_1 \wedge \ldots \wedge \omega_m = \sqrt{g(t)}\,dV_o$$

where

$$g(t) = \det((g_{ij}(t)))$$

and so, at the point p

$$\frac{d}{dt}\, dV_t\Big|_{t=0} = \frac{d}{dt}\sqrt{g(t)}\Big|_{t=0} dV_o = \frac{1}{2}\frac{dg}{dt}(0)dV_o \ .$$

We now need a lemma from linear algebra.

LEMMA 1 - Let $A(t) = ((a_{ij}(t)))$; $t \in I$ be a smooth family of $m \times m$ matrices such that $A(0) =$ $=$ identity. Then

$$\frac{d}{dt}\det(A(t))\Big|_{t=0} = \text{trace}\ (A'(0)).$$

Proof: Each $A(t)$ can be considered as the matrix of a linear transformation $A(t): \mathbb{R}^m \to \mathbb{R}^m$ with respect to the canonical basis $e_1 = (1,0,\ldots,0),\ldots, e_m =$ $= (0,\ldots,0,1)$. Let W be an alternating m-form on $\mathbb{R}^m$ such that $W(e_1,\ldots,e_m) = 1$. Then $\det(A(t)) =$ $= W(A(t)e_1,\ldots,A(t)e_m)$, and

$$\frac{d}{dt}\det(A(t))\Big|_{t=0} = \sum_{k=1}^{m} W(A(t)e_1,\ldots,A'(t)e_k,\ldots,A(t)e_m)\Big|_{t=0}$$
$$= \sum_{k=1}^{m} W(e_1,\ldots,A'(0)e_k,\ldots,e_m)$$
$$= \sum_{k=1}^{m} A'_{kk}(0) = \text{trace}\,(A'(0)).$$

This proves the lemma.

Thus, at p we have

$$\frac{d}{dt}\,dV_t\Big|_{t=0} = \frac{1}{2}\sum_{k=1}^{m}\frac{dg_{kk}}{dt}(0)dV_o$$

We now extend $\mathcal{E}_1,\ldots,\mathcal{E}_m$ over $I\times$ (neighborhood of $p)\subset I\times M$ in the usual fashion and note that $[\frac{\partial}{\partial t},\mathcal{E}_k] = 0$ for $k = 1,\ldots,m$. Let $\tilde{E},\tilde{\mathcal{E}}_1,\ldots,\tilde{\mathcal{E}}_m$ denote the images of these vector fields along the mapping F. Then $g_{kk}(t) = \langle f_{t_*}\mathcal{E}_k, f_{t_*}\mathcal{E}_k\rangle = \langle\tilde{\mathcal{E}}_k,\tilde{\mathcal{E}}_k\rangle$ at $F(t,p)$, and

$$\frac{dg_{kk}}{dt}(t) = \tilde{E}\langle\tilde{\mathcal{E}}_k,\tilde{\mathcal{E}}_k\rangle = 2\langle\bar{\nabla}_{\tilde{E}}\tilde{\mathcal{E}}_k,\tilde{\mathcal{E}}_k\rangle = 2\langle\bar{\nabla}_{\tilde{\mathcal{E}}_k}\tilde{E},\tilde{\mathcal{E}}_k\rangle$$
$$= 2[\tilde{\mathcal{E}}_k\langle\tilde{E},\tilde{\mathcal{E}}_k\rangle - \langle\tilde{E},\bar{\nabla}_{\tilde{\mathcal{E}}_k}\tilde{\mathcal{E}}_k\rangle]\ .$$

Hence, at p we have (using (2) above) that

$$\sum_{k=1}^{m}\frac{1}{2}\frac{dg_{kk}}{dt}(0) = -\langle E,K\rangle + \sum_{k=1}^{m}\mathcal{E}_k\langle E,\mathcal{E}_k\rangle\ .$$

It remains only to show that the sum on the right is equal to $(d{*}\omega)_p(\mathcal{E}_1,\ldots,\mathcal{E}_m)$. However, by definition $\omega = \sum_k\langle E,\mathcal{E}_k\rangle\omega_k$, and therefore

$$*\omega = \sum_{k=1}^{m} (-1)^{k+1} \langle E,\varepsilon_k\rangle \, \omega_1 \wedge\ldots\wedge \hat{\omega}_k \wedge\ldots\wedge \hat{\omega}_m \, .$$

Recall that for any $(m-1)$-form Ω on M we have

$$d\Omega(\varepsilon_1,\ldots,\varepsilon_m) = \sum_{k=1}^{m} (-1)^{k+1} \varepsilon_k \Omega(\varepsilon_1,\ldots,\hat{\varepsilon}_k,\ldots,\varepsilon_m) + \\ + \sum_{i<j} (-1)^{i+j} \Omega([\varepsilon_i,\varepsilon_j],\varepsilon_1,\ldots,\hat{\varepsilon}_i,\ldots,\hat{\varepsilon}_j,\ldots,\varepsilon_m)$$

. Letting $\Omega = *\omega$ and using the fact that $[\varepsilon_i,\varepsilon_j]_p = (\nabla_{\varepsilon_i}\varepsilon_j)_p - (\nabla_{\varepsilon_j}\varepsilon_i)_p = 0$, one can easily complete the computation, and, thus, the proof of the theorem. □

REMARK 1 - If we restrict the variation above to be normal, that is, if we require E to be everywhere normal to the immersion, then $\omega \equiv 0$ and the formula remains valid <u>without the condition at the boundary</u>.

REMARK 2 - In the case that M is non-compact or non-orientable one can use the above formula by restricting attention to compactly supported variations and local area functions.

In terms of the above description, we have now shown that K does represent the gradient of the area function. Thus, an immersion $f: M \to \bar{M}$ is "critical for the area function" or "stationary with respect to area" if $K = 0$. By Theorem 1.1 this condition holds if and only if for all smooth variations f_t of f, the function

$A(t) = \text{area}(f_t)$ is constant to first order. The equations $K = 0$ are simply the Euler-Lagrange equations for the variation of area (cf. Courant-Hilbert [1], Ch. IV). They represent an elliptic, non-linear system of partial differential equations. Any immersion satisfying the condition $K = 0$ will be called a minimal immersion or minimal submanifold.

As a basis for subsequent discussion we now examine the equation for minimal submanifolds in certain important contexts and discuss a number of examples.

§3. Minimal submanifolds in euclidean space

Let M be a connected riemannian m-manifold. By the Laplace-Beltrami operator on M we mean a map $\Delta: C^\infty(M) \to C^\infty(M)$ defined in any of the following equivalent ways. Let $p \in M$ and $f \in C^\infty(M)$; then:

(a) If $\mathcal{E}_1,\dots,\mathcal{E}_m \in \mathfrak{X}_p$ are pointwise orthonormal, then

$$(1.3) \qquad \Delta f = \sum_{k=1}^{m} \{\mathcal{E}_k\mathcal{E}_k f - (\nabla_{\mathcal{E}_k}\mathcal{E}_k)f\}$$

in a neighborhood of p.

(b) If $(X^1,\dots,X^m)$ are local coordinates at p, then in the coordinate neighborhood

(1.4) $$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

where the metric $ds^2 = \Sigma g_{ij} dx^i dx^j$, the matrix $((g^{ij})) = ((g_{kl}))^{-1}$ and $g = \det((g_{ij}))$.

(c) $\Delta f = -*d*df$.

(We leave the equivalence of (a), (b) and (c) as an exercise.) The operator Δ is an invariant of the riemannian structure of M and generalizes the usual laplacian in euclidean space. A function $f \in C^\infty(M)$ such that $\Delta f = 0$ is called a harmonic function. It was proven by Hopf [1] that any harmonic function f on M satisfies a strict maximum (and corresponding minimum) principle, namely: *If f assumes a local maximum (or minimum) at a point $p \in M \sim \partial M$, then f = constant.*

Let $\mathbb{R}^n$ denote the space of n-tuples of real numbers, $(X_1,\ldots,X_n)$, with the riemannian metric $ds^2 = dX_1^2 + \ldots + dX_n^2$. ($\mathbb{R}^n$ = euclidean space with a distinguished coordinate system.) As in elementary calculus, we make, for any point $p \in \mathbb{R}^n$, the canonical identification $T_p(\mathbb{R}^n) = \mathbb{R}^n$ by translation. Using this identification, we have the following

PROPOSITION 1 - *Let $\psi: M \to \mathbb{R}^n$ be an isometric immersion and let K be the mean curvature vector* field of ψ. Then

$$\Delta\psi = K$$

where $\Delta\psi = (\Delta\psi_1,\ldots,\Delta\psi_n)$.

Proof: Let $p \in M$ and choose $\mathcal{E}_1,\ldots,\mathcal{E}_m \in \mathfrak{X}_p$ as in (a).

Then for each k, $\mathcal{E}_k\psi = \mathcal{E}_k$ (actually $= \psi_*\mathcal{E}_k$) and $\mathcal{E}_k\mathcal{E}_k\psi = \bar{\nabla}_{\mathcal{E}_k}\mathcal{E}_k$ where $\bar{\nabla}$ denotes the euclidean connection. Hence, $\Delta f = \sum_k \{\mathcal{E}_k\mathcal{E}_k\psi - (\nabla_{\mathcal{E}_k}\mathcal{E}_k)\psi\}$ =
$= \sum_k \{\bar{\nabla}_{\mathcal{E}_k}\mathcal{E}_k - \nabla_{\mathcal{E}_k}\mathcal{E}_k\} = \sum_k (\bar{\nabla}_{\mathcal{E}_k}\mathcal{E}_k)^N = K$. □

COROLLARY 1 - Let $\psi: M \to \mathbb{R}^n$ be an isometric immersion. Then ψ is a minimal immersion if and only if ψ is harmonic.

Thus, in the induced metric the minimal surface equations are simple, linear ones; namely $\Delta\psi = 0$. Note, however, that as the immersion changes, the induced metric and, thus, the operator Δ change.

We can now deduce a second important corollary from Proposition 1. For each pair of vectors $v,w \in \mathbb{R}^n$ we have defined a half-space $H_{v,w} = \{v+x \in \mathbb{R}^n : \langle x,w\rangle \leq 0\}$. Recall that for any set X in $\mathbb{R}^n$, the convex hull of X is the set

$$C(X) = \bigcap\{H_{v,w} : v,w \in \mathbb{R}^n \text{ and } X \subset H_{v,w}\} .$$

This is the smallest closed, convex set containing X. Assume now that $\psi: M \to R^n$ is an isometric minimal immersion, and for each $v, w \in \mathbb{R}^n$ consider the function $f_{v,w}: M \to \mathbb{R}$ given by

$$f_{v,w}(x) = \langle \psi(x) - v, w \rangle .$$

By Corollary 1 this function is harmonic in M. Thus applying the Hopf maximum principle to these functions we have

COROLLARY 2 - *Let* $\psi: M \to \mathbb{R}^n$ *be a minimal immersion where* M *is compact. Then setting* $M^o = M \sim \partial M$ *we have*

$$\psi(M) \subset C[\psi(\partial M)]$$

and if $\psi(M)$ *lies in no proper affine subspace*,

$$\psi(M^o) \subset C[\psi(\partial M)]^o .$$

In particular, if M *is compact without boundary, no minimal immersions of* M *into* $\mathbb{R}^n$ *exist*.

The second half of this corollary can also be proven by using the following identity.

LEMMA 2 - *Let* $f \in C^\infty(M)$. *Then*

$$\frac{1}{2}\Delta f^2 = f\Delta f + |\nabla f|^2$$

where locally $|\nabla f|^2 = \sum_{k=1}^{m} (\varepsilon_k f)^2$ *for orthonomal vector fields* $\varepsilon_1, \ldots, \varepsilon_m$.

Proof: Let $p \in M$ and choose orthonormal fields $\varepsilon_1, \ldots, \varepsilon_m$

so that $(\nabla_{\mathcal{E}_i}\mathcal{E}_j)_p = 0$ for all i,j. Then $\frac{1}{2}\Delta f^2(p) =$
$= \frac{1}{2}\sum_k \mathcal{E}_k\mathcal{E}_k f^2(p) = \Sigma\, \mathcal{E}_k(f\mathcal{E}_k f)(p) = \Sigma[f\mathcal{E}_k\mathcal{E}_k f(p)+(\mathcal{E}_k f)^2(p)]$
$= f\Delta f(p) + |\nabla f|^2(p)$. □

§4. Minimal submanifolds in the euclidean sphere

Let $\bar{M} \subset \mathbb{R}^n$ be an embedded submanifold, and for any $p \in \bar{M}$ and $X \in T_p(\mathbb{R}^n)$ let X^T denote the orthogonal projection of X onto $T_p(\bar{M})$. Suppose now that $\psi: M \to \bar{M} \subset \mathbb{R}^n$ is an immersion with mean curvature vector fields K in $\bar{M}$ and K^* in $\mathbb{R}^n$. Then

$$K = (K^*)^T = (\Delta\psi)^T . \tag{1.6}$$

This follows from Proposition 1 and the fact that $K =$ $= \Sigma(\bar{\nabla}_{\mathcal{E}_k}\mathcal{E}_k)^N = \Sigma((\nabla^*_{\mathcal{E}_k}\mathcal{E}_k)^T)^N = (\Sigma(\nabla^*_{\mathcal{E}_k}\mathcal{E}_k)^N)^T = (K^*)^T$ where $\bar{\nabla}$, ∇^* are the connections on $\bar{M}$ and $\mathbb{R}^n$ respectively. Setting $\bar{M} = S^n = \{x \in \mathbb{R}^{n+1}: |x| = 1\}$ we then have

PROPOSITION 2 - Let M be a riemannian m-manifold and let $\psi: M \to S^n \subset \mathbb{R}^{n+1}$ be an isometric immersion. Then ψ is a minimal immersion into S^n if and only if

$$\Delta\psi = -m\psi \tag{1.7}$$

Proof: By equation (1.6) above we see that ψ is minimal if and only if for all $p \in M$, $\Delta\psi(p)$ is parallel to the normal to S^n at $\psi(p)$, i.e., if and only if $\Delta\psi = \lambda\psi$ for some $\lambda \in C^\infty(M)$. However, from Lemma 1.2 and the condition that $|\psi|^2 = 1$ we see that if $\Delta\psi = \lambda\psi$, then $0 = \frac{1}{2}\Delta|\psi|^2 = \langle\psi,\Delta\psi\rangle + |\nabla\psi|^2 = \lambda|\psi|^2 + |\nabla\psi|^2 = \lambda + |\nabla\psi|^2$. Hence, $\lambda = -|\nabla\psi|^2 = -\sum_k \langle\mathcal{E}_k\psi,\mathcal{E}_k\psi\rangle = -\sum_k |\psi_*\mathcal{E}_k|^2 = -m$, and the proposition is proved. □

Thus we see that the minimal immersions of a differentiable manifold M into S^n are just those immersions whose coordinate functions in the ambient euclidean space are eigenfunctions of the Laplace-Beltrami operator in the induced metric with eigenvalue = -dim(M). Moreover, we have the following useful fact. For each $r > 0$, let

$$(1.9) \qquad S^n(r) = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : \sum_k x_k^2 = r^2\}\ .$$

PROPOSIITION 3 - (T. Takahashi [1]). <u>Let</u> M <u>be a riemannian m-manifold and</u> $\psi: M \to \mathbb{R}^{n+1}$ <u>an isometric immersion such that</u>

$$\Delta\psi = -\lambda\psi$$

<u>for some constant</u> $\lambda \neq 0$. <u>Then:</u>

(a) $\lambda > 0$.

(b) $\psi(M) \subset S^n(r)$ <u>where</u> $r^2 = \frac{m}{\lambda}$.

(c) <u>The immersion</u> $\psi: M \to S^n(r)$ <u>is minimal</u>.

<u>Proof</u>: From Proposition 2 we have that $\Delta\psi = -\lambda\psi = K$, and therefore at any point $p \in M$ the vector $\psi(p)$ is normal to the immersion. Hence, for any tangent vector field X on M we have $X\cdot\langle\psi,\psi\rangle = 2\langle X\cdot\psi,\psi\rangle = 2\langle\psi_*X,\psi\rangle (=2\langle X,\psi\rangle) = 0$, and it follows that $|\psi|^2 =$ constant $\overset{\text{def.}}{=} r^2$. Then, as above, we have $0 = \frac{1}{2}\Delta|\psi|^2 = = \langle\psi,\Delta\psi\rangle + |\nabla\psi|^2 = -\lambda r^2 + m$, and so $\lambda = m/r^2 > 0$. The minimality of ψ follows immediately from equation (1.6). □

This proposition is particularly useful in studying isometric minimal immersions of riemannian symmetric spaces into spheres since it shows that such immersions correspond precisely to the isometric immersions into $\mathbb{R}^n$ which can be achieved by eigenfunctions of the Laplace-Beltrami operator (with the same non-zero eigenvalue). For example we have

COROLLARY 3 - <u>Let</u> G/H <u>be a riemannian homogeneous space where</u> G <u>is a compact Lie group and where the isotropy representation of</u> H (<u>on the tangent space at the point</u> $e\cdot H \in G/H$) <u>is irreducible. Let</u> $E_\lambda = = \{\varphi\in C^\infty(G/H): \Delta\varphi = -\lambda\varphi\}$ <u>be a non-trivial eigenspace of the Laplace-Beltrami operator, and introduce on</u> E_λ <u>an</u>

inner product invariant under the natural action $\varphi \mapsto g_*\varphi = \varphi \circ g$ of G on E_λ. Choose an orthonormal basis $\varphi_1,\ldots,\varphi_N$ for E_λ in this inner product. Then, for an appropriate real number $\alpha \neq 0$ the mapping

$$\Psi = (\alpha\varphi_1,\ldots,\alpha\varphi_N)$$

is an isometric minimal immersion $\Psi: G/H \to S^{N-1}(r)$ for some $r > 0$.

Proof: Consider the map $\Phi = (\varphi_1,\ldots,\varphi_N)$ and note that the metric induced by Φ on G/H is G-invariant as follows. For each $g \in G$ there is an orthogonal $N\times N$ matrix Θ_g such that $g^*\Phi = \Theta_g\Phi$. The induced metric on G/H has the form $ds^2 = \sum_k d\varphi_k \circ d\varphi_k \overset{\text{def.}}{=} (d\Phi,d\Phi)$ where "$\circ$" denotes symmetric tensor product. Thus, for each $g \in G$, $g^*ds^2 = (g^*d\Phi,g^*d\Phi) = (dg^*\Phi,dg^*\Phi) = (d\Theta_g\Phi,d\Theta_g\Phi) = (d\Phi,d\Phi) = ds^2$. It then follows from the irreducibility of the action of H that for some $\alpha \in R$, $\alpha ds^2 =$ the given metric on G/H Kobayashi-Nomizu [1]. (All G-invariant metrics on G/H must be multiples of one another.) Applying Proposition 1.3 now completes the proof. □

This method produces many examples of isometric immersions of "nice" spaces into spheres. In particular, it shows that by using spherical harmonics it is possible to write down many isometric minimal immersions of euclidean spheres $S^n(r)$ into $S^N(1)$. For example the map of $S^2(\sqrt{3}) = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 3\}$ into $S^4(1)$ given by

$$\psi(x,y,z) = (xy, xz, yz, \tfrac{1}{2}(x^2-y^2), \tfrac{1}{2\sqrt{3}}(x^2+y^2-2z^2))$$

is such an immersion. It represents an isometric minimal embedding of the real projective plane with curvature $\frac{1}{3}$ into S^4, and is called the Veronese surface.

The very interesting question of whether the embeddings of $S^n(r) \to S^N(1)$ given in this manner are the only isometric minimal immersions has been studied in detail by M. do Carmo and N. Wallach [1].

§5. Totally geodesic submanifolds

By a totally geodesic (t.g.) submanifold of a riemannian manifold we mean a submanifold whose second fundamental form is everywhere zero. Clearly, any such submanifold is minimal. However, the converse is true only in dimension one. (One-dimensional minimal submanifolds are geodesics.) The totally geodesic submanifolds are characterized as follows. Let M be a submanifold of the riemannian manifold $\bar{M}$. Then M is totally geodesic if and only if every geodesic in the (induced) riemannian structure of M is also a geodesic in $\bar{M}$. This follows immediately from the fact that for any tangent vector fields X and Y on M,

$$B_{X,Y} = \bar{\nabla}_X Y - \nabla_X Y = 0 . \tag{1.8}$$

This equation also shows that if M is totally geodesic, then the riemannian curvature tensor of M agrees everywhere with the one induced from $\bar{M}$.

The totally geodesic submanifolds are clearly the most natural ones with respect to the riemannian structure. Unfortunately, in the general situation such submanifolds do not exist except in dimension 1. However, in the nicest spaces, symmetric spaces, there do exist many such submanifolds of higher dimension, and here they play a central role in the study of the minimal submanifolds of these spaces. In $\mathbb{R}^n$, the t.g. submanifolds are the affine subspaces, $\{x + c: x \in \mathbb{R}^k\}$, where $\mathbb{R}^k$ is a linear subspace of $\mathbb{R}^n$ and $c \in \mathbb{R}^n$. In S^n, the t.g. submanifolds are the "great spheres" $S^k = S^n \cap \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1}$ where $\mathbb{R}^{k+1}$ is a linear subspace. In complex projective n-space $\mathbb{C}P^n$ with the Fubini - Study metric the t.g. submanifolds are the "linear subspaces" $\mathbb{C}P^k$; $k = 1,\ldots,n$, and real projective spaces $\mathbb{R}P^k$; $k = 1,\ldots,n$. For a description of these see the following section.

It should be pointed out here that, in general, each component of the fixed-point set of an isometry of a riemannian manifold is a totally geodesic submanifold Kobayashi [1].

§6. Kählerian geometry and Wirtinger's Inequality

A second important class of minimal submanifolds is the class of complex submanifolds of Kählerian spaces. Since a number of examples, important for our subsequent discussion, can be constructed from this category, we shall treat certain aspects of Kähler geometry here in detail.

For completeness we shall begin with the notion of a complex manifold. Let $\mathbb{C}^m$ denote the complex vector space of m-tuples of complex numbers. Of course, $\mathbb{C}^m \cong \cong R^{2m}$ in a natural way. Thus, we define a complex m-manifold to be a real 2m-dimensional differentiable manifold M together with an atlas of local charts $\{\psi_\alpha\}_{\alpha\in A}$, where $\psi_\alpha\colon U_\alpha =$ (an open set in $M) \to \mathbb{C}^m$, such that the coordinate changes $\psi_\beta^{-1}\circ\psi_\alpha\colon \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ are all biholomorphisms. Note that at each point $Z \in \mathbb{C}^m$ we have a natural map of $T_Z(\mathbb{C}^m)$ into itself given by multiplication by $i(=\sqrt{-1})$. By definition, a map $\Phi\colon \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic if and only if at every point $d\Phi \circ i = i \circ d\Phi$

(i.e., if the differential is everywhere complex linear). Hence, on a complex manifold M the coordinate changes preserve this notion of multiplication by i, and thus, we have defined at each point $p \in M$ a linear map $J_p: T_p(M) \to T_p(M)$ such that

$$(1.10) \qquad J_p^2 = -\mathbb{1}$$

In local coordinates $(z^1, \ldots, z^m) = (x^1+iy^1, \ldots, x^m+iy^m)$, J is given by $J(\partial/\partial x^i) = \partial/\partial y^i$ and $J(\partial/\partial y^i) = -\partial/\partial x^i$. J is a globally defined, C^∞-tensor field of type (1,1) called the almost complex structure of M. One can easily verify in local coordinates that for any tangent vector fields X and Y on M, we have

$$(1.11) \qquad T_{X,Y} \overset{\text{def.}}{=} [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] = 0$$

NOTE. A deep theorem of Newlander and Nirenberg [1] (See also Hörmander [1]) states that any C^∞-manifold which admits a tensor field J of type (1,1) satisfying (1.10) and (1.11) has an atlas of charts making it a complex manifold with J its associated almost complex structure.

NOTE. Suppose V is a real vector space with an endomorphism J where $J^2 = -\mathbb{1}$. Then V admits a basis of the type $\{e_1, Je_1, \ldots, e_m, Je_m\}$, and any two such bases differ by a transformation of positive determinant. Thus, J determines an orientation of V. We see, therefore, that an complex manifold is not only orientable, but comes equipped with a canonical orientation.

EXAMPLES: (a) $\mathbb{C}^m$ itself.

(b) m-dimensional complex projective space, $\mathbb{C}P^m$. As a topological space $\mathbb{C}P^m$ is the quotient space $(\mathbb{C}^{m+1} \sim \{0\})/\cong$ where $\cong$ is the following equivalence relation: For $Z, Z' \in \mathbb{C}^{m+1} \sim \{0\}$ we say $Z \cong Z'$ if and only if there exists $\alpha \in \mathbb{C}$ such that $Z = \alpha Z'$. Thus, $\mathbb{C}P^m$ is the space of all "complex lines", i.e., all one-dimensional complex linear subspaces, of $\mathbb{C}^{m+1}$. We give a set of local complex coordinates for $\mathbb{C}P^m$ as follows. Let $\pi\colon (\mathbb{C}^{m+1} \sim \{0\}) \to \mathbb{C}P^m$ be the natural projection, and define for each $\alpha = 0,\ldots,m$ a map $\varphi_\alpha\colon \mathbb{C}^m \to \mathbb{C}^{m+1} \sim \{0\}$ by

$$\varphi_\alpha(z_\alpha^0,\ldots,z^{\alpha-1},z^{\alpha+1},\ldots,z_\alpha^m) =$$

$$= (z_\alpha^0,\ldots,z^{\alpha-1},1,z^{\alpha+1},\ldots,z_\alpha^m) .$$

Then $\pi \circ \varphi_\alpha$ is a homeomorphism of $\mathbb{C}^m$ with $U_\alpha = \pi \circ \varphi_\alpha(\mathbb{C}^m)$, and we define the coordinate chart $\psi_\alpha\colon U_\alpha \to \mathbb{C}^m$ by $(\pi \circ \varphi_\alpha)^{-1}$. It is easy to see that the coordinate changes are biholomorphic. In fact $\psi_\beta \circ \psi_\alpha^{-1}$ is just given by:

$$\begin{cases} z_\beta^k = \dfrac{z_\alpha^k}{z_\alpha^\beta}\,; & \text{if } k \neq \alpha,\beta \\ z_\beta^\alpha = \dfrac{1}{z_\alpha^\beta} \end{cases}$$

in the set where $z_\alpha^\beta \neq 0$.

NOTE. The coordinates $(Z^0,\ldots,Z^m)$ of $\mathbb{C}^{m+1}$ are called homogeneous coordinates for the projective space $\mathbb{C}P^m$. Every non-singular complex linear transformation of the homogeneous coordinates gives rise to a diffeomorphism of $\mathbb{C}P^m$ which is also called a linear transformation. Every (k+1)-dimensional subspace V of $\mathbb{C}^{m+1}$ projects to a submanifold $\mathbb{C}P^k = \pi(V \sim \{0\})$ called a k-dimensional linear submanifold of $\mathbb{C}P^m$. Let $R^{m+1} \subset C^{m+1}$ be the real subspace fixed under complex conjugation of homogeneous coordinates. If $V \subset R^{m+1}$ is a real (k+1)-dimensional subspace, then $\pi(V \sim \{0\})$ is a submanifold (diffeomorphic to real projective k-space) called a real linear submanifold.

We now want to consider riemannian metrics on a complex manifold M. Of course we want the riemannian structure to be "compatible" with the given complex structure, and so we shall require our riemannian metric g to satisfy the following two conditions.

(H) For all $p \in M$ and all $X, Y \in T_p(M)$

$$g(JX, JY) = g(X, Y).$$

In this case g is called a <u>hermitian metric</u>.

(K) For all tangent vector fields X and Y on M,

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y) = 0$$

where ∇ is the riemannian connection.

The first condition says that, at every point, J should be an isometry of the tangent space. (Natural enough!) The second condition states that the field J should be globally parallel in the riemannian connection. This condition, while also quite natural, is strong and, as we shall see, in the compact case it imposes severe resctrictions on the topology of the manifold. If a metric g on a complex manifold M satisfies conditions (H) and (K), it is called <u>Kählerian</u>, and the riemannian manifold (M, g) is called a <u>Kähler manifold</u>.

Suppose now that M is a complex manifold with a hermitian metric g. Then in the usual way we can expand the metric g at each point p to a complex-valued, "sesquilinear" form h by setting

$$h(X, Y) = g(X, Y) + i\omega(X, Y)$$

where

$$\omega(X,Y) = g(X,JY)$$

for all $X,Y \in T_p(M)$. Observe that since g is hermitian, $\omega(X,Y) = g(X,JY) = g(JY,X) = g(J^2Y,JX) = -\omega(Y,X)$, and thus ω is a globally defined exterior 2-form on M. As we shall see shortly, ω is of central importance in the study of Kähler manifolds and, as a consequence, is called the <u>fundamental 2-form</u> or <u>Kähler form</u> of M. Our first observation is the following.

LEMMA 4 - <u>Let</u> M <u>be a complex manifold with a hermitian metric</u> g. <u>Then</u> g <u>is Kählerian if and only if</u> $d\omega = 0$.

<u>Proof</u>: Consider a point $p \in M$. Choose any vectors $X_1,X_2,X_3 \in T_p(M)$ and extend them to local fields $\tilde{X}_1,\tilde{X}_2,\tilde{X}_3$ such that $(\nabla_{\tilde{X}_i}\tilde{X}_j)_p = 0$ for $i,j = 1,2,3$. Then $[\tilde{X}_i,\tilde{X}_j]_p = 0$ also for $i,j = 1,2,3$, and we have that $d\omega_p(X_1,X_2,X_3) = X_1\omega(\tilde{X}_2,\tilde{X}_3) - X_2\omega(\tilde{X}_1,\tilde{X}_3) + X_3\omega(\tilde{X}_1,\tilde{X}_2)$. Using the fact that $\omega(X,Y) = g(X,JY)$ we then obtain the following tensor equation.

$$d\omega(X_1,X_2,X_3) = g(X_2,(\nabla_{X_1}J)(X_3)) - g(X_1,(\nabla_{X_2}J)(X_3)) + g(X_1,(\nabla_{X_3}J)(X_2)).$$

From this it is clear that if g is Kählerian then $d\omega = 0$.

For the converse we proceed as follows. We note first that since $J^2 = -1$, $(\nabla_X J)J = -J(\nabla_X J)$ for any tangent vector X. Using this fact and the equation above, a straightforward computation shows that

$$g(X_2, (\nabla_{X_1} J)(X_3)) =$$
$$d\omega(X_1, X_2, X_3) - d\omega(X_1, JX_2, JX_3) - g(X_1, JT_{X_2, X_3})$$

where the integrability tensor $T_{X,Y} = 0$. (See equation (1.11).) The converse statement follows immediately. □

The first consequence of Lemma 1.3 is a topological one

COROLLARY 4 - Let M be a compact, Kähler manifold of complex dimension m. Then

$$H^{2k}(M;\mathbb{R}) \neq 0$$

for $k = 0, \ldots, m$.

Proof: Let ω be the fundamental 2-form of M. Then for each $k \geq 0$, $d\omega^k = k\,d\omega \wedge \omega \wedge \ldots \wedge \omega = 0$, and thus, ω^k represents a real cohomology class $[\omega^k]$ of dimension $2k$ in the sense of deRham. Moreover, it follows from deRham's theorem that

$$[\omega^k] = [\omega] \cup \ldots \cup [\omega] \quad (k \text{ - times}) \tag{1.12}$$

where "$\cup$" represents the cup-product in $H^*(M;\mathbb{R})$.

Observe now that since M is a complex manifold, it has a canonical orientation. In fact, an easy computation shows that $\frac{1}{m!}\omega^m$ = the volume form of M. Hence, letting $[M] \in H_{2m}(M;\mathbb{R})$ denote the fundamental cycle, we have

$$[\omega^m]([M]) = \int_M \omega^m = m!\cdot\text{Volume}(M) \neq 0 .$$

Thus, $\omega^m \neq 0$, and therefore from the ring isomorphism (1.12), $\omega^k \neq 0$ for $k = 1,\ldots,m$. □

We shall now inspect two important Kähler manifolds.

EXAMPLE 1.1 is $\mathbb{C}^m = \{(Z^1,\ldots,Z^m): Z^k \in \mathbb{C}\}$ with the metric $|dZ|^2 = \sum_k |dZ^k|^2 = \sum_k[(dX^k)^2+(dY^k)^2]$ where $Z^k = X^k + iY^k$. The Kähler form is then

$$\omega = \frac{i}{2}\sum_k dZ^k \wedge d\bar{Z}^k = \sum_k dX^k \wedge dY^k$$

EXAMPLE 1.2 is $\mathbb{C}P^m$ with the following metric. Let $(Z^0,\ldots,Z^m)$ be fixed homogeneous coordinates for $\mathbb{C}P^m$ and let $\pi\colon (\mathbb{C}^{m+1} \sim \{0\}) \to \mathbb{C}P^m$ be the canonical projection. In $\mathbb{C}^{m+1} \sim \{0\}$ we then define an "almost metric" by

(1.13) $$ds_o^2 = 4\frac{|Z \wedge dZ|^2}{|Z|^4}$$

where $|Z \wedge dZ|^2 = \sum_{i<j} |Z^i dZ^j - Z^j dZ^i|^2 = |Z|^2 |dZ|^2 - |(Z,dZ)|^2$, $(Z,W) = \sum_k Z^k \bar{W}^k$ and $|Z|^2 = (Z,Z)$. In other words $ds_0^2 = \sum_{i,j} g_{ij} dZ^i d\bar{Z}^j$ where

$$(1.14) \qquad g_{ij} = \frac{4(\delta_{ij}|Z|^2 - \bar{Z}^i Z^j)}{|Z|^4} = 4 \frac{\partial}{\partial Z^i} \frac{\partial}{d\bar{Z}^j} \log(|Z|^2)$$

From expression (1.13) it is clear that ds_0^2 is invariant under multiplication by non-zero complex scalars. Furthermore, from the same expression one sees that at a point $Z \in \mathbb{C}^{m+1} \sim \{0\}$ the restriction of ds_0^2 to the space normal to the complex line $\ell_Z = \{\alpha Z : \alpha \in \mathbb{C}\}$ is positive definite. In particular, if $0 \neq W \in T_Z(\mathbb{C}^{m+1})$ is normal to ℓ_Z, then $(Z, dZ(W)) = (Z,W) = 0$ and $ds_0^2(W,W) = (4/|Z|^2)|dZ|^2(W,W) > 0$. Hence, the form ds_0^2 projects to a Hermitian metric ds^2, called the Fubini-Study metric, on $\mathbb{C}P^m$ ($ds_0^2 = \pi^* ds^2$). To represent this metric in the coordinates (Z_α^k) given above, we simply pull ds_0^2 back by the map φ_α.

In the same manner we can express the Kähler form of the metric ds^2 as $\omega_0 = \pi^*\omega$ in $\mathbb{C}^{m+1} \sim \{0\}$ by the formula

$$\omega_0 = \frac{i}{2} \sum_{i,j} g_{ij} dZ^i \wedge d\bar{Z}^j$$

This equation can be derived by using the equation $\omega_0(X,Y) = ds_0^2(X,iY)$. We now consider the operators

(1.15)
$$\partial(\cdot) = \sum_{k=0}^{m} dZ^k \wedge \frac{\partial}{\partial Z^k}(\cdot)$$
$$\bar\partial(\cdot) = \sum_{k=0}^{m} d\bar Z^k \wedge \frac{\partial}{\partial \bar Z^k}(\cdot)$$

acting on complex valued, exterior differential forms in $\mathbb{C}^{m+1}$. It is straight-forward to verify that $\partial^2 = \bar\partial^2 = 0$, $d = \partial + \bar\partial$ and thus $\partial\bar\partial = -\bar\partial\partial$. In terms of these operators we can write

(1.16)
$$\omega_0 = 4\, \partial\bar\partial \log|Z|^2 .$$

It now follows immediately that $d\omega_0 = 0$ and, since $d \circ \pi^* = \pi^* \circ d$, that therefore $d\omega = 0$. Thus the metric ds^2 on $\mathbb{C}P^m$ is Kählerian.

NOTE. It also follows from (1.13) that any unitary transformation of the homogeneous coordinates $(Z^0,\ldots,Z^m)$ projects to an isometry of $\mathbb{C}P^m$. Moreover, so does the transformation $(Z^0,\ldots,Z^m) \to (\bar Z^0,\ldots,\bar Z^m)$.

NOTE. The real and complex linear subspaces of $\mathbb{C}P^m$ are totally geodesic in this metric. This follows easily from the fact that they are fixed point sets of isometries.

NOTE. With respect to Corollary 4 it is interesting to note that $H^{2k}(\mathbb{C}P^m;\mathbb{R}) = \mathbb{R}$ for $k = 0,\dots,m$ and all other groups are zero. In fact $H^*(\mathbb{C}P^m;\mathbb{R})$ is precisely the algebra over $\mathbb{R}$ generated by the element $[\omega]$.

Our purpose here is to study submanifolds, and so we need one more (natural) definition. Let $\bar{M}$ be a complex manifold. Then by a <u>complex submanifold</u> of $\bar{M}$ we mean an immersion $\psi: M \to \bar{M}$ of a complex manifold M satisfying either of the following equivalent conditions.

(a) The representations of ψ in local complex coordinates are holomorphic.

(b) For all $p \in M$

$$\bar{J} \circ d\psi_p = d\psi_p \circ J .$$

Locally, an embedded submanifold $M \subset \bar{M}$ has the structure of a complex submanifold if and only if for each $p \in M$, the tangent space $T_p(M) \subset T_p(\bar{M})$ is J-invariant.

From the above examples and the following lemma we will be able to immediately write down a great number of Kähler manifolds.

LEMMA 5 - <u>Every complex submanifold of a Kähler manifold is Kählerian in the induced metric.</u>

<u>Proof</u>: Since we may work locally, we can consider for convenience a small, embedded complex submanifold

$M \subset \bar{M}$. Note that the complex structure of M is the one inherited from $\bar{M}$, say J. Then, if X and Y are tangent vector fields on M, we have $\nabla_X(JY) = (\bar{\nabla}_X JY)^T = (J\bar{\nabla}_X Y)^T = J(\bar{\nabla}_X Y)^T = J\nabla_X Y$. Thus, M is Kählerian. □

Our first reason for considering Kähler manifolds is the following.

LEMMA 6 - <u>Every complex submanifold of a Kähler manifold is minimal.</u>

<u>Proof</u>: Consider $M \subset \bar{M}$ as in Lemma 5. Then for tangent vector fields X and Y on M we have $(\bar{\nabla}_X JY)^N = (J\bar{\nabla}_X Y)^N = J(\bar{\nabla}_X Y)^N$ since the normal spaces are J-invariant. Hence,

$$B_{X,JY} = JB_{X,Y} = B_{JX,Y},$$

that is, B is complex-linear. Recall that for any X we have $\langle X,X\rangle = \langle JX,JX\rangle$. Furthermore, for any X, JX is perpendicular to X since $\langle X,JX\rangle = \omega(X,X) = 0$. Thus, at any $p \in M$ it is possible to choose inductively a set of local, pointwise orthonormal vector fields of the form $\varepsilon_1, J\varepsilon_1, \varepsilon_2, J\varepsilon_2, \ldots, \varepsilon_m, J\varepsilon_m$, where $m = \dim_{\mathbb{C}}(M)$. Thus, at p we have $K = \sum_k (B_{\varepsilon_k,\varepsilon_k} + B_{J\varepsilon_k,J\varepsilon_k}) = \sum_k (B_{\varepsilon_k,\varepsilon_k} + J^2 B_{\varepsilon_k,\varepsilon_k}) = 0$. □

Note that the minimality here is very strong in the sense that for any normal vector ν, the eigenvalues of the symmetric bilinear form $\langle B_{X,Y}, \nu\rangle_p$ appear in pairs: $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_m, -\lambda_m$. This is equivalent to the fact that all the odd symmetric functions of the eigenvalues, not just the trace function, vanish.

Our second reason for studying Kähler manifolds follows from this elementary, but quite remarkable fact.

PROPOSITION 4 - (Wirtinger's Inequality). _Let_ $\bar{M}$ _be a Kähler manifold and_ $M \subset \bar{M}$ _any 2m-dimensional, oriented, real submanifold. At any point_ $p \in M$ _let_ dV_p _denote the volume form of the induced metric on_ M. _Then the restriction of the_ m^{th} _power_ $\omega^m = \omega \wedge \ldots \wedge \omega$ _of the Kähler form of_ $\bar{M}$ _to_ $T_p(M)$ _satisfies_

$$\frac{\omega^m}{m!} \leqq dV_p \,, \tag{1.17}$$

and equality holds if and only if $T_p(M)$ _is a complex subspace of_ $T_p(M)$, _with the canonical orientation._

Proof: We first observe that for any two unit vectors $X, Y \in T_p(\bar{M})$

$$\omega(X,Y)^2 = \langle X, JY\rangle^2 \leqq |X|^2|JY|^2 = |X|^2|Y|^2 = 1,$$

and equality holds if an only if $X = \pm JY$, i.e., if and only if X and Y span a one-dimensional complex subspace of $T_p(\bar{M})$. We now consider the form ω restricted to $T_p(M)$. By elementary linear algebra there exists an oriented, orthonormal basis $\varepsilon_1,\ldots,\varepsilon_{2m}$ of $T_p(M)$ in which ω is represented by a matrix of the form:

$$\begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & & \bigcirc \\ & \begin{matrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{matrix} & & \\ & & \ddots & \\ \bigcirc & & & \begin{matrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{matrix} \end{pmatrix}$$

where $\lambda_k = \omega(\varepsilon_{2k-1},\varepsilon_{2k})$ for $k = 1,\ldots,m$. Thus, letting $\omega_1,\ldots,\omega_{2m}$ be the 1-forms dual to $\varepsilon_1,\ldots,\varepsilon_{2m}$, we have

$$\omega = \sum_{k=1}^{m} \lambda_k\, \omega_{2k-1} \wedge \omega_{2k} ,$$

and so,

$$\omega^m = (m!)\lambda_1\cdots\lambda_m\, \omega_1 \wedge\cdots\wedge \omega_{2m}$$

$$= (m!)\lambda_1\cdots\lambda_m\, dV_0 .$$

Hence $|\omega^m| \leqq (m!)dV_0$, and equality holds $\Leftrightarrow |\lambda_1\cdots\lambda_m| = 1 \Leftrightarrow \omega(\varepsilon_{2k-1},\varepsilon_{2k})^2 = 1$ for $k = 1,\ldots,m \Leftrightarrow \varepsilon_{2k-1} = \pm J\varepsilon_{2k}$ for $k = 1,\ldots,m \Leftrightarrow T_p(M)$ is J-invariant. Equality holds without the absolute value sign exactly when the orientation agrees with the one induced by J. $\square$

COROLLARY 5 - Let $\bar{M}$ be any Kähler manifold and let $\psi: M \to \bar{M}$ be a complex submanifold where M is compact with boundary (possibly empty) and where $\dim_{\mathbb{R}} M = 2m$. Then the volume of M in the induced metric is less than or equal to the volume of any other 2m-dimensional submanifold which is homologous to M in $\bar{M}$.

Proof: Let $\psi': M' \to \bar{M}$ be any 2m-dimensional submanifold homologous to $\psi: M \to \bar{M}$. Then since $d\omega^m = 0$, $\int_M \psi^*\omega^m = \int_{M'} \psi'^*\omega^m$. Hence, by Proposition 1.4 we have

$$\text{Volume}(M) = \int_M dV = \frac{1}{m!}\int_M \psi^*\omega^m = \frac{1}{m!}\int_{M'} \psi'^*\omega^m$$

$$\leq \int_{M'} dV' = \text{Volume}(M'). \quad \square$$

Note that the submanifold M' will have area equal to M if and only if $\psi': M' \to \bar{M}$ is also a complex submanifold.

Thus we see that every compact, complex submanifold of a Kähler manifold is automatically the solution to a Plateau problem. Namely, such a submanifold $\psi: M \to \bar{M}$ has least area among all manifolds lying in the same homology class of $H_{2m}(\bar{M}, \psi(\partial M); Z)$. If, for example, $\bar{M} = \mathbb{C}^n$ $(=\mathbb{R}^{2n})$, this means that ψ minimizes area among all submanifolds in $\mathbb{C}^n$ having the same boundary.

Incidentally, we note that if $\partial M = 0$, then the submanifold ψ represents a non-trivial homology class of $\bar{M}$. This fact is sufficiently interesting that we shall write it down as a corollary.

COROLLARY 6 - Let $\bar{M}$ be any Kähler manifold and suppose $\psi: M \to \bar{M}$ is a compact complex submanifold of $\bar{M}$ with $\partial M = \emptyset$ and $\dim_{\mathbb{R}}(M) = 2m$. Then if $[M]$ represents the fundamental cycle of M and ψ_* is the natural map on homology, we have that $\psi_*[M] \neq 0$ in $H_{2m}(\bar{M};\mathbb{Z})$. Furthermore, $H^{2k}(\bar{M};\mathbb{R}) \neq 0$ for $k = 0,\ldots,m$.

Proof: Let ω be the Kähler form of $\bar{M}$. Then

$$[\omega^m](\psi_*[M]) = \int_M \psi^*\omega^m = (m!)\mathrm{Volume}(M) \neq 0. \quad \text{Thus,}$$

$\psi_*([M]) \neq 0$, and $[\omega^m] = [\omega] \cup \ldots \cup [\omega] \neq 0$. □

Before proceeding to examples it should be pointed out that both Corollaries 5 and 6 can be considerably strengthened. It is, first of all, possible to replace "compact, complex submanifold" with the notion of "complex analytic subvariety," i.e., a compact subset of M which is described in local complex coordinates as the zeroes of holomorphic functions. Such a variety is a submanifold with singularities of codimension 2. (The singularities do not contribute to the homological boundary nor do they enter in the computation of the volume.)

Secondly, it is possible to show that complex subvarieties minimize in a class of objects much more general than regular submanifolds, namely the class of <u>integral currents</u>. (See Chapter 3). This theorem is due to Herbert Federer [1] who is also responsible for the particular proof given here of Wirtinger's inequality.

§7. Some important examples

As a result of our remarks on Kähler manifolds we can now give a number of examples of minimal submanifolds.

CASE 1. Consider the Kähler manifold $\mathbb{C}^n = \mathbb{R}^{2n}$ with the euclidean metric. By the above, any complex submanifold of $\mathbb{C}^n$ is minimal in the underlying euclidean space, and furthermore, if the complex submanifold is compact with boundary, it represents an absolute minimum in area. A great number of such manifolds can be described by choosing complex polynomials $p_1(z_1,\dots,z_n)$, $\dots,p_{n-m}(z_1,\dots,z_n)$ and setting

$$(1.18) \qquad V^m = \{z \in \mathbb{C}^n: p_1(z) = \dots = p_{n-m}(z) = 0\}.$$

By the Complex Analytic Implicit Function Theorem (cf. Gunning Rossi, [1]), this complex subvariety V will be a complex submanifold if for all $Z \in V$

$$\operatorname{rank}_{\mathbb{C}}\left(\left(\frac{\partial p_i}{\partial Z_j}(Z)\right)\right) = n-m .$$

Of course one can also give submanifolds directly by choosing polynomials $p_1(Z_1,\ldots,Z_m),\ldots,p_n(Z_1,\ldots,Z_m)$ where $m < n$ and defining $\psi\colon \mathbb{C}^m \to \mathbb{C}^n$ by $\psi(Z) = (p_1(Z),\ldots,p_n(Z))$. Then ψ is an immersion if for all $Z \in \mathbb{C}^m$, $\operatorname{rank}_{\mathbb{C}}\left(\left(\frac{\partial p_i}{\partial Z_j}\right)\right) = m$.

As a special case, let $\mathcal{R}$ be a Riemann surface and suppose $\varphi_1,\ldots,\varphi_n$ are holomorphic functions on $\mathcal{R}$ such that $\sum_{k=1}^{n} |\varphi_k'(z)| \neq 0$ for all $z \in \mathcal{R}$. Then the map $\varphi = (\varphi_1,\ldots,\varphi_n)\colon \mathcal{R} \to \mathbb{C}^n$ is a minimal immersion into $\mathbb{R}^{2n}$. Moreover, as we shall see below, for every (two-dimensional) minimal surface in $\mathbb{R}^n$, there is a unique, associated holomorphic surface in $\mathbb{C}^n$ with the same metric.

Consider now the map $\psi\colon \mathbb{C} \to \mathbb{C}^2$ given by

$$\psi(Z) = (Z^3, Z^2) . \tag{1.19}$$

In the set $\mathbb{C} \sim \{0\}$, ψ is an immersion and represents an embedded minimal surface in $\mathbb{R}^4$. At $Z = 0$, the map has a singularity (which is geometric and not the result of a bad parameterization). However, by using the Wirtinger Inequality and proceeding as above one can immediately verify that the "surface" $\psi|\Delta\colon\Delta \to \mathbb{R}^4$

($\Delta = \{Z: |Z| \leqq 1\}$) represents the surface of least area among all surfaces in $\mathbb{R}^4$ having $\Gamma = \psi[\partial\Delta]$ as boundary. Thus, we see that there is no hope for having regular solutions to Plateau's problem at least for codimension greater than 1. (This represents, of course, a special but easily verified case of Federer's result mentioned above.)

CASE 2. Consider $\mathbb{C}^{n+1}$ as a set of homogeneous coordinates for $\mathbb{C}P^n$ as above. Let $V^{k+1} \subset \mathbb{C}^{n+1}$ be given as in Equation (1.18) where each of the polynomials p_j is homogeneous (i.e., for each j there is an integer $m_j \geqq 0$ such that for all $\alpha \in \mathbb{C}$ and all $Z \in \mathbb{C}^{n+1}$, $p_j(\alpha Z) = \alpha^{m_j+1} p_j(Z)$). Note that by the homogeneity, if $Z \in V$, then $\alpha Z \in V$ for all $\alpha \in \mathbb{C}$. It follows that if $V \sim \{0\}$ is a regular submanifold of $\mathbb{C}^{n+1}$, then $V - \{0\}$ projects to a (compact) complex submanifold $\bar{V}^k$ of $\mathbb{C}P^n$. By the above, the homology class of $\bar{V}$ in $\mathbb{C}P^n$ is nontrivial and $\bar{V}$ representes a manifold of least area among all manifolds (in fact among all rectifiable cycles) in that homology class.

The homology of $\mathbb{C}P^n$ is

$$H_i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0,2,4,\ldots,2n \\ 0 & \text{otherwise} \end{cases}$$

where $H_{2k}(\mathbb{C}P^n;\mathbb{Z})$ is generated by the linear subspace $[\mathbb{C}P^k]$. The homology class of $\bar{V}$ is thus $d[\mathbb{C}P^k]$ where d is the algebraic degree of $\bar{V}$. Thus, the regular manifolds $\bar{V}_d^k$ represented by the equations

$$(1.20) \qquad \begin{cases} Z_0^d + \cdots + Z_k^d = 0 \\ Z_{k+1} = 0 \\ \vdots \\ Z_n = 0 \end{cases}$$

show that every homology class in $\mathbb{C}P^n$ is represented by an embedded submanifold of least volume. Unfortunately, this beautiful situation does not hold in general (Thom, [1]). To get a correct theorem we must replace the notion of manifold by the more general concept of an integral current. (See Federer [2].)

CASE 3. Let $V^{k+1} \subset \mathbb{C}^{n+1}$ be a homogeneous algebraic variety as in Case 2 and assume that $V^k \sim \{0\}$ is a regular submanifold. Let $S^{2n+1} = \{Z \in \mathbb{C}^n : |Z|^2 = 1\}$. Then $M_V = V \cap S^{2n+1}$ is <u>a compact, minimal submanifold of</u> S^{2n+1}.

This fact is a consequence of the following more general one. By a regular cone in $\mathbb{R}^n$ we mean a set $C \subset \mathbb{R}^n$ such that: $C \sim \{0\}$ is a manifold, and for every

$x \in C$ and every $\alpha \in \mathbb{R}^+$, we have $\alpha \cdot x \in C$. Note that if $C \subset \mathbb{R}^n$ is a regular cone, then $M_C = S^{n-1} \cap C$ is a submanifold. Furthermore, if $M \subset S^{n-1}$ is a submanifold, then the set $C(M) = \{x \in \mathbb{R}^n: x \in M \text{ and } \alpha \in R^+\}$ is a regular cone. Clearly, $C(M_C) = C$ and $M_{C(M)} = M$. The fact of interest here is the following.

LEMMA 7 - $M \subset S^{n-1}$ _is a minimal submanifold of_ S^{n-1} _if and only if_ $C(M) \sim \{0\}$ _is a minimal submanifold of_ $\mathbb{R}^n$.

The proof is straightforward and left as an exercise for the reader.

We now see that it is possible to construct a great many compact, minimal submanifolds of codimension-2 in spheres (by using, say, Equations (1.20)). In particular we can construct minimal 3-manifolds in S^5 as follows. Let $V_d \subset \mathbb{C}^3$ be the variety given by the equation

$$Z_0^d + Z_1^d + Z_2^d = 0 .$$

The image of V_d in $\mathbb{C}P^2$ is a compact, orientable surface of genus $g = \frac{1}{2}(d-1)(d-2)$. (See, for example, Milnor [2, pg 85].) The restricted projection map $\pi: S^5 \to \mathbb{C}P^2$ is a fibre bundle with fibre S^1. Thus, the manifold $M_d = S^5 \cap V_d$ is a minimal 3-manifold which is topologically a circle bundle over a surface of genus $\frac{1}{2}(d-1)(d-2)$.

Thus we see that it is possible to achieve minimal embeddings of quite complicated 3-manifolds into S^5, and, by using the totally geodesic inclusion $S^5 \subset S^n$, into higher dimensional spheres. Note, however, that this method gives us no examples of minimal surfaces in S^3 or $\mathbb{R}^3$, nor does it give examples of any minimal submanifolds of codimension one. Examples of this type, and theorems concerning them will be discussed in Chapter III (See also Lawson [3].)

§8. The second variational formula

In §1 we discussed the fact that a minimal immersion $f: M \to \bar{M}$ represents a critical point for the area function on the space of all immersions of M into $\bar{M}$. At such points it is natural to ask whether f actually represents a local minimum for the area function. That is, is it true that for every smooth variation $f_t: M \to \bar{M}$, we have $\text{Area}(f) \leq \text{Area}(f_t)$ for all t in some neighborhood of 0? To answer this question it is usually sufficient to consider only the second derivatives of the area function. The purpose of this section is to derive a formula which relates these second derivatives to fundamental geometric invariants of the immersion.

The first of these invariants is the so-called Laplace operator in the normal bundle, defined as follows. Let $f: M \to \bar{M}$ be a minimal immersion where M is a compact manifold with (possibly empty) boundary ∂M, and let us set notation as in §1. Observe that for any $X \in \mathfrak{X}_M$ and any normal vector field , we can define a new normal vector field ∇_X by the formula

$$(1.20) \qquad \nabla_X = (\bar{\nabla}_X)^N .$$

It is straigtforward to verify that the resulting map $\nabla_X: \Gamma(N(M)) \to \Gamma(N(M))$, where $\Gamma(N(M))$ denotes the space of all C^∞ normal vector fields, satisfies the following relations:

(1)' $\qquad \nabla_{fX+gY}\nu = f\,\nabla_X\nu + g\,\nabla_X\nu$

(2)' $\qquad \nabla_X(f\nu) = (Xf)\nu + f\,\nabla_X\nu$

(3)' $\qquad X\langle\nu,\mu\rangle = \langle\nabla_X\nu,\mu\rangle + \langle\nu,\nabla_X\mu\rangle$

for all $f,g \in C^\infty(M)$, $X,Y \in \mathfrak{X}_M$ and $\nu,\mu \in \Gamma(N(M))$. The operator ∇ is called a <u>connection</u> in the normal bundle of M, and using this connection we define the <u>Laplacian</u> $\Delta: \Gamma(N(M)) \to \Gamma(N(M))$ by setting, at each $p \in M$,

$$(1.21) \qquad \Delta\nu(p) = \sum_{j=1}^{m} (\nabla_{\mathcal{E}_j}\nabla_{\mathcal{E}_j}\nu - \nabla_{\nabla_{\mathcal{E}_j}\mathcal{E}_j}\nu)(p)$$

where $\mathcal{E}_1,\ldots,\mathcal{E}_m \in \mathfrak{X}_p$ are local, pointwise orthonormal tangent vector fields. It is easy to see that the definition is independent of the choice of fields $\mathcal{E}_j$, and so $\Delta\nu$ is again a smooth normal vector field.

We now consider Δ on the space of normal vector fields which correspond to compactly supported variations. Let $\Gamma_o(N(M))$ denote the subspace of all $\nu \in \Gamma(N(M))$ which vanish on ∂M and have compact support in M, and introduce on $\Gamma_o(N(M))$ the inner product

$$(\nu,\mu) = \int_M \langle\nu,\mu\rangle dV .$$

LEMMA 8 - <u>The Laplacian, restricted to</u> $\Gamma_o(N(M))$, <u>is a symmetric, negative semi-definite operator</u>. <u>If</u> $\partial M \neq \emptyset$, <u>it is in fact negative definite</u>.

<u>Proof</u>: Let $\nu,\mu \in \Gamma_o(N(M))$, and for $p \in M$, choose pointwise orthonormal fields $\mathcal{E}_1,\ldots,\mathcal{E}_m \in \mathfrak{X}_p$ such that $(\nabla_{\mathcal{E}_i}\mathcal{E}_j)_p = 0$ for all i,j. Then, at p we have

$$\sum_{j=1}^{m} \mathcal{E}_j\langle\nabla_{\mathcal{E}_j}\nu,\mu\rangle = \sum_{j=1}^{m} (\langle\nabla_{\mathcal{E}_j}\nabla_{\mathcal{E}_j}\nu,\mu\rangle + \langle\nabla_{\mathcal{E}_j}\nu,\nabla_{\mathcal{E}_j}\mu\rangle)$$

$$= \langle\Delta\nu,\mu\rangle + \langle\nabla\nu,\nabla\mu\rangle .$$

If we now define on M a 1-form θ by setting $\theta(X) =$

$= \langle \nabla_X \nu, \mu \rangle$, the above equation tells us that

$$d(*\theta) = (\langle \Delta\nu, \mu \rangle + \langle \nabla\nu, \nabla\mu \rangle) dV .$$

Integrating and using the fact that $*\theta | \partial M \equiv 0$, we have

$$(\Delta\nu, \mu) = -(\nabla\nu, \nabla\mu) \overset{\text{def.}}{=} -\int_M \langle \nabla\nu, \nabla\mu \rangle dV .$$

The result follows immediately. □

The second geometric invariant we shall consider involves curvature. Recall that for vectors $X, Y \in T_p(\bar{M})$ we define a curvature transformation $\bar{R}_{X,Y}: T_p(\bar{M}) \to T_p(\bar{M})$ by

$$\bar{R}_{X,Y} Z = (\bar{\nabla}_{\tilde{X}} \bar{\nabla}_{\tilde{Y}} \tilde{Z} - \bar{\nabla}_{\tilde{Y}} \bar{\nabla}_{\tilde{X}} \tilde{Z} - \bar{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z})_p$$

where $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}_p$ are any local fields extending X, Y, and Z. A similar transformation $R_{X,Y}: T_p(M) \to T_p(M)$ is, of course, defined for the submanifold M, and the two are related by the following generalized Gauss curvature formula.

(1.22)
$$\begin{aligned} &\langle \bar{R}_{X,Y} Z, W \rangle - \langle R_{X,Y} Z, W \rangle \\ &= \langle B_{X,Z}, B_{Y,W} \rangle - \langle B_{X,W}, B_{Y,Z} \rangle \end{aligned}$$

for all $X, Y, Z, W \in T_p(M)$, where B is the second fundamental form of M in $\bar{M}$. To establish this formula we note that since $\nabla_X Y = (\bar{\nabla}_X Y)^T$, we have

$$\langle \bar{R}_{X,Y}Z,W\rangle - \langle R_{X,Y}Z,W\rangle =$$

$$= \langle \bar{\nabla}_X(\bar{\nabla}_Y Z)^N,W\rangle - \langle \bar{\nabla}_Y(\bar{\nabla}_X Z)^N,W\rangle$$

$$= -\langle (\bar{\nabla}_Y Z)^N,\bar{\nabla}_X W\rangle + \langle (\bar{\nabla}_X Z)^N,\bar{\nabla}_Y W\rangle$$

$$= -\langle B_{Y,Z},B_{X,W}\rangle + \langle B_{X,Z},B_{Y,W}\rangle .$$

At each point $p \in M$ we now construct a linear transformation $\bar{\mathcal{R}}: N_p(M) \to N_p(M)$ by setting

$$\bar{\mathcal{R}}(\nu) = \sum_{j=1}^{m} (\bar{R}_{e_j,\nu}e_j)^N$$

where $e_1,\ldots,e_m$ form an orthonormal basis of $T_p(M)$. Note that from the basic symmetries of the Riemann curvature tensor (cf. Milnor [1, page 53]), we have

$$\langle \bar{\mathcal{R}}(\nu),\mu\rangle = \langle \nu,\bar{\mathcal{R}}(\mu)\rangle .$$

Thus $\bar{\mathcal{R}}$ represents a smooth section of the bundle $SN(M) \subset \mathrm{Hom}(N(M),N(M))$ whose fiber at p is the space of symmetric endomorphisms of $N_p(M)$.

The final geometric invariant we wish to consider involves the second fundamental form B. Recall that at $p \in M$, B is a symmetric bilinear form on $T_p(M)$ with values in $N_p(M)$, i.e., $B_p \in \mathrm{Hom}(ST_p(M),N_p(M))$ where $ST_p(M)$ is the set of symmetric endomorphisms of $T_p(M)$. Since $ST_p(M)$ inheirits a natural inner product, we can speak of the transposed mapping ${}^tB_p \in \mathrm{Hom}(N_p(M),ST_p(M))$, and define $\mathcal{B}_p: N_p(M) \to N_p(M)$ by

$$\mathcal{B} = B \circ {}^tB.$$

Note that for $\mu,\nu \in N_p(M)$, $\langle \mathcal{B}(\nu),\mu\rangle = \langle B \circ {}^tB(\nu),\mu\rangle =$
$= \langle {}^tB(\nu), {}^tB(\mu)\rangle = \Sigma\langle {}^tB(\nu), e_i\otimes e_j\rangle\langle e_i\otimes e_j, {}^tB(\mu)\rangle =$
$= \Sigma\langle \nu, B_{e_i,e_j}\rangle\langle \mu, B_{e_i,e_j}\rangle$ where $\{e_1,\ldots,e_m\}$ is any orthonormal basis of $T_p(M)$. In particular,

$$\langle \mathcal{B}(\nu),\mu\rangle = \langle \nu, \mathcal{B}(\mu)\rangle.$$

We are now in a position to state the main result of this section. Let $F\colon I\times M \to \tilde{M}$ be a smooth variation of the minimal immersion $f\colon M \to \tilde{M}$, and suppose that the variation vector field $E = F_* \frac{\partial}{\partial t}\Big|_{t=0}$ is a normal vector field with compact support. ($E \in \Gamma_o(N(M))$.) If $A(t)$ is the area function, defined as in §1, then we have.

THEOREM 1' (The second variational formula).

$$\frac{d^2A}{dt^2}\Big|_{t=0} = \int_M \langle -\Delta E + \overline{\mathcal{R}}(E) - \mathcal{B}(E), E\rangle\, dV.$$

<u>Proof</u>: From Theorem 1 and the fact that the mean curvature K is identically zero for $t = 0$, we have that

$$\frac{d^2A}{dt^2}\Big|_{t=0} = -\int_M \frac{d}{dt}\langle K,E\rangle\Big|_{t=0}\, dV.$$

We now fix a point $p \in M$, choose vector fields $\mathcal{E}_1,\ldots,\mathcal{E}_m \in \mathfrak{X}_p$ as in the proof of Theorem 1, and set $g_{ij}(t) = \langle f_{t*}\mathcal{E}_i, f_{t*}\mathcal{E}_j\rangle$. To simplify notation we shall denote $f_{t*}\mathcal{E}_j$ simply by $\mathcal{E}_j$ whenever the meaning is clear. A straightforward calculation shows that at any time t

$$K = \sum_{i,j=1}^{m} g^{ij}(t)(\bar{\nabla}_{\mathcal{E}_i}\mathcal{E}_j)^N$$

where $g^{ij}(t)$ denotes the inverse matrix to $g_{ij}(t)$. It follows that $\langle K,E\rangle = \Sigma\, g^{ij}(t)\langle\bar{\nabla}_{\mathcal{E}_i}\mathcal{E}_j,E\rangle$, and, since $K \equiv 0$ and $g_{ij}(0) = \delta_{ij}$, we have

$$(1.23) \qquad \frac{d}{dt}\langle K,E\rangle\Big|_{t=0} = \sum_{i,j=1}^{m} \frac{dg^{ij}}{dt}(0)\langle\bar{\nabla}_{\mathcal{E}_i}\mathcal{E}_j,E\rangle + \sum_{i=1}^{m} \langle\bar{\nabla}_E\bar{\nabla}_{\mathcal{E}_i}\mathcal{E}_i,E\rangle .$$

Differentiating the identity $\sum_j g^{ij}(t)g_{jk}(t) \equiv \delta^i_k$ shows that $(dg^{ij}/dt)(0) = -(dg_{ij}/dt)(0) = -E\langle\mathcal{E}_i,\mathcal{E}_j\rangle =$
$= -(\langle\bar{\nabla}_E\,\mathcal{E}_i,\mathcal{E}_j\rangle + \langle\mathcal{E}_i,\bar{\nabla}_E\mathcal{E}_j\rangle) = -(\langle\bar{\nabla}_{\mathcal{E}_i}E,\mathcal{E}_j\rangle + \langle\mathcal{E}_i,\bar{\nabla}_{\mathcal{E}_j}E\rangle) =$
$= 2\langle B_{\mathcal{E}_i,\mathcal{E}_j},E\rangle$ where we have used the fact that

$$(1.24) \qquad \bar{\nabla}_E\mathcal{E}_j - \bar{\nabla}_{\mathcal{E}_j}E = [E,\mathcal{E}_j] = 0.$$

It follows immediately that the first sum in (1.23) is equal to

$$(1.25) \qquad 2\sum_{i,j=1}^{m} \langle B_{\mathcal{E}_i,\mathcal{E}_j},E\rangle^2 = 2\langle \mathcal{B}(E),E\rangle .$$

We now observe that

$$\langle \bar\nabla_E \bar\nabla_{\mathcal{E}_j} \mathcal{E}_j, E\rangle = \langle \bar R_{E,\mathcal{E}_j} \mathcal{E}_j, E\rangle + \langle \bar\nabla_{\mathcal{E}_j} \bar\nabla_E \mathcal{E}_j, E\rangle =$$

$$= -\langle \bar R_{\mathcal{E}_j, E} \mathcal{E}_j, E\rangle + \mathcal{E}_j \langle \bar\nabla_{\mathcal{E}_j} E, E\rangle - \|\bar\nabla_{\mathcal{E}_j} E\|^2.$$

(Note the extensive use of (1.24)). However,

$$\sum_{j=1}^{m} \|\bar\nabla_{\mathcal{E}_j} E\|^2 = \sum_j \|(\bar\nabla_{\mathcal{E}_j} E)^N\|^2 + \sum_j \|(\bar\nabla_{\mathcal{E}_j} E)^T\|^2$$

$$= \sum_j \|\nabla_{\mathcal{E}_j} E\|^2 + \sum_{i,j} \langle \bar\nabla_{\mathcal{E}_j} E, \mathcal{E}_i\rangle^2 = \|\nabla E\|^2 + \langle \mathcal{B}(E), E\rangle .$$

Finally, if we define the one-form ω on M by the equation $\omega(X) = \langle \bar\nabla_X E, E\rangle$, then at p,

$$\delta\omega = \sum_{j=1}^{m} \mathcal{E}_j \langle \nabla_{\mathcal{E}_j} E, E\rangle .$$

It follows that the second sum in (1.23) is

$$- \|\nabla E\|^2 - \langle \bar{\mathcal{R}}(E), E\rangle - \langle \mathcal{B}(E), E\rangle + \delta\omega .$$

Combining this with (1.25), integrating over M and using the fact that $(\Delta E, E) = - \int_M \|\nabla E\|^2$ gives the result. □

Theorem 1' has several immediate consequences. Let us consider the symmetric differential operator

$$\mathcal{L} = - \Delta + \tilde{\mathcal{R}} - \mathcal{B}$$

defined in $\Gamma_o(N(M))$. If M is compact (with boundary), the operator $\mathcal{L}$ is strongly elliptic. The general theory of such operators shows that $\mathcal{L}$ can be diagonalized on

$\Gamma_o(N(M))$ with eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots \to \infty$$

where each eigenspace V_{λ_j} is finite dimensional.

If one considers a minimal immersion as a critical point of the area function on the space of immersions $M \to \bar{M}$ fixing the boundary, then the quadratic form

$$I(E) = \int_M \langle \mathcal{L}(E), E \rangle = (\mathcal{L}(E), E)$$

is the Hessian form of second derivatives of the area function at this point. In analogy with standard critical point theory we define:

$$\text{index}(M) = \dim\left(\bigoplus_{\lambda<0} V_\lambda \right)$$

$$\text{nullity}(M) = \dim\,(V_o).$$

Note that if M represents a manifold of least area with respect to its boundary, then $\text{index}(M) = 0$.

By definition a normal field $E \in \Gamma_o(N(M))$ lies in V_o if and only if $\mathcal{L}(E) \equiv 0$. Any such field is called a <u>Jacobi field</u>.

One of the fundamental theorems concerning Jacobi fields is the following. Let $f\colon M \to \bar{M}$ be a compact minimal submanifold with boundary, and let c_t be a contraction of M into itself. In particular, assume c_t, $t \geq 0$ is a smooth family of diffeomorphisms of M

into M such that:

(i) c_o = identity

(ii) $c_t(M) \subset c_s(M)$ for $t > s$

(iii) $\lim_{t\to\infty} \mathrm{Vol}(c_t(M)) = 0$.

Write $M_t = c_t(M)$ and consider the minimal immersion $f|M_t$. The main result is the following (cf. Simons [1]).

THEOREM - (Morse, Simons, Smale)

$$\mathrm{index}(M) = \sum_{t>0} \mathrm{nullity}(M_t).$$

Note that one result of this is that $\mathrm{nullity}(M_t) = 0$ except for a finite number of t's.

The importance of this result is that it allows the index of M also to be computed in terms of solutions to the equation $\mathcal{L}(E) = 0$. For a discussion of some of the applications of this to the case of geodesics, see Milnor [1].

CHAPTER II

THE CLASSICAL PLATEAU PROBLEM

The study of minimal surfaces has a long and rich history dating from the experiments of the Belgian physicist J. Plateau in 1847 [1]. He showed that by the laws of surface tension the soap film formed by dipping a wire form in a soap solution represented a surface which was stable with respect to area. That is, under slight deformations the soap film always became larger. Mathematicians were soon able to give a local description of such surfaces as we have done above. However, a mathematical proof that for every "wire" there exists a "soap film" proved to be a far more difficult task. As we shall see, simply to find a viable mathematical formulation of the problem was a non-trivial undertaking. The first real solution to the problem was finally given in 1930 by Jessie Douglas [1] and T. Rado [1]. Douglas work was important both for the result and for the method of proof. For that reason we shall present his solution here in detail. The basic reference for this work is the excellent book of Courant [2]. Any reader looking for more elaboration than is found here should begin with this book.

While much was proved in the 1930's about the existence of solutions to Plateau's problem, it was not until the 1960's that the questions of interior and boundary regularity of the solutions were settled. We shall present here a survey of these results and refer the reader to the journals for details.

§1. The solution of Douglas-Rado

The first difficulty in trying to attack the Plateau problem is to find a workable formulation of the question. Roughly speaking, of course, it is: "Given a Jordan curve Γ in $\mathbb{R}^n$, find a surface in $\mathbb{R}^n$ of least area having Γ as boundary." However, a little thought on this problem (and perhaps some experimentation with soap films) quickly shows that the topological type of Γ may be quite complicated. Moreover, minimal surfaces of various topological types with Γ as boundary may exist (while only one may represent an absolute minimum in area.) A good example of this, due to W. Fleming [1], is illustrated in Figure 1. Intuition as well as approximating soap film experiments indicate that there is a mi-

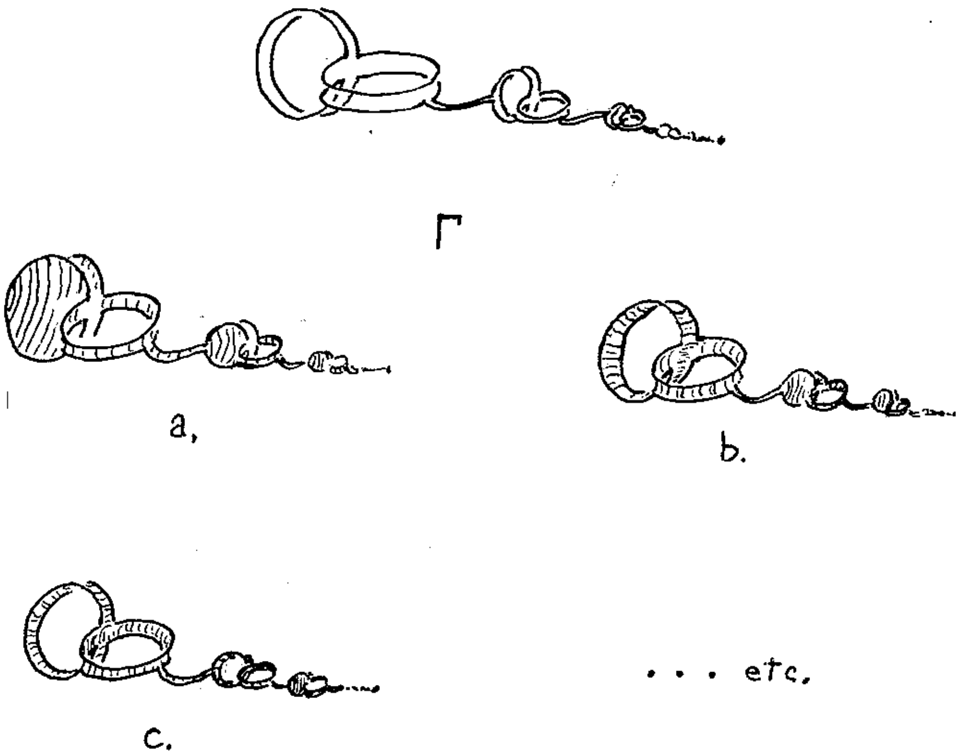


Figure 1

nimal surface (a) of the type of the disk, having Γ as boundary. Moreover, there is a minimal surface (b), homeomorphic to the disk with one handle attached, which has smaller area. The surface (c), a disk with two handles, has even smaller area, etc. The surface of absolutely least area is infinitely connected. In view of this we shall restrict our attention to trying to find a surface Σ of least area and of prescribed topological type, the simplest being that of the disk.

We want to consider our "surfaces" here to be mappings of two-manifolds into $\mathbb{R}^n$. However, there is no reason why these mappings should be immersions. In fact, from the example given by Equation (1.19) we see that in general even our solution surfaces will have points of irregularity. However, to preserve the notion of area among all the competing surfaces in our problem we shall require that the maps be piecewise continuously differentiable.

Our problem can now be formulated as follows. Let $\Gamma \subset \mathbb{R}^n$ $(n \geqq 2)$ be a Jordan curve, i.e., a subset homeomorphic to the circle, and set $\Delta = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$. A mapping $\psi: \Delta \to \mathbb{R}^n$ is called <u>piecewise</u> C^1 if it is continuous and if, except along $\partial\Delta$ and along a finite number of regular C^1-arcs and points in Δ^0, ψ is of class C^1. A continuous map $b: \partial\Delta \to \Gamma$ is called <u>montone</u> if for each $p \in \Gamma$ the set $b^{-1}(p)$ is connected. We now define the class of competing surfaces

$$(2.1) \qquad X_\Gamma = \{\psi : \Delta \to \mathbb{R}^n : \psi \text{ is piecewise } C^1 \text{ and } \psi|\partial\Delta \text{ is a monotone parameterization of } \Gamma\}.$$

We then define the area function $A: X_\Gamma \to \mathbb{R}^+ \cup \{\infty\}$ by the following (generally improper) integral.

$$(2.2) \qquad A(\psi) = \iint_{\Delta} |\psi_x \wedge \psi_y| dxdy$$

where $|\psi_x \wedge \psi_y|^2 = |\psi_x|^2 \cdot |\psi_y|^2 - \langle \psi_x, \psi_y \rangle^2$. The precise statement of our problem is now <u>to find a</u> $\psi \in X$ <u>such that</u> $A(\psi) = \mathfrak{a}_\Gamma$ <u>where</u>

$$(2.3) \qquad \mathfrak{a}_\Gamma = \inf_{\psi \in X_\Gamma} A(\psi)$$

Note that we have one minor complication. For this problem to be interesting we should know that $\mathfrak{a}_\Gamma < \infty$. If Γ is rectifiable this is true, but in general it will have to be assumed. An example of a curve Γ with $\mathfrak{a}_\Gamma = \infty$ can be constructed in the following manner. Let $\Gamma_0 \subset \mathbb{R}^3$ be a plane circle and string onto Γ_0 a number of "beads", i.e., solid tori equally spaced along Γ_0 as shown. Construct a new curve Γ_1 from Γ_0 by wrapping around each torus in succession like a coil, as in Figure 2. String a new succesion of (smaller) beads on Γ_1 and repeat the process to obtain Γ_2. Let $\Gamma_\infty = \lim_n \Gamma_n$. For an appropriate choice of parameters (size of beads, number of beads, etc.) we will have $\mathfrak{a}_{\Gamma_\infty} = \infty$. To see this, estimate the area of the surface required to span each helical coil.

The question now is how to solve the problem we have posed. Geometric intuition would suggest that we take a sequence of surfaces $\{\psi_n\}_{n=1}^{\infty}$ such that $A(\psi_n) \to a_\Gamma$, and try to show that some subsequence must converge

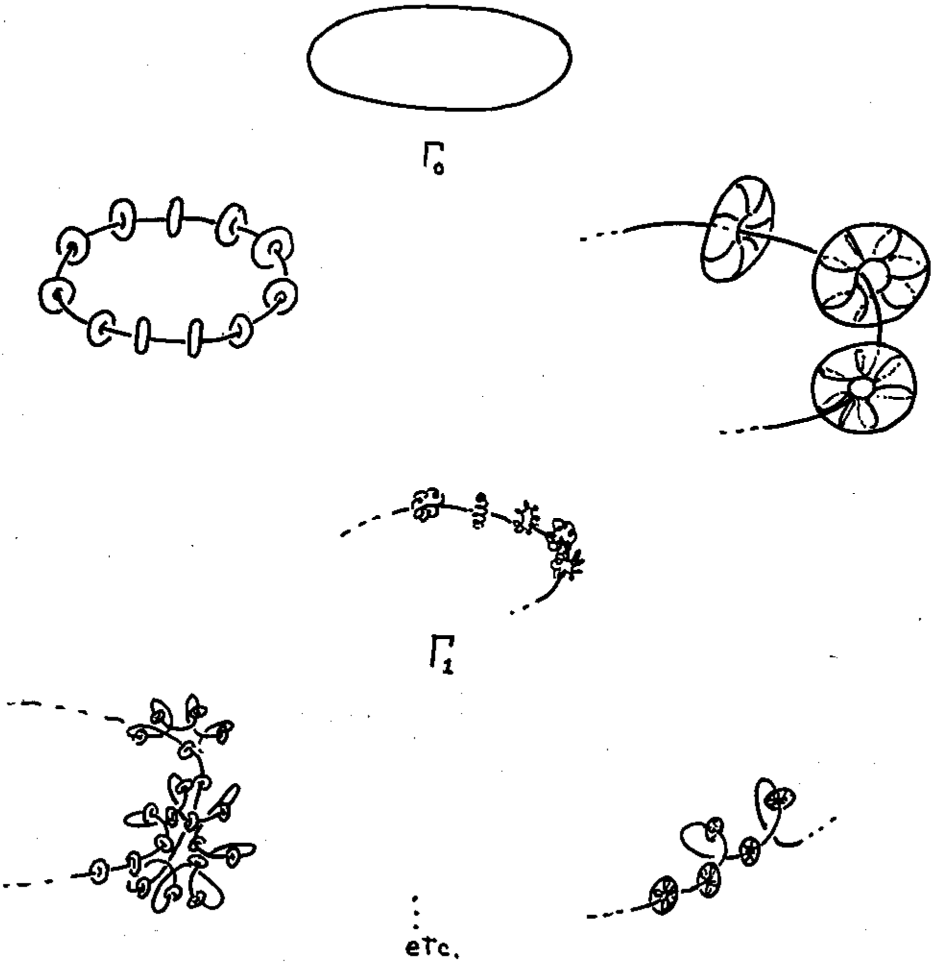


Figure 2

to a solution. However, we must deal in this case not only with the geometric images of the surfaces but with the way they are parameterized. The area integral (2.2) is clearly invariant under even piecewise C^1-reparameterizations. Thus, suppose that $\{\psi_n\}$ were a sequence of C^1-functions in X_Γ such that $\psi_n \to \psi \in X_\Gamma$ and $\nabla\psi_n \to \nabla\psi$ uniformly on compact subsets of Δ^0, and such that $A(\psi_n) \to \mathcal{Q}_\Gamma$. For each n let $d_n: \Delta \to \Delta$ be a piecewise smooth homeomorphism with the property that $d_n(re^{i\theta}) = re^{in\theta}$ for $0 \leq \theta \leq \frac{1}{n}(2\pi - \frac{1}{n})$. Then the sequence $\psi_n' = \psi_n \circ d_n$ would again be minimizing, i.e., $A(\psi_n') \to \mathcal{Q}_\Gamma$. However, no subsequence could converge even to a continuous function, because the d_n's converge pointwise to a discontinuous map.

The lesson here is that we must somehow control the parameterizations of our minimizing sequences of surfaces. In the one-dimensional case, i.e., curves in a riemannian manifold, this is done by minimizing the <u>energy integral</u> (cf. Milnor, [1].) Here minimizing sequences of curves tend to geodesics which not only minimize the length integral but which are forced to be parameterized by a multiple of arc-length. Physically, one could think of a rubber band lying on a surface with its end-points fixed. This band not only minimizes its length but also minimizes its potential energy by stretching itself uniformly along the surface. The analogous situation holds for surfaces. The corresponding energy integral in the so-called Dirichlet integral

$$(2.4) \qquad D(\psi) = \int_\Delta\int(|\psi_x|^2 + |\psi_y|^2)dxdy \ .$$

As we shall now see, the functions in X_Γ which minimize the Dirichlet integral not only minimize area but must have natural parameterizations, namely, conformal ones, which correspond to a tight, or least energy, spreading of the surface over the geometric configuration of least area.

To begin we note that for any two vectors $v,w \in R^n$ we have $|v \wedge w|^2 = |v|^2|w|^2 - \langle v,w\rangle^2 \leq |v|^2|w|^2 \leq$ $\leq \frac{1}{4}(|v|^2 + |w|^2)^2$ where equality holds if and only if $|v| = |w|$ and $\langle v,w\rangle = 0$. It follows immediately that for any $\psi \in X_\Gamma$,

$$(2.5) \qquad A(\psi) \leq \frac{1}{2} D(\psi)$$

where equality holds if and only if

$$(2.6) \qquad |\psi_x| = |\psi_y| \quad \text{and} \quad \langle\psi_x,\psi_y\rangle = 0$$

almost everywhere in Δ. Any mapping satisfying (2.6) is called almost conformal. Wherever $|\psi_x| > 0$, such a map is conformal, or angle-preserving, and induces a metric on

Δ of the form

$$ds^2 = F(dx^2 + dy^2) \tag{2.7}$$

where $F = |\psi_x| = |\psi_y|$. Under condition (2.7) the parameters (x,y) are called isothermal coordinates for the surface. A theorem of fundamental importance for us is the following.

THEOREM IC - (The existence of isothermal coordinates.)

Let $\psi: \Delta \to R^n$ be a continuous map such that $\psi|\Delta^O$ is an immersion of class C^k, $1 \leqq k \leqq \infty$ (or real analytic). Then there exists a homeomorphism $d: \Delta \to \Delta$ where $d|\Delta^O$ is of class C^k (or real analytic) such that the reparameterized mapping $\psi' = \psi \circ d$ is conformal.

For a proof of this we refer the reader to Morrey [1, pg. 366] or to Chern [1].

We now consider a Jordan curve $\Gamma \subset \mathbb{R}^n$ and define

$$d_\Gamma = \inf_{\psi \in X_\Gamma} D(\psi) . \tag{2.8}$$

Then from Theorem IC we have the following

LEMMA 1 - $$\alpha_\Gamma = \frac{1}{2} d_\Gamma .$$

Proof: From Equation (2.5) we clearly have that $\alpha_\Gamma \leqq \frac{1}{2} d_\Gamma$.

It remains only to prove the reverse inequality. To do this, let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence from X_Γ such that $A(\psi_n) \to \mathfrak{a}_\Gamma$. By making appropriate approximations we may assume each ψ_n is class C^1 in Δ^0. We shall now reparameterize each ψ_n so that the new map $\bar{\psi}_n$ satisfies $D(\bar{\psi}_n) \leqq 2A(\bar{\psi}_n) + \frac{2}{n}$. This will prove the lemma. We begin by considering for each $r \geqq 0$ the extended map $\psi_{n,r}: \Delta \to \mathbb{R}^{n+2}$ given by

$$\psi_{n,r}(x,y) = (\psi_n(x,y), rx, ry) .$$

(For $r \neq 0$, $\psi_{n,r} \notin X_\Gamma$, however, it is an immersion.) It is clear that $A(\psi_{n,r})$ depends continuously on r. Thus, there exists some $\varepsilon > 0$ such that $|A(\psi_{n,\varepsilon}) - A(\psi_n)| < \frac{1}{n}$. However, by Theorem IC we may reparameterize $\psi_{n,\varepsilon}$ so that the new map $\bar{\psi}_{n,\varepsilon}$ is conformal. This map can be expressed as

$$\bar{\psi}_{n,\varepsilon}(x,y) = (\bar{\psi}_n(x,y), \varepsilon u(x,y), \varepsilon v(x,y))$$

where $\bar{\psi}_n \in X_\Gamma$. In fact, $\bar{\psi}_n(x,y) = \psi_n(u(x,y),v(x,y))$ is just a reparameterization of ψ_n, and so $A(\psi_n) = A(\bar{\psi}_n)$. Using now the fact that $\bar{\psi}_{n,\varepsilon}$ is conformal we have that $D(\bar{\psi}_n) \leqq D(\bar{\psi}_{n,\varepsilon}) = 2A(\bar{\psi}_{n,\varepsilon}) = 2A(\psi_{n,\varepsilon}) \leqq 2A(\psi_n) + \frac{2}{n} = 2A(\bar{\psi}_n) + \frac{2}{n}$. □

We now have immediately the following

COROLLARY 1 - Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve with $\mathfrak{a}_\Gamma < \infty$. Then for any $\psi \in X_\Gamma$

$$D(\psi) = d_\Gamma \Leftrightarrow A(\psi) = \mathfrak{a}_\Gamma \text{ and } \psi \text{ is almost conformal.}$$

Hence, to solve the Plateau problem it is sufficient to find a function $\psi \in X_\Gamma$ which minimizes the Dirichlet integral. However, to do this we can use harmonic function theory and, in particular:

DIRICHLET'S PRINCIPLE - Let $b\colon \partial\Delta \to \mathbb{R}^n$ be a continuous map and define

$$X_b = \{\psi\colon \Delta \to R^n\colon \ \psi \text{ is piecewise } C^1 \text{ and } \psi|\partial\Delta = b\} .$$

Assume that the number

$$d_b = \inf_{\psi \in X_b} D(\psi)$$

is finite. Then there exists a unique function $\psi_b \in X_b$ such that $D(\psi_b) = d_b$. The function ψ_b is harmonic in Δ^0 and represents the solution to the boundary value problem: $\nabla^2\psi = 0$, $\psi|\partial\Delta = b$.

Proof: Recall that if a function $u\colon \Delta \to \mathbb{R}^n$ is harmonic in Δ^0, then for every r, $0 < r < 1$, we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, u(re^{i\theta})\,d\theta \tag{2.9}$$

for all z with $|z| < r$. It follows easily that if $\{u_n\}_{n=1}^{\infty}$ is a sequence of harmonic functions in Δ which converge to a harmonic function u uniformly on compact subsets of Δ^0, then $\nabla u_n \to \nabla u$ uniformly on compact subsets of Δ^0. This fact, referred to here as Harnack's Principle, gives us the following

LEMMA 2 - *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of harmonic functions in Δ converging to a harmonic function u uniformly on compact subsets of Δ^0. Then*

$$D(u) \leqq \liminf_n D(u_n) \ .$$

To see this let K be a compact subset of Δ^0 and denote by D_K the Dirichlet integral over K. Then by Harnack's Principle, $D_K(u) = \lim D_K(u_n) \leqq \liminf D(u_n)$. The lemma follows by letting K approach Δ.

We now let the vectors a_k, b_k; $k = 0,1,2,\ldots$ denote the Fourier coefficients of b and define for each $n \geqq 0$ the harmonic function

$$\text{(2.10)} \qquad \psi_n(re^{i\theta}) = \frac{a_o}{2} + \sum_{k=1}^{n} r^k(a_k \cos k\theta + b_k \sin k\theta) \ .$$

These partial sums converge uniformly to a harmonic function ψ on compact subsets of Δ^0. Furthermore, ψ extends to a continuous function on Δ such that $\psi|\partial\Delta =$ $= b$. (This is the "Abel convergence" of the Fourier series.) We claim that $D(\psi) = d_b$. To see this choose any $\Phi \in X_b$ with $D(\Phi) < \infty$, and for each n set $\Phi_n = \Phi - \psi_n$. Then for each n, we have that

$$D(\Phi) = D(\Phi_n) + D(\psi_n) + 2D(\Phi_n, \psi_n)$$

where

$$D(\Phi_n, \psi_n) = \int_\Delta\int \left(\left\langle \frac{\partial \Phi_n}{\partial x}, \frac{\partial \psi_n}{\partial x}\right\rangle + \left\langle \frac{\partial \Phi_n}{\partial y}, \frac{\partial \psi_n}{\partial y}\right\rangle\right) dxdy \ .$$

However, by Green's identity, which holds even though Φ is only piecewise C^1, we have

$$D(\Phi_n, \psi_n) = \int_0^{2\pi} \left\langle \Phi_n, \frac{\partial \psi_n}{\partial r}\right\rangle \Big|_{r=1} d\theta - \int_\Delta\int \langle \Phi_n, \nabla^2 \psi_n\rangle dxdy \ .$$

Of course, $\nabla^2\psi_n = 0$. Moreover, the first term on the right can be expressed, by using (2.10), as a combination of the first $2n$ Fourier coefficients of the function $b - \psi_n|\partial\Delta$, each of which is zero. Thus, $D(\Phi_n, \psi_n) = 0$. It then follows that $D(\psi_n) = D(\Phi) - D(\Phi_n) \leqq D(\Phi)$, and so, by Lemma 2

$$D(\psi) \leqq \lim_n \inf D(\psi_n) \leqq D(\Phi) \ .$$

We have now proved that the function $\psi \in X_b$ is harmonic in Δ^0 and $D(\psi) = d_b$. It remains to prove that ψ is unique. To see this let ψ' be any function in X_b with $D(\psi') = d_b$ and set $u = \psi - \psi'$. Note that for all real numbers ε we have

$$D(\psi) \leqq D(\psi+\varepsilon u) = D(\psi) + 2\varepsilon D(\psi,u) + \varepsilon^2 D(u) .$$

It follows that $D(\psi,u) = 0$ and therefore

$$D(\psi') = D(\psi+u) = D(\psi) + D(u) .$$

Hence, $D(u) = 0$ and so $\nabla u = 0$. Since $u|\partial\Delta = 0$ we have $u = \psi-\psi' = 0$, and the uniqueness in proved. □

For further details concerning Dirichlet's Principle we refer the reader to Courant [2] or Morrey [1].

Our requirements for solving the Plateau problem are now vastly simplified. We want to minimize the integral $D(\psi)$ over the class X_Γ. However, for each <u>fixed parameterization</u> $b: \partial\Delta \to \Gamma$ we know that there exists a unique function $\psi_b \in X_b \subset X_\Gamma$ such that $D(\psi_b) = d_b$. However, for <u>different parameterizations</u> b of Γ we will, in general have <u>different values</u> of d_b. Hence it remains only for us to find a parameterization b of Γ such that $d_b = d_\Gamma$.

To find such a minimal parameterization we shall choose a sequence $\{b_n\}_{n=1}^{\infty}$ such that $\lim d_{b_n} = d_\Gamma$ and show that there exists a uniformly convergent subsequence. To do this we will need to normalize the mappings b_n. This normalization will correspond to normalizing the maps ψ_{b_n}. For this we first note the following.

LEMMA 3 - The Dirichlet integral $D(\psi)$ is invariant under conformal transformations of the disk.

The proof of this is straightforward and is left to the reader.

Recall that by a conformal transformation of Δ it is possible to map any given three points of $\partial\Delta$ to any other three, distinct points of $\partial\Delta$. Moreover, having prescribed the images of three such points the conformal transformation is uniquely determined (cf. Ahlfors, [1]). We now normalize our surfaces as follows. We choose three distinct points $p_1, p_2, p_3 \in \Gamma$ and three distinct points $z_1, z_2, z_3 \in \partial\Delta$, and we define

$$X_\Gamma' = \{\psi \in X_\Gamma : \psi(z_k) = p_k \text{ for } k=1,2,3\}.$$

By Lemma 3 we have that $\inf\{D(\psi) : \psi \in X_\Gamma'\} = d_\Gamma$, and thus we may solve the Plateau problem by minimizing in this somewhat smaller class. However, for this class we have the following important fact.

PROPOSITION 1 - Let M be a constant $> d_\Gamma$. Then the family of functions

$$\mathfrak{F} = \{\psi|\partial\Delta : \psi \in X_\Gamma' \text{ and } D(\psi) \leq M\}$$

is equicontinuous on $\partial\Delta$. Thus, by Arzelà's theorem $\mathfrak{F}$ is compact in the topology of uniform convergence.

Proof: Let $\psi \in X_T$ such that $D(\psi) \leqq M$. For each $z \in \mathbb{R}^2$ and each $r > 0$ we define C_r to be the intersection of Δ with the circle of radius r about the point z, and we denote by s the arc length parameter on C_r. We then claim that for each positive number $\delta < 1$ there exists a number ρ (depending on ψ) with $\delta \leqq \rho \leqq \sqrt{\delta}$ such that

$$(2.11) \qquad \int_{C_r} |\psi_s|^2 \, ds \leqq \frac{\varepsilon(\delta)}{\rho}$$

where

$$\varepsilon(\delta) = \frac{2M}{\log(1/\delta)} .$$

To see this we consider the integral

$$I \overset{\text{def.}}{=} \int_{\delta}^{\sqrt{\delta}} \int_{C_r} |\psi_s|^2 ds dr \leqq D(\psi) \leqq M ,$$

and express I as

$$I = \int_{\delta}^{\sqrt{\delta}} p(r) \tfrac{1}{r} dr$$

where

$$p(r) = r \int_{C_r} |\psi_s|^2 \, ds .$$

We assume for the moment that ψ has continuous first derivatives in Δ^O. Then by the Mean Value Theorem (for the measure $d(\log r)$) we have that there exists a number ρ, with $\delta \leqq \rho \leqq \sqrt{\delta}$, such that

$$I = p(\rho) \int_{\delta}^{\sqrt{\delta}} d(\log r) = p(\rho)\, \frac{1}{2} \log(\tfrac{1}{\delta}) .$$

Thus, $p(\rho) \leqq 2M/\log(1/\delta)$ as claimed.

If ψ is only piecewise smooth we proceed as follows. Let $\mathfrak{G} \subset \Delta$ be the set where ψ is of class C^1 ($\mathfrak{G} = \Delta \sim$ a finite number of smooth arcs and points), and let $\{K_n\}_{n=1}^{\infty}$ be a monotone increasing sequence of closed, polyhedral domains converging to $\mathfrak{G}$. Then for each n, there exists a number ρ_n such that Equation (2.11) holds. Here, C_r is defined as $K_n \cap \{$the circle of radius r about $z\}$. By passing to a subsequence we may assume that the numbers ρ_n converge to some number ρ. By passing to the limiting case we then establish Equation (2.11) in general.

It now follows from (2.11) and the Schwarz Inequality that for each positive number $\delta < 1$, there exists a number ρ with $\delta \leqq \rho \leqq \sqrt{\delta}$ such that

$$\ell(C_r)^2 \leqq 2\pi\mathcal{E}(\delta) \tag{2.12}$$

where $\ell(C_r) = \int_{C_r} |\psi_s|\, ds =$ the length of the curve $\psi|C_r$.

We shall use this last inequality together with the geometry of Γ to establish the equicontinuity of $\cdot$ Suppose we are given a number $e > 0$. Then by a straightforward topological argument we see that there exists a number $d > 0$ such that for all $p, p' \in \Gamma$ with $0 < |p-p'| < d$, one of the two components of $\Gamma \sim \{p,p'\}$ will have diameter $<e$. (Recall that for a set $X \subset R^n$, diameter $(X) = \{\sup|q-q'| : q,q' \in X\}$.) We now choose a $\delta < 1$ such that $\sqrt{2\pi\varepsilon(\delta)} < d$, and such that for any $z \in \partial\Delta$ we have $|z_k - z| > \sqrt{\delta}$ for at least two of the points z_1, z_2, z_3. We assume without loss that the number e is less than $\min_{i \neq j} \{|p_i - p_j|\}$. Then for any $z \in \partial\Delta$, there exists by (2.12) a number ρ with $\delta \leqq \rho \leqq \sqrt{\delta}$ such that $\ell(C_\rho) < d$. The boundary of the disk $\partial\Delta$ is now divided by C_ρ into two arcs: a "small" arc A' containing z and its compliment A'' containing two of the points z_1, z_2, z_3. The corresponding arcs on Γ are $\bar{A}'$ and $\bar{A}''$, <u>one</u> of which has diameter $<e$ because $\ell(C) \leqq d$. However, since $\bar{A}''$ contains at least two of the points p_1, p_2, p_3, its diameter is $>e$. Hence, diameter $(\bar{A}') < e$. This immediately implies that for $|z'-z| < \delta$ in $\partial\Delta$, we have

$$|\psi(z) - \psi(z')| < e$$

where δ was chosen independently of z, z' and ψ. This establishes the equicontinuity of the family of functions.$\square$

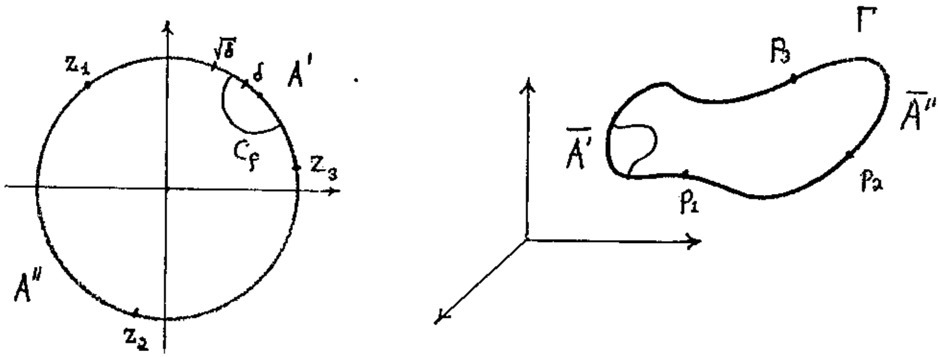


Figure 3

We can now complete the solution to the Plateau problem. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence from $\mathfrak{F}$ such that $\lim_n d_{b_n} = d_\Gamma$. By Proposition 1 there exists a subsequence $\{b_{n_j}\}_{j=1}^{\infty}$ which converges uniformly to some $b \in \mathfrak{F}$. By Lemma 2 we have

$$D(\psi_b) \leqq \liminf_j D(\psi_{b_{n_j}}) = d_\Gamma.$$

Consequently, $D(\psi_b) = d_\Gamma$ and we have proven the following theorem proved for $n=3$ also by T. Rado.

THEOREM 1 - (J. Douglas). <u>Let</u> Γ <u>be a Jordan curve in</u> $\mathbb{R}^n$ <u>such that</u> $\mathfrak{a}_\Gamma < \infty$. <u>Then there exists a continuous map</u> $\psi : \Delta \to \mathbb{R}^n$ <u>such that</u>

1) $\psi|\partial\Delta$ <u>maps</u> $\partial\Delta$ <u>monotonically onto</u> Γ,
2) $\psi|\Delta^o$ <u>is harmonic and almost conformal</u>,

3) $D(\psi) = d_\Gamma$ and $A(\psi) = \mathfrak{a}_\Gamma$

The function ψ of Theorem 1 will be called a solution to the Plateau problem for Γ. The existence of such solutions in the generality above is quite remarkable if one considers that, for example, in $\mathbb{R}^3$ Γ can be knotted and non-tame.

We suggest as a good exercise the proof that in the case $n = 2$, Theorem 1 is the Riemann Mapping Theorem for domains bounded by Jordan curves.

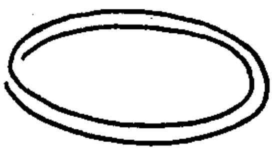
§2. Generalizations

An immediate question which comes to mind is: can the methods of §1 be applied to obtain solutions to a more general Plateau problem? There are two specific directions in which this is true.

C.B. Morrey [1, pg. 389] proved that Theorem 1 remains true if we replace euclidean space R^n by any complete riemannian manifold which is metrically well behaved at infinity. For example, it holds in any compact, or any homogeneous Riemannian manifold.

Douglas, Courant and others did much work in establishing the existence of solutions to the Plateau problem in R^n with higher topological type. Note that

Theorem 1 establishes the existence of a surface of least area among all surfaces homeomorphic to a disk. As pointed out before, however, this may not represent a surface of least area among surfaces of all topological types. An easy example is found by considering a curve Γ constructed as follows. Begin with a double covering of a plane circle of radius r, and pull the two branches apart slightly, to obtain a Jordan curve. The solution for Γ given by Theorem 1 will have area $\approx 2 \times \{\text{area of the original disk bounded by the circle}\} = 2\pi r^2$. (To see this, project the solution back into the plane.) However, there clearly exists a surface of the type of a Möbius band, having area $\approx 2\pi r\varepsilon$ where the branches are separated by a distance $\varepsilon > 0$. Intuitively, one feels there



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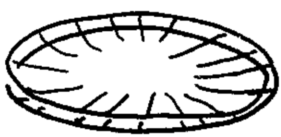
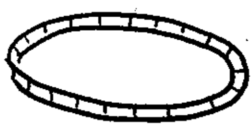


Douglas Solution.



Möbius band.

should be a minimal surface of the type of the Möbius band.

Naively, one may formulate the following general problem. Given a system Γ of k Jordan curves $\gamma_1,\ldots,\gamma_n$ in $\mathbb{R}^n$, find a surface of least area among all surfaces in $\mathbb{R}^n$ of prescribed Euler characteristic and prescribed character of orientability, having Γ as boundary.

It is immediately apparent that the question is, in general, ill posed. Consider, for example, a plane circle Γ in $\mathbb{R}^2 \subset \mathbb{R}^n$. By Corollary I.2 the only minimal surface in $\mathbb{R}^n$ with this boundary is the disk $\Delta = C(\Gamma) \subset \mathbb{R}^2$. If we tried to solve the problem for a disk with a handle attached, we would find that any minimizing sequence would, in the limit, pinch the handle off. Another example can be found by considering two

Figure 4

parallel, plane circles of radius 1, with the same center line in $\mathbb{R}^3$. If they are sufficiently close, there is a minimizing surface, homeomorphic to an anulus, with this boundary, namely part of the well known catenoid. (See Chapter III.) When the circles become a certain distance apart, the catenoid is a local, but not global minimum. At great distances apart (past the "conjugate" distance) the catenoid is not even stable. The surface of least area in the last two cases is simply the disjoint union of two plane disks. A minimizing sequence of maps of the annulus would degenerate as shown in the figure.

Figure 5

A more realistic formulation of the problem considered above would be the following. Let Σ and Σ' be compact (not necessarily connected) surfaces whose boundaries each have k-components. Then Σ is said to have <u>higher topological type</u> than Σ' if the Euler characteristic $\chi(\Sigma) < \chi(\Sigma')$. For each such surface Σ and each Γ as above, we define $X_{\Gamma,\Sigma} = \{\psi : \Sigma \to \mathbb{R}^n : \psi$ is continuous, $\psi|\mathrm{int}(\Sigma)$ is piecewise C^1 and $\psi|\partial\Delta$ is a monotone parameterization of $\Gamma\}$, and we set

$$G_{\Gamma,\Sigma} = \inf_{\psi \in X_{\Gamma,\Sigma}} A(\psi) .$$

PROBLEM - Let $\Gamma = \gamma_1 \cup \ldots \cup \gamma_k \subset \mathbb{R}^n$ be a system of k Jordan curves and let Σ be a compact 2-manifold whose boundary has k-components. Assume that $\mathcal{a}_{\Gamma,\Sigma} < \infty$ and $\mathcal{a}_{\Gamma,\Sigma} < \mathcal{a}_{\Gamma,\Sigma'}$ for all Σ' of lower topological type. Does there exist a map $\psi \in X_{\Gamma,\Sigma}$ such that:

1.) $A(\psi) = \mathcal{a}_{\Gamma,\Sigma}$,

2.) $\psi|\text{int}(\Sigma)$ is real analytic,

3.) ψ is almost conformal and harmonic with respect to some conformal structure on Σ ?

This problem has been solved, and we refer the interested reader to Courant [2] for the details.

Note that the hypothesis of this problem are at times easy to verify as in the first example above and in the following. Let γ_1, γ_2 be two linked, plane circles. The solution with Euler characteristic 2 is two plane disks intersecting along an arc α. By cutting and pasting along α we can get a surface of Euler characteristic 0 (an annulus), which has smaller area.

Theorem 1 and its generalizations represent, in many ways, a very satisfactory solution to the problem we originally posed. Nonetheless, there still remain some basic questions about the geometric behavior of the minimizing surfaces. We shall take these questions up in the following sections.

§3. The interior regularity of the solution

The first question we consider is whether the mapping ψ given by Theorem 1 is, in fact, an immersion in Δ^0. We begin by noting that since ψ is harmonic,

$$\frac{d}{d\bar{z}} \frac{d}{dz} \psi = 0$$

where $d/dz = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$. Hence, the $\mathbb{C}^n$-valued function

$$(2.13) \qquad \varphi \overset{\text{def.}}{=} \frac{d\psi}{dz} = (\varphi_1, \ldots, \varphi_n)$$

is holomorphic in Δ^0. Furthermore, we see easily that $4\varphi^2 = 4 \Sigma \varphi_k^2 = |\psi_x|^2 - |\psi_y|^2 - 2i\langle \psi_x, \psi_y \rangle$, and $4|\varphi|^2 = 4 \Sigma |\varphi_k|^2 = |\psi_x|^2 + |\psi_y|^2$. Since ψ is almost conformal, we then have

$$(2.14) \qquad \varphi^2 = 0$$

$$(2.15) \qquad |\varphi|^2 = F$$

where the metric induced by ψ is of the form

$$(2.16) \qquad ds^2 = 2F|dz|^2 \quad .$$

The mapping ψ will be regular at precisely those points where $F \neq 0$. Equation (2.15) and the fact that φ is holomorphic therefore show us the following.

LEMMA 4 - The solution surface ψ given in Theorem 1 is an immersion everywhere in Δ^0 except possibly at isolated points.

These points of irregularity will be called branch points of the minimal surface.

From example (1.19) we see that branch points do exist on solutions to the Plateau problem in R^n, at least for $n \geqq 4$. Moreover, if we consider mappings $\psi \in X_\Gamma$ which have only properties a) and b) of Theorem 1, then branch points can appear on these even in $\mathbb{R}^3$. An example of such a surface is given by the mapping

$$\Psi(z) = (\mathrm{Re}\{z^2-\tfrac{1}{2}z^4\},\ \mathrm{Im}\{z^2-\tfrac{1}{2}z^4\},\ \mathrm{Im}\{\tfrac{4}{3}z^3\}). \tag{2.17}$$

In a neighborhood of $\psi(0)$, this can be visualized geometrically as follows. Let ℓ_1, ℓ_2 be two straight line segments of lengths 1 and $1+\varepsilon$ respectively, which meet at an angle $2\pi/3$ in $\mathbb{R}^3$. Join the free ends by a curve, which does not lie in the (ℓ_1,ℓ_2)-plane, to form a closed Jordan curve Γ. Span Γ by a minimal surface Σ. We then reflect Σ about the arc ℓ_2, reflect again about the image of ℓ_1, again about the image of ℓ_2 ($= \ell_1$), etc. After five reflections the surface "closes" and produces a minimal surface with one branch point, at the ℓ_1,ℓ_2-vertex. (See Figure 6.) The surface intersects itself along three straight lines meeting in this vertex.

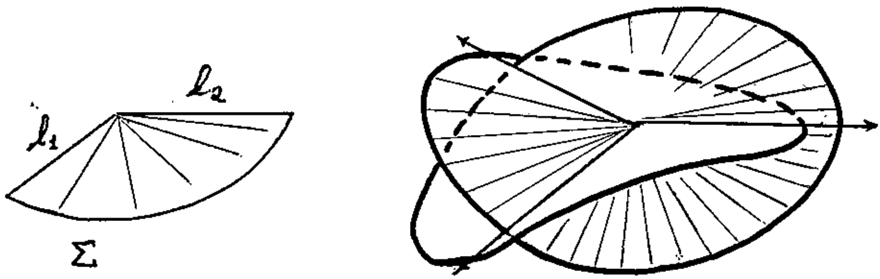


Figure 6

At first sight the surface Ψ appears to be minimizing in a neighborhood of zero, thus giving solutions to the Plateau problem in $\mathbb{R}^3$ with branch points. However, Robert Osserman has shown that, in fact, this is not true. To see why, we first observe that

$$\Psi(x,0) = \Psi(-x,0) \tag{2.18}$$

for all x. This means we can make a reparameterization of the disk Δ as follows. For some $\varepsilon > 0$, we make a cut in the disk Δ along the interval $\{(x,0): -\varepsilon < x < \varepsilon\}$ and add along this cut both an "upper" and a "lower" boundary. By means of a piecewise smooth mapping f we then "reseal" the disk such that points $(x,0)$, $(-x,0)$ of the upper boundary are identified, the points $(x,0)$, $(-x,0)$ of the lower boundary are identified, and the points $(-\mathcal{E},0)$, $(\mathcal{E},0)$ are identified. (See Figure 7.) Because of (2.18), the

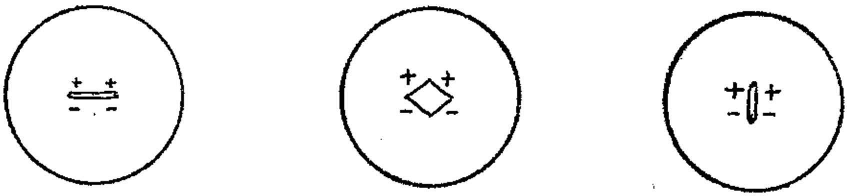


Figure 7

mapping $\Psi \circ f^{-1}$ is a well defined reparameterization of the surface and is still piecewise-smooth. Along the curve of self-intersection $\Psi(x,0)$; $-\mathcal{E} \leqq x \leqq \mathcal{E}$ (or $\Psi \circ f^{-1}(0,y)$; $-\mathcal{E} \leqq y \leqq \mathcal{E}$) the new parameterization behaves differently. The old parameterization Ψ mapped a regular surface (near $(x,0)$) transversely through a regular surface (near $(-x,0)$). The new one, $\Psi \circ f^{-1}$, maps two creased surfaces (near $(0,y)$ and near $(0,-y)$) which intersect along the crease. By deforming one of the creases of $\Psi \circ f^{-1}$ we can now easily reduce the area of the surface while maintaining the boundary and staying in the class of piecewise-smooth maps.

Figure 8

Carefully studying the general behavior of branch points on minimal surfaces in $\mathbb{R}^3$ and applying the above construction, Osserman went on to prove the following remarkable fact.

THEOREM 2 (Osserman [7]). Every solution to the Plateau problem in $\mathbb{R}^3$ is free of branch points, i.e., is a regularly immersed surface.

The result is local in nature and therefore holds both for the solutions of Theorem 1 and its generalizations.

REMARK - Osserman's original proof actually applied only to geometric or "honest" branch points. It failed to handle the case of "false" branch points, that is, points where the surface image is regular but the parameterization has a singularity. Such a singularity could be introduced, for example, by starting with a regular surface $\psi\colon \Delta \to \mathbb{R}^n$ and defining $\varphi(z) = \psi(z^k)$ for some $k > 1$. The first complete argument for the non-existence of false points was given by Gulliver [1]. The problem is surprizingly difficult. In fact, consideration of exactly what must be involved in the solution has lead to the development of a deep and quite delicate theory of branched immersions (cf. Gulliver, Osserman and Royden [1]).

It has also been shown by Gulliver [1] that the solution to a large class of variational problems (including the Plateau problem) in general riemannian 3-manifolds are free of branch points.

§4. The regularity of the solution at the boundary

Our next question is roughly this. When the Jordan curve $\Gamma \subset \mathbb{R}^n$ is well behaved, is each solution to the Plateau problem for Γ correspondingly well behaved at the boundary?

To begin we note that for each solution $\psi\colon \Delta \to \mathbb{R}^n$, the map $\psi|\partial\Delta$ is a homeomorphism of $\partial\Delta$ with Γ and not just a monotone map. The proof of this can be found in Courant [1, pp. 63-64].

Suppose now that Γ is a regular curve of class C^k for $1 \leqq k \leqq \infty$ or $k = \omega$. This means that for each $p \in \Gamma$ there exists a neighborhood U of p in $\mathbb{R}^n$ and a diffeomorphism $\varphi: U \to D^n = \{x \in \mathbb{R}^n: |x| < 1\}$ of class C^k such that $\varphi(\Gamma \cap U) =$ the straight line $\{(x_1, 0, \dots, 0): -1 < x_1 < 1\}$. The natural question is whether, under this assumption, each solution to the Plateau problem for Γ is of class C^k on the closed disk.

If, for example, Γ contains a subarc γ which is itself a straight line, then we can apply the classical reflection principle:

PROPOSITION 2 - Each solution to the Plateau problem for Γ can be analytically continued as a minimal surface (with branch points) by reflection across γ.

Proof: Let $\psi: \Delta \to \mathbb{R}^n$ be a solution for Γ and set $\Delta^+ = \{z \in \Delta: \operatorname{Im}(z) \geqq 0\}$. By a conformal transformation we may change ψ to a mapping $\psi: \Delta^+ \to \mathbb{R}^n$ such that $\psi[\Delta^+ \cap (x\text{-axis})] = \gamma$. By a change of coordinates in $\mathbb{R}^n$, we may assume that γ is a part of the line: $x_2 = x_3 = \dots = x_n = 0$. Hence, we have $\psi_2(x,0) = \dots = \psi_n(x,0) = 0$ for $-1 \leqq x \leqq 1$, and by the reflection principle for harmonic function (cf. Ahlfors, [1]), we may continue $\psi_2,\ldots,\psi_n$ as harmonic functions into the whole disk by setting

$$\psi_k(x,y) = -\psi_k(x,-y)$$

for each k. Therefore, each of the functions $\varphi_k = \partial\psi_k/\partial z$; $2 \leq k \leq n$ extends to a holomorphic function in Δ which is purely imaginary on the x-axis. It then follows from equation (2.14) that

$$\varphi_1^2 = -(\varphi_2^2 + \ldots + \varphi_n^2) \; ,$$

where the right hand side of this expression is real ≥ 0 on the x-axis. The limiting values of the function φ_1 along the x-axis are, therefore, real; and by the reflection principle for holomorphic functions, we can extend φ_1 analytically to the whole disk by setting $\varphi_1(\bar{z}) = \bar{\varphi}_1(z)$. Integrating $(\psi_1(z) = \mathrm{Re}\{\int_0^z \varphi_1(z)dz\})$, we see that ψ_1 has been extended to the disk by the relation

$$\psi_1(x,y) = \psi_1(x,-y) \; .$$

The extended map $\psi : \Delta \to \mathbb{R}^n$ is clearly still a minimal surface. □

NOTE. Proposition 2 does not use the absolute minimality of ψ and actually holds for any minimal surface whose boundary contains the straight line γ.

In view of Proposition 2 it was natural to first consider the case where Γ is a real analytic arc. This problem was mentioned, for example, by Courant [2]. It was not, however, until 1959 that the following theorem was proved.

THEOREM 3 - (H. Lewy [1]). Let $\Gamma \subset \mathbb{R}^n$ be a real analytic arc and suppose that $\psi: \Delta \to \mathbb{R}^n$ is a solution to the Plateau problem for Γ. Then ψ can be extended analytically, as a minimal surface, across the boundary.

Then in 1968 the following was proven.

THEOREM 4 - (S. Hildebrandt [1]). Let $\Gamma \subset \mathbb{R}^n$ be a Jourdan curve of class $C^{k,\alpha}$ for $4 \leqq k \leqq \infty$, ω and $0 \leqq \alpha < 1$.* Then each solution of the Plateau problem $\psi: \Delta \to \mathbb{R}^n$ for Γ is of class $C^{k,\alpha}$ at the boundary.

Hildebrandt's proof uses the theory of elliptic partial differential equation and therefore generalizes to a number of other situations (solutions to the equati n for constant mean curvature in $\mathbb{R}^3$, minimal

* $C^{n,\alpha}$ means class C^n with the n^{th} derivatives being α-Holder continuous.

surfaces in riemannian manifolds (See Heinz and Hildebrandt [1].), etc.).

The following year, using the more elementary techniques of complex function theory, several people gave independent proofs of a slightly improved version of Theorem 4 for the case of $\mathbb{R}^n$.

THEOREM 4' - (Kinderlehrer [1], Lesley [1], Nitsche [3], and Warschawski [1].) Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve of class $C^{n,\alpha(t)}$ for $1 \leq n \leq \infty$ or ω and where $\alpha(t)$ is a modulus of continuity for the n^{th} derivative. Then each solution to the Plateau problem $\psi: \Delta \to \mathbb{R}^n$ for Γ is of class $C^{n,\alpha(t)}$ at the boundary.

These later proofs of the theorem are shorter, more elementary and give better results in $\mathbb{R}^n$, but they don't have the power of generalization that Hildebrandt's methods do.

REMARK 1 - Theorems 3, 4, and 4' are all local in nature. That is, if Γ is of class $C^{n,\alpha}$ in a neighborhood of $p \in \Gamma$, then ψ is of class $C^{n,\alpha}$ in the corresponding neighborhood of $\partial\Delta$. These theorems, furthermore, do not require that ψ be an absolute minimum, but only that it be almost conformal and harmonic.

Note, incidentally, that for $n = 2$. Theorems 4 and 4' are simply the classical theorem of Kellogg concerning the boundary regularity of a conformal map of Δ onto a domain in C bounded by a Jordan curve.

§5. The uniqueness of the solution

Certainly one of the most natural questions to ask is whether the solutions to the Plateau problem, given by Theorem 1, are unique. In general they are not, as can be seen by considering the curve Γ in Figure 9. This curve has an obvious symmetry of degree 2. If the solution of Theorem 1 for Γ were (geometrically) unique,

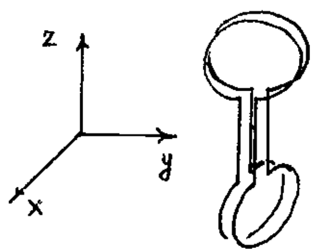


Figure 9

its image would be invariant under this symmety. Consider the coordinate function z restricted to this surface. The parallel planes z = constant each intersect Γ in at most four points. It therefore follows from Proposition 5 (of the next section) that each critical point of z in the interior of the surface is non-degenerate and of index 1. There are four critical points of z restricted to the boundary, two at the top and two at the bottom. By Theorem 3 the surface is analytic at these points. Moreover the surface is regular and transverse to the planes z = constant at these points, for if they were not, one could use the techniques of the proof of Proposition 5 to show that there would be points of the surface both above and below these planes, and thus outside the convex hull of Γ. The methods of Morse Theory (cf. Milnor [1]) then show that z has exactly one critical point in Δ^{0}. To see this explicitly, let $\psi: \Delta \to \mathbb{R}^3$ be the surface and consider the sets $M_a = (z \circ \psi)^{-1}((-\infty, a])$, as a increases from $-\infty$. Just after a passes the first critical value, we have that M_a is diffeomorphic to two half-disks (and has the homotopy type of two points). The manifold $M_\infty = \Delta$ is then obtained by attaching a one handle for each critical point in the interior of Δ. It follows that there can be only one such critical point, as claimed.

Figure 10

By the assumed uniqueness of the solution, this critical point must be located at the center of symmetry of Γ. It is not difficult to see that the surface is therefore of the form shown in Figure 11a. By projecting

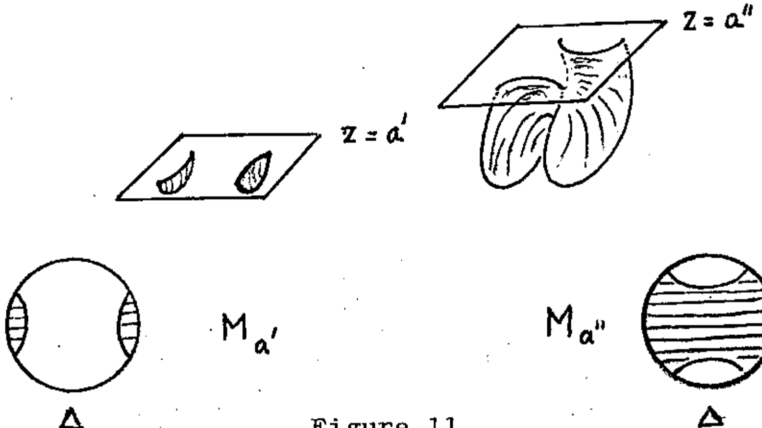


Figure 11

onto the plane parallel to the top disks and then onto the plane parallel to the bottom disks, one can see that the area of this surface is $\geqq 4 \times$ (area of one disk). However, it is easy to find surfaces of smaller area (cf.,

Figure 11b), and so the solution to the Plateau problem (of Theorem 1) is not geometrically unique.

Another example of a contour for which there is non-uniqueness is sketched in Figure 12. For details see Lévy [1].

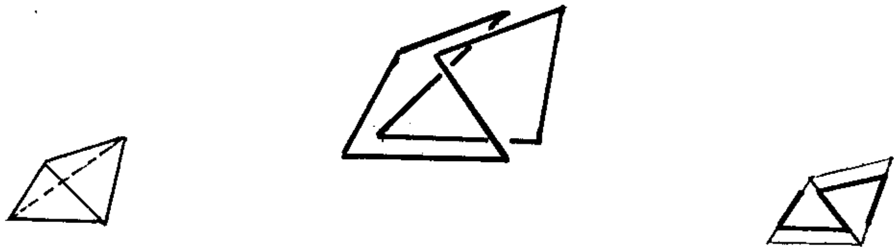


Figure 12

Note that for a given Jordan curve Γ there may also be non-minimizing minimal surfaces with Γ as boundary. In fact, Morse and Tompkins [1] showed that whenever Γ is a boundary of non-uniqueness as above, such surfaces must exist. An entire (beautiful) theory of unstable minimal surfaces was developed by M. Morse, M. Shiffman and C. Tompkins. We refer the reader to Courant [2] for an introduction and bibliography.

One may wonder whether there are curves Γ with more than two solutions to the Plateau problem. If we take an infinite sequence of surves of the type in Figure 9, where each is one half the size of its predecessor, and then connect them as in Figure 1, we will obtain a

Jordan curve which bounds a non-denumerable infinity of minimal surfaces. (See Courant [2], Lemma 3.3.) For each sequence of "up's" and "down's" we get a distinct surface homeomorphic to a disk. In general, of course, these surfaces will not be minimizing. However, Paul Lévy [1] showed that, in fact, there exist rectifiable Jordan curves in R^3 for which the number of geometrically distinct solutions to the Plateau problem (cf. Theorem 1) is also non-denumerably infinite.

The natural question now is whether boundaries of uniqueness are in some sense generic. A partial answer is given in the following Proposition, due essentially to Frederick Almgren. Let $\mathcal{B}$ denote the set of Jordan curves in R^n. We topologize $\mathcal{B}$ with a metric d as follows. For $\Gamma, \Gamma' \in \mathcal{B}$, $d(\Gamma,\Gamma') = \inf\{\|\gamma-\gamma'\|_\infty\}$ where $\|\cdot\|_\infty$ is the usual sup-norm and where γ and γ' run over all possible homeomorphisms of S^1 with Γ and Γ' respectively.

PROPOSITION 3 - The set of Jordan curves for which the solution to the Plateau problem is unique is dense in $\mathcal{B}$.

Proof: The polygonal Jordan arcs are dense in $\mathcal{B}$, and so it is sufficient to show that we can approximate any curve of this type, arbitrarily closely, with an arc of uniqueness. Let Γ, therefore, be a polygonal Jordan curve. Let $v \in \mathbb{R}^n$ be a non-zero vector which is not perpendicular to any of the sides of Γ, and consider the hyperplanes $H_s = \{x \in \mathbb{R}^n: \langle x,v \rangle = s\}$ for $s \in \mathbb{R}$. Let $s_o = \max\{s: H_s \cap \Gamma \neq \emptyset\}$. Then $H_{s_o} \cap \Gamma$ consists of a finite number of vertices of Γ. Without loss of generality we may assume that the angle of each vertex is a rational multiple of 2π. Thus, near each vertex, any solution $\psi: \Delta \to \mathbb{R}^n$ of the Plateau problem for Γ can be continued as an analytic, branched minimal surface, by the Reflection Principle. It follows that for $\varepsilon > 0$ sufficiently small, the hyperplane $H_{s_o-\varepsilon}$ will intersect the surface ψ in embedded analytic curves near the vertices $H_{s_o} \cap \Gamma$.

Consider now the surface $\psi|\Delta_\varepsilon$ where $\Delta_\varepsilon = \{z \in \Delta: \langle \psi(z),v \rangle \leq s_o - \varepsilon\}$ and set $\Gamma_\varepsilon = \psi(\partial\Delta_\varepsilon)$. We claim that Γ_ε is a curve of uniqueness. Clearly $\psi|\Delta_\varepsilon$ is a solution to the Plateau problem for Γ_ε. Suppose that $\varphi: \Delta_\varepsilon \to \mathbb{R}^n$ were another. By Theorem 3, φ is analytic along the segments $\psi(\Delta) \cap H_{s_o-\varepsilon}$. We continue the surface φ to an area minimizing surface for Γ by gluing on $\psi|(\Delta \sim \Delta_\varepsilon)$ with a piecewise smooth reparameterization. If the tangent spaces of the two pieces (which are defined everywhere except possibly at isolated points) do not agree at each point of the "seam", it is possible to make a deformation to a surface of smaller area. Hence, the surface $\varphi(\Delta_\varepsilon)$ is a C^1 continuation of the surface $\psi(\Delta \sim \Delta_\varepsilon)$. From the minimizing property it is, in fact, an analytic continuation. Thus $\varphi = \psi$ in Δ_ε, and we have uniqueness. Clearly, as $\varepsilon \to 0$, we have that $\Gamma_\varepsilon \to \Gamma$ in $\mathcal{B}$, and the proof is complete. □

One might now conjecture that the curves of uniqueness form an open set in $\mathcal{B}$. Unfortunately, they do not, and in fact, we have the following.

PROPOSITION 4 - <u>The set of Jordan curves for which the solution to the Plateau problem is not unique is dense in</u> $\mathcal{B}$.

<u>Proof</u>: Let $\Gamma \in \mathcal{B}$ be a polygonal curve and let H_s be a sequence of hyperplanes chosen as above. Let Γ_0 be the curve illustrated in Figure 9 and for $\varepsilon > 0$ set $\varepsilon\Gamma_0 = \{\varepsilon x : x \in \Gamma_0\}$. Let $p \in \Gamma \cap H_{s_0}$ be an extreme vertex. Then for each $\varepsilon > 0$ we can translate the curve $\varepsilon\Gamma_0 \subset \mathbb{R}^3 \subset \mathbb{R}^n$ to the half-space $\{x \in \mathbb{R}^n : \langle x,v\rangle > s_0\}$ and attach it to Γ at p by a pair of straight lines of length ε. (See Figure 13.) Call the new curve Γ_ε. By Courant [2, Lemma 3.3] there are two minimal surfaces with boundary Γ_ε, one of which, by the methods of Lévy [1], will be a minimum. Using the continuity of harmonic

surfaces with respect to the boundary, one can easily adjust Γ_ε so that the two solutions have the same (minimum) area. (Change the size of one set of disks in $\varepsilon\Gamma_0$.) Again, as $\varepsilon \to 0$, $\Gamma_\varepsilon \to \Gamma$ in $\mathcal{B}$. □

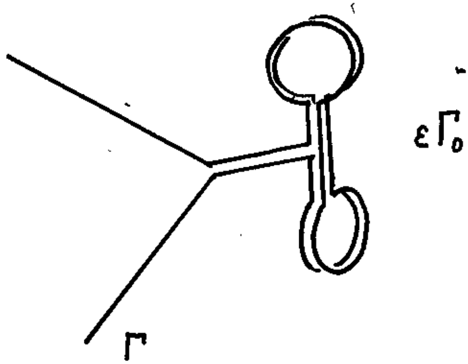


Figure 13

It is an open question whether the curves of uniqueness are generic, that is, whether they form a set of second category in $\mathcal{B}$. It would also be interesting to know if, in the set $\mathcal{B}_1$ of regular Jordan curves of class C^1 with an appropriate "C^1-topology", the set of curves of uniqueness is open and dense.

§6. Conditions for the solution to be one-to-one

The solution surfaces given in Theorem 1 will, in general, have self-intersections. In fact, if Γ is, say, a knot in $\mathbb{R}^3$, every solution must have sef-intersections. Nonetheless, it is reasonable to expect that if Γ is not too badly behaved, the minimizing surfaces for Γ will be embedded. Following the work of T. Rado we shall give a set of geometric conditions on Γ which will gaurantee that the solution of Theorem 1 is not only embedded, but free of branch points and unique.

More specifically, we shall prove the following.

THEOREM 5 - (Rado [2]). If the Jordan curve $\Gamma \subset \mathbb{R}^n$ has a one-to-one, convex projection onto a plane $\mathbb{R}^2 \subset \mathbb{R}^n$, then the solution to the Plateau problem for Γ is unique, free of branch points, and can be expressed as the graph of a function $f: \mathbb{R}^2 \to \mathbb{R}^{n-2}$.

Proof: Let $\pi: \mathbb{R}^n \to \mathbb{R}^2$ be the (perpendicular) projection map, and let $\psi: \Delta \to \mathbb{R}^n$ be a solution to the Plateau problem for Γ. Since $\pi(\Gamma)$ is convex, we clearly have that $\pi[\mathbf{C}(\Gamma)^0] = \mathbf{C}(\pi[\Gamma])^0$ = the domain D bounded by $\pi[\Gamma]$ in $\mathbb{R}^2$. ($\mathbf{C}(X)$ = convex hull of X.) Therefore, by Corollary I.2 we have that $\pi \circ \psi: \Delta^0 \to D$ and $\pi \circ \psi | \partial\Delta$ is a homeomorphism of $\partial\Delta$ with $\partial D = \pi[\Gamma]$. We shall prove that the differential $d(\pi \circ \psi)$ is non-singular at every point of Δ^0, and thus, $\pi \circ \psi$ is a local diffeomorphism. This implies, by a monodromy argument, that $\pi \circ \psi$ is a homeomorphism of Δ with $\bar{D}$

(and a diffeomorphism in the interior). Consequently, the map ψ is non-singular and one-to-one. Furthermore, the surface can be expressed as the graph of a function $f: \mathbb{R}^2 \to \mathbb{R}^{n-2}$, i.e., as the set $\{(x,y,f_3(x,y),\dots,f_n(x,y)): (x,y) \in \bar{D}\}$. Taking the Euler-Lagrange equations of the area integral for f, we see that f must satisfy the equation:

$$(2.19)\qquad (1+|f_y|^2)f_{xx} - 2\langle f_x,f_y\rangle f_{xy}+(1+|f_x|^2)f_{yy} = 0 .$$

Uniqueness is then proven by applying the maximum principle (cf. Courant and Hilbert [1, p. 326 ff.]) to the coordinate functions of f.

To establish the claim made above, we shall prove a fact of independent interest. For each non-zero vector $v \in \mathbb{R}^n$ and each real number s, we define the hyperplane

$$H_{v,s} = \{x \in \mathbb{R}^n : \langle x,v\rangle = s\} .$$

We then say that the hyperplane $H_{v,s}$ has <u>k^{th}-order contact with the surface</u> $\psi: \Delta \to \mathbb{R}^n$ at $p \in \Delta^0$ if

1) $\langle \psi(p),v\rangle = s$

2) $\langle \frac{\partial^\ell \psi}{\partial x^i \partial y^j}(p),v\rangle = 0$; for $0 \leq i + j = \ell \leq k$.

Observe that, since ψ is almost conformal, either

$\frac{\partial\psi}{\partial x}(p)$ and $\frac{\partial\psi}{\partial y}(p)$ are linearly independent or they both vanish. If they vanish, then every hyperplane $H_{v,s}$ which contains $\psi(p)$ has 1^{st}-order contact with ψ at p. Otherwise, $H_{v,s}$ has 1^{st}-order contact with ψ at p if and only if it contains the tangent plane to ψ at p.

The fact we shall need to establish our claim is the following.

PROPOSITION 5 - *Let $\psi: \Delta \to \mathbb{R}^n$ be a solution to the Plateau problem for the curve Γ, and suppose that the hyperplane $H_{v,s}$ has k^{th}-order contact with ψ at $p \in \Delta^0$. Then, if $H_{v,s} \not\supset \Gamma$, the set $\Gamma \cap H_{v,s}$ has at least $2k+2$ components.*

Proof: Consider the harmonic function f on Δ given by $f = \langle \psi, v\rangle - s$. Let $p = (x_0, y_0)$ and set $\xi = x - x_0$, $\eta = y - y_0$. Since f is real analytic, it can be expressed in a neighborhood of p as

$$f(\xi,\eta) = \sum_{j=m}^{\infty} P_j(\xi,\eta)$$

where each P_j is a homogeneous polynomial of degree j, and where $m \geq k+1$ with $P_m \not\equiv 0$. From the fact that $\Delta f = 0$ it follows that $\Delta P_m = 0$. Thus, setting $\zeta = \xi + i\eta$, we have

$$P_m(\zeta) = a\mathrm{Re}\{\zeta^m\} + b\mathrm{Im}\{\zeta^m\}$$
$$= c\mathrm{Re}\{e^{i\theta}\zeta^m\}$$

for constants a, b, c, θ. It follows that any sufficiently small neighborhood of p is divided by the zero set $Z = \{q \in \Delta: f(q) = 0\}$, like a pie, into $2m$ regions, such that in any two adjacent regions f has opposite signs.

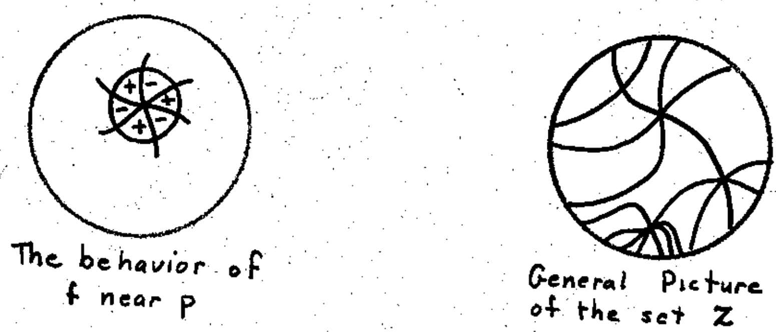


Figure 14

We assert that the set $C = \Delta \sim Z$ meets the boundary $\partial\Delta$ in at least $2m$ components. In particular each component C_α of C must have the property that $\partial C_\alpha \cap \partial\Delta$ contains an open interval of $\partial\Delta$. If this were not so, we would have $f|\partial C_\alpha = 0$, and by the maximum principle for harmonic functions f would vanish identically on the region C_α. It follows from analyticity that $f \equiv 0$, and so $H_{v,s} \supset \Gamma$ contrary to assumption.

It is now easy to see that each of the wedge-like regions of C in a neighborhood of p must belong to a distinct component of C. Hence, C has at least $2m$ components. This implies that $\partial C \cap \partial \Delta$ and, therefore, also $\partial \Delta \sim C = \partial \Delta \cap Z$ have at least $2m$ components. Since $\psi | \partial \Delta$ is monotone, the proposition follows immediately. □

To complete the proof of the theorem we proceed as follows. Let $p \in \Delta^0$ and suppose that the rank of $d(\pi \circ \psi)$ at p is $\leqq 1$. Then there exists a line $\ell \subset \mathbb{R}^2$ through the point $\pi \circ \psi(p)$ such that ℓ contains the vectors $(\pi \circ \psi)_x(p) = \pi(\psi_x(p))$ and $(\pi \circ \psi)_y(p) = \pi(\psi_y(p))$. We see, therefore, that the hyperplane $H = \pi^{-1}(\ell)$ has first order contact with ψ at p. However, $H \cap \Gamma$ = two points, since $\pi(\Gamma)$ is convex and $\pi | \Gamma$ is one-to-one. This contradicts Proposition 1, and we conclude that $d(\pi \circ \psi)$ has rank 2 throughout Δ^0. This completes the proof. □

There are some further consequences of Proposition 5 worth noting here.

COROLLARY 2 - *Let* Γ *be a Jordan curve in* $\mathbb{R}^n$ *and suppose that through every point* $p \in C(\Gamma)^0$ *there passes a hyperplane* H *such that* $H \cap \Gamma$ *has at* most 3 components. Then every solution to the Plateau problem for Γ (of the type of the disk) is free of branch points.

By Theorem 2 this statement is superfluous for solutions to the Plateau problem in R^3. However, for general minimal surfaces in R^n it is non-trivial.

COROLLARY 3 - Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve and suppose $\psi: \Delta \to \mathbb{R}^n$ is a solution to the Plateau problem for Γ. Suppose that Γ is not contained in any hyperplane of $\mathbb{R}^n$. Then through every point $\psi(p)$, for $p \in \Delta^0$, there passes a hyperplane H such that $H \cap \Gamma$ has at least $2[\frac{n+1}{2}]$ components.

Proof: Let $p \in \Delta^0$ and consider the vectors

$$(2.20) \qquad \frac{\partial^{i+j}\psi}{\partial x^i \, \partial y^j}(p);\ 1 \leq i + j \leq m .$$

Since $\nabla^2\psi \equiv 0$, all the k^{th} order derivative of ψ are identically equal to $\pm \partial^k\psi/\partial x^k$ or $\pm \partial^k\psi/\partial^{k-1}x\partial y$. Consequently, the linear space spanned by the vectors in (2.20) has dimension $\leq \min\{2m,n\}$. Thus, if $2m \leq n-1$, there exists a hyperplane H through $\psi(p)$ containing these vectors, i.e., having m^{th}-order contact with ψ at p. The result now follows from Proposition 5. □

CHAPTER III

COMPLETE MINIMAL SURFACES IN $\mathbb{R}^n$

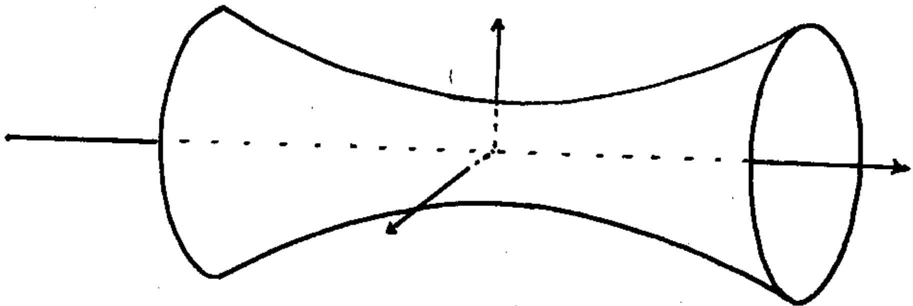
In the last chapter we examined minimal surfaces with boundary in $\mathbb{R}^n$ and obtained what might be considered a local existence theory for minimizing surfaces. It is natural, from a geometric viewpoint, to pass now to infinite minimal surfaces and try to study their global geometric behavior (in analogy, say, with studying the behavior of infinite geodesics). Or course, we already know that no minimal surface in $\mathbb{R}^n$ can be compact, so we shall restrict attention to those (non-compact) minimal surfaces which are complete in the induced metric.

§1. Some examples

That there exist a sufficient number of such surfaces to make the study interesting will soon be evident. However, as reference points, we mention here some important examples in $\mathbb{R}^3$.

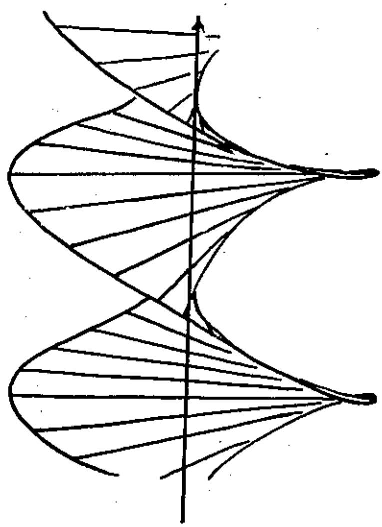
EXAMPLE 1 - The Plane, $\mathbb{R}^2 \subset \mathbb{R}^3$.

EXAMPLE 2 - The Catenoid. This is described by revolving the catenary $z = \cosh(x)$ about the x-axis in (x,y,z)-space.



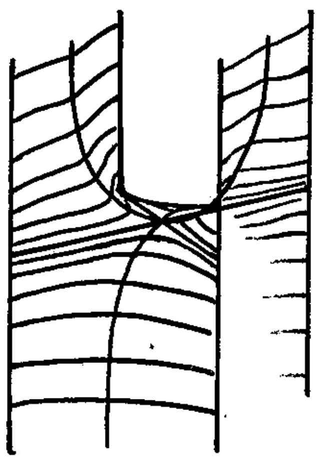
$$z^2 + y^2 = \cosh^2(x)$$

EXAMPLE 3 - The Helicoid. This is generated by revolving a line, perpendicular to the z-axis, about the z-axis, while moving the line in the z-direction, both at constant speed.



$$y \tan z = x$$

EXAMPLE 4 - <u>Scherk's surface</u>. This is a doubly periodic surface, invariant by the translations $(x,y,z) \mapsto (x + 2\pi, y, z)$ and $(x,y,z) \mapsto (x, y + 2\pi, z)$. The interior of a fundamental domain of the surface can be expressed as the graph of the function $z = \log(\cos x/\cos y)$ in the square: $|x| < \pi/2$ and $|x| < \pi/2$. This function goes to ∞ as $(x,y) \to (\pm\pi/2, y)$ for $|y| < \pi/2$ and goes to $-\infty$ as $(x,y) \to (x, \pm\pi/2)$ for $|x| < \pi/2$. The resulting surface assumes the four lines $|x| = |y| = \pi/2$ as boundary. The surface can now be continued indefinitely by reflection (cf. Proposition II.2).



$$\cos y \, e^z = \cos x$$

EXAMPLE 5 - <u>Enneper's surface</u>. This surface is given analytically by the equations:

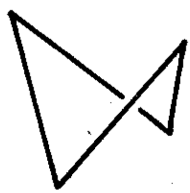
$$x = \mathrm{Re}\{w - \tfrac{1}{3} w^3\}$$

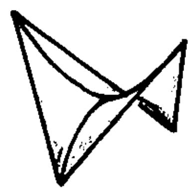
$$y = \mathrm{Re}\{i(w + \tfrac{1}{3} w^3)\}$$

$$z = \mathrm{Re}\{w^2\}$$

where w ranges over the complex plane $\mathbb{C}$.

EXAMPLE 6 - Schwarz surfaces. These surfaces can be described as follows. Let Γ be a 4-sided polygonal curve in $\mathbb{R}^3$ such that each vertex angle is of the form $\pi/(k_i+1)$, where $k_1,\ldots,k_4$ are positive integers. Let $\Sigma = \psi(\Delta)$ be the solution to the Plateau problem for Γ. By Theorem II.5 and Proposition II.2 this is a unique, embedded surface, regular at the boundary, and can be continued indefinitely as a regular minimal surface by reflection across the boundary edges. If $k_1 = \ldots = k_4 = 2$ and if





the edges are of equal length, the resulting surface is an embedded, triply-periodic surface. Its conjugate surface (See below.) is obtained similarly from symmetric polygon where $k_1 = k_3 = 1$, and $k_2 = k_4 = 2$.

This surface is also triply periodic and embedded.

NOTE. A.H. Schoen has discovered that "between" these two (i.e., as an associate surface of these two) there lies a third embedded surface, having a 3-dimensional skew lattice of symmetries and containing no straight lines. (See the picture in Osserman [5].)

§2. Non-parametric surfaces; the Bernstein Theorem

One of the simplest ways to express a surface in $\mathbb{R}^n$ is in non-parametric form, that is to say, as the graph

$$\mathcal{G}_f = \{(x,y,f_3(x,y),\dots,f_n(x,y)): (x,y) \in U\}$$

of a function $f: U \subset \mathbb{R}^2 \to \mathbb{R}^{n-2}$. Any regular surface can be locally expressed in this manner, by using the tangent planes for example. By computing the Euler-Lagrange equations for the area integral

$$A(f) = \iint_U \sqrt{(1+|f_x|^2)(1+|f_y|^2) - \langle f_x, f_y \rangle^2}\, dxdy$$

we see that the surface $\mathcal{G}_f$ is minimal if and only if

$$(3.1) \qquad (1+|f_y|^2)f_{xx} - 2\langle f_x, f_y\rangle f_{xy} + (1+|f_x|^2)f_{yy} = 0$$

Note that if $\mathcal{G}_f$ is minimal and U is convex, then, by Theorem II.5 and the maximum principle, $\mathcal{G}_f$ is the unique solution to the Plateau problem for $\partial\mathcal{G}_f$. Therefore, every <u>regular minimal surface in</u> $\mathbb{R}^n$ <u>is locally minimizing</u>.

Equation (3.1) shows that the study of non-parametric minimal surfaces is, in fact, a topic in quasi-linear partial differential equations. A great deal of interesting work has been done in this area (for surfaces of dimension 2 <u>and</u> higher), and we refer the reader to Osserman [5], [6] for a discussion and an extensive bibliography.

In keeping with our global point of view, we shall restrict our attention here to the following question: What can be said about a non-parametric surface $\mathcal{G}_f$ when f is defined over the entire (x,y)-plane?

In 1915, S. Bernstein proved the following for the case of surfaces in $\mathbb{R}^3$.

THEOREM 1 - (Bernstein [3]). *If* $f: \mathbb{R}^2 \to \mathbb{R}$ *is a function whose graph* $\mathcal{G}_f \subset \mathbb{R}^3$ *is a minimal surface, then* f *is linear*.

This result is a beautiful, non-trivial example of a global theorem in non-linear, partial differential equations, and much work has been devoted to trying to generalize it. Since we shall prove one such generalization later in this chapter, we omit a proof here. (For the interested reader, an elegant proof of this case can be found in Osserman [5].)

Observe that Theorem 1, as stated, does not hold in higher dimensions. In fact, if $\varphi: \mathbb{C} \to \mathbb{C}$ is any entire holomorphic function, then the graph

$$\{(z,\varphi(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\}$$

of φ in $\mathbb{C}^2 = \mathbb{R}^4$ is a minimal surface, as pointed out in Chapter I, §6. (A classification of all possibilities for the case $n = 4$ can be found in Osserman [5, pp. 40-42].) At first this may seem to say that no generalization for 2-dimensional surfaces in $\mathbb{R}^n$ exists. This is far from true, but to find the proper statement it is necessary to express Theorem 1 in a slightly different way, namely: If Σ is a complete minimal surface in $\mathbb{R}^3$ all of whose normals make an acute angle with some fixed direction, then Σ is a plane.

This last statement turns out to be a special case of a much more spectacular, and more geometric theorem. However, before discussing that theorem we need to establish certain basic concepts.

§3. General minimal surfaces; the Gauss map

Let $\psi: M \to \mathbb{R}^n$ be an immersion of a connected, orientable 2-manifold. (For cases where M is non-orientable we pass to the 2-sheeted, orientable covering surface.) By Theorem IC, Ch. II, we know that at each $p \in M$ there exist local coordinates (x,y) in which the metric induced by ψ has the form

$$(3.2) \qquad ds^2 = 2F|dz|^2$$

where $z = x + iy$. Clearly the coordinate transformation between any two such coordinate systems is either conformal or anticonformal. By orientability we may chose an atlas of such coordinates where the transformations are, in fact, conformal, thereby making M into a Riemann surface (or complex manifold of dimension 1). In view of this, we can (and will) assume that M is a Riemann sur-

<u>face and the map</u> $\psi: M \to \mathbb{R}^n$ <u>is conformal</u>.

Assume now that the immersion ψ is minimal. Let z be a local coordinate on M, where the induced metric is given by (3.2). Then the Laplace-Beltrami operator is expressed in these coordinates by

$$\Delta = \frac{2}{F}\frac{d}{dz}\frac{d}{d\bar{z}} \tag{3.3}$$

where $\frac{d}{dz} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Hence, since $\Delta\psi = 0$ (Corollary I.1), we have

$$\frac{d}{dz}\frac{d}{d\bar{z}}\psi = 0 . \tag{3.4}$$

We now consider the local $\mathbb{C}^n$-valued function $\varphi = (\varphi_1,\ldots,\varphi_n)$ defined by

$$\varphi = \frac{d\psi}{dz} . \tag{3.5}$$

From Equation (3.4) it is clear that each φ_k is holomorphic in z. Furthermore, since $4\varphi^2 = |\psi_x|^2 - |\psi_y|^2 - 2i\langle\psi_x,\psi_y\rangle$ and since the induced metric is given by $g_{xx} = |\psi_x|^2 = 2F$, $g_{yy} = |\psi_y|^2 = 2F$, $g_{xy} = \langle\psi_x,\psi_y\rangle = 0$, we have

$$\varphi^2 = \sum_{k=1}^{n} \varphi_k^2 = 0 \tag{3.6}$$

$$|\varphi|^2 = \sum_{k=1}^{n} |\varphi_k|^2 = F \tag{3.7}$$

Equation (3.6) is exactly the condition that ψ be conformal.

Observe now that if we make a change of (conformal) coordinates $w = w(z)$ on M, the new function $\tilde{\varphi}(w) = d\psi/dw$ satisfies

$$(3.8) \qquad \tilde{\varphi}(w(z)) = \varphi(z) \cdot \cancel{\frac{dw}{dz}} \, \frac{dz}{dw}$$

Thus,

$$(3.9) \qquad \alpha = (\alpha_1, \ldots, \alpha_n)$$

where $\alpha_k = \varphi_k dz$, is a set of n holomorphic differential forms, globally defined on M.

In particular, it follows from (3.8) that at each point $p \in M$, the vector φ gives a complex line in $\mathbb{C}^n$, defined independently of the choice of local coordinates. (Note that $\varphi \neq 0$ by (3.7).) From this and from Equation (3.6) we see that we get a well defined, holomorphic mapping

$$(3.10) \qquad \Phi: M \to Q_{n-2} \subset \mathbb{C}P^{n-1}$$

into complex propective space $\mathbb{C}P^{n-1}$, such that the image lies in the algebraic subvariety Q_{n-2}, which is given in homogeneous coordinates by the equation

$$z_1^2 + \ldots + z_n^2 = 0 \,.$$

In order to interpret this map geometrically, we first observe that the manifold Q_{n-2} is naturally diffeomorphic to the Grassmann manifold $G^o(2,n) = SO(n)/SO(2) \times SO(n)$ of oriented 2-planes in $\mathbb{R}^n$. To see this, let $\mathbb{R}^2 \subset \mathbb{R}^n$ be a 2-dimensional, linear subspace and choose an (oriented) basis $\{v_1, v_2\}$ for $\mathbb{R}^2$ such that $|v_1| = |v_2|$ and $\langle v_1, v_2\rangle = 0$. Then the associated complex n-vector $w = v_1 + iv_2$ satisfies the equation $w^2 = 0$. Furthermore, if $\{v_1', v_2'\}$ is a similar basis with the same orientation, then its corresponding complex vector $w' = \alpha \cdot w$ for some $\alpha \in \mathbb{C} \sim \{0\}$. Hence, to each $\mathbb{R}^2 \subset \mathbb{R}^n$ we have associated a unique point in manifold Q_{n-2}. This correspondence is bijective and easily seen to be a diffeomorphism if $G^o(2,n)$ is given the usual differentiable structure as a homogeneous space.

Note that if $\psi: M \to \mathbb{R}^n$ is a surface as above, and if $z = x + iy$ is a local coordinate at $p \in M$, then $\{\psi_x(p), \psi_y(p)\}$ form just such a basis for $\psi_* T_p(M) \subset \mathbb{R}^n$. Hence,

$$\bar{\varphi}(p) = \frac{d\psi}{dz}(p) = \psi_x(p) + i\psi_y(p)$$

is an exact expression of the above correspondence. The resulting map

$$\bar{\Phi}: M \to Q_{n-2}$$

is called the <u>Gauss map of the surface</u>.

Note that an immersion $\psi: M \to \mathbb{R}^n$ is minimal if and only if its Gauss map is antiholomorphic. Since holomorphic functions are more commonly considered than anti-holomorphic ones, we shall, for minimal surfaces, replace $\bar{\Phi}$ by the conjugate map Φ (of Equation (3.10)) and from here on refer to this second map as the Gauss map.

REMARK 1 - In the case $n = 3$, it is more common to define the Gauss map of a surface $\psi: M \to \mathbb{R}^3$ by assigning to each point $p \in M$ the unit normal vector

$$(3.11) \qquad N(p) = \frac{\psi_x \wedge \psi_y}{|\psi_x \wedge \psi_y|}(p) \in S^2 \subset \mathbb{R}^3 \quad .$$

The above map $\bar{\Phi}$ instead assigns to p the tangent plane

$$[\psi_x(p) + i\psi_y(p)] \in Q_1 \subset \mathbb{C}P^2$$

which is perpendicular to $N(p)$. However, one can easily check that the map $S^2 \to Q_1$ which assigns to each $e_3 \in S^2$ the plane $[e_1 + ie_2] \in Q_1$, such that $\{e_1, e_2, e_3\}$ is a positively oriented, orthonormal basis of $\mathbb{R}^3$, is a conformal diffeomorphism. Thus, in $\mathbb{R}^3$ we also have that a surface $\psi: M \to \mathbb{R}^3$ is minimal if and only if the (other) Gauss map $N: M \to S^2$ is anticonformal.

Observe now that if $\psi: M \to \mathbb{R}^n$ is a minimal surface whose Gauss map is expressed canonically by the differential forms $\alpha_1,\ldots,\alpha_n$ (cf. Equation (3.9)), then the original surface can be recaptured by setting

$$\psi_k(z) = 2\mathrm{Re}\{\int_0^z \alpha_k\}; \quad k = 1,\ldots,n \,. \tag{3.12}$$

In fact, from this point of view we have a method for generating all minimal immersions of M into $\mathbb{R}^n$. Let $\alpha_1,\ldots,\alpha_2$ be any n holomorphic differentials which satisfy the relations: $\Sigma\, \alpha_k^2 = 0$ (i.e., locally, $\alpha_k = \varphi_k(z)dz$ and $\Sigma\, \varphi_k^2 = 0$) and $\Sigma|\alpha_n|^2 > 0$, (and which ? have no real periods on M.) Then the mapping $\psi = (\psi_1,\ldots,\psi_n)$ whose coordinate functions are given by (3.12) will be a minimal immersion. In this way, it is easy to explicitly write down a great number of minimal surfaces in R^n.

For the case $n = 3$, in particular, it is possible to give a simple description of all solutions to the equation $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$. Assume that we do not have the case $\alpha_1 \equiv i\alpha_2$ and $\alpha_3 \equiv 0$. (Here $\psi(M)$ is a plane. This case is easily avoided by a rotation of coordinates in $\mathbb{R}^3$.) We than define a holomorphic differential w and a meromorphic function g on M by

$$\begin{cases} w = \alpha_1 - i\alpha_2 \\ g = \dfrac{\alpha_3}{\alpha_1 - i\alpha_2} \end{cases} \,. \tag{3.13}$$

Locally, if $\alpha_k = \varphi_k\,dz$, then $w = fdz$ where f is a holomorphic function and

(3.13)'
$$\begin{cases} f = \varphi_1 - i\varphi_2 \\ g = \dfrac{\varphi_3}{\varphi_1 - i\varphi_2} \end{cases}$$

We then have that

(3.14)
$$\begin{cases} \alpha_1 = \frac{1}{2}\,(1-g^2)w \\ \alpha_2 = \frac{i}{2}\,(1+g^2)w \\ \alpha_3 = gw \end{cases}$$

Conversely, let $g\colon M \to \mathbb{C}$ be a meromorphic function and let w be a holomorphic differential on M such that whenever g has a pole of order m at a point $p \in M$, w has a zero at p of order $2m$. Then, if the α_k''s defined by (3.14) have no real periods, Equation (3.12) defines a minimal immersion $\psi\colon M \to \mathbb{R}^3$.

The equations (3.14) are called the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$. This representation makes it easy to write down an enormous number of complete minimal surfaces in $\mathbb{R}^3$. For example, if we set $M = \mathbb{C}$, $w = dz$ and $g(z) = z$, we get Enneper's surface. If we set $M = \mathbb{C} \sim \{0\}$, $w = (1/z^2)dz$ and $g(z) = z$ we get the catenoid.

It will be convenient, for later use, to make some geometric observations about the Weierstrass representation. Choose a local coordinate z on M, and write $\alpha_k = \varphi_k dz$ and $w = f dz$ as above. Then the metric on M has the form $ds^2 = 2F|dz|^2$ where by (3.7) and (3.14)

$$F = (1 + |g|^2)^2 |f|^2 \tag{3.15}$$

Observe that the function g can be thought of as a conformal map $g: M \to \mathbb{C} \cup \{\infty\} = S^2$. In this sense, g <u>is exactly the Gauss map of the surface in the sense of Remark 1</u>. In particular, let $N: M \to S^2 \subset \mathbb{R}^3$ be the map defined by (3.11) and suppose $\pi: S^2 \sim \{(0,0,1)\} \to \mathbb{R}^2$ is stereographic projection into the (x,y)-plane. Then

$$g = \pi \circ N . \tag{3.16}$$

To see this, we note that $\partial\psi/\partial x - i\partial\psi/\partial y = 2(\varphi_1,\varphi_2,\varphi_3)$, and therefore,

$$\frac{\partial\psi}{\partial x} \times \frac{\partial\psi}{\partial y} = 4\mathrm{Im}\,\{(\varphi_2\bar\varphi_3,\ \varphi_3\bar\varphi_1,\ \varphi_1\bar\varphi_2)\} =$$

$$= (1+|g|^2)|f|^2\ (2\mathrm{Re}(g), 2\mathrm{Im}(g), |g|^2-1).$$

Hence,

$$N = \frac{1}{1+|g|^2}\,(2\mathrm{Re}(g),\ 2\mathrm{Im}(g),\ |g|^2 - 1)$$

and

$$\pi \circ N = (\mathrm{Re}(g),\ \mathrm{Im}(g)) .$$

(The fact that g is conformal, and not anticonformal, is due to the fact that the projection π is here anti-conformal.)

Equation (3.16) means that <u>the poles of</u> g <u>occur exactly at those points</u> $p \in M$ <u>where</u> $N(p) = (0,0,1)$. Thus, if the Gauss map N omits at least one point of S^2 we may, by making a rotation of coordinates, assume that g has no poles on M (and, therefore, w also has no zeroes). *

Let us return now to the general case of a minimal surface $\psi: M \to \mathbb{R}^n$ in $\mathbb{R}^n$. Recall that if in a local coordinate z on M the metric is expressed as $ds^2 = 2F|dz|^2$, the Gauss curvature K of the surface is given by

(3.17) $$K = -\frac{1}{F}\frac{d}{dz}\frac{d}{d\bar{z}}\log F .$$

We then have from (3.7) that, in terms of the functions $\varphi = \partial\psi/\partial z$, K can be expressed as

(3.18) $$K = -\frac{|\varphi \wedge \varphi'|^2}{|\varphi|^6}$$

where $|\varphi \wedge \varphi'|^2 = |\varphi|^2|\varphi'|^2 - |\langle\varphi,\varphi'\rangle|^2 = \sum_{i<j} |\varphi_i\varphi_j' - \varphi_j\varphi_i'|^2$.

We introduce on $\mathbb{C}P^{n-1}$ the Fubini-Study metric

(3.19) $$ds^2 = 2\frac{|Z \wedge dZ|^2}{|Z|^4}$$

(cf. Chapter I, §5). We have renormalized the metric here (a factor of 2 instead of 4) so that the induced metric on the quadric $\mathbb{Q}_1$ is of constant curvature 1. The equivalence, $S^2 \approx \mathbb{Q}_1$, of Remark 1 is now an isometry. Each of the linear subspaces $\mathbb{C}P^1 \subset \mathbb{C}P^{n-1}$ now has curvature 2, and volume 2π.

It is easy to see that the metric $d\sigma^2$ induced on M by the Gauss map $\Phi: M \to \mathbb{Q}_{n-2} \subset \mathbb{C}P^{n-1}$ has the form

$$d\sigma^2 = 2G|dz|^2 \tag{3.20}$$

where

$$G = \frac{|\varphi \wedge \varphi'|^2}{|\varphi|^4} .$$

Hence, as a generalization of the classical case in $\mathbb{R}^3$, we have

$$K = -\frac{d\sigma^2}{ds^2} .$$

Letting $C(\psi)$ denote the total curvature of M and $A(\Phi)$ the area induced by the Gauss map, we see that therefore

$$C(\psi) = -A(\Phi) . \tag{3.21}$$

It follows from Equation (3.18) that $K \leq 0$ and that either $K \equiv 0$ or $K = 0$ at isolated points. We observe that the case $K \equiv 0$ is quite special.

LEMMA 1 - <u>Every flat minimal surface in</u> $\mathbb{R}^n$ <u>is a plane</u>.

<u>Proof</u>: If $K \equiv 0$, then by (3.18), $\varphi \wedge \varphi' \equiv 0$. Therefore, $\varphi'(z) = h(z)\varphi(z)$ where h is a holomorphic function. It follows that $\varphi(z) = e^{\int^z h(z)dz} \cdot C = \beta(z) \cdot C$ where $C = (C_1, \ldots, C_n)$ is a constant vector in $\mathbb{C}^n$. The lemma now follows from Equation (3.12). □

NOTE. Lemma 1 can also be proved by observing that from the Gauss curvature equation, the surface must be totally geodesic. This second argument also shows that every flat k-dimensional minimal submanifold of $\mathbb{R}^n$ is a k-plane.

§4. <u>Conjugate minimal surfaces</u>

Let $\psi: M \to \mathbb{R}^n$ be a minimal surface where M is simply-connected. (If M is not simply-connected, we pass to the universal covering surface of M.) By the Koebe Uniformization Theorem (cf. Springer [1]), M is conformally equivalent either to the complex plane $\mathbb{C}$ or the unit disk $\Delta^o = \{z \in \mathbb{C} : |z| < 1\}$. Hence, M has a global parameter z in which the Gauss map can be expressed by

$$\varphi = (\varphi_1, \ldots, \varphi_n) = \frac{d\psi}{dz} .$$

The original immersion can be written as

$$\psi = 2\mathrm{Re}\{ \int^z \varphi dz\} .$$

For each θ, $0 \leq \theta < \pi$, we define the mapping $\psi_\theta : M \to \mathbb{R}^n$ by

$$(3.22) \qquad \psi_\theta = 2\mathrm{Re}\{ e^{i\theta} \int^z \varphi dz\} .$$

Since $\varphi_\theta = d\psi_\theta/dz = e^{i\theta}\varphi$, we have that $\varphi_\theta^2 = 0$ and $|\varphi_\theta|^2 = |\varphi|^2$. It follows that <u>each of the maps</u> ψ_θ <u>is a minimal immersion isometric to</u> ψ. They are called the <u>associate minimal surfaces</u> to ψ. The particular immersion $\psi_{\pi/2}$ is called the <u>conjugate minimal surface</u>, since its coordinate functions are the harmonic conjugates of those of ψ.

While the associate surfaces are all isometric, they are usually not congruent. The classic example of two conjugate surfaces in $\mathbb{R}^3$ are the catenoid and the helicoid. A picture of the family of associate surfaces joining them can be found in Struik [1].

§5. The generalized Bernstein theorem according to Osserman

One way to phrase the classical Bernstein Theorem given in §2 is: Any complete minimal surface in $\mathbb{R}^3$, all of whose normals are contained in a fixed, open hemisphere of S^2, is a plane. It was conjectured by Nirenberg that this statement might generalize to the following: The normals to a complete minimal surface in $\mathbb{R}^3$, which is not a plane, are dense in S^2. A proof of this was given by R. Osserman in 1959. The theorem was then generalized by Osserman, and also by S.S. Chern, to surfaces in higher dimensional euclidean space as follows.

THEOREM 2 - *If the normals to a complete minimal surface in $\mathbb{R}^n$ omit a neighborhood of some direction, then the surface is a plane.*

Before proceeding to a proof we shall interpret this statement in terms of the Gauss map defined in §3. Let $\psi: M \to \mathbb{R}^n$ be a minimal surface, and let $v = (v_1, \ldots, v_n)$ be any non-zero vector in $\mathbb{R}^n$. Then for any $p \in M$, v is normal to the surface at $\psi(p)$ if and only if $\langle v, \psi_x(p)\rangle = \langle v, \psi_y(p)\rangle = 0$, that is, if and only if

(3.23) $$\langle v, \varphi(p)\rangle = 0 ,$$

where $\varphi = \partial\psi/\partial z$ is the Gauss map. Conversely, the normals to M omit a neighborhood of v if and only if there exists an $\varepsilon > 0$ such that at every point

$$\frac{\langle v,\psi_x\rangle^2}{|v|^2|\psi_x|^2} \geqq \varepsilon^2 \quad \text{and} \quad \frac{\langle v,\psi_y\rangle^2}{|v|^2|\psi_y|^2} \geqq \varepsilon^2 ,$$

that is,

(3.24) $$\frac{|\langle v,\varphi\rangle|^2}{|v|^2|\varphi|^2} \geqq \varepsilon^2 .$$

(Note that (3.23) and (3.24) are independent of the length of v and of the local coordinates chosen on M.)

Equation (3.23) can be interpreted by saying that the image of p under the Gauss map $\Phi: M \to Q_{n-2} \subset \mathbb{C}P^{n-1}$ lies in the hyperplane (linear subspace of codimension-one) of $\mathbb{C}P^{n-1}$, determined by the equation

$$v_1 z_1 + \dots + v_n z_n = 0$$

in homogeneous coordinates. Similarly, Equation (3.24) says that the Gaussian image omits a neighborhood of this hyperplane.

We shall therefore rephrase our theorem as follows.

THEOREM 2' - <u>If</u> $\psi: M \to \mathbb{R}^n$ <u>is a complete minimal surface</u>, which is not a plane, then the Gauss image $\Phi(M)$ of M in $\mathbb{C}P^{n-1}$ meets a dense set of hyperplanes.

NOTE. This is slightly stronger than the first statement. It says that if $\psi(M)$ is not a plane, then $\Phi(M)$ comes arbitrarily close to any hyperplane $\Sigma\, v_k z_k = 0$, not just those where $v_1, \ldots, v_n$ are real.

Proof: Let $v = (v_1, \ldots, v_n)$ be a unit vector in $\mathbb{C}^n$, and assume that the Gauss map satisfies Equation (3.24). We shall show that the Gauss map must be constant.

We begin by lifting ψ to the universal, conformal covering surface $\tilde{M}$ of M. By the Koebe Uniformization Theorem, $\tilde{M}$ is conformally equivalent to either the sphere S^2, the open disk $\Delta^o = \{z \in \mathbb{C} : |z| < 1\}$, or the plane $\mathbb{C}$. The first case is ruled out by Corollary I.1. (Every harmonic function on S^2 is constant.) The second case is eliminated by the following

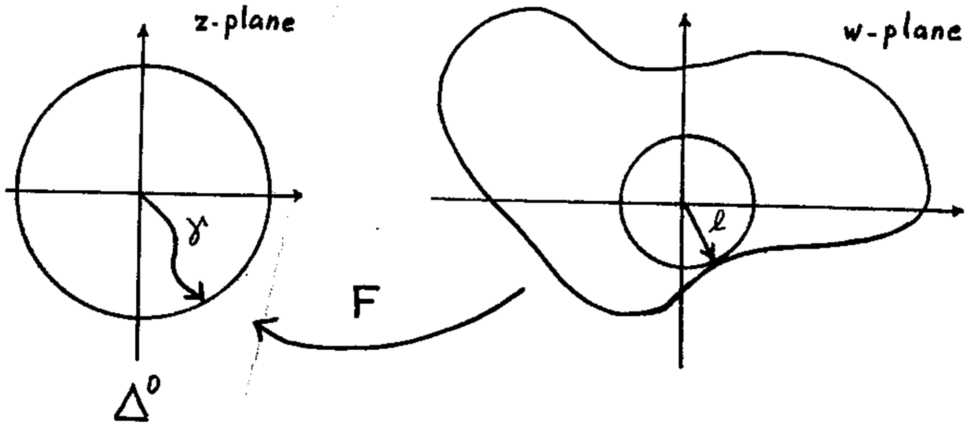
LEMMA 2 - (Osserman [4]). Let $\psi: \Delta^o \to \mathbb{R}^n$ be a minimal surface whose Gauss map $\varphi = d\psi/dz$ satisfies (3.24) for some vector $v \in \mathbb{C}^n \sim \{0\}$. Then this surface is not complete.

To prove the lemma we will show that the distance from 0 to $\partial\Delta$, in the metric induced by ψ on Δ^o, is finite.

Recall that this metric has the form $ds^2 = 2|\varphi|^2|dz|^2$ where by (3.24), $|\varphi|^2 \leqq (1/\varepsilon^2)|\langle v,\varphi\rangle|^2$. Hence, if γ is any smooth curve in Δ, beginning at 0 and going to the boundary, then

$$(3.25) \qquad \text{length}(\gamma) = \int_\gamma ds = \int_\gamma \sqrt{2}|\varphi||dz| \leqq \frac{\sqrt{2}}{\varepsilon}\int_\gamma |\langle v,\varphi\rangle||dz|$$

We now consider the holomorphic function $w(z) = \int_0^z \langle v,\varphi(z)\rangle dz$. Since $\langle v,\varphi\rangle$ is never zero in Δ^o (by (3.24)), this function has an inverse $z = F(w)$ defined in a disk about the point $w = 0$. Let R be the radius of the largest such disk. Clearly, $R < \infty$, since otherwise F would be a bounded entire function and, therefore, constant by Liouville's Theorem. Hence, there exists a point w_o with $|w_o| = R$ such that F cannot be extended to a neighborhood of w_o. Let $\ell = \{tw_o : 0 \leq t < 1\}$ and set $\gamma = F(\ell)$. Then γ is an analytic curve in Δ^o which begins at 0 and goes to the boundary.



Moreover, we have that

$$\int_{\gamma} |\langle v,\varphi\rangle|\,|dz| = \int_{\ell} |dw| = R .$$

It follows from (3.25) that γ has finite length and, therefore, the metric in Δ^o is not complete.

Note that if the metric on M is complete, then so is the one on the universal covering space $\tilde{M}$. We may conclude from the lemma, therefore, that M is conformally equivalent to $\mathbb{C}$.

Consider now the holomorphic mapping $\varphi = (\varphi_1,\ldots,\varphi_n)\colon \mathbb{C} \to \mathbb{C}^n$ (the lift of $\varphi = \partial\psi/\partial z$ to $\tilde{M}$). For each k, the entire function $\varphi_k/\langle v,\varphi\rangle$ satisfies the inequality

$$\left|\frac{\varphi_k}{\langle v,\varphi\rangle}\right|^2 \leqq \frac{|\varphi|^2}{|\langle v,\varphi\rangle|^2} \leqq \frac{1}{\varepsilon^2}$$

by (3.24), and is therefore identically equal to a constant c_k. It follows that $\varphi = \langle v,\varphi\rangle(c_1,\ldots,c_n)$, and so $\varphi \wedge \varphi' \equiv 0$. By (3.18) the Gauss curvature is everywhere zero, and thus by Lemma 1, the surface is a plane. □

The basic idea of the above proof is that if M is a complete minimal surface in $\mathbb{R}^n$ whose normals are bounded away from some direction, then M is conformally large, i.e., $M \cong \mathbb{C}$. One can now apply the Liouville Theorem to the Gauss map and conclude that M is flat.

A second approach to understanding this theorem can be found in the following, more delicate result due to Osserman [4] (and stated here without proof.)

THEOREN 3 - _Let_ $\psi: M \to \mathbb{R}^n$ _be any minimal surface all of whose normals make an angle of at least_ α _with some fixed direction. Let_ d _be the distance from a fixed point_ $p \in M$ _to the boundary of_ M, _in the induced metric. Then the Gauss curvature_ K_p _at_ p _satisfies_

$$|K_p| \leq \frac{1}{d^2} \frac{16(n-1)}{\sin^4(\alpha)} .$$

Let us restrict our attention now to minimal surfaces in $\mathbb{R}^3$. Note that if $\psi: M \to \mathbb{R}^3$ is minimal, then its Gauss map $\Phi: M \to Q_1 \cong S^2$ is an open mapping (since it is holomorphic). Consequently, when the surface is complete, the set $S^2 \sim \Phi(M)$ is closed and nowhere dense. One might conjecture that this set is, in fact, finite. Whether this conjecture is true or not is still unknown. However, as we shall now see, it is very close to being true.

DEFINITION - A Riemann surface M is called <u>hyperbolic</u> if there exists a non-constant, negative subharmonic function on M. A closed set $C \subset S^2$ is then said to have <u>positive logarithmic capacity</u> if $S^2 \sim C$ is hyperbolic, and <u>zero logarithmic capacity</u> otherwise. Finite subsets of S^2 have zero logarithmic capacity, but the converse is not true.

A full discussion of this subject can be found in Ahlfors and Sario [1]. For our purposes, however, we will need only the following characterization.

LEMMA 3 - *Let* D *be a domain in the complex plane* $\mathbb{C} = S^2 \sim \{\infty\}$. *Then* $S^2 \sim D$ *has zero logarithmic capacity if and only if the function* $\log(1+|z|^2)$ *has no harmonic majorant in* D.

Proof: If there exists a harmonic function h in D such that $\log(1+|z|^2) < h$, then $-h$ is a negative harmonic function on D. If h is constant, then D is bounded and the function $\bar{h} = \mathrm{Re}\{z-R\}$, for R sufficiently large, is non-constant, negative and harmonic on D. It follows that D is hyperbolic and $S^2 \sim D$ has positive logarithmic capacity.

Suppose, on the other hand, that D is hyperbolic. Then there exists (see Ahlfors and Sario [1, IV 6 and IV 22].) for each $\zeta \in D$ a Green's function with a pole at ζ, that is, a positive function $G_\zeta(z)$ with

$$G_\zeta(z) + \log|z-\zeta| = h(z)$$

where h is harmonic in D. Since G_ζ is positive, we have

$$\log|z-\zeta| < h(z) .$$

Observe also that the function $\log((1+|z|^2)/|z-\zeta|^2)$ is continuous on $S^2 \sim D$ and thus assumes a maximum $M < \infty$ on this set. It follows that for any $z' \in \partial D$ we have

$$\overline{\lim_{z \to z'}} \ [\log(1+|z|^2) - 2h(z)]$$

$$\leq \overline{\lim_{z \to z'}} [\log(1+|z|^2) - 2 \log|z-\zeta|] \leq M.$$

Applying the maximum principle to the subharmonic function $\log(1+|z|^2) - 2h(z)$ then shows that $\log(1+|z|^2) \leq$ $\leq 2h(z) + M$. □

We are now in a position to prove

THEOREM 4 - (Osserman [2]). <u>A complete minimal surface in</u>

$\mathbb{R}^3$ is either a plane or its normals assume every direction in S^2 with the possible exception of a set of logarithmic capacity zero.

Proof: Let $\psi: M \to \mathbb{R}^3$ be a complete minimal surface where, by passing to the universal covering surface, we assume that M is simply connected. Since M is conformally equivalent to either Δ^o or $\mathbb{C}$, we have a global conformal parameter z. Let $f(z)$ and $g(z)$ be the functions given in the Weierstrass representation (3.13)'.

Recall that the meromorphic function g is just the stereographic projection of the Gauss normal map, from the point $(0,0,1)$, into the (x,y)-plane. (See Equation (3.16) and the attending discussion.) Assume that the Gauss map omits at least one point $p \in S^2$. Then by a rotation of coordinates we may assume that $p = (0,0,1)$ and, therefore, that g is holomorphic in M. Set $D = g(M) \subset \mathbb{C}$ and note that since the surface is not a plane, g is non-constant and D is open. We want to show that D is not hyperbolic.

If $M \cong \mathbb{C}$, then by the Picard Theorem, $\mathbb{C} \sim D$ contains at most one point, and the theorem holds. If $M \cong \Delta^o$, we proceed as follows. Suppose D is hyperbolic. Then by Lemma 3 there exists a harmonic function h with $\log(1+|z|^2) < h(z)$ in D. Since g is holomorphic, $h \circ g$ is harmonic and equals the real part of a holomorphic function H in Δ^o. The function $G = f e^H$ is then holomorphic and non-zero in Δ^o. (Recall that f is zero only at the poles of g.) By considering the mapping $w(z) = \int_0^z G(z)dz$ it now follows, exactly as in the proof of Lemma 2, that there exists smooth curve γ in Δ^o, beginning at 0 and going to the boundary, such that

$$\int_\gamma |G(z)||dz| < \infty .$$

However, from Equation (3.15) we see that in the metric on Δ^o

$$\text{length}(\gamma) = \int_\gamma \sqrt{2}|f|(1+|g|^2)|dz| \leqq \int_\gamma \sqrt{2}|G(z)||dz| < \infty .$$

This contradicts the completenss of the surface and proves the theorem. □

It is known that for any set of k points in S^2, where $1 \leqq k \leqq 4$, there exists a complete minimal surface in $\mathbb{R}^3$ whose Gauss map omits exactly this set of points. (See Osserman [5, pg. 72].) As examples, note that the Gauss map of Enneper's surface omits one point, that of the catenoid, two points, and that of Scherk's surface, four points $((\pm1,0,0)$ and $(0,\pm1,0))$. It is still unknown whether there exists a complete minimal surface in $\mathbb{R}^3$ whose Gauss map omits a larger set.

Theorem 4 has been generalized to minimal surfaces in $\mathbb{R}^n$ by chern and Osserman [1]. We refer the reader to their paper for a statement and proof.

§6. Complete minimal surfaces of finite total curvature

Since the Gauss curvature of a minimal surface $\psi: M \to \mathbb{R}^n$ is always non-positive, it makes sense to speak of its total curvature

$$C(\psi) = \int_M \int KdA .$$

The complete minimal surfaces for which this number is finite are the simplest ones geometrically, and this section will be devoted to their study.

As noted in §3, saying that a minimal surface has finite total curvature is equivalent to saying that its Gauss image has finite volume (counting multiplicities). However, when the surface is complete even more can be said, and for this we need the following notion.

DEFINITION - Let $\psi: M \to \mathbb{R}^n$ be a minimal surface. Then the Gauss map $\Phi: M \to Q_{n-2}$ is called algebraic if

(a) M is conformally equivalent to a compact Riemann surface M' punctured at a finite number of points $p_1, \ldots, p_r$.

(b) The map Φ extends to a holomorphic mapping $\Phi': M' \to Q_{n-2}$.

LEMMA 4 - Let $\psi: M \to \mathbb{R}^n$ be a minimal surface whose Gauss map is algebraic. Then:

(1) $C(\psi) = -2\pi N$ for some integer $N \geq 1$.

(2) The Gauss map Φ intersects at most N-times enery hyperplane of $\mathbb{C}P^{n-1}$ which does not contain it.

Proof: Let ω be the Kähler form of $\mathbb{C}P^{n-1}$ (with the metric (3.19)). Let M' and Φ' be as given in the definition above. Then from Proposition I.4 and the fact that $d\omega = 0$, it follows that the area $A(\Phi')$ of M' in the metric induced by Φ' satisfies

$$A(\Phi') = \iint_{M'} \Phi'^{*}\omega = \iint_{\mathbb{C}P^1} \omega = N \cdot \text{area}(\mathbb{C}P^1) =$$

$$= 2\pi N$$

where N is the homology degree of $\Phi': M' \to \mathbb{C}P^{n-1}$. Part (a) now follows from equation (3.21). (Clearly, $A(\Phi) = A(\Phi')$.)

Part (b) can now be deduced using the intersection theory of Algebraic Topology and the fact that $\Phi'(M')$ meets any hyperplane with positive intersection number because it is complex analytic. We shall give, however, a more elementary, analytic proof.

We consider $M = M' \sim \{p_1,\ldots,p_r\}$ and express the Gauss map Φ, canonically, in terms of the differentials $\alpha_1,\ldots,\alpha_n$ (cf. (3.9)). The fact that Φ is algebraic means that $\alpha_1,\ldots,\alpha_n$ extend to meromorphic differentials on M'. If one of the differentials α_k has a pole at some point $p \in M'$, we can regularize this representation of Φ' near p by dividing by the highest order pole of the α's at p. Note that this does not affect the map into $\mathbb{C}P^{n-1}$. Similarly, if the α's have a common zero of some order at p, we can divide out by this zero. Thus, the extended α's give a well defined mapping into $\mathbb{C}P^n$ and by the above procedures we can always give locally a <u>regular geometric representation</u> of the map in homogeneous coordinates.

Let $v = (v_1,\ldots,v_n) \in \mathbb{C}^n \sim \{0\}$ and consider the differential

$$\alpha = v_1\alpha_1 + \cdots + v_n\alpha_n \quad .$$

Note that the hyperplane $\mathbb{C}P_v^{n-2}$, determined by the equation $\Sigma\, v_k z_k = 0$, contains $\Phi'(M')$ if and only if

$\alpha \equiv 0$. Suppose $\alpha \not\equiv 0$. Then Φ' has an intersection or order k with $\mathbb{C}P_v^{n-2}$ at $p \in M'$ if and only if the differential α, after possibly being altered to make the representation at p regular geometric, has a zero of order k at p. The sum of all intersection orders over M' is the total order of intersection $N(\Phi', v)$ of Φ' with $\mathbb{C}P_v^{n-2}$.

Suppose that α and α' are two such non-zero differentials corresponding to v and v'. Then α/α' is a meromorphic function on M' and thus has the same number of zeros as poles (cf. Springer [1]). Since we have taken a quotient, the regularization procedures needed to compute the intersection numbers cancel, and it is not difficult to see from here that $N(\Phi', v) = N(\Phi', v')$. Consequently, for all v such that $\mathbb{C}P_v^{n-2} \not\supset \Phi'(M')$, we have $N(\Phi', v) = \text{constant} = \bar{N}$.

The fact that $\bar{N} = N$ follows from the integral geometric formula (See Santalò [1].)

$$A(\Phi') = 2\pi \int_{\mathbb{C}P^{n-1}} N(\Phi', v)\,dv \tag{3.26}$$

where dv is the volume element of the Fubini-Study metric normalized so that

$$\int_{\mathbb{C}P^{n-1}} dv = 1 . \qquad \square$$

We can now make an interpretation of finite total curvature in terms of the Gauss map. This theorem (also Theorem 6) was first proved by Osserman [4] in $\mathbb{R}^3$ and extended by Chern and Osserman [1] to $\mathbb{R}^n$.

THEOREM 5 - <u>Let</u> $\psi: M \to \mathbb{R}^n$ <u>be a complete minimal surface. Then the total curvature</u> $C(\psi)$ <u>is finite if and only if the Gauss map is algebraic.</u>

We shall preface the proof of Theorem 5 with a sequence of preparatory lemmas.

LEMMA 5 - <u>Let</u> D <u>be a domain in the complex plane</u> $\mathbb{C}$, <u>and let</u> $ds^2 = \lambda^2|dz|^2$ <u>be a complete riemannian metric on</u> D. <u>If there exists a harmonic function</u> h <u>in</u> D <u>such that</u>

$$\log \lambda \leqq h ,$$

<u>then either</u> $D = \mathbb{C}$ <u>or</u> $D = \mathbb{C} \sim \{p\}$ <u>for some</u> $p \in \mathbb{C}$.

<u>Proof</u>: Consider the metric $d\tilde{s}^2 = \tilde{\lambda}^2|dz|^2$ in D where $\tilde{\lambda} = e^h$. Then $d\tilde{s}^2$ is also complete since for any path γ in D,

$$\int_\gamma \tilde{\lambda}\,|dz| \geqq \int_\gamma \lambda\,|dz| .$$

Let $\tilde{D}$ be the universal covering space of D. Then the lift of h to $\tilde{D}$ can be written as the real part of a holomorphic function H. Consider the function

$$w(z) = \int_{z_o}^{z} e^{H(\zeta)} d\zeta$$

on $\tilde{D}$. Since $|dw/dz| = |e^H| = e^h = \tilde{\lambda}$, the mapping $w\colon \tilde{D} \to \mathbb{C}$ has an inverse defined in a maximum disk $\Delta(R) = \{w\colon |w| < R\}$ about $w = 0$. It follows from the completeness of $d\tilde{s}^2$ (as in the proof of Lemma 2) that $R = \infty$. Hence, $w(z)$ is onto and one-to-one, i.e. $\tilde{D} \cong \mathbb{C}$. The lemma now follows by applying the Picard Theorem to the (holomorphic) covering map $\pi\colon \tilde{D} \to D \subset \mathbb{C}$. □

LEMMA 6 - *Let* D *be an annular domain* $0 < r_1 < |z| < r_2 \leq \infty$, *and let* $ds^2 = \lambda^2 |dz|^2$ *be a metric on* D *such that:*

(a) $\log \lambda \leq h$ *for some harmonic function* h *on* D,

(b) *each path* $z(t)$; $0 \leq t < 1$ *in* D *with* $\lim_{t\to 1} |z(t)| = r_2$ *has infinite length.*

Then, $r_2 = \infty$.

Proof: Suppose $r_2 < \infty$. Then by a conformal map of type $z \to c\cdot z$, some $c \in \mathbb{R}$, we may assume $r_1 < 1/r_2 < 1 < r_2$. Set $D' = \{z\colon 1/r_2 < |z| < r_2\}$ and note that the metric $\mu^2 |dz|^2$, where $\mu(z) = \lambda(z)\lambda(1/z)$, is complete in D'. Furthermore, $\log \mu(z) \leq h(z) + h(1/z)$.

Hence, by Lemma 5, either $D = \mathbb{C}$ or $D = \mathbb{C} \sim \{0\}$, contradicting the fact, $r_2 < \infty$. □

LEMMA 7 - Let $D \subset \mathbb{C}$ be a hyperbolic domain, and let $ds^2 = \lambda^2 |dz|^2$ be a complete metric in D whose Gauss curvature K satisfies

(a) $K \leqq 0$

(b) $\iint_D |K| dA < \infty$

Then there exists a harmonic function h on D such that $\log \lambda \leqq h$.

Proof: Observe that assumptions (a) and (b) can be rewritten as

(a') $\nabla^2 \log \lambda \geqq 0$

(b') $\iint_D \nabla^2 \log \lambda \quad dxdy < \infty$.

Since D is hyperbolic, there exists for each $\zeta \in D$ a Green's function $G_\zeta > 0$ in $D \sim \{\zeta\}$ such that

$$G_\zeta(z) + \log |z-\zeta| = H_\zeta(z)$$

where H_ζ is harmonic throughout D. (See Ahlfors and Sario [1, IV 6 and IV 22].) Set

$$U(\zeta) = \frac{1}{2\pi} \iint_D G_\zeta(z) \nabla^2 \log \lambda(z) dxdy$$

which exists and is $\geqq 0$ by (a') and (b'). By Poisson's formula we have $\nabla^2 U = -\nabla^2 \log \lambda$, and therefore $h = U + \log \lambda$ is harmonic in D. Since $U \geqq 0$, $h \geqq \log \lambda$. □

We now have the following

PROPOSITION 1 - Let M be a complete riemannian 2-manifold whose Gauss curvature K satisfies

(a) $K \leqq 0$

(b) $\int\int_M |K| dA < \infty$.

Then M is conformally equivalent to a compact Riemann surface punctured at a finite number of points.

Proof: By a theorem of Huber [1], condition (b) implies that M is finitely connected. Hence there exists a compact region $M_o \subset M$, bounded by a finite number of regular Jordan curves $\gamma_1, \ldots, \gamma_r$, such that each component M_j of $M \sim M_o$ can be conformally mapped onto the annulus $D_j = \{z \in \mathbb{C}: 1 < |z| < r_j\}$ where γ_j corresponds to $|z| = 1$. (See Ahlfors and Sario [1, I 44 D and II 3 B].) The region D_j is hyperbolic since $\mathrm{Re}\{1 - \frac{1}{z}\} < 0$. Furthermore, the metric on D_j induced from M_j, satisfies the conditions of Lemma 7, and therefore also the conditions of Lemma 6. It follows that each $r_j = \infty$. Let $\bar{D}_j = D_j \cup \{\infty\}$ $(\subset S^2 = \mathbb{C} \cup \{\infty\})$. Then by means of the maps $M_j \xrightarrow{\approx} D_j$ we can conformally attach the disks $\bar{D}_j$ to M, and thereby produce a compact Riemann surface $M' \supset M$ such that $M \sim M' = \{p_1,\ldots,p_r\}$.

□

<u>Proof of Theorem 5</u>: If the Gauss map is algebraic, then

the total curvature is finite by Lemma 4. On the other hand, if the total curvature is finite, then by Proposition 1 the Riemann surface $M = M' \sim \{p_1,\ldots,p_r\}$ where M' is a compact Riemann surface. Let $\Delta_j = \{z \in \mathbb{C}: |z| < 1\}$ be a local coordinate system for M' with 0 corresponding to p_j, and express the Gauss map in $\Delta_j \sim \{0\}$ by the functions $\varphi = d\psi/dz = (\varphi_1,\ldots,\varphi_n)$. It remains to show that the functions $\varphi_1,\ldots,\varphi_n$ have at most a pole at 0. The differentials $\alpha_k = \varphi_k dz$ will then extend to meromorphic differentials on M', and the map Φ will extend to a holomorphic map of M' (via the "regularization" procedure discussed in the proof of Lemma 4).

Suppose that one of the functions φ_k had an essential singularity at $z = 0$. Then for almost all $v \in \mathbb{C}^n$, the function $\langle v,\varphi\rangle = \Sigma\, v_k\varphi_k$ would have an essential singularity at $z = 0$. Hence, for almost all v, the function $\langle v,\varphi\rangle$ would assume the value 0 infinitely often in every neighborhood of 0. Since the φ's are holomorphic in $\Delta_j \sim \{0\}$ and $\Sigma|\varphi_k|^2 = F > 0$, this means the intersection number $N(\Phi, v) = \infty$ for almost all v. Applying formula (3.26), we would then get that $A(\Phi') = -C(\psi) = -\infty$, contrary to our assumption. □

From Theorem 5 and Lemma 4 we now have

COROLLARY 1 - Let $\psi: M \to \mathbb{R}^n$ be a complete minimal surface. Then the total curvature

$$C(\psi) = -2\pi N$$

for some integer N, $0 \leq N \leq \infty$.

NOTE. If $n = 3$. Then $C(\psi) = -4\pi N$, $0 \leq N \leq \infty$, because when $C(\psi)$ is finite, the extended Gauss map $\Phi': M' \to Q_1$ is a finite branched covering. Hence, $A(\Phi') = N \operatorname{Area}(Q_1) = 4\pi N$.

While Theorem 5 is interesting in its own right, it also allows us to make some deep statements about surfaces of finite total curvature.

Before proceeding to these theorems we need to establish an important technical lemma. Let $\psi: M \to \mathbb{R}^n$ be a complete minimal surface with algebraic Gauss map. Let $M = M' \sim \{p_1, \ldots, p_r\}$ as above, and for each j, let $\Delta_j = \{z \in \mathbb{C}: |z| < 1\}$ be a coordinate system for M' with 0 corresponding to p_j. Let $\varphi = (\varphi_1, \ldots, \varphi_n) = d\psi/dz$ be the canonical expression of the Gauss map in

$\Delta_j \sim \{0\}$. Since the φ_k's are meromorphic at 0, we have

$$(3.27) \qquad |\varphi|^2 = \sum_{k=1}^{n} |\varphi_k|^2 \sim \frac{c}{|z|^{2m}}$$

near zero, where $c > 0$ and m is an integer. Since the metric $ds^2 = 2|\varphi|^2|dz|^2$ is complete on $\Delta_j \sim \{0\}$, we must have $m \geqq 1$.

LEMMA 8 - $m \geqq 2$.

Proof: Suppose $m = 1$. Then for suitable constants $c_1,\ldots,c_n \in \mathbb{C}$ (not all zero) we have that $f_k = \varphi_k - c_k/z$ is holomorphic at zero. Consequently, the function

$$\begin{aligned} \mathrm{Re}\{c_k \log z\} &= \mathrm{Re}\{\int^z (\varphi_k - f_k)dz\} = \\ &= \psi_k(z) - \mathrm{Re}\{\int^z f_k dz\} \end{aligned}$$

is a well defined, harmonic function at $z = 0$. It follows that each c_k is real. However, by Equation (3.6) we must have $\Sigma\, c_k^2 = 0$, and therefore $c_1 = \ldots = c_n = 0$. Thus, $m \leqq 0$, contrary to assumption. □

It is a well known theorem of Cohn-Vossen that if M is a complete, riemannian 2-manifold of finite total curvature and finite Euler characteristic χ, then

$\int\int_M KdA \leqq 2\pi\chi$. For minimal surfaces a stronger statement can be made. This was proved in Osserman [4] for n=3 and in Chern-Osserman [1] for general n.

THEOREM 6 - Let $\psi: M \to \mathbb{R}^n$ be a complete minimal surface with Euler characteristic χ and with r ends (i.e. boundary components). Then

$$C(\psi) \leqq 2\pi(\chi - r) .$$

Proof: If $C(\psi) = -\infty$, there is nothing to prove. If $C(\psi)$ is finite, then by Theorem 1 we may consider $M = M' \sim \{p_1,\ldots,p_r\}$ as above, where M' is a compact surface of genus γ. It is straightforward to see that $\chi = 2-2\gamma-r$.

Consider the meromorphic differentials $\alpha_k = \varphi_k dz = (d\psi_k/dz)dz$ on M', and for each j let m_j denote the maximum order of the poles of $\alpha_1,\ldots,\alpha_n$ at p_j. One can see easily that for a suitable choice of constants $c_1,\ldots,c_n \in \mathbb{C}$, the differential

$$\alpha = c_1\alpha_1 + \ldots + c_n\alpha_n$$

will have a pole of order exactly m_j at each point p_j. Riemann's relation (cf. Springer [1]) now says that if α is a meromorphic differential on M' with N zeroes and P poles, then $N-P = 2\gamma - 2$. It follows that

$$N = \sum_{j=1}^{r} m_j + 2\gamma - 2 \geqq 2r + 2\gamma - 2 = r - \chi$$

since, by Lemma 8, each $m_j \geq 2$. However, since $N = N(\Phi', c)$ where $c = (c_1, \ldots, c_n)$, we have $C(\psi) = -2\pi N$, and the theorem follows. □

Note that for minimal surfaces the inequality of Cohn-Vossen can never be an equality, since $r \geq 1$.

Focusing our attention again in R^3, we can now prove a deep result related to the discussion in §5.

THEOREM 7 - (Osserman [5]). Let $\psi: M \to \mathbb{R}^3$ be a complete minimal surface of finite total curvature. Then if the Gauss map omits more than 3 points of S^2, $\psi(M)$ is a plane.

Proof: Since the total curvature is finite we may assume $M = M' \sim \{p_1, \ldots, p_r\}$ where M' is a compact Riemann surface of genus γ. Let g and $f\,dz$ be the meromorphic function and the holomorphic differential appearing in the Weierstrass representation of the surface. The function g represents the Gauss map and therefore extends to a meromorphic function on M', i.e. to a holomorphic map

$$g: M' \to S^2 = \mathbb{C} \cup \{\infty\} .$$

It follows from (3.16) that by making a rotation of coordinates in $\mathbb{R}^3$, we may assume that

(a) g has simple poles on M'

(b) g assumes finite values at $p_1, \ldots, p_r$.

The map $g: M' \to S^2$ is an N-sheeted, branched covering of S^2. In terms of local coordinates, $g(z)$ is a local homeomorphism near $z = 0$ if $\frac{dg}{dz}(0) \neq 0$. Otherwise, g behaves locally like the mapping $z \to z^{1+k}$, where k is the order of the zero of dg/dz at 0. (Note that since the poles of g are simple, g is regular at these points.) One sees from this that, counting multiplicity, g assumes every value exactly N-times.

Let n be the total order of branching of g. Then n is the number of zeros of the meromorphic differential $\omega = (dg/dz)dz$ on M'. Since g has N simple poles, ω has N double poles. Thus, by Riemann's relation, we have $n - 2N = 2\gamma - 2$, or

(3.28) $$n = 2(N + \gamma - 1).$$

The differential fdz has double zeros exactly at the poles of g. Furthermore, if z is a local coordinate at p_j on M', where p_j corresponds to $z = 0$, then by Lemma 8

$$|\varphi|^2 = |f|^2(1+|g|^2)^2 \sim \frac{c}{|z|^{2m_j}}$$

where $2 \leqq m_j < \infty$. Thus, fdz extends to a meromorphic differential on M' having a pole at p_j of order $m_j \geqq 2$ for $j = 1,\ldots,r$ (and no others). From Riemann's relation we have

$$2N - \sum_{j=1}^{r} m_j = 2\gamma - 2,$$

and therefore,

$$r + \gamma - 1 \leqq N . \tag{3.29}$$

Suppose now that $g|M$ omits k points $q_1,\ldots, q_k \in S^2$. Then $g^{-1}(\{q_1,\ldots,q_k\}) \subset \{p_1,\ldots,p_r\}$, and each q_i has, counting multiplicities, exactly N preimages. At p_j, g assumes its value to some multiplicity $1+O_j$. Hence,

$$k \cdot N \leqq \sum_{j=1}^{r} (1+O_j) = r + \sum_{j=1}^{r} O_j .$$

However, ΣO_j = the order of branching at $\{p_1,\ldots,p_r\}$ $\leqq n$, and therefore,

$$k \cdot N \leqq r + n . \tag{3.30}$$

Combining (3.28) and (3.30) shows that

$$k \cdot N - r \leqq 2(N + \gamma - 1) .$$

Adding (3.29) gives

(3.31) $$1 - \gamma \leqq (3-k)\cdot N ,$$

and rewriting (3.29) as $r - N \leqq 1 - \gamma$, we have

(3.32) $$r \leqq (4-k)\cdot N .$$

Since M is not compact, $r \geqq 1$, and we must have $k < 4$. □

COROLLARY 2 - _Let_ $\psi: M \to \mathbb{R}^3$ _be a complete minimal surface of finite total curvature. Then if the normals to_ M _omit three directions, the genus of_ M' _is_ $\geqq 1$ _and_

$$\left| \iint_M K dA \right| \geqq 12\pi .$$

Proof: Using the above notation, we have $r \geqq k = 3$. It follows from (3.31) that $\gamma \geqq 1$, and from (3.32) that $N \geqq 3$. To complete the proof we recall that $\iint_M K dA = -4\pi N$. □

We now consider the case where the total curvature is a minimum.

COROLLARY 3 - _A complete minimal surface in_ $\mathbb{R}^3$ _whose total curvature is_ -4π _is either Enneper's surface or the Catenoid._ (_Hence, these surfaces are completely characterized by their total curvature and their fundamental group._)

Proof: Let $\psi: M \to \mathbb{R}^3$ be the surface and adopt the notation above. Since $N = 1$, $g: M' \to S^2$ is a conformal homeomorphism, and M is conformally equivalent to $S^2 \sim \{p_1,\ldots,p_r\}$. However, by (3.29), $1 = N \geq r + \gamma - 1 = r - 1$. Thus, $r = 1$ or 2.

If $r = 1$, then $M \cong \mathbb{C}$. In fact, we may identify M with $g(M) \subset S^2$ and assume $S^2 \sim g(M) = \{\infty\}$. Then $g(z) = z$, and it follows that the holomorphic differential $f(z)dz$ has no zeroes in $\mathbb{C}$. Since fdz extends to a meromorphic differential on S^2, f is a polynomial in z and therefore constant. We have shown that ψ is Enneper's surface.

Suppose $r = 2$. Again we may identify M with $g(M) \subset S^2$ and assume $S^2 \sim g(M) = \{\infty, z_o\}$. Then $g(z) = z$ and the differential $f(z)dz$ satisfies:

(i) $f(z)$ is a rational function in z.

(ii) $f(z)$ has no zeros or poles in $\mathbb{C} \sim \{z_o\}$.

(iii) $f(z)$ has a pole of order ≥ 2 at z_o.

It follows that $f(z) = c/(z-z_o)^m$ where $c \neq 0$ and $m \geq 2$. Since M is complete we must have that for any curve γ going to infinity,

$$\int_\gamma |f|(1+|g|^2)|dz| = \int \frac{|c|}{|z-z_o|^m}(1+|z|^2)|dz| = \infty .$$

It follows that $m = 2$ or 3.

We now show that $m = 2$ and $z_o = 0$. Let $\varphi_k = d\psi/dz_k$ for $k = 1,2,3$. Then each φ_k must have a purely imaginary period at z_o. Using Equations (3.14) we have:

$$\int_{|z-z_o|=\varepsilon} \varphi_1(z)dz = \frac{c}{2} \int_{|z-z_o|=\varepsilon} \frac{(1-z^2)}{(z-z_o)^m}\, dz =$$

$$= \begin{cases} -2\pi i c z_o & \text{if } m = 2 \\ -\pi i c & \text{if } m = 3 \end{cases}$$

$$\int_{|z-z_o|=\varepsilon} \varphi_2(z)dz = \frac{2c}{2} \int_{|z-z_o|=\varepsilon} \frac{(1+z^2)}{(z-z_o)^m}\, dz =$$

$$= \begin{cases} -2\pi c z_o & \text{if } m = 2 \\ -\pi c & \text{if } m = 3 \end{cases}$$

Since $c \neq 0$, we conclude that $m = 2$ and $z_o = 0$. Therefore, $f(z) = c/z^2$ and the resulting surface is the catenoid. □

CHAPTER IV

CLASSIFICATION OF MINIMAL SURFACES

In this chapter we shall be concerned with the following two questions:

(i) Given a minimal surface $\psi: M \to \mathbb{R}^n$, how many non-congruent minimal surfaces are there in $\mathbb{R}^{n+m}$ (any m) which are isometric to ψ.

(ii) Given an analytic metric ds^2 on a Riemann surface M, under what conditions can ds^2 be realized as the metric induced by a minimal immersion $\psi: M \to \mathbb{R}^n$.

Throughout this section we shall assume that M is simply connected, by passing, if necessary, to the universal covering surface of M.

Our first observation will be with regard to question (i). Let $\psi: M \to \mathbb{R}^n$ be a minimal surface, and let $\tilde{\psi}: M \to \mathbb{R}^n$ be the conjugate surface, that is, the surface whose k^{th} component function $\tilde{\psi}_k$ is the harmonic conjugate of ψ_k. We now regard $\mathbb{C}^n$ as $\mathbb{R}^n \times i\mathbb{R}^n$ and define a new map $\Psi: M \to \mathbb{C}^n$ by

$$\Psi = \frac{1}{\sqrt{2}}(\psi, \tilde{\psi}) . \tag{4.1}$$

One can easily see that Ψ is minimal and isometric to ψ (cf. III.3). Moreover, the mapping Ψ is holomorphic. Consequently, to each minimal surface $\psi: M \to \mathbb{R}^n$ there is associated a holomorphic curve $\Psi: M \to \mathbb{C}^n$ which is isometric to ψ.

The question now is to what extent is this surface unique. Suppose $\psi: M \to \mathbb{R}^n$ and $\varphi: M \to \mathbb{R}^m$ are isometric minimal surfaces. Then are the associated holomorphic maps Ψ and Φ congruent? The answer is yes. We begin the proof with

PROPOSITION 2 - Let $\varphi: \Delta \to \mathbb{C}^n$ and $\psi: \Delta \to \mathbb{C}^{n+m}$ be holomorphic mappings of the unit disk such that

$$|\varphi|^2 = |\psi|^2 ,$$

and consider $\mathbb{C}^n \subset \mathbb{C}^{n+m}$ as the first n coordinates. Then there exists a unitary transformation $U: \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ such that

$$\psi = U\varphi .$$

Proof: If $|\varphi|^2 \equiv 0$, the theorem is trivial, so we may assume that at some point, say $z = 0$, $|\varphi(0)| \neq 0$. The proof will proceed by induction on n.

Suppose $n = 1$. Then we have $|\varphi|^2 = |\varphi_1|^2 = \sum_{k=1}^{m+1} |\psi_k|^2$. Dividing by $|\varphi_1|^2$ in a neighborhood of zero we obtain

$$1 = \sum_{k=1}^{m+1} |\psi_k/\varphi_1|^2 \; . \tag{4.2}$$

Applying the operator $(d/dz)(d/d\bar{z})$ to Equation (4.2) we get

$$0 = \sum_{k=1}^{m+1} |(\psi_k/\varphi_1)'|^2 \; .$$

It follows that in a neighborhood of zero, $\psi_k/\varphi_1 =$ = constant = c_k where $\Sigma|c_k|^2 = 1$. Thus, by analyticity, $\psi_k = c_k\varphi_1$ in Δ for each k. Let $((a_{kj}))$ be an $(m+1) \times (m+1)$ unitary matrix such that $a_{k1} = c_k$ for $k = 1,\ldots,m+1$. Then

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \vdots \\ \psi_{m+1} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} \\ a_{2,1} & \cdots & a_{2,m+1} \\ \vdots & \cdots & \vdots \\ a_{m+1,\cdot} & \cdots & a_{m+1,m+1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

as claimed.

Suppose now that $\varphi = (\varphi_1,\ldots,\varphi_n,0,\ldots,0)$ has n linearly independent coordinates and that the theorem is true for maps φ with fewer linearly independent coordinates. We may further assume by making a unitary change of the first n coordinates, that $\varphi_1(0) \neq 0$ and $\varphi_j(0)=$

$= 0$ for $j > 1$. We have that

$$\sum_{j=1}^{n} |\varphi_j|^2 = \sum_{k=1}^{n+m} |\psi_k|^2 .$$

Dividing by $|\varphi_1|^2$ as above we get

$$(4.3) \qquad 1 + \sum_{j=2}^{n} |\Phi_j|^2 = \sum_{k=1}^{n+m} |\Psi_k|^2$$

where $\Phi_j = \varphi_j/\varphi_1$ and $\Psi_k = \psi_k/\varphi_1$. Taking $(d/dz)(d/d\bar{z})$ gives

$$\sum_{j=2}^{n} |\Phi_j'|^2 = \sum_{k=1}^{n+m} |\Psi_k'|^2 .$$

Hence, by induction there exists an $(n+m) \times (n+m)$ unitary matrix $((c_{kj}))$ such that

$$\Psi_k' = \sum_{j=2}^{n} c_{kj} \Phi_j' \quad ; \quad k = 1,\ldots,n+m .$$

Integrating, we have

$$\Psi_k = c_{k1}' + \sum_{j=2}^{n} c_{kj} \Phi_j \; ; \quad k = 1,\ldots,n+m$$

and therefore,

$$\psi_k = c_{k1}' \varphi_1 + \sum_{j=2}^{n} c_{kj} \varphi_j \quad ; \quad k = 1,\ldots,n+m$$

throughout Δ, where the c_{k1}''s are constants. However, using (4.3) we see that

$$\sum_k |\Psi_k|^2 = \sum_k |c_{k1}'|^2 + 2 \sum_{k,j} \mathrm{Re}\{\overline{c_{k1}'}\, c_{kj} \Phi_j\} + \sum_j |\Phi_j|^2 =$$
$$= 1 + \Sigma |\Phi_j|^2 .$$

Since $\Phi_2(0) = \ldots = \Phi_n(0) = 0$, we have

$$\sum_k |C'_{k1}|^2 = 1$$

$$\sum_{k,j} \operatorname{Re}\{\overline{C}'_{k1} C_{kj} \Phi_j\} = 0 .$$

Moreover, since $f = \sum_{k,j} \overline{C}'_{k1} C_{kj} \Phi_j$ is holomorphic and $\operatorname{Re}(f) = 0$, we have $f = 0$. Therefore, from the linear independence of $\varphi_1,\ldots,\varphi_n$ (and, thus, of $\Phi_2,\ldots,\Phi_n$) we conclude that

$$\sum_{k=1}^{n+m} \overline{C}'_{k1} C_{kj} = 0 \ ; \quad j = 2,\ldots,n .$$

We now let $U = ((a_{kj}))$ be any $(n+m)\times(n+m)$ unitary matrix such that $a_{k1} = C'_{k1}$ and $a_{kj} = c_{kj}$ for $j = 2,\ldots,n$ and $k = 1,\ldots,n+m$. Then $\psi = U\varphi$. □

NOTE. The above clearly continues to hold if we replace Δ by any connected Riemann surface M. Furthermore, the same proof goes through for holomorphic functions φ and ψ of several complex variables.

COROLLARY 1 - Let $\psi: M \to \mathbb{C}^{n+m}$ and $\varphi: M \to \mathbb{C}^n \subset \mathbb{C}^{n+m}$ be isometric holomorphic curves. Then there exists a holomorphic isometry $F: \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ (i.e., a unitary transformation plus a translation) such that

$$\psi = F \circ \varphi .$$

<u>Proof</u>: By assumption, the induced metrics on M are equal, that is,

$$ds^2 = 2|\psi'|^2\,|dz|^2 = 2|\varphi'|^2\,|dz|^2\,.$$

Thus $|\psi'|^2 = |\varphi'|^2$, and by Proposition 2 there is a unitary transformation U of $\mathbb{C}^{n+m}$ such that $\psi' = U\varphi'$. Hence, $\psi = U\varphi + c$ for $c \in \mathbb{C}^n$. □

REMARK 1 - Corollary 1 is a special case of a general theorem due to E. Calabi [1]. Calabi's theorems also show that a statement analogous to this corollary holds for curves in complex projective space.

We have now established the following central result.

THEOREM 1 - <u>In each class of isometric non-congruent minimal surfaces in euclidean space there exists exactly one holomorphic curve</u>.

With this theorem in mind, we shall fix a holomorphic immersion $\Psi: M \to \mathbb{C}^m$ and proceed to describe the space $\mathcal{J}(\Psi)$ of all non-congruent minimal immersions $\psi: M \to \mathbb{R}^n$ which are isometric to Ψ. To begin we normalize our immersions by fixing a point $p_o \in M$ and assuming that $\Psi(p_o) = 0$, $\psi(p_o) = 0$. We furthermore assume that neither of the images $\Psi(M)$ or $\psi(M)$ is contained in a proper linear subspace. Notice that under this assumption the integer m becomes an invariant of the immersion Ψ, and thus, by Theorem 8, also an invariant of each isometric minimal surface ψ.

OBSERVATION 1 - $2m \geqq n \geqq m$.

To see this, note that since Ψ and ψ are isometric, $2|d\psi/dz|^2 \equiv |\Psi'|^2$. Since ψ is minimal, the map $d\psi/dz\colon M \to \mathbb{C}^n$ is holomorphic, and thus by Proposition 2, there is a unitary $N\times N$ matrix U, where $N = \max\{n,m\}$, such that

$$\frac{\partial\psi}{\partial z} = U\cdot\frac{1}{\sqrt{2}}\Psi' \quad . \tag{4.4}$$

(Here we have added zeros to one of the vectors to make it of length N.) Since the components of Ψ' are linearly independent over $\mathbb{C}$, we must have $n \geqq m$. Since $\psi = 2\mathrm{Re}\{\int_{p_o}^{z} \partial\psi/\partial z)dz\}$, it follows from (4.4) that $\psi = \sqrt{2}\ \mathrm{Re}\{U\Psi\} = \frac{1}{\sqrt{2}}(U\Psi + \overline{U\Psi})$. The components of ψ are linearly independent over $\mathbb{R}$, and so $2m \geqq n$.

Let us denote by S the $n\times m$ matrix comprising the first m columns of U. From the above discussion we immediately conclude the following

OBSERVATION 2 - $\psi = \sqrt{2}\ \mathrm{Re}\{S\Psi\}$ where S in an $n\times m$ complex matrix such that:

(1) $\bar{S}^t S = \mathbf{1}_m$

(2) The $n\times 2m$ matrix $(S,\bar{S})$ has rank n.

(3) $\Psi'(z)^t\ S^t S\ \Psi'(z) \equiv 0$, that is, the quadratic form on $\mathbb{C}^m$ given by $S^t S$ annihilates all the tangent vectors to Ψ.

Part (3) of this observation corresponds to the fact that $(\partial\psi/\partial z)^2 = \Psi'^t\ S^t S\ \Psi' \equiv 0$.

NOTE. In the special case " $\psi = \Psi$ ", i.e., when $n = 2m$ and $\psi = (\mathrm{Re}\,\Psi, \mathrm{Im}\,\Psi)$, the matrix S, in $m\times m$ block form, is

$$S = \frac{1}{\sqrt{2}}\binom{1}{-i}.$$

OBSERVATION 3 - Two normalized immersions ψ_1, ψ_2 are congruent iff there is an orthogonal $n\times n$ matrix, O, such that $\psi_1 = O\psi_2$, or equivalently,

$$S_1 = O\ S_2\ .$$

OBSERVATION 4 - For any complex $n\times m$ matrix S with properties (1), (2) and (3) above, the map $\psi = \sqrt{2}\ \mathrm{Re}\{S\Psi\}$ is an isometric minimal immersion of M into $\mathbb{R}^n$, which does not lie in any linear subspace.

Putting together the above remarks we obtain a complete description of $\mathcal{I}(\Psi)$ entirely in terms of the matrices S.

We shall now examine an alternative presentation of this space which is due to E. Calabi. Let us replace the $n \times m$ matrix S by $P = S^t S$. Note that two matrices S_1, S_2, equivalent in the sense of Observation 3, give rise to the same P. We shall translate conditions (1), (2) and (3) above into conditions on P. To do this we first consider the $2m \times 2m$ matrices:

$$\rho = \begin{pmatrix} \mathbb{1}_m & \bar{P} \\ P & \mathbb{1}_m \end{pmatrix} = \overline{(S,\bar{S})}^t \cdot (S,\bar{S})$$

$$\rho' = \begin{pmatrix} \mathbb{1}_m & -\bar{P} \\ -P & \mathbb{1}_m \end{pmatrix} = \overline{(S,-\bar{S})}^t \cdot (S,-\bar{S}).$$

Note that we have used equation (1). By condition (2) we know that ρ and ρ' have rank n. Observe, furthermore, that both ρ and ρ' are hermitian symmetric and positive semi-definite, and that

$$\rho\rho' = \rho'\rho = \begin{pmatrix} \mathbb{1}_m - \bar{P}P & 0 \\ 0 & \mathbb{1}_m - P\bar{P} \end{pmatrix}.$$

Hence, $\mathbb{1}_m - \bar{P}P$ is positive semi-definite. It is straightforward to check that $V = \binom{V_1}{V_2} \in \mathbb{C}^m \oplus \mathbb{C}^m$ satisfies $\rho V = 0$ if and only if $V_2 = -PV_1$ and $(1_m - \bar{P}P)V_1 = 0$. Thus, $\dim(\ker \rho) = \dim(\ker(\mathbb{1}_m - \bar{P}P)) = 2m-n$. We conclude that $\mathbb{1} - \bar{P}P$ has rank $n-m$.

Conversely, suppose P is an $m\times m$ complex matrix such that $P = P^t$ and such that $\mathbb{1}_m - P\bar{P} \geqq 0$ with rank $n-m$. Then by essentially reversing the above process it is possible to find a complex $n\times m$ matrix S, unique up to multiplication on the left by an orthogonal matrix, such that $\bar{S}^t S = \mathbb{1}_m$, $\mathrm{rank}(S,\bar{S}) = n$, and $S^t S = P$. Summing up we have the following result.

THEOREM 2 - (E. Calabi [3]). The space $\mathcal{J}(\Psi)$ of non-congruent minimal immersions $\psi: M \to \mathbb{R}^n$ which are isometric to a given holomorphic immersion $\Psi: M \to \mathbb{C}^n$ is naturally described as the set of all complex symmetric $m\times m$ matrices P such that:

(i) $\mathbb{1}_m - P\bar{P} \geqq 0$

and

(ii) $(\Psi')^t P\Psi' \equiv 0$ (i.e., P annihilates all tangent lines to the curve Ψ.)

Furthermore, let n be the dimension of the smallest affine subspace containing $\psi(M)$ where ψ is a minimal immersion corresponding to P. Then

$$n - m = \mathrm{rank}\ (\mathbb{1}_m - P\bar{P})\ .$$

In particular, $m \leq n \leq 2m$ and $n = 2m$ iff $\mathbb{1}_m - P\bar{P} > 0$.

Theorems 1 and 2 together show that each space of isometric minimal immersions can be naturally embedded as a linear variety in the closure of the Siegel domain $\mathfrak{D} = \{P: P = P^t \text{ and } \mathbb{1}_m - P\bar{P} > 0\}$. The point $P = 0$ always lies in the variety and corresponds to the unique holomorphic immersion.

EXAMPLE 1 - Consider $\Psi: \mathbb{C} \to \mathbb{C}^3$ given by $\Psi(z)^t = (z, z^2/\sqrt{2}, z^3/3)$. Then $(\Psi')^t P \Psi' \equiv 0$ if and only if

$$P = c \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some $c \in \mathbb{C}$. Clearly $\mathbb{1} - P\bar{P} \geqq 0$ if and only if $1 \geqq |c|^2$, and so

$$\mathcal{J}(\Psi) \cong \Delta .$$

Note that the points $c \in \Delta^o$ correspond to minimal surfaces which lie fully in $\mathbb{R}^6$, and the points $c \in \partial\Delta$ correspond to minimal surfaces in $\mathbb{R}^3$. If $c_1 = e^{i\theta} c_2$, then the minimal immersions are associate (cf. §4); thus, $\partial\Delta$ parameterizes a single family of associate surfaces in $\mathbb{R}^3$. As we shall see later this picture holds quite generally for minimal immersions in 3-space.

A more geometric description of the space of isometric immersions can be given by reinterpreting Theorem 2 in terms of the Gauss map. If $\Psi: M \to \mathbb{C}^m$ and $\psi: M \to \mathbb{R}^n$ are as above, we have seen that $m \leq n \leq 2m$.

Write $\psi_o = (\text{Re}\,\Psi, \text{Im}\,\Psi): M \to \mathbb{R}^{2m}$ and consider $\mathbb{R}^n \subsetneq \mathbb{R}^{2m}$ in the usual way. Then if we set $\varphi = \partial\psi/\partial z$ and $\varphi_o = \partial\psi_o/\partial z = \frac{1}{2}(\Psi', -i\Psi')$, we have $|\varphi|^2 \equiv |\varphi_o|^2$, and there must be a unitary $2m \times 2m$ matrix U such that $\varphi = U\varphi_o$. In particular, the correponding Gauss maps $\Phi, \Phi_o: M \to Q_{2m-2}$ are congruent in $\mathbb{C}P^{2m-1}$. (See the note following Proposition 2.) However, these maps may not be congruent as submanifolds of Q_{2m-2}. On fact, this will be true if and only if U can be replaced by $e^{i\theta}\cdot O$ where O is a real orthogonal matrix. Recall from §4 that two minimal immersions whose Gauss maps differ by a constant factor $e^{i\theta}$ are called associate. In light of this we denote by $\tilde{\mathcal{J}}(\psi_o)$ the space of non-congruent, non-associate minimal immersions of M into euclidean space, which are isometric to ψ_o. Similarly, we let $\mathcal{J}(\varphi_o)$ denote the non-congruent holomorphic immersions of M into Q_{2m-2}, which are isometric to φ_o. Then there is a natural one-to-one correspondence:

$$\tilde{\mathcal{J}}(\psi_o) \cong \mathcal{J}(\varphi_o) .$$

Observe that the set $\mathcal{J}(\varphi_o)$ can be interpreted as the collection of all essentially distinct ways of moving the curve $\varphi_o(M) \subset \mathbb{C}P^{2m-1}$ into the submanifold Q_{2m-2}. This embeds $\mathcal{J}(\varphi_o)$ as a subset of the homogeneous space $U(2m)/O(2m)$.

We now turn our attention to the second question posed at the beginning of this chapter, namely: when is a given analytic metric $ds^2 = F|dz|^2$ on a Riemann surface M induced by some minimal immersion $\psi: M \to \mathbb{R}^n$? Of course, by Theorem 6 this happens exactly when ds^2 is induced by an (essentially unique) holomorphic map $\Psi: M \to \mathbb{C}^m$. Therefore, our question can be reduced locally to asking <u>under what conditions on</u> F <u>do there exist holomorphic functions</u> $\varphi_1,\ldots,\varphi_m$ <u>such that</u>

$$(4.5) \qquad F = \sum_{k=1}^{m} |\varphi_k|^2 .$$

To answer this latter question we shall first derive some necessary conditions. Let $\varphi = (\varphi_1,\ldots,\varphi_m)$ where $\varphi_1,\ldots,\varphi_m$ are linearly independent, analytic functions on the disk Δ; and for $z,w \in \mathbb{C}^m$ set $\langle z,w\rangle = \Sigma\, z_k\overline{w}_k$. Then, if $F = \langle\varphi,\varphi\rangle$, we consider, for each integer $k > 0$, the $k\times k$ hermitian matrix $\mathfrak{F}^k = ((\mathfrak{F}_{p,q}))_{p,q=0}^{k-1}$ where

$$\mathfrak{F}_{p,q} = \frac{\partial^p}{\partial z^p}\frac{\partial^q}{\partial \bar{z}^q} F = \langle \varphi^{(p)}, \varphi^{(q)}\rangle ,$$

and observe that

$$(4.6) \qquad \det \mathfrak{F}^k = |\varphi \wedge \varphi' \wedge \ldots \wedge \varphi^{(k-1)}|^2 .$$

Consequently, if we define

$$F_k = \det \mathfrak{F}^k \; ; \quad k = 1,2,3,\ldots.$$

then one necessary condition for F to be of the form $\langle\varphi,\varphi\rangle$ is that

$$(4.7) \qquad \begin{cases} F_k \geqq 0 \quad (\text{but not} \equiv 0\,) & k \leqq m \\ F_k \equiv 0 & k > m \end{cases}$$

in Δ.

It turns out that this condition is also sufficient, that is, if F is a positive, real analytic function in Δ, which satisfies the conditions (4.7) then there is a holomorphic C^m-valued function φ such that $F = \langle\varphi,\varphi\rangle$. This is proven as follows. F has a convergent power series expansion at 0 of the form

$$F = \Sigma\, A_{p,q}\, z^p\, \bar{z}^q$$

where $A_{p,q} = (1/p!q!)\, \mathfrak{F}_{p,q}(0)$. If we denote by A^k the $k\times k$ hermitian matrix $((A_{p,q}))_{p,q=0}^{k-1}$, then

$$\det A^k = \Big(\prod_{p=0}^{k-1} p!\Big)^{-2} F_k \,.$$

Therefore, condition (4.7) tells us that the infinite hermitian matrix $A = A^\infty$ is positive semi-definite of rank m. We claim that, as a result, there are m "infinite" vectors

$$a_r = \begin{pmatrix} a_{r1} \\ a_{r2} \\ \vdots \end{pmatrix} \quad ; \quad r = 1,\ldots,m$$

such that

(4.8) $$A = a_1\bar{a}_1^t + \ldots + a_m\bar{a}_m^t \quad .$$

To see this we first note that such a statement is certainly true for each $k \times k$ hermitian matrix A^k where $k \geqq m$. In fact, if $v_1,\ldots,v_k$ are orthonormal eigenvectors of A^k with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_m > 0 = = \lambda_{m+1} = \ldots = \lambda_k$, then $A^k = \Sigma\, \lambda_i\, v_i\, \bar{v}_i^t$, and we need only set $a_i = \sqrt{\lambda_i}\, v_i$ for $i = 1,\ldots,m$. Note, however, that the set of vectors $\{a_1,\ldots,a_m\}$ is not unique. If $b_i = \Sigma\, \alpha_{ij} a_j$ where $((\alpha_{ij}))$ is a unitary $m \times m$ matrix, then $A^k = \Sigma\, b_i\bar{b}_i^t$. Thus, we can normalize the a_i's by assuming that the first σ_i entries of the column vector a_i vanish, where $\sigma_{i+1} > \sigma_i$, and that the first non-vanishing entry a_{i,σ_i+1} is real and positive. It is easy to see that under this normalization the vectors, which we denote $a_1^k,\ldots,a_m^k$, are <u>unique</u>. It follows that a_i^k is obtained from a_2^{k+1} by dropping the last entry, that is, a_i^{k+1} is a "continuation" of a_i^k. Therefore, if we define $a_i = \lim_k a_i^k$, we have $A = \Sigma\, a_i\bar{a}_i^t$ as claimed.

From (4.8) and the convergence of $\Sigma A_{p,q} z^p \bar{z}^q$ it follows that each series

$$\varphi_i(z) = \sum_{p=0}^{\infty} a_{ip} z^p ,$$

for $i = 1,\ldots,m$, is convergent. Thus each φ_2 is holomorphic, and $F = \Sigma|\varphi_2|^2$. If we set $\psi = \int\varphi$, then ψ is a holomorphic map into $\mathbb{C}^n$ with induced metric $ds^2 = F|dz|^2$.

We have now established that condition (4.7) is sufficient to guarantee that in a neighborhood of every point there is a holomorphic map into $\mathbb{C}^m$ which induces the given metric. However, by the uniqueness of Theorem I we see that this immersion can be continued in a well defined way along any curve emanating from a given point. A standard monodromy argument then gives the following result due to E. Calabi [2].

THEOREM 3 - (Intrinsic characterization of minimal surfaces). Let $ds^2 = 2F|dz|^2$ be a real analytic metric on a simply-connected Riemann surface M. Then ds^2 is induced by a minimal immersion into euclidean space and, in particular by a linearly full holomorphic immersion $\Psi: M \to \mathbb{C}^m$ if and only if the functions $F_k = \det[((\partial^p/\partial z^p \, \partial^q/\partial \bar{z}^q F))^k_{p,q=1}]$ satisfy:

$$F_k \geqq 0 \quad \text{and not} \quad \equiv 0 \quad \text{for} \quad k \leqq n$$

$$F_k \equiv 0 \quad \text{for} \quad k > m .$$

REMARK 2 - The functions F_k can actually be generated recursively by the formula

$$F_{k+1} = \frac{F_k^2}{F_{k-1}} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log F_k \tag{4.9}$$

where $F_1 = F$ and $F_o = 1$. This fact is straightforward to check and makes computations simpler. The Equations (4.9) make it clear that the conditions on the metric ds^2 in Theorem 3 are independent of the local coordinate representation.

This last theorem has, as a special consequence, the following classical result.

COROLLARY 2 - (Ricci-Curbastro). <u>Let</u> M <u>be a simply-connected surface with a</u> (<u>class</u> C^4) <u>riemannian metric</u> ds^2 <u>having Gauss curvature</u> $K < 0$. <u>Then a necessary and sufficient condition that</u> ds^2 <u>be induced by a minimal immersion</u> $\psi: M \to \mathbb{R}^3$ <u>is that the metric</u>

$$d\hat{s}^2 = \sqrt{-K}\, ds^2$$

<u>be flat</u> (i.e. <u>have Gauss curvature</u> $\hat{K} \equiv 0$).

<u>Proof</u>: From the discussion at the beginning of this chapter it is clear that a metric $ds^2 = 2F|dz|^2$ on M is induced by a minimal immersion into $\mathbb{R}^3$ if and only if there is a holomorphic mapping $\varphi: M \to \mathbb{C}^3$ such that:

(i) $\quad |\varphi|^2 = F$

(ii) $\quad \varphi^2 = 0$

($\varphi = \partial\psi/\partial z$, and $\psi = 2\mathrm{Re}\{\int \varphi dz\}$ where $\psi: M \to \mathbb{R}^3$ is the minimal immersion.)

Condition (ii) is equivalent to the fact that $\Phi = \pi \circ \varphi$, where $\pi: \mathbb{C}^3 \sim \{0\} \to \mathbb{C}P^2$, maps M onto the quadric $Q_1 \cong S^2$. (The Gauss map is onto the unit sphere.) Consider the metric

$$d\sigma^2 = 2\frac{|\varphi \wedge \varphi'|^2}{|\varphi|^4}|dz|^2$$

induced by Φ from $\mathbb{C}P^2$. By formula (3.17) the curvature of this metric is

$$K_\Phi = -\frac{|\varphi|^4}{|\varphi \wedge \varphi'|^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log\frac{|\varphi \wedge \varphi'|^2}{|\varphi|^4}$$

$$= -\left(\frac{|\varphi|^4}{|\varphi \wedge \varphi'|^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\log\frac{|\varphi \wedge \varphi'|^2}{|\varphi|^2}\right) + 1$$

since $\partial/\partial z\, \partial/\partial \bar{z} \log |\varphi|^2 = |\varphi \wedge \varphi'|^2/|\varphi|^4$. It follows that

$$(4.10) \qquad K_\Phi \equiv 1 \iff \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log \frac{|\varphi \wedge \varphi'|^2}{|\varphi|^2} \equiv 0 .$$

On the other hand if $ds^2 = 2|\varphi|^2|dz|^2$, then by (3.18) its Gauss curvature is $K = -|\varphi \wedge \varphi'|^2/|\varphi|^6$. Consequently, $d\hat{s}^2 = \sqrt{-K}\, ds^2 = (|\varphi \wedge \varphi'|/|\varphi|)|dz|^2$, and so

$$(4.11) \qquad \hat{K} \equiv 0 \iff \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log \frac{|\varphi \wedge \varphi'|^2}{|\varphi|^2} \equiv 0 .$$

Putting this together, we have that if ds^2 comes from a minimal immersion into $\mathbb{R}^3$, then the Gauss curvature of the Gauss image, $K_\Phi \equiv 1$, and so $\hat{K} \equiv 0$.

Conversely, suppose that we are given $ds^2 = 2F|dz|^2$ with $\hat{K} \equiv 0$. Then we set $F_o = 1$, $F_1 = F$ and define F_k for $k > 1$ by formulas (4.9). We note that

$$F_2 = F^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log F = -KF^3 > 0$$

since $K < 0$. The condition $\hat{K} \equiv 0$ is equivalent to

$$(4.12) \qquad \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log(-KF^2) \equiv 0 .$$

From (4.12) and (3.17) we have

$$F_3 = \frac{F_2^2}{F}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log F_2 = \frac{F_2^2}{F}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log F = -F^6 K^3 ,$$

and so

$$F_4 \equiv 0 .$$

Equations (4.12) and (3.17) also show that F is real analytic. Therefore, by Theorem 3 there is a holomorphic map $\varphi: M \to \mathbb{C}^3$ such that $ds^2 = 2|\varphi|^2|dz|^2$. From (4.10) and (4.11) we then conclude that the image $\Phi = \pi \circ \varphi$ of φ in complex projective space has constant curvature 1. However, the remark following Theorem 1 implies that, up to isometries of $\mathbb{C}P^2$, Φ is a mapping into Q_1. That is, after possibly changing φ by a unitary transformation, we have $\varphi^2 \equiv 0$. This completes the proof. □

REMARK 3 - In general the metric on a minimal surface in $\mathbb{R}^3$ has $K < 0$ except at isolated points.

However, if we weaken the condition on curvature in Corollary 2 to allow such points, <u>the result continues to hold if and only if the metric is real analytic</u>. If the metric is analytic the above proof works. If not there are counterexamples in Lawson [3, Rmk 12.1]. A more direct proof of this corollary can be obtained by using the condition $\hat{K} \equiv 0$ to construct a traceless second fundamental form satisfying the Gauss-curvature and Mainardi-Codazzi equations. (cf. Lawson [3, §12].)

A great deal of interesting work has been done on minimal surfaces satisfying the condition $\hat{K} \equiv 0$, notably by M. Pinl [1]. We point out here one interesting fact which follows from the work above.

COROLLARY 5 - (Lawson [1]). Let $\psi: M \to \mathbb{R}^n$ be a simply connected minimal surface whose metric satisfies the Ricci condition $\hat{K} \equiv 0$ of Corollary 2 away from the zeros of the curvature. Then there is an isometric minimal immersion $\psi_o: M \to \mathbb{R}^3$, and a number $\beta \in [0,2\pi]$ such that

$$\psi = \psi_\beta \overset{\text{def.}}{=} \cos\beta\ \psi_o \oplus \sin\beta\ \psi_o^*: M \to \mathbb{R}^6 \subset \mathbb{R}^n$$

where ψ_o^* is the immersion conjugate to ψ_o. Furthermore, up to congruence, every minimal immersion which is isometric to ψ, is associate to one of the surfaces ψ_β.

In particular:

(a) If $n \leqq s$, then $\psi(M) \subseteq \mathbb{R}^3$.

(b) Any two isometric minimal immersion of M into $\mathbb{R}^3$ are associate.

Proof: If $\psi(M)$ is a plane, the statement is easily proved, so we assume it is not. Then the existence of an isometric minimal immersion $\psi_o: M \to \mathbb{R}^3$ can be deduced from Corollary 2 and the analyticity of the metric. (See Remark 3.) Let $\varphi_o = d\psi_o/dz$ and recall that $\varphi_o^2 \equiv 0$.

By Theorem 2 the space, $\mathcal{J}(\psi)$, of all non-congruent minimal immersions isometric to ψ corresponds to the space of complex symmetric 3x3 matrices P such that $\varphi_o^t P\varphi_o \equiv 0$ and $\mathbb{1}_3 - P\bar{P} \geqq 0$. If $P = c\mathbb{1}_3$ where $|c| \leqq$ $\leqq 1$, then P satisfies these conditions. Conversely, if $\varphi_o^t P\varphi_o \equiv 0$ for some P, then $P = cI$ since otherwise, $\varphi_o(M) \subset \{z \in \mathbb{C}^3 : z^t z = z^t Pz = 0\}$ = the union of a finite number of complex lines passing through the origin, and therefore, $\varphi_o \wedge \varphi_o' \equiv 0$, i.e., $\psi(M)$ is a plane. Since, $\mathbb{1}_3 - P\bar{P} \geqq 0$, $1 \geqq |c|^2$. Hence, as in Example 1

$$\mathcal{J}(\psi) \cong \bar{\Delta} .$$

Each circle $S^1(r) = \{c: |c| = r\} \subset \Delta$ corresponds to a family of associate immersions. The immersions are linearly full in some affine 6-space if and only if $r < 1$. The family of surfaces corresponding to $S^1(1) = \partial\Delta$ lies in 3-space. □

In summary we see that given an analytic metric ds^2 on a Riemann surface M, it is intrinsically decidable whether that metric is induced by a minimal immersion into euclidean space. If it is, then, in fact, there exists a holomorphic immersion $\Psi: M \to \mathbb{C}^m$ which induces this metric, and Ψ is unique up to isometries of $\mathbb{C}^m$. The space of all the other isometric, but non-congruent, minimal immersions, $\mathcal{J}(\Psi)$, can be described as the set of all $m\times m$ complex symmetric matrices P such that $\mathbb{1}_m - P\bar{P} \geq 0$ and $(\Psi')^t P\Psi' \equiv 0$.

We observe that in "most cases", that is, for most holomorphic curves Ψ, there are no interesting isometric minimal surfaces. In particular, $\mathcal{J}(\Psi) = \{0\}$ unless Ψ' satisfies a non-trivial polynomial equation of degree 2. (More generally, $\dim_{\mathbb{C}} \mathcal{J}(\Psi) =$ the number of linearly independent quadratic polynomials which annihilate Ψ.) In light of this fact it would be interesting to have a characterization of those minimal surfaces $\psi: M \to \mathbb{R}^{2m}$ which are actually holomorphic curves in $\mathbb{C}^m \cong \mathbb{R}^{2m}$.

We begin by deriving some necessary conditions. Suppose $\Psi: M \to \mathbb{C}^m$ is holomorphic, and write $\psi = (\mathrm{Re}\,\Psi, \mathrm{Im}\,\Psi): M \to \mathbb{R}^m \times \mathbb{R}^m$. Then,

$$\varphi = \frac{\partial \psi}{\partial z} = (\Psi, -i\Psi) ,$$

and so the Gauss map $\Phi: M \to Q_{2m-2}$ actually takes its values in the <u>linear</u> space $\mathbb{C}P^{m-1} \subset Q_{2m-2}$ defined by the m equations

$$(4.13) \qquad \begin{cases} Z_1 = i\, Z_{m+1} \\ \vdots \qquad \vdots \\ Z_m = i\, Z_{2m} \end{cases}$$

in homogeneous coordinates.

We now make an observation about the geometry of the Grassmannian Q_{2m-2}.

LEMMA 1 - <u>A linear subspace of</u> Q_{2m-2} <u>is maximal if and only if it has dimension</u> m-1. <u>Furthermore, any two maximal linear subspaces of</u> Q_{2m-2} <u>are congruent, that is, they differ by an orthogonal transformation of homogeneous coordinates</u>.

<u>Proof</u>: Let $\mathbb{C}P^{\ell-1} \subset \mathbb{C}P^{2m-1}$ be represented in homogeneous coordinates by an ℓ-dimensional, complex linear subspace $V \subset \mathbb{C}^{2m}$ with basis $\varepsilon_1,\ldots,\varepsilon_\ell$. We now observe that $\mathbb{C}P^{\ell} \subset Q_{2m-2}$ if and only if for all $a \in V$, $a\cdot a = \Sigma\, a_i a_j\, \varepsilon_i\cdot\varepsilon_j = 0$. Thus, $\mathbb{C}P^{\ell} \subset Q_{2m-2}$ if and only if

$$(4.14) \qquad \varepsilon_i\cdot\varepsilon_j = 0 \quad \text{for all } i,j \;.$$

(It follows that $a\cdot b = 0$ for all $a,b \in V$. Such a linear subspace of $\mathbb{C}^{2m}$ is called <u>totally isotropic</u>.)

Without loss of generality we may assume that $\varepsilon_1,\ldots,\varepsilon_\ell$ are orthonormal in the hermitian inner product $\langle a,b\rangle = a\cdot\bar{b}$. If we then write $\varepsilon_j = e_j + if_j$ for $e_j, f_j \in \mathbb{R}^{2m}$, condition (4.14) becomes equivalent to the fact that $\{e_1,\ldots,e_\ell,\, f_1,\ldots,f_\ell\}$ form an orthonormal basis of V over $\mathbb{R}$. It is therefore clear that V is a maximal totally isotropic subspace of $\mathbb{C}^m$ if and only if $\dim_{\mathbb{C}} V = m$.

Furthermore, suppose we had two such maximal spaces with bases $\mathcal{E}_j = e_j + if_j$; $j = 1,\ldots,m$ and $\mathcal{E}'_j = e'_j + if'_j$; $j = 1,\ldots,m$ as above. The letting O be the orthogonal $2m \times 2m$ matrix such that $e'_j = Oe_j$ and $f_j = Of'_j$ for all j, we have $\mathcal{E}_j = O\mathcal{E}'_j$, and the two associated projective spaces are congruent in the Grassmannian, as claimed. □

Recall that we are interested in when a minimal immersion $\psi: M \to \mathbb{R}^{2m}$ is "holomorphic". Actually, we have to be careful in asking this question because there are a number of inequivalent ways of making R^{2m} into a complex vector space. In particular, we can define an <u>orthogonal complex structure</u> on $\mathbb{R}^{2m}$ to be an orthogonal transformation $J: \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ such that $J^2 = -\mathbb{1}$. Given J we can make $\mathbb{R}^{2m}$ into an m-dimensional complex vector space by defining scalar multiplication as the map $\mathbb{C} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ where $(x+iy,v) \longmapsto xv + yJ(v)$. Conversely, if $\mathbb{R}^{2m}$ is a vector space over $\mathbb{C}$ such that multiplication by i preserves the length of vectors, then we can define $Jv = iv$. Note that there are many such structures; in fact, the set of inequivalent ones corresponds naturally to the homogeneous space $O(2m)/U(m)$.

From Lemma 9 above we can deduce the following result.

PROPOSITION 2 - Let $\psi: M \to \mathbb{R}^{2m}$ be a minimal immersion with Gauss map $\Phi: M \to \mathbb{Q}_{2m-2}$. Then there exists an orthogonal complex structure on $\mathbb{R}^{2m}$ with respect to which ψ is holomorphic if and only if the Gaussian image $\Phi(M)$ lies in a linear subspace of $\mathbb{Q}_{2m-2}$.

This latter condition is equivalent to the fact that

$$\frac{\partial^p \psi}{\partial z^p} \cdot \frac{\partial^q \psi}{\partial z^q} = 0$$

for all $p, q \geq 1$.

Proof: It was observed above that if ψ is holomorphic then the condition on the Gauss map is satisfied. If, conversely, the condition on $\Phi(M)$ is satisfied, we consider a maximal linear subspace of $\mathbb{Q}_{2m-2}$ containing $\Phi(M)$. By Lemma 1 there exists a orthogonal transformation O mapping this space onto the one given in homogeneous coordinates by Equations (4.13). We then consider the map $J_o: \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ whose matrix in block form is given by

$$J_o \cong \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix},$$

and define an orthogonal complex structure $J = O^t J_o O$.

In homogeneous coordinates the Gauss map is represented by $\varphi = \partial\psi/\partial z$. Thus, from Equations (4.13) we have

$$J \frac{\partial \psi}{\partial z} = i \frac{\partial \psi}{\partial z}$$

which is an exact expression of the Cauchy-Riemann equations for ψ with respect to the complex structure J. □

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