LECTURES ON HAMILTONIAN SYSTEMS

R. CLARK ROBINSON

MONOGRAFIAS DE MATEMÁTICA

1) Alberto Azevedo & Renzo Piccinini — Introdução à Teoria dos Grupos

2) Nathan M. Santos — Vetores e Matrizes

3) Manfredo P. Carmo — Introdução à Geometria Diferencial Global

4) Jacob Palis Jr. — Sistemas Dinâmicos

5) João Pitombeira de Carvalho — Introdução à Álgebra Linear

6) Pedro Fernandez — Introdução à Teoria das Probabilidades

7) R. Clark Robinson — Lectures on Hamiltonian systems

8) Manfredo P. Carmo — Notas de Geometria Riemaniana

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§0 - INTRODUCTION

These lectures are an introduction to Hamiltonian systems. The main emphasis is on properties that occur generically, i.e. for most Hamiltonian systems.

§1 contains the basic linear theory. §2 contains some facts about normalizing symplectic matrices, which is used in §3 to study generic bifurcation of eigenvalues for one parameter families of symplectic transformations. §4 contains the basic nonlinear theory. If the reader is primarily interested in the nonlinear theory it is possible to read through Proposition 1.6 and the definition of N-elementary and then ship to §4, §5 contains a proof of Darboux's Theorem and related theorems. These are proved using the method developed by Moser and Weinstein. §6 contains the results about generic properties of Hamiltonian systems. We prove that generically closed orbits lie on one parameter families. Using this fact the theorems in §3 tell something about how the derivative of the Poincaré map varies as the orbit varies along this family. Also C^1 generically the periodic points are dense in the nonwandering set. Finally we show that any Hamiltonian system that has a closed orbit with an eigenvalue of absolute value one is not structurally stable. §8 gives a proof of the general density theorem stated in §6. §7 states the theorems in transversality necessary for the proof of §8.

The following books and papers also give introductions into Hamiltonian systems and treat some subjects not covered here: [A1], [A3, Chapter 4], [A5], [B1], [M6], [S1], and [S11].

§1 - BASIC LINEAR THEORY

For the linear theory we start with a pair (V^{2n},ω) where V^{2n} is an even dimensional real vector space and ω is an alternating bilinear two form on V. To ω we associate a map, $\bar{\omega}:V \to V^*$, from V so its dual space defined by $\bar{\omega}(v) = i_v\omega = \omega(v,\cdot)$ where i_v is the interior product of ω with v. The rank of ω is the dimension of the image of $\bar{\omega}$. ω is nondegenerate if ω has rank $2n$, i.e. $\bar{\omega}$ is an isomorphism. The pair (V,ω) is called a (real) symplectic space if ω is a nondegenerate alternating bilinear two form on V.

We usually identify V with R^{2n} by picking a basis, $u_1,\ldots,u_{2n}$. We associate to ω the matrix $A = (a_{ij})$ where $a_{ij} = \omega(u_i,u_j)$. Then $\omega(x,y) = x^tAy$ where we write vectors as columns and x^t is the transpose. Since ω is alternating, A is skewsymmetric. The standard skewsymmetric matrix is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $n\times n$ identity matrix. A basis for which ω has J as its matrix is called a symplectic basis.

LEMMA 1.1 - Given a symplectic space (V^{2n},ω) there exists a symplectic basis.

<u>Proof</u>: Pick any vector $u_1 \in V$. Since ω is nondegenerate there exists a vector u_{n+1} such that $\omega(u_1,u_{n+1}) \neq 0$. By scalar multiplication we can make $\omega(u_1,u_{n+1}) = 1$. Let $\langle u_1,u_{n+1}\rangle$ be the space spanned by these vectors.

Let

$$V_2 = \{v \in V: \omega(v,u_1) = 0 = \omega(v,u_{n+1})\} = \langle u_1,u_{n+1}\rangle^{\perp} .$$

V_2 has dimension $2n-2$, $\omega|V_2$ is nondegenerate, and $V = \langle u_1,u_{n+1}\rangle \oplus V_2$ since ω is nondegenerate on V. By induction on the dimension there exists a symplectic basis of V_2, $u_2,\dots,u_n,u_{n+2},\dots,u_{2n}$. Then $u_1,\dots,u_{2n}$ is a symplectic basis of V.

Q.E.D.

LEMMA 1.2 - Let (V^{2n},ω) be a symplectic space. Let $u_1,\dots,u_{p+q},u_{n+1},\dots,u_{n+p}$ be independent vectors such that $\omega(u_j,u_{j+n}) = 1$ and $\omega(u_j,u_k) = 0$ $j \neq k \pm n$. Then there exist vectors which complete these to a symplectic basis.

<u>Proof</u>: We prove by induction on q that we can find u_j $j = n+p+1,\dots,n+p+q$. If $q = 1$ then the construction is just the one in Lemma 1.1. Assume the

construction is possible for $q-1$. Take the subspaces

$B = \langle u_1,\ldots,u_{p+q},u_{n+1},\ldots,u_{n+p}\rangle^{\perp} = \{v \in V\colon \omega(v,u_j) = 0$

$j = 1,\ldots,p+q,n+1,\ldots,n+p\}$ and

$C = \langle u_1,\ldots,u_p,u_{p+2},\ldots,u_{p+q},u_{n+1},\ldots,u_{n+p}\rangle^{\perp}$. By the independence of the vectors and nondegeneracy of ω, $C \supsetneq B$. Take $u_{n+p+1} \in C-B$ such that $\omega(u_{p+1},u_{n+p+1}) = 1$. By induction we can find u_j $j = n+p+2,\ldots,n+p+q$.

Now use the construction of Lemma 1.1 to complete the basis.

Q.E.D.

To (V,ω) there is associated a Lie group of linear maps that preserve ω, $Sp(V)$. A transformation $A \in Sp(V)$ is called a symplectic transformation (or matrix). For $A \in Sp(V)$ $(A^*\omega)(u,v) = \omega(Au,Av) = \omega(u,v)$ or in matrix notation with a symplectic basis $A^tJA = J$.
Let $A_s \in Sp(R^{2n})$ be a differentiable curve of matrices with $A_o = I$. Let $B = \frac{d}{ds} A_s\big|_{s=0}$. Then $0 = \frac{d}{ds}J =$
$= \frac{d}{ds} A_s{}^tJA_s = B^tJ + JB$, or $\omega(Bu,v) + \omega(u,Bv) = 0$.
B is called an infinitesimally symplectic transformation (or matrix). The set of the infinitesimally symplectic transformations is labeled $sp(V)$. It is the Lie algebra associated to the Lie group $Sp(V)$.

LEMMA 1.3 - If $A \in Sp(R^{2n})$ then $\det A = I$.

Proof: By taking determinants of $A^tJA = J$ we get $(\det A)^2 = 1$. That is not enough. Define a volume on V, Ω, by $\Omega(v_1,\ldots,v_{2n}) = \det(\omega(v_i,v_j))$.
$(A^*\Omega)(v_1,\ldots,v_{2n}) = \Omega(Av_1,\ldots,Av_{2n}) = \det(\omega(Av_i,Av_j)) =$
$= \det(\omega(v_i,v_j)) = \Omega(v_1,\ldots,v_{2n})$. But one definition of the determinant is $A^*\Omega = (\det A)\Omega$. Therefore $\det A = 1$.

Q.E.D.

LEMMA 1.4 - Let $A \in Sp(R^{2n})$ and $p(x) = \det(A-xI)$. Then $p(x) = x^{2n}\, p(1/x)$.

Proof: $p(x) = \det(A-xI) = \det(JAJ^{-1}-xI) = \det((A^t)^{-1}-xI) =$
$= \det(A^{-1}-xI) = (-x)^{2n}\,(\det A^{-1})\,\det(-I/x+A) =$
$= x^{2n}\, p(1/x)$.

Q.E.D.

PROPOSITION 1.5 - If λ is an eigenvalue of $A \in Sp(R^{2n})$ then $1/\lambda$, $1/\bar{\lambda}$, and $\bar{\lambda}$ also are eigenvalues with the same multiplicity. In particular both 1 and -1 have even multiplicity if they are eigenvalues.

Proof: This result follows directly from Lemma 1.4.

PROPOSITION 1.6 - If λ is an eigenvalue of $B \in sp(R^{2n})$ then $-\lambda$, $-\bar{\lambda}$, and $\bar{\lambda}$ also are eigenvalues with the same multiplicity. In particular if 0 is an eigenvalue then it has even multiplicity.

Proof: This result is proved in a similar way as the above result. We leave the proof to the reader.

Often, we want to talk about multiplicative independence of the eigenvalues for a symplectic matrix. However λ and $1/\lambda$ are not independent. To have independence we need to take half of the eigenvalues. The principal eigenvalues of a symplectic matrix are those n eigenvalues with (i) absolute value strictly greater than one or (ii) absolute value equal to one and imaginary part greater than or equal to zero. In order to get n principal eigenvalues we take exactly half the eigenvalues equal to 1 or -1.

A symplectic matrix is called N-elementary $(N > 0)$ if the principal eigenvalues $\lambda_1,\ldots,\lambda_n$ satisfy the following condition: if $\lambda_1^{p_1} \ldots \lambda_n^{p_n} = 1$ with $\Sigma_{j=1}^{n} |p_j| \leq N$ then $p_j = 0$ for all j. It is called elementary if it is N-elementary for all $N > 0$. If a matrix is N-elementary then no eigenvalue is a j^{th} root of unity for $0 < j \leq N$. In particular, a matrix is 1-elementary if and only if 1 is not an eigenvalue. A matrix is 2-elementary if and only if it does not have any multiple eigenvalues.

Let λ be a real eigenvalue of $B \in Sp(R^{2n})$. The eigenspace of λ is defined to be the set of vectors u

such that $(B-\lambda I)u = 0$. The root space of λ is defined to be the set of vectors such that $(B-\lambda I)^m u = 0$ for some integer $m > 0$. If λ is not real replace $(B-\lambda I)$ with $(B^2 - \lambda B - \bar{\lambda}B + \lambda\bar{\lambda}I)$ in both definitions. This gives a real subspace associated to both λ and $\bar{\lambda}$.

LEMMA 1.7 - Let $\lambda = a + ib$ be an eigenvalue of $B \in Sp(R^{2n})$ with $b \neq 0$. Let u be in the eigenspace of λ. Then there exists another vector v in the eigenspace such that $Bu = au + bv$ and $Bv = -bu+av$.

Proof: Let $v = 1/b(Bu-au)$. $Bu = au+bv$. $0 = B^2u - 2aBu + (a^2+b^2)u = b^2u - abv + bBv$ so $Bv = -bu + av$.

Q.E.D.

The following propositions show ω is nondegenerate on the root space.

LEMMA 1.8 - Let λ_1 and λ_2 be eigenvalues for $B \in Sp(R^{2n})$ with $\lambda_1\lambda_2 \neq 1$ and $\lambda_1\bar{\lambda}_2 \neq 1$. Then the root spaces of λ_1 and λ_2 are perpendicular with respect to ω, i.e. if v_i is in the root space of λ_i then $\omega(v_1,v_2) = 0$.

Proof: See [M5]. Extend B to act linearly on $\mathbb{C}^{2n}$ and extend ω to be an alternating bilinear form. Using the Jordan form for B (with λ_i's off the diagonal as well as on the diagonal in the blocks corresponding to λ_i) there exists a complex basis $u_1,\ldots,u_{2n}$ such that $B^p u_j = \lambda_j P_j(p)$ where $P_j(p)$ is a nonzero vector whose components are polynomials in p. This is not necessarily a symplectic basis. Also we have relabeled the eigenvalues so u_j is the root space of λ_j. The components of $P_j(p)$ are polynomials in p since for fixed q the binomial coefficient $\binom{p}{q}$ is a polynomial in p. Then $\omega(u_j,u_k) = \omega(B^p u_j, B^p u_k) = (\lambda_j\lambda_k)^p\ \omega(P_j(p),P_k(p))$. If $\lambda_j\lambda_k \neq 1$ it follows that $\omega(u_j,u_k) = 0$. In the real root space corresponding to λ_j and λ_k it follows the vectors must be perpendicular with respect to ω.

Q.E.D.

COROLLARY 1.9 - Let λ be an eigenvalue of $B \in Sp(R^{2n})$. then ω is nondegenerate on the subspace spanned by vectors in the root space of λ and $1/\lambda$. In particular if $|\lambda| = 1$ then ω is nondegenerate on the root space of λ.

COROLLARY 1.10 - Let λ be an eigenvalue of $B \in Sp(R^{2n})$ with $|\lambda| = 1$ and $\lambda \neq \pm 1$. If $\omega(u,Bu) = 0$ for a nonzero vector in the root space then $\omega(v,Bv)$ is both strictly positive and strictly negative on the root space. Thus $\omega(v,Bv)$ is either positive definite, negative

negative definite, or assume both signs on the root space of an eigenvalue λ of absolute value one with $\lambda \neq \pm 1$.

Proof: Assume $\omega(u,Bu) = 0$. Since $B^2u - u \neq 0$ and ω is nondegenerate on the root space, there exists a vector y such that $\omega(By, B^2u-u) \neq 0$. Let c be a scalar. Then $\omega(y+cu, B(y+cu)) = \omega(y,By) + c\omega(y,Bu) + c\omega(u,By) + c^2\omega(u,Bu) = \omega(y,By) + c(\omega(By,B^2u) - \omega(By,u)) = \omega(y,By) + c\omega(By,B^2u-u)$. Thus $\omega(y+cu, B(y+cu))$ assumes both signs on the root space for different values of c.

Q.E.D.

Using Corollary 1.10 we give names to the three possible cases. Let λ be an eigenvalue of $B \in Sp(R^{2n})$ with $|\lambda| = 1$ and $\lambda \neq \pm 1$. λ is said to be of positive type (resp. negative type) if $\omega(u,Bu) > 0$ (resp. < 0) for all nonzero vectors in the root space. λ is said to be of mixed type if $\omega(u,Bu) > 0$ and $\omega(v,Bv) < 0$ for u and v both in the root space of λ.

Both positive and negative rotations on a plane have the same eigenvalues. However if ω is nondegenerate on this plane it induces an orientation which distinguishes the two cases, i.e. if $\omega(u,v) = 1$, $Bu = au + bv$, $Bv = -bu + av$ then $\omega(u,Bu) = \omega(v,Bv) = b$.

Next we prove the root space equals the eigenspace for an eigenvalue of positive or negative type. This proposition is false for eigenvalues of mixed type. See the examples in §2.

LEMMA 1.11 - Let $B \in Sp(R^{2n})$. Assume $V \subset R^{2n}$ is an invariant subspace for B and $\omega|V$ is non-degenerate. Then $V^{\perp} = \{u: \omega(u,v) = 0$ for all $v \in V\}$ is invariant by B.

<u>Proof</u>: Assume $V^{\perp}$ is not invariant. Then there exists $u \in V^{\perp}$ such that $Bu = x+v$ with $x \in V^{\perp}$ and $v \in V$. Let $v' \in V$ be such that $\omega(v,Bv') \neq 0$. Then $0 = \omega(u,v') = \omega(Bu,Bv') = \omega(v,Bv') \neq 0$. Contradiction.

Q.E.D.

PROPOSITION 1.12 - Let λ be an eigenvalue for $B \in Sp(R^{2n})$ with $|\lambda| = 1$, $\lambda \neq \pm 1$, and of positive or negative type. Then the eigenspace of λ equals the root space.

<u>Proof</u>: Let P be the root space and V be the eigenspace of λ. $\omega|P$ is nondegenerate by Corollary 1.9. Using Lemma 1.7 the reader can check that $\omega|V$ is non-degenerate. By Lemma 1.11 $V^{\perp} \cap P$ is invariant by B. $B|(V^{\perp} \cap P)$ must have an eigenvector u. But all the eigenvectors for λ lie in V. Therefor $V^{\perp} \cap P = \{0\}$.

Q.E.D.

§2 - NORMAL FORMS FOR SYMPLECTIC MATRICES

In this section we give an explicit normal form for matrices with eigenvalues of multiplicity at most two. See [M5] and [W2] for the general case. These results are used in the next section to study generic properties of one parameter families of symplectic matrices.

We are given a symplectic transformation B on a real symplectic space (R^{2n},ω). By Lemma 1.8 and Corollary 1.9 we can find a symplectic basis $v_1,\dots,v_{2n}$ such that each vector v_j lies in a root space for B. ω is nondegenerate on the subspace generated by the root spaces of λ and λ^{-1}. Thus it is sufficient to look at these subspaces one at a time. In the following $\lambda = a+ib$ is an eigenvalue of $B \in Sp(R^{2n})$.

PROPOSITION 2.1 - Let $\lambda = a+ib$ be an eigenvalue of absolute value one for which the eigenspace equals the root space, V. Then there exists a symplectic basis for the root space such that $B|V$ has as its matrix

$$\begin{pmatrix} aI_1 & 0 & -bI_1 & 0 \\ 0 & aI_2 & 0 & bI_2 \\ bI_1 & 0 & aI_1 & 0 \\ 0 & -bI_2 & 0 & aI_2 \end{pmatrix}$$

where I_1 is the $j\times j$ identity matrix, I_2 is the $k\times k$ identity matrix, and λ has multiplicity $j+k = q$.

Proof: We can take $b > 0$ by using λ or $\bar{\lambda}$.

Assume $\omega(u,Bu) > 0$ for some vector $u \in V$. Then by Lemma 1.7 there exists $v \in V$ such that $Bu = au+\sigma bv$, $Bv = -\sigma bu+av$ where $\sigma = \omega(u,v)$. In fact we can take $\sigma = \pm 1$ by scalar multiplication. By reordering u and v we can assume $\omega(u,v) = +1$. Let $u_1 = u$, $u_{q+1} = v$. Look at $\langle u_1,u_{q+1}\rangle^{\perp} = \{v \in V: \omega(v,u_1) = \omega(v,u_{q+1}) = 0\}$. It must be invariant by Lemma 1.11. $\langle u_1,u_{q+1}\rangle^{\perp}$ has dimension two less than V. Continuing by induction we can find $u_1,\ldots,u_j,u_{q+1},\ldots,u_{q+j}$ such that $Bu_i = au_i + bu_{q+i}$, $B u_{q+i} = -bu_i + a u_{q+i}$, and such that $\omega(v,Bv) \le 0$ $v \in \langle u_1,\ldots,u_j,u_{q+1},\ldots,u_{q+j}\rangle^{\perp} = V_1^{\perp}$. By Corollary 1.10 B must be of negative type on $V_1^{\perp}$.

Now proceed to pick pairs u_i,u_{i+q} such that $\omega(u_i,u_{i+q}) = 1$ and $Bu_i = au_i - bu_{i+q}$, $Bu_{i+q} = bu_i +$ $+ a u_{i+q}$. By induction we can complete the basis.

Q.E.D.

REMARK - The above construction could be used for a different proof by Proposition 1.12.

COROLLARY 2.2 - Let $\lambda = a \pm ib$ be an eigenvalue of absolute value one, multiplicity two, of mixed type, and for which the eigenspace equals the root space. Then there exists a symplectic basis for the root space, $v_1,\dots,v_4$ with $\omega(v_1,v_3) = 1$ and $\omega(v_2,v_4) = 1$ as usual, and such that B has as its matrix

$$\begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

Proof: Let $u_1,\dots,u_4$ be the basis that put B in the form of Proposition 2.1 with $I_1 = I_2 = (1)$.

Let

$$\begin{aligned} v_1 &= u_1 + u_2 \\ v_2 &= u_3 - u_4 \\ v_3 &= \tfrac{1}{2}(u_3+u_4) \\ v_4 &= \tfrac{1}{2}(u_2-u_1) \end{aligned}$$

The reader can check the details.

Q.E.D.

PROPOSITION 2.3 - Let $\lambda = a + ib$ be an eigenvalue of absolute value one, multiplicity two, of mixed type, and for which the eigenspace does not equal the root space. Then there exists a symplectic basis such that B has as its matrix

$$\begin{pmatrix} A & cA \\ 0 & A \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and $c = \pm 1$.

<u>Proof</u>: See [M5] for a proof that includes higher multiplicity. Take two eigenvectors u and v such that $Bu = au+bv$ and $Bv = -bu+av$. If $\omega(u,v) \neq 0$ then $\langle u,v\rangle^{\perp}$ would be invariant and the root space would equal the eigenspace. Thus $\omega(u,v) = 0$. Let $u_1 = u$, $u_2 = v$, and pick u_3 and u_4 that complete the symplectic basis. In terms of this basis B equals

$$\begin{pmatrix} a & -b & x_1 & x_2 \\ b & a & x_3 & x_4 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

Let $v_1 = u_1$, $v_2 = u_2$, $v_3 = u_3 + y_1u_1 + y_2u_2$, and $v_4 = u_4 + y_3u_1 + y_4u_2$. To be a simplectic basis $\omega(v_3,v_4) = y_2-y_3 = 0$. Then

$Bv_3 = u_1(x_1+ay_1-by_2) + u_2(x_3+by_1+ay_2) + au_3 + bu_4 =$

$$= u_1(x_1-2by_2) + u_2(x_3+by_1-by_4) + av_3 + bv_4$$

$$Bv_4 = u_1(x_2+ay_2-by_4) + u_2(x_4+y_2b+ay_4) - bu_3+au_4 =$$
$$= u_1(x_2+by_1-by_4) + u_2(x_4+2by_2) - bv_3 + av_4.$$

We want $x_1 - 2by_2 = x_4 + 2by_2$ and $x_3 + b(y_1-y_4) = -x_2 - b(y_1-y_4)$ or $y_2 = (x_1-x_4)/4b$ and $y_4-y_1 = (x_2+x_3)/2b$. After making these choices B has the form

$$\begin{pmatrix} a & -b & z_1 & -z_2 \\ b & a & z_2 & z_1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

$0 = \omega(Bv_3,Bv_4) = 2\, z_2a - 2\, bz_1$ so $z_1/z_2 = a/b$ or $z_1 = ca$, $z_2 = cb$. By taking the basis dv_1, dv_2, $(1/d)v_3$, $(1/d)v_4$ with $d = (|c|)^{1/2}$ we can make $c = \pm 1$.

Q.E.D.

PROPOSITION 2.4 - Let $\lambda = \pm 1$ be an eigenvalue of multiplicity two. Then there exists a symplectic basis such that B has as it matrix

$$\begin{pmatrix} \lambda & c \\ 0 & \lambda \end{pmatrix}$$

where $c = -1$, 0, or 1.

The proof is left to the reader.

PROPOSITION 2.5 - Let λ be an eigenvalue of absolute value value different from one. Then given any basis of the root space of λ, there exists a basis of the root space of $1/\lambda$ such that the combined basis is symplectic. In terms of this basis B has the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where $A_2^{-1} = A_1^t$. In particular A_1 can be taken in Jordan form.

<u>Proof</u>: Take any basis $u_1,\ldots,u_n$ of the root space of λ.

By Lemma 1.8, $\omega(u_j,u_k) = 0$ for all j and k. By Lemma 1.2 we can complete this to a symplectic basis. In fact we can always pick vectors from the root space of $1/\lambda$ to complete the basis. Then the matrix of B has the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

Since $B^tJB = J$, a direct computation shows $A_1^tA_2 = I$.

Q.E.D.

§3 - GENERIC BIFURCATION OF EIGENVALUES FOR LINEAR SYSTEMS

In this section we study generic properties of C^1 maps from the closed interval $I = [-1,1]$ to $Sp(R^{2n})$, $C^1(I,Sp(R^{2n}))$. Using Theorem 6.1, Theorem 4.5, and Proposition 4.7 below, these result can be interpreted in the nonlinear setting. The matrices correspond the to derivative of the Poincaré map restricted to an energy surface. The parameter in I corresponds to moving along a one parameter family of closed orbits.

Let $T = \{B \in Sp(R^{2n}): B$ has a multiple eigenvalue$\}$ $= \{B \in Sp(R^{2n}): B$ is not 2-elementary$\}$. Let $T' = \{B \in T:$ i) B has an eigenvalue of multiplicity at least three or ii) B has two eigenvalues $\lambda_1 \neq \lambda_2$ of multiplicity two with $|\lambda_1|, |\lambda_2| \geq 1$ and $\operatorname{Im} \lambda_1, \operatorname{Im} \lambda_2 \geq 0\}$.
T and T' are both semi-algebraic subset of $Sp(R^{2n})$ so there exist submanifolds of $Sp(R^{2n})$, $T_1,\ldots,T_k$ such that $T' = \bigcup_{j=1}^{q} T_j$ and $T = \bigcup_{j=1}^{k} T_j$. See [A2].
T is closed so by the Thom Transversality Theorem, $G = \{A \in C^1(I,Sp(R^{2n})): A$ is transverse to all the $T_j\}$ is dense and open in $C^1(I;Sp(R^{2n}))$. Also see Theorems 7.1 and 7.2 below. An element $A \in G$ is called a <u>generic</u>

curve of symplectic matrices, or just a generic curve. The result of this section completely describe what can occur for $A \in G$ and $A(0) \in T$. These results are directly connected with the parametric stability studied by Krein, Moser, Diliberto, and Coppel and Howe. See [D1], [C1], [C2], [C3], [M5]. However as noted below the results are slightly different.

LEMMA 3.1 - Let $A \in G$. If $A(0) \in T$ then $A(0) \in T-T'$.

Proof: By the normal forms of §2 the interior of T is empty. Thus all the T_j have at least codimension one. Also by these normal forms the interior of T' in T is empty so T' has codimension one in T and codimension two in $Sp(R^{2n})$. Thus any $A \in G$ can not intersect any of the $T_1, \ldots, T_q$ where $T' = \bigcup_{j=1}^{q} T_j$.

Q.E.D.

REMARK - The codimensions probably also follow from the fact that $T \neq Sp(R^{2n})$ and $T' \neq T$ using general facts about algebraic subvarieties.

THEOREM 3.2 - I) Let $B \in T$ have a multiple eigenvalue λ such that 1. the root space of λ equals its eigenspace or 2. $|\lambda| \neq 1$ and $\operatorname{Im} \lambda \neq 0$. Then there does not exist a $A \in G$ such that $A(0) = B$, i.e. B is not in the image of any generic curve $A \in G$.

II. Let $B \in T - T'$ have a multiple eizenvalue λ such that 1. $|\lambda| = 1$ or $\mathrm{Im}\,\lambda = 0$ and 2. the root space of λ does not equal its eigenspace. Then there exists $A \in G$ such that $A(0) = B$.

REMARK - Let $A \in G$ be such that $A(0) \in T - T'$. Then by the theorem the root space does not equal the eigenspace. The reader can check that the linear diffeomorphism is not Liapounov stable at the origin. Or see [M5]. Also the case of a mixed eigenvalue in I above is not parametrically stable and the case of a eigenvalue of positive type is parametrically stable even though both do not occur generically.

We state the last result before giving the proofs.

THEOREM 3.3 - Let $A \in G$ be such that $A(0) \in T$. Then there exists $\delta > 0$ and eigenvalues $\lambda(e)$ and $\mu(e)$ of $A(e)$ such that one of the following occurs:

I. (i) $\lambda(0) = \mu(0)^{-1}$, (ii) $|\lambda(e)| = |\mu(e)| = 1$ and $\lambda(e) \neq \mu(e)^{-1}$ for $-\delta \le e < 0$ (resp. $0 < e \le \delta$), and (iii) $\lambda(e) = \mu(e)^{-1}$ and $|\lambda(e)| > 1$ for $0 < e \le \delta$ (resp. $-\delta \le e < 0$) or

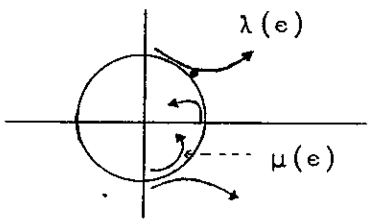
II. (i) $\lambda(0) = 1$ or -1, (ii) $|\lambda(e)| = 1$ and $\lambda(e) \neq \pm 1$ for $-\delta \le e < 0$ (resp. $0 < e \le \delta$) and

(iii) $\lambda(e)$ is real and $\neq \pm 1$ for $0 < e \leq \delta$ (resp. $-\delta \leq e < 0$) or

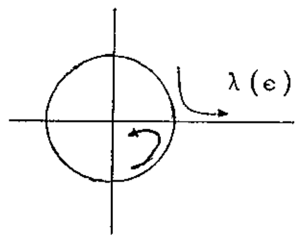
III. (i) $\lambda(0) = \mu(0)^{-1}$, (ii) $\mathrm{Im}\,\lambda(e) = \mathrm{Im}\,\mu(e) = 0$ and $\lambda(e) \neq \mu(e)^{-1}$ for $-\delta \leq e < 0$ (resp. $0 < e \leq \delta$) and (iii) $\lambda(e) = \mu(e)^{-1}$ and $\mathrm{Im}\,\lambda(e) > 0$ for $0 < e \leq \delta$ (resp. $-\delta \leq e < 0$).

In pictures we have one of the following cases of change of eigenvalues:

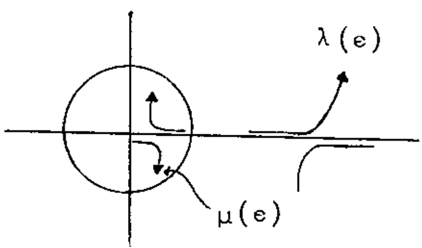
(i)



(ii)



(iii)



Proof of Theorem 3.2 - By Corollary 1.10 and Proposition 1.12 there are only the following cases:

I 1. λ is an eigenvalue of positive or negative type of multiplicity two

2. λ is an eigenvalue of mixed type of multiplicity two and its eigenspace equals its root space

3. λ is a real multiple eigenvalue and its eigenspace equals its root space

4. λ is a multiple eigenvalue and $|\lambda| \neq 1$ and $\operatorname{Im} \lambda \neq 0$.

II 1. λ is a mixed eigenvalue and its eigenspace does not equal its root space

2. λ is a real multiple eigenvalue and its eigenspace does not equal its root space.

Case I1.

Let $B \in T - T'$ have a positive eigenvalue of multiplicity two. It suffices to show T has codimension ≥ 2 at B. We show this fact by constructing a two dimensional subspace in the tangent space to $Sp(R^{2n})$ at B that is not tangent to T.

By Propositions 2.1 and 1.12 there exists a symplectic basis $u_1, \ldots, u_{2n}$ such that the matrix of B has the following form:

$$\begin{pmatrix} aI & 0 & -bI & 0 \\ 0 & A_1 & 0 & A_2 \\ bI & 0 & aI & 0 \\ 0 & A_3 & 0 & A_4 \end{pmatrix}$$

where I is the 2×2 identity matrix. Let $a = \cos e$, $b = \sin e$, $a(t) = \cos(e+t)$, and $b(t) = \sin(e+t)$. Let $B(t)$ be the transformation whose matrix equals

$$\begin{pmatrix} a(t) & 0 & 0 & -b(t) & 0 & 0 \\ 0 & a(-t) & 0 & 0 & -b(-t) & 0 \\ 0 & 0 & A_1 & 0 & 0 & A_2 \\ b(t) & 0 & 0 & a(t) & 0 & 0 \\ 0 & b(-t) & 0 & 0 & a(t) & 0 \\ 0 & 0 & A_3 & 0 & 0 & A_4 \end{pmatrix}$$

$B(0) = B$ but $B(t) \notin T$ for small $t \neq 0$. We now use the symmetry of B to construct a whole plane of such matrices. Let $C(d)$ have the following matrix

$$\begin{pmatrix} \begin{matrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & I \end{matrix} & 0 \\ 0 & \begin{matrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & I \end{matrix} \end{pmatrix}$$

where $c = \cos(d/2)$ and $s = \sin(d/2)$. Let $B(t,d) = C(d)B(t)C(-d)$. Note that $B(0,d) = B$ for all d. Also $C(d)$ is a symplectic matrix so conjugation by it represents a symplectic change of basis. Direct computation shows that the derivative of $B(t,d)$ with respect to t at $t = 0$ has the following form:

$$B'(0,d) = C(d)\, B'(0)\, C(-d) =$$

$$\cos(d)\begin{pmatrix} -b & 0 & 0 & -a & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & -b & 0 & 0 \\ 0 & -a & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \sin(d)\begin{pmatrix} 0 & -b & 0 & 0 & -a & 0 \\ -b & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & -b & 0 \\ a & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{tB'(0,d) : d \in [0,2\pi],\ t \in R\}$ forms a plane in the tangent space of $Sp(R^{2n})$ at B. Conjugation by $C(d)$ preserves the form of B, so if $B'(0)$ is not tangent to T at B then none of the matrices $t\,B'(0,d)$ can be tangent.

Assume $B'(0)$ is tangent to T. Then there would exist a curve $D(t)$ in T such that $D(0) = B$ and $D'(0) = B'(0)$. Four of the eigenvalues of $B'(0)$ are $\pm b \pm ia$ and the root space of $D(0)$ equals its eigenspace. By a lemma in [C2] four of the eigenvalues of $D(t)$ would have to be $(a+ib) + t(-b+ia) + \Theta(t)$, $(a+ib) + t(b-ia) +$

$+\theta(t)$ and their complex conjugates. The other eigenvalues would have to be distinct. Clearly the eigenvalues of $D(t)$ can not remain multiple so $D(t)$ can not lie in T.

We have shown there exists a two dimensional plane in the tangent space of $Sp(R^{2n})$ at B that does not lie in the tangent space of T at B. Thus T has codimension two at B.

<u>Case A2</u>.

Let $B \in T-T'$ have a mixed eigenvalue whose root space equals its eigenspace. By Proposition 2.2 there exists a symplectic basis for which B equals

$$\begin{pmatrix} a & -b & 0 & & & \\ b & a & 0 & & 0 & \\ 0 & 0 & A_1 & 0 & 0 & A_2 \\ & & 0 & a & -b & 0 \\ & 0 & 0 & b & a & 0 \\ & & A_3 & 0 & 0 & A_4 \end{pmatrix}$$

Form $B(t)$ by multiplying the first and second row by t and the $n+1$ and $n+2$ row by $1/t$. Let $C(d)$ equal

$$\begin{pmatrix} c & 0 & 0 & -s & 0 & 0 \\ 0 & c & 0 & 0 & -s & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ s & 0 & 0 & c & 0 & 0 \\ 0 & s & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

with $c = \cos(d/2)$ and $s = \sin(d/2)$. Let $B(t,d) = C(d)B(t)C(-d)B(1,d) = B$ for all d. An argument as before shows none of the $B'(1,d)$ are tangent to T.

Case I3.

If λ is equal to ± 1 the root space only has dimension two otherwise it is similar to case I2. If λ is real and $\neq \pm 1$ the proof is very similar to case I1. We omit the details.

Case I4.

Let $B \in T - T'$ have a multiple eigenvalue $\lambda = a+ib$ with $|\lambda| \neq 1$ and $\operatorname{Im} \lambda \neq 0$. If the root space equals the eigenspace then a proof similar to the above cases shows T has codimension two. Thus we can assume they are not equal. The essential ideas are contained in the case when $n = 4$ so we look at that case.

By Proposition 2.5 there exists a symplectic basis $u_1, \ldots, u_8$ such that B equals

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

with $A_1^{-1} = A_2^t$ and A_1 equal to

$$\begin{pmatrix} a & -b & a & -b \\ b & a & b & a \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

Set $A(x,y)$ equal to

$$\begin{pmatrix} x & -y & x & -y \\ y & x & y & x \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}$$

and

$$B(x,y) = \begin{pmatrix} A(x,y) & 0 \\ 0 & A(x,y)^{t-1} \end{pmatrix}$$

Define $F: R^2 \times Sp(R^8) \to T \subset Sp(R^8)$ by $F(x,y,C) = C^{-1}B(x,y)C$. F is analytic. Taking the derivative,

$DF(x,y,I)\cdot(u,v,E) = B(x,y)E - EB(x,y) + DB(x,y)(u,v)$.

The kernel of $DF(x,y,I)$ is four dimensional and is contained in the tangent space to $Sp(R^8)$. It is made up of matrices

$$\begin{pmatrix} D & E & & \\ 0 & D & 0 & \\ & 0 & -D^t & 0 \\ & & -E^t & -D^t \end{pmatrix}$$

where D and E are 2×2 matrices in the space spanned by I and J, i.e. they are the general product of a rotation and an expansion. The derivative at the point (x,y,C) is conjugate to the one at (x,y,I) so its kernel has the same dimension:

$$DF(x,y,C)(u,v,E) = (C^{-1}\,B(x,y)C)C^{-1}E - C^{-1}E(C^{-1}\,B(x,y)C) + \\ + \; C^{-1}DB(x,y).(u,v)C \;.$$

The the image of F has codimension two.

Next we show F is onto a neighborhood of B in T. Let $A(t)$ be a differentiable curve in T such that $A(0) = B$. Let γ be a small closed curve around the multiple eigenvalue for $A(0)$ and hence around the multiple eigenvalue for $A(t)$ for small t. Let $2\lambda(t) = \frac{1}{2\pi i}\int_\gamma \zeta\,\frac{p_t'(\zeta)}{p_t(\zeta)}d\zeta$ where $p_t(x)$ is the characteristic polynomial for A_t. Then $\lambda(t)$ is the multiple eigenvalue for $A(t)$ and it varies differentiably. The lemma below shows that there is a differentiable curve of matrices $C(t)$ such that $C(0) = I$ and $C(t)A(t)C(t)^{-1} = B(\text{Re}\,\lambda(t),\ \text{Im}\,\lambda(t))$. Thus $A(t) = F(\text{Re}\,\lambda(t),\ \text{Im}\,\lambda(t),\ C(t))$. Since T is a union of manifolds this shows F is onto a neighborhood of B in T. This neighborhood has codimension two. We are done except the lemma.

LEMMA 3.4 - Let $A(t)$ have an eigenvalue $\lambda(t)$ of multiplicity exactly two, with $|\lambda(t)| \neq 1$, with $\operatorname{Im} \lambda(t) \neq 0$, and with the root space not equal to the eigenspace. Then there exists $C(t)$ such that $C(0) = I$ and $C(t)A(t)C(t)^{-1} = B(\operatorname{Re} \lambda(t), \operatorname{Im} \lambda(t))$.

REMARK - What this ammounts to showing is that the change of coordinates to Jordan form can be choosen differentiably when the multiplicity remains constant.

<u>Proof</u>: Let $P_t = \frac{1}{2\pi i} \int_{\gamma} (\zeta I - A(t))^{-1} d\zeta$.

If γ is a small closed curve that surrounds just this one eigenvalue, $\lambda(t)$, then P_t is a projection onto the root space of $\lambda(t)$ for $A(t)$. See [R5]. Take $v = u_3 - iu_4$ $v(t) = P_t v$, and $u(t) = (A(t) - \lambda(t))v(t)$. Then $u(0) =$ $= u_1 - iu_2$. By continuity $u(t) \neq 0$ for small t. Direct computation shows that $u(t)$ is an eigenvector of $A(t)$ for $\lambda(t)$. Let $u_1(t) = \operatorname{Re} u(t)$, $u_2(t) = -\operatorname{Im} u(t)$, $u_3(t) = \operatorname{Re} v(t)$, and $u_4(t) = -\operatorname{Im} v(t)$. Then $u_j(0) = u_j$ is part of the symplectic basis we started with and the $u_j(t)$ vary differentiably. We can complete these vectors to a symplectic basis $u_j(t)$ where the other vectors lie in the root space of $\lambda(t)^{-1}$. Let $C(t)$ be the change of basis from the $u_j(t)$ to the u_j basis. Then $C(t)A(t)C(t)^{-1} = B(\operatorname{Re} \lambda(t), \operatorname{Im} \lambda(t))$ and $C(0) = I$. Q.E.D.

Case II1.

Let $B \in T-T'$ have a mixed eigenvalue of multiplicity two whose eigenspace does not equal its root space. For simplicity assume $2n=4$. By Proposition 2.3 there exists a symplectic basis for which B equals

$$\begin{pmatrix} a & -b & ca & -cb \\ b & a & cb & ca \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}$$

with $c = \pm 1$ and $a,b \neq 0$. Define a curve $B(t)$, through B by $B(t)$ equals

$$\begin{pmatrix} a(t) & -b(t) & ca(t) & -cb(t) \\ b(t) & a(t) & cb(t) & ca(t) \\ 0 & 0 & a(t) & -b(t) \\ 0 & 0 & b(t) & a(t) \end{pmatrix}$$

with $a(t) = \cos(d+t)$, $b(t) = \sin(d+t)$, $a(0) = a$, and $b(0) = b$. Define $F:R \times Sp(R^4) \to T$ by $F(t,C) =$ $= C^{-1} B(t)C$. F is analytic. An argument as in Case I4 shows F is onto a neighborhood of B in T.

$DF(t,I)(s,E) = B(t)E - EB(t) + s\,B'(t)$. A direct computation shows that its kernel is two dimensional and is spanned by the matrices

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix and J is the 2×2 standard skew symmetric matrix. Thus the image of F has codimension one. A neighborhood of B in T has codimension one.

Case II 2.

We omit the proof in this case. The proof should be divided into the cases when $\lambda = \pm 1$ or when it does not.

Q.E.D.

To prove Theorem 3.3 we first give a lemma.

LEMMA 3.5 - Let $A \in G$ be such that $A(0) \in T$. Then in every neighborhood of $A(0)$ in $Sp(R^{2n})$ there are matrices $B_1 \in Sp(R^{2n}) - T$ with the same number of eigenvalues of absolute value one and the same number of real eigenvalues and there are matrices $B_2 \in Sp(R^{2n}) - T$ with either fewer eigenvalues of absolute value one or fewer real eigenvalues.

§4 - BASIC NONLINEAR THEORY

The equations for the flow of a Hamiltonian vector field are just generalizations to manifolds of Hamilton's equations of motion in mechanics. To put these equations on a manifold we start with an even dimensional manifold, M^{2n}, and a closed two form ω, i.e. the exterior derivative of ω is zero, $d\omega = 0$. ω is called <u>nondegenerate</u> if for each point $p \in M$ the map $\bar{\omega}_p : T_pM \to T_p^*M$ is an isomorphism from the tangent space at p to the cotangent space at p, $\bar{\omega}_p(v) = i_v\omega = \omega(v,\cdot)$. A <u>symplectic manifold</u> is a pair (M^{2n},ω) where ω is a closed nondegenerate two form on M. Local coordinates on M, $x^1,\ldots,x^{2n}$, such that $\omega = \Sigma_{j=1}^{n} dx^j \wedge dx^{j+n}$ are called <u>symplectic coordinates</u>. We prove below in Theorem 5.1, that there exist symplectic coordinates in a neighborhood of every point in M. This is the nonlinear analogue of Lemma 1.1.

Given a C^{r+1} real valued function on M, $H \in C^{r+1}(M,R)$, its exterior derivative is a C^r section of the cotangent bundle, $dH \in \Gamma^r(T^*M)$. $\bar{\omega}$ induces an isomorphism between $\Gamma^r(TM) = \mathfrak{X}(M)$ and $\Gamma^r(T^*M)$. Associated to dH is a vector field X_H. X_H is called a <u>Hamiltonian vector field</u>. The set of all such Hamiltonian vector fields is denoted $\mathfrak{X}_H^r(M)$. Let us see what this looks like in symplectic coordinates. If $H(x^1,\dots,x^{2n})$ is a real valued function then $dH = \frac{\partial H}{\partial x^1}\, dx^1 + \dots + \frac{\partial H}{\partial x^{2n}}\, dx^{2n}$ and $X_H = \frac{\partial H}{\partial x^{n+1}} \frac{\partial}{\partial x^1} + \dots + \frac{\partial H}{\partial x^{2n}} \frac{\partial}{\partial x^n} - \frac{\partial H}{\partial x^1} \frac{\partial}{\partial x^{n+1}} - \dots - \frac{\partial H}{\partial x^n} \frac{\partial}{\partial x^{2n}}$. If $x^1(t),\dots,x^{2n}(t)$ is a trajectory of X_H then $\frac{dx^1}{dt} = \frac{\partial H}{\partial x^{n+1}},\dots,\frac{dx^n}{dt} = \frac{\partial H}{\partial x^{2n}}, \frac{dx^{n+1}}{dt} = -\frac{\partial H}{\partial x^1},\dots, \frac{dx^{2n}}{dt} = -\frac{\partial H}{\partial x^n}$. These are just Hamilton's equations of motion.

A <u>symplectic diffeomorphism</u> is a diffeomorphism, f, on a symplectic manifold, (M,ω), such that f preserves ω, i.e. $f^*\omega = \omega$ or $\omega(u,v) = \omega(Df(m)u, Df(m)v)$ for all $m \in M$ and $u,v \in T_mM$.

EXAMPLE 4.1 - The positions of n-particles in R^3 is described by $(x^1,\dots,x^{3n}) \in R^{3n}$ where $(x^{3j+1}, x^{3j+2}, x^{3j+3})$ describes the position of the j^{th} particle. Let their velocities be $(v^1,\dots,v^{3n})$ and their masses $m_1,\dots,m_{3n}$ where $m_{3j+1} = m_{3j+2} = m_{3j+3}$. Let $p^j = m_j v^j$ be the linear momentum. Then Newton's equations are

$$\frac{dx^j}{dt} = p^j/m_j \qquad \text{and} \qquad \frac{dp^j}{dt} = F_j(x^1,\dots,x^{3n})$$

where $(F_1,\ldots,F_{3n})$ is the total force acting on the particles. If the forces are assumed to be conservative we get a potential energy $V(x^1,\ldots,x^{3n})$ such that $F_j = -\dfrac{\partial V}{\partial x^j}$. The kenetic energy of the system is $K(p^1,\ldots,p^{3n}) = \sum_j (p^j)^2/2m_j$. Let $H(x,p) = V(x) + K(p)$. The equations of motions can be given by

$$\frac{dx^j}{dt} = \frac{\partial H}{\partial p^j} \quad \text{and} \quad \frac{dp^j}{dt} = -\frac{\partial H}{\partial x^j}\,.$$

If we let $\omega = \Sigma\, dx^j \wedge dp^j$ then the Hamiltonian vector field X_H just gives the above equations of motion for the particles.

Let $r^j = (x^{3j+1}, x^{3j+2}, x^{3j+3})$. Then for most laws of attraction $V(r^1,\ldots,r^n) = \pm\infty$ when $r^j = r^k$ for $j \neq k$. Therefore V is not well defined on all of R^{3n}. Let $W = \{(r^1,\ldots,r^n)\in R^{3n} : r^j \neq r^k \text{ for } j \neq k\}$. Let $M = T^*W = W \times R^{3n}$. Let ω be as above. Then (M,ω) is a symplectic manifold and $H:M \to R$ is a real valued function. X_H then describes the motion. It is not clear from this presentation why the momentum should be considered a covector while velocity is a vector. However we have the following theorem.

THEOREM 4.2 - Let W be a manifold and $M = T^*W$ its cotangent bundle. Then there exists a canonical two form ω on M such that (T^*W,ω) is a symplectic manifold. In fact $\omega = -d\theta$ where θ is a one form.

<u>Proof</u>: We have the following diagram

$$\begin{tikzcd}
& T(T^*W) \arrow[dl, "\pi"', "\rho"] \arrow[dr, "T\pi^*"] & \\
T^*W \arrow[dr, "\pi^*"] & & TW \arrow[dl] \\
& W &
\end{tikzcd}$$

$\pi\colon T(T^*W) \to T^*W$ is the natural projection from the tangent space. $\pi^*\colon T^*W \to W$ is the natural projection and $T\pi^*\colon T(T^*W) \to TW$ is its derivative. Fixing $x \in W$, T_x^*W is a linear space. There is a natural projection from its tangent space to itself $\rho\colon T(T_x^*W) \to T_x^*W$. This extends to a projection $\rho\colon TT^*W \to T^*W$. If $(x,p,u,v) \in TT^*W$ then $\pi(x,p,u,v) = (x,p)$, $\rho(x,p,u,v) = (x,v)$, and $T\pi^*(x,p,u,v) = (x,u)$. Let $\langle\ ,\ \rangle_x\colon T_xW \times T_x^*W \to R$ be the natural pairing.

Define a one form θ and a two form ω on $M = T^*W$ by $\theta(U) = \langle T\pi^*U, \pi U\rangle$ and $\omega(U_1,U_2) = \langle T\pi^*U_1, \rho U_2\rangle - \langle T\pi^*U_2, \rho U_1\rangle$. Then $\theta(x,p,u,v) = \langle u,p\rangle_x$

and $\omega((x,p,u_1,v_1), (x,p,u_2,v_2)) = \langle u_1,v_2\rangle - \langle u_2,v_1\rangle$. ω is obviously nondegenerate and alternating.

Let us look at ω in local coordinates. Take coordinates $x^1,\ldots,x^n$ on W. $\frac{\partial}{\partial x^i}(x) \in T_xW$. These coordinates induce coordinates $v^1,\ldots,v^n$ on the fibers of TW by $v^j(\frac{\partial}{\partial x^i}) = \delta_{ij}$, i.e. is one if and only if $i = j$. These coordinates on TW induce coordinates on T^*W, $x^1,\ldots,x^n$, $p^1,\ldots,p^n$, such that for $p_x \in T^*_x W$ and $v_x \in T_xW$, $\langle v_x,p_x\rangle = \Sigma^n_{j=1} p^j(p_x)\, v^j(v_x)$. In these coordinates $\Theta = \Sigma^n_{j=1} p^j dx^j$ and $\omega = \Sigma^n_{j=1} dx^j \wedge dp^j$. $\omega = -d\Theta$.

Q.E.D.

EXAMPLE 4.3 - We use the theorem to show how the first example can be explained in a more invariant fashion. This includes a discussion of the geodesic flow for a metric on both the tangent bundle and on the cotangent bundle.

Let g be a Riemannian metric on a manifold W, i.e. for each $x \in W$ $g_x : T_xW \times T_xW \to R$ is a positive definite symmetric bilinear map. Let $K:TW \to R$ be the associated quadratic function, $K(x,v) = \frac{1}{2} g_x(v,v)$. Let $V:W \to R$ be a potential function. We extend $V:TW \to R$ by letting $V(v) = V(\pi v)$ where $\pi:TW \to W$. Let $L:TW \to R$ be given by $L = K-V$. L is called the <u>Lagrangian</u>.

Taking the derivative of L along the fibers we get $D_2L(x,v)u = D_2K(x,v)u = g_x(v,u)$ for $u \in T_xW$. Therefore $D_2L(x,v)$ is a linear map from T_xW to R or $D_2L(x,v)$ is a covector in T_x^*W. $D_2L(x,v) = g_x(v,\cdot) = \bar{g}(v)$. Since g_x is nondegenerate $\bar{g}_x$ is a linear isomorphism between the fibers T_xW and T_x^*W. In fact $\bar{g}$ is a diffeomorphism from TW to T^*W. This diffeomorphism is called the <u>Legendre transformation</u>. Let $A:TW \to R$ be defined by $A(x,v) = D_2L(x,v)v = \bar{g}(v)v = g_x(v,v) = 2K(x,v)$ for $v \in T_xW$. A is called the <u>action</u>. Let $E = A-L = K+V:TW \to R$. E is called the <u>energy</u>.

Since the canonical two form ω is on T^*W we use D_2L to transfer E to T^*W. Let $H:T^*W \to R$ be defined by $H(x,p) = E(x,\bar{g}_x^{-1}p) = K(x,\bar{g}_x^{-1}p) + V(x)$. Let X_H be the associated Hamiltonian vector field on T^*W, $X_H \in \mathfrak{X}_H(T^*W)$. This is just the procedure we used in Example 4.1 to define the motion of n particles in R^3. $K(x,\bar{g}_x^{-1}(p))$ is a quadratic function on the fibers of T^*W. Thus H is the sum of an (arbitrary) quadratic function plus a potential.

We want to show that in the present case, where the Riemannian metric need not be independent of the base point $x \in W$, the motion still corresponds to the Euler Lagrange equations for motion on W under the influence of the potential V.

Let $(x(t),p(t))$ be a trajectory of X_H in T^*W. Take local coordinates on TW and T^*W as in the above theorem, $x^1,\dots,x^n$, $v^1,\dots,v^n$, $p^1,\dots,p^n$. $\omega = \Sigma\, dx^j \wedge dp^j$.

The path $(x(t),p(t))$ satisfies

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p^i} \quad \text{and} \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial x^i} .$$

Let G_x be the matrix representing g_x in these local coordinates, i.e. $g_x(u,v) = u^t\, G_x v$ where tangent vectors are columns and u^t is the transpose (a row). Then $\bar{g}_x(v) = p = v^t G_x$ is a row, $v^t = p\, G_x^{-1}$, and $p^i = p^i(\bar{g}_x(v)) = p.e_i = D_2\, L(x,v).e_i = \frac{\partial L}{\partial x^i}(x,v)$. $H(x,p) =$ $= \frac{1}{2}\, pG_x^{-1}\, G_x\, G_x^{-1}\, p^t + V(x) = \frac{1}{2}\, p\ \ G_x^{-1}\, p^t + V(x)$. $\frac{\partial H}{\partial p^i}(x,p) = p\, G_x^{-1}\, e_i = v.e_i = v^i$. Thus $\frac{dx^i}{dt} = v^i$ the first set of equations we want. $\frac{\partial H}{\partial x^i}(x,p) = \frac{\partial V}{\partial x^i}(x) +$ $+ \frac{1}{2}\, p\, G_x^{-1}\, \frac{\partial G_x}{\partial x_i}\, G_x^{-1}\, p^t - \frac{1}{2}\, p\, \frac{\partial G_x^{-1}}{\partial x^i} G_x\, G_x^{-1}\, p^t -$ $- \frac{1}{2}\, p\, G_x^{-1}\, G_x\, \frac{\partial G_x^{-1}}{\partial x^i}\, p^t = \frac{\partial E}{\partial x^i}(x,v) - p\, G_x^{-1}\, \frac{\partial G_x}{\partial x^i}\, G_x^{-1}\, p^t =$ $= \frac{\partial E}{\partial x^i}(x,v) - \frac{\partial A}{\partial x^i}(x,v) = -\frac{\partial L}{\partial x^i}(x,v)$. Combing this last calculation with representation of p^i above we get $\frac{dp^i}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x,v)\right) = -\frac{\partial H}{\partial x^i}(x,p) = \frac{\partial L}{\partial x^i}(x,v)$ or

g.

$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x,v)\right) = \frac{\partial L}{\partial x^i}(x,v)$. This is the second set of equations we want. Thus $\bar{g}^{-1}(x(t),p(t))$ satisfy the Euler Lagrange equations of motion under the influence of a potential V. In particular if $V = 0$ then the curves $x(t)$ are geodesics on W for the metric g_x. We have constructed the "geodesic flow" on T^*W.

Instead of transfering E to T^*W, we could have pulled back the two form to TW, $\varphi = \bar{g}^*\omega$. Let X_E be the Hamiltonian vector field for E using the two form φ. The reader can check that $(x(t),v(t))$ is a trajectory for X_E iff $\bar{g}(x(t),v(t))$ is a trajectory for X_H. We could have used this procedure to construct the geodesic flow on TW. See [L2, p. 109].

For a more general and complete discussion of the above procedures see [A1].

Q.E.D.

REMARK - A more invariant derivation of Lagrange equations from Hamilton's equations is as follows:

Let $p = D_2\, L(x,v)$. Then $H(x,p) = H(x,D_2\, L(x,v)) =$
$= A(x,v) - L(x,v) = D_2\, L(x,v).v - L(x,v)$. Taking the derivative with respect to v we get

$D_2\, H(x,p)\, D_{22}\, L(x,v) = D_{22}\, L(x,v).v + D_2\, L(x,v) -$
$- D_2\, L(x,v) = D_{22}\, L(x,v).v$. Thus $D_2\, H(x,p) D_{22}\, L(x,v) =$

$= D_{22}\ L(x,v).v.$ $v \in T_x W$ and $D_2\ H(x,p) \in T_x^{**}\ W.$ Since $D_{22}\ L(x,v)$ is invertible the usual identification shows $v = D_2\ H(x,p).$ $\frac{dx}{dt} = D_2\ H(x,p) = v.$ Next taking the derivative of the same equation above with respect to x holding v fixed we get $D_1\ H(x,p) + D_2\ H(x,p) D_{21}\ L(x,v) =$ $= D_{21}\ L(x,v).v - D_1\ L(x,v).$ As above $D_2\ H(x,p) D_{21}\ L(x,v) =$ $= D_{21}\ L(x,v).v$ so $D_1 H(x,p) = -D_1\ L(x,v).$ Then $\frac{d}{dt}(D_2\ L(x,v)) = \frac{dp}{dt} = -D_1\ H(x,p) = D_1\ L(x,v).$

Q.E.D.

Next we prove that energy and the two form are preserved by the flow.

THEOREM 4.4 - Let $X_H \in \mathfrak{X}_H(M)$ have a flow $\varphi(t,m) = \varphi_t(m)$. Then $H\ \varphi(t,n) = H(m)$ and $\varphi_t^*\ \omega = \omega$. The fact that $\varphi_t^*\ \omega = \omega$ is equivalent to the fact that $D_2\ \varphi(t,m): T_m M \to T_m M$ is a symplectic transformation, i.e. preserves ω.

<u>Proof</u>: Fix $m \in M$ and let $k(t) = H \circ \varphi(t,m)$.

$k'(t) = DH(\varphi(t,m)).X_H(\varphi(t,m)) = \omega(X_H(\varphi(t,m)), X_H(\varphi(t,m))) = 0$ since ω is alternating. Therefore $k(t) = k(0)$.

To prove $\varphi_t^*\ \omega = \omega$ we first look at the Lie derivative. Let $X = X_H$. $L_X \omega = i_X\ d\omega + d\ i_X \omega = 0 + d^2 H = 0$. See [A1] for the formula for the Lie derivative.

Then

$$\frac{d}{ds}(\varphi_s^* \omega)(m)\Big|_{s=t} = \varphi_t^* \frac{d}{ds} \varphi_s^* \omega\Big|_{s=0} (m) = (\varphi_t^* L_X\omega)(m) = 0.$$

$\varphi_o = id$ so $\varphi_o^*\omega = \omega$. Therefore $\varphi_t^*\omega = \omega$ for all t.

It is instructive to write out the second proof in symplectic coordinates. Writing $X = X_H$ as a column vector and its parials in the various columns we get $DX = \begin{pmatrix} A & B \\ C & D' \end{pmatrix}$ where $A = \left(\frac{\partial^2 H}{\partial x^j \partial x^{i+n}}\right)$, $B = \left(\frac{\partial^2 H}{\partial x^{j+n}\,\partial x^{i+n}}\right)$, $C = \left(-\frac{\partial^2 H}{\partial x^j\,\partial x^i}\right)$, and $D' = \left(-\frac{\partial^2 H}{\partial x^{j+n}\,\partial x^i}\right)$. A direct computation show that since $D' = -A^t$ and B and C are symmetric, it follows that $(DX)^t J + J(DX) = 0$, or $\omega(DX\cdot,\cdot) + \omega(\cdot, DX\cdot) = 0$. Let $g(t)(u,v) = (\varphi_t^*\omega)(m)(u,v) =$
$= \omega(\varphi_t(m))(D\,\varphi_t(m)u, D\,\varphi_t(m)v)$. $g'(t)(u,v) =$
$= \omega(\varphi_t(m)(\frac{d}{dt}\,D\varphi_t(m)u,\ D\varphi_t(m)v) +$
$+ \omega(\varphi_t(m))(D\,\varphi_t(m)u,\ \frac{d}{dt}\,D\,\varphi_t(m)v)$ since ω is constant in these local coordinates. $g'(t)(u,v) =$
$= \omega(\varphi_t(m))(DX\cdot D\varphi_t u,\ D\varphi_t\ v) + \omega(\varphi_t(m))(D\varphi_t u, DX\cdot D\varphi_t v) = 0$
by above.

Q.E.D.

Given a closed orbit γ for a Hamiltonian vector field X_H, we want to show there is a one parameter family of closed orbits nearby under very general conditions. Take a transversal to γ (surface of section)

Σ. Σ is a $2n-1$ dimensional local submanifold. Locally if $x^1,\ldots,x^{2n}$ are coordinates with $X_H = \frac{\partial}{\partial t}$ then let $\Sigma = \{(x^1,\ldots,x^{2n}):x^1 = 0\}$. By following the flow of X_H we define a local diffeomorphism from Σ to Σ called the Poincaré map. Let $\varphi(t,x)$ be the flow of X_H at time t. Using the implicit function theorem there is a $t(x)$ such that $\varphi(t(x),x) \in \Sigma$ where $t(x)$ is the period. $\Theta(x) = \varphi(t(x),x)$. $\Theta:\Sigma_1 \to \Sigma$ where Σ_1 is a neighborhood in Σ. Thus Θ is as differentiable as X_H. Locally on Σ there exists a vector field Y such that $\omega(X_H,Y) = 1$. Y is like $\frac{\partial}{\partial H}$. Define $\pi:T\Sigma \to T\Sigma$ by $\pi v_x = v_x - Y(x)\omega(X_H(x),v_x)$. π is a projection onto vectors tangent to the energy surface.

THEOREM 4.5 (One parameter closed orbit theorem).

Let γ be a closed orbit, $\Theta:\Sigma_1 \to \Sigma$ and $\pi:T\Sigma \to T\Sigma$ as above, and $m = \Sigma \cap \gamma$. Let $\Sigma' = \Sigma \cap H^{-1}H(m)$.

If $\pi \circ D\Theta(m) - \pi:T_m\Sigma \to T_m\Sigma'$ is a surjection then γ lies on a one parameter family of closed orbits that are fixed points of Θ.

<u>Proof</u>: Take local coordinates $x^1,\ldots,x^{2n}$ such that

$$X_H = \frac{\partial}{\partial x^1} \quad\text{and}\quad x^{n+1} = H. \quad dH(m) \neq 0 \quad\text{since } X_H(m) \neq 0.$$

Let $\Sigma = \{x : x^1(x) = 0\}$. Let $p:\Sigma \to R^{2n-2}$ be defined by $p(x) = (x^2(x),\ldots,x^n(x),x^{n+2}(x),\ldots,x^{2n}(x))$.

Let $\psi:\Sigma_1 \to R^{2n-2}$ be defined by $\psi(x) = p\circ\Theta(x)-p(x)$. $D\psi'(m)$ is a surjection by the assumption. By the implicit function theorem there is a one dimensional set in Σ_1 that are taken to zero by ψ. For these points $p\circ\Theta(x) = p(x)$. By Theorem 4.4 $H\Theta(x) = H(x)$ so $\Theta(x) = x$.

Q.E.D.

If $\pi\circ D\Theta(m)-\pi:T_m\Sigma \to T_m\Sigma'$ is a surjection the closed orbit is called <u>0-elementary</u>. The reader can check if 1 is not an eigenvalue of $D\Theta(m):T_m\Sigma' \to T_m\Sigma'$ then the orbit is 0-elementary, i.e. if $D\Theta(m)|T_m\Sigma$ 1-elementary then the orbit is 0-elementary.

PROPOSITION 4.6 - With the assumptions of Theorem 4.5, the one dimensional set of fixed points of Θ is either parameterized locally by H or by the period $t(x)$ of the closed orbit.

<u>Proof</u>: Assume the one dimensional set is not parameterized by H, i.e. the one dimensional set is tangent to the energy surface $H^{-1}(e_o)$. Thus there is a vector $u_2 \in T_m\Sigma'$ such that $D\Theta(m)u_2 = u_2$, or $D_2\varphi(t(m),m)u_2 = u_2 + aX_H(m)$. The reader can check that $Dt(m).u_2 = -a$. Let $u_1 = X_H$ and u_{n+1} be such that $\omega(u_1,u_{n+1}) = 1$ and $\omega(u_2,u_{n+1}) = 0$. Complete these vectors to a symplectic basis of T_mM as in Lemma 1.1, $u_1,\ldots,u_{2n}$.

Since $D_2\varphi(t(m),m)$ preserves the two form ω a calculation shows that $D_2\varphi(t(m),m)\ \Sigma_{j=1}^{2n} b^j u_j + (b^{2+n} - ab^{1+n})u_{n+2}$ + other terms. Thus $(\pi \circ DO(m)-\pi)\Sigma_{j=2}^{2n} b^j u_j = -ab^{n+1}u_{n+2} +$ + other terms. For $\pi \circ D\theta(m)-\pi$ to be a surjection it is necessary that $-a \neq 0$.

Q.E.D.

The idea of this proposition is used to continue the one parameter family of closed orbits on computers. It energy does not parameterize the orbits then the computer switches and uses the period.

PROPOSITION 4.7 - Let γ be a closed orbit, $\theta:\Sigma_1 \to \Sigma$ be the Poincaré map, $m = \Sigma \cap \gamma$, $e = H(m)$, and $\Sigma' = \Sigma_1 \cap H^{-1}(e)$. Then $\theta' = \theta|\Sigma'$ is a symplectic diffeomorphism, i.e. preserves $\omega|\Sigma'$.

<u>Proof</u>: Look at $D_2\varphi(t(m),m)$. Then $v_1, v_2 \in T_m\Sigma' \subset T_mM$

$D_2\varphi(t(m),m)v_i = a_i X_H(m) + Av_i$ where $A = D\theta'(m)$, i.e. its image in tangent to $H^{-1}(e)$. Then $\omega(v_1,v_2) =$ $= \omega(a_1X_H + Av_1,\ a_2X_H + Av_2) = \omega(Av_1,Av_2)$.

Q.E.D.

EXAMPLE 4.8 - We now give a local example near an elliptic fixed point. We will show later all flows near elliptic fixed points look like this formally, i.e. their C^∞ jets look like this example at their fixed points.

Let $\rho_j = \frac{1}{2}((x^j)^2 + (x^{j+n})^2)$. Let $i = (i_1,\ldots,i_n) \in (Z^+)^n$, i.e. an n-tuple of integers ≥ 0. Let $\rho^i = \rho_1^{i_1}\cdots\rho_n^{i_n}$ be the product. Let H be given by a power series in the ρ_j, $H(x^1,\ldots,x^{2n}) = \sum_{i\in(Z^+)^n} a(i)\rho^i$.

$$\frac{\partial H}{\partial x^j} = x^j \frac{\partial H}{\partial \rho_j} \quad \text{and} \quad \frac{\partial H}{\partial x^{j+n}} = x^{j+n}\frac{\partial H}{\partial \rho_j} .$$

$$X_H = x^{n+1}\frac{\partial H}{\partial \rho_1}\frac{\partial}{\partial x^1} + \cdots + x^{2n}\frac{\partial H}{\partial \rho_n}\frac{\partial}{\partial x^n} - x^1\frac{\partial H}{\partial \rho_1}\frac{\partial}{\partial x^{n+1}} - \cdots - x^n\frac{\partial H}{\partial \rho_n}\frac{\partial}{\partial x^{2n}} .$$

Thus $X_H(0) = 0$ and zero is a fixed point. Looking at the vector X_H as a column and taking partials we get that $DX_H(0) = \begin{pmatrix} 0 & A \\ -A & 0\end{pmatrix}$ where $A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n\end{pmatrix}$ and $a_j = a(0,\ldots,1,\ldots,0) = \frac{\partial H}{\partial \rho_j}(0)$. Thus the eigenvalues of $DX_H(0)$ are $\pm i\, a_j$ $j = 1,\ldots,n$. Zero is an elliptic fixed point.

$$\frac{d\rho_j}{dt} = x^j\frac{dx^j}{dt} + x^{j+n}\frac{dx^{j+n}}{dt} = x^j x^{j+n}\left(\frac{\partial H}{\partial \rho_j} - \frac{\partial H}{\partial \rho_j}\right) = 0 .$$

Thus the flow preserves the functions ρ_j. A neighborhood of the origin is filled with invariant tori so the origin is Liapounov stable.

If we replace ρ_j by $x^j x^{j+n}$ we would get a hyperbolic fixed point with eigenvalues $\pm a_j$.

Q.E.D.

PROPOSITION 4.9 (Generating functions): Take local symplectic coordinates $x^1,\ldots,x^n,y^1,\ldots,y^n$.

Let G be a real values function of $2n$ variables, $G(x^1,\ldots,x^n,\eta^1,\ldots,\eta^n)$ with $G(0) = 0$, $DG(0) = 0$, and $D^2G(0) = 0$. Let $f = (f_1,\ldots,f_{2n})$ be the map defined by $\xi^i = f_i(x,y)$ and $\eta^i = f_{i+n}(x,y)$ where

$$y^i = \eta^i + \frac{\partial G}{\partial x^i}(x,\eta) \quad \text{and} \quad \xi^i = x^i + \frac{\partial G}{\partial \eta^i}(x,\eta) .$$

Then f is a local symplectic diffeomorphism.

<u>Proof</u>: Since G starts with terms of order three it is possible to solve the equations $y^i = \eta^i + \frac{\partial G}{\partial x^i}$ and $\xi^i = x^i + \frac{\partial G}{\partial \eta^i}$ to give ξ^i and η^i as functions of x and y. Then $Df(0) = id$ so f is a local diffeomorphism.

$$dy^i = d\eta^i + \sum_j\left(\frac{\partial^2 G}{\partial x^j \partial x^i}\, dx^j + \frac{\partial^2 G}{\partial \eta^j \partial x^i}\, d\eta^j\right) ,$$

$$d\xi^i = dx^i + \sum_j\left(\frac{\partial^2 G}{\partial x^j \partial \eta^i}\, dx^j + \frac{\partial^2 G}{\partial \eta^j \partial \eta^i}\, d\eta^j\right) ,$$

$$dx^i \wedge dy^i = dx^i \wedge d\eta^i + \sum_j\left(\frac{\partial^2 G}{\partial x^j \partial x^i}\, dx^i \wedge dx^j + \frac{\partial^2 G}{\partial \eta^j \partial x^i}\, dx^i \wedge d\eta^j\right) ,$$

$$d\xi^i \wedge d\eta^i = dx^i \wedge d\eta^i + \sum_j\left(\frac{\partial^2 G}{\partial x^j \partial x^i}\, dx^j \wedge d\eta^i + \frac{\partial^2 G}{\partial \eta^j \partial \eta^i}\, d\eta^j \wedge d\eta^i\right) ,$$

Summing on i the reader can check that

$$\sum_{i=1}^{n} dx^i \wedge dy^i = \sum_{i=1}^{n} d\xi^i \wedge d\eta^i .$$

Q.E.D.

Next we show Example 4.8 is more general than it appears. We show that there exist symplectic coordinates such that the derivatives at the origin look like Example 4.8.

THEOREM 4.10 (Birkhoff normal form) - Let X_H be a C^∞ Hamiltonian vector field with an elliptic fixed point with the eigenvalues of $DX_H(0)$ additively independent over the integers. Then there exist C^∞ symplectic coordinates about the fixed point $x^1,\ldots,x^n$, $y^1,\ldots,y^n$ such that $H(x,y) = H(0) + \sum_{i\in(Z^+)^n} a(i)\rho^i + O(\infty)$ where $\rho^i = \rho_1^{i_1}\cdots\rho_n^{i_n}$, $\rho_j = (x^j)^2 + (y^j)^2$, $a(i)$ is a real coefficient, and $O(\infty)$ has all its derivatives at the origin equal to zero. Note the series may not converge but its sum with $O(\infty)$ gives a C^∞ function.

Proof: For a more detailed proof than we give see [M6]. Also a more general theorem is given there. The theorem is only about the derivatives of H at the origin, so it is sufficient to work with formal power series, i.e. series that may diverge. These can always be extended to give C^∞ functions with the same derivatives at the origin. We represent H by $H_2 + \ldots + H_n + \ldots$ where H_n is a homogeneous polynomial of degree n.

Take symplectic coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$

such that $H_2(x,y) = \frac{1}{2}(\lambda_1 \rho_1 + \ldots + \lambda_n \rho_n)$ where $\rho_j = x_j^2 + y_j^2$ and $\pm i\lambda_j$ are the eigenvalues of $DX_H(0)$. These coordinates exist since the eigenvalues are distinct. We assume coordinates have been found so that $H_2 + \ldots + H_{n-1}$ are in the correct form. We take a generating function G_n which is a homogeneous polynomial of degree n. Define a change of coordinates by $y_j = \eta_j + \frac{\partial G_n}{\partial x_j}(x,\eta)$, $\xi_j = x_j + \frac{\partial G_n}{\partial \eta_j}(x,\eta)$. We want to show we can pick G_n so $H_2 + \ldots + H_n$ are in the desired form in the ξ,η coordinates. $x_j = \xi_j - \frac{\partial G_n}{\partial \eta_j}(\xi,\eta) + \ldots$, $y_j = \eta_j + \frac{\partial G_n}{\partial \xi_j}(\xi,\eta) + \ldots$ where the dots represent higher order terms.

Then $H_2(x,y) = \sum_j \frac{1}{2} \lambda_j((\xi_j - \frac{\partial G_n}{\partial \eta_j} + \ldots)^2 + (\eta_j + \frac{\partial G_n}{\partial \xi_j} + \ldots)^2)$

$= H_2(\xi,\eta) + \Sigma\, \lambda_j(\eta_j \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \eta_j})G_n + \ldots,$

$H(x,y) = H_2(\xi,\eta) + \ldots + H_n(\xi,\eta) + \Sigma\, \lambda_j(\eta_j \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \eta_j})G_n + \ldots$

Thus we can elliminate all the terms of order n except the kernel of the operator $D = \Sigma\, \lambda_j(\eta_j \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \eta_j})$.

Looking in the associated complex coordinates $\zeta_j = \xi_j + i\,\eta_j$ and $\bar{\zeta}_j = \xi_j - i\,\eta_j$, $D = \Sigma\lambda_j(\zeta_j \frac{\partial}{\partial \zeta_j} - \bar{\zeta}_j \frac{\partial}{\partial \bar{\zeta}_j})$.

Monic polynomials $\zeta^k \bar{\zeta}^p = \zeta_1^{k_1} \ldots \zeta_n^{k_n} \bar{\zeta}_1^{p_1} \ldots \bar{\zeta}_n^{p_n}$ are eigenvectors and they are in the kernel exactly when $k_j = p_j$ for all j, $D\zeta^k\bar{\zeta}^p = \zeta^k\bar{\zeta}^p\, \Sigma(k_j - p_j)\lambda_j$. Thus we can elliminate all terms except for polynomials in the

$\rho_j = \zeta_j \bar{\zeta}_j = \xi_j^2 + \eta_j^2$.

Q.E.D.

The corresponding theorem for symplectic diffeomorphisms is also true.

THEOREM 4.11 (Birkhoff) - Let g be a C^∞ symplectic diffeomorphism with an elementary elliptic fixed point, i.e. all the eigenvalues are of absolute value one and are multiplicatively independent over the integers. Then there exist C^∞ symplectic coordinates $x^1,\ldots,x^{2n}$ such that

$$g_j(x) = x^j \cos L - x^{j+n} \sin L + O(\infty)$$

$$g_{j+n}(x) = x^j \sin L + x^{j+n} \cos L + O(\infty)$$

where $L = \Sigma_j\, a(j)\, \rho^j + O(\infty)$, $j = (j_1,\ldots,j_n) \in (Z^+)^n$, $\rho^j = \rho_1^{j_1}\cdots\rho_n^{j_n}$, $\rho_j = (x^j)^2 + (x^{j+n})^2$, and all the $O(\infty)$ have all their derivatives at the origin equal to zero.

The best proof is indicated in [S10]. The formal diffeomorphism associated to g is embedded in a formal vector field as the time one map. As in [M6], the formal vector field must be a formal Hamiltonian vector field. Take coordinates as in Theorem 4.10 that normalized the vector field. These coordinates normalize the time one map. This proof is probably what Birkhoff meant in [B2] but it is not clear. A direct proof of Theorem 4.11 is contained in [S9] and [S1].

Birkhoff worked with analytic diffeomorphisms and analytic change of coordinates. He hoped the formal power series would actually converge. Siegal proved that generically the change of coordinates would not converge. See [M6] and [M4]. We can see a differentiable analogue from Theorem 6.1 below. That theorem proves generically there are only a countable number of periodic points for a symplectic diffeomorphism. If a differentiable change of coordinates existed the elliminated that $O(\infty)$ terms then all points would lie on invariant tori. Some of these tori would be completely filled with periodic points so there would be uncountably many periodic points. Thus the change of coordinates can not converge for a generic set.

Kolmorgorov proved that if the matrix $a_{ij} = \frac{\partial^2 L}{\partial \rho_i \partial \rho_j}$ has nonzero determinant then the tori near the origin with "highly irrational rotations" exist and form a set of positive measure. Thus the tori with periodic points generically do not exist but those with irrational rotations do exist. For a summary of this work see [S11], [A3], [A4], [A5], [M7], [M9].

§5 - DARBOUX'S THEOREM AND RELATED RESULTS

The proofs given here arise from an idea Moser in [M8]. They were adapted to Darboux's theorem by A. Weinstein in [W1] and generalized to prove Morse's lemma in [P1].

THEOREM 5.1 (Darboux) - Let (M^{2n},ω) be a symplectic manifold and $p \in M$. Then there exist symplectic coordinates on M in a neighborhood U of p, i.e. there exist $g = (x^1,\ldots,x^{2n}): U \subset M \to R^{2n}$ such that $g\,\omega = \Sigma_{j=1}^{n} dx^j \wedge dx^{j+n}$.

Proof: By Lemma 1.1 we can pick a basis of vectors in T_pM such that $\omega(p)$ has J as its matrix in this one fiber. By exponentiating down there vectors we get coordinates $\psi = (y^1,\ldots,y^{2n})$ in a neighborhood of p, $\psi(p) = 0$. $(\psi_*\omega)(y) = \sum_{jk} a_{jk}(y)dx^j \wedge dx^k$ and $(a_{jk}(0)) = J$. Let Ω_1 be the two form given by $\Sigma_{j=1}^{n} dy^j \wedge dy^{j+n}$ defined in a neighborhood of 0. Let $\Omega_o = \psi_*\omega$, $\Delta = \Omega_1 - \Omega_o$, and $\Omega_t(y) = \Omega_o(y) + t(\Omega_1(y) - \Omega_o(y))$. $\Omega_t(0) = \Omega_o(0)$ for all t so there

is a neighborhood U' of the origin such that $\Omega_t(y)$ is nondegenerate for $y \in U'$ and $0 \le t \le 1$. Let $U = \varphi^{-1}(U')$. $d\Delta = d\Omega_t = 0$ for all t. By Poincaré Lemma there exists a one form α defined in U' such that $\Delta = d\alpha$. Let $X_t(y)$ be the time dependent vector field defined by $X_t(y) = -\bar{\Omega}_t^{-1}\alpha(y)$. i.e. $\Omega_t(X_t(y),\cdot) = -\alpha(y)$. $\alpha(y) \in T_y^* R^{2n}$. Let $\varphi_t(y)$ be the flow of the time dependent vector field X_t, i.e. $\varphi_o(y) = y$ and $\frac{d}{ds}\varphi_s(y)\big|_{s=t} = X_t(\varphi_t(y))$ for all t. φ_o = identity so $\varphi_o^*\Omega_o = \Omega_o$. $\frac{d}{ds}(\varphi_s^*\Omega_s)(y)\big|_{s=t} = \varphi_t^*\frac{d}{ds}\Omega_s\big|_{s=t} + \varphi_t^*\frac{d}{ds}\varphi_s^*\Omega_t\big|_{s=t}$ $= \varphi_t^*\Delta + \varphi_t^*(-\Delta) = 0$. Thus $\varphi_1^*\Omega_1 = \Omega_o$.
Let $g = \varphi_1 \circ \psi = (x^1,\ldots,x^{2n})$. Then $g^*\Omega_1 = \omega$.

Q.E.D.

COROLLARY 5.2 - Let V^{q+2k} be a C^∞ and ω a closed two form such that it has constant rank $2k$, i.e. the dimension of the image of $\bar{\omega}(p):T_pV \to T_p^*V$ is independent of p and is equal to $2k$. Let $p \in V$. Then there exist coordinates on M in a neighborhood U of p, $g = (x^1,\ldots,x^{2k+q})$, such that
$g_*\omega = \sum_{j=1}^{k} dx^j \wedge dx^{j+k}$.

<u>Proof</u>: We prove the corollary by induction on q. If $q = 0$ this result is just Theorem 5.1. Assume the corollary is true for $q-1$. There exists a $2k+q-1$ dimen-

sional subspace of T_pV on which $'\omega(p)$ has rank $2k$. Let V' be a $2k+q-1$ dimensional submanifold of V tangent to this subspace. Then in a neighborhood of p, $\omega|V'$ has rank $2k$. By the induction hypothesis there exist coordinates $h = (y^1,\dots,y^{2k+q-1})$ on V' such that $h_*\omega|V' = \Sigma_{j=1}^{k} dy^j \wedge dy^{j+k}$. Let $\ker(\omega) = \{u \in TV: \tilde{\omega}(u) = 0 \in T^*V\}$. In a neighborhood of p there exists a section of $\ker(\omega)$, $Y \in \Gamma(\ker(\omega)) \subset \subset \mathfrak{X}(V) = \Gamma(TV)$, such that $Y(m) \notin T_mV'$ for $m \in V'$. Let $\varphi(t,m)$ be the flow of Y. There exists a neighborhood W of V' in V and a differentiable function $s: W \to R$ such that $\varphi(s(m),m) \in V'$. Let $x^j(m) = y^j \circ \varphi(s(m),m)$ $j = 1,\dots,2k+q-1$ and $x^{2k+q}(m) = -s(m)$. Let $g = (x^1,\dots,x^{2k+q})$. Since $Y \in \ker(\omega)$ as forms on V $(g_*\omega)(x) = \Sigma_{j=1}^{k} dx^j \wedge dx^{j+k}$ for $x \in g(V')$. The Lie derivative of ω with respect to Y is zero, $L_Y\omega = i_Y\, d\omega + di_Y\omega = 0$. In the local coordinates $\Sigma_{j=1}^{k} dx^j \wedge dx^{j+k}$ is a constant form so its Lie derivative with respect to g_*Y is zero. Thus $g_*\omega = \Sigma_{j=1}^{k} dx^j \wedge dx^{j+k}$ along the flow of Y and hence in a neighborhood of V'.

Q.E.D.

COROLLARY 5.3 - Let (M^{2n},ω) be a symplectic manifold. Let V^{2k+q} be a submanifold such that $\omega|V$ has constant rank $2k$. Let $p \in V$. Then there exist symplectic coordinates in a neighborhood U of p in M, $g = (x^1, \ldots, x^{2n})$, such that

$$V \cap U = \{m: x^j(m) = 0 \quad j = k+q+1, \ldots, n, \ n+k+1, \ldots, 2n\},$$

Proof: By Corollary 5.2 there exist coordinates on a contractible neighborhood W in V, $h = (y^1, \ldots, y^{k+q}, y^{n+1}, \ldots, y^{n+k})$, such that $h_*(\omega|V) = \Sigma_{j=1}^{k} dx^j \wedge dx^{j+n}$. Let Y_j be vector fields on W defined by $h_* Y_j = \frac{\partial}{\partial y^j}$.

Applying the construction of Lemma 1.2 to cross sections of the tangent bundle of M on W, we get there exists $Y_j \in \Gamma(T_W M)$ $j = k+q+1, \ldots, n, n+k+q+1, \ldots, 2n$ such that for each $m \in W$ the set $\{Y_j(m): j=1, \ldots, 2n\}$ forms a symplectic basis of $T_m M$.

By the proof of the existence of tubular neighborhoods [L2, p.75] there exist C^∞ coordinates on a neighborhood U in M, $y^1, \ldots, y^{2n}$ that are extensions of the coordinates on W and such that $Y_j(m) = \frac{\partial}{\partial y^j}(m)$ $j = 1, \ldots, 2n$ for $m \in W$. Let $h = (y^1, \ldots, y^{2n})$. Now apply the proof of Theorem 5.1 to find symplection coordinates, $g = \varphi_1 \circ h = (x^1, \ldots, x^{2n})$. Since $(h_*\omega)(y) = \Sigma_{j=1}^{n} dy^j \wedge dy^{j+n}$ for $y \in h(W)$ it follows that $\varphi_1(y) = y$ for $y \in hW$ so $g(W)$ has the form stated in the corollary. Q.E.D.

This last corollary can be used to find local symplectic coordinates that give stable, center-stable, or center-manifolds as flat planes in a neighborhood of a fixed point.

THEOREM 5.4 (Symplectic flow box coordinates) - Let $X_H \in \mathfrak{X}_H(M)$ and let $p \in M$ be such that $X_H(p) \neq 0$. Then there exist symplectic coordinate in a neighborhood of p, $g = (x^1, \ldots, x^{2n})$, such that $x^{n+1} = H$ and $g_* X_H = \dfrac{\partial}{\partial x^1}$.

<u>Proof</u>: Take a transversal Σ^{2n-1} to X_H at p.

$X_H(p) \neq 0$ implies $dH_p \neq 0$ so the energy surface is regular in a neighborhood of p. ω must have rank $2n-2$ on $H^{-1}(e)$ since it was nondegenerate on M. Therefore $\dim(\ker(\omega | H^{-1}(e))) = 1$. $\widetilde{\omega}(X_H) = dH$ so $X_H \in \ker(\omega | H^{-1}(e))$. Since X_H is transverse to Σ it follows that $\omega | \Sigma'$ is nondegenerate where $\Sigma' = \Sigma \cap H^{-1}(e_o)$, $H(m) = e_o$. The reader can check that is possible to take $y^{n+1} = H$ in the proof of Corollary 5.2 to get coordinates $h = (y^2, \ldots, y^{2n})$ on Σ such that $h_*(\omega | \Sigma) = \Sigma_{j=1}^{n} dy^j \wedge dy^{j+n}$.

Take the flow of X_H to give coordinates in a neighborhood of Σ i.e. these exists $s(m)$ such that $\varphi(s(m), m) \in \Sigma$.

Let $x^j(m) = y^j \circ \varphi(s(m),m)$ for $j \neq 1$ and $x^1 = -s(m)$. Let $g = (x^1,\ldots,x^{2n})$. $g_*X_H = \frac{\partial}{\partial x^1}$ by construction. For $m \in g(\Sigma)$ $g_*\omega(m) = \Sigma_{j=1}^n dx^j \wedge dx^{j+n}$ since $g_*\omega(m)(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^j}) = dH \cdot \frac{\partial}{\partial x^j}$ is 1 or 0 as $j = n+1$, $j \neq n+1$. But $g_*\omega$ and $\Sigma_{j=1}^n dx^j \wedge dx^{j+n}$ have Lie derivative with respect to g_*X_H equal to zero. Therefore they are equal in a neighborhood of Σ.

Q.E.D.

§6 - GENERAL DENSITY THEOREM FOR HAMILTONIAN SYSTEMS

In this section we state the general density theorem for Hamiltonian systems. This theorem is analogous to the Kupka-Smale Theorem for ordinary vector fields. Its proof is given in §8 after the transversality theorems necessary for its proof are given in §7. Also in this section we give other theorems related to generic properties of Hamiltonian systems. See [R1] and [R2].

By Proposition 4.7, the Poincaré map near a closed orbit is a map from an odd dimensional manifold, Σ^{2n-1}, to itself such that when it is restricted to an energy surface, Σ_e^{2n-2}, it is a local symplectic diffeomorphism. Thus the essential details of the behavior near a closed orbit is contained in a study of maps $f: I \times M \to I \times M$ with $I = [-1,1]$ a closed interval, M a symplectic manifold, $f(t,m) = (t, f_t(m))$, and with f_t a symplectic diffeomorphism for each t. Such maps are called <u>one parameter families of symplectic diffeomorphisms</u>. The set of all such maps is denoted by $\mathrm{Sym}^r(I \times M)$. The results we state about $\mathrm{Sym}^r(I \times M)$ have analogous results in $\mathfrak{X}_H^r(M)$.

For the next statements $f(t,m) = (t,f_t(m))$ is always a family of symplectic diffeomorphisms. A point (t,m) has prime period p if $f^p(t,m) = (t,m)$ but $f^k(t,m) \neq (t,m)$ for $0 < k < p$. A point (t,m) with prime period p is called 0-elementary if $D(\pi \circ f^p)(t,m) - D\pi(t,m): T_{(t,m)}(I \times M) \to T_m M$ is onto where $\pi: I \times M \to M$ is the natural projection. (Note for Hamiltonian vector fields there is no natural projection on the energy surface). If a periodic point is 0-elementary then by Theorem 4.5 there exists a one dimensional manifold in $I \times M$, $\{(t(s),m(s))\}$, such that all the points have prime period p. Let $E = \{(q,A): q \in M$ and $A: T_q M \to T_q M$ is a linear symplectic transformation$\}$. Let $W^N = \{(q,A) \in E$: A is not N-elementary$\}$. Let $g: R \to E$ be defined by $g(s) = (m(s), Df_{t(s)}{}^p(m(s)))$. f is said to have property H2-N at a point (t,m) of prime period p if

i) (t,m) is a 0-elementary periodic point and

ii) the map $g: R \to E$ defined above crosses W^N transversally. In other words (t,m) does not have to be N-elementary but all but a finite number of nearby periodic points are N-elementary. Here we are calling a periodic point N-elementary for $N > 0$ if its derivative along the energy surface is N-elementary. Note that if a periodic point is 1-elementary (1 is not an eigenvalue of $Df_t{}^p$)

then it is 0-elementary.

THEOREM 6.1 - Let $2 \leq r \leq \infty$, $\mathcal{B}^r(p,N) = \{f \in \mathrm{Sym}^r(I\times M):$ f has property H2-N at all points of prime period $\leq p\}$, and $\mathcal{B}^r = \bigcap_{p,N} \mathcal{B}^r(p,N)$. Then $\mathcal{B}^r(p,N)$ and $\mathcal{B}^r$ are residual subsets of $\mathrm{Sym}^r(I\times M)$. A residual subset is dense.

If a periodic point is not 2-elementary then there is a multiple eigenvalue of $Df_t^p(m)$. The analysis of §3 shows what is happening to $Df_t^p(m)$ as it moves along the family of periodic points near a non-2-elementary periodic point. If a periodic point is not 1-elementary then 1 is an eigenvalue of $Df_t^p(m)$. The next proposition says what is happening near a non-1-elementary periodic point.

PROPOSITION 6.2 - Let $f \in \mathcal{B}^2(p,1)$ have a point (t,m) of period p that is not 1-elementary but is 0-elementary. Let $\{(t(s),m(s))\}$ be the set of points nearby of period p where $t(0) = t$. Then $t(s)$ has a quadratic maximum or minimum at 0, i e. $t'(0) = 0$ and $t''(0) \neq 0$. Also for every $k \geq p$ there is a neighborhood $U(k) \subset I\times M$ such that the only points in $U(k)$ of period $\leq k$ are $\{(t(s),m(s))\} \cap U(k)$.

<u>Proof</u>: For the proof of the first part see [R2, §IV B].

The idea is as follows. Assume $t''(0) = 0$. Let $v(s)$

be a vector tangent to the family of periodic points. Let $u(s) = \pi \cdot v(s) \in T_{m(s)}M$. Then $u(0) = v(0)$ and $Df_{t(s)}^{p}(m(s)) \cdot u(s) - u(s)$ almost equals zero. Using this fact we prove 1 is close enough to being an eigenvalue that $Df_{t(s)}^{p}(m(s))$ can not be transverse to matrices that are not 1-elementary.

For the second part it follows from general arguments that if k is not a multiple of p then there exists such a $U(k)$. Now assume $k = qp$. Let $A = Df^{p}(t,m)$ and $B = Df_{t}^{p}(m)$. Then $\pi \circ A - \pi$ is onto. Look at $\pi \circ A^{q} - \pi$. Its image contains the image of $(B^{q-1} + \ldots + I)(\pi A - \pi)$. $B^{q-1} + \ldots + I$ is an isomorphism because otherwise B would have an eigenvalue that was a q^{th} root of unity but not equal to 1. $\pi \circ A - \pi$ is onto. Thus the composition is onto.

Q.E.D.

When $\dim M = 2$, K. Meyer has a complete analysis of the generic cases of branching of periodic points at non-N-elementary periodic points. $\dim M = 2$ implies that non-N-elementary periodic points have eigenvalues that are N^{th} roots of unit. Assume the non-N-elementary point has period p. Then there exist two families of points of period Np branching off at the point. One family has hyperbolic periodic points and the other family has elliptic periodic points. See [M3]. It would seem that his results could be combined with Theorem 6.1 to give results in higher dimensions. To do this the work in [T1] should be used.

[R3] proves that generically all but a countable number of stable manifolds are transverse in the energy surfaces. We refer the reader to this paper for these results.

Another generic property is the density of periodic points in the nonwandering set. Let $f \in \mathrm{Sym}(I \times M)$. $m \in I \times M$ is <u>nonwandering</u> if for every neighborhood U of m there exists an integer $k > 0$ such that $f^k(U) \cap U \neq \emptyset$. Let $\Omega_c(f) = \{m \in I \times M:\ m$ is nonwandering and the trajectory of m is contained in a compact subset of $I \times M\}$. The second condition avoids the necessity of assuming M is compact. Let $\mathrm{Per}(f) = \{m \in I \times M:\ m$ is a periodic point of $f\}$. Then the C^1 Closing Lemma of Pugh gives the following result.

PROPOSITION 6.3 - There exists a residual subset $R \subset \mathrm{Sym}^1(I \times M)$ such that for $f \in R$ closure $\mathrm{Per}(f) =$ closure $\Omega_c(f)$.

For a proof of this result see [P5] and [R2]. "The result is unknowing in $\mathrm{Sym}^r(I \times M)$ for $r \geq 2$.

f is said to be <u>structurally stable</u> if there exists a neighborhood U such that for all $g \in U$ there exists a homeomorphism $h: I \times M \to I \times M$ such that $gh = hf$. f is said to be <u>P-stable</u> if there exists a neighborhood U such that for $g \in U$ there exists a homeomorphism from the periodic points of f onto the periodic points of g, $h: \mathrm{Per}(f) \to \mathrm{Per}(g)$, such that $gh = hf|\mathrm{Per}(f)$. Obviously if f is not P-stable then it is not structurally stable.

THEOREM 6.4 - Let $f \in \mathrm{Sym}^r(I \times M)$ for $1 \leq r \leq \infty$ have a point (t,m) of period p with an eigenvalue of $Df_t^p(m)$ of absolute value one. Then f is not P-stable.

<u>Proof</u>: We give the proof for $r < \infty$. Also see related statements in [M6] in the analytic setting.

Assume f is P-stable. Let $\mathfrak{h}$ be a neighborhood of f such that f is P-conjugate to everything in $\mathfrak{h}$. Take $g \in \mathfrak{h} \cap \mathcal{B}^\infty$ such that g has a periodic point (t,m) with eigenvalues $\lambda_1, \ldots, \lambda_q, \bar{\lambda}_1, \ldots, \bar{\lambda}_q$ of absolute value one. Since $g \in \mathcal{B}^\infty$, g has only a countable number of periodic points. Let $P \subset T_m M$ be the linear subspace spanned by the root spaces of $\lambda_1, \ldots, \lambda_q$. The dimension of P is $2q$. $\omega|P$ is nondegenerate by Corollary 1.9.

There exists a local center manifold C tangent to P and invariant by g_t. See [A1]. C has dimension $2q$. $\omega|C$ is nondegenerate in a neighborhood of m. By Corollary 5.3 there exist symplectic coordinates $y^1,\dots,y^{2n}$ such that $C = \{x \in M: y^j(x) = 0 \quad j = q+1,\dots,n,n+q+1,\dots, 2n\}$. By Proposition 4.11 there exist dymplectic coordinates $x^1,\dots,x^{2n}$ with $x^j = y^j \quad j = q+1,\dots,n,n+q+1,\dots,2n$ and for s near t, $g_j(s,x) = h_j(s,x) + \Theta(r)$ for $j = 1,\dots,q,\ n+1,\dots,n+q$ where $h_j(s,\cdot)$ is a polynomial in $\rho_k = (x^k)^2 + (x^{k+n})^2 \quad k = 1,\dots,q$ and $\Theta(r)$ is C^∞ and has its first r derivatives equal to zero at points $(s,0)$. By a perturbation of g we can assume the determinant of the coefficients of the quadratic terms in h_s are nonzero. As in Example 4.9 $h_t(x)$ will have tori near the origin filled with periodic points.

Let $h_j(s,x) = g_j(s,x)$ for the other j. All the derivatives of h equal those of g at points $(s,0)$ for small s. Let $\rho(s,x)$ be a bump function in a neighborhood of $(t,0)$ in the local coordinates. Assume $\rho(s,x) \equiv 1$ in a smaller neighborhood of $(t,0)$. Let $\rho_\lambda(y) = \rho(x/\lambda)$. Let $h_\lambda(s,x) = (1 - \rho_\lambda(s,x))g(s,x) + \rho_\lambda(s,x)h(s,x)$ By standard calculations $h_\lambda \to g$ in the C^r topology. Thus for small λ, $h_\lambda \in \mathfrak{h}$. But h_λ has whole tori filled with periodic points in a neighborhood of $(t,0)$. Thus there can not even exist a homeomorphism from $\mathrm{Per}(h_\lambda)$ to $\mathrm{Per}(g)$ let alone one the conjugates the maps. Thus h_λ and g are not P-conjugate. Thus f can not be P-conjugate to both h_λ and g. Contradiction.

Q.E.D.

§7 - TRANSVERSALITY THEOREMS

Let M and N be second countable Banach manifolds, N finite dimensional and $V \subset N$ a submanifold. $f: M \to N$ is said to be *transverse to V at x* if either $f(x) \notin V$ or $f(x) \in V$ and $Df(x) \cdot T_x M + T_{f(x)} V = T_{f(x)} N$. Thus the image of f crosses V in a way that locally can not be destroyed by a small perturbation. F is said to be *transverse to V* if it is transverse to V at all points $x \in M$. By the implicit function theorem if f is transverse to V then $f^{-1}(V)$ is a submanifold of M whose codimension equals the codimension of V in N.

Let A be a topological space and $F: A \to C^r(M,N)$ a point set map. $C^r(M,N)$ is not given a topology here. The *evaluation map* $ev(F): A \times M \to N$ is defined by $ev(F)(f,x) = (Ff)(x)$. Let $F^1: A \to C^{r-1}(TM,TN)$ be defined by $(F^1 f)(x,v) = D(F^1 f)(x)v$. F is called a *C^1 pseudo-representation* if $ev(F^1): A \times TM \to TN$ is continuous. Let A be a Banach manifold F is called a *C^r representation* if $ev(F): A \times M \to N$ is C^r.

A subspace $R \subset A$ is called *residual* if there exist a countable set of open dense subsets $R_n \subset A$ such that $\bigcap_n R_n \subset R$. By Baire category theorem if A is a metric space then a residual subset is dense. A topological space A is said to be a <u>Baire space</u> if all residual subsets are dense.

THEOREM 7.1 - Let A be a topological space, M and N finite dimensional second countable manifolds. Let V be a closed subset of N that is the finite union of C^1 submanifolds of N. Let K be a compact subset of M, and $F: A \to C^1(M,N)$ be a C^1 pseudo-representation. Then the subset $R = \{f \in A: F(f)$ is transverse to V at points in $K\}$ is open in A.

THEOREM 7.2 (Abraham) - Let A,M,N be second countable Banach manifolds and M and N be finite dimensional. Let $F: A \to C^r(M,N)$ be a C^r representation and V a submanifold of N. Assume $r \geq \max\{1, \dim M - \operatorname{codim} V+1\}$, and the evaluation map of F is transverse to V. Then the subset $R = \{f \in A: F(f)$ is transverse to $V\}$ is residual in A (hence dense).

The most important assumption is that the evaluation map is transverse to V. A loose interpretation of this assumption is that if $f(x) \in V$ then we can construct a perturbation of f that is transverse to V at points near x. The theorem then globalizes this result to give a perturbation that is transverse to V at all points.

The idea of the proof is that since the evaluation map is transverse to V, $W = ev(F)^{-1}(V)$ is a submanifold of $A \times M$ of finite codimension. Then look at the projection $\pi: A \times M \to A$ restricted to W. Next it is shown Sards theorem applies so a residual set of points are regular values of $\pi: W \to A$. Finally, the regular values f can be shown to exactly equals those f such that $F(f)$ is transverse to V.

For detailed proofs of both of the above theorems see [A2].

Finally we need a statement about how transverse intersections change under perturbation. The following theorem is a simplified version of [A2, Theorem 20.2].

THEOREM 7.3 - Let A be a topological space, M and N finite dimensional second countable manifolds. Let V be a closed submanifold of N, K a compact submanifold of M with boundary, $F: A \to C^1(M,N)$ a C^1 pseudorepresentation, and $f \in A$ such that $F(f): \mathrm{int}(K) \to N$ and $F(f): \partial K \to N$ are transverse to V. Then the function $g \mapsto (F(g)^{-1}V) \cap K$ is continuous at f. Here we put the Hausdorff metric on the distance between two compact sets

$$d(A,B) = \max\{\sup\{d(a,B):a\in A\},\ \sup\{d(b,A):b\in B\}\}.$$

§8 - PROOF OF THEOREM 6.1

Fix N. Define $F_j : \mathrm{Sym}^r(I \times M) \to C^1(I \times M, M \times M)$ by $F_j(f)(t,m) = (m, f_t^j(m))$. Let $W = \{(m,m) : m \in M\} \subset M \times M$. Then $F_j(f)(t,m) \in W$ if and only if the point has period j. The reader can check that $F_j(f)$ is transverse to W at (t,m) if and only if (t,m) is a 0-elementary periodic point. If $F_j(f)$ is transverse to W then $F_j(f)^{-1}(W) \subset$ $\subset I \times M$ is a submanifold of the same codimension as W, 2n, so it is one dimensional. This is just Proposition 4.7.

Define $G_j : \mathrm{Sym}^r(I \times M) \to C^1(I \times M, E)$ by $G_j(g)(t,m) =$ $= (F_j(g)(t,m), Dg_t^j(m))$. E is the bundle $\{(m,p,A) : m,p \in M$ and $A : T_mM \to T_pM$ is a linear symplectic transformation$\}$. Let $W^N = \{(m,m,A) \in E : A$ is not N-elementary$\}$. W^N is closed. By [A2, 30.4] W^N is locally the union of submanifolds W^{Nh} $h = 1,\ldots,h_N$ since it is a semialgebraic set in each fiber. W^N has empty interior in each fiber so it has codimension at least 1 in each fiber. W has codimension 2n. Combining this two facts we get that each W^{Nh} has at least codimension 2n+1. If $F_j(f)$ is transverse to W and $G_j(f)$

is transverse to the W^{Nh} then the map $\pi_3 \circ G_j(f):F_j(f)^{-1}(W) \to Sp(R^{2n})$ is transverse to non-N-elementary matrices. This fact is just condition (ii) of the definition of H2-N.

A periodic point of period p that is 0-elementary but has an i^{th} root of unit as an eigenvalue has 1 as an eigenvalue of $Df_t^{pi}(m)$. $F_{pi}(f)$ would not be transverse to W. The results of K. Meyer [M3] show that the points of period pi probably do not form a manifold but bifurcate forming a branched manifold. The perturbations defined below can not be used to make F_{pi} transverse since they only apply to the prime period. Therefore we need to modify the proof of the standard Kupka-Smale Theorem.

Let V be a compact submanifold with boundary of M (M is not assumed to be compact). Let $\mathcal{S}^r(p,N,V) = \{f \in Sym(I\times M)$: if $m \in V$ is a point of prime period k and $k \le p$ then $F_k(f)$ is transverse to W at m and $G_k(f)$ is transverse to all the W^{Nh} at m. Morevver if m is on the boundary of V then $F_k(m)|\partial V$ is transverse to $W\}$. We show below $\mathcal{S}^r(p,N,V)$ is residual. By taking the intersection over a countable number of such V whose union is all of $I\times M$ we get that $\mathcal{S}^r(p,N)$ is residual. Now we fix N and V and let $\mathcal{S}^r(p,N,V) = R(p)$.

Let d be a distance between points of $I \times M$ derived from a Riemannian metric. Also let d stand for the induced minimum distance from a point to a closed set and the Hausdorff distance between two compact sets,

$$d(A,B) = \sup\{d(a,B), d(b,A): a \in A,\ b \in B\} .$$

For $g \in \mathrm{Sym}(I \times M)$ and $p \in \mathbb{Z}^+$ let $\Gamma(g,p) = \{m \in V \subset I \times M:$ m is a point of period $\le p$ for $g\}$. Let $C(g,p,j) = \{m \in V \subset I \times M: d(m, \Gamma(g,p)) \ge 2^{-j}\}$. For $g \in \mathrm{Sym}(I \times M)$, $j \in \mathbb{Z}^+$, and using induction on $p \in \mathbb{Z}^+$ define $\Gamma(g,p,j) = \{m \in V \subset I \times M:$ m is a point of period k for g with $k \le p$ and $d(m, \Gamma(g,k-1,j)) \ge 2^{-j}\}$.

We claim the map $g \to \Gamma(g,p)$ is continuous at points of $R(p)$. Let $f \in R(p)$. Take $\epsilon = 2^{-j}$. By Theorem 7.3 there exists a neighborhood P_1 of f such that for $g \in P_1$ $\sup\{d(m, \Gamma(g,p)): m \in \Gamma(f,p,j+1)\} \le 2^{-(j+1)}$. Thus $\sup\{d(m,\Gamma(g,p)): m \in \Gamma(f,p)\} \le 2^{-j}$. By opanness of nonintersection of $F_j(g)$ $C(f,p,j) \cap W = \emptyset$, there exists a neighborhood P_2 of f such that for $g \in P_2$ $\Gamma(g,p) \cap C(f,p,j) = \emptyset$. Thus for $g \in P_2$ $\sup\{d(m,\Gamma(f,p)): m \in \Gamma(g,p)\} \le 2^{-j}$. For $g \in P_1 \cap P_2$, $d(\Gamma(f,p), \Gamma(g,p)) \le 2^{-j}$.

Assume the result has been proven for $p-1$ and $R(p-1)$ is residual. Take $f \in R(p-1)$ and a neighborhood $P(f)$ such that for $g \in P(f)$, $d(\Gamma(f,p-1), \Gamma(g,p-1)) \le$

$\leq 2^{-(j+1)}$. Let $R(f,p,j) = \{g \in P(f): F_p(g)$ and $F_p(g)|\partial V$ are transverse to W and $G_p(g)$ is transverse to all the W^{Nh} at points of $C(f,p-1,j)\}$. $R(f,p,j)$ is open by Theorem 7.1. We prove below it is dense in $P(f)$.
Let $R(p,j) = \cup\{R(f,p,j): f \in R(p-1)\}$ and $R_1(p) = \cap\{R(p,j): j \in \mathbb{Z}^+\} \cap R(p-1)$. By induction $R(p-1)$ is residual so $R(p,j)$ is dense and open and $R_1(p)$ is residual. But for $g \in P(f)$, $C(g,p-1,j) \subset C(f,p-1,j+1)$. Thus $R(p,j+1) \subset \{g: F_p(g)$ and $F_p(g)|\partial V$ are transverse to W and $G_p(g)$ is transverse to all the W^{Nh} at points of $C(g,p-1,j)\}$. Thus $R_1(p) = R(p)$. All is left to prove is the density of $R(f,p,j)$ in $P(f)$.

PROPOSITION 8.1 - Let $g \in \mathrm{Sym}^r(I\times M)$ and U_1 be a compact subset of $I\times M$ such that g has no points of period $\leq p-1$ in U_1. Let ∂V be a submanifold of $I\times M$. Then the set $R = \{h \in \mathrm{Sym}^r(I\times M): F_p(h)$ and $F_p(h)|\partial V$ is transverse to W and $G_p(h)$ is transverse to the W^{Nh} at points of $U_1\}$ is dense at g.

REMARK: In the above application $U_1 = C(f,p-1,j)$.

<u>Proof</u>: Let U_2 be a compact neighborhood of U_1. Let

$\mathbb{B} = \{X \in \mathfrak{X}^r(I\times M)$: $X(m) = 0$ for $m \in (I\times M-U_2)\}$ and for each $t \in I$ $X(t,\cdot) \in \mathfrak{X}^r_H(M)$, i.e. $X(t,\cdot)$ is tangent to the energy surfaces and on each energy surface it is a Hamiltonian vector field}. Let $\varphi_1(X)$ be the time one map of the flow of $X \in \mathbb{B}$. Define $K:B \to \mathrm{Sym}(I\times M)$ by $K(X) = \varphi_1(X) \circ g$. The reader can check that $F_p \circ K$ and $G_p \circ K$ are C^2 and C^1 representations respectively. Claims a) and b) prove their evaluations are transverse to W and W^{Nh} respectively on $\mathbb{Z} \times U_1$ where Z is a neighborhood of 0 in $\mathbb{B}$. By Theorem 7.2, $Y = \{X \in \mathbb{Z}: F_p \circ K$ is transverse to W and $G_p \circ K$ is transverse to all the W^{Nh} at points of $U_1\}$ is dense in $\mathbb{Z}$. Thus $K(Y)$ is dense at g. This proves R is dense at g.

CLAIM a - $ev(F_p \circ K)$ is transverse to W on a neighborhood of $0 \times U_1$.

Proof: Since transversality is an open condition we only need to prove transversality on $0 \times U_1$. Let $F_p = F$. Assume $F(g)(t,m) \in W$. Take symplectic coordinates $(x^1,\ldots,x^{2n}) = x$ on M near m. Let $b: I\times M \to \mathbb{R}$ be a bump function whose support is in the coordinate neighborhood of (t,m) and is identically equal to one in a smaller neighborhood of (t,m). Let $L(t,x^1,\ldots,x^{2n})$ be independent of t and a linear function in the x^j. Let c be a real number. Let $cbL(t,m) = cb(t,m)L(t,x(m))$, $cbL: I\times M \to R$. Let X_{cbL} is the associated Hamiltonian vector field on each level set. $X_{cbL} \in B$. Identify points to their image under $x = (x^1,\ldots,x^{2n})$. For c small

$$K(X_{cbL})(t,m) = (t,m + cX_L(m)) \; .$$

$$\frac{d}{dc} K(X_{cbL})(t,m) = (0,X_L(m)) \; .$$

L is an arbitrary linear function so $X_L(m)$ is an arbitrary vector in T_mM. Thus these perturbations span T_mM. $ev(F \circ K)$ is transverse to W at (t,m).

CLAIM b - $ev(G_p \circ K)$ is transverse to W^{Nh} on a neighborhood of $0 \times U_1$.

<u>Proof</u>: Let $G = G_p$. Assume $G_p(g)(t,m) \in W^{Nh}$. Let X_{cLb} be as before but now L is a homogeneous quadratic function.

In local coordinates

$$X_{cbL}(s,y) = c\begin{pmatrix} 0 \\ \dfrac{\partial}{\partial x^{i+n}} bL \\ \\ -\dfrac{\partial}{\partial x^{i}} bL \end{pmatrix}_{i=1,\ldots,n}$$

$$DX_{cbL}(t,m) = c \begin{pmatrix} 0 & \cdots & & & & 0 \\ \vdots & \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^{i+n}} L & \frac{\partial}{\partial} & \frac{\partial}{\partial x^{j+n}} & \frac{\partial}{\partial x^{i+n}} L \\ 0 & -\frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^i} L & \frac{\partial}{\partial} & -\frac{\partial}{\partial x^{j+n}} & \frac{\partial}{\partial x^i} L \end{pmatrix}$$

As L varies over all quadratic functions the $2n \times 2n$ submatrix varies over all infinitesimally symplectic matrices

$$D(K(X_{cbL}))(m) = e^c \exp(DX_{bL}(m))\ Dg(t,m),$$

$$\frac{d}{dc} D(K(X_{cbL}))(m) = \exp(DX_{bL}(m))\ Dg(t,m).$$

The exponential of infinitesimally symplectic matrices is onto the symplectic matrices*. Thus these perturbations together with those of Claim a show that $ev(G \circ K)$ is transverse to all the W^{Nh} on $0 \times U_1$.

Q.E.D.

* The exponential of infinitesimally symplectic matrices is onto a neighborhood of the identity in the set of all symplectic matrices.

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