Global and Local Aspects of Levi-flat Hypersurfaces

Publicações Matemáticas

Global and Local Aspects of Levi-flat Hypersurfaces

Jiří Lebl
Oklahoma State University

Arturo Fernández Pérez
UFMG

impa

$30^{\underline{o}}$ Colóquio Brasileiro de Matemática

Impresso no Brasil / Printed in Brazil
Capa: Noni Geiger / Sérgio R. Vaz

30º Colóquio Brasileiro de Matemática

- Aplicações Matemáticas em Engenharia de Produção - Leonardo J. Lustosa e Fernanda M. P. Raupp
- Boltzmann-type Equations and their Applications - Ricardo Alonso
- Dissipative Forces in Celestial Mechanics - Sylvio Ferraz-Mello, Clodoaldo Grotta-Ragazzo e Lucas Ruiz dos Santos
- Economic Models and Mean-Field Games Theory - Diogo A. Gomes, Levon Nurbekyan and Edgard A. Pimentel
- Generic Linear Recurrent Sequences and Related Topics - Letterio Gatto
- Geração de Malhas por Refinamento de Delaunay - Marcelo Siqueira, Afonso Paiva e Paulo Pagliosa
- **Global and Local Aspects of Levi-flat Hypersurfaces - Arturo Fernández Pérez e Jiří Lebl**
- Introdução às Curvas Elípticas e Aplicações - Parham Salehyan
- Métodos de Descida em Otimização Multiobjetivo - B. F. Svaiter e L. M. Graña Drummond
- Modern Theory of Nonlinear Elliptic PDE - Boyan Slavchev Sirakov
- Novel Regularization Methods for Ill-posed Problems in Hilbert and Banach Spaces - Ismael R. Bleyer e Antonio Leitão
- Probabilistic and Statistical Tools for Modeling Time Series - Paul Doukhan
- Tópicos da Teoria dos Jogos em Computação - O. Lee, F. K. Miyazawa, R. C. S. Schouery e E. C. Xavier
- Topics in Spectral Theory - Carlos Tomei

Distribuição: IMPA
Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ
E-mail: ddic@impa.br
http://www.impa.br

ISBN: 978-85-244-0407-8

Preface

The present work is a short advanced course to be given as a series of five lectures at 30th Brazilian Mathematical Colloquium - 30º Colóquio Brasileiro de Matemática. The aim is to present the current state of the art within Levi-flat hypersurfaces for advanced graduate students and researchers. We have tried to balance accesibility with depth, and we wish to present both some of the currently known results and some of the techniques used.

The chapter 1 is devoted to give a rough sketch the basics of several complex variables and CR geometry as it relates to real-analytic Levi-flat hypersurfaces. We present Cartan's theorem for smooth real analytic Levi-flat hypersurfaces, this theorem establishes the existence of a trivial normal form near smooth points.

In chapter 2, we present the complete list of quadratic Levi-flat hypersurfaces proved by D. Burns and X. Gong. They also gave a rigid normal form for Levi-flat hypersurface with a Morse singularity. Using the theory of holomorphic foliations we give an extension of this rigid normal form. Furthermore, we state a new rigid normal forms of Levi-flat hypersurfaces with special singularities.

A Levi-flat hypersurface is equipped with a foliation by complex leaves, called the Levi-foliation (not necessarily a restriction of a holomorphic foliation). For a non-singular Levi-flat hypersurface M, Cartan's local trivialization provides an explicit extension of the Levi-foliation to a non-singular holomorphic foliation in a neighborhood of M. The situation becomes much more complicated when one considers singularities. For a singular hypersurface, the Levi-foliation does not extend in general unless the singularity is small, for instance, there exist Levi-flat hypersurfaces whose Levi-foliations only extend to k-webs in the ambient space. The chapter 3 is devoted to study the extension problem for the Levi-foliation of a real analytic Levi-flat hypersurface. This problem has been considered by E. Bedford and P. De Bartolomeis in the C^∞-smooth case. In the singular case, the problem was partially solved. We present a result that provides conditions for extension of the Levi-foliation to a k-web. Recently R. Shafikov and A. Sukhov improved the current state of the art of this problem and this result is presented in the last part of this chapter.

In chapter 4, we disuss what is known about the structure of the singular locus of a real-analytic Levi-flat hypersurface. It is known for example that it is Levi-flat in the proper sense. We will show how Segre varieties are used in the proof including a discussion of the classical lemma of Diederich-Fornæss.

In chapter 5, we consider Levi-flat hypersurfaces globally in projective space. It is known that such hypersurfaces must be singular when the dimension is 3 or more by a theorem of Lins Neto. Chow's theorem says that a complex subvariety of projective space is algebraic. A Levi-flat analogue of such a theorem does not hold in general, without additional hypotheses. We present a version of Chow's theorem for Levi-flat hypersurfaces with compact leaves.

The first named author is grateful to the Pontíficia Universidad Católica del Perú - PUCP for their hospitality, and the support of CAPES - Bolsista da CAPES - Brasilia/Brasil.

Notations

1. $\mathcal{O}_n$: the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$.
2. $\mathcal{O}(U)$: set of holomorphic functions in the open set $U \subset \mathbb{C}^n$.
3. $\mathcal{O}_n^* = \{f \in \mathcal{O}_n | f(0) \neq 0\}$.
4. $\mathcal{O}^*(U) = \{f \in \mathcal{O}(U) | f(z) \neq 0, \forall z \in U\}$.
5. $\mathcal{M}_n = \{f \in \mathcal{O}_n | f(0) = 0\}$ the maximal ideal of $\mathcal{O}_n$.
6. $\mathcal{A}_n$: the ring of germs at $0 \in \mathbb{C}^n$ of complex valued real analytic functions.
7. $\mathcal{A}_{n\mathbb{R}}$: the ring of germs at $0 \in \mathbb{C}^n$ of real valued real analytic functions. Note that $F \in \mathcal{A}_n$ is in $\mathcal{A}_{n\mathbb{R}}$ if and only if $F = \bar{F}$.
8. $\mathsf{Diff}(\mathbb{C}^n, 0)$: the group of germs at $0 \in \mathbb{C}^n$ of holomorphic diffeomorphisms $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with the operation of composition.
9. $j_0^k(f)$: the k-jet at $0 \in \mathbb{C}^n$ of $f \in \mathcal{O}_n$.

Contents

Introduction

Complex analysis is the study of holomorphic functions, and the most basic question about holomorphic functions is the study of domains where such functions are defined. Natural domains for holomorphic functions have what is called pseudoconvex boundary, and a Levi-flat hypersurface is "pseudoconvex from both sides." Levi-flat hypersurfaces have no local biholomorpic invariants unless they are singular. Singular Levi-flat hypersurfaces appear in many contexts, for example the zero set of the real-part of a holomorphic function, or as an invariant set of a holomorphic foliation. While singular Levi-flat hypersurfaces have many properties of complex subvarieties, they have a more complicated geometry and inherit many pathologies from general real-analytic subvarieties. Their study is on the interface of studying real and complex subvarieties. It is a rich field mostly unexplored, with many questions still unanswered (or unasked).

These notes explore the techniques from holomorphic foliation theory, CR geometry, the geometry of algebraic and analytic subvarieties of complex spaces, and several complex variables in general in the study of Levi-flat hypersurfaces. In the problem of extending the Levi-foliation, it is necessary understand the connection between k-webs and partial differential equations of first order. We will present several important examples showing various phenomenona appearing in the the study of singular real-analytic Levi-flat hypersurfaces. These notes show the interconnection of several areas in mathematics and a rich field of study.

Chapter 1

Non-singular Levi-flat hypersurfaces and theorem of Cartan

In this first chapter we give a rough sketch the basics of several complex variables and CR geometry as it relates to real-analytic Levi-flat hypersurfaces.

1.1 Holomorphic functions

Let $\mathbb{C}^n$ be the complex Euclidean space. Write $z = (z_1, z_2, \ldots, z_n)$ for the coordinates of $\mathbb{C}^n$. For $\mathbb{R}^n$ we write coordinates as $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by letting $z = x + iy$. The *complex conjugate* is then $\bar{z} = x - iy$. The z are the *holomorphic coordinates* and $\bar{z}$ the *antiholomorphic coordinates*.

Define the *Wirtinger operators*

$$\frac{\partial}{\partial z_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Definition 1.1.1. Let $U \subset \mathbb{C}^n$ be an open set. Suppose $f \colon U \to \mathbb{C}$ is a locally bounded function whose first partial derivatives exist and

f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

Then f is *holomorphic.*

That is, f is holomorphic if it is holomorphic in each variable separately as a function of one variable. Similarly as in one variable Cauchy formula prove that a holomorphic function is infinitely differentiable and has a power series expansion around every point.

Because of the above formalism, we often write smooth function on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as $f(z, \bar{z})$ as if they were functions of z and $\bar{z}$ as separate variables. Then f is holomorphic if it does not depend on the $\bar{z}$ variable. It turns out this formalism is actually valid if f is a polynomial or a real-analytic function. That is, any polynomial in x and y on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ can be written as a polynomial in z and $\bar{z}$ using

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}.$$

The polynomial f is then holomorphic if we can write it as a function of z only.

In particular, any real-analytic function (a function that can be locally written as a power series in x and y) can be locally written as a power series in z and $\bar{z}$. This has a really important consequence. A real-analytic function can be *complexified.* That is, if $f(z, \bar{z})$ is real-analytic near zero in $\mathbb{C}^n$, then for some small neighbourhood of 0 in $(z, \xi) \in \mathbb{C}^n \times \mathbb{C}^n$, the power series for f with $\bar{z}$ replaced with ξ converges, and we obtain a holomorphic function $F(z, \xi)$ such that $f(z, \bar{z}) = F(z, \bar{z})$. It is not difficult to show that F is the unique such holomorphic function, and so we abuse notation somewhat and simply write $f(z, \xi)$ when needed. Similarly, when real-analytic functions are concerned it is common to pretend that $\bar{z}$ is an independent variable, although we must be careful to keep convergence issues in mind.

Complexification plays a key role in CR geometry and the study of Levi-flat hypersurfaces in particular.

It turns out that not every domain is a natural domain of definition for holomorphic functions. Let us mention a famous example.

Theorem 1.1.2 (Hartogs phenomenon)**.** *Let $U \subset \mathbb{C}^n$ be a domain, $n \geq 2$, and let $K \subset\subset U$ be a compact set such that $U \setminus K$ is connected. If $f \colon U \setminus K \to \mathbb{C}$ is holomorphic, then there exists a unique holomorphic $F \colon U \to \mathbb{C}$ such that $F|_{U\setminus K} = f$.*

For example, let $B(p, r)$ denote the open ball centered at $p \in \mathbb{C}^n$ and radius r. Every holomorphic function on $B(0, 1) \setminus \overline{B(0, \frac{1}{2})}$ extends to a holomorphic function on $B(0, 1)$. It is a problem with the geometry of the boundary on the inside ball. In particular, the domain is strictly concave along the entire inside boundary.

Though it is not simply convexity that is the correct concept here as holomorphic functions do not see all of convexity. Let us make this precise. Before we do, we need to see what kind of changes of coordinates we are allowing.

Definition 1.1.3. Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and suppose we have the map $f \colon U \to V$. Write $f = (f_1, f_2, \dots, f_m)$. The map f is a *holomorphic map* if f_j is holomorphic for all j.

If $m = n$ and furthermore f is one-to-one, onto, and f^{-1} is holomorphic[1], then f is said to be *biholomorphic*.

1.2 CR geometry

Let $M \subset \mathbb{C}^n$ be a real smooth hypersurface. That is, suppose that near each point $p \in M$, there is a neighbourhood U of p in $\mathbb{C}^n$, and a smooth function $r \colon U \to \mathbb{R}$ with $dr \neq 0$ on U such that $M \cap U = \{z : r(z, \bar{z}) = 0\}$. We are interested in what geometric information is preserved under a biholomorphic change of coordinates.

$\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ by $z = x + iy$, and so the tangent spaces $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. Write

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_n}\Big|_p \right\}.$$

Simply replace all the real coefficients with complex ones.

[1] It can be proved that this requirement is automatically true.

Once we do that we notice that $\frac{\partial}{\partial z_j}\Big|_p$, and $\frac{\partial}{\partial \bar{z}_j}\Big|_p$ are both in $\mathbb{C} \otimes T_p\mathbb{C}^n$, and in fact:

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

Define

$$T_p^{(1,0)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p \right\}, \qquad \text{and}$$
$$T_p^{(0,1)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

Vectors in $T_p^{(1,0)}\mathbb{C}^n$ are the *holomorphic vectors* and vectors in $T_p^{(0,1)}\mathbb{C}^n$ are the *antiholomorphic vectors*. We decompose the full tangent space as

$$\mathbb{C} \otimes T_p\mathbb{C}^n = T_p^{(1,0)}\mathbb{C}^n \oplus T_p^{(0,1)}\mathbb{C}^n.$$

A holomorphic function is one that vanishes on $T_p^{(0,1)}\mathbb{C}^n$. It can be checked that a holomorphic map pushes $T_p^{(1,0)}\mathbb{C}^n$ vectors to $T_{f(p)}^{(1,0)}\mathbb{C}^n$ vectors, and $T_p^{(0,1)}\mathbb{C}^n$ vectors to $T_{f(p)}^{(0,1)}\mathbb{C}^n$ vectors.

Let $M \subset \mathbb{C}^n$ be a smooth hypersurface as above. Define

$$T_p^{(1,0)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(1,0)}M\big), \qquad \text{and}$$
$$T_p^{(0,1)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(0,1)}M\big).$$

Decompose $\mathbb{C} \otimes T_pM$ as

$$\mathbb{C} \otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p.$$

The B_p is a one-dimensional space sometimes called the "bad direction."

It is not hard to show the following useful proposition.

Proposition 1.2.1. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. After a translation and rotation via a unitary matrix, $p = 0$ and near the origin M is written in variables $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ as*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

with the $\varphi(0)$ *and* $d\varphi(0) = 0$*. Consequently*

$$T_0^{(1,0)} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\Big|_0 \right\},$$
$$T_0^{(0,1)} M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_0, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_0 \right\},$$
$$B_0 = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial (\operatorname{Re} w)}\Big|_0 \right\}.$$

Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface. and suppose r is a defining function for M.

If for all nonzero $X_p \in T_p^{(1,0)} \partial U$,

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}\Big|_p,$$

we have

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \geq 0,$$

then M is said to be *pseudoconvex*[2] at p.

For $X_p \in T_p^{(1,0)} \partial U$, the expression

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p$$

is called the *Levi-form* at p. So M is pseudoconvex at $p \in \partial U$ if the Levi-form is positive semidefinite at p.

1.3 Smooth Levi-flat hypersurfaces

The hypersurface M is said to be *Levi-flat* if M is "pseudoconvex from both sides", that is, the Levi-form vanishes identically (it is both positive and negative semidefinite.

[2]It is really the "domain" $r < 0$ that is pseudoconvex rather than M.

In the setting of the proposition above we can write M near 0 as

$$\operatorname{Im} w = Q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w),$$

where Q is the quadratic part and E is of order 3. The real valued Q can be written as

$$Q(z,\bar{z}) = z^* A z + z^t B z + \overline{z^t B z}$$

for $(n-1)$ by $(n-1)$ matrices A and B where A is Hermitian. A holomorphic change of variables taking w to $w + iz^t Bz$ changes the equation to

$$\operatorname{Im} w = z^* A z + E(z,\bar{z},\operatorname{Re} w),$$

where again E is of order 3. The matrix A is then the Levi-form.

Therefore for a Levi-flat M, $A = 0$. Our goal (Cartan's theorem) is to show that we can also remove E, as long as M is real-analytic. Before we prove this theorem let us write M in a slightly different way. If M is real-analytic, we can regard z, $\bar{z}$, w, and $\bar{w}$ as separate variables. Then applying the holomorphic implicit function theorem we can write M near zero as

$$\bar{w} = Q(z,\bar{z},w),$$

for a holomorphic function Q that vanishes to second order. We could also solve for w rather than $\bar{w}$ by simply taking a complex conjugate. The function Q also satisfies $\bar{w} = Q(z,\bar{z},\bar{Q}(\bar{z},z,\bar{w}))$. Here $\bar{Q}$ is as usual the complex conjugate of the function Q.

Let us talk about the CR vector fields on M, that is, vector fields in $T^{(0,1)}M$. To find a basis we find vector fields in $T^{(0,1)}\mathbb{C}^n$ that vanish on the function $\bar{w} - Q(z,\bar{z},w)$. It is easy to see that the following vector fields will give us the basis of $T^{(0,1)}M$ at each point.

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \frac{\partial Q}{\partial \bar{z}_j}\frac{\partial}{\partial \bar{w}}.$$

Proposition 1.3.1. *If $M \subset \mathbb{C}^n$ is a real-analytic smooth hypersurface. Suppose that $f\colon M \to \mathbb{C}$ is a real-analytic* CR function. *For every $p \in M$, there exists a holomorphic function F defined in a neighbourhood of p such that F equals f on M.*

The proposition is not true if either f or M are only smooth rather than real-analytic.

Sketch of proof. First, because f is real-analytic it is possible to show that we can complexify f to an analytic function of z, $\bar{z}$, and w. To see why we can do this, for each fixed z, we get a real-analytic function on a real-analytic curve in $\mathbb{C}$ and therefore we can extend f to a holomorphic function of w analytically by simply letting w range over not just points corresponding to points of M, but all w in a neighbourhood. So far we have not used the fact that f is CR. Computing

$$0 = L_j f = \frac{\partial f}{\partial \bar{z}_j} + \frac{\partial Q}{\partial \bar{z}_j}\frac{\partial f}{\partial \bar{w}} = \frac{\partial f}{\partial \bar{z}_j}.$$

and f is therefore holomorphic in z and w. □

When M is Levi-flat, then a direct computation below shows that $T^{(0,1)}M \oplus T^{(1,0)}M$ is involutive. That is, take

$$\begin{aligned}
[L_j, \overline{L_k}] &= \left(\frac{\partial}{\partial \bar{z}_j} + \frac{\partial Q}{\partial \bar{z}_j}\frac{\partial}{\partial \bar{w}}\right)\left(\frac{\partial}{\partial z_k} + \frac{\partial \bar{Q}}{\partial z_k}\frac{\partial}{\partial w}\right) \\
&\quad - \left(\frac{\partial}{\partial z_k} + \frac{\partial \bar{Q}}{\partial z_k}\frac{\partial}{\partial w}\right)\left(\frac{\partial}{\partial \bar{z}_j} + \frac{\partial Q}{\partial \bar{z}_j}\frac{\partial}{\partial \bar{w}}\right) \\
&= \left(\frac{\partial^2 \bar{Q}}{\partial z_k \partial \bar{z}_j} + \frac{\partial Q}{\partial \bar{z}_j}\frac{\partial^2 \bar{Q}}{\partial z_k \partial \bar{w}}\right)\frac{\partial}{\partial w} \\
&\quad - \left(\frac{\partial^2 Q}{\partial \bar{z}_j \partial z_k} + \frac{\partial \bar{Q}}{\partial z_k}\frac{\partial^2 Q}{\partial \bar{z}_j \partial w}\right)\frac{\partial}{\partial \bar{w}}
\end{aligned}$$

Then since $\frac{\partial^2 Q}{\partial \bar{z}_j \partial z_k}$ is zero at the origin as the Levi-form vanishes at the origin, and $\frac{\partial \bar{Q}}{\partial z_k}$ vanishes at the origin as well. As M is Levi-flat similar calculation works at every point. You can see the connection of the Levi-form and the CR vector fields. It is easy to see that the commutators $[L_j, L_k]$ and $[\overline{L_j}, \overline{L_k}]$ are also in the space $T^{(0,1)}M \oplus T^{(1,0)}M$. And hence $T^{(0,1)}M \oplus T^{(1,0)}M$ is involutive. Frobenius theorem then implies that M is foliated by submanifolds whose complexified tangent space is $T^{(0,1)}M \oplus T^{(1,0)}M$, in fact these are complex submanifolds. This foliation is called the *Levi-foliation.*

We can now sketch a proof of Cartan's theorem.

Theorem 1.3.2 (Cartan)**.** *If $M \subset \mathbb{C}^n$ is a Levi-flat real-analytic smooth hypersurface, then near each point $p \in M$, there exist local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p such that M near p is given by*

$$\operatorname{Im} w = 0.$$

The leaves of the Levi-foliation are given by

$$\{(z, w) : w = t\} \quad \text{for some } t \in \mathbb{R}.$$

Sketch of proof. It is enough to find a holomorphic function defined near p with nonvanishing derivative that is real valued on M near p.

To define such a function, we take the real-analytic Levi-foliation defined above and near p we find a real-valued real-analytic f that is constant along leaves, and has nonvanishing derivative. It is automatically a CR function since it must be killed by the CR vector fields as those point along the leaves. It extends, and voilá. □

We now find that a real-analytic Levi-flat M is always locally given by

$$\operatorname{Im} w = 0,$$

in suitable holomorphic coordinates. In particular there are no local biholomorphic invariants in the nonsingular real-analytic case. The sets $\{w = t\}$ for various t are the leaves of the Levi-foliation.

Conversely, given any holomorphic function f, the set

$$\{z : \operatorname{Im} f(z) = 0\},$$

that is, a zero set of a so-called *pluriharmonic* function is Levi-flat for all points where the derivative does not vanish (and hence where we obtain a real-analytic hypersurface).

Finally let us mention a very important feature of complexification with respect to Levi-flat hypersurfaces. Suppose that M is a Levi-flat hypersurface defined near the origin and ρ is a real-analytic function also defined in a neighbourhood of the origin such that ρ vanishes on M. If we comlexify $\rho(z, \bar{z})$ the set

$$\{z : \rho(z, 0) = 0\}$$

vanishes on the leaf of the Levi-foliation through the origin. To see this fact, take the real valued function f we found above in the proof of Cartan's theorem. That is $df \neq 0$ and M near zero is given by $\operatorname{Im} f = 0$. This function $\operatorname{Im} f$ generates the ideal of (germs at zero) of functions vanishing on M. Therefore there is some real-analytic function $a(z, \bar{z})$ such that near zero

$$\rho(z, \bar{z}) = a(z, \bar{z})\big(\operatorname{Im} f(z)\big) = a(z, \bar{z})\frac{f(z) - \bar{f}(\bar{z})}{2i}$$

If we complexify and set $\bar{z} = 0$ we get

$$\rho(z, 0) = a(z, 0)\frac{f(z)}{2i}.$$

And $f(z) = 0$ gives the leaf of the Levi-foliation through zero.

In fact if $d\rho \neq 0$ (then ρ is usually called a *defining function* for M), then a must not vanish at 0, and $\rho(z, 0) = 0$ is in fact exactly the leaf of the foliation through zero.

Chapter 2

Singular Levi-flat hypersurfaces and their local normal form

We study normal forms of singular Levi-flat hypersurfaces. Under certain conditions, there are partial local normal forms for real analytic Levi-flat hypersurfaces. The problem of classification of normal forms for Levi-flat hypersurfaces is analogous to problem of classification of holomorphic functions studied by Vladímir Arnold [A1] in Singularity Theory. According to D. Burns and X. Gong, the normal forms are known completely for the quadratic Levi-flat hypersurfaces [BG]. Furthermore in [BG] existence and convergence of a rigid normal form in the case of a generic (Morse) singularity was proved.

Here, using theory of holomorphic foliations, we give an extension of result of Burns-Gong [BG]. To do this, we give some definitions about singular Levi-flat hypersurfaces and then establish some results on holomorphic foliations tangent to Levi-flat hypersurfaces. Finally, we use these results to prove some normal forms of Levi-flat hypersurfaces with special singularities.

Let $M \subset U \subset \mathbb{C}^n$ be a real analytic variety of real codimension one, where U is an open set. Let M^* denote its *regular part*, that is, the smooth part of M of highest dimension — near each point

$p \in M^*$, the variety M is a real-analytic manifold of real codimension one. For each $p \in M^*$, there is a unique complex hyperplane L_p contained in the tangent space T_pM^*. This defines a real analytic distribution

$$L : p \mapsto L_p \subset T_pM^*$$

of complex hyperplanes in TM^*, known as the *Levi distribution*. When L is integrable in the sense of Frobenius, we say that M is a *Levi-flat hypersurface*. The resulting foliation, denoted by $\mathcal{L}$, is known as the *Levi-foliation*.

As has been seen in Chapter 1, this concept goes back to E. Cartan (see for instance theorem 1.3.2); Cartan's theorem assures that there are local holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ around $p \in M^*$ such that $M^* = \{\mathrm{Im}(z_n) = 0\}$. As a consequence, the leaves of the Levi-foliation $\mathcal{L}$ have local equations $z_n = c$, for $c \in \mathbb{R}$. From the global viewpoint, they are complex manifolds of codimension one immersed in U. Cartan's local trivialization provides an intrinsic way to extend the Levi-foliation to a non-singular holomorphic foliation in a neighborhood of M^*. Locally, we extend $\mathcal{L}$ to a neighborhood of $x \in M^*$ as the foliation having, in the coordinates $(z_1, \ldots, z_n)$, horizontal leaves $z_n = c$, for $c \in \mathbb{C}$. Since M^* has real codimension 1, this is the unique possible local extension of $\mathcal{L}$, so that these local extensions glue together yielding a foliation defined in whole neighborhood of M^*. Nevertheless, it is not true in general that $\mathcal{L}$ extends to a holomorphic foliation in a neighborhood of $\overline{M^*}$, even if singularities are admitted. There are examples of Levi-flat hypersurfaces whose Levi-foliations extend to k-webs in the ambient space, see for instance [Br1], [Fe2] and [SS]. We explain it in the next chapter.

2.1 The complexification

Given $F \in \mathcal{A}_n$, we can write its Taylor series at $0 \in \mathbb{C}^n$ as

$$F(z) = \sum_{\mu,\nu} F_{\mu\nu} z^\mu \bar{z}^\nu, \tag{2.1}$$

where $F_{\mu\nu} \in \mathbb{C}$, $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, $z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n}$, $\bar{z}^\nu = \bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}$. When $F \in \mathcal{A}_{n\mathbb{R}}$, the coefficients $F_{\mu\nu}$ satisfy

$$\bar{F}_{\mu\nu} = F_{\nu\mu}.$$

The complexification $F_{\mathbb{C}} \in \mathcal{O}_{2n}$ of F is defined by the series

$$F_{\mathbb{C}}(z,w) = \sum_{\mu,\nu} F_{\mu\nu} z^{\mu} w^{\nu}. \tag{2.2}$$

Note that if the series in (2.1) converges in the polydisk $D_r^n = \{z \in \mathbb{C}^n : |z_j| < r\}$ then the series in (2.2) converges in the polydisk D_r^{2n}. Moreover, $F(z) = F_{\mathbb{C}}(z, \bar{z})$ for all $z \in D_r^n$.

Remark 2.1.1. The complexification does not depends on the coordinate system, in the sense that if $\varphi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ then there exists an unique $\varphi_{\mathbb{C}} \in \mathsf{Diff}(\mathbb{C}^{2n}, 0)$ such that

$$(F \circ \varphi)_{\mathbb{C}} = F_{\mathbb{C}} \circ \varphi_{\mathbb{C}} \tag{2.3}$$

In fact, if $\varphi(x) = \sum_{\sigma} \varphi_{\sigma} x^{\sigma}$ is the Taylor series of φ and $\varphi_{\mathbb{C}}(x,y) = (\varphi(x), \bar{\varphi}(y))$, where $\bar{\varphi}(y) = \sum_{\sigma} \overline{\varphi_{\sigma}} x^{\sigma}$ then relation (2.3) is satisfied for all $F \in \mathcal{A}_n$.

Let $M = \{F(z) = 0\}$ be a germ of Levi-flat hypersurface at $0 \in \mathbb{C}^n$, where $F \in \mathcal{A}_{n\mathbb{R}}$. Let us define the singular set of M by

$$\text{sing}(M) = \{F(z) = 0\} \cap \{dF(z) = 0\}.$$

The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}} = \{F_{\mathbb{C}} = 0\}$ and its regular part is $M_{\mathbb{C}}^* = M_{\mathbb{C}} \backslash \{dF_{\mathbb{C}} = 0\}$. Clearly $M_{\mathbb{C}}$ defines a germ of complex subvariety of dimension $2n-1$ at $0 \in \mathbb{C}^{2n}$. On the other hand, since the Levi distribution is defined by

$$L_p := Ker(\partial F(p)) \subset T_p M^* = Ker(dF(p)) \quad \text{for any} \quad p \in M^*,$$

we have *Levi 1-form* $\eta := i(\partial F - \bar{\partial} F)$ restricted to M^* define the Levi-foliation $\mathcal{L}$ on M^*. The complexification $\eta_{\mathbb{C}}$ of η can be written as

$$\eta_{\mathbb{C}} = i \sum_{j=1}^{n} \left(\frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j - \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j \right) = i \sum_{\mu,\nu} (F_{\mu\nu} w^{\nu} d(z^{\mu}) - F_{\mu\nu} z^{\mu} d(w^{\nu})).$$

The integrability condition of $\eta = i(\partial F - \bar{\partial} F)|_{M^*}$ implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable. Therefore $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*} = 0$ defines a holomorphic foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called *the complexification of $\mathcal{L}$.*

Remark 2.1.2. Let $\eta = i(\partial F - \bar{\partial} F)$ and $\eta_{\mathbb{C}}$ be as before. Then $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$, respectively. Set $\alpha = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j$ and $\beta = \sum_{j=1}^n \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j$. Hence $dF_{\mathbb{C}} = \alpha + \beta$ and $\eta_{\mathbb{C}} = i(\alpha - \beta)$, so that

$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = 2i\alpha|_{M^*_{\mathbb{C}}} = -2i\beta|_{M^*_{\mathbb{C}}} \tag{2.4}$$

In particular, $\alpha|_{M^*_{\mathbb{C}}}$ and $\beta|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}_{\mathbb{C}}$.

The results to continuation can be found in [CL]. We prove it here for completeness.

Lemma 2.1.3. *Let $h \in \mathcal{O}_n$, $h \neq 0$, $h(0) = 0$. Suppose that h is not a power in $\mathcal{O}_n$. Then* $\mathrm{Im}(h)$ *and* $\mathrm{Re}(h)$ *are irreducible in $\mathcal{A}_{n\mathbb{R}}$.*

Proof. We will prove that $\mathrm{Im}(h)$ is irreducible. Since $\mathrm{Re}(h) = \mathrm{Im}(i \cdot h)$ we will get also that $\mathrm{Re}(h)$ is irreducible. Let $h(z) = \sum_\mu h_\mu z^\mu$ be the Taylor series of h and $\bar{h}(w) := \sum_\mu \bar{h}_\mu w^\mu$, $w \in (\mathbb{C}^n, 0)$. Note that $\mathrm{Im}(h) = \frac{i}{2}\Big(h(z) - \bar{h}(\bar{z})\Big)$. Let $H := \frac{2}{i}\,\mathrm{Im}(h)$, so that the complexification $H_{\mathbb{C}}$ of H can be written as

$$H_{\mathbb{C}}(z, w) = h(z) - \bar{h}(w).$$

Suppose by contradiction that $\mathrm{Im}(h)$ is reducible in $\mathcal{A}_{n\mathbb{R}}$. In this case, we can write $H(z) = \phi(z) \cdot \varphi(z)$, where $\phi, \psi \in \mathcal{A}_{n\mathbb{R}}$ and $\phi(0) = \psi(0) = 0$. Let $\phi_{\mathbb{C}}$ and $\psi_{\mathbb{C}}$ be the complexifications of ϕ and ψ, respectively. Since $H = \phi \cdot \psi$, by complexification of the Taylor series in both members, we get $H_{\mathbb{C}}(z, w) = \phi_{\mathbb{C}} \cdot \psi_{\mathbb{C}}$, so that $H_{\mathbb{C}}$ is reducible.

Since the germ h is not a power, there exist $\epsilon > 0$ and a representative of h, denoted by the same letter, which is holomorphic in a polydisk D^n_r around 0 such that the fiber $h^{-1}(c)$ is connected for all $c \in \mathbb{C}$ with $0 < |c| < \epsilon$ (cf. [MM]). This implies that, if $w_0 \in D^n_r$ is such that $0 < |\bar{h}(w_0)| < \epsilon$ then

$$H_{w_0}(z) := H_{\mathbb{C}}(z, w_0) = h(z) - \bar{h}(w_0)$$

is irreducible in D^n_r. Let $0 < \delta \leq r$ be such that $|\bar{h}(w)| < \epsilon$ if $|w| < \delta$ and denote $V := D^n_\delta \setminus \bar{h}^{-1}(0)$. We have concluded that if $w \in V$ then $H_w(z)$ is irreducible in D^n_δ.

In fact, for a fixed $w_0 \in D^n_\delta$ set $\phi_{w_0}(z) = \phi_{\mathbb{C}}(z, w_0)$ and $\phi_{w_0}(z) = \psi_{\mathbb{C}}(z, w_0)$. From $H_{\mathbb{C}} = \phi_{\mathbb{C}} \cdot \psi_{\mathbb{C}}$ we get

$$H_w = \phi_w \cdot \psi_w$$

and this implies that for all $w \in V$, either $\phi_w \in \mathcal{O}^*(D^n_\delta)$, or $\psi_w \in \mathcal{O}^*(D^n_\delta)$. Since $0 \in \mathbb{C}^n$ is in the closure of the open set V, we can assume that there exists a sequence $(w_n)_{n\geq 1}$ in V such that $\lim\limits_{n\to\infty} w_n = 0$ and $\psi_{w_n} \in \mathcal{O}^*(D^n_\delta)$. Since $\lim\limits_{n\to\infty} \psi_{w_n} = \psi_0$ in compact sets, we must have $\psi_0 \equiv 0$, because $\psi_0(0) = \psi_{\mathbb{C}}(0,0) = 0$.

On the other hand, the complexification $\psi_{\mathbb{C}}$ of $\psi \in \mathcal{A}_{n\mathbb{R}}$ satisfies

$$\psi_{\mathbb{C}}(z, w) = \overline{\psi_{\mathbb{C}}(\bar{w}, \bar{z})},$$

and so $\psi \equiv 0$ implies that $\overline{\psi_{\mathbb{C}}(0, \bar{z})} = \psi_{\mathbb{C}}(z, 0) = 0$ for all z, and hence $\psi_z(0) = 0$ for all z, which is a contradiction with $\psi_{w_n} \neq 0$. This contradiction proves that $H_{\mathbb{C}}$ is irreducible. □

Now we prove the following lemma.

Lemma 2.1.4. *If F is irreducible in $\mathcal{A}_{n\mathbb{R}}$ and $M = \{F = 0\}$ has real codimension one then $F_{\mathbb{C}}$ is irreducible in $\mathcal{O}_{2n}$.*

Proof. Suppose that $F_{\mathbb{C}} = g_1^{k_1} \dots g_r^{k_r}$, $g_j \in \mathcal{O}_{2n}$, $k_j \geq 1$, $g_j(0) = 0$ for all $1 \leq j \leq r$. Then $F = G_1^{k_1} \dots G_r^{k_r}$, where $G_j(z) = g_j(z, \bar{z})$, $1 \leq j \leq r$. Note that $g_j = G_{j\mathbb{C}}$ for all $1 \leq j \leq r$. Since $F = \bar{F}$, we get

$$F = G_1^{k_1} \dots G_r^{k_r} = \bar{G}_1^{k_1} \dots \bar{G}_r^{k_r}$$

so $G_1^{k_1} \dots G_r^{k_r} = \bar{G}_1^{k_1} \dots \bar{G}_r^{k_r}$, this implies that $r = 2a + b$ and after, reordering the indexes, we can assume that there exist units $U_1, \dots, U_a, V_1 \dots V_b \in \mathcal{O}_{2n}$ such that $G_{2j-1\mathbb{C}} = U_j(\bar{G}_{2j})_{\mathbb{C}}$, $k_{2j-1} = k_{2j}$ if $1 \leq j \leq a$ and $G_{2a+j\mathbb{C}} = V_j(\bar{G}_{2a+j})_{\mathbb{C}}$ if $1 \leq j \leq b$. In particular, we can write $F = G \cdot H$ where

$$G = (G_1\bar{G}_1)^{k_1} \dots (G_a\bar{G}_a)^{k_a} \qquad H = UG_{2a+1}^{k_{2a+1}} \dots G_r^{k_r}$$

and U is an unit in $\mathcal{A}_n$. Note that $H \in \mathcal{A}_{n\mathbb{R}}$ because $F, G \in \mathcal{A}_{n\mathbb{R}}$. Since F is irreducible in $\mathcal{A}_{n\mathbb{R}}$ we have two possibilities:

- $a = 1$, $k_1 = 1$, $b = 0$ and $H(0) \neq 0$. This implies

$$F = H|G_1|^2 = H(\mathrm{Re}(G_1)^2 + \mathrm{Im}(G_1)^2),$$

and so $M = \{\mathrm{Re}(G_1) = \mathrm{Im}(G_1) = 0\}$. But this would imply that M has real codimension ≥ 2, a contradiction.

- $a = 0$, $b = r$ and $(\bar{G}_j)_{\mathbb{C}} = V_j \cdot G_{j\mathbb{C}}$ for all $j = 1, \ldots, r$. In this case, we get $\bar{G}_j = U_j G_j$, where $U_j(z) = V_j(z, \bar{z})$ is an unit. Therefore $|U_j| = 1$ and so $U_j = e^{i\alpha_j}$, $\alpha \in \mathcal{A}_{n\mathbb{R}}$. If we set $h_j = e^{i\alpha^j/2} \cdot G_j$, then we get $\bar{h}_j = h_j$ and $h_j \in \mathcal{A}_{n\mathbb{R}}$. Hence, we can write $F = U h_1^{k_1} \ldots h_b^{k_b}$, where $U \in \mathcal{A}_{n\mathbb{R}}$, $U(0) \neq 0$. Since F is irreducible $b = 1$ and $k_1 = 1$. Finally, $F_{\mathbb{C}}$ is irreducible in $\mathcal{O}_n$.

This ends the proof. □

2.2 Levi-flat hypersurfaces and holomorphic foliations

This section is devoted to state some results about Levi-flat hypersurfaces invariant by holomorphic foliations.

Definition 2.2.1. Let $\mathcal{F}$ and $M = \{F = 0\}$ be germs at $(\mathbb{C}^n, 0)$, $n \geq 2$, of a codimension one singular holomorphic foliation and of a real Levi-flat hypersurface, respectively. We say that M is *invariant by* $\mathcal{F}$ (or that $\mathcal{F}$ *leaves* M *invariant*) if the leaves of the Levi-foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

Definition 2.2.2. The *algebraic dimension* of $\mathrm{sing}(M)$ is the complex dimension of the singular set of $M_{\mathbb{C}}$.

In the proof of main result of this chapter, we will use the following result due to Cerveau-Lins Neto [CL]. This result is fundamental to find normal forms of Levi-flat hypersurfaces and essentially assures that if the singularities of M are sufficiently small (in the algebraic sense) then M is given by the zeros of the real part of a holomorphic function.

Theorem 2.2.3 (Cerveau-Lins Neto [CL]). *Let $M = \{F = 0\}$ be a germ of an irreducible real analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi 1-form $\eta := i(\partial F - \bar{\partial} F)$. Assume that the algebraic dimension of $\operatorname{sing}(M)$ is less than or equal to $2n-4$. Then there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, if one of the following conditions is fulfilled:*

1. *$n \geq 3$ and $cod_{M^*_{\mathbb{C}}}(\operatorname{sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 3$.*
2. *$n \geq 2$, $cod_{M^*_{\mathbb{C}}}(\operatorname{sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral f such that $M = \{\operatorname{Re}(f) = 0\}$.

Now we state key lemmas. Suppose that $F_{\mathbb{C}}$ converges in a neighborhood of

$$\bar{\triangle} = \{(z, w) \in \mathbb{C}^4 : |z|, |w| \leq 1\} := \bar{B} \times \bar{B},$$

set $V := M^*_{\mathbb{C}} \setminus \operatorname{sing}(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$ and denote by L_p the leaf of $\mathcal{L}_{\mathbb{C}}$ through p, where $p \in V$. Consider the holomorphic vector fields X and Y on B defined by

$$X = \frac{\partial F_{\mathbb{C}}}{\partial z_2} \partial_{z_1} - \frac{\partial F_{\mathbb{C}}}{\partial z_1} \partial_{z_2} \quad \text{and} \quad Y = \frac{\partial F_{\mathbb{C}}}{\partial w_2} \partial_{w_1} - \frac{\partial F_{\mathbb{C}}}{\partial w_1} \partial_{w_2}.$$

Since $X(F_{\mathbb{C}}) = Y(F_{\mathbb{C}}) \equiv 0$, they are tangent to the leaves of $F_{\mathbb{C}}$ and in particular to $M_{\mathbb{C}}$. Denote by $\mathcal{G}$ and $\mathcal{H}$ the foliations by curves defined by X and Y on $M^*_{\mathbb{C}}$, respectively.

Lemma 2.2.4 (Cerveau-Lins Neto [CL]). *In above situation, we have the following:*

1. *$X|_{M^*_{\mathbb{C}}}$ and $Y|_{M^*_{\mathbb{C}}}$ are tangent to $\mathcal{L}_{\mathbb{C}}$. In particular, if we denote by L^g_q and L^h_q the leaves of $\mathcal{G}$ and $\mathcal{H}$ through $q \in V$ then $L^g_q \subset L_q$ and $L^h_q \subset L_q$.*
2. *X and Y are linearly independent along V. In particular, for any $p \in V$ the leaves L^g_p and L^h_p intersect transversaly in L_p at p: $T_pL_p = T_pL^g_p \oplus T_pL^h_p$.*

3. *For any* $(z_0, w_0) \in V$ *the leaf* L_p^g *(resp.* L_p^h*) is contained in*

$$A_p^g := \{(z_0, w_0) \in V : F_{\mathbb{C}}(z, w_0) = 0\}$$

(resp. $A_p^h := \{(z_0, w_0) \in V : F_{\mathbb{C}}(z, w_0) = 0\}$*). In particular,* L_p^g *and* L_p^h *are closed in* $V = M_{\mathbb{C}}^* \setminus \mathrm{sing}(\eta|_{M_{\mathbb{C}}^*})$*.*

Proof. First we prove item (1), this following from the fact that $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M_{\mathbb{C}}^*} = -\beta|_{M_{\mathbb{C}}^*}$ and since $i_X(\alpha) \equiv 0$ we have that if $p \in V$ then $X(p) \in T_p\mathcal{L}_{\mathbb{C}}$, where $T_p\mathcal{L}_{\mathbb{C}} := T_pL_p$. Hence $L_p^g \subset L_p$. Similarly, $i_Y(\beta) \equiv 0$, so that $L_p^h \subset L_p$. To prove item (2), observe that since $p \in V$, either $\frac{\partial F_{\mathbb{C}}}{\partial z_1}(p) \neq 0$ or $\frac{\partial F_{\mathbb{C}}}{\partial z_2}(p) \neq 0$, which implies that $X(p) \neq 0$. Similarly, $Y(p) \neq 0$. Therefore, $X(p) \wedge Y(p) \neq 0$, and the vector fields X and Y are linearly independent along V. Item (3) follows from $X(F) = X(w_1 - w_{01}) = X(w_2 - w_{02}) = 0$ that $L_p^g \subset A_p^g$. Let S_p^g be the irreducible component of A_p^g which contains p. For any $q \in V$ we have that, either $\frac{\partial F_{\mathbb{C}}}{\partial z_1}(q) \neq 0$ or $\frac{\partial F_{\mathbb{C}}}{\partial z_2}(q) \neq 0$. This implies that $\dim_{\mathbb{C}}(S_p^g) = 1$. On the other hand, $L_p^g \subset S_p^g$ and $\dim_{\mathbb{C}}(L_p^g) = 1$, so that $L_p^g = S_p^g$, by definition of leaf. Hence. L_p^g is closed in V. Similarly, L_p^h is closed in V. □

Lemma 2.2.5 (Cerveau-Lins Neto [CL]). *For any* $p \in V$*, the leaf* L_p *of* $\mathcal{L}_{\mathbb{C}}$ *is closed in* $M_{\mathbb{C}}^*$*.*

Proof. Fix $p = (z_0, w_0) \in V$ and set

$$N_p := \bigcup_{q \in L_p^g} L_q^g \subset V.$$

We assert that N_p is closed in V and $N_p = L_p$. In fact, let $(p_n = (z_n, w_n))_{n \geq 1}$ be a sequence in N_p such that $\lim_{n\to\infty} p_n = q = (z', w') \in V$. Recall that we are working in $\triangle = B \times B$. In particular, $z', w' \in B$. By lemma 2.2.4 item (3), we get $(z_0, w_n) \in L_p^h$, for all $n \geq 1$, and $(z_0, w_n) \to (z_0, w')$. Again by lemma 2.2.4 item (3), we have $(z', w') \in L^g_{(z_0, w')}$. Hence, $q \in N_p$. This proves the assertion. Finally, we assert that $L_p = N_p$. In fact, lemma 2.2.4 item (1) implies that $N_p \subset L_p$. Since N_p is closed in V, it is also closed in $L_p \subset V$. On the other hand, it follows from lemma 2.2.4 item (2) that N_p is open in L_p. Hence, $N_p = L_p$, because L_p is connected. □

Normal form in $\mathbb{C}^n$	Singular set
$(\operatorname{Im} z_1)^2 = 0$	empty
$\operatorname{Im}(z_1^2 + z_2^2 + \cdots + z_k^2) = 0$	$\mathbb{C}^{n-k}$
$z_1^2 + 2\lambda z_1\bar{z}_1 + \bar{z}_1^2 = 0 \ \big(\lambda \in (0,1)\big)$	$\mathbb{C}^{n-1}$
$(\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0$	$\mathbb{R}^2 \times \mathbb{C}^{n-2}$
$\lvert z_1\rvert^2 - \lvert z_2\rvert^2 = 0$	$\mathbb{C}^{n-2}$

Table 2.1: Quadratic Levi-flat hypersurfaces

2.3 Normal forms

The systematic local study of singular Levi-flat real analytic hypersurfaces was iniciated by Burns and Gong. Here we state the classification of normal forms of quadratic Levi-flat hypersurfaces due to Burns and Gong.

Theorem 2.3.1 (Burns-Gong [BG]). *The table 2.1 is a complete list of holomorphic equivalence class of Levi-flat quadratic real hypersurfaces in $\mathbb{C}^n$.*

The following result is the first rigid normal form of singular Levi-flat hypersurfaces.

Theorem 2.3.2 (Burns-Gong [BG]). *Let M be a germ of real analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, defined by*

$$\operatorname{Re}(z_1^2 + \ldots + z_n^2) + H(z, \bar{z}) = 0$$

with $H(z,\bar{z}) = O(|z|^3)$, $H(z,\bar{z}) = \overline{H}(\bar{z}, z)$. Then there exists a holomorphic coordinate system $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that

$$M = \{\operatorname{Re}(x_1^2 + \ldots + x_n^2) = 0\}.$$

This result can be viewed as a Morse's lemma for Levi-flat hypersurfaces and it is a normal form in the case of a generic (Morse) singularity. In order to prove a generalization of theorem 2.3.2, we need some definitions.

Definition 2.3.3. Two germs $f, g \in \mathcal{O}_n$ are right equivalent if there exists $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ such that $f \circ \phi^{-1} = g$.

The local algebra of $f \in \mathcal{O}_n$ is by definition

$$A_f := \mathcal{O}_n / \langle \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle.$$

We denote by $\mu(f,0) := \dim A_f$ the Milnor number of f at $0 \in \mathbb{C}^n$. This number is finite if and only if 0 is an isolated singularity of f. Morse lemma can now be rephrased by saying that if $0 \in \mathbb{C}^n$ is an isolated singularity of f with Milnor number $\mu(f,0) = 1$ then f is right equivalent to its second jet. The next lemma is a generalization of Morse's lemma. We refer to [AGV], pp. 121.

Lemma 2.3.4. *Suppose $0 \in \mathbb{C}^n$ is an isolated singularity of $f \in \mathcal{M}_n$ with Milnor number μ. Then f is right equivalent to $j_0^{\mu+1}(f)$.*

In [Fe], using theory of holomorphic foliations, theorem 2.3.2 was generalized in this following sense.

Theorem 2.3.5. *Let $M = \{F = 0\}$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 2$, be a germ of irreducible real analytic function such that*

1. *$F(z_1, \ldots, z_n) = \mathrm{Re}(P(z_1, \ldots, z_n)) + H(z, \bar{z})$, where P is a homogeneous polynomial of degree k with an isolated singularity at $0 \in \mathbb{C}^n$ and $H(z, \bar{z}) = O(|z|^{k+1})$, $H(z, \bar{z}) = \overline{H}(\bar{z}, z)$.*

2. *The Milnor number of P at $0 \in \mathbb{C}^n$ is μ.*

3. *M is Levi-flat.*

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(M) = \{\mathrm{Re}(h) = 0\}$, where $h(z)$ is a polynomial of degree $\mu + 1$ and $j_0^k(h) = P$.

Proof. The idea is to use theorem 2.2.3 to prove that there exists a germ $f \in \mathcal{O}_n$ such that the foliation $\mathcal{F}$ defined by $df = 0$ is tangent to M and $M = \{\mathrm{Re}(f) = 0\}$. Note that $\mathcal{F}$ can viewed as an extension to a neighborhood of $0 \in \mathbb{C}^n$ of the Levi-foliation $\mathcal{L}$ on M^*.

Suppose for a moment that $M = \{\mathrm{Re}(f) = 0\}$ and let us conclude the proof. Without lost of generality, we can suppose that f is not a power in $\mathcal{O}_n$. In this case $\mathrm{Re}(f)$ is irreducible by lemma 2.1.3. This implies that $\mathrm{Re}(f) = U.F$, where $U \in \mathcal{A}_{n\mathbb{R}}$ and $U(0) \neq 0$.

Let $\sum_{j\geq k} f_j$ be the Taylor series of f, where f_j is a homogeneous polynomial of degree j, $j \geq k$. Then

$$\mathrm{Re}(f_k) = j_0^k(\mathrm{Re}(f)) = j_0^k(U.F) = U(0).\,\mathrm{Re}(P(z_1,\ldots,z_n)).$$

Hence $f_k(z_1,\ldots,z_n) = U(0).P(z_1,\ldots,z_n)$. We can suppose that $U(0) = 1$, so that

$$f(z) = P(z) + h.o.t \tag{2.5}$$

In particular, $\mu = \mu(f,0) = \mu(P,0)$, $f \in \mathcal{M}_n$, because P has isolated singularity at $0 \in \mathbb{C}^n$. Hence by lemma 2.3.4, f is right equivalent to $j_0^{\mu+1}(f)$, that is, there exists $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ such that $h := f \circ \phi^{-1} = j_0^{\mu+1}(f)$. Therefore, $\phi(M) = \{\mathrm{Re}(h) = 0\}$ and this will conclude the proof of theorem 2.3.5.

Let us prove that we can apply theorem 2.2.3. We can write

$$F(z) = \mathrm{Re}(P(z_1,\ldots,z_n)) + H(z_1,\ldots,z_n),$$

where $H : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ is a germ of real analytic function and $j_0^k(H) = 0$. For simplicity, we assume that P has real coefficients. Then we get the complexification

$$F_{\mathbb{C}}(z,w) = \frac{1}{2}(P(z) + P(w)) + H_{\mathbb{C}}(z,w)$$

and $M_{\mathbb{C}} = \{F_{\mathbb{C}}(z,w) = 0\} \subset (\mathbb{C}^{2n}, 0)$. Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$, we get $sing(M_{\mathbb{C}}) = \{0\}$, and so the algebraic dimension of $sing(M)$ is 0.

On the other hand, the complexification of $\eta = i(\partial F - \bar{\partial} F)$ is

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}).$$

Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$. Now we compute $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with

$$\alpha = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j := \frac{1}{2}\sum_{j=1}^{n}(\frac{\partial P}{\partial z_j}(z) + A_j)dz_j$$

and

$$\beta = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j := \frac{1}{2}\sum_{j=1}^{n}(\frac{\partial P}{\partial w_j}(w) + B_j)dw_j,$$

where $\displaystyle\frac{1}{2}\sum_{j=1}^{n} A_j dz_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial z_j} dz_j$ and $\displaystyle\frac{1}{2}\sum_{j=1}^{n} B_j dw_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial w_j} dw_j$.

Then $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so

$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = (\eta_{\mathbb{C}} + idF_{\mathbb{C}})|_{M^*_{\mathbb{C}}} = 2i\alpha|_{M^*_{\mathbb{C}}} = -2i\beta|_{M^*_{\mathbb{C}}}. \tag{2.6}$$

In particular, $\alpha|_{M^*_{\mathbb{C}}}$ and $\beta|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}_{\mathbb{C}}$. Therefore $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$ can be splited in two parts. Let $M_1 = \{(z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0$ for some $j = 1, \ldots, n\}$ and $M_2 = \{(z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0$ for some $j = 1, \ldots, n\}$, note that $M^*_{\mathbb{C}} = M_1 \cup M_2$; if we denote by

$$X_1 := M_1 \cap \left\{ \frac{\partial P}{\partial z_1}(z) + A_1 = \ldots = \frac{\partial P}{\partial z_n}(z) + A_n = 0 \right\}$$

and

$$X_2 := M_2 \cap \left\{ \frac{\partial P}{\partial w_1}(w) + B_1 = \ldots = \frac{\partial P}{\partial w_n}(w) + B_n = 0 \right\},$$

then $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = X_1 \cup X_2$. Since $P \in \mathbb{C}[z_1, \ldots, z_n]$ has an isolated singularity at $0 \in \mathbb{C}^n$, we conclude that $cod_{M^*_{\mathbb{C}}} sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = n$.

If $n \geq 3$, we can directly apply theorem 2.2.3 and the proof ends. In the case $n = 2$, we are going to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

We begin by a blow-up at $0 \in \mathbb{C}^4$. Let $F(x,y) = \mathrm{Re}(P(x,y)) + H(x,y)$ and $M = \{F(x,y) = 0\}$ Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(x,y,z,w) = \frac{1}{2}P(x,y) + \frac{1}{2}P(z,w) + H_{\mathbb{C}}(x,y,z,w).$$

We take the exceptional divisor $D := \mathbb{P}^3$ of the blow-up $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$ with homogeneous coordinates $[x : y : z : w]$, $(x,y,z,w) \in \mathbb{C}^4 \backslash \{0\}$. The intersection of the strict transform $\tilde{M}_{\mathbb{C}}$ of $M_{\mathbb{C}}$ by π with the divisor $D = \mathbb{P}^3$ is the surface

$$Q = \{[x : y : z : w] \in \mathbb{P}^3 : P(x,y) + P(z,w) = 0\},$$

which is an irreducible smooth surface. Consider for instance the chart $(W, (t,u,z,v))$ of $\tilde{\mathbb{C}}^4$ where

$$\pi(t,u,z,v) = (t.z, u.z, z, v.z) = (x,y,z,w).$$

We have

$$F_{\mathbb{C}} \circ \pi(t,u,z,v) = z^k \left(\frac{1}{2} P(t,u) + \frac{1}{2} P(1,v) + z H_1(t,u,z,v) \right),$$

where $H_1(t,u,z,v) = H(tz,uz,z,vz)/z^{k+1}$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W = \left\{ \frac{1}{2} P(t,u) + \frac{1}{2} P(1,v) + z H_1(t,u,z,v) = 0 \right\}$$

and so $Q \cap W = \{z = P(t,u) + P(1,v) = 0\}$. On the other hand, as we have seen in (3.2), the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = \frac{1}{2} \frac{\partial P}{\partial x} dx + \frac{1}{2} \frac{\partial P}{\partial y} dy + \frac{\partial H_{\mathbb{C}}}{\partial x} dx + \frac{\partial H_{\mathbb{C}}}{\partial y} dy.$$

In particular, we get

$$\pi^*(\alpha) = z^{k-1} \left(\frac{1}{2} \frac{\partial P}{\partial x}(t,u) z dt + \frac{1}{2} \frac{\partial P}{\partial y}(t,u) z du + \frac{1}{2} k P(t,u) dz + z\theta \right),$$

where $\theta = \pi^*(\frac{\partial H_{\mathbb{C}}}{\partial x} dx + \frac{\partial H_{\mathbb{C}}}{\partial y} dy)/z^k$. Hence, $\tilde{\mathcal{L}}_{\mathbb{C}}$ is defined by

$$\alpha_1 = \frac{1}{2} \frac{\partial P}{\partial x}(t,u) z dt + \frac{1}{2} \frac{\partial P}{\partial y}(t,u) z du + \frac{1}{2} k P(t,u) dz + z\theta. \qquad (2.7)$$

Since $Q \cap W = \{z = P(t,u) + P(1,v) = 0\}$, we see that Q is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. In particular, $S := Q \backslash sing(\tilde{\mathcal{L}}_{\mathbb{C}})$ is a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Fix $p_0 \in S$ and a transverse section Σ through p_0. Let $G \subset \mathsf{Diff}(\Sigma, p_0)$ be the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Since $\dim(\Sigma) = 1$, we can think that $G \subset \mathsf{Diff}(\mathbb{C}, 0)$. Let us prove that G is finite and linearizable.

At this part we use that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed (see lemma 2.2.5). Let $G' = \{f'(0) : f \in G\}$ and consider the homomorphism $\phi : G \to G'$ defined by $\phi(f) = f'(0)$. We assert that ϕ is injective. In fact, assume that $\phi(f) = 1$ and by contradiction that $f \neq id$. In this case $f(z) = z + a.z^{r+1} + \ldots$, where $a \neq 0$. According to [L], the pseudo-orbits of this transformation accumulate at $0 \in (\Sigma, 0)$, contradicting that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed. Now, it suffices to prove that any element $g \in G$ has finite order (cf. [MM]). In fact, if $\phi(g) = g'(0)$ is a root of unity then g has finite order because ϕ is injective. On the other hand, if $g'(0)$ was not a root of unity then g would have pseudo-orbits accumulating at $0 \in (\Sigma, 0)$ (cf. [L]). Hence, all transformations of G have finite order and G is linearizable.

This implies that there is a coordinate system w on $(\Sigma, 0)$ such that $G = \langle w \to \lambda w \rangle$, where λ is a d^{th}-primitive root of unity (cf. [MM]). In particular, $\psi(w) = w^d$ is a first integral of G, that is $\psi \circ g = \psi$ for any $g \in G$.

Let Z be the union of the separatrices of $\mathcal{L}_{\mathbb{C}}$ through $0 \in \mathbb{C}^4$ and $\tilde{Z}$ be its strict transform under π. The first integral ψ can be extended to a first integral $\varphi : \tilde{M}_{\mathbb{C}} \backslash \tilde{Z} \to \mathbb{C}$ be setting

$$\varphi(p) = \psi(\tilde{L}_p \cap \Sigma),$$

where $\tilde{L}_p$ denotes the leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$ through p. Since ψ is bounded (in a compact neighborhood of $0 \in \Sigma$), so is φ. It follows from Riemann extension theorem that φ can be extended holomorphically to $\tilde{Z}$ with $\varphi(\tilde{Z}) = 0$. This provides the first integral and finishes the proof of theorem 2.3.5. □

Remark 2.3.6. The rigid normal form of Burns-Gong is a corollary of theorem 2.3.5. In fact, let us consider

$$M = \{\mathrm{Re}(P) + H = 0\}$$

such that M is Levi-flat hypersurface at $0 \in \mathbb{C}^n$ and $P(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2$. The Milnor number of P is 1, then by theorem 2.3.5 there exists a germ of biholomrphism ϕ of $(\mathbb{C}^n, 0)$ such that $\phi(M) = \{\mathrm{Re}(h) = 0\}$, where h is a polynomial of degree 2 and $j_0^2(h) = P$, so that $h = P$. In particular, M is bihomorphic to

$$\{\mathrm{Re}(z_1^2 + \ldots + z_n^2) = 0\}.$$

Now we state some general facts about rigid normal forms of Levi-flat hypersurfaces with quasihomogeneous singularities.

Definition 2.3.7. A germ $f \in \mathcal{O}_n$ is said to be quasihomogeneous of degree d with indices $\alpha_1, \ldots, \alpha_n$ if for any $\lambda \in \mathbb{C}^*$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$ we have

$$f(\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n) = \lambda^d f(z_1, \ldots, z_n).$$

The index α_s is also called the weight of the variable z_s.

Definition 2.3.8. The Newton support of germ $f = \sum a_{ij}x^i y^j$ is defined as $supp(f) = \{(i,j) : a_{ij} \neq 0\}$.

Definition 2.3.9. A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be quasihomogeneous of degree d with indices $\alpha_1, \ldots, \alpha_n$, if for any $\lambda \in \mathbb{C}$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$, we have

$$f(\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n) = \lambda^d f(z_1, \ldots, z_n).$$

The index α_s is also called the weight of the variable z_s.

In the above situation, if $f = \sum a_k x^k$, $k = (k_1, \ldots, k_n)$, $x^k = x_1^{k_1} \ldots x^{k_n}$, then $supp(f) \subset \Gamma = \{k : a_1 k_1 + \ldots + a_n k_n = d\}$. The set Γ is called the diagonal. Usually one takes $\alpha_i \in \mathbb{Q}$ and $d = 1$.

One can define the quasihomogeneous filtration of the ring $\mathcal{O}_n$. It consists of the decreasing family of ideals $\mathcal{A}_d \subset \mathcal{O}_n$, $\mathcal{A}_{d'} \subset \mathcal{A}_d$ for $d < d'$. Here $\mathcal{A}_d = \{Q$: degrees of monomials from $supp(Q)$ are $deg(Q) \geq d\}$; (the degree is quasihomogeneous).

When $\alpha_1 = \ldots = \alpha_n = 1$, this filtration coincides with the usual filtration by the usual degree.

Definition 2.3.10. A function f is called semiquasihomogeneous if $f = Q + F'$, where Q is quasihomogeneous of degree d of finite multiplicity and $F' \in \mathcal{A}_{d'}$, $d' > d$.

We will use the following result of Arnold [A].

Theorem 2.3.11. *Let f be a semiquasihomogeneous function, $f = Q + F'$ with quasihomogeneous Q of finite multiplicity. Then f is right equivalent to the function $Q + \sum_j c_j e_j(z)$, where $e_1, \ldots, e_s$ are the elements of the monomial basis of the local algebra A_Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.*

Example 2.3.12. If $f = Q + F'$ is semiquasihomogeneous and $Q(x,y) = x^2 y + y^k$, then f is right equivalent to Q. Indeed, the base of the local algebra

$$\mathcal{O}_2/(xy, x^2 + ky^{k-1})$$

is $1, x, y, y^2, \ldots, y^{k-1}$ and lies below the diagonal Γ. Here $\mu(Q, 0) = k + 1$.

We have the following generalization.

Theorem 2.3.13. *Let $M = \{F = 0\}$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 3$, be a germ of irreducible real analytic function such that*

1. *$F(z_1, \ldots, z_n) = \mathrm{Re}\big(Q(z_1, \ldots, z_n)\big) + H(z_1, \ldots, z_n)$, where Q is a quasihomogeneous polynomial of degree d with an isolated singularity at $0 \in \mathbb{C}^n$.*

2. *M is Levi-flat.*

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\phi(M) = \{\mathrm{Re}(Q(z) + \sum_j c_j e_j(z)) = 0\},$$

where $e_1, \ldots, e_s$ are the elements of the monomial basis of the local algebra of Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.

Sketch of proof. It is easily seen that $sing(M_{\mathbb{C}}) = \{0\}$ and

$$cod_{M^*_{\mathbb{C}}} sing(\mathcal{L}_{\mathbb{C}}) \geq 3.$$

The argument is essentially the same of the proof of theorem 2.3.5. In this way, there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, moreover $\mathcal{F}_M$: $dh = 0$, $h(z) = Q(z) + h.o.t$ and $M = \{\mathrm{Re}(h) = 0\}$. According to theorem 2.3.11, there exists $\phi \in \mathsf{Diff}(\mathbb{C}^n, 0)$ such that $h \circ \phi^{-1}(w) = Q(w) + \sum_k c_k e_k(w)$, where c_k and e_k as above. Hence

$$\phi(M) = \{\mathrm{Re}(Q(w) + \sum_k c_k e_k(w)) = 0\}.$$

□

We give some applications of theorem 2.3.5.

Example 2.3.14. Let $Q(z, w) = z^2 w + w^3$, we look that Q has an isolated singularity at $0 \in \mathbb{C}^2$ and Milnor number $\mu(Q, 0) = 4$. According to [AGV] pp. 184, any germ $f(z, w) = z^2 w + w^3 + h.o.t$ is right equivalent to $z^2 w + w^3$. In particular, if

$$F(z, w) = \mathrm{Re}(z^2 w + w^3) + h.o.t \in \mathcal{A}_{2\mathbb{R}}$$

and $M = \{F(z,w) = 0\}$ is Levi-flat at $0 \in \mathbb{C}^2$, then theorem 2.3.5 implies that there exists a holomorphic change of coordinate such that

$$M = \{\mathrm{Re}(z^2 w + w^3) = 0\}.$$

Example 2.3.15. If $Q(z,w) = z^5 + w^5$ then $f(z,w) = Q(z,w) + h.o.t$ is right equivalent to $z^5 + w^5 + c.z^3 w^3$, where $c \neq 0$ is a constant (cf. [AGV], pp. 194). Let $F(z,w) = \mathrm{Re}(z^5 + w^5) + h.o.t \in \mathcal{A}_{2\mathbb{R}}$ be such that $M = \{F(z,w) = 0\}$ is Levi-flat, therefore theorem 2.3.5 implies that there exists a holomorphic change of coordinate such that

$$M = \{\mathrm{Re}(z^5 + w^5 + c.z^3 w^3) = 0\}.$$

2.4 Levi-flat with Arnold singularities

In the same spirit the purpose now is establish the existence of rigid normal forms of Levi-flat hypersurfaces with Arnold type singularities. More precisely, these normal forms are for Levi-flat hypersurfaces which are defined by the vanishing of the real part of A_k, D_k, E_k *singularities*. One motivation for considering A_k, D_k, E_k singularities is the following: When we consider the problem of classification of holomorphic germs f with an isolated singularity at $0 \in \mathbb{C}^n$, with respect to holomorphic changes of coordinates the list starts with the famous A_k, D_k, E_k *singularities or simple singularities*, see for instance Arnold's papers [A], [A1]:

Type	Normal form	Conditions
A_k	$z_1^2 + z_2^{k+1} + \ldots + z_n^2,$	$k \geq 1$
D_k	$z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2,$	$k \geq 4$
E_6	$z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2$	
E_7	$z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2$	
E_8	$z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2$	

Table 2.2: A_k, D_k, E_k singularities

Several characterizations of the A_k, D_k, E_k singularities are well-known, see for instance [D]. The following result is an Arnold's type result for singular Levi-flat hypersurfaces.

Theorem 2.4.1 (Fernández-Pérez [Fe4]). *Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of irreducible real analytic Levi-flat hypersurface. Suppose that F is of one of the following types:*

(a) $F(z) = \mathrm{Re}(z_1^2 + z_2^{k+1} + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where $k \geq 2$ and*

$$H(z,\bar{z}) = O(|z|^{k+2})\,,\ H(z,\bar{z}) = \overline{H}(\bar{z}, z).$$

(b) $F(z) = \mathrm{Re}(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where $k \geq 5$ and*

$$H(z,\bar{z}) = O(|z|^{k})\,,\ H(z,\bar{z}) = \overline{H}(\bar{z}, z).$$

(c) $F(z) = \mathrm{Re}(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where*

$$H(z,\bar{z}) = O(|z|^{5})\,,\ H(z,\bar{z}) = \overline{H}(\bar{z}, z).$$

(d) $F(z) = \mathrm{Re}(z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where*

$$H(z,\bar{z}) = O(|z|^{5})\,,\ H(z,\bar{z}) = \overline{H}(\bar{z}, z).$$

(e) $F(z) = \mathrm{Re}(z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2) + H(z, \bar{z})$, *where*

$$H(z,\bar{z}) = O(|z|^{6})\,,\ H(z,\bar{z}) = \overline{H}(\bar{z}, z).$$

Then there exists a germ of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

(a) $\varphi(M) = (\mathrm{Re}(z_1^2 + z_2^{k+1} + z_3^2 + \ldots + z_n^2) = 0)$,

(b) $\varphi(M) = (\mathrm{Re}(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0)$,

(c) $\varphi(M) = (\mathrm{Re}(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0)$,

(d) $\varphi(M) = (\mathrm{Re}(z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0)$,

(e) $\varphi(M) = (\mathrm{Re}(z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0)$, *respectively.*

We find the following list:

Type	Normal form	Conditions
A_k	$\mathrm{Re}(z_1^2 + z_2^{k+1} + \ldots + z_n^2) = 0$	$k \geq 1$
D_k	$\mathrm{Re}(z_1^2 z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2) = 0$	$k \geq 4$
E_6	$\mathrm{Re}(z_1^4 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	
E_7	$\mathrm{Re}(z_1^3 z_2 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	
E_8	$\mathrm{Re}(z_1^5 + z_2^3 + z_3^2 + \ldots + z_n^2) = 0$	

Table 2.3: Levi-flat hypersurfaces with A_k, D_k, E_k singularities

The proof of this theorem is similar to theorem 2.3.5. The idea is to apply theorem 2.2.3, weighted blow-up at quasihomogeneous singularities and Arnold rigidity theorem for A_k, D_k, E_k singularities.

2.5 Isolated line singularity

The aim of this section is to state some normal forms for local real-analytic Levi-flat hypersurfaces defined by the vanishing of real part of holomorphic functions with an *isolated line singularity* (for short: ILS).

The main motivation for this is a result due to Dirk Siersma, who introduced in [Si] the class of germs of holomorphic functions with an ILS. More precisely, let $\mathcal{O}_{n+1} := \{f : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}\}$ be the ring of germs of holomorphic functions and let m be its maximal ideal. If $(x, y) = (x, y_1, \ldots, y_n)$ denote the coordinates in $\mathbb{C}^{n+1}$ and consider the line $L := \{y_1 = \ldots = y_n = 0\}$, let $I := (y_1, \ldots, y_n) \subset \mathcal{O}_{n+1}$ be its ideal and denote by $\mathcal{D}_I$ the group of local analytic isomorphisms $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ for which $\varphi(L) = L$. Then $\mathcal{D}_I$ acts on I^2 and for $f \in I^2$, the tangent space of (the orbit of) f with respect to this action is the ideal defined by

$$\tau(f) := m.\frac{\partial f}{\partial x} + I.\frac{\partial f}{\partial y}$$

and the codimension of (the orbit) of f is $c(f) := \dim_{\mathbb{C}} \frac{I^2}{\tau(f)}$.

A line singularity is a germ $f \in I^2$. An ILS is a line singularity f such that $c(f) < \infty$. Geometrically, $f \in I^2$ is an ILS if and only if the singular locus of f is L and for every $x \neq 0$, the germ of (a

Type	Normal form	Conditions
A_∞	$y_1^2 + y_2^2 + \ldots + y_n^2$	
D_∞	$xy_1^2 + y_2^2 + \ldots + y_n^2$	
$J_{k,\infty}$	$x^k y_1^2 + y_1^3 + y_2^2 + \ldots + y_n^2$	$k \geq 2$
$T_{\infty,k,2}$	$x^2 y_1^2 + y_1^k + y_2^2 + \ldots + y_n^2$	$k \geq 4$
$Z_{k,\infty}$	$xy_1^3 + x^{k+2} y_1^2 + y_2^2 + \ldots + y_n^2$	$k \geq 1$
$W_{1,\infty}$	$x^3 y_1^2 + y_1^4 + y_2^2 + \ldots + y_n^2$	
$T_{\infty,q,r}$	$xy_1y_2 + y_1^q + y_2^r + y_3^2 \ldots + y_n^2$	$q \geq r \geq 3$
$Q_{k,\infty}$	$x^k y_1^2 + y_1^3 + xy_2^2 + y_3^2 \ldots + y_n^2$	$k \geq 2$
$S_{1,\infty}$	$x^2 y_1^2 + y_1^2 y_2 + y_3^2 + \ldots + y_n^2$	

Table 2.4: Isolated Line singularities

representative of) f at $(x, 0) \in L$ is equivalent to $y_1^2 + \ldots + y_n^2$. In a certain sense ILS are the first generalization of isolated singularities. Dirk Siersma proved the following result. (The topology on $\mathcal{O}_{n+1}$ is introduced as in [D, pp. 145]).

Theorem 2.5.1 (Siersma [Si]). *A germ $f \in I^2$ is D_I-simple (i.e. $c(f) < \infty$ and f has a neighborhood in I^2 which intersects only a finite number of D_I-orbits) if and only if f is D_I-equivalent to one the germs in the table 2.4.*

The singularities in theorem 2.5.1 are analogous of the A_k, D_k, E_k singularities due to Arnold [A1]. A new characterization of simple ILS have been proved by A. Zaharia [Z].

Theorem 2.5.2 (Fernández-Pérez [Fe5]). *Let $M = \{F = 0\}$ be a germ of an irreducible real-analytic hypersurface on $(\mathbb{C}^{n+1}, 0)$, $n \geq 3$. Suppose that*

1. *$F(x, y) = \mathcal{R}e(P(x, y)) + H(x, y)$, where $P(x, y)$ is one of the germs of the Table 1.*
2. *$M = \{F = 0\}$ is Levi-flat.*
3. *$H(x, 0) = 0$ for all $x \in (\mathbb{C}, 0)$ and $j_0^k(H) = 0$, for $k = \deg(P)$.*

Then there exists a biholomorphism $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ preserving L such that

$$\varphi(M) = \{\mathcal{R}e(P(x, y)) = 0\}.$$

This result is a Siersma's type theorem for singular Levi-flat hypersurfaces. We remark that the function H is of course restricted by the assumption that M is Levi flat.

Now, if $\varphi(M) = \{\mathcal{R}e(P(x,y)) = 0\}$, where P is a germ with an ILS at L then $\mathrm{sing}(M) = L$. In other words, M is a Levi-flat hypersurface with an ILS at L. If $P(x,y)$ is the germ A_∞, one have that theorem 2.5.2 is true in the case $n = 2$.

Theorem 2.5.3 (Fernández-Pérez [Fe5]). *Let $M = \{F = 0\}$ be a germ of an irreducible real-analytic Levi-flat hypersurface on $(\mathbb{C}^3, 0)$. Suppose that F is defined by*

$$F(x,y) = \mathcal{R}e(y_1^2 + y_2^2) + H(x,y),$$

where H is a germ of real-analytic function such that $H(x,0) = 0$ and $j_0^k(H) = 0$ for $k = 2$. Then there exists a biholomorphism $\varphi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ preserving L such that $\varphi(M) = \{\mathcal{R}e(y_1^2 + y_2^2) = 0\}$.

The above result should be compared to theorem 2.3.2. This result can be viewed as a Morse's Lemma for Levi-flat hypersurfaces with an ILS at L. The problem of normal forms of Levi-flat hypersurfaces in $\mathbb{C}^3$ with an ILS seems difficult in the other cases and remains open.

Chapter 3

The Levi-foliation and its extension

We consider the *holomorphic extension problem for the Levi-foliation of a Levi-flat real analytic hypersurface* in a neighborhood of $0 \in \mathbb{C}^n$. Let M be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of a real analytic Levi-flat hypersurface and as before, we say that $p \in M$ is a regular point, if M is a real analytic submanifold of dimension $2n-1$ near p. The union of all regular points form a regular part denoted by M^*. Let $\text{sing}(M)$ be the set of singular points of M, points near which M have dimension strictly smaller that $\dim_{\mathbb{R}} M^*$. We remark that this convention is different from the usual definition of a regular point in semi-analytic geometry. In general, the singular set of a real-analytic subvariety M in a complex manifold is defined as the set of points near which M is not a real-analytic submanifold (of any dimension) and "in general" has structure of a semi-analytic set, see for instance [Ło]. The classical example of the Whitney umbrella shows that an irreducible real analytic set does not always have pure dimension, see the example 3.0.9. By definition the Levi-foliation $\mathcal{L}$ on M only is defined on M^*. Therefore, we address to provide conditions for the Levi-foliation extends holomorphically on some neighborhood of M in the ambient.

The problem has been considered by E. Bedford and P. De Bartolomeis in [BD], in the case that M is smooth and C^∞. They proved that the Levi-foliation cannot be extended *"in general"*. It can be extended if M is nonsingular real analytic Levi-flat hypersurface. In fact, its local structure is very well understood, according to E. Cartan, around each $p \in M$ we can find local holomorphic coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $M = \{\mathrm{Im}(z_n) = 0\}$. In particular, the Levi-foliation $\mathcal{L}$ is defined by $dz_n|_M = 0$ and extends on a neighborhood of p to a holomorphic foliation with leaves $\{z_n = c\}$, $c \in \mathbb{C}$. It is easily seen that these local extensions glue together, giving a codimension one holomorphic foliation $\mathcal{F}$ on some neighborhood of M. When M is a real analytic submanifold of higher codimension, the problem was solved by C. Rea [Re].

We are interested when M is an irreducible real analytic Levi-flat hypersurface with singularities. In this case, it is known that the Levi-foliation does not necessarily extend to a holomorphic foliation, because the singular set of the hypersurface could be of higher dimension. For instance, in [Br] is given an example where the Levi-foliation extends to a holomorphic web.

Example 3.0.4 (Brunella)**.** Let (z, w) be the coordinates of $\mathbb{C}^2$ such that $z = x + iy$, $w = s + it$ and consider the real hypersurface M defined by

$$M = \{(z, w) \in \mathbb{C}^2 : t^2 = 4(y^2 + s)y^2\}. \tag{3.1}$$

Then M is Levi-flat with $\mathrm{sing}(M) = \{t = y = 0\}$. The leaves of $\mathcal{L}$ on M^* are the complex curves

$$L_c = \{w = (z + c)^2 : \mathrm{Im}\, z \neq 0\}, \quad c \in \mathbb{R}.$$

Note that $\mathrm{sing}(M)$ is a generic submanifold of real dimension two. It is not difficult see that $\mathcal{L}$ extend to 2-web $\mathcal{W}$ defined by

$$dw^2 - 4wdz^2 = 0.$$

Moreover, $\mathcal{W}$ has multivalued first integral $f(z, w) = z \pm \sqrt{w}$, that is, a meromorphic correspondence which is constant along the leaves of $\mathcal{W}$. Note also that $M = \{F = 0\}$, where

$$F = det \begin{pmatrix} z^2 - w & 2z & 1 & 0 \\ 0 & z^2 - w & 2z & 1 \\ \bar{z}^2 - \bar{w} & 2\bar{z} & 1 & 0 \\ 0 & \bar{z}^2 - \bar{w} & 2\bar{z} & 1 \end{pmatrix}$$

Therefore, the extension problem for the Levi-foliation $\mathcal{L}$ of a real analytic Levi-flat hypersurface M consists of proving the existence of a codimension one holomorphic foliation $\mathcal{F}$ or a holomorphic k-web $\mathcal{W}$ defined on some neighborhood of M such that any leaf of $\mathcal{L}$ on M^* is also a leaf of $\mathcal{F}$ or $\mathcal{W}$ respectively.

In very general terms, a codimension one k-web is a collection of k codimension one holomorphic foliations in *"general position"*. We can also think k-webs as first order differential equations of degree k. See the books [Ce] and [PP] for the basic language and background about holomorphic foliations and webs.

In this chapter, we will use the Segre varieties associated to M. This tool has been used by B. Segre [Se], Webster [W], Diederich-Fornaess [DF], Burns-Gong [BG], Lebl [Le2] and many others. More precisely, given a real analytic hypersurface M defined by $\{F = 0\}$, we shall denote by Σ_p the *Segre variety* defined by $\{F(z, \bar{p}) = 0\}$, and by Σ'_p by the union of all branches of Σ_p which are contained in M. An important property is that the Segre varieties contains the leaves of the Levi-foliation.

Proposition 3.0.5. *Let L_p be the leaf of $\mathcal{L}$ through $p \in M^*$, then L_p is an irreducible component of (Σ_p, p) and $\Sigma'_p = L_p$.*

Proof. Since $p \in M^*$, E. Cartan's theorem assures that there exists a holomorphic coordinate system such that near of p, M is given by $\{\mathrm{Re}(z_n) = 0\}$ and p is the origin. In this coordinates system the foliation $\mathcal{L}$ is defined by $dz_n|_{M^*} = 0$. In particular, $L_0 = \{z_n = 0\}$ and obviously $\{z_n = 0\}$ is a branch of Σ_0. Furthermore, L_0 is the unique germ of complex variety of pure dimension $n-1$ at 0 which is contained in M. Hence $\Sigma'_0 = L_0$. $\square$

Let $p \in \mathrm{sing}(M)$, we say that p is a *Segre degenerate singularity* if Σ_p is of dimension n, that is, $\Sigma_p = (\mathbb{C}^n, p)$. Otherwise, we say that p is a *Segre nondegenerate singularity.*

Suppose that M is defined by $\{F = 0\}$ in a neighborhood of p, observe that p is a degenerate singularity of M if $z \longmapsto F_{\mathbb{C}}(z, \bar{p})$ is identically zero. We remark that if V is a germ of complex variety of dimension $n-1$ contained in M then for $p \in V$, we have $(V, p) \subset (\Sigma_p, p)$. In particular, if there exists distinct infinitely many complex varieties of dimension $n-1$ through $p \in M$ then p is a Segre degenerate singularity.

In order to clarify ideas, we give more examples and comments some known results.

Example 3.0.6. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a non-constant holomorphic function, where U is an open subset and set $M = \{\text{Im}(f) = 0\}$. Then M is Levi-flat and $\mathcal{L} : df|_{M^* \cap U} = 0$ extends to holomorphic foliation $\mathcal{F} : df = 0$.

Cerveau-Lins Neto's theorem (see theorem 2.2.3) asserts that a local real analytic Levi-flat hypersurface M with a sufficiently small singular set is given by the zeros of the imaginary part of a holomorphic function. In particular, the Levi-foliation $\mathcal{L}$ on M^* extends to a holomorphic foliation on some neighborhood of M. Most recently, an analogous result was obtained by Lebl [Le2], see the next chapter for more details.

Now, it is known that if there exists a meromorphic function defined in a neighborhood of M constant on the leaves of M^* then the Levi-foliation extends also to a holomorphic foliation (cf. [Le1]).

Example 3.0.7. The hypersurface defined by

$$M = \{(z, w) \in \mathbb{C}^2 : z\bar{w} - z\bar{w} = 0\}$$

is Levi-flat, the leaves of $\mathcal{L}$ are the complex lines

$$L_c : z + cw = 0, \quad c \in \mathbb{R}.$$

The Levi-foliation $\mathcal{L}$ extends to holomorphic foliation $\mathcal{F}$ with meromorphic first integral $f(z, w) = z/w$.

However, not every Levi-flat hypersurface can be obtained as in Example 3.0.6 or as the zeros of the real part of a meromorphic function, there are other types, like the following examples.

Example 3.0.8. Consider $(z, w) \in \mathbb{C}^2$ and $z = x + iy$, $w = s + it$ and let M be defined by

$$M = \{(z, w) \in \mathbb{C}^2 : x^2 + y^3 = 0\}.$$

Then M is a real analytic Levi-flat hypersurface in $\mathbb{C}^2$, it is foliated by the complex lines parallel to w-axis. Burns and Gong [BG] proved that this Levi-flat cannot be given in the form $\{\mathrm{Re}(f) = 0\}$, where f is meromorphic or holomorphic function.

Example 3.0.9 ([Le3])**.** Let (z, w) be the coordinates of $\mathbb{C}^2$ and write $z = x + iy$ and $w = s + it$. Consider the hypersurface M defined by

$$M = \{sy^2 - txy - t^2 = 0\} \tag{3.2}$$

This hypersurface is Levi-flat, because its regular part is foliated by the complex lines given by

$$w = cz + c^2 \qquad \text{where} \quad c \in \mathbb{R}.$$

Indeed, setting $z = \zeta$ and $w = c\zeta + c^2$ gives $\overline{M^*}$ as an image of $\mathbb{C} \times \mathbb{R}$. Note that M has a lower-dimensional totally-real component given by $\{t = 0, y = 0\}$, "half of it" (for s sufficiently negative) sticking out og the hypersurface as the "umbrella handle". In the complement of $\{y = 0\}$, the hypersurface M is a graph: $s = x(\frac{t}{y}) + (\frac{t}{y})^2$. The set $\{t = 0, y = 0, s < \frac{x^2}{4}\}$ is not in the closure of M^*. Therefore, not all of M is a union of complex lines, only the closure M^* is such a of complex lines. Notice that when $x = 0$, we obtain the standard Withney umbrella in $\mathbb{R}^3$, see Figure 3.0.9.

The singular set $\mathrm{sing}(M) = \{t = 0, y = 0, s \geq \frac{x^2}{4}\}$ is of real dimension 2, and it is a generic manifold at points arbitrarily near the origin. The segre varieties of M are given by

$$\Sigma_{(z_0,w_0)} = \{w^2 + wz\bar{z}_0 - w\bar{z}_0^2 - 2w\bar{w}_0 - z^2\bar{w}_0 + z\bar{w}_0\bar{z}_0 + \bar{w}_0^2 = 0\}.$$

We see that $\Sigma_{(0,0)} = \{w = 0\}$, and the origin is a nondegenerate singularity of M. Moreover, the Levi-foliation does not extend as a singular holomorphic foliation of a neighborhood of the origin. The Levi-foliation does extend as a holomorphic 2-web given by

$$dw^2 + zdzdw - wdz^2 = 0.$$

Its multivalued first integral is obtained by applying the quadratic formula to $w = cz + c^2$.

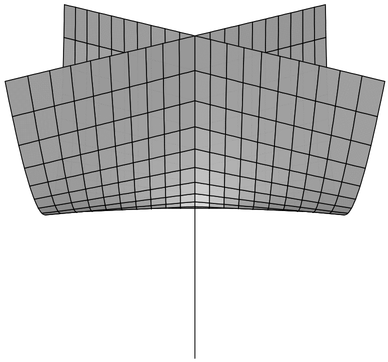


Figure 3.1: Whitney umbrella

Similar examples of Levi-flat hypersurfaces can be obtained as follows.

Example 3.0.10. ([CL]) For instance, consider the Levi-flat hypersurface defined by $\{F = 0\}$, where

$$F = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ \bar{f}_0 & \bar{f}_1 & \bar{f}_2 & 0 \\ 0 & \bar{f}_0 & \bar{f}_1 & \bar{f}_2 \end{pmatrix}$$

and $f_0, f_1, f_2 \in \mathcal{O}_2$, $f_0(0) = f_1(0) = f_2(0) = 0$. The leaves of $\mathcal{L}$ are

$$f_0(z) + c \cdot f_1(z) + c^2 \cdot f_2(z) = 0, \quad c \in \mathbb{R}.$$

In this case $\mathcal{L}$ extends to 2-web given by the implicit differential equation $\Omega = 0$, where

$$\Omega = det \begin{pmatrix} f_0 & f_1 & f_2 & 0 \\ 0 & f_0 & f_1 & f_2 \\ df_0 & df_1 & df_2 & 0 \\ 0 & df_0 & df_1 & df_2 \end{pmatrix}$$

The goal of this chapter is to give sufficient conditions for the Levi-foliation to extend to a holomorphic web. We prove the following result.

Theorem 3.0.11 (Fernández-Pérez). *Let M be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of an irreducible real analytic Levi-flat hypersurface with Levi-foliation $\mathcal{L}$. Suppose that* $\text{sing}(M)$ *is a generic real submanifold of dimension $2n-2$ near points where it is a CR real analytic submanifold and only contain Segre nondegenerate singularities. Then there exists a germ at $0 \in \mathbb{C}^n$ of a codimension one holomorphic k-web $\mathcal{W}$, $k \geq 2$, such that every leaf of $\mathcal{L}$ on M^* is a leaf of $\mathcal{W}$.*

The above theorem was motivated by example 3.0.4. According to theorem 4.2.1, if the singular set of a Levi-flat hypersurface is a generic CR-manifold then it is necessarily of dimension $2n-2$. Therefore the hypothesis on the dimension of singular set of M can be removed.

We will assume that $\text{sing}(M) \cap \overline{M^*} \neq \emptyset$, because $0 \in \overline{M^*}$ (otherwise, the germ of M at 0 would not be a germ of hypersurface as prescribed by theorem 3.0.11, here $\overline{M^*}$ mean the relative closure in M). Then we will prove that in a generic point p of $\text{sing}(M)$, $\Sigma_p^{'}$ is a complex variety with k irreducible components, these irreducible components will be the leaves of a k-web defined in some neighborhood of M.

Remark 3.0.12. Given M a singular real analytic Levi-flat hypersurface in $\mathbb{C}^n$, $n \geq 2$, we note that around points where $\text{sing}(M)$ is a real or complex submanifold, we must have $\text{dim}_{\mathbb{R}} \text{sing}(M) \leq 2n-2$. If $\text{dim}_{\mathbb{R}} \text{sing}(M) < 2n-4$, for $n \geq 3$ or $\text{dim}_{\mathbb{R}} \text{sing}(M) = 2n-4$, for $n \geq 2$, plus additional hypotheses, it is proved that the Levi-foliation $\mathcal{L}$ extends to a holomorphic foliation (cf. [CL], [Le2]). Theorem 3.0.11 establish that occurs when $\text{sing}(M)$ is generic real submanifold of dimension $2n-2$ that only contain Segre nondegenerate singularities.

Let us consider generic submanifolds of $\mathbb{C}^n$. Let H be a real analytic submanifold in $\mathbb{C}^n$. Take $p \in H$ and consider the complexified tangent space $\mathbb{C} \otimes T_pH$. Let $T_p^{0,1}H \subset \mathbb{C} \otimes T_pH$ be space of antiholomorphic vectors at p where an antiholomorphic vector is a tangent vector which can be written in terms of the basis $\frac{\partial}{\partial \bar{z}_j}$, $1 \leq j \leq n$. A real submanifold H is said to be *CR-submanifold* (for Cauchy-Riemann) if $T_p^{0,1}H$ has constant dimension at all points of H. For a CR-submanifold, $\dim T_p^{0,1}H$ will be called the CR dimension of H. CR-submanifolds with CR dimension 0 are called totally real.

A real submanifold $H \subset \mathbb{C}^n$ is generic if near every $p \in H$ there is local defining function $\rho = (\rho_1, \ldots, \rho_d)$ such that the complex differentials $\partial\rho_1, \ldots, \partial\rho_d$ are $\mathbb{C}$ linearly independent near p. The following results are classical and can be found in [BSE].

Proposition 3.0.13. *Let H be a real submanifold of $\mathbb{C}^n$. The following equivalence holds:*

1. *H is CR-submanifold if and only if $\dim_{\mathbb{R}}(T_pH \cap iT_pH)$ independent of $p \in H$.*
2. *H is totally real if and only if $T_pH \cap iT_pH = 0$ for all $p \in H$.*
3. *H is generic if and only if $T_pH + iT_pH = T_p\mathbb{C}^n$ for all $p \in H$.*

Theorem 3.0.14. *Let $H \subset \mathbb{C}^n$ be a generic submanifold and let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function where U is a connected open set such that $H \cap U \neq \emptyset$, and further suppose that $f(H \cap U) = \{0\}$, that is f is zero when restricted to H. Then in fact $f \equiv 0$ on U.*

The above result implies the following lemma.

Lemma 3.0.15. *A generic submanifold $H \subset \mathbb{C}^n$ is not contained in and does not contain any proper complex variety of pure dimension $n-1$ in $\mathbb{C}^n$.*

3.1 Holomorphic foliations and webs

A germ of singular codimension one holomorphic foliation $\mathcal{F}$ is an equivalence class $[\omega]$ of germs of holomorphic 1-forms in $\Omega^1(\mathbb{C}^n, 0)$ modulo multiplication by elements of $\mathcal{O}^*(\mathbb{C}^n, 0)$ such that any representative ω is integrable ($\omega \wedge d\omega = 0$) and with singular set $\text{sing}(\mathcal{F}) = \{p \in (\mathbb{C}^n, 0) : \omega(p) = 0\}$ of codimension at least two.

Consider now an arbitrary germ of a holomorphic foliation $\mathcal{F}$ at an isolated singularity $0 \in \mathbb{C}^2$. A *separatrix* of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is the germ at $0 \in \mathbb{C}^2$ of an irreducible curve which is invariant by $\mathcal{F}$.

Recall that a germ of foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is *dicritical* if it has infinitely many analytic separatrices through the origin. Otherwise it is called *non-dicritical*.

An analogous definition can be made for codimension one k-webs. A germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of codimension one k-web $\mathcal{W}$ is an equivalence class $[\omega]$ of germs of k-symmetric 1-forms, that is sections of $Sym^k\Omega^1(\mathbb{C}^n, 0)$, modulo multiplication by $\mathcal{O}^*(\mathbb{C}^n, 0)$ such that a suitable representative ω defined in a connected neighborhood U of the origin satisfies the following conditions:

1. The zero set of ω has codimension at least two.
2. The 1-form ω, seen as a homogeneous polynomial of degree k in the ring $\mathcal{O}_n[dx_1, \dots, dx_n]$, is square-free.
3. (Brill's condition) For a generic $p \in U$, $\omega(p)$ is a product of k linear forms.
4. (Frobenius's condition) For a generic $p \in U$, the germ of ω at p is the product of k germs of integrable 1-forms.

Both conditions (3) and (4) are automatic for germs at $0 \in \mathbb{C}^2$ of webs and non-trivial for germs at $0 \in \mathbb{C}^n$ when $n \geq 3$.

We can also consider webs as closures of meromorphic multi-sections. Let us denote $\mathbb{P} := \mathbb{P}T^*(\mathbb{C}^n, 0)$ the projectivization of the cotangent bundle of $(\mathbb{C}^n, 0)$ and $\pi : \mathbb{P}T^*(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ the natural projection. Over a point p the fiber $\pi^{-1}(p)$ parametrizes the one-dimensional subspaces of $T^*_p(\mathbb{C}^n, 0)$. On $\mathbb{P}$ there is a canonical codimension one distribution, the so called contact distribution $\mathcal{D}$. Its description in terms of a system of coordinates $x = (x_1, \dots, x_n)$ of $(\mathbb{C}^n, 0)$ goes as follows: let $dx_1, \dots, dx_n$ be the basis of $T^*(\mathbb{C}^n, 0)$ associated to the coordinate system $(x_1, \dots, x_n)$. Given a point $(x, y) \in T^*(\mathbb{C}^n, 0)$, we can write $y = \sum_{j=1}^n y_j dx_j$, $(y_1, \dots, y_n) \in \mathbb{C}^n$. In this way, if $(y_1, \dots, y_n) \neq 0$ then we set $[y] = [y_1, \dots .y_n] \in \mathbb{P}^{n-1}$ and $(x, [y]) \in (\mathbb{C}^n, 0) \times \mathbb{P}^{n-1} \cong \mathbb{P}$. In the affine coordinate system $y_n \neq 0$ of $\mathbb{P}$, the distribution $\mathcal{D}$ is defined by $\alpha = 0$, where

$$\alpha = dx_n - \sum_{j=1}^{n-1} p_j dx_j, \qquad p_j = -\frac{y_j}{y_n} \qquad (1 \leq j \leq n-1). \tag{3.3}$$

The 1-form α is called the contact form.

Let us consider $S \subset \mathbb{P}$ a subvariety, not necessarily irreducible, but of pure dimension n. Let $\pi_S : S \to (\mathbb{C}^n, 0)$ be the restriction to S of the projection π. Suppose also that S satisfies the following conditions:

1. The image under π of every irreducible component of S has dimension n.

2. The generic fiber of π intersects S in k distinct smooth points and at these the differential $d\pi_S : T_pS \to T_{\pi(p)}(\mathbb{C}^n, 0)$ is surjective. Note that $k = \deg(\pi_S)$.

3. The restriction of the contact form α to the smooth part of every irreducible component of S is integrable. We denote $\mathcal{F}_S$ the foliation defined by $\alpha|_S = 0$.

We can define a germ $\mathcal{W}$ at $0 \in \mathbb{C}^n$ of k-web as a triple $(S, \pi_S, \mathcal{F}_S)$. This definition is equivalent to one given in the previous paragraph. Such S is called the variety associated to $\mathcal{W}$. A leaf of the web $\mathcal{W}$ is, by definition, the projection on $(\mathbb{C}^n, 0)$ of a leaf of $\mathcal{F}_S$.

Denote by $\Delta_{\mathcal{W}}$ the discriminant of $\mathcal{W}$. Recall that for every $p \in (\mathbb{C}^n, 0)\backslash\Delta_{\mathcal{W}}$, there exists a coordinate system near of p such that the germ of $\mathcal{W}$ at p is given by $\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_k$, in other words, $\mathcal{W}$ is a superposition of k germs of smooth foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. We refer [PP] Section 1.3 for more details on holomorphic webs.

Let $\mathcal{F}$ and M be germs at $0 \in \mathbb{C}^n$ of a codimension one singular holomorphic foliation and of a real analytic Levi-flat hypersurface, respectively.

Definition 3.1.1. We say that $\mathcal{F}$ and M are tangent, if the leaves of the Levi-foliation $\mathcal{L}$ on M are also leaves of $\mathcal{F}$.

Definition 3.1.2. A meromorphic (holomorphic) function h is called a meromorphic (holomorphic) first integral for $\mathcal{F}$ if its indeterminacy (zeros) set is contained in $\mathrm{sing}(\mathcal{F})$ and its level hypersurfaces contain the leaves of $\mathcal{F}$.

Cerveau and Lins Neto proved a nice result on holomorphic foliations tangent to real analytic Levi-flat hypersurfaces.

Theorem 3.1.3 (Cerveau-Lins Neto [CL]). *Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of holomorphic codimension one foliation tangent to a germ of an irreducible real analytic hypersurface M. Then $\mathcal{F}$ has a non-constant meromorphic first integral. In the case $n = 2$ we have:*

(a) If $\mathcal{F}$ is dicritical then it has a non-constant meromorphic first integral f/g, where $f, g \in \mathcal{O}_2$ and $f(0) = g(0) = 0$.

(b) If $\mathcal{F}$ is non-dicritical then it has a non-constant holomorphic first integral.

Recently, Marco Brunella [Br1] gives a new proof using some general principles of analytic geometry. In the case of webs, similarly, we say that a germ of k-web $\mathcal{W}$ is tangent to a Levi-flat hypersurface M, if any leaf of the Levi-foliation on M is a leaf of $\mathcal{W}$. A *nondicritical k-web version* of theorem 3.1.3 can be found in [Fe1].

3.2 Proof of Theorem 3.0.11

We use the following lemma of [Fe2]

Lemma 3.2.1. *Let M, $\operatorname{sing}(M)$ be as in Theorem 3.0.11. We have the following:*

(a) *If $p \in \operatorname{sing}(M)$ and V is a germ of an irreducible complex variety of pure dimension $n-1$ contained in M through p. Then there exists a germ of leaf of $\mathcal{L}$ such that its closure is V.*

(b) *The foliation $\mathcal{L}$ cannot be extended to a holomorphic foliation $\mathcal{F}$.*

(c) *Let V be the closure of a leaf of $\mathcal{L}$. Suppose that V cuts transversely $\operatorname{sing}(M)$. Then $V \cap \operatorname{sing}(M)$ is a real analytic variety of dimension $2n-3$.*

Let us prove theorem 3.0.11. Let M be a germ at $0 \in \mathbb{C}^n$ of Levi-flat hypersurface and $p \in \operatorname{sing}(M)$. We can assume that $\operatorname{sing}(M)$ is a generic submanifold of real dimension $2n-2$ near p and that M is defined by $\{F = 0\}$ in a neighborhood of p.

According to a result of Burns-Gong (cf. [BG], lemma 3.2 or lemma 4.2.3), Σ'_p is non-empty and since $\operatorname{sing}(M)$ only contain Segre nondegenerate singularities, Σ'_p must be a complex variety of pure dimension $n-1$. As $\operatorname{sing}(M)$ is generic, there is no branch of Q'_p which contains $\operatorname{sing}(M)$ (see lemma 3.0.15). Furthermore because $\operatorname{sing}(M)$ is generic, there is no branch of Σ'_p lies in $\operatorname{sing}(M)$. Then there exists $q \in \operatorname{sing}(M)$ such that Σ'_q intersect M^*. We set $p = q$

and works in a neighborhood of p. Now, take $x \in M^* \cap \Sigma'_p$, since there exists a unique irreducible component of Σ'_p through x, we have Σ'_x is a branch of Q'_p. We can pick x in an irreducible component of $M^* \cap (\Sigma'_p)_{reg}$ such that p is in the closure of this component. Lemma 3.2.1, item (a) imply that the closure of L_x (the leaf of $\mathcal{L}$ through x) is a branch of Σ'_p.

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a real analytic curve such that $\gamma(0) = x$, $\{\gamma\} \subset M^*$ and γ is transverse to $\mathcal{L}$ on M^*. The function $t \longmapsto F_{\mathbb{C}}(p, \overline{\gamma(t)})$ is not identically zero, because otherwise M could contain a Segre degenerate singularity. By complexification of variable t, we obtain a germ of holomorphic function

$$g(z, w) = F_{\mathbb{C}}(z, \overline{\gamma(\bar{w})}), \tag{3.4}$$

where $z \in (\mathbb{C}^n, p)$ and $w \in (\mathbb{C}, 0)$. Applying the Weierstrass Preparation theorem to g, we have

$$g(z, w) = u(z, w)(w^k + a_{k-1}(z)w^{k-1} + \ldots + a_0(z)), \tag{3.5}$$

where $u(0, 0) \neq 0$ and $a_j \in \mathcal{O}(\mathbb{C}^n, p)$, for all $0 \leq j \leq k-1$.

Let $\mathcal{W}_p$ be the k-web which is obtained by the elimination of w in the system given by

$$\begin{cases} a_0 + w \cdot a_1 + w^2 \cdot a_2 + \ldots + w^{k-1} \cdot a_{k-1} + w^k = 0 \\ da_0 + w \cdot da_1 + w^2 \cdot da_2 + \ldots + w^{k-1} \cdot da_{k-1} = 0. \end{cases}$$

Observe that $\mathcal{W}_p$ is a k-web defined on some neighborhood of p and $k \geq 2$ by lemma 3.2.1, item (b). In any point ζ outside of $\Delta_{\mathcal{W}_p}$ (the discrminant of $\mathcal{W}_p$), we have $V_1, \ldots, V_k$ leaves of $\mathcal{W}_p$ through ζ. It follows from (3.4) that the leaves of $\mathcal{W}_p$ are contained in Σ_ζ and by the construction in M too. Hence, lemma 3.2.1, item (a) implies that any leaf of $\mathcal{L}$ is a leaf of $\mathcal{W}_p$, in others words, $\mathcal{W}_p$ is a local extension of $\mathcal{L}$.

These local constructions are sufficiently canonical to be patched together, when p varies on $\overline{M^*}$. In fact, let $\pi : \mathbb{P}T^*(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be the natural projection. Denote by $\pi_p : S_p \to (\mathbb{C}^n, p)$, the restriction to S_p of π, where S_p is the variety associated to $\mathcal{W}_p$. On S_p, we have the foliation $\mathcal{F}_p$ associated to $\mathcal{W}_p$ which is defined by the restriction to S_p of the contact 1-form on $\mathbb{P}T^*(\mathbb{C}^n, 0)$. Denote by M' the lifting of M^* by π and take a point $q \in \overline{M'}$ such that $\pi(q) = p$ then it is proved in [Br] that there exist, in a germ of neighborhood U_q of q, a germ of a real analytic variety N_q of dimension $2n-1$ containing $M^{'} \cap U_q$. Furthermore, we can consider U_q sufficiently small and assume that $N_q \subset S_p \cap U_q$. N_q is Levi-flat because each irreducible component contains a Levi-flat piece. The above construction implies that $\mathcal{F}_p$ is tangent to N_q. Suppose that we have two connected neighborhoods U_1 and U_2 such that $M^{'} \cap U_1 \cap U_2 \neq \emptyset$. Furthermore, assume there exists holomorphic 1-forms ω_j that define the foliation $\mathcal{F}_{p_j}$, for each $j = 1, 2$. If $S_{p_1} \subset U_1$, $S_{p_2} \subset U_2$ are as above then $S_{p_1} \cap U_1 \cap U_2$ and $S_{p_2} \cap U_1 \cap U_2$ have some common irreducible components containing $M^{'} \cap U_1 \cap U_2$, so that S_{p_1} and S_{p_2} can be glued by identifying those components. Note also that $N_{q_1} \cap U_1 \cap U_2$ and $N_{q_2} \cap U_1 \cap U_2$ can be glued and in this way we obtain a Levi-flat hypersurface N. On the other hand, if we can show that ω_1 is proportional to ω_2 then we have a foliation on $S_{p_1} \cap S_{p_2}$. Since $\mathcal{W}_{p_1}$ and $\mathcal{W}_{p_2}$ are tangent to M, we have $\mathcal{F}_{p_j}$ is tangent to $N \cap U_1 \cap U_2$, for each $j = 1, 2$. This implies that in some neighborhood of $N^* \cap U_1 \cap U_2$, ω_1 and ω_2 are proportional. N^* is a real hypersurface, thus they are proportional in whole neighborhood as they are holomorphic. In this way, we obtain a complex variety S and a holomorphic foliation $\mathcal{F}$ on S tangent to N. Finally, we obtain a k-web enjoying all the properties stated in theorem 3.0.11.

3.3 Shafikov-Sukhov theorem

In 2014, R. Shafikov and A. Sukhov [SS] presented the following result.

Theorem 3.3.1 (Shafikov-Sukhov)**.** *Let $M \subset \Omega$ be an irreducible Levi-flat real analytic hypersurface in a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, and $0 \in \overline{M^*}$. Assume that al least one of the following conditions holds:*

1. *$0 \in M$ is not a dicritical singularity.*
2. *M is a real algebraic hypersurface.*

Then there exists a neighborhood U of the origin and a singular holomorphic k-web $\mathcal{W}$ in U such that $\mathcal{W}$ extends the Levi-foliation $\mathcal{L}$.

Furthemore, $\mathcal{W}$ admits a multiple-valued meromorphic first integral in U.

Here a singular point p of a Levi-flat hypersurface is called *dicritical* if infinitely many leaves of the Levi-foliation have p in their closure. It is easily seen that a dicritical singularity is a degenerate singularity. We remark that Shafikov-Sukhov's theorem remove the condition of CR generic in the statement of theorem 3.0.11. The following example give a singular Levi-flat hypersurface which satisfies condition (1) of theorem 3.3.1 but not those of theorem 3.0.11.

Example 3.3.2 (Shafikov-Sukhov)**.** Consider the set given in $\mathbb{C}^2$ by the equation

$$M = \{(z,w) \in \mathbb{C}^2 : (|w|^2(z+\bar{z})^2 - (w+\bar{w}))^2 - 4|w|^2 = 0\},$$

we have M is a Levi-flat hypersurface, because M^* is foliated by complex curves

$$w = \frac{1}{(z+c)^2} \quad \text{where} \quad c \in \mathbb{R}. \tag{3.6}$$

The Segre varieties are given by

$$\Sigma_{(z_0,w_0)} = \{(z,w) \in \mathbb{C}^2 : (w\bar{w}_0(z+\bar{z}_0) - (w-\bar{w}_0))^2 - 4w\bar{w}_0 = 0\}.$$

Since $\Sigma_{(0,0)} = \{w = 0\}$, the origin is Segre nondegenerate singularity, in particular is a nondicritical singularity. Note that $\mathrm{sing}(M) = \Sigma_{(0,0)}$ and hence we can not apply theorem 3.0.11.

From 3.6, we deduce that the Levi-foliation extends to a 2-web $\mathcal{W}$ defined by

$$\left(\frac{dw}{dz}\right)^2 = 4w^3.$$

Moreover, the map $f(z,w) = z \pm \frac{1}{\sqrt{w}}$ is a first integral for $\mathcal{W}$.

Remark 3.3.3. The extension problem of the Levi-foliation of a real analytic Levi-flat hypersurface M remains open in the case of dicritcal or Segre degenerate sigularities, even if M has an isolated singularity.

Chapter 4

The singular set of Levi-flat hypersurfaces

Let $M \subset U \subset \mathbb{C}^n$ be a real-analytic Levi-flat subvariety of codimension 1. Let M^* be the set of points near which M is a nonsingular real-analytic hypersurface, let M_{reg} be the set of *regular* points, that is, points where M is a real-analytic submanifold of some dimension, and let M_s denote the singular locus, that is, the set $M \setminus M_{reg}$.

4.1 Examples

We would like to say that the singular set is Levi-flat in the sense that the Levi-form (or Levi-map) vanishes. In the real-analytic case this definition is equivalent to the following: We say a real-analytic submanifold M is *Levi-flat* if near every point, there exist local holomorphic coordinates such that M is given by

$$\operatorname{Im} z_1 = \cdots = \operatorname{Im} z_j = 0, \quad z_{j+1} = \cdots = z_{j+k} = 0, \tag{4.1}$$

for some $j, k = 0, 1, \ldots n$ with $j + k \leq n$ (where we interpret $j = 0$ and $k = 0$ appropriately). Note that this definition includes complex submanifolds as Levi-flat. We say M is *generic* if $k = 0$, that is, if M is defined only by the first set of equations.

We wish to study the CR structure of the set M_s. First, Burns and Gong [BG] classified the Levi-flat quadrics, that is hypersurfaces defined by a quadratic polynomial. So let us look at their singularities.

Normal form in $\mathbb{C}^n$	Singular set
$(\operatorname{Im} z_1)^2 = 0$	empty
$\operatorname{Im}(z_1^2 + z_2^2 + \cdots + z_k^2) = 0$	$\mathbb{C}^{n-k}$
$z_1^2 + 2\lambda z_1 \bar{z}_1 + \bar{z}_1^2 = 0 \ \big(\lambda \in (0,1)\big)$	$\mathbb{C}^{n-1}$
$(\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0$	$\mathbb{R}^2 \times \mathbb{C}^{n-2}$
$\lvert z_1\rvert^2 - \lvert z_2\rvert^2 = 0$	$\mathbb{C}^{n-2}$

Notice that the singularity can be a complex variety of any dimension. The quadrics demonstrate the types of singularities we can get. The first is a nonsingular hypersurface, which is not really a quadric. The second is a hypersurface defined by

$$\operatorname{Im} f(z) = 0,$$

for a holomorphic function f. In this case, it is not a hard exercise to prove that the singularity is precisely the set where $df(z) = 0$ (see also [BG]). In particular, the singularity is always a complex analytic subvariety of complex codimension 1.

The third hypersurface is a union of two nonsingular hypersurface meeting along a leaf. The singularity is therefore the leaf and hence complex analytic.

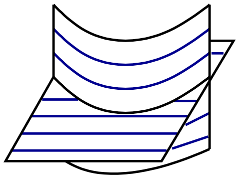
The fourth hypersurface is a union of two nonsingular hypersurfaces where the intersection is transverse to the foliation. The singular set is a generic Levi-flat hypersurface.

The last hypersurface is one where all leaves go through a single point and therefore the singular set is complex hypersurface of complex codimension 2.

An intersection of two smooth Levi-flats could also be a CR singular manifold. For example,

$$\big(\operatorname{Re}(z_2 - z_1^2)\big)\big(\operatorname{Im} z_2\big) = 0.$$

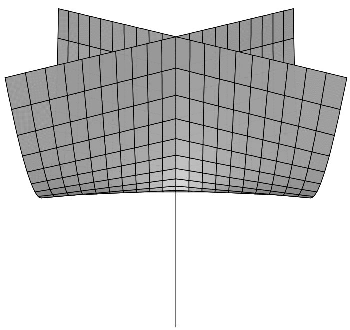
The singular set is defined by $z_2 = \operatorname{Re} z_1^2$, which is not a CR submanifold at the origin (a Bishop surface with infinite Bishop invariant in this case). What happens is that the foliations do not meet transversally at one point.



What could also happen is an "umbrella"-type hypersurface. Such examples were given in [Br, Le1, Le3] (See also Example 3.0.4). First let us start with the Whitney umbrella. Let $(z, w) \in \mathbb{C}^2$, $z = x + iy$, $w = s + it$. Let M be given by

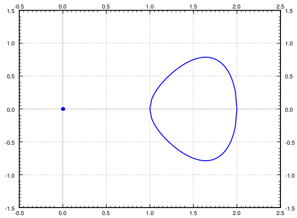
$$sy^2 - txy - t^2 = 0.$$

The subvariety is irreducible at the origin. If we look at $x = 0$ we obtain the standard Whitney umbrella in $\mathbb{R}^3$:



Here $M^* \subsetneq M_{reg}$. The "umbrella-handle" is a subset of the totally-real submanifold $\{t = 0, y = 0\}$. Part of this submanifold is the singular set, where the two sheets meet, and part is the handle. The handle is part of the regular points of the subvariety, but not of M^*. The set $\overline{M^*}$ is a union of complex lines, it is the image of $\mathbb{C} \times \mathbb{R}$ under the map $(\xi, c) \mapsto (z = \xi, w = c\xi + c^2)$. In this case the set M_s is not a subvariety, it is a so called semi-algebraic set.

It is also possible to obtain an umbrella with a complex handle. Take the algebraic subvariety in $\mathbb{R}^2$ given by $y^2 + x^2(x-1)(x-2) = 0$:



This subvariety is irreducible algebraically, and has two topological components: the origin and a compact loop. We look at the complex cone over this set, that is we replace $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$. Then we bihomogenize by adding w and $\bar{w}$ to obtain

$$\left(\frac{z\bar{w} - \bar{z}w}{2i}\right)^2 w^2\bar{w}^2 + \left(\frac{z\bar{w} + \bar{z}w}{2}\right)^2 \left(\frac{z\bar{w} + \bar{z}w}{2} - w\bar{w}\right) \left(\frac{z\bar{w} + \bar{z}w}{2} - 2w\bar{w}\right) = 0$$

The variety is a complex cone (a union of complex lines through the origin), it is irreducible, the singularity is the origin alone, and there is an umbrella handle that is a complex line $z = 0$. In this case $\dim_{\mathbb{R}} M_s = 0$, and M has a component (the handle) of real dimension 2. The subvariety is also irreducible (analytically) because the original curve was irreducible algebraically and the set is a complex cone.

4.2 Singularity is Levi-flat

Now that we have seen the difficulties, let us give what is known about the singular set. Currently we cannot say much about the "umbrella handle" above. Furthermore, from a point of view of geometry it is important to understand the structure of the singularity of $\overline{M^*}$, which can be thought as the hypersurface part of M. This set is a so-called semianalytic set.

Theorem 4.2.1 (L. [Le2])**.** *Let $U \subset \mathbb{C}^n$ be an open set and let $M \subset U$ be a (closed) Levi-flat subvariety of codimension 1. Then* *the singular set* $(\overline{M^*} \cap U)_s$ *is Levi-flat near points where it is a CR real-analytic submanifold.*

Furthermore, if $(\overline{M^*} \cap U)_s$ *is a generic submanifold, then* $(\overline{M^*} \cap U)_s$ *is a generic Levi-flat submanifold of dimension* $2n - 2$*.*

The theorem is mainly about large singularities as small singularities are easy to handle. Either by the theorem of Cerveau-Lins Neto, or by the analogous theorem in [Le2]:

Theorem 4.2.2 (Lebl [Le2])**.** *Let* $U \subset \mathbb{C}^n$ *be an open set and let* $M \subset U$ *be a Levi-flat real-analytic subvariety that is irreducible as a germ at* $p \in \overline{M^*} \cap U$*. If either*

1. $\dim M_s = 2n - 4$ *and* p *is not dicritical, or*

2. $\dim M_s < 2n - 4$*,*

Then the Levi-foliation extends to a singular holomorphic foliation of a neighbourhood of p*.*

Once the foliation extends, the singularity of M will in fact be complex analytic as it will essentially be either a leaf of the foliation or the singularity of the foliation.

The theorem is mainly about large singularities as small singularities are easy to handle.

A key lemma in the above theorem which is also useful in other contexts is the existence of leaves through singular points that lie in $\overline{M^*}$. The lemma follows from the work of Fornæss. A weaker version of this lemma in the singular case appeared in Burns-Gong [BG], where the leaf was only required to lie in M.

Lemma 4.2.3 (Lebl [Le2])**.** *Let* $M \subset U \subset \mathbb{C}^n$ *be a singular Levi-flat hypersurface. If* $p \in \overline{M^*} \cap U$*, then there exists a neighborhood* V *of* p *and a complex subvariety* $X \subset V$ *of dimension* $n - 1$ *such that* $X \subset \overline{M^*} \cap V$ *and* $p \in X$*.*

It is possible that there is more than one complex variety in M. It is even possible for some of these complex subvarieties to lie in M but not $\overline{M^*}$. For example, the umbrella subvariety that is a cone of the loop plus a point above.

The lemma also says that $\overline{M^*}$ (but perhaps not M) divides its ambient space into pseudoconvex parts (just like a nonsingular Levi-flat). To see this fact, note that any complex hypersurface $X \subset \overline{M^*}$ through p is defined by the vanishing of a single holomorphic function h near p. Take the function $\frac{1}{h}$ and you obtain a holomorphic function that cannot be extended past p. To see that M does not have this property, look at the Whitney umbrella example. The handle is a totally-real submanifold, and any holomorphic function must extend across the handle, so for any neighbourhood U, one component of $U \setminus M$ is not pseudoconvex. On the other hand $U \setminus \overline{M^*}$ is pseudoconvex.

Let us return to proving the theorem. The theorem follows by treating different cases. First when the singularity is small we obtain an extension of the holomorphic foliation, in which case the theorem is not difficult to prove (we obtain a complex subvariety for the singular set). The difficult case is if the singularity is large.

In this case we use Segre varieties, which were introduced in the 1930s by Beniamino Segre, and applied with great success in CR geometry of real-analytic sets by Diederich, Fornaess, Webster and others in the late 1970s.

Let us give a very brief introduction to the use of Segre varieties in studying the singularity, and a very rough sketch of the proof as these ideas can be useful elsewhere. Suppose that M is defined in some polydisc $U \subset \mathbb{C}^n$ by $r(z, \bar{z}) = 0$. Suppose that U is small enough such that r complexifies to $U \times U$, that is, $r(z, \xi)$ is a holomorphic function of $U \times U$. For a point $p \in U$ define the *Segre variety*

$$\Sigma_p = \{z \in U : r(z, \bar{p}) = 0\}. \tag{4.2}$$

That is, we treat z and $\bar{z}$ as separate variables (we *complexify*) and then we look at the trace of the complexified M with $\bar{z} = \bar{p}$.

For $M = \{\operatorname{Im} z_1 = 0\}$ and p the origin, the Segre variety Σ_p is $\{z_1 = 0\}$, that is, the leaf at the origin. In fact, if $X \subset M$ is a complex subvariety and $p \in X$, then (even if M is not Levi-flat)

$$X \subset \Sigma_p,$$

at least locally near p. This is not hard to prove: Suppose $f \colon \Delta \subset \mathbb{C} \to X$ is an holomorphic nonzero function defined on a disc Δ with $f(0) = p$. We have

$$r\big(f(\xi), \bar{f}(\bar{\xi})\big) = 0.$$

for all $\xi \in \Delta$. We an assume that Δ is small enough so that the equation complexifies and so we get

$$0 = r\big(f(\xi), \bar{f}(0)\big) = r\big(f(\xi), \bar{p}\big).$$

Therefore $f(\Delta) \subset \Sigma_p$. One can always fill such an X with such "analytic discs" to prove that $X \subset \Sigma_p$ (at least near p).

Therefore, if M is Levi-flat and $p \in M^*$, the set Σ_p has a branch that agrees with the leaf of the Levi-foliation through p.

Unless of course $r(z, \bar{p}) \equiv 0$, in which case Σ_p is not a proper variety. Points where $r(z, \bar{p}) \equiv 0$ for every defining function are called *Segre-degenerate*. For example: the cone $z_1\bar{z}_1 + z_2\bar{z}_2 = 0$ is Segre-degenerate at the origin. The set of Segre-degenerate points turns out to be a small set, and so we can avoid them. Suppose for now that Σ_p is always a hypersurface.

Take a path $\gamma\colon (-\epsilon, \epsilon) \to M^*$ and study the set

$$\bigcup_{t \in (-\epsilon, \epsilon)} \Sigma_{\gamma(t)} = \{z : r(z, \overline{\gamma(t)}) = 0, t \in (-\epsilon, \epsilon)\}$$

We obtain a nontrivial part of M (plus maybe other points as Σ_p need not be contained in M).

One can apply the Weierstrass preparation theorem $r(z, \overline{\gamma(t)})$ to rewrite it as a polynomial in t. Then outside of the discriminant set we can check if any of the roots are real. We obtain possibly a union of several nonsingular Levi-flats that contain the set we swept out.

If one thinks about Weierstrass preparation theorem as a generalization of implicit function theorem, then the above is essentially an attempt to carry through a calculation to come up with a "multivalued" Cartan's theorem from the first chapter. That is, we are looking for the real roots of a multivalued holomorphic function t.

4.3 Diederich-Fornæss Lemma

A key ingredient in the above work, and one of the most useful results about real-analytic subvarieties containing complex subvarieties is the following lemma that allows one to extend complex subvarieties of real-subvarieties to a fixed neighbourhood.

Lemma 4.3.1 (Diederich-Fornæss [DF])**.** *Let $X \subset U \subset \mathbb{C}^n$ be a real-analytic subvariety and $p \in X$. Then there exists a neighbourhood $p \in \widetilde{U} \subset U$ such that if $(Y, q) \subset (X, q)$ is a germ of a complex subvariety for $q \in \widetilde{U}$, then there exists a closed complex subvariety $\widetilde{Y} \subset \widetilde{U}$ such that $\widetilde{Y} \subset X$ and $(\widetilde{Y}, q) = (Y, q)$.*

This lemma is one of the best illustrations of the use of Segre varieties.

Sketch of proof. Let us consider only the hypersurface case, that is suppose there is a single real analytic function ρ defining function. We can also assume that $p = 0$. Suppose that the complexified ρ converges in some polydisc $P \times P$ around 0. We can suppose that Y is a complex submanifold not necessarily closed in P. By a very similar argument as above it is not difficult to show that for $\rho(z, \bar{w}) = 0$ for w near z if w and z are both in Y.

Let Σ_ξ denote the segre variety of ρ in P. Define

$$W' := \bigcap_{z \in Y} \Sigma_z, \qquad W := \bigcap_{w \in W'} \Sigma_w. \tag{4.3}$$

Now suppose $\xi \in W'$ and $\omega \in Y$, then $\rho(\xi, \overline{\omega}) = 0$ by definition. Next, $\rho(\omega, \overline{\xi}) = 0$ by reality of ρ. Hence $Y \subset W \subset W'$. Furthermore if $z \in W$, then $\rho(z, \bar{z}) = 0$, so $W \subset X$. The sets W' and W are (closed) complex analytic subvarieties of $P = \widetilde{U}$. $\square$

A similar but more refined argument proves Lemma 4.2.3. Let us mention a couple of other related applications of this lemma. One by Diederich-Fornæss says that no compact real-analytic subvariety of $\mathbb{C}^n$ contains a germ of a complex subvariety (In particular, all Levi-flats in $\mathbb{C}^n$ are non-compact). Another is that a real-analytic subvariety which is a complex submanifold at smooth points is in fact a complex analytic subvariety.

4.4 Open questions

Many questions are still left open. Firstly the theorem above is only the first step. We really want a whole Levi-flat stratification.

Question 4.4.1. *Given a Levi-flat subvariety X. Is there a Levi-flat stratification? That is, can X be expressed as a locally finite union of Levi-flat submanifolds, and can we also make them glue together in a nice way.*

One case that is not understood are singularities of dimension $2N - 3$. They seem too large for foliations, but too small to arise in the intersection of two smooth Levi-flats.

Question 4.4.2. *Given a Levi-flat hypersurface M in $\mathbb{C}^N$, does there exist a nontrivial (irreducible in particular) example where M_s is of dimension $2N - 3$?*

Despite Levi-flats being essentially just a family of complex hypersurfaces, they have many of the pathological properties of real-analytic subvarieties. Therefore, the correct direction may be to study semianalytic Levi-flat sets rather than subvarieties, as is common in real analytic geometry. The theorem in this section goes in this direction in studying $\overline{M^*}$ rather than M.

Question 4.4.3. *Are all notions of a degenerate singularity the same? In particular is Segre-degenerate equivalent to there being infinitely many complex hypersurfaces in M through the point?*

Chapter 5

Levi-flat hypersurfaces in complex projective space

5.1 Projective space

As in one complex variable, it is often convenient to "compactify" $\mathbb{C}^n$ in a way which preserves the complex structure. That is, we wish to obtain the natural compact complex manifold that contains $\mathbb{C}^n$. When $n = 1$, this compactification is the Riemann sphere which we denote by $\mathbb{P}^1$. In general, the correct compactification is the complex projective space:

Definition 5.1.1. Let $Z \in \mathbb{C}^{n+1}$ be coordinates. Let $\sim$ denote the following equivalence on the set $\mathbb{C}^{n+1} \setminus \{0\}$:

$$Z \sim W$$

if $Z = \lambda W$ for some $\lambda \in \mathbb{C}$. Denote by $[Z]$ the equivalence class of Z. Then define the *complex projective space* as the set of these equivalence classes:

$$\mathbb{P}^n \overset{\text{def}}{=} (\mathbb{C}^{n+1} \setminus \{0\})/\sim = \bigl\{[Z] : Z \in \mathbb{C}^{n+1} \setminus \{0\}\bigr\}.$$

Note that sometimes $\mathbb{P}^n$ is denoted by $\mathbb{CP}^n$.

The complex projective space is the space of complex lines through the origin in $\mathbb{C}^{n+1}$, and it is often convenient to keep this more informal definition in mind.

We write a point in $\mathbb{P}^n$ as $[Z_0, Z_1, \ldots, Z_n]$, starting the index at 0 for convenience. These coordinates are called the *homogeneous coordinates* for $\mathbb{P}^n$. We use square brackets to indicate this is an element of an equivalence class.

To see the manifold structure of $\mathbb{P}^n$, take U_j be the set of points of $\mathbb{P}^n$ corresponding to $Z_j \neq 0$. Note that if $Z_j \neq 0$ for one representative of the equivalence class, it is true for all representatives, and so we get a well-defined set. Without loss of generality now assume $j = 0$. Notice that there is exactly one representative of the equivalence class in U_0 such that $Z_0 = 1$. Therefore each point is uniquely represented by

$$[1, z_1, \ldots, z_n]$$

for each $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We thus obtain a coordinate chart for $\mathbb{P}^n$, and the corresponding coordinates $z_1, \ldots, z_n$ are called the *inhomogeneous coordinates*. On U_0 we let $\mathbb{P}^n$ inherit the complex structure from $\mathbb{C}^n$, that is, we consider the natural inclusion map

$$(z_1, \ldots, z_n) \mapsto [1, z_1, \ldots, z_n]$$

as a biholomorphism. As an exercise, one should check that when we take U_j for $j \neq 0$, then the transition maps are biholomorphisms (they will be linear).

Complex and real-analytic subvarieties on $\mathbb{P}^n$ are defined in the usual way as we would on any manifold. Natural subvarieties of $\mathbb{P}^n$ to study are the *algebraic* subvarieties.

Definition 5.1.2. Let $Z = (Z_0, \ldots, Z_n)$ denote the variables. A polynomial $P(Z)$ is *homogeneous* of degree d, if $P(\lambda Z) = \lambda^d P(Z)$ for all Z and all $\lambda \in \mathbb{C}$. Similarly, a polynomial $P(Z, \bar{Z})$ is *bihomogeneous* of bidegree (d, k), if $P(\lambda Z, \bar{\lambda}\bar{Z}) = \lambda^d \bar{\lambda}^k P(Z, \bar{Z})$ for all Z and all $\lambda \in \mathbb{C}$.

The zero set of homogeneous (resp. bihomogeneous) polynomials is a *complex cone*. That is take the zero set $X = P^{-1}(0) \subset \mathbb{C}^{n+1}$ for a homogeneous or a bihomogeneous polynomial P. If $Z \in X$, then λZ

is also in X for all λ. That is, X is a union of complex lines through the origin. Let $\sigma\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the natural projection, that is,

$$\sigma(Z_0, \ldots, Z_n) \overset{\text{def}}{=} [Z_0, \ldots, Z_n].$$

Then if $X \subset \mathbb{C}^{n+1}$ is a union of complex lines through the origin, via a very slight abuse of notation[1] denote by $\sigma(X)$ the corresponding set in $\mathbb{P}^n$. On the other hand, given $Y \subset \mathbb{P}^n$ define

$$\tau(Y) \overset{\text{def}}{=} \sigma^{-1}(Y) \cup \{0\},$$

to be the corresponding complex cone in $\mathbb{C}^{n+1}$.

Definition 5.1.3. A subset $X \subset \mathbb{P}^n$ is a *complex algebraic subvariety* (resp. *real algebraic subvariety*) if there exist finitely many homogeneous (resp. bihomogeneous) polynomials $P_1, \ldots, P_k$ such that

$$X = \sigma\big(P_1^{-1}(0) \cap \ldots \cap P_k^{-1}(0)\big).$$

That is, X is defined by the simultaneous vanishing of P_1 through P_k.

5.2 Chow's theorem

One of the most fundamental results about varieties in projective space is Chow's theorem [Ch].

Theorem 5.2.1 (Chow). *If $X \subset \mathbb{P}^n$ is a complex analytic subvariety, it is a complex algebraic subvariety.*

It is important to note that Chow's theorem does not hold for real-analytic subvarieties. It may be good to sketch a proof of Chow's theorem to see what goes wrong in the real-analytic case.

Sketch of proof. It is very easy to see that a subvariety (real or complex) $X \subset \mathbb{P}^n$ induces the subvariety $\sigma^{-1}(X) \subset \mathbb{C}^{n+1} \setminus \{0\}$. In the complex case, the theorem of Remmert-Stein (see e.g. [Wh]) implies that $\sigma^{-1}(X) \cup \{0\} = \tau(X)$ is a subvariety (no such result holds for

[1] Really we mean $\sigma(X \setminus \{0\})$.

real-analytic subvarieties). For simplicity let us suppose that $\tau(X)$ is of codimension 1, and therefore there is a holomorphic function f defined in a neighbourhood of the origin in $\mathbb{C}^{n+1}$, such that the zero set of f is $\tau(X)$. Write the decomposition into homogeneous parts

$$f(Z) = \sum_{j=0}^{\infty} f_j(Z)$$

Then as $\tau(X)$ is a complex cone, if $Z \in \tau(X)$, then $\lambda Z \in \tau(X)$ for all $\lambda \in \mathbb{C}$. So

$$0 = f(\lambda Z) = \sum_{j=0}^{\infty} f_j(\lambda Z) = \sum_{j=0}^{\infty} \lambda^j f_j(Z)$$

As this is true for all λ, we get $f_j(Z) = 0$ for all j. Consequently, $\tau(X)$ is defined by the vanishing of homogeneous polynomials, and so is an algebraic subvariety. □

The difference in the real case is that $\tau(X)$ can fail to be a subvariety at the origin. If $\tau(X)$ is a real subvariety at the origin, a similar proof shows that it is a real algebraic subvariety, see also [Le1, Le3].

5.3 Hypersurfaces with compact leaves

Let $M \subset \mathbb{P}^n$ be a real-analytic Levi-flat subvariety of codimension 1. As before, we say M is Levi-flat if M^* is Levi-flat. Suppose that a leaf $L \subset M^*$ of the Levi-foliation is closed in M^*. Then the closure $\overline{L}$ in $\mathbb{P}^n$ is a compact complex subvariety: it is closed in M and a complex analytic subvariety by Diederich-Fornæss, and hence algebraic by Chow's theorem. We then identify L with $\bar{L}$ and say that L is a compact leaf.

If M is algebraic, that is given by a polynomial equation, then all the leaves are compact. This fact follows easily by the fact that the Segre variety using the polynomial equations is automatically a compact subset of $\mathbb{P}^n$. The other direction is not true. Let us without proof note that in [Le3] it was shown that there exists a singular Levi-flat subvariety $X \subset \mathbb{P}^2$, that is a union of complex hyperplanes (that is, with compact leaves), such that the only polynomial vanishing on X is the zero polynomial, in particular, X is not algebraic. The example shows that studying Levi-flat hypersurfaces cannot be done purely from an algebraic perspective.

Let us give a simple proposition however.

Proposition 5.3.1. *Let $M \subset \mathbb{P}^n$, $n \geq 2$, be a real-analytic Levi-flat subvariety of codimension 1. Suppose that M has two distinct compact leaves, then $\dim_{\mathbb{R}} M_s \geq 2n-4$. In particular, such a subvariety is always singular.*

Proof. The two compact leaves L_1 and L_2 are complex subvarieties of (complex) codimension 1 in $\mathbb{P}^n$ and therefore their intersection $L_1 \cap L_2$ is a subvariety of (complex) codimension 2 in $\mathbb{P}^n$. The subvariety $L_1 \cup L_2 \subset M$ is a complex analytic variety with the points $L_1 \cap L_2$ belonging to its singularity. A nonsingular Levi-flat real-analytic hypersurface contains a unique nonsingular complex hypersurface, therefore $L_1 \cap L_2 \subset M_s$ and we are done. □

Example 5.3.2. The subvariety in $\mathbb{P}^2$ given by

$$Z_1 \bar{Z}_1 - Z_2 \bar{Z}_2 = 0,$$

the complex cone has a unique singular point $[1, 0, 0]$ and hence of real dimension $2n - 4 = 0$.

Example 5.3.3. Let us give a very intresting and useful example of a hypersurface M. Take M to be the closure of the set of points $[Z_0, Z_1, Z_2]$ in $\mathbb{P}^2$ where

$$Z_0 + Z_1 t + Z_2 t^2, \quad \text{for some } t \in \mathbb{R}.$$

This set is a semialgebraic set (defined by algebraic equalities and inequalities). We can easily compute the polynomial equation giving the closure of the set using the quadratic formula (we leave this to the reader). M is necessarily Levi-flat and algebraic, and hence has compact leaves. In fact the leaves are complex lines in $\mathbb{P}^2$. In the next section we will talk about extensions of the foliation, in this example, the foliation does not extend, though it does extend to a 2-web.

An interesting feature of this hypersurface is that while it has compact leaves, it has no degenerate singularity. Each leaf only ever meets finitely many leaves (clearly at most two), otherwise we would get quadratic polynomials that vanish at more than 2 points. The singularity is of dimension 2.

Recently, the second author proved in [Le3] that for a subvariety induced by complex hyperplanes (like both examples above), the singularity is either of dimension $2n-2$ or $2n-4$.

5.4 Extending the foliation

One of the tools in studying Levi-flat hypersurfaces is the foliation, and the extension of the foliation to a holomorphic codimension one foliation. As we have seen the foliation does not always extend, even locally. But in certain cases we can extend the foliation globally.

Let $M \subset \mathbb{P}^n$ be a real-analytic Levi-flat subvariety of codimension 1. Let $\mathcal{F}$ be a possibly singular holomorphic foliation of codimension one of a neighbourhood of $\overline{M^*}$ in $\mathbb{P}^n$, which extends the Levi-foliation of M. That is, there exists a covering of M by open sets $\{U_\iota\}$ of $\mathbb{P}^n$ with the following property, in each U_ι there exists a holomorphic one-form ω_ι with $d\omega_\iota \wedge \omega_\iota = 0$ (integrable one-form so defines a foliation locally), and locally the leaves of the Levi-foliation of M are leaves of $\mathcal{F}$. If $U_\iota \cap U_\kappa \neq \emptyset$, then ω_ι and ω_κ must be proportional at every point of the intersection. The extension at a nonsingular point of M can be stated as follows: if $\operatorname{Im} f = 0$ defines M locally for a holomorphic f, then df is proportional to $\mathcal{F}$ at points of M. The points where ω_ι vanishes is the singular set of $\mathcal{F}$.

We say the foliation extends to $\mathbb{P}^n$ if we can find $\{U_\iota\}$ with the above property that covers all of $\mathbb{P}^n$.

Lins Neto proved the following lemma [LN]. The proof of Lins Neto only considers the nonsingular M. The key point of the proof is that the components of $\mathbb{P}^n \setminus M$ (or $\mathbb{P}^n \setminus \overline{M^*}$ in the singular case) are Stein manifolds (manifolds biholomorphic to closed submanifolds of some $\mathbb{C}^K$). This follows from a theorem of Takeuchi saying that any pseudoconvex subset of $\mathbb{P}^n$ which is not the whole $\mathbb{P}^n$ is Stein. In the case as we state it, it then follows from Lemma 4.2.3, which says that $\overline{M^*}$ divides space into pseudoconvex pieces.

Lemma 5.4.1 (Lins Neto)**.** *Suppose that $M \subset \mathbb{P}^n$ is a real-analytic*

Levi-flat subvariety of codimension 1, $n \geq 2$. Suppose that for each $p \in \overline{M^*}$, there exists a neighbourhood U of p and a holomorphic foliation $\mathcal{F}_p$ in U that extends the Levi-foliation of $M \cap U$.

Then there exists a possibly singular holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^n$ that extends the Levi-foliation of M.

The proof is essentially a Hartogs like extension theorem for foliations in Stein manifolds. That is, foliations extend through compact sets in Stein manifolds. The way to prove this is to note that if we write the foliation at each point in terms of some set of coordinates (Stein manifolds can be properly embedded in $\mathbb{C}^N$ for some N) we can divide by one of the coefficients. We obtain a global meromorphic one-form, and we can extend meromorphic functions.

The foliation of $\mathbb{P}^n$ can be given as a single one-form with polynomial coefficients. The proof of this is essentially the same as Chow's theorem. That is, there exists a one-form

$$\sum_{j=0}^{n} F_j(Z) dZ_j$$

where F_j are homogeneous polynomials, and this one-form gives the foliation on all of $\mathbb{P}^n$. We can also think of it as foliation on $\mathbb{C}^{n+1}$. The study of Levi-flat hypersurfaces of $\mathbb{P}^n$ whose foliation extends is then the study of the invariant sets of such algebraic one-forms, which is by no means trivial.

The lemma shows that the key is extending the foliation locally. We do have to assume local irreducibility, otherwise we would only obtain a k-web.

Theorem 5.4.2. *Suppose $M \subset \mathbb{P}^n$ is a locally irreducible Levi-flat subvariety of codimension 1 satisfying conditions of Theorem 2.2.3 or Theorem 4.2.2 at each point $p \in \overline{M^*}$. In particular for example if the dimension of the singularity is $2n-4$ and the the singularity is nondicritical, or if the singularity is of dimension $2n-5$ or less. Then the foliation extends to a holomorphic foliation of $\mathbb{P}^n$.*

5.5 Theorem of Lins Neto and the analogue of Chow's theorem

By extending Lins Neto uses the lemma above to prove the following well-known result.

Theorem 5.5.1 (Lins Neto)**.** *There exists no nonsingular real-analytic Levi-flat hypersurface $M \subset \mathbb{P}^n$ for $n \geq 3$.*

The proof of this theorem follows by first extending the foliation to a foliation of $\mathbb{P}^n$. Such a foliation must have a singularity S of complex dimension $n-2$. When $n \geq 3$, S is a complex subvariety of positive dimension and $\mathbb{P}^n \setminus M$ are Stein, then S must intersect M, which is a contradiction to M being nonsingular.

A related question is an analogue of the theorem of Chow for Levi-flats. As we mentioned above, there exist nonalgebraic Levi-flats even with compact leaves. The key point is the extensibility of the foliation. In [Le1] the following theorem was proved.

Theorem 5.5.2. *Let $M \subset \mathbb{P}^n$, $n \geq 2$, be an irreducible Levi-flat subvariety of codimension 1 with infinitely many compact leaves, and assume that the Levi-foliation of M extends to a foliation of a neighbourhood of $\overline{M^*}$.*

Then, there exists a global rational function $R\colon \mathbb{P}^n \to \mathbb{C}$ and a real-algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $M \subset \overline{R^{-1}(S)}$. In particular, M is semialgebraic; it is contained in an algebraic Levi-flat subvariety.

It should be noted that an algebraic Levi-flat always has all leaves compact. The theorem arose from trying to construct a non-algebraic Levi-flat by pulling back a non-algebraic curve in $\mathbb{C}$ to $\mathbb{P}^n$ via a rational function R. Unfortunately by the theorem such a pullback cannot be a real analytic subvariety of $\mathbb{P}^n$ for $n \geq 2$. The trouble will occur at the point p where both the numerator and denominator of R vanish. At p the pullback of a non-algebraic subvariety cannot be contained in any real-analytic subvariety of any neighbourhood of p.

5.6 Some open questions

Let us look at some open questions.

Question 5.6.1. *Does there or does there not exist a nonsingular Levi-flat in* $\mathbb{P}^2$*?*

Despite several published papers claiming this result (in varius regularities), this question is still open.

Question 5.6.2. *Does there exist a singular Levi-flat hypersurface of* $\mathbb{P}^n$*,* $n \geq 2$*, with no compact leaves?*

Clearly such a hypersurface would not be algebraic. The question is related to the following question.

Question 5.6.3. *Is it enough to require that the Levi-foliation of* M *extends to* $\mathbb{P}^n$ *to obtain algebraicity of* M*?*

Such a question would implicitly also prove nonexistence of nonsingular Levi-flats.

Question 5.6.4. *Does there exist* $M \subset \mathbb{P}^n$ *that is irreducible (does not contain a smaller hypersurface) such that* $\dim_{\mathbb{R}} M_s = 2n - 3$*?*

We can also relax the requirement of extending the Levi-foliation to a holomorphic foliation, but to a so-called k-web:

Question 5.6.5. *Given a singular Levi-flat hypersurface of* $M \subset \mathbb{P}^n$*,* $n \geq 2$*, does the Levi-foliation extend to a holomorphic* k*-web of* $\mathbb{P}^n$*?*

This question is solved in many cases, see [Fe3, Le2, SS].

Bibliography

[A] V.I Arnold: *Normal forms of functions in the neighborhood of degenerate critical points.* I. Uspehi Mat. Nauk 29, no. 2(176) (1974), 11-49.

[A1] V.I. Arnold: *Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities.* Funkcional. Anal. i Priložen. **6**, no. 4 (1972), 3-25.

[AGV] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko: Singularities of Differential Maps, Vol. I, Monographs in Math., vol. 82, Birkhäuser, 1985.

[BD] E. Bedford, P. DE Bartolomeis: *Levi-flat hypersurfaces which are not holomorphically flat*, Proc. Amer. Math. Soc., **81** (1981), 575–578.

[BSE] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild: Real submanifolds in complex space and their mappings. Princeton Mathematical Series, 47. Princeton University Press, Princeton, NJ, 1999.

[BG] D. Burns, X. Gong. *Singular Levi-flat real analytic hypersurfaces*, Amer. J. Math., **121** (1999), 23–53.

[Br] M. Brunella: *Singular Levi-flat hypersurfaces and codimension one foliations.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) vol. VI (2007), no. 4, 661–672.

[Br1] M. Brunella: *Some remarks on meromorphic first integrals.* Enseign. Math. (2), 58 (3–4) (2012), 315–324.

[CL] D. Cerveau, A. Lins Neto: *Local Levi-flat hypersurfaces invariants by a codimension one holomorphic foliation*, Amer. J. Math., **133**, no. 3 (2011), 677–716.

[Ca] E. Cartan: *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes.* Ann. Mat. Pura Appl. 11 (1933), no. 1, 17–90.

[Ce] D. Cerveau: *Feuilletages holomorphes de codimension 1. Réduction des singularités en petites dimensions et applications.* Panor. Synthèses 8 (1999), 11–47.

[Ch] W. Chow: *On compact complex analytic varieties*, Amer. J. Math., **71** (1949), 893–914.

[DF] K. Diederich, J. E. Fornaess: *Pseudoconvex domains with real-analytic boundary.* Ann. of Math. 107 (1978), 371–384.

[D] A.H. Durfee: *Fifteen characterizations of rational double points and simple critical points.* Enseign. Math. (2) 25, no. 1-2 (1979), 131-163.

[Fe] A. Fernández-Pérez: *On normal forms of singular Levi-flat real analytic hypersurfaces.* Bull. Braz. Math. Soc., **42** (2011), no. 1, 75–85.

[Fe1] A. Fernández-Pérez: *On Levi-flat hypersurfaces tangent to holomorphic webs.* Ann. Sci. Toulouse Math (6) 20(3) (2011), 581–597.

[Fe2] A. Fernández-Pérez: *On Levi-flat hypersurfaces with generic real singularities.* J. Geom. Anal. **23** (2013), 2020–2033.

[Fe3] A. Fernández-Pérez: *Levi-flat hypersurfaces tangent to projective foliations.* J. Geom. Anal. **24** (2014), 1959–1970.

[Fe4] A. Fernández-Pérez: *Normal forms of Levi-flat hypersurfaces with Arnold type singularities.* Ann. Sc. Norm. Super. Pisa Cl. Sci. Vol VIII (3) (2014), 745–774.

[Fe5] A. Fernández-Pérez: *On normal forms for Levi-flat hypersurfaces with an isolated line singularity.* Arkiv för Matematik., 53 (1) (2015), 65–78.

[Le1] J. Lebl: *Algebraic Levi-flat hypervarieties in complex projective space.* J. Geom. Anal. **22** (2012), no. 2, 410–432. doi: 10.1007/s12220-010-9201-9

[Le2] J. Lebl: *Singular set of a Levi-flat hypersurface is Levi-flat.* Math. Ann. **355** (2013). no. 3. 1177–1199.

[Le3] J. Lebl: *Singular Levi-flat hypersurfaces in complex projective space induced by curves in the Grassmannian.* To appear in Int. J. Math. Preprint http://arxiv.org/pdf/1407.5913v3.pdf

[LN] A. Lins Neto: *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier (Grenoble), (4) **49** (1999), 1369–1385.

[L] F. Loray: *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux.* Avaliable in http://hal.archives-ouvertures.fr/ccsd-00016434

[Ło] S. Łojasiewicz: *On semi-analytic and subanalytic geometry.* Panorama Math. Banach Center Publ. **34** (1995), 89–104.

[MM] J.F. Mattei, R. Moussu: *Holonomie et intégrales premières.* Ann. Ec. Norm. Sup. **13** (1980), 469–523.

[Re] C. Rea:*Levi-flat submanifolds and holomorphic extensions of foliations.* Ann. Sc. Norm. Super. Pisa Cl. Sci. 3^e serie, tome 26 (1972), no. 3, 661–672.

[SS] R. Shafikov, A. Sukhov: *Germs of singular Levi-flat hypersurfaces and holomorphic foliations.* To appear in Commentarii Mathematici Helvetici.

[PP] J.V. Pereira, L. Pirio: An invitation to web geometry. From Abel's addition theorem to the algebraization of codimension one webs. Publicações Matemáticas do IMPA, 27° Cóloquio Brasileiro de Matemática, Instituto Nacional de Matemática Pura e Aplicada IMPA. Rio de Janeiro, 2009.

[Si] D. Siersma: *Isolated line singularity.* Proc. of Symposia in Pure Math. (2) **40** (1983), 485–496.

[Se] B. Segre: *Intorno al problem di Poincaré della rappresentazione pseudoconform.* Rend. Acc. Lincei, **13** (1931), 676–683

[W] S. M. Webster: *On the mapping problem for algebraic real hypersurfaces.* Invent. Math. **43** (1977), 53–68.

[Wh] H. Whitney: *Complex analytic varieties*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972.

[Z] A. Zaharia: *Characterizations of simple isolated line singularities.* Canad. Math. Bull. Vol. 42 **4** (1999), 499-506.