Generic Linear Recurrent Sequences and Related Topics

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A mia moglie Sheila

e ai nostri **10** *figli,*

Giuseppe e Giuliano

Preface

This book contains an expanded version of the material presented in the short course of the same name given at IMPA during the *XXX Colóquio Brasileiro de Matemática*. The aim of these notes is to introduce and develop the elementary theory of *generic linear recurrent sequences* to show how it supplies a natural unified framework for many seemingly unrelated subjects. Among them: traces of an endomorphism and the Cayley-Hamilton theorem, Generic Linear ODEs and their Wronskians, the exponential of a matrix with indeterminate entries (revisiting Putzer's method dating back to 1966), universal decomposition algebras of a polynomial into the product of two monic polynomials of fixed smaller degree, vertex operators obtained via Schubert calculus tools (Giambelli's formula) inspired by previous work by the author and by Laksov and Thorup over the past decade. The emphasis is put on the characterization of decomposable tensors of an exterior power of a free abelian group of possibly infinite rank. An alternative way is described for deducing the expression of the vertex operators employed in the description of the Kadomtsev-Petshiasvilii (KP) hierarchy.

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I am also indebted to my scientific collaborators, Inna Scherbak and Parham Salehyan, who have generously shared with me many of the ideas contained in this book and to Maxim Kazarian who first suggested to me that there should be a relationship between derivations of an exterior algebra and the boson-fermion correspondence. Special thanks go to Jorge Cordovez and Simon G. Chiossi whose help was crucial during the preparation of these notes. Many thanks are due to Martin Gulbrandsen and Andrea Ricolfi, who hosted some preliminary talks on this subject at Stavanger University; to Igor Mencattini, Alex Kasman and Carlos Rito for stimulating discussions on PDEs; to Alessandro Ardizzoni and Paolo Saracco for enlightening discussions on infinite exterior powers; to Antonio Nigro and Stefano Barbero for sharp criticism and advice; to Prof. Louis Rowen for precious indirect suggestions not to mention his tenacious encouragement.

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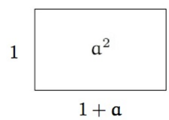
Sangano, 18 Maggio 2015

Contents

Introduction

Originally proposed to model the reproduction of rabbits, the Fibonacci[1] sequence $(1, 1, 2, 3, 5, 8, \ldots)$, whose two initial conditions are equal to 1 and the n-th term ($n \geq 3$) is the sum of the preceding two, is perhaps the most popular example of *Linear Recurrent Sequence* (LRS in the following). For instance, the italian artist Mario Merz[2] (see e.g. http://fondazionemerz.org/en/mario-merz/) realized several neon lights models of the first few Fibonacci's numbers. One of them, *Il volo dei numeri*[3], was placed some years ago on the spire of the Mole Antonelliana, the tower symbol of the italian town of Torino, which nowadays hosts the italian National Cinema Museum (http://www.museocinema.it/).

Fibonacci's numbers obey to the same recursive law enjoyed by the powers $(1, a, a^2, \ldots)$ of one of the two possible roots $(1 \pm \sqrt{5})/2$ of the polynomial $X^2 - X - 1$, whence the equality $a^2 = 1(1 + a)$



A *golden rectangle* with $a = \frac{1}{2} + \frac{\sqrt{5}}{2}$

that points its kinship with the famous *golden ratio*, which the ancient Greek architects used to design the planimetry of their temples.

[1] Leonardo Pisano, known as Fibonacci (Pisa,1170–1240) wrote the famous *Liber Abaci* in 1202 where the number zero appeared for the first time. Its name came from "zephyrus", which is a wind blowing from the west.

[2] Mario Merz, Milano 1925–2003, painter and sculptor who used poor materias for his artworks.

[3] *Numbers flying.*

Historical curiosities aside, Fibonacci's numbers supply an example of a $\mathbb{Z}$-valued LRS of *order* 2, with *characteristic polynomial* $X^2 - X - 1$. In a broader sense, a LRS generalizes the sequence $(1, a, a^2, \ldots)$ of the powers of the roots of a given monic polynomial $P := X^r - e_1(P)X^{r-1} + \ldots + (-1)^r e_r(P)$ of degree r, with coefficients in a commutative ring A with unit. A sequence $\mathbf{m} := (m_0, m_1, \ldots)$ of elements of an A-module M is a LRS with *characteristic polynomial* P if the relation $m_{j+r} - e_1(P)m_{j+r-1} + \ldots + (-1)^r e_r(P)m_j = 0$ holds for all $j \geq 0$: the degree of P is said to be the *order* of the LRS.

The sequence $(\mathbb{1}_M, f, f^2, \ldots)$ of the powers of an endomorphism of a free A-module of rank r is an example of a LRS with characteristic polynomial $P_f(X) := \det(X\mathbb{1}_M - f)$, due to the celebrated theorem by Cayley and Hamilton, revisited and substantially generalized in Chapter 2. One further relevant instance of LRS is the sequence $(y, y', y'', \ldots)$ of the derivatives of the solutions to a linear ODE with constant coefficients, whose elementary theory is phrased in Chapter 1 within a purely algebraic language, which amounts to construct the D-module associated to a generic linear ordinary differential operator of order r (see e.g. [12, Chapter 6] and [40, Example 1.2.4]).

As it is easy to guess, there is an enormous deal of literature (e.g. [2, 9, 10, 68, 39, 57, 67, 76, 78, 79] just to quote some) and many excellent expository books (such as [3, 16, 42]) concerning Linear Recurrent Sequences. It is hard to add anything substantial to this subject without taking the risk to be trivial and, in fact, these notes will not pursue such an ambitious goal. Rather, they will focus on the elementary notion of *generic LRS* just as a pretext to make an interdisciplinary journey to visit a few, and just a little, amusing mathematical landscapes whose snapshots we believe could be put into a common picture frame.

The characteristic polynomial of a generic M-valued LRS of order r is, by definition, the *generic monic polynomial* $\mathtt{p}_r(X) := X^r - e_1X^{r-1} + \ldots + (-1)^r e_r \in B_r[X]$. Here B_r denotes the ring of polynomials with integral coefficients in the r indeterminates $(e_1, \ldots, e_r)$. The letter "B" used in the notation reflects the fact that if one denotes by B_∞ the polynomial ring $\mathbb{Z}[e_1, e_2, \ldots]$ in the infinitely many indeterminates $(e_1, e_2, \ldots)$, then $B := B_\infty \otimes_{\mathbb{Z}} \mathbb{C}$ can be interpreted as the *bosonic Fock representation* of the Heisenberg oscillator algebra, i.e.

the *Weyl affinization* [44, p. 51] of the trivial one-dimensional complex Lie algebra (see the historical remark 0.3.3). Such analogy is not that audacious as it may seem. Along our path through generic LRS, we shall encounter and recognize the approximation of the same *vertex operators* that describe the celebrated Kadomtsev-Petshiasvili (KP) Hierachy in terms of the Plücker embedding of an infinite Grassmannian (Chapters 4 and 5). Their expression is essentially deduced by invoking Schubert Calculus arguments (mainly Giambelli's formula) phrased as, e.g., in the papers [22] and [58, 59], basically exploiting what nowadays people refer to as the Satake identification of $H^*(G(r,n))$ with $\bigwedge^r H^*(\mathbb{P}^{n-1})$ [34, 37] (see also [23, p. ix]).

Below we shall briefly describe, with a little more detail, the main topics covered in the exposition and how the notes are organized.

Chapter 0 is a Prologue, whence the choice to distinguish its numbering from that of the "official" part of the exposition. It aims to draw a quick and non-technical expository path through the Korteweg and de Vries non-linear PDE, that model the solitary waves observed by John Scott Russel in 1834, and its generalization, due to Kadomtsev and Petviashvili, for applications to plasma physics. Although it does not seem immediately related with the main subject of the notes, the chapter culminates with the appearance, as a kind of a *Deus ex-machina*, of certain vertex operators acting on a polynomial ring in infinitely many indeterminates, able to encode the full system of PDEs known under the name of KP hierarchy. These arise as compatibility conditions for another system of infinitely many PDEs and thanks to the work of Sato [74] and Date-Jimbo-Kashiwara-Miwa [13, 14] they can be phrased as Plücker equations for the Grassmann cone of the decomposable tensors in an infinite wedge power of an infinite–dimensional vector space.

The elementary linear algebraic roots of such geometric interpretation will be made explicit in **Chapter 4**, by characterizing decomposable tensors in the r-th exterior power of a free abelian group of countable rank, so that the limit for $r \to \infty$ returns the equation of the KP hierarchy as displayed in Section 0.4.

Chapter 1 contains a quick introduction to the elementary theory of generic LRS with values in a module M over a B_r-algebra. The main piece of information one gains from this chapter is the universal expression of the solution to the Cauchy problem for linear ODEs with constant coefficients and with analytic forcing term (Corollary 1.6.6). This is achieved by means of a distinguished sequence $(u_i)_{i\in\mathbb{Z}}$ of B_r-valued generic LRS, introduced first in [32] and then generalized in [28]. Such a sequence will reveal itself to be an extremely useful formal tool, although not indispensable, to determine the explicit expression of the vertex operators we alluded to. In particular it will be used in Chapter 5 to construct certain infinite exterior powers as limit of finite exterior powers of modules of generic LRS of finite order.

As for **Chapter 2**, it revisits the classical theorem by Cayley and Hamilton through the (re-)definition of the traces (or the principal invariants) of an endomorphism of a free module in terms of derivations of its exterior algebra, in the sense of [22, 23, 31]. Its standard formulation, *each endomorphism is a root of its own characteristic polynomial*, turns out to be a special case of a more general vanishing statement, involving the whole exterior algebra. The purpose of this chapter is to lay out the pre-requisites for the sequel of the story. In due course, applications will be shown to the exponential of a matrix without using the Jordan canonical form, simplifying methods by Putzer [72, 1966], Leonard [63, 1996] and Liz [62, 1998], and to the explicit determination of prime integrals of linear ODEs of order n that miss the derivative of order $n-1$. The latter leads, as in Example 2.5.6, to a cubic generalization of the popular formula $\cos^2 x + \sin^2 x = 1$, easily extendable to higher degrees (with the help of a computer).

Chapter 3 and 4 form the core of the notes. In order to keep the exposition as self-contained as possible, the well-known natural $\mathbb{Z}$-module isomorphism between B_r and the r-th exterior power of a free abelian group of infinite countable rank is proven anew. Such an identification can be seen either as a kind of toy version of the so-called *boson-fermion correspondence* [5, 54, 47, 71, 44, 64] or as the essential algebraic content of Giambelli's formula in classical Schubert Calculus, once one identifies the cohomology of the Grassmannian $G(r,n)$, *à la Satake* (see Remark 3.6.12), with the r-th exterior power of $\mathbb{P}^{n-1}$, as in [22, 23, 31] and [58, 59]. See also the recent [43]. The main content of Theorem 4.5.3 is the announced formula, encoding the Plücker quadrics cutting out $G(r, n)$ in its own Plücker embedding. The pleasant feature of the formula is that, as r, n go to ∞, it produces precisely the KP Hierarchy. The proof of such a formula (reproduced as in [30]) is based on several ingredients that we believe are interesting by their own means. A crucial one consists in equipping a free $\mathbb{Z}$-module M_0, of infinite countable rank, with a structure of free B_r-module of rank r, denoted by M_r, for all $r \geq 1$, with the effect of turning its basis into a generic LRS. This is achieved by applying the Cayley-Hamilton theorem to the characteristic polynomial operator, in the sense of Chapter 2, associated to the shift endomorphism mapping each element b_i of an (ordered) basis of M_0 to b_{i+1}. It turns out that such traces coincide with the endomorphisms of the exterior algebra used in the paper [22] to rephrase Schubert calculus via derivations.

The traces of the shift endomorphism of step -1 also play a crucial role in the theory and they are the counterpart in the finite-dimensional setting of the partial derivatives involved in the expression of the vertex operators. It is worth to remark that to study decomposable tensors in $\bigwedge^r M_0$ one is led to consider the interaction of $\bigwedge^r M_r$ with $\bigwedge^{r-1} M_{r-1}$ and $\bigwedge^{r+1} M_r$, whose different structures as modules of rank 1 (over B_r, B_{r-1} and B_{r+1} respectively) is lost in the limit $r \to \infty$. The fact that a formula, living in the realm of finite Grassmannians, recovers the KP Hierarchy in the limit for $r \to \infty$, reveals that the latter's embryo is contained in the classical Plücker embedding equations for the finite Grassmannian, as Alex Kasman [51, 35, 53] pointed out in his work from a different point of view.

Chapter 5 is yet another take on the material of **Chapter 4**, albeit from a more concrete point of view, due to the identification of M_0 with the $\mathbb{Z}$-module spanned by generic LRS of finite order. Using the distinguished basis introduced in Chapter 1, numbered by decreasing indices, one may constructs a suitable infinite exterior power, by wedging all together its elements. The latter are interpreted as a fermionic Fock space like those described e.g. in the book [47]. The vertex operators showing up in the Prologue make their return in this chapter as well, by suitably generalizing and/or modifying, where necessary, the statement and the proofs exposed in Chapter 4.

Chapter 0

Prologue

0.1 The KdV equation

0.1.1 The KdV equation is a non-linear *Partial Differential Equation* (PDE) empirically deduced by the dutch mathematicians Diederik Johannes Korteweg (1848–1941) and Gustav de Vries (1866–1934) [56] to model the dynamics of *solitary waves*, or *solitons*, observed for the first time in 1834 by John Scott Russel [73] who was observing two horses rapidly pulling a boat in a narrow channel. For more on this picturesque story see e.g. [51, p. 45]. The most general form of the KdV equation would be $af_t + bff_x + cf_{xxx} = 0$, for arbitrary complex constants a, b, c, but, mostly for pedagogical reasons and to be more adherent to the many excellent expositions on the subject, like [70, 5] or [47, p. 75], we shall write it as follows:

$$4\frac{\partial f}{\partial t} - 12f\frac{\partial f}{\partial x} - \frac{\partial^3 f}{\partial x^3} = 0, \tag{1}$$

where f is sought in the class of C^3-functions defined in a neighborhood of the real plane (x, t).

In spite of the empirical origin, equation (1) reveals several interesting features, not to speak of its amazing relationship with many topics in algebraic geometry. It can be written in a number of equivalent ways and to find exact explicit solutions is not that difficult, even without being experts in PDEs.

A natural and standard way to deal with (1) is to look for solutions of the form $p := p(x-ct)$, the constant "c" being interpreted as the speed of the "wave" motion:

$$f := -p(x-ct) + K, \tag{2}$$

where K is an arbitrary constant free to vary according to the convenience. Substitution of (2) into (1) gives:

$$p''' - 12pp' - 4cp' + 12Kp' = 0 \tag{3}$$

and writing K as $c/3$, equation (3) gets rephrased in the simpler form $p''' = 12pp'$, that begs for being integrated once, giving

$$p'' = 6(p)^2 - \frac{1}{2}g_2, \tag{4}$$

where $-g_2/2$ is an arbitrary constant. Multiplying (4) by p':

$$p''p' = 6(p)^2p' + \frac{1}{2}g_2p'$$

a further integration yields:

$$(p')^2 = 4p^3 - g_2p - g_3. \tag{5}$$

At the cost of looking for solutions among complex functions of one complex variable, equation (5) is satisfied by the famous *Weierstrass $\wp$-function*:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{\mathbf{0}\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \tag{6}$$

where $\Lambda := \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ is a *lattice* [1] in $\mathbb{C}$. The coefficients g_2 and g_3 depend on the lattice Λ, which can be identified with a complex number with positive imaginary part, and are indeed *modular forms*[2] of weight 4 and 6 respectively. In fact

$$g_2 := g_2(\Lambda) = 60G_4 \quad \text{and} \quad g_3 := g_3(\Lambda) = 140G_6$$

[1] A discrete subgroup Λ of $\mathbb{C}$ such that $\dim_{\mathbb{R}} \Lambda \otimes_{\mathbb{Z}} \mathbb{R} = 2$.

[2] A *modular form* is a complex valued function f, defined on the space $\mathbb{H}$ of the complex numbers with positive imaginary part, such that $f(gz) = (cz+d)^k f(z)$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z})$. See the exciting survey e.g. [77]

where, for $k > 1$

$$G_{2k}(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{\mathbf{0}} \lambda^{-2k} \tag{7}$$

is the famous *Eisenstein* modular form of *weight* $2k$ [75, p.157]. Conversely, for general values of $g_2, g_3 \in \mathbb{C}$, there exists a *lattice* Λ_{g_2,g_3} in the complex plane, depending on g_2 and g_3, such that (6) satisfies (5) for $\Lambda = \Lambda_{g_2,g_3}$. In fact, an equivalence class of lattices modulo the action of the group $Sl_2(\mathbb{Z})$ is parameterized by a point τ of the Poincaré half–plane $\mathbb{H}$ and the *j-invariant* $j : \mathbb{H} \to \mathbb{C}$

$$j(\tau) = \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}$$

is surjective, i.e. for each pair $g_2, g_3 \in \mathbb{C}$ there exists τ such that the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ does provide, via the Weierstrass $\wp$-function , the parameterization of (5). We have so found solutions to (1) of the form

$$f := -\wp_\Lambda(x - ct) + \frac{c}{3},$$

called *periodic solutions*. Using elliptic functions to solve (1) suggests another kind of substitution. See below.

0.1.2 The g-dimensional *Siegel generalized domain* is the set of all $g \times g$ hermitian matrices with positive definite imaginary part:

$$\mathbb{H}^g := \{\Omega \in \mathbb{C}^{g\times g} \,|\, \Omega^T = \overline{\Omega}, \operatorname{Im}(\Omega) > 0\}$$

where ^{T} denotes transposition and $^-$ complex conjugation. If $g = 1$ then $\mathbb{H} := \mathbb{H}^1$ consists of the complex numbers with positive imaginary part. Clearly, the matrix $\Lambda_\Omega := (\mathbb{1}_{g\times g}, \Omega) \in \mathbb{C}^{g\times 2g}$, where $\mathbb{1}_{g\times g}$ is the $g\times g$ identity matrix, defines a g-dimensional lattice[3] in $\mathbb{C}^g$. Recall that a *principally polarized abelian variety* is a pair (X, Θ) where X is $\mathbb{C}^g/\Lambda_\Omega$ and Θ is an ample divisor class which is the fundamental class of the zero locus of the θ-function defined on $\mathbb{C}^g$:

$$\theta_\Omega(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp \pi\sqrt{-1}(\mathbf{n}^T \cdot \Omega \cdot \mathbf{n} + 2\mathbf{n}^T \cdot \mathbf{z}). \tag{8}$$

[3]That is, a discrete abelian subgroup Λ of $\mathbb{C}^g$ such that $\dim_{\mathbb{R}}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) = 2g$.

The theta function as defined by (8) is not invariant by the action of the lattice Λ_Ω on $\mathbb{C}^g$. However the equality:

$$\theta_\Omega(\mathbf{z}+\mathbf{n}+\Omega\cdot\mathbf{m}) = \exp 2\pi\sqrt{-1}\left(-\frac{1}{2}\mathbf{m}^T\cdot\Omega\cdot\mathbf{m}-\mathbf{m}^T\cdot\mathbf{z}\right)\theta_\Omega(\mathbf{z}).$$

shows that its zero locus is well defined modulo Λ_Ω. If $g=1$ we have

$$\theta_\tau(\mathbf{z}) = \sum_{n\in\mathbb{Z}} \exp \pi\sqrt{-1}(n^2\tau+2nz) \tag{9}$$

where $\Im(\tau)>0$. The theta function (9) is related to the Weierstrass $\wp$-function associated to the lattice $\Lambda_\tau := (1,\tau)$ according to [75, pp. 155–156]:

$$\wp_{\Lambda_\tau}(z) = -\frac{d^2}{dz^2}\log\theta_\tau\left(z+\frac{1}{2}(1+\tau)\right)+k$$

and this last remark suggests that solutions to (1) can be sought in terms of θ functions on an elliptic curve rather than in terms of the Weierstrass $\wp$-function.

This last remark is perhaps the origin of the so-called *Hirota trick*. It consists to look for solutions to the KdV in the form

$$f = \frac{\partial^2}{\partial x^2}\log v(x,t), \tag{10}$$

where $v=v(x,t)$ is a sufficiently regular function with no zero in the considered domain. Substitution of (10) into (1) gives:

$$\begin{aligned}
0 &= 4\frac{\partial^3\log(v)}{\partial x^2\partial t} - 12\frac{\partial^2\log(v)}{\partial x^2}\cdot\frac{\partial^3\log(v)}{\partial x^3} - \frac{\partial^5\log(v)}{\partial x^5} = \\
&= \frac{\partial}{\partial x}\left(4\frac{\partial}{\partial t}\left(\frac{\partial\log(v)}{\partial x}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{\partial\log v}{\partial x}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{\partial\log v}{\partial x}\right)\right) \\
&= \frac{\partial}{\partial x}\left(4\frac{\partial}{\partial t}\left(\frac{v_x}{v}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{v_x}{v}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{v_x}{v}\right)\right),
\end{aligned}$$

from which

$$4\frac{\partial}{\partial t}\left(\frac{v_x}{v}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{v_x}{v}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{v_x}{v}\right) - \gamma = 0, \tag{11}$$

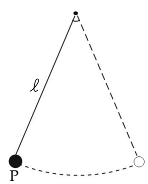
where $\gamma = \gamma(t)$ is an arbitary function which does not depend on x. A further expansion of (11) gives:

$$\frac{\gamma v^2 + 4v_x v_t + 3v_{xx}^2 - 4v_t v_{xxx} - 4v v_{xt} + v v_{xxxx}}{v^2} = 0.$$

Clearing the denominators, one finally obtains the *Hirota bilinear form of the KdV* equation:

$$\gamma v^2 + 4v_x v_t + 3v_{xx}^2 - 4v_t v_{xxx} - 4v v_{xt} + v v_{xxxx} = 0. \tag{12}$$

0.1.3 Example. There are many classical and elementary problems of mathematical physics whose solution invoke the use of elliptic functions. An example is provided by the equations of the motion of a rigid body with a fixed point, due to Euler, which, for sake of exercise, have been explicitly solved in detail in [26]. The most classical is perhaps that of the *simple pendulum*.



Its linearized dynamics (in a neighborhood of the stable equilibrium point) is described by the classical linear ODE of the *harmonic oscillator*

$$\ddot{\theta} + \omega^2 \theta = 0 \tag{13}$$

which possesses the *prime integral* [4]: $\dot{\theta}^2 + \omega^2\theta^2 = 2E$, where E is a constant called the *total energy* [5]. The non-linearized dynamics of the simple pendulum is

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{14}$$

It possesses a prime integral as well,

$$\dot{\theta}^2 - 2\omega^2 \cos\theta = 2E, \tag{15}$$

[4] A function which is constant on the integral curves of a vector field.

[5] More precisely it is the total energy divided by the mass of the material point P and by the length ℓ of the supposed massless string.

obtained by multiplying by $\dot\theta$ both sides of (14) and integrating. Reparameterizing the cosine by means of $u = \tan(\theta/2)$ one gets

$$\cos\theta = \frac{1-u^2}{1+u^2} \qquad \text{and} \qquad \dot\theta = \frac{2\dot u}{1+u^2}. \tag{16}$$

Substituting into (15) and simplifying:

$$2\dot u^2 - \omega^2(1-u^2)(1+u^2) - E(1+u^2)^2 = 0, \tag{17}$$

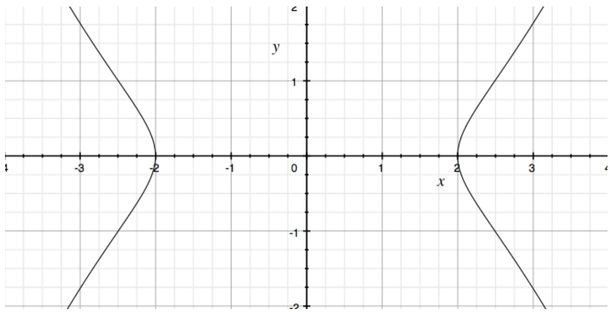
which we shall rewrite in the form

$$2\dot u^2 = \beta(u^2+1)(u^2-\alpha^2) \tag{18}$$

where, for generic values of E and ω^2, we have put $\beta := E - \omega^2$ and denoted by α a square root of

$$\frac{E+\omega^2}{\omega^2 - E}.$$

Putting $v =: \dot u$, the equation $2v^2 - \beta(u^2+1)(u^2-\alpha^2) = 0$ describes (the affine part of a) double covering of the projective line with 4 ramification points ($\pm\sqrt{-1}$ and $\pm\alpha$), i.e. it defines an *elliptic curve*.



The graph in the (u, v) plane of the real part of the curve
$$2v^2 - \beta(u^2+1)(u^2-\alpha^2) = 0$$
$$(\beta = 1/6, \alpha = 2)$$

The solution of (17) can be obtained via a standard sequence of steps. First change the variable u, by acting on $\mathbb{P}^1$ via some element of $Sl_2(\mathbb{C})$, to send one of the ramification points to ∞ and another one to 0. A further change of variable will be performed to kill the degree 2 term, in order to obtain the elliptic curve in the *Weierstrass canonical form* (5). One finally reaches the sought canonical form via the Möbius transformation

$$X := \frac{\beta}{24}\frac{(5\alpha^2-1)u+5\alpha+\alpha^3}{u-\alpha} \tag{19}$$

which maps $\alpha \mapsto \infty$. Inversion of 19 gives

$$u = \alpha \cdot \frac{24X+5\beta+\alpha^2\beta}{24X-\beta-5\alpha^2\beta}.$$

which substituted into (17) gives

$$\dot{X}^2 = 4X^3 - g_2X^2 - g_3 \tag{20}$$

after clearing the denominator in any neighborhood of t where it does not vanishes, having set

$$g_2 := \frac{\beta^2}{48}(1-14\alpha^2+\alpha^4) \tag{21}$$

and

$$g_3 := \frac{\beta^3(\alpha^2-1)}{1728}\left(\alpha^4+34\alpha^2+1\right). \tag{22}$$

Thus, in the variable X, the solution to (18) is given by

$$X = \wp_{\Lambda_g}(z(t)),$$

where Λ_g is any lattice such that $(\wp_{\Lambda_g}(t), \wp'_{\Lambda_g}(t))$ gives the parameterization of (20) with g_2 and g_3 given respectively by (21) and (22), i.e.

$$\theta(t) = 2\arctan\left(\alpha \cdot \frac{24\wp_{\Lambda_g}(z(t))+5\beta+\alpha^2\beta}{24\wp_{\Lambda_g}(z(t))-\beta-5\alpha^2\beta}\right)$$

is the explicit solution of (14), which has so been linearized. The parameter $z = z(t)$ is the equation of the linear flow, which has unitary speed, i.e. $z(t) = z + a$, where a is a constant.

0.2 The KP equation

0.2.1 In the Seventies, Kadomtsev and Petviashvili [50] generalized the KdV equation motivated by application to plasma physics. Their turned it into a seemingly more complicated one:

$$3\frac{\partial^2 f}{\partial^2 y} - \frac{\partial}{\partial x}\left(4\frac{\partial f}{\partial t} - 12f\cdot\frac{\partial f}{\partial x} - \frac{\partial^3 f}{\partial^3 x}\right) = 0. \tag{23}$$

Solutions are now functions in three variables (x, y, t). It is apparent that each solution of the KdV equation (constant along y) is a solution of (23). If Kadomtsev and Petviashvili were algebraic geometers, their generalization of (1) could seem rather an attempt to further promote the many algebraic beauties behind it. Once (23) is turned into a bilinear form, using the same Hirota trick (10) employed for the KdV equation, it is very easy to find families of exact solutions, if one looks for them in the form

$$f := \frac{\partial^2}{\partial x^2}(\log w)$$

where $w := w(x, y, t)$. Then

$$0 = 3\frac{\partial^2}{\partial^2 x}\frac{\partial^2 \log w}{\partial^2 y} - \frac{\partial^2}{\partial x^2}\left(4\frac{\partial}{\partial t}\left(\frac{w_x}{w}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{w_x}{w}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{w_x}{w}\right)\right) =$$

$$= \frac{\partial^2 \log w}{\partial^2 y} - \left(4\frac{\partial}{\partial t}\left(\frac{w_x}{w}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{w_x}{w}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{w_x}{w}\right)\right),$$

from which:

$$3\frac{\partial^2 \log w}{\partial^2 y} - 4\frac{\partial}{\partial t}\left(\frac{w_x}{w}\right) - 6\left(\frac{\partial}{\partial x}\left(\frac{w_x}{w}\right)\right)^2 - \frac{\partial^3}{\partial x^3}\left(\frac{w_x}{w}\right) + \gamma_1 x + \gamma_2 = 0.$$

Here $\gamma_i := \gamma_i(y, t)$ $(i = 1, 2)$ is an arbitrary function that independs on the variable x. Thus:

$$\frac{1}{w^2}\Big(-3w_y^2 + 4w_t w_x + 3w_{xx}^2 - 4w_x w_{xxx} + 3ww_{yy} - 4ww_{xt} +$$

$$+ \quad ww_{xxxx}\Big) + \gamma_1 x + \gamma_2 = 0,$$

that in turn can be rephrased in the form

$$ww_{xxxx} + 3w_{yy}w - 4ww_{xt} + 3w_{xx}^2 - 3w_y^2 + 4w_t w_x - 4w_x w_{xxx} +$$
$$- \quad (\gamma_1 x + \gamma_2)w^2 = 0. \tag{24}$$

Equation (24) is the most general form of the *Hirota bilinear form* of the KP equation.

0.2.2 We want to limit ourselves to the case $\gamma_1 = \gamma_2 = 0$ which, due to its exceptional importance, we write once again:

$$ww_{xxxx} + 3w_{yy}w - 4ww_{xt} + 3w_{xx}^2 - 3w_y^2 + 4w_t w_x - 4w_x w_{xxx} = 0. \tag{25}$$

An important peculiarity of (25) is that the sum of the coefficients is zero, a fact enabling to find in an easy way many solutions. Imitating Kasman [51] we give the following

0.2.3 Definition. *A function* $w := w(x, y, t)$ *is* nicely weighted *if* $w_{xx} = w_y$ *and* $w_{xxx} = w_t$.

0.2.4 Proposition. *Any nicely weighted function is a solution of* (25).

Proof. If w is nicely weighted, the following equalities

$$w_{xxxx} = \frac{\partial^2 w_{xx}}{\partial x^2} = \frac{\partial^2 w_y}{\partial x^2} = \frac{\partial w_{xx}}{\partial y} = \frac{\partial w_y}{\partial y} = w_{yy}$$

hold. Similarly one has $w_{xx}^2 = w_y^2$ and the nicely weighted function is a solution of (25). ∎

It is very easy to produce nicely weighted functions. The most obvious is $w = \exp(x\lambda + y\lambda^2 + t\lambda^3)$. In fact

$$\frac{\partial^n w}{\partial x^n} = \lambda^n w, \qquad \frac{\partial^n w}{\partial y^n} = \lambda^{2n} w, \qquad \frac{\partial^n w}{\partial t^n} = \lambda^{3n} w.$$

In this case, however $f = (\log w)_{xx} = (x\lambda + y\lambda^2 + t\lambda^3)_{xx} = 0$, and then not that interesting, as trivial solution of the original KP equation. To produce plenty non–trivial solutions one may however consider the family of functions g_i defined by

$$1 + \sum_{i\geq 1} g_i(x, y, t)\lambda^i = \exp(x\lambda + y\lambda^2 + t\lambda^3).$$

Notice that

$$\sum_{i\geq 0}\frac{\partial g_i}{\partial x}\lambda^i = \lambda\exp(x\lambda + y\lambda^2 + t\lambda^3) = \sum_{i\geq 0} g_i\lambda^{i+1}$$

from which $\frac{\partial g_i}{\partial x} = g_{i-1}$. Similarly

$$\frac{\partial g_i}{\partial y} = g_{i-2} = \frac{d^2 g_i}{\partial x^2} \quad \text{and} \quad \frac{\partial g_i}{\partial t} = g_{i-3} = \frac{\partial^3 g_i}{\partial x^3}.$$

In other words, each g_i is nicely weighted and as such is a solution of the Hirota equation (25). For instance

$$g_3 = \frac{x^3}{3!} + xy + t$$

is nicely weighted as one can easily check. It follows that

$$f := \frac{\partial^2}{\partial x^2}\log\left(\frac{x^3}{3!} + xy + t\right) = \frac{3(12tx - x^4 - 12y^2)}{(6t + 6x + x^3)^2}$$

is a solution to the KP equation (23), as the brave reader can patiently check.

0.3 Vertex Operators

0.3.1 Let $B := \mathbb{Q}[x_1, x_2, \ldots]$ be the algebra of polynomials in the infinitely many indeterminates $(x_1, x_2, \ldots)$ with rational coefficients. For each $i > 0$, the multiplication by x_i and the partial derivative

$$\partial_i := \frac{\partial}{\partial x_i}$$

are $\mathbb{Q}$-endomorphisms of B. They form indeed a Lie algebra, since

$$[\partial_i, x_j] = \delta_{ij}$$

Let

$$B[[z^{-1}, z]] := \{\sum_{i\in\mathbb{Z}} a_i z^i \mid a_i \in B\}$$

be the *formal Laurent series* with B-coefficients in a further indeterminate z. The following maps

$$\Gamma(z) := \exp\left(\sum_{i\geq 1} x_i z^i\right) \exp\left(-\sum_{i\geq 1} \frac{1}{iz^i}\frac{\partial}{\partial x_i}\right) : B \to B[[z^{-1}, z]] \tag{26}$$

and

$$\Gamma^\vee(z) := \exp\left(-\sum_{i\geq 1} x_i z^i\right) \exp\left(\sum_{i\geq 1} \frac{1}{iz^i}\frac{\partial}{\partial x_i}\right) : B \to B[[z^{-1}, z]] \tag{27}$$

are known in the literature as *vertex operators* (see [47]) [6]. It is easily seen that $\Gamma(z)$ and $\Gamma^\vee(z)$ map B to $B((z)) := B[z^{-1}, z]]$. Each element of B is in fact a polynomial in finitely many variables $x_{i_1}, \ldots, x_{i_r}$ and then it suffices to check the statement on each monomial. To this purpose we first observe that

$$\Gamma(z)x_j = \exp\left(\sum_{i\geq 1} x_i z^i\right)\left(x_j - \frac{1}{jz^j}\right) \in B((z))$$

and that

$$\Gamma^\vee(z)x_j = \exp\left(\sum_{i\geq 1} -x_i z^i\right)\left(x_j + \frac{1}{jz^j}\right) \in B((z)).$$

Secondly, we observe that

$$G(z) := \exp\left(-\sum_{i\geq 1} \frac{1}{iz^i}\frac{\partial}{\partial x_i}\right) \quad \text{and} \quad G^\vee(z) := \exp\left(\sum_{i\geq 1} \frac{1}{iz^i}\frac{\partial}{\partial x_i}\right)$$

are ring homomorphisms $B \to B[z^{-1}]$, because both $G(z)$ and $G^\vee(z)$ are the exponential of a derivation (Cf. Lemma 5.5.1). Thus $\Gamma(z)$ and $\Gamma^\vee(z)$ map indeed any polynomial to $B((z))$.

[6]Our present definition of $\Gamma^\vee(z)$ is $1/z$ times that used e.g. in [47], but in Chapter 5 the standard conventions will be restored.

0.3.2 Example. The image of $x_1^2+x_1x_2+x_2^2$ is

$$\Gamma(z)(x_1^2+x_1x_2+x_2^2)=$$

$$\begin{aligned}
&= \exp(\sum_{i\geq 1} x_i z^i)\left[\left(x_1-\frac{1}{z}\right)^2+\left(x_1-\frac{1}{z}\right)\left(x_2-\frac{1}{2z}\right)+\right.\\
&+ \left.\left(x_2-\frac{1}{2z}\right)^2\right]\\
&= \exp(\sum_{i\geq 1} x_i z^i)\left(x_1^2+x_1x_2+x_2^2-\frac{5}{2}\frac{x_1}{z}-2\frac{x_2}{z}+\frac{7}{4z^2}\right)\\
&= \exp(\sum_{i\geq 1} x_i z^i)\left(x_1^2+x_1x_2+x_2^2-\frac{1}{z}\left(\frac{5}{2}x_1-2x_2\right)+\frac{7}{4z^2}\right).
\end{aligned}$$

0.3.3 Historical Remark. It is often convenient to allow not just polynomials, but also elements of the completion $\widehat{B}:=\mathbb{Q}[[x_1,x_2,\ldots]]$. In fact a differential operator on B can be identified with linear map $B\to\widehat{B}$ (Cf. [48, p.]). Examples of such operators are the multiplication by x_n as well as $(T_af)(x_1,x_2,\ldots)=f(x_1+a_1,x_2+a_2,\ldots)$, which by Taylor's formula is nothing but

$$(T_af)(x_1,x_2,\ldots)=\left(\exp\sum_{i\geq 1} a_i\frac{\partial}{\partial x_i}\right)f.$$

Vertex operators of the form (26) and (27) arise naturally in the representation theory of the affine *Kac-Moody algebras*.

The easiest example of Kac-Moody algebra is the Weyl affinization $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of the traceless 2×2 complex matrices. Its support is the vector space $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t^{-1},t]\oplus\mathbb{C}\mathbf{k}$ and the Lie bracket are defined as

$$[a\otimes t^m, b\otimes t^n]=[a,b]\otimes t^{m+n}+m\cdot\mathrm{tr}(ab)\delta_{m,-n}\mathbf{k}$$

for all $a,b\in\mathfrak{sl}_2(\mathbb{C})$ and $m,n\in\mathbb{Z}$, while $\mathbf{k}$ is a central element, i.e. it commutes with all the elements of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$. The same definition applied to the trivial one-dimensional Lie algebra $\mathbb{C}$ gives rise to the oscillator Heisenberg Algebra $\mathcal{H} := \mathbb{C}[t^{-1}, t]$ with commutation relations $[t^m, t^n] = m\delta_{m,-n}\mathbf{k}$. In 1978 Lepowsky and Wilson [61] found a concrete realization of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ by representing it on the space $\mathbb{C}[x_{\frac{1}{2}}, x_{\frac{3}{2}}, \ldots]$. Let Y_j be the coefficient of z^j ($j \in \frac{1}{2}\mathbb{Z}$) in the expansion of of the *vertex operator*

$$Y := \exp\left(\sum_{n\in\frac{1}{2}\mathbb{N}^*} \frac{x_n}{n} z^n\right) \exp\left(-2 \sum_{n\in\frac{1}{2}\mathbb{N}^*} \frac{1}{z^n} \frac{\partial}{\partial x_n}\right).$$

The main theorem in Lepowsky-Wilson [61] claims that

$$\mathbb{C}\cdot 1 \oplus \bigoplus_{n\in\frac{1}{2}\mathbb{N}^*} \mathbb{C}\cdot x_n \oplus \bigoplus_{n\in\frac{1}{2}\mathbb{N}^*} \mathbb{C}\cdot \frac{\partial}{\partial x_n} \oplus \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathbb{C}Y_j$$

is a Lie algebra of operators on $\mathbb{C}[y_j]_{j\in\frac{1}{2}\mathbb{N}^*}$, with respect to the usual commutator, which is isomorphic to the affine Kac-Moody algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$. A few years later such a result was generalized in the milestone paper [48], where the so-called *basic representation* of an "Euclidean Lie algebra" $\mathfrak{g}$ is realized as a ring of operators on $B \otimes \mathbb{C} := \mathbb{C}[x_1, x_2, \ldots]$ spanned by the identity, the *annihilation* and *creation* operators (namely multiplication by x_n and differentiation with respect to x_n, $n \in \mathbb{N}^*$) and the homogeneous components of operators of the form

$$\exp\left(\sum_{j\geq 1} \mu_{ij} x_j\right) \exp\left(\sum_{j\geq 0} \nu_{ij} \frac{\partial}{\partial x_j}\right).$$

In their article the authors recall that a differential operator on $B_{\mathbb{C}} := B \otimes_{\mathbb{Q}} \mathbb{C}$ can be seen as a linear map $D : B_{\mathbb{C}} \mapsto \widehat{B_{\mathbb{C}}}$ and in Corollary 3.1 they show that if $[x_i, A] = a_i A$ and $[\partial/\partial x_i, A] = b_i A$ then $A = C\exp(\sum_i a_i x_i)\exp(-\sum_{i\geq 1} b_i(\partial/\partial x_i))$, where C is a constant. Taking $A = T_a$ with

$$a = \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3} \cdots\right)$$

a simple exercise shows that

$$[x_j, T_a] = -\frac{1}{jz^j}T_a.$$

Moreover

$$\left[\frac{\partial}{\partial x_j}, T_a\right] = z^j T_a$$

which yields, for instance, formula (26). The strategy employed in Chapter 4 and 5 to determine expressions (26) and (27) is different and is based on Schubert Calculus as formulated e.g. in [22] or [58, 59] via derivation on an exterior algebra. It amounts to compute a "finite" approximation of $\Gamma(z)$ and $\Gamma^\vee(z)$ on a ring B_r in r-indeterminates, that can be seen as the ratio of two characteristic polynomials. See Section 2 and 4.

0.4 The KP Hierarchy via Vertex Operators

Although the use of the same symbol for different meanings should be frowned upon, it belongs to the tradition to denote by τ both a point of the Poincaré half–plane as well as a solution to the *KP-hierarchy*. We hope that respecting the tradition will cause no confusion.

0.4.1 Definition. *A* "tau" *function for the* KP Hierarchy *is an element τ of B solving the equation*

$$\mathrm{Res}_{z=0}\Gamma^\vee(z)\tau \otimes \Gamma(z)\tau = 0.$$

where $\Gamma(z)$ and $\Gamma^\vee(z)$ are as in (26) and (27) respectively.

0.4.2 Here is another reason why (0.4.1) is geometrically significant. Let V be a $\mathbb{Q}$-vector space of (countable) infinite dimension with basis $(b_i)_{i\in\mathbb{Z}}$. All elements of V are finite linear combinations of $(b_i)_{i\in\mathbb{Z}}$. By the *vacuum vector* of *total charge* i one means the expression

$$\Phi_i := b_i \wedge b_{i-1} \wedge b_{i-2} \ldots, \qquad i \in \mathbb{Z}.$$

If $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition of length at most r, let us denote by $\Phi_{i+\boldsymbol{\lambda}}$ the "excitation" of Φ_i via $\boldsymbol{\lambda}$:

$$\Phi_{i+\boldsymbol{\lambda}} = b_{i+\lambda_1} \wedge b_{i-1+\lambda_1} \wedge \ldots \wedge b_{i-r+1+\lambda_r} \wedge \Phi_{i-r}$$

and let

$$F_i = \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}} \mathbb{Q}\cdot\Phi_{i+\boldsymbol{\lambda}},$$

where $\mathcal{P}$ is the set of all of the partitions (Cf. Section 3.1.1). Following [47], we call *fermionic Fock space* of total charge i the vector space F_i. The space F_i can be seen as an infinite exterior power of an infinite dimensional vector space, whose construction we shall sketch in Section 4.6. Since $F_i \cap F_j = 0$ if $i \neq j$, it is customary to set $\bigwedge^{\infty/2} V := \bigoplus_{i\in\mathbb{Z}} F_i$. For practical purposes it works as an ordinary exterior power of a vector space of sufficiently high dimension. In particular, each monomial $\Phi_{i+\boldsymbol{\lambda}}$ changes sign whenever two vectors in the exterior monomial are exchanged.

Denote by $Gl_\infty(\mathbb{Q})$ the group of all automorphisms $\mathcal{M}$ of V such that $\mathcal{M}b_j = b_j$ for all but finitely many j. Such a group (see [47]) can be identified with the group of all the invertible matrices $\mathcal{M} := (a_{ij})_{i,j\in\mathbb{Z}}$ such that $a_{ij} - \delta_{ij} = 0$ for all but finitely many entries. Hence one may think of $Gl_\infty(\mathbb{Q})$ as an invertible matrix $(a_{ij}) \in Gl_n(\mathbb{Q})$ embedded in an infinite array, where all the off–diagonal entries are 0 but finitely many and all the elements along the diagonal $i = j$ are 1 but finitely many.

$$\begin{array}{ccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\ldots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & a_{i,i} & a_{i,i+1} & a_{i,i+2} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & a_{i+2,i} & a_{i+2,i+1} & a_{i+2,i+2} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

– a matrix $(a_{ij}) \in Gl_3(\mathbb{Q})$ embedded in $Gl_\infty(\mathbb{Q})$ –

0.4.3 It is well known that the ring $B = \mathbb{Q}[x_1, x_2, \ldots]$ possesses a basis parameterized by the set of all of the partitions (see e.g. Chapter 3). In fact each element of B is a finite linear combination of certain *Schur Polynomials*:

$$B := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}} \mathbb{Q}\cdot S_{\boldsymbol{\lambda}}(\mathbf{x}),$$

where $\mathbf{x} := (x_1, x_2, \ldots)$ and, by definition,

$$\begin{cases} \sum_{j\geq 0} S_j(\mathbf{x})z^j & = & \exp(\sum_{j\geq 1} x_j z^j), \\ \\ S_{\boldsymbol{\lambda}}(\mathbf{x}) & = & \det(S_{\lambda_j - j + i}(\mathbf{x})). \end{cases}$$

What is important is that Schur polynomials are parametrized by partitions. This gives rise to an obvious vector space isomorphism $\varphi_i : F_i \to B$, by taking the $\mathbb{Q}$-linear extension of the set-theoretical bijection $\Phi_{i+\boldsymbol{\lambda}} \mapsto S_{\boldsymbol{\lambda}}(\mathbf{x})$. In particular $\Phi_i \mapsto 1$. Following [47], the isomorphism φ_i will be said *boson-fermion correspondence*: it associates to each element of F_i a polynomial living in B. The group $Gl_\infty(\mathbb{Q})$ acts on F_i via the *determinant representation* introduced by Kac and Peterson [46]: for $\mathcal{M} \in Gl_\infty(\mathbb{C})$, let

$$\det(\mathcal{M})\Phi_{i+\boldsymbol{\lambda}} = \mathcal{M}b_{i+\lambda_1} \wedge \ldots \wedge \mathcal{M}b_{i-r+1+\lambda_r} \wedge \mathcal{M}b_{i-r} \wedge \mathcal{M}b_{i-r-1} \ldots$$

Via the boson-fermion correspondence, the group $Gl_\infty(\mathbb{Q})$ acts on B as well, via $\mathcal{M}\cdot S_{\boldsymbol{\lambda}}(\mathbf{x}) = \varphi_i(\det(\mathcal{M})\Phi_{i+\boldsymbol{\lambda}})$. A linear combination $\sum_{\boldsymbol{\lambda}\in\mathcal{P}} a_{\boldsymbol{\lambda}}\Phi_{i+\boldsymbol{\lambda}} \in F_i$, where $a_{\boldsymbol{\lambda}} = 0$ for all but finitely many $\boldsymbol{\lambda}$, is *decomposable* if there exist finitely many vectors $v_1, \ldots, v_r \in V$ such that

$$\begin{aligned} \sum_{\boldsymbol{\lambda}\in\mathcal{P}} a_{\boldsymbol{\lambda}}\Phi_{i+\boldsymbol{\lambda}} &= v_1 \wedge v_2 \wedge \ldots v_r \wedge b_{i-r} \wedge b_{i-r-1} \wedge b_{i-r-2} \wedge \ldots \\ &= v_1 \wedge v_2 \wedge \ldots v_r \wedge \Phi_{i-r} = \det(\mathcal{M})\Phi_i, \end{aligned}$$

where $\mathcal{M}$ is the unique element of $Gl_\infty(\mathbb{Q})$ such that $\mathcal{M}b_{i-j+1+\lambda_j} = v_j$ for $1 \leq j \leq r$ and $\mathcal{M}b_j = b_j$ otherwise. The locus of polynomials of B that do correspond to decomposable tensors of F_i (for all $i \in \mathbb{Z}$) are then in the $Gl_\infty(\mathbb{Q})$-orbit Ω of $1 \in B$.

From now on, to fix the ideas, choose $i = 0$ and look for equations describing Ω in B. An arbitrary element of $\xi \in F_0$ is a finite linear combination:

$$\xi = \sum_{i=1}^{n} a_i \tilde{\eta}_i$$

where $a_i \in \mathbb{Q}$ and $\tilde{\eta}_i \in F_0$. By definition of F_0, there is a big enough integer r and $\eta_1, \eta_2, \ldots, \eta_n \in \bigwedge^r V$ such that for all $1 \leq i \leq n$

$$\tilde{\eta}_i = \eta_i \wedge \Phi_{-r} := \eta_i \wedge b_{-k} \wedge b_{-k-1} \wedge b_{-k-2} \wedge \ldots$$

Thus ξ is decomposable if and only if $\sum_{i=1}^{n} a_i \eta_i$ is a decomposable vector in $\bigwedge^r V$. One then invokes Theorem 4.1.4 guaranteeing that $\eta \in \bigwedge^r V$ is decomposable if and only if

$$\sum_{i \in \mathbb{Z}} (b_i \wedge \eta) \otimes (\beta_i \lrcorner \eta) = 0, \tag{28}$$

where $\beta_i(b_j) = \delta_{ij}$ and $\beta_i \lrcorner \eta$ is the *contraction* of η against $\beta_i \in V^\vee$. The sum (28) makes sense: it is obviously finite because $\beta_i \lrcorner \eta = 0$ for all but finitely many i. Define the formal Laurent series

$$\mathbf{b}(z) = \sum_{i \in \mathbb{Z}} b_i z^i \qquad \text{and} \qquad \boldsymbol{\beta}(z) = \sum_{i \in \mathbb{Z}} \beta_i z^{-i}.$$

A simple inspection shows that (28) holds if and only if

$$\mathrm{Res}_{z=0} \frac{1}{z} (\mathbf{b}(z) \wedge \eta) \otimes (\boldsymbol{\beta}(z) \lrcorner \eta) = 0. \tag{29}$$

Using the boson-fermion correspondence $\varphi_0 : F_0 \to B$, in both Chapter 4 and 5, we will show that equation (29) translates into

$$\mathrm{Res}_{z=0} \Gamma(z) \varphi_0(\eta) \otimes \Gamma^\vee(z) \varphi_0(\eta) = 0. \tag{30}$$

It follows that a “tau” function τ corresponds to a decomposable tensor in F_0 if and only if (30) holds and (30) can be seen as the set of Plücker equation defining the *Grassmann cone* of F_0 in B.

0.4.4 Equation (30) takes places in the tensor product $B\otimes B\cong\mathbb{Q}[\mathbf{x}',\mathbf{x}'']$ where $\mathbf{x}'=(x_1',x_2',\ldots)$ and $\mathbf{x}''=(x_1'',x_2'',\ldots)$ and so $\tau\in B$ is a "tau" function if and only if the residue of

$$\exp\left(\sum_{i\geq 1}(x_i'-x_i'')z^i\right)\exp\left(-\sum_{i\geq 1}\frac{1}{iz}\left(\frac{\partial}{\partial x_i'}-\frac{\partial}{\partial x_i''}\right)\right)\tau(\mathbf{x}')\tau(\mathbf{x}'') \tag{31}$$

at $z=0$ vanishes.

0.4.5 At this point, the best for the reader is to see the details in either [47, pp. 72–75] or the introduction of [49] or [5, Section 4]. We summarize here the arguments explained in those references just for sake of self-containedness. Perform a change of variable, putting

$$x_i'=x_i-y_i\qquad\text{and}\qquad x_i''=x_i+y_i$$

to write (31) in the form

$$\exp\left(\sum_{i\geq 1}(-2y_iz^i)z^i\right)\exp\left(-\sum_{i\geq 1}\frac{1}{iz^i}\left(\frac{\partial}{\partial y_i}\right)\right)\tau(\mathbf{x}-\mathbf{y})\tau(\mathbf{x}+\mathbf{y}),$$

where $\mathbf{y}:=(y_1,y_2,\ldots)$ i.e., using the definition of the polynomial expression S_j:

$$\left(\sum_{i\geq 1}S_i(-2\mathbf{y})z^i\cdot\sum_{i\geq 1}S_j(\tilde{\partial}_{\mathbf{y}})z^{-j}\right)\tau(\mathbf{x}-\mathbf{y})\tau(\mathbf{x}+\mathbf{y}) \tag{32}$$

where $\tilde{\partial}_{\mathbf{y}}=\left(\frac{\partial}{\partial y_1},\frac{1}{2}\frac{\partial}{\partial y_2},\frac{1}{3}\frac{\partial}{\partial y_3},\ldots\right)$. The residue at $z=0$ of (32) is

$$\sum_{i\geq 1}S_i(-2\mathbf{y})S_{i+1}(\tilde{\partial}_{\mathbf{y}})\tau(\mathbf{x}-\mathbf{y})\tau(\mathbf{x}+\mathbf{y})$$

equation which can be written, equivalently,

$$\sum_{i\geq 1}S_i(-2\mathbf{y})S_{i+1}(\tilde{\partial}_{\boldsymbol{\zeta}})\tau(\mathbf{x}-\mathbf{y}-\boldsymbol{\zeta})\tau(\mathbf{x}-\mathbf{y}+\boldsymbol{\zeta})_{|\boldsymbol{\zeta}=0},$$

where $\boldsymbol{\zeta} := (\zeta_1, \zeta_2, \ldots)$ are auxiliary variables. Using the Taylor formula for polynomials, we can then conclude that τ corresponds to an element of the $Gl_\infty(\mathbb{Q})$-orbit of $1 \in B$ if and only if

$$\sum_{i\geq 1} S_i(-2\mathbf{y})S_{i+1}(\tilde{\partial}_{\boldsymbol{\zeta}}) \exp\left(\sum_{j\geq 1} y_j \frac{\partial}{\partial \zeta_j}\right) \tau(\mathbf{x}-\boldsymbol{\zeta})\tau(\mathbf{x}+\boldsymbol{\zeta})_{|\boldsymbol{\zeta}=0} = 0. \tag{33}$$

In particular, all the coefficients of the expressions $y_{i_1}^{j_1} y_{i_2}^{j_2} \ldots y_{i_k}^{j_k}$ in the expansion of (33) in formal power series of $\mathbf{y}$-monomials, must vanish. It is easy to check that the coefficients of y_1 and of y_2 are identically 0 (Cf. [47, pp. 72–74], while the vanishing of the coefficient of y_3 gives the equation

$$\begin{aligned} &\left(\frac{1}{3}\frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} - \frac{1}{4}\frac{\partial^2}{\partial \zeta_2^2} + \right. \\ - \quad & \left. \frac{1}{12}\frac{\partial^4}{\partial \zeta_1^4} - \frac{1}{2}\left(\frac{\partial}{\partial \zeta_4} + \frac{\partial^3}{\partial \zeta_1^2 \partial \zeta_2}\right)\right) \tau(\mathbf{x}+\boldsymbol{\zeta})\tau(\mathbf{x}-\boldsymbol{\zeta})_{|u=0} = 0 \end{aligned}$$

which is equivalent to

$$\left(\frac{\partial^4}{\partial \zeta_1^4} + 3\frac{\partial^2}{\partial \zeta_2^2} - 4\frac{\partial^2}{\partial \zeta_1 \partial \zeta_3}\right) \tau(\mathbf{x}+\boldsymbol{\zeta})\tau(\mathbf{x}-\boldsymbol{\zeta})_{|u=0} = 0, \tag{34}$$

because all the partial derivatives of odd order of the product $\tau(\mathbf{x}-\boldsymbol{\zeta})\tau(\mathbf{x}+\boldsymbol{\zeta})$, with respect to ζ_i, $i \geq 1$, vanish at $\boldsymbol{\zeta} = (0, 0, \ldots)$. We leave to the patient reader the task of expanding the first member of (34) to show that then $\tau(\mathbf{x})$ must satisfy the Hirota bilinear form (25) of the KP equation.

The main purpose of Chapter 4 will be to explain why the embryo of the KP Hierarchy is contained in the ideal which define any finite Grassmannian in its Plücker embedding.

Chapter 1

Linear Recurrent Sequences

1.1 Sequences in A-modules

1.1.1 Let $M^{\mathcal{S}}$ be the A-module of all the maps $\mathbf{m} : \mathcal{S} \to M$, where $\emptyset \neq \mathcal{S} \subseteq \mathbb{Z}$, i.e. the A-module of the M-valued sequences defined over $\mathcal{S}$. The image of $s \in \mathcal{S}$ is denoted by m_s. We shall be concerned mostly with the cases $\mathcal{S} = \mathbb{N}$ and $\mathcal{S} = \mathbb{Z}$. If $\mathcal{S} = \mathbb{N}$, the elements of $M^{\mathbb{N}}$ will be represented through the list of its images $(m_0, m_1, \ldots)$. Let

$$M[[t]] := \Big\{ \sum_{j \geq 0} m_j t^j \,|\, m_j \in M \Big\} \tag{1.1}$$

be the A-module of *formal power series* with coefficients in M, where t is any indeterminate over A. If $M = A$, then $A[[t]]$ is an A-algebra with respect to the usual product

$$\sum_{i \geq 0} a_i t^i \cdot \sum_{j \geq 0} a_j t^j = \sum_{k \geq 0} \left(\sum_{i=0}^{k} a_i a_{k-i} \right) t^k, \qquad a_i \in A. \tag{1.2}$$

The obvious generalization of (1.2) endows $M[[t]]$ with a structure of $A[[t]]$-module:

$$\sum_{i\geq 0} a_i t^i \cdot \sum_{j\geq 0} m_j t^j = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i m_{k-i}\right) t^k, \qquad (a_i, m_j)\in A\times M.$$

1.1.2 If $P \in A[X]$ is a polynomial of degree n, where X is another indeterminate over A, denote by $(-1)^i e_i(P)$ the coefficient of X^{n-i}, $0 \leq i \leq n$:

$$P = e_0(P)X^n - e_1(P)X^{n-1} + \ldots + (-1)^n e_n(P).$$

The polynomial P is *monic* if $e_0(P) = 1$.

1.1.3 Examples.

i) Let $P := x^4 - 3X^2 + \sqrt{2}X - 3 \in \mathbb{Z}[\sqrt{2}]$. Then $e_0(P) = 1$, $e_1(P) = 0$, $e_2(P) = -3$, $e_3(P) = -\sqrt{2}$ and $e_4(P) = -3$.

ii) If $P \in A[X]$ splits as the product of r distinct linear factors:

$$P = (X - x_1)\cdot \ldots \cdot (X - x_r)$$

then $e_1(P), e_2(P), \ldots, e_r(P)$ are precisely the r elementary symmetric polynomials in the roots $(x_1, \ldots, x_n)$ of P. This motivates the notation we have chosen to denote the coefficients of a polynomial.

1.1.4 Definition. *A sequence* $\mathbf{m} := (m_0, m_1, \ldots) \in M^{\mathbb{N}}$ *is a* linear recurrent sequence *(LRS) of order r if there exists a polynomial P of degree r such that the equality*

$$m_{r+j} - e_1(P)m_{r+j-1} + \ldots + (-1)^r e_r m_j = 0 \tag{1.3}$$

holds for all $j \geq 0$.

The polynomial P occurring in Definition 1.1.4 is said to be the *characteristic polynomial* of the linear recurrent sequence and (1.3) is the *Linear Recurrent Relation* enjoyed by the given LRS. Expression (1.3) is also said *linear difference equation*. All the linear recurrent sequences associated to the same polynomial form the module of solutions to a homogeneous *linear difference equation*.

1.1.5 Examples

i) Let $\alpha \in A$ be a root of $P := X^2 - e_1(P)X + e_2(P) \in A[X]$. Then

$$(1, \alpha, \alpha^2, \alpha^3, \ldots)$$

satisfies the LRR of degree 2 having P as characteristic polynomial.

ii) Let $\mathcal{M}$ be a square $n \times n$ matrix with coefficients in some commutative associative $\mathbb{Z}$-algebra. By abuse of notation denote by $(-1)^j e_j(\mathcal{M})$ the coefficient of X^{n-j} in the polynomial

$$P_{\mathcal{M}}(X) := \det(X \cdot \mathbb{1}_n - \mathcal{M}) \qquad (1.4)$$

of $\mathcal{M}$. According to *Cayley-Hamilton theorem* (see Theorem 2.4.8), the equality $P_{\mathcal{M}}(\mathcal{M}) = 0$ holds, i.e. $\mathcal{M}$ is a root of (1.4). Hence $(\mathbb{1}_n, \mathcal{M}, \mathcal{M}^2, \ldots)$ is a LRR of order n, with $P_{\mathcal{M}}(X)$ as *characteristic polynomial*.

iii) Let $M = A = \mathbb{Z}$ and $P_f(X) = X^2 - X - 1$. The unique LRS $(f_0, f_1, \ldots)$ having characteristic polynomial P_f and initial conditions $f_0 = f_1 = 1$ is the *Fibonacci sequence*[1]:

$$(1, 1, 2, 3, 5, \ldots).$$

The following equality holds:

$$\sum_{n \geq 0} f_n t^n = \frac{1}{1 - t - t^2}.$$

i.e. the n-th coefficient of the formal Taylor expansion of the last side is precisely f_n.

1.1.6 Let $M((t)) := M[t^{-1}, t]]$ be the A-module of the M-valued *formal Laurent series*, i.e.

$$M((t)) := \{ \sum_{j \geq -i} m_j t^j \mid m_j \in M, i \in \mathbb{N} \}.$$

[1] To people who have skipped the introduction, we recall that Leonardo Pisano, known as Fibonacci (Pisa,1170–1240), wrote the famous *Liber Abaci* in 1202 where the number zero appeared for the first time. Its name came from "zephyrus", which is a wind blowing from the west.

Each element of $M((t))$ has only finitely many non-zero coefficients of negative powers of t. A quick check shows that the kernel of the A-epimorphism $M((t)) \mapsto M[[t]]$ mapping $\sum_{j\geq -i} m_j t^j$ to its "holomorphic" part $\sum_{j\geq 0} m_j t^j$ is the submodule $t^{-1}M[t^{-1}]$. Let

$$\varrho_M : \frac{M((t))}{t^{-1}M[t^{-1}]} \to M[[t]] \tag{1.5}$$

be the induced isomorphism. Define

$$D^j\mathbf{m}(t) := \varrho_M\left(\frac{\mathbf{m}(t)}{t^j} + t^{-1}M[t^{-1}]\right), \qquad j \geq 0. \tag{1.6}$$

More concretely:

$$D^j \sum_{i\geq 0} m_i t^i = \sum_{i\geq 0} m_{i+j} t^i.$$

Clearly $D^j \in \mathrm{End}_A(M[[t]])$ is the j-th iteration of the composition of $D := D^1$ with itself. We set $D^0 = \mathbb{1}_M$, the identity endomorphism of M. Via the identification $M[[t]] \cong M^{\mathbb{N}}$, the endomorphism D^j is nothing but the shift operator $m_i \mapsto m_{i+j}$.

1.1.7 Example. The unique solution of the equation $D\mathbf{a}(t) = \mathbf{a}(t)$ with initial condition a_0 is

$$\mathbf{a}(t) = \frac{a_0}{1-t} = a_0(1 + t + t^2 + \ldots).$$

In other words $a_0/(1-t)$ is related to D as $\exp(a_0 t)$ is related to ∂_t, the derivative with respect to t. See Section 1.6 below.

1.2 Generic Polynomials

1.2.1 From now on $(e_1, \ldots, e_r)$ will denote a finite sequence of indeterminates over $\mathbb{Z}$. Let B_r be the polynomial ring in the indeterminates $e_1, \ldots, e_r$ with integral coefficients

$$B_r := \mathbb{Z}[e_1, \ldots, e_r].$$

Define the two polynomials:

$$E_r(t) := 1 - e_1 t + \ldots + (-1)^r e_r t^r \tag{1.7}$$

and the *generic monic polynomial of degree* r:

$$\mathtt{p}_r(X) := X^r - e_1X^{r-1} + \ldots + (-1)^r e_r \tag{1.8}$$

where t and X are two indeterminates over B_r.

In the rings $B_r((t)) := B_r[t^{-1}, t]]$ and $B_r((X)) := B_r[X^{-1}, X]]$ of *formal Laurent series* in t and X respectively, the polynomials $E_r(t)$ and $\mathtt{p}_r(X)$ are obviously related:

$$E_r(t) = t^r \mathtt{p}_r\left(\frac{1}{t}\right) \qquad \text{and} \qquad \mathtt{p}_r(X) := X^r E_r\left(\frac{1}{X}\right).$$

Let $H_r(t) := \sum_{n\in\mathbb{Z}} h_n t^n \in B_r((t))$ given by

$$H_r(t) := \frac{1}{E_r(t)} = 1 + \sum_{n\geq 1}(E_r(t) - 1)^n. \tag{1.9}$$

By definition $h_j = 0$ if $j < 0$ and $h_0 = 1$, while h_i is an explicit polynomial expression in $(e_1, \ldots, e_r)$ which is homogeneous of degree i, once each indeterminate e_i is given weight i. For example:

$$h_1 = e_1, \qquad h_2 = e_1^2 - e_2, \qquad h_3 = e_1^3 - 2e_1e_2 + e_3, \ldots$$

In general, h_n can be computed recursively using the equation $H_r(t)E_r(t) = 1$, equivalent to (1.9). Indeed $E_r(t)\sum_{n\geq 0} h_n t^n = 1$ if and only if $h_0 = 1$ and the coefficient of t^n, in the left hand side, vanishes for all $n \geq 1$, i.e.:

$$h_n - e_1h_{n-1} + \ldots + (-1)^n e_n h_0 = 0, \tag{1.10}$$

with the usual convention that $e_n = 0$ if $n \geq r$. Using such relations one can show (see e.g. [65]) that

$$h_n := \det(e_{j-i+1})_{0\leq i,j\leq n}.$$

1.2.2 The sequence H_r can be alternatively defined through the equality

$$\sum_{j\geq 0} \frac{h_j}{X^{r+j}} = \frac{1}{\mathtt{p}_r(X)},$$

from which

$$h_j = \mathrm{Res}_{X=0}\, \frac{X^{j-1}}{\mathtt{p}_r(X)},$$

where the *residue* of a formal Laurent series $\sum_{j\in\mathbb{Z}} a_j X^j$ is by definition a_{-1}. See also [58, 59].

1.2.3 Note that the ring B_r can be thought of as a quotient of a polynomial ring with infinitely many indeterminates, considering the unique $\mathbb{Z}$-module homomorphism

$$\pi : \mathbb{Z}[X_1, X_2, \ldots] \longrightarrow B_r,$$

mapping $X_i \mapsto h_i$. It is clearly an epimorphism, because any element of B_r is a polynomial in $e_1, \ldots, e_r$ and each e_i is a weighted homogeneous polynomial expression of degree i in $h_1, \ldots, h_i$. The quotient

$$\mathbb{Z}[h_1, h_2, \ldots,] := \frac{\mathbb{Z}[X_1, X_2 \ldots,]}{\ker(\pi)}$$

will be denoted by $\mathbb{Z}[H_r]$. So $B_r := \mathbb{Z}[e_1, \ldots, e_r]$ may be also seen as $\mathbb{Z}[H_r]$, where H_r denotes the sequence $(h_1, h_2, \ldots)$.

1.3 Generic Linear Recurrent Sequences

1.3.1 In this section M will be assumed to be a module over a B_r-algebra A, fixed once and for all. Define linear maps $\mathtt{U}_i : M[[t]] \to M$ via the equality:

$$\sum_{i\geq 0} \mathtt{U}_i(\mathbf{m}(t))t^i := E_r(t)\mathbf{m}(t), \qquad (\mathbf{m}(t) \in M[[t]]). \tag{1.11}$$

Comparing the coefficients of t^i on both sides of (1.11) we obtain $\mathtt{U}_0(\mathbf{m}(t)) = m_0$ and, for $i > 0$:

$$\mathtt{U}_i(\mathbf{m}(t)) = m_i + \sum_{j=1}^{r} (-1)^j e_j m_{i-j}, \tag{1.12}$$

with the convention that $e_j = 0$ if $j > r$. So, for example,

$$\mathtt{U}_0(\mathbf{m}(t)) = m_0, \quad \mathtt{U}_1(\mathbf{m}(t)) = m_1 - e_1 m_0,$$

$\mathtt{U}_2(\mathbf{m}(t))=m_2-e_1m_1+e_2m_0,\ \mathtt{U}_3(\mathbf{m}(t))=m_3-e_1m_2+e_2m_1-e_3m_0,\ldots$

1.3.2 The fundamental sequence. In the following, a special role will be played by the sequence $(u_i)_{i\in\mathbb{Z}}$ of elements of $B_r[[t]]$ defined by

$$u_i := D^iH_r(t) \qquad \text{and} \qquad u_{-i} := t^iH_r(t), \tag{1.13}$$

for all $i \geq 0$. In particular $u_0 = H_r(t)$. By the very definition of the endomorphism D of $A[[t]]$ (Cf. (1.6) for $M = A$), it follows that for all $(i,j) \in \mathbb{N}\times\mathbb{Z}$

$$D^iu_j = u_{i+j},$$

an equality whose quick check is left to the reader.

1.3.3 Proposition. *Each* $\mathbf{m}(t) \in M[[t]]$ *admits the unique expansion*

$$\mathbf{m}(t) = \sum_{j\geq 0}\mathtt{U}_j(\mathbf{m}(t))u_{-j}. \tag{1.14}$$

In particular, if $M = A$*,*

$$\mathtt{U}_j(u_{-i}) = \delta_{ji}. \tag{1.15}$$

Proof. Re-write equality (1.12) by inverting $E_r(t)$ in $B_r[[t]]$:

$$\begin{aligned}\mathbf{m}(t) &= \sum_{j\geq 0}\mathtt{U}_j(\mathbf{m}(t))\frac{t^j}{E_r(t)}\\ &= \sum_{j\geq 0}\mathtt{U}_j(\mathbf{m}(t))t^jH_r(t) = \sum_{j\geq 0}\mathtt{U}_j(\mathbf{m}(t))u_{-j}.\end{aligned}$$

In particular $t^j = E_r(t)H_r(t)t^j = E_r(t)u_{-j} = \sum_{i\geq 0}\mathtt{U}_i(u_{-j})t^i$ if $M = A$ and (1.15) follows. ∎

1.3.4 Let X be an indeterminate over A. If $M := A[X]$, consider the formal power series

$$\frac{1}{1-Xt} := \sum_{n\geq 0}X^nt^n \in A[X][[t]]. \tag{1.16}$$

1.3.5 Proposition. *For $j \geq 0$:*

$$\mathtt{U}_j\left(\frac{1}{1-Xt}\right) = \mathtt{p}_j(X) := X^j - e_1X^{j-1} + \ldots + (-1)^r e_r X^{j-r}$$

by agreeing that $X^i = 0$ if $i < 0$.

Proof. In fact

$$\begin{aligned}\sum_{j\geq 0}\mathtt{U}_j\left(\frac{1}{1-Xt}\right)t^j &= \frac{E_r(t)}{1-Xt} = (1-\sum_{j=1}^{r}(-1)e_it^i)\sum_{i\geq 0}X^nt^n \\ &= \sum_{j\geq 0}(X^j - e_1X^{j-1} + \ldots + (-1)^r e_r X^{j-r})t^j,\end{aligned}$$

and the claim follows. ∎

1.3.6 Definition. *The sequence $(m_0, m_1, \ldots) \in M^{\mathbb{N}}$ is a* generic Linear Recurrence Sequence *(generic LRS) of order r if*

$$U_{r+j}(\mathbf{m}(t)) := m_{r+j} - e_1m_{r+j-1} + \ldots + (-1)^r e_r m_j = 0$$

for all $j \geq 0$.

The r-tuple $(m_0, m_1, \ldots, m_{r-1})$ is said to be the *initial data* of the generic LRS. The characteristic polynomial of the generic LRS is clearly $\mathtt{p}_r(X)$ – the generic polynomial of degree r. Within the language of M-valued formal power series, we can say that $\mathbf{m}(t) := \sum_{j\geq 0} m_jt^j$ is a generic LRS if and only if $\mathtt{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$. If $D := D^1 \in End_A(M[[t]])$ is like in (1.6), let $\mathtt{p}_r(D)$ be the endomorphism of $M[[t]]$ obtained by evaluating $\mathtt{p}_r(X)$ at D.

1.3.7 Proposition. *The formal power series $\mathbf{m}(t) \in M[[t]]$ is a generic LRS of order r if and only if $\mathtt{p}_r(D)\mathbf{m}(t) = 0$.*

Proof. In fact

$$\begin{aligned}\mathtt{p}_r(D)\mathbf{m}(t) &= (D^r - e_1D^{r-1} + \ldots + (-1)^r e_r)\sum_{j\geq 0} m_jt^j \\ &= \sum_{j\geq 0}(m_{r+j} - e_1m_{r+j-1} + \ldots + (-1)^r e_r m_j)t^j \\ &= \sum_{j\geq 0}\mathtt{U}_{r+j}(\mathbf{m}(t))t^j.\end{aligned}$$

Thus $\mathtt{p}_r(D)\mathbf{m}(t) = 0$ if and only if $\mathtt{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$. ∎

1.3.8 Let K_r be the B_r-submodule $\ker \mathtt{p}_r(D)$ of $B_r[[t]]$, i.e.

$$K_r := \{u \in B_r[[t]] \mid \mathtt{p}_r(D)u = 0\} \tag{1.17}$$

and define in addition $K_r(A) = K_r \otimes_{\mathbb{Z}} A$ and $K_r(M) := K_r(A) \otimes_A M$.

1.3.9 Proposition. *Each $\mathbf{m}(t) \in K_r(M)$ can be uniquely written as*

$$\mathbf{m}(t) = u_0\mathtt{U}_0(\mathbf{m}(t)) + u_{-1}\mathtt{U}_1(\mathbf{m}(t)) + \ldots + u_{-r+1}\mathtt{U}_{-r+1}(\mathbf{m}(t)). \tag{1.18}$$

Proof. By formula (1.14):

$$\mathbf{m}(t) = \sum_{j \geq 0} \mathtt{U}_j(\mathbf{m}(t))u_{-j}.$$

If $\mathbf{m}(t) \in K_r(M)$ then $\mathtt{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$, and (1.18) follows. ∎

1.3.10 Corollary. *The r-tuple $(u_0, u_{-1}, \ldots, u_{-r+1})$ is an A basis of $K_r(A)$.*

Proof. Each $\mathbf{a}(t) \in K_r(A)$ can be written as $\sum_{j=0}^{r-1} \mathtt{U}_j(\mathbf{a}(t))u_{-j}$ by virtue of Proposition 1.3.9. In addition $\mathtt{U}_{r+j}(u_{-i}) = \delta_{i,r+j}$ which is 0 in the range $0 \leq i \leq r-1$, i.e. $u_{-i} \in K_r(A)$. ∎

1.3.11 Remark. What about u_j if the index j does not run in the range $-r+1 \leq j \leq 0$? First of all, if $j > 0$, then $u_j \in K_r(A)$ for all B_r-algebras A. In fact

$$\mathtt{p}_r(D)u_j = \mathtt{p}_r(D)D^j u_0 = D^j \mathtt{p}_r(D)u_0 = 0,$$

where in the most left equality we used the first of (1.13). In particular

$$u_j = \mathtt{U}_0(u_j)u_0 + \mathtt{U}_1(u_j)u_{-1} + \ldots + \mathtt{U}_{r-1}(u_j)u_{-r+1}, \tag{1.19}$$

where

$$\mathtt{U}_i(u_j) = h_{i+j} - e_1 h_{i+j-1} + \ldots + (-1)^{i-1} e_{i-1} h_j. \tag{1.20}$$

If $j < -r+1$, then $j = -r-i$ for some $i \geq 0$ and then

$$\mathtt{p}_r(D)u_{-r-i} = t^i. \tag{1.21}$$

In fact

$$\mathtt{p}_r(D)u_{-r} = \mathtt{p}_r\left(\frac{1}{t}\right)t^r u_0 = E_r(t)u_0 = E_r(t)H_r(t) = 1,$$

and consequently:

$$\mathtt{p}_r(D)u_{-r-i} = \mathtt{p}_r(D)t^i u_{-r} = t^i \mathtt{p}_r(D)u_{-r} = t^i,$$

which proves (1.21).

1.3.12 Observation. We notice that $\mathbf{m}(t)$ is a generic LRS if and only if (Cf. 1.1.6):

$$\varrho_M\left(\mathtt{p}_r\left(\frac{1}{t}\right)\frac{\mathbf{m}(t)}{t} + t^{-1}M[t^{-1}]\right) = 0$$

in $M((t))$. In this case

$$\begin{aligned} \mathbf{m}(t) &= \mathtt{U}_0(\mathbf{m}(t))u_0 + \mathtt{U}_1(\mathbf{m}(t))tu_0 + \ldots + \mathtt{U}_{r-1}(\mathbf{m}(t))t^{r-1}u_0 \\ &= \mathtt{U}_0(\mathbf{m}(t))D^{r-1}u_{-r+1} + \mathtt{U}_1(\mathbf{m}(t))D^{r-2}u_{-r+1} + \ldots \\ &+ \mathtt{U}_{r-1}(\mathbf{m}(t))u_{-r+1} \end{aligned}$$

and $$\mathtt{U}_j(\mathbf{m}(t)) = \mathrm{Res}_t\left(\mathtt{p}_j\left(\frac{1}{t}\right)\frac{\mathbf{m}(t)}{t}\right), \qquad (0 \leq j \leq r-1).$$

The situation is analogous to that of M-valued formal distributions $M[[z^{-1}, z, w^{-1}, w]]$ belonging to the kernel of the multiplication by a power of $z-w$. In fact, if $\mathbf{m}(w,z) \in M[[z^{\pm 1}, w^{\pm 1}]]$, then $(z-w)^r\mathbf{m}(z,w) = 0$ if and only if

$$\mathbf{m}(z,w) =$$

$$= c^0(w)\delta(z-w) + c^{(1)}(w)\partial_w\delta(z-w) + \ldots + c^{(r-1)}(w)\partial_w^{(r-1)}\delta(z-w)$$

where $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n$ is the formal Dirac δ-function

$$\partial_w^{(j)}\delta(z-w) = \frac{1}{j!}\frac{\partial^j\delta(z-w)}{\partial w^j}$$

and

$$c^{(j)}(w) = Res_z(z-w)^j\mathbf{m}(z,w).$$

See [44, p. 17] or [45, Theorem 1.6].

1.4 Cauchy Problems for Generic LRS

1.4.1 Definition. *Two M-valued formal power series $\mathbf{m}(t), \widetilde{\mathbf{m}}(t)$ share the same* initial conditions *modulo (t^r) if $\mathbf{m}(t) - \widetilde{\mathbf{m}}(t) \in t^r M[[t]]$.*

In particular $m_i = \widetilde{m}_i$ for $0 \le i \le r-1$. An easy induction shows that $\mathbf{m}(t), \widetilde{\mathbf{m}}(t)$ have the same initial conditions modulo t^r if and only if $\mathtt{U}_i(\mathbf{m}(t)) = \mathtt{U}_i(\widetilde{\mathbf{m}}(t))$ for $0 \le i \le r-1$.

1.4.2 Lemma. *If $\mathbf{m}(t) \in K_r(M)$ and $\mathtt{U}_i(\mathbf{m}(t)) = 0$ for $0 \le i \le r-1$, then $\mathbf{m}(t) = 0$.*

Proof. Obvious. In fact by hypothesis all the coefficients of the expansion (1.14) of $\mathbf{m}(t)$ vanish. ■

If $\mathbf{f} := (f_0, f_1, \ldots)$ is a further sequence of indeterminates over A, set $A[\mathbf{f}] := A[f_0, f_1, \ldots]$ and $M[\mathbf{f}] := M \otimes_A A[\mathbf{f}]$. Form $\mathbf{f}(t) = \sum_{j\ge 0} f_j t^j$. Extend $\mathtt{p}_r(D)$ to an endomorphism of $M[\mathbf{f}][[t]]$ in the obvious way.

1.4.3 Proposition (Cauchy Theorem for generic LRS). *Let $\widetilde{\mathbf{m}}(t) \in M[[t]] \subseteq M[\mathbf{f}][[t]]$. Then*

$$\mathbf{m}(t) = \mathtt{U}_0(\widetilde{\mathbf{m}}(t))u_0 + \ldots + \mathtt{U}_{-r+1}(\widetilde{\mathbf{m}}(t))u_{-r+1} + \sum_{j\ge 0} u_{-r-j} f_j \quad (1.22)$$

is the unique element of $\mathtt{p}_r(D)^{-1}(\mathbf{f}(t))$ sharing the same initial conditions of $\widetilde{\mathbf{m}}(t)$.

Proof. Apply $\mathtt{p}_r(D)$ to both sides of 1.22, exploiting its "M-linearity"

$$\mathtt{p}_r(D)\mathbf{m}(t) = \sum_{j=0}^{r-1} \mathtt{U}_j(\widetilde{\mathbf{m}}(t))\mathtt{p}_r(D)u_{-j} + \sum_{j\ge 0} f_j \mathtt{p}_r(D)u_{-r-j}.$$

The equality $\mathtt{p}_r(D)u_{-j} = 0$ for $0 \le j \le r-1$ together with (1.21) imply $\mathtt{p}_r(D)\mathbf{m}(t) = \sum_{j\ge 0} f_j t^j$ as required. It is clear that (1.22) shares the same initial condition as $\widetilde{\mathbf{m}}(t)$. Were $\mathbf{m}'(t)$ another element of $\mathtt{p}_r(D)^{-1}(\mathbf{f}(t))$ with the same initial conditions as $\widetilde{\mathbf{m}}(t)$, one would obtain $\mathbf{m}(t) - \mathbf{m}'(t) \in K_r(M)$, with the same initial conditions as the null series $0 \in M[[t]]$. Hence $\mathbf{m}(t) = \mathbf{m}'(t)$ because of Lemma 1.4.2. ■

1.4.4 Universality. Formula (1.22) is universal in the following sense. Let A be any $\mathbb{Z}$-algebra, M any A-module and $P \in A[X]$ be any monic polynomial of degree r. Let $\mathbf{m}(t)$ and $\mathbf{g}(t) := \sum_{j\geq 0} g_j t^j$ be M-valued formal power series. Then the unique element of $P(D)^{-1}(\mathbf{g}(t))$ sharing the same initial conditions as $\mathbf{m}(t)$ is

$$\sum_{j=0}^{r-1} \mathtt{U}_j(\mathbf{m}(t))u_{-j} + \sum_{j\geq 0} g_j u_{-r-j},$$

where A is regarded as a $B_r[\mathbf{f}]$-module through the unique $\mathbb{Z}$-algebra homomorphism mapping $e_i \mapsto e_i(P)$ and $f_j \mapsto g_j$.

1.4.5 Corollary. *The following Cauchy sequence*

$$0 \longrightarrow K_r(M) \hookrightarrow M[[t]] \xrightarrow{\mathtt{p}_r(D)} M[[t]] \longrightarrow 0 \tag{1.23}$$

is exact.

Proof. The map $K_r(M) \hookrightarrow M[[t]]$ is the inclusion. It is then obvious that (1.23) is a complex. In addition the map $\mathtt{p}_r(D)$ is surjective, as a consequence of Proposition 1.4.3. ■

1.5 Generic LRS via Formal Distributions

1.5.1 Sequences of elements of the A-module M may be identified with the A-module $\mathrm{Hom}_A(A[X], M) \cong M \otimes A[X]^\vee$. In fact the map

$$\begin{cases} \psi_M : & \mathrm{Hom}_A(A[X], M) & \longrightarrow & M[[t]] \\ & v & \longmapsto & \sum_{i\geq 0} v(X^i)t^i \end{cases} \tag{1.24}$$

sending $v \mapsto \sum_{i\geq 0} v(X^i)t^i$ is obviously an A-module homomorphism. Then M-valued formal power series can be seen as formal distributions, namely linear map defined on $A[X]$, viewed as a space of "test functions" (the polynomials).

For $f \in A[X]$, consider the *multiplication-by-f* homomorphism of A-modules:

$$A[X] \xrightarrow{f} A[X].$$

Its pullback $f^* : \mathrm{Hom}_A(A[X], M) \to \mathrm{Hom}_A(A[X], M)$, given by $(f^*v)(g) = v(fg)$, induces on $\mathrm{Hom}_A(A[X], M)$ a structure of $A[X]$-module. If $f \in A[X]$ and $v \in \mathrm{Hom}_A(A[X], M)$ we shall simply write fv instead of f^*v if no confusion is likely.

If $M = A$, a *convolution product* "$\star$" is defined on $A[X]^\vee$ by requiring that the map (1.24)

$$(A[X]^\vee, \star) \longrightarrow (A[[t]], \cdot)$$

is a ring isomorphism. An easy check shows that $\mathrm{Hom}_A(A[X], M)$ is then an $(A[X]^\vee, \star)$-module isomorphic to the $A[[t]]$-module $M[[t]]$. Considering the map $\mathtt{p}_r(X) : \mathrm{Hom}_A(A[X], M) \to \mathrm{Hom}_A(A[X], M)$, the module of the generic *Linear Recurrent Sequences* can be identified with $\ker(\mathtt{p}_r(X)\cdot)$. In fact $\mathtt{p}_r(X)v = 0$ if and only if

$$(\mathtt{p}_r(X)v)(X^n) = 0,$$

for all $n \geq 0$, i.e. if and only if

$$\begin{aligned} 0 &= v(\mathtt{p}_r(X)X^n) = v(X^{n+r} - e_1X^{n+r-1} + \ldots + (-1)^r e_r X^n) \\ &= v(X^{n+r}) - e_1 v(X^{n+r-1}) + \ldots + (-1)^r e_r v(X^n). \end{aligned}$$

In other words $(v(1), v(X), \ldots, v(X^{r-1}), v(X^r), \ldots)$ or, equivalently, $\sum_{i\geq 0} v(X^i)t^i$ is a LRS. Let $A[\xi] := A[X]/(\mathtt{p}_r(X))$, where $\xi := X + \mathtt{p}_r(X)$. Elementary commutative algebra say that the sequence of $A[X]$-modules and homomorphisms

$$0 \longrightarrow \ker \mathtt{p}_r(X) \longrightarrow A[X] \longrightarrow A[\xi] \longrightarrow 0$$

is exact. Moreover it is split. A section $s : A[\xi] \to A[X]$ is given by

$$s(f + \mathtt{p}_r(X)) = r(f),$$

where $r(f)$ is the remainder of the euclidean division of f by $\mathtt{p}_r(X)$. As the contravariant functor $\mathrm{Hom}_A(-, M)$ is exact on split exact sequences, we obtain the *Cauchy Exact Sequence*:

$$0 \to \mathrm{Hom}_A(A[\xi], M) \longrightarrow \mathrm{Hom}_A(A[X], M) \longrightarrow \mathrm{Hom}_A(A[X], M) \to 0.$$

It says that the Cauchy problem $\mathtt{p}_r(X)v = w$ with fgiven initial conditions $v(i) = a_i$, $0 \leq i \leq r-1$ has a solution and this is unique.

1.5.2 For all $j \geq 0$, let

$$v_{-j}(X^n) = h_{n-j} \qquad \text{and} \qquad v_j = X^j v_0,$$

i.e. $\psi_A(v_j) = u_j$ for all $j \geq 0$. Each $v \in A[X]^\vee$ can be uniquely written as

$$v = \sum_{j \geq 0} \mathtt{U}_j(v) v_{-j},$$

where by definition $U_j(v) = U_j(\psi_A(v))$. Moreover v corresponds to a LRS if and only if $v \in \ker \mathtt{p}_r(X)$. An A-basis for $v \in \ker \mathtt{p}_r(X)$ is precisely

$$(v_0, v_{-1}, \ldots, v_{-r+1}) = (X^{r-1} v_{-r+1}, \ldots, X v_{-r+1}, v_{-r+1}),$$

and then

$$v \in \ker \mathtt{p}_r(X) \iff v = \sum_{j=0}^{r-1} U_{r-1-j}(v) X^j v_{-r+1}. \tag{1.25}$$

1.6 The Formal Laplace Transform and Linear ODEs

1.6.1 Let M be a module over any $\mathbb{Z}$-algebra A. The A-linear map map $\mathtt{L} : M[[t]] \to M[[t]]$ defined as

$$\mathtt{L}(\sum_{j \geq 0} m_j t^j) = \sum_{j \geq 0} j! m_j t^j, \tag{1.26}$$

will be said *formal Laplace transform*. If A contains the rational numbers, the formal Laplace transform is invertible:

$$\mathtt{L}^{-1}\left(\sum_{j \geq 0} m_j t^j\right) = \sum_{j \geq 0} m_j \frac{t^j}{j!}.$$

Define $\partial_t : M[[t]] \to M[[t]]$ as:

$$\partial_t \mathbf{m}(t) = \sum_{j \geq 0} (j+1) m_{j+1} t^j.$$

1.6.2 Proposition.

$$\mathtt{L}\partial_t = D\mathtt{L}. \tag{1.27}$$

In particular, $\mathtt{L}$ *maps* $\ker \mathtt{p}_r(\partial_t)$ *to* $K_r(M)$.

Proof. Just matter of routine calculation.

$$\mathtt{L}\partial_t \sum_{j\geq 0} m_j t^j = \sum_{j\geq 0}(j+1)! m_{j+1} t^j = D \sum_{j\geq 0} j! m_j t^j = D\mathtt{L} \sum_{j\geq 0} m_j t^j.$$

In particular $\mathtt{L} \circ \partial_t^i = D^i \circ \mathtt{L}$ and thus $\mathtt{L}$ maps $\ker \mathtt{p}_r(\partial_t)$ to $K_r(M)$. ■

If A contains the rational numbers, then $\mathtt{L}$ is invertible. In this case $\partial_t = \mathtt{L}^{-1} D\mathtt{L}$, by induction $\partial_t^i = \mathtt{L}^{-1} D^i \mathtt{L}$ and $\ker \mathtt{p}_r(\partial_t)$ is isomorphic to $K_r(M)$. In other words, to determine $\ker \mathtt{p}_r(\partial_t)$ it suffices to determine $K_r(M)$.

1.6.3 From now on we assume $M = A$. Then ∂_t is a derivation of $A[[t]]$, i.e.

$$\partial_t(\mathbf{a}(t)\mathbf{b}(t)) = \partial_t \mathbf{a}(t)\mathbf{b}(t) + \mathbf{a}(t)\partial_t \mathbf{b}(t)$$

a well known equality which is left as an exercise.

1.6.4 Lemma. *For all* $\mathbf{f}(t) \in A[[t]]$ *the expansion*

$$\mathbf{f}(t) = \sum_{j\geq 0} \mathtt{U}_j(\mathtt{L}(\mathbf{f}(t))\mathtt{L}^{-1}(u_{-j}) \tag{1.28}$$

holds.

Proof. In fact $\mathtt{L}(\mathbf{f}(t)) = \sum_{j\geq 0} \mathtt{U}_j(\mathtt{L}(\mathbf{f}(t))u_{-j}$. Taking the inverse transform of both members, and using A-linearity, formula (1.28) follows. ■

1.6.5 Given $\mathbf{f}(t) \in A[[t]]$, the pre-image $\mathtt{p}_r(\partial_t)^{-1}(\mathbf{f}(t))$ is, by definition, the set of the solutions to the generic linear *Ordinary Differential Equation* (ODE)

$$y^{(r)} - e_1 y^{(r-1)} + \ldots + (-1)^r e_r y = \mathbf{f}(t), \tag{1.29}$$

where $y^{(i)} := \partial_t^i y$. We say that $y(t) \in A[[t]]$ shares the same initial conditions of $\varphi \in A[[t]]$ modulo t^r if $y(t) - \varphi(t) \in t^r A[[t]]$. Thus Proposition 1.4.3 has the following important

1.6.6 Corollary. *The unique element of* $\mathtt{p}_r(\partial_t)^{-1}(\mathbf{f}(t))$ *sharing the same initial conditions of* φ *modulo* t^r *is:*

$$y = \sum_{i=0}^{r-1} \mathtt{U}_i(\mathtt{L}(\varphi))\mathtt{L}^{-1}(u_{-i}) + \sum_{j\geq 0} j! f_j \mathtt{L}^{-1}(u_{-r-j}). \tag{1.30}$$

Proof. Just apply $\mathtt{p}_r(\partial_t) = \mathtt{L}^{-1}\mathtt{p}_r(D)\mathtt{L}$ to both sides of (1.30):

$$\begin{aligned}\mathtt{p}_r(\partial_t)y = \mathtt{L}^{-1}\mathtt{p}_r(D)\mathtt{L}\left(\sum_{i=0}^{r-1} \mathtt{U}_i(\mathtt{L}(\varphi))\mathtt{L}^{-1}(u_{-i})\right) + \\ + \quad \mathtt{L}^{-1}\mathtt{p}_r(D)\mathtt{L}\left(\sum_{j\geq 0} j! f_j \mathtt{L}^{-1}(u_{-r-j})\right)\end{aligned}$$

Using the A-linearity of L and the fact that $\mathtt{p}_r(D)u_{-i} = 0$ of $0 \leq i \leq r-1$, the last member is:

$$\begin{aligned}&\mathtt{L}^{-1}\left(\sum_{i=0}^{r-1} \mathtt{U}_i(\mathtt{L}(\varphi))\mathtt{p}_r(D)(u_{-i})\right) + \mathtt{L}^{-1}\mathtt{p}_r(D)\left(\sum_{j\geq 0} j! f_j u_{-r-j}\right) \\ = \quad & 0 \quad + \mathtt{L}^{-1}(L(\mathbf{f}(t)) = L^{-1}\left(\sum_{j\geq 0} j! f_j t^j\right) \\ & = \sum_{j\geq 0} f_j t^j = \mathbf{f}(t).\end{aligned}$$

The unicity is obvious. ∎

1.6.7 Example. It is worth to see a couple of examples. Suppose one wants to solve the linear ODE

$$y'' + \omega^2 y = t^n \tag{1.31}$$

Write $n+2$ in the form $4p+q$ where $p \in \mathbb{N}$ and $q \in \{0,1,2,3\}$, and rewrite (1.31) in the form

$$y'' + \omega^2 y = t^{4p+q-2}, \qquad (4p+q \geq 2)$$

Solution of (1.6.7) amounts to solve

$$(D^2+\omega^2)L(y) = (4p+q-2)!t^{4p+q-2}.$$

There is a unique morphism $B_2 \to \mathbb{Q}$ sending $e_1 \mapsto 0$ and $e_2 \mapsto \omega^2$. Let $\phi = a_0 + a_1 t \in \mathbb{Q}[t]$ and look for the unique solution such that $y - \phi = 0 \mod t^2$. Then

$$u_0 = H_2(t) = \frac{1}{1+\omega^2 t^2} = 1 + \sum_{j\geq 1} (-1)^j \omega^{2j} t^{2j}$$

and

$$u_{-1} = t u_0 = \frac{1}{\omega}\left(\omega t + \sum_{j\geq 1}(-1)^j \omega^{2j+1} t^{2j+1}\right).$$

The solution to the Cauchy problem is then

$$L(y) = a_0 u_0 + a_1 u_{-1} + (4p+q-2)! u_{-4p-q}$$

Now

$$\begin{aligned}
u_{-4p-q} &= t^{4p+q} u_0 = \frac{1}{\omega^{4p+q}} (\omega t)^{4p+q} u_0 \\
&= \frac{1}{3}(2q-1)(q-1)(q-3)u_0 + \frac{1}{3\omega} q(q-2)(q-4)\omega u_{-1} \\
&+ \frac{1}{6} q(q-1)(q-2)\omega t - \frac{1}{2} q(q-1)(q-3) \\
&- \sum_{a=0}^{p} (-1)^a \omega^{2a+q} t^{2a+q},
\end{aligned}$$

a formula that the reader can easily check by exercise. Since

$$L^{-1}(u_0) = \cos(\omega t) \qquad \text{and} \qquad L^{-1}(u_{-1}) = \frac{1}{\omega}\sin(\omega t)$$

one finally obtains

$$y = a_0 \cos \omega t + \frac{a_1}{\omega} \sin \omega t + (4p+q-2)! L^{-1}(u_{4p+q}),$$

where

$$\begin{aligned}
L^{-1}(u_{-4p-q}) &= L^{-1}(t^{4p+q}u_0) = \frac{1}{\omega^{4p+q}}L^{-1}((\omega t)^{4p+q}u_0) \\
&= \frac{1}{\omega^{4p+q}}\left(\frac{(2q-1)(q-1)(q-3)}{3}\cos\omega t\right. \\
&+ \frac{q(q-2)(q-4)}{3}\sin\omega t \\
&- \sum_{a=0}^{p}(-1)^{2a+q}\omega^{2a+q}\frac{t^{2a+q}}{(2a+q)!} \\
&+ \left.\frac{1}{6}q(q-1)(q-2)\omega t - \frac{1}{2}q(q-1)(q-3)\right).
\end{aligned}$$

1.6.8 Example. In Example 1.6.7, put $p = 1$ and $q = 3$ to solve

$$y'' + \omega^2 y = t^5.$$

Then

$$\begin{aligned}
y &= a_0\cos\omega t + \frac{a_1}{\omega}\sin\omega t + \frac{5!}{\omega^7}\left(\omega t - \frac{\omega^3t^3}{3!} + \frac{\omega^5t^5}{5!} - \sin\omega t\right) \\
&= a_0\cos\omega t + \left(\frac{a_1}{\omega} - \frac{5!}{\omega^7}\right)\sin\omega t + \frac{5!}{\omega^6}\left(t - \frac{\omega^2t^3}{3!} + \frac{\omega^4t^5}{5!}\right).
\end{aligned}$$

Chapter 2

Cayley-Hamilton Theorem Revisited

2.1 Exterior Algebras of Free Modules

2.1.1 Throughout this chapter, and unless otherwise stated, M will denote a module over a commutative ring A and $\bigwedge M$ its *exterior algebra*. The latter is the quotient of the *tensor algebra* $T(M)$ modulo the bilateral ideal generated by all the elements of the form $m \otimes m$, $m \in M$. Let $T(M) \to \bigwedge M$ be the canonical epimorphism: the image of $m_1 \otimes m_2$ is usually denoted by $m_1 \wedge m_2$. The graduation of $T(M)$ induces a graduation on $(\bigwedge M, \wedge)$:

$$\bigwedge M = \bigoplus_{j \geq 0} \bigwedge^j M.$$

The degree j piece $\bigwedge^j M$ of $\bigwedge M$ is called the j-th *exterior power* of M. Notice that $(\bigwedge M, \wedge)$ is a *superalgebra*, i.e. it possesses a $\mathbb{Z}/2\mathbb{Z}$ graduation:

$$\bigwedge M = (\bigwedge M)_{\overline{0}} \oplus (\bigwedge M)_{\overline{1}}$$

where

$$(\bigwedge M)_{\overline{0}} = \bigoplus_{j\geq 0} \bigwedge^{2j} M \qquad \text{and} \qquad (\bigwedge M)_{\overline{1}} = \bigoplus_{j\geq 0} \bigwedge^{2j+1} M$$

so that the $\wedge$-product is *super-commutative*. In fact if $\alpha \in \bigwedge^i M$ and $\beta \in \bigwedge^j M$, then

$$\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha.$$

2.1.2 We will mainly work with free A-modules of finite or countable rank:

$$M = \bigoplus_{0\leq i<n} A \cdot b_i,$$

where $n \in \mathbb{N}$ or $n = \infty$ and $\mathcal{B} = (b_0, b_1, \ldots)$ is an A-basis of M. In this case $\bigwedge^k M$ can be defined as the A–linear span of $\bigwedge^k \mathcal{B} := (b_{i_0} \wedge \ldots \wedge b_{i_{k-1}})_{0\leq i_0<i_1<\ldots<i_{k-1}}$ and

$$b_{i_{\sigma(0)}} \wedge \ldots \wedge b_{i_{\sigma(k-1)}} = \mathrm{sgn}(\sigma) b_{i_0} \wedge \ldots \wedge b_{i_{k-1}},$$

for all permutations σ on k elements. If $\mathrm{rk}_A M = n$, then

$$\mathrm{rk}_A \bigwedge^k M = \binom{n}{k}. \tag{2.1}$$

The *exterior algebra* of any such a module is just $\bigwedge M = \bigoplus_{j\geq 0} \bigwedge^j M$ with $\wedge$-product given by juxtaposition:

$$(b_{i_1} \wedge \ldots \wedge b_{i_h}) \wedge (b_{j_1} \wedge \ldots \wedge b_{j_k}) = b_{i_1} \wedge \ldots \wedge b_{i_h} \wedge b_{j_1} \wedge \ldots \wedge b_{j_k}$$

and where by convention $\bigwedge^0 M = A$ and $\bigwedge^1 M := M$. In addition $\bigwedge^j M = 0$ if $j > \mathrm{rk}_A M$, while no exterior power vanishes if M has infinite rank.

2.2 Derivations on Exterior Algebras

2.2.1 Let $\bigwedge M$ be the exterior algebra of an arbitrary A-module. If $\mathcal{D} := (D_0, D_1, \ldots)$ is a sequence of endomorphisms of $\bigwedge M$, let $\mathcal{D}(t) :=$

$\sum_{i\geq 0} D_i t^i \in \mathrm{End}_A(\bigwedge M)[[t]]$. The multiplication

$$\mathcal{D}(t)\widetilde{\mathcal{D}}(t) = \sum_{n\geq 0}\big(\sum_{i=0} D_i \circ \widetilde{D}_{n-i}\big)t^n$$

defines a structure of $\mathrm{End}_A(\bigwedge M)$-algebra. If $\mathcal{D}(t) \in \mathrm{End}_A(\bigwedge M)[[t]]$, it induces an A-homomorphism

$$\mathcal{D}(t) : \bigwedge M \longrightarrow \bigwedge M[[t]],$$

indicated in the same way by abuse of notation, that maps $\alpha \mapsto \mathcal{D}(t)\alpha = D_0\alpha + D_1\alpha \cdot t + \ldots \in \bigwedge M[[t]]$. Endow the module $\bigwedge M[[t]]$ with the algebra structure:

$$\sum_{i\geq 0}\alpha_i t^i \wedge \sum_{j\geq 0}\beta_j t^j = \sum_{k\geq 0}\big(\sum_{i+j=k}(\alpha_i \wedge \beta_j)\big)t^k.$$

2.2.2 Definition. *A* derivation *of $\bigwedge M$ is a $\wedge$-algebra homomorphism $\mathcal{D}(t) : \bigwedge M \to \bigwedge M[[t]]$, i.e. for all $\alpha, \beta \in \bigwedge M$*

$$\mathcal{D}(t)(\alpha \wedge \beta) = \mathcal{D}(t)\alpha \wedge \mathcal{D}(t)\beta.$$

In [22], the set of all the derivations of $\bigwedge M$ was denoted by $HS(\bigwedge M)$ (from the initials of Hasse and Schmidt[1])

2.2.3 Proposition. *$HS(\bigwedge M)$ is a subalgebra of $\mathrm{End}_A(\bigwedge M)$, i.e. if $\mathcal{D}(t), \widetilde{\mathcal{D}}(t) \in HS(\bigwedge M)$ then $\mathcal{D}(t)\widetilde{\mathcal{D}}(t)$ is a derivation.*

Proof. Using the hypothesis that $\mathcal{D}(t)$ and $\widetilde{\mathcal{D}}(t)$ are derivations:

$$\begin{aligned}
\mathcal{D}(t)\widetilde{\mathcal{D}}(t)(\alpha \wedge \beta) &= \mathcal{D}(t)(\widetilde{\mathcal{D}}(t)(\alpha \wedge \beta)) \\
&= \mathcal{D}(t)(\widetilde{\mathcal{D}}(t)\alpha \wedge \widetilde{\mathcal{D}}(t)\beta) \\
&= \mathcal{D}(t)\widetilde{\mathcal{D}}(t)\alpha \wedge \mathcal{D}(t)\widetilde{\mathcal{D}}(t)\beta \\
&= \mathcal{D}(t)\widetilde{\mathcal{D}}(t)\,\alpha \wedge \mathcal{D}(t)\widetilde{\mathcal{D}}(t)\,\beta.
\end{aligned}$$ ∎

[1] If C is a ring and A any C-algebra, a Hasse Schmidt derivation is a C-homomorphism $\phi : A \to A[[t]]$ such that $\phi(ab) = \phi(a)\phi(b)$ (Cf. [66, p. 207]) We do the same by replacing A with the exterior algebra of a module with respect to the "$\wedge$" product.

2.2.4 Proposition. *The series* $\mathcal{D}(t) := \sum_{i\geq 0} D_i t^i$ *is a* derivation *on* $\bigwedge M$ *if and only if* D_k *satisfies the Leibniz's rule for all* $k \geq 0$*:*

$$D_k(\alpha \wedge \beta) = \sum_{i=0}^{k} D_i\alpha \wedge D_{k-i}\beta. \tag{2.2}$$

Proof. Consider the equality

$$\begin{aligned}\sum_{k\geq 0} D_k(\alpha\wedge\beta)t^k &= \left(\sum_{k\geq 0} D_k t^k\right)(\alpha\wedge\beta)\\ &= \sum_{i\geq 0} D_i\alpha\cdot t^i \wedge \sum_{j\geq 0} D_j\beta\cdot t^j\\ &= \sum_{k\geq 0}\sum_{i=0}^{k}(D_i\alpha\wedge D_{k-i}\beta)t^k. \end{aligned} \tag{2.3}$$

Comparing the coefficient of $t^k$of the first and the last side of (2.3) gives precisely (2.2). The converse obviously holds. ∎

2.2.5 Definition. *An element* $\mathcal{D}(t) \in \mathrm{End}_A(\bigwedge M)[[t]]$ *is* invertible *if there exists* $\overline{\mathcal{D}}(t) \in End_A(\bigwedge M)[[t]]$ *such that* $\mathcal{D}(t)\overline{\mathcal{D}}(t) = \overline{\mathcal{D}}(t)\mathcal{D}(t) = \mathbb{1}$*, where* $\mathbb{1}$ *is the identity endomorphism of* $\bigwedge M$*.*

2.2.6 Proposition. *The series* $\mathcal{D}(t) \in \mathrm{End}_A(\bigwedge M)[[t]]$ *is invertible if and only if* D_0 *is an* A*-automorphism of* $\bigwedge M$*.*

Proof. If $\mathcal{D}(t)$ is invertible, obviously D_0 is invertible. Conversely, suppose D_0 is invertible. We look for an inverse of the form $\overline{\mathcal{D}}(t) = 1 - \overline{D}_1 t + \overline{D}_2 t^2 - \ldots$. Write $\mathcal{D}(t) = D_0(1 + D'_1 t + D'_2 t^2 + \ldots)$, where $D'_i = D_0^{-1} D_i$. Then the equation

$$(1 + D'_1 t + D'_2 t^2 + \ldots)(1 - \overline{D}_1 t + \overline{D}_2 t^2 + \ldots) = 1$$

enables to find $\overline{D}_i$ as polynomial in $1, D_1, \ldots, D_i$. Thus $\overline{\mathcal{D}}(t) := D_0^{-1}(1 - \overline{D}_1 t + \overline{D}_2 t^2 + \ldots)$ is the required inverse. ∎

2.2.7 Proposition. *The inverse of an invertible derivation is a derivation.*

Proof. Let $\overline{\mathcal{D}}(t)$ be the inverse in $\bigwedge M[[t]]$ of $\mathcal{D}(t)$.

$$\begin{aligned}\overline{\mathcal{D}}(t)(\alpha\wedge\beta) &= \overline{\mathcal{D}}(t)(\mathcal{D}(t)\overline{\mathcal{D}}(t)\alpha\wedge\mathcal{D}(t)\overline{\mathcal{D}}(t)\beta)\\ &= (\overline{\mathcal{D}}(t)\mathcal{D}(t))(\overline{\mathcal{D}}(t)\alpha\wedge\overline{\mathcal{D}}(t)\beta)\\ &= \overline{\mathcal{D}}(t)\alpha\wedge\overline{\mathcal{D}}(t)\beta.\end{aligned}$$ ■

2.3 The Trace Operators Polynomials

2.3.1 Suppose now that M is a free A-module of finite rank r. To each $f \in End_A(M)$ we attach a sequence $\overline{D}_1(f), \overline{D}_2(f), \ldots, \overline{D}_r(f)$ of endomorphisms of $\bigwedge M$ satisfying the following properties:

i) $\bigwedge^i M \subseteq \ker \overline{D}_j(f)$ whenever $0 \leq i < j$;

ii) $\overline{D}_1(f)m = f(m)$ for all $m \in M \cong \bigwedge^1 M \subseteq \bigwedge M$;

iii) The *characteristic polynomial operator*

$$\overline{\mathcal{D}}(f;t) := 1 - \overline{D}_1(f)t + \ldots + (-1)^r\overline{D}_r(f)t^r \in \mathrm{End}_A(\bigwedge M)[t] \tag{2.4}$$

is a derivation, namely $\overline{\mathcal{D}}(f;t)(\alpha\wedge\beta) = \overline{\mathcal{D}}(f;t)\alpha\wedge\overline{\mathcal{D}}(f;t)\beta$, for all $\alpha, \beta \in \bigwedge M$.

2.3.2 It is easy to see that $\overline{D}_i(f)$ are uniquely determined by (2.2) and the properties i), ii), iii) above. For instance, if $m_1, m_2 \in M$

$$\begin{aligned}\overline{D}_2(f)(m_1\wedge m_2) &= \overline{D}_2(f)m_1\wedge m_2 + \overline{D}_1(f)m_1\wedge\overline{D}_1(f)m_2 +\\ &+ m_1\wedge\overline{D}_2(f)m_2\\ &= \overline{D}_1(f)m_1\wedge\overline{D}_1(f)m_2 = f(m_1)\wedge f(m_2),\end{aligned}$$

where in the first equality we used iii) to apply Leibniz's rule, in the second equality we used i) for the vanishing of $\overline{D}_2(f)m_i$ and, finally, the inital conditions prescribed by ii).

2.3.3 The *characteristic polynomial operator* $\overline{\mathcal{D}}(f;t)$ given by (2.4) is an invertible derivation. Let $\mathcal{D}(f,t)$ be its inverse:

$$\mathcal{D}(f;t) = \sum_{i\geq 0} D_j(f)t^j = \frac{1}{\overline{\mathcal{D}}(f;t)}, \tag{2.5}$$

where $D_0 = id_{\bigwedge M}$.

2.3.4 Remark. The equality $\overline{\mathcal{D}}(f;t)\mathcal{D}(f;t) = 1$ holding by definition implies

$$D_k(f) - \overline{D}_1(f)D_{k-1}(f) + \ldots + (-1)^k\overline{D}_k(f) = 0$$

for all $k \geq 1$.

2.3.5 Since $\bigwedge^r M$ is free of rank 1, there exist

$$e_1(f), e_2(f), \ldots, e_r(f) \in A \tag{2.6}$$

such that

$$\overline{D}_i(f)\zeta = e_i(f)\zeta, \qquad \forall\ 0 \neq \zeta \in \bigwedge^r M.$$

Let

$$E_r(f,t) = 1 - e_1(f)t + \ldots + (-1)^r e_r(f)t^r \in A[t]. \tag{2.7}$$

Similarly, let us denote by $h_i(f)$ the eigenvalues of $D_i(f)_{|\bigwedge^r M}$:

$$h_i(f)\zeta := D_i(f)\zeta, \qquad 0 \neq \zeta \in \bigwedge^r M.$$

Because of (2.5), one has the equality

$$E_r(f,t)H_r(f,t) = 1,$$

where we set

$$H_r(f,t) := \sum_{i\in\mathbb{Z}} h_i(f)t^i,$$

adopting the convention $h_j(f) = 0$ if $j < 0$. The eigenvalue $e_r(f)$ is classically said to be the *determinant* of f, while $e_1(f)$ is the *trace*. We shall call $e_i(f)$, $1 \leq i \leq r$, the *i-th trace* of f.

2.3.6 Proposition. *The equality*

$$D_i(f)(m) = f^i(m) \tag{2.8}$$

holds for all $m \in M$.

Proof. By induction on $i \geq 1$, the case $i = 0$ being obvious. If $i = 1$, one has

$$D_1(f)(m) = \overline{D}_1(f)(m) = f(m),$$

because $D_1(f) = \overline{D}_1(f)$ and by 2.3.1, item ii). Assuming the equality (2.8) for all $1 \leq j \leq i-1$:

$$D_i(f)(m) = (\overline{D}_1(f)D_{i-1}(f) + \ldots - (-1)^i\overline{D}_i(f))(m).$$

Induction, and the fact that $\overline{D}_i$ vanishes on M for $i > 1$, implies:

$$\begin{aligned} D_i(f)(m) &= \overline{D}_1(f)D_{i-1}(f)(m) = \overline{D}_1(f)(f^{i-1}(m)) \\ &= f(f^{i-1}(m)) = f^i(m). \end{aligned}$$

■

2.3.7 Recall that the multinomial coefficient

$$\binom{j}{i_1, \ldots, i_r} = \frac{j!}{i_1!i_2!\ldots i_r!}$$

is the coefficient of $t_1^{i_1} \cdot \ldots \cdot t_r^{i_r}$ in the expansion of $(t_1 + \ldots + t_r)^j$. In particular if $r = 2$ one recovers the classical definition of binomial coefficient. The following Lemma will be useful in the sequel and is very much related with the paper [11].

2.3.8 Lemma. *For each $\alpha, \beta \in \bigwedge M$, the* Newton binomial formula *holds:*

$$D_1(f)^i(\alpha \wedge \beta) = \sum_{i_1+i_2=i} \binom{i}{i_1, i_2} D_1(f)^{i_1}\alpha \wedge D_1(f)^{i_2}\beta \tag{2.9}$$

Proof. Is a matter of a simple induction on the integer i. One has:

$$D_1(f)(\alpha \wedge \beta) = D_1(f)\alpha \wedge \beta + \alpha \wedge D_1(f)\beta$$

and (2.9) holds for $i = 1$. Suppose now that (2.9) holds for all $1 \leq j \leq i-1$. Then

$$\begin{aligned} D_1(f)^i(\alpha \wedge \beta) &= D_1(f)\big(D_1(f)^{i-1}(\alpha \wedge \beta)\big) \\ &= D_1(f) \sum_{i_1+i_2=i-1} \binom{i-1}{i_1, i_2} D_1(f)^{i_1}(\alpha) \wedge D_1(f)^{i_2}(\beta) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1+i_2=i-1} \binom{i-1}{i_1, i_2} \big(D_1(f)^{i_1+1}(\alpha) \wedge D_1(f)^{i_2}(\beta) \\
&+ \ D_1(f)^{i_1}(\alpha) \wedge D_1(f)^{i_2+1}(\beta) \big) \\
&= \sum_{i_1+i_2=i} \binom{i}{i_1, i_2} D_1(f)^{i_1}(\alpha) \wedge D_1(f)^{i_2}(\beta),
\end{aligned}
$$

having used well known properties of the binomial coefficients. ∎

As a consequence we have:

2.3.9 Proposition. *The formula*

$$
D_1(f)^i(m_1 \wedge \ldots \wedge m_r) = \sum_{i_1+\ldots+i_r=i} \binom{i}{i_1 \ldots i_r} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}(m_r) \tag{2.10}
$$

holds for all $r \geq 2$.

Proof. The case $r = 2$ follows from (2.9) for $\alpha = m_1$, $\beta = m_2$ and the fact that $D_1(f)^j(m) = \overline{D}_1^j(f)(m) = f^j(m)$ if $m \in M$. Suppose now that formula (2.10) holds for all $\bigwedge^s M$ with $2 \leq s \leq r-1$. Then

$$
\begin{aligned}
& D_1(f)^i(m_1 \wedge \ldots \wedge m_{r-1} \wedge m_r) = \\
&= \sum_{j+i_r=i} \binom{i}{j, i_r} D_1(f)^j(m_1 \wedge \ldots \wedge m_{r-1}) \wedge D_1(f)^{i_r} m_r = \\
&= \sum_{j+i_r=i} \binom{i}{j, i_r} \sum_{i_1+\ldots+i_{r-1}=j} \binom{j}{i_1, \ldots, i_{r-1}} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}(m_r) = \\
&= \sum_{i_1+\ldots+i_r=i} \binom{i}{i_1, \ldots i_r} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}(m_r)
\end{aligned}
$$

as desired. ∎

2.3.10 Corollary. If $f \in End_A(M)$ and $\mathrm{rk}_A M = r$, then

$$
e_1(f)^i(m_1 \wedge \ldots \wedge m_r) = \sum_{i_1+\ldots+i_r=i} \binom{i}{i_1, \ldots . i_r} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}(m_r).
$$

Proof. Obvious, since $D_1(f)(m_1 \wedge \ldots \wedge m_r) = e_1(f)(m_1 \wedge \ldots \wedge m_r)$, by definition of $e_1(f)$. ∎

2.3.11 Proposition. *Consider* $f - t \cdot 1_M \in End_{A[t]}(M \otimes_A A[t])$. *Then*

$$e_r(f - t \cdot 1_M) = E_r(f, t)$$

Proof. If $0 \neq \zeta \in \bigwedge^r M$, then

$$\begin{aligned}
& e_r(f - t \cdot 1_M)m_1 \wedge \ldots \wedge m_r = \overline{D}_r(f - t \cdot 1_M)m_1 \wedge \ldots \wedge m_r \\
= \ & (f - t \cdot 1_M)m_0 \wedge \ldots \wedge (f - t \cdot 1_M)m_{r-1} \\
= \ & \sum_{j=0}^{r} (-1)^j \Big(\sum_{\substack{i_1 + \ldots + i_r = j \\ 0 \leq i_k \leq 1}} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}(m_r) \Big) t^j \\
= \ & \sum_{j=0}^{r} (-1)^j \overline{D}_j(f)(m_1 \wedge \ldots \wedge m_r) t^j \\
= \ & \sum_{j=0}^{r} (-1)^j e_j(f)(m_1 \wedge \ldots \wedge m_r) t^j = E_r(f, t) m_1 \wedge \ldots \wedge m_r.
\end{aligned}$$ ∎

2.4 Generic Cayley-Hamilton Theorem

The content of this section is among the outputs of the collaboration with Inna Scherbak begun in [32, 33].

2.4.1 In the following, the endomorphism f of M considered in Section 2.3.1 shall be understood. Thus, we shall write $D_i, \overline{D}_i$ instead of $D_i(f)$ and $\overline{D}_i(f)$. We shall also write $H_r(t) = \sum_{i \geq 0} h_i t^i$ and $E_r(t) = 1 + \sum_{i \geq 1} (-1)^i e_i t^i$ (convention: $e_i = 0$ if $i > r$). Our arbitrary endomorphism f is so identified with the restriction of $D_1 = \overline{D}_1$ to $M \cong \bigwedge^1 M$. The equality

$$E_r(t)\mathcal{D}(t) = \sum_{i \in \mathbb{Z}} \mathtt{U}_i(\mathcal{D}(t)) t^i \tag{2.11}$$

defines $\mathtt{U}_i(\mathcal{D}(t)) \in End_A(\bigwedge M)$ for $i \in \mathbb{N}$. Equality (2.11) implies that $\mathtt{U}_j(\mathcal{D}(t)) = 0$ if $j < 0$. Moreover:

$$\begin{aligned} \mathtt{U}_0(\mathcal{D}(t)) &= D_0 = 1, \\ \mathtt{U}_1(\mathcal{D}(t)) &= D_1 - e_1 D_0, \\ &\vdots \\ \mathtt{U}_r(\mathcal{D}(t)) &= D_r - e_1 D_{r-1} + \ldots + (-1)^r e_r D_0. \end{aligned}$$

2.4.2 Lemma. *The endomorphism* $\mathtt{U}_j(\mathcal{D}(t))$ *vanishes on* $\bigwedge^r M$ *for all* $j \geq 1$.

Proof. Relation $E_r(t)H_r(t) = 1$ implies that

$$h_j - e_1 h_{j-1} + \ldots + (-1)^r e_r h_{j-r} = 0$$

for all $j \geq 1$. Now, if $0 \neq \zeta \in \bigwedge^r M$:

$$\mathtt{U}_j(\mathcal{D}(t))\zeta = (h_j - e_1 h_{j-1} + \ldots + (-1)^r e_r h_{j-r})\zeta = 0. \qquad \blacksquare$$

2.4.3 Warning. In general, $\mathtt{U}_j(\mathcal{D}(t))$ does not vanish on the whole $\bigwedge M$ for $0 \leq j < r$. For example, let $M = \mathbb{R}^2$ and f be any invertible matrix. If $\mathtt{U}_1(f) = 0$, then

$$(f - e_1(f)\mathbb{1})v = 0$$

for all $v \in \mathbb{R}^2$, i.e. $e_1(f)$ (the trace of f) is an eigenvalue of f. Then f should have a null eigenvalue contradicting the invertibility.

In spite of its simplicity, the following statement is very important and one of our key tool to discover vertex operators in elementary context as well as to compute them (See Theorem 4.3.1).

2.4.4 Lemma. *The* integration by parts *formula holds:*

$$\mathcal{D}(t)\alpha \wedge \beta = \mathcal{D}(t)(\alpha \wedge \overline{\mathcal{D}}(t)\beta). \tag{2.12}$$

Proof. In fact

$$\mathcal{D}(t)\alpha \wedge \beta = \mathcal{D}(t)\alpha \wedge \mathcal{D}(t)\overline{\mathcal{D}}(t)\beta = \mathcal{D}(t)(\alpha \wedge \overline{\mathcal{D}}(t)\beta),$$

using the fact that $\mathcal{D}(t)$ and $\overline{\mathcal{D}}(t)$ are inverse of each other and that $\mathcal{D}(t)$ is a derivation. ∎

Taking the coefficient of t^j on both sides of (2.12), for each $j \geq 1$, formula (2.12) implies the following more explicit identity,

$$\begin{aligned} D_j\alpha\wedge\beta &= D_j(\alpha\wedge\beta) - D_{j-1}(\alpha\wedge\overline{D}_1\beta) + \ldots + (-1)^j\alpha\wedge\overline{D}_j\beta \\ &= \sum_{i=0}^{j}(-1)^i D_{j-i}(\alpha\wedge\overline{D}_i\beta). \end{aligned} \tag{2.13}$$

2.4.5 Corollary.

$$[\mathtt{U}_i(\mathcal{D}(t))\alpha]\wedge\beta = \sum_{j=0}^{i}\mathtt{U}_{i-j}(\mathcal{D}(t))(\alpha\wedge\overline{D}_j\beta). \tag{2.14}$$

Proof. In fact, invoking (1.11)

$$\sum_{i\geq 0}(\mathtt{U}_i(\mathcal{D}(t))\alpha)t^i\wedge\beta = E_r(t)\mathcal{D}(t)\alpha\wedge\beta. \tag{2.15}$$

Using integration by parts (2.12):

$$\begin{aligned} &= E_r(t)(\mathcal{D}(t)(\alpha\wedge\overline{\mathcal{D}}(t)\beta)) = \sum_{j\geq 0}\mathtt{U}_j(\mathcal{D}(t))(\alpha\wedge\overline{\mathcal{D}}(t)\beta)t^j \\ &= \sum_{k\geq 0}\mathtt{U}_k(\mathcal{D}(t))(\alpha\wedge\sum_{j\geq 0}(-1)^j\overline{D}_j(t)\beta)t^{j+k} \\ &= \sum_{i\geq 0}\left(\sum_{j=0}^{i}\mathtt{U}_{i-j}(\mathcal{D}(t))(\alpha\wedge\overline{D}_{i-j}(t)\beta)\right)t^i. \end{aligned} \tag{2.16}$$

Comparing the coefficients of t^i in the l.h.s. of (2.15) and the r.h.s. of (2.16), gives precisely equality (2.14). ∎

Using Lemma 2.4.4 and Corollary 2.4.5 we can finally prove

2.4.6 Theorem (Cayley-Hamilton). *The endomorphism $\mathtt{U}_{r+j}(\mathcal{D}(t))$ vanishes on the whole exterior algebra $\bigwedge M$.*

Proof. It is obvious that $\mathtt{U}_{r+j}(\mathcal{D}(t))\alpha = 0$ for $\alpha\in\bigwedge^0 M := A$ and for $\alpha\in\bigwedge^r M$, due to Lemma 2.4.2. Assume then $\alpha\in\bigwedge^i M$, for

$0 < i < r$, and let us show that $\mathtt{U}_{r+j}(\mathcal{D}(t))\alpha = 0$ for all $j \geq 0$. To prove this, it suffices to prove that $\mathtt{U}_{r+j}(\mathcal{D}(t))\alpha \wedge \beta = 0$ for all $\beta \in \bigwedge^{r-i} M$. Using (2.14):

$$\begin{aligned}\mathtt{U}_{r+j}(\mathcal{D}(t))\alpha \wedge \beta &= \sum_{k=0}^{r-1} \mathtt{U}_{r+j-k}(\mathcal{D}(t))(\alpha \wedge \overline{D}_k\beta) \\ &+ \sum_{k=r}^{r+j} \mathtt{U}_{r+j-k}(\mathcal{D}(t))(\alpha \wedge \overline{D}_k\beta).\end{aligned}$$

But $\overline{D}_k\beta = 0$ for all $k \geq r$, because $\beta \in \bigwedge^{r-i} M$ and $r - i < r$, by item i) of 2.3.1. Moreover, for each $k \geq 0$, $\alpha \wedge \overline{D}_k\beta \in \bigwedge^r M$, hence $\mathtt{U}_{r+j-k}(\mathcal{D}(t))$ vanishes on it. ∎

2.4.7 Recall, by Remark 2.3.4, that

$$D_j = D_{j-1}\overline{D}_1 - D_{j-2}\overline{D}_2 + \ldots - (-1)^j\overline{D}_j.$$

Since $\overline{D}_k m = 0$ for all $k \geq 2$ and all $m \in M$ (Cf. 2.3.1 item i)), the equality $D_j m = \overline{D}_1 D^{j-1} m = D_1 D^{j-1} m$ holds. An easy induction then shows $D_j m = D_1^j m$ for all $m \in M$.

2.4.8 Corollary. (Cayley–Hamilton Theorem, classical form)

$$\mathtt{p}_r(D_1) := D_1^r - e_1 D_1^{r-1} + \ldots + (-1)^r e_r = 0.$$

Proof. In fact

$$\begin{aligned} & (D_1^r - e_1 D_1^{r-1} + \ldots + (-1)^r e_r)(m) \\ = \ & (D_r - e_1 D_{r-1} + \ldots + (-1)^r e_r)m = 0,\end{aligned}$$

where last equality is due to Theorem 2.4.6. ∎

2.5 The Exponential of an Endomorphism.

Notation as in the previous sections. Let $f \in \mathrm{End}_A(M)$ and $e_i(f) \in A$ be its i-th trace. Equip A with the B_r-module structure induced by the unique $\mathbb{Z}$-algebra homomorphism $B_r \to A$ mapping $e_i \mapsto e_i(f)$. If t is an indeterminate over A, by $(\mathbb{1}_M - tf)^{-1} \in End_A(M)[[t]]$ we mean the formal power series $\mathbb{1} + ft + f^2t^2 + \ldots$

2.5.1 Proposition. *The following equality holds (notation as in 1.3.5):*

$$(\mathbb{1}_M - tf)^{-1} = \mathbb{1}_M u_0 + \mathtt{p}_1(f)u_{-1} + \ldots + \mathtt{p}_{r-1}(f)u_{-r-1}. \quad (2.17)$$

Proof. In fact:

$$E_r(t)(\mathbb{1}_M - tf)^{-1} = \sum_{i\geq 0} \mathtt{U}_i((\mathbb{1}_M - tf)^{-1})t^i = \sum_{i\geq 0} \mathtt{p}_i(f)t^i. \quad (2.18)$$

According to Cayley–Hamilton Theorem 2.4.8 , $\mathtt{p}_{r+i}(f) = 0$ for all $i \geq 0$, and thus the right hand side of (2.18) is a finite sum

$$E_r(t)(\mathbb{1}_M - tf)^{-1} = \sum_{i=0}^{r-1} \mathtt{p}_i(f)t^i.$$

Dividing both sides by $E_r(t)$ and using the definition of u_{-i}, we finally obtain (2.17). ∎

We note in passing that

$$\det[(\mathbb{1} - tf)^{-1}] = \frac{1}{e_r(\mathbb{1}_M - tf)} = \frac{1}{E_r(f,t)} = H_r(f,t) = u_0 \otimes 1_A.$$

2.5.2 Definition. *Assume that A is a $\mathbb{Q}$-algebra. Then* $\exp(tf)$ *is the formal power series defined by:*

$$\begin{aligned} \exp(tf) &= L^{-1}((\mathbb{1}_M - tf)^{-1}) = L^{-1}(1 + ft + f^2t^2 + \ldots) \\ &= \sum_{j\geq 0} \frac{f^j t^j}{j!} \in End_A(M)[[t]]. \end{aligned} \quad (2.19)$$

2.5.3 Proposition. *One has:*

$$\exp(tf) = \mathtt{p}_0(f)L^{-1}(u_0) + \ldots + \mathtt{p}_{r-1}(f)L^{-1}(u_{-r+1}). \quad (2.20)$$

Proof. In fact

$$\begin{aligned} \exp(tf) &= L^{-1}((\mathbb{1}_M - tf)^{-1}) \\ &= L^{-1}(\sum_{j=0}^{r-1} \mathtt{p}_j(f)u_{-j}) = \sum_{j=0}^{r-1} \mathtt{p}_j(f)L^{-1}(u_{-j}), \end{aligned}$$

as desired. ∎

The finite sum expression (2.20) to write $\exp(tf)$ permits to identify it with an endomorphism of $A[[t]] \otimes_A M$. The latter is a free $A[[t]]$-module of rank r. It is then meaningful to speak of its determinant. The following is a well known result, here spelled in a purely algebraic context.

2.5.4 Proposition. *The determinant of the exponential is the exponential of the trace:*

$$e_r(\exp(tf)) = \exp(e_1(f)t).$$

Proof. By definition:

$$\begin{aligned} e_r(\exp(tf))m_1 \wedge \ldots \wedge m_r : \quad &= \quad \overline{D}_r(\exp(tf))(m_1 \wedge \ldots \wedge m_r) \\ &= \overline{D}_1(\exp(tf))m_1 \wedge \ldots \wedge \overline{D}_1(\exp(tf))m_r \\ &= \exp(tf)m_1 \wedge \ldots \wedge \exp(tf)m_r. \end{aligned} \tag{2.21}$$

However, (2.21) is nothing but:

$$\sum_{i_1 \geq 0} f^{i_1}(m_1)\frac{t^{i_1}}{i_1!} \wedge \ldots \wedge \sum_{i_r \geq 0} f^{i_r}(m_1)\frac{t^{i_r}}{i_r!}$$

which is in turn equal to

$$\sum_{j \geq 0} \frac{t^j}{j!} \sum_{i_1+\ldots+i_r=j} \binom{j}{i_1, \ldots, i_r} f^{i_1}(m_1) \wedge \ldots \wedge f^{i_r}, (m_r)$$

that is, using Corollary 2.3.10:

$$= \sum_{j \geq 0} e_1^j(f)\frac{t^j}{j!}(m_1 \wedge \ldots \wedge m_r) = \exp(e_1(f)t)m_1 \wedge \ldots \wedge m_r,$$

proving the desired formula. ∎

Formula (2.20) and Proposition 2.5.4 give rise to some interesting identities.

2.5.5 Example. Let

$$f := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

thought of as endomorfism of $M := A^2$ where $A := \mathbb{Q}[a,b,c,d]$. We make A into a B_2-algebra through the unique homomorphism mapping $e_1 \mapsto a+d$ and $e_2 := \det(f) = ad - bc$. Let $u_0 := H_2(t)$ and $u_{-1} = tH_2(t)$. Then

$$(\mathbb{1}_M - tf)^{-1} = \mathbb{1}_M u_0 + (f - e_1\mathbb{1}_M)u_{-1} = \begin{pmatrix} u_0 - du_{-1} & bu_{-1} \\ cu_{-1} & u_0 - au_{-1} \end{pmatrix}$$

and

$$\begin{aligned} \exp(tf) &= L^{-1}((\mathbb{1}_M - tf)^{-1}) \\ &= \mathbb{1}_M L^{-1}(v_0) + (f - e_1\mathbb{1}_M)L^{-1}(v_{-1}) \\ &= \begin{pmatrix} v_0 - dv_{-1} & bv_{-1} \\ cv_{-1} & v_0 - av_{-1} \end{pmatrix}, \end{aligned}$$

where by simplicity we set $v_i := L^{-1}(u_i)$. Then

$$e_2(\exp(tf)) = v_0^2 - e_1 v_0 v_{-1} + e_2 v_{-1}^2.$$

On the other hand $e_2(\exp(tf)) = \exp(e_1(f)t) = \exp(e_1 t)$, from which the identity

$$v_0^2 - e_1 v_0 v_{-1} + e_2 v_{-1}^2 = \exp(e_1 t).$$

In particular, if $e_1 = 0$ we have the relation

$$v_0^2 + e_2 v_{-1}^2 = 1.$$

which, if $e_2 = 1$ is the classical formula $\cos^2 t + \sin^2 t = 1$ (Cf. [25, p. 64] for additional details). Taking into account that $\partial_t v_{-1} = v_0$ and putting $y = v_{-1}$ we obtain

$$(y')^2 + e_2 y^2 = 1$$

which is a prime integral of $y'' + e_2 y = 0$.

2.5.6 Example. It is possible to obtain similar formulas for higher order square matrices. For instance let $\mathcal{M}$ be a 3×3 matrix having e_1, e_2 and e_3 as traces. Then

$$\exp(t\mathcal{M}) = v_0 \mathbb{1}_{3\times 3} + v_{-1}(\mathcal{M} - e_1 \mathbb{1}_{3\times 3}) + v_{-2}(\mathcal{M}^2 - e_1\mathcal{M} + e_2 \mathbb{1}_{3\times 3}),$$

where $v_i = L^{-1}(u_i)$, i.e. (v_0, v_{-1}, v_{-2}) is the canonical basis of solutions to the generic linear ODE $y''' - e_1 y'' + e_2 y' - e_3 = 0$. Then a tedious but simple computation shows that

$$\det(\exp(t\mathcal{M})) = \exp(e_1 t)$$

can be explicitly phrased as:

$$\begin{aligned}\exp(e_1 t) = {} & v_0^3 - 2e_1 v_0^2 v_{-1} + (e_1^2 + e_2) v_0^2 v_{-2} + (e_2 + e_1^2) v_0 v_{-1}^2 + \\ & - (e_1 e_2 + 3e_3) v_0 v_{-1} v_{-2} + e_1 e_3 v_0 v_{-2}^2 + (e_3 - e_1 e_2) v_{-1}^3 + \\ & + (e_1 e_3 + e_2^2) v_{-1}^2 v_{-2} + (e_1 e_2^2 - 2e_2 e_3 - 2e_1^2 e_3) v_{-1} v_{-2}^2 + e_3^2 v_{-2}^3.\end{aligned}$$

If $e_1 = 0$ one obtains

$$\begin{aligned} v_0^3 \quad + & \quad e_2 v_0^2 v_{-2} + e_2 v_0 v_{-1}^2 - 3e_3 v_0 v_{-1} v_{-2} + e_3 v_{-1}^3 + \\ + & \quad e_2^2 v_{-1}^2 v_{-2} - 2e_2 e_3 v_{-1} v_{-2}^2 + e_3^2 v_{-2}^3 = 1. \end{aligned}$$

Putting $e_2 = 0$ and $e_3 = -1$, one deduce the relation

$$v_0^3 + 3v_0 v_{-1} v_{-2} - v_{-1}^3 + v_{-2}^3 = 1 \tag{2.22}$$

between the canonical solutions of the linear ODE $y''' + y = 0$. Formula (2.22) can be then seen as a cubic generalization of the popular relation $\cos^2 t + \sin^2 t = 1$ enjoyed by the canonical solutions $v_0 = \cos t$ and $v_{-1} = \sin t$ of the second order linear ODE $y'' + y = 0$. Setting $y := v_{-2}$, equation (2.22) gives

$$(y'')^3 + 3y''y' - (y')^3 + y^3 = 1. \tag{2.23}$$

Notice that differentiating (2.23) gives

$$(y''' + y)((y'')^2 + yy') = 0.$$

In other words the prime integral (2.23) is obtained from $y''' + y = 0$ via multiplication by the factor $(y'')^2 + yy'$.

Chapter 3

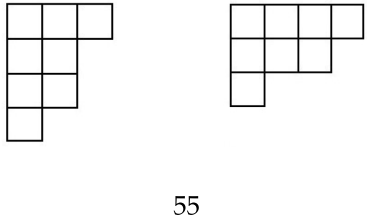
Exterior Algebra of a Free Abelian Group

3.1 Schur Polynomials.

3.1.1 Recall that a *partition* is a monotonic non-increasing sequence of non negative integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$$

All the *parts* λ_i are zero but finitely many. Let $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots)$ be a partition. Its *length* $\ell(\boldsymbol{\lambda})$ is the number of its non zero parts. Its *weight* is $|\boldsymbol{\lambda}| = \sum_i \lambda_i$. Let $\mathcal{P}$ be the set of all of partitions. Then $\boldsymbol{\lambda} \in \mathcal{P}$ is a partition of the integer $w := |\boldsymbol{\lambda}|$. The *Young diagramme* of a partition $(\lambda_1, \ldots, \lambda_r)$ is an array $Y(\boldsymbol{\lambda})$ of left justified rows of boxes, such that the first row has λ_1 boxes, ..., the r-th has λ_r-boxes. Below is depicted the Young diagramme of the partition $(3, 2, 2, 1)$ and of its *conjugated* $(4, 3, 1)$



The number of partitions of a given integer is ruled by a generating function essentially due to *Euler*:

$$\sum_{k\geq 0} p(k)q^k = \frac{1}{\prod_{k\in\mathbb{N}}(1-q^k)}.$$

3.1.2 Denote by $\mathcal{P}_r$ the set of partitions of length $k \leq r$: if the length is strictly less than r we may, according to the convenience, add a string of $r-k$ zeros. To each partition $\boldsymbol{\lambda} \in \mathcal{P}_r$ we associate an element $\Delta_{\boldsymbol{\lambda}}(H_r) \in B_r$ to be computed as follows:

$$\Delta_{\boldsymbol{\lambda}}(H_r) = \det(h_{\lambda_j - j + i})_{1\leq i,j\leq r}.$$

More explicitly:

$$\Delta_{\boldsymbol{\lambda}}(H_r) := \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2-1} & \cdots & h_{\lambda_r-r+1} \\ h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r} \end{vmatrix}.$$

In practical terms, one allocates $h_{\lambda_1}, \ldots, h_{\lambda_r}$ along the principal diagonal from top down. Then, *above* each h_{λ_i}, in the same column, the index decreases by one unit for each row and *below* h_{λ_i}, in the same column, increases by one unit per row. The element $\Delta_{\boldsymbol{\lambda}}(H_r) \in B_r$ is called *Schur polynomial* associated to the partition $\boldsymbol{\lambda}$. Using Schur determinants one can check that

$$e_j = \Delta_{(1^j)}(H_r),$$

where by (1^j) denotes the partition $\underbrace{(1,\ldots,1)}_{j\text{ times}}$ with j parts equal to 1.

3.1.3 Example. Let $\boldsymbol{\lambda}$ be the partition $(2,2,1)$ of the integer 5. Then, according to 3.1.5:

$$\Delta_{(2,2,1)}(H_r) = \begin{vmatrix} h_2 & h_1 & 0 \\ h_3 & h_2 & 1 \\ h_4 & h_3 & h_1 \end{vmatrix} = h_1h_2^2 - h_1^2h_3 - h_2h_3 + h_1h_4$$

3.1.4 Exercise. Show that if $1 \leq r \leq 2$ then $\Delta_{(2,2,1)}(H_r) = 0$.

3.1.5 The ring B_r is a graded ring. Giving the indeterminate e_i weight i ($\leq i \leq r$) each monomial

$$e_1^{i_1}e_2^{i_2}\cdot\ldots\cdot e_r^{i_r}$$

has degree $i_1 + 2i_2 + \ldots + ri_r$. To each such a monomial one may associate a partition $\tilde{\boldsymbol{\lambda}} = (1^{i_1}, 2^{i_2}, \ldots, r^{i_r})$, having i_1 parts equal to 1, i_2 parts equal to 2, ..., i_r parts equal to r. The transpose of the Young diagramme of any such a partition is the Young diagramme of a partition $\boldsymbol{\lambda}$ of length at most r and of the same weight. By abuse of notation we shall write $e^{\boldsymbol{\lambda}}$ to denote the monomial $e_1^{i_1}e_2^{i_2}\cdot\ldots\cdot e_r^{i_r}$, which has degree $|\boldsymbol{\lambda}| = i_1 + 2i_2 + \ldots + ri_r$, in $e_1, \ldots, e_r$. Then

$$B_r = \bigoplus_{k\geq 0}(B_r)_k$$

where $(B_r)_k = \bigoplus_{|\boldsymbol{\lambda}|=k}\mathbb{Z}e^{\boldsymbol{\lambda}}$.

3.1.6 Example Let $r = 3$. Then $e^{(411)} = e_1^3e_3$, $e^{(2,2,2)} = e_3^2$, $e^{(321)} = e_1e_2e_3$, $e^{(111)} = e_3$, $e^{(3)} = e_1^3$.

3.2 Shift Endomorphisms

From this section onwards, a free abelian group $M_0 := \bigoplus_{i\in\mathbb{N}}\mathbb{Z}b_i$ (e.g. $:= \mathbb{Z}[X]$) will be fixed once and for all. Denote by $\mathcal{B}_0$ the basis $(b_0, b_1, \ldots)$ of M_0. Imitating (and abusing) the notation of Section 1.1.6 in another (but related) context, we shall denote by D_1 the shift endomorphism of M_0 given by $D_1b_j = b_{j+1}$.

3.2.1 Let $M_{0,n} := \bigoplus_{i=0}^{n-1}\mathbb{Z}b_i$: it is a submodule of M_0, direct summand of $D_nM_0 := D_1^nM_0 = \sum_{i\geq n}\mathbb{Z}\cdot b_i$. We aim to make M_0, $M_{0,n}$ and D_nM_0 into modules over B_r, with the purpose of studying the locus of *decomposable tensors* in both $\bigwedge^r M_0$ and in $\bigwedge^r M_{0,n}$, as in the forthcoming Chapter 4.

By $D_i\mathcal{B}_0$ we denote the basis $(b_i, b_{i+1}, \ldots)$ of D_iM_0 (in particular $D_0\mathcal{B}_0 = \mathcal{B}_0$). For all $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_r$ let

$$\mathbf{b}_{r,i+\boldsymbol{\lambda}} := b_{i+\lambda_r}\wedge b_{i+1+\lambda_{r-1}}\wedge\ldots\wedge b_{i+r-1+\lambda_1} \in \bigwedge^r D_iM_0 \qquad (3.1)$$

and, accordingly,

$$\mathbf{b}_{r,i} := \mathbf{b}_{r,i+(0)} = b_i \wedge b_{i+1} \wedge \ldots \wedge b_{i+r-1} \in \bigwedge^r D_i M_0.$$

3.2.2 If $i = 0$, we set $\mathbf{b}_{r,\boldsymbol{\lambda}} := \mathbf{b}_{r,0+\boldsymbol{\lambda}}$, so that

$$\mathbf{b}_{r,0} = b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}.$$

The *weight* of $\mathbf{b}_{r,\boldsymbol{\lambda}}$ is by definition $|\boldsymbol{\lambda}| := \sum_{i=1}^r \lambda_i$. Let

$$(\bigwedge^r M_0)_w := \oplus_{|\boldsymbol{\lambda}|=w} \mathbb{Z} \cdot \mathbf{b}_{r,\boldsymbol{\lambda}} \tag{3.2}$$

be the sub-module of $\bigwedge^r M_0$ generated by monomials of weight w. The *weight graduation* of $\bigwedge^r M_0$ is:

$$\bigwedge^r M_0 = \bigoplus_{w \geq 0} (\bigwedge^r M_0)_w. \tag{3.3}$$

3.2.3 For our purposes it is also important to consider the shift endomorphism of step -1:

$$D_{-1} b_j := \begin{cases} b_{j-1} & \text{if} \quad j \geq 1, \\ 0 & \text{if} \quad j = 0. \end{cases} \tag{3.4}$$

The homomorphisms $D_{\pm 1}$ have degree ± 1 with respect to the weight graduation (3.3) of $M_0 = \bigwedge^1 M_0$ ($wt(b_j) = j$) and D_{-1} is *locally nilpotent*, i.e. for all $m \in M_0$ there exists $j := j(m)$ such that $(D_{-1})^i m = 0$ for all $i \geq j$.

3.2.4 Denote by $\overline{\mathcal{D}}_+(z), \overline{\mathcal{D}}_-(z) \in \mathrm{End}_{\mathbb{Z}}(\bigwedge M_0)[z^{-1}]$ the *characteristic polynomial series* associated to D_1 and D_{-1} respectively, defined precisely like in Section 2.3.1. Recall that, in particular, this means $\overline{\mathcal{D}}_\pm(z)(\alpha \wedge \beta) = \overline{\mathcal{D}}_\pm(z)\alpha \wedge \overline{\mathcal{D}}_\pm(z)\beta$).

We shall also need the series

$$\mathcal{D}_+(z) := \sum_{i \geq 0} D_i z^i \qquad \text{and} \qquad \mathcal{D}_-(z) := \sum_{i \geq 0} D_{-i} z^{-i},$$

such that $\overline{\mathcal{D}}_+(z)\mathcal{D}_+(z) = \mathbb{1} \in End_{\mathbb{Z}}(\bigwedge M_0)[[z]]$ and $\overline{\mathcal{D}}_-(z)\mathcal{D}_-(z) = \mathbb{1} \in End_{\mathbb{Z}}(\bigwedge M_0)[[z^{-1}]]$. Observe that for each $\alpha \in \bigwedge M_0$ we have

$\mathcal{D}_-(z)\alpha \in \bigwedge M_0[z]$. Furthermore, once one declares that the degrees of $\overline{D}_j$ and D_j are j, D_i is an explicit homogeneous polynomial expression of degree i in $(\overline{D}_1, \overline{D}_2, \overline{D}_3, \ldots)$. This is because the equality $\overline{\mathcal{D}}_+(z)\mathcal{D}_+(z) = \mathbb{1}$ implies the identities

$$D_i + \textstyle\sum_{j=1}^{i}(-1)^j\overline{D}_j D_{i-j} = 0.$$

The same holds verbatim for D_{-i}: it is a homogeneous polynomial expression of degree $-i$ in $(\overline{D}_{-1}, \overline{D}_{-2}, \ldots)$, once one declares that the degree of both $\overline{D}_{-j}$ and D_{-j} is $-j$. The definition makes obvious that D_i (resp. $\overline{D}_i, \overline{D}_{-i}, D_{-i}$) are $\mathbb{Z}$–homomorphisms of $\bigwedge M_0$, homogeneous of degree zero, i.e. each degree $\bigwedge^r M_0$ of the exterior algebra is an invariant submodule for them. When restricted to $\bigwedge^r M_0$, they are homogeneous of degree $\pm i$ with respect to the weight graduation (3.3). Let $\mathcal{A}(\mathcal{D}_+)$ be the minimal $\mathbb{Z}$-sub-algebra of $End_{\mathbb{Z}}(\bigwedge M_0)$ containing $\mathcal{D}_+ := (D_1, D_2, \ldots)$. Clearly $\overline{D}_j \in \mathcal{A}(\mathcal{D}_+)$ for all $j \geq 0$. It turns out that $\mathcal{A}(\mathcal{D}_+)$ is a commutative sub-algebra of $\bigwedge M_0$, by virtue of Proposition 3.2.5 below.

3.2.5 Proposition. *For all $\alpha \in \bigwedge M_0$ and all $i, j \geq 0$:*

$$D_i D_j \alpha = D_j D_i \alpha. \tag{3.5}$$

Proof. We know that $D_i D_j m = D_1^{i+j} m = D_j D_i m$, for all $i, j \geq 0$ and all $m \in M$. Suppose (3.5) holds for all $\alpha \in \bigwedge^i M_0$ and all $1 \leq i \leq r$. Writing $\alpha \in \bigwedge^{r+1} M$ as $\sum_{finite} m_i \wedge \beta_i$, it suffices to check the commutativity of D_i and D_j on elements of the form $m \wedge \beta$. Now:

$$\begin{aligned} D_i D_j (m \wedge \beta) &= D_i\Big(\sum_{j_1=0}^{j} D_{j_1} m \wedge D_{j-j_1}\beta\Big) \\ &= \sum_{i_1=0}^{i}\sum_{j_1=0}^{j} D_{i_1} D_{j_1} m \wedge D_{i-i_1} D_{j-j_1}\beta. \end{aligned} \tag{3.6}$$

By the inductive hypothesis, the last side of (3.6) is equal to

$$\begin{aligned} &\sum_{j_1=0}^{j}\sum_{i_1=0}^{i} D_{j_1} D_{i_1} m \wedge D_{j-j_1} D_{i-i_1}\beta \\ = \;& D_j\Big(\sum_{i_1=0}^{i} D_{i_1} m \wedge D_{i-i_1}\beta\Big) = D_j D_i (m \wedge \beta). \quad \blacksquare \end{aligned}$$

3.2.6 Proposition. *For all integers $\lambda \geq 0$ and $\alpha \in \bigwedge M_0$*

$$D_\lambda(b_i \wedge \alpha) = b_i \wedge D_\lambda \alpha + D_{\lambda-1}(b_{i+1} \wedge \alpha). \tag{3.7}$$

Proof. The best way to prove (3.7) is to use integration by parts (2.12). One has:

$$\begin{aligned} b_i \wedge \mathcal{D}_+(z)\alpha &= \mathcal{D}_+(z)(\overline{\mathcal{D}}_+(z)b_i \wedge \alpha) \\ &= \mathcal{D}_+(z)(b_i \wedge \alpha) - \mathcal{D}_+(z)(b_{i+1} \wedge \alpha) \end{aligned}$$

from which

$$\mathcal{D}_+(z)(b_i \wedge \alpha) = b_i \wedge \mathcal{D}_+(z)\alpha + \mathcal{D}_+(z)(b_{i+1} \wedge \alpha). \tag{3.8}$$

Formula (3.7) then follows by taking the coefficient of z^λ on both sides of 3.8. ∎

3.2.7 Corollary. *The following equalities hold for all $k \geq 1$:*

i) $\overline{\mathcal{D}}_+(z)(b_0 \wedge \ldots \wedge b_{k-1}) \wedge b_k = b_0 \wedge b_1 \wedge \ldots \wedge b_k$;

ii) $D_\lambda(b_0 \wedge \ldots \wedge b_{k-1} \wedge b_k \wedge \alpha) = b_0 \wedge \ldots \wedge b_{k-1} \wedge D_\lambda(b_k \wedge \alpha)$;

iii) $D_\lambda(b_0 \wedge \ldots \wedge b_{r-1}) = b_0 \wedge \ldots \wedge b_{r-2} \wedge b_{r-1+\lambda}$.

Proof. To prove item i), we argue by induction. For $k = 1$, the property is true:

$$(\overline{\mathcal{D}}_+(z)b_0) \wedge b_1 = (b_0 - b_1 z) \wedge b_1 = b_0 \wedge b_1.$$

Suppose it holds for all positive integers less or equal than $k-1$. Then

$$(\overline{\mathcal{D}}_+(z)b_0 \wedge \ldots \wedge b_{k-1}) \wedge b_k = \overline{\mathcal{D}}_+(z)(b_0 \wedge \ldots \wedge b_{k-2}) \wedge \overline{\mathcal{D}}_+(z)b_{k-1} \wedge b_k$$

$$= \overline{\mathcal{D}}_+(z)(b_0 \wedge \ldots \wedge b_{k-2}) \wedge (b_{k-1} - b_k z) \wedge b_k = b_0 \wedge \ldots \wedge b_{k-2} \wedge b_{k-1} \wedge b_k,$$

as desired.

To prove item ii) we use integration by parts (2.12) and i):

$$b_0 \wedge \ldots \wedge b_{k-1} \wedge \mathcal{D}_+(z)(b_k \wedge \alpha) \tag{3.9}$$

$$= \mathcal{D}_+(z)(\overline{\mathcal{D}}_+(z)(b_0 \wedge \ldots \wedge b_{k-1}) \wedge b_k \wedge \alpha)$$

$$= \mathcal{D}_+(z)(b_0 \wedge \ldots \wedge b_k \wedge \alpha). \tag{3.10}$$

Equating the coefficients of z^λ of (3.9) and (3.10) gives ii). Item iii) follows from ii) putting $\alpha = b_{r-1}$. ∎

3.3 The B_r-Module Structure of $\bigwedge^r M_0$

Define a B_r-module structure on $\bigwedge^r M_0$ by imposing the equality

$$e_i \cdot \alpha = \overline{D}_i\alpha, \qquad \forall \alpha \in \bigwedge^r M_0. \tag{3.11}$$

By construction $\bigwedge^r M_0$ is an eigen-module of all $\overline{D}_j$ with e_j as eigenvalue. Let $\mathbb{Z}[X] := \mathbb{Z}[X_1, X_2, \ldots]$ and $\mathrm{ev}_{\mathcal{D}}$ be the composition

$$\mathbb{Z}[\mathbf{X}] \xrightarrow{X_i \mapsto h_i} B_r \xrightarrow{\mathrm{ev}_{\overline{\mathcal{D}}_+}} End_A(\bigwedge^r M_0),$$

which is the unique $\mathbb{Z}$-algebra homomorphism mapping $X_i \mapsto D_i$. For $P \in \mathbb{Z}[\mathbf{X}]$ let $P(\mathcal{D}_+) := \mathrm{ev}_{\mathcal{D}_+}(P) \in End_{\mathbb{Z}}(\bigwedge^r M_0)$. Call $\mathcal{A}_r(\mathcal{D}_+)$ its image. Clearly $\mathcal{A}_r(\mathcal{D}_+)$ is the restriction to $\bigwedge^r M_0$ of the algebra $\mathcal{A}(\mathcal{D}_+)$, introduced in 3.2.4. By abuse of notation we have denoted by the same symbol the restriction of D_i to $\bigwedge^r M_0$.

3.3.1 Proposition. *The ring $\mathcal{A}_r(\mathcal{D}_+)$ coincides with $\mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r] := \mathrm{ev}_{\overline{\mathcal{D}}_+}(\mathbb{Z}[X_1, \ldots, X_r])$, where we have denoted by the same symbol $\overline{D}_i$ its restriction to $\bigwedge^r M_0$.*

Proof. Since $\mathcal{A}_r(\mathcal{D}_+)$ contains $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_r$, the inclusion $\mathcal{A}_r(\mathcal{D}_+) \supseteq \mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r]$ is clear. In addition $\mathcal{A}_r(\mathcal{D}_+)$ is the restriction of a polynomial algebra generated by $(\overline{D}_1, \overline{D}_2, \ldots)$, which implies the reversed inclusion $\mathcal{A}_r(\mathcal{D}_+) \subseteq \mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r]$. In particular $\mathcal{A}_r(\mathcal{D}_+)$ is the minimal commutative sub-algebra of $\mathrm{End}_{\mathbb{Z}}(\bigwedge^r M_0)$ containing $(D_1, D_2, \ldots)$ because $\mathbb{Z}[D_1, D_2, \ldots]$ is. ■

We have the natural $\mathbb{Z}$-module evaluation homomorphism:

$$\begin{cases} \mathrm{ev}_{\mathbf{b}_{r,0}} & : & B_r & \longrightarrow & \bigwedge^r M_0 \\ & & P(e_1, \ldots, e_r) & \longmapsto & P(\overline{\mathcal{D}}_+)\mathbf{b}_{r,0} \end{cases} \tag{3.12}$$

3.3.2 Proposition. *The map (3.12) is a $\mathbb{Z}$-module epimorphism.*

Proof. Given that D_i is a weighted homogeneous polynomial expression in $\overline{D}_1, \ldots, \overline{D}_r$, of degree i, to prove the claim amounts to show that for all $\boldsymbol{\lambda} \in \mathcal{P}_r$ there exists $G_{\boldsymbol{\lambda}} \in \mathbb{Z}[X_1, X_2, \ldots]$ such that

$$\mathbf{b}_{r,\boldsymbol{\lambda}} = G_{\boldsymbol{\lambda}}(\mathcal{D}_+)\mathbf{b}_{r,0}. \tag{3.13}$$

We argue by induction on the length of the partition. If $\boldsymbol{\lambda} = (\lambda)$ we have

$$\mathbf{b}_{(\lambda)} = b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1+\lambda} = D_\lambda(b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}),$$

because of 3.2.7, item ii). In this case $G_{(\lambda)}(D) = D_\lambda$. Assume now that the property holds for all partitions of length at most $k-1 \leq r-1$. Then

$$\begin{aligned} & b_0 \wedge \ldots \wedge b_{r-k+\lambda_k} \wedge \ldots \wedge b_{r-1+\lambda_1} \\ = \; & D_{\lambda_k}(b_0 \wedge \ldots \wedge b_{r-k}) \wedge b_{r-k+1+\lambda_{k-1}} \wedge \ldots \wedge b_{r-1+\lambda_1} \\ = \; & \sum_{i=0}^{\lambda_k} (-1)^i D_{\lambda_k - i}(\mathbf{b}_{r-k,0} \wedge \overline{D}_i(b_{r-k+1+\lambda_{k-1}} \wedge \ldots \wedge b_{r-1+\lambda_1})). \end{aligned}$$

Each term $\mathbf{b}_{r-k,0} \wedge \overline{D}_i(b_{r-k+1+\lambda_{k-1}} \wedge \ldots \wedge b_{r-1+\lambda_1})$ is an integral linear combination of elements of the form $a_j \mathbf{b}_{r,\boldsymbol{\lambda}_{ij}}$ where $\boldsymbol{\lambda}_{ij}$ is a partition of length at most $k-1$. By the inductive hypothesis, there exists $G_i(\mathcal{D}_+)$ such that

$$\mathbf{b}_{r-k,0} \wedge \overline{D}_i(b_{r-k+1+\lambda_{k-1}} \wedge \ldots \wedge b_{r-1+\lambda_1}) = G_i(\mathcal{D}_+)\mathbf{b}_{r,0}.$$

Thus

$$G_{\boldsymbol{\lambda}}(\mathcal{D}_+) = \sum_{i=0} (-1)^i D_{\lambda_k - i} G_i(\mathcal{D}_+) \in \mathcal{A}_r(\mathcal{D}_+)$$

is a polynomial with the required property. ∎

Denote by $G_{\boldsymbol{\lambda}}(H_r) \in B_r$ the eigenvalue of $G_{\boldsymbol{\lambda}}(\mathcal{D}_+)$ having the whole $\bigwedge^r M_r$ as eigen-module. In spite we shall reprove this fact by our formalism, the following result is well known: see e.g. [65]. It basically says that any $\mathbb{Z}$-polynomial ring in r indeterminates possesses a $\mathbb{Z}$-basis parameterized by partitions of length at most r.

3.3.3 Theorem. *The data $(G_{\boldsymbol{\lambda}}(H_r) \,|\, |\boldsymbol{\lambda}| = w)$ form a $\mathbb{Z}$-basis for $(B_r)_w$. In particular the epimorphism (3.12) is a $\mathbb{Z}$-module isomorphism.*

Proof. It is clear that $G_{\boldsymbol{\lambda}}(H_r) \in B_r$ for all $\boldsymbol{\lambda} \in \mathcal{P}_r$. Recall (Cf. 3.1.5) that $(e^{\boldsymbol{\lambda}} \,|\, |\boldsymbol{\lambda}| = w)$ is a $\mathbb{Z}$-basis of $(B_r)_w$. We contend that

$$(G_{\boldsymbol{\lambda}}(H_r) \,|\, |\boldsymbol{\lambda}| = w)$$

are linearly independent over the integers. In fact $\sum_{|\boldsymbol{\lambda}|=i} a_{\boldsymbol{\lambda}} G_{\boldsymbol{\lambda}}(H_r) = 0$ implies

$$0 = \sum_{|\boldsymbol{\lambda}|=w} a_{\boldsymbol{\lambda}} G_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \sum_{|\boldsymbol{\lambda}|=w} a_{\boldsymbol{\lambda}} \mathbf{b}_{r,\boldsymbol{\lambda}},$$

which in turn forces all $a_{\boldsymbol{\lambda}}$ being zero, because $(\mathbf{b}_{r,\boldsymbol{\lambda}} \mid |\boldsymbol{\lambda}| = w)$ is a $\mathbb{Z}$-basis of $(\bigwedge^r M_0)_w$. It remains to show that $B_r = \bigoplus_{|\boldsymbol{\lambda}|=w} \mathbb{Z} G_{\boldsymbol{\lambda}}(H_r)$. To this purpose we observe that both $(e_{\boldsymbol{\lambda}})_{|\boldsymbol{\lambda}|=w}$ and $(G_{\boldsymbol{\lambda}}(H_r))_{|\boldsymbol{\lambda}|=w}$ are bases of the $\mathbb{Q}$-vector space $(B_r \otimes_{\mathbb{Z}} \mathbb{Q})_w$. Thus:

$$e^{\boldsymbol{\lambda}} = \sum_{|\boldsymbol{\mu}|=w} a_{\boldsymbol{\lambda}\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(H_r)$$

where $a_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{Q}$. Evaluating the left hand side at $\overline{\mathcal{D}}_+$ and applying to $\mathbf{b}_{r,0}$ gives:

$$\overline{D}_1^{i_1} \ldots \overline{D}_r^{i_r} \mathbf{b}_{r,0} = \sum_{|\boldsymbol{\mu}|=w} a_{\boldsymbol{\lambda}\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(H_r)\mathbf{b}_{r,0}$$

Since the left hand side is an integral linear combination of the basis elements of $(\bigwedge^r M_0)_w$ so is the right hand side, i.e. $a_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{Z}$. ■

3.3.4 Corollary. *The map $\mathcal{A}_r(\mathcal{D}_+) \cong \mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r] \to \bigwedge^r M_0$ mapping $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+) \mapsto \mathbf{b}_{r,\boldsymbol{\lambda}}$ is a $\mathbb{Z}$-module isomorphism.*

Proof. By its very construction, the map $\mathrm{ev}_{\mathbf{b}_{r,0}} : B_r \to \bigwedge^r M_0$ factorizes through $\mathrm{ev}_{\mathcal{D}_+} : B_r \to \mathcal{A}_r(\mathcal{D}_+)$ and the claimed isomorphism follows. ■

3.4 The B_r-Module Structure of M_0.

Our next goal is to construct out of M_0 a free B_r-module M_r of rank r such that $\bigwedge^r M_r := B_r \otimes_{\mathbb{Z}} \bigwedge^r M_0$. This will enable us to prove the equality $G_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(H_r)$, a Giambelli's like formula.

3.4.1 Lemma.

1. *Let $m \in M_0$ such that $m \wedge \alpha = 0$ for all $\alpha \in \bigwedge^{r-1} M_0$. Then $m = 0$.*

2. *If $m_1, m_2 \in M_0$ are such that $m_1 \wedge \alpha = m_2 \wedge \alpha$ for all $\alpha \in \bigwedge^{r-1} M_0$, then $m_1 = m_2$*

Proof. Let

$$m = a_0 b_{i_0} + a_1 b_{i_1} + \ldots + a_{i_k} b_{i_{k-1}} \in M_0$$

expressed as a finite linear combinations of elements of the basis $\mathcal{B}_0$ of M_0. For each $0 \leq j \leq k-1$, choose a monomial $\alpha_j \in \bigwedge^{r-1} \mathcal{B}_0$ such that $b_{i_j} \wedge \alpha_j \neq 0$. Then

$$m \wedge \alpha_j = a_0 b_{i_0} \wedge \alpha_j + a_1 b_{i_1} \wedge \alpha_j + \ldots + a_{i_k} b_{i_{k-1}} \wedge \alpha_j$$

is a $\mathbb{Z}$-linear combination of linear independent elements of $\bigwedge^r \mathcal{B}_0$, such that $b_j \wedge \alpha_j \neq 0$. This implies $a_j = 0$, for all $0 \leq j \leq k-1$, i.e. $m = 0$.

Item ii) follows immediately from i): the hypothesis is equivalent to $(m_1 - m_2) \wedge \alpha = 0$ for all $\alpha \in \bigwedge^{r-1} M_0$, whence $m_1 - m_2 = 0$. ∎

3.4.2 Basing on Lemma 3.4.1, for all $1 \leq i \leq r$ and $m \in M_0$ define

$$e_i m$$

as the unique element of $M_r := B_r \otimes_{\mathbb{Z}} M_0$ such that

$$e_i m \wedge \alpha = e_i(m \wedge \alpha) = \overline{D}_i(m \wedge \alpha). \tag{3.14}$$

3.4.3 Proposition. *The shift $D_1 : M_0 \to M_0$ is B_r-linear and then extends to an endomorphism of M_r.*

Proof. It suffices to show that

$$D_1(e_i m) = e_i D_1 m, \qquad \forall 1 \leq i \leq r.$$

In fact, using integration by parts (2.13) for $j = 1$:

$$\begin{aligned}
(D_1 e_i m) \wedge \alpha &= D_1(e_i m \wedge \alpha) - e_i m \wedge D_1 \alpha \\
&= e_i(D_1 m \wedge \alpha) + e_i m \wedge D_1 \alpha - e_i m \wedge D_1 \alpha \\
&= (e_i D_1 m) \wedge \alpha,
\end{aligned}$$

and the claim is proven. ∎

3.4.4 Proposition. *Equation (3.14) defines on M_0 a structure of free B_r-module of rank r (denoted by M_r in the following).*

Proof. Since D_1 is B_r-linear and $\overline{\mathcal{D}}_+(z)$ is the characteristic polynomial operator (i.e. $\overline{D}_1, \ldots, \overline{D}_r$ are the traces endomorphims, induced by D_1, of the B_r-module $\bigwedge^r M_r$), by the theorem of Cayley and Hamilton we have:

$$\mathtt{p}_r(D_1)b_j = 0,$$

i.e. $(b_0, b_1, \ldots, b_{r-1}, b_r)$ are surely linearly dependent over B_r.

Consider the evaluation map $\mathrm{ev}_{b_0} : B_r[X] \to M_r$ defined by $P(X) \to P(D_1)b_0$. It is clearly surjective. We claim that if P is any non zero polynomial of degree $\leq r-1$, then $P(D_1)b_0 \neq 0$. To see this, let

$$P(X) = a_0X^i + a_1X^{i-1} + \ldots + a_i, \qquad a_i, \in B_r,$$

with $a_0 \neq 0$ and $0 \leq i \leq r-1$, so that:

$$P(D_1)b_0 = a_0b_i + a_1b_{i-1} + \ldots + a_ib_0.$$

Thus

$$(P(D_1)b_0) \wedge b_0 \wedge \ldots \wedge b_{i-1} \wedge \widehat{b_i} \wedge b_{i+1} \wedge \ldots \wedge b_{r-1} = \pm a_0\mathbf{b}_{r,0} \neq 0,$$

i.e. $P(X)$ does not belong to the kernel of ev_{b_0}. Suppose now that $P \in \ker \mathrm{ev}_{b_0}$. Then $\deg(P) \geq r$. Write P as $\mathtt{p}_r(X)Q(X)+r(X)$, where $\deg(r(X)) < r-1$. Then

$$0 = P(D_1)b_0 = Q(D_1)\mathtt{p}_r(D_1)b_0 + r(D_1)b_0 = r(D_1)b_0$$

from which $r(X) = 0$, i.e $\ker \mathrm{ev}_{b_0} = \mathtt{p}_r(X)$. In other words $M_r := B_r \otimes_{\mathbb{Z}} M_0$, with the module structure defined by (3.14), is a free B_r-module of rank r isomorphic to $B_r[X]/(\mathtt{p}_r(X))$ and, moreover,

$$\bigwedge^r M_r = B_r \otimes \bigwedge^r M_0,$$

where in the last side the B_r-module structure is given by (3.11). ∎

3.5 Giambelli's Formula

3.5.1 Recall that the *residue* $\mathrm{Res}(g)$ of a Laurent series $g = \sum_{i\le n} g_i X^i$ ($n \in \mathbb{Z}$) is the coefficient g_{-1} of X^{-1}. The following definition is due to Laksov and Thorup [58, 59]:

3.5.2 Definition. *The* residue *of an ordered r-tuple*

$$g_i := \sum_{j\le n_i} g_{ij} X^j, \qquad 0 \le i \le r-1.$$

of Laurent series with B_r-coefficients is:

$$\mathrm{Res}(g_0, g_1, \ldots, g_{r-1}) :=$$

$$= \begin{vmatrix} \mathrm{Res}(g_0) & \mathrm{Res}(g_1) & \ldots & \mathrm{Res}(g_{r-1}) \\ \mathrm{Res}(Xg_0) & \mathrm{Res}(Xg_1) & \ldots & \mathrm{Res}(Xg_{r-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Res}(X^{r-1}g_0) & \mathrm{Res}(X^{r-1}g_1) & \ldots & \mathrm{Res}(X^{r-1}g_{r-1}) \end{vmatrix}. \tag{3.15}$$

Clearly $\mathrm{Res}(g_0, g_1, \ldots, g_{r-1})$ is B_r-multilinear and alternating:

$$\mathrm{Res}(g_{\sigma(0)}, g_{\sigma(1)}, \ldots, g_{\sigma(r-1)}) = \mathrm{sgn}(\sigma)\mathrm{Res}(g_0, g_1, \ldots, g_{r-1}),$$

where $\sigma \in S_r$ (the permutations on r elements) and $\mathrm{sgn}(\sigma)$ is its parity ± 1. If at least one among the g_j is a polynomial, the j-th row of the determinant (3.15) vanishes, which causes

$$\mathrm{Res}(g_0, g_1, \ldots, g_{r-1}) = 0.$$

The following result is also due to Laksov and Thorup:

3.5.3 Lemma. *Let $f(X) = X^\lambda + a_1 X^{\lambda-1} + \ldots + a_\lambda$ be a monic polynomial of degree λ with coefficients in a B_r-algebra A. Then, for all $1 \le i \le r$*

$$\begin{aligned} \mathrm{Res}\left(\frac{X^{i-1}f(X)}{\mathrm{p}_r(X)}\right) &= h_{i-r+\lambda} + a_1 h_{i-r+\lambda-1} + \ldots + a_\lambda h_{i-r} \\ &= \sum_{j=0}^{\lambda} a_j h_{i-r+\lambda-j}. \end{aligned} \tag{3.16}$$

Proof. Recall that

$$\frac{1}{\mathtt{p}_r(X)} = \sum_{n\geq 0} h_n X^{-r-n}.$$

The equality $X^{i-1}f(X) = \sum_{j=0}^{\lambda} a_j X^{\lambda+i-1-j}$ gives:

$$\begin{aligned}\frac{X^{i-1}f(X)}{\mathtt{p}_r(X)} &= \sum_{j=0}^{\lambda} a_j X^{\lambda+i-1-j} \sum_{n\geq 0} h_n X^{-r-n} \\ &= \sum_{p\in\mathbb{Z}} \left(\sum_{j+n=\lambda+i-1-r-p} a_j h_n \right) X^p \end{aligned} \tag{3.17}$$

The sought for residue is the coefficient of X^{-1} in (3.17), i.e:

$$\mathrm{Res}\left(\frac{X^{i-1}f(X)}{\mathtt{p}_r(X)}\right) = \sum_{j+n=\lambda+i-r} a_j h_n = \sum_{j=0}^{\lambda} a_j h_{i-r+\lambda-j}. \qquad \blacksquare$$

3.5.4 Theorem. *Let $f_0, f_1, \ldots, f_{r-1}$ be polynomials. Then*

$$\begin{aligned}&f_0(D_1)b_0 \wedge f_1(D_1)b_0 \wedge \ldots \wedge f_{r-1}(D_1)b_0 \\ =&\mathrm{Res}\left(\frac{f_{r-1}(X)}{\mathtt{p}_r(X)}, \frac{f_{r-2}(X)}{\mathtt{p}_r(X)}, \ldots, \frac{f_0(X)}{\mathtt{p}_r(X)}\right) b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}. \end{aligned} \tag{3.18}$$

Proof. Both member of (3.18) are B_r-multilinear and alternating in $f_0, f_1, \ldots, f_{r-1}$. If one of the f_j is divisible by $\mathtt{p}_r(X)$ one has $f_j := q_j(X)\mathtt{p}_r(X)$, but then $f_j(D_1)b_0 = q_j(D_1)\mathtt{p}_r(D_1)b_0 = 0$. Then both sides of (3.18) vanish if one of f_j is divisible by $\mathtt{p}_r(X)$. Moreover they are equal when $f_i(X) = X^i$. In fact the first member would be just $b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}$ while

$$\mathrm{Res}\left(\frac{X^{r-1}}{\mathtt{p}_r(X)}, \frac{X^{r-2}}{\mathtt{p}_r(X)}, \ldots, \frac{1}{\mathtt{p}_r(X)}\right) = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ h_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{r-1} & h_{r-2} & \ldots & 1 \end{vmatrix} = 1$$

Then equality (3.18) holds. $\blacksquare$

3.5.5 Corollary. *One has:*

$$\mathbf{b}_{r,\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)\mathbf{b}_{r,0}.$$

Proof. In fact

$$\begin{aligned}
\mathbf{b}_{r,\boldsymbol{\lambda}} &= D_1^{\lambda_r} b_0 \wedge D_1^{1+\lambda_{r-1}} b_0 \wedge \ldots \wedge D_1^{r-1+\lambda_r} b_0 \\
&= \operatorname{Res}\left(\frac{X^{\lambda_r}}{\mathrm{p}_r(X)}, \frac{X^{1+\lambda_{r-1}}}{\mathrm{p}_r(X)}, \ldots, , \frac{X^{r-1+\lambda_1}}{\mathrm{p}_r(X)}\right) b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1} \\
&= \Delta_{\boldsymbol{\lambda}}(H_r) b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}.
\end{aligned}$$

Last equality holds because $\Delta_{\boldsymbol{\lambda}}(H_r)$ is eigenvalue of $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)$. ■

3.5.6 Corollary. *The Schur polynomials $\Delta_{\boldsymbol{\lambda}}(H_r)$ form a $\mathbb{Z}$-basis of B_r.*

Proof. It follows from 3.3.3 and the fact that $G_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(H_r)$. ■

3.5.7 Notation. Corollary 3.5.6 says that the map

$$\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \mathbf{b}_{r,\boldsymbol{\lambda}}$$

extends to a $\mathbb{Z}$-module isomorphism $B_r \to \bigwedge^r M$. The inverse isomorphism $\bigwedge^r M \to B_r$ will be denoted as

$$\alpha \mapsto \frac{\alpha}{\mathbf{b}_{r,0}} \tag{3.19}$$

to mean the unique element of B_r which multiplied by $\mathbf{b}_{r,0}$ gives back α.

3.6 Application to Modules of Finite Rank

3.6.1 Consider the submodule $D_n M_0 := D_1^n M_0$ of M_0 generated by $(b_{n+j} = D_1^{n+j} b_0)_{j \geq 0}$. Then

$$M_0 = M_{0,n} \oplus D_n M_0$$

where $M_{0,n} := \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot b_i$. The truncation map

$$\begin{cases} \gamma_{0,n} & : & M_0 & \longrightarrow & M_{0,n} \\ & & \sum_{i\geq 0} n_i b_i & \longmapsto & \sum_{i=0}^{n-1} n_i b_i \end{cases} \qquad (n_i \in \mathbb{Z})$$

is a homomorphism of abelian groups with kernel $D_n M_0$. We have a canonical graded epimorphism

$$\bigwedge \gamma_{0,n} : \bigwedge M_0 \longrightarrow \bigwedge M_{0,n} \tag{3.20}$$

defined by

$$\mathbf{b}_{r,\boldsymbol{\lambda}} \longmapsto \gamma_{r,n}(\mathbf{b}_{r,\boldsymbol{\lambda}}) := \gamma_{0,n}(b_{\lambda_r}) \wedge \ldots \wedge \gamma_{0,n}(b_{-r+1+\lambda_1})$$

From now on, and for sake of notational brevity, we set:

$$\gamma_{r,n} := \bigwedge^r \gamma_{0,n}. \tag{3.21}$$

3.6.2 Proposition. *The kernel of the map* (3.20) *is precisely the bilateral ideal*

$$\bigwedge M_0 \wedge D_n M_0 := \{\mathbf{b}_{r,\boldsymbol{\lambda}} \,|\, \lambda_1 \geq n - r + 1\}$$

generated by $D_n M_0$ in $\bigwedge M_0$.

Proof. It suffices to prove that $\ker \gamma_{r,n} = \bigwedge^{r-1} M_0 \wedge D_n M_0$. It is clear that if $\lambda_1 \geq n - r + 1$ then

$$b_{\lambda_r} \wedge b_{1+\lambda_{r-1}} \wedge \ldots \wedge b_{r-1+\lambda_1} \in \ker(\gamma_{r,n}).$$

Conversely, suppose that $\gamma_{r,n}(\mathbf{b}_{r,\boldsymbol{\lambda}}) = 0$. As $\gamma_{0,n}$ is an isomorphism when restricted to $M_{0,n}$, such a vanishing implies that $r - j + \lambda_j \geq n$ for at least one $j \in \{0, 1, \ldots, r-1\}$, forcing the inequality $\lambda_1 \geq n - r + 1$. Thus

$$\sum a_{\boldsymbol{\lambda}} \mathbf{b}_{r,\boldsymbol{\lambda}} \in \ker(\gamma_{r,n}), \qquad (0 \neq a_{\boldsymbol{\lambda}} \in \mathbb{Z}), \tag{3.22}$$

if and only if all the summands belong to $\bigwedge^{r-1} M_0 \wedge D_n M_0$. ∎

3.6.3 We have just proved that $\gamma_{r,n}$ induces an isomorphism

$$\frac{\bigwedge^r M_0}{\bigwedge^{r-1} M_0 \wedge D_n M_0} \longrightarrow \bigwedge^r M_{0,n}.$$

For sake of brevity denote by $\overline{\mathrm{ev}}_{\mathbf{b}r,0}$ the composition $B_r \to \bigwedge^r M_{0,n}$ of the evaluation map (3.12) with $\gamma_{r,n}$. Let

$$B_{r,n} := \frac{B_r}{\ker \mathrm{ev}_{\mathbf{b}_{r,0}}}.$$

Then the epi-morphism $B_r \to \bigwedge^r M_{0,n}$ factorizes as

$$B_r \xrightarrow{\pi_{r,n}} B_{r,n} \xrightarrow{\mathrm{ev}_{\mathbf{b}_{r,0}}} \bigwedge^r M_{0,n},$$

where $\pi_{r,n}$ is the canonical projection and and the second arrow is an isomorphism which, by a reasonable abuse of notation, is denoted as that occurring in formula (3.12). The following diagramme is obviously commutative:

$$\begin{array}{ccc} B_r & \xrightarrow{\pi_{r,n}} & B_{r,n} \\ {\scriptstyle \mathrm{ev}_{\mathbf{b}_{r,0}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathbf{b}_{r,0}}} \\ \bigwedge^r M_0 & \xrightarrow{\gamma_{r,n}} & \bigwedge^r M_{0,n} \end{array} \tag{3.23}$$

for all $r \in \mathbb{N}^*$.

3.6.4 Proposition. *We have:* $\ker(\overline{\mathrm{ev}}_{\mathbf{b}_{r,0}}) = (h_{n-r+1}, \ldots, h_n)$.

Proof. If $j \geq 0$ then $h_{n-r+1+j} \in \ker \overline{\mathrm{ev}}_{\mathbf{b}_{r,0}}$. In fact

$$\gamma_{r,n}(h_{n-r+1+j} b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1}) =$$

$$= \bigwedge^r \gamma_{0,n}(b_0 \wedge b_1 \wedge \ldots \wedge b_{n+j}) = b_0 \wedge \ldots \wedge b_{r-2} \wedge \gamma_{0,n}(b_{n+j}) = 0.$$

Conversely, suppose that $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} \Delta_{\boldsymbol{\lambda}}(H_r) \in B_r$ belongs to $\ker \overline{\mathrm{ev}}_{\mathbf{b}_{r,0}}$. Then

$$0 = \gamma_{r,n}(\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} \Delta_{\boldsymbol{\lambda}}(H_r) \mathbf{b}_{r,0}) = \sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} \cdot \gamma_{r,n}(\mathbf{b}_{r,\boldsymbol{\lambda}}),$$

which implies $\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^{r-1} M_0 \wedge D_n M_0$ because of Proposition 3.6.2. So, all the partitions occurring in the sum have $\lambda_1 \geq n-r+1$. In the case $\lambda_1 = n-r+1$ then

$$\Delta_{\boldsymbol{\lambda}}(H_r) \in (h_{n-r+1}, \ldots, h_n).$$

We contend that h_{n+1+j} belongs to $(h_{n-r+1}, \ldots, h_n)$ for all $j \geq 0$. Using the relation

$$h_{n+1+j} + \sum_{i=1}^{r} (-1)^i e_i h_{n+1+j-i} = 0,$$

it is apparent that $h_{n+1+j} \in (h_{n+j}, \ldots, h_{n+j-r})$. By induction $\ker(\overline{\mathrm{ev}}_{\mathbf{b}_{r,0}}) = (h_{n-r+1}, \ldots, h_n)$ as claimed. ■

3.6.5 Corollary. *The $\mathbb{Z}$-algebra homomorphism*

$$B_{r,n} \to \frac{\mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r]}{(D_{n-r+1}, \ldots, D_n)}$$

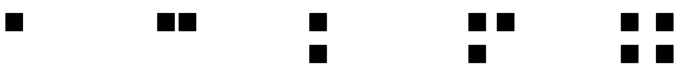
defined by $e_i \mapsto \overline{D}_i$ is an isomorphism.

Proof. Due to the fact that the evaluation maps $\mathrm{ev}_{\mathbf{b}_{r,0}} : B_r \to \bigwedge^r M_0$ factorizes through $\mathbb{Z}[\overline{D}_1, \ldots, \overline{D}_r]$ and that $P(H_r)\mathbf{b}_{r,0} = 0$ if and only if $P(\mathcal{D}_+)\mathbf{b}_{r,0} = 0$. ■

3.6.6 Notation as in 3.6.3. It turns out that $\pi_{r,n}(h_{n-r+j}) = 0$ for all $j \geq 1$. Let $H_{r,n}$ be the sequence

$$\pi_{r,n}(H_r) = (1, h_1, \ldots, h_{n-r}, 0, 0, \ldots), \tag{3.24}$$

and $\mathcal{P}_{r,n}$ be the subset of the partitions of length at most r whose Young diagramme is contained in a $r(n-r)$ rectangle.



(1) (2) (1,1) (2,1) (2,2)

The non-null partitions of $\mathcal{P}_{2,2}$.

Thus $B_{r,n} := \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbb{Z}\Delta_{\boldsymbol{\lambda}}(H_{r,n})$ and is obviously isomorphic to the sub-module $\bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} \mathbb{Z}\Delta_{\boldsymbol{\lambda}}(H_r)$ of B_r. The equality $\pi_{r,n}\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\pi_{r,n}H_r)$ holds because $\pi_{r,n}$ is a ring epimorphism. Notice in addition that $H_{r,n} = \pi_{r,n}(H_r)$ is defined by the equality

$$E_r(t)\sum_{i=0}^{n-r+1} h_i t^i = 1,$$

holding in $B_{r,n} = \pi_{r,n}(B_r)$. In fact $\pi_{r,n}\Delta_{\boldsymbol{\lambda}}(H_r) = 0$ implies $\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^{r-1} M_0 \wedge D_n M_0$, i.e. $\Delta_{\boldsymbol{\lambda}}(H_r) \in (h_{n-r+1}, \ldots, h_n)$, that is $\lambda_1 \geq n-r+1$ and then the Young diagram of $\boldsymbol{\lambda}$ cannot be contained in $r(n-r)$ rectangle.

3.6.7 Proposition. *The map $\gamma_{r,n} : \bigwedge^r M_0 \to \bigwedge^r M_{0,n}$ is a B_r–$B_{r,n}$ module homomorphism, i.e.*

$$\gamma_{r,n}(\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0}) = (\pi_{r,n}\Delta_{\boldsymbol{\lambda}}(H_r))\mathbf{b}_{r,0} = \Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,0}. \tag{3.25}$$

Proof. If $\boldsymbol{\lambda} \notin \mathcal{P}_{r,n}$ then $\lambda_1 \geq n-r+1$, then both members of (3.25) vanish. If $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, instead

$$\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^r M_{0,n}$$

i.e. $\mathbf{b}_{r,\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,0} = \pi_{r.n}\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0}$. ∎

3.6.8 The $\mathbb{Z}$-algebra homomorphism $\pi_{r,n} : B_r \to B_{r,n}$ has a natural extension to an epi-morphism $B_r((z)) \to B_{r,n}((z))$ denoted by the the same symbol by abuse of notation. The algebra $B_{r,n}$ can be also viewed as the quotient:

$$B_{r,n} = \frac{\mathbb{Z}[e_1, \ldots, e_r, h_1, \ldots, h_{n-r}]}{(E_r(t)(1 + h_1 t + \ldots + h_{n-r}t^{n-r}) - 1)}. \tag{3.26}$$

Notation (3.26) means that the quotient is taken with respect to the ideal generated by the coefficients of positive degree in the expansion $E_r(t)(1 + h_1 t + \ldots + h_{n-r}t^{n-r})$.

3.6.9 Corollary. *The isomorphism $B_{r,n} \mapsto \bigwedge^r M_{0,n}$ makes $\bigwedge^r M_{0,n}$ into a free $B_{r,n}$-module $\bigwedge^r M_{r,n}$ of rank 1 generated by $\mathbf{b}_{r,0}$, such that for each $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$*

$$\mathbf{b}_{r,\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,0}$$

Proof. For all $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, define the $B_{r,n}$-module structure through

$$\Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,\boldsymbol{\mu}} = \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,\boldsymbol{\mu}},$$

which is well defined for $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ the map $\Delta_{\boldsymbol{\lambda}}(H_{r,n}) \mapsto \Delta_{\boldsymbol{\lambda}}(H_r)$ is a section $B_{r,n} \to B_r$. Moreover for all $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$:

$$\mathbf{b}_{r,\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,0}. \qquad \blacksquare$$

3.6.10 Corollary. *Let $B_{r,n} \otimes_{\mathbb{Z}} \bigwedge^r M_0$ be the $B_{r,n}$-structure of $\bigwedge^r M_{0,n}$ described in Corollary 3.6.9. Then $M_{0,n}$ can be equipped with a structure of free $B_{r,n}$-module of rank r, $M_{r,n}$, such that $\bigwedge^r M_{r,n} = B_{r,n} \otimes \bigwedge^r M_{0,n}$.*

Proof. The epimorphism $M_0 \to M_{0,n}$ induces a B_r-module structure on $M_{0,n}$ which factorizes through $B_{r,n}$. Hence $M_{0,n}$ can be given a structure of free $B_{r,n}$-module of rank r that we denote by $M_{r,n}$. Clearly $\bigwedge^r M_{r,n} = B_{r,n} \otimes \bigwedge^r M_{0,n}$. $\blacksquare$

3.6.11 Example. The factorization $B_r \to B_{r,n} \to \bigwedge^r M_{0,n}$ allows to simplify some computations. For instance, suppose one wants to compute

$$\Delta_{(3,1)}(H_2)b_1 \wedge b_2$$

in $\bigwedge^2 M_0$. One has, on one hand:

$$\begin{aligned}(h_3h_1 - h_4)(b_0 \wedge b_2) &= (D_3D_1 - D_4)(b_0 \wedge b_2)\\ &= D_3(b_1 \wedge b_2 + b_0 \wedge b_3) - b_0 \wedge b_6\\ &= b_1 \wedge b_5 + b_2 \wedge b_4 + b_1 \wedge b_5 + b_0 \wedge b_6 - b_0 \wedge b_6\\ &= 2b_1 \wedge b_5 + b_2 \wedge b_4.\end{aligned}$$

On the other hand, the partitions $(3,1)$ and (1) are both contained in a 2×3 rectangle, and then computations can be performed in the $B_{2,5}$-module $\bigwedge^2 M_{0,5}$, where $h_4 = 0$. In this case

$$\begin{aligned}(h_3h_1 - h_4)(b_0 \wedge b_2) &= h_3h_1(b_0 \wedge b_2)\\ &= D_3(b_1 \wedge b_2 + b_0 \wedge b_3)\\ &= 2b_1 \wedge b_5 + b_2 \wedge b_4,\end{aligned}$$

having used the fact that $\gamma_{0,5}(b_6) = 0$.

3.6.12 Remark. There is $\mathbb{Z}$-algebra isomorphism between $B_{r,n}$ and $H^*(G(r,\mathbb{C}^n),\mathbb{Z})$, the singular cohomology of the Grassmann variety parameterizing r-dimensional subspaces of $\mathbb{C}^n$. The isomorphism is given explicitly by $h_i \mapsto \sigma_i := c_i(\mathcal{Q}_r)$, where $\mathcal{Q}_r$ is the universal quotient bundle over the Grassmannian. The cohomology of the grassmannian is a free $\mathbb{Z}$-module of rank $\binom{n}{k}$ generated by $\{\sigma_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_{r,n}\}$, the Poincaré dual of the classes of the closure of the cells associated to a complete flag of $\mathbb{C}^n$. The isomorphism holds because of Giambelli's formula $\sigma_{\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (1, \sigma_1, \sigma_2, \ldots)$ (Cf. [**?**]GH or [20, p. 271]).

Another argument is based on the known fact that in the cohomology ring of the Grassmannian, the polynomial X^n universally factorizes into the product of two monic polynomials of degree r and $n-r$ (Cf. Section 3.7 below). Then $H^*(\mathbb{P}^{n-1}) = H^*(G(1,\mathbb{C}^n)) = B_{1,n} = \mathbb{Z}[e_1]/(e_1^n)$ and the hyperplane class corresponds to $e_1 \in B_{1,n}$. Capping with the fundamental class

$$\cap\, [G(r,n)] : H^*(G(r,\mathbb{C}^n),\mathbb{Z}) = B_{r,n} \longrightarrow H_*(G(r,\mathbb{C}^n)) \qquad (3.27)$$

gives the Poincaré isomorphism between the cohomology and the homology of the Grassmannian. The $\mathbb{Z}$-module isomorphisms $B_{1,n} \cong M_{0,n}$ and $B_{r,n} \longrightarrow \bigwedge^r M_{0,n}$ show that there is a natural isomorphism between $\bigwedge^r H_*(\mathbb{P}^{n-1},\mathbb{Z})$ and $H^*(G(r,n),\mathbb{Z})$. This fact was used in [22] (see also [23]) to describe Schubert Calculus in Grassmannians via derivations on a Grassmann algebra. Composing the isomorphism $\bigwedge^r H_*(\mathbb{P}^{n-1},\mathbb{Z}) \longrightarrow H^*(G(r,n),\mathbb{Z})$ with the Poincaré isomorphism (3.27), yields what in [34] and [37] is named the *Satake identification*

$$\begin{cases} Sat \quad : & \bigwedge^r H^*(\mathbb{P}^{n-1}) & \longrightarrow & H^*(G(r,n)) \\ & \sigma_{\lambda_1+r-1} \wedge \sigma_{\lambda_2+r-2} \wedge \cdots \wedge \sigma_{\lambda_r} & \longrightarrow & \sigma_{\boldsymbol{\lambda}}, \end{cases}$$

where $\sigma_{\lambda_i+r-i} = (\sigma_1^{\lambda_i+r-i})$ is the special Schubert class of $\mathbb{P}^{n-1} = G(1,n)$, namely the class of a linear space of dimension $n-r+i-\lambda_i$. The arguments used there are based on the "take the span map"

$$\underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{\text{r times}} \dashrightarrow G(r,n),$$

and on the fact that the cohomology groups of $\mathbb{P}^{n-1}$ and $G(r,n)$ are representations of $Gl(n,\mathbb{C})$: $H^*(\mathbb{P}^{n-1})$ is the standard representation of $Gl(H^*(\mathbb{P}^{n-1})) = GL(n,\mathbb{C})$ and $H^*(G(r,n))$ is the r-th wedge power of the standard representation. Using a quantum version of the Satake identification essentially as in [34, p. 47–48] one obtains quantum Pieri's formulas substantially as in [22, Corollary 2.7].

3.7 Universal Factorizations of Polynomials

3.7.1 If A is any ring and $P(X) \in A[X]$ any monic polynomial of degree n, there is a unique A-algebra $A_{P;(r,n)}$, up to isomorphism, such that the polynomial P, regarded as element of $A_{P;(r,n)}[X]$, can be written as the product of two monic polynomials $P_r, P_{n-r} \in A_{P;(r,n)}[X]$ of degree r and $n-r$ respectively, satisfying the following universal property. If A' is any A-algebra where P decomposes into the product of two monic polynomials $Q_r, Q_{n-r} \in A_{P;(r,n)}[X]$, there is a unique algebra homomorphism mapping the coefficients of P_r to those of Q_r which maps the coefficients of P_{n-r} to those of Q_{n-r}. Such a distinguished algebra is called in [58] the *universal factorization algebra* of P into the product of two monic polynomials of degree r and s.

3.7.2 Proposition. *The $\mathbb{Z}$-algebra $B_{r,n}$ is the universal factorization algebra of X^n into the product of two monic polynomials of degree r and $n-r$ respectively.*

Proof. In the ring $B_{r,n}[[t]]$ we have the following equality:

$$\frac{1}{E_r(t)} = 1 + h_1 t + \ldots + h_{n-r} t^r. \tag{3.28}$$

In fact, on one hand the inverse of $E_r(t)$ is $\sum_{n\geq 0} h_n t^n$ but, on the other hand, $h_{n-r+1+j} = 0$ in the ring $B_{r,n}$, for all $j \geq 0$. This proves (3.28). Putting $X = 1/t$ one has

$$\begin{aligned} X^n &= X^n E_r(t)(1 + h_1 t + \ldots + h_{n-r} t^r) \\ &= (X^r - e_1 X^{r-1} + \ldots + (-1)^r e_r)(X^{n-r} + h_1 X^{r-1} + \ldots + h_r). \end{aligned}$$

Last equality proves that X^n decomposes in $B_{r,n}$ as the product of two monic polynomials of degree r and $n-r$ respectively. To prove the universality, let A be any $\mathbb{Z}$-algebra where X^n decomposes as

$$X^n = P_r(X)P_{n-r}(X) \in A[X].$$

Let us write $P_r(X)$ as

$$X^r - e_1(P_r)X^{r-1} + \ldots + (-1)^r e_r(P_r), \qquad e_i(P_r) \in A$$

and

$$P_{n-r}(X) = X^{n-r} + h_1(P_{n-r})X^{n-r-1} + \ldots + h_{n-r}(P).$$

As $e_1, \ldots, e_r$ generate $B_{r,n}$ as $\mathbb{Z}$-algebra, there is a unique homomorphism $B_{r,n} \mapsto A_{P;(r,n)}$, mapping $e_i \mapsto e_i(P_r)$. Since $h_1 = e_1$ and $h_1(P_{n-r}) = e_1(P_r)$ the homomorphism maps $h_1 \mapsto h_1(P_{n-r})$. Assume that $h_1, \ldots, h_{j-1}$ are mapped to $h_1(P_{n-r}), \ldots, h_{j-1}(P_{n-r})$. Then

$$\begin{aligned} h_j &= e_1 h_{j-1} + \ldots + (-1)^r e_r h_{j-r} \mapsto \sum_{i=1}^{r} (-1)^{i+1} e_i(P_r) h_{j-i}(P_{n-r}) \\ &= h_j(P_{n-r}) \end{aligned}$$

as desired. ∎

3.7.3 More generally, let $P(X) = X^n - a_1 X^{n-1} + \ldots + a_n \in \mathbb{Z}[X]$ and, for each $i \in 0, 1, \ldots, r-1$, let $v_{nq+i} = P(D_1)^q b_i$. Then $(v_0, v_1, \ldots)$ is a $\mathbb{Z}$-basis of M_0 (the matrix of basis change is triangular with all 1 along the diagonal). Notice that $M_{0,n} = \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot b_i = \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot v_i$ and so the basis (v_i) supply us with another projection $\gamma_P : M_0 \to M_{0,n}$ defined by $\sum_{j \geq 0} y_j v_j = \sum_{j=0}^{n-1} y_j v_j$. Thus there is an induced epimorphism $B_r \to \bigwedge^r M_{0,n}$ obtained by composing the isomorphism $B_r \to \bigwedge^r M_0$ with the epimorphism $\bigwedge^r \gamma_P : \bigwedge^r M_0 \to \bigwedge^r M_{0,n}$ which factors through a $\mathbb{Z}$-algebra $B_{r,P}$ isomorphic to $\bigwedge^r M_{0,n} \cong \bigwedge^r (M_0/\mathtt{p}(D_1)M_0)$. We contend that $B_{r,P}$ is precisely the universal factorization algebra of the polynomial P as the product of two monic polynomials of degree r and $n-r$. To see this let us consider a new sequence of element of B_r, $(h'_1, h'_2, \ldots)$, defined by

$$E_r(t) \sum_{n \geq 0} h'_n t^n = (1 - a_1 t + \ldots + (-1)^n a_n t^n). \tag{3.29}$$

Keeping into account the relation $E_r(t)\sum_{n\geq 0} h_n t^n$, an easy check shows that:

$$h'_1 = h_1 - a_1, \qquad h'_2 = h_2 - a_1 h_1 + a_2, \quad \ldots,$$

$$\ldots \quad h'_n = h_n - a_1 h_{n-1} + \ldots + (-1)^n a_n.$$

In general $h'_j = h_j + \sum_{i=1}^{n} a_i h_{j-i}$ ($h_k = 0$ if $k < 0$). Let now $\mathcal{D}'(z) := \sum_{j\geq 0} D'{}_j z^j$ be the unique derivation of $\bigwedge M_0$ such that $D'{}_j v_i = v_{i+j}$. Now

$$D'_{n-r+j} v_0 \wedge \ldots \wedge v_{r-1} = v_0 \wedge \ldots \wedge v_{r-2} \wedge v_{n+j}$$

$$= b_0 \wedge \ldots \wedge b_{r-2} \wedge (a_1 b_{n-1+j} - a_2 b_{n-2+j} + \ldots - (-1)^n a_n b_{n+j})$$

$$(h_{n-r+j} - a_1 h_{n-r+j-1} + \ldots + (-1)^{n-r+1} a_n h_{-r+j}) b_0 \wedge \ldots \wedge b_{r-1}$$

$$= h'_{n-r+j} b_0 \wedge \ldots \wedge b_{r-1}$$

and then we have, basically arguing as in 3.6.3:

$$B_{r,P} = \frac{B_r}{(h'_{n-r+1}, \ldots, h'_n)}.$$

3.7.4 Proposition. *The algebra $B_{r,P}$ is the universal factorization algebra of the polynomial $P(X)$ as the product of two monic polynomials $P_r(X)$ and $P_{n-r}(X)$.*

Proof. It is clear that the polynomial $P(X)$ factorizes in $B_{r,P}$ as

$$P(X) = (X^r - e_1 X^{r-1} + \ldots + (-1)^r e_r)(X^{n-r} + h'_1 X^{n-r-1} + \ldots + h'_r) \tag{3.30}$$

In fact relation (3.29) reads in $B_{r,P}$ as

$$(1 - e_1 t + \ldots + (-1)^r e_r t^r)(1 + h'_1 t + \ldots + h'_{n-r} t^{n-r})$$

$$= (1 - a_1 t + \ldots + (-1)^n a_n t^n)$$

i.e., putting $X = 1/t$ precisely (3.30). It remains to check the universality for which one argues exactly as in Proposition 3.7.2. ∎

Chapter 4

Decomposable Tensors in Exterior Powers

In order not to loose the reader's attention, it is good to make clear since now that the purpose of this chapter is to prove a formula which detects the locus of decomposable tensors in the $\mathbb{Z}$-module $\bigwedge^r M_0$ (Theorem 4.5.3). Were M_0 a $\mathbb{C}$-vector space, the formula would encode all the quadratic equations defining the Plücker embedding of the complex Grassmannian $G(r, \mathbb{C}^n)$. The reason why one should appreciate this equivalent phrasing of the Plücker embedding is that the limit for $r, n \to \infty$ gives precisely the equation of the KP hierarchy in the form mentioned in Section 0.4. This Chapter is the output of a joint research with P. Salehyan, begun in [28] and prosecuted in the forthcoming [30].

4.1 Wedging, Contracting and Decomposing

The purpose of this section is to recall a well known criterion to establish the decomposability of an element of $\bigwedge^r M_0$.

4.1.1 Wedging and Contracting. Each $m \in M_0$ induces a *wedging homomorphism*

$$\begin{cases} m\wedge & : & \bigwedge^k M_0 & \longrightarrow & \bigwedge^{k+1} M_0 \\ & & \alpha & \longmapsto & m \wedge \alpha. \end{cases}$$

On a dual side, let $M_0^\vee := \mathrm{Hom}_{\mathbb{Z}}(M_0, \mathbb{Z})$ and $(\beta_j)_{j \geq 0}$ be the basis of $M_0^\vee$ dual of $\mathcal{B}_0$, i.e. such that $\beta_j(b_i) = \delta_{ij}$. Each $\mu \in M_r^\vee$ induces a *contraction homomorphism*

$$\mu\lrcorner : \bigwedge^r M_0 \to \bigwedge^{r-1} M_0.$$

For our purposes, it suffices to define it by induction as follows. If $m \in M_0 := \bigwedge^1 M_0$, let

$$\mu\lrcorner m = \mu(m).$$

Then, supposing $\mu\lrcorner$ being defined for all $\alpha \in \bigwedge^j M_0$, for $j < r$, we set:

$$\begin{aligned} \mu\lrcorner b_{i_0} \wedge (b_{i_1} \wedge \ldots \wedge b_{i_{r-1}}) &= \mu(b_{i_1}) b_{i_1} \wedge \ldots \wedge b_{i_{r-1}} + \\ &- b_{i_0} \wedge \mu\lrcorner(b_{i_1} \wedge \ldots \wedge b_{i_{r-1}}). \end{aligned}$$

4.1.2 Example. Let $\mu \in M_0^\vee$. Its contraction against $\bigwedge^3 M_0$ is:

$$\mu\lrcorner(b_{i_0} \wedge b_{i_1} \wedge b_{i_2}) = \mu(b_{i_0}) b_{i_1} \wedge b_{i_2} - b_{i_0} \wedge \mu\lrcorner(b_{i_1} \wedge b_{i_2}). \tag{4.1}$$

Now:

$$\mu\lrcorner(b_{i_1} \wedge b_{i_2}) = \mu(b_{i_1}) b_{i_2} - b_{i_1} \wedge \mu\lrcorner b_{i_2} = \mu(b_{i_1}) b_{i_2} - \mu(b_{i_2}) b_{i_1}. \tag{4.2}$$

Finally, substituting (4.2) into (4.1):

$$\mu\lrcorner b_{i_0} \wedge b_{i_1} \wedge b_{i_2} =$$

$$= \mu(b_{i_0}) b_{i_1} \wedge b_{i_2} - \mu(b_{i_1}) b_{i_1} \wedge b_{i_2} + \mu(b_{i_2}) b_{i_0} \wedge b_{i_1}.$$

4.1.3 Definition. *A tensor $\alpha \in \bigwedge^r M_0$ is* divisible *by $m \in M_0$ if $\alpha \wedge m = 0$; it is* decomposable *if there exist $m_1, \ldots, m_r$ such that $\alpha = m_1 \wedge \ldots \wedge m_r$.*

Clearly, $\alpha \in \bigwedge^r M_0$ is decomposable if the kernel of the map $M_0 \to \bigwedge^{r+1} M_0$ sending $m \mapsto \alpha \wedge m$ has rank r. We have the following characterization:

4.1.4 Theorem. *The r-vector α is decomposable if and only if*

$$\sum_{i \geq 0} (\alpha \wedge b_i) \otimes (\beta_i \lrcorner \alpha) = 0. \tag{4.3}$$

4.1.5 Remark. Notice that (4.3) is a finite sum. In fact α is a finite $\mathbb{Z}$-linear combination $a_{\boldsymbol{\lambda}} \mathbf{b}_{r,\boldsymbol{\lambda}}$ and then $\beta_j \lrcorner \alpha = 0$ for all but finitely many j.

Proof of Theorem 4.1.4. First notice that α is *divisible* by m if and only if

$$0 = \alpha \wedge m = \alpha \wedge \sum_{i=0}^{r-1} \beta_i(m) b_i = \sum_{i=0}^{r-1} (\alpha \wedge b_i) \beta_i(m) = 0. \tag{4.4}$$

Suppose that α is decomposable, i.e. that $\alpha = m_0 \wedge \ldots \wedge m_{r-1}$ for some $m_i \in M_0$ $(0 \leq i \leq r-1)$:

$$\begin{aligned} &\sum_{i=0}^{r-1} (\alpha \wedge b_i) \otimes \beta_i \lrcorner (m_0 \wedge \ldots \wedge m_{r-1}) = \\ = \quad &\sum_{j=0}^{r-1} (-1)^j \sum_{i=0}^{r-1} (\alpha \wedge b_i) \beta_i(m_j) m_0 \wedge \ldots \wedge \widehat{m}_j \wedge \ldots \wedge m_{r-1} = 0, \end{aligned}$$

where the last vanishing is due to (4.4) applied to each sum $\sum_{i=1}^{r} (\alpha \wedge b_i) \beta_i(m_j)$, $\leq j \leq r-1$. The notation "$\widehat{m}_j$" means that m_j is omitted.

Conversely suppose that (4.3) holds. We claim that α is divisible. If $\alpha = 0$ this is clear. If $\alpha \neq 0$ then $\beta_i \lrcorner \alpha$ cannot vanish for all i, and there must exist $\xi \in (\bigwedge^{r-1} M_0)^\vee$ such that $a_i := \xi(\beta_i \lrcorner \alpha) \neq 0$ for some i. Thus

$$\sum_{i=0}^{r-1} a_i (\alpha \wedge b_i) = 0$$

is a non trivial linear relation and then α is divisible by the initial remark of the proof. Write $\alpha = m_1 \wedge \alpha_1$, for some $m_1 \in M$ and

$\alpha_1 \in \bigwedge^{r-1} M$. Then

$$0 = \sum_{i\geq 0}(b_i \wedge m_1 \wedge \alpha_1) \otimes (\beta_i \lrcorner (m_1 \wedge \alpha_1)) =$$

$$\sum_{i\geq 0}(b_i \wedge m_1 \wedge \alpha_1) \otimes (\beta_i(m_1)\alpha_1 - m_1 \wedge \beta_i \lrcorner \alpha) =$$

$$\sum_{i\geq 0}(b_i \wedge m_1 \wedge \alpha_1) \otimes (\beta_i(m_1)\alpha_1) + \sum_{i\geq 0}(m_1 \wedge b_i \wedge \alpha_1) \otimes (m_1 \wedge (\beta_i \lrcorner \alpha_1)) \quad (4.5)$$

i.e.

$$\sum_{i\geq 0}(m_1 \wedge b_i \wedge \alpha_1) \otimes (m_1 \wedge (\beta_i \lrcorner \alpha_1)) = 0 \quad (4.6)$$

because the first summand of (4.5) is zero, due to (4.4). Since (4.6) holds if and only if $\sum_{i\geq 0}(b_i \wedge \alpha_1) \otimes (\beta_i \lrcorner \alpha_1) = 0$, then α_1 is divisible and can be in turn written as $\alpha_1 = m_2 \wedge \alpha_2$ for some $m_2 \in M$ and $\alpha_2 \in \bigwedge^{r-2} M$. Iterating the process supplies $m_1, m_2, \ldots, m_r \in M$ such that $\alpha = m_1 \wedge \ldots \wedge m_r$. ∎

4.1.6 Proposition. *The epi-morphism* (3.21)*,* $\gamma_{r,n} : \bigwedge^r M_0 \to \bigwedge^r M_{0,n}$*, maps decomposable tensors onto decomposable tensors.*

Proof. Suppose $\alpha = m_1 \wedge \ldots \wedge m_r \in \bigwedge^r M_0$. Then $\wedge^r \gamma_{0,n}(\alpha) = \gamma_{0,n}(m_1) \wedge \ldots \wedge \gamma_{0,n}(m_r)$. Conversely, if $\alpha \in \bigwedge^r M_{0,n}$ is decomposable, then $\alpha := m_1 \wedge \ldots \wedge m_r \in \bigwedge^r M_{0,n} \subseteq \bigwedge^r M_0$, and the claim follows because $\gamma_{r,n}$ restricted to $\bigwedge^r M_{0,n}$ is the identity . ∎

4.1.7 Corollary. *The tensor* $\alpha \in \bigwedge^r M_{0,n}$ *is decomposable if and only if*

$$\sum_{i=0}^{n-1}(b_i \wedge \alpha) \otimes (\beta_i \lrcorner \alpha) = 0. \quad (4.7)$$

Proof. Because of the inclusion $\bigwedge^r M_{0,n} \subseteq \bigwedge^r M_0$, if α is decomposable then equality (4.3) holds. However, since α is a linear combination of $\mathbf{b}_{r,\boldsymbol{\lambda}}$, with $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, all the tensor factors $\beta_j \lrcorner \alpha$ vanish for

$j \geq n$, whence formula (4.7). Conversely, if $\alpha \in \bigwedge^r M_{0,n}$ satisfies (4.7) then

$$0 = \sum_{i=0}^{n-1} (b_i \wedge \alpha) \otimes (\beta_i \lrcorner \alpha) + \sum_{i \geq n} (b_i \wedge \alpha) \otimes (\beta_i \lrcorner \alpha),$$

because $\beta_i \lrcorner \alpha = 0$ if $i \geq n$. ∎

4.2 Further Properties of $\overline{\mathcal{D}}_-(z)$

4.2.1 Recall from 3.2.4 the definition of characteristic polynomial operators $\overline{\mathcal{D}}_\pm(z)$ associated to the shift operators $D_{\pm 1}$ and their inverse $\mathcal{D}_\pm(z)$ in $\mathrm{End}_\mathbb{Z}(\bigwedge M)[[z^{\pm 1}]]$ respectively. For each $r \geq 1$ and $j \in \mathbb{Z}$ we have $\mathbb{Z}$-module endomorphisms $D_j, \overline{D}_j : B_r \to B_r$ defined as follows:

$$(D_j P)\mathbf{b}_{r,0} := D_j(P\mathbf{b}_{r,0}) \qquad \text{and} \qquad (\overline{D}_j P)\mathbf{b}_{r,0} := \overline{D}_j(P\mathbf{b}_{r,0}). \tag{4.8}$$

In spite of the apparent symmetry of the expression (4.8), the behaviour of $D_j, \overline{D}_j$ is quite different depending on j being negative or positive.

4.2.2 Lemma. *For all $j, n \geq 0$ and $r \geq 1$ we have:*

$$D_{-j} h_n = h_{n-j}. \tag{4.9}$$

Proof. Let (n) be the partition $(n, \underbrace{0, \ldots, 0}_{r-1 \text{ times}})$

Then

$$\begin{aligned}
(D_{-j}h_n)\mathbf{b}_{r,0} &= D_{-j}(h_n \mathbf{b}_{r,0}) = D_{-j}\mathbf{b}_{r,(n)} = D_{-j}(\mathbf{b}_{r-1,0} \wedge b_{r-1+n}) = \\
&= \sum_{i=0}^{j} D_{-j+i}\mathbf{b}_{r-1,0} \wedge D_{-i} b_{r-1+n} = \\
&= \mathbf{b}_{r-1,0} \wedge b_{r-1+n-j} = h_{n-j}\mathbf{b}_{r,0}
\end{aligned}$$

whence (4.9). ∎

4.2.3 Lemma. *Let $n \geq 0$ and $r \geq 1$. For each $j \geq 0$:*

$$\overline{D}_{-2-j}h_n = 0.$$

Proof. In fact

$$\begin{aligned}
(\overline{D}_{-2-j}h_n)\mathbf{b}_{r,0} &= \overline{D}_{-2-j}(h_n\mathbf{b}_{r,0}) = \overline{D}_{-2-j}\mathbf{b}_{r,(n)} = \\
&= \overline{D}_{-2-j}(\mathbf{b}_{r-1,0} \wedge b_{r-1+n}) = \\
&= \mathbf{b}_{r-1,0} \wedge \overline{D}_{-2-j}b_{r-1+n} + \overline{D}_{-1}\mathbf{b}_{r-1,0} \wedge b_{r+n} \\
&= \overline{D}_{-1}\mathbf{b}_{r-1,0} \wedge b_{r-2+n},
\end{aligned}$$

because $\overline{D}_{-2-j}$ vanishes on $\bigwedge^1 M_0 = M_0$ and

$$\overline{D}_{-1}\mathbf{b}_{r-1,0} = (\overline{D}_{-1}h_0)\mathbf{b}_{r-1,0} = (D_{-1}h_0)\mathbf{b}_{r-1,0} = 0$$

because $\overline{D}_{-1} = D_{-1}$ and Lemma 4.2.2. ∎

4.2.4 Warning. Although the expression D_jh_n makes perfectly sense in B_r for all $r \geq 1$ and positive j, its output does depend on the integer r. A uniform formula like (4.9) is not available in this case. For example, in B_1 one has

$$D_jh_n = \frac{D_jb_n}{b_0} = \frac{b_{n+j}}{b_0} = h_{n+j}$$

while in B_2:

$$\begin{aligned}
D_jh_n &= \frac{D_j(b_0 \wedge b_{1+n})}{b_0 \wedge b_1} = \frac{b_0 \wedge b_{1+n+j} + D_{j-1}(b_1 \wedge b_{1+n})}{b_0 \wedge b_1} = \\
&= h_{n+j} + \frac{D_{j-1}(b_1 \wedge b_{1+n})}{b_0 \wedge b_1} \neq h_{n+j}.
\end{aligned}$$

4.2.5 Corollary. *The following equality holds in B_r for all $r \geq 1$:*

$$\overline{\mathcal{D}}_-(z)\,h_n = h_n - \frac{h_{n-1}}{z}.$$

Proof. By definition

$$\overline{\mathcal{D}}_-(z)h_n = \sum_{j\geq 0}\frac{\overline{D}_{-j}h_n}{z^j}.$$

Now use equality $\overline{D}_{-1} = D_{-1}$ and lemmas 4.2.2 and 4.2.3. ∎

4.2.6 Corollary. *For all $r \geq 1$:*

$$\mathcal{D}_-(z)\, h_n = \sum_{j \geq 0} \frac{h_{n-j}}{z^j}.$$

Proof. Recall that $h_i = 0$ if $i < 0$. By definition

$$\mathcal{D}_-(z) h_n = \sum_{j \geq 0} \frac{D_{-j} h_n}{z^j}$$

and then the formula follows by Lemma 4.2.2. ∎

4.2.7 Proposition. *The operator $\overline{\mathcal{D}}_-(z)$ commutes with taking $\Delta_{\boldsymbol{\lambda}}$:*

$$\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_r), \tag{4.10}$$

where $\overline{\mathcal{D}}_-(z)H_r = (1, \overline{\mathcal{D}}_-(z)h_1, \overline{\mathcal{D}}_-(z)h_2, \ldots)$.

Proof. By definition:

$$\begin{aligned}
&\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \overline{\mathcal{D}}_-(z)\mathbf{b}_{r,\boldsymbol{\lambda}} = \\
&= \overline{\mathcal{D}}_-(z)b_{\lambda_r} \wedge \overline{\mathcal{D}}_-(z)b_{1+\lambda_{r-1}} \wedge \ldots \wedge \overline{\mathcal{D}}_-(z)b_{r-1+\lambda_1} = \\
&= \left(b_{\lambda_r} - \frac{b_{\lambda_r - 1}}{z}\right) \wedge \left(b_{1+\lambda_{r-1}} - \frac{b_{\lambda_{r-1}}}{z}\right) \wedge \ldots \wedge \left(b_{r-1+\lambda_1} - \frac{b_{\lambda_r - 2+\lambda_1}}{z}\right) = \\
&= f_0(D_1)b_0 \wedge f_1(D_1)b_0 \wedge \ldots \wedge f_{r-1}(D_1)b_0
\end{aligned} \tag{4.11}$$

where

$$f_j(X) = X^{j+\lambda_{r-j}} - \frac{X^{j+\lambda_{r-j}-1}}{z}, \qquad 0 \leq j \leq r-1.$$

By Lemma 3.5.3

$$\operatorname{Res}\left(\frac{X^{i-1} f_{r-j}(X)}{\mathrm{p}_r(X)}\right) = h_{\lambda_j - j + i} - \frac{h_{\lambda_j - j + i - 1}}{z} = \overline{\mathcal{D}}_-(z) h_{\lambda_j - j + i}.$$

By applying Theorem 3.5.4:

$$\begin{aligned}
\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) &= \operatorname{Res}\left(\frac{f_{r-1}(X)}{\mathrm{p}_r(X)}, \ldots, \frac{f_0(X)}{\mathrm{p}_r(X)}\right) = \\
&= \det(\overline{\mathcal{D}}_-(z) h_{\lambda_j - j + i}) = \Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_r)).
\end{aligned}$$

∎

4.2.8 Corollary. *For all r-tuple $(h_{i_1},\dots h_{i_r})\in B_r^r$*

$$\overline{\mathcal{D}}_-(z)(h_{i_1}\cdot\ldots\cdot h_{i_r})=\overline{\mathcal{D}}_-(z)h_{i_1}\cdot\ldots\cdot\overline{\mathcal{D}}_-(z)h_{i_r}.$$

Proof. Each product $h_{i_1}\cdot\ldots\cdot h_{i_r}$ is a linear combination of Schur polynomials:

$$h_{i_1}\cdot\ldots\cdot h_{i_r}=\sum_{\boldsymbol{\lambda}\in\mathcal{P}_r}a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(H_r).$$

where $a_{\boldsymbol{\lambda}}=0$ for all but finitely many $\boldsymbol{\lambda}\in\mathcal{P}_r$. Thus

$$\begin{aligned}\overline{\mathcal{D}}_-(z)(h_{i_1}\cdot\ldots\cdot h_{i_r}) &= \sum_{\boldsymbol{\lambda}}a_{\boldsymbol{\lambda}}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r)=\\ &= \sum_{\boldsymbol{\lambda}}a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_r)=\\ &= \overline{\mathcal{D}}_-(z)h_{i_1}\cdot\ldots\cdot\overline{\mathcal{D}}_-(z)h_{i_r}\end{aligned}$$

as desired. ∎

4.2.9 Warning. The $\mathbb{Z}$-module homomorphism $\overline{\mathcal{D}}_-(z):B_r\to B_r((z))$ is not a ring homomorphism. To see this, take $r=1$. Then

$$\overline{\mathcal{D}}_-(z)h_2=h_2-\frac{h_1}{z}.$$

On the other hand, for $r=1$, $h_2=h_1^2$. However

$$\begin{aligned}\overline{\mathcal{D}}_-(z)h_1\cdot\overline{\mathcal{D}}_-(z)h_1 &= \left(h_1-\frac{1}{z}\right)^2=\\ &= h_2-\frac{2h_1}{z}+\frac{1}{z^2}\neq\overline{\mathcal{D}}_-(z)h_2.\end{aligned}$$

4.2.10 Proposition. *The operator $\mathcal{D}_-(z)$ commutes with taking $\Delta_{\boldsymbol{\lambda}}$.*

$$\mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r)=\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r),\tag{4.12}$$

where $\mathcal{D}_-(z)H_r=(1,\mathcal{D}_-(z)h_1,\mathcal{D}_-(z)h_2,\dots)$.

Proof.

$$
\begin{aligned}
(\mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r))\mathbf{b}_{r,0} &= \mathcal{D}_-(z)(b_{\lambda_r}\wedge b_{1+\lambda_{r-1}}\wedge\ldots\wedge b_{r-1+\lambda_1}) \\
&= \mathcal{D}_-(z)b_{\lambda_r}\wedge\ldots\wedge\mathcal{D}_-(z)b_{r-1+\lambda_1} = \\
&= f_0(D_1)b_0\wedge f_1(D_1)b_0\wedge\ldots\wedge f_{r-1}(D_1)b_0,
\end{aligned}
$$

where

$$
f_j(X) = \sum_{p=0}^{j+\lambda_{r-j}} \frac{X^{j+\lambda_{r-j}-p}}{z^p}
$$

By Lemma 3.5.3

$$
\mathrm{Res}\left(\frac{X^{i-1}f_{r-j}(X)}{\mathtt{p}_r(X)}\right) = \sum_{p=0}^{\lambda_j-j+i}\frac{h_{\lambda_j-j+i-p}}{z^p} = \mathcal{D}_-(z)h_{\lambda_j-j+i}.
$$

In conclusion:

$$
\mathrm{Res}\left(\frac{f_{r-1}(X)}{\mathtt{p}_r(X)}, \frac{f_{r-2}(X)}{\mathtt{p}_r(X)}, \ldots, \frac{f_0(X)}{\mathtt{p}_r(X)}\right) = \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r),
$$

as claimed. ∎

The following is the analogous of Corollary 4.2.8 for which we omit the completely analogous proof.

4.2.11 Corollary. *For all r-tuple $(h_{i_1},\ldots h_{i_r})\in B_r^r$*

$$
\mathcal{D}_-(z)(h_{i_1}\cdot\ldots\cdot h_{i_r}) = \mathcal{D}_-(z)h_{i_1}\cdot\ldots\cdot\mathcal{D}_-(z)h_{i_r}. \qquad \blacksquare
$$

4.3 The Vertex-like Operator $\Gamma_r(z)$

We have seen that an arbitrary exterior power $\bigwedge^r M_0$ of a free abelian group M_0 (Cf. Section 3.2) can be naturally made into a free B_r-module $\bigwedge^r M_r$ of rank 1 generated by $\mathbf{b}_{r,0}$. The purpose of this section is to study the map $B_r \to B_{r+1}((z))$ given by the unique $\mathbb{Z}$-linear extension of:

$$
\Gamma_r(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{(\sum_{j\geq 0} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}})\otimes 1_{B_r}}{\mathbf{b}_{r+1,0}}.
$$

If $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, then $\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^r M_{r,n}$. The main result to be proven in this section is

4.3.1 Theorem.

$$\Gamma_r(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{z^r(\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r+1}))}{E_{r+1}(z)}, \tag{4.13}$$

where by $\overline{\mathcal{D}}_-(z)H_{r+1}$ *we mean the sequence* $(1, \overline{\mathcal{D}}_-(z)h_1, \overline{\mathcal{D}}_-(z)h_2, \ldots)$

4.3.2 Lemma. *For all* $r \geq 1$ *and all* $\boldsymbol{\lambda} \in \mathcal{P}_r$*:*

$$b_0 \wedge \overline{D}_r b_{r,\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_{r+1}) b_0 \wedge b_1 \wedge \ldots \wedge b_r. \tag{4.14}$$

Proof. It is a consequence of the definition of $\overline{D}_r$ and Corollary 3.5.5:

$$\begin{aligned} b_0 \wedge \overline{D}_r b_{r,\boldsymbol{\lambda}} &= b_0 \wedge \overline{D}_r(b_{\lambda_r} \wedge \ldots \wedge b_{r-1+\lambda_1}) \\ &= b_0 \wedge b_{1+\lambda_r} \wedge \ldots \wedge b_{r+\lambda_1} = \\ &= \Delta_{\boldsymbol{\lambda}}(H_{r+1}) b_0 \wedge b_1 \wedge \ldots \wedge b_r. \end{aligned}$$ ■

4.3.3 Proposition. *For all* $\alpha \in \bigwedge^r M_r$*:*

$$\frac{1}{z^r} b_0 \wedge \overline{\mathcal{D}}_+(z)\alpha = \overline{\mathcal{D}}_-(z)(b_0 \wedge \overline{D}_r \alpha). \tag{4.15}$$

Proof. It suffices to prove (4.15) for $\alpha = \mathbf{b}_{r,\boldsymbol{\lambda}}$. We have

$$\overline{\mathcal{D}}_+(z)\mathbf{b}_{r,\boldsymbol{\lambda}} = \mathbf{b}_{r,\boldsymbol{\lambda}} - \overline{D}_1 \mathbf{b}_{r,\boldsymbol{\lambda}} z + \ldots + (-1)^r \overline{D}_r \mathbf{b}_{r,\boldsymbol{\lambda}} z^r. \tag{4.16}$$

Now

$$\begin{aligned} b_0 \wedge \overline{D}_i \mathbf{b}_{r,\boldsymbol{\lambda}} &= b_0 \wedge \overline{D}_i(b_{\lambda_r} \wedge b_{1+\lambda_{r-1}} \wedge \ldots \wedge b_{r-1+\lambda_1}) = \\ &= b_0 \wedge \sum b_{\lambda_r+i_1} \wedge b_{1+\lambda_{r-1}+i_2} \wedge \ldots \wedge b_{r-1+\lambda_1+i_r}, \end{aligned} \tag{4.17}$$

where the sum is over all $(i_1, \ldots, i_r)$ such that $0 \leq i_j \leq 1$ and $\sum i_j = i$. Putting $j_\ell = 1 - i_\ell$, so that $0 \leq j_\ell \leq 1$ and $\sum j_\ell = r - i$, formula (4.17) can be rewritten as:

$$\sum b_0 \wedge b_{1+\lambda_r-j_1} \wedge b_{2+\lambda_{r-1}-j_2} \wedge \ldots \wedge b_{r+\lambda_1-j_r} \tag{4.18}$$

summing over all $(j_1,\ldots,j_r)$ such that $0\leq j_\ell\leq 1$ and $\sum j_\ell = r-i$. Thus (4.18) is precisely the definition of

$$\overline{D}_{i-r}\mathbf{b}_{r+1,\boldsymbol{\lambda}} = \overline{D}_{i-r}(b_0\wedge\overline{D}_r b_{r,\boldsymbol{\lambda}}). \tag{4.19}$$

Plugging the right hand side of (4.19) into (4.16) one obtains

$$\frac{1}{z^r}b_0\wedge\overline{\mathcal{D}}_+(z)\alpha = \sum_{j=0}^{r}\frac{(-1)^j\overline{D}_{j-r}}{z^{r-j}}(b_0\wedge\overline{D}_r\alpha) = \overline{\mathcal{D}}_-(z)(b_0\wedge\overline{D}_r\alpha)$$

as desired. ∎

4.3.4 Proof of Theorem 4.3.1. We first notice that for all $\alpha\in\bigwedge^r M_r$:

$$\sum_{j\geq 0} b_j z^j\wedge\alpha = \mathcal{D}_+(z)b_0\wedge\alpha = \mathcal{D}_+(z)(b_0\wedge\overline{\mathcal{D}}_+(z)\alpha),$$

last equality due to integration by parts, formula (2.12). Using the fact that $\bigwedge^{r+1}M_{r+1}$ is eigen-module for $\mathcal{D}_+(z)$ and Proposition 4.3.3, with eigenvalue $H_{r+1}(z) = 1/E_{r+1}(z)$:

$$\sum_{j\geq 0} b_j z^j\wedge\alpha = \frac{z^r}{E_{r+1}(z)}\overline{\mathcal{D}}_-(z)(b_0\wedge\overline{D}_r\alpha). \tag{4.20}$$

Putting now $\alpha = \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0}$ in equation (4.20) and using Lemma 4.3.2

$$\sum_{j\geq 0} b_j z^j\wedge\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} \quad = \quad \frac{z^r}{E_{r+1}(z)}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r+1})\mathbf{b}_{r+1,0},$$

as desired. ∎

4.3.5 Corollary. *The following equality holds:*

$$\Gamma_r(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{z^r\Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_{r+1})}{E_{r+1}(z)}.$$

Proof. Just apply Proposition 4.2.7 to the numerator of (4.13). The commutation is allowed because $\boldsymbol{\lambda}$ is a partition of length at most r and, then, of length at most $r+1$ (Cf. Remark 4.4.4 below). ∎

4.3.6 Corollary. *Let $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ and $\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^r M_{r,n}$. Then*

$$\sum_{j=0}^{n-1} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}} = \frac{1}{E_{r+1}(z)} \overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r+1,n}).$$

Proof. We use diagramme (3.23) for $r+1$. We have

$$\begin{aligned}\sum_{j=0}^{n-1} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}} \quad + \quad \sum_{j\geq n} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}} &= \sum_{j\geq 0} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}} \\ &= \frac{1}{E_{r+1}(z)} \Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z) H_{r+1})\mathbf{b}_{r+1,0}. \end{aligned} \tag{4.21}$$

Applying the homomomorphism $\gamma_{r+1,n} := \bigwedge^{r+1}\gamma_{0,n}$ to the first and last side of (4.21) and using Proposition 3.6.7, one obtains:

$$\sum_{j=0}^{n-1} b_j z^j \wedge \mathbf{b}_{r,\boldsymbol{\lambda}} = \frac{1}{E_{r+1}(z)} \Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z) H_{r+1,n})\mathbf{b}_{r+1,0}. \qquad \blacksquare$$

4.4 The Vertex-like Operator $\Gamma_r^\vee(z)$

Denote by $\boldsymbol{\beta}\left(z^{-1}\right)$ the formal power series $\sum_{j\geq 0}\beta_j z^{-j}$.

4.4.1 Definition. *Let $\Gamma_r^\vee(z) : B_r \to B_{r-1}((z))$ be the unique $\mathbb{Z}$-linear extension of the map*

$$\Gamma_r^\vee(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{\boldsymbol{\beta}(z^{-1}) \lrcorner \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0}}{\mathbf{b}_{r-1,0}}. \tag{4.22}$$

4.4.2 Lemma. *For each $\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^r M_r$:*

$$\boldsymbol{\beta}\left(z^{-1}\right) \lrcorner \mathbf{b}_{r,\boldsymbol{\lambda}} =$$

$$= \frac{1}{z^{r-1}} \begin{vmatrix} z^{-\lambda_1} & z^{-\lambda_2+1} & \dots & z^{-\lambda_r+r-1} \\ h_{\lambda_1+1} & h_{\lambda_2} & \dots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \dots & h_{\lambda_r} \end{vmatrix} b_0 \wedge b_1 \wedge \ldots \wedge b_{r-2}. \tag{4.23}$$

(Sketch of) Proof. In fact

$$\boldsymbol{\beta}\left(z^{-1}\right)\lrcorner b_{\lambda_r}\wedge b_{-1+\lambda_{r-1}}\wedge\ldots\wedge b_{r-1+\lambda_1} =$$

$$\begin{aligned} &= \sum_{j=0}^{r-1}\frac{1}{z^{j+\lambda_{r-j}}}(-1)^j\mathbf{b}_{r-1,\boldsymbol{\lambda}_j} \\ &= \frac{1}{z^{r-1}}\sum_{j=0}^{r-1}\frac{1}{z^{\lambda_{r-j}+1-(r-j)}}\mathbf{b}_{r-1,\boldsymbol{\lambda}_j} \end{aligned} \tag{4.24}$$

where $\boldsymbol{\lambda}_j$ denotes the partition of length at most $r-1$ obtained by removing the j-th part from $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Now

$$\mathbf{b}_{r-1,\boldsymbol{\lambda}_j} = \Delta_{\boldsymbol{\lambda}_j}(H_{r-1})\mathbf{b}_{r-1,0}$$

and so (4.24) is equivalent to:

$$\frac{1}{z^{r-1}}\sum_{j=0}^{r-1}\frac{\mathbf{b}_{r-1,\boldsymbol{\lambda}_j}}{z^{\lambda_{r-j}+1-(r-j)}} = \frac{1}{z^{r-1}}\sum_{j=0}^{r-1}(-1)^j\frac{\Delta_{\boldsymbol{\lambda}_j}(H_{r-1})}{z^{\lambda_{r-j}+1-(r-j)}}\mathbf{b}_{r-1,0}. \tag{4.25}$$

A quick inspection shows that (4.25) is precisely the expansion of the last side of (4.23). ■

4.4.3 Theorem. *The following equality holds:*

$$\Gamma_r^\vee(z)\Delta_{\boldsymbol{\lambda}}(H_r) := \frac{\boldsymbol{\beta}\left(z^{-1}\right)\lrcorner\mathbf{b}_{r,\boldsymbol{\lambda}}}{\mathbf{b}_{r,0}} = \frac{E_{r-1}(z)}{z^{r-1}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1}) = \tag{4.26}$$

$$= \frac{E_{r-1}(z)}{z^{r-1}}\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} & \mathcal{D}_-(z)h_{\lambda_2-1} & \ldots & \mathcal{D}_-(z)h_{\lambda_r-r+1} \\ \mathcal{D}_-(z)h_{\lambda_1+1} & \mathcal{D}_-(z)h_{\lambda_2} & \ldots & \mathcal{D}_-(z)h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_-(z)h_{\lambda_1+r-1} & \mathcal{D}_-(z)h_{\lambda_2+r-2} & \ldots & \mathcal{D}_-(z)h_{\lambda_r} \end{vmatrix} \tag{4.27}$$

Proof. Recall from Section 1.3.2 the definition (1.13) of $u_j \in B_{r-1}[[z]]$:

$$u_j = \sum_{n\geq 0} h_{n+j}z^n$$

for $j \geq 0$, where $H_{r-1}(z)E_{r-1}(z) = 1$. Since

$$E_r(z)u_j = \mathtt{U}_0(u_j) + \mathtt{U}_1(u_j)z + \ldots + \mathtt{U}_{r-1}(u_j)z^{r-1}, \tag{4.28}$$

where each $\mathtt{U}_i(u_j)$ is a B_{r-1}–linear combination of $h_j, h_{j+1}, \ldots, h_{j+i-2}$, the determinant

$$\begin{vmatrix} E_{r-1}(z)u_{\lambda_1+1} & E_{r-1}(z)u_{\lambda_2} & \ldots & E_{r-1}(z)u_{\lambda_r+r-2} \\ h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r} \end{vmatrix} \tag{4.29}$$

vanishes by skew-symmetry. Notice that

$$\frac{1}{z^\lambda} = \frac{E_{r-1}(z)H_{r-1}(z)}{z^\lambda} = E_{r-1}(z)\left(\mathcal{D}_-(z)h_\lambda + zu_{\lambda+1}\right),$$

from which

$$\frac{1}{z^{r-1}}\begin{vmatrix} z^{-\lambda_1} & z^{-\lambda_2+1} & \ldots & z^{-\lambda_r+r-1} \\ h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r} \end{vmatrix} =$$

$$= \frac{E_{r-1}(z)}{z^{r-1}}\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} + zu_{\lambda_1+1} & \ldots & \mathcal{D}_-(z)h_{\lambda_r-r+1} + zu_{\lambda_r-r+2} \\ h_{\lambda_1+1} & \ldots & h_{\lambda_r+r-2} \\ \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & \ldots & h_{\lambda_r} \end{vmatrix} =$$

$$= \frac{E_{r-1}(z)}{z^{r-1}}\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} & \mathcal{D}_-(z)h_{\lambda_2-1} & \ldots & \mathcal{D}_-(z)h_{\lambda_r-r+1} \\ h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r} \end{vmatrix} \tag{4.30}$$

the last equality due to the vanishing of (4.29).

To conclude the proof observe that for all $1 \leq i, j \leq r$:

$$\mathcal{D}_-(z)h_{\lambda_j-j+i} = h_{\lambda_j-j+i} + \mathcal{D}_-(z)h_{\lambda_j-j+i-1}$$

and thus, exploiting once again the skew-symmetry of the determinant, expression (5.26) is equivalent to

$$= \frac{E_{r-1}(z)}{z^{r-1}} \begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} & \mathcal{D}_-(z)h_{\lambda_2-1} & \dots & \mathcal{D}_-(z)h_{\lambda_r-r+1} \\ \mathcal{D}_-(z)h_{\lambda_1+1} & \mathcal{D}_-(z)h_{\lambda_2} & \dots & \mathcal{D}_-(z)h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_-(z)h_{\lambda_1+r-1} & \mathcal{D}_-(z)h_{\lambda_2+r-2} & \dots & \mathcal{D}_-(z)h_{\lambda_r} \end{vmatrix}$$

which proves that, according to the Definition 4.4.1,

$$\Gamma_r^\vee(z) = \frac{E_{r-1}(z)}{z^{r-1}} \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1}),$$

as desired. ∎

4.4.4 Remark. It is important to notice that, unlike the case regarding $\Gamma_r(z)$ (Cf. Corollary 4.3.5), here, in general,

$$\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1}) \neq \mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r-1}).$$

For instance, $\Delta_{\boldsymbol{\lambda}}(H_1) = 0$ for all partitions of length 2. Thus

$$\begin{aligned} 0 &= \mathcal{D}_-(z)\Delta_{(11)}(H_1) = \mathcal{D}_-(z)(h_1^2 - h_1^2) \neq \\ &\neq \Delta_{(11)}(\mathcal{D}_-(z)H_1) = \begin{vmatrix} h_1 + \frac{1}{z} & 1 \\ h_2 & h_1 \end{vmatrix} = \frac{h_1}{z} \end{aligned}$$

However $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1}) = \mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r-1})$ if $\boldsymbol{\lambda}$ has length at most $r-1$ as a consequence of Proposition 4.2.10.

4.4.5 Corollary. *Let $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$. Then the equality*

$$\Gamma_r^\vee(z)\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \frac{E_{r-1}(z)}{z^{r-1}} \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1,n})\mathbf{b}_{r,0} \tag{4.31}$$

holds in $\bigwedge^{r-1} M_0$.

Proof. First of all, by virtue of Theorem (4.4.3), one has:

$$\frac{E_{r-1}(z)}{z^{r-1}} \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1})\mathbf{b}_{r-1,0} \quad = \quad \sum_{j\geq 0} \beta_j(z^{-1}) \lrcorner \mathbf{b}_{r,\boldsymbol{\lambda}}$$

$$= \sum_{j=0}^{n-1} \beta_j(z^{-1}) \lrcorner \mathbf{b}_{r,\boldsymbol{\lambda}},$$

where in the last equality we used the fact that if $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$ then $\beta_j \lrcorner \mathbf{b}_{r,\boldsymbol{\lambda}} = 0$ for all $j \geq n - r + 1$. Now one uses the fact that if $\boldsymbol{\lambda} \in \mathcal{P}_{r,n}$, then $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1})\mathbf{b}_{r-1,0}$ lands in $\bigwedge^{r-1} M_{0,n}$ and is equal to $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1,n})\mathbf{b}_{r-1,0}$, basically by Proposition 3.6.7. ■

4.5 Plücker Equations for Grassmann Cones

Let $P(H_r) \in B_r$ (i.e. the element of B_r obtained by evaluating $P \in \mathbb{Z}[\mathbf{X}]$ at $X_i = h_i$). Then $P(H_r) = \sum a_{\boldsymbol{\lambda}} \Delta_{\boldsymbol{\lambda}}(H_r)$. When does P corresponds to a decomposable tensor $P(\mathcal{D}_+)\mathbf{b}_{r,0} \in \bigwedge^r M_0$? The trick to ease computations is to use the B_r-module $\bigwedge^r M_r$ constructed out of $\bigwedge^r M_0$.

4.5.1 Theorem. *A polynomial $P(H_r) \in B_r$ corresponds to a decomposable tensor of $\bigwedge^r M_0$ if and only if*

$$\mathrm{Res}_{z=0} \Gamma_r(z)P(H_r) \otimes \frac{\Gamma_r^\vee(z)P(H_r)}{z} = 0. \tag{4.32}$$

Proof. By Theorem 4.1.4, the tensor $\alpha := P(\mathcal{D}_+)\mathbf{b}_{r,0}$ is decomposable if and only if the residue at $z = 0$ in the expansion

$$\frac{1}{z} \sum_{i \geq 0} (b_i z^i \wedge P(\mathcal{D}_+)\mathbf{b}_{r,0}) \otimes (z^{-i} \beta_i \lrcorner P(\mathcal{D}_+)\mathbf{b}_{r,0})$$

vanishes. Now

$$\sum_{i \geq 0} (b_i z^i \wedge P(\mathcal{D}_+)\mathbf{b}_{r,0}) \otimes \frac{1}{z} \sum_{j \geq 0} (\beta_j \lrcorner P(\mathcal{D}_+)\mathbf{b}_{r,0}) \otimes_{\mathbb{Z}} 1_{B_r} =$$

$$= \Gamma_r(z)P(H_r)\mathbf{b}_{r+1,0} \otimes \frac{\Gamma_r^\vee(z)P(H_r)\mathbf{b}_{r-1,0}}{z}$$

and so the residue at $z = 0$ of the last side vanishes if and only if (4.32) holds. ■

4.5.2 An arbitrary $P(H_r) \in B_r$ is of the form:

$$P(H_r) = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_r} a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(H_r), \qquad (a_{\boldsymbol{\lambda}} \in \mathbb{Z})$$

where all $a_{\boldsymbol{\lambda}} = 0$ for all but finitely many $\boldsymbol{\lambda} \in \mathcal{P}_r$. Substituting in (4.32) the explicit expressions (4.13) and (4.26) of $\Gamma_r(z)$ and $\Gamma_r^\vee(z)$ respectively, we have proven the following

4.5.3 Theorem. *The $\mathbb{Z}$-linear combination $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(H_r) \in B_r$ corresponds to a decomposable tensor in $\bigwedge^r M_0$ if and only if*

$$\mathrm{Res}_{z=0} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}E_{r-1}(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1}) \otimes \frac{\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H_{r+1})}{E_{r+1}(z)} = 0. \tag{4.33}$$

■

4.5.4 Corollary. *Let $P(H_r) := \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(H_r)$. The polynomial $P(H_r)$ corresponds to a decomposable tensor if and only if*

$$\mathrm{Res}_{z=0} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}E_{r-1}(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_{r-1,n}) \otimes \frac{\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H_{r+1,n})}{E_{r+1}(z)} = 0. \tag{4.34}$$

Proof. In fact $\alpha := P(H_r)\mathbf{b}_{r,0} \in \bigwedge^r M_{r,n}$ by hypothesis. Thus we may apply Corollaries 4.3.6 and 4.4.5. ■

This formula can be written in a more intelligible form once one identifies the tensor product of polynomial rings with a bigger polynomial ring (Cf. 0.4.4)

$$B_{r-1} \otimes B_{r+1} = \mathbb{Z}[e'_1, \ldots, e'_{r-1}, e''_1, \ldots, e''_{r+1}].$$

We denote by $E'_{r-1}(z) = 1 - e'_1z + \ldots + (-1)^{r-1}e'_{r-1}z^{r-1}$ and by $E''_{r+1}(z) = 1 - e''_1z + \ldots + (-1)^{r+1}e''_{r+1}z^{r+1}$. Similarly we let $H'_{r-1}(z) = \sum_{n\geq 0} h'_nz^n$ and $H''(z) = \sum_{n\geq 0} h''_nz^n$ as being the inverse of $E'_{r-1}(z)$ and $E''_{r+1}(z)$ in $B'_r[[z]]$ and $B''_r[[z]]$ respectively. Formula (4.33) can be then written as

$$\mathrm{Res}_{z=0}\frac{E'_{r-1}(z)}{E''_{r+1}(z)} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H'_{r-1}) \cdot \overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H''_{r+1}) = 0$$

4.5.5 Corollary. *The polynomial*

$$\sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(H_{r,n}) \in B_{r,n}$$

corresponds to a decomposable tensor in $\bigwedge^r M_{0,n}$ *if and only if*

$$\mathrm{Res}_{z=0}\frac{E'_{r-1}(z)}{E''_{r+1}(z)}\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{P}_{r,n}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H'_{r-1,n})\cdot\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H''_{r+1,n}) = 0$$

Proof. Let $\alpha = \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{r,n}} a_{\boldsymbol{\lambda}}\mathbf{b}_{r,\boldsymbol{\lambda}} \in \bigwedge^r M_{0,n}$ be decomposable. Then $\Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0} = \Delta_{\boldsymbol{\lambda}}(H_{r,n})\mathbf{b}_{r,0}$ and because of the inclusion $\bigwedge^r M_{0,n} \subseteq \bigwedge^r M_0$, we have

$$0 = \mathrm{Res}_{z=0}\frac{E'_{r-1}(z)}{E''_{r+1}(z)}\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H'_{r-1})\cdot\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H''_{r+1})$$

$$= \mathrm{Res}_{z=0}\frac{E'_{r-1}(z)}{E''_{r+1}(z)}\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} a_{\boldsymbol{\lambda}}a_{\boldsymbol{\mu}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H'_{r-1,n})\cdot\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\mu}}(H''_{r+1,n}). \blacksquare$$

4.5.6 Example. Let

$$\begin{aligned} P(H_2): &= a_0 + a_1h_1 + a_2h_2 \\ &+ a_{11}\Delta_{(11)}(H_2) + a_{21}\Delta_{(21)}(H_2) + a_{22}\Delta_{(22)}(H_2) \in B_2. \end{aligned}$$

The goal is to determine conditions on the coefficients $(a_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_{2,4})$ ensuring that $P(H_2)\mathbf{b}_{2,0}$ is a decomposable tensor in $\bigwedge^2 M_0$. To this purpose we use the B_2-module structure of $\bigwedge^2 M_2 := B_2 \otimes \bigwedge^2 M_0$. Since $P(H_2)\mathbf{b}_{2,0} \in \bigwedge^2 M_{2,4}$, the B_2-module structure of $\bigwedge^2 M_2$ can be factored through that of $B_{2,4}$. Let us compute

$$\Gamma_2^\vee(z)P(H_{2,4}) \in B_{1,4}((z))$$

according to the recipe (4.31). We have:

$$\begin{aligned} P(\mathcal{D}_-(z)H_{1,4}) &= a_0 + a_1\mathcal{D}_-(z)h_1 + a_2\mathcal{D}_-(z)h_1^2 \\ &+ a_{11}\Delta_{(11)}(\mathcal{D}_-(z)H_1) + a_{21}\Delta_{(21)}(\mathcal{D}_-(z)H_1) \\ &+ \Delta_{(22)}(\mathcal{D}_-(z)H_1) \end{aligned}$$

from which

$$\Gamma_2^\vee(z)P(H_{2,4}) =$$
$$= \frac{1-e_1z}{z}\left(a_0+a_1h_1+a_2h_1^2+\frac{1}{z}(a_1+a_2h_1)+\frac{a_2}{z^2}\right)\in B_1((z)).$$

On the other hand

$$\begin{aligned} P(\overline{\mathcal{D}}_-(z)H_{3,4}) &= \sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}\Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_{3,4}) \\ = & \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{2,4}} a_{(\lambda_1,\lambda_2)}\det(h_{\lambda_j-j+i-1}-h_{\lambda_j-j+i}z^{-1})\in B_{3,4}((z)) \end{aligned}$$

and so the equality below

$$\Gamma_2(z)P(H_2)=\frac{z^2}{E_3(z)}\sum_{\boldsymbol{\lambda}} a_{(\lambda_1,\lambda_2)}\det(h_{\lambda_j-j+i-1}-h_{\lambda_j-j+i}z^{-1})$$

holds in $B_{3,4}((z))$. We now restrict the output above in the case of $B_{2,4}$. We have

$$B_{3,4}=\frac{\mathbb{Z}[e_1,e_2,e_3]}{(h_2,h_3,h_4)}\cong\frac{\mathbb{Z}[y]}{(y^4)}$$

where we put $y=e_1+(h_2,h_3,h_4)$. Indeed, the relation $h_2=0$ implies that $e_2=e_1^2$. Moreover $h_3-e_1h_2+e_2h_1+e_3=0$ together with $h_2=h_3=0$ yields $e_3=h_1e_2=e_1^3$. In addition $h_4-e_1h_3+e_2h_2-e_3h_1=0$, whence $e_3h_1=e_1^4=0 \mod (h_2,h_3,h_4)$. Similarly

$$B_{1,4}=\frac{\mathbb{Z}[x]}{(h_4)}=\frac{\mathbb{Z}[x]}{(x^4)}$$

where we put $x=e_1+(h_4)$. In fact the relation

$$(1-e_1z)(1+h_1z+h_2z^2+h_3z^3)=1,$$

holding in $B_{1,4}$, says that $h_1=e_1$, $h_2=e_1^2$, $h_3=e_1^3$ and $e_1^4=0$. So we have

$$\begin{aligned}\Gamma_2^\vee(z)P(H_{2,4}) &= \frac{1-xz}{z}\left(a_0+a_1\left(x+\frac{1}{z}\right)+a_2\left(x^2+\frac{x}{z}+\frac{1}{z^2}\right)+\right.\\ &+ \left. a_{11}\frac{x}{z}+a_{21}\left(\frac{x}{z^2}+\frac{x^2}{z}\right)+a_{22}\frac{x^2}{z^2}\right)\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(z)P(H_{2,4}) = & \; z^2(1+yz)\left[a_0 + a_1\left(y-\frac{1}{z}\right) - a_2\frac{y}{z} + \right. \\ & \left. + \; a_{11}\left(y^2 - \frac{y}{z} + \frac{1}{z^2}\right) + a_{21}\left(\frac{y}{z^2} - \frac{y^2}{z}\right) + a_{22}\frac{y^2}{z^2}\right].\end{aligned}$$

Finally, after some computations, that the author did with the help of `CoCoA` [1]

$$\mathrm{Res}_{z=0}\frac{1}{z}\Gamma_2(z)P(H'_{2,4}))(\Gamma_2^\vee(z)P(H''_{2,4})) =$$

$$\begin{gathered}(-a_{11}a_2 + a_1a_{21} - a_0a_{22})x^3 + (a_{11}a_2 - a_1a_{21} + a_0a_{22})x^2y - \\ (a_{11}a_2 - a_1a_{21} + a_0a_{22})xy^2 + (a_{11}a_2 - a_1a_{21} + a_0a_{22})y^3 \\ = (a_{11}a_2 - a_1a_{21} + a_0a_{22})(y^3 - y^2x + x^2y - x^3)\end{gathered}$$

which is identically zero if and only if the Plücker equation

$$a_{11}a_2 - a_1a_{21} + a_0a_{22} = 0 \tag{4.35}$$

holds.

4.5.7 Remark. If $V = M_{0,4} \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^4$ and

$$\sum_{2\geq\lambda_1\geq\lambda_2\geq 0} a_{(\lambda_1,\lambda_2)} b_{\lambda_2} \wedge b_{1+\lambda_1} \in \bigwedge^2 \mathbb{C}^4,$$

equation (4.35) is the equation of the *Klein quadric* in $\mathbb{P}^5$ whose zero locus corresponds to points of the Grassmann variety $G(2,4)$ parameterizing 2-dimensional subspaces of $\mathbb{C}^4$.

4.5.8 Example. In the similar way adopted in Example 4.5.6, one can find the locus of polynomials of $B_2 := \mathbb{Z}[e_1, e_2]$

$$P(H_2) := \sum_{\boldsymbol{\lambda}\in\mathcal{P}_{2,5}} a_{\boldsymbol{\lambda}} \Delta_{\boldsymbol{\lambda}}(H_2)$$

corresponding to decomposable tensors in $\bigwedge^2 M_0$. As before, to perform computations, one can use the $B_{2,5}$-module $\bigwedge^2 M_{2,5}$. In $B_{1,5}$

we put $h'_1 = x$ and in $B_{3,5}$ we set $h''_1 = y_1$ and $h''_2 = y_2$. Thus the equation of decomposable tensors is given by the vanishing of the residue at $z = 0$ of

$$\frac{1}{z}(1-xz)(1+y_1z+y_2z^2)\sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{P}_{2,5}}\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H'_{1,5})\Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H''_{3,5}).$$

Executing computation precisely as in Example 4.5.6, one finds

$$\begin{array}{ll} & (a_{11}a_{20}-a_{10}a_{21}+a_{00}a_{22})P_3(x,\mathbf{y}) \\ + & (a_{11}a_{30}-a_{10}a_{31}+a_{00}a_{32})P_4(x,\mathbf{y}) \\ + & (a_{21}a_{30}-a_{20}a_{31}+a_{00}a_{33})P_5(x,\mathbf{y}) \\ + & (a_{22}a_{30}-a_{20}a_{32}+a_{10}a_{33})P_6(x,\mathbf{y}) \\ + & (a_{22}a_{31}-a_{21}a_{32}+a_{11}a_{33})P_7(x,\mathbf{y}) = 0 \end{array}$$

where to short notation we have set $\mathbf{y} = (y_1, y_2)$ and

$$\begin{array}{rcl} P_3(x,\mathbf{y}) & := & y_1^3-2y_1y_2-xy_1^2+xy_2+x^2y_1-x^3; \\ P_4(x,\mathbf{y}) & := & y_1^2y_2-y_2^2-y_1y_2x+y_2x^2-x^4; \\ P_5(x,\mathbf{y}) & := & y_1y_2^2-y_2^2x+y_2x^3-y_1x^4; \\ P_6(x,\mathbf{y}) & := & y_2^3-y_2^2x^2+y_1y_2x^3-y_1^2x^4+y_2x^4; \\ P_7(x,\mathbf{y}) & := & y_2^3x-y_1y_2^2x^2+y_1^2y_2x^3-x^3y_2^2-2x^4y_1y_2. \end{array}$$

The expression above vanishes if and only if the coefficients of the forms of degree $3,4,5,6,7$ in (x, y_1, y_2) vanish. One easily recognize in such coefficients the five Pfaffians of the 5×5 skew-sym- metric matrix

$$\begin{pmatrix} 0 & a_{00} & a_{10} & a_{20} & a_{30} \\ -a_{00} & 0 & a_{11} & a_{21} & a_{31} \\ -a_{10} & -a_{11} & 0 & a_{22} & a_{32} \\ -a_{20} & -a_{21} & -a_{22} & 0 & a_{33} \\ -a_{30} & -a_{31} & -a_{32} & -a_{33} & 0 \end{pmatrix}$$

as it had to be (Cf. e.g. [69]).

4.6 The Infinite Exterior Power

We have seen that there is a $\mathbb{Z}$-module isomorphism $B_r \to \bigwedge^r M_0$ mapping $\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto \mathbf{b}_{r,\boldsymbol{\lambda}}$. Notice that replacing the finite sequence of indeterminates $(e_1, e_2, \ldots, e_r)$ with an infinite sequence $(e_1, e_2, \ldots)$ it makes sense to consider the polynomial ring

$$B_\infty = \mathbb{Z}[e_1, e_2, \ldots]$$

in infinitely many indeterminates. The latter, following [65], is the projective limit of B_r in the category of graded ring. It can be explicitly constructed as follows. Recall that each B_r is a graded ring by weight:

$$(B_r)_w = \bigoplus_{|\boldsymbol{\lambda}|=w} \mathbb{Z}\Delta_{\boldsymbol{\lambda}}(H_r)$$

and that there is a $\mathbb{Z}$-module isomorphism $(B_r)_w \to (\bigwedge^r M)_w$ mapping $\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto \Delta_{\boldsymbol{\lambda}}(H_r)\mathbf{b}_{r,0}$. For all $s \geq r$ there is a diagramme of $\mathbb{Z}$-module homomorphism

$$\begin{array}{ccc} (\bigwedge^s M_0)_w & \longrightarrow & (\bigwedge^r M_0)_w \\ \downarrow & & \downarrow \\ (B_s)_w & \longrightarrow & (B_r)_w \end{array} \tag{4.36}$$

whose horizontal arrows are the epi-morphisms mapping

$$\Delta_{\boldsymbol{\lambda}}(H_s) \mapsto \Delta_{\boldsymbol{\lambda}}(H_r) \qquad \text{and} \qquad \mathbf{b}_{s,\boldsymbol{\lambda}} \mapsto \mathbf{b}_{r,\boldsymbol{\lambda}} \tag{4.37}$$

if $\boldsymbol{\lambda} \in \mathcal{P}_r$ and 0 otherwise. The epi-morphisms (4.37) have a section. The former is given by $\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto \Delta_{\boldsymbol{\lambda}}(H_s)$ and the latter by $\mathbf{b}_{r,\boldsymbol{\lambda}} \mapsto \mathbf{b}_{s,\boldsymbol{\lambda}}$.

Each diagramme like (4.36) factorizes through $(B_t)_w \to (\bigwedge^t M_0)_w$ for all $s \geq t \geq r \geq 1$:

$$\begin{array}{ccccc} (\bigwedge^s M_0)_w & \longrightarrow & (\bigwedge^t M_0)_w & \longrightarrow & (\bigwedge^r M_0)_w \\ \downarrow & & \downarrow & & \downarrow \\ (B_s)_w & \longrightarrow & (B_t)_w & \longrightarrow & (B_r)_w \end{array}$$

Hence the data $((B_s)_w \to (\bigwedge^s M_0)_w, \rho_{sr}, \pi_{rs})$ is an inverse system ($\rho_{ts}\rho_{sr} = \rho_{tr}$ and $\pi_{ts}\pi_{sr} = \pi_{tr}$, for all $s \geq t \geq r$) and one can then take the inverse limit $(B_\infty)_w \longrightarrow (\bigwedge^\infty M)_w$, where clearly $\bigwedge^\infty M)_w$ is a notation for $\lim_{\leftarrow}(B_r)_w \mathbf{b}_{r,0} = \lim_{\leftarrow}(\bigwedge^r M_0)_w$. One so define

$$\bigwedge^\infty M_0 := \sum_{w\geq 0}(\bigwedge^\infty M_0)_w = \sum_{w\geq 0} B_\infty \cdot \mathbf{b}_{\infty,0}$$

where $\mathbf{b}_{\infty,0} := b_0 \wedge b_1 \wedge b_2 \wedge \ldots$.

In Chapter 5, we shall identify $M_r := M_0 \otimes_{\mathbb{Z}} B_r$ as in Section 3.4 with the $\mathbb{Z}$-module of generic linear recurrent sequences of order r. This will suggest a formalism whose underlying idea is that of embedding an r-th exterior power inside an infinite exterior power of a module of countable infinite rank. As r goes to ∞, a module which is isomorphic to $\bigwedge^\infty M_0$ defined above will be recovered. The vertex-like operators met in this chapter will appear to be a prototypical version of the vertex operators properly said, whose expression will be computed, yielding exactly that encountered in the Section 0.3.1 of the Prologue and in the literature on the subject.

Chapter 5

Vertex Operators via Generic LRS

The purpose of this chapter is to sketch a construction of the infinite wedge power of a module of infinite rank using as a model for M_r the B_r-module K_r defined in Section 1.3.8. The infinite wedge power will be seen as the limit of $\bigwedge^r K_r$ for $r \to \infty$. There is a rich literature which is concerned with the infinite exterior power of a in infinite dimensional vector space. Beside the pioneristic work by Kac and Peterson [46], the reader may look at e.g. [6, 47, 54, 64].

5.1 Preliminaries

5.1.1 For some integer $r \geq 1$, fixed once and for all, let $E_r(t), H_r(t) \in B_r[[t]]$ as in Section 1.3. Recall the sequence $(u_i)_{i\in\mathbb{Z}}$

$$u_i = \sum_{n\geq 0} h_{n+j}t^n,$$

with the usual convention $h_k = 0$ if $k < 0$. Also recall that u_i is a generic LRS for all $i \geq -r+1$. The map $b_i \mapsto u_{i-r+1}$ gives a natural model for the free abelian group M_0, considered in Chapter 4, with the $\mathbb{Z}$-module

$$\bigoplus_{i\geq -r+1} \mathbb{Z}\cdot u_i. \tag{5.1}$$

In this case, the module $M_r = B_r \otimes_{\mathbb{Z}} M_0$ constructed in Section 3.3 is precisely the free B_r-module K_r generated by $(u_0, \dots, u_{-r+1})$, as defined in (1.17). Clearly $\bigwedge^r K_r$ is a free B_r-module of rank 1 generated by

$$\mathbf{u}_{r,0} := u_0 \wedge u_{-1} \wedge \ldots \wedge u_{-r+1}$$

and the map

$$\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto \mathbf{u}_{r,\boldsymbol{\lambda}} := u_{\lambda_1} \wedge u_{-1+\lambda_2} \wedge \ldots \wedge u_{-r+1+\lambda_r}$$

is a $\mathbb{Z}$-module isomorphism $B_r \to \bigwedge^r K_r$. Let

$$V_r := \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\cdot u_i = \bigoplus_{i\leq -r} \mathbb{Z}\cdot u_i \oplus \bigoplus_{i\geq -r+1} \mathbb{Z}\cdot u_i.$$

The reason for the subscript r is to keep track of the B_r–module structure of V_r, induced by the basis $u_i \in B_r$. The completion of V_r with respect to the topology for which $\{t^N u_0\}$ is a fundamental system of neighborhoods of 0 is $B_r[[t]]$. In fact $V_r \otimes_{\mathbb{Z}} B_r = B_r[t]u_0$, due essentially to Proposition 1.3.3, and then the completion of $V_r \otimes_{\mathbb{Z}} B_r$ with respect to $t^N u_0$ is $B_r[[t]]u_0 = B_r[[t]]$. Define the two shift endomorphisms $D_{\pm 1} \in End_{\mathbb{Z}}(B_r \otimes_{\mathbb{Z}} V_r)$ of step ± 1, as in Chapter 4, namely $D_1 u_j = u_{j+1}$ and $D_{-1} u_j = u_{j-1}$. The role played by D_1 and D_{-1} is not as symmetric as it may seem. In fact D_1 is B_r-linear by construction (it is precisely the endomorphism (1.6)) while D_{-1} is not. To see this, notice that for all $i \geq 1$, $u_{i-1} \in K_r := \ker \mathtt{p}_r(D)$ and that

$$u_{i-1} = D_{-1}(u_i) = D_{-1}(\sum_{j=0}^{r} \mathtt{U}_j(u_i) u_{-j}) \neq \sum_{j=0}^{r} \mathtt{U}_j(u_i) u_{-j-1} \notin \ker \mathtt{p}_r(D).$$

5.2 Semi-Infinite Exterior Powers

5.2.1 Notation. For all $i, j \in \mathbb{Z}$ and $\boldsymbol{\lambda} \in \mathcal{P}_r$ let:

i) $\mathbf{u}_{r,i+\boldsymbol{\lambda}} := u_{i+\lambda_1} \wedge u_{i-1+\lambda_2} \wedge \ldots \wedge u_{i-r+1+\lambda_r} \in \bigwedge^r V_r$

ii) $\Phi^r_j := u_j \wedge u_{j-1} \wedge u_{j-2} \wedge \ldots$

iii) $\Phi^r_{i+\boldsymbol{\lambda}} := \mathbf{u}_{r,i+\boldsymbol{\lambda}} \wedge \Phi^r_{i-r} := u_{i+\lambda_1} \wedge \ldots \wedge u_{i-r+1+\lambda_r} \wedge \Phi^r_{i-r}$

Following [47] and others references, the expression $\Phi^r_{i+\boldsymbol{\lambda}}$ will be called *semi-infinite exterior monomial*. For the limited purposes of this exposition it will be considered no more than a notation. For all $i \in \mathbb{Z}$, we consider the $\mathbb{Z}$-module:

$$F^r_i := \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_r} \mathbb{Z} \cdot \Phi^r_{i+\boldsymbol{\lambda}}. \tag{5.2}$$

5.2.2 Let $\mathbf{u}_{r,\boldsymbol{\lambda}} := \mathbf{u}_{r,0+\boldsymbol{\lambda}}$. There is an obvious canonical $\mathbb{Z}$-module isomorphism between $\bigwedge^r K_r$ and F^r_0 given by

$$\mathbf{u}_{r,\boldsymbol{\lambda}} \mapsto \Phi^r_{0+\boldsymbol{\lambda}} := \mathbf{u}_{r,\boldsymbol{\lambda}} \wedge \Phi^r_{-r},$$

which amounts to the identification $F^r_0 \cong \bigwedge^r K_r \wedge \Phi^r_{-r}$. So to speak, F^r_0 can be seen as a way to embed the exterior power of a module of finite rank inside an infinite wedge power of a module of infinite rank. Notice that $F^r_i \cap F^r_j = \{\mathbf{0}\}$ if $i \neq j$.

5.2.3 For $(i,j) \in \mathbb{N} \times \mathbb{Z}$ define

$$\overline{D}_i \Phi^r_j := u_{j+1} \wedge u_j \wedge \ldots \wedge u_{j-i+2} \wedge \Phi^r_{j-i}. \tag{5.3}$$

and

$$\overline{D}_{-i} \Phi^r_j = 0 \tag{5.4}$$

For example

$$\overline{D}_1 \Phi^r_0 = u_1 \wedge \Phi^r_{-1} = u_1 \wedge u_{-1} \wedge u_{-2} \wedge \ldots$$

$$\overline{D}_2 \Phi^r_0 = u_1 \wedge u_0 \wedge \Phi^r_{-2} = u_1 \wedge u_0 \wedge u_{-2} \wedge u_{-3} \wedge \ldots$$

In particular

$$\overline{D}_+(z) \Phi^r_j := \sum_{i \geq 0} \overline{D}_i \Phi^r_j z^j \in F^r_j[[z]]$$

is well defined. Equalities (5.3) and (5.4) enable us to extend the definition 2.3.1 of the characteristic polynomial operator to that of *characteristic polynomial series* $\overline{\mathcal{D}}_+(z) = \sum_{j \geq 0} \overline{D}_j z^j \in End_{\mathbb{Z}}(\bigwedge V_r)[[z]]$

and $\overline{\mathcal{D}}_-(z)=\sum_{j\geq 0}\overline{D}_{-j}z^{-j}\in End_{\mathbb{Z}}(\bigwedge V_r)[[z^{-1}]]$, associated to the shifts D_1 and D_{-1} respectively. They are defined over F_i^r by setting:

$$\begin{aligned}\overline{\mathcal{D}}_+(z)\Phi^r_{i+\boldsymbol{\lambda}} &= \overline{\mathcal{D}}_+(z)(\mathbf{u}_{r,i+\boldsymbol{\lambda}}\wedge\Phi^r_{i-r})=\\ &= \overline{\mathcal{D}}_+(z)\mathbf{u}_{r,i+\boldsymbol{\lambda}}\wedge\overline{\mathcal{D}}_+(z)\Phi^r_{i-r}\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}\overline{\mathcal{D}}_-(z)\Phi^r_{i+\boldsymbol{\lambda}} &= \overline{\mathcal{D}}_-(z)(\mathbf{u}_{r,i+\boldsymbol{\lambda}}\wedge\Phi^r_{i-r})=\\ &= \overline{\mathcal{D}}_-(z)\mathbf{u}_{r,i+\boldsymbol{\lambda}}\wedge\overline{\mathcal{D}}_-(z)\Phi^r_{i-r}=\\ &= \overline{\mathcal{D}}_-(z)\mathbf{u}_{r,i+\boldsymbol{\lambda}}\wedge\Phi^r_{i-r}\end{aligned}\tag{5.6}$$

In other words $\overline{\mathcal{D}}_\pm(z)$ are derivations of the exterior algebra $\bigwedge V_r$.

5.2.4 Remark. Let $\boldsymbol{\lambda}$ be a partition of length at most k. Let us show that (5.5) implies:

$$\overline{D}_i\Phi^r_{j+\boldsymbol{\lambda}}=\overline{D}_i(\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\Phi_{j-k})=\sum_{p=0}^{i}\overline{D}_{i-p}\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\overline{D}_p\Phi_{j-k}.\tag{5.7}$$

In fact

$$\overline{\mathcal{D}}_+(z)\Phi^r_{j+\boldsymbol{\lambda}}=\sum_{i\geq 0}\overline{D}_i\Phi^r_{j+\boldsymbol{\lambda}}z^i$$

On the other hand

$$\overline{\mathcal{D}}_+(z)\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\overline{D}_+(z)\Phi^r_{j-k}=\sum_{i_1\geq 0}\overline{D}_{i_1}\mathbf{u}_{r,i+\boldsymbol{\lambda}}z^{i_1}\wedge\sum_{i_2\geq 0}\overline{D}_{i_2}\Phi_{r,j-k}z^{i_2}\tag{5.8}$$

and so $\overline{D}_i(\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\Phi_{j-k})$ is the coefficient of z^i on the right hand side of (5.8) , which is precisely the right hand side of (5.7). ∎

Define

$$\mathcal{D}_+(z)=\frac{1}{\overline{\mathcal{D}}_+(z)}\in End_{\mathbb{Z}}(\bigwedge V_r)[[z]]$$

and

$$\mathcal{D}_-(z)=\frac{1}{\overline{\mathcal{D}}_-(z)}\in End_{\mathbb{Z}}(\bigwedge V_r)[[z^{-1}]]$$

Clearly $\mathcal{D}_{\pm}(z)$ are derivations of the exterior algebra $\bigwedge V_r$ as well, being formal inverse of $\overline{\mathcal{D}}_{\pm}(z^{\pm})$ and (2.2) also holds.

5.2.5 Proposition. *For all* $(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}$*:*

a) $\overline{D}_j \Phi^r_i = u_{i+1} \wedge \overline{D}_{j-1}\Phi^r_{i-1}$*;*

b) $u_j \wedge \overline{D}_i \Phi^r_{j-1} = 0$ *for all* $i \geq 1$*;*

c) *If* $\boldsymbol{\lambda}$ *has length* $k \geq 1$*,*

$$\overline{D}_{-i}\Phi^r_{j,\boldsymbol{\lambda}} = \overline{D}_{-i}(u_{j+\lambda_1} \wedge \ldots \wedge u_{j-k+1+\lambda_k}) \wedge \Phi^r_{j-k}.$$

Proof. Item a) follows immediately from (5.3); To prove b) we apply a):

$$u_j \wedge \overline{D}_i \Phi^r_{j-1} = u_j \wedge (u_j \wedge \overline{D}_{i-1}\Phi^r_{i-1}) = (u_j \wedge u_j) \wedge \overline{D}_{i-1}\Phi^r_{i-1} = 0$$

Item c) is a consequence of (5.6). ∎

The lemma below generalizes Corollary 3.2.7, although unfortunately its proof does not.

5.2.6 Lemma. *For all* $\mu \in \bigwedge V_r$ *and* $j \geq 0$

$$D_j(\mu \wedge u_i \wedge \Phi^r_{i-1}) = D_j(\mu \wedge u_i) \wedge \Phi^r_{i-1}$$

Proof. The property is obviously true for $j = 0$. For $j = 1$:

$$\begin{aligned} D_1(\mu \wedge u_i \wedge \Phi^r_{i-1}) &= D_1(\mu \wedge u_i) \wedge \Phi^r_{i-1} + \mu \wedge u_i \wedge D_1\Phi^r_{i-1} = \\ &= D_1(\mu \wedge u_i) \wedge \Phi^r_{i-1} + \mu \wedge u_i \wedge u_i \wedge \Phi^r_{i-2} = \\ &= D_1(\mu \wedge u_i) \wedge \Phi^r_{i-1} \end{aligned}$$

Assume the property true for all $1 \leq k \leq j-1$ and recall that the equality $\mathcal{D}_+(z)\overline{\mathcal{D}}_+(z) = 1$ implies:

$$\sum_{k=0}^{j} (-1)^k D_j \overline{D}_{j-k} = 0. \tag{5.9}$$

Then

$$D_j(\mu \wedge u_i \wedge \Phi^r_{i-1}) = \sum_{k=1}^{j} (-1)^{k+1} \overline{D}_k D_{j-k}(\mu \wedge u_i \wedge \Phi^r_{i-1})$$

and by the inductive hypothesis:

$$\begin{aligned} D_j(\mu\wedge u_i\wedge\Phi^r_{i-1}) &= \sum_{k=1}^{j-1}(-1)^{k+1}\overline{D}_k(D_{j-k}(\mu\wedge u_i)\wedge\Phi^r_{i-1})+ \\ &+ (-1)^{j+1}\overline{D}_j(\mu\wedge u_i\wedge\Phi^r_{i-1}) = \\ &= \sum_{k=1}^{j}(-1)^{k+1}\sum_{p=0}^{k}\overline{D}_{k-p}D_{j-k}(\mu\wedge u_i)\wedge\overline{D}_p\Phi^r_{i-1} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{j}(-1)^{k+1}\overline{D}_kD_{j-k}(\mu\wedge u_i)\wedge\Phi^r_{i-1}+ \\ &+ \sum_{p=1}^{j}\sum_{k=p}^{j}(-1)^{k+1}\overline{D}_{k-p}D_{j-k}(\mu\wedge u_i)\wedge\overline{D}_p\Phi^r_{i-1} \end{aligned}$$

Because of (5.9), the second summand in the last equality vanishes for all $1\leq p\leq j$ and the proposition follows. ∎

5.2.7 Corollary.

$$D_j\Phi^r_i = u_{i+j}\wedge\Phi^r_{i-1}$$

Proof. In fact, by 5.2.6,

$$D_j\Phi^r_i = D_j(u_i\wedge u_{i-1})\wedge\Phi^r_{i-2} = u_{i+j}\wedge u_{i-1}\wedge\Phi^r_{i-2} = u_{i+j}\wedge\Phi^r_{i-1}. \blacksquare$$

By Corollary 5.2.7 it makes sense to consider the formal power series

$$\mathcal{D}_+(z)\Phi^r_j = \sum_{i\geq 0}u_{j+i}\wedge\Phi^r_{j-1}z^i\in F^r_j[[z]].$$

Thus, invoking (5.5):

5.2.8 Proposition. *The following equality holds*

$$\begin{aligned} \mathcal{D}_+(z)\Phi^r_{j+\boldsymbol{\lambda}} &= \mathcal{D}_+(z)(\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\Phi^r_{j-r}) \\ &= \mathcal{D}_+(z)\mathbf{u}_{r,j+\boldsymbol{\lambda}}\wedge\mathcal{D}_+(z)\Phi^r_{j-r} \end{aligned} \tag{5.10}$$

Proof. In fact

$$\mathcal{D}_+(z)(\mathbf{u}_{r,j+\boldsymbol{\lambda}} \wedge \Phi^r_{j-r})$$

$$\begin{aligned} &= \mathcal{D}_+(z)(\overline{\mathcal{D}}_+(z)\mathcal{D}_+(z)\mathbf{u}_{r,j+\boldsymbol{\lambda}} \wedge \overline{\mathcal{D}}_+(z)\mathcal{D}_+(z)\Phi^r_{j-r}) \\ &= \mathcal{D}_+(z)\overline{\mathcal{D}}_+(z)(\mathcal{D}_+(z)\mathbf{u}_{r,j+\boldsymbol{\lambda}} \wedge \mathcal{D}_+(z)\Phi^r_{j-r}) \\ &= \mathcal{D}_+(z)\mathbf{u}_{r,j+\boldsymbol{\lambda}} \wedge \mathcal{D}_+(z)\Phi^r_{j-r}. \end{aligned}$$ ∎

5.2.9 Similarly to Section 3.2.4, we denote by $\mathcal{A}(\mathcal{D}_+)$ the sub-algebra of $\mathrm{End}_{\mathbb{Z}}(F^r_i)$ generated by

$$(1, D_1, D_2, \dots)$$

An element of $\mathcal{A}(\mathcal{D}_+)$ is a polynomial expression in $D_1, D_2, \dots$ Arguing analogously to the proof of Proposition 3.2.5, it follows that $\mathcal{A}(\mathcal{D}_+)$ is a commutative sub-algebra of $\mathrm{End}_{\mathbb{Z}}(F^r_i)$. Moreover, in general, $\mathcal{A}(\mathcal{D}_+)$ is isomorphic to $\mathbb{Z}[\overline{D}_1, \overline{D}_2, \dots,]$. It turns out that Lemma 2.4.4 can be extended to

5.2.10 Lemma (Integration by parts). *The equality:*

$$\mu \wedge D_j\Phi^r_i = D_j(\mu\wedge\Phi^r_i) - D_{j-1}(\overline{D}_1\mu\wedge\Phi^r_i) + \dots + (-1)^j\overline{D}_j\mu\wedge\Phi^r_i \quad (5.11)$$

holds for all $\mu \in \bigwedge V_r$.

Proof. We use the equality

$$\mu \wedge \mathcal{D}_+(z)\Phi^r_i = \mathcal{D}_+(z)(\overline{\mathcal{D}}_+(z)\mu \wedge \Phi^r_i) \quad (5.12)$$

like in Lemma 2.4.4. The sought for expression $\mu \wedge D_j\Phi^r_i$ is then the coefficient of z^j in the expansion of the right hand side of (5.12), which is precisely the right hand side of (5.11). ∎

5.2.11 Proposition. *The $\mathbb{Z}$-module F^r_i is a free $\mathcal{A}(\mathcal{D}_+)$-module of rank 1 generated by Φ^r_i.*

Proof. We shall prove that

$$\Phi^r_{i+\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)\Phi^r_i.$$

for all $\boldsymbol{\lambda} \in \mathcal{P}$ (not necessarily of length at most r). Lemma 5.2.6 implies that for any choice of $i_1, \ldots, i_k$

$$D_{i_1} D_{i_2} \ldots D_{i_r} \Phi_i^r = D_{i_1} \ldots D_{i_r}(u_i \wedge \ldots \wedge u_{i-k+1}) \wedge \Phi_{i-k}^r$$

As a consequence

$$\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)\Phi_i^r = \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)(u_i \wedge \ldots \wedge u_{i-k+1}) \wedge \Phi_{i-k}^r$$

and then we have reduced the Proposition to the same situation of Corollary 3.5.5 applied to the module $M_0 := \bigoplus_{j \geq 0} \mathbb{Z} b_j$ where $b_j = u_{i-r+1+j}$. The reason why there is no torsion is that $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)$ is a basis of $\mathbb{Z}[\overline{D}_1, \overline{D}_2, \ldots]$ and then the map $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+) \mapsto \Phi_{i+\boldsymbol{\lambda}}^r$ is a $\mathbb{Z}$-module isomorphism. ■

5.2.12 Remark. If $r = \infty$, the module structure stated in Proposition 5.2.11 is called in [47] *boson-fermion correspondence*.

5.2.13 Proposition. *The space F_0^r is an eigenspace of $\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)$ with eigenvalue $\Delta_{\boldsymbol{\lambda}}(H_r)$.*

Proof. We have, by (1.14):

$$\begin{aligned}
& u_{\lambda_1} \wedge \ldots \wedge u_{-r+1+\lambda_r} \wedge \Phi_{-r}^r = \\
= \; & \sum_{j=0}^{r-1} \mathtt{U}_j(u_{\lambda_1}) u_{-j} \wedge \ldots \wedge \sum_{j=0}^{r-1} \mathtt{U}_j(u_{-r+1+\lambda_r}) u_{-j} \wedge \Phi_{-r}^r = \\
= \; & \begin{vmatrix} \mathtt{U}_0(u_{\lambda_1}) & \mathtt{U}_0(u_{-1+\lambda_2}) & \cdots & \mathtt{U}_0(u_{-r+1+\lambda_r}) \\ \mathtt{U}_1(u_{\lambda_1}) & \mathtt{U}_1(u_{-1+\lambda_2}) & \cdots & \mathtt{U}_1(u_{-r+1+\lambda_r}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathtt{U}_{r-1}(u_{\lambda_1}) & \mathtt{U}_{r-1}(u_{-1+\lambda_2}) & \cdots & \mathtt{U}_{r-1}(u_{-r+1+\lambda_r}) \end{vmatrix} \Phi_0^r = \\
= \; & \Delta_{\boldsymbol{\lambda}}(H_r)\Phi_0^r = \Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)\Phi_0^r,
\end{aligned}$$

where we have used the equality $u_0 \wedge u_{-1} \wedge \ldots \wedge u_{-r+1} \wedge \Phi_{-r}^r = \Phi_0^r$.

5.2.14 Remark. An alternative proof of 5.2.13, consists in observing that $u_{-i+1+\lambda_i} = D_{r-i+\lambda_i} u_{-r+1}$ and then

$$u_{\lambda_1} \wedge \ldots \wedge u_{-r+1+\lambda_r} \wedge \Phi_{-r}^r$$

$$
\begin{aligned}
&= D_{r-1-\lambda_1}u_{-r+1}\wedge\ldots\wedge D_{\lambda_r}u_{-r+1}\wedge\Phi^r_{-r}\\
&= Res\left(\frac{D_{r-1-\lambda_1}}{\mathrm{p}_r(D)},\ldots,\frac{D_{\lambda_r}}{\mathrm{p}_r(D)}\right)u_0\wedge\ldots\wedge u_{-r+1}\wedge\Phi^r_{-r}\\
&= \Delta_{\boldsymbol{\lambda}}(H_r)\Phi^r_0.
\end{aligned}
$$

having applied 3.5.4

5.2.15 Comparing module structures. For all $i\in\mathbb{Z}$ the map $D_i: V_r\to V_r$ is a $\mathbb{Z}$-module isomorphism. For each $j\in\mathbb{Z}$ it induces an isomorphism:

$$\sum_{k\geq 0}\mathbb{Z}u_{j-k}\longrightarrow\sum_{k\geq 0}\mathbb{Z}u_{j+i-k}$$

and hence a determinant map

$$\ell^i: F^r_j\to F^r_{j+i}$$

mapping $\Phi^r_{j+\boldsymbol{\lambda}}\mapsto\Phi^r_{j+i+\boldsymbol{\lambda}}$. We use it to induce on $\bigwedge^{\infty/2}V_r:=\bigoplus_{i\in\mathbb{Z}}F^r_i$ a structure of free $B_r(\ell):=B_r[\ell,\ell^{-1}]$-module of rank 1 generated by Φ^r_0. Indeed we have

$$\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_+)\Phi^r_i=\Phi^r_{i+\boldsymbol{\lambda}}=\ell^i\Phi^r_{\boldsymbol{\lambda}}=\ell^i\Delta_{\boldsymbol{\lambda}}(H_r)\Phi^r_0.$$

Define operators $X_r(z): F^r_i\to F^r_{i+1}$ and $X^\vee_r(z): F^r_i\to F^r_{i-1}[[z^{-1},z]]$ as follows:

$$X_r(z)\wedge\Phi^r_{i+\boldsymbol{\lambda}}=\sum_{j\in\mathbb{Z}}z^ju_j\wedge\Phi^r_{i+\boldsymbol{\lambda}}\tag{5.13}$$

and

$$X^\vee_r(z)\lrcorner\Phi^r_{i+\boldsymbol{\lambda}}=\sum_{j\in\mathbb{Z}}z^{-j}u^\vee_j\lrcorner\Phi^r_{i+\boldsymbol{\lambda}}\tag{5.14}$$

where $u^\vee_j\in V^\vee_r$ is defined by $u^\vee_j(u_i)=\delta_{ji}$.

5.2.16 Definition. *Let* $\Gamma_r(z),\Gamma^\vee_r(z): B_r(\ell)\to B_r(\ell)[[z]]$ *defined by*

$$\Gamma_r(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r)=\frac{(X_r(z)\wedge\Phi^r_{i+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi^r_0}\tag{5.15}$$

and

$$\Gamma^\vee_r(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r)=\frac{(X^\vee_r(z)\lrcorner\Phi^r_{i+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi^r_0}\tag{5.16}$$

They will be said truncated *vertex operators to the order* r.

The terminology is suggested by the fact that they are truncation of the vertex operators $\Gamma_\infty(z), \Gamma^\vee_\infty(z)$ arising in the representation theory of the Heisenberg algebra. Our next goal is to compute explicitly $\Gamma_r(z)$ and $\Gamma^\vee_r(z)$ and to do this we are going to exploit the B_r-module structure of F^r_0. To this purpose observe that

$$\begin{aligned} X_r(z)\wedge\Phi^r_{i+\boldsymbol{\lambda}} &= \sum_{j\in\mathbb{Z}} z^j u_j\wedge\Phi^r_{i+\boldsymbol{\lambda}} = \\ &= z^{i+1}\sum_{j\in\mathbb{Z}} z^{j-i-1}D_{i+1}u_{j-i-1}\wedge\ell^{i+1}\Phi^r_{-1+\boldsymbol{\lambda}} = \\ &= z^{i+1}\sum_{j\in\mathbb{Z}} z^{j-i-1}\ell^{i+1}(u_{j-i-1}\wedge\Phi^r_{-1+\boldsymbol{\lambda}}) = \\ &= \ell^{i+1}z^{i+1}\sum_{j\in\mathbb{Z}} u_j z^j\wedge\Phi^r_{-1+\boldsymbol{\lambda}}. \end{aligned}$$

Thus, to compute $\Gamma_r(z)$ it is sufficient to analyze the formal power series $\sum_{j\in\mathbb{Z}} z^j u_j\wedge\Phi^r_{-1+\boldsymbol{\lambda}}$. The computation of $\Gamma^\vee_r(z)$ can also be reduced to a special case basing on the following easy

5.2.17 Exercise. Show that

$$z^{-i}\ell^i(X^\vee_r(z)\lrcorner\Phi^r_{j+\boldsymbol{\lambda}}) = X^\vee_r(z)\lrcorner\Phi^r_{i+j+\boldsymbol{\lambda}} \tag{5.17}$$

According to Exercise 5.2.17

$$X^\vee_r(z)\lrcorner\Phi^r_{i+\boldsymbol{\lambda}} = \ell^{i-1}z^{-i+1}X^\vee_r(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}}$$

and then to determine the expression of $\Gamma^\vee_r(z)$ it sufficient to analyze the formal power series $X^\vee_r(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}}$.

5.2.18 Exercise. Prove that

$$X_r(z) = \frac{1}{E_r(z)}\sum_{j\geq 0}\left(\frac{t}{z}\right)^n = \frac{z}{E_r(z)}\cdot i_{z,t}\frac{1}{z-t}, \tag{5.18}$$

where if $f(t,z)\in\mathbb{Z}[t^{\pm 1}, z^{\pm 1}, (t-z)^{\pm 1}]$, by $i_{z,t}(f(z,t))$ one denotes its expansion in powers of t/z (Cf. [44, p. 16]). (*Hint.* Multiply $\sum_{j\in\mathbb{Z}} u_j z^j$ by $E_r(z)$ and then use the fact that $\mathrm{p}_r(D)u_{-j-r} = t^j$ and $\mathrm{p}_r(D)u_{-r+1+j} = 0$ for all $j\geq 0$).

5.3 The Truncated Operator $\Gamma_r(z)$

We begin the section with the analogue of Proposition 4.3.3.

5.3.1 Lemma. *Let $\boldsymbol{\lambda}$ be a partition of length $k \leq r$. Then, for each $0 \leq j \leq k$:*

$$u_{-k} \wedge \overline{D}_j \Phi^r_{-1+\boldsymbol{\lambda}} = (-1)^k \overline{D}_{-k+j} \Phi^r_{0+\boldsymbol{\lambda}}.$$

Proof. Write $\Phi^r_{-1+\boldsymbol{\lambda}}$ as $u_{-1+\lambda_1} \wedge \ldots \wedge u_{-1+\lambda_k} \wedge \Phi^r_{-k-1}$. Then

$$\begin{aligned} \overline{D}_j \Phi^r_{-1+\boldsymbol{\lambda}} &= \overline{D}_j (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) \wedge \Phi^r_{-k-1} + \\ &+ \sum_{i \geq 1} \overline{D}_{j-i} (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) \wedge \overline{D}_i \Phi^r_{-k-1}. \end{aligned}$$

As $u_{-k} \wedge \overline{D}_i \Phi^r_{-k-1} = 0$ for $i > 0$, due to 5.2.5 item b), it follows that

$$u_{-k} \wedge \overline{D}_j \Phi^r_{-1+\boldsymbol{\lambda}} = u_{-k} \wedge \overline{D}_j (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) \wedge \Phi^r_{-k-1}.$$

Now

$$\overline{D}_j (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) = \sum u_{-1+\lambda_1+j_1} \wedge \ldots \wedge u_{-k+\lambda_k+j_k},$$

where the sum is taken over all the k-tuples such that $0 \leq j_i \leq 1$ and $\sum j_i = j$. Then:

$$\begin{aligned} & u_{-k} \wedge \overline{D}_j (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) = \\ = \ & (-1)^k \overline{D}_j (u_{-1+\lambda_1} \wedge \ldots \wedge u_{-k+\lambda_k}) \wedge u_{-k} \\ = \ & (-1)^k \sum u_{-1+\lambda_1+j_1} \wedge \ldots \wedge u_{-k+\lambda_k+j_k} \wedge u_{-k} \\ = \ & (-1)^k \sum u_{\lambda_1-(1-j_1)} \wedge \ldots \wedge u_{-k+1+\lambda_k-(1-j_k)} \wedge u_{-k}. \end{aligned} \tag{5.19}$$

Putting $s_i := 1 - j_i$, so that $0 \leq s_i \leq 1$ and $\sum s_i = k - j$, last side of (5.19) can be written as

$$\begin{aligned} & (-1)^k \sum u_{\lambda_1-s_1} \wedge \ldots \wedge u_{-k+1+\lambda_k-s_k} \wedge u_{-k} = \\ = \ & (-1)^k \overline{D}_{j-k} (u_{\lambda_1} \wedge \ldots \wedge u_{-k+1+\lambda_k}) \wedge u_{-k} \end{aligned}$$

In conclusion:

$$\begin{aligned} & u_{-k}\wedge \overline{D}_j\Phi^r_{-1+\boldsymbol{\lambda}}= \\ = \quad & (-1)^k\overline{D}_{j-k}(u_{\lambda_1}\wedge\ldots\wedge u_{-k+1+\lambda_k})\wedge u_{-k}\wedge\Phi^r_{-k-1} \\ = \quad & (-1)^k\overline{D}_{j-k}(u_{\lambda_1}\wedge\ldots\wedge u_{-k+1+\lambda_k}\wedge u_{-k}\wedge\Phi^r_{-k-1}) \\ = \quad & (-1)^k\overline{D}_{j-k}\Phi^r_{0+\boldsymbol{\lambda}}. \end{aligned}$$

■

5.3.2 Exercise. Let $\boldsymbol{\lambda}$ be a partition of length $0\le k\le r$ and let $0\le j\le k$. Then

$$t^j\wedge\Phi^r_{-1+\boldsymbol{\lambda}}=(-1)^{k-j}e_{k-j}u_{-k}\wedge\Phi^r_{-1+\boldsymbol{\lambda}}=(-1)^j\overline{D}_{-j}\Phi^r_{0,\boldsymbol{\lambda}}.$$

(*Hint*. Use the fact that $\mathtt{p}_r(D)u_{-j-r}=t^j$).

5.3.3 Corollary. For all $\boldsymbol{\lambda}\in\mathcal{P}_k$:

$$\frac{1}{z^k}\left(u_{-k}\wedge\overline{\mathcal{D}}_+(z)\Phi^r_{-1+\boldsymbol{\lambda}}\right)=\overline{\mathcal{D}}_-(z)\Phi^r_{0+\boldsymbol{\lambda}}$$

Proof. In fact, basing on Lemma 5.3.1:

$$\begin{aligned} u_{-k}\wedge\overline{\mathcal{D}}_+(z)\Phi^r_{-1+\boldsymbol{\lambda}} &= u_{-k}\wedge\sum_{j\ge 0}(-1)^j\overline{D}_j\Phi^r_{-1+\boldsymbol{\lambda}}= \\ &= \sum_{j\ge 0}(-1)^{j+k}z^j\overline{D}_{j-k}\Phi^r_{0+\boldsymbol{\lambda}}= \\ &= z^k\sum_{j\ge 0}(-1)^{j+k}z^{j-k}\overline{D}_{j-k}\Phi^r_{0+\boldsymbol{\lambda}}= \\ &= z^k\overline{\mathcal{D}}_-(z)\Phi^r_{0+\boldsymbol{\lambda}}, \end{aligned}$$

that proves the claim. ■

5.3.4 Theorem.

$$\Gamma_r(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r)=\frac{X_r(z)\wedge\Phi^r_{i+\boldsymbol{\lambda}}}{\Phi^r_0}=\frac{\ell^{i+1}z^{i+1}}{E_r(z)}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r).$$

Proof. By definition

$$X_r(z)\wedge\Phi^r_{-1+\boldsymbol{\lambda}}=\sum_{j\in\mathbb{Z}}z^ju_j\wedge u_{-1+\lambda_1}\wedge\ldots\wedge u_{-r+\lambda_r}\wedge u_{-r+1}\wedge\ldots$$

which is equal to

$$= \sum_{j\geq -r} z^j u_j \wedge \Phi^r_{-1+\boldsymbol{\lambda}} = \frac{1}{z^r}\mathcal{D}_+(z)u_{-r} \wedge \Phi^r_{-1+\boldsymbol{\lambda}}. \tag{5.20}$$

Since $\mathcal{D}_+(z)$ behaves as a derivation on the exterior algebra (Cf. Proposition 5.2.8), last side of (5.20) can be written as

$$\mathcal{D}_+(z)(u_{-r} \wedge \overline{\mathcal{D}}_+(z)\Phi^r_{-1+\boldsymbol{\lambda}})$$

which, invoking Corollary 5.3.3, is in turn equal to

$$\mathcal{D}_+(z)(\overline{\mathcal{D}}_-(z)\Phi^r_{0+\boldsymbol{\lambda}}) = \frac{1}{E_r(z)}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r)\Phi^r_0.$$

Thus

$$\begin{aligned}\Gamma_r(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) &= \frac{X_r(z)\wedge\Phi^r_{i+\boldsymbol{\lambda}}}{\Phi^r_0} = \ell^{i+1}z^{i+1}\frac{X_r(z)\wedge\Phi^r_{-1+\boldsymbol{\lambda}}}{\Phi^r_0} \\ &= \frac{\ell^{j+1}z^{j+1}}{E_r(z)}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r).\end{aligned}$$

■

5.3.5 Corollary.

$$\Gamma_r(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{\ell^{i+1}z^{i+1}}{E_r(z)}\Delta_{\boldsymbol{\lambda}}(\overline{\mathcal{D}}_-(z)H_r).$$

Proof. Due to Proposition 4.2.7, which proves that $\overline{\mathcal{D}}_-(z)$ commutes with taking $\Delta_{\boldsymbol{\lambda}}$. ■

5.3.6 Corollary.

$$\Gamma_r(z)\ell^{-1}h_n = \frac{1}{E_r(z)}\left(h_n - \frac{h_{n-1}}{z}\right),$$

with the convention that $h_j = 0$ *if* $j < 0$.

Proof.

$$\Gamma_r(z)\ell^{-1}h_n = \frac{1}{E_r(z)}\overline{\mathcal{D}}_-(z)h_n = \frac{1}{E_r(z)}\left(h_n - \frac{h_{n-1}}{z}\right).$$

■

5.4 The Truncated Operator $\Gamma_r^\vee(z)$

The purpose of this section is to prove the following analogous of Theorem 4.4.3.

5.4.1 Theorem. *Let $\Gamma_r^\vee(z)$ as in formula (5.16). Then, for all $i \in \mathbb{Z}$:*

$$\Gamma_r^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) = \ell^{i-1}z^{-i}E_r(z)\mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) \tag{5.21}$$

Notice that because of Proposition 4.2.10 formula (5.21) is equivalent to

$$\Gamma_r^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) = \ell^{i-1}z^{-i}E_r(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r) \tag{5.22}$$

The proof of Theorem 5.4.1 will be split into the proofs of some lemmas, analogous to Lemma 4.4.2.

5.4.2 Lemma. *Let $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_r)$ be any partition of length at most r. Then:*

$$\begin{aligned}\frac{(zX_r^\vee(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi^r_0} &= \begin{vmatrix} z^{-\lambda_1} & z^{1-\lambda_2} & \ldots & z^{r-1-\lambda_r} \\ h_{\lambda_1+1} & h_{\lambda_2} & \ldots & h_{\lambda_r+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \ldots & h_{\lambda_r} \end{vmatrix} + \\ &+ \quad (-1)^r z^r\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r).\end{aligned} \tag{5.23}$$

where as usual $h_j = 0$ if $j < 0$.

Proof. The proof of equality (5.23) is straightforward. In fact:

$$z\cdot X_r^\vee(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}} =$$

$$\begin{aligned}
&= z\cdot X_r^\vee(z)\lrcorner(u_{1+\lambda_1}\wedge u_{\lambda_2}\wedge\ldots\wedge u_{-r+1+\lambda_r}\wedge u_{-r}\wedge\Phi^r_{1,-r-1})\\
&= z^{-\lambda_1}\Delta_{(\lambda_2,\ldots,\lambda_r)}(H_r) - z^{1-\lambda_2}\Delta_{(\lambda_1+1,\lambda_3,\ldots,\lambda_r)}(H_r)\\
&+ \ldots + (-1)^{r-1}z^{r-1+\lambda_r}\Delta_{\lambda_1,\ldots,\lambda_{r-1}}(H_r)\\
&- (-1)^{r-1}z^r\Delta_{(\lambda_1-1\ldots,\lambda_r-1)}(H_r)\\
&= \sum_{j=1}^r(-1)^{j-1}z^{j-1+\lambda_j}\Delta_{(\lambda_1+1,\ldots,\lambda_{j-1}+1,\lambda_{j+1},\ldots,\lambda_r)}(H_r)+
\end{aligned}$$

$$+ \quad (-1)^r z^r \Delta_{(\lambda_1-1\dots,\lambda_r-1)}(H_r).$$

The first summand of (5.24) is precisely the determinant occurring in (5.23), while the second determinant $\Delta_{(\lambda_1-1,\dots,\lambda_r-1)}(H_r)$ is precisely $\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r)$. We remark that no power of z with exponent bigger than r can occur in the expansion above, because its coefficient would be the Schur determinant associated to H_r and to a partition of length bigger than r, that vanishes. ∎

The proposition below is the analogous of Theorem 4.4.3. The same kind of proof gets rid of $(-1)^r z^r\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r)$, the additional summand occurring in formula (5.23).

5.4.3 Lemma.

$$\frac{(zX_r^\vee(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi^r_0} = E_r(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r). \tag{5.24}$$

Proof. As in the proof of Lemma 4.4.2, one first observe that

$$E_r(z)u_j = \mathtt{U}_0(u_j) + \mathtt{U}_1(u_j)z + \ldots + \mathtt{U}_{r-1}(u_j)z^{r-1}, \tag{5.25}$$

where each $\mathtt{U}_i(u_j)$ is equal to h_{j+i} plus a B_r–linear combination of $h_{j+i-1}, h_{j+i-2}, \dots, h_i$. Then

$$\begin{aligned}
\frac{1}{z^{\lambda_i-i+1}} &= \frac{E_r(z)H_r(z)}{z^{\lambda_i-i+1}} = E_r(z)\left(\mathcal{D}_-(z)h_{\lambda_i-i+1} + zu_{\lambda_i-i+1}\right) = \\
&= E_r(z)\mathcal{D}_-(z)h_{\lambda_i-i+1} + zE_r(z)u_{\lambda_i-i+1} = \\
&= E_r(z)\mathcal{D}_-(z)h_{\lambda_i-i+1} + \sum_{j=1}^{r}\mathtt{U}_{j-1}(u_{\lambda_i-i+1})z^j.
\end{aligned}$$

Substituting into the displayed determinant on the right hand side of (5.23) and using skew-symmetry, one obtains

$$= E_r(z)\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} + h_{\lambda_1+r}z^r & \dots & \mathcal{D}_-(z)h_{\lambda_r-r+1} + h_{\lambda_r+1}z^r \\ h_{\lambda_1+1} & \dots & h_{\lambda_r+r-2} \\ \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & \dots & h_{\lambda_r} \end{vmatrix}$$

$$= E_r(z)\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} & \dots & \mathcal{D}_-(z)h_{\lambda_r-r+1} \\ h_{\lambda_1+1} & \dots & h_{\lambda_r+r-2} \\ \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & \dots & h_{\lambda_r} \end{vmatrix} + (-1)^{r-1}z^r\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r) \tag{5.26}$$

Thus the right hand side of (5.23) is equal to

$$E_r(z)\begin{vmatrix} \mathcal{D}_-(z)h_{\lambda_1} & \dots & \mathcal{D}_-(z)h_{\lambda_r-r+1} \\ h_{\lambda_1+1} & \dots & h_{\lambda_r+r-2} \\ \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} & \dots & h_{\lambda_r} \end{vmatrix}$$

due to the cancelation of $(-1)^{r-1}z^r\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r)$ with the summand $(-1)^r z^r\overline{D}_r\Delta_{\boldsymbol{\lambda}}(H_r)$. To conclude the proof observe that for all $1 \leq i, j \leq r$:

$$\mathcal{D}_-(z)h_{\lambda_j-j+i} = h_{\lambda_j-j+i} + \mathcal{D}_-(z)h_{\lambda_j-j+i-1}$$

and thus, exploiting once again the skew-symmetry of the determinant, expression (5.26) is equivalent to $E_r(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r)$, as desired. ∎

Proof of Theorem 5.4.1. By definition, and using Exercise 5.2.17

$$\begin{aligned} \Gamma_r^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) &= \frac{(X_r^\vee(z)\lrcorner\Phi^r_{i+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi_0^r} \\ &= \ell^{i-1}z^{-i+1}\frac{(X_r^\vee(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi_0^r}. \end{aligned}$$

Thus

$$\Gamma_r^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) = \ell^{i-1}z^{-i}\frac{(zX_r^\vee(z)\lrcorner\Phi^r_{1+\boldsymbol{\lambda}})\otimes 1_{B_r}}{\Phi_0^r},$$

which, by Lemma 5.4.3, gives:

$$\begin{aligned} \Gamma_r^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_r) &= \ell^{i-1}z^{-i}E_r(z)\Delta_{\boldsymbol{\lambda}}(\mathcal{D}_-(z)H_r) \\ &= \ell^{i-1}z^{-i}E_r(z)\mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r), \end{aligned}$$

where in the last equality we used Proposition 4.2.10. ∎

5.5 The Vertex Operators $\Gamma(z)$ and $\Gamma^{\vee}(z)$

This final section is devoted to deduce the expression of the vertex operators used to describe the KP hierarchy, introduced in Section 0.3.1, as a limit for $r \to \infty$ of the truncations $\Gamma_r(z)$ and $\Gamma_r^{\vee}(z)$ previously computed in this chapter. We begin by recalling some well known auxiliary results. Let k be any commutative ring with unit and A any k-algebra. A *k-derivation* $\mathfrak{D}$ on A is a k-module endomorphism of A satisfying the Leibniz rule:

5.5.1 Lemma. *If $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ is a sequence of k-derivations of A and $\mathfrak{D}(z) = \sum_{j\geq 1} \mathfrak{D}_j z^j$, where z is a formal variable, then*

$$\exp(\sum_{j\geq 1} \mathfrak{D}_j z^j) : A \to A[[z]]$$

is a k-algebra homomorphism.

Proof. If $a, b \in A$, it is immediate to see that

$$\mathfrak{D}(z)(ab) = \mathfrak{D}(z)a \cdot b + a \cdot \mathfrak{D}(z)b,$$

just by applying Leibniz rule to $\mathfrak{D}_j$, for all $j \geq 1$. The equality

$$\mathfrak{D}(z)^n(ab) = \sum_{j=0}^{n} \binom{n}{i} \mathfrak{D}(z)^i a \cdot \mathfrak{D}(z)^{n-i} b$$

is matter of a straightforward induction. It follows that

$$\begin{aligned}
\exp(\mathfrak{D}(z))(ab) &= \sum_{n\geq 0} \frac{1}{n!} \mathfrak{D}(z)^n(ab) = \\
&= \sum_{n\geq 0} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \mathfrak{D}(z)^i a \cdot \mathfrak{D}(z)^{n-i} b = \\
&= \sum_{i\geq 0} \frac{\mathfrak{D}(z)^i a}{i!} \sum_{j\geq 0} \frac{\mathfrak{D}(z)^j b}{j!} = \\
&= \exp(\mathfrak{D}(z)a) \cdot \exp(\mathfrak{D}(z)b)
\end{aligned}$$

as desired. ∎

5.5.2 Proposition. *If $\psi(z) := 1 + \sum_{j\geq 1}\psi_j z^j : A \to A[[z]]$ is a k-algebra homomorphism, then there exists a unique sequence*

$$\mathfrak{D} := (\mathfrak{D}_1, \mathfrak{D}_2, \ldots).$$

of k-derivations of A such that $\psi(z) = \exp(\mathfrak{D}(z))$.

Proof. Define

$$\sum_{j\geq 1}\mathfrak{D}_j z^j = \log(1 + (\psi(z) - 1)) = -\sum_{n\geq 0}\frac{1}{n}(\sum_{i\geq 0}\psi_i z^i)^n$$

Then $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ is a sequence of k-derivations of A and clearly $\psi(z) = \exp(\mathfrak{D}(z))$. ∎

5.5.3 If $r = \infty$, the elements $(h_1, h_2, \ldots)$ of the sequence H_∞ are algebraically independents. In particular $B_\infty = \mathbb{Z}[h_1, h_2, \ldots]$. Clearly, the sequences E_∞ and H_∞ generate also $B := B_\infty \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. $B = \mathbb{Q}[E_\infty] = \mathbb{Q}[H_\infty]$. See [29] and [19, p. 4].

5.5.4 Lemma. *The $\mathbb{Q}$-vector spaces homomorphisms $\overline{\mathcal{D}}_-(z), \mathcal{D}_-(z) : B \to B[z^{-1}]$, extending by $\mathbb{Q}$-linearity those of Section 4.2, i.e.*

$$\overline{\mathcal{D}}_-(z)h_n = h_n - \frac{h_{n-1}}{z} \qquad \text{and} \qquad \mathcal{D}_-(z)h_n = \sum_{i=0}^{n}\frac{h_{n-i}}{z^i}$$

are ring homomorphisms.

Proof. The ring B is generated as a $\mathbb{Q}$-algebra by $H_\infty := (h_1, h_2, \ldots)$. It suffices to prove that for all $s \geq 1$ and each $1 \leq i_1 < i_2 < \ldots < i_s$,

$$\overline{\mathcal{D}}_-(z)(h_{i_1}\cdot \ldots \cdot h_{i_s}) = \overline{\mathcal{D}}_-(z)h_{i_1}\cdot \ldots \cdot \overline{\mathcal{D}}_-(z)(h_{i_s}) \tag{5.27}$$

and similarly

$$\mathcal{D}_-(z)(h_{i_1}\cdot \ldots \cdot h_{i_s}) = \mathcal{D}_-(z)h_{i_1}\cdot \ldots \cdot \mathcal{D}_-(z)(h_{i_s}) \tag{5.28}$$

Let $w = i_1 + \ldots + i_s$ and let $r > i_s$. Then in $(B_r)_w$ we may apply Corollaries 4.2.8 and 4.2.11, i.e. (5.27) and (5.28) hold for all $r > s$ and hence it holds in $B_w := \bigoplus_{|\boldsymbol{\lambda}|=w}\mathbb{Z}\Delta_{\boldsymbol{\lambda}}(H_\infty)$ as well, the latter being the projective limit of all the $\mathbb{Q}[H_r]_w$ taken with respect to the projection maps $\mathbb{Q}[H_r]_w \mapsto \mathbb{Q}[H_s]_w$ (4.37), for all $r \geq s$. ∎

5.5.5 Working over the rationals, it is meaningful to consider the auxiliary sequence $(x_1, x_2, \ldots)$ of elements of B defined by:

$$\exp(\sum_{i\geq 1} x_i t^i) = \frac{1}{E_\infty(t)} = \sum_{n\geq 0} h_n t^n. \tag{5.29}$$

Then each h_n can be regarded as a function of $(x_1, x_2, \ldots)$. The first few terms of H_∞ as polynomial expressions of the x_is are:

$$h_1 = x_1, \quad h_2 = \frac{x_1^2}{2} + x_2, \quad h_3 = \frac{x_1^3}{3!} + x_1x_2 + x_3, \ldots$$

(Cf. [47, p. 59] and Section 0.4.3 where the h_is are called S_i. We are rather using Macdonald notation [65] for the complete symmetric polynomials). Clearly $B = \mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, x_2, \ldots,]$ as well.

5.5.6 Lemma. *For each* $(n, j) \in \mathbb{Z} \times \mathbb{N}^*$*:*

$$\frac{\partial h_n}{\partial x_j} = h_{n-j}. \tag{5.30}$$

Proof. In fact

$$\begin{aligned}
\sum_{n\geq 0} \frac{\partial h_n}{\partial x_j} t^n &= \frac{\partial}{\partial x_j} \sum_{n\geq 0} h_n t^n = \frac{\partial}{\partial x_j} \exp(\sum_{i\geq 0} x_i t^i) = \\
&= t^j \exp(\sum_{i\geq 0} x_i t^i) = \sum_{n\geq 0} h_{n-j} t^n,
\end{aligned}$$

whence (5.30), by equating the coefficients of the same power of t. ∎

We can finally prove the following

5.5.7 Lemma.

$$\overline{\mathcal{D}}_-(z) = \exp\left(-\sum_{i\geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i}\right) \tag{5.31}$$

and

$$\mathcal{D}_-(z) = \exp\left(\sum_{i\geq 1} \frac{1}{iz^i} \frac{\partial}{\partial x_i}\right). \tag{5.32}$$

Proof. We have:

$$\overline{\mathcal{D}}_-(z)h_n = h_n - \frac{h_{n-1}}{z} = \left(1 - \frac{1}{z}\frac{\partial}{\partial x_1}\right)h_n.$$

Writing the right hand side as the exponential of its logarithm:

$$\overline{\mathcal{D}}_-(z)h_n = \exp\left(-\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial^i}{\partial x_1^i}\right)h_n = \exp\left(-\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right)h_n,$$

where the last equality is due to the relation $\partial^i h_n/\partial x_1^i = \partial h_n \partial x_i$, inferred from (5.30). Now, by Lemma 5.5.1, the right hand side of (5.31) is a ring homomorphism $B \to B[[z]]$, because is the exponential of the first order differential operator $-\sum_{i\geq 1}(iz^i)^{-1}\partial/\partial x_i$. Since $H_\infty := (h_1, h_2, \ldots)$ generate B as a $\mathbb{Q}$-algebra, and both members of (5.31) coincide when evaluated at h_n, for all $n \geq 0$, they do coincide. The proof of (5.32) is similar, and is based on the equality

$$\mathcal{D}_-(z)h_n = \sum_{j=0}^{n}\frac{h_{n-j}}{z^j} = \left(\frac{1}{1 - \frac{1}{z}\frac{\partial}{\partial x_1}}\right)h_n = \exp\left(\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right)h_n.$$

which implies (5.32), given that its right hand side is the exponential of the derivation and hence a ring homomorphism. ■

5.5.8 Corollary. *The following two equalities hold:*

$$\Gamma_\infty(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_\infty) = \ell^{i+1}z^{i+1}\exp(\sum_{i\geq 1}x_i z^i)\cdot\exp\left(-\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right),$$

$$\Gamma_\infty^\vee(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_\infty) = \ell^{i-1}z^{-i}\exp(-\sum_{i\geq 1}x_i z^i)\cdot\exp\left(\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right).$$

Proof. In fact

$$\Gamma_\infty(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_\infty) = \frac{\ell^{i+1}z^{i+1}}{E_\infty(z)}\overline{\mathcal{D}}_-(z)\Delta_{\boldsymbol{\lambda}}(H_\infty),$$

and

$$\Gamma^\vee_\infty(z)\ell^i\Delta_{\boldsymbol{\lambda}}(H_\infty) = \ell^{i-1}z^{-i}E_\infty(z)\mathcal{D}_-(z)\Delta_{\boldsymbol{\lambda}}(H_\infty).$$

The claim then follows by definition (5.29) of the sequence $\mathbf{x} := (x_1, x_2 \ldots)$, by (5.31) and (5.32). ∎

Define the operator $R(z) : B(\ell) \to B(\ell)[z]$ which sends any polynomial $f(\mathbf{x}, \ell)$ to $R(z)f(\mathbf{x}, \ell) = \ell z f(\mathbf{x}, \ell z)$. Let $R(z)^{-1}$ be its inverse:

$$R(z)^{-1}f(\mathbf{x}, \ell) = \ell^{-1}z^{-1}f(\mathbf{x}, \ell z^{-1}).$$

Then we have (re)proven the following:

5.5.9 Theorem (Cf. [47, Theorem 5.1]).

$$\Gamma(z) = R(z)\exp(\sum_{i\geq 1} x_i z^i)\exp\left(-\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right)$$

and

$$\Gamma^\vee(z) = R(z)^{-1}\exp(-\sum_{i\geq 1} x_i z^i)\exp\left(\sum_{i\geq 1}\frac{1}{iz^i}\frac{\partial}{\partial x_i}\right).$$ ∎

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