Eigenvalues
on Riemannian Manifolds

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Eigenvalues on Riemannian Manifolds

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Chapter 1

Eigenvalue problems on Riemannian manifolds

1.1 Introduction

Let (M, g) be an n-dimensional Reimannian manifold with boundary (possibly empty). The most important operator on M is the Laplacian Δ. In local coordinate system $\{x_i\}_{i=1}^n$, the Laplacian is given by

$$\Delta = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} \frac{\partial}{\partial x_j} \right)$$

where (g^{ij}) is the inverse matrix $(g_{ij})^{-1}$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are the coefficients of the Riemannian metric in the local coordinates, and $G = det(g_{ij})$. In local coordinates, the Riemannian measure dv on (M, g) is given by

$$dv = \sqrt{G} dx_1 ... dx_n.$$

Let $\phi \in C^\infty(M)$ and set

$$||\phi||_1^2 = \int_M |\nabla \phi|^2 + \int_M |\phi|^2.$$

Here and in the future, the integrations on M are always taken with respect to the Riemannian measure on M. Let us denote by $H_1^2(M)$ and $\overset{o}{H}{}_1^2(M)$ the completion of $C^\infty(M)$ and $C_0^\infty(M)$ with respect to $\|\ \|$. The theory of Sobolev spaces tells us that $H_1^2(M) = \overset{o}{H}{}_1^2(M)$ when M is complete. Our purpose is to study some eigenvalue problems associated to the Laplacian operator on a compact manifold M. When $\partial M = \emptyset$, we consider the *closed eigenvalue problem*:

$$\Delta u + \lambda u = 0. \tag{1.1}$$

When $\partial M \neq \emptyset$, we are interested in the following eigenvalue problems.

- *The Dirichlet problem*:

$$\begin{cases} \Delta u = \lambda u \quad \text{in} \quad M, \\ u|_{\partial M} = 0. \end{cases} \tag{1.2}$$

- *The Neumannn problem*:

$$\begin{cases} \Delta u = \lambda u \quad \text{in} \quad M, \\ \frac{\partial u}{\partial \nu}\big|_{\partial M} = 0, \end{cases} \tag{1.3}$$

where ν is the unit outward normal to ∂M.

- *The clamped plate problem*:

$$\begin{cases} \Delta^2 u = \lambda u \quad \text{in} \quad M, \\ u|_{\partial M} = \frac{\partial u}{\partial \nu}\big|_{\partial M} = 0, \end{cases} \tag{1.4}$$

- *The buckling problem*:

$$\begin{cases} \Delta^2 u = -\lambda \Delta u \quad \text{in} \quad M, \\ u|_{\partial M} = \frac{\partial u}{\partial \nu}\big|_{\partial M} = 0, \end{cases} \tag{1.5}$$

- *The eigenvalue problem of poly-harmonic operator*:

$$\begin{cases} (-\Delta)^l u = -\lambda u \quad \text{in} \quad M, \\ u|_{\partial M} = \frac{\partial u}{\partial \nu}\big|_{\partial M} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial M} = 0, \ l \geq 2. \end{cases} \tag{1.6}$$

- *The buckling problem of arbitrary order*:

$$\begin{cases} (-\Delta)^l u = -\lambda \Delta u \quad \text{in} \quad M, \\ u|_{\partial M} = \frac{\partial u}{\partial \nu}\big|_{\partial M} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial M} = 0, \ l \geq 2. \end{cases} \tag{1.7}$$

- *The Steklov problem of second order*:

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad M, \\ \frac{\partial u}{\partial \nu} = \lambda u \quad \text{on} \quad \partial M. \end{cases} \tag{1.8}$$

- *The Steklov problem of fourth order*:

$$\begin{cases} \Delta^2 u = 0 \quad \text{in} \quad M, \\ u = \Delta u - \lambda \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M. \end{cases} \tag{1.9}$$

Let us denote by λ_1 the first non-zero eigenvalue of the above problems. We can arrange the eigenvalues of these problems as follows:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

For many reasons in Mathematics and Physics, it is important to obtain nice estimates for the $\lambda's$. We will concentrate our attention on this problem. Let us list some basic facts in this direction.

Theorem 1.1 (Weyl's asymptotic formula, [97]). *In each of the eigenvalue problems (1.1), (1.2), (1.3), let $N(\lambda)$ be the number of eigenvalues, counted with multiplicity, $\leq \lambda$. Then*

$$N(\lambda) \sim \omega_n |M| \lambda^{n/2} / (2\pi)^n \tag{1.10}$$

as $\lambda \to \infty$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $|M|$ is the volume of M. In particular,

$$\lambda_k^{n/2} \sim \{(2\pi)^n / \omega_n\} k / |M| \tag{1.11}$$

as $\lambda \to +\infty$.

There are similar asymptotic formulas for the other eigenvalue problems above (Cf. [1], [79], [80]).

Define a space H as follows:

For the closed eigenvalue problem (1.1),

$$H = \left\{ f \in H_1^2(M) \middle| \int_M f = 0 \right\}. \tag{1.12}$$

For the Dirichlet eigenvalue problem (1.2),

$$H = \overset{o}{H}{}_1^2(M). \tag{1.13}$$

For the Neumann eigenvalue problem (1.3),

$$H = \left\{ f \in H_1^2(M) \middle| \int_M f = 0 \right\}. \tag{1.14}$$

A fundamental fool in the theory of eigenvalues is the

Mini-Max principle. We can find a countable orthonormal basis $\{f_i\}, f_i \in C^\infty(M)$ for the problems (1.1), (1.2) and (1.3) such that

$$\begin{aligned} \lambda_1 &= \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in H \right\}, \\ \lambda_i &= \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in H, \int_M f f_j = 0, j = 1, \cdots, i-1 \right\}. \end{aligned} \tag{1.15}$$

In particular, we have the

Poincaré inequality:

$$\int_M |\nabla f|^2 \geq \lambda_1 \int_M f^2, \ \forall f \in H. \tag{1.16}$$

For other eigenvalue problems above, similar mini-max principles also hold.

Theorem 1.2 (The Co-Area formula, [17]). *Let M be a compact Riemannian manifold with boundary, $f \in H^1(M)$. Then for any non-negative function g on M,*

$$\int_M g = \int_{-\infty}^{\infty} \left(\int_{\{f=\sigma\}} \frac{g}{|\nabla f|} \right) d\sigma \tag{1.17}$$

1.2 Some estimates for the first eigenvalue of the Laplacian

In this section, we will prove some estimates for the first eigenvalue of the Laplacian.

Theorem 1.3 ([73]). *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$. Then the first non-zero eigenvalue of the closed eigenvalue problem (1.1) of M satisfies $\lambda_1(M) \geq n$.*

The proof of Theorem 1.3 can be carried out by substituting a first eigenfunction into the Bochner formula and integrating on M the resulted equality (see the proof of theorem 1.6 below).

An important classical result about eigenvalue is the following

Theorem 1.4 (Cheng's Comparison Theorem, [18]). *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature satisfying $\mathrm{Ric}_M \geq (n-1)c$ and let $B_R(p)$ be an open geodesic ball of radius R around a point p in M, where $R < \pi/\sqrt{c}$, when $c > 0$. Then the first eigenvalue of the Dirichlet problem (1.2) of $B_R(p)$ satisfies*

$$\lambda_1(B_R(p)) \leq \lambda_1(B_R(c)), \tag{1.18}$$

with equality holding if and only if $B_R(p)$ is isometric to $B_R(c)$, where $B_R(c)$ is a geodesic ball of radius R in a complete simply connected Riemannian manifold of constant curvature c and of dimension n.

An immediate application of Cheng's eigenvalue comparison theorem is a rigidity theorem for compact manifolds of positive Ricci curvature.

Theorem 1.5 (The Maximal Diameter Theorem, [18]). *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$. If the diameter of M satisfies $d(M) \geq \pi$, then M is isometric to an n-dimensional unit sphere.*

Proof. Take two points $p, q \in M$ so that $d(p,q) \geq \pi$; then $B_{\pi/2}(p) \cap B_{\pi/2}(q) = \emptyset$. Let f and g be the first eigenfunctions corresponding to the first Dirichlet eigenvalues of $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$, respectively. We extend f and g on the whole M by setting $f|_{M\setminus B_{\pi/2}(p)} = g|_{M\setminus B_{\pi/2}(q)} = 0$ and take two non-zero constants a and b such that

$$\int_M (af + bg) = 0$$

Observe that the first Dirichlet eigenvalue of an n-dimensional unit hemisphere is n. The mini-max principle and Cheng's comparison theorem then imply that

$$\begin{aligned}
n &\leq \lambda_1(M) \\
&\leq \frac{\int_M |\nabla(af+bg)|^2}{\int_M (af+bg)^2} \\
&= \frac{a^2 \int_{B_{\pi/2}(p)} |\nabla f|^2 + b^2 \int_{B_{\pi/2}(q)} |\nabla g|^2}{a^2 \int_{B_{\pi/2}(p)} f^2 + b^2 \int_{B_{\pi/2}(q)} g^2} \\
&= \frac{a^2 \lambda_1(B_{\pi/2}(p)) \int_{B_{\pi/2}(p)} f^2 + b^2 \lambda_1(B_{\pi/2}(q)) \int_{B_{\pi/2}(q)} g^2}{a^2 \int_{B_{\pi/2}(p)} f^2 + b^2 \int_{B_{\pi/2}(q)} g^2} \\
&\leq \frac{n a^2 \int_{B_{\pi/2}(p)} f^2 + n b^2 \int_{B_{\pi/2}(q)} g^2}{a^2 \int_{B_{\pi/2}(p)} f^2 + b^2 \int_{B_{\pi/2}(q)} g^2} = n.
\end{aligned}$$

We conclude from the equality case of the mini-max principle and Cheng's comparison theorem that each of $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$ is isometric to the n-dimensional unit hemisphere and

$$M = \overline{B_{\pi/2}(p)} \cup \overline{B_{\pi/2}(q)}$$

Consequently, M is isometric to a unit n-sphere.

The maximal diameter theorem can be also used to prove the Obata theorem below.

Theorem 1.6 ([76]). *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$. If the first non-zero eigenvalue of the closed eigenvalue problem (1.1) of M is n, then M is isometric to a unit n-sphere.*

Proof. Let f be a first eigenfunction corresponding to the first eigenvalue n of M. From the Bochner formula, we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla f|^2 &= |\nabla^2 f|^2 + \langle\nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}(\nabla f, \nabla f) \\ &\geq \frac{(\Delta f)^2}{n} - n|\nabla f|^2 + (n-1)|\nabla f|^2 = nf^2 - |\nabla f|^2.\end{aligned} \tag{1.19}$$

Integrating on M and noticing $\int_M (nf^2 - |\nabla f|^2) = 0$, we conclude that the inequalities in 1.19 should take equality sign. Thus, we have

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla f|^2 + f^2) &= \frac{1}{2}\Delta|\nabla f|^2 + \frac{1}{2}\Delta f^2 \\ &= nf^2 - |\nabla f|^2 + f\Delta f + |\nabla f|^2 = 0\end{aligned}$$

and so $|\nabla f|^2 + f^2$ is a constant. Without lose of generality, we can assume that $|\nabla f|^2 + f^2 = 1$ and so

$$\frac{|\nabla f|}{\sqrt{1-f^2}} = 1.$$

Let p and q be points of M such that $f(p) = -f(q) = -1$ and take a unit speed minimizing geodesic $\gamma : [0, l] \to M$ from p to q. Integrating the above equation along γ, we obtain

$$l = \int_\gamma ds = \int_\gamma \frac{|\nabla f|}{\sqrt{1-f^2}} ds \geq \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \pi.$$

It then follows from the maximal diameter theorem that M is isometric to an unit n-sphere.

Remark 1.1. Let M^n be a compact Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$ and nonempty boundary. If the mean curvature of ∂M is nonnegative, then the first Dirichlet eigenvalue of M satisfies $\lambda_1 \geq n$ with equality holding if and only if M^n is isometric to an n-dimensional unit hemisphere [82]. Similarly, if the boundary of M is convex, then the first non-zero Neumann eigenvalue of M must satisfy $\lambda_1 \geq n$ with equality holding if and only if M^n is isometric to an n-dimensional unit hemisphere [34, 100].

We now prove another rigidity theorem using the techniques of eigenvalues.

Theorem 1.7 ([101]). *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$ and let N be a closed minimal hypersurface which divides M into two disjoint open domains Ω_1 and Ω_2. If there exists a point $p \in M$ such that $d(p,N)$, the distance from p to N, is no less than $\pi/2$, then the pair (M,N) is isometric to the pair $(\mathbb{S}^n(1), \mathbb{S}^{n-1}(1)$, being $\mathbb{S}^n(1)$ the unit n-sphere.*

Proof. Assume without lose of generality that $p \in \Omega_1$. We know from $d(p,N) \geq \pi/2$ that $B_{\pi/2}(p) \subset \Omega_1$. It then follows from the domain monotonicity [17] that the first Dirichlet eigenvalues of $B_{\pi/2}(p)$ and Ω_1 satisfy

$$\lambda_1\left(B_{\pi/2}(p)\right) \geq \lambda_1(\Omega_1). \tag{1.20}$$

On the other hand, Cheng's comparison theorem tells us that

$$\lambda_1\left(B_{\pi/2}(p)\right) \leq n \tag{1.21}$$

and Reilly's estimate implies that $\lambda_1(\Omega_1) \geq n$. Thus, the inequalities in (1.20) and (1.21) should be equalities. Consequently, $B_{\pi/2}(p) = \Omega_1$ is isometric to an n-dimensional unit hemisphere and so $N = \partial\Omega_1 = \mathbb{S}^{n-1}(1)$ is totally geodesic. It then follows from a result of [39] that Ω_2 is also isometric to an n-dimensional unit hemisphere. □

Let λ_1 be the least nontrivial eigenvalue of an n-dimensional compact manifold M and let ϕ be the corresponding eigenfunction. By multiplying with a constant it is possible to assume that

$$a-1 = \inf_M \phi; \ \ a+1 = \sup_M \phi$$

where $0 \leq a(\phi) < 1$ is the median of ϕ.

Suppose that M^n is a compact manifold without boundary of nonnegative Ricci curvature and of diameter d. Li-Yau [72] showed that the first nontrivial eigenvalue satisfies

$$\lambda_1 \geq \frac{\pi^2}{(1+a)d^2}$$

and conjectured that

$$\lambda_1 \geq \frac{\pi^2}{d^2}. \tag{1.22}$$

Li-Yau's conjecture was proved by Zhong and Yang in [103]. Let us provide a proof of (1.22) given by Li in [71].

Lemma 1.1. The function

$$z(u) = \frac{2}{\pi}\left(\arcsin(u) + u\sqrt{1-u^2}\right) - u$$

defined on [-1, 1] satisfies

$$uz' + z''(1-u^2) + u = 0; \tag{1.23}$$

$$z'^2 - 2zz'' + z' \geq 0; \tag{1.24}$$

$$2z - uz' + 1 \geq 0; \tag{1.25}$$

and

$$1 - u^2 \geq 2|z|. \tag{1.26}$$

Proof. Differentiating yields

$$z' = \frac{4}{\pi}\sqrt{1-u^2} - 1, \;\; z'' = \frac{-4u}{\pi\sqrt{1-u^2}}.$$

Thus (1.23) is satisfied.

To see (1.24), we note that

$$z'^2 - 2zz'' + z' = \frac{4}{\pi\sqrt{1-u^2}}\left\{\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin u\right) - (1+u^2)\right\}.$$

Since the right hand side is an even function, it suffices to check that

$$\frac{4}{\pi}\left(\sqrt{1-u^2} + u\arcsin u\right) - (1+u^2) \geq 0$$

on [0, 1]. It is easy to see that

$$\frac{d}{du}\left\{\frac{4}{\pi}\left(\sqrt{1-u^2}+u\arcsin u\right)-(1+u^2)\right\}=\frac{4}{\pi}\arcsin u-2u$$

which is nonpositive on [0, 1]. Hence

$$\begin{aligned}&\frac{4}{\pi}\left(\sqrt{1-u^2}+u\arcsin u\right)-(1+u^2)\\ \geq\ &\left[\frac{4}{\pi}\left(\sqrt{1-u^2}+u\arcsin u\right)-(1+u^2)\right]\Bigg|_{u=1}\\ =\ &0.\end{aligned}$$

Inequality (1.25) follows easily because

$$2z-uz'+1=\frac{4}{\pi}\arcsin u+1-u\geq 0.$$

To see (1.26), let us consider the cases $-1\leq u\leq 0$ and $0\leq u\leq 1$ separately. It is clearly that the inequality is valid at -1, 0 and 1. Setting

$$f(u)=1-u^2-\frac{4}{\pi}\left(\arcsin u+u\sqrt{1-u^2}\right)+2u;$$

then

$$f'=-2u-\frac{4}{\pi}(2\sqrt{1-u^2})+2,$$

$$f''=-2+\frac{8u}{\pi\sqrt{1-u^2}},$$

and

$$f'''=\frac{8}{\pi(1-u^2)^{3/2}}.$$

When $-1\leq u\leq 0, f''\leq 0$. Hence $f(u)\geq\min\{f(-1),f(0)\}=0$. For the case $0\leq u\leq 1, f'''\geq 0$. Thus

$$f'\leq\max\{f'(0),f'(1)\}=\max\left\{2-\frac{8}{\pi},0\right\}=0.$$

Therefore $f(u) \geq f(1)$ which proves (1.26).

Lemma 1.2. *Suppose M is a compact manifold without boundary of nonnegative Ricci curvature. Assume that a nontrivial eigenfunction ϕ corresponding to the eigenvalue λ is normalized so that for $0 \leq a < 1, a+1 = \sup_M \phi$ and $a - 1 = \inf_M \phi$. If $u = \phi - a$, then*

$$|\nabla|^2 \leq \lambda(1-u^2) + 2a\lambda z(u) \tag{1.27}$$

where

$$z(u) = \frac{2}{\pi}\left(\arcsin u + u\sqrt{1-u^2}\right) - u. \tag{1.28}$$

Proof. We need only to prove an estimate similar to (1.27) for $u = \epsilon(\phi - a)$ where $0 < \epsilon < 1$. The lemma will follow by letting $\epsilon \to 0$. By the definition of u; we have

$$\Delta u = -\lambda(u + \epsilon a)$$

with $-\epsilon \leq u \leq \epsilon$. We may assume $a > 0$. Consider the function

$$Q = |\nabla u|^2 - c(1-u^2) - 2a\lambda z(u),$$

We can choose c large enough so that $\sup_M Q = 0$. The lemma follows if $c \leq \lambda$; for a sequence of $\epsilon \to 1$, hence we may assume that $c > \lambda$.

Let the maximizing point of Q be x_0. We claim that $|\nabla u(x_0)| > 0$ since otherwise $\nabla u(x_0) = 0$ and

$$0 = Q(x_0) = -c(1-u^2)(x_0) - 2a\lambda z(x_0) \leq -(c - a\lambda)(1-\epsilon^2)$$

by (1.26), which is a contradiction.

Differentiating in the e_i direction gives

$$\frac{1}{2}Q_i = u_j u_{ji} + cuu_i - a\lambda z' u_i. \tag{1.29}$$

We can assume at x_0 that $u_(x_0) = |\nabla u(x_0)|$ and using $Q_i = 0$, we have

$$u_{ji}u_{ji} \geq u_{11}^2 = (cu - a\lambda z')^2. \tag{1.30}$$

Differentiating again, using the commutation formula, $Q(x_0) = 0$, (1.26), (1.29), and (1.30), we get

$$\begin{aligned} 0 &\geq \frac{1}{2}\Delta Q(x_0) \qquad (1.31) \\ &= u_{ji}u_{ji} + u_j(\Delta u)_j + \mathrm{Ric}(\nabla u, \nabla u) + (c - a\lambda z'') + (cu - a\lambda z')\Delta u \\ &\geq (cu - a\lambda z')^2 + (c - \lambda - a\lambda z'')(c(1-u^2) + 2a\lambda z) \\ &\quad -\lambda(cu - a\lambda z')(u + \epsilon a) \\ &= -ac\lambda((1-u^2)z'' + uz' + \epsilon u) + a^2\lambda^2(-2zz'' + z'^2 + \epsilon z') \\ &\quad +a\lambda(c-\lambda)(-uz' + 2z + 1) + (c-\lambda)(c - a\lambda). \end{aligned}$$

However by (1.23), (1.24), and (1.25), we conclude that

$$\begin{aligned} 0 &\geq ac\lambda(1-\epsilon)u - a^2\lambda^2(1-\epsilon)z' + (c-\lambda)(c-a\lambda) \qquad (1.32) \\ &\geq -ac\lambda(1-\epsilon) - a^2\lambda^2(1-\epsilon)\left(\frac{4}{\pi} - 1\right) + (c-\lambda)(c-a\lambda) \\ &\geq -(c+\lambda)\lambda(1-\epsilon) + (c-\lambda)^2. \end{aligned}$$

This implies that

$$c \leq \lambda \left\{ \frac{2 + (1-\epsilon) + \sqrt{(1-\epsilon)(9-\epsilon)}}{2} \right\}.$$

Taking $\epsilon \to 0$ one gets the desired estimate.

Theorem 1.8. ([103]) *Suppose M is a compact manifold without boundary whose Ricci curvature is nonnegative. Let a $u \geq 0$ be the median of a normalized first eigenfunction with $a + 1 = \sup_M \phi$ and $a - 1 = \inf_M \phi$; and let d be the diameter. Then the first non-zero eigenvalue of M satisfies*

$$d^2\lambda_1 \geq \pi^2 + \frac{6}{\pi}\left(\frac{\pi}{2} - 1\right)^4 a^2 \geq \pi^2(1 + 0.02a^2). \qquad (1.33)$$

Proof. Let $u = \phi - a$ and let γ be the shortest geodesic from the minimizing point of u to the maximizing point with length at most

d. Integrating the gradient estimate (1.27) along this segment with respect to arc-length and using oddness, we have

$$\begin{aligned} d\lambda^{1/2} &\geq \int_\gamma ds \\ &\geq \int_\gamma \frac{|\nabla u| ds}{\sqrt{1-u^2+2az}} \\ &\geq \int_0^1 \left\{ \frac{1}{\sqrt{1-u^2+2az}} + \frac{1}{\sqrt{1-u^2-2az}} \right\} du \\ &\geq \int_0^1 \frac{1}{\sqrt{1-u^2}} \left\{ 2 + \frac{3a^2z^2}{1-u^2} \right\} du \\ &\geq \pi + 3a^2 \left(\int_0^1 \frac{z}{\sqrt{1-u^2}} \right)^2 \\ &= \pi + \frac{3a^2}{\pi^2} \left(\frac{\pi}{2} - 1 \right)^4 . \end{aligned}$$

Remark 1.2. It has been shown by Hang-Wang [40] that if the equality holds in (1.33) then M is isometric to a circle.

Remark 1.3. Let M^n be a compact manifold with smooth boundary and nonnegative Ricci curvature. Suppose that the second fundamental form of M is nonnegative. Then the first nontrivial eigenvalue of the Laplacian with Neumann boundary conditions also satisfies the inequality (1.27). The proof runs the same as Lemma 1.1 except that the possibility of the maximum of the test function Q at the boundary must be handled. In fact, the boundary convexity assumption implies that the maximum of Q cannot occur on the boundary.

Chapter 2

Isoperimetric inequalities for eigenvalues

2.1 Introduction

In this chapter, we will prove some isoperimetric inequalities for eigenvalues on manifolds which have always been important problems in geometric analysis. Owing to the limitation on the materials, we only select some of the results in the area. For more interesting results, we refer to [3] , [8], [17] and the references therein. The isoperimetric inequalities to be proved are : the Faber-Krahn inequality for the first eigenvalue of the Dirichlet eigenvalue; the Szegö-Weinberger inequality for the first nontrivial Neumann eigenvalue; the Hersch theorem for the first closed eigenvalue on a compact Riemannian surface of genus zero; the Ashbaugh-Benguria theorem; etc. For the convenience of later use, we recall now the notion of spherically symmetric rearrangement. Suppose that f is a bounded measurable function on the bounded measurable set $\Omega \subset \mathbb{R}^n$. Consider the distribution function $\mu_f(t)$ defined by

$$\mu_f(t) = |\{x \in \Omega||f(x)| > t\}| \tag{2.1}$$

where $|\cdot|$ denotes Lebesgue measure. The distribution function can be viewed as a function from $[0,\infty)$ to $[0, |\Omega|]$ and is nonincreasing. The decreasing rearrangement f^* of f, is the inverse of μ_f and is defined by

$$f^*(s) = \inf\{t \geq 0 | \mu_f(t) < s\}. \tag{2.2}$$

It is a nonincreasing function on $[0, |\Omega|]$. For a bounded measurable set $\Omega \subset \mathbb{R}^n$, its spherical rearrangement Ω^* is defined as the ball centered at the origin having the same measure as Ω. The spherically (symmetric) decreasing rearrangement $f^\star : \Omega^* \to \mathbb{R}$ is defined by

$$f^\star(x) = f^*(C_n|x|^n) \text{ for } x \in \Omega^\star \tag{2.3}$$

where $C_n = \pi^{n/2}/\Gamma\left(\frac{n}{2}+1\right)$ is the volume of the unt ball in $\mathbb{R}^n$. An important fact we will use is that

$$\int_\Omega f^2 = \int_0^{|\Omega|} (f^*(s))^2 ds = \int_{\Omega^*} (f^\star)^2. \tag{2.4}$$

It is known that for any function f in the Sobolev space $H_0^1(\Omega)$, $f^\star \in H_0^1(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla f^\star|^2 \leq \int_\Omega |\nabla f|^2. \tag{2.5}$$

For two nonnegative measurable functions f and g on Ω we have

$$\int_\Omega fg \leq \int_{\Omega^*} f^\star g^\star. \tag{2.6}$$

Let us recall the notion of *spherically (symmetric) increasing rearrangement*, which we denote by a lower $\star$. The definition is almost identical to that of spherically decreasing rearrangement, except that $g_\star$ should be radially increasing (in the weak sense) on Ω^*. In this case, we have

$$\int_\Omega fg \geq \int_{\Omega^*} f^\star g_\star. \tag{2.7}$$

2.2 The Faber-Krahn Inequality

In this section, we will prove the Faber-Krahn inequality which is one of the oldest isoperimetric inequalities for an eigenvalue.

Theorem 2.1 (Faber-Krahn [36],[61]). *For a bounded domain $\Omega \subset \mathbb{R}^n$, the first Dirichlet eigenvalue satisfies*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \tag{2.8}$$

with equality if and only if $\Omega = \Omega^*$.

Proof. Let u_1 be a first Dirichlet eigenfunction for Ω. We have from (2.4), (2.5) and the mini-max principle that

$$\begin{aligned} \lambda_1(\Omega) &= \frac{\int_\Omega |\nabla u_1|^2}{\int_\Omega u_1^2} \\ &= \frac{\int_\Omega |\nabla u_1|^2}{\int_{\Omega^*} (u_1^*)^2} \\ &\geq \frac{\int_{\Omega^*} |\nabla u_1|^2}{\int_{\Omega^*} (u_1^*)^2} \\ &\geq \lambda_1(\Omega^*). \end{aligned} \tag{2.9}$$

For the characterization of the case of equality, we refer to [56]. □

The Faber-Krahn inequality is valid for more general manifolds. Let M be an n-dimensional complete Riemannian manifold and for a fixed $\kappa \in \mathbb{R}$, let $\mathbb{M}_\kappa$ be the complete simply connected n-dimensional space form of constant sectional curvature κ. To each bounded domain Ω in M, associate the geodesic ball D in $\mathbb{M}_\kappa$ satisfying

$$|\Omega| = |D|. \tag{2.10}$$

If $\kappa > 0$ then only consider those Ω for which $|\Omega| < |\mathbb{M}_\kappa|$.

Theorem 2.2. *If, for all such Ω in M, equality (2.10) implies the isoperimetric inequality*

$$|\partial\Omega| \geq |\partial D|, \tag{2.11}$$

with equality in (2.11) if and only if Ω *is isometric to* D*, then we also have, for every bounded domain* Ω *in* M*, that equality (2.10) implies the inequality for the first Dirichlet eigenvalue*

$$\lambda_1(\Omega) \geq \lambda_1(D), \tag{2.12}$$

with equality holding if and only if Ω *is isometric to* D*.*

For a proof of Theorem 2.2, we refer to [17].

2.3 The Szegö-Weinberger Inequality

In this section, we prove the Szegö-Weinberger inequality which is a counterpart to the first non-zero Neumann eigenvalue of the Faber-Krahn inequality.

Theorem 2.3 ([96]). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*. Then the first non-zero Neumann eigenvalue of* Ω *satisfies*

$$\lambda_1(\Omega) \leq \lambda_1(\Omega^*) \tag{2.13}$$

with equality holding if and only if $\Omega = \Omega^*$*.*

Proof. Let R be the radius of Ω^* and let g be the solution of the equation

$$\left\{ \begin{array}{l} g'' + \frac{n-1}{r} g' - \frac{n-1}{r^2} g + \lambda_1(\Omega^*) g = 0 \\ g(0) = 0, g'(R) = 0 \end{array} \right. \tag{2.14}$$

By a topological argument, we can take as trial functions P_i, such that $\int_\Omega P_i = 0$ for $i = 1, \cdots, n$, with

$$P_i(x) = h(r) \frac{x_i}{r},$$

where the $x_i's$ are Cartesian coordinates, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n, r = |x|$, and

$$h(r) = \left\{ \begin{array}{l} g(r) \text{ for } 0 \leq r \leq R \\ g(R) \text{ for } \ r \geq R. \end{array} \right.$$

Observe that by an appropriate choice of sign, $g(r)$ is increasing on [0,R] and hence that h is everywhere nondecreasing for $r \geq 0$. By substituting our trial functions P_i into the mini-max inequality for λ_1, we find

$$\lambda_1(\Omega) \int_\Omega P_i^2 \leq \int_\Omega |\nabla P_i|^2.$$

Summing this in i for $1 \leq i \leq n$, we arrive at

$$\begin{aligned} \lambda_1(\Omega) &\leq \frac{\int_\Omega \sum_{i=1}^n |\nabla P_i|^2}{\int_\Omega \sum_{i=1}^n P_i^2} \\ &= \frac{\int_\Omega \left(h'(r)^2 + \frac{n-1}{r^2} h(r)^2\right)}{\int_\Omega h(r)^2} \\ &= \frac{\int_\Omega A(r)}{\int_\Omega h(r)^2} \end{aligned} \tag{2.15}$$

where

$$A(r) = h'(r)^2 + \frac{n-1}{r^2} h(r)^2. \tag{2.16}$$

$A(r)$ is easily seen to be decreasing for $0 \leq r \leq R$ by differentiating and using the differential equation (2.14). One finds

$$A'(r) = -2(\lambda_1(\Omega^*) h h' + (n-1)(rh' - h)^2/r^3) < 0, \; 0 < r < R \tag{2.17}$$

Also, $A(r) = (n-1)g(R)^2/r^2$ for $r \geq R$ shows that A is decreasing for $r > R$. Since A is continuous for all $r \geq 0$, it is also decreasing there. Observe that

$$\int_\Omega A(r) \leq \int_{\Omega^*} A(r) \tag{2.18}$$

since the volumes integrated over are the same in both cases, while in passing from the left to right hand sides we are exchanging integrating over $\Omega \setminus \Omega^*$ for integrating over $\Omega^* \setminus \Omega$ which are sets of equal volume. Since A is (strictly) decreasing this clearly increases the value of the integral unless $\Omega = \Omega^*$, when equality obtains. Similarly we find that

$$\int_\Omega h(r)^2 \geq \int_{\Omega^*} h(r)^2 \tag{2.19}$$

since h is nondecreasing. Thus we arrive at

$$\lambda_1(\Omega) \leq \frac{\int_{\Omega^*} A(r)}{\int_{\Omega^*} h(r)^2} = \lambda_1(\Omega^*), \tag{2.20}$$

since each P_i is precisely a Neumann eigenfunction of Δ with eigenvalue $\lambda_1(B_R)$ for the domain $B_R = \Omega^*$. This completes the proof of the Szegö-Weinberger inequality, including the characterization of the case of equality.

2.4 The Ashbaugh-Benguria Theorem

In this section we consider the sharp upper bound for λ_2/λ_1 for the Dirichlet eigenvalue problem proved by Ashbaugh-Benguria. In 1955 and 1956, Payne, Pólya and Weinberger [77], [78], proved that

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for} \quad \Omega \subset \mathbb{R}^2$$

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{\text{disk}} = \frac{j_{1,1}^2}{j_{0,1}^2}$$

with equality if and only if Ω is a disk and where $j_{p,k}$ denotes the k^{th} positive zero of the Bessel function $J_p(t)$. For general dimension $n \geq 2$, the analogous statements are

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for} \quad \Omega \subset \mathbb{R}^n,$$

and the *PPW conjecture*

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{n-\text{ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2}, \tag{2.21}$$

with equality if and only if Ω is an n-ball. This important conjecture was proved by Ashbaugh-Benguria (see [5], [6], [7]).

We proceed now with the proof of (2.21). Let us start from the variational principle for λ_2

$$\lambda_2(\Omega) = \min_{\phi \in H_0^1(\Omega), 0 \neq \phi \perp u_1} \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2}, \tag{2.22}$$

which, by integration by parts, leads to

$$\begin{aligned} &\lambda_2(\Omega) - \lambda_1(\Omega) \\ \leq \quad & \frac{\int_\Omega |\nabla P|^2 u_1^2}{\int_\Omega P^2 u_1^2}, P u_1 \in H_0^1(\Omega), \int_\Omega P u_1^2 = 0, P \neq 0. \end{aligned} \tag{2.23}$$

To get the isoperimetric result out of (2.23), one must make very special choices of the function P, in particular, choices for which (2.23) is an equality if Ω is a ball. Thus we shall use n trial functions $P = P_i$, such that $\int_\Omega P_i u_1^2 = 0$ for $i = 1, \cdots, n$ where

$$P_i = g(r) \frac{x_i}{r} \tag{2.24}$$

and

$$g(r) = \begin{cases} f(r) = \text{“right” radial function on } B_R \text{ for } 0 \leq r \leq R, \\ f(R) \quad \text{for } r \geq R. \end{cases} \tag{2.25}$$

The right R in this case turns out to be the unique R such that $\lambda_1(B_R) = \lambda_1(\Omega)$. Substituting P_i into (2.23) and summing on i, we find

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_\Omega B(r) u_1^2}{\int_\Omega f(r)^2 u_1^2} \tag{2.26}$$

where

$$B(r) = f'(r)^2 + \frac{n-1}{r^2} f(r)^2. \tag{2.27}$$

Now the equation (2.26) does not depend on the $P_i's$ and so we are in a position to define the function f. The idea is to take f as a properly quotient of Bessel functions so that the equality occur if Ω is a ball in $\mathbb{R}^n$. This motivates the choice of :

$$f(r) = w(\gamma r), \tag{2.28}$$

where

$$w(x) = \begin{cases} \frac{j_{n/2}(\beta x)}{j_{n/2-1}(\alpha x)}, \text{ if } 0 \leq x < 1, \\ w(1) \equiv \lim_{x \to 1} w(x), \text{ if } x \geq 1, \end{cases} \tag{2.29}$$

with $\alpha = j_{n/2-1,1}$, $\beta = j_{n/2,1}$ and $\gamma = \sqrt{\lambda_1}/\alpha$.

Lemma 2.1. *The equality occurs in (2.26) when Ω is a ball with λ_1 as the first Dirichlet eigenvalue and f is given by (2.28).*

Proof. If S_1 is a closed ball of $\mathbb{R}^n$ of appropriate radius centered in the origin in which the problem

$$\begin{cases} \Delta z = -\lambda z \text{ in } S_1, \\ z|_{\partial S_1} = 0. \end{cases} \tag{2.30}$$

has λ_1 as the first eigenvalue, then

$$S_1 = \{x \in \mathbb{R}^n; |x| \leq \alpha/\sqrt{\lambda_1} = 1/\gamma\}.$$

The second eigenvalue of the problem (2.30) is $\tilde{\lambda_2} = \frac{\beta^2}{\alpha^2}\lambda_1$. The first eigenfunction of S_1 is

$$z(x) = c|x|^{1-n/2} j_{n/2-1}(\sqrt{\lambda_1}|x|),$$

and the eigenfunctions corresponding to $\tilde{\lambda_2}$ are:

$$f_i(x) = c|x|^{1-n/2} j_{n/2}(\sqrt{\tilde{\lambda_2}}|x|)\frac{x_i}{|x|}, \ i = 1, \cdots, n,$$

where c is a non-zero constant.

Let

$$Q(r) = \begin{cases} \frac{j_{n/2}\left(\sqrt{\tilde{\lambda_2}}r\right)}{j_{n/2-1}(\sqrt{\lambda_1}r)}, \text{ if } 0 \leq r < 1/\gamma, \\ \lim_{r \to 1/\gamma} \frac{j_{n/2}\left(\sqrt{\tilde{\lambda_2}}r\right)}{j_{n/2-1}(\sqrt{\lambda_1}r)}, \text{ if } r \geq 1/\gamma, \end{cases} \tag{2.31}$$

Observe that $Q(r) = w(\gamma r) = g(r)$ and let

$$Q_i(x) = Q(|x|)\frac{x_i}{|x|} = g(r)\frac{x_i}{|x|},$$

then

$$\int_{S_1} Q_i z^2 = 0, \ i = 1, \cdots, n$$

and $Q_i z$ are eigenfunctions of $\tilde{\lambda_2}$. Thus, we have

$$\tilde{\lambda_2} = \frac{\int_{S_1} |\nabla(Q_i z)|^2}{\int_{S_1} (Q_i z)^2}, \ i = 1, \cdots, n.$$

Summing over i and simplifying, we get

$$\tilde{\lambda_2} - \lambda_1 = \frac{\int_{S_1} \left((f'(r))^2 + (n-1)\frac{f^2(r)}{r^2}\right) z^2}{\int_{S_1} f^2(r) z^2}. \tag{2.32}$$

This completes the proof of Lemma 3.1.

Substituting (2.28) into (2.26), we get

$$\lambda_2 - \lambda_1 \leq \frac{\lambda_1 \int_\Omega B(\gamma r) u_1^2}{\int_\Omega w^2(\gamma r) u_1^2} \tag{2.33}$$

where

$$B(x) = (w'(x))^2 + (n-1)\frac{w^2(x)}{x^2}. \tag{2.34}$$

From the definition of w and the properties of Bessel functions one can prove that $w(t)$ is nondecreasing and $B(t)$ is non-increasing. Therefore, we have

$$\int_\Omega B(\gamma r) u_1^2 \leq \int_{\Omega^*} B(\gamma r)^* (u_1^*)^2 \leq \int_{\Omega^*} B(\gamma r)(u_1^*)^2 \tag{2.35}$$

and

$$\int_\Omega w(\gamma r)^2 u_1^2 \geq \int_{\Omega^*} w(\gamma r)_* (u_1^*)^2 \leq \int_{\Omega^*} w(\gamma r)(u_1^*)^2 \tag{2.36}$$

In order to continue the proof, we need a result of Chiti: If c is chosen so that

$$\int_\Omega u_1^2 = \int_{\Omega^*} u_1^2 = \int_{S_1} z^2, \tag{2.37}$$

then

$$\int_{\Omega^*} f(r)(u_1^*)^2 \geq \int_{S_1} f(r) z^2, \tag{2.38}$$

if f is increasing, and the reverse inequality if f is decreasing. It follows from (2.38) and the monotonicity properties of B and w that

$$\int_{\Omega^*} B(\gamma r)(u_1^*)^2 \leq \int_{S_1} B(\gamma r) z^2 \tag{2.39}$$

and

$$\int_{\Omega^*} w(\gamma r)(u_1^*)^2 \geq \int_{S_1} w(\gamma r)^2 z^2. \tag{2.40}$$

Combining (2.33), (2.35), (2.36), (2.39), (2.40), and using the definition of z, we finally get

$$\lambda_2 - \lambda_1 \leq \frac{\lambda_1 \int_{S_1} B(\gamma r) z^2}{\alpha^2 \int_{S_1} w^2(\gamma r) z^2} = \frac{\lambda_1}{\alpha^2}(\beta^2 - \alpha^2). \tag{2.41}$$

From here the inequality

$$\frac{\lambda_2}{\lambda_1} \leq \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \tag{2.42}$$

follows immediately. Also, it is clear from the proof that the equality holding in (2.42) if and only if Ω is a ball.

2.5 The Hersch Theorem

In 1974, Hersch proved an isoperimetric inequality for the first non-trivial eigenvalue on the 2-dimensional sphere $\mathbb{S}^2$.

Theorem 2.5 ([48]). *For any metric on $\mathbb{S}^2$, the first non-trivial eigenvalue satisfies*

$$\lambda_1 \leq \frac{8\pi}{A(\mathbb{S}^2)}. \tag{2.43}$$

Proof. For any metric $d\tilde{s}^2$ on $\mathbb{S}^2$, we can construct a conformal map $\phi : (\mathbb{S}^2, d\tilde{s}^2) \to (\mathbb{S}^2, ds_0^2)$, here ds_0^2 denotes the standard metric on $\mathbb{S}^2$. From the mini-max principle, we have

$$\lambda_1 = \inf_{\int_{\mathbb{S}^2} f d\tilde{v}=0} \frac{\int_{\mathbb{S}^2} |\nabla f|^2 d\tilde{v}}{\int_{\mathbb{S}^2} f^2 d\tilde{v}}, \tag{2.44}$$

where $d\tilde{v}$ is the area element with respect to $d\tilde{s}^2$. Take the coordinate functions $x^i (i = 1, 2, 3)$ on $(\mathbb{S}^2, ds_0^2)$; then $x^i \circ \phi, i = 1, 2, 3$, are functions on $(\mathbb{S}^2, d\tilde{s}^2)$.

Observe that ϕ is a conformal map and that in the case of surfaces, the Dirichlet integral of a function is a conformal invariant. Thus we have

$$\int_{\mathbb{S}^2} |\nabla(x^i \circ \phi)|^2 d\tilde{v} = \int_{\mathbb{S}^2} |\nabla x^i|^2 dv = -\int_{\mathbb{S}^2} x^i \Delta x^i = 2\int_{\mathbb{S}^2} (x^i)^2 = \frac{8\pi}{3}.$$

Since

$$Area(\mathbb{S}^2) = \int_{\mathbb{S}^2} d\tilde{v} = \sum_{i=1}^{3} \int_{\mathbb{S}^2} (x^i \circ \phi)^2 d\tilde{v},$$

there exists at least one i such that

$$\int_{\mathbb{S}^2} (x^i \circ \phi)^2 d\tilde{v} \geq \frac{Area(\mathbb{S}^2)}{3}.$$

Also, we can choose ϕ satisfying $\int_{\mathbb{S}^2} x^i \circ \phi d\tilde{v} = 0$ [85]. Thus

$$\lambda_1 \leq \frac{\int_{\mathbb{S}^2} |\nabla(x^i \circ \phi)|^2 d\tilde{v}}{\int_{\mathbb{S}^2} (x^i \circ \phi)^2 d\tilde{v}} \leq \frac{8\pi}{A(\mathbb{S}^2)}. \tag{2.45}$$

For the discussion of equality case, we refer to [48]. □

Remark 2.1. $\mathbb{S}^2$ is a Riemann surface of genus $g = 0$. For Riemannian surface Σ_g of genus $g > 0$, Yang-Yau obtained a similar result.

Theorem 2.6 ([99]). *For any metric on Σ_g, the first eigenvalue satisfies*

$$\lambda_1 \leq \frac{8\pi(1+g)}{|\Sigma_g|} \tag{2.46}$$

Remark 2.2. Hersch's theorem can't be generalized directly to the case of higher dimensions [86]. That is, one can't expect that

$$\lambda_1 \mathrm{Vol}(M)^{2/n} \leq C,$$

with a constant depending only on n. It must depend also on other geometric invariants of M.

Here is an interesting application of Hersch's theorem.

Theorem 2.7 ([19]). *Suppose that M is homeomorphic to $\mathbb{S}^2$ and ϕ_1, ϕ_2, ϕ_3 are three first eigenfunctions such that their square sum is a constant. Then M is actually isometric to a sphere with constant sectional curvature.*

Proof. The assumption of Theorem 2.7 says that

$$\begin{cases} \Delta\phi_i + \lambda_1(M)\phi_i = 0, & i = 1, 2, 3, \\ \sum_{i=1}^3 \phi_i^2 = c, & c \text{ is a constant.} \end{cases}$$

Thus,

$$\begin{aligned} 0 &= \Delta\left(\sum_{i=1}^3 \phi_i^2\right) = 2\sum_{i=1}^3 |\nabla\phi_i|^2 + 2\sum_{i=1}^3 \phi_i \Delta\phi_i \\ &= 2\sum_{i=1}^3 |\nabla\phi_i|^2 - 2\lambda_1(M)\sum_{i=1}^3 \phi_i^2 \end{aligned}$$

which gives

$$\sum_{i=1}^3 |\nabla\phi_i|^2 = c\lambda_1(M). \tag{2.47}$$

Taking the Laplacian of both sides of (2.47) and using the Bochner formula, we get

$$\begin{aligned}
0 &= \frac{1}{2}\sum_{i=1}^{3}\Delta|\nabla\phi_i|^2 \qquad (2.48)\\
&= \sum_{i=1}^{3}|\nabla^2\phi_i|^2 + \sum_{i=1}^{3}\nabla\phi_i\cdot\nabla(\Delta\phi_i) + \sum_{i=1}^{3}\mathrm{Ric}(\nabla\phi_i,\nabla\phi_i)\\
&= \sum_{i=1}^{3}|\nabla^2\phi_i|^2 - \lambda_1(M)^2\sum_{i=1}^{3}\phi_i^2 + K\sum_{i=1}^{3}|\nabla\phi_i|^2\\
&\geq \sum_{i=1}^{3}\frac{|\Delta\phi_i|^2}{2} - c\lambda_1(M)^2 + Kc\lambda_1(M)\\
&= -\lambda_1(M)^2/2 + Kc\lambda_1(M),
\end{aligned}$$

where K is the sectional curvature of M. Thus we have

$$\lambda_1(M) \geq 2K. \qquad (2.49)$$

Integrating (2.49) and using the Gauss-Bonnet formula, we have

$$\lambda_1(M) \times area(M) \geq 8\pi. \qquad (2.50)$$

Combining (2.50) and Hersch's theorem we know that M is a 2-sphere. □

We have an isoperimetric upper bound for the first eigenvalue of the Laplacian of a closed (compact without boundary) hypersurface embedded in $\mathbb{R}^n$.

Theorem 2.8 ([92]). *Let M be a connected closed hypersurface embedded in $\mathbb{R}^n(n \geq 3)$. Let Ω be the region bounded by M. Denote by V and A the volume of Ω and the area of M, respectively. Then the first non-zero eigenvalue λ_1 of the Laplacian acting on functions on M satisfies*

$$\lambda_1 \leq \frac{(n-1)A}{nV}\left(\frac{\omega_n}{V}\right)^{1/n}. \qquad (2.51)$$

with equality holding if and only if M is an $(n-1)$-sphere.

Proof. Let us denote by $x_1, \cdots, x_n$, the coordinate functions on $\mathbb{R}^n$. By choosing the coordinates origin properly, we can assume that

$$\int_M x_i = 0, \quad i = 1, \cdots, n.$$

Since M cannot be contained in any hyperplane, each x_i is not a constant function, $i = 1, \cdots, n$. It follows from the Poincaré inequality that for each fixed $i \in \{1, \cdots, n\}$

$$\lambda_1 \int_M x_i^2 \leq \int_M |\nabla x_i|^2,$$

with equality holding if and only if $\Delta x_i = -\lambda_1 x_i$.

Summing over i from 1 to n, we get

$$\lambda_1 \int_M \sum_{i=1}^n x_i^2 \leq \int_M \sum_{i=1}^n |\nabla x_i|^2 = \int_M (n-1) = (n-1)A, \qquad (2.52)$$

with equality if and only if

$$\Delta x_i = -\lambda_1 x_i, \quad \forall i \in \{1, \cdots, n\}. \qquad (2.53)$$

Take a ball B in $\mathbb{R}^n$ of radius R centered at the origin so that $\mathrm{vol}(B) = V$; then

$$R = \left(\frac{V}{\omega_n}\right)^{1/n}.$$

By using the weighted isoperimetric inequality proved in [14], we have

$$\begin{aligned} \int_M \sum_{i=1}^n x_i^2 &\geq \int_{\partial B} \sum_{i=1}^n x_i^2 \qquad (2.54) \\ &= \mathrm{area}(\partial B) \cdot R^2 \\ &= nV \left(\frac{V}{\omega_n}\right)^{1/n}. \end{aligned}$$

Substituting (2.54) into (2.52), one gets (2.51). If the equality holds in (2.51), then the inequalities (2.52) and (2.54) must take equality sign.

It follows that the position vector $x = (x_1, \cdots, x_n)$ when restricted on M satisfies

$$\Delta x =: (\Delta x_1, \cdots, \Delta x_n) = -\lambda_1(x_1, \cdots, x_n).$$

Also, it is well known that

$$\Delta x = (n-1)\overrightarrow{H},$$

where $\overrightarrow{H}$ is the mean curvature vector of M. Consider now the function $g = |x|^2 : M \to \mathbb{R}$. Observing that $\overrightarrow{H}$ is normal to M, we infer that

$$wf = 2\langle w, x\rangle = -\frac{2(n-1)}{\lambda_1}\langle w, \overrightarrow{H}\rangle = 0, \quad \forall w \in \chi(M).$$

Thus f is a constant function and so M is a hypersphere. $\square$

Chapter 3

Universal Inequalities for Eigenvalues

3.1 Introduction

Payne, Pólya and Weinberger proved that the Dirichlet eigenvalues of the Laplacian for $\Omega \subset \mathbb{R}^2$ satisfy the bound [77], [77].

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \cdots \tag{3.1}$$

This result easily extends to $\Omega \subset \mathbb{R}^n$ as

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \cdots \tag{3.2}$$

Many interesting generalizations of (1.3) have been done during the past years, e. g., in [3], [4], [9], [20], [21], [22], [23], [25], [26], [27], [29], [30], [33], [41], [42], [43], [44], [45], [46], [47], [50], [52], [68], [69], [90], [98]. In 1991, Yang [98] proved the following much stronger result:

Theorem 3.1. *The Dirichlet eigenvalues of the Laplacian of* $\Omega \subset \mathbb{R}^n$ *satisfy the inequality*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_{k+1}-\left(1+\frac{4}{n}\right)\lambda_i\right) \leq 0, \quad \text{for } k=1,2,\cdots. \tag{3.3}$$

The inequality (3.3), as observed by Yang himself, and as later proved, e. g., in [3], [4], [9], is the strongest of the classical inequalities that are derived following the scheme devised by Payne-Pólya-Weinberger. Yang's inequality provided a marked improvement for eigenvalues of large index. Recently, some Yang type inequalities on eigenvalues of the clamped plate problem, the buckling problem, the polyharmonic operator and some other type eigenvalue problems have been proved. This chapter is devoted to prove some of the universal inequalities in this subarea. Since the method in proving Yang's inequality has been widely generalized in obtaining various universal inequalities for eigenvalues, we end this section by proving Yang's inequality.

Proof of Theorem 3.1. Let u_k be the orthonormal eigenfunction corresponding to the k^{th} eigenvalue λ_k, i.e. u_k satisfies

$$\begin{cases} \Delta u_k = -\lambda_k u_k, \quad \text{in } \Omega \\ u_k|_{\partial\Omega} = 0, \\ \int_\Omega u_i u_j = \delta_{ij}. \end{cases} \tag{3.4}$$

Let $x_1, \cdots, x_n$ be the standard coordinate functions in $\mathbb{R}^n$. For any fixed $p = 1, \cdots, n$, put $g = x_p$ and define ϕ_i by

$$\phi_i = g u_i - \sum_{j=1}^{k} a_{ij} u_j, \quad a_{ij} = \int_\Omega g u_i u_j = a_{ji}. \tag{3.5}$$

It is easy to see that

$$\int_\Omega \phi_i u_j = 0, \quad \text{for } \ i,j = 1, \cdots, k. \tag{3.6}$$

Letting

$$b_{ij} = \int_\Omega u_j \nabla g \cdot \nabla u_i,$$

from Green's formula, we derive

$$\lambda_j a_{ij} = \int_\Omega g(-\Delta u_j)u_i = -2b_{ij} + \lambda_i a_{ij}$$

and so

$$2b_{ij} = (\lambda_i - \lambda_j)a_{ij}. \tag{3.7}$$

Since

$$\Delta\phi_i = -\lambda_i g u_i + 2\nabla g \cdot \nabla u_i + \sum_{j=1}^{k} \lambda_j a_{ij} u_j,$$

we have

$$\int_\Omega |\nabla\phi_i|^2 = \lambda_i \int_\Omega \phi_i^2 - 2\int_\Omega \phi_i \nabla g \cdot \nabla u_i. \tag{3.8}$$

On the other hand, from the definition of ϕ_i, (3.5) and (3.6), we derive

$$\begin{aligned} & -2\int_\Omega \phi_i \nabla g \cdot \nabla u_i && (3.9) \\ = \quad & -2\int_\Omega g\nabla g \cdot u_i \nabla u_i + 2\sum_{j=1}^{k} a_{ij} \int_\Omega u_j \nabla g \cdot \nabla u_i \\ = \quad & 1 + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2. \end{aligned}$$

From the mini-max principle, we obtain

$$(\lambda_{k+1} - \lambda_i)\int_\Omega \phi_i^2 \le 1 + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{ij}^2. \tag{3.10}$$

Multiplying (3.9) by $(\lambda_{k+1} - \lambda_i)^2$ and taking sum on i from 1 to k, we obtain

$$\begin{aligned} & \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 + \sum_{i,j=1}^{k}(\lambda_i - \lambda_j)(\lambda_{k+1} - \lambda_i)^2 a_{ij}^2 \qquad (3.11) \\ & = -2\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_\Omega \phi_i \nabla g \cdot \nabla u_i. \end{aligned}$$

By $a_{ij} = a_{ji},\ b_{ij} = -b_{ji}$, we have

$$\begin{aligned} &-2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega \phi_i \nabla g\cdot\nabla u_i \\ &=\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 - 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)b_{ij}^2 \equiv w. \end{aligned} \tag{3.12}$$

Multiplying (3.10) by $(\lambda_{k+1}-\lambda_i)^2$ and taking sum on i from 1 to k, we infer

$$\begin{aligned} &\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^3\int_\Omega \phi_i^2 \\ &\leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 - 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)b_{ij}^2 = w. \end{aligned}$$

From $\int_\Omega u_i\phi_j = 0$ for all $i, j = 1, \cdots, k$, we have, for arbitrary constants d_{ij},

$$\begin{aligned} & w^2 \\ = \ & \left(-2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega \phi_i\nabla g\cdot\nabla u_i\right)^2 \\ \leq \ & 4\sum_{i=1}^{k}\int_\Omega(\lambda_{k+1}-\lambda_i)^3\phi_i^2 \\ & \times\sum_{i=1}^{k}\int_\Omega\left((\lambda_{k+1}-\lambda_i)^{1/2}\nabla g\cdot\nabla u_i - \sum_{j=1}^{k}d_{ij}u_j\right)^2 \\ \leq \ & 4w\sum_{i=1}^{k}\int_\Omega\left((\lambda_{k+1}-\lambda_i)|\nabla g\cdot\nabla u_i|^2 + \left(\sum_{j=1}^{k}d_{ij}u_j\right)^2\right. \\ & \left. -2\sum_{j=1}^{k}d_{ij}(\lambda_{k+1}-\lambda_i)^{1/2}u_j\nabla g\cdot\nabla u_i\right). \end{aligned}$$

Then we have

$$\begin{aligned} w &\le 4\sum_{i=1}^{k}\int_{\Omega}(\lambda_{k+1}-\lambda_i)\left(\frac{\partial u_i}{\partial x_p}\right)^2 \\ &\quad +4\left(-2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^{1/2}b_{ij}+\sum_{i,j=1}^{k}d_{ij}^2\right). \end{aligned}$$

Putting $d_{ij}=(\lambda_{k+1}-\lambda_i)^{1/2}b_{ij}$, we obtain

$$w \le 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}\left(\frac{\partial u_i}{\partial x_p}\right)^2 - 4\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)b_{ij}^2 \quad (3.13)$$

and so we infer

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}\left(\frac{\partial u_i}{\partial x_p}\right)^2. \quad (3.14)$$

Summing over p, we obtain

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_{\Omega}|\nabla u_i|^2 \quad (3.15) \\ &= \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i. \end{aligned}$$

□

Yang's inequality has been generalized to bounded domains in complete submanifolds in Euclidean space. That is, we have

Theorem 3.2 ([20], [41]). *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M^n isometrically immersed in $\mathbb{R}^N$. Then the Dirichlet eigenvalues of the Laplacian of Ω satisfy the inequality*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_k)^2 \le \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_k)(\lambda_i+\frac{n^2}{4}||H||^2), \quad (3.16)$$

where H is the mean curvature vector field of M^n and $||H||^2 = \sup_\Omega |H|^2$.

3.2 Eigenvalues of the Clamped Plate Problem

Let us generalize Yang's method to prove universal inequalities for eigenvalues of the clamped plate problem on Riemannian manifolds.

Theorem 3.2 ([94]). *Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded domain with smooth boundary in M. Denote by ν the outward unit normal of $\partial\Omega$ and let λ_i the i-th eigenvalue of the problem:*

$$\begin{cases} \Delta^2 u = \lambda u \quad \text{in} \ \ \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0. \end{cases} \tag{3.17}$$

i) If M is isometrically immersed in $\mathbb{R}^m$ with mean curvature vector $\mathbf{H}$, then

$$\begin{aligned} &\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \qquad (3.18)\\ \leq \ &\frac{1}{n}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(n^2H_0^2+(2n+4)\lambda_i^{1/2}\right)\right\}^{1/2} \\ &\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(n^2H_0^2+4\lambda_i^{1/2}\right)\right\}^{1/2}, \end{aligned}$$

where $H_0 = \sup_\Omega |\mathbf{H}|$.

ii) If there exists a function $\phi : \Omega \to \mathbb{R}$ and a constant A_0 such that

$$|\nabla\phi| = 1, \ \ |\Delta\phi| \leq A_0, \quad \text{on } \Omega, \tag{3.19}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \tag{3.20}$$
$$\leq \left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(A_0^2+4A_0\lambda_i^{1/4}+6\lambda_i^{1/2}\right)\right\}^{1/2}$$
$$\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(2\lambda_i^{1/4}+A_0\right)^2\right\}^{1/2}.$$

iii) If there exists a function $\psi : \Omega \to \mathbb{R}$ and a constant B_0 such that

$$|\nabla\psi| = 1,\ \Delta\psi = B_0, \quad \text{on } \Omega, \tag{3.21}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(6\lambda_i^{1/2}-B_0^2)\right\}^{1/2} \tag{3.22}$$
$$\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\lambda_i^{1/2}-B_0^2\right)\right\}^{1/2}.$$

iv) If Ω admits an eigenmap $f=(f_1,f_2,\cdots,f_{m+1}) : \Omega \to \mathbb{S}^m(1)$ corresponding to an eigenvalue μ, that is,

$$\Delta f_\alpha = -\mu f_\alpha,\ \alpha = 1,\cdots,m+1, \quad \sum_{\alpha=1}^{m+1} f_\alpha^2 = 1,$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(6\lambda_i^{1/2}+\mu\right)\right\}^{1/2} \tag{3.23}$$
$$\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\lambda_i^{1/2}+\mu\right)\right\}^{1/2},$$

where $\mathbb{S}^m(1)$ *is the unit* m*-sphere.*

Theorem 3.2 can be deduced from a general result.

Lemma 3.1 ([89]). *Let* $\lambda_i, i = 1, \cdots,$ *be the* i*-th eigenvelue of the problem (3.17) and* u_i *the orthonormal eigenfunction corresponding to* λ_i*, that is,*

$$\begin{cases} \Delta^2 u_i = \lambda_i u_i \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = \frac{\partial u_i}{\partial \nu}\Big|_{\partial\Omega} = 0, \\ \int_M u_i u_j = \delta_{ij}, \quad \forall \ \ i, \ j = 1, 2, \cdots . \end{cases} \tag{3.24}$$

Then for any smooth function $h : \Omega \to \mathbb{R}$ *and any* $\delta > 0$*, we have*

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega u_i^2 |\nabla h|^2 \\ \leq \ \ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega \{u_i^2 (\Delta h)^2 - 2u_i |\nabla h|^2 \Delta u_i \\ &\qquad + 4((\nabla h \cdot \nabla u_i)^2 + u_i \Delta h \nabla h \cdot \nabla u_i)\} \\ &+ \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \int_\Omega \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2} \right)^2 . \end{aligned} \tag{3.25}$$

Proof of Lemma 3.1. For $i = 1, \cdots, k$, consider the functions $\phi_i : \Omega \to \mathbb{R}$ given by

$$\phi_i = h u_i - \sum_{j=1}^{k} r_{ij} u_j,$$

where

$$r_{ij} = \int_\Omega h u_i u_j.$$

Since $\phi_i|_{\partial\Omega} = \frac{\partial \phi_i}{\partial \nu}\Big|_{\partial\Omega} = 0$ and

$$\int_\Omega u_j \phi_i = 0, \quad \forall \ i, j = 1, \cdots, k,$$

it follows from the mini-max inequality that

$$\lambda_{k+1} \leq \frac{\int_\Omega \phi_i \Delta^2 \phi_i}{\int_\Omega \phi_i^2}. \tag{3.26}$$

We have

$$\begin{aligned} & \int_\Omega \phi_i \Delta^2 \phi_i \qquad\qquad (3.27) \\ = & \int_\Omega \phi_i \left(\Delta^2(hu_i) - \sum_{j=1}^{k} r_{ij}\lambda_j u_j \right) \\ = & \int_\Omega \phi_i \Delta^2 (hu_i) \\ = & \lambda_i ||\phi_i||^2 - \sum_{j=1}^{k} r_{ij} s_{ij} \\ & + \int_\Omega hu_i(\Delta(u_i\Delta h) + 2\Delta(\nabla h \cdot \nabla u_i) + 2\nabla h \cdot \nabla(\Delta u_i) + \Delta h \Delta u_i), \end{aligned}$$

where $||\phi_i||^2 = \int_\Omega \phi_i^2$ and

$$s_{ij} = \int_\Omega u_j(\Delta(u_i\Delta h) + 2\Delta(\nabla h \cdot \nabla u_i) + 2\nabla h \cdot \nabla(\Delta u_i) + \Delta h \Delta u_i).$$

Multiplying the equation $\Delta^2 u_i = \lambda_i u_i$ by hu_j, we have

$$hu_j \Delta^2 u_i = \lambda_i hu_i u_j. \tag{3.28}$$

Changing the roles of i and j, one gets

$$hu_i \Delta^2 u_j = \lambda_j hu_i u_j. \tag{3.29}$$

Subtracting (3.28) from (3.29) and integrating the resulted equation on Ω, we get

$$\begin{aligned}
& (\lambda_j - \lambda_i) r_{ij} \qquad (3.30)\\
= & \int_\Omega (hu_i \Delta^2 u_j - hu_j \Delta^2 u_i)\\
= & \int_\Omega (\Delta(hu_i)\Delta u_j - \Delta(hu_j)\Delta u_i)\\
= & \int_\Omega ((u_i \Delta h + 2\nabla h \cdot \nabla u_i)\Delta u_j - (u_j \Delta h + 2\nabla h \cdot \nabla u_j)\Delta u_i)\\
= & \int_\Omega u_j (\Delta(u_i \Delta h) + 2\Delta(\nabla h \cdot \nabla u_i) + \Delta h \Delta u_i + 2\nabla(\Delta u_i) \cdot \nabla h)\\
= & s_{ij},
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int_\Omega hu_i (\Delta(u_i \Delta h) + 2\Delta(\nabla h \cdot \nabla u_i) + 2\nabla h \cdot \nabla(\Delta u_i) + \Delta h \Delta u_i)\\
= & \int_\Omega (u_i^2 (\Delta h)^2 + 4(|\nabla h \cdot \nabla u_i|^2 + u_i \Delta h \nabla h \cdot \nabla u_i) - 2u_i |\nabla h|^2 \Delta u_i).
\end{aligned}$$

It follows from (3.26), (3.27) and (3.30) that

$$\begin{aligned}
& (\lambda_{k+1} - \lambda_i) ||\phi_i||^2 \qquad (3.31)\\
\leq & \int_\Omega (u_i^2 (\Delta h)^2 + 4(|\nabla h \cdot \nabla u_i|^2 + u_i \Delta h \nabla h \cdot \nabla u_i) - 2u_i |\nabla h|^2 \Delta u_i)\\
& + \sum_{j=1}^{k} (\lambda_i - \lambda_j) r_{ij}^2.
\end{aligned}$$

Set

$$t_{ij} = \int_\Omega u_j \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2} \right); \qquad (3.32)$$

then $t_{ij} + t_{ji} = 0$ and

$$\begin{aligned}
&\int_\Omega (-2)\phi_i \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right) \qquad (3.33)\\
= &\int_\Omega (-2hu_i \nabla h \cdot \nabla u_i - u_i^2 h \Delta h) + 2\sum_{j=1}^k r_{ij} t_{ij}\\
= &\int_\Omega u_i^2 |\nabla h|^2 + 2\sum_{j=1}^k r_{ij} t_{ij}.
\end{aligned}$$

Multiplying (3.33) by $(\lambda_{k+1} - \lambda_i)^2$ and using the Schwarz inequality, we get

$$\begin{aligned}
&(\lambda_{k+1} - \lambda_i)^2 \left(\int_\Omega u_i^2 |\nabla h|^2 + 2\sum_{j=1}^k r_{ij} t_{ij}\right) \qquad (3.34)\\
= \ &(\lambda_{k+1} - \lambda_i)^2 \int_M (-2)\phi_i \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right)\\
= \ &(\lambda_{k+1} - \lambda_i)^2 \int_M (-2)\phi_i \left(\left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right) - \sum_{j=1}^k t_{ij} u_j\right)\\
\leq \ &\delta(\lambda_{k+1} - \lambda_i)^3 ||\phi_i||^2\\
&+ \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \int_M \left|\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2} - \sum_{j=1}^k t_{ij} u_j\right|^2\\
= \ &\delta(\lambda_{k+1} - \lambda_i)^3 ||\phi_i||^2\\
&+ \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left(\int_\Omega \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right)^2 - \sum_{j=1}^k t_{ij}^2\right)
\end{aligned}$$

Substituting (3.31) into (3.34) and summing over i from 1 to k and noticing $r_{ij} = r_{ji},\ t_{ij} = -t_{ji}$, we get

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2|\nabla h|^2 - 2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}t_{ij} \\
\leq\ & \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\delta\int_\Omega(u_i^2(\Delta h)^2 + 4((\nabla h\cdot\nabla u_i)^2 + u_i\Delta h\nabla h\cdot\nabla u_i) \\
& -2u_i|\nabla h|^2\Delta u_i) + \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\left((\nabla h\cdot\nabla u_i)^2 + \frac{u_i\Delta h}{2}\right)^2 \\
& -\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)\delta(\lambda_i-\lambda_j)^2 r_{ij}^2 - \sum_{i,j=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}t_{ij}^2,
\end{aligned}$$

which implies (3.25). □

Proof of Theorem 3.2. Let $\{u_i\}_{i=1}^{\infty}$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the problem (3.17).

i) Let $x_\alpha, \alpha = 1, \cdots, m$, be the standard coordinate functions of $\mathbb{R}^m$. Taking $h = x_\alpha$ in (3.25) and summing over α, we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m}\int_\Omega u_i^2|\nabla x_\alpha|^2 \\
\leq\ & \delta\sum_{i=1}^{k+1}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m}\int_\Omega(u_i^2(\Delta x_\alpha)^2 + 4((\nabla x_\alpha\cdot\nabla u_i)^2 \\
& +u_i\Delta x_\alpha\nabla x_\alpha\cdot\nabla u_i) - 2u_i|\nabla x_\alpha|^2\Delta u_i) \\
& +\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\sum_{\alpha=1}^{m}\int_\Omega\left(\nabla x_\alpha\cdot\nabla u_i + \frac{u_i\Delta x_\alpha}{2}\right)^2,
\end{aligned} \tag{3.35}$$

Since M is isometrically immersed in $\mathbb{R}^m$, we have

$$\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2 = n$$

which implies that

$$\sum_{\alpha=1}^{m} \int_\Omega u_i^2 |\nabla x_\alpha|^2 = n \tag{3.36}$$

Also, we have

$$\Delta(x_1, \cdots, x_m) \equiv (\Delta x_1, \cdots, \Delta x_m) = n\mathbf{H}, \tag{3.37}$$

$$\sum_{\alpha=1}^{m} (\nabla x_\alpha \cdot \nabla u_i)^2 = \sum_{\alpha=1}^{m} (\nabla u_i(x_\alpha))^2 = |\nabla u_i|^2 \tag{3.38}$$

and

$$\sum_{\alpha=1}^{m} \Delta x_\alpha \nabla x_\alpha \cdot \nabla u_i = \sum_{\alpha=1}^{m} \Delta x_\alpha \nabla u_i(x_\alpha) = n\mathbf{H} \cdot \nabla u_i = 0. \tag{3.39}$$

Substituting (3.36)-(3.39) into (3.35), we get

$$\begin{aligned} & n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \qquad (3.40) \\ \leq \ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega (n^2 u_i^2 |\mathbf{H}|^2 + 4|\nabla u_i|^2 - 2n u_i \Delta u_i) \\ & + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \int_\Omega \left(|\nabla u_i|^2 + \frac{n^2 u_i^2 |\mathbf{H}|^2}{4} \right) \\ \leq \ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 (n^2 H_0^2 + (2n+4)\lambda_i^{1/2}) \\ & + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left(\lambda_i^{1/2} + \frac{n^2 H_0^2}{4} \right). \end{aligned}$$

Here in the last inequality, we have used the fact that $|\mathbf{H}| \leq H_0$ and

$$\int_\Omega |\nabla u_i|^2 = -\int_\Omega u_i \Delta u_i \leq \left(\int_\Omega u_i^2 \right)^{1/2} \left(\int_\Omega (\Delta u_i)^2 \right)^{1/2} = \lambda_i^{1/2} \tag{3.41}$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/2} + \frac{n^2H_0^2}{4}\right)}{\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 (n^2H_0^2 + (2n+4)\lambda_i^{1/2})} \right\}^{1/2},$$

one gets (3.18).

ii) Substituting $h=\phi$ into (3.25) and using (3.19) and the Schwarz inequality, we get

$$\begin{aligned} &\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \qquad (3.42)\\ \leq\ & \delta \sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega \{u_i^2(\Delta\phi)^2 + 4((\nabla\phi\cdot\nabla u_i)^2 + u_i\Delta\phi\nabla\phi\cdot\nabla u_i)\\ & -2u_i\Delta u_i\} + \sum_{i=1}^k \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \left(\nabla\phi\cdot\nabla u_i + \frac{u_i\Delta\phi}{2}\right)^2\\ \leq\ & \delta \sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega (A_0^2u_i^2 + 4(|\nabla u_i|^2 + A_0|u_i||\nabla u_i|) - 2u_i\Delta u_i)\\ & + \sum_{i=1}^k \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega \left(|\nabla u_i|^2 + A_0|u_i||\nabla u_i| + \frac{A_0^2u_i^2}{4}\right). \end{aligned}$$

Substituting (3.41) and

$$\int_\Omega |u_i||\nabla u_i| \leq \left(\int_\Omega u_i^2\right)^{1/2}\left(\int_\Omega |\nabla u_i|^2\right)^{1/2} \leq \lambda_i^{1/4}$$

into (3.42), we get

$$\begin{aligned} &\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2\\ \leq\ & \delta \sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 (A_0^2 + 4A_0\lambda_i^{1/4} + 6\lambda_i^{1/2})\\ & + \sum_{i=1}^k \frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\lambda_i^{1/4} + \frac{A_0}{2}\right)^2. \end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/4}+\frac{A_0}{2}\right)^2}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(A_0^2+4A_0\lambda_i^{1/4}+6\lambda_i^{1/2})}\right\}^{1/2},$$

we obtain (3.20).

iii) Introducing $h=\psi$ into (3.25) and using (3.21), we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\
\leq\ &\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega(B_0^2u_i^2+4(|\nabla u_i|^2+B_0u_i\nabla\psi\cdot\nabla u_i)-2u_i\Delta u_i)\\
&+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_\Omega\left(|\nabla u_i|^2+B_0u_i\nabla\psi\cdot\nabla u_i+\frac{B_0^2u_i^2}{4}\right)\\
\leq\ &\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(6\lambda_i^{1/2}-B_0^2)+\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\lambda_i^{1/2}-\frac{B_0^2}{4}\right),
\end{aligned}$$

where in the last inequality, we have used the fact that

$$\int_\Omega u_i\langle\nabla\psi,\nabla u_i\rangle=-\frac{1}{2}\int_\Omega u_i^2\Delta\psi=-\frac{B_0^2}{2}.$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/2}-\frac{B_0^2}{4}\right)}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(6\lambda_i^{1/2}-B_0^2)}\right\}^{1/2},$$

we obtain (3.22).

iv) Taking the Laplacian of the equation

$$\sum_{\alpha=1}^{m+1}f_\alpha^2=1$$

and using the fact that

$$\Delta f_\eta = -\mu f_\eta,\ \eta = 1, \cdots, m+1,$$

we have

$$\sum_{\eta=1}^{m+1} |\nabla f_\eta|^2 = \mu.$$

It then follows by taking $h = f_\eta$ in (3.25) and summing over η that

$$\begin{aligned}
&\mu \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\
\leq\ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega \left(\mu^2 u_i^2 + 4 \sum_{\alpha=1}^{m+1} (\nabla f_\alpha \cdot \nabla u_i)^2 - 2\mu u_i \Delta u_i \right) \\
&+ \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \int_\Omega \left(\sum_{\alpha=1}^{m+1} (\nabla f_\alpha \cdot \nabla u_i)^2 + \frac{\mu^2 u_i^2}{4} \right) \\
\leq\ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 (\mu^2 + 6\mu \lambda_i^{1/2}) \\
&+ \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left(\mu \lambda_i^{1/2} + \frac{\mu^2}{4} \right).
\end{aligned}$$

We get (3.23) by taking

$$\delta = \left\{ \frac{\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i^{1/2} + \frac{\mu}{4} \right)}{\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(6\lambda_i^{1/2} + \mu \right)} \right\}^{1/2}.$$

□

Here are some examples of manifolds supporting the functions on the whole manifolds as stated in items ii)-v) of Theorem 1.1.

Example 3.1. Let M be an n-dimensional Hadamard manifold with Ricci curvature satisfying $\mathrm{Ric}_M \geq -(n-1)c^2,\ c \geq 0$ and let

$\gamma : [0, +\infty) \to M$ be a geodesic ray, namely a unit speed geodesic with $d(\gamma(s), \gamma(t)) = t - s$ for any $t > s > 0$. The Busemann function b_γ corresponding to γ defined by

$$b_\gamma(x) = \lim_{t \to +\infty} (d(x, \gamma(t)) - t)$$

satisfies $|\nabla b_\gamma| \equiv 1$(Cf. [10], [49]). Also, it follows from Theorem 3.5 in [84] that $|\Delta b_\gamma| \leq (n-1)c^2$ on M. Thus any Hadamard manifold with Ricci curvature bounded below supports functions satisfying (3.19).

Example 3.2. Let (N, ds_N^2) be a complete Riemannian manifold and define a Riemannian metric on $M = \mathbb{R} \times N$ by

$$ds_M^2 = dt^2 + \eta^2(t) ds_N^2, \tag{3.43}$$

where η is a positive smooth function defined on $\mathbb{R}$ with $\eta(0) = 1$. The manifold (M, ds_M^2) is called a warped product and denoted by $M = \mathbb{R} \times_\eta N$. It is known that M is a complete Riemannian manifold.

Set $\eta = e^{-t}$ and consider the warped product $M = \mathbb{R} \times_{e^{-t}} N$. Define $\psi : M \to \mathbb{R}$ by $\psi(x, t) = t$. One can show that

$$|\nabla \psi| = 1, \quad \Delta \psi = 1 - n. \tag{3.44}$$

That is, a warped product manifold $M = \mathbb{R} \times_{e^{-t}} N$ admits functions satisfying (3.21).

Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. Using the upper half-space model, $\mathbb{H}^n$ is given by

$$\mathbb{R}_+^n = \{(x_1, x_2, \cdots, x_n) | x_n > 0\} \tag{3.45}$$

with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2} \tag{3.46}$$

One can check that the map $\Phi : \mathbb{R} \times_{e^{-t}} \mathbb{R}^{n-1}$ given by

$$\Phi(t, x) = (x, e^t)$$

is an isometry. Therefore, $\mathbb{H}^n$ admits a warped product model, $\mathbb{H}^n = \mathbb{R} \times_{e^{-t}} \mathbb{R}^{n-1}$.

Example 2.3. Any compact homogeneous Riemannian manifold admits eigenmaps to some unit sphere for the first positive eigenvalue of the Laplacian [70].

3.3 Eigenvalues of the Polyharmonic Operator

The method of proving universal bounds for eigenvalues of the clamped plate problem can be generalized to the eigenvalue problem of polyharmonic operators.

Theorem 3.3 ([55]). *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty) Let l be a positive integer and let λ_i, $i = 1, \cdots$, be the i-th eigenvalue of the problem (1.7) and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,*

$$\begin{cases} (-\Delta)^l u_i = \lambda_i u_i \quad \text{in } M, \\ u_i|_{\partial M} = \frac{\partial u_i}{\partial \nu}\big|_{\partial M} = \cdots = \frac{\partial^{l-1} u_i}{\partial \nu^{l-1}}\Big|_{\partial M} = 0, \\ \int_M u_i u_j = \delta_{ij}, \quad \text{for any} \quad i,\ j = 1, 2, \cdots . \end{cases} \tag{3.47}$$

Then for any function $h \in C^{l+2}(M) \cap C^{l+1}(\partial M)$ and any positive integer k, we have

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_M u_i^2 |\nabla h|^2 \qquad (3.48) \\ \leq\ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_M h u_i \left((-\Delta)^l (h u_i) - \lambda_i h u_i \right) \\ & + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left\| \langle \nabla h, \nabla u_i \rangle + \frac{u_i \Delta h}{2} \right\|^2, \end{aligned}$$

where δ is any positive constant.

Proof. For $i = 1, \cdots, k$, consider the functions $\phi_i : M \to \mathbb{R}$ given by

$$\phi_i = h u_i - \sum_{j=1}^{k} r_{ij} u_j, \tag{3.49}$$

where

$$r_{ij} = \int_M h u_i u_j. \tag{3.50}$$

Since

$$\phi_i|_{\partial M} = \left.\frac{\partial \phi_i}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1}\phi_i}{\partial \nu^{l-1}}\right|_{\partial M} = 0$$

and

$$\int_M u_j\phi_i = 0, \quad \forall\, i, j = 1, \cdots, k,$$

it follows from the mini-max inequality that

$$\begin{aligned} & \lambda_{k+1}\int_M \phi_i^2 \\ \leq \; & \int_M \phi_i(-\Delta)^l\phi_i \\ = \; & \lambda_i||\phi_i||^2 + \int_M \phi_i\left((-\Delta)^l\phi_i - \lambda_i h u_i\right) \\ = \; & \lambda_i||\phi_i||^2 + \int_M \phi_i\left((-\Delta)^l(hu_i) - \lambda_i h u_i\right) \\ = \; & \lambda_i||\phi_i||^2 + \int_M hu_i\left((-\Delta)^l(hu_i) - \lambda_i h u_i\right) - \sum_{j=1}^k r_{ij}s_{ij}, \end{aligned} \tag{3.51}$$

where

$$s_{ij} = \int_M \left((-\Delta)^l(hu_i) - \lambda_i h u_i\right) u_j.$$

Notice that if $u \in C^{l+2}(M) \cap C^{l+1}(\partial M)$ satisfies

$$u|_{\partial M} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1}u}{\partial \nu^{l-1}}\right|_{\partial M} = 0, \tag{3.52}$$

then

$$\begin{aligned} u|_{\partial M} &= \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots \\ &= \Delta^{m-1}u\big|_{\partial M} = \nabla(\Delta^{m-1}u)\big|_{\partial M} = 0, \quad \text{when} \quad l = 2m \end{aligned}$$

and

$$\begin{aligned} u|_{\partial M} &= \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots = \Delta^{m-1}u\big|_{\partial M} \\ &= \nabla(\Delta^{m-1}u)\big|_{\partial M} = \Delta^m u|_{\partial M} = 0, \quad \text{when} \quad l = 2m+1. \end{aligned}$$

We can then use integration by parts to conclude that

$$\int_M u_j(-\Delta)^l(hu_i) = \int_M hu_i(-\Delta)^l(u_j) = \lambda_j r_{ij},$$

which gives

$$s_{ij} = (\lambda_j - \lambda_i)r_{ij}. \tag{3.53}$$

Set

$$p_i(h) = (-\Delta)^l(hu_i) - \lambda_i hu_i;$$

then we have from (3.51) and (3.53) that

$$\begin{aligned}(\lambda_{k+1} - \lambda_i)||\phi_i||^2 &\leq \int_M \phi_i p_i(h) \qquad (3.54)\\ &= \int_M hu_i p_i(h) + \sum_{j=1}^k (\lambda_i - \lambda_j)r_{ij}^2.\end{aligned}$$

Set

$$t_{ij} = \int_M u_j \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right); \tag{3.55}$$

then $t_{ij} + t_{ji} = 0$ and

$$\int_M (-2)\phi_i \left(\nabla h \cdot \nabla u_i + \frac{u_i \Delta h}{2}\right) = w_i + 2\sum_{j=1}^k r_{ij}t_{ij}, \tag{3.56}$$

where

$$w_i = \int_M (-hu_i^2 \Delta h - 2hu_i \nabla h \cdot \nabla u_i). \tag{3.57}$$

Multiplying (3.56) by $(\lambda_{k+1}-\lambda_i)^2$ and using the Schwarz inequality and (3.54), we get

$$\begin{aligned}
& (\lambda_{k+1}-\lambda_i)^2\left(w_i+2\sum_{j=1}^{k} r_{ij}t_{ij}\right)\\
= \;& (\lambda_{k+1}-\lambda_i)^2\int_M (-2)\phi_i\left(\left(\nabla h\cdot\nabla u_i+\frac{u_i\Delta h}{2}\right)-\sum_{j=1}^{k} t_{ij}u_j\right)\\
\leq \;& \delta(\lambda_{k+1}-\lambda_i)^3||\phi_i||^2\\
& +\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_M\left|\nabla h\cdot\nabla u_i+\frac{u_i\Delta h}{2}-\sum_{j=1}^{k} t_{ij}u_j\right|^2\\
= \;& \delta(\lambda_{k+1}-\lambda_i)^3||\phi_i||^2\\
& +\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\left\|\nabla h\cdot\nabla u_i+\frac{u_i\Delta h}{2}\right\|^2-\sum_{j=1}^{k} t_{ij}^2\right)\\
\leq \;& \delta(\lambda_{k+1}-\lambda_i)^2\left(\int_M h u_i p_i(h)+\sum_{j=1}^{k}(\lambda_i-\lambda_j)r_{ij}^2\right)\\
& +\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\left\|\nabla h\cdot\nabla u_i+\frac{u_i\Delta h}{2}\right\|^2-\sum_{j=1}^{k} t_{ij}^2\right).
\end{aligned}$$

Summing over i and noticing $r_{ij}=r_{ji},\ t_{ij}=-t_{ji}$, we infer

$$\begin{aligned}
& \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 w_i-2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}t_{ij}\\
\leq \;& \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M h u_i p_i(h)\\
& +\sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left\|\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right\|^2\\
& -\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)\delta(\lambda_i-\lambda_j)^2 r_{ij}^2-\sum_{i,j=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta}t_{ij}^2. \qquad \square
\end{aligned}$$

Using (3.48), we can obtain universal inequalities for eigenvalues of the problem (1.7) when M is a bounded domain in $\mathbb{R}^n$ or $\mathbb{S}^n(1)$.

Theorem 3.4 ([55]). *Let Ω be a bounded domain in $\mathbb{R}^n$ and Denote by λ_i the i-th eigenvelue of the eigenvalue problem:*

$$\begin{cases} (-\Delta)^l u = \lambda u \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.58}$$

Then we have

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq\ & \left(\frac{4l(n+2l-2)}{n^2}\right)^{1/2} \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{(l-1)/l}\right)^{1/2} \\ & \times \left(\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^{1/l}\right)^{1/2} \end{aligned} \tag{3.59}$$

Proof. Let $x_1, x_2, \cdots, x_n$ be the standard Euclidean coordinate functions of $\mathbb{R}^n$. Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i of the problem (3.58), $i = 1, \cdots;$ then

$$(-\Delta)^l (x_\alpha u_i) = \lambda_i x_\alpha u_i + 2l(-1)^l \nabla x_\alpha \cdot \nabla(\Delta^{l-1} u_i) \tag{3.60}$$

Taking $h = x_\alpha$ in (3.48), we infer for any $\delta > 0$ that

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq\ & \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega 2l(-1)^l x_\alpha u_i \nabla x_\alpha \cdot \nabla(\Delta^{l-1} u_i) \\ & + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|\nabla x_\alpha \cdot \nabla u_i\|^2. \end{aligned}$$

Summing over α, we have

$$\begin{aligned} & n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 & (3.61)\\ \leq\ & 2l\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{n}\int_\Omega(-1)^l x_\alpha u_i\nabla x_\alpha\cdot\nabla(\Delta^{l-1}u_i)\\ & +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega|\nabla u_i|^2. \end{aligned}$$

By induction, we infer

$$\int_\Omega u_i(-\Delta)^k u_i \leq \lambda_i^{k/l}, k=1,\cdots,l. \qquad (3.62)$$

Since

$$\Delta^{l-1}(x_\alpha u_i)=2(l-1)\nabla x_\alpha\cdot\nabla(\Delta^{l-2}u_i)+x_\alpha\Delta^{l-1}u_i,$$

we have

$$\begin{aligned} & \int_\Omega x_\alpha u_i\nabla x_\alpha\cdot\nabla(\Delta^{l-1}u_i) & (3.63)\\ =\ & \int_\Omega x_\alpha u_i\Delta^{l-1}\nabla x_\alpha\cdot\nabla u_i\\ =\ & \int_\Omega \Delta^{l-1}(x_\alpha u_i)\nabla x_\alpha\cdot\nabla u_i\\ =\ & \int_\Omega \left(2(l-1)\nabla x_\alpha\cdot\nabla(\Delta^{l-2}u_i)+x_\alpha\Delta^{l-1}u_i\right)\nabla x_\alpha\cdot\nabla u_i. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \int_\Omega x_\alpha u_i\nabla x_\alpha\cdot\nabla(\Delta^{l-1}u_i) & (3.64)\\ =\ & -\int_\Omega \Delta^{l-1}u_i\ \mathrm{div}(x_\alpha u_i\nabla x_\alpha)\\ =\ & -\int_\Omega \Delta^{l-1}u_i(|\nabla x_\alpha|^2u_i+x_\alpha\nabla x_\alpha\cdot\nabla u_i). \end{aligned}$$

Combining (3.63) and (3.64), we obtain

$$\begin{aligned}
&\int_{\Omega} x_\alpha u_i \nabla x_\alpha \cdot \nabla(\Delta^{l-1}u_i) \qquad (3.65)\\
= &\int_M \left\{(l-1)\nabla x_\alpha \cdot \nabla(\Delta^{l-2}u_i)\nabla x_\alpha \cdot \nabla u_i - \frac{1}{2}\Delta^{l-1}u_i|\nabla x_\alpha|^2 u_i\right\}
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{\alpha=1}^{n}\int_{\Omega}(-1)^l x_\alpha u_i \nabla x_\alpha \cdot \nabla(\Delta^{l-1}u_i) \qquad (3.66)\\
= &\int_{\Omega}(-1)^l\left\{(l-1)\nabla(\Delta^{l-2}u_i)\cdot\nabla u_i - \frac{n}{2}u_i\Delta^{l-1}u_i\right\}\\
= &\left(l-1+\frac{n}{2}\right)\int_{\Omega}u_i(-\Delta)^{l-1}u_i\\
\leq &\left(l-1+\frac{n}{2}\right)\lambda_i^{(l-1)/l}.
\end{aligned}$$

Substituting (3.62) and (3.66) into (3.61), one gets

$$\begin{aligned}
&n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \qquad (3.67)\\
\leq\ &l(n+2l-2)\delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{(l-1)/l} + \frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{1/l}.
\end{aligned}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{1/l}}{l(n+2l-2)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{(l-1)/l}}\right\}^{1/2},$$

we get (3.59). □

Let l be a positive integer and for $p = 0, 1, 2, ...,$ define the polynomials $F_p(t)$ inductively by

$$\begin{cases} F_0(t) = 1, \quad F_1(t) = t - n,\\ F_p(t) = (2t-2)F_{p-1}(t) - (t^2+2t-n(n-2))F_{p-2}(t), p = 2, \cdots. \end{cases}$$

$$\tag{3.68}$$

Set

$$F_l(t) = t^l - a_{l-1}t^{l-1} + \cdots + (-1)^{l-1}a_1 t + (-n)^l. \tag{3.69}$$

Theorem 3.4 ([55]). *Let λ_i be the i-th eigenvalue of the eigenvalue problem:*

$$\begin{cases} (-\Delta)^l u = \lambda u \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a compact domain in $\mathbb{S}^n(1)$. Then we have

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq \quad & \frac{1}{n}\left\{\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \left(a_{l-1}^{+}\lambda_i^{\frac{l-1}{l}} + \cdots + a_1^{+}\lambda_i^{\frac{1}{l}} + a_0^{+}\right)\right\}^{1/2} \\ & \times \left\{\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left(n^2 + 4\lambda_i^{1/2}\right)\right\}^{1/2}, \end{aligned} \tag{3.70}$$

where $a_j^{+} = \max\{0, a_j\}$.

Proof. As before, let $x_1, x_2, \cdots, x_{n+1}$ be the standard coordinate functions of $\mathbb{R}^{n+1}$; then

$$\mathbb{S}^n(1) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{\alpha=1}^{n+1} x_\alpha^2 = 1\}.$$

It is well known that

$$\Delta x_\alpha = -n x_\alpha, \quad \alpha = 1, \cdots, n+1. \tag{3.71}$$

Taking the Laplacian of the equation $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$ and using (3.71), we get

$$\sum_{\alpha=1}^{n+1} |\nabla x_\alpha|^2 = n. \tag{3.72}$$

Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i, $i = 1, 2, \cdots$. For any $\delta > 0$, by taking $h = x_\alpha$ in (3.48), we have

$$\begin{aligned} &\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2|\nabla x_\alpha|^2 \\ \leq\ & \delta\sum_{i=1}^{k+1}(\lambda_{k+1}-\lambda_i)^2\int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i)-\lambda_i x_\alpha u_i) \\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left\|\nabla x_\alpha\cdot\nabla u_i+\frac{u_i\Delta x_\alpha}{2}\right\|^2 \end{aligned}$$

Taking sum on α from 1 to $n+1$ and using (3.72), we get

$$\begin{aligned} &n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \qquad (3.73)\\ \leq\ & \delta\sum_{i=1}^{k+1}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i)-\lambda_i x_\alpha u_i) \\ &+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^{n+1}\left\|\nabla x_\alpha\cdot\nabla u_i+\frac{u_i\Delta x_\alpha}{2}\right\|^2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\sum_{\alpha=1}^{n+1}\left\|\nabla x_\alpha\cdot\nabla u_i+\frac{u_i\Delta x_\alpha}{2}\right\|^2 \qquad (3.74)\\ =\ & \int_\Omega\sum_{\alpha=1}^{n+1}\left((\nabla x_\alpha\cdot\nabla u_i)^2-n\nabla x_\alpha\cdot\nabla u_i u_i x_\alpha+\frac{n^2u_i^2x_\alpha^2}{4}\right) \\ =\ & \frac{n^2}{4}+\int_\Omega|\nabla u_i|^2 \\ \leq\ & \frac{n^2}{4}+\lambda_i^{1/l}. \end{aligned}$$

For any smooth functions f, g on Ω, we have from the Bochner formula that

$$\begin{aligned} & \Delta(\nabla f \cdot \nabla g) \\ = \; & 2\nabla^2 f \cdot \nabla^2 g + \nabla f \cdot \nabla(\Delta g) + \nabla g \cdot \nabla(\Delta f) \\ & +2(n-1)\nabla f \cdot \nabla g, \end{aligned} \tag{3.75}$$

where

$$\nabla^2 f \cdot \nabla^2 g = \sum_{s,t=1}^{n} \nabla^2 f(e_s, e_t) \nabla^2 g(e_s, e_t),$$

being $e_1, \cdots, e_n$ orthonormal vector fields locally defined on Ω. Since

$$\nabla^2 x_\alpha = -x_\alpha I,$$

we infer from (3.75) by taking $f = x_\alpha$ that

$$\begin{aligned} & \Delta(\nabla x_\alpha \cdot \nabla g) \\ = \; & -2x_\alpha \Delta g + \nabla x_\alpha \cdot \nabla(\Delta g) + (n-2)\nabla x_\alpha \cdot \nabla g \\ = \; & -2x_\alpha \Delta g + \nabla x_\alpha \cdot \nabla((\Delta + (n-2))g). \end{aligned} \tag{3.76}$$

For each $q = 0, 1, \cdots$, thanks to (3.71) and (3.76), there are polynomials B_q and C_q of degrees less than or equal to q such that

$$\Delta^q(x_\alpha g) = x_\alpha B_q(\Delta) g + 2\nabla x_\alpha \cdot \nabla(C_q(\Delta) g). \tag{3.77}$$

It is obvious that

$$B_0 = 1, \quad B_1 = t - n, \quad C_0 = 0, \quad C_1 = 1. \tag{3.78}$$

It follows from (3.71), (3.76) and (3.77) that

$$\begin{aligned} \Delta^q(x_\alpha g) &= \Delta(\Delta^{q-1}(x_\alpha g)) \\ &= \Delta(x_\alpha B_{q-1}(\Delta) g + 2\nabla x_\alpha \cdot \nabla(C_{q-1}(\Delta) g)) \\ &= x_\alpha((\Delta - n)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta))g \\ &\quad +2\nabla x_\alpha \cdot \nabla((B_{q-1}(\Delta) + (\Delta + (n-2))C_{q-1}(\Delta))g). \end{aligned} \tag{3.79}$$

Thus, for any $q = 2, \cdots$, we have

$$B_q(\Delta) = (\Delta - n)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta), \tag{3.80}$$

$$C_q(\Delta) = B_{q-1}(\Delta) + (\Delta + (n-2))C_{q-1}(\Delta). \tag{3.81}$$

Consequently,

$$\begin{aligned} & B_q(\Delta) & (3.82) \\ = \; & (2\Delta - 2)B_{q-1}(\Delta) - (\Delta + n - 2)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta) \\ = \; & (2\Delta - 2)B_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n-2))B_{q-2}(\Delta) \\ & +4\Delta[B_{q-2}(\Delta) + (\Delta + n - 2)C_{q-2}(\Delta) - C_{q-1}(\Delta)] \\ = \; & (2\Delta - 2)B_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n-2))B_{q-2}(\Delta), \quad q = 2, \cdots . \end{aligned}$$

Since (3.78) and (3.82) hold, we know that $B_q = F_q, \;\; \forall q = 0, 1, \cdots$. It follows from (3.77) and the divergence theorem that

$$\begin{aligned} & \int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) & (3.83) \\ = \; & \int_\Omega x_\alpha u_i \left((-1)^l \left(x_\alpha B_l(\Delta)u_i + 2\nabla x_\alpha \cdot \nabla(C_l(\Delta)u_i)\right) - \lambda_i x_\alpha u_i\right) \\ = \; & \int_\Omega x_\alpha u_i \left((-1)^l \left(x_\alpha(\Delta^l - a_{l-1}\Delta^{l-1} + \cdots + (-n)^l)u_i \right.\right. \\ & \left.\left. +2\nabla x_\alpha \cdot \nabla(C_l(\Delta)u_i)\right) - \lambda_i x_\alpha u_i\right) \\ = \; & \int_\Omega (-1)^l x_\alpha u_i \left(x_\alpha(-a_{l-1}\Delta^{l-1} + \cdots + (-n)^l)u_i + 2\nabla x_\alpha \cdot \nabla(C_l(\Delta)u_i)\right) \end{aligned}$$

Summing on α, one has

$$\begin{aligned} & \int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) & (3.84) \\ = \; & \int_\Omega u_i(-1)^l(-a_{l-1}\Delta^{l-1} + \cdots + (-n)^l a_0)u_i \\ = \; & a_{l-1}\int_\Omega u_i(-\Delta)^{l-1}u_i + \cdots + a_1\int_\Omega u_i(-\Delta)u_i + n^l\int_\Omega u_i^2 \\ \leq \; & a_{l-1}^+\lambda_i^{(l-1)/l} + \cdots + a_1^+\lambda_i^{1/l} + n^l. \end{aligned}$$

Substituting (3.74) and (3.84) into (3.73), we infer

$$\begin{aligned} & n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\ \leq\ & \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(a_{l-1}^{+}\lambda_i^{(l-1)/l}+\cdots+a_1^{+}\lambda_i^{1/l}+n^l) \\ & +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/l}+\frac{n^2}{4}\right). \end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/l}+\frac{n^2}{4}\right)}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(a_{l-1}^{+}\lambda_i^{(l-1)/l}+\cdots+a_1^{+}\lambda_i^{1/l}+n^l)}\right\}^{1/2},$$

we get (3.70). □

3.4 Eigenvalues of the Buckling Problem

Let $\Omega\subset\mathbb{R}^n$ and consider the problem

$$\begin{cases} \Delta^2 u=-\lambda\Delta u, \\ u|_{\partial\Omega}=\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0 \end{cases} \tag{3.85}$$

which is used to describe the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary.

Payne, Pólya and Weinberger [77] proved

$$\lambda_2/\lambda_1<3 \quad \text{for } \Omega\subset\mathbb{R}^2.$$

For $\Omega\subset\mathbb{R}^n$ this reads

$$\lambda_2/\lambda_1<1+4/n.$$

Subsequently Hile and Yeh [51] reconsidered this problem obtaining the improved bound

$$\frac{\lambda_2}{\lambda_1}\leq\frac{n^2+8n+20}{(n+2)^2} \qquad \text{for } \Omega\subset\mathbb{R}^n.$$

Ashbaugh [3] proved :

$$\sum_{i=1}^{n} \lambda_{i+1} \leq (n+4)\lambda_1. \tag{3.86}$$

This inequality has been improved to the following form [54]:

$$\sum_{i=1}^{n} \lambda_{i+1} + \frac{4(\lambda_2 - \lambda_1)}{n+4} \leq (n+4)\lambda_1.$$

Cheng and Yang introduced a new method to construct trial functions for the problem (3.85) and obtained the following universal inequality [28]:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4(n+2)}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \tag{3.87}$$

It has been proved in [88] that for the problem (1.5) if M is a bounded connected domain in an n-dimensional unit sphere, then the following inequality holds

$$\begin{aligned} & 2\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq \ & \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(\delta\lambda_i + \frac{\delta^2(\lambda_i - (n-2))}{4(\delta\lambda_i + n - 2)} \right) \\ & + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{(n-2)^2}{4} \right), \end{aligned} \tag{3.88}$$

where δ is any positive constant.

The inequality (3.87) has been generalized to eigenvalues of buckling problem of arbitrary orders. That is, we have

Theorem 3.5 ([54]). *Let $l \geq 2$ and let λ_i be the i-th eigenvalue of the following eigenvalue problem:*

$$\begin{cases} (-\Delta)^l u = -\lambda \Delta u \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.89}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$. Then for $k = 1, \cdots$, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \tag{3.90}$$
$$\leq \frac{2(2l^2+(n-4)l+2-n)^{1/2}}{n}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{(l-2)/(l-1)}\right\}^{1/2}$$
$$\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{1/(l-1)}\right\}^{1/2}.$$

Before proving theorem 3.5, let us recall a method of constructing trial functions developed by Cheng-Yang (Cf. [28], [54]). Let M be an n-dimensional complete submanifold in an m-dimensional Euclidean space $\mathbb{R}^m$. Denote by $\cdot$ the canonical metric on $\mathbb{R}^m$ as well as that induced on M. Let Δ and ∇ be the Laplacian and the gradient operator of M, respectively. Let Ω be a bounded connected domain of M with smooth boundary $\partial\Omega$ and let ν be the outward unit normal vector field of $\partial\Omega$. For functions f and g on Ω, the *Dirichlet inner product* $(f,g)_D$ of f and g is given by

$$(f,g)_D = \int_\Omega \nabla f \cdot \nabla g.$$

The Dirichlet norm of a function f is defined by

$$||f||_D = \{(f,f)_D\}^{1/2} = \left(\int_\Omega |\nabla f|^2\right)^{1/2}.$$

Consider the eigenvalue problem

$$\begin{cases} (-\Delta)^l u = -\lambda\Delta u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{l-1}u}{\partial\nu^{l-1}}\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.91}$$

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

denote the successive eigenvalues, where each eigenvalue is repeated according to its multiplicity.

Let u_i be the i-th orthonormal eigenfunction of the problem (3.91) corresponding to the eigenvalue λ_i, $i = 1, 2, \cdots$, that is, u_i satisfies

$$\begin{cases} (-\Delta)^l u_i = -\lambda_i \Delta u_i \quad \text{in} \quad \Omega, \\ u_i|_{\partial\Omega} = \frac{\partial u_i}{\partial \nu}\big|_{\partial\Omega} = \cdots \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\Big|_{\partial\Omega} = 0, \\ (u_i, u_j)_D = \int_\Omega \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}, \quad \forall\ i, j. \end{cases} \tag{3.92}$$

For $k = 1, \cdots, l$, let ∇^k denote the k-th covariant derivative operator on M, defined in the usual weak sense via an integration by parts formula. For a function f on Ω, the squared norm of $\nabla^k f$ is defined as

$$\left|\nabla^k f\right|^2 = \sum_{i_1, \cdots, i_k = 1}^{n} \left(\nabla^k f(e_{i_1}, \cdots, e_{i_k})\right)^2, \tag{3.93}$$

where $e_1, \cdots, e_n$ are orthonormal vector fields locally defined on Ω. Define the Sobolev space $H_l^2(\Omega)$ by

$$H_l^2(\Omega) = \{f : \ f, \ |\nabla f|, \cdots, \left|\nabla^l f\right| \in L^2(\Omega)\}.$$

Then $H_l^2(\Omega)$ is a Hilbert space with respect to the norm $||\cdot||_{l,2}$:

$$||f||_{l,2} = \left(\int_\Omega \left(\sum_{k=0}^{l} |\nabla^k f|^2\right)\right)^{1/2}. \tag{3.94}$$

Consider the subspace $H_{l,D}^2(\Omega)$ of $H_l^2(\Omega)$ defined by

$$H_{l,D}^2(\Omega) = \left\{ f \in H_l^2(\Omega) : \ f|_{\partial\Omega} = \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega} = \cdots \frac{\partial^{l-1} u}{\partial \nu^{l-1}}\bigg|_{\partial\Omega} = 0 \right\}.$$

The operator $(-\Delta)^l$ defines a self-adjoint operator acting on $H_{l,D}^2(\Omega)$ with discrete eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$ for the buckling problem (3.91) and the eigenfunctions $\{u_i\}_{i=1}^\infty$ defined in (3.92) form a complete orthonormal basis for the Hilbert space $H_{l,D}^2(\Omega)$. If $\phi \in H_{l,D}^2(\Omega)$ satisfies $(\phi, u_j)_D = 0$, $\forall j = 1, 2, \cdots, k$, then the Rayleigh-Ritz inequality tells us that

$$\lambda_{k+1} ||\phi||_D^2 \leq \int_\Omega \phi(-\Delta)^l \phi. \tag{3.95}$$

For vector-valued functions $F = (f_1, f_2, \cdots, f_m)$, $G = (g_1, g_2, \cdots, g_m)$: $\Omega \to \mathbb{R}^m$, we define an inner product (F, G) by

$$(F, G) = \int_\Omega \sum_{\alpha=1}^m f_\alpha g_\alpha.$$

The norm of F is given by

$$||F|| = (F, F)^{1/2} = \left\{ \int_\Omega \sum_{\alpha=1}^m f_\alpha^2 \right\}^{1/2}.$$

Let $\mathbf{H}_1^2(\Omega)$ be the Hilbert space of vector-valued functions given by

$$\begin{aligned} & \mathbf{H}_1^2(\Omega) \\ = & \left\{ F = (f_1, \cdots, f_m) : \Omega \to \mathbb{R}^m; f_\alpha, |\nabla f_\alpha| \in L^2(\Omega), \alpha = 1, \cdots, m \right\} \end{aligned}$$

with norm

$$||F||_1 = \left(||F||^2 + \int_\Omega \sum_{\alpha=1}^m |\nabla f_\alpha|^2 \right)^{1/2}.$$

Observe that a vector field on Ω can be regarded as a vector-valued function from Ω to $\mathbb{R}^m$. Let $\mathbf{H}_{1,D}^2(\Omega) \subset \mathbf{H}_l^2(\Omega)$ be a subspace of $\mathbf{H}_l^2(\Omega)$ spanned by the vector-valued functions $\{\nabla u_i\}_{i=1}^\infty$, which form a complete orthonormal basis of $\mathbf{H}_{1,D}^2(\Omega)$. For any $f \in H_{l,D}^2(\Omega)$, we have $\nabla f \in \mathbf{H}_{1,D}^2(\Omega)$ and for any $X \in \mathbf{H}_{1,D}^2(\Omega)$, there exists a function $f \in H_{l,D}^2(\Omega)$ such that $X = \nabla f$.

Proof of Theorem 3.5. With notations as above, we consider now the special case that Ω is a bounded domain in $\mathbb{R}^n$. Let us decompose the vector-valued functions $x_\alpha \nabla u_i$ as

$$x_\alpha \nabla u_i = \nabla h_{\alpha i} + W_{\alpha i}, \tag{3.96}$$

where $h_{\alpha i} \in H_{2,D}^l(\Omega)$, $\nabla h_{\alpha i}$ is the projection of $x_\alpha \nabla u_i$ in $\mathbf{H}_{1,D}^2(\Omega)$ and $W_{\alpha i} \perp \mathbf{H}_{1,D}^2(\Omega)$. Thus we have

$$W_{\alpha i}|_{\partial\Omega} = 0, \ \int_\Omega W_{\alpha i} \cdot \nabla u = 0, \ \forall\ u \in H_{l,D}^2(\Omega) \tag{3.97}$$

and from the discussions in [28] and [88] we know that

$$\text{div } W_{\alpha i} = 0. \tag{3.98}$$

For each $\alpha = 1, \cdots, n$, $i = 1, \cdots, k$, consider the functions $\phi_{\alpha i} : \Omega \to \mathbb{R}$, given by

$$\phi_{\alpha i} = h_{\alpha i} - \sum_{j=1}^{k} a_{\alpha ij} u_j, \tag{3.99}$$

where

$$a_{\alpha ij} = \int_\Omega x_\alpha \nabla u_i \cdot \nabla u_j = a_{\alpha ji}. \tag{3.100}$$

We have

$$\phi_{\alpha i}|_{\partial\Omega} = \left.\frac{\partial \phi_{\alpha i}}{\partial \nu}\right|_{\partial\Omega} = \cdots \left.\frac{\partial^{l-1} \phi_{\alpha i}}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0, \tag{3.101}$$

$$(\phi_{\alpha i}, u_j)_D = \int_\Omega \nabla \phi_{\alpha i} \cdot \nabla u_j = 0, \quad \forall j = 1, \cdots, k. \tag{3.102}$$

It follows from the Rayleigh-Ritz inequality that

$$\begin{aligned} & \lambda_{k+1} \int_\Omega |\nabla \phi_{\alpha i}|^2 \\ \leq \quad & \int_D \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i}, \alpha = 1, \cdots, n, i = 1, \cdots, k. \end{aligned} \tag{3.103}$$

It is easy to see that

$$(-\Delta)^l \phi_{\alpha i} = (-1)^l \Delta^{l-1} \left(u_{i,\alpha} + x_\alpha \Delta u_i\right) + \sum_{j=1}^{k} a_{\alpha ij} \lambda_j \Delta u_j$$

and so

$$
\begin{aligned}
& \int_\Omega \phi_{\alpha i}(-\Delta)^l \phi_{\alpha i} \qquad (3.104)\\
= & \int_\Omega \phi_{\alpha i}(-1)^l \Delta^{l-1}\left(u_{i,\alpha} + x_\alpha \Delta u_i\right)\\
= & \int_\Omega h_{\alpha i}(-1)^l \Delta^{l-1}\left(u_{i,\alpha} + x_\alpha \Delta u_i\right) - \sum_{j=1}^k a_{\alpha ij}\int_\Omega u_j(-\Delta)^l h_{\alpha i}\\
= & \int_\Omega \Delta h_{\alpha i}(-1)^l \Delta^{l-2}\left(u_{i,\alpha} + x_\alpha \Delta u_i\right) - \sum_{j=1}^k a_{\alpha ij}\int_\Omega h_{\alpha i}(-\Delta)^l u_j\\
= & \int_\Omega \Delta h_{\alpha i}(-1)^l((\Delta^{l-2}u_i)_{,\alpha} + \Delta^{l-2}(x_\alpha \Delta u_i)) + \sum_{j=1}^k \lambda_j a_{\alpha ij}\int_\Omega h_{\alpha i}\Delta u_j\\
= & \int_\Omega (-1)^l(u_{i,\alpha} + x_\alpha \Delta u_i)((\Delta^{l-2}u_i)_{,\alpha} + \Delta^{l-2}(x_\alpha \Delta u_i))\\
& -\sum_{j=1}^k \lambda_j a_{\alpha ij}\int_\Omega \langle \nabla h_{\alpha i}, \nabla u_j\rangle\\
= & \int_\Omega (-1)^l(u_{i,\alpha} + x_\alpha \Delta u_i)((2l-3)(\Delta^{l-2}u_i)_{,\alpha} + x_\alpha \Delta^{l-1}u_i))\\
& -\sum_{j=1}^k \lambda_j a_{\alpha ij}\int_\Omega \langle \nabla h_{\alpha i}, \nabla u_j\rangle\\
= & \int_\Omega (-1)^l((2l-3)\{u_{i,\alpha}(\Delta^{l-2}u_i)_{,\alpha} + x_\alpha \Delta u_i(\Delta^{l-2}u_i)_{,\alpha})\\
& +u_{i,\alpha}x_\alpha \Delta^{l-1}u_i + x_\alpha^2 \Delta u_i \Delta^{l-1}u_i\} - \sum_{j=1}^k \lambda_j a_{\alpha ij}^2
\end{aligned}
$$

Since

$$\Delta^{l-1}(x_\alpha u_i) = 2(l-1)(\Delta^{l-2}u_i)_{,\alpha} + x_\alpha \Delta^{l-1}u_i,$$

we have

$$\begin{aligned}\int_\Omega x_\alpha u_i(\Delta^{l-1}u_i)_{,\alpha} &= \int_\Omega x_\alpha u_i \Delta^{l-1}u_{i,\alpha} \\ &= \int_\Omega \Delta^{l-1}(x_\alpha u_i)u_{i,\alpha} \\ &= \int_\Omega \left(2(l-1)(\Delta^{l-2}u_i)_{,\alpha} + x_\alpha \Delta^{l-1}u_i\right) u_{i,\alpha}.\end{aligned} \tag{3.105}$$

On the other hand, it holds

$$\int_\Omega x_\alpha u_i(\Delta^{l-1}u_i)_{,\alpha} = -\int_\Omega \Delta^{l-1}u_i(u_i + x_\alpha u_{i,\alpha}). \tag{3.106}$$

Combining (3.105) and (3.106), we obtain

$$\begin{aligned}&\int_\Omega x_\alpha u_i(\Delta^{l-1}u_i)_{,\alpha} \\ = &\int_M \left\{(l-1)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha} - \frac{1}{2}u_i\Delta^{l-1}u_i\right\}\end{aligned} \tag{3.107}$$

Hence

$$\begin{aligned}\int_\Omega x_\alpha u_{i,\alpha}\Delta^{l-1}u_i &= -\int_\Omega u_i(\Delta^{l-1}u_i + x_\alpha(\Delta^{l-1}u_i)_{,\alpha}) \\ &= -\int_M \left\{(l-1)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha} + \frac{1}{2}u_i\Delta^{l-1}u_i\right\}\end{aligned} \tag{3.108}$$

and consequently, we have

$$\begin{aligned}\int_\Omega x_\alpha \Delta u_i(\Delta^{l-2}u_i)_{,\alpha} &= \int_\Omega x_\alpha \Delta u_i \Delta^{l-2}u_{i,\alpha} \\ &= \int_\Omega \Delta^{l-2}(x_\alpha \Delta u_i)u_{i,\alpha} \\ &= \int_\Omega u_{i,\alpha}\left(2(l-2)(\Delta^{l-2}u_i)_{,\alpha} + x_\alpha\Delta^{l-1}u_i\right) \\ &= \int_\Omega \left\{(l-3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha} - \frac{1}{2}u_i\Delta^{l-1}u_i\right\}.\end{aligned} \tag{3.109}$$

Also, one has

$$\begin{aligned} & \int_\Omega u_i x_\alpha^2 \Delta u_i \\ = & -\int_\Omega x_\alpha^2 |\nabla u_i|^2 - 2\int_\Omega x_\alpha u_i u_{i,\alpha} \\ = & -\int_\Omega x_\alpha^2 |\nabla u_i|^2 + \int_\Omega u_i^2, \end{aligned} \tag{3.110}$$

$$\begin{aligned} & \int_\Omega x_\alpha^2 \Delta u_i \Delta^{l-1} u_i \\ = & \int_\Omega u_i \Delta(x_\alpha^2 \Delta^{l-1} u_i) \\ = & \int_\Omega u_i (2\Delta^{l-1} u_i + x_\alpha^2 \Delta^l u_i + 4x_\alpha (\Delta^{l-1} u_i)_{,\alpha}) \\ = & \int_\Omega u_i (2\Delta^{l-1} u_i + (-1)^{l-1} \lambda_i x_\alpha^2 \Delta u_i + 4x_\alpha (\Delta^{l-1} u_i)_{,\alpha}). \end{aligned} \tag{3.111}$$

Combining (3.107), (3.110) and (3.111), we get

$$\begin{aligned} \int_\Omega x_\alpha^2 \Delta u_i \Delta^{l-1} u_i = & \; 4(l-1)\int_\Omega (\Delta^{l-2} u_i)_{,\alpha} u_{i,\alpha} \\ & + (-1)^{l-1}\lambda_i \left\{ -\int_\Omega x_\alpha^2 |\nabla u_i|^2 + \int_\Omega u_i^2 \right\}. \end{aligned} \tag{3.112}$$

Substituting (3.109), (3.111) and (3.112) into (3.104), one gets

$$\begin{aligned} & \int_\Omega \phi_{\alpha i} (-\Delta)^l \phi_{\alpha i} \\ = & \int_\Omega (-1)^l \left\{ (-l+1) u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{,\alpha} u_{i,\alpha} \right\} \\ & + \lambda_i \left\{ \int_\Omega x_\alpha^2 |\nabla u_i|^2 - \int_\Omega u_i^2 \right\} - \sum_{j=1}^k \lambda_j a_{\alpha ij}^2. \end{aligned} \tag{3.113}$$

It is easy to see that

$$||x_\alpha \nabla u_i||^2 = ||\nabla h_{\alpha i}||^2 + ||W_{\alpha i}||^2 \tag{3.114}$$

and

$$||\nabla h_{\alpha i}||^2 = ||\nabla \phi_{\alpha i}||^2 + \sum_{j=1}^{k} a_{\alpha ij}^2, \tag{3.115}$$

Combining (3.103), (3.113), (3.114) and (3.115), we infer

$$\begin{aligned} & (\lambda_{k+1} - \lambda_i)||\nabla \phi_{\alpha i}||^2 && (3.116)\\ \leq \; & \int_\Omega (-1)^l \left\{(-l+1)u_i\Delta^{l-1}u_i + (2l^2-4l+3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha}\right\} \\ & -\lambda_i(||u_i||^2 - ||W_{\alpha i}||^2) + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{\alpha ij}^2, \end{aligned}$$

Observe that $\nabla(x_\alpha u_i) = u_i \nabla x_\alpha + x_\alpha \nabla u_i \in \mathbf{H}^2_{1,D}(\Omega)$. For $A_{\alpha i} = \nabla(x_\alpha u_i - h_{\alpha i})$, we have

$$u_i \nabla x_\alpha = A_{\alpha i} - W_{\alpha i} \tag{3.117}$$

and so

$$||u_i||^2 = ||u_i \nabla x_\alpha||^2 = ||W_{\alpha i}||^2 + ||A_{\alpha i}||^2.$$

Because of $(\nabla u_{i,\alpha}, W_{\alpha i}) = 0$, it follows that

$$\begin{aligned} 2||u_{i,\alpha}||^2 &= -2\int_\Omega A_{\alpha i} \cdot \nabla u_{i,\alpha} \\ &\leq \lambda_i^{1/(l-1)}||A_{\alpha i}||^2 + \frac{1}{\lambda_i^{1/(l-1)}}||\nabla u_{i,\alpha}||^2 \end{aligned}$$

which gives

$$-\lambda_i ||A_{\alpha i}||^2 \leq -2\lambda_i^{\frac{l-2}{l-1}}||u_{i,\alpha}||^2 + \lambda_i^{\frac{l-3}{l-1}}||\nabla u_{i,\alpha}||^2 \tag{3.118}$$

Introducing (3.118) into (3.116), we get

$$\begin{aligned} & (\lambda_{k+1} - \lambda_i)||\nabla \phi_{\alpha i}||^2 && (3.119)\\ \leq \; & \int_\Omega (-1)^l \left\{(-l+1)u_i\Delta^{l-1}u_i + (2l^2-4l+3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha}\right\} \\ & -2\lambda_i^{(l-2)/(l-1)}||u_{i,\alpha}||^2 + \lambda_i^{(l-3)/(l-1)}||\nabla u_{i,\alpha}||^2 \\ & + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{\alpha ij}^2, \end{aligned}$$

Since

$$\begin{aligned}
& -2\int_\Omega x_\alpha \nabla u_i \cdot \nabla u_{i,\alpha} \\
= \;& 2\int_\Omega u_{i,\alpha}^2 + 2\int_\Omega x_\alpha u_{i,\alpha}\Delta u_i \\
= \;& 2\int_\Omega u_{i,\alpha}^2 + 2\int_\Omega u_i\Delta(x_\alpha u_{i,\alpha}) \\
= \;& 2\int_\Omega u_{i,\alpha}^2 + 2\int_\Omega u_i x_\alpha(\Delta u_i)_{,\alpha} + 4\int_\Omega u_i\nabla x_\alpha \cdot \nabla u_{i,\alpha} \\
= \;& 2\int_\Omega u_{i,\alpha}^2 - 2\int_\Omega \Delta u_i(u_i + x_\alpha u_{i,\alpha}) - 4\int_\Omega u_{i,\alpha}\mathrm{div}(u_i\nabla x_\alpha) \\
= \;& 2\int_\Omega u_{i,\alpha}^2 + 2 - 2\int_\Omega x_\alpha u_{i,\alpha}\Delta u_i - 4\int_\Omega u_{i,\alpha}^2 \\
= \;& -2\int_\Omega u_{i,\alpha}^2 + 2 + 2\int_\Omega \nabla u_i \cdot \nabla(x_\alpha u_{i,\alpha}) \\
= \;& 2 + 2\int_\Omega x_\alpha \nabla u_i \cdot \nabla u_{i,\alpha},
\end{aligned}$$

we have

$$-2\int_\Omega x_\alpha \nabla u_i \cdot \nabla u_{i,\alpha} = 1. \tag{3.120}$$

Set

$$d_{\alpha ij} = \int_\Omega \nabla u_{i,\alpha} \cdot \nabla u_j;$$

then $d_{\alpha ij} = -d_{\alpha ji}$ and we get

$$\begin{aligned}
1 \;&=\; -2\int_\Omega x_\alpha \nabla u_i \cdot \nabla u_{i,\alpha} \qquad (3.121)\\
&=\; -2\int_\Omega \nabla h_{\alpha i} \cdot \nabla u_{i,\alpha} \\
&=\; -2\int_\Omega \nabla \phi_{\alpha i} \cdot \nabla u_{i,\alpha} - 2\sum_{j=1}^k a_{\alpha ij} d_{\alpha ij}.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(1+2\sum_{j=1}^{k} a_{\alpha ij}d_{\alpha ij}\right) \qquad (3.122)\\
= \quad &(\lambda_{k+1}-\lambda_i)^2\left(-2\nabla\phi_{\alpha i}, \left(\nabla u_{i,\alpha}-\sum_{j=1}^{k} d_{\alpha ij}\nabla u_j\right)\right)\\
\leq \quad &\delta(\lambda_{k+1}-\lambda_i)^3||\nabla\phi_{\alpha i}||^2+\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\left(||\nabla u_{i,\alpha}||^2-\sum_{j=1}^{k} d_{\alpha ij}^2\right),
\end{aligned}
$$

where δ is any positive constant. Substituting (3.119) into (3.122), we get

$$
\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(1+2\sum_{j=1}^{k} a_{\alpha ij}d_{\alpha ij}\right)\\
\leq \quad &\delta(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega(-1)^l\{(-l+1)u_i\Delta^{l-1}u_i\right.\\
&+(2l^2-4l+3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha}\}\\
&\left.-2\lambda_i^{\frac{l-2}{l-1}}||u_{i,\alpha}||^2+\lambda_i^{\frac{l-3}{l-1}}||\nabla u_{i,\alpha}||^2+\sum_{j=1}^{k}(\lambda_i-\lambda_j)a_{\alpha ij}^2\right)\\
&+\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\left(||\nabla u_{i,\alpha}||^2-\sum_{j=1}^{k} d_{\alpha ij}^2\right),
\end{aligned}
$$

Summing on i from 1 to k, we infer

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 - 2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)a_{\alpha ij}d_{\alpha ij} \\
\leq\ & \delta\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega(-1)^l\{(-l+1)u_i\Delta^{l-1}u_i\right.\right. \\
&+(2l^2-4l+3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha}\} \\
&\left.-2\lambda_i^{(l-2)/(l-1)}||u_{i,\alpha}||^2+\lambda_i^{(l-3)/(l-1)}||\nabla u_{i,\alpha}||^2\right) \\
&\left.-\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{\alpha ij}^2\right) \\
&+\frac{1}{\delta}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)||\nabla u_{i,\alpha}||^2-\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)d_{\alpha ij}^2\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
\leq\ & \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega(-1)^l\left\{(-l+1)u_i\Delta^{l-1}u_i\right.\right. \\
&\left.+(2l^2-4l+3)(\Delta^{l-2}u_i)_{,\alpha}u_{i,\alpha}\right\} \\
&\left.-2\lambda_i^{\frac{l-2}{l-1}}||u_{i,\alpha}||^2+\lambda_i^{\frac{l-3}{l-1}}||\nabla u_{i,\alpha}||^2\right) \\
&+\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)||\nabla u_{i,\alpha}||^2,
\end{aligned}$$

Taking sum for α from 1 to n, we get

$$\begin{aligned}
& n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
\leq\ & \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega(-1)^l\left\{n(-l+1)u_i\Delta^{l-1}u_i\right.\right. \\
& \left.+(2l^2-4l+3)\nabla(\Delta^{l-2}u_i)\cdot\nabla u_i\right\} \\
& \left.-2\lambda_i^{\frac{l-2}{l-1}}+\lambda_i^{\frac{l-3}{l-1}}\sum_{\alpha=1}^{n}||\nabla u_{i,\alpha}||^2\right) \\
& +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^{n}||\nabla u_{i,\alpha}||^2 \\
=\ & \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(-2\lambda_i^{\frac{l-2}{l-1}}+\lambda_i^{\frac{l-3}{l-1}}\sum_{\alpha=1}^{n}||\nabla u_{i,\alpha}||^2\right. \\
& \left.+(2l^2+(n-4)l+3-n)\int_\Omega u_i(-\Delta)^{l-1}u_i\right) \\
& +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^{n}||\nabla u_{i,\alpha}||^2.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{\alpha=1}^{k}||\nabla u_{i,\alpha}||^2 &= -\int_\Omega\sum_{\alpha=1}^{k}u_{i,\alpha}\Delta u_{i,\alpha} \\
&= -\int_\Omega\sum_{\alpha=1}^{k}u_{i,\alpha}(\Delta u_i)_{,\alpha} \\
&= \int_\Omega\sum_{\alpha=1}^{k}u_{i,\alpha\alpha}\Delta u_i \\
&= \int_\Omega(\Delta u_i)^2 \\
&= \int_\Omega u_i\Delta^2u_i,
\end{aligned}$$

where $u_{i,\alpha\alpha} = \frac{\partial^2 u_i}{\partial x_\alpha^2}$. Therefore,

$$\begin{aligned} & n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\ \leq\ & \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(-2\lambda_i^{\frac{l-2}{l-1}}+\lambda_i^{\frac{l-3}{l-1}}\int_\Omega u_i\Delta^2 u_i\right. \\ & \left.+(2l^2+(n-4)l+3-n)\int_\Omega u_i(-\Delta)^{l-1}u_i\right) \\ & +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_\Omega u_i\Delta^2 u_i. \end{aligned}$$

Observe that

$$\int_\Omega u_i(-\Delta)^{l-1}u_i \leq \lambda_i^{\frac{l-2}{l-1}}, \quad \int_\Omega u_i\Delta^2 u_i \leq \lambda_i^{\frac{1}{l-1}}.$$

Thus we have

$$\begin{aligned} & n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\ \leq\ & \delta(2l^2+(n-4)l+2-n)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-2}{l-1}} \\ & +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l-1}}. \end{aligned}$$

Taking

$$\delta = \frac{\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l-1}}\right\}^{1/2}}{\left\{(2l^2+(n-4)l+2-n)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{\frac{l-2}{l-1}}\right\}^{1/2}},$$

we get (3.90). □

Recently, Cheng-Yang have strengthened the inequality (3.87) to the following form:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4\left(n+\frac{4}{3}\right)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i. \tag{3.123}$$

The inequality (3.90) has been improved by Cheng-Qi-Wang-Xia very recently in [24]:

$$\begin{aligned} & n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\ \leq\ & \sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2\left(2l^2+\left(n-\frac{14}{3}\right)l+\frac{8}{3}-n\right)\lambda_i^{\frac{l-2}{l-1}} \\ & +\sum_{i=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\lambda_i^{\frac{1}{l-1}}, \end{aligned} \tag{3.124}$$

where $\{\delta_i\}_{i=1}^{k}$ is any positive non-increasing monotone sequence.

For eigenvalues of the buckling problem on spherical domains, the inequality (3.88) has also been improved (Cf. [28])) and generalized to buckling problem of arbitrary orders (Cf. [54], [24]).

Chapter 4

Pólya Conjecture and Related Results

4.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and consider the eigenvalue problem of the Dirichlet Laplacian on Ω:

$$\begin{cases} \Delta u + \lambda u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

In 1960, Pólya [81] showed that for any "plane covering domain" Ω in $\mathbb{R}^2$ (those that tile $\mathbb{R}^2$) the Weyl asymptotic relation (1.) is in fact a one-sided inequality (his proof also works for $\mathbb{R}^n$-covering domains) and conjectured, for any bounded domain $\Omega \subset \mathbb{R}^n$, the inequality

$$\lambda_k \geq C(n) \left(\frac{k}{|\Omega|} \right)^{2/n} \quad \forall k, \text{ with } C(n) = \frac{(2\pi)^2}{\omega_n^{\frac{2}{n}}}. \tag{4.2}$$

Pólya's conjecture has been a central problem about eigenvalues and many important developments have been made during the past years. In 1982, Li-Yau [72] showed the lower bound

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nkC(n)}{n+2} \left(\frac{k}{|\Omega|} \right)^{2/n} \tag{4.3}$$

which yields an individual lower bound on λ_k in the form

$$\lambda_k \geq \frac{nC(n)}{n+2} \left(\frac{k}{|\Omega|} \right)^{2/n}. \tag{4.4}$$

Similar bounds for eigenvalues with Neumann boundary conditions have been proved in [62], [63] and [65]. It was pointed out in [66] that (4.3) also follows from an earlier result by Berezin [11] by the Legendre transformation. The inequality (4.3) has been improved by Melas:

Theorem 4.1 ([75]). *For any bounded domain $\Omega \subset \mathbb{R}^n$ and any $k \geq 1$, the eigenvalues of the problem (4.1) satisfy the inequality*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nkC(n)}{n+2} \left(\frac{k}{|\Omega|} \right)^{2/n} + d(n)k\frac{|\Omega|}{I(\Omega)} \tag{4.5}$$

where the constant d_n depends only on the dimension and $I(\Omega) = \min_{a\in\mathbf{R}^n} \int_\Omega |x-a|^2 dx$ is the "moment of inertia" of Ω.

Proof. Fix a $k \geq 1$ and let $u_1, \cdots, u_k$ be an orthonormal set of eigenfunctions of (4.1) corresponding to the set of eigenvalues $\lambda_1, \cdots, \lambda_k$. We consider the Fourier transform of each eigenfunction

$$f_j(\xi) = \hat{u}_j(\xi) = (2\pi)^{-n/2} \int_\Omega u_j(x) e^{ix\cdot\xi} dx. \tag{4.6}$$

¿From Plancherel's Theorem, we know that $f_1, \cdots, f_k$ is an orthonormal set in $\mathbb{R}^n$. Bessel's inequality implies that for every $\xi \in \mathbb{R}^n$

$$\sum_{j=1}^{k} |f_j(\xi)|^2 \leq (2\pi)^{-n} \int_\Omega |e^{ix\cdot\xi}| dx = (2\pi)^{-n} |\Omega| \tag{4.7}$$

and

$$\sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \leq (2\pi)^{-n} \int_\Omega |ixe^{ix\cdot\xi}| dx = (2\pi)^{-n} I(\Omega). \tag{4.8}$$

Since $u_j|_{\partial\Omega} = 0$ it is easy to see that

$$\int_{\mathbb{R}^n} |\xi|^2 |f_j(\xi)|^2 d\xi = \int_\Omega |\nabla u_j|^2 = \lambda_j. \tag{4.9}$$

Setting

$$F(\xi) = \sum_{j=1}^{k} |f_j(\xi)|^2; \tag{4.10}$$

then $0 \leq F(\xi) \leq (2\pi)^{-n}|\Omega|$,

$$\begin{aligned} |\nabla F(\xi)| &\leq 2(\sum_{j=1}^{k} |f_j(\xi)|^2)^{1/2} (\sum_{j=1}^{k} |\nabla f_j(\xi)|^2)^{1/2} \\ &\leq 2(2\pi)^{-n}\sqrt{|\Omega| I(\Omega)}, \ \forall \xi \in \mathbb{R}^n, \end{aligned} \tag{4.11}$$

$$\int_{\mathbb{R}^n} F(\xi) d\xi = k \tag{4.12}$$

and

$$\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{j=1}^{k} \lambda_j. \tag{4.13}$$

Let $F^*(\xi) = \phi(|\xi|)$ denote the decreasing radial rearrangement of F. By approximating F we may assume that the function $\phi : [0; +\infty) \to [0, (2\pi)^{-n}|\Omega|]$ is absolutely continuous. Setting $\mu(t) = |\{F^* > t\}| = |\{F > t\}|$ the co-area formula (1.17) implies that

$$\mu(t) = \int_t^{(2\pi)^{-n}|\Omega|} \int_{\{F=s\}} |\nabla F|^{-1} d\sigma_s ds. \tag{4.14}$$

Observe that F^* is radial and so $\mu(\phi(s)) = |\{F^* > \phi(s)\}| = \omega_n s^n$ which gives $n\omega_n s^{n-1} = \mu'(\phi(s))\phi'(s)$ a.e. It follows from (4.11), (4.14) and the isoperimetric inequality that

$$\begin{aligned} -\mu'(\phi(s)) &= \int_{\{F=\phi(s)\}} |\nabla F|^{-1} d\sigma_s \\ &\geq \rho^{-1} |\{F = \phi(s)\}| \\ &\geq \rho^{-1} n\omega_n s^{n-1} \end{aligned} \tag{4.15}$$

and so for almost every s

$$-\rho \le \phi'(s) \le 0, \tag{4.16}$$

where

$$\rho = 2(2\pi)^{-n}\sqrt{|\Omega| I(\Omega)}.$$

From (4.12) and (4.13), we have

$$\begin{aligned} k = \int_{\mathbb{R}^n} F(\xi) d\xi &= \int_{\mathbb{R}^n} F^*(\xi) d\xi \\ &= n\omega_n \int_0^\infty s^{n-1}\phi(s) ds \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \sum_{j=1}^k \lambda_j &= \int_{\mathbb{R}^n} |xi|^2 F(\xi) d\xi \\ &\ge \int_{\mathbb{R}^n} |xi|^2 F^*(\xi) d\xi \\ &= n\omega_n \int_0^\infty s^{n+1}\phi(s) ds. \end{aligned} \tag{4.18}$$

We need an elementary lemma.

Lemma 4.1([75]) *Let* $n \ge 1, \rho, A > 0$ *and* $\psi : [0, +\infty) \to [0, +\infty)$ *be decreasing (and absolutely continuous) such that*

$$-\rho \le \phi'(s) \le 0, \ \int_0^\infty s^{n-1}\psi(s) ds = A. \tag{4.19}$$

Then

$$\int_0^\infty s^{n+1}\psi(s) ds \ge \frac{1}{n+2}(nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}} + \frac{A\psi(0)^2}{6(n+2)\rho^2}. \tag{4.20}$$

Applying Lemma 4.1 to the function ϕ with $A = (n\omega_n)^{-1}k, \ \rho = 2(2\pi)^{-n}\sqrt{|\Omega| I(\Omega)}$ we get in view of (4.12) and (4.13) that

$$\sum_{j=1} \lambda_j \ge \frac{n}{n+2}\omega_n^{-\frac{2}{n}} k^{\frac{n+2}{n}} \phi(0)^{-\frac{2}{n}} + \frac{ck\phi(0)^2}{(n+2)\rho^2} \tag{4.21}$$

where c is any constant such that $0 < c < \frac{1}{6}$. Observe that $0 < \phi(0) \leq (2\pi)^{-n}|\Omega|$ and that if R is such that $\omega_n R^n = |\Omega|$, then $I(\Omega) \geq \int_{B(R)} |x|^2 dx = \frac{n\omega_n R^{n+2}}{n+2}$ and so

$$\begin{aligned} \rho &\geq 2(2\pi)^{-n}\sqrt{\frac{n}{n+2}\omega_n^{-\frac{2}{n}}|\Omega|^{\frac{n+2}{n}+1}} \\ &\geq (2\pi)^{-n}\omega_n^{-\frac{1}{n}}|\Omega|^{\frac{n+1}{n}}. \end{aligned} \tag{4.22}$$

Let us choose c independent of n satisfying $c < (2\pi)^2\omega_n^{-\frac{1}{n}}$. It is easy to see that the function

$$g(t) = \frac{n}{n+2}\omega_n^{-\frac{2}{n}}k^{\frac{n+2}{n}}t^{-\frac{2}{t}} + \frac{ckt^2}{(n+2)\rho^2}$$

is decreasing on $(0, (2\pi)^{-n}|\Omega|]$. We can replace $\phi(0)$ by $(2\pi)^{-n}|\Omega|$ in (4.21) which gives the inequality (4.5). $\square$

4.2 The Kröger's Theorem

Let λ_k be the k^{th} eigenvalue for the Neumann boundary value problem with respect to the Laplace operator on a bounded domain with piecewise smooth boundary in $\mathbb{R}^n$. Pólys conjecture states that $\lambda_k \leq C_n \left(\frac{k}{|\Omega|}\right)^{\frac{2}{n}}$. With respect to this conjecture, Kröger proved the following result.

Theorem 4.2 ([62]). *The first $k+1$ Neumann eigenvalues of a a bounded domain Ω with piecewise smooth boundary in $\mathbb{R}^n$ satisfy the inequality*

$$\lambda_{k+1} \leq inf_{r>2\pi\left(\frac{nk}{\alpha_{n-1}|\Omega|}\right)^{1/n}} \frac{\frac{r^{n+2}}{n+2}(\alpha_{n-1}|\Omega|) - (2\pi)^n \sum_{j=1}^k \lambda_j}{\frac{r^n}{n}\alpha_{n-1}|\Omega| - (2\pi)^n k} \tag{4.24}$$

where α_{n-1} denotes the area of the $(n-1)$-unit sphere in $\mathbb{R}^n$.

Taking

$$r = 2\pi\left(\frac{n(k+1)}{\alpha_{n-1}|\Omega|}\right)^{1/n},$$

one gets

Corollary 4.1. *Under the assumptions of the Theorem the following inequality holds for every k:*

$$\sum_{j=1}^{k} \lambda_j \leq \frac{n}{n+2}(2\pi)^2 \left(\frac{1}{n}\alpha_{n-1}|\Omega|\right)^{-\frac{2}{n}} k^{\frac{n+2}{n}} \tag{4.25}$$

Proof of Theorem 4.2. Let $\{\phi_j\}_{j=1}^k$ be the set of orthonormal eigenfunctions for the eigenvalues $\lambda_1, ..., \lambda_k$. Fix $z \in \mathbb{R}^n$ and consider $h_z : \mathbb{R} \to \mathbb{C}$ given by $h_z(y) = e^{iy\cdot z}$. Letting $a_j = \int_\Omega h_z(x)u_j(x)dx$; then the projection of $h_z(y) \equiv e^{iz\cdot y}$ into the subspace of $L^2(\Omega)$ spanned by $\phi_1, ..., \phi_k$ can be written as

$$\begin{aligned} h_z(y)|_{span\{u_i\}_{i=1}^k} &= \sum_{j=1}^{k}\left(\int_\Omega h_z(x)u_j(x)dx\right)u_j(y) \quad (4.26) \\ &= \sum_{j=1}^{k} a_j u_j(y). \end{aligned}$$

It is clear that the function $g_z(y) \equiv h_z(y) - \sum_{j=1}^k a_j u_j(y)$ is orthogonal to $u_j, j = 1, \cdots, k$. Thus we have

$$\lambda_{k+1} \leq \frac{\int_\Omega |\nabla g_z(y)|^2 dy}{\int_\Omega g_z^2 dy}. \tag{4.27}$$

Elementary computation sows that

$$\int_\Omega |\nabla g_z(y)|^2 dy = |z|^2|\Omega| - \sum_{j=1}^{k} |a_j|^2 \lambda_j \tag{4.28}$$

and

$$\int_\Omega g_z^2 dy = |\Omega| - \sum_{j=1}^{k} |a_j|^2. \tag{4.29}$$

Assuming that $r > 2\pi\left(\frac{nk}{\alpha_{n-1}|\Omega|}\right)^{1/n}$; then

$$\lambda_{k+1} \leq \frac{\int_{B_r}\left(\int_\Omega |\nabla g_z(y)|^2 dy\right) dz}{\int_{B_r}\left(\int_\Omega g_z(y)^2 dy\right) dz} \tag{4.30}$$

Since

$$\begin{aligned} & \int_{B_r}\left(\int_\Omega |\nabla g_z(y)|^2 dy\right) dz \\ = & \int_{B_r} |z|^2|\Omega| dz - \int_{B_r}\sum_{j=1}^{k}\lambda_j |a_j|^2 dz \\ = & \frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - \sum_{j=1}^{k}\lambda_j\int_{B_r}|a_j|^2 dz \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} & \int_{B_r}\left(\int_\Omega (g_z(y))^2 dy\right) dz \\ = & \int_{B_r} |\Omega| dz - \int_{B_r}\sum_{j=1}^{k}|a_j|^2 dz \\ = & \frac{r^{n}\alpha_{n-1}|\Omega|}{n} - \sum_{j=1}^{k}\int_{B_r}|a_j|^2 dz, \end{aligned} \tag{4.32}$$

we get

$$\lambda_{k+1} \leq \frac{\frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - \sum_{j=1}^{k}\lambda_j\int_{B_r}|a_j|^2 dz}{\frac{r^{n}\alpha_{n-1}|\Omega|}{n} - \sum_{j=1}^{k}\int_{B_r}|a_j|^2 dz}. \tag{4.33}$$

Recall that the Fourier transform $\hat{u_j}$ of u_j is given by

$$\hat{u_j}(z) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{iy\cdot z}u_j(y)dy$$

and we know from the Plancherel Theorem that $\int_{\mathbb{R}^n}|\widehat{u_j}|^2(z)dz =$

$\int_{\mathbb{R}^n} |u_j|^2(y)dy$. Hence we have

$$\begin{aligned}\int_{B_r}\left|\int_\Omega h_z u_j dy\right|^2 dz &\leq \int_{\mathbb{R}^n}\left|\int_\Omega h_z u_j dy\right|^2 dz\\ &= (2\pi)^n \int_{\mathbb{R}^n} |\widehat{u_j}(z)|^2 dz\\ &= (2\pi)^n \int_{\mathbb{R}^n} |u_j(y)|^2 dy\\ &= (2\pi)^n.\end{aligned}$$

Let us prove by induction that

$$\lambda_{k+1} \leq \frac{\frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - (2\pi)^n \sum_{j=1}^k \lambda_j}{\frac{r^n\alpha_{n-1}|\Omega|}{n} - (2\pi)^n k}. \tag{4.34}$$

Suppose that (4.34) is true for $k-1$, that is

$$\lambda_k \leq \frac{\frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - (2\pi)^n \sum_{j=1}^{k-1} \lambda_j}{\frac{r^n\alpha_{n-1}|\Omega|}{n} - (2\pi)^n (k-1)}.$$

We then have that

$$\lambda_k \leq \frac{\frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - (2\pi)^n \sum_{j=1}^k \lambda_j}{\frac{r^n\alpha_{n-1}|\Omega|}{n} - (2\pi)^n k} = \frac{A}{B},$$

where

$$A = \frac{r^{n+2}\alpha_{n-1}|\Omega|}{n+2} - (2\pi)^n \sum_{j=1}^k \lambda_j$$

and

$$B = \frac{r^n\alpha_{n-1}|\Omega|}{n} - (2\pi)^n k.$$

Setting $C_l = (2\pi)^n - \int_{B_r} |a_l|^2 dz$ and observing

$$\frac{r^n\alpha_{n-1}|\Omega|}{n} - (2\pi)^n k > 0,$$

we get

$$\begin{aligned}\lambda_{k+1} &\leq \frac{A+\sum_{l=1}^{k}\lambda_l C_l}{B+\sum_{l=1}^{k}C_l}\\ &\leq \frac{A+\lambda_k\sum_{l=1}^{k}C_l}{B+\sum_{l=1}^{k}C_l}\\ &\leq \frac{A+\frac{A}{B}\sum_{l=1}^{k}C_l}{B+\sum_{l=1}^{k}C_l}\\ &= \frac{A}{B}.\end{aligned}$$

This completes the proof of Theorem 4.2. □

4.3 A generalized Pólya conjecture by Cheng-Yang

In [31], Cheng-Yang investigated eigenvalues of the Dirichlet Laplacian on a bounded domain in an n-dimensional complete Riemannian manifold M and proposed a generalized Pólya conjecture.

Cheng-Yang's Conjecture ([31]). *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant $c(M,\Omega)$, which only depends on M and Ω such that eigenvalue λ_i's of the eigenvalue problem*

$$\begin{cases}\Delta u=-\lambda u \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0.\end{cases} \tag{4.35}$$

satisfy

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i+c(M,\Omega)\geq\frac{n}{n+2}\frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}k^{\frac{2}{n}},\ k=1,\cdots, \tag{4.36}$$

$$\lambda_i+c(M,\Omega)\geq\frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}k^{\frac{2}{n}},\ k=1,\cdots, \tag{4.37}$$

Remark 4.1. Cheng-Yang believed that in the above conjecture, when M is the unit sphere $\mathbb{S}^n(1), c(M,\Omega) = \frac{n^2}{4}$, when M is the hyperbolic space $\mathbb{H}^n(-1), c(M,\Omega) = -\frac{(n-1)^2}{4}$ and when M is a complete minimal submanifold in $\mathbb{R}^N, c(M,\Omega) = 0$. Cheng-Yang obtained a partial solution to the above conjecture.

Theorem 4.3 ([31]). *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant $H^2 > 0$, which only depends on M and Ω such that eigenvalue λ_i's of the problem (4.35) satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}k^{\frac{2}{n}},\ k = 1,\cdots, \quad (4.38)$$

Moreover, when M is the unit sphere $\mathbb{S}^n(1)$, one can take $H_0^2 = \frac{n^2}{4}$ and when M is a complete minimal submanifold in $\mathbb{R}^N$, one can take $H_0^2 = 0$.

A crucial result in the proof of Theorem 4.3 is the following

Lemma 4.2 ([25]). *Let $\mu_1 \leq \cdots \leq \mu_{k+1}$ be any non-negative real numbers satisfying*

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i). \quad (4.39)$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i,\ T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2,\ F_k = \left(1+\frac{2}{t}\right)G_k^2 - T_k. \quad (4.40)$$

Then, we have

$$F_{k+1} \leq C(t,k)\left(\frac{k+1}{k}\right)^{\frac{4}{t}}F_k, \quad (4.41)$$

where t is any positive real number and

$$C(t,k) = 1 - \frac{1}{3t}\left(\frac{k}{k+1}\right)^{\frac{4}{t}} \frac{\left(1+\frac{2}{t}\right)\left(1+\frac{4}{t}\right)}{(k+1)^3} < 1. \tag{4.42}$$

Proof of Theorem 4.3. From Nashs theorem, we know that M can be isometricly immersed into a Euclidean space $\mathbb{R}^N$, that is, there exists an isometric immersion:

$$\phi : M \to \mathbb{R}^N. \tag{4.43}$$

Thus, M can be seen as a complete submanifold isometricly immersed into $\mathbb{R}^N$. We denote by $|H|$ the mean curvature of the immersion ϕ. From (3.16), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i + \frac{n^2}{4}\sup_{\Omega}|H|^2). \tag{4.44}$$

Since eigenvalues are invariants of isometries, the above inequality holds for any isometric immersion from M into a Euclidean space. Let us define

$$\Phi = \{\phi; \phi \text{ is an isometric immersion from } M \text{ into a Euclidean space}\}.$$

Putting

$$H_0 = \inf_{\phi\in\Phi}\sup_{\Omega}|H|^2;$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i + \frac{n^2}{4}H_0^2). \tag{4.45}$$

Letting $\mu_i = \lambda_i + \frac{n^2}{4}H_0^2$; then

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i). \tag{4.46}$$

From theorem 2.1 with $t = n$ of [25], we have

$$F_{k+1} \leq C(n,k)\left(\frac{k+1}{k}\right)^{\frac{1}{n}} F_k \leq \left(\frac{k+1}{k}\right)^{\frac{1}{n}} F_k.$$

Therefore, we infer

$$\frac{F_{k+1}}{(k+1)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

For any positive integers l and k, we have

$$\frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

From the Weyl's asymptotic formula (1.10)

$$\lim_{l\to\infty} \frac{\lambda_l}{l^{\frac{2}{n}}} = \frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}},$$

we get

$$\lim_{l\to\infty} \frac{\frac{1}{\lambda_l}\sum_{i=1}^{l}\lambda_i}{l^{\frac{2}{n}}} = \frac{n}{n+2}\frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}$$

and

$$\lim_{l\to\infty} \frac{\frac{1}{\lambda_l}\sum_{i=1}^{l}\lambda_i^2}{l^{\frac{4}{n}}} = \frac{n}{n+4}\frac{16\pi^4}{(\omega_n|\Omega|)^{\frac{4}{n}}}.$$

Hence

$$\lim_{l\to\infty} \frac{F_{k+l}}{(k+l)^{\frac{4}{n}}} = \frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n|\Omega|)^{\frac{4}{n}}}.$$

According to (4.46), we have, for any positive integer k,

$$\frac{F_k}{k^{\frac{4}{n}}} \geq \frac{2n}{(n+2)(n+4)}\frac{16\pi^4}{(\omega_n|\Omega|)^{\frac{4}{n}}}. \tag{4.47}$$

Since

$$F_k = \left(1 + \frac{2}{n}\right) G_k^2 - T_k \leq \frac{2}{n} G_k^2,$$

we get

$$\frac{2}{n}\frac{G_k^2}{k^{\frac{4}{n}}} \geq \frac{F_k}{k^{\frac{4}{n}}} \geq \frac{2n}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n|\Omega|)^{\frac{4}{n}}}$$

which implies (4.38).

In order to finish the proof of Theorem 4.3, we need only to observe that $\mathbb{S}^n(1)$ can be seen as a compact hypersurface in $\mathbb{R}^{n+1}$ with mean curvature 1 and that a complete minimal submanifold in $\mathbb{R}^N$ has mean curvature $|H| = 0$. $\square$

4.4 Another generalized Pólya conjecture

Let Ω be a bounded domain in $\mathbb{R}^n$ and let L be the elliptic operator of order $2t$ defined by

$$Lu = \sum_{m=r+1}^{t} a_{m-r}(-\Delta)^m u, \quad u \in C^\infty(\Omega)$$

where $r \geq 0$ is an integer, a_{m-r}'s are constants with $a_{m-r} \geq 0$, $r+1 \leq m \leq t$, $a_{t-r} = 1$, t a fixed positive integer. Consider the following eigenvalue problem about L which is important in the study of various branches of mathematics, such as differential equations, differential geometry and mathematical physics:

$$\begin{cases} Lu = \lambda(-\Delta)^r u, & u \in C^\infty(\Omega), \\ (\partial/\partial\nu)^j u|_{\partial\Omega} = 0, & j = 0, 1, 2, \cdots, t-1. \end{cases} \tag{4.48}$$

Let

$$0 < \lambda_{1,r} \leq \lambda_{2,r} \leq \cdots \leq \lambda_{k,r} \leq \cdots \to \infty. \tag{4.49}$$

be the eigenvalues of the problem (4.48). In [58], the following conjecture was proposed :

Generalized Pólya Conjecture. *The eigenvalues* $\lambda_{k,r}$, $k = 1, 2, \cdots$, *of the eigenvalue problem (1.7) of the operator* L *satisfies the inequality*

$$\lambda_{k,r} \geq \sum_{m=1}^{t-r} a_m C^m \left(\frac{k}{|\Omega|} \right)^{2m/n}. \tag{4.50}$$

With respect to the above generalized Pólya conjecture, Ku-Ku-Tang showed in [58] that if r is even, then

$$\lambda_{k,r} \geq \sum_{m=1}^{t-r} \frac{n a_m}{n+2m} C^m \left(\frac{k}{|\Omega|} \right)^{2m/n}, \quad k = 1, 2, \cdots. \tag{4.51}$$

This section provides comparison theorems between the k-th eigenvalues of the problem (4.1) and that of the problem (4.48) which shows that if the Pólya conjecture (4.2) is true then so is the generalized Pólya conjecture (4.50).

Theorem 4.3 ([93]). *Let* M *be an* $n(\geq 2)$*-dimensional compact Riemannian manifold with boundary. Denote by* Δ *the Laplacian operator on* M *and let* L *be the elliptic operator given by*

$$Lu = \sum_{m=r+1}^{t} a_{m-r} (-\Delta)^m u, \ u \in C^\infty(M), \tag{4.52}$$

where $r \geq 0$ *is an integer,* a_{m-r}*'s are constants with* $a_{m-r} \geq 0$, $r + 1 \leq m \leq t$, $a_{t-r} = 1$, t *a fixed positive integer. Consider the following eigenvalue problems:*

$$\begin{cases} Lu = \lambda(-\Delta)^r u, & u \in C^\infty(M), \\ (\partial/\partial\nu)^j u|_{\partial M} = 0, & j = 0, 1, 2, \cdots, t-1, \end{cases} \tag{4.53}$$

$$\begin{cases} (-\Delta)^{r+1} u = \Lambda(-\Delta)^r u, & u \in C^\infty(M), \\ (\partial/\partial\nu)^j u|_{\partial M} = 0, & j = 0, 1, 2, \cdots, r, \end{cases} \tag{4.54}$$

$$\begin{cases} -\Delta u = \lambda u \quad \text{in} \quad M, \\ u|_{\partial M} = 0. \end{cases} \tag{4.55}$$

Denote by

$$0 < \lambda_{1,r} \le \lambda_{2,r} \le \cdots \to \infty \tag{4.56}$$

$$0 < \Lambda_{1,r} \le \Lambda_{2,r} \le \cdots \to \infty \tag{4.57}$$

and

$$0 < \lambda_1 < \lambda_2 \le \cdots \to \infty \tag{4.58}$$

the successive eigenvalues for (4.53), (4.54) and (4.55), respectively. Here each eigenvalue is repeated according to its multiplicity. Then for any $k = 1, 2, \cdots$, we have

$$\lambda_{k,r} \ge a_1\Lambda_{k,r} + a_2\Lambda_{k,r}^2 + \cdots + a_{t-(r+1)}\Lambda_{k,r}^{t-(r+1)} + \Lambda_{k,r}^{t-r} \tag{4.59}$$

and

$$\Lambda_{k,r} \ge \lambda_k. \tag{4.60}$$

Remark 1.1. If M is a bounded domain in $\mathbb{R}^n$, we know from the inequality

$$\lambda_{k,r} \ge a_1\lambda_k + a_2\lambda_k^2 + \cdots + a_{t-(r+1)}\lambda_k^{t-(r+1)} + \lambda_k^{t-r} \tag{4.61}$$

which is a combination of (4.59) and (4.60) that if the Pólya conjecture (4.2) is true then so is the generalized Pólya conjecture (4.50).

Proof of Theorem 4.3. Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions of the problem (4.53) corresponding to $\{\lambda_{i,r}\}_{i=1}^k$, that is,

$$\begin{cases} Lu_i = \lambda_{i,r}(-\Delta)^r u_i, \text{ in } M, \\ u_i|_{\partial M} = \cdots = \left.\frac{\partial^{t-1}u_i}{\partial\nu^{t-1}}\right|_{\partial M} = 0, \\ \int_M u_i(-\Delta)^r u_j = \delta_{ij}, \ i, j = 1, \cdots, k. \end{cases}$$

Similarly, let $\{v_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions of the problem (4.54) corresponding to $\{\Lambda_{i,r}\}_{i=1}^k$, that is,

$$\begin{cases} (-\Delta)^{r+1}v_i = \Lambda_{i,r}(-\Delta)^r v_i \quad \text{in} \quad M, \\ v_i|_{\partial M} = \cdots = \left.\frac{\partial^{r}v_i}{\partial\nu^{r}}\right|_{\partial M} = 0, \\ \int_M v_i(-\Delta)^r v_j = \delta_{ij}, \ i, j = 1, \cdots, k. \end{cases}$$

Let $w = \sum_{j=1}^{k} \alpha_j u_j \neq 0$ be such that

$$\int_M w(-\Delta)^r v_j = 0, \quad \forall j = 1, \cdots, k-1. \tag{4.62}$$

Such an element w exists because $\{\alpha_j | 1 \leq j \leq k\}$ is a non-trivial solution of a system of $(k-1)$-linear equations

$$\sum_{j=1}^{k} \alpha_j \int_M u_j (-\Delta)^r v_i = 0, \quad 1 \leq i \leq k-1, \tag{4.63}$$

in k unknowns. Notice that if $u \in C^\infty(M)$ satisfies

$$u|_{\partial M} = \frac{\partial u}{\partial \nu}\bigg|_{\partial M} = \cdots = \frac{\partial^{t-1} u}{\partial \nu^{t-1}}\bigg|_{\partial M} = 0,$$

then

$$\begin{aligned} u|_{\partial M} &= \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots = \\ &= \Delta^{p-1} u\big|_{\partial M} = \nabla(\Delta^{p-1} u)\big|_{\partial M} = 0, \qquad \text{when} \quad t = 2p \end{aligned}$$

and

$$\begin{aligned} u|_{\partial M} &= \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots = \Delta^{k-1} u\big|_{\partial M} \\ &= \nabla(\Delta^{p-1} u)\big|_{\partial M} = \Delta^p u|_{\partial M} = 0, \qquad \text{when} \quad t = 2p+1. \end{aligned}$$

Observe that $\int_M w(-\Delta)^r w \neq 0$. In fact, from divergence theorem, we have

$$\int_M w(-\Delta)^r w = \begin{cases} \int_M (\Delta^{r/2} w)^2, & \text{if } r \text{ is even}, \\ \int_M \left|\nabla(\Delta^{(r-1)/2} w)\right|^2, & \text{if } r \text{ is odd}. \end{cases}$$

Thus, if $\int_M w(-\Delta)^r w = 0$, then $\Delta^{r/2} w = 0$ when r is even and $\Delta^{(r-1)/2} w = 0$ when r is odd. It then follows from the maximum principle for harmonic functions that $\Delta^{r/2-1} w = 0$ when r is even and $\Delta^{(r-1)/2-1} w = 0$ when r is odd. Continuing this process, we conclude that $w = 0$. This is a contradiction. Thus $\int_M w(-\Delta)^r w \neq 0$. Let us assume without loss of generality that

$$\int_M w(-\Delta)^r w = 1. \tag{4.64}$$

Hence we infer from the Rayleigh-Ritz inequality that

$$\Lambda_{k,r} \leq \int_M w(-\Delta)^{r+1} w \tag{4.65}$$

We *claim* that for any $j = 1, \cdots, t-r$,

$$\left(\int_M w(-\Delta)^{r+j} w\right)^{j+1} \leq \left(\int_M w(-\Delta)^{r+j+1} w\right)^{j}. \tag{4.66}$$

Let us first prove that (4.66) holds when $j = 1$. In fact, if $r = 2h$ is even, then one deduces from the divergence theorem and the Hölder's inequality that

$$\begin{aligned} & \left(\int_M w(-\Delta)^{r+1} w\right)^2 \\ = & \left(\int_M (-\Delta)^h w(-\Delta)^{h+1} w\right)^2 \\ \leq & \left(\int_M ((-\Delta)^h w)^2\right)\left(\int_M ((-\Delta)^{h+1} w)^2\right) \\ = & \int_M w(-\Delta)^{r+2} w. \end{aligned}$$

On the other hand, if $r = 2h+1$ is odd, then

$$\begin{aligned}
&\left(\int_M w(-\Delta)^{r+1} w\right)^2 \\
= &\left(\int_M ((-\Delta)^{h+1} w)(-\Delta)((-\Delta)^h w)\right)^2 \\
= &\left(\int_M \nabla((-\Delta)^{h+1} w)\nabla((-\Delta)^h w)\right)^2 \\
\leq &\left(\int_M |\nabla((-\Delta)^{h+1} w|^2\right)\left(\int_M |\nabla((-\Delta)^h w)|^2\right) \\
= &\left(\int_M (-\Delta)^h w(-\Delta)^{h+1} w\right)\left(\int_M (-\Delta)^{h+1} w(-\Delta)^{h+2} w\right) \\
= &\left(\int_M w(-\Delta)^r w\right)\left(\int_M w(-\Delta)^{2h+3} w\right) \\
= &\int_M w(-\Delta)^{r+2} w.
\end{aligned}$$

Thus (4.66) holds when $j = 1$. Suppose now that (4.66) holds for $j-1$, that is

$$\left(\int_M w(-\Delta)^{r+j-1} w\right)^j \leq \left(\int_M w(-\Delta)^{r+j} w\right)^{j-1}. \tag{4.67}$$

When $r+j$ is even, we have

$$\begin{aligned}
&\int_M w(-\Delta)^{r+j} w \qquad (4.68)\\
= &\int_M \Delta^{(r+j)/2-1} w \Delta\left(\Delta^{(r+j)/2} w\right) \\
= &-\int_M \nabla\left(\Delta^{(r+j)/2-1} u_i\right)\nabla\left(\Delta^{(r+j)/2} u_i\right) \\
\leq &\left(\int_M \left|\nabla\left(\Delta^{(r+j)/2-1} w\right)\right|^2\right)^{1/2}\left(\int_M \left|\nabla\left(\Delta^{(r+j)/2} w\right)\right|^2\right)^{1/2} \\
= &\left(\int_M w(-\Delta)^{r+j-1} w\right)^{1/2}\left(\int_M w(-\Delta)^{r+j+1} w\right)^{1/2},
\end{aligned}$$

On the other hand, when $r+j$ is odd,

$$\begin{aligned} & \int_M w(-\Delta)^{r+j} w \\ = & \int_M (-\Delta)^{(r+j-1)/2} w(-\Delta)^{(r+j+1)/2} w \\ \leq & \left(\int_M \left((-\Delta)^{(r+j-1)/2} w\right)^2\right)^{1/2} \left(\int_M \left((-\Delta)^{(r+j+1)/2} w\right)^2\right)^{1/2} \\ = & \left(\int_M w(-\Delta)^{r+j-1} w\right)^{1/2} \left(\int_M w(-\Delta)^{r+j+1} w\right)^{1/2}. \end{aligned} \tag{4.69}$$

Thus we always have

$$\begin{aligned} & \int_M w(-\Delta)^{r+j} w \\ \leq & \left(\int_M (-\Delta)^{r+j-1} w\right)^{1/2} \left(\int_M w(-\Delta)^{r+j+1} w\right)^{1/2}. \end{aligned} \tag{4.70}$$

Combining (4.67) and (4.70), we know that (4.66) is true for j. Using (4.67) repeatedly, we get

$$\int_\Omega w(-\Delta)^{r+1} w \leq \left(\int_M w(-\Delta)^{r+s} w\right)^{1/s}, \ s=1,\cdots,t-(r+1).$$

which, combining with (4.65), gives

$$\Lambda_{k,r}^s \leq \int_M w(-\Delta)^{r+s} w, \quad s=1,2,\cdots,t-(r+1).$$

Thus we have

$$\begin{aligned} & a_1\Lambda_{k,r} + a_2\Lambda_{k,r}^2 + \cdots + a_{t-(r+1)}\Lambda_{k,r}^{t-(r+1)} + \Lambda_{k,r}^{t-r} \\ \leq & \int_M w(a_1(-\Delta)^{r+1} + a_2(-\Delta)^{r+2} + \cdots + (-\Delta)^t) w \\ = & \int_M wLw = \sum_{i,j=1}^k \eta_i\eta_j \int_M u_i L u_j \\ = & \sum_{i,j=1}^k \eta_i\eta_j \int_M u_i \lambda_{j,r}(-\Delta)^r u_j = \sum_{i=1}^k \eta_i^2 \lambda_{i,r} \leq \lambda_{k,r}, \end{aligned}$$

where in the last equality, we have used the fact that

$$\sum_{i=1}^{k} \eta_i^2 = \int w(-\Delta)^r w = 1.$$

This proves (4.59).

In order to prove (4.60), let us take a set of orthonormal eigenfunctions $\{z_i\}_{i=1}^k$ of the problem (4.55) corresponding to $\{\lambda_i\}_{i=1}^k$, that is,

$$\begin{cases} \Delta z_i = -\lambda_i z_i \quad \text{in} \quad M, \\ z_i|_{\partial M} = 0, \\ \int_M z_i z_j = \delta_{ij}, \ i, j = 1, \cdots, k. \end{cases} \tag{4.71}$$

Let $\xi = \sum_{j=1}^k \beta_j v_j$ be such that

$$\int_M \xi^2 = 1 \ \text{ and } \ \int_M \xi z_j = 0, \quad \forall j = 1, \cdots, k-1. \tag{4.72}$$

It follows from the Rayleigh-Ritz inequality that

$$\lambda_k \le \int_M \xi(-\Delta \xi) \tag{4.73}$$

Using the same arguments as in the proof of (4.59), we have

$$\left(\int_M \xi(-\Delta)^j \xi\right)^{j+1} \le \left(\int_M \xi(-\Delta)^{j+1} \xi\right)^j, \quad j = 1, \cdots, r. \tag{4.74}$$

Thus we have

$$\lambda_k \le \left(\int_M \xi(-\Delta)^r \xi\right)^{\frac{1}{r}} = \left(\sum_{j=1}^k \beta_j^2\right)^{\frac{1}{r}} \tag{4.75}$$

On the other hand, taking $j = r$ in (4.73), we get

$$\left(\int_M \xi(-\Delta)^r \xi\right)^{r+1} \le \left(\int_M \xi(-\Delta)^{r+1} \xi\right)^r, \tag{4.76}$$

which implies that

$$\begin{aligned}
\sum_{j=1}^{k}\beta_j^2 &\leq \left(\int_M \xi(-\Delta)^{r+1}\xi\right)^{\frac{r}{r+1}} \qquad (4.77)\\
&= \left(\int_M \sum_{i,j=1}^{k}\beta_i\beta_j v_i(-\Delta)^{r+1}v_j\right)^{\frac{r}{r+1}}\\
&= \left(\int_M \sum_{i,j=1}^{k}\beta_i\beta_j v_i\Lambda_{j,r}(-\Delta)^{r}v_j\right)^{\frac{r}{r+1}}\\
&= \left(\sum_{j=1}^{k}\Lambda_{j,r}\beta_j^2\right)^{\frac{r}{r+1}}\\
&\leq \Lambda_{k,r}^{\frac{r}{r+1}}\left(\sum_{j=1}^{k}\beta_j^2\right)^{\frac{r}{r+1}}.
\end{aligned}$$

Thus we have

$$\sum_{j=1}^{k}\beta_j^2 \leq \Lambda_{k,r}^{r}. \qquad (4.78)$$

Combining (4.75) and (4.78), one gets (4.60). □

Chapter 5

The Steklov eigenvalue problems

5.1 Introduction

Let M be an n-dimensional compact Riemannian manifold with boundary. The Stekloff problem is to find a solution of the equation

$$\begin{cases} \Delta u = 0 \text{ in } M, \\ \frac{\partial u}{\partial \nu} = pu \text{ on } \partial M, \end{cases} \tag{5.1}$$

where p is a real number. This problem was first introduced by Steklov for bounded domains in the plane in [87]. His motivation came from physics. The function u represents the steady state temperature on M such that the flux on the boundary is proportional to the temperature. Problem (5.1) is also important in conductivity and harmonic analysis as it was initially studied by Calderón (Cf. [15]). This connection arises because the set of eigenvalues for the Steklov problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map. This map associates to each function u defined on the boundary ∂M, the normal derivative of the harmonic function on M with boundary data u. The Steklov eigenvalue problem has appeared in quite a few physical fields, such as fluid mechanics, electromagnetism, elasticity, etc., and received increasing attention [53], [64]. It has applications, for instance, in the investigation of surface waves [12], the analysis of stability of mechanical oscillators immersed in a viscous fluid [32], and the study of the vibration modes of a structure in contact with an incompressible fluid [13]. Numerical methods have been developed for this problem. For instance, its optimal error estimates of linear finite element approximations have been obtained in [2]. Interesting estimates for eigenvalues of the Steklov problem have been obtained some of which will be introduced in this chapter.

5.2 Estimates for the Steklov eigenvalues

In this section, we prove some estimates for the Steklov eigenvalues. Let us recall firstly the Reilly's formula. Let M be n-dimensional compact manifold M with boundary ∂M. We denote by $\langle , \rangle$ the Riemannian metric on M as well as that induced on ∂M. Let ∇ and Δ be the connection and the Laplacian on M, respectively. Let ν be the unit outward normal vector of ∂M. The shape operator of ∂M is given by $S(X) = \nabla_X \nu$ and the second fundamental form of ∂M is defined as $II(X,Y) = \langle S(X), Y\rangle$, here $X, Y \in T\partial M$. The eigenvalues of S are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H = \frac{1}{n-1}\mathrm{tr}\, S$, here $\mathrm{tr}\, S$ denotes the trace of S. For a smooth function f defined on an n-dimensional compact manifold M with boundary ∂M, the following identity holds if $u = \frac{\partial f}{\partial \nu}\Big|_{\partial M}$, $z = f|_{\partial M}$ and Ric denotes the Ricci tencor of M, then (see [82], p. 46):

$$\begin{aligned} &\int_M \left((\Delta f)^2 - |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f)\right) \\ = &\int_{\partial M} \left(((n-1)Hu + 2\overline{\Delta} z)u + II(\overline{\nabla} z, \overline{\nabla} z)\right). \end{aligned} \tag{5.2}$$

Here $\nabla^2 f$ is the Hessian of f; $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

Theorem 5.1 ([35]). *Let M be an n-dimensional compact connected Riemannian manifold with non-negative Ricci curvature and* *boundary. Assume that the principal curvatures of ∂M are bounded from below by a positive constant c. Then the first non-zero eigenvalue of the problem (5.1) satisfies $p_1 > \frac{c}{2}$.*

Proof. Let f be the first eigenfunction of the Steklov problem (5.1). Setting $z = f|_{\partial M}, u = \frac{\partial u}{\partial \nu}\big|_{\partial M}$ and using Reilly's formula we find after integration by parts that

$$0 > -\int_M |\nabla^2 f|^2 \geq -2\int_{\partial M} \overline{\nabla} z \cdot \overline{\nabla} u + c\int_{\partial M} |\overline{\nabla} z|^2.$$

$$0 > -2p_1 \int_{\partial M} |\overline{\nabla} z|^2 + c\int_{\partial M} |\overline{\nabla} z|^2. \tag{5.3}$$

Since $z \neq$ constant, otherwise f=constant on M which is a contradiction, we have

$$\int_{\partial M} |\overline{\nabla} z|^2 > 0. \tag{5.4}$$

Thus (5.3) implies that $p_1 > c/2$.

In view of Theorem 5.1, Escobar conjectured that under the same conditions as in Theorem 5.1, $p_1 \geq c$.

Theorem 5.2. *Let the conditions be as in Theorem 5.1. Then we have*

i) The non-zero eigenvalue of the Laplacian of ∂M satisfies

$$\lambda_1 \geq (n-1)c^2 \tag{5.5}$$

with equality holding if and only if M is isometric to a ball of radius $1/c$ in $\mathbb{R}^n$ ([102]).

ii) The non-zero eigenvalue of the Steklov problem (5.1) satisfies

$$p_1 \leq \frac{\sqrt{\lambda_1}}{(n-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right) \tag{5.6}$$

Moreover, the equality holds in (5.6) if and only if M is isometric to a ball of radius $1/c$ in $\mathbb{R}^n$ ([89]).

Proof. Let z be an eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the Laplacian of ∂M:

$$\overline{\Delta} z + \lambda_1 z = 0.$$

Let f in $C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta f = 0 \text{ in } M, \\ f|_{\partial M} = z. \end{cases} \tag{5.7}$$

It then follows from (5.2) and the nonnegativity of the Ricci curvature of M that

$$\begin{aligned} 0 &\geq \int_M \left((\Delta f)^2 - |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f)\right) \qquad (5.8) \\ &= \int_{\partial M} \left(((n-1)Hh + 2\overline{\Delta} z)h + II(\overline{\nabla} z, \overline{\nabla} z)\right) \\ &\geq \int_{\partial M} (-2(-\lambda_1 z)u + (n-1)cu^2 + c|\nabla z|^2) \\ &\geq \int_{\partial M} (2\lambda_1 zu + (n-1)cu^2 + c\lambda_1 z^2) \\ &\geq \int_{\partial M} \left\{(n-1)c\left(u + \frac{\lambda_1 z}{(n-1)c}\right)^2 + \left(c\lambda_1 - \frac{\lambda_1^2}{(n-1)c}\right) z^2\right\} \\ &\geq \int_{\partial M} \left\{\left(c\lambda_1 - \frac{\lambda_1^2}{(n-1)c}\right) z^2\right\}. \end{aligned}$$

Thus we have

$$c\lambda_1 - \frac{\lambda_1^2}{(n-1)c} \leq 0$$

or

$$\lambda_1 \geq (n-1)c^2.$$

If M is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$, it is well known that $\lambda_1(M) = (n-1)c^2$. Now we assume conversely that $\lambda_1(M) = (n-1)c^2$. In this case the inequalities in (5.8) must take equality sign. In particular, we have

$$\nabla^2 f = 0, \ H = c, \ u = -\frac{\lambda_1 z}{(n-1)c} = -cz. \tag{5.9}$$

From $\nabla^2 f = 0$, we know that $|\nabla f|^2$ is a constant and is not zero since f is not a constant. Without loss of generality, we can assume $|\nabla f|^2 = 1$. Thus for any point $p \in M$, we have

$$1 = |\nabla f|^2(p) = |\nabla z^2|(p) + u^2(p). \tag{5.10}$$

It follows from (5.10) by integration and $u = -cz$, that

$$\begin{aligned} A(\partial M) &= \int_{\partial M} (|\nabla z|^2 + u^2) \\ &= \int_{\partial M} (\lambda_1 z^2 + u^2) \\ &= \int_{\partial M} (nu^2). \end{aligned} \tag{5.11}$$

On the other hand, from

$$\frac{1}{2}\Delta(f^2) = |\nabla f|^2 + f\Delta f = 1 \tag{5.12}$$

and the divergence theorem we have

$$V(M) = \int_M \frac{1}{2}\Delta(f^2) = -\int_{\partial M} zu = \int_{\partial M} \frac{u^2}{c},$$

which, combining with (5.11), gives

$$H = c = \frac{1}{n} \cdot \frac{A(\partial M)}{V(M)}. \tag{5.13}$$

It then follows from a result in [83] that M is isometric to a ball in $\mathbb{R}^n$. Since $\lambda_1(\partial M) = (n-1)c^2$, the radius of M is easily seen to be $\frac{1}{c}$. This proves item i) of Theorem 5.1.

Now let us prove (5.6). Let f and z be as in the proof of item i). We have from the Rayleigh-Ritz inequality that (Cf. [59])

$$p_1 \leq \frac{\int_{\partial M} h^2}{\int_M |\nabla f|^2} \tag{5.14}$$

and

$$p_1 \leq \frac{\int_M |\nabla f|^2}{\int_{\partial M} z^2}, \tag{5.15}$$

which gives

$$p_1^2 \leq \frac{\int_{\partial M} h^2}{\int_{\partial M} z^2}. \tag{5.16}$$

It then follows by substituting f into the Reilly's formula that

$$\begin{aligned} 0 &\geq \int_M \left((\Delta f)^2 - |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f)\right) \quad (5.17) \\ &\geq (n-1)c\int_{\partial M} h^2 - 2\lambda_1 \int_{\partial M} hz + c\int_{\partial M} |\overline{\nabla} z|^2 \\ &= (n-1)c\int_{\partial M} h^2 - 2\lambda_1 \int_{\partial M} hz + c\lambda_1 \int_{\partial M} z^2 \\ &\geq (n-1)c\int_{\partial M} h^2 - 2\lambda_1 \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}} + c\lambda_1 \int_{\partial M} z^2. \end{aligned}$$

Hence, we have

$$\left(\int_{\partial M} h^2\right)^{\frac{1}{2}} \leq \frac{\sqrt{\lambda_1}}{(n-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right)\left(\int_{\partial M} z^2\right)^{\frac{1}{2}}.$$

Hence

$$p_1 \leq \frac{\sqrt{\lambda_1}}{(n-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right).$$

Assume now that

$$p_1 = \frac{\sqrt{\lambda_1}}{(n-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right).$$

Then the inequalities in (5.17) should take equality sign. We infer therefore

$$\nabla^2 f = 0, \qquad H = c \tag{5.18}$$

and

$$h = \frac{\sqrt{\lambda_1}}{(n-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right) z. \tag{5.19}$$

Take a local orthonormal fields $\{e_i\}_{i=1}^{n-1}$ tangent to ∂M. We infer from (5.18) and (5.19) that

$$\begin{aligned} 0 &= \sum_{i=1}^{n-1} \nabla^2 f(e_i, e_i) = \overline{\Delta} z + (n-1)Hh \qquad (5.20) \\ &= -\lambda_1 z + (n-1)c \cdot \frac{\sqrt{\lambda_1}}{(n-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}\right) z, \end{aligned}$$

which gives $\lambda_1 = (n-1)c^2$ and so M is isometric to an n-dimensional Euclidean ball of radius $\frac{1}{c}$. □

We consider now a fourth order Steklov eigenvalue problem on an n-dimensional compact connected Riemannian manifold $(M, \langle , \rangle)$ given by

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in} \quad M, \qquad (5.21) \\ &u = \Delta u - q\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M, \end{aligned}$$

wher q is a real number. Let q_1 be the first non-zero eigenvalue of the problem (5.21). As pointed by Kuttler [63], q_1 is the sharp constant for a priori estimates for the Laplace equation

$$\Delta v = 0 \quad \text{in} \quad M, \qquad v = g \quad \text{on} \quad \partial M, \qquad (5.22)$$

where $g \in L^2(\partial M)$.

It has been proven by Payne that if $\Omega \subset \mathbf{R}^2$ is a bounded convex domain with smooth boundary then $q_1(\Omega) \geq 2\rho_0$ with equality holding if and only if Ω is a disk, where ρ_0 is the minimum geodesic curvature of $\partial\Omega$. This Payne's theorem has been extended to higher dimensional Euclidean domains by Ferrero, Gazzola and Weth [38].

Theorem 5.3 ([91]). *Let $(M, \langle , \rangle)$ be an $n(\geq 2)$-dimensional compact connected Riemannian manifold with boundary ∂M and non-negative Ricci curvature. Assume that the mean curvature of M is bounded below by a positive constant c. Let q_1 be the first eigenvalue of the following Stekloff eigenvalue problem :*

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in} \quad M, \qquad (5.23) \\ &u = \Delta u - q\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M. \end{aligned}$$

Then $q_1 \geq nc$ with equality holding if and only if M is isometric to a ball of radius $\frac{1}{c}$ in $\mathbb{R}^n$.

Proof. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of the problem (5.23), that is

$$\begin{aligned} &\Delta^2 w = 0 \quad \text{in} \quad M, \\ &w = \Delta w - q_1 \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial M. \end{aligned} \tag{5.24}$$

Set $\eta = \frac{\partial w}{\partial \nu}\big|_{\partial M}$; then

$$q_1 = \frac{\int_M (\Delta w)^2}{\int_{\partial M} \eta^2}. \tag{5.25}$$

Substituting w into Reilly's formula, we have

$$\begin{aligned} &\int_M \left\{(\Delta w)^2 - |\nabla^2 w|^2\right\} \\ = \quad &\int_M \mathrm{Ric}(\nabla w, \nabla w) + \int_{\partial M} (n-1) H \eta^2 \\ \geq \quad &(n-1)c \int_{\partial M} \eta^2. \end{aligned} \tag{5.26}$$

The Schwarz inequality implies that

$$|\nabla^2 w|^2 \geq \frac{1}{n} (\Delta w)^2 \tag{5.27}$$

with equality holding if and only if $\nabla^2 w = \frac{\Delta w}{n}\langle , \rangle$. Combining (5.25)-(5.27), we have $q_1 \geq nc$. This completes the proof of the first part of Theorem 1.2. Assume now that $q_1 = nc$. In this case, the inequalities (5.26) and (5.27) must take equality sign. In particular, we have

$$\nabla^2 w = \frac{\Delta w}{n} \langle , \rangle. \tag{5.28}$$

Take an orthornormal frame $\{e_1, \cdots, e_{n-1}, e_n\}$ on M such that when restricted to ∂M, $e_n = \nu$. From $0 = \nabla^2 w(e_i, e_n)$, $i = 1, \cdots, n-1$, and $w|_{\partial M} = 0$, we conclude that $\eta = \rho = const.$ and so $\Delta w|_{\partial M} =$

$q_1\eta = nc\rho$ is also a constant. Since (5.26) takes equality sign and η is constant, we infer that $H \equiv c$. Also, we conclude from the fact that Δw is a harmonic function on M and the maximum principle that Δw is constant on M. Suppose without loss of generality that $\Delta w = 1$ and so we have

$$\nabla^2 w = \frac{1}{n}\langle , \rangle. \tag{5.29}$$

It then follows by deriving (5.29) covariantly that $\nabla^3 w = 0$ and from the Ricci identity,

$$R(X,Y)\nabla w = 0, \tag{5.30}$$

for any X, Y tangent vector to M, where R is the curvature tensor of M. From the maximum principle w attains its minimum at some point x_0 in the interior of M. From (5.29) it follows that

$$\nabla w = \frac{1}{n} r \frac{\partial}{\partial r}, \tag{5.31}$$

where r is the distance function to x_0. Using (5.30), (5.31), Cartan's theorem and $w|_{\partial M} = 0$, we conclude that M is an Euclidean ball whose center is x_0, and f is given by

$$w(x) = \frac{1}{n}(|x - x_0|^2 - b^2)$$

in M, b being the radius of the ball. Since the mean curvature of ∂M is c, the radius of the ball is $\frac{1}{c}$. □

For more recent developments about Steklov eigenvalues, we refer to [37, 74, 95] and the references therein.

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