Asymptotic Models for Surface and Internal Waves

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Chapter 1

Introduction

"Wovon man nicht sprechen kann, darüber muss man schweigen"
Ludwig Wittgenstein

The (surface or internal) water wave equations are too complicated to hope to describe the long time dynamics of the solutions except in trivial situations (perturbations of the flat surface). A natural idea is to "zoom" at some specific regimes of wavelengths, amplitudes, steepness,..., in order to derive asymptotic models that will describe interesting dynamics. One has first to define one or several "small" parameters and then to expand ad hoc quantities with respect to them. A similar situation occurs in other physical contexts, nonlinear optics, plasma physics,...

Actually this idea goes back to Lagrange (1781) who derived the water waves system for potential flows and obtained at the first order approximation the linear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - gh\frac{\partial^2 u}{\partial x^2} = 0,$$

where g is the constant of gravity and h the mean depth of the fluid layer.

It is only nearly one century after that Boussinesq derived in 1871 a weakly nonlinear dispersive model, the Korteweg-de Vries equation.

This explains why most of nonlinear dispersive equations or systems (see a definition below) such as the Korteweg -de Vries (KdV), Kadomtsev-Petviashvili (KP), Benjamin-Ono (BO), Boussinesq, nonlinear Schrödinger NLS), Davey-Stewartson (DS),... are not derived from first principles but through some asymptotic expansion from more complicated systems (water waves system, nonlinear Maxwell equations, plasmas equations...). Consequently, they are not supposed to be "good" models for all time. This fact leads to mathematical questions that are not in general addressed when considering them as just mathematical objects. For instance the classical dichotomy *local well-posedness* versus *finite time blow-up* should often be replaced by questions on *long time existence* (with respect to some parameters). To answer those questions using the methods of dispersive equations (even the more sophisticated ones) seem to be insufficient...

The situation for classical one-way propagation waves (KdV, KP, BO..) where local well-posedness in sufficiently large classes combined with the conservation of "charge" and energy implies *global* well-posedness **does not generalize** to the more (physically) relevant two-ways models which are *systems* and do not possess (in general) useful conserved quantities.

Moreover in many relevant water waves models, the dispersion is "weak" and cannot be efficiently used to derive the (linear) dispersive estimates that are a basic tool to study the Cauchy problem for "strongly" dispersive equations such as the KdV, NLS, KP, BO equations we were alluding to.

The aim of these Notes is an attempt to describe some aspects of classical and non classical dispersive equations and systems from the point of view of asymptotic models, that is by trying to keep in mind the origin of the equations. We will thus barely touch topics that have an interest *per se* for the equations as mathematical objects but relatively little for our viewpoint, for instance the issues of obtaining the solvability of the Cauchy problem in the largest possible space. We refer to the books [162, 230, 44, 50] for an extensive treatment of those issues.

We will consider mainly *dispersive* waves. For linear equations or systems posed in the whole space $\mathbb{R}^n$, this means that the solutions of the Cauchy problem corresponding to localized initial data disperse (in the sup norm) in large times. For constant coefficients equations this is easily expressed in terms of plane wave solutions $e^{i(x\cdot\xi-\omega t)}$, $\xi\in\mathbb{R}^n$. This gives the *dispersion relation*, that writes $G(\omega,\xi)=0$ and leads to one or several equations of the form $\omega = \omega(\xi)$. The equation is said *dispersive* if $\omega(\xi)$ is real and det $\left|\frac{\partial^2 \omega}{\partial_{\xi_i} \partial_{\xi_j}}\right| \not\equiv 0$. This means that the group velocity depends on the wave numbers, that is different Fourier modes travel at different speeds and thus wave packets tend to disperse.

To give a precise sense to this notion, consider for instance a linear equation

$$\begin{cases} i\frac{\partial u}{\partial t} + P(D)u = 0, \quad u = u(x,t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \\ u(\cdot,0) = u_0, \end{cases} \tag{1.1}$$

where $P(D)$ is defined in Fourier variables by

$$\widehat{P(D)u}(\xi) = p(\xi)\hat{u}(\xi),$$

p being a real valued, homogeneous of degree d function, that is

$$p(\lambda\xi) = \lambda^d p(\xi), \quad \lambda > 0, \quad \xi \in \mathbb{R}^n.$$

We moreover assume that the function

$$G(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi + ip(\xi)} d\xi$$

is bounded. [1]

The solution of (1.1) is given (using Fourier transform) as a convolution

$$u(\cdot,t) = G_t \star_x u_0, \quad \text{where} \quad G_t(x) = \int_{\mathbb{R}^n} e^{itp(\xi)+ix\cdot\xi} d\xi.$$

Using the homogeneity of p one has

$$G_t(x) = t^{-n/d} G(x),$$

which with the boundedness of G and a classical convolution estimate yields

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \leq \frac{C}{t^{n/d}} ||u_0||_{L^1}, \quad t \neq 0. \tag{1.2}$$

[1] We will see later how the Van der Corput lemma can be used to prove the boundedness of such oscillatory integrals.

This expresses that a sufficiently localized initial data leads to a solution that disperse as $t \to \infty$ as $\frac{1}{t^{n/d}}$.

A typical example is when $P(D) = \Delta$ (the linear Schrödinger equation) for which

$$G(x) = C_1 e^{i|x|^2/4}$$

leading to a decay in $t^{-n/2}$ of solutions emanating from a localized initial data.

Actually estimates like (1.2) are fundamental to derive the "$L^p - L^q$" Strichartz estimates that are used to solve the Cauchy problems for many semilinear dispersive equations such as the semilinear Schrödinger equations.

The importance of water waves equations and more generally dispersive equations should not be underestimated. As V. E. Zakharov expresses in the Introduction of [250] :

> In spite of the fact that the mathematical aspect of wave propagation is one of the classical subjects of Mathematical Physics, the theory of surface waves for many decades was an isolated island, just weakly connected with the main continent, the theory of sound and the theory of electromagnetic waves. One of the reasons for this was a dispersion. In contrary to the light and to the sound, the waves on the surface of an incompressible fluid are strongly dispersive. Their phase velocity depends essentially on a wave number. Another reason was a belief that the theory of surface waves is not a normal subject of pure mathematics. The basic equations describing waves on a surface of an ideal fluid in their classical formulation are neither ordinary nor partial differential equations. They look like an orphan in a society of normal PDE equations, like the Maxwell equations or linearized Navier-Stokes equations describing the ordinary waves. Nevertheless, the theory of surface waves became a cradle of the modern theory of waves in nonlinear dispersive media. It was Stokes who formulated the concept of a progressive stationary wave and calculated the nonlinear correction to dispersion relation. Another fundamental concept of modern nonlinear physics, the soliton, was also born in the theory of surface waves. The isolation of the theory of surface

> waves was broken in the fifties and sixties of this century. The fast development of plasma and solid state physics showed that a strong dispersion is a common thing for waves in real media, and non-dispersive sound and light waves are just very special exclusions in the world of waves, which mostly are strongly dispersive. In the last three decades the surface waves became a subject of intense study.

The Notes are roughly divided into three parts as follows. The two first Chapters are introductory. We give generalities on the Cauchy problem for infinite dimensional equations, emphasizing the notion of well/ill-posedness. We also present standard facts on the *compactness* method which might have been forgotten by the younger generation! As an application we treat in some details and as a paradigm the case of the Burgers equation since the method adapts at once to its skew-adjoint perturbations.

Finally we recall various classical tools: interpolation, oscillatory integrals,.. which will be of frequent use in the subsequent chapters.

Part two consisting of three Chapters is devoted to the presentation of the equations of both surface and internal waves, together with the derivation of their asymptotic models. The aim is to show how various, classical and less classical, many often dispersive, equations and systems can be rigorously derived as asymptotic models.

In Part three consisting of eight Chapters, we treat various mathematical problems related to some of the asymptotic models. Our choice is somewhat arbitrary, but we have tried to cover topics that have not be considered in books, and to maintain as possible some links with the origin of the models.

One advantage of the theory of PDE's with respect to other domains of Mathematics is that it benefits of a continuous flux of new problems arising for the real world. We hope that these Notes will show the richness of the mathematical problems stemming from the modeling of water waves and that they will inspire further work.

Part of the material of this book has its origin in joint works with various colleagues and friends. I thus express my heartful thanks to Anne de Bouard, Matania Ben-Artzi, Jerry Bona, Min Chen, Vassilis Dougalis, Jean-Michel Ghidaglia, Philippe Guyenne, Christian Klein, Herbert Koch, David Lannes, Felipe Linares, Dimitri Mitsotakis, Luc Molinet, Didier Pilod, Gustavo Ponce, Roger Temam, Nikolay Tzvetkov, Li Xu for our friendly and fruitful collaboration.

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Notations

The Fourier transform in $\mathbb{R}^n$ will be denoted $\mathfrak{F}$ or $\hat{\cdot}$.

The norm of Lebesgue spaces $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, will be denoted $|\cdot|_p$. We will mainly use the L^2 based Sobolev spaces $H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n); (1+|\xi^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^n)\}$ with their standard norms $|(1+|\cdot|^2)^{s/2}\hat{f}|_2$ denoted by $||\cdot||_s$ or $||\cdot||_{H^s}$. The Schwartz space of rapidly decaying C^∞ functions is denoted $\mathscr{S}(\mathbb{R}^n)$ and its dual (tempered distributions) $\mathscr{S}'(\mathbb{R}^n)$.

Chapter 2

The Cauchy problem

2.1 Generalities

Solving the Cauchy problem locally in time is the first step of our understanding of nonlinear PDE's. Note however that -though it might be very difficult- one should not overestimate its importance to understand the dynamics of the underlying equations or systems. After all, this step is nothing more than the equivalent of the Cauchy-Lipchitz theorem for ODE's and it tells nothing on the long time dynamics of the solution and actually there is little hope for most physically relevant systems to describe this dynamics. This is the reason why (starting actually from Lagrange [153] who derived the water wave system and from it the linear wave equation in the linear, infinite long wave regime)[1] one is led to "zoom" at a specific domain of amplitudes, frequencies,...in order to derive simpler asymptotic equations or systems which approximate the solutions of the original system on relevant time scales. We will precise this strategy in the subsequent chapters.

[1] Lagrange derived the linear wave equation for the horizontal velocity u of the wave

$$u_{tt} - ghu_{xx} = 0 \tag{2.1}$$

where h is a typical depth of the fluid and g the modulus of acceleration. This equation gives the correct order of magnitude for the speed of propagation of a tsunami in the ocean...

Nevertheless the fact that one has to work in infinite dimensional spaces leads to difficulties even in the definition of a *well-posed* problem. This fact was first put forward by J. Hadamard in the beginning of the twentieth century and we recall his classical example of an *ill-posed problem.*

We aim to solving the Cauchy problem in the upper half-plane for the Laplace equation :

$$\begin{cases} u_{tt} + u_{xx} = 0, \quad \text{in} \quad D = \{(x,t),\, x \in \mathbb{R},\, t > 0\} \\ u(x,0) = 0, \quad u_t(x,0) = f(x). \end{cases} \tag{2.2}$$

By Schwarz reflexion principle, the data f has to be analytic if u is required to be continuous on $\bar{D}$. We consider the sequence of initial data $\phi_n,\ n \in \mathbb{N}$:

$$\phi_n(x) = e^{-\sqrt{n}}\, n \sin(nx),\ \phi_0(x) \equiv 0\,.$$

It is easily checked that for any $k \geq 0$, $\phi_n \to 0$ in the C^k norm. In fact for any $\varepsilon > 0$, there exist $N_{\varepsilon,k} \in \mathbb{N}$ such that

$$\sup_x \sum_{j \leq k} |\phi_n^{(j)}(x)| \leq \varepsilon \quad \text{if } n \geq N_{\varepsilon,k}\,.$$

Note that ϕ_n oscillates more and more as $n \to +\infty$. On the other hand one finds by separation of variables that for any $n \in \mathbb{N}$, the Cauchy problem (2.2) with $f = \phi_n$ has the unique solution

$$v_n(x,t) = e^{-\sqrt{n}} \sin(nx)\ \sinh(nt)\ ,$$

and of course $v_0(x,t) \equiv 0$.

Things seem going well but for any $t_0 > 0$ (even arbitrary small), and any $k \in \mathbb{N}$,

$$\sup_x |v_n^{(k)}(x,t_0)| = n^k e^{-\sqrt{n}} \sinh(nt_0) \to +\infty$$

as $n \to +\infty$. In other words, the map $T\ :\ \varphi_n \to v_n(\cdot,t_0)$. is not continuous in any C^k topology. This catastrophic instability to short waves is called *Hadamard instability*. It is totally different from instability phenomenon one encounters in ODE problems, *eg* the exponential growth in time of solutions.

Consider for instance the PDE

$$\begin{cases} u_t + c\,u_x - u = 0 \\ u(x,0) = e^{inx}, \quad n \in \mathbb{N}. \end{cases} \tag{2.3}$$

The initial data is thus a plane wave of wave length $\frac{1}{n}$. The solution is $u_n(x,t) = e^t e^{in(x-ct)}$. It is clear that for any *reasonable* norm, $\|u_n(\cdot,t)\|$ grows exponentially to infinity as $t \to +\infty$. However, if one restricts to an interval $[0,T]$, the same *reasonable* norms are uniformly controlled in t with respect to those of the initial data.

Remark 2.1. The fact that norms in an infinite dimensional space are not equivalent implies of course that the asymptotic behavior of solutions to PDE's depends on the topology as shows the elementary but striking example.

Let consider the Cauchy problem

$$\begin{cases} u_t + xu_x = 0 \\ u(x,0) = u_0, \end{cases} \tag{2.4}$$

where $u_0 \in \mathscr{S}(\mathbb{R})$.

The L^p norms of the space derivatives $u_x^{(k)} = \partial_x^k u$ of the solution $u(x,t) = u_0(xe^{-t})$ are

$$|u_x^{(k)}|_p = e^{t(\frac{1}{p}-k)}|u_0^{(k)}|_p.$$

For instance, the L^∞ norm of u is constant while the L^p norms, $1 \le p < \infty$ grow exponentially. On the other hand all the homogeneous Sobolev norms $\dot{W}^{k,p}$, $k \ge 1$, $p > 1$ decay exponentaily to zero.

One can state a general concept of a *well-posed problem* for any PDE problem $(\mathscr{P})$. Let be given three topolological vector spaces (most often Banach spaces!) U, V, F, with $U \subset V$. Let f be the vector of data (initial conditions, boundary data, forcing terms,...) and u be the solution of $(\mathscr{P})$. One says that $(\mathscr{P})$ is *well-posed* (in the considered functional framework) if the three following conditions are fullfilled

1 - For any $f \in F$, there exists a solution $u \in U$ of $\mathscr{P}$.

2 - This solution is unique in U.

3 - The mapping $f \in F \mapsto u \in V$ is continuous from F to V.

To be more specific, consider for instance scalar [2] Cauchy problems of

[2]The same considerations apply of course to systems

type

$$\begin{cases} \partial_t u = u'(t) = iLu(t) + F(u(t)), \\ u(0) = u_0. \end{cases} \tag{2.5}$$

Here $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. L is a skew-adjoint operator defined in Fourier variables by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where the symbol p is a real function (not necessary a polynomial). F is a nonlinear term depending on u and possibly on its space derivatives. The linear part of (2.5) thus generates a unitary group $S(t)$ in $L^2(\mathbb{R}^n)$ (and in all Sobolev spaces) which is unitary equivalent to $\hat{u_0} \mapsto e^{itp(\xi)}\hat{u_0}$.

Classical examples involve the nonlinear Schrödinger equation (NLS), where here u is complex-valued

$$iu_t + \Delta u \pm |u|^p u = 0,$$

the *generalized* Korteweg- de Vries equation

$$u_t + u^p u_x + u_{xxx} = 0,$$

or the Benjamin-Ono equation (BO)

$$u_t + uu_x - \mathscr{H} u_{xx} = 0,$$

where $\mathscr{H}$ is the Hilbert transform, and many of the classical nonlinear dispersive equations.

Definition 2.1. The Cauchy problem (2.5) is said to be (locally) well posed -in short LWP- for data in $H^s(\mathbb{R}^n)$ if for any bounded set B in $H^s(\mathbb{R}^n)$ there exist $T > 0$ and a Banach space X_T continuously embedded into $C([-T,+T];H^s(\mathbb{R}^n))$ such that for any $u_0 \in B$ there exists a unique solution u of (2.5) in the class X_T. Moreover, the flow map

$$u_0 \mapsto u(t)$$

from B to $H^s(\mathbb{R}^n)$ is continuous.

Remark 2.2. 1. One might add the *persistency* property : if $u_0 \in H^{s'}(\mathbb{R}^n)$ with $s' > s$ then the corresponding solution belongs to the corresponding class X_T with the same T.

2. In some *critical* cases, T does not depend only on B (that is only on the H^s norm of u_0), but on u_0 itself in a more complicated way.

3. We will see below that for hyperbolic equations or systems (for instance the Burgers equation) one cannot have a better definition (that is the flow map cannot be smoother).

4. Of course very relevant equations or systems arising in the theory of water waves cannot be written under the simple form (2.5). This is the case for instance for the Green-Nagdhi or full dispersion systems (see Chapter 4) but the notion remains the same.

A natural way to prove LWP, inspired from the ODE case is to try to implement a Picard iterative scheme on the integral *Duhamel* formulation of (2.5), that is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds. \tag{2.6}$$

where as already mentioned $S(t)$ denotes the unitary group in $L^2(\mathbb{R}^n)$ generated by L.

We are thus reduced to finding a functional space $X_\tau \subset C([-\tau,+\tau]; H^s(\mathbb{R}^n))$, $\tau > 0$, such that for any bounded $B \subset H^s(\mathbb{R}^n)$, there exists $T > 0$ such that for any $u_0 \in B$, the right hand side of (2.6) is a contraction in a suitable ball of X_T.

It is only in very special situations that the choice $X_\tau = C([-\tau,+\tau]; H^s(\mathbb{R}^n))$ is possible, for instance when F is lipschitz if $s = 0$, or in the case of NLS when $n = 1$ and $s > 1/2$ (exercice!).

So, in this approach, the crux of the matter is the choice of an appropriate space X_τ. This can be carried out by using various *dispersive* estimates (see Chapter on the KdV equation) or by using a *Bourgain type* space (see [96] and below for a short description).

This method has the big advantage (on a compactness one that we will describe below for instance) of providing “for free” the uniqueness of the solution, the strong continuity in time and the “smoothness” of the flow (actually the only limitation of the smoothness of the flow is that of the smoothness of the nonlinearity).

This leads us to the

Definition 2.2. The Cauchy problem (2.5) is said (locally) semilinearly well-posed for data in $H^s(\mathbb{R}^n)$ if it is LWP in the previous sense and if moreover the flow map is smooth (at least locally lipschitz).

The Cauchy problem is said quasilinearly well-posed if it is well-posed, the flow map being only continuous.

Remark 2.3. The Cauchy problems which can be solved by a Picard iterative scheme on the Duhamel formulation with a suitable functional setting are thus semilinearly well-posed. As we noticed before, the Cauchy problems associated to quasilinear hyperbolic equations or systems (such as the Burgers equation) are only quasilinearly well-posed.

For dispersive equations, the situation is a bit more involved. For instance the KdV equation leads to a semilinear Cauchy problem, while the BO equation leads to a quasilinear one (see [185] and the discussion in Chapter VII).

2.2 The compactness method

We recall here some classical results which were very popular in the seventies but which might less known to the younger generation.

The rough idea is to construct approximate solutions, by regularizing the equation, the data or the unknown (for instance by truncating high frequencies) and then to get a priori bounds on those approximate solutions. The fact that closed balls in infinite dimensional normed spaces are not relatively compact gives obviously some trouble.

The starting technical point of the method is thus to look for conditions insuring that a bounded sequence u_n in $L^p(0,T;B)$, $1 \leq p \leq \infty$, where B is a Banach space, is relatively compact in $L^p(0,T;B)$.

This will allow to pass to the limit on a subsequence of approximate solutions and to get hopefully a solution to the problem.

As a simple example we will treat in some details the case of the Cauchy problem for the Burgers equation and some of its skew-adjoint perturbations.

The Aubin-Lions theorem

We will prove here a basic compactness result well suited to treat PDE problems.

Before stating it we recall two standard compactness results in normed spaces (see [211] chapter 7) :

1. A normed space is reflexive if and only if its unit ball is weakly compact.

2. For a normed space E, the weak* topology on its dual E' is that of the simple convergence on E. The unit ball B' of E' is weakly* compact.

Theorem 2.3. *Let $B_0 \subset B \subset B_1$, with B_i reflexive, $i = 0,1$ be three Banach spaces with $B_0 \subset B$ compact. Let*

$$W = \{v \in L^{p_0}(0,T;B_0), v'$$

$$= \frac{dv}{dt} \in L^{p_1}(0,T;B_1), 0 < T < \infty, 1 < p_i < \infty, i = 0,1\},$$

equipped with its natural Banach norm.

Then the embedding

$$W \subset L^{p_0}(0,T;B)$$

is compact.

Proof. Let (v_n) be a bounded sequence in W. Since B_i is reflexive and $1 < p_i < \infty$, $i = 0,1$, $L^{p_i}(0,T;B_i)$ is reflexive. One can thus extract a subsequence (v_μ) such that $v_\mu \rightharpoonup v$ in W weakly. One can assume that $v = 0$, reducing thus to showing (possibly after extraction of a subsequence):

$$v_n \rightharpoonup 0 \text{ in } W \text{ weak} \Longrightarrow v_n \longrightarrow 0 \text{ in } L^{p_0}(0,T;B) \text{ strong.}$$

Let admit for the moment the

Lemma 2.4. *Let three Banach spaces $B_0 \subset B \subset B_1$, with $B_0 \subset B$ compact.*

Then, for any $\eta > 0$, there exists $C_\eta > 0$ such that for all $v \in B_0$,

$$||v||_B \leq \eta ||v||_{B_0} + C_\eta ||v||_{B_1}.$$

It results from the lemma that for any $\eta > 0$, there exists $d_\eta > 0$ such that

$$||v_n||_{L^{p_0}(0,T;B)} \leq \eta ||v_n||_{L^{p_0}(0,T;B_0)} + d_\eta ||v_n||_{L^{p_0}(0,T;B_1)}.$$

Let $\varepsilon > 0$ be fixed. Since $||v_n||_{L^{p_0}(0,T;B_0)} \leq C$, one has

$$||v_n||_{L^{p_0}(0,T;B)} \leq \frac{\varepsilon}{2} + d_\eta ||v_n||_{L^{p_0}(0,T;B_1)},$$

provided we take η such that $\eta C \leq \frac{\varepsilon}{2}$.

One is thus reduced to proving that

$$v_n \to 0 \text{ in } L^{p_0}(0,T;B_1) \text{ strongly.} \tag{2.7}$$

By Sobolev embedding theorem in dimension 1, $W \subset C^0([0,T],B_1)$, and thus $||v_n(t)||_{B_1} \leq C$, $\forall t \in [0,T]$. From Lebesgue theorem, (2.7) will be established if one proves that

$$v_n(s) \to 0 \text{ in } B_1 \text{ strong}, \forall s \in [0,T] \tag{2.8}$$

As s plays no special role, we are reduced to proving that

$$v_n(0) \to 0,\ B_1 \text{ strong.}$$

Let then $w_n(t) = v_n(\lambda t)$, $\lambda > 0$ to be determined. One has

$$\begin{cases} v_n(0) = w_n(0), \\ ||w_n||_{L^{p_0}(0,T;B_0)} \leq C_1 \lambda^{-\frac{1}{p_0}}, \\ ||w_n'||_{L^{p_1}(0,T;B_1)} \leq C_2 \lambda^{1-\frac{1}{p_1}}. \end{cases} \tag{2.9}$$

Let $\phi \in C^1([0,T]),\ \phi(0) = -1,\ \phi(T) = 0$. It results

$$w_n(0) = \int_0^T (\phi w_n)' dt = \beta_n + \gamma_n,$$

$$\beta_n = \int_0^T \phi w_n' dt,\ \gamma_n = \int_0^T \phi' w_n dt.$$

From (2.9) we deduces

$$||v_n(0)||_{B_1} \leq ||\beta_n||_{B_1} + ||\gamma_n||_{B_1} \leq C_3 \lambda^{1-\frac{1}{p_1}} + ||\gamma_n||_{B_1}.$$

If $\varepsilon > 0$ is fixed, we choose λ such that $C_3 \lambda^{1-\frac{1}{p_1}} \leq \frac{\varepsilon}{2}$ and (2.7) will be established provided ones prove that $\gamma_n \to 0$ in B_1 strongly.

But $w_n \rightharpoonup 0$ in $L^{p_0}(0,T;B_0)$ weak (λ is fixed and one may assume that it is ≤ 1). Thus $\gamma_n \rightharpoonup 0$ in B_0 weak. Since $B_0 \subset B_1$ is compact, one deduces that $\gamma_n \to 0$ in B_1 strong.

Proof of lemma. By contradiction. Assume thus that there exist $\eta > 0$, $v_n \in B_0$, $c_n \to +\infty$, such that

$$||v_n||_B \geq \eta ||v_n||_{B_0} + c_n ||v_n||_{B_1}.$$

Let $w_n = \frac{v_n}{||v_n||_{B_0}}$. One has thus

$$||w_n||_B \geq \eta + c_n ||w_n||_{B_1}$$

and

$$||w_n||_B \leq C ||w_n||_{B_0} \leq C.$$

One deduces

$$||w_n||_{B_1} \to 0. \tag{2.10}$$

But since $||w_n||_{B_0} = 1$ and $B_0 \subset B$ is compact, one can extract a subsequence w_μ converging strongly in B to w.

Since $||w_\mu - w||_{B_1} \leq C ||w_\mu - w||_B$, one has $w = 0$ from (2.10), which is absurd since $||w_\mu||_B \geq \eta$. □

2.2.1 Application to PDE problems

The considerations above apply in general as follows. In order to solve (for instance) a Cauchy problem posed in $\mathbb{R}^n$, one constructs (by smoothing the equation, or the unknown function,..) a sequence of approximated solutions (u_m), bounded in a space $L^\infty(0,T;H^s(\mathbb{R}^n)), s > 0$.

We recall that the space $L^\infty(0,T;H^s(\mathbb{R}^n)), s > 0$ is the dual of $L^1(0,T;H^{-s}(\mathbb{R}^n))$. By the weak $\star$ compactness of the closed balls of the dual of a normed space, it thus results that, modulo extraction of a subsequence, $u_m \rightharpoonup u$ in $L^\infty(0,T;H^s(\mathbb{R}^n))$ weak*.

On the other hand, the equation provides a bound on $(u'_m) = (\frac{d}{dt}u_m)$ in $L^\infty(0,T;H^{s-d}(\mathbb{R}^n))$. Aubin-Lions theorem implies that for any bounded subset $B \subset \mathbb{R}^n$, and all $p \geq 1$, there exists a subsequence (still denoted (u_m)) such that for any integer k such that $k < [s]$ the derivatives of order $k, \partial^k(u_m)$

converge to $\partial^k u$ in $L^p(0,T;L^2(B))$ strongly and almost everywhere in $B \times [0,T]$. By the Cantor diagonal process (write $\mathbb{R}^n = \bigcup_{N\in\mathbb{N}^\star} B_N$ where $B_N = B(0,N)$), one may assume that $\partial^k(u_m)$ converges to $\partial^k u$ almost everywhere in $\mathbb{R}^n \times [0,T]$. This argument suffices in general to pass to the limit in the nonlinear terms (the convergence in the linear terms does not pose any problem thanks to the weak convergence). Since all convergences also hold in the sense of distributions one obtains a solution $u \in L^\infty(0,T;H^s(\mathbb{R}^n))$ with $\frac{du}{dt} \in L^\infty(0,T;H^{s-d}(\mathbb{R}^n))$.

In order to prove that the trace $u(.,0)$ makes sense in $H^s(\mathbb{R}^n)$ one uses a classical result of W. Strauss.

We first recall that if Y is a Banach space, $C_w([0,T];Y)$ denotes the subspace of $L^\infty(0,T;Y)$ of functions which are continuous from $[0,T]$ in Y equipped with the weak topology. One has ([222]):

Theorem 2.5. *Let V and Y be two Banach spaces, V reflexive, the embedding $V \subset Y$ being continuous and dense. Then*

$$L^\infty(0,T;V) \cap C_w([0,T];Y) = C_w([0,T];V)$$

Proof. Let u be in the LHS space. It suffices to prove that there exists a constant M such that :

$$u(t) \in V \, and \, |u(t)|_V \leq M, \forall t \in [0,T]. \tag{2.11}$$

In fact, if (2.11) is true, one can extract from any converging sequence $t_n \to t_0$ in $[0,T]$ a subsequence t_m such that $u(t_m)$ converges weakly in V. Since u is weakly continuous with values in V, the limit must be $u(t_0)$. Thus $u(t_n) \rightharpoonup u(t_0)$ weakly in V.

To show (2.11), one considers a regularizing sequence $\eta_\varepsilon(t)$ in the usual way : let $\eta_0(t)$ an even, positive function, $\mathscr{C}^\infty$ compactly supported and with integral 1; one then defines for $\varepsilon > 0$, $\eta_\varepsilon(t) = \varepsilon^{-1}\eta_0(t/\varepsilon)$. Let us consider $0 < t < T$ such that $(\eta_\varepsilon * u)(t) \in V$ for ε small enough. Let M be the norm of u in $L^\infty(0,T;V)$. Then

$$|(\eta_\varepsilon * u)(t)|_V \leq \int \eta_\varepsilon(s)|u(t-s)|_V ds \leq M.$$

There exists therefore a "subsequence" of ε such that $(\eta_\varepsilon * u)(t)$ converges weakly in V. On the other hand, for all v in the dual V' of V,

$$((\eta_\varepsilon * u)(t) - u(t), v) \to 0 \quad \text{when} \quad \varepsilon \to 0,$$

since $(u(s), v)$ is a continuous function of s. It results that $u(t) \in V$,

$$(\eta_\varepsilon * u)(t) \rightharpoonup u(t)$$

weakly in V, and that $|u(t)|_V \leq M$.

To prove (2.11) in the case $t = 0$, one applies a similar argument with $(\eta_\varepsilon * u)(t)$ replaced by $(\eta_\varepsilon * u)(\varepsilon)$; for $t = T$, one considers $(\eta_\varepsilon * u)(T - \varepsilon)$. □

Corollary 2.6. *Let V and W be Banach spaces, V reflexive, both contained in a vectorial space, such that $V \cap W$ is dense in V and W. If $u \in L^\infty(0,T;V)$ and $u' = \frac{du}{dt} \in L^1(0,T;W)$, there exists a weakly continuous function from $[0,T]$ with values in V which is equal to u almost everywhere.*

Proof. The space $Y = V + W$ satisfies the conditions of Theorem 2.5. The hypotheses on u imply in particular that $u \in L^1(0,T;Y)$ and $u' \in L^1(0,T;Y)$. u is thus weakly continuous with values in Y, and Theorem 2.5 applies. □

In the present situation, one applies Strauss Theorem with $V = H^s(\mathbb{R}^n)$ and $Y = H^{s-d}(\mathbb{R}^n)$ (notice that by Sobolev in dimension 1 one already knows that $u \in C([0,T],Y)$).

Remark 2.4. Contrary to a method based on an iterative scheme *à la* Picard, the compactness method does not provide either the (possible) uniqueness of the solution, neither the strong continuity in time, neither the continuity of the flow. One has to establish those properties separately.

The uniqueness (when it holds!) is usually obtained by a direct argument based on Gronwall's lemma.

In some situations (for instance for symmetric hyperbolic systems) on can prove the strong continuity in time, (i.e that $u \in C([0,T],V)$) by using the uniqueness in $L^\infty(0,T;V)$ and the fact that the equation is reversible in time (see for instance [236] Chapter 16).

One can also prove the strong time continuity and the continuity of the flow map by an approximation process (the "Bona-Smith trick", [36], see below).

2.2.2 J. Simon Theorem.

We mention here for the sake of completeness a more general compactness result. In [215] J. Simon caracterizes the relatively compact subsets of $L^p(0,T;B)$ where $1 \leq p \leq \infty$ and B is a Banach space.

The result is the following:

Theorem 2.7. *Let $F \subset L^p(0,T;B)$. F is relatively compact in $L^{p_0}(0,T;B)$ for $1 < p < \infty$, or in $C(0,T;B)$ if $p = \infty$ if and only if*

$$\{\int_{t_1}^{t_2} f(t)dt : f \in F\} \textit{ is relatively compact in } B,\ \forall\, 0 < t_1 < t_2 < T.$$

$$||\tau_h f - f||_{L^p(0,T-h;B)} \to 0, \textit{ when } h \to 0, \textit{ uniformly for } f \in F.$$

(we have denoted $\tau_h f(t) = f(t+h)$ for $h > 0$).

If one has a second Banach space X such that $X \subset B$ is compact, Theorem 2.7 implies:

Theorem 2.8. *Let $F \subset L^p(0,T,B)$ with $1 \leq p \leq \infty$. One assumes that F is bounded in $L^1_{loc}(0,T;X)$ and that*

$$||\tau_h f - f||_{L^p(0,T-h,B)} \to 0, \textit{ when } h \to 0, \textit{ uniformly for } f \in F.$$

Then F is relatively compact in $L^p(0,T,B)$ (and then $C(0,T;B)$ if $p = \infty$).

One can deduce useful *sufficient* conditions of compactness. For instance

Proposition 2.9. *Let B and X two Banach spaces such that $X \subset B$ is compact and let m be an integer.*

Let F be a bounded subset of $W^{-m,1}_{loc}(0,T;X)$ such that $\partial F/\partial t = \{\partial f/\partial t : f \in F\}$ be bounded in $L^1(0,T;B)$. Then F is relatively compact in $L^p(0,T;B)$, $1 \leq p < \infty$.

Let F bounded in $W^{-m,1}_{loc}(0,T;X)$ with $\partial F/\partial t$ bounded in $L^r(0,T;B)$, where $r > 1$. Then F is relatively compact in $C(0,T;B)$.

The article [215] contains many variants of this result.

2.3 The Burgers equation and related equations.

This example is elementary but typical of the compactness method; it adapts easily to more general contexts such as the symmetrizable quasilinear hyperbolic systems. [3]

We recall that $|.|_p$ will denote the norm in L^p, $1 \leq p \leq +\infty$.

We consider the Cauchy problem:

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(x,0) = u_0, \end{cases} \tag{2.12}$$

where $u_0 \in B_R = \{v \in H^s(\mathbb{R}), ||v||_{H^s} \leq R\}$, $s > \frac{3}{2}$.

We approximate the equation by "truncating the high frequencies". Let $\chi \in C_0^\infty(\mathbb{R})$ even, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[-1,+1]$.

We introduce for $j \in \mathbb{N}$ the operator $\chi_j = \chi(j^{-1}D)$ (notation of Fourier multipliers), $\widehat{\chi_j(f)}(\xi) = \chi(j^{-1}\xi)\hat{f}(\xi)$.

It is clear that $\chi_j : H^s(\mathbb{R}) \to H^r(\mathbb{R})$ is continuous for all $s, r \in \mathbb{R}$, and that it is self-adjoint and commutes with all derivations (of integer order or not).

We approximate (2.12) by

$$\begin{cases} \partial_t U^j + \chi_j(U^j \chi_j(\partial_x U^j)) = 0, \\ U^j(x,0) = u_0, \end{cases} \tag{2.13}$$

The properties of χ_j imply easily that

$$U \mapsto \chi_j(U\chi_j(\partial_x U))$$

is locally Lipschitz on $H^s(\mathbb{R})$, $s > \frac{3}{2}$. Cauchy-Lipschitz theorem in Banach spaces implies the existence of a maximal solution $U^j \in C([0,T_j]; H^s(\mathbb{R}))$.

The following estimates show that T_j depends only on R and provide bounds on U^j.

One applies $D^s = (I - \partial_x^2)^{\frac{s}{2}}$ to (2.13) setting $D^s U^j = V^j$:

$$\partial_t V^j + \chi_j(U^j \chi_j(\partial_x V^j)) = \chi_j([U^j, D^s]\chi_j(\partial_x U^j)), \tag{2.14}$$

[3] We will see in Chapter 10 an approach via characteristics which explicits the finite time blow-up of the solution.

and one takes the L^2 scalar product of (2.14) with V^j to obtain (we do not indicate the t dependence):

$$\frac{1}{2}\frac{d}{dt}|V^j|_2^2+\int_{\mathbb{R}}\chi_j(U^j\chi_j(\partial_x V^j))V^j dx=\int_{\mathbb{R}}\chi_j([U^j,D^s]\chi_j(\partial_x U^j))V^j dx. \tag{2.15}$$

Using that χ_j is self-adjoint and commutes with ∂_x, the integral in the LHS of (2.15) writes

$$\frac{1}{2}\int_{\mathbb{R}}U^j\partial_x[\chi_j(V^j)]^2dx=-\frac{1}{2}\int_{\mathbb{R}}\partial_x(U^j)[\chi_j(V^j)]^2dx. \tag{2.16}$$

We recall the Kato-Ponce commutator lemma [130] :

Lemma 2.10. *Let $s>0$ and $1<p<+\infty$ and $D^s=(I-\Delta)^{\frac{s}{2}}$. There exists $C>0$ such that*
$|D^s(fg)-f(D^sg)|_p\le C(|\nabla f|_\infty|D^{s-1}g|_p+|D^sf|_p|g|_\infty)$.

One deduces from (2.15), (2.16) and from Lemma (2.10) (applied with $f=U^j$ and $g=\chi_j(\partial_xU^j)$) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|V^j|_2^2 &\le \frac{1}{2}|\partial_x\chi_j(U^j)|_\infty\int_{\mathbb{R}}(V^j)^2dx+\\ &+C|V^j|_2\{|\partial_xU^j|_\infty|D^{s-1}\chi_j(\partial_xU^j)|_2+|D^sU^j|_2|\chi_j(\partial_xU^j)|_\infty\}\\ &\le C|V^j|_2\{|\partial_xU^j|_\infty|V^j|_2+|V^j|_2|\chi_j(\partial_xU^j)|_\infty\}\\ &=C|V^j|_2^2\{|\partial_xU^j|_\infty+|\chi_j(\partial_xU^j)|_\infty\}\end{aligned}$$

We therefore deduce with Gronwall' s lemma that, for all $t\in[0,T_j)$:

$$||U^j(.,t)||_{H^s}\le||U_0||_{H^s}\exp(C||\partial_xU^j||_{L^1(0,T_j;\,L^\infty(\mathbb{R}))}+||\chi_j(\partial_xU^j)||_{L^1(0,T_j;\,L^\infty(\mathbb{R}))}).$$

Since $s>\frac{3}{2}$, Sobolev theorem implies that the argument in the exponential is majorized by

$$CT_j||U^j||_{L^\infty(0,T_j;\,H^s(\mathbb{R}))}.$$

On the other hand, $y_j(t)=||V^j(t)||_2^2$ is majorized by the solution of

$$y'(t)=Cy(t)^{\frac{3}{2}},\ y(0)=||u_0||_{H^s}^2$$

on its maximal interval of existence. The solution of this Cauchy problem being $y(t) = \frac{4y_0}{(2-FCty(0)^{\frac{1}{2}})^2}$,

we deduce that $T_j = T$ does not depend on j, that $T = O(\frac{1}{||u_0||_{H^s}})$ and one has the bound

$$||U^j||_{L^\infty(0,T;\,H^s(\mathbb{R}))} \leq C(||u_0||_{H^s(\mathbb{R})}). \tag{2.17}$$

This is the key estimate. Using the equation, one deduces that $\partial_t U^j$ is bounded (at least!) in $L^\infty(0,T;\,L^2(\mathbb{R}))$ and thus converges to $\partial_t U$ in $L^\infty(0,T;\,L^2(\mathbb{R}))$ weak* and in $L^2(0,T;\,L^2(\mathbb{R}))$ weakly. As a consequence, (modulo extraction of a subsequence still denoted U_j), U_j converges in $L^\infty(0,T;\,H^s(\mathbb{R}))$ weak-star to U. Aubin-Lions theorem insures that, for any interval $I_m = (-m,m)$, $m \in \mathbb{N}$, a subsequence $U_{(j,m)}$ converges in $L^2(0,T;\,L^2(I_m))$ strongly and almost everywhere. Cantor diagonal procedure then implies that another subsequence (still denoted U_j) converges to U almost everywhere in $\mathbb{R} \times (0,T)$.

It is then easy, using estimate (2.17) to show with Lebesgue dominated convergence theorem that $(U^j)^2$ converges to U^2 (at least !) in the sense of distributions and thus that $\partial_x (U^j)^2$ converges to $\partial_x U^2$. Passing to the limit in the linear terms giving no trouble because of the weak convergences, one deduces that U satisfies (2.12) in the sense of distributions and in fact in $L^\infty(0,T;\,H^{s-1}(\mathbb{R}))$. Strauss result above implies that $U \in C_w([0,T];\,H^s(\mathbb{R}))$ and thus that the initial data is taken into account in $H^s(\mathbb{R})$. In fact we have a much more precise result as we will see below.

Uniqueness in the class $L^\infty(0,T;\,H^s(\mathbb{R}))$ is very easily obtained. Let u and v be two solutions corresponding to the same initial data $u_0 \in H^s(\mathbb{R})$ and let $w = u - v$. w satisfies the equation

$$w_t + wu_x + vw_x = 0.$$

Taking the L^2 scalar product with w one obtains after an integration by parts (justified thanks to the regularity of the solutions) :

$$\frac{1}{2}\frac{d}{dt}|w(.,t)|_2^2 + \int_{\mathbb{R}} [w^2(x,t)u_x(x,t) - \frac{1}{2}v_x(x,t)w^2(x,t)]dx = 0,$$

so that

$$\frac{1}{2}\frac{d}{dt}|w(.,t)|_2^2 \leq C\int_{\mathbb{R}} w^2(x,t)dx$$

and one concludes with Gronwall's lemma.

Remark 2.5. One has a persistency result, immediate consequence of the proof above : if $u_0 \in H^r(\mathbb{R})$ with $r > s$, the corresponding solution belongs to $L^\infty(0,T;\, H^r(\mathbb{R}))$, on the same interval $[0,T)$.

We will use freely this property in the following argument ("Bona-Smith trick") which proves the strong continuity in time and the continuity of the flow map.

Let $\rho \in \mathscr{S}(\mathbb{R})$, $\hat{\rho} \in C_0^\infty(\mathbb{R})$, $\hat{\rho}(\xi) \equiv 1$ in a neighborhood of 0. For $\varepsilon > 0$ we denote $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\frac{x}{\varepsilon})$.

We consider then the solution of (2.12) corresponding to an initial data $u_0 \in B_R$. We denote u^ε the solution of (2.12) with initial data $u_{0\varepsilon} = \rho \star u_0$.

Since

$$||u_{0\varepsilon}||_{H^s} \leq C||u_0||_{H^s}, \quad \varepsilon \in (0,1],$$

u^ε satisfies the bounds

$$\begin{cases} ||u_x^\varepsilon||_{L^1(0,T;\,L^\infty)} \leq C, \\ ||u^\varepsilon||_{L^\infty(0,T;\,H^s)} \leq C, \end{cases} \tag{2.18}$$

on the existence interval of de u. Let us also notice that from the persistency property, the equation and Sobolev embedding theorem in dimension one, that u^ε belongs (in particular) to $C([0,T];\, H^s(\mathbb{R}))$. We will prove that u^ε is a Cauchy sequence in this space. For $\varepsilon > \varepsilon' > 0$ we set $v = u^\varepsilon - u^{\varepsilon'}$. One easily checks that

$||v(0)||_2 = O(\varepsilon^s)$ and $||v(0)||_{H^s} = o(1)$ as $\varepsilon \to 0$.

On the other hand, v satisfies the equation

$$2v_t + (u_x^\varepsilon + u_x^{\varepsilon'})v + (u^\varepsilon + u^{\varepsilon'})v_x = 0. \tag{2.19}$$

Taking the L^2 scalar product of (2.19) with v and using the bounds (2.18) on u^ε and $u^{\varepsilon'}$, one obtains the estimate

$$|v(.,t)|_2^2 \leq C\varepsilon^s, \tag{2.20}$$

for t in the interval of existence $[0, T = T(R))$ of u.

Proceeding as in the existence proof, we apply D^s to (2.19), then takes the L^2 scalar product with $D^s v$ and uses Kato-Ponce lemma to end with :

$$\frac{d}{dt}||v(.,t)||_{H^s}^2 \leq C\{||u^\varepsilon(.,t)||_{H^s} + ||u^{\varepsilon'}(.,t)||_{H^s}\}||v(.,t)||_{H^s}^2$$
$$+C\{||u^\varepsilon(.,t)||_{H^{s+1}} + ||u^{\varepsilon'}(.,t)||_{H^{s+1}}\}||v(.,t)||_{H^{s-1}}||v(.,t)||_{H^s}.$$

One deduces from (2.18) that, for $0 \leq t < T$,

$$||u_\varepsilon(.,t)||_{H^{s+1}} \leq C||u_0^\varepsilon||_{H^{s+1}} \leq C\varepsilon^{-1}. \tag{2.21}$$

On the other hand, a classical interpolation inequality [4] and (2.20) imply that

$$||v(.,t)||_{H^{s-1}} \leq ||v(.,t)||_2^{\frac{1}{s}} ||v(.,t)||_{H^s}^{1-\frac{1}{s}} \leq C\varepsilon||v(.,t)||_{H^s}^{1-\frac{1}{s}} \tag{2.22}$$

Gathering those inequalities one deduces with Gronwall's lemma that

$$||u^\varepsilon(.,t) - u^{\varepsilon'}(.,t)||_{H^s} = o(1) \tag{2.23}$$

when $\varepsilon \to 0$.

A similar argument leads to the continuity of the flow map $u_0 \mapsto u(.,t)$ on bounded subsets of H^s. It suffices to consider u_0 as above and a sequence u_{0n} converging to u_0 in $H^s(\mathbb{R})$; we denote u_n the corresponding solutions. One proves that

$$||u^\varepsilon(.,t) - u^{\varepsilon'}(.,t)||_{H^s} \leq C||u_{0n} - u_0||_{H^s} + o(1) \tag{2.24}$$

as $\varepsilon \to 0$.

One easily deduces with (2.23) the continuity of the flow.

Remark 2.6. From Sobolev embedding theorem we deduce that u, u_x and u_t are continuous on $[0,T] \times \mathbb{R}$, and thus that u is C^1 and satisfies the equation in the classical sense.

Remark 2.7. It is straightforward to check that all the considerations above apply *mutatis mutandi*, with minor modifications, to linear *skew-adjoint* perturbations of the Burgers equation, for instance to the Korteweg- de Vries

$$u_t + uu_x + u_{xxx} = 0$$

or Benjamin-Ono

$$u_t + uu_x - \mathscr{H}u_{xx} = 0,$$

equations.

Recall that $\mathscr{H}$ is the Hilbert transform, $\mathscr{H} = . \star\ vp(\frac{1}{x})$, $\widehat{\mathscr{H}f}(\xi) = -i\text{sign}\,\xi \hat{f}(\xi)$.

[4] If $s = (1-\theta)s_1 + \theta s_2$, $0 \leq \theta \leq 1$, then $||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^{s_1}(\mathbb{R}^n)}^{(1-\theta)} ||f||_{H^{s_2}(\mathbb{R}^n)}^{\theta}$

In those two examples, the equation is of course no more satisfied in the classical sense and the result is far from being optimal as for the minimal regularity assumptions on the initial data.

One can actually obtain *global* results with the following observation made in [215, 1] (this could also result from the Brezis-Galloüet inequality [46] : assuming that one has local well-posedness in $H^s(\mathbb{R}^n), s > \frac{n}{2}$ and a global a priori bound in $H^{\frac{n}{2}}(\mathbb{R}^n)$, then one has global well-posedness in $H^s(\mathbb{R}^n), s > \frac{n}{2}$.

Consider for example the Cauchy problem for the Benjamin-Ono equation :

$$u_t + uu_x - \mathscr{H} u_{xx} = 0, \quad u(\cdot,0) = u_0. \tag{2.25}$$

We have the following global existence result.

Theorem 2.11. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then there exists a unique solution u of* (2.25) *such that $u \in C([0,+\infty; H^s(\mathbb{R}))$.*

Proof. We will just prove the key a priori bound, the computations below can be justified on a smooth local solution emanating from a regularization of u_0.

We recall first that the Benjamin-Ono equation being integrable by the Inverse Scattering method possesses an infinite number of conserved quantities. Besides the "trivial" ones

$$I_0(u) = \int_{\mathbb{R}} u^2(x,t)dx = \text{const.}, \quad I_1(u)$$

$$= \int_{\mathbb{R}} [\frac{1}{2} D^{1/2} u(x,t)^2 - \frac{1}{6} u^3(x,t)]dx = \text{const.},$$

where D^s is defined by $D^s u(x) = \mathfrak{F}^{-1} |\xi|^s \hat{u}(\xi)$, it has in particular the invariants (see [1])

$$I_2(u) = \int_{\mathbb{R}} [\frac{1}{4} u^4 - \frac{3}{2} u^2 \mathscr{H}(u_x) + 2u_x^2]dx,$$

$$I_3(u) = \int_{\mathbb{R}} [\{-\frac{1}{5} u^5 + [\frac{4}{3} u^3 \mathscr{H}(u_x) + u^2 \mathscr{H}(uu_x)]$$

$$-[2u\mathscr{H}(u_x)^2 + 6uu_x^2] - 4u\mathscr{H}(u_{xxx})\}dx.$$

We leave as an exercise that I_0, I_1 provide a global a priori $H^{\frac{1}{2}}(\mathbb{R})$ bound, I_0, I_1, I_2 a global a priori $H^1(\mathbb{R})$ bound, and I_0, I_1, I_2, I_3 a global a priori $H^{\frac{3}{2}}(\mathbb{R})$ bound.

□

Lemma 2.12. *For any $T > 0$ there exists $C(T, ||u_0||_{3/2})||u_0||_s)$, such that*

$$||u||_{L^\infty(0,T;H^s(\mathbb{R}))} \leq C(T, ||u_0||_{3/2})||u_0||_s. \tag{2.26}$$

Proof. We take the L^2 scalar product of (2.25) by $D^s u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |D^s u|^2 dx + \int_{\mathbb{R}} D^s(uu_x) D^s u\, dx = 0. \tag{2.27}$$

We observe now (see [204] for instance that for $s \geq \gamma = 1 > \frac{3}{2}$ and $u, v \in H^s(\mathbb{R})$, there exists a positive constant $c(\gamma, s) = c'(s)/\sqrt{\gamma - \frac{1}{2}}$ such that

$$|D^s(uv) - uD^s v|_2 \leq c(\gamma, s)(||u||_s ||v||_\gamma + ||u||_{\gamma+1}||v||_{s-1}).$$

Since $s > \frac{3}{2}$ there exists $\eta > 0$ such that $s > \frac{3}{2} + \eta$. We apply the elementary commutator estimate above with $v = u_x$ and $\gamma = \frac{1}{2} = \eta$ to obtain that

$$|D^s(uu_x) - uD^s u_x|_0 \leq \frac{c}{\sqrt{\eta}}||u||_{3/2+\eta}||u||_s. \tag{2.28}$$

On the other hand,

$$(uD^s_x, D^s u) = -\frac{1}{2}(u_x D^s u, D^s u),$$

and since (at it is easily checked using Fourier transforms) for any $\eta > 0$

$$|u_x|_\infty \leq \frac{c}{\sqrt{\eta}}||u||_{3/2+\eta},$$

one derives the inequality

$$|(uD^s u_x, D^s u)| \leq \frac{c}{\sqrt{\eta}}||u||_{3/2+\eta}|D^s u|_2^2. \tag{2.29}$$

We deduce from (2.28) and (2.29) that for any $\eta > 0$ such that $\frac{3}{2} + \eta < s$,

$$|(D^s(uu_x), D^s u)| \leq \frac{c}{\sqrt{\eta}}||u||_{3/2+\eta}||u||_s^2. \tag{2.30}$$

Since $H^{3/2+\eta}(\mathbb{R}) = [H^s(\mathbb{R}), H^{3/2}(\mathbb{R})]_\theta$ with $\theta = 1 - \frac{2\eta}{(2s-3)} =: 1 - 2\gamma\eta$, we have the interpolation inequality

$$||u||_{3/2+\eta} \leq c||u||_s^{2\gamma\eta}||u||_{3/2}^{1-2\gamma\eta},$$

where c does not depend on η.

We also recall the a priori bound

$$||u||_{L^\infty(\mathbb{R}_+;H^{3/2}(\mathbb{R}))} \leq ca(||u_0||_{3/2}).$$

Combining all those inequalities yields

$$\frac{1}{2}\frac{d}{dt}||u||_s^2 \leq C\left(\frac{1}{\sqrt{\eta}}||u||_s^{2+2\eta}\right)$$

for all $\eta > 0$ such that $\frac{3}{2} + \eta < s$, where $C = [ca(||u_0||_{3/2})]^{1-2\gamma\eta}$.

One thus deduces that

$$||u(\cdot,t)||_s^2 \leq y(t),$$

where $y(t)$ is the solution of the differential equation

$$y'(t) = \frac{C}{\sqrt{\eta}}[y(t)]^{1+\delta\eta}, \quad y(0) = ||u_0||_s^2,$$

on its maximal interval of existence $[0, T(\eta))$, where $\delta = 1/(2s-3)$.

One finds easily that $y(t) = (||u_0||_s^{-2\delta\eta} - \delta\sqrt{\eta}Ct)^{-1/\delta\eta}$, whence

$$T(\eta) = \frac{1}{\delta C\sqrt{\eta}}||u_0||_s^{-2\delta\eta} \to +\infty, \quad \text{as } \eta \to 0.$$

For any fixed $T > 0$, we can choose $\eta > 0$ so small that $T < \frac{1}{2}T(\eta)$. It then follows that for $0 \leq t \leq T$,

$$y(t) \leq c(T; ||u_0||_{3/2})||u_0||_s^2,$$

achieving the proof of Lemma 2.12.

□

Remark 2.8. As we will see in Chapter 4, many equations obtained as asymptotic models in a suitable regime write

$$u_t + u_x + \varepsilon(uux - Lu_x) = 0, \tag{2.31}$$

where L is a skew-adjoint linear operator and $\varepsilon > 0$ is a small parameter. The method of proof developed above gives for the Cauchy problem an interval of existence of order $[0, O(\frac{1}{\varepsilon}))$, that is existence on the "hyperbolic time" $\frac{1}{\varepsilon}$.

Remark 2.9. The method above applies also ([236]) , with some extra technical difficulties to *symmetric or symmetrizable hyperbolic quasilinear systems* in $\mathbb{R}^n$ of the form

$$A(x,t,U)\partial_t U + \sum_{j=1}^{n} A_j(x,t,U)\partial_{x_j} U + B(x,t,U) = 0,$$

where A, A_j are real symmetric $p \times p$ smooth matrices, A being moreover positive definite, uniformly with respect to (x,t,U), that is there exists $C > 0$ such that

$$(A(x,t,\xi),\xi) \geq C|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \mathbb{R}^n, t \in \mathbb{R}.$$

One also assume that B is smooth.

In general, A arises from *symmetrization* of a non symmetric system, the idea being then to replace the L^2 scalar products in the energy estimates by

$$(w, S(x,t,U)w)_{L^2},$$

where the symmetrizer S is a symmetric, positive definite matrix.

As example we consider the one-dimensional version of the Saint-Venant system that we will encounter in Chapter 4 as a model of certain surface water waves:

$$\begin{cases} \partial_t \eta + \mathrm{div}(\mathbf{u} + \eta \mathbf{u}) & = & 0 \\ \partial_t \mathbf{u} + \nabla \eta + \frac{1}{2}\nabla|\mathbf{u}|^2 & = & 0, \end{cases} \tag{2.32}$$

where $\mathbf{u} = (u_1, u_2)$ and η are functions of $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$.

This system writes in space dimension one :

$$\begin{cases} \partial_t \eta + u_x + (u\eta)_x & = & 0 \\ \partial_t u + \eta_x + uu_x & = & 0, \end{cases} \tag{2.33}$$

Setting $U = (\eta, u)^T$, (2.33) writes also

$$\partial_t U + C(U)U_x = 0,$$

where

$$C(U) = \begin{pmatrix} u & 1+\eta \\ 1 & u \end{pmatrix}$$

is of course not symmetric. However, setting

$$S(U) = \begin{pmatrix} 1 & 0 \\ 0 & 1+\eta \end{pmatrix},$$

we notice that

$$A_1(U) = S(U)C(U) = \begin{pmatrix} u & 1+\eta \\ 1+\eta & u(1+\eta) \end{pmatrix}$$

is symmetric. We thus get the system

$$A(U)\partial_t U + A_1(U)U_x = 0,$$

which is of the desired form, provided we restrict to the region where $1+\eta > 0$, which is physically meaningful, this condition ensuring that the "non cavitation" : the free surface does not touch the bottom.

Exercise : give the details of the proof for local well-posedness with initial data in $H^s(\mathbb{R})$, $s > \frac{3}{2}$.

Remark 2.10. Again if one considers the Saint-Venant system involving a small parameter $\varepsilon > 0$

$$\begin{cases} \partial_t \eta + \operatorname{div}(\mathbf{u} + \varepsilon \eta \mathbf{u}) & = & 0 \\ \partial_t \mathbf{u} + \nabla \eta + \dfrac{\varepsilon}{2} \nabla |\mathbf{u}|^2 & = & 0, \end{cases} \tag{2.34}$$

the interval of existence for the Cauchy problem will be of order $[0, O(\frac{1}{\varepsilon}))$.

We conclude this Section by proving that the flow map associated to the Burgers equation is no more regular than continuous. This displays the "quasilinear" character of the associated Cauchy problem, a property that is typical of quasilinear symmetric hyperbolic systems but that we will also encounter for some nonlinear dispersive equations, a typical example being the Benjamin-Ono equation.

The following result is based on the property of finite speed of propagation of the Burgers solutions and is due to T. Kato [132].

Theorem 2.13. *The flow map associated to the Cauchy problem for the Burgers equation is not (locally) Hölder continuous in H^s, $s > \frac{3}{2}$, for any Hölder exponent. In particular it is not (locally) Lipschitzian.*

Proof. We recall (classical result obtained by the method of characteristics, see Chapter 10) that for any C^1 initial data, bounded together with its derivative, that is $u_0 \in C_b^1(\mathbb{R})$ (2.12) has a unique C^1 solution which is defined for

$$t < \frac{1}{||\partial_x u_0||_\infty}. \tag{2.35}$$

It is implicitly defined by

$$u = u_0(x - tu) \tag{2.36}$$

Let $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi \equiv 1$ for $|x| \leq 2$ and $\alpha \in \mathbb{R}$ such that

$$s - \frac{3}{2} < \alpha < s - \frac{1}{2}.$$

In order to avoid technical details, we will restrict to the case where s is *integer* ≥ 2.

We consider the sequence of initial data

$$u_0^\lambda(x) = (\lambda + x_+^{\alpha+1})\phi(x), \quad -1 \leq \lambda \leq 1, \tag{2.37}$$

where $x_+ = \sup(x, 0)$.

It is clear (choice of $\alpha > s - \frac{3}{2}$) that $u_0^\lambda \in H^s(\mathbb{R}) \subset C_b^1(\mathbb{R})$, the norms H^s being uniformly bounded in λ.

From (2.35), there exists $T > 0$ such that the solutions u^λ of (2.12) associated to u_0^λ exist for $|t| \leq T$, for all $\lambda \in [-1, +1]$. Choosing T small enough, one can furthermore assume that

$$|tu^\lambda(x,t)| \leq 1, \quad x \in \mathbb{R}, \quad |t| \leq T, \quad |\lambda| \leq 1. \tag{2.38}$$

Lemma 2.14.

$$u^\lambda(x,t) = \lambda + (x - \lambda t)_+^{\alpha+1} p(t(x - \lambda t)_+^\alpha, \quad |x| \leq 1, \quad |\lambda| \leq 1,$$

where $p(z)$ is a power series in z with $p(0) = 1$ with a stricltly positive radius of convergence.

Proof. Set $y = x - tu^\lambda(x,t)$. One deduces from (2.38) that $|y| \leq 2$ for $|x| \leq 1$, $|t| \leq 1$ and $|\lambda| \leq 1$. So we have $\phi(y) = 1$. It then results from (2.36) and (2.37) that

$$u^\lambda(x,t) = u_0^\lambda(y) = \lambda + y_+^{\alpha+1}.$$

One therefore deduces successively, for t small enough the sequence of equalities :

$$y = x - \lambda t - t y_+^{\alpha+1}, \quad (1 + t y_+^\alpha) y = x - \lambda t, \quad (1 + t y_+^\alpha) y_+ = (x - \lambda t)_+.$$

One gets the result by solving the last equation. □

Coming back to the proof of the Theorem 2.13, Lemma 2.14implies that for t small enough,

$$(u^\lambda - u^0)(x,t) = \lambda + (x - \lambda t)_+^{\alpha+1} - x_+^{\alpha+1} + ... \tag{2.39}$$

$$\partial_x^s(u^\lambda - u^0)(x,t) = (\alpha+1)...(\alpha - s + 2)[(x - \lambda t)_+^{\alpha-s+1} - x_+^{\alpha-s+1}] + ... \tag{2.40}$$

for $|x| \leq 1$, the ... indicating the higher order terms.

Observe now that

$$\begin{aligned} ||u^\lambda(.,t) - u(^0(.,t)||_{H^s} &\geq ||\partial_x^s(u^\lambda(.,t) - u(^0(.,t))||_2 \\ &\geq [\int_{-1}^1 |\partial_x^s(u^\lambda - u^0)(x,t)|^2]^{\frac{1}{2}} \end{aligned} \tag{2.41}$$

One easily checks that the contribution of the first term in the RHS of (2.40) in the RHS of (2.41) is of order at least $|\lambda t|^{\alpha-s+\frac{3}{2}}$.The contribution of the remaining terms is of higher order. It results that there exists $c > 0$ such that

$$||u^\lambda(.,t) - u(^0(.,t)||_{H^s} \geq c|\lambda t|^{\alpha-s+\frac{3}{2}}, \tag{2.42}$$

for $|t|$ and $|\lambda|$ sufficiently small.

On the other hand, $u_0^\lambda - u^0 = \lambda\phi$ so that $||u_0^\lambda - u_0||_{H^s} = |\lambda| ||\phi||_{H^s}$.

Since one can choose α such that $\alpha - s + \frac{3}{2} > 0$ is arbitrarily small and have again u_0 in H^s, (2.42) proves that the mapping $u_0 \mapsto u(.,t)$ for $t \neq 0$ cannot be Hölderian for any prescribed exponent.

□

Chapter 3

Varia. Some classical facts

We recall here some useful tools that we will freely use in the subsequent chapters.

3.1 Vector-valued distributions

If X is a Banach space, we denote (see [212], [165])

$\mathscr{D}'(0,T;X) = \mathscr{L}(\mathscr{D}(]0,T[;X)$ equipped with the topology of uniform convergence on bounded sets of $\mathscr{D}(]0,T[)$ the space of distributions on $]0,T[$ with values in X.

If $T \in \mathscr{D}'(0,T;X)$, its distribution derivative is thus defined as

$$\frac{\partial T}{\partial t}(\phi) = -T(\frac{d\phi}{dt}), \quad \forall \phi \in \mathscr{D}(]0,T[).$$

To $f \in L^p(0,T;X)$, corresponds a distribution, still denoted f on $]0,T[$ with values in X, by

$$f(\phi) = -\int_0^T f(t)\phi(t)dt, \quad \phi \in \mathscr{D}(]0,T[).$$

Observe (vectorial version of Sobolev embedding theorem in one dimension), that if $f \in L^p(0,T;X)$ and $\frac{\partial f}{\partial t} \in L^p(0,T;X)$, then, after possibly a modification on a subset of measure zero of, $]0,T[$, that f is continuous from $[0,T]$ to X.

The following result is useful (see a proof in [232], Chapitre III, 1).

Lemma 3.1. *Let V and H be two Hilbert spaces, the inclusion being continuous and dense $V \subset H$. One thus can identify H to its dual and write $V \subset H \equiv H' \subset V'$. Then, if $w \in L^2(0,T;V)$ and $w' = \frac{dw}{dt} \in L^2(0,T;V')$,*

$$\frac{1}{2}\frac{d}{dt}|w(t)|_H^2 = \langle w(t), w'(t)\rangle, \quad a.e.\ t.$$

For instance, if $u \in L^2(0,T,H_0^1(\Omega))$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0,$$

one can justify the energy equality :

$$\frac{1}{2}\frac{d}{dt}|u(.,t)|_{L^2} + |\nabla u(.,t)|_{L^2}^2 = 0$$

by taking the (H_0^1, H^{-1}) duality of the equation with u.

3.2 Some interpolation results and applications

We consider here the triple $(X,\mathscr{A},\mu)$, where the set X is equipped with the σ-algebra $\mathscr{A}$ and with the measure μ.

We shall denote $L^p(X,\mathscr{A},\mu)$ the associated Lebesgue spaces. We consider also another triple $(Y,\mathscr{B},\nu)$.

We state the Riesz-Thorin interpolation theorem :

Theorem 3.2. *Assume that $1 \le p_0 \ne p_1 \le \infty$, $1 \le q_0 \ne q_1 \le \infty$.*

If $T \in \mathscr{L}(L^{p_0}(X,\mathscr{A},\mu), L^{q_0}(Y,\mathscr{B},\nu))$ with norm M_0 and

$T \in \mathscr{L}(L^{p_1}(X,\mathscr{A},\mu), L^{q_1}(Y,\mathscr{B},\nu))$ with norm M_1,

then $T \in \mathscr{L}(L^{p_\theta}(X,\mathscr{A},\mu), L^{q_\theta}(Y,\mathscr{B},\nu))$ with norm M_θ,

where $M_\theta \le M_0^{1-\theta} M_1^\theta$ and

$$\tfrac{1}{p_\theta} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1},\ \ \tfrac{1}{q_\theta} = \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1},\ \ \theta \in (0,1).$$

We refer to [220], [27] for the proof, based on Phragmen-Lindelöf, itself based on the three lines theorem of Hadamard.

We now indicate some important consequences. Let $X = Y = \mathbb{R}^n$ eand $\mu = \nu = dx$, the Lebesgue measure. We say that a map $T \in \mathscr{L}(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ is of type (p,q) and we will denote $f \star g$ the convolution of f by g. The following result is known as Young theorem.

Theorem 3.3. *Let* $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 \le p,q \le \infty$, *with* $\frac{1}{p} + \frac{1}{q} \ge 1$. *Then* $f \star g \in L^r(\mathbb{R}^n)$ *where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *Moreover,* $||f \star g||_r \le ||f||_p ||g||_q$.

Proof. For g fixed, let $T_g f = f \star g$. From Minkowsky inequality, $||T_g f||_q \le ||g||_q ||f||_1$. On the other hand Hölder inequality implies that $||T_g f||_\infty \le ||g||_q ||f||_{q'}$.

T_g is thus of type $(1,q)$ with norm $\le ||g||_q$ and of type $(q', +\infty)$ with norm $\le ||g||_q$. From Riesz-Thorin theorem, T_g is of type (p,r) with $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q'} = 1 - \frac{\theta}{q}$ et $\frac{1}{r} = \frac{1-\theta}{q} + 0 = \frac{1}{q} + (1 - \frac{\theta}{q}) - 1 = \frac{1}{p} + \frac{1}{q} - 1$, with norm $\le ||g||_q$.

□

Another consequence is the Hausdorff-Young theorem (we denote $\hat{f}$ the Fourier transform of f).

Theorem 3.4. *Let* $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. *Then* $\hat{f} \in L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$, *with*

$$||\hat{f}||_{p'} \le ||f||_p.$$

Proof. It is a classical fact that the Fourier transform is of type $(1,\infty)$ and $(2,2)$, with norms 1. The result follows readily from Riesz-Thorin theorem.

□

We will need a generalization of Theorem 3.3 that necessitates the use of Lebesgue weak- L^p spaces which we define now in the content of a measured space (U,μ) (it will be $(\mathbb{R}^n, dx)$ for us actually).

Definition 3.5. For any measurable function$f$ on U, almost everywhere, we define its distribution function as

$$m(\sigma, f) = \mu(\{x : |f(x) > \sigma\}).$$

For $1 \leq p < \infty$, the weak- L^p space, denoted L^p_w, consists in the $f's$ such that

$$||f||_{L^p_w} = \sup_{\sigma} \sigma m(\sigma, f)^{1/p} < \infty.$$

Remark 3.1. 1. For $1 \leq p < \infty$, the weak-L^p space is only quasi- normed. In fact, the inequality $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$ implies that

$$||f+g||_{L^p_w} \leq 2(||f||_{L^p_w} + ||g||_{L^p_w}).$$

2. Obviously :

$$||f||_{L^p_w} \leq ||f||_{L^p}.$$

3. It is easily checked that the function f defined on $\mathbb{R}^n$ by $f(x) = 1/|x|^{\frac{n}{p}}$ belongs to $L^p_w(\mathbb{R}^n, dx)$ but does not belong to any $L^r(\mathbb{R}^n)$, for any $1 \leq r \leq +\infty$.

4. The L^p_w spaces are particular examples of two- parameters space, L^{pq}, $1 \leq p, q \leq +\infty$, the Lorentz spaces. The space L^p_w is the Lorentz space $L^{p,\infty}$.

A linear map $T : L^p \to L^q_w$ is said to be bounded if $||Tf||_{L^p_w} \leq C||f||_{L^p}$. The infimum on all $C's$ is by definition the norm of T. We will denote then $T \in \mathscr{L}(L^p, L^q_w)$.

We state now the Marcinkiewicz interpolation theorem. A proof can be found in [27].

Theorem 3.6. *Let* $1 \leq p_0 \neq p_1 \leq \infty$, $1 \leq q_0 \neq q_1 \leq \infty$, $\quad 0 < \theta < 1$ *and*

$$T \in \mathscr{L}(L^{p_0}(U,\mu), L^{q_0}_w(V,\nu)), \text{ with norm } M_0,$$

$$T \in \mathscr{L}(L^{p_1}(U,\mu), L^{q_1}_w(V,\nu)), \text{ with norm } M_1.$$

Set

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and assume that$p \leq q$.

Then $T \in \mathscr{L}^p(U,\mu), L^q(V,\nu))$.

Another useful result is Hunt interpolation theorem (see [198]).

Theorem 3.7. *Let $1 \leq p_0 \neq p_1 \leq \infty$, $1 \leq q_0 \neq q_1 \leq \infty$. We assume that T is linear continuous from $L^{p_0}(U,\mu)$ to $L^{q_0}(V,\nu)$ and from $L^{p_1}(U,\mu)$ to $L^{q_1}(V,\nu)$.*

Then, if p and q are defined as in Theorem 3.6, T extends to a linear continuous map from $L^p_w(U,\mu)$ into $L^q_w(V,\nu)$.

Moreover, $||Tf||_{L^q_w} \leq C||f||_{L^p_w}$.

One deduces from Hunt and Marcinkiewicz theorems the generalized Young inequality, see [198] :

Theorem 3.8. *Let $1 < p,q < \infty$ and $f \in L^p(\mathbb{R}^n)$, $g \in L^q_w(\mathbb{R}^n)$.*

Then $f \star g \in L^r(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ et

$$||f \star g||_{L^r} \leq C||f||_{L^p}||g||_{L^q_w}.$$

Proof. Fix $f \in L^p(\mathbb{R}^n)$ and set $T_f = f \star \cdot$. From Young's inequality, $T_f \in \mathscr{L}(L^q(\mathbb{R}^n), L^r(\mathbb{R}^n))$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

One takes successively $q = 1$ then $q = p'$ proving that

$||T_f g||_p \leq ||f||_p||g||_1$ and $||T_f g||_\infty \leq ||f||_p||g||_{p'}$ and from Hunt's theorem,

$$||T_f g||_{L^r_w} \leq C||g||_{L^q_w},$$

for $1 < q < p'$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$.

Fix now $g \in L^q(\mathbb{R}^n)$. From what precedes, $T_g : L^p(\mathbb{R}^n) \to L^r_w(\mathbb{R}^n)$ is continuous and from Marcinkiewicz theorem, $f \star g \in L^r(\mathbb{R}^n)$ with the desired estimate. □

The following corollary, a direct consequence of the above considerations is known as the Hardy-Littlewood-Sobolev inequality.

Corollary 3.9. *Let $0 < \alpha < n$, $1 \leq p < q < +\infty$ avec $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Set $I_\alpha(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$. Then

$$||I_\alpha(f)||_{L^q(\mathbb{R}^n)} \leq C(\alpha,p,n)||f||_{L^p(\mathbb{R}^n)}.$$

Proof. It suffices to apply Young generalized inequality, recalling that the function ϕ defined by $\phi(x) = \frac{1}{x^{n-\alpha}}$ belongs $L_w^{n/(n-\alpha)}(\mathbb{R}^n)$. □

Two excellent references on the theory of interpolation are the books [27] and [220].

3.3 The Van der Corput lemma

The Van der Corput lemma is a basic tool to evaluate oscillatory integrals that occur often in the analysis of linear dispersive equations.

We follow here the excellent treatment in [219]. We want to study the behavior for large positive λ of the oscillatory integral

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)dx, \tag{3.1}$$

where the phase ϕ is a smooth real-valued function and the amplitude ψ is complex -valued and smooth. We first consider the case where the phase has no stationary points (critical points).

Lemma 3.10. *Assume that ϕ and ψ are smooth, that ψ has compact support in (a,b) and that $\phi'(x) \neq 0$ for all $x \in [a,b]$.*

Then

$$I(\lambda) = O(\lambda^{-N}) \quad as \quad \lambda \to +\infty$$

Proof. Let D denote the differential operator

$$Df(x) = (i\lambda\phi'(x))^{-1} \cdot \frac{df}{dx}$$

and let ${}^tD$ its transpose,

$${}^tDf(x) = -\frac{d}{dx}\left(\frac{f}{i\lambda\phi'(x)}\right).$$

Then $D^N(e^{i\lambda\phi}) = e^{i\lambda\phi}$ for avery N and by integration by parts,

$$\int_a^b e^{i\lambda\phi}\psi dx = \int_a^b D^N(e^{i\lambda\phi})\psi dx = \int_a^b e^{i\lambda\phi} \cdot ({}^tD)^N(\psi)dx$$

which implies clearly $|I(\lambda)| \leq A_N\lambda^{-N}$.

□

The Van der Corput lemma deals with the case where ϕ has critical points.

Proposition 3.11. *Suppose that ϕ is real-valued and smooth on (a,b), and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k\lambda^{-1/k}$$

holds when:

(i) $k \geq 2$, or

(ii) $k = 1$ and ϕ' is monotonic.

The bound c_k is independent of ϕ and λ.

Proof. We first prove (ii). One has

$$\int_a^b e^{i\lambda\phi}dx = \int_a^b D(e^{i\lambda\phi})dx = \int_a^b e^{i\lambda\phi}\cdot {}^tD(1)dx + (i\lambda\phi')^{-1}e^{i\lambda\phi}|_a^b$$
$$\leq \lambda^{-1}\int_a^b \left|\frac{d}{dx}\left(\frac{1}{\phi'}\right)\right|dx = \lambda^{-1}\left|\int_a^b \frac{d}{dx}\left(\frac{1}{\phi'}\right)dx\right|,$$

where we have used that $\frac{d}{dx}\left(\frac{1}{\phi'}\right) = \frac{\phi''}{-\phi'2}$ has a sign.

The last expression equals

$$\lambda^{-1}\left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right| \leq \frac{1}{\lambda}.$$

This gives the conclusion with $c_1 = 3$.

We now prove (ii) by induction on k. Supposing that the case k is known we may assume (replacing possibly ϕ by $-\phi$) that

$$\phi^{(k+1)} \geq 1 \quad \text{for all } x \in [a,b].$$

Let $x = c$ the unique point in $[a,b]$ where $|\phi^{(k)}(x)|$ achieves its minimum value. If $\phi^{(k)}(c) = 0$ then, outside some interval $(c-\delta, c+\delta)$ we have $|\phi^{(k)}(x)| \geq \delta$ (and of course, $\phi'(x)$ is monotonic in the case $k = 1$). We decompose the integral as

$$\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b.$$

By the inductive hypothesis,

$$\left|\int_a^{c-\delta} e^{i\lambda\phi}dx\right| \le c_k(\lambda\delta)^{-1/k}.$$

Similarly,

$$\left|\int_{c+\delta}^{b} e^{i\lambda\phi}dx\right| \le c_k(\lambda\delta)^{-1/k}.$$

Since $\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi}dx\right| \le 2\delta$, we conclude that

$$\left|\int_a^{b} e^{i\lambda\phi}dx\right| \le \frac{2c_k}{(\lambda\delta)^{1/k}} + 2\delta.$$

If $\phi^{(k)}(c) \neq 0$, and so c is one of the endpoints of $[a,b]$, a similar argument shows that the integral is majored by $c_k(\lambda\delta)^{1/k} + \delta$. In both situations, the case $k+1$ follows by taking

$$\delta = \lambda^{-1/(k+1)},$$

which proves the claim with $c_{k+1} = 2c_k + 2$. Since $c_1 = 3$, we have $c_k = 5\cdot 2^{k-1} - 2$.

□

As a consequence, we obtain the following result on integrals of type (3.1). We do not assume that ψ vanishes near the endpoints of $[a,b]$.

Corollary 3.12. *Under the assumptions on ϕ in Proposition 3.11,*

$$\left|\int_a^{b} e^{i\lambda\phi(x)}\psi(x)dx\right| \le c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|dx\right].$$

Proof. We write (3.1) as $\int_a^b F'(x)\psi(x)dx$, with

$$F(x) = \int_a^x e^{i\lambda\phi(t)}dt.$$

Then we integrate by parts, and use the estimate

$$|F(x)| \le c_k\lambda^{1/k}, \quad \text{for } x \in [a,b],$$

obtained in Van der Corput lemma. □

As example of application of the van der Corput Lemma, we will, following [162] (see [137] for a more general result) indicate how it can be used to prove that the function (“half derivative of the Airy function”) defined by

$$F(x)=\int_{\mathbb{R}}|\xi|^{1/2}e^{i(x\xi+\xi^3)}d\xi,$$

belongs to $L^\infty(\mathbb{R})$.

Proposition 3.13. *Let $\beta\in[0,1/2]$ and*

$$I_\beta(x)=\int_{-\infty}^{\infty}|\eta|^\beta e^{i(x\eta+\eta^3)}d\eta.$$

Then $I_\beta\in L^\infty(\mathbb{R})$.

Proof. Let us first fix $\phi_0\in C^\infty(\mathbb{R})$ with

$$\phi_0(\eta)=\begin{cases}1, & |\eta|>2\\ 0, & |\eta|\le 3/2.\end{cases}$$

Since the function $F_1(\eta)=(1-\phi_0)(\eta)e^{i\eta^3}|\eta|^\beta\in L^1(\mathbb{R})$, its Fourier transform belongs to $L^\infty(\mathbb{R})$ and it suffices to consider

$$\tilde{I_\beta}(x)=\int_{-\infty}^{\infty}e^{i(x\eta+\eta^3)}|\eta|^\beta\phi_0(\eta)d\eta.$$

- For $x\ge -3$, the phase $\Phi_x(\eta)=x\eta+\eta^3$ satisfies

$$|\phi_x'(\eta)|=|x+3\eta^2|\ge(|x|/2+\eta^2)\quad\text{on the support of }\phi_0,$$

and the result follows by integration by parts.

- For $x<-3$, one takes $(\phi_1,\phi_2)\in C_0^\infty(\mathbb{R})\times C^\infty(\mathbb{R})$ such that

$\phi_1(\eta)+\phi_2(\eta)=1$ with

$$\text{Supp }\phi_1\subset A=\{\eta,|x+3\eta^2|\le|x|/2\},$$

$$\phi_2\equiv 0\quad\text{in}\quad B=\{\eta,|x+3\eta^2|<|x|/3.$$

We set

$$\tilde{I}_{\beta}^{j}(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}|\eta|^{\beta}\phi_0(\eta)\phi_j(\eta)d\eta, \quad j = 1,2.$$

so that

$$|\tilde{I_\beta}(x)| \leq |\tilde{I_\beta}^1(x)| + |\tilde{I_\beta}^2(x)|.$$

When $\phi_2(\eta) \neq 0$, the triangle inequality implies that

$$|\phi_x'(\eta)| = |x+3\eta^2| \geq \frac{1}{6}(|x|+\eta^2).$$

One integrate by parts to obtain

$$|\tilde{I}_\beta^2(x)| = \left| \int_{-\infty}^{\infty} \frac{|\eta|^\beta}{\phi_x'(\eta)} \phi_0(\eta)\phi_2(\eta) \frac{d}{d\eta}[e^{i(x\eta+\eta^3}]d\eta \right| \leq C.$$

If now $\eta \in A$, we have $|x|/2 \leq 3\eta^2 \leq 3|x|/2$ and

$$\left| \frac{d^2\phi_x}{d\eta^2}(\eta) \right| = 6|\eta| \geq |x|^{1/2}.$$

Corollary 3.12 together with the definition of ϕ_0, ϕ_1 then implies that there exists a constant c independent of $x > -3$ such that

$$|\tilde{I}_\beta^1(x)| = \left| \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3}|\eta|^\beta \phi_0(\eta)\phi_1(\eta)d\eta \right| \leq c|x|^{-1/4}|x|^{\beta/2}$$

and the proof is complete.

□

We will encounter in Chapter 9 another typical application of the Van der Corput lemma.

Chapter 4

Surface water waves

Most of the equations or systems that will be considered in those Notes can be derived from many physical systems, showing their universal nature as a kind of *normal forms*[1]. For the sake of simplicity we will consider only their derivation from the water wave system, for one or two layers of fluid. We consider in this Chapter the case of surface waves.

4.1 The Euler equations with free surface

We recall here briefly the derivation of the water wave system, that is the Euler equations with a free surface. A much more complete discussion of these topics can be found in the excellent book by D. Lannes [155] which contains also an extensive treatment of the justification of the asymptotic models. Historical aspects of water waves and other fluid mechanics topics can be found in [69].

We consider a irrotational flow of an incompressible, inviscid (non viscous) fluid (say water) submitted to the gravitation field. The final goal is to describe the evolution of surface gravity (or gravity-capillary waves if surface tension effects are taken into consideration). One assume that the bottom of the fluid layer is flat and that the fluid cannot penetrates it. [2]

[1] In analogy with the ODE theory where a normal form describes the dynamics in the neighborhood of a critical point

[2] Of course it is important in most oceanographic situations to consider non flat bottoms

The flows will be bi or tri-dimensional, $d = 1,2$ denoting the horizontal dimension of the flow. We will denote $X = x$ or $X = (x,y)$ the horizontal variables and z the vertical variable. The height of the fluid at rest is h and the free surface will be parametrized by $z = \zeta(X,t)$, t being the time variable. The fluid domain is thus $\Omega(t) =: \{(X,z) - h < z < \zeta(X,t)\}$.

We denote $\mathbf{v} = (V,w)$ the velocity field, p the (scalar) pressure field, $\mathbf{g} = (0,0,-g)$ the acceleration of gravity, ρ the volumic mass of the fluid.

The Euler equations (1755) are a set of equations based on conservation principles.

1. Conservation of mass.

$$\rho_t + \text{div}\ (\rho \mathbf{v}) = 0 \tag{4.1}$$

which by incompressibility reduces to

$$\text{div}\ \mathbf{v} = 0. \tag{4.2}$$

2. Newton's law which implies here (assuming that the forces are just due to the isotropic pressure and to exterior forces)

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla_{X,z} \mathbf{v} = -\frac{1}{\rho} \nabla_{X,z} p + \mathbf{g}. \tag{4.3}$$

We have denoted $\nabla_{X,z}$ the gradient with respect to (X,z), ∇ the gradient with respect to X.

Moreover the irrotationality condition curl $\mathbf{v} = 0$ implies that there exists a velocity potential ϕ such that $\mathbf{v} = \nabla_{X,z}\phi$.

We have to impose boundary conditions on the upper (free) surface and on the bottom. Both surfaces should be "bounding", that is no fluid particle should cross them. For a surface given implicitely by the equation $\Sigma(X,z,t) = 0$ this fact expresses that the material derivative vanishes identically, that is

$$\Sigma_t + \mathbf{v} \cdot \nabla_{X,z} \Sigma = 0.$$

For the lower (resp. upper) surfaces one has $\Sigma(X,z,t) = z + h$, (resp. $\Sigma(X,z,t) = z - \zeta(X,t)$,) and the bounding condition yields the two equations

$$v_3 = 0 \quad \text{at} \quad z = -h \tag{4.4}$$

---

but this leads to technical difficulties that we want to avoid here.

$$\zeta_t = (-\nabla\zeta, 1)^T \cdot \mathbf{v}_{|z=\zeta(X,t)} \tag{4.5}$$

Finally, we need a boundary condition at the free surface. In case of a air-water interface and for a not too thin layer, one can ignore the effects of surface tension and then the pressure at the free surface is a constant, equal to the air pressure. The pressure p being defined up to an additive constant, one may assume

$$p = 0 \quad \text{at} \quad z = \zeta(X,t). \tag{4.6}$$

The free surface Euler system (4.2)- (4.3)- (4.4)- (4.5)- (4.6) can be greatly simplified for potential flows. First the incompressibility condition (4.2) writes

$$\Delta_{X,z}\phi = 0 \quad \text{in} \quad \Omega(t). \tag{4.7}$$

Next, (4.3) implies that

$$\nabla_{X,z}(\phi_t + \frac{1}{2}|\nabla_{X,z}\phi|^2 + \frac{p}{\rho} + gz) = 0 \quad -h \le z \le \zeta(X,t), \tag{4.8}$$

which yields after integration

$$\phi_t + \frac{1}{2}|\nabla_{X,z}\phi|^2 + \frac{p}{\rho} + gz = f(t) \quad -h \le z \le \zeta(X,t). \tag{4.9}$$

By changing ϕ by $\phi + \int_0^t f(s)ds$ one may assume that $f \equiv 0$. Taking the trace at $z = \zeta(X,t)$ and using (4.6) one gets

$$\phi_t + \frac{1}{2}|\nabla_{X,z}\phi|^2 + gz = 0, \quad z = \zeta(X,t). \tag{4.10}$$

Lastly the boundary conditions (4.4) and (4.5) write in terms of ϕ :

$$\partial_z\phi = 0 \quad \text{at} \quad z = 0 \tag{4.11}$$

and

$$\partial_t\zeta + \nabla\zeta \cdot \nabla\phi_{|z=\zeta(X,t)} = \partial_z\phi_{|z=\zeta(X,t)}. \tag{4.12}$$

The system (4.7)-(4.10)-(4.11)-(4.12) is the formulation of the Euler equations with free boundary which was established in 1781 by Lagrange [153]. A drawback is that the Laplace equation is posed in the moving domain $\Omega(t)$ which is an unknown of the problem.

To overcome this difficulty we recall the strategy of Zakharov [248] and Craig-Sulem [64, 65] which leads to a system posed on a fixed domain, and eventually to the Hamiltonian form of the system.

The idea is to use as independent variables the elevation ζ and the trace of the potential ϕ at the free surface, that is $\psi(X,t) = \phi(X,\zeta(X,t),t)$.

When ψ is known one reconstruct ϕ by solving the Laplace equation in $\Omega(t)$ with Neumann condition on the bottom and the Dirichlet condition $\phi = \psi$ on the free surface.

The linear map

$$Z(\zeta) : \psi \to \partial_z \phi_{|z=\zeta}$$

is called the Dirichlet-Neumann operator. It is easily checked that if $\Gamma = \{z = \zeta\}$ is a smooth surface, then $Z(\zeta) \in \mathscr{L}(H^{3/2}(\mathbb{R}^{d-1}), H^{1/2}(\mathbb{R}^{d-1}))$.

A precise functional setting will be given for similar operators in the next Chapter.

Using that

$$\partial_t \psi = \partial_t \phi_{|z=\zeta} + \partial_t \zeta Z(\zeta)\psi \quad \text{and} \quad \nabla\psi = \nabla\phi_{z=\zeta} + Z(\zeta)\psi\nabla\zeta,$$

it is easily checked that one can express (4.11) and (4.12) as

$$\begin{cases} \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}(1+|\nabla\zeta|^2)(Z(\zeta)\psi)^2 = 0 \\ \partial_t \zeta + \nabla\psi \cdot \nabla\zeta - (1+|\nabla\zeta|^2)Z(\zeta)\psi = 0, \end{cases} \tag{4.13}$$

which is the Zakharov-Craig-Sulem formulation of the water wave problem.

Remark 4.1. Another definition of the Dirichlet-Neumann operator leading to a slightly different but equivalent formulation is

$$G[\zeta] : \psi \to \sqrt{1+|\nabla\zeta|^2}\partial_n \phi_{|z=\zeta} \tag{4.14}$$

where ∂_n denotes the unit normal derivative to the free surface.

Remark 4.2. When surface tension effects are taken into account, (4.6) should be replaced by

$$p = \sigma\nabla.\left(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\right),$$

where $\sigma \geq 0$ is the surface tension parameter. This leads to the gravity-capillary water waves system

$$\begin{cases} \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}(1+|\nabla\zeta|^2)(Z(\zeta)\psi)^2 + \frac{\sigma}{\rho}K(\zeta) = 0 \\ \partial_t \zeta + \nabla\psi\cdot\nabla\zeta - (1+|\nabla\zeta|^2)Z(\zeta)\psi = 0, \end{cases} \tag{4.15}$$

where $K(\zeta) = \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}$.

Remark 4.3. Although this will not be used in the sequel, one should recall that Zakharov [248] used the above formulation to derive the *Hamiltonian* form of the water waves system. More precisely (4.15) can be written as

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\zeta H \\ \partial_\psi H \end{pmatrix},$$

where the Hamiltonian $H = K + E$ is the sum of kinetic and potential energy,

$$\begin{aligned} K = \frac{1}{2}\int_{\mathbb{R}}^{d}\int_{-h}^{\zeta(X)} |\nabla_{X,z}\phi(X,z)|^2 dzdX &= \text{(by Green's formula)} \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \psi Z[\zeta]\psi dX, \end{aligned}$$

and

$$E = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 dX.$$

Solving the Cauchy problem for (4.13) is not an easy task. We refer to S. Wu [241, 242] D. Lannes [155, 12], Iguchi [114],T. Alazard, N. Burq, C. Zuily [4, 5, 6, 7, 8], M.Ming, P. Zhang, Z. Zhang [178, 179] for local well-posedness results in various functional settings and to [243, 244], [90, 91, 9] for global or almost global well-posedness with small initial data.

In any case those results do not provide information on the dynamics of solutions on relevant time scales. To this purpose one should focus on specific regimes and try to derive and justify asymptotic models which will describe the dynamics in those regime.

To have an insight on the nature of (4.13) let us have a look at the linearization of (4.13) at the trivial solution $(\zeta, \psi) = (0,0)$, that is the system

$$\begin{cases} \partial_t \psi + g\zeta = 0 \\ \partial\zeta - Z(0)\psi = 0. \end{cases} \tag{4.16}$$

The solution of Laplace equation in the strip $\mathbb{R}^2\times[-h,0]$ with boundary conditions $\phi(X,0)=\psi$ and $\partial_z\phi_{|z=-h}=0$ is

$$\phi(X,z)=\frac{\cosh((z+h)|D|)}{\cosh(h|D|)}\psi,$$

which implies that $Z(0)\psi=\partial_z\phi_{|z=0}=|D|\tanh(h|D|)\psi$, so that (4.16) writes

$$\partial_t^2\zeta+g|D|\tanh(h|D|)\zeta=0. \tag{4.17}$$

A plane wave $\cos(\mathbf{k}\cdot X-\omega t), \mathbf{k}\in\mathbb{R}^d$, is solution if and only if

$$\omega^2=g|\mathbf{k}|\tanh(|\mathbf{k}|h), \tag{4.18}$$

which is the *dispersion relation* of surface gravity waves.

Where surface tension is taken into account, the dispersion relation becomes

$$\omega^2=(g+\frac{\sigma}{\rho}|\mathbf{k}|^2)|\mathbf{k}|\tanh(|\mathbf{k}|h). \tag{4.19}$$

In the infinite depth case ($h\to+\infty$) , (4.17) reduces to

$$\partial_t^2\zeta+g|D|\zeta=0, \tag{4.20}$$

which can be seen as the product of two nonlocal Schrödinger type equations

$$(i\partial_t+g^{1/2}|D|^{1/2})(i\partial_t-g^{1/2}|D|^{1/2})\zeta=0.$$

On the other hand, when $|\mathbf{k}|h\ll 1$ so that $\omega(\mathbf{k})\simeq\sqrt{gh}$, (4.17) reduces formally to the linear wave equation

$$\partial_t^2\zeta-gh\Delta\zeta=0.$$

It is amazing that Lagrange already derived this equation in the very same context in 1781. Also it is worth noticing that this simple linear PDE retains some relevant physics : in case of a tsunami propagating in an ocean of mean depth 4 km, it predicts a velocity of about $\sqrt{4000\times 10}$ m/s $=720$ k/h, which is the correct order of magnitude for such a wave in the ocean.

4.2 Asymptotic models

Because of the complexity of the system, it is almost hopeless to get rigorous informations on the long time dynamics of the water wave equations except in trivial situations such as the perturbation of the state of rest. [3].

One can however, by restricting the range of wavelength, amplitude, steepness,...get information on the dynamics of water waves on physically relevant time scales. The resulting asymptotic systems are mathematically simpler and allow to perform easier numerical simulations. This method is in fact general and can be applied in various physical contexts (nonlinear optics, plasmas physics,...). It leads to canonical equations or systems, which are in some sense universal.

We explain here how to obtain asymptotic models from the full water wave system (4.13). A thorough and complete discussion can be found in [155].

We first define parameters that will be used to obtain a dimensionless form of the equations, and-when small- to derive the asymptotic models.

a will denote a typical amplitude of the wave, *h* the mean depth of the fluid layer, λ a typical horizontal wave length (we will not distinguish for the moment the *x* and *y* variables in the horizontal two-dimensional case).

We now adimensionalize the variables and unknowns as follows

$$\tilde{\zeta} = \frac{\zeta}{a}, \quad \tilde{z} = \frac{z}{h}, \quad \tilde{X} = \frac{X}{\lambda}$$

$$\tilde{t} = \frac{t}{\lambda/\sqrt{gh}}, \quad \tilde{\phi} = \frac{\phi}{a\lambda\sqrt{g/h}}.$$

We introduce two important dimensionless parameters

$$\varepsilon = \frac{a}{h} \quad \text{and} \quad \mu = \left(\frac{h}{\lambda}\right)^2$$

which measure respectively the *amplitude* and the *shallowness* of the flow.

The adimensionalized version of the water waves system write in those

[3] this leads however to interesting and difficult mathematical problems (see for instance [9] and the references therein)

variables (dropping the tilde) :

$$\begin{cases} \mu\Delta\phi + \partial_z^2\phi = 0, \quad -1 < z < \varepsilon\zeta \\ \partial_z\phi = 0, \quad z = 1, \\ \partial_t\phi + \zeta + \dfrac{\varepsilon}{2}|\nabla\phi|^2 + \dfrac{\varepsilon}{2\mu}|\partial_z\phi|^2 = 0, \quad z = \varepsilon\zeta, \\ \partial_t\zeta + \varepsilon\nabla\zeta\cdot\nabla\phi - \dfrac{1}{\mu}\partial_z\phi = 0, \quad z = \varepsilon\zeta. \end{cases} \tag{4.21}$$

When $\varepsilon = 0$, the nonlinear terms vanish while when $\mu = 0$, the two first equations yield $\partial_z\phi = 0$ and the dispersive terms vanish.

We define (always dropping the tilde)

$$\psi(X,t) = \phi(X, \varepsilon\zeta(X,t),t)) \quad \text{and} \quad Z_\mu(\varepsilon\zeta)\psi = \partial_z\phi_{|z=\varepsilon\zeta},$$

allowing to write the adimensionalized version of (4.13)

$$\begin{cases} \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \dfrac{\varepsilon}{2}\left(\dfrac{1}{\mu} + \varepsilon^2|\nabla\zeta|^2\right)(Z_\mu(\varepsilon\zeta)\psi)^2 = 0 \\ \partial_t\zeta + \varepsilon\nabla\psi\cdot\nabla\zeta - \left(\dfrac{1}{\mu} + \varepsilon^2|\nabla\zeta|^2\right)Z_\mu(\varepsilon\zeta)\psi = 0. \end{cases} \tag{4.22}$$

When one (or both) parameter ε or μ is small, one can find asymptotic expansions of $Z_\mu(\varepsilon\zeta)\cdot$ and thus replace it in (4.22) by simpler (often local) operators. Expansions with respect to μ (supposed to be small) amounts to studying how the solution of a linear elliptic equation depends on the boundary.

Of course several regimes can be considered. We recall a few typical ones.

1. Let $S = \frac{\varepsilon}{\mu}$, the Stokes number. When $\varepsilon, \mu \ll 1$ with $S \sim 1$, one has the *Boussinesq regime* which eventually will lead to the Korteweg-de Vries equation.

2. When $\mu \ll 1$ and $\varepsilon \sim 1$, say $\varepsilon = 1$, (thus $S \gg 1$) one has the *nonlinear regime*.

3. One could also define the *steepness parameter* $\varepsilon = \frac{h}{\lambda}$. The case $\varepsilon \ll 1$, (which allows *large* amplitudes, that is $\varepsilon \sim 1$) leads to *full-dispersion models*.

We describe briefly now the classical systems that are obtained in those regimes (see [12, 155] for details).

4.2.1 The Boussinesq regime

Since here $\varepsilon \sim \mu \ll 1$ we can take $\varepsilon = \mu$. The strategy is first to strengthen the fluid domain to the flat strip $\{-1 < z < 0\}$ and to look for a WKB expansion for the velocity potential

$$\phi_{\text{app}} = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$$

where

$$\phi_{0|z=0} = \psi, \quad \phi_{j|z=0} = 0\,, j = 1,2, \quad \phi_{j|z=-1} = 0,\ j = 0,1,2.$$

One finds (see [155])

$$Z_\varepsilon(\varepsilon\zeta) = -\varepsilon\Delta\psi - \varepsilon^2(\frac{1}{3}\Delta^2\psi + \zeta\Delta\psi) + O(\varepsilon^3).$$

Reporting in (4.22) one obtains a first Boussinesq system

$$\begin{cases} \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 = 0(\varepsilon^2) \\ \partial_t\zeta + \varepsilon\nabla\psi\cdot\nabla\zeta + \Delta\psi + \varepsilon(\dfrac{1}{3}\Delta^2\psi + \zeta\Delta\psi) = 0(\varepsilon^2). \end{cases} \tag{4.23}$$

Setting $U = \nabla\psi$, taking the gradient of the first equation and dropping the $0(\varepsilon^2)$ terms, one gets

$$\begin{cases} \partial_t U + \nabla\zeta + \dfrac{\varepsilon}{2}\nabla|U|^2 = 0, \\ \partial_t\zeta + \nabla\cdot U + \varepsilon(\nabla\cdot(\zeta U) + \dfrac{1}{3}\Delta\nabla\cdot U) = 0. \end{cases} \tag{4.24}$$

Remark 4.4. 1. Since by definition $U = \nabla\phi_{|z=\varepsilon\zeta}$, the expansion of $Z_\varepsilon(\varepsilon\zeta)$ implies that $U = \nabla_{|z=\varepsilon\zeta} + 0(\varepsilon^2)$ and is therefore an approximation of the vertical component of the velocity at the interface.

2. The system (4.24) has the big shortcoming of being linearly ill-posed to short waves. In fact the linearization at $(\zeta, U) = (0,0)$ is equivalent to the equation $\zeta_{tt} - \Delta\zeta + \frac{1}{3}\Delta^2\zeta = 0$ and plane wave solutions $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ should satisfy

$w(\mathbf{k})^2 = |\mathbf{k}|^2(1 - \frac{1}{3}|\mathbf{k}|^2)$ which is not real for large $|\mathbf{k}|$.

One can nevertheless obtain an equivalent 3-parameters *family* of Boussinesq systems with better mathematical properties. One parameter arises from considering an approximation of the horizontal component of the velocity at the height $z = \theta - 1$. It turns out that such an approximation is given by

$$U_\theta = (1 + \frac{\varepsilon}{2}(1-\theta^2)\Delta)U + 0(\varepsilon^2).$$

Two more parameters are introduced by a double use of the so-called BBM (Benjamin-Bona-Mahony [20]) trick, that is to write

$$\partial_t U = -\nabla\zeta + O(\varepsilon) \quad \text{and} \quad \partial_t \zeta = -\nabla \cdot U + O(\varepsilon),$$

leading to

$$\begin{cases} \partial_t U = (1-\mu)\partial_t U - \mu\nabla\zeta + O(\varepsilon) \\ \nabla \cdot U = \lambda\nabla \cdot U - (1-\lambda)\partial_t\zeta + O(\varepsilon). \end{cases} \tag{4.25}$$

Plugging these relations into (4.24) one obtains (dropping the error terms $O(\varepsilon^2)$ and denoting $U = U_\varepsilon$,) the $S_{\theta,\lambda,\mu}$ family of Boussinesq systems ([34, 35, 30])

$$\begin{cases} \partial_t U + \nabla\zeta + \varepsilon(\frac{1}{2}\nabla|U|^2 + a\Delta\nabla\zeta - b\Delta\partial_t U) = 0 \\ \partial_t\zeta + \nabla \cdot U + \varepsilon(\nabla\cdot(\zeta U) + c\Delta\nabla\cdot U - d\Delta\partial_t\zeta) = 0, \end{cases} \tag{4.26}$$

where

$$a = \frac{1-\theta^2}{2}\mu, \quad b = \frac{1-\theta^2}{2}(1-\mu),$$

$$c = (\frac{\theta^2}{2} - \frac{1}{6})\lambda, \quad d = (\frac{\theta^2}{2} - \frac{1}{6})(1-\lambda),$$

so that $a+b+c+d = \frac{1}{3}$.

Remark 4.5. 1. The Boussinesq systems can be conveniently classified according to the linearization at the null solutions which display their dispersive properties [35]. More precisely, the dispersion matrix writes in Fourier variables,

$$\widehat{A}(\xi_1,\xi_2) = i\begin{pmatrix} 0 & \frac{\xi_1(1-\varepsilon a|\xi|^2)}{1+\varepsilon b|\xi|^2} & \frac{\xi_2(1-\varepsilon a|\xi|^2)}{1+\varepsilon b|\xi|^2} \\ \frac{\xi_1(1-\varepsilon c|\xi|^2)}{1+\varepsilon d|\xi|^2} & 0 & 0 \\ \frac{\xi_2(1-\varepsilon c|\xi|^2)}{1+\varepsilon d|\xi|^2} & 0 & 0 \end{pmatrix}.$$

The corresponding non zero eigenvalues are

$$\lambda_\pm(\xi) = \pm i|\xi|\left(\frac{(1-\varepsilon a|\xi|^2)(1-\varepsilon c|\xi|^2)}{(1+\varepsilon d|\xi|^2)(1+\varepsilon b|\xi|^2)}\right)^{\frac{1}{2}}.$$

One deduces that the Boussinesq systems are linearly well-posed when $b \geq 0,\ d \geq 0$ and $a \leq 0,\ c \leq 0$, (or $a = c$).

2. Although all equivalent (see below for a precise notion), the Boussinesq systems have different mathematical properties because of the various behavior of the dispersion (that is of the $\lambda'_\pm s$) for large frequencies. In particular, $\lambda_\pm(\xi)$ can have order $3, 2, 1, 0, -1, -2$ depending on the values of (a,b,c,d).

Remark 4.6. When surface tension is taken into account, one obtains a similar class of Boussinesq systems, [68] with a changed into $a - T$ where $T \geq 0$ is the Bond number which measures surface tension effects. The constraint on the parameters a,b,c,d is the same and the conditions for linear well-posedness read now

$$\begin{cases} a - T \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \\ \text{or} \\ a - T = c > 0, \quad b \geq 0, \quad d \geq 0. \end{cases} \tag{4.27}$$

The Boussinesq systems can be simplified when one restricts the allowed motion. First for a one dimensional wave propagating to the right and weakly modulated, one looks for u and ζ of the form

$$u(x,t) = \tilde{u}(x-t,\varepsilon t), \quad \zeta(x,t) = \tilde{\zeta}(x-t,\varepsilon t).$$

Denoting $\tau = \varepsilon t$ and $\xi = x - t$, and considering firstly the original Boussinesq system (4.24) $\tilde{u}$ and $\tilde{\zeta}$ satisfy

$$\begin{cases} \varepsilon\partial_\tau\tilde{u} - \partial_\xi\tilde{u} + \partial_\xi\tilde{\zeta} + \varepsilon\tilde{u}\partial_\xi\tilde{u} = 0 \\ \varepsilon\partial_\tau\tilde{\zeta} - \partial_\xi\tilde{\zeta} + \partial_\xi\tilde{u} + \varepsilon(\partial_\xi(\tilde{\zeta}\tilde{u}) + \frac{1}{3}\partial_\xi^2\tilde{u}) = 0 \end{cases} \tag{4.28}$$

One thus has $\partial_\xi\tilde{u} = \partial_\xi\tilde{\zeta} + 0(\varepsilon)$ and one can replace $\tilde{u}$ by $\tilde{\zeta}$ in the dispersive and nonlinear terms, the resulting error being $0(\varepsilon^2)$. Adding then the equation and neglecting the $0(\varepsilon^2)$ terms one arrives to the Korteweg-de Vries equation

$$\partial_\tau\tilde{\zeta} + \frac{1}{6}\partial_\xi^3\tilde{\zeta} + \frac{3}{2}\tilde{\zeta}\partial_\xi\tilde{\zeta} = 0.$$

One easily check that starting from a $S_{\theta,\lambda,\mu}$ system would have led to

$$\partial_\tau\tilde{\zeta} + \frac{a+b+c+d}{2}\partial_\xi^3\tilde{\zeta} + \frac{3}{2}\tilde{\zeta}\partial_\xi\tilde{\zeta} = 0,$$

that is to the same equation since $a+b+c+d = \frac{1}{3}$.

4.2.2 The weakly transverse "KP" regime

Another interesting motion is the *weakly transverse one*. We assume that the scales in two horizontal variables (x,y) are different, more precisely, we introduce a new scaling, denoting again by a a typical amplitude, h the mean depth, λ_1 the typical wavelength along the longitudinal direction x and λ_2, the wavelength along the transverse direction y. We still denote $\varepsilon = \frac{a}{h} \ll 1$ but we will assume now that

$$\frac{h^2}{\lambda_1^2} \sim \varepsilon, \quad \frac{h^2}{\lambda_2^2} \sim \varepsilon^2.$$

The KP equation is obtained by decomposing the elevation as the sum of two counter propagating waves following the ansatz

$$\zeta_\varepsilon^{KP}(x,y,t) = \frac{1}{2}(\zeta_+(x-t,\sqrt{\varepsilon}y,\varepsilon t) + \zeta_-(x-t,\sqrt{\varepsilon}y,\varepsilon t)).$$

Neglecting the $O(\varepsilon^2)$ terms, one checks that $\zeta_\pm(X,Y,\tau)$ solves the Kadomtsev-Petviashvili II (KP II) equation :

$$\partial_\tau \zeta_\pm \pm \frac{1}{2}\partial_X^{-1}\partial_Y^2\zeta_\pm \pm \frac{1}{6}\partial_X^3\zeta_\pm \pm \frac{3}{2}\zeta_\pm\partial_X\zeta_\pm = 0. \qquad (4.29)$$

When strong surface tension effects are present (Bond number $> \frac{1}{3}$ one obtains the KP I equation (the sign of the $\frac{1}{2}\partial_X^{-1}\partial_Y^2\zeta_\pm$ term is $\mp$).

It turns out that the precision of the KP approximation is much worse than the KdV or Boussinesq one : in general $o(1)$ instead of $O(\varepsilon^2 t)$ (see [155, 159]). It has been established rigorously in [154], [159] that the KP II equation yields a poor error estimate when used as an asymptotic model of the water wave system. Roughly speaking, the error estimates with the relevant solution of the (Euler) water waves system writes :

$$||U_{Euler} - U_{KP}|| = o(1).$$

So the error is $o(1)$, ($O(\sqrt{\varepsilon})$ with some additional constraint) instead of $O(\varepsilon^2 t)$, which should be the optimal rate in this regime (as it is the case for the KdV, Boussinesq, equations or systems.). Nevertheless the KP II equation reproduces (qualitatively) observed features of the water wave theory. For instance the well-known picture below displays the interaction of two oblique "line solitary waves" in the Oregon coast which shows a striking resemblance with the so-called KP II 2-soliton.

Figure 1: Interaction of line solitons. Oregon coast

This is due to the singularity of the dispersion relation $\xi_1^3 + \frac{\xi_2^2}{\xi_1}$ at $\xi_1 = 0$ which of course is not present in the original water wave dispersion. This implies a constraint on the x-mean of u which is not physical (see however Chapter 9 below).

One can however derive a five parameters family of *weakly transverse* Boussinesq systems which are consistent with the Euler system, do not suffer from the unphysical zero-mass constraint and have the same precision as the isotropic ones (see [36]) :

$$\begin{cases} \partial_t v + \partial_x \zeta + \varepsilon(a\partial_x^3 \zeta - b\partial_x^2 \partial_t v + v\partial_x v + \frac{1}{2} w\partial_x w) + \frac{1}{2}\varepsilon^{3/2} w\partial_y w = 0 \\ \partial_t w + \sqrt{\varepsilon}\partial_y \zeta + \varepsilon(-e\partial_x^2 \partial_t w + w\partial_y w + \frac{1}{2} v\partial_x w) + \varepsilon^{3/2}(f\partial_x^2 \partial_y \zeta + \frac{1}{2} v\partial_y v) = 0 \\ \partial \zeta + \partial_x v + \sqrt{\varepsilon}\partial_y w + \varepsilon(v\partial_x \zeta + \zeta\partial_x v + c\partial_x^3 v - d\partial_x^2 \partial_t \zeta) \\ +\varepsilon^{3/2}(w\partial_y \zeta + \zeta\partial_y w + g\partial_x^2 \partial_y w) = 0, \end{cases} \tag{4.30}$$

where $\nabla\psi = (v, w)$ and

$$\begin{cases} a = \frac{1-\theta^2}{2}\mu, \quad b = \frac{1-\theta^2}{2}(1-\mu), \\ c = (\frac{\theta^2}{2} - \frac{1}{6})\lambda, \quad d = (\frac{\theta^2}{2} - \frac{1}{6})(1-\lambda) \\ e = \frac{1}{2}(1-\sigma^2)(1-\nu), \quad f = \frac{1}{2}(1-\sigma^2)\nu \\ g = (\frac{\sigma^2}{2} + \frac{1}{6}) - (\frac{\theta^2}{2} - \frac{1}{6})(1-\lambda), \end{cases} \tag{4.31}$$

$$0 \le \theta \le 1, \quad \lambda, \mu, \sigma, \nu \in \mathbb{R}.$$

A similar system for capillary-gravity waves has been obtained in [180].

Remark 4.7. The $\partial_x^{-1}\partial_y^2 u$ term of the KP equation comes from the strong uni-directionalization made in its derivation. In the weakly transverse Boussinesq systems, the uni-directionalization is less strong (it is modelled by the introduction of the larger transverse wavelength in the nondimensionalization); this yields less disastrous consequences: the zero mass constraints are replaced by the possible growth in the second component of the velocity, taken into account by the assumption that $\sqrt{\varepsilon}\partial_y \psi$ is bounded (see [159]).

The family of a, b, c, d isotropic Boussinesq systems cannot be used in the "KP scaling" considered here. In fact, as observed in [159], these isotropic systems do not provide a good convergence rate in the present case: indeed, it is shown in [30] that the error made by these models in approximating the full water-waves equations is of size $\varepsilon^2 C(||\zeta^0||_{H^s}, ||\nabla\psi^0||_{H^s})t$, for s large enough, and where (ζ^0,ψ^0) is the initial condition. For weakly transverse initial conditions of the form

$$\zeta^0(x,y) = \tilde{\zeta}^0(x,\sqrt{\varepsilon}y), \quad \psi^0(x,y) = \tilde{\psi}^0(x,\sqrt{\varepsilon}y),$$

with $\tilde{\zeta}^0$ and $\tilde{\psi}^0$ bounded in $H^s(\mathbb{R}^2)$,, the error estimates of [30] are therefore $\varepsilon^2 C(\varepsilon^{-1/4})t$ and may thus grow to infinity when $\varepsilon \to 0$, justifying the introduction of the weakly transverse systems which give the correct error estimate $O(\varepsilon^2 t)$.

For the sake of completeness we recall the original derivation of Kadomtsev and Petviashvili [122] that displays the universal nature of the KP approximation. The argument is a linear one. Actually the (formal) analysis in [122] consists in looking for a *weakly transverse* perturbation of the one-dimensional transport equation

$$u_t + u_x = 0. \tag{4.32}$$

As observed in [130] the correction to (4.32) due to weak transverse effects is independent of the dispersion in x and is related only to the finite propagation speed properties of the transport operator $M = \partial_t + \partial_x$. Recall that M gives rise to one-directional waves moving to the right with speed one; i.e., a profile $\varphi(x)$ evolves under the flow of M as $\varphi(x-t)$. A weak transverse perturbation of $\varphi(x)$ is a two-dimensional function $\psi(x,y)$ close to $\varphi(x)$, localized in the frequency region $\left|\frac{\eta}{\xi}\right| \ll 1$, where ξ and η are the Fourier modes corresponding to x and y, respectively. We thus look for a two-dimensional perturbation $\widetilde{M} = \partial_t + \partial_x + \omega(D_x, D_y)$ of M such that, similarly to above, the profile of $\psi(x,y)$ does not change much when evolving under the flow of $\widetilde{M}$. Here $\omega(D_x,D_y)$ denotes the Fourier multiplier with symbol the real function $\omega(\xi,\eta)$. Natural generalizations of the flow of M in two dimensions are the flows of the wave operators $\partial_t \pm \sqrt{-\Delta}$ which enjoy the finite propagation speed property.

Since, by a Taylor expansion of the dispersion relation $\omega(\xi,\eta) = \sqrt{\xi^2+\eta^2} = \pm\xi(1+\frac{\eta^2}{\xi^2})^{1/2} \sim \pm(\xi + \frac{1}{2}\frac{\eta^2}{\xi})$ of the two-dimensional linear wave equation assuming $|\xi|$ and $\frac{|\eta|}{|\xi|} \ll 1$, we deduce that

$$\partial_t + \partial_x + \frac{1}{2}\partial_x^{-1}\partial_y^2 \sim \partial_t \pm \sqrt{-\Delta},$$

which leads to the correction $\omega(D_x, D_y) = \frac{1}{2}\partial_x^{-1}\partial_y^2$ in (4.32) thus to the equation

$$u_t + u_x + \frac{1}{2}\partial_x^{-1}u_{yy} = 0. \tag{4.33}$$

Here the operator ∂_x^{-1} is defined via Fourier transform,

$$\widehat{\partial_x^{-1}f}(\xi) = \frac{i}{\xi_1}\widehat{f}(\xi), \text{ where } \xi = (\xi_1, \xi_2).$$

Of course when the transverse effects are two-dimensional, the correction is $\frac{1}{2}\partial_x^{-1}\Delta_\perp$, where $\Delta_\perp = \partial_y^2 + \partial_z^2$.

Remark 4.8. Equation (4.33) is reminiscent of the *linear diffractive pulse equation*

$$2u_{tx} = \Delta_y u,$$

where Δ_y is the Laplace operator in the transverse variable y, studied in [11].

The same formal procedure is applied in [122] to the KdV equation (10.24), assuming that the transverse dispersive effects are of the same order as the x-dispersive and nonlinear terms, yielding the KP equation in the form

$$u_t + u_x + uu_x + (\frac{1}{3} - T)u_{xxx} + \frac{1}{2}\partial_x^{-1}u_{yy} = 0. \tag{4.34}$$

where T is the Bond number which measures the surface tension effects.

By change of frame and scaling, (4.34) reduces to

$$u_t + uu_x + u_{xxx} \pm \partial_x^{-1}u_{yy} = 0$$

with the $+$ sign (KP II) when $T < \frac{1}{3}$ and the $-$ sign (KP I) when $T > \frac{1}{3}$.

Note however that $T > \frac{1}{3}$ corresponds to a layer of fluid of depth smaller than 0.46 cm, and in this situation viscous effects due to the boundary layer at the bottom cannot be ignored. One could then say that “the KP I equation does not exist in the context of water waves”, but it appears naturally in other contexts, for instance in the long wave approximation of the Gross-Pitaevskii equation, see [54].

Of course the same formal procedure could also be applied to *any* one-dimensional weakly nonlinear dispersive equation of the form

$$u_t + u_x + f(u)_x - Lu_x = 0,\ x \in \mathbb{R},\ t \geq 0, \tag{4.35}$$

where $f(u)$ is a smooth real-valued function (most of the time polynomial) and L a linear operator taking into account the dispersion and defined in Fourier variable by

$$\widehat{Lu}(\xi) = p(\xi)\mathfrak{F}u(\xi), \tag{4.36}$$

where the symbol $p(\xi)$ is real-valued. The KdV equation corresponds for instance to $f(u) = \frac{1}{2}u^2$ and $p(\xi) = -\xi^2$. Examples with a fifth order dispersion in x are considered in [3], [125], [126].

This leads to a class of generalized KP equations

$$u_t + u_x + f(u)_x - Lu_x + \frac{1}{2}\partial_x^{-1}u_{yy} = 0, \; x \in \mathbb{R}, \; t \geq 0. \tag{4.37}$$

The Cauchy problem for the KP I type equation associated to a fifth order KdV equation is studied in [205].

4.2.3 The Camassa-Holm regime

This *medium amplitude* regime corresponds to $\mu \ll 1$ and $\varepsilon = O(\sqrt{\mu})$. It is therefore more nonlinear than the Boussinesq one. For one-directional one-dimensional waves it leads (see[155]) to the Camassa-Holm equation :

$$U_t + \kappa U_x + 3UU_x - U_{xxt} = 2U_xU_{xx} + UU_{xxx}, \tag{4.38}$$

with $\kappa \in \mathbb{R}$. We refer to [59] for a rigorous derivation.

4.2.4 The Saint-Venant (nonlinear) regime

In this regime, one has $\varepsilon \sim 1$ and $\mu \ll 1$. One can prove (see [155]) that the Dirichlet-Neumann operator has the expansion

$$Z_\mu(\zeta)\psi = -\mu(1+\zeta)\Delta\psi + 0(\mu^2),$$

leading to the system

$$\begin{cases} \partial_t\psi + \zeta + \frac{1}{2}|\nabla\psi|^2 = O(\mu) \\ \partial_t\zeta + \nabla\psi \cdot \nabla + (1+\zeta)\Delta\psi = O(\mu). \end{cases} \tag{4.39}$$

Setting $U = \nabla\psi$ and dropping the $O(\mu)$ terms we obtain the Saint-Venant system

$$\begin{cases} \partial_t U + \nabla\zeta + \frac{1}{2}\nabla|U|^2 = 0 \\ \partial_t \zeta + \nabla\cdot U + \nabla\cdot(\zeta U) = 0. \end{cases} \tag{4.40}$$

Note that this system contains no dispersive terms and is purely hyperbolic (in the domain $1+\zeta > 0$).

One obtains dispersive terms by going one order further in the expansion of $Z_\mu(\zeta)\psi$. This leads to the Green-Naghdi system [101] (see [12, 13] for a complete rigorous derivation).

$$\begin{cases} (h+\mu\mathfrak{T}[h])\partial_t U + h\nabla\zeta + h(U\cdot\nabla)U + \frac{\mu}{3}\nabla(h^3\mathfrak{D}_U \operatorname{div} U) = 0 \\ \partial_t \zeta + \nabla\cdot U + \nabla\cdot(\zeta U) = 0, \end{cases} \tag{4.41}$$

where $h = 1+\zeta$, $\mathfrak{T}[h] = -\frac{1}{3}\nabla(h^3\nabla\cdot U)$ and $\mathfrak{D}_U = -(U\cdot\nabla U) + \operatorname{div} U$.

Remark 4.9. 1. One recovers the Saint-Venant system when $\mu = 0$.

=

2. The choice $\varepsilon = \sqrt{\mu}$ instead of $\varepsilon = 1$ would have led to the Serre system (see [12, 13]).

4.2.5 The full dispersion regime.

In this regime, the *shallowness parameter* $\mu = \frac{h^2}{\lambda^2}$ is allowed to be large (deep water) but the *steepness parameter* $\tilde{\varepsilon} = \varepsilon\sqrt{\mu}$ is supposed to be small. This leads to the *Full dispersion* system, derived formally in [172, 173, 55] (see [155] for a rigorous derivation):

$$\begin{cases} \partial_t \zeta - \mathscr{T}_\mu U + \tilde{\varepsilon}(\mathscr{T}_\mu(\zeta\nabla\mathscr{T}_\mu U) + \nabla\cdot(\zeta U)) = 0, \\ \partial_t U + \nabla\zeta + \tilde{\varepsilon}\left(\frac{1}{2}\nabla|U|^2 - \nabla\zeta\,\mathscr{T}_\mu\nabla\zeta\right) = 0, \end{cases} \tag{4.42}$$

where $\mathscr{T}_\mu$ is the Fourier multiplier defined by

$$\widehat{\mathscr{T}_\mu U}(\xi) = -\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|}(i\xi)\cdot\widehat{U}(\xi).$$

In the infinite depth case, (4.42) simplifies to

$$\begin{cases} \partial_t \zeta - \mathscr{H} V + \tilde{\varepsilon}(\mathscr{H}(\zeta \nabla \mathscr{H} V) + \nabla \cdot (\zeta V)) = 0, \\ \partial_t V + \nabla \zeta + \tilde{\varepsilon} \left(\frac{1}{2} \nabla |V|^2 - \nabla \zeta \mathscr{H} \nabla \zeta \right) = 0, \end{cases} \tag{4.43}$$

where $\mathscr{H} = -\frac{1}{|D|} \nabla^T$.

4.3 The modulation regime

This regime is somewhat different from the previous ones since the asymptotic models approximate *wave packets*, that is fast oscillating waves whose amplitude is slowly varying. One obtains equations or systems of *nonlinear Schrödinger* type, the real part of the unknown being an approximation of the slowly varying amplitude of the wave.

The first derivation of a nonlinear Schrödinger equation (NLS) as equation of the envelope of wave trains with slowly varying amplitudes was performed in the pioneering paper [21].

The formal derivation of the nonlinear Schrödinger equation in this regime in the context of water waves has been obtained by Zakharov in [248]. Benney and Roskes [22] derived the so-called Benney-Roskes system. [4] Davey and Stewartson [70] and Djordjevic and Redekopp [71] when surface tension is included (see also Ablowitz and Segur [2]) derived the Davey-Stewartson systems in the context of water waves. A rigorous derivation (in the sense of consistency) of the Davey-Stewartson system is made in [64, 65]. A complete rigorous treatment is given [155] and the sketch below follows closely this reference.

One starts from the water waves equations on the form

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} \mathscr{G} \psi = 0, \\ \partial_t \psi + \zeta + \frac{\tilde{\varepsilon}}{2} |\nabla \psi|^2 - \tilde{\varepsilon} \frac{(\frac{1}{\sqrt{\mu}} \mathscr{G} \psi + \tilde{\varepsilon} \nabla \zeta \cdot \nabla \psi)^2}{2(1 + \tilde{\varepsilon}^2 |\nabla \zeta|^2)} = 0, \end{cases} \tag{4.44}$$

[4] A similar system was derived by Zakharov and Rubenchik [252] as a "universal" system to describe the interaction of short and long waves.

where the version $\mathscr{G}$ of the Dirichlet-Neumann operator is defined by

$$\mathscr{G}[\zeta]\psi = \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}\phi_{|z=\zeta}.$$

and, as in the Full dispersion regime, $\tilde{\varepsilon} = \varepsilon\sqrt{\mu} = \frac{\text{typical amplitude}}{\text{typical horizontal scale}} = \frac{a}{\lambda}$ denotes again the steepness parameter.

The linearization of (4.44) around the trivial solution is

$$\partial_t U + \mathscr{A}_0(D)U = 0, \quad \text{with} \quad \mathscr{A}_0(D) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mu}}\mathscr{G}_0(D) \\ 1 & 0 \end{pmatrix}, \tag{4.45}$$

where $\mathscr{G}_0(D) = \sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)$ and $U = (\zeta,\psi)^T$.

Equation (4.45) admits real-valued plane wave solutions

$$U(X,t) = \begin{pmatrix} i\omega\psi_{01} \\ \psi_{01} \end{pmatrix} e^{i\theta} + \text{complex conjugate},$$

where ψ_{01} is a constant and

$$\theta = X\cdot\mathbf{k} - \omega t, \quad \omega = \omega(\mathbf{k}) = (|\mathbf{k}|\tanh(\sqrt{\mu}|\mathbf{k}|))^{1/2}.$$

The idea is to look for approxlmate solutions of (4.44) under the form of wave packets.

We will drop the tilde on the ε's for the rest ot the Section.

$$U(X,t) = \begin{pmatrix} i\omega\psi_{01}(\varepsilon X,\varepsilon t) \\ \psi_{01}(\varepsilon X,\varepsilon t) \end{pmatrix} e^{i\theta} + \text{complex conjugate}. \tag{4.46}$$

Since the nonlinearities in (4.44) do not preserve the structure of (4.46) and create higher order harmonics, one must in fact look for approximate solutions of a more general form, that is

$$U_{\text{app}}(X,t) = U_0(X,t) + \varepsilon U_1(X,t) + \varepsilon^2(X,t), \tag{4.47}$$

where the leading term U_0 is the sum of a wave packet similar to (4.46) and of a non oscillating term necessary to describe the creation of a mean mode by nonlinear interaction of oscillating modes. More precisely,

$$U_0(X,t) = \begin{pmatrix} i\omega\psi_{01}(\varepsilon X,\varepsilon t) \\ \psi_{01}(\varepsilon X,\varepsilon t) \end{pmatrix} e^{i\theta} + \text{c.c.} + \begin{pmatrix} 0 \\ \psi_{00}(\varepsilon X,\varepsilon t) \end{pmatrix}.$$

The corrector terms U_1, U_2 are sought under the form

$$U_1(X,t) = \begin{pmatrix} \zeta_{11}(\varepsilon X,\varepsilon t)e^{i\theta} + \zeta_{12}(\varepsilon X,\varepsilon t)e^{2i\theta} + c.c. + \zeta_{10}(\varepsilon X,\varepsilon t) \\ \psi_{11}(\varepsilon X,\varepsilon t)e^{i\theta} + \psi_{12}(\varepsilon X,\varepsilon t)e^{2i\theta} + c.c. \end{pmatrix},$$

$$U_2(X,t) = \sum_{n=1}^{3} U_{2n}(\varepsilon X,\varepsilon t)e^{in\theta} + c.c. + U_{20}(\varepsilon X,\varepsilon t).$$

The strategy is now to plug the ansatz into (4.44) and to cancel the leading order terms in ε.

This involves a multiscale expansion of the Dirichlet-Neumann operator and leads eventually to the following *Full dispersion Benney-Roskes system* (see [155]), where $\tau = \varepsilon t$ and the spatial derivatives are taken with respect to the slow space variable $\tilde{X} = \varepsilon X$.

$$\begin{cases} \partial_\tau \psi_{01} + i\dfrac{\omega(\mathbf{k}+\varepsilon D) - \omega(\mathbf{k})}{\varepsilon}\psi_{01} \\ \quad + \varepsilon i[\mathbf{k}\cdot\nabla\psi_{00} + \dfrac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta_{10} + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_{01}|^2]\psi_{01} = 0 \\ \partial_\tau \zeta_{10} - |D|\dfrac{\tanh(\varepsilon\sqrt{\mu}|D|)}{\varepsilon}\psi_{00} = 2\omega\mathbf{k}\cdot\nabla|\psi_{01}|^2, \\ \partial_\tau \psi_{00} + \zeta_{10} = -|\mathbf{k}|^2(1-\sigma^2)|\psi_{01}|^2, \end{cases} \tag{4.48}$$

where $\sigma = \tanh(\sqrt{\mu}|\mathbf{k}|)$ and $\alpha = -\frac{9}{8\sigma^2}(1-\sigma^2)^2$.

The classical Benney-Roskes system, derived formally in [22], in [252] in the context of acoustic waves, in [193] for plasma waves and in [253] as a “universal model” for interaction of short and long waves is deduced from (4.48) by approximating the nonlocal operator $i\omega(\mathbf{k}+\varepsilon D)$ by its Taylor expansion in ε at order two, that is by the differential operator $i\omega + \varepsilon\nabla\omega(\mathbf{k})\cdot\nabla - \varepsilon^2\frac{i}{2}\nabla\mathscr{H}_\omega(\mathbf{k})\nabla$.

The resulting system takes a simple form if one assume (without loss of generality) that $\mathbf{k}$ is oriented along the x-axis, that is $\mathbf{k} = |\mathbf{k}|\mathbf{e}_x$. Following [155] we also use the notations

$$\omega(\mathbf{k}) = \tilde{\omega}(|\mathbf{k}|), \quad \text{with} \quad \tilde{\omega}(r) = (r\tanh(\sqrt{\mu}r))^{1/2},$$

$$\omega = \tilde{\omega}(|\mathbf{k}|), \quad \omega' = \tilde{\omega}'(|\mathbf{k}|), \quad \omega'' = \tilde{\omega}''(|\mathbf{k}|)$$

and obtain

$$\begin{cases} \partial_\tau \psi_{01} + \omega' \partial_x \psi_{01} - i\varepsilon \frac{1}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi_{01} \\ \quad + \varepsilon i[|\mathbf{k}|\partial_x \psi_{00} + \frac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta_{10} + 2\frac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_{01}|^2]\psi_{01} = 0 \\ \partial_\tau \zeta_{10} + \sqrt{\mu}\Delta\psi_{00} = -2\omega|\mathbf{k}|\partial_x|\psi_{01}|^2, \\ \partial_\tau \psi_{00} + \zeta_{10} = -|\mathbf{k}|^2(1-\sigma^2)|\psi_{01}|^2. \end{cases} \tag{4.49}$$

We will study the local Cauchy problem for Benney-Roskes systems in Chapter 13.

One can derived a simplified system from the Benney-Roskes system using the fact that at leading order, Ψ_{01} travels at the group velocity $c_g = \nabla\omega(\mathbf{k})$. We refer to [155] for details. This amounts in particular in replacing ∂_τ in the two last equations of (4.48) by $-c_g\nabla$, where ∇ denotes here the gradient with respect to the variable $x - c_g t$.

One finally arrives at (see [155])

$$\begin{cases} \partial_\tau \psi_{01} - \frac{i}{2}\left(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2\right)\psi_{01} + i(\beta\partial_x\psi_{00} + 2\frac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha}|\psi_{01}|^2)\psi_{01} = 0 \\ [(\sqrt{\mu} - \omega'^2)\partial_x^2 + \sqrt{\mu}\partial_y^2]\psi_{00} = -2\omega\beta\partial_x|\psi_{01}|^2, \end{cases} \tag{4.50}$$

where

$$\beta = |\mathbf{k}|(1 + (1-\sigma^2)\frac{\omega'|\mathbf{k}|}{2\omega}, \quad \tilde{\alpha} + \frac{1}{4}(1-\sigma^2)^2,$$

α as previously while ζ_{10} is given by

$$\zeta_{10} = \tilde{\omega}'(|\mathbf{k}|)\partial_x\psi_{00} - |\mathbf{k}|^2(1-\sigma^2)|\psi_{01}|^2.$$

The system (4.50) belongs to the family of Davey-Stewartson systems

$$\begin{cases} i\partial_\tau + a\partial_x^2\psi + b\partial_y^2\psi = (\nu_1|\psi|^2 + \nu_2\partial_x\phi)\psi, \\ c\partial_x^2\phi + \partial_y^2\phi = -\delta\partial_x|\psi|^2, \end{cases} \tag{4.51}$$

where $b > 0$ (after a possible change of unknown) and $\delta > 0$. Those conditions are satisfied by (4.50). The nature of (4.51) depends on the sign of

a and c, leading in particular to the so-called DS I and DS II systems (see Chapter 12).

Using the terminology in [92], (4.50) can be classified into four types :

- elliptic-elliptic if (sign a, sign c) = (+1,+1)
- hyperbolic-elliptic if (sign a, sign c) = (-1,+1)
- elliptic-hyperbolic if (sign a, sign c) = (+1,-1)
- hyperbolic-hyperbolic if (sign a, sign c) = (-1,-1).

For the gravity water waves problem, $a = \frac{1}{2}\omega'' < 0$ since $\omega'' = \tilde{\omega}''(|\mathbf{k}|)$ and $\tilde{\omega}''$ defined by $\tilde{\omega}(r) = (r\tanh(\sqrt{\mu}r))^{1/2}$ is concave.

We also have that $c = \sqrt{\mu} - \omega'^2 > 0$. In variables with dimensions, this is equivalent to $M < 1$ where $M = c_g/\sqrt{gh}$ is the Mach number, with $c_g = \frac{d}{dr}\sqrt{gr\tanh(hr)}_{|r=|\mathbf{k}|}$ the group velocity in dimensional form.

As noticed in [215, 71] the Mach number M can be ≥ 1 in presence of surface tension, and then $c \leq 0$.

The Davey-Stewartson systems with $c<0$ ("DS I type Davey-Stewartson systems") have quite different mathematical properties than the "DS II type" (when $c > 0$.) In particular the equation for ϕ is hyperbolic and one should instead of a Dirichlet condition prescribes a radiation condition of type

$$\phi(x,y,t) \to 0 \quad \text{as} \quad x+y,\ x-y \to +\infty.$$

This implies that in the term $\partial_x \phi\, \psi$ which can be written as $R(|\psi|^2)\psi$, the nonlocal operator R has order one (and not zero as in the elliptic-elliptic or hyperbolic-elliptic cases).

Remark 4.10. The derivation of the Davey-Stewartson system breaks down when $c_g = \sqrt{gh}$ ([71, 155]). This corresponds to a long-wave/short-wave resonance in which the group velocity of the short (capillary) wave matches the phase velocity of the long (gravity) wave. A different scaling and analysis are required and one obtains a coupled pair of equations of type (see [71])

$$\begin{cases} iA_\tau + \lambda A_{\xi\xi} = BA \\ B_\tau = -\alpha(|A|^2)_\xi. \end{cases} \tag{4.52}$$

The Cauchy problem for elliptic-hyperbolic or hyperbolic-hyperbolic [5] Davey Stewartson systems is studied in [163]. More results for the elliptic-hyperbolic case are proven in [106, 53]. All these results are established under a smallness condition on the initial data.

For purely gravity waves one thus encounters only the hyperbolic-elliptic Davey-Stewartson systems. Some mathematical results will be presented in Chapter 2.

[5] Note that the hyperbolic-hyperbolic Davey-Stewartson systems do not seem to arise in any physical context.

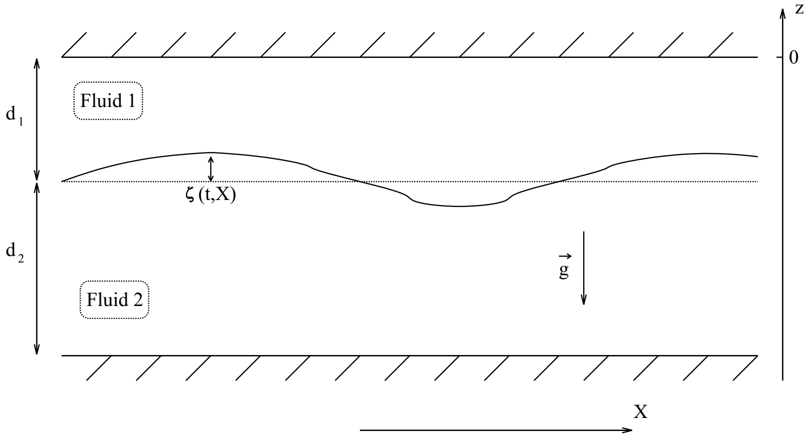
Chapter 5

Internal waves

5.1 The internal waves system

Oceans are often stratified in two (or several) layers of different densities, due to differences of salinity or temperature (see [107]). Gravity waves are generated at the interface (s) and the goal of this Chapter is present a brief overview of the mathematical modeling of those phenomena. We refer to [32, 203] (and to [62] for a different approach) for a more complete description and we just recall the equations and the different regimes. We will consider only the two-layers case with a flat bottom.

Here is the geometry of the two-layer system with flat bottom and rigid lid.



5.1.1 The Equations

As in Figure 1, the origin of the vertical coordinate z is taken at the rigid top of the two-fluid system. Assuming each fluid is incompressible and each flow irrotational, there exists velocity potentials Φ_i $(i = 1,2)$ associated to both the upper and lower fluid layers which satisfy

$$\Delta_{X,z}\Phi_i = 0 \quad \text{in } \Omega_t^i \tag{5.1}$$

for all time t, where Ω_t^i denotes the region occupied by fluid i at time t, $i = 1,2$. As above, fluid 1 refers to the upper fluid layer whilst fluid 2 is the lower layer (see again Figure 1). Assuming that the densities ρ_i, $i = 1,2$, of both fluids are constant, we also have two Bernoulli equations, namely,

$$\partial_t\Phi_i + \frac{1}{2}|\nabla_{X,z}\Phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in } \Omega_t^i, \tag{5.2}$$

where g denotes the acceleration of gravity and P the pressure inside the fluid. These equations are complemented by two boundary conditions stating that the velocity must be horizontal at the two rigid surfaces $\Gamma_1 := \{z = 0\}$ and $\Gamma_2 := \{z = -d_1 - d_2\}$, which is to say

$$\partial_z\Phi_i = 0 \quad \text{on} \quad \Gamma_i, \qquad (i = 1,2). \tag{5.3}$$

Finally, as mentioned earlier, it is presumed that the interface is given as the graph of a function $\zeta(t,X)$ which expresses the deviation of the interface from its rest position $(X,-d_1)$ at the spatial coordinate X at time t.

As in the case of surface waves, the interface $\Gamma_t := \{z = -d_1 + \zeta(t,X)\}$ between the fluids is taken to be a bounding surface, or equivalently it is assumed that no fluid particle crosses the interface. This condition, written for fluid i, is expressed by the relation $\partial_t\zeta = \sqrt{1+|\nabla\zeta|^2}v_n^i$, where v_n^i denotes the upwards normal derivative of the velocity of fluid i at the surface. Since this equation must of course be independent of which fluid is being contemplated, it follows that the normal component of the velocity is continuous at the interface. The two equations

$$\partial_t\zeta = \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_1 \quad \text{on} \quad \Gamma_t, \tag{5.4}$$

and

$$\partial_n\Phi_1 = \partial_n\Phi_2 \quad \text{on} \quad \Gamma_t, \tag{5.5}$$

with

$$\partial_n := \mathbf{n}\cdot\nabla_{X,z} \quad \text{and} \quad \mathbf{n} := \frac{1}{\sqrt{1+|\nabla\zeta|^2}}(-\nabla\zeta,1)^T$$

follow as a consequence. A final condition is needed on the pressure to close this set of equations, namely,

$$P \text{ is continuous at the interface.} \tag{5.6}$$

When surface tension between the two layers is taken into account (see Cung The Anh [66]), the continuity of pressure across the interface should then be replaced by

$$P_1 - P_2 = \sigma\nabla.(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}),$$

where $\sigma \geq 0$ is the surface tension parameter.

Though the case $\sigma = 0$ is physically very relevant, $\sigma > 0$ plays a major role to establish the well-posedness of the full system (see D. Lannes [158]).

5.1.2 Transformation of the Equations

In this subsection, a new set of equations is deduced from the internal-wave equations (5.1)-(5.6). Introduce the trace of the potentials Φ_1 and Φ_2 at the interface,

$$\psi_i(t,X) := \Phi_i(t,X,-d_1+\zeta(t,X)), \qquad (i=1,2).$$

One can evaluate Eq. (5.2) at the interface and use (5.4) and (5.5) to obtain a set of equations coupling ζ to ψ_i $(i=1,2)$, namely

$$\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_i = 0 \quad (5.7)$$

$$\rho_i\Big(\partial_t\psi_i + g\zeta + \frac{1}{2}|\nabla\psi_i|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_i)+\nabla\zeta\cdot\nabla\psi_i)^2}{2(1+|\nabla\zeta|^2)}\Big) = -P \quad (5.8)$$

where in (5.7) and (5.8), $(\partial_n\Phi_i)$ and P are both evaluated at the interface $z=-d_1+\zeta(t,X)$. Notice that $\partial_n\Phi_1$ is fully determined by ψ_1 since Φ_1 is uniquely given as the solution of Laplace's equation (5.1) in the upper fluid domain, the Neumann condition (5.3) on Γ_1 and the Dirichlet condition $\Phi_1=\psi_1$ at the interface. Following the formalism introduced for the study of surface water waves in [65, 64, 248], we can therefore define the Dirichlet-Neumann operator $G[\zeta]\cdot$ by

$$G[\zeta]\psi_1 = \sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_1)_{|_{z=-d_1+\zeta}}.$$

Similarly, one remarks that ψ_2 is determined up to a constant by ψ_1 since Φ_2 is given (up to a constant) by the resolution of the Laplace equation (5.1) in the lower fluid domain, with Neumann boundary conditions (5.3) on Γ_2 and $\partial_n\Phi_2=\partial_n\Phi_1$ at the interface (this latter being provided by (5.5)). It follows that ψ_1 fully determines $\nabla\psi_2$ and we may thus define the operator $\mathbf{H}[\zeta]\cdot$ by

$$\mathbf{H}[\zeta]\psi_1 = \nabla\psi_2.$$

Using the continuity of the pressure at the interface expressed in (5.6), we may equate the left-hand sides of $(5.8)_1$ and $(5.8)_2$ using the operators $G[\zeta]$ and $\mathbf{H}[\zeta]$ just defined. This yields the equation

$$\partial_t(\psi_2-\gamma\psi_1) + g(1-\gamma)\zeta + \frac{1}{2}\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \mathscr{N}(\zeta,\psi_1) = 0$$

where $\gamma=\rho_1/\rho_2$ and

$$\mathscr{N}(\zeta,\psi_1) := \frac{\gamma\big(G[\zeta]\psi_1+\nabla\zeta\cdot\nabla\psi_1\big)^2 - \big(G[\zeta]\psi_1+\nabla\zeta\cdot\mathbf{H}[\zeta]\psi_1\big)^2}{2(1+|\nabla\zeta|^2)}.$$

Taking the gradient of this equation and using (5.7) then gives the system of equations

$$\begin{cases} \partial_t \zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1-\gamma)\nabla\zeta \\ \qquad + \frac{1}{2}\nabla\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathscr{N}(\zeta,\psi_1) = 0, \end{cases} \tag{5.9}$$

for ζ and ψ_1. This is the system of equations that will be used in the next sections to derive asymptotic models.

Remark 5.1. More precise definitions of the operators $G[\zeta]$ and $\mathbf{H}[\zeta]$ will be presented in Subsection 5.1.3 and in Section **??**.

Remark 5.2. Setting $\rho_1 = 0$, and thus $\gamma = 0$, in the above equations, one recovers the usual surface water-wave equations written in terms of ζ and ψ as in [65, 64, 248].

5.1.3 Non-Dimensionalization of the Equations

The asymptotic behaviour of (5.9) is more transparent when these equations are written in dimensionless variables. Denoting by a a typical amplitude of the deformation of the interface in question, and by λ a typical wavelength, the following dimensionless indendent variables

$$\widetilde{X} := \frac{X}{\lambda}, \quad \widetilde{z} := \frac{z}{d_1}, \quad \widetilde{t} := \frac{t}{\lambda/\sqrt{gd_1},},$$

are introduced. Likewise, we define the dimensionless unknowns

$$\widetilde{\zeta} := \frac{\zeta}{a}, \quad \widetilde{\psi}_1 := \frac{\psi_1}{a\lambda\sqrt{g/d_1}},$$

as well as the dimensionless parameter‘s

$$\gamma := \frac{\rho_1}{\rho_2}, \quad \delta := \frac{d_1}{d_2}, \quad \varepsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2}.$$

Though they are redundant, it is also notationally convenient to introduce two other parameter‘s ε_2 and μ_2 defined as

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \qquad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

Remark 5.3. The parameters ε_2 and μ_2 correspond to ε and μ with d_2 rather than d_1 taken as the unit of length in the vertical direction.

Before writing (5.9) in dimensionless variables, a dimensionless Dirichlet-Neumann operator $G^\mu[\varepsilon\zeta]\cdot$ is needed, associated to the non-dimensionalized upper fluid domain

$$\Omega_1 = \{(X,z) \in \mathbb{R}^{d+1}, -1+\varepsilon\zeta(X) < z < 0\}.$$

Throughout the discussion, it will be presumed that this domain remains connected, so there is a positive value H_1 such that

$$1-\varepsilon\zeta \geq H_1 \quad \text{on} \quad \mathbb{R}^d. \tag{5.10}$$

Definition 5.1. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (5.10) is satisfied and let $\psi_1 \in H^{3/2}(\mathbb{R}^d)$. If Φ_1 is the unique solution in $H^2(\Omega_1)$ of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 & \text{in } \Omega_1, \\ \partial_z\Phi_1{}_{|_{z=0}} = 0, & \Phi_1{}_{|_{z=-1+\varepsilon\zeta(X)}} = \psi_1, \end{cases} \tag{5.11}$$

then $G^\mu[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$ is defined by

$$G^\mu[\varepsilon\zeta]\psi_1 = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_1{}_{|_{z=-1+\varepsilon\zeta}} + \partial_z\Phi_1{}_{|_{z=-1+\varepsilon\zeta}}.$$

Remark 5.4. Another way to approach G^μ is to define

$$G^\mu[\varepsilon\zeta]\psi_1 = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,\partial_n\Phi_1{}_{|_{z=-1+\varepsilon\zeta}}$$

where $\partial_n\Phi_1{}_{|_{z=-1+\varepsilon\zeta}}$ stands for the upper conormal derivative associated to the elliptic operator $\mu\Delta\Phi_1 + \partial_z^2\Phi_1$.

In the same vein, one may define a dimensionless operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ associated to the non-dimensionalized lower fluid domain

$$\Omega_2 = \{(X,z) \in \mathbb{R}^{d+1}, -1-1/\delta < z < -1+\varepsilon\zeta(X)\},$$

where it is assumed as in (5.10) that there is an $H_2 > 0$ such that

$$1+\varepsilon\delta\zeta \geq H_2 \quad \text{on} \quad \mathbb{R}^d. \tag{5.12}$$

Definition 5.2. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (5.10) and (5.12) are satisfied, and suppose that $\psi_1 \in H^{3/2}(\mathbb{R}^d)$ is given. If the function Φ_2 is the unique solution (up to a constant) of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \Omega_2, \\ \partial_z\Phi_2|_{z=-1-1/\delta} = 0, & \partial_n\Phi_2|_{z=-1+\varepsilon\zeta(X)} = \frac{1}{(1+\varepsilon^2|\nabla\zeta|^2)^{1/2}}G^\mu[\varepsilon\zeta]\psi_1, \end{cases} \tag{5.13}$$

then the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ is defined on ψ_1 by

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\Phi_2|_{z=-1+\varepsilon\zeta}) \in H^{1/2}(\mathbb{R}^d).$$

Remark 5.5. In the statement above, $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta}$ stands here for the upwards conormal derivative associated to the elliptic operator $\mu\Delta\Phi_2 + \partial_z^2\Phi_2$,

$$\sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_2|_{z=-1+\varepsilon\zeta} + \partial_z\Phi_2|_{z=-1+\varepsilon\zeta}.$$

The Neumann boundary condition of (5.13) at the interface can also be stated as $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = \partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$.

Remark 5.6. Of course, the solvability of (5.13) requires the condition $\int_\Gamma \partial_n\Phi_2 d\Gamma = 0$ (where $d\Gamma = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}dX$ is the Lebesgue measure on the surface $\Gamma = \{z = -1+\varepsilon\zeta\}$). This is automatically satisfied thanks to the definition of $G^\mu[\varepsilon\zeta]\psi_1$. Indeed, applying Green's identity to (5.11), one obtains

$$\int_\Gamma \partial_n\Phi_2 d\Gamma = \int_\Gamma \partial_n\Phi_1 d\Gamma = -\int_{\Omega_1}(\mu\Delta\Phi_1 + \partial_z^2\Phi_1) = 0.$$

Example 5.3. The operators $G^\mu[\varepsilon\zeta]\cdot$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ have explicit expressions when the interface is flat (i.e. when $\zeta = 0$). In that case, taking the horizontal Fourier transform of the Laplace equations (5.11) and (5.13) transforms them into ordinary differential equations with respect to z which can easily be solved to obtain

$$G^\mu[0]\psi = -\sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)\psi \quad \text{and} \quad \mathbf{H}^{\mu,\delta}[0]\psi = -\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\frac{\sqrt{\mu}}{\delta}|D|)}\nabla\psi.$$

The equations (5.9) can therefore be written in dimensionless variables as

$$\begin{cases} \partial_{\tilde t}\widetilde{\zeta} - \frac{1}{\mu} G^{\mu}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 & = & 0, \\ \partial_{\tilde t}\big(\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 - \gamma\nabla\widetilde{\psi}_1\big) + (1-\gamma)\nabla\widetilde{\zeta} & & \\ \quad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1|^2 - \gamma|\nabla\widetilde{\psi}_1|^2\big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\widetilde{\zeta},\widetilde{\psi}_1) & = & 0, \end{cases} \tag{5.14}$$

where $\mathcal{N}^{\mu,\delta}$ is defined for all pairs (ζ,ψ) smooth enough by the formula

$$\mathcal{N}^{\mu,\delta}(\zeta,\psi) := \mu\frac{\gamma\big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi\big)^2 - \big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\mathbf{H}^{\mu,\delta}[\zeta]\psi\big)^2}{2(1+\mu|\nabla\zeta|^2)}.$$

When surface tension is present, one just adds $-\frac{\sigma}{\rho_2}\nabla K(\widetilde{\zeta})$ to the LHS of the second equation, where $K(\widetilde{\zeta}) = (\frac{\nabla\widetilde{\zeta}}{\sqrt{1+|\nabla\widetilde{\zeta}|^2}})$.

We will later derive models describing the asymptotics of the non-dimensionalized equations (5.14) in various physical regimes corresponding to different relationships among the dimensionless parameter‘s ε, μ and δ.

Remark 5.7. Linearizing the equations (5.14) around the rest state, one finds the equations

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}G^{\mu}[0]\psi_1 & = & 0, \\ \partial_t\big(\mathbf{H}^{\mu,\delta}[0]\psi_1 - \gamma\nabla\psi_1\big) + (1-\gamma)\nabla\zeta & = & 0. \end{cases}$$

The explicit formulas in Example 5.3 thus allow one to calculate the linearized dispersion relation

$$\omega^2 = (1-\gamma)\frac{|\mathbf{k}|}{\sqrt{\mu}}\frac{\tanh(\sqrt{\mu}|\mathbf{k}|)\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}{\tanh(\sqrt{\mu}|\mathbf{k}|)+\gamma\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}; \tag{5.15}$$

corresponding to plane-wave solutions $e^{i\mathbf{k}\cdot X - i\omega t}$. In particular, the expected instability is found when $\gamma > 1$, corresponding to the case wherein the heavier fluid lies over the lighter one. One also checks that the classical dispersion relation

$$\omega^2 = \frac{1}{\sqrt{\mu}}|\mathbf{k}|\tanh(\sqrt{\mu}|\mathbf{k}|)$$

for surface water waves is recovered when $\gamma = 0$ and $\delta = 1$.

Thus the linearization around the rest state is Hadamard well posed when $\gamma \leq 1$ but the linearization around a state presenting a discontinuity of the horizontal velocities at the interface leads to Kelvin-Helmholtz instabilities (see [49, 113] and the experiments in [234]), possibly stabilized by surface tension ([115, 158]).

It is worth to consider the related problem of the *linear* stability of horizontal shear flows. It is well known (see [49], §100), that for horizontal shear flows with constant horizontal velocities U_1 and U_2 the flat interface develops instabilities for perturbations in the direction of streaming having wave numbers $\mathbf{k}$ such that (in variables with dimension)

$$|\mathbf{k}| \geq k_{min} = \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1 \rho_2 (U_2 - U_1)^2} \tag{5.16}$$

This formula shows in particular the stabilizing role of gravity on long waves. However, the surface tension effects are very weak or negligible in oceanographic situations. Furthermore, one observes “stable” waves both in experiments and in oceans (see the above pictures) in relevant, long enough, time scales.

In a recent paper which we will comment on in Section 6, D. Lannes [158] studies the Cauchy problem for the two-layer system with surface tension under a condition which extends both the classical Taylor sign condition and the Chandrasekhar condition. This allows for sufficiently large time existence and justification of some of the asymptotic models.

On the other hand, in presence of surface tension, the dispersion relation of the two-layer system reads :

$$\omega^2 = (1 - \gamma + \varepsilon\sqrt{\mu}\nu|\mathbf{k}|^2)\frac{|\mathbf{k}|}{\sqrt{\mu}} \frac{\tanh(\sqrt{\mu}|\mathbf{k}|)\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}{\tanh(\sqrt{\mu}|\mathbf{k}|) + \gamma\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)},$$

where

$$\nu = \frac{\sigma}{\rho_2 \lambda^2}.$$

5.2 Asymptotic models

As in the case of surface waves one has to derive asymptotic models to describe the long time dynamics in various regimes. There are however complications here due to the large number of parameters. We will only sketch the description of the relevant regimes and write down the asymptotic systems they lead to. We refer to [32, 203] for details.

Here is a summary of the different asymptotic regimes investigated in this paper. It is convenient to organize the discussion around the parameters ε and $\varepsilon_2 = \varepsilon\delta$ (the nonlinearity, or amplitude, parameters for the upper and lower fluids, respectively), and in terms of μ and $\mu_2 = \frac{\mu}{\delta^2}$ (the long-wavelength parameters for the upper and lower fluids). Notice that the assumptions made about δ are therefore implicit.

The interfacial wave is said to be of *small amplitude* for the upper fluid layer (resp. the lower layer) if $\varepsilon \ll 1$ (resp. $\varepsilon_2 \ll 1$) and the upper (resp. lower) layer is said to be *shallow* if $\mu \ll 1$ (resp. $\mu_2 \ll 1$). This terminology is consistent with the usual one for surface water waves (recovered by taking $\rho_1 = 0$ and $\delta = 1$). In the discussion below, the notation *regime 1/regime 2* means that the wave motion is such that the upper layer is in regime 1 (small amplitude or shallow water) and the lower one is in regime 2.

1. The small-amplitude/small-amplitude regime: $\varepsilon \ll 1$, $\varepsilon_2 \ll 1$. This regime corresponds to interfacial deformations which are small for both the upper and lower fluid domains. Various sub-regimes are defined by making further assumptions about the size of μ and μ_2.

2. The Full Dispersion /Full Dispersion (FD/FD) regime: $\varepsilon \sim \varepsilon_2 \ll 1$ and $\mu \sim \mu_2 = O(1)$ (and thus $\delta \sim 1$). In this regime, the shallowness parameters are not small for either of the fluid domains, and the full dispersive effects must therefore be kept for both regions; the asymptotic model corresponding to this situation is

$$\begin{cases} \partial_t \zeta + \frac{1}{\sqrt{\mu}} \frac{\nabla}{|D|} \cdot \big(\frac{\mathbf{T}_\mu \mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big) \\ \qquad + \frac{\varepsilon_2}{\sqrt{\mu}} \frac{\nabla}{|D|} \cdot \big(\frac{\mathbf{T}_\mu \mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} B(\zeta, \frac{\mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v}) \big) \\ \qquad - \varepsilon \nabla \cdot \big(\zeta \frac{\mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big) + \varepsilon |D| \mathbf{T}_\mu \big(\zeta \frac{\nabla}{|D|} \cdot \big(\frac{\mathbf{T}_\mu \mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big) \big) = 0 \\ \partial_t \mathbf{v} + (1-\gamma) \nabla \zeta \\ \qquad + \frac{\varepsilon}{2} \nabla \big(\big| \frac{\mathbf{T}_\mu}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big|^2 - \gamma \big| \frac{\mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big|^2 \big) + \varepsilon \frac{\gamma - 1}{2} \nabla \big(\frac{\nabla}{|D|} \cdot \big(\frac{\mathbf{T}_\mu \mathbf{T}_{\mu_2}}{\gamma \mathbf{T}_{\mu_2} + \mathbf{T}_\mu} \mathbf{v} \big) \big)^2 = 0, \end{cases} \tag{5.17}$$

where, $\mathbf{T}_\mu = \tanh(\sqrt{\mu}|D|)$, $\mathbf{T}_{\mu_2} = \tanh(\sqrt{\mu_2}|D|)$ and the bilinear mapping

$B(\cdot,\cdot)$ is given by

$$\begin{aligned} B(\zeta,\nabla\psi_1) = & \sqrt{\mu_2}\frac{|D|}{\tanh(\sqrt{\mu_2}|D|)}\Pi\big[\zeta(1+\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)})\nabla\psi_1\big] \\ + & \sqrt{\mu_2}\nabla\big[(1+\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)})(\zeta\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\Delta\psi_1)\big]. \end{aligned} \tag{5.18}$$

3. The Boussinesq / Full dispersion (B/FD) regime: $\mu \sim \varepsilon \ll 1$, $\mu_2 \sim 1$. This regime corresponds to the case where the flow has a Boussinesq structure in the upper part (and thus dispersive effects of the same order as nonlinear effects), but with a shallowness parameter not small in the lower fluid domain. This configuration occurs when $\delta^2 \sim \varepsilon$, that is, when the lower region is much larger than the upper one. A further analysis of the asymptotic model yields a three-parameter family of equivalent systems

$$\begin{cases} \big(1-\mu b\Delta\big)\partial_t\zeta + \frac{1}{\gamma}\nabla\cdot\big((1-\varepsilon\zeta)\mathbf{v}_\beta\big) \\ \qquad -\frac{\sqrt{\mu}}{\gamma^2}|D|\coth(\sqrt{\mu_2}|D|)\nabla\cdot\mathbf{v}_\beta \\ +\frac{\mu}{\gamma}\Big(a-\frac{1}{\gamma^2}\coth^2(\sqrt{\mu_2}|D|)\Big)\Delta\nabla\cdot\mathbf{v}_\beta = 0 \\ (1-\mu d\Delta)\partial_t\mathbf{v}_\beta + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}_\beta|^2 + \mu c(1-\gamma)\Delta\nabla\zeta = 0, \end{cases} \tag{5.19}$$

where $\mathbf{v}_\beta = (1-\mu\beta\Delta)^{-1}\mathbf{v}$ and the constants a, b, c and d are defined by

$$a = \frac{1}{3}(1-\alpha_1-3\beta), \qquad b = \frac{1}{3}\alpha_1, \qquad c = \beta\alpha_2, \qquad d = \beta(1-\alpha_2),$$

with $\alpha_1 \geq 0$, $\beta \geq 0$ and $\alpha_2 \leq 1$.

Remark 5.8. The dispersion relation associated to (5.19) is

$$\begin{aligned} \omega^2 = & \frac{1-\gamma}{\gamma}|\mathbf{k}|^2(1-\mu c|\mathbf{k}|^2) \\ \times & \frac{1-\frac{\sqrt{\mu}}{\gamma}|\mathbf{k}|\coth(\sqrt{\mu_2}|\mathbf{k}|) - \mu|\mathbf{k}|^2\big(a-\frac{1}{\gamma^2}\coth^2(\sqrt{\mu_2}|\mathbf{k}|)\big)}{(1+\mu b|\mathbf{k}|^2)(1+\mu d|\mathbf{k}|^2)}, \end{aligned}$$

and (5.19) is therefore linearly well-posed when $b,d \geq 0$ and $a,c \leq 0$. Notice that in the case $\alpha_1 = \alpha_2 = \beta = 0$, one has $a = \frac{1}{3}$ and $b = c = d = 0$ and the corresponding system is thus linearly ill-posed. The freedom to choose a well-posed model is just one of the advantages of a three-parameter family of formally equivalent systems. The same remark has already been made about the Boussinesq systems for wave propagation in the case of surface gravity waves [34, 30]).

It has been shown in [67] that the linearly well-posed B/FD systems are also *locally nonlinearly well-posed* provided one at least of the coefficients b, d is strictly positive.

4. The Boussinesq/Boussinesq (B/B) regime: $\mu \sim \mu_2 \sim \varepsilon \sim \varepsilon_2 \ll 1$. In this regime, one has $\delta \sim 1$ and the flow has a Boussinesq structure in both the upper and lower fluid domains. Here again, a three-parameter family of asymptotic systems is obtained.

$$\left\{ \begin{array}{l} \Big(1-\mu b\Delta\Big)\partial_t\zeta + \dfrac{1}{\gamma+\delta}\nabla\cdot\mathbf{v}_\beta + \varepsilon\dfrac{\delta^2-\gamma}{(\gamma+\delta)^2}\nabla\cdot(\zeta\mathbf{v}_\beta) + \mu a\nabla\cdot\Delta\mathbf{v}_\beta = 0 \\ \Big(1-\mu d\Delta\Big)\partial_t\mathbf{v}_\beta + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon}{2}\dfrac{\delta^2-\gamma}{(\delta+\gamma)^2}\nabla|\mathbf{v}_\beta|^2 + (1-\gamma)\mu c\Delta\nabla\zeta = 0, \end{array}\right. \tag{5.20}$$

where $\mathbf{v}_\beta = (1-\mu\beta\Delta)^{-1}\mathbf{v}$, and where the coefficients a,b,c,d are provided by

$$a = \frac{(1-\alpha_1)(1+\gamma\delta) - 3\delta\beta(\gamma+\delta)}{3\delta(\gamma+\delta)^2}, \quad b = \alpha_1\tfrac{1+\gamma\delta}{3\delta(\gamma+\delta)},$$
$$c = \beta\alpha_2, \qquad d = \beta(1-\alpha_2),$$

with $\alpha_1 \geq 0$, $\beta \geq 0$ and $\alpha_2 \leq 1$.

Remark 5.9. Taking $\gamma = 0$ and $\delta = 1$ in the Boussinesq/Boussinesq equations (5.20), reduces them to the system

$$\left\{ \begin{array}{l} \big(1-\mu\dfrac{\alpha_1}{3}\Delta\big)\partial_t\zeta + \nabla\cdot((1+\varepsilon\zeta)\mathbf{v}) + \mu\dfrac{1-\alpha_1-3\beta}{3}\nabla\cdot\Delta\mathbf{v} = 0 \\ \big(1-\mu\beta(1-\alpha_2)\Delta\big)\partial_t\mathbf{v} + \nabla\zeta + \dfrac{\varepsilon}{2}\nabla|\mathbf{v}|^2 + \mu\beta\alpha_2\Delta\nabla\zeta = 0, \end{array}\right.$$

which is exactly the family of formally equivalent Boussinesq systems derived in [**?**, 30].

Remark 5.10. The dispersion relation associated to (5.20) is

$$\omega^2 = (1-\gamma)|\mathbf{k}|^2\frac{(\frac{1}{\gamma+\delta} - \mu a|\mathbf{k}|^2)(1-\mu c|\mathbf{k}|^2)}{(1+\mu b|\mathbf{k}|^2)(1+\mu d|\mathbf{k}|^2)}.$$

It follows that (5.20) is linearly well-posed when $a,c \le 0$ and $b,d \ge 0$. The system corresponding to $\alpha_1 = \alpha_2 = \beta = 0$ is ill-posed (one can check that $a = \frac{1+\gamma\delta}{3\delta(\gamma+\delta)^2} > 0$). This system corresponds to a Hamiltonian system derived in [62] (see their formula (5.10)). As mentioned before, the present, three-parameter family of systems allows one to circumvent the problem of ill-posedness without the need of taking into account higher-order terms in the expansion, as in [62]).

The Boussinesq/Boussinesq systems (5.20) are essentially similar to the familly of classical Boussinesq systems for surface waves which were derived in [34], [30] and we have seen in the Surface waves Section. It was proved in [35], [72] (see also [67]) that the linearly well-posed systems are also locally nonlinearly well-posed in suitable Sobolev classes. Moreover they are Hamiltonian systems when $b = d$.

5. The Shallow Water/Shallow Water (SW/SW) regime: $\mu \sim \mu_2 \ll 1$. This regime, which allows relatively large interfacial amplitudes ($\varepsilon \sim \varepsilon_2 = O(1)$), does not belong to the regimes singled out above. The structure of the flow is then of shallow water type in both regions; in particular, the asymptotic model

$$\begin{cases} \partial_t \zeta + \frac{1}{\gamma+\delta} \nabla \cdot \big(h_1 \mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta} \varepsilon\delta\zeta](h_2 \mathbf{v}) \big) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta \\ \qquad + \frac{\varepsilon}{2} \nabla \Big(\big| \mathbf{v} - \frac{\gamma}{\gamma+\delta} \mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta} \varepsilon\delta\zeta](h_2\mathbf{v}) \big|^2 \\ - \frac{\gamma}{(\gamma+\delta)^2} \big| \mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta} \varepsilon\delta\zeta](h_2\mathbf{v}) \big|^2 \Big) = 0, \end{cases} \tag{5.21}$$

where $h_1 = 1 - \varepsilon\zeta$, $h_2 = 1 + \varepsilon\delta\zeta$, and the operator $\mathfrak{Q}$ is defined in Lemma **??** below is a nonlinear, but non-dispersive system, given in (5.21), which degenerates into the usual shallow water equations when $\gamma = 0$ and $\delta = 1$. It is very interesting in this case that a non-local term arises when $d = 2$. Such a nonlocal term does not appear in the one-dimensional case, nor in the two-dimensional shallow water equations for surface waves.

Lemma 5.4. *Assume that* $\zeta \in L^\infty(\mathbb{R}^d)$ *is such that* $|\varepsilon_2\zeta|_\infty < 1$. *Let also* $W \in L^2(\mathbb{R}^d)^d$. *Then* *i. One can define the mapping $\mathfrak{Q}[\varepsilon_2\zeta]$ as*

$$\mathfrak{Q}[\varepsilon_2\zeta]: \quad \begin{array}{ccc} L^2(\mathbb{R}^d)^d & \to & L^2(\mathbb{R}^d)^d \\ U & \mapsto & \sum_{n=0}^{\infty}(-1)^n(\Pi(\varepsilon_2\zeta\Pi\cdot))^n(\Pi U) \end{array}$$

ii. There exists a unique solution $V \in L^2(\mathbb{R}^d)^d$ to the equation

$$\nabla\cdot(h_2V) = \nabla\cdot W, \qquad (h_2 = 1+\varepsilon_2\zeta)$$

such that $\Pi V = V$ and one has $V = \mathfrak{Q}[\varepsilon_2\zeta]W$;
iii- If moreover $\zeta \in H^s(\mathbb{R}^d)$ and $W \in H^s(\mathbb{R}^d)^d$ ($s > d/2+1$) then $\mathfrak{Q}[\varepsilon_2\zeta]W \in H^s(\mathbb{R}^d)^d$ and

$$|\mathfrak{Q}[\varepsilon_2\zeta]W|_{H^s} \leq C(|\varepsilon_2\zeta|_{H^s}, \frac{1}{1-|\varepsilon_2\zeta|_\infty})|W|_{H^s}.$$

Remark 5.11. Taking $\gamma = 0$ and $\delta = 1$ in the SW/SW equations (5.21) yields the usual shallow water equations for surface water waves (recall that it follows from Lemma 5.4 that $\nabla\cdot[(1-\varepsilon\zeta)\mathfrak{Q}[-\varepsilon\zeta]((1+\varepsilon\zeta)\mathbf{v})] = \nabla\cdot((1+\varepsilon\zeta)\mathbf{v})$).

Remark 5.12. In the one-dimensional case $d = 1$, one has

$$\frac{1}{\gamma+\delta}\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta](h_2\mathbf{v}) = \frac{h_2}{\delta h_1+\gamma h_2}$$

and the equations (5.21) take the simpler form

$$\begin{cases} \partial_t\zeta + \partial_x\big(\frac{h_1h_2}{\delta h_1+\gamma h_2}\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{\varepsilon}{2}\partial_x\big(\frac{(\delta h_1)^2-\gamma h_2^2}{(\delta h_1+\gamma h_2)^2}|\mathbf{v}|^2\big) = 0, \end{cases}$$

which coincides of course with the system (5.26) of [62]. The presence of the nonlocal operator $\mathfrak{Q}$, which does not seem to have been noticed before, appears to be a purely two dimensional effect.

1. The Shallow Water/Small Amplitude (SW/SA) regime: $\mu \ll 1$ and $\varepsilon_2 \ll 1$. In this regime, the upper layer is shallow (but with possibly large surface deformations), and the surface deformations are small for the lower layer (but it can be deep). Various sub-regimes arise in this case also.

2. The Shallow Water/Full dispersion (SW/FD) regime: $\mu \sim \varepsilon_2^2 \ll 1$, $\varepsilon \sim \mu_2 \sim 1$. The dispersive effects are negligible in the upper fluid, but the full dispersive effects must be kept in the lower one to get

$$\begin{cases} \partial_t \zeta + \frac{1}{\gamma}\nabla\cdot(h_1\mathbf{v}) - \frac{\sqrt{\mu}}{\gamma^2}\nabla\cdot\big(h_1|D|\coth(\sqrt{\mu_2}|D|)\Pi(h_1\mathbf{v})\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla\big[|\mathbf{v}|^2 - 2\frac{\sqrt{\mu}}{\gamma}\mathbf{v}\cdot\big(|D|\coth(\sqrt{\mu_2}|D|)\Pi(h_1\mathbf{v})\big)\big] = 0, \end{cases} \tag{5.22}$$

where $h_1 = 1-\varepsilon\zeta$ and $\Pi = -\frac{\nabla\nabla^T}{\Delta}$.

3. The Intermediate Long Waves (ILW) regime: $\mu \sim \varepsilon^2 \sim \varepsilon_2 \ll 1$, $\mu_2 \sim 1$. In this regime, the interfacial deformations are also small for the upper fluid (which is not the case in the SW/FD regime). This allows some simplifications, and it is possible (see (5.23)) to derive a one-parameter family of equivalent systems.

These depend upon the parameter α and have the form

$$\begin{cases} [1+\sqrt{\mu}\frac{\alpha}{\gamma}|D|\coth(\sqrt{\mu_2}|D|)]\partial_t\zeta + \frac{1}{\gamma}\nabla\cdot((1-\varepsilon\zeta)\mathbf{v}) \\ \qquad -(1-\alpha)\frac{\sqrt{\mu}}{\gamma^2}|D|\coth(\sqrt{\mu_2}|D|)\nabla\cdot\mathbf{v} = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0. \end{cases} \tag{5.23}$$

Remark 5.13. The ILW equation derived in [**?**, 133] is obtained as the unidirectional limit of the one dimensional ($d = 1$) version of (5.23) – see for instance §5.5 of [62].

4. The Benjamin-Ono (BO) regime: $\mu \sim \varepsilon^2 \ll 1$, $\mu_2 = \infty$. For completeness, we investigate the Benjamin-Ono regime, characterized by the assumption $\delta = 0$ (the lower layer is of infinite depth). Taking $\mu_2 = \infty$ in (5.23) leads one to replace $\coth(\sqrt{\mu_2}|D|)$ by 1. The following two-dimensional generalization of the system (5.31) in [62] emerges in this

situation.

$$\begin{cases} [1+\sqrt{\mu}\frac{\alpha}{\gamma}|D|]\partial_t\zeta + \frac{1}{\gamma}\nabla\cdot((1-\varepsilon\zeta)\mathbf{v}) - (1-\alpha)\frac{\sqrt{\mu}}{\gamma^2}|D|\nabla\cdot\mathbf{v} = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0. \end{cases} \tag{5.24}$$

Neglecting the $O(\sqrt{\mu}) = O(\varepsilon)$ terms, one finds that ζ must solve a wave equation (with speed $\sqrt{\frac{1-\gamma}{\gamma}}$). Thus, in the case of horizontal dimension $d = 1$, any interfacial perturbation splits up at first approximation into two counter-propagating waves. If one includes the $O(\sqrt{\mu}, \varepsilon)$ terms, one obtains the one-parameter family

$$(1+\sqrt{\mu}\frac{\alpha}{\gamma}|\partial_x|)\partial_t\zeta + c\partial_x\zeta - \varepsilon\frac{3}{4}c\partial_x\zeta^2 - \frac{\sqrt{\mu}}{2\gamma}c(1-2\alpha)|\partial_x|\partial_x\zeta = 0, \tag{5.25}$$

of *regularized Benjamin-Ono equations* (see **??**). Here, $c = \sqrt{\frac{1-\gamma}{\gamma}}$. The usual Benjamin-Ono equation is recovered by taking $\alpha = 0$. (5.25).

The range of validity of these regimes is summarized in the following table.

	$\varepsilon = O(1)$	$\varepsilon \ll 1$
$\mu = O(1)$	Full equations	$\delta \sim 1$: FD/FD eq'ns
$\mu \ll 1$	$\delta \sim 1$: SW/SW eq'ns $\delta^2 \sim \mu \sim \varepsilon_2^2$: SW/FD eq'ns	$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon$: B/FD eq'ns $\mu \sim \varepsilon$ and $\delta \sim 1$: B/B eq'ns $\delta^2 \sim \mu \sim \varepsilon^2$: ILW eq'ns $\delta = 0$ and $\mu \sim \varepsilon^2$: BO eq'ns

Remark 5.14. The small amplitude/shallow water regime is not considered here. It corresponds to the situation where the upper fluid domain is much larger than the lower one, which is more of an atmospheric configuration than an oceanographic case.

Remark 5.15. The two-layer system is an idealization. In the actual ocean the density varies rapidly but continuously between two or more layers of nearly constant densities in a narrow layer called the pycnocline. We do not know of a rigorous and systematic derivation of asymptotic models for this situation. In [151] (see also [171] when shear is taken into account), one-dimensional weakly nonlinear models are formally derived. They write

$$u_t + c_0 u_x + \alpha u u_x - c_0 \partial_x^2 \int_{-\infty}^{\infty} G(x-\xi)u(\xi,t)d\xi, \tag{5.26}$$

where

$$G(x) = \frac{\beta_1}{2H_1}\{\coth\left(\frac{\pi x}{2H_1}\right) - \text{sign }(x)\} + \frac{\beta_1}{2H_2}\{\coth\left(\frac{\pi x}{2H_2}\right) - \text{sign }(x)\}$$

where H_1 and H_2 are the dimensionless (scaled by wave length) depths of the two layers and the constants α and c_0 depend on the density stratification $\rho(z)$. Note that (5.26) is reminiscent of the ILW equation obtained as the unidirectional limit of the one dimensional (d = 1) and one directional version of (5.23). We refer to [10] for mathematical issues concerning (6.3) (and its extension of a system derived in [166, 167] for the transfer of energy between waves running along neighbouring pycnoclines): Cauchy problem, existence and stability of solitary waves.

It would be interesting to derive rigorously asymptotic models in this situation and to investigate the limit as the width of the pycnocline tends to zero, that is the discontinuous limit of a continuous stratification.

Chapter 6

Justification of the asymptotic equations

6.1 Justification of the asymptotic equations

As we have seen in the previous chapters, the asymptotic equations or systems derived from the Euler equations with free surface (or surfaces) are supposed to describe the long time dynamics of the original systems in specific regimes.

A not too bad analogy is the convergence theory of finite difference schemes (Lax-Richtmeyer theorem):

A finite difference scheme approximating a PDE problem $\mathscr{P}$ is convergent if and only if

- $\mathscr{P}$ is well-posed.
- The scheme is consistent with $\mathscr{P}$. This is not a dynamical notion (just algebraic).
- The scheme is stable. This is a dynamical notion (well-posedness).

For the water waves asymptotic models the full rigorous justification involves four steps that are in the same spirit.

6.1.1 Consistency

The various approximations of the non local Dirichlet-Neumann or interface operators lead to *consistent systems* in the sense of the following definitions

Definition 6.1. The water wave equations (4.13) are consistent with a system S of of $d+1$ equations for ζ and $\mathbf{v}$ if for all sufficiently smooth solutions (ζ, ψ_1), the pair $\zeta, \mathbf{v}$ solves S up to a small residual called the *precision* of the asymptotic model.

It can be rigorously established (see [32]) that the internal-wave equations (5.14) are *consistent* with the asymptotic models for $(\zeta, \mathbf{v})$ described in Chapter 5 in the following precise sense.

Definition 6.2. The internal wave equations (5.14) are *consistent* with a system S of $d+1$ equations for ζ and $\mathbf{v}$ if for all sufficiently smooth solutions (ζ, ψ_1) of (5.14) such that (5.10) and (5.12) are satisfied, the pair $(\zeta, \mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1)$ solves S up to a small residual called the *precision* of the asymptotic model.

Remark 6.1. 1. The rigorous approximations of the non local operators provides Sobolev type norms to evaluate the residual terms.

2. The definition of consistency does not require the well-posedness of the water wave or the internal wave equations, neither of the asymptotic systems, and of course does not fully justify the asymptotic systems. It does not involve any Dynamics of the underlying systems and as aforementioned is similar to the notion of consistency for a finite difference scheme (which of course needs not to be convergent, even if it is consistent!).

On the other hand, the other properties involves the dynamics of the systems and ate the counterparts of stability for a finite difference scheme (and of the well-posedness of the system the scheme is supposed to approximate).

6.1.2 Long time existence for the Euler systems

This is the more difficult step, since even the *local*-wellposedness of the corresponding Cauchy problems gives serious difficulties.

The water wave systems

A pioneering result is the work by W. Craig [61] who treated the one dimensional water waves problem in the KdV scaling. For most asymptotic regimes excluding the modulation one, long time existence for the water wave problem has been established in [12].

Long time existence in the modulation regime (the *Schrödinger* one) has been established in [233] in the infinite depth case. The finite depth case seems to be open.

The Cauchy problem for the two-layer system

The Cauchy problem for the two-layer system leads to serious difficulties because of the possible Hadamard type instabilities due to shearing at the interface.

We sketch here the solution given by D. Lannes [158] to the apparent paradox mentioned in the Introduction : the presence of Kelvin-Helmholtz instabilities for the two-layer system in absence of surface tension, while *stable* (in the sense of "observable on a relevant time scale") internal waves are observed in the ocean or in the laboratory despite the fact that surface tension effects are physically negligible.

To sum up, the solution of the aforementionned "paradox" is, as expressed in [158], that "*the Kelvin-Helmholtz instabilities appear above the frequency threshold for which surface tension is relevant, while the main (observable) part of the wave is located below this threshold. The Kelvin-Helmholtz instabilities are regularized by surface tension, while the main part of the wave is unaffected by it. This is gravity that prevent Kelvin-Helmholtz instabilities for low fequencies*".

This scenario is rigorously proved in [158] and we sketch below its precise mathematical expression.

We will use here the notation $-$ (resp. $+$) to index the quantities related to the upper (resp. lower) layer, so that for instance $\rho^+ > \rho^-$.

An important result in the analysis of [158] is a new nonlinear criterion for the stability of two-fluids interfaces. It expresses a condition on the jump of the vertical derivative of the pressure P across the interface and reads

$$-\partial_z P^{\pm}{}_{|z=\zeta} > \frac{1}{4}\frac{(\rho^+\rho^-)^2}{\sigma(\rho_+ + \rho^-)^2}\mathfrak{c}(\zeta)|\omega|_\infty^4, \tag{6.1}$$

where $\omega = V^{\pm}\,_{|z=\zeta}$ is the jump of the horizontal velocity at the interface, $\mathfrak{c}(\zeta)$ is a constant that depends on the geometry of the problem and that can be estimated quite precisely and σ is the surface tension coefficient.

This criterion is somewhat reminiscent, (but stronger) of the classical Rayleigh-Taylor criterion for surface water waves (thus $\rho^- = 0$) which writes in absence of surface tension and in infinite depth,

$$-\partial_z P\,_{|\text{surface}} > 0. \tag{6.2}$$

We recall that the linearization of the Rayleigh-Taylor criterion around the rest state is just $g > 0$ where g is the vertical acceleration of gravity.

Both quantities $-\partial_z P^{\pm}\,_{|z=\zeta}$ and $|\omega|_\infty$ are not easy to evaluate experimentally. D. Lannes establishes a *practical* criterion in the shallow water regime (the wave length of the perturbation is large relatively to the depths of the layers). In this regime one can use the hydrostatic approximation with a fairly good precision, that is evaluate the jump in the vertical derivative of the pressure by

$$-\partial_z P \sim (\rho^+ - \rho^-)\, g$$

One can also prove that ω has a typical size of order $\frac{a}{H}\sqrt{g'H}$, where g' is the *reduced gravity*, $g' = \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-}\, g$, a a typical amplitude of the interface and $H = (\rho^+ + \rho^-)\frac{H^+H^-}{\rho^+H^- + \rho^-H^+}$. Plugging those expressions into (6.1) shows the relevance of the dimensionless parameter $\mathscr{T}$,

$$\mathscr{T} = \frac{(\rho^+\rho^-)^2}{(\rho^+ + \rho^-)^3}\frac{a^4}{H^2}\frac{g'}{4\sigma}.$$

This leads to a very simple practical stability criterion in the shallow water regime,

$$\mathscr{T} \ll 1: \quad \text{Stable configuration;} \quad \mathscr{T} \gg 1: \quad \text{Unstable configuration.}$$

When $\mathscr{T} \sim 1$, one should consider the exact criterion (6.1).

The stability criterion (6.1) (or its practical version above) ensures the existence of a "stable" solution of the two-layer system, that is a solution that exists on a time scale which does not vanish as the surface tension parameter tends to zero, and which is uniformly bounded from below with respect to the physical parameters ε and μ. This leads in particular to the *full justification* of the SW/SW system studied in Section 4.

One also finds in [158] interesting comparisons of the theoretical results with the experiments of [176], [123], [102].

6.1.3 Large time existence for the asymptotic models

The aim is here to establish the well-posedness of the asymptotic models on time scales at least as large as the ones on which the solutions of the original systems are proved to exist. As we already noticed, this question is a trivial one when the asymptotic models are, in the case of unidirectional waves, scalar equations for which the global existence can be established. This is not so however for most of two-directional waves systems, and this has to be done case by case. We will go back to this issue in Chapter 11, where we will study in some detail three systems of surface or internal waves.

6.1.4 Stability

Assuming that both the original system and the asymptotic model are well-posed on the relevant time scales, it remains to prove the optimal error estimates between the exact solution and the approximate one provided by the asymptotic system. This is done in detail in [156]. For instance, for the (a,b,c,d) Boussinesq systems for surface waves, this error is shown in [30] to be of order $\varepsilon^2 t$.

Chapter 7

The Cauchy problem for the KdV and the BBM equation

7.1 The Cauchy problem for the KdV equation

We will state and essentially prove here a global well-posedness result for the Korteweg -de Vries equation due to Kenig Ponce and Vega [139]. Although it does not give the optimal result for the regularity of the initial data, it implies the global well-posedness in $H^1(\mathbb{R})$ which is required to study the dynamics of the equation (in particular the orbital stability of the solitary waves). We recall that as far as the rigorous connexion with the water wave system is concerned, only a local well-posedness in H^s with s large is required, this implies well-posedness on time intervals of length $O(\frac{1}{\varepsilon})$, see Section 2.3 of Chapter 2.

The result we will set is also interesting in that it is a typical example of how to use the dispersive properties of the linear group to get non trivial results on the Cauchy problem.

We recall that the L^p norm, $1 \leq p \leq +\infty$ is denoted $|\cdot|_p$.

For $\alpha \in \mathbb{R}$, we set

$$D_x^\alpha f(x) = c_\alpha \int_{-\infty}^{\infty} e^{ix\xi} |\xi|^\alpha \hat{f}(\xi,t) d\xi$$

where $\hat{f}$ denotes the Fourier transform with respect to x.

We first recall the dispersive properties of the Airy group $S(t) = e^{t\partial_x^3}$ (not all of them will be used in the sequel). We refer for instance to the book [162] for a presentation of various dispersive estimates, in particular the ones used here.

1. Unitarity.

$$|S(t)u_0|_2 = |u_0|_2, \ \forall t \in \mathbb{R}. \tag{7.1}$$

2. Dispersion (consequence of $Ai \in L^\infty(\mathbb{R})$ where the Airy function is defined here by$Ai = \mathfrak{F}^{-1}(e^{i\xi^3})$).

$$|S(t)u_0|_\infty \le \frac{C}{t^{1/3}} |u_0|_1, \ t \neq 0. \tag{7.2}$$

3. Strichartz estimates (consequence of **2.**)

For any admissible pair (q,r), that is $2 \le r \le +\infty$ and $\frac{1}{q} = \frac{1}{3}(\frac{1}{2} - \frac{1}{r})$, one has for $\Lambda f(t) = \int_0^t e^{(t-s)\partial_x^3} f(s) ds$,

- $||S(t)f||_{L_t^q(L_x^r)} \le C||f||_2$.[1]
- $||\Lambda f||_{L_t^q(L_x^r} \le C||f||_{L_t^{q_1'} L_x^{r_1'}}$, for any couple of admissible pairs $(q,r), (q_1, r_1)$.
- $||\Lambda f||_{L_t^\infty(L_x^2)} \le ||f||_{L_t^{q'} L_x^{r'}}$.
- $||\Lambda f||_{L_t^q(L_x^r)} \le C||f||_{L_t^1 L_x^2}$.

4. Strichartz estimates with smoothing [137] . They are consequence of $|D_x|^{1/2} Ai \in L^\infty(\mathbb{R})$) which implies with a *complex interpolation* result of Stein ([220])

$$|D_x S(t) u_0|_{2/(1\theta)} \le C|t|^{-\theta(\alpha+1)/3} |u_0|_{2/(1+\theta)},$$

[1] L_t^q means $L^q(\mathbb{R}_t)$. L_x^r means $L^r(\mathbb{R}_x)$. L_T^q means $L^q(-T,T)$.

for any $(\alpha,\theta) \in [0,1]\times[0,\frac{1}{2}]$.

For any (α,θ) as above and $(q,p) = (\frac{6}{\theta(\alpha+1)}, \frac{\alpha}{1-\theta})$,

$\bullet ||D_x^{\alpha\theta/2} S(t)u_0||_{L_t^q L_x^p} \le C|u_0|_2$.

$\bullet || \int_{-\infty}^{\infty} D_x^{\alpha\theta} S(t-s) g(\cdot,s)ds||_{L_t^q(L_x^p)} \le C||g||_{L_t^{q'}L_x^{p'}}$.

$\bullet || \int_0^t D_x^{\alpha\theta} S(t-s) g(\cdot,s)ds||_{L_T^q L_x^p} \le C||g||_{L_T^{q'}L_x^{p'}}$.

Remark 7.1. For an operator L with symbol $p(\xi)$,the smoothing for the free group is given (in Fourier space) by the square root $|H(p(\xi))|^{1/2}$of the hessian determinant [137], thus no such *global* smoothing effect is true for the Schrödinger group.

5. Local smoothing.

$\bullet ||\partial_x S(t)u_0||_{L_x^\infty(L_t^2)} \le C||u_0||_2$.

$\bullet ||\partial_x^2 \int_0^t S(t-s)f(s)ds||_{L_x^\infty L_t^2} \le C||f||_{L_x^1 L_t^2}$.

Remark 7.2. (i) In term of L^2 space in x the smoothing is *local* : for almost every t, $S(t)u_0 \in L^2_{\text{loc}}(\mathbb{R})$. It cannot be global since $S(t)$ is unitary in $L^2(\mathbb{R})$.

(ii) For general symbols $p(\xi)$ the local gain of smoothness is given by $|\nabla p(\xi)|^{1/2}$.

6. Maximal function estimate (see [138]).
Let $s > \frac{3}{4}$. Then, $\forall u_0 \in H^s(\mathbb{R})$, $\forall \rho > \frac{3}{4}$

$\bullet ||S(t)u_0||_{L_x^2 L_T^\infty} \le C(1+T)^\rho ||u_0||_{H^s}$.

7. For u_0 in the homogeneous Sobolev space $\dot{H}^{\frac{1}{4}}(\mathbb{R}) = \{f \in \mathfrak{S}',\ |\xi|^{\frac{1}{4}}\hat{f}(\xi) \in L^2_\xi(\mathbb{R})\}$,

$\bullet ||S(t)u_0||_{L_x^4 L_t^\infty} \le C|D_x^{1/4}u_0|_2$.

Let us sketch the proof. The estimate is equivalent to

$$||D_x^{-1/4}S(t)u_0||_{L^4_xL^\infty_t} \le C|u_0|_2,$$

and by duality to

$$||\int_{-\infty}^{\infty} D_x^{-1/4}S(t)g(\cdot,t)||_{L^2_x} \le C||g||_{L_x^{4/3}L_t^1}.$$

Squaring the left hand side and applying a trick used in the proof of Strichartz estimates (the so-called P.Tomas argument), we arrive at the integrals are taken from minus to plus infinity:

$$||\int_{-\infty}^{\infty} D_x^{-1/4}S(t)g(\cdot,t)||^2_{L^2_x} = \iint g(x,t)\int D_x^{-1/2}S(t-s)g(x,s)dsdxdt.$$

We are thus reduced to proving

$$||\int_{-\infty}^{\infty} D_x^{-1/2}S(t-s)g(x\cdot,s)ds||_{L^4_xL^\infty_t} \le c||g||_{L_x^{4/3}L_t^1}. \tag{7.3}$$

This will be a consequence of the

Lemma 7.1.

$$\left|\int_{-\infty}^{\infty} \frac{e^{i(x\xi+t\xi^3)}}{|\xi|^{1/2}}d\xi\right| = |I_t(x)| \le \frac{C}{|x|^{1/2}}.$$

Proof of lemma.

• $t=0$. It suffices to recall that the Fourier transform of the distribution (in $\mathbb{R}^n$) $\frac{1}{|x|^\alpha}$, $0<\alpha<1$, is equal to $\frac{C_{n,\alpha}}{|\xi|^{n-\alpha}}$.

• $t \neq 0$. We observe by scaling that

$$I_t(x) = \frac{1}{t^{1/6}}I_1\left(\frac{x}{t^{1/3}}\right)$$

and it suffices to prove that

$$|I_1(x)| \le \frac{C}{|x|^{1/2}}.$$

This results from the van der Corput lemma, by a proof similar to that we used in Chapter 3 to prove that the function of x

$$\int_{\mathbb{R}} |\xi|^{1/2} e^{i(x\xi+\xi^3)} d\xi$$

is bounded.

From the lemma, it results that

$$|| \int_{-\infty}^{\infty} D_x^{-1/2} S(t-s) g(x\cdot, s) ds ||_{L_x^4 L_t^\infty} \leq \frac{c}{|x|^{1/2}} \star \int_{-\infty}^{\infty} |g(\cdot, s) ds.$$

Noticing that $\frac{c}{|x|^{1/2}} \in L_w^2(\mathbb{R})$ (7.3) follows since by the generalized Young convolution inequality

$$L_w^2 \star L^{4/3} \subset L^4.$$

We can now state a LWP result for the KdV equation.

Theorem 7.2. *Let $s > \frac{3}{4}$ and $u_0 \in H^s(\mathbb{R})$. Then there exist $T = T(||u_0||_{H^s}) >$ 0 ($T(\rho \to +\infty$ when $\rho \to 0$) and a unique solution u of the corresponding Cauchy problem for the KdV equation such that*

(i) $u \in C([-T, T]; H^s(\mathbb{R}))$

(ii) $\partial_x u \in L_T^4(L_x^\infty)$

(iii) $||D_x^s \frac{\partial u}{\partial x}||_{L_x^\infty(L_T^2)} < +\infty$

(iv) $||u||_{L_x^2(L_T^\infty)} < +\infty$

Moreover :

• $\forall T' \in (0, T)$*, there exists a neighborhood $\mathfrak{V}$ of u_0 in H^s such that the mapping $v_0 \mapsto v$ from $\mathfrak{V}$ into the class above with T' instead of T is lipschitz.*

• *If $u_0 \in H^{s'}(\mathbb{R})$ with $s' > s$, the above result holds true with s' instead of s, on the same time interval $(-T, T)$.*

$\bullet |u(\cdot,t)|_2 = |u_0|_2, \; \forall t \in (-T,T)$.

When $s \geq 1$ one has the additional properties:

The energy $E(t) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u(x,t)|^2 dx - \frac{1}{6}\int_{\mathbb{R}} u^3(x,t)dx$ is independent of t on $(-T,T)$ and the solution is globally defined.

Remark 7.3. Theorem 7.2 provides the ad hoc framework to justify fully theH^1 orbital stability of the KdV solitary wave (see [19, 26].

Proof. We will a complete proof only in the case $s = 1$ which corresponds to the space$H^1(\mathbb{R})$, the *energy space* for the KdV equation.

(i) We let the conservation of the L^2 norm as an exercice.

(ii) Conservation of energy and globalization.

The conservation of energy is obtained formally by multiplying the equation by $u_{xx} + \frac{1}{2}u^2$ and integration by parts. The argument can be made rigorous by smoothing the initial data by $u_{0\varepsilon} = u_0 \star \rho_\varepsilon \in H^\infty(\mathbb{R})$ where $\rho_\varepsilon(x) = \rho(x/\varepsilon)$, $\rho \in \mathfrak{D}(\mathbb{R})$, $\rho \geq 0$, $\int_{\mathbb{R}} \rho(x)dx = 1$ and then passing to the limit when $\varepsilon \to 0$.

To get the *a priori* H^1 bound which leads to global-well posedness we write first using the conservation of energy

$$\int_{\mathbb{R}} u_x^2(x,t)dx = 2E(0) + \frac{1}{3}\int_{\mathbb{R}} u^3(x,t)dx.$$

Then

$$\begin{aligned} &|\int_{\mathbb{R}} u^3(x,t)dx| \leq |u(\cdot,t)|_\infty \, |u(\cdot,t)|_2^2 = |u(\cdot,t)|_\infty |u_0|_2^2 \\ &\leq C|u(\cdot,t)|_2^{1/2}|u_x(\cdot,t)|_2^{1/2}|u_0|_2^2 = |u_0|_2^{3/2}|u_x(\cdot,t)|_2^{1/2}, \end{aligned} \tag{7.4}$$

where we have used the well-known inequality for $f \in H^1(\mathbb{R})$

$$|f|_\infty \leq \sqrt{2}|u_x|_2^{1/2}|u|_2^{1/2}.$$

Using Young's inequality

$$ab \leq \frac{1}{2\varepsilon}a^{\frac{1}{\alpha}} + \frac{\varepsilon}{2}b^{\frac{1}{1-\alpha}}, \text{ for any } \varepsilon > 0, \; 0 < \alpha < 1,$$

one deduces (with $\alpha = \frac{1}{4}$) that

$$|\int_{\mathbb{R}} u^3(x,t)dx| \leq \varepsilon |u(\cdot,t)|_2^2 + C(\varepsilon, |u_0|_2),$$

and thus that

$$||u(\cdot,t)||_{H^1} \leq C(||u_0||_{H^1}).$$

(iii) Local well-posedness.

Definition of the space X_T.
For $f : \mathbb{R} \times [-T,T] \to |R$, we set

$$\lambda_1^T(f) = \max_{t\in[-T,T]} ||f(\cdot,t)||_{H^1} \tag{7.5}$$

$$\lambda_2^T(f) = \left(\int_{-T}^{T} |f_x(\cdot,t)|_\infty^4 dt\right)^{\frac{1}{4}} \tag{7.6}$$

$$\lambda_3^T(f) = ||\frac{\partial^2 f}{\partial x^2}||_{L_x^\infty(L_T^2)} \tag{7.7}$$

$$\lambda_4^T(f) = (1+T)^{-\rho}||f||_{L_x^2(L_T^\infty)} \text{ for a fixed } \rho > \frac{3}{4} \tag{7.8}$$

$$\Lambda^T(f) = \max_{j=1,\dots,4} \lambda_j^T(f). \tag{7.9}$$

Finally,

$$X_T = \{f \in C([-T,T];H^1(\mathbb{R})),\ \Lambda^T(f) < +\infty\}.$$

The linear estimates above insure that if $u_0 \in H^1(\mathbb{R})$, then $S(t)u_0 \in X^T$, $\forall T > 0$.

For $v_0 \in H^1(\mathbb{R})$ we denote $u = \phi_{u_0}(v) = \phi(v)$ the unique solution of

$$u_t + u_{xxx} = -vv_x,\ u(x,0) = u_0(x)$$

or, in the Duhamel representation

$$u(t) = S(t)u_0 - \int_0^t S(t-s)vv_x(s)ds. \tag{7.10}$$

We shall prove that there exist $T = T(||u_0||_{H^1}) > 0$ and $a = a(||u_0||_{H^1}) > 0$ such that if $v \in X_T^a = \{f \in X_T, \Lambda^T(f) \le a\}$, then $u = \Phi(v) \in X_T^a$ and $\phi : X_T^a \to X_T^a$ is a contraction.

One proves first that

$$||\partial_x(vv_x)||_{L_x^2 L_T^2} + ||vv_x||_{L_x^2 L_T^2} \le C(1+T)^\rho(\Lambda^T(v))^2. \tag{7.11}$$

In fact,

$$\begin{aligned}
||vv_x||_{L_x^2 L_T^2} &= \left(\int_{-T}^{T}\int_{-\infty}^{\infty} |v|^2|v_x|^2 dxdt\right)^{\frac{1}{2}} \\
&\le \left(\int_{-T}^{T} |v_x(\cdot,t)|_\infty^2 dt\right)^{\frac{1}{2}} \sup_{t\in[-T,T]} |v(\cdot,t)|_2 \\
&\le \left(\int_{-T}^{T} |v_x(\cdot,t)|_\infty^2 dt\right)^{\frac{1}{2}} \sup_{t\in[-T,T]} ||v(\cdot,t)||_{H^1}^2 \qquad (7.12)\\
&\le \text{(by Hölder in time)} \\
&\le CT^{\frac{1}{4}} ||v_x||_{L_T^4(L_x^\infty)} \lambda_1^T \\
&\le CT^{\frac{1}{4}} \lambda_2^T \lambda_1^T \\
&\le C(1+T)^\rho(\Lambda^T(v))^2.
\end{aligned}$$

On the other hand, again by Hölder inequality in time,

$$\begin{aligned}
||\partial_x(vv_x)||_{L_x^2 L_T^2} &\le \left(\int_{-T}^{T} |v_x(\cdot,t)|_\infty^2 |v_x(\cdot,t)|_2^2 dt\right)^{\frac{1}{2}} + \left(\int_{-T}^{T} |v(\cdot,t)v_{xx}(\cdot,t)|^2 dxdt\right)^{\frac{1}{2}} \\
&\le CT^{\frac{1}{4}} ||v_x||_{L_T^4 L_x^\infty} \sup_{t\in[-T,T]} |v_{xx}(\cdot,t)|_2 + ||v||_{L_x^2 L_T^\infty} ||v_{xx}||_{L_x^\infty L_T^2} \\
&\le CT^{\frac{1}{4}} \lambda_2^T(v)\lambda_1^T(v) + C(1+T)^\rho \lambda_4^T(v)\lambda_3^T(v) \\
&\le C(1+T)^\rho(\Lambda^T(v))^2
\end{aligned} \tag{7.13}$$

which achieves to prove (11.6).

Using the Duhamel formulation (7.10), the linear estimates on $S(t)u_0$, (11.6) and Hölder inequality in time, one gets

$$\begin{aligned}\Lambda^T(u) &\leq C||u_0||_{H^s} + C\int_{-T}^{T} ||vv_x||_{H^s}(t)dt \\ &\leq C||u_0||_{H^s} + CT^{1/2}\left(\int_{-T}^{T} ||vv_x||_{H^s}^2(t)dt\right)^{1/2} \\ &\leq C||u_0||_{H^s} + CT^{1/2}(1+T)^{\rho}(\Lambda^T v))^2.\end{aligned} \tag{7.14}$$

Now we choose $a = 2C||u_0||_{H^s}$, then T such that

$$4CT^{1/2}(1+T)^{\rho}a < 1.$$

Verify (exercice) that if $v \in X_T^a$, then $u = \phi(v) \in C([-T,T];H^s(\mathbb{R}))$. One then concludes that $\phi : X_T^a \to X_T^a$.

One can establish in a similar way that

$$\Lambda^T(\phi(v) - \phi(\tilde{v})) \leq CT^{1/2}(1+T)^{\rho}\{\Lambda^T(v) + \Lambda^T(\tilde{v})\}\Lambda(v-\tilde{v}), \tag{7.15}$$

and that for $T_1 \in (0,T)$,

$$\begin{aligned}\Lambda^{T_1}(\phi_{u_0}(v) - \phi_{\tilde{u_0}}(\tilde{v} \leq C||u_0 - \tilde{u_0}||_{H^1} \\ + CT_1^{1/2}(1+T_1)^{\rho}\Lambda^{T_1}(v-\tilde{v})\{\Lambda_{T_1}(v) + \Lambda^{T^1}(\tilde{v})\}.\end{aligned} \tag{7.16}$$

We have thus proven (see (7.14), (7.15)) the existence of a unique $u \in X_T^a$ such that $\phi_{u_0}(u) \equiv u$, that is

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(uu_x)(s)ds.$$

Moreover (7.16), together with our choice of T and a shows that for any $T_1 \in (0,T)$ the map $\tilde{u_0} \mapsto \tilde{u}$ on a neighborhood $\mathfrak{V}$ of u_0 in H^1 depending on T_1 to $X_{T_1}^a$ is lipschitz.

One easily checks that u satisfies the KdV equation (at least) in the distribution sense.

To prove uniqueness in X_T, we suppose that $w \in X_{T_1}$ for some $T_1 \in (0,T)$ is a solution of the KdV equation. The argument leading to (7.14) shows that for some $T_2 \in (0,T_1)$, $w \in X_{T_2}^a$. Hence our choice of T implies that $w \equiv u$ in $\mathbb{R} \times [-T_2,T_2]$. By reapplying the argument this result can be extended to the whole interval $[-T,T]$. □

Remark 7.4. The general case $s > \frac{3}{4}$ is proven by similar arguments. An extra technical difficulty arises when one evaluates a term like $D_x^s(vv_x)$ when s is not an integer. For $\frac{3}{4} < s < 1$ one must use the following fractional Leibniz formula (see [139]):

Theorem 7.3. *Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Let p, p_1, p_2, $q, q_1, q_2 \in (1,+\infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$||D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f||_{L_x^p L_T^q} \le c||D_x^{\alpha_1} f||_{L_x^{p_1} L_T^{q_1}} ||D_x^{\alpha_2} g||_{L_x^{p_2} L_T^{q_2}}.$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = +\infty$ is allowed.

Remark 7.5. The method we used in proving the theorem is robust. Similar arguments (with extra technical difficulties) can be applied to the resolution of the Cauchy problem with data in $H^{s_k}(\mathbb{R})$ for the generalized KdV equations

$$u_t + u^k u_x + u_{xxx} = 0, \; k = 1, 2, \ldots$$

They give essentially optimal results as far as the value of s_k is concerned for $k \ge 2$. On the other hand since the method relies heavily on dispersive properties of $e^{t\partial_x^3}$ on $\mathbb{R}$ it cannot be applied in the periodic case, for instance when $u_0 \in H^1(\mathbb{T})$. One should then use Bourgain's method which implements a Picard scheme with a totally different choice of the space X_T (see [96] and below).

This method applies as well in the case of nonperiodic initial data, leading to LWP for the KdV equation with data in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$ (the solution is in fact global...).

The (rough) idea of the method is to use a *classical* space for $S(-t)u$, that is use to implement the contraction argument a (possibly localized in time) space X with norm $||u||_X = ||S(-t)u||_H$, where H is a *classical* space (Sobolev, Lebesgue, Besov,..). A typical choice for the KdV equation is

$$X^{s,b} = \{f \in \mathscr{S}'(\mathbb{R}^2); \int_{\mathbb{R}^2} (1+|\tau - \xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\xi,\tau)|^2 d\xi d\tau < +\infty\}.$$

Here ξ (resp. τ) is the Fourier dual variable of x (resp. t).

Denoting by $< \cdot >= (1+|\cdot|^2)^{\frac{1}{2}}$ the "Japanese bracket", one sees that

$$X^{s,b} = \{u; S(-t)u \in H^{s,b}\},$$

where

$$H^{s,b} = \{u; || < \xi >^s < \tau >^b \hat{u}||_{L^2} < +\infty\},$$

where $\hat{u}$ is the Fourier transform of u with respect to x and t.

This choice of space somehow evacuates the free evolution term and all the difficulties are concentrated on a *bilinear estimate* which writes for the KdV equation (that is $S(t) = e^{t\partial_x^3}$) :

Theorem 7.4. *(i) Let* $s > -\frac{3}{4}$. *There exists* $b > \frac{1}{2}$ *such that if* $v \in X^{s,b}$, *then* $v\partial_x v \in X^{s,b-1}$ *and*

$$||v\partial_x v||_{X^{s,b-1}} \leq c||v||^2_{X^{s,b}}.$$

(ii) For $s \leq -\frac{3}{4}$, *the above bilinear estimate is false for any* $b \in \mathbb{R}$.

Note that s measures the regularity in space and thus the bilinear estimate “regains” the derivative in the nonlinear term by loosing a degree of regularity in time. See [140] for details.

7.1.1 Remarks on the Inverse Scattering Transform method

As we already emphasized, the global solvability of the Cauchy problem does not give any qualitative information on the solutions. We comment here briefly on the Inverse Scattering method see [86, 160] which allows to describe very precisely the long time behavior of the KdV solutions.

We follow the treatment in Schuur [210] and consider the KdV equation written in the form

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(\cdot,0) = u_0. \tag{7.17}$$

We recall that (7.17) has the classical *soliton* solution

$$u(x,t) = -2k_0^2 \text{sech}\,[k_0(x - x_0 - 4k_0^2 t],$$

where k_0, x_0 are constants and $\text{sech} = 1/\cosh$.

The crux of the method is the direct and inverse spectral problems for the Schrödinger operator

$$L(t) = -\frac{d^2\psi}{dx^2} + u(\cdot,t)\psi,$$

considered as an unbounded operator in $L^2(\mathbb{R})$. We thus consider the spectral problem

$$\psi_{xx} + (k^2 - u(x,t))\psi = 0, \quad -\infty < x < +\infty.$$

Given $u_0 = u(\cdot,0)$ sufficiently smooth and decaying at $\pm\infty$, say in the Schwartz space $\mathscr{S}(\mathbb{R})$, one associates to $L(0)$ its spectral data, that is a finite possibly empty) set of negative eigenvalues $-\kappa_1^2 < -\kappa_2^2 \cdots < -\kappa_N^2$, together with right normalization coefficients c_j^r and right reflection coefficients $b_r(k)$ (see [210] for precise definitions and properties of those objects).

The spectral data consists thus in the collection of $\{b_r(k), \kappa_j, c_j^r\}$. It turns out that if $u(x,t)$ evolves according to the KdV equation, the scattering data evolves in a very simple way :

$$\kappa_j(t) = \kappa_j.$$

$$c_j^r(t) = c_j^r \exp(4\kappa_j^3 t), \quad j = 1, 2, \cdots, N$$

$$b_r(k,t) = b_r(k)\exp(8ik^3 t), \quad -\infty < k < +\infty.$$

The potential $u(x,t)$ is recovered as follows. Let

$$\Omega(\xi;t) = 2\sum_{j=1}^{N} [c_j^r(t)]^2 e^{-2\kappa_j \xi} + \frac{1}{\pi}\int_{-\infty}^{\infty} b_r(k,t) e^{2ik\xi}\, dk.$$

One then solve the linear integral equation (Gel'fand-Levitan equation):

$$\beta(y;x,t) + \Omega(x+y;t) + \int_0^{\infty} \Omega(x+y+z;t)\beta(z;x,t)\,dz = 0, \quad y>0,\ x\in\mathbb{R},\ t>0. \tag{7.18}$$

The solution of the Cauchy problem (7.17) is then given by

$$u(x,t) = -\frac{\partial}{\partial x}\beta(0^+;x,t), \quad x\in\mathbb{R}, \quad t>0.$$

One obtains explicit solutions when $b_r = 0$. A striking case is obtained when the scattering data are $\{0, \kappa_j, c_j^r(t)\}$. It corresponds to the so-called $N-soliton$ solution according to its asymptotic behavior obtained by Tanaka [229]:

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} \left| u_d(x,t) - \sum_{p=1}^{N}\left(-2\kappa_p^2 \operatorname{sech}^2[\kappa_p(x - x_p^+ - 4\kappa_p^2 t)]\right)\right| = 0, \tag{7.19}$$

where

$$x_p^+ = \frac{1}{2\kappa_p} \log \left\{ \frac{[c_p^r]^2}{2\kappa_p} \prod_{l=1}^{p-1} \left(\frac{\kappa_l - \kappa_p}{\kappa_l + \kappa_p} \right)^2 \right\}.$$

In other words, $u_d(x,t)$ appears for large positive time as a sequence of N solitons, with the largest one in the front, uniformly with respect to $x \in \mathbb{R}$.

For $u_0 \in \mathscr{S}(\mathbb{R})^2$, the solution of (7.17) has the following asymptotics

$$\sup_{x \geq -t^{1/3}} |u(x,t)| = O(t^{-2/3}), \quad \text{as } t \to \infty \tag{7.20}$$

in the absence of solitons (that is when $L(0)$ has no negative eigenvalues) and

$$\sup_{x \geq -t^{1/3}} |u(x,t) - u_d(x,t)| = O(t^{-1/3}), \quad \text{as } t \to \infty \tag{7.21}$$

in the general case, the N in u_d being the number of negative eigenvalues of $L(0)$. One has moreover the convergence result

$$\lim_{t \to +\infty} \sup_{x \geq -t^{1/3}} \left| u(x,t) - \sum_{p=1}^{N} \left(-2\kappa_p^2 \text{sech}^2 [\kappa_p (x - x_p^+ - 4\kappa_p^2 t)] \right) \right| = 0. \tag{7.22}$$

In both cases, a "dispersive tail" propagates to the left.

Remark 7.6. The shortcoming of those remarkable results is of course that they apply only to the integrable KdV equation (and also to the modified KdV equation)

$$u_t + 6u^2 u_x + u_{xxx} = 0.$$

However, though they are of reach of "classical" PDE methods, they give hints on the behavior of other, non integrable, equations whose dynamics could be in some sense similar.

7.2 The Cauchy problem for the BBM equation

The BBM equation can be viewed as an ODE in any $H^s(\mathbb{R})$ space, $s \geq 0$ which makes the local Cauchy theory easy. The global theory for $0 \leq s < 1$ however needs a little care.

[2]This condition can be weakened, but a decay property is always needed.

Observing that the Fourier transform of $\phi(\xi) = \frac{\xi}{1+\xi^2}$ is equal to $F(x) = C \operatorname{sign} x e^{-|x|}$ the Cauchy problem for the BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad u(\cdot,0) = u_0 \tag{7.23}$$

is equivalent to

$$u_t + F \star_x \left(u + \frac{u^2}{2}\right) = 0, \quad u(\cdot,0) = u_0. \tag{7.24}$$

Theorem 7.5. *Let $s \geq 0$ and $u_0 \in H^s(\mathbb{R})$. Then* (7.24) *or* (7.23) *has a unique solution $u \in C([0,\infty; H^s(\mathbb{R}))$.*

Moreover, if $s \geq 1$ the energy

$$\int_{\mathbb{R}^n} [u^2(x,t) + u_x(x,t)^2]dx \tag{7.25}$$

is independent of time.

Proof. We give details for the case $0 \leq s \leq 1$ and refer to [38] for a complete proof.

Lemma 7.6. *For any $s \geq 0$ the map $\Phi : u \to F \star (u + \frac{u^2}{2})$ belongs to $C^1(H^s(\mathbb{R}), H^s(\mathbb{R}))$.*

Proof. One has

$$D\Phi(u) \cdot v = \frac{d}{d\varepsilon}[F \star (u + \varepsilon v + \frac{(u+\varepsilon v)^2}{2})]_{|\varepsilon=0} = F \star (v + uv)$$

and we just have to prove that the map $v \to F \star (v + uv)$ belongs to $\mathscr{L}(H^s(\mathbb{R}), H^s(\mathbb{R}))$, $0 \leq s \leq 1$.

This results from the convolutions inequalities

$$|F \star (v_x + uv)|_2 \leq C(|F|_1 |v_x|_2 + |F|_2 |uv|_1 \leq C(|F|_1 |v|_2 + |F|_2 |u|_2 |v|_2,$$

$$\begin{aligned} |F \star (v + (uv)_x)|_2 &\leq C(|F|_1 |v|_2 + |F|_2 \, |(uv)_x|_1 \\ &\leq C(|F|_1 |v_x|_2 + |F|_2(|u_x|_2 |v|_2 + |u|_2 |v_x|_2) \end{aligned}$$

and interpolating between L^2 and H^1..

This proves that Φ is differentiable from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. A similar argument proves that Φ is actually C^1 (observe that $D\Phi(u) \cdot v$ is a sum of a continuous linear map in u and a bilinear continuous map in (u,v)). □

By the Cauchy-Lipschitz theorem in Banach spaces, (7.24) possesses for any $u_0 \in H^s(\mathbb{R}), 0 \leq s \leq 1$, a unique solution $u \in C([0, T(||u_0||_s)]$ with $T(||u_0||_s) \to +\infty$ as $||u_0||_s \to 0$.

The fact that this solution is global when $s = 1$ results trivially from the conservation law (7.25).

To prove that the solution is global in L^2 (actually in $H^s, s \geq 0$), we sketch the argument of [38].

Fix $T > 0$ and let $u_0 \in H^s(\mathbb{R})$. In what follows we will use the notation $X_T^s = \{f \in C([0,T]; H^s(\mathbb{R}))\}$ equipped with the standard norm. We want to prove that the unique local solution of (7.23) lies in X_T^s and that u depends continuously upon u_0. By the local well-posedness result this is true if the initial data is small enough in H^s. We write $u_0 = v_0 + w_0$ according to the decomposition

$$\widehat{u_0}(\xi) = \chi_{|\xi| \geq N} \widehat{u_0}(\xi) + \chi_{|\xi| \leq N} \widehat{u_0}(\xi),$$

with $N > 0$ large enough in order that the solution of (7.23) with initial data v_0 belongs to X_T. This is possible since

$$\int_{|\xi| \geq N} (1 + |\xi|^2)^s |\hat{u_0}(\xi)|^2 d\xi$$

is as small as we wish for N large enough.

We consider the Cauchy problem

$$\begin{cases} w_t + w_x - w_{xxt} + w w_x + (vw)_x = 0, \\ w(\cdot, 0) = w_0. \end{cases} \tag{7.26}$$

If there exists a solution of (7.26) in X_T^s, then $v + w$ will solve (7.23) in X_T and the result will be established. Note that $w_0 \in H^r(\mathbb{R})$ for any r and in particular $w_0 \in H^1(\mathbb{R})$.

One establishes easily following the lines of the local H^s theory that, v being fixed, (7.26) has a unique local solution in X_τ^1 for some positive τ and it suffices to prove that we can extend this solution up to $\tau = T$. This is established by an a priori bound on w. Multiplying (7.26) by w and integrating by parts we get

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}} [w^2(x,t) + w_x^2(x,t))] dx \right) - \int_{\mathbb{R}} v(x,t) w(x,t) w_x(x,t) dx = 0.$$

One obtains by Sobolev and Hölder inequalities

$$\|\int_{\mathbb{R}} v(x,t)w(x,t)w_(x,t)dx\| \leq C||w(\cdot,t)||_1|v(\cdot,t)|_2$$

and we deduce with Gronwall's lemma that

$$||w(\cdot,t||_1 \leq ||w_0||_1 \exp\left(\int_0^t |v(\cdot,t')dt'\right)$$

proving that $w(\cdot,t)$ exists up to T.

The proof of Theorem 7.5 is complete. □

Chapter 8

Some semilinearly ill-posed dispersive equations

8.1 Some semilinearly ill-posed dispersive equations

Deciding whether a nonlinear dispersive equation is semilinearly well-posed is somewhat subtle. To start with we consider the case of the Benjamin-Ono equation following [185].

8.1.1 "Ill-posedness" of the Benjamin-Ono equation

We thus consider the Cauchy problem

$$\begin{cases} u_t - Hu_{xx} + uu_x = 0, \ (t,x) \in \mathbb{R}^2, \\ u(0,x) = \phi(x). \end{cases} \tag{8.1}$$

Setting $S(t) = e^{itH\partial_x^2}$, we write (8.1) as an integral equation:

$$u(t) = S(t)\phi - \int_0^t S(t-t')(u_x(t')u(t'))dt'. \tag{8.2}$$

The main results is the following

Theorem 8.1. *Let $s \in \mathbb{R}$ and T be a positive real number. Then there does not exist a space X_T continuously embedded in $C([-T,T], H^s(\mathbb{R}))$ such that there exists $C > 0$ with*

$$\|S(t)\phi\|_{X_T} \leq C\|\phi\|_{H^s(\mathbb{R})}, \quad \phi \in H^s(\mathbb{R}), \tag{8.3}$$

and

$$\left\| \int_0^t S(t-t') \left[u(t')u_x(t')\right] dt' \right\|_{X_T} \leq C\|u\|_{X_T}^2, \quad u \in X_T. \tag{8.4}$$

Note that (8.3) and (8.4) would be needed to implement a Picard iterative scheme on (8.2), in the space X_T. As a consequence of Theorem 14.1 we can obtain the following result.

Theorem 8.2. *Fix $s \in \mathbb{R}$. Then there does not exist a $T > 0$ such that* (8.1) *admits a unique local solution defined on the interval $[-T,T]$ and such that the flow-map data-solution*

$$\phi \longmapsto u(t), \quad t \in [-T,T],$$

for (8.1) *is C^2 differentiable at zero from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.*

Remark 8.1. This result implies that the Benjamin-Ono equation is "quasilinear". It has been precised in [150] where it is shown that the flow map cannot even be locally Lipschitz in H^s for $s \geq 0$.

Proof of Theorem 14.1

Suppose that there exists a space X_T such that (8.3) and (8.4) hold. Take $u = S(t)\phi$ in (8.4). Then

$$\left\| \int_0^t S(t-t') \left[(S(t')\phi)(S(t')\phi_x)\right] dt' \right\|_{X_T} \leq C\|S(t)\phi\|_{X_T}^2.$$

Now using (8.3) and that X_T is continuously embedded in $C([-T,T], H^s(\mathbb{R}))$ we obtain for any $t \in [-T,T]$ that

$$\left\| \int_0^t S(t-t') \left[(S(t')\phi)(S(t')\phi_x)\right] dt' \right\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}^2. \tag{8.5}$$

We show that (8.13) fails by choosing an appropriate ϕ.

Take ϕ defined by its Fourier transform as[1]

$$\widehat{\phi}(\xi) = \alpha^{-\frac{1}{2}} 1_{I_1}(\xi) + \alpha^{-\frac{1}{2}} N^{-s} 1_{I_2}(\xi), \quad N \gg 1, \ 0 < \alpha \ll 1,$$

where I_1, I_2 are the intervals

$$I_1 = [\alpha/2, \alpha], \ \ I_2 = [N, N+\alpha].$$

Note that $\|\phi\|_{H^s} \sim 1$. We will use the next lemma.

Lemma 8.3. *The following identity holds:*

$$\begin{aligned} & \int_0^t S(t-t') \Big[(S(t')\phi)(S(t')\phi_x) \Big] \, dt' \\ = \quad & c \int_{\mathbb{R}^2} e^{ix\xi + itp(\xi)} \, \xi \, \hat{\phi}(\xi_1) \hat{\phi}(\xi - \xi_1) \frac{e^{it(p(\xi_1)+p(\xi-\xi_1)-p(\xi))} - 1}{p(\xi_1) + p(\xi - \xi_1) - p(\xi)} d\xi d\xi_1, \end{aligned}$$

where $p(\xi) = \xi|\xi|$.

Proof of Lemma 8.6. Taking the inverse Fourier transform with respect to x, it is easily seen that

According to the above lemma,

$$\int_0^t S(t-t') \left[(S(t')\phi)(S(t')\phi_x) \right] dt' = c(f_1(t,x) + f_2(t,x) + f_3(t,x)),$$

where, from the definition of ϕ, we have the following representations for

[1] The analysis below works as well for $\mathscr{R}e\,\phi$ instead of ϕ (some new harmless terms appear).

f_1, f_2, f_3:

$$
\begin{aligned}
f_1(t,x) &= \frac{c}{\alpha}\int_{\substack{\xi_1\in I_1\\ \xi-\xi_1\in I_1}} \xi\, e^{ix\xi+it\xi|\xi|}\frac{e^{it(\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|)}-1}{\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|}d\xi d\xi_1,\\
f_2(t,x) &= \frac{c}{\alpha N^{2s}}\int_{\substack{\xi_1\in I_2\\ \xi-\xi_1\in I_2}} \xi\, e^{ix\xi+it\xi|\xi|}\frac{e^{it(\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|)}-1}{\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|}d\xi d\xi_1,\\
f_3(t,x) &= \frac{c}{\alpha N^{s}}\int_{\substack{\xi_1\in I_1\\ \xi-\xi_1\in I_2}} \xi\, e^{ix\xi+it\xi|\xi|}\frac{e^{it(\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|)}-1}{\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|}d\xi d\xi_1\\
&\quad+\frac{c}{\alpha N^{s}}\int_{\substack{\xi_1\in I_2\\ \xi-\xi_1\in I_1}} \xi\, e^{ix\xi+it\xi|\xi|}\frac{e^{it(\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|)}-1}{\xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|}d\xi d\xi_1.
\end{aligned}
$$

Set

$$\chi(\xi,\xi_1) := \xi_1|\xi_1|+(\xi-\xi_1)|\xi-\xi_1|-\xi|\xi|.$$

Then clearly

$$
\begin{aligned}
\mathscr{F}_{x\mapsto\xi}(f_1)(t,\xi) &= \frac{c\,\xi e^{it\xi|\xi|}}{\alpha}\int_{\substack{\xi_1\in I_1\\ \xi-\xi_1\in I_1}}\frac{e^{it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)}d\xi_1,\\
\mathscr{F}_{x\mapsto\xi}(f_2)(t,\xi) &= \frac{c\,\xi e^{it\xi|\xi|}}{\alpha N^{2s}}\int_{\substack{\xi_1\in I_2\\ \xi-\xi_1\in I_2}}\frac{e^{it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)}d\xi_1,\\
\mathscr{F}_{x\mapsto\xi}(f_3)(t,\xi) &= \frac{c\,\xi e^{it\xi|\xi|}}{\alpha N^{s}}\left(\int_{\substack{\xi_1\in I_1\\ \xi-\xi_1\in I_2}}\frac{e^{it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)}d\xi_1+\int_{\substack{\xi_1\in I_2\\ \xi-\xi_1\in I_1}}\frac{e^{it\chi(\xi,\xi_1)}-1}{\chi(\xi,\xi_1)}d\xi_1\right).
\end{aligned}
$$

Since the supports of $\mathscr{F}_{x\mapsto\xi}(f_j)(t,\xi)$, $j=1,2,3$, are disjoint, we have

$$\left\|\int_0^t S(t-t')\left[(S(t')\phi)(S(t')\phi_x)\right]dt'\right\|_{H^s(\mathbb{R})} \geq \|f_3(t,\cdot)\|_{H^s(\mathbb{R})}.$$

We now give a lower bound for $\|f_3(t,\cdot)\|_{H^s(\mathbb{R})}$. Note that for $(\xi_1,\xi-\xi_1)\in I_1\times I_2$ or $(\xi_1,\xi-\xi_1)\in I_2\times I_1$ one has $|\chi(\xi,\xi_1)| = 2|\xi_1(\xi-\xi_1)| \sim \alpha N$.

Hence it is natural to choose α and N so that $\alpha N = N^{-\varepsilon}$, $0 < \varepsilon \ll 1$. Then

$$\left| \frac{e^{it\chi(\xi,\xi_1)} - 1}{\chi(\xi,\xi_1)} \right| = |t| + O(N^{-\varepsilon})$$

for $\xi_1 \in I_1$, $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$, $\xi - \xi_1 \in I_1$. Hence for $t \neq 0$,

$$\|f_3(t,\cdot)\|_{H^s(\mathbb{R})} \gtrsim \frac{N N^s \alpha \alpha^{\frac{1}{2}}}{\alpha N^s} = \alpha^{\frac{1}{2}} N.$$

Therefore we arrive at

$$1 \sim \|\phi\|^2_{H^s(\mathbb{R})} \geq \|f_3(t,\cdot)\|_{H^s(\mathbb{R})} \geq \alpha^{\frac{1}{2}} N \sim N^{\frac{1-\varepsilon}{2}},$$

which is a contradiction for $N \gg 1$ and $\varepsilon \ll 1$. This completes the proof of Theorem 14.1.

8.1.2 Proof of Theorem 8.2

Consider the Cauchy problem

$$\begin{cases} u_t - Hu_{xx} + uu_x = 0, \\ u(0,x) = \gamma\phi, \quad \gamma \ll 1, \quad \phi \in H^s(\mathbb{R}) . \end{cases} \tag{8.6}$$

Suppose that $u(\gamma,t,x)$ is a local solution of (8.6) and that the flow map is C^2 at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. We have successively

$$u(\gamma,t,x) = \gamma S(t)\phi + \int_0^t S(t-t')u(\gamma,t',x)u_x(\gamma,t',x)dt'$$

$$\frac{\partial}{\partial\gamma}(0,t,x) = S(t)\phi(x) =: u_1(t,x)$$

$$\frac{\partial^2 u}{\partial\gamma^2}(0,t,x) = -2\int_0^t S(t-t')\left[(S(t')\phi)(S(t')\phi_x)\right]dt'.$$

The assumption of C^2 regularity yields

$$\left\| \int_0^t S(t-t')\left[(S(t')\phi)(S(t')\phi_x)\right]dt' \right\|_{H^s(\mathbb{R})} \lesssim \|\phi\|^2_{H^s(\mathbb{R})}.$$

But the above estimate is (8.13), which has been shown to fail in section 8.1.1.

Remark 8.2. The previous results are in fact valid in a more general context. We consider now the class of equations

$$u_t + uu_x - Lu_x = 0, \quad u(0,x) \;=\; \phi(x), \quad (t,x) \in \mathbb{R}^2, \tag{8.7}$$

where L is defined via the Fourier transform

$$\widehat{Lf}(\xi) = \omega(\xi)\hat{f}(\xi).$$

Here $\omega(\xi)$ is a continuous real-valued function. Set $p(\xi) = \xi\,\omega(\xi)$. We assume that $p(\xi)$ is differentiable and such that, for some $\gamma \in \mathbb{R}$,

$$|p'(\xi)| \lesssim |\xi|^\gamma, \quad \xi \in \mathbb{R}. \tag{8.8}$$

The next theorem shows that (8.7) shares the bad behavior of the Benjamin–Ono equation with respect to iterative methods.

Theorem 8.4. *Assume that* (8.8) *holds with* $\gamma \in [0,2[$. *Then the conclusions of Theorems* 14.1*,* 8.2*, are valid for the Cauchy problem* (8.7)*.*

The proof follows the considerations of the previous section. The main point in the analysis is that for $\xi_1 \in I_1$, $\xi - \xi_1 \in I_2$ one has

$$|p(\xi_1) + p(\xi - \xi_1) - p(\xi)| \lesssim \alpha N^\gamma, \qquad \alpha \ll 1, \quad N \gg 1.$$

We choose α and N such that $\alpha N^\gamma = N^{-\varepsilon}$, $0 < \varepsilon \ll 1$. We take the same ϕ as in the proof of Theorem 14.1 and arrive at the lower bound

$$1 \sim \|\phi\|^2_{H^s(\mathbb{R})} \geq \alpha^{\frac{1}{2}} N = N^{1-\frac{\gamma+\varepsilon}{2}},$$

which fails for $0 < \varepsilon \ll 1$, $\gamma \in [0,2[$.

Here we give several examples where Theorem 8.4 applies.

- Pure power dispersion:

$$\omega(\xi) = |\xi|^\gamma, \quad 0 \leq \gamma < 2.$$

This dispersion corresponds to a class of models for vorticity waves in the coastal zone (see [213]). It is interesting to notice that the case $\gamma = 2$ corresponds to the KdV equation which can be solved by iterative methods as we have s(see [138]). Therefore Theorem 8.4 is sharp for a pure power dispersion. However, the Cauchy problem corresponding to $1 \leq \gamma < 2$ has been proven in [**?**] to be locally well-posed by a compactness method combined with sharp estimates on the linear group for initial data in $H^s(\mathbb{R})$, $s \geq (9-3\gamma)/4$.

• Perturbations of the Benjamin-Ono equation:

$\omega(\xi) = (|\xi|^2+1)^{\frac{1}{2}}$. This case corresponds to an equation introduced by Smith [217] for continental shelf waves.

$\omega(\xi) = \xi \ \coth\ (\xi)$. This corresponds to the Intermediate long wave equation, cf. [151, 10, 1] and Chapter 5 Section 2.

The "ill-posedness" of the Benjamin -Ono equation is due to the "bad" interactions of small and large frequencies.

The mechanism is different for the KP I equation. Recall that the KP equations are

$$u_t + uu_x + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0,$$

the $+$ sign corresponding to the KP II equation, the $-$ sign to the KP I equation.

8.1.3 Ill-posedness of the KP I equation

The KP I equation is also semilinearly ill-posed, though for different reasons.

Consider the initial value problem for the KP-I equation

$$\begin{cases} (u_t + u_{xxx} + uu_x)_x - u_{yy} = 0, \\ u(0,x,y) = \phi(x,y). \end{cases} \tag{8.9}$$

We have the following result.

Theorem 8.5. *Let $(s_1,s_2) \in \mathbb{R}^2$ (resp. $s \in \mathbb{R}$). Then there exists no $T>0$ such that* ((8.14)) *admits a unique local solution defined on the interval $[-T,T]$ and such that the flow-map*

$$S_t : \phi \longmapsto u(t), \quad t \in [-T,T]$$

for (8.14) is C^2 differentiable at zero from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$, (resp. from $H^s(\mathbb{R}^2)$ to $H^s(\mathbb{R}^2)$).

Remark 8.3. As in the case of the Benjamin-Ono equation, Theorem 8.14 implies that one cannot solve the KPI equation by iteration on the Duhamel formulation for data in Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ or $H^s(\mathbb{R}^2)$, for any value of s, s_1, s_2.

This is in contrast with the KP II equation which can be solved by Bourgain's method, thus by iteration for data in $H^s(\mathbb{R}^2)$, $s \geq 0$, see [45].[2]

Remark 8.4. It has been proved in [150] that the flow map cannot be uniformly continuous in the energy space.

Proof. We merely sketch it (see [187] for details). Let

$$\sigma(\tau,\xi,\eta) = \tau - \xi^3 - \frac{\eta^2}{\xi},$$

$$\sigma_1(\tau_1,\xi_1,\eta_1) = \sigma(\tau_1,\xi_1,\eta_1),$$
$$\sigma_2(\tau_1,\xi,\eta_1,\tau_1,\xi_1,\eta_1) = \sigma(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1).$$

We then define

$$\chi(\xi,\xi_1,\eta,\eta_1) := 3\xi\xi_1(\xi-\xi_1) - \frac{(\eta\xi_1-\eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.$$

Note that $\chi(\xi,\xi_1,\eta,\eta_1) = \sigma_1+\sigma_2-\sigma$. The "resonant" function $\chi(\xi,\xi_1,\eta,\eta_1)$ plays an important role in our analysis. The "large" set of zeros of $\chi(\xi,\xi_1,\eta,\eta_1)$ is responsible for the ill-posedness issues. In contrast, the corresponding resonant function for the KP II equation is

$$\chi(\xi,\xi_1,\eta,\eta_1) := 3\xi\xi_1(\xi-\xi_1) + \frac{(\eta\xi_1-\eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.$$

Since it is essentially the sum of two squares, its set of zeroes is small and this is the key point to establish the crucial bilinear estimate in Bourgain' s method ([45]).

We consider $H^{s_1,s_2}(\mathbb{R}^2)$ and will indicate later the modifications for $H^s(\mathbb{R}^2)$.

Consider the Cauchy problem

$$\begin{cases} (u_t+u_{xxx}+uu_x)_x - u_{yy} = 0, \\ u(0,x,y) = \gamma\phi(x,y), \quad \gamma\in\mathbb{R}, \end{cases} \tag{8.10}$$

[2] and even in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 \geq \frac{1}{3}$, $s_2 \geq 0$, see [228].

where $\phi \in H^{s_1,s_2}(\mathbb{R}^2)$, $(s_1\, s_2) \in \mathbb{R}^2$ will be chosen later. Suppose that $u(\gamma,t,x,y)$ solves (8.10). We fix $t \neq 0$, $t \in [-T,T]$ such that S_t is C^2. We have

$$u(\gamma,t,x,y) = \gamma U(t)\phi(x,y) - \int_0^t U(t-t')u(\gamma,t',x,y)u_x(\gamma,t',x,y)dt', \tag{8.11}$$

where $U(t) = \exp(-t(\partial_x^3 - \partial_x^{-1}\partial_y^2))$. Note that

$$\begin{aligned} \frac{\partial u}{\partial \gamma}(0,t,x,y) &= U(t)\phi(x,y) := u_1(t,x,y), \\ \frac{\partial^2 u}{\partial \gamma^2}(0,t,x,y) &= -2\int_0^t U(t-t')u_1(t',x,y)\partial_x u_1(t',x,y)dt' := u_2(t,x,y). \end{aligned}$$

We could define formally, in a similar fashion $u_k(t,x,y)$ as $\frac{\partial^k u}{\partial \gamma^k}(0,t,x,y)$, $k = 3,4,\cdots$, and thus taking into account that $u(0,t,x,y) = 0$, write a formal Taylor expansion

$$u(\gamma,t,x,y) = \frac{\gamma}{1!}u_1(t,x,y) + \frac{\gamma^2}{2!}u_2(t,x,y) + \frac{\gamma^3}{3!}u_3(t,x,y) + \cdots \tag{8.12}$$

The term $u_k(t,x,y)$ would correspond to the k-th iteration in an iterative method applied to (8.11). Here the assumption of C^2 regularity of S_t yields

$$u(\gamma,t,x,y) = \gamma u_1(t,x,y) + \gamma^2 u_2(t,x,y) + o(\gamma^2)$$

and

$$\|u_2(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \lesssim \|\phi\|^2_{H^{s_1,s_2}(\mathbb{R}^2)}. \tag{8.13}$$

We look for data ϕ such that (8.13) fails. We define ϕ by its Fourier transform as [3]

$$\widehat{\phi}(\xi,\eta) = \alpha^{-\frac{3}{2}} 1_{D_1}(\xi,\eta) + \alpha^{-\frac{3}{2}} N^{-s_1-2s_2} 1_{D_2}(\xi,\eta).$$

Here the positive parameters N and α are such that $N \gg 1$, $\alpha \ll 1$ and D_1, D_2 are the rectangles in $\mathbb{R}^2_{\xi,\eta}$:

$$D_1 = [\alpha/2,\alpha] \times [-6\alpha^2, 6\alpha^2], \;\; D_2 = [N, N+\alpha] \times [\sqrt{3}N^2, \sqrt{3}N^2 + \alpha^2].$$

[3]The analysis below works as well for $\mathscr{R}e\,\phi$ instead of ϕ (some new harmless terms appear).

We note that $\|\phi\|_{H^{s_1,s_2}} \sim 1$. We have

$$\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(u_1)(t,\xi,\eta) = \exp(it(\xi^3 + \frac{\eta^2}{\xi}))\widehat{\phi}(\xi,\eta)$$

and hence

$$u_1(t,x,y) = \frac{1}{4\pi^2}\alpha^{-\frac{3}{2}}\int_{D_1}\exp(ix\xi + iy\eta + it(\xi^3 + \frac{\eta^2}{\xi}))d\xi d\eta$$

$$+\frac{1}{4\pi^2}\alpha^{-\frac{3}{2}}N^{-s_1-2s_2}\int_{D_2}\exp(ix\xi + iy\eta + it(\xi^3 + \frac{\eta^2}{\xi}))d\xi d\eta$$

The next lemma is in [187] used in order to compute $u_2(t,x,y)$. We omit its proof.

Lemma 8.6. *The following identity holds*

$$\int_0^t U(t-t')F(t',x,y)dt'$$

$$= c\int_{\mathbb{R}^3} e^{ix\xi+iy\eta+it(\xi^3+\frac{\eta^2}{\xi})}\frac{e^{it(\tau-\xi^3-\frac{\eta^2}{\xi})}-1}{\tau-\xi^3-\frac{\eta^2}{\xi}}\widehat{F}(\tau,\xi,\eta)d\tau d\xi d\eta$$

whenever both terms are well defined.

Using Lemma 8.6 we obtain the following representation for $u_2(t,x,y)$

$$u_2(t,x,y) = c\int_{\mathbb{R}^3}\xi\exp(ix\xi + iy\eta$$

$$+it(\xi^3 + \frac{\eta^2}{\xi}))\frac{e^{it(\tau-\xi^3-\frac{\eta^2}{\xi})}-1}{\tau-\xi^3-\frac{\eta^2}{\xi}}(\widehat{u_1}\star\widehat{u_1})(\tau,\xi,\eta)d\tau d\xi d\eta.$$

Since $\widehat{u_1}(\tau,\xi,\eta) = \delta(\tau-\xi^3-\frac{\eta^2}{\xi})\widehat{\phi}(\xi,\eta)$ (δ stays for Dirac delta function) we obtain that $(\widehat{u_1}\star\widehat{u_1})(\tau,\xi,\eta)$ is equal to

$$\int_{-\infty}^{\infty}\delta(\tau-\xi_1^3-\frac{\eta_1^2}{\xi_1}-(\xi-\xi_1)^3$$

$$-\frac{(\eta-\eta_1)^2}{\xi-\xi_1})\widehat{\phi}(\xi_1,\eta_1)\widehat{\phi}(\xi-\xi_1,\eta-\eta_1)d\xi_1 d\eta_1.$$

$$\Phi(t,x,y,\xi,\xi_1,\eta,\eta_1) := \xi\ e^{ix\xi+iy\eta+it(\xi^3+\frac{\eta^2}{\xi})}\ \frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}.$$

Then the second iteration $u_2(t,x,y)$ can be splitted in three parts

$$u_2(t,x,y) = c(f_1(t,x,y)+f_2(t,x,y)+f_3(t,x,y)).$$

where

$$f_1(t,x,y) = \frac{c}{\alpha^3}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_1}} \Phi(t,x,y,\xi,\xi_1,\eta,\eta_1)d\xi d\eta d\xi_1 d\eta_1$$

$$f_2(t,x,y) = \frac{c}{\alpha^3 N^{2(s_1+2s_2)}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_2}} \Phi(t,x,y,\xi,\xi_1,\eta,\eta_1)d\xi d\eta d\xi_1 d\eta_1$$

$$f_3(t,x,y) = \frac{c}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_2}} \Phi(t,x,y,\xi,\xi_1,\eta,\eta_1)d\xi d\eta d\xi_1 d\eta_1$$

$$+ \frac{c}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_1}} \Phi(t,x,y,\xi,\xi_1,\eta,\eta_1)d\xi d\eta d\xi_1 d\eta_1$$

Now we easily compute the Fourier transforms with respect to (x,y)

$$\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_1)(t,\xi,\eta) = \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1$$

$$\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_2)(t,\xi,\eta) = \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{2(s_1+2s_2)}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1$$

$$\begin{aligned}\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_3)(t,\xi,\eta) &= \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_1\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1\\ &+ \frac{c\,\xi e^{it(\xi^3+\frac{\eta^2}{\xi})}}{\alpha^3 N^{s_1+2s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_2\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}d\xi_1 d\eta_1\end{aligned}$$

Actually the main contribution to the H^{s_1,s_2} norm of $u_2(t,\cdot,\cdot)$ is given by f_3. For a fixed t the supports of $\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_1)(t,\xi,\eta)$, $\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_2)(t,\xi,\eta)$, $\mathscr{F}_{(x,y)\mapsto(\xi,\eta)}(f_3)(t,\xi,\eta)$ are disjoint and therefore

$$\|u_2(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \geq \|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)}.$$

Hence a lower bound for $\|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)}$ is needed. We shall make use of the following Lemma.

Lemma 8.7. *Let*

$$(\xi_1,\eta_1)\in D_1, \quad (\xi-\xi_1,\eta-\eta_1)\in D_2$$

or

$$(\xi_1,\eta_1)\in D_2, \quad (\xi-\xi_1,\eta-\eta_1)\in D_1$$

Then

$$|\chi(\xi,\xi_1,\eta,\eta_1)| \lesssim \alpha^2 N.$$

Proof of Lemma 8.7.
Let first $(\xi_1,\eta_1)\in D_2$ and $(\xi-\xi_1,\eta-\eta_1)\in D_1$.

Fix $(\xi_1,\eta_1)\in D_2$. Let $\xi\in\mathbb{R}$ be such that $\xi-\xi_1\in[\alpha/2,\alpha]$. We look for $\eta^\star(\xi,\xi_1,\eta_1)$ such that $\chi(\xi,\xi_1,\eta^\star(\xi,\xi_1,\eta_1),\eta_1)=0$ and $|\eta^\star(\xi,\xi_1,\eta_1)-\eta_1|\le 6\alpha^2$. Solving $\chi(\xi,\xi_1,\eta,\eta_1)=0$ for fixed (ξ_1,η_1,ξ) we set

$$\eta^\star(\xi,\xi_1,\eta_1):=\eta_1+\frac{\eta_1(\xi-\xi_1)-\sqrt{3}\xi\xi_1(\xi-\xi_1)}{\xi_1}.$$

Now we bound $|\eta^\star(\xi,\xi_1,\eta_1)-\eta_1|$:

$$\begin{aligned}|\eta^\star(\xi,\xi_1,\eta_1)-\eta_1| &= \frac{|(\xi-\xi_1)(\eta_1-\sqrt{3}\xi\xi_1)|}{|\xi_1|}\\ &\le \frac{|\xi-\xi_1|}{|\xi_1|}|\eta_1-\sqrt{3}\xi_1^2-\sqrt{3}\xi_1(\xi-\xi_1)|.\end{aligned}$$

Recall that η_1 ranges in $[\sqrt{3}N^2,\sqrt{3}N^2+\alpha^2]$ and ξ_1 in $[N,N+\alpha]$. Hence

$$\sqrt{3}\xi_1^2\in[\sqrt{3}N^2,\sqrt{3}N^2+2\sqrt{3}N\alpha+\sqrt{3}\alpha^2]$$

and moreover

$$|\eta_1-\sqrt{3}\xi_1^2|\le 2\sqrt{3}N\alpha+\sqrt{3}\alpha^2.$$

Therefore

$$|\eta^\star(\xi,\xi_1,\eta_1)-\eta_1|\le\frac{\alpha}{N}(2\sqrt{3}N\alpha+\sqrt{3}\alpha^2+\sqrt{3}\alpha(N+\alpha))\le 6\alpha^2,$$

provided $N\gg 1$.

Hence we can write for $(\xi_1,\eta_1)\in D_2$ and $(\xi-\xi_1,\eta-\eta_1)\in D_1$

$$\begin{aligned}\chi(\xi,\xi_1,\eta,\eta_1) &= \chi(\xi,\xi_1,\eta^\star(\xi,\xi_1,\eta_1),\eta_1)\\ &\quad+(\eta-\eta^\star(\xi,\xi_1,\eta_1))\frac{\partial\chi}{\partial\eta}(\xi,\xi_1,\bar\eta,\eta_1),\end{aligned}$$

where $\bar\eta\in[\eta,\eta^\star(\xi,\xi_1,\eta_1)]$. Thus

$$\chi(\xi,\xi_1,\eta,\eta_1)=-(\eta-\eta^\star(\xi,\xi_1,\eta_1))\frac{2\xi_1(\bar\eta\xi_1-\eta_1\xi)}{\xi\xi_1(\xi-\xi_1)}.$$

Therefore

$$\begin{aligned}
|\chi(\xi,\xi_1,\eta,\eta_1)| &= |2\xi_1(\eta-\eta^\star(\xi,\xi_1,\eta_1))|\left|\frac{(\bar{\eta}-\eta_1)\xi_1-\eta_1(\xi-\xi_1)}{\xi\xi_1(\xi-\xi_1)}\right| \\
&\lesssim \alpha^2 N\left(\frac{|\eta_1(\xi-\xi_1)|}{|\xi\xi_1(\xi-\xi_1)|}+\frac{|(\bar{\eta}-\eta_1)\xi_1|}{|\xi\xi_1(\xi-\xi_1)|}\right) \\
&\lesssim \alpha^2 N\left(\frac{\sqrt{3}N^2+\alpha^2}{N^2}+\frac{6\alpha^2}{N\cdot\frac{\alpha}{2}}\right) \\
&\lesssim \alpha^2 N\left(\frac{\sqrt{3}N^2+1}{N^2}+\frac{12}{N}\right) \\
&\lesssim \alpha^2 N,
\end{aligned}$$

provided $N \gg 1$.

Let now $(\xi_1,\eta_1)\in D_1$ and $(\xi-\xi_1,\eta-\eta_1)\in D_2$. Since $(\xi_1,\eta_1)=(\xi-(\xi-\xi_1),\eta-(\eta-\eta_1))\in D_1$ the previous argument yields

$$|\chi(\xi,\xi-\xi_1,\eta,\eta-\eta_1)|\lesssim \alpha^2 N.$$

The proof of Lemma 8.7 is complete by observing that

$$\chi(\xi,\xi_1,\eta,\eta_1)=\chi(\xi,\xi-\xi_1,\eta,\eta-\eta_1).$$

□

Choose now α and N so that $\alpha^2 N = N^{-\varepsilon}$, where $0<\varepsilon\ll 1$ and $N\gg 1$. Then due to Lemma 8.7 we obtain

$$\left|\frac{e^{-it\chi(\xi,\xi_1,\eta,\eta_1)}-1}{\chi(\xi,\xi_1,\eta,\eta_1)}\right| = |t| + O(N^{-\varepsilon})$$

for $(\xi_1,\eta_1)\in D_1$,$(\xi-\xi_1,\eta-\eta_1)\in D_2$ or $(\xi_1,\eta_1)\in D_2$,$(\xi-\xi_1,\eta-\eta_1)\in D_1$. Hence

$$\|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \gtrsim \frac{N N^{s_1+2s_2}\,\alpha^3\,\alpha^{\frac{3}{2}}}{\alpha^3 N^{s_1+2s_2}} = \alpha^{\frac{3}{2}} N.$$

Therefore we arrive at

$$\begin{aligned} 1 \sim \|\phi\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} &\gtrsim \|u_2(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \\ &\geq \|f_3(t,\cdot,\cdot)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \gtrsim \alpha^{\frac{3}{2}} N \sim N^{\frac{1}{4}-\frac{3\varepsilon}{4}}. \end{aligned}$$

Contradiction for $N \gg 1$ and $\varepsilon \ll 1$. This completes the proof of Theorem 8.5 in H^{s_1,s_2}. The proof in $H^s(\mathbb{R}^2)$ is exactly the same with ϕ defined as

$$\widehat{\phi}(\xi,\eta) = \alpha^{-\frac{3}{2}} 1_{D_1}(\xi,\eta) + \alpha^{-\frac{3}{2}} N^{-2s} 1_{D_2}(\xi,\eta).$$

□

8.1.4 Local well-posedness of the KP-I equation via compactness arguments

In this section we shall prove local existence results for the KP-I equation, based on the classical compactness method we have discussed in Section. This method is quite general and does not use the dispersive nature of the equation. One first solves a regularized version of the KP-I equation involving a small parameter. Then one passes to the limit as the parameter tends to zero using a suitable a priori estimate on the solution. Consider the Cauchy problem for the KP-I equation

$$\begin{cases} (u_t + u_{xxx} + uu_x)_x - u_{yy} = 0, \quad (t,x,y) \in \mathbb{R}^3, \\ u(0,x,y) = \phi(x,y), \end{cases} \tag{8.14}$$

For $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we define the Sobolev spaces $H^s_{-k}(\mathbb{R}^2)$ as follows

$$H^s_{-k}(\mathbb{R}^2) = \left\{ u \in \mathscr{S}'(\mathbb{R}^2) : \|u\|_{H^s_{-k}} < \infty \right\},$$

where

$$\|u\|_{H^s_{-k}} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\xi|^{-k})^2 (1+|\xi|^2+|\eta|^2)^s \|\widehat{u}(\xi,\eta)|^2 d\xi d\eta \right)^{1/2}.$$

$H^s_0(\mathbb{R}^2)$ will be denoted by $H^s(\mathbb{R}^2)$. Similarly, for s_1, s_2 real and $k \in \mathbb{N}$, we define the anisotropic Sobolev spaces $H^{s_1,s_2}_{-k}(\mathbb{R}^2)$ as follows

$$H^{s_1,s_2}_{-k}(\mathbb{R}^2) = \left\{ u \in \mathscr{S}'(\mathbb{R}^2) : \|u\|_{H^{s_1,s_2}_{-k}} < \infty \right\},$$

where

$$\|u\|_{H^{s_1,s_2}_{-k}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\xi|^{-k})^2\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}|\widehat{u}(\xi,\eta)|^2 d\xi d\eta\right)^{1/2}.$$

We set $H^{s_1,s_2}(\mathbb{R}^2) = H_0^{s_1,s_2}(\mathbb{R}^2)$.
We recall a result of Iório and Nunes (cf. [118]) (see also [237], and indicate briefly a somewhat shorter proof.

Theorem 8.8. *The Cauchy problem (8.14) is locally well-posed for data in $H^s_{-1}(\mathbb{R}^2)$, provided $s > 2$.*

Proof of Theorem 8.8. It is based on a compactness method and relies on commutator estimates and Sobolev embedding theorems. Let $\varepsilon > 0$ and $\phi_\varepsilon \in \partial_x(C_0^\infty(\mathbb{R}^2))$ such that ϕ_ε converges to ϕ in $H^s_{-1}(\mathbb{R}^2)$ as ε tends to zero. We consider a regularized version of (8.14)

$$(u_t^\varepsilon + \varepsilon\Delta^2 u_t^\varepsilon + u_{xxx}^\varepsilon + u^\varepsilon u_x^\varepsilon)_x - u_{yy}^\varepsilon = 0, \tag{8.15}$$

where Δ is the Laplace operator on $\mathbb{R}^2$. By standard Picard iteration arguments the Cauchy problem associated to (8.15) has a unique solution $u^\varepsilon \in C([0,T];H^k(\mathbb{R}^2))$, where k is any fixed integer. Here T depends on ε and $\|\phi\|_{H^k(\mathbb{R}^2)}$. In fact T can be chosen independent of ε. Taking the H^s scalar product of (8.15) with u^ε yields

$$\frac{d}{dt}\left(\|u^\varepsilon(t)\|_{H^s}^2 + \varepsilon\|\Delta u^\varepsilon(t)\|_{H^s}^2\right) + (u^\varepsilon(t)u_x^\varepsilon(t), u^\varepsilon(t))_s = 0 \tag{8.16}$$

where $(.,.)_s$ connotes the H^s inner product. By classical arguments involving some commutator estimates proved in [130], it yields

$$\begin{aligned}\frac{d}{dt}\|u^\varepsilon(t)\|_{H^s}^2 &\lesssim \|\nabla u^\varepsilon(t)\|_{L^\infty}\|u^\varepsilon(t)\|_{H^s}^2 \\ &\lesssim \|u^\varepsilon(t)\|_{H^{2+}}\|u^\varepsilon(t)\|_{H^s}^2.\end{aligned}$$

Similarly we obtain

$$\frac{d}{dt}\|\partial_x^{-1}u^\varepsilon(t)\|_{H^s}^2 \lesssim \|\partial_x^{-1}u^\varepsilon(t)\|_{H^s}\|u^\varepsilon(t)\|_{H^s}\|u^\varepsilon(t)\|_{H^{1+}}.$$

From the previous inequalities one deduces by a standard compactness method that (8.14) possesses, for each

$$\phi \in H^s_{-1}(\mathbb{R}^2), \quad s > 2$$

a unique solution (in distributional sense) which satisfies

$$u \in L^\infty(0,T;H^s_{-1}(\mathbb{R}^2)),$$

where $T = T(\|\phi\|_{H^{2+}(\mathbb{R}^2)})$. Further it can be shown by the regularization technique of Bona-Smith (cf. [36] and Chapter 2, Section 3) that

$$u \in C([0,T];H^s_{-1}(\mathbb{R}^2))$$

and depends continuously on the data ϕ. This completes the proof of Theorem 8.8. □

Remark 8.5. The previous theorem is sufficient (as it gives existence on a time interval of length $\frac{1}{\varepsilon}$ in the weakly transverse Boussinesq regime) to justify the KP approximation (with a "bad" error estimate though). One can nevertheless prove the *global wellposedness*of the KPI Cauchy problem. The first global well-posedness result for arbitrary large initial data in a suitable Sobolev type space was obtained by Molinet, Saut and Tzvetkov [188]. The solution is uniformly bounded in time and space.

The proof is based on a rather sophisticated compactness method and uses the first invariants of the KP I equation to get global in time bounds. It is worth noticing that, while the recursion formula in [253] gives formally a infinite number of invariants, except the first ones, those invariants do not make sense for functions belonging to $L^2(\mathbb{R}^2)$ based Sobolev spaces.

For instance, the invariant which should control $||u_{xxx}(.,t)||_{L^2}$ contains the L^2 norm of $\partial_x^{-1}\partial_y(u^2)$ which does not make sense for a non zero function u in the Sobolev space $H^3(\mathbb{R}^2)$. Actually (see [188]), one checks easily that if $\partial_x^{-1}\partial_y(u^2) \in L^2(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2} \partial_y(u^2)dx = \partial_y \int_{\mathbb{R}^2} u^2 dx \equiv 0$, $\forall y \in \mathbb{R}$, which, with $u \in L^2(\mathbb{R}^2)$, implies that $u \equiv 0$. Similar obstructions occur for the higher order "invariants".

One is thus led to introduce a *quasi-invariant* (by skipping the non defined terms) which eventually will provide the desired bound. There are also serious technical difficulties to *justify* rigorously the conservation of the meaningful invariants along the flow and to control the remainder terms

The result of [188] was extended by C. Kenig [132] (who considered initial data in a larger space), and by Ionescu, Kenig and Tataru [117] who proved that the KP I equation is globally well-posed in the energy space $Y = \{u \in L^2(\mathbb{R}^2); u_x, \partial_x^{-1} u_y \in L^2(\mathbb{R}^2)\}$.

Note that the solution constructed in Theorem 8.8 satisfies equation (8.14) in the space $C([0,T]; H^{s-4}(\mathbb{R}^2))$. In addition the energy is well-defined and conserved. For data ϕ which belong only to $H^s(\mathbb{R}^2)$, $s > 2$, we will obtain solutions satisfying the KP-I equation in a weaker sense by considering the integral equation corresponding to the KP-I equation

$$u(t,x,y) = U(t)\phi(x,y) - \int_0^t U(t-t')\left[u(t',x,y)u_x(t',x,y)\right] dt', \quad (8.17)$$

where $U(t) = \exp(-t(\partial_x^3 - \partial_x^{-1}\partial_y^2))$ is the unitary group defining the free KP-I evolution. We have the following result.

Theorem 8.9. *The integral equation (8.17) is locally well-posed for data in $H^s(\mathbb{R}^2)$, provided $s > 2$.*

Proof of Theorem 8.9. Define $H^\infty_{-1}(\mathbb{R}^2)$ as

$$H^\infty_{-1}(\mathbb{R}^2) := \bigcap_{s \in \mathbb{R}} H^s_{-1}(\mathbb{R}^2).$$

Let $\phi \in H^s(\mathbb{R}^2)$ with $s > 2$. From Lemma 3.2 of [183] there exists a sequence $\{\phi^\varepsilon\} \subset H^\infty_{-1}(\mathbb{R}^2)$ converging to ϕ in $H^s(\mathbb{R}^2)$ as ε tends to zero. Using Theorem 8.8 there exists a unique solution $u^\varepsilon \in C([0,T_\varepsilon]; H^\infty_{-1}(\mathbb{R}^2))$ of (8.14) with data ϕ^ε. As in the proof of Theorem 8.8 we obtain that the sequence $\{u^\varepsilon\}$ is bounded in $L^\infty(0,T; H^s(\mathbb{R}^2))$, where T only depends on $\|\phi\|_{H^s(\mathbb{R}^2)}$. Since $u^\varepsilon \in C([0,T]; H^s(\mathbb{R}^2))$, u^ε satisfies the following integral equation on the time interval $[0,T]$

$$u^\varepsilon(t) = U(t)\phi^\varepsilon - \frac{1}{2}\int_0^t U(t-t')\partial_x(u^\varepsilon(t'))^2 dt' \quad (8.18)$$

Set

$$v^\varepsilon(t) = \frac{1}{2}\int_0^t U(t-t')\partial_x(u^\varepsilon(t'))^2 dt'.$$

Note that the sequence $\{v^\varepsilon\}$ is bounded in $L^\infty(0,T;H^{s-1}(\mathbb{R}^2))$. We can write

$$2\partial_t(v^\varepsilon(t)) = \partial_x(u^\varepsilon(t))^2 - \int_0^t (\partial_x^3 - \partial_x^{-1}\partial_y^2)U(t-t')\partial_x(u^\varepsilon(t'))^2 dt'.$$

We observe that the sequence $\{\partial_t(v^\varepsilon)\}$ is bounded in $L^\infty(0,T;H^{s-4}(\mathbb{R}^2))$. Therefore from Aubin-Lions compactness theorem one obtains

$$v^\varepsilon \longrightarrow v, \quad \text{in} \quad L^2_{loc}((0,T)\times\mathbb{R}^2)$$

Since

$$U(t)\phi^\varepsilon \longrightarrow U(t)\phi, \quad \text{in} \quad L^\infty((0,T);H^s(\mathbb{R}^2))$$

we infer that

$$u^\varepsilon \longrightarrow u, \quad \text{in} \quad L^2_{loc}((0,T)\times\mathbb{R}^2). \tag{8.19}$$

Note that $L^\infty(0,T;H^{s-1}(\mathbb{R}^2))$ being an algebra, $\{(u^\varepsilon)^2\}$ is bounded in $L^\infty(0,T;H^{s-1}(\mathbb{R}^2))$ and thus

$$(u^\varepsilon)^2 \rightharpoonup u^2, \quad \text{weakly in } L^2(0,T;H^{s-1}(\mathbb{R}^2)),$$

where we have identified the limit thanks to (8.19). Hence for a fixed t one has

$$U(t-t')\partial_x(u^\varepsilon(t'))^2 \rightharpoonup U(t-t')\partial_x(u(t')^2), \quad \text{weakly in } L^2(0,T;H^{s-2}(\mathbb{R}^2))$$

and therefore

$$\int_0^t U(t-t')\partial_x(u^\varepsilon(t'))^2 dt' \rightharpoonup \int_0^t U(t-t')\partial_x(u(t'))^2 dt', \text{ weakly in } H^{s-2}(\mathbb{R}^2).$$

In particular

$$\frac{1}{2}\int_0^t U(t-t')\partial_x(u^\varepsilon(t'))^2 dt' \longrightarrow \frac{1}{2}\int_0^t U(t-t')\partial_x(u(t'))^2 dt'$$

in $\mathscr{D}'((0,T)\times\mathbb{R}^2)$ which implies

$$v(t) = \frac{1}{2}\int_0^t U(t-t')\partial_x(u(t'))^2 dt'.$$

The uniqueness is obtained by Gronwall lemma arguments using the above approximation of the initial data. The strong continuity in time of the solution and the continuity of the flow -map in $H^s(\mathbb{R}^2)$ are again consequence of Bona-Smith argument together with Lemma 3.2 of [183]. This completes the proof of Theorem 8.9. □

Let us emark that by differentiating (8.17) first with respect to x and then with respect to t we obtain readily

$$\partial_t \partial_x u + \partial_x(uu_x) + \partial_x^4 u - \partial_y^2 u = 0 \quad \text{in} \quad C([0,T];H^{s-4}(\mathbb{R}^2)).$$

However the identity $\partial_t \partial_x u = \partial_x \partial_t u$ holds only in a very weak sense, for example in $\mathscr{D}'((0,T)\times\mathbb{R}^2)$. Thus the KP-I equation is satisfied only in the sense of distributions. Note also that the energy can be meaningless here. Nevertheless the L^2 conservation law is valid. This can be seen by using the approximation of the initial data performed in the proof of Theorem 8.9.

Chapter 9

The constraint problem for the KP equation

When a KP type equation is written as

$$u_t + uu_x - Lu_x \pm \partial_x^{-1}\partial_y^2 u = 0, \tag{9.1}$$

it is implicitly assumed that the operator $\partial_x^{-1}\partial_y^2$ is well defined, which a priori imposes a constraint on the solution u, which, in some sense, has to be an x-derivative. This is achieved, for instance, if $u \in \mathscr{S}'(\mathbb{R}^2)$ is such that

$$\xi_1^{-1}\,\xi_2^2\,\widehat{u}(t,\xi_1,\xi_2) \in \mathscr{S}'(\mathbb{R}^2)\,, \tag{9.2}$$

thus in particular if $\xi_1^{-1}\,\widehat{u}(t,\xi_1,\xi_2) \in \mathscr{S}'(\mathbb{R}^2)$. Another possibility to fulfill the constraint is to write u as

$$u(t,x,y) = \frac{\partial}{\partial x} v(t,x,y), \tag{9.3}$$

where v is a continuous function having a classical derivative with respect to x, which, for any fixed y and $t \neq 0$, vanishes when $x \to \pm\infty$. Thus one has

$$\int_{-\infty}^{\infty} u(t,x,y)dx = 0, \qquad y \in \mathbb{R},\ t \neq 0, \tag{9.4}$$

in the sense of generalized Riemann integrals. Of course the differentiated version of (9.1), namely

$$(u_t + u_x + uu_x - Lu_x)_x + \partial_y^2 u = 0, \tag{9.5}$$

can make sense without any constraint of type (9.2) or (9.4) on u, and so does the Duhamel integral representation of (9.1),

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(u(s)u_x(s))ds, \tag{9.6}$$

where $S(t)$ denotes the (unitary in all Sobolev spaces $H^s(\mathbb{R}^2)$) group associated with (9.1),

$$S(t) = e^{-t(\partial_x - L\partial_x + \partial_x^{-1}\partial_y^2)} . \tag{9.7}$$

In view of the above discussion, all the results established for the Duhamel form of KP-type equations (e.g., those of Bourgain [45] and Saut and Tzvetkov [204]) do not need any constraint on the initial data u_0. It is then possible (see, for instance [186]) to check that the solution u will satisfy (9.5) in the distributional sense but not a priori the integrated form involving the operator $\partial_x^{-1}\partial_y$.

On the other hand, a constraint has to be imposed when using the Hamiltonian formulation of the equation. In fact, the Hamiltonian for (9.5) is

$$\frac{1}{2}\int \left[-uLu + (\partial_x^{-1}u_y)^2 + u^2 + \frac{u^3}{3} \right] \tag{9.8}$$

and the Hamiltonian associated with (9.6) is

$$\frac{1}{2}\int \left[(\partial_x^{-1}u_y)^2 + u^2 + \frac{u^3}{3} \right] . \tag{9.9}$$

Therefore, the global well-posedness results for KP-I obtained in [187, 132] do need that the initial data satisfy (in particular) the constraint $\partial_x^{-1}\partial_y u_0 \in L^2(\mathbb{R}^2)$, and this constraint is preserved by the flow. Actually, the global results of [187, 132] make use of the next conservation law of the KP-I equation whose quadratic part contains the L^2-norms of u_{xx}, u_y, and $\partial_x^{-2}u_{yy}$. The constraint $\partial_x^{-2}\partial_{yy}u_0 \in L^2(\mathbb{R}^2)$ is thus also clearly needed, and one can prove that it is preserved by the flow.

The linear case

We consider two-dimensional linear KP-type equations

$$(u_t - Lu_x)_x + u_{yy} = 0, \quad u(0,x,y) = \varphi(x,y) , \tag{9.10}$$

where

$$\widehat{Lf}(\xi) = \varepsilon|\xi|^{\alpha}\widehat{f}(\xi), \quad \xi \in \mathbb{R}, \tag{9.11}$$

where $\varepsilon = 1$ (KP-II-type equations) or $\varepsilon = -1$ (KP-I-type equations). We denote by G the fundamental solution

$$G(t,x,y) = \mathscr{F}^{-1}_{(\xi,\eta)\to(x,y)}\left[e^{it(\varepsilon\xi|\xi|^{\alpha}-\eta^2/\xi)}\right].$$

A priori, we have only that $G(t,\cdot,\cdot) \in \mathscr{S}'(\mathbb{R}^2)$. Actually, for $t \neq 0$, $G(t,\cdot,\cdot)$ has a very particular form which is the main result of this section.

Theorem 9.1. *Suppose that $\alpha > 1/2$ in* (9.11)*. Then for $t \neq 0$,*

$$G(t,\cdot,\cdot) \in C(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2).$$

Moreover, for $t \neq 0$, there exists

$$A(t,\cdot,\cdot) \in C(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2) \cap C^1_x(\mathbb{R}^2)$$

($C^1_x(\mathbb{R}^2)$ denotes the space of continuous functions on $\mathbb{R}^2$ which have a continuous derivative with respect to the first variable) such that

$$G(t,x,y) = \frac{\partial A}{\partial x}(t,x,y).$$

In addition, for $t \neq 0$, $y \in \mathbb{R}$, $\varphi \in L^1(\mathbb{R}^2)$,

$$\lim_{|x|\to\infty} (A \star \varphi)(t,x,y) = 0.$$

As a consequence, the solution of (9.10) *with data $\varphi \in L^1(\mathbb{R}^2)$ is given by*

$$u(t,\cdot,\cdot) \equiv S(t)\varphi = G \star \varphi$$

and

$$u(t,\cdot,\cdot) = \frac{\partial}{\partial x}\big(A \star \varphi\big).$$

One therefore has

$$\int_{-\infty}^{\infty} u(t,x,y)\,dx = 0 \quad \forall y \in \mathbb{R},\ \forall t \neq 0$$

in the sense of generalized Riemann integrals.

Remark 9.1. It is worth noticing that the result of Theorem 9.1 is related to the infinite speed of propagation of the KP free evolutions. Let us also notice that the assumption $\alpha > 1/2$ can be relaxed if we assume that a sufficient number of derivatives of φ belong to L^1. Such an assumption is, however, not natural in the context of the KP equations.

Remark 9.2. In the case of the classical KP-II equation ($\alpha = 2$, $\varepsilon = +1$), Theorem 9.1 follows from an observation of Redekopp [197]. Namely, one has

$$G(t,x,y) = -\frac{1}{3t}\operatorname{Ai}(\zeta)\operatorname{Ai}'(\zeta),$$

where Ai is the Airy function and

$$\zeta = c_1\frac{x}{t^{1/3}} + c_2\frac{y^2}{t^{4/3}}$$

for some real constants $c_1 > 0$ and $c_2 > 0$. Thus $G(t,x,y) = \frac{\partial}{\partial x}A(t,x,y)$ with

$$A(t,x,y) = -\frac{1}{6c_1 t^{2/3}}\operatorname{Ai}^2\Big(c_1\frac{x}{t^{1/3}} + c_2\frac{y^2}{t^{4/3}}\Big)$$

and

$$u = \frac{\partial A}{\partial x}\star\varphi = \frac{\partial}{\partial x}(A\star\varphi),$$

which proves the claim for the KP-II equation (the fact that $\lim_{|x|\to\infty}A(t,x,y) = 0$ results from a well-known decay property of the Airy function). A similar explicit computation does not seem to be valid for the classical KP-I equation or for KP-type equations with general symbols.

Proof of Theorem 9.1. We will consider only the case $\varepsilon = 1$ in (9.10). The analysis in the case $\varepsilon = -1$ is analogous. It is plainly sufficient to consider only the case $t > 0$. We have

$$G(t,x,y) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{i(x\xi+y\eta)+it(\xi|\xi|^{\alpha}-\eta^2/\xi)}d\xi d\eta\,, \tag{9.12}$$

where the last integral has the usual interpretation of a generalized Riemann integral. We first check that $G(t,x,y)$ is a continuous function of x and y. By the change of variables

$$\eta' = \frac{t^{1/2}}{|\xi|^{1/2}}\eta,$$

we obtain

$$\begin{aligned}
G(t,x,y) &= \frac{c}{t^{1/2}}\int_{\mathbb{R}_\xi}|\xi|^{1/2}\left(\int_{\mathbb{R}_\eta}e^{i(y/t^{1/2})|\xi|^{1/2}\eta - i\operatorname{sgn}(\xi)\eta^2}d\eta\right)e^{ix\xi+it\xi|\xi|^\alpha}d\xi\\
&= \frac{c}{t^{1/2}}\int_{\mathbb{R}}e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}|\xi|^{1/2}e^{iy^2\xi/4t}\,e^{ix\xi+it\xi|\xi|^\alpha}d\xi\\
&= \frac{c}{t^{1/2}}\int_{\mathbb{R}}e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}\,|\xi|^{1/2}e^{i\xi(x+y^2/4t)}\,e^{it\xi|\xi|^\alpha}d\xi\\
&= \frac{c}{t^{\frac{1}{2}+\frac{3}{2(\alpha+1)}}}\int_{\mathbb{R}}e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}\,|\xi|^{1/2}\\
&\quad\times\quad \exp\left(i\xi\left(\frac{x}{t^{\frac{1}{\alpha+1}}}+\frac{y^2}{4t^{\frac{\alpha+2}{\alpha+1}}}\right)\right)e^{i\xi|\xi|^\alpha}d\xi\,.
\end{aligned}$$

Let us define

$$H(\lambda) = c\int_{\mathbb{R}}e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}\,|\xi|^{1/2}e^{i\lambda\xi}\,e^{i\xi|\xi|^\alpha}d\xi\,.$$

Then H is continuous in λ. We will consider only the worst case $\lambda \le 0$. The phase $\varphi(\xi) = i(\lambda\xi + \xi|\xi|^\alpha)$ then has two critical points $\pm\xi_\alpha$, where $\xi_\alpha = \left(\frac{\mu}{\alpha+1}\right)^{1/\alpha}$, $\mu = -\lambda$. We write, for $\varepsilon > 0$ small enough

$$H(\lambda) = \int_{-\infty}^{-\xi_\alpha-\varepsilon} + \int_{-\xi_\alpha-\varepsilon}^{\xi_\alpha+\varepsilon} + \int_{\xi_\alpha+\varepsilon}^{\infty} := I_1(\lambda)+I_2(\lambda)+I_3(\lambda)\,.$$

Clearly $I_2(\lambda)$ is a continuous function of λ. We consider only $I_3(\lambda)$,

$$\begin{aligned}
I_3(\lambda) &= c\int_{\xi_\alpha+\varepsilon}^{\infty}\frac{\xi^{1/2}}{\varphi'(\xi)}\frac{d}{d\xi}\left[e^{\varphi(\xi)}\right]d\xi = c\left[\frac{\xi^{1/2}e^{\varphi(\xi)}}{\lambda+\xi^\alpha(\alpha+1)}\right]_{\xi_\alpha+\varepsilon}^{\infty}\\
&+ c\int_{\xi_\alpha+\varepsilon}^{\infty}\left[\frac{1}{2(\lambda+\xi^\alpha(\alpha+1))\xi^{1/2}} - \frac{\alpha(\alpha+1)\xi^{\alpha-1/2}}{(\lambda+(\alpha+1)\xi^\alpha)^2}\right]e^{\varphi(\xi)}d\xi,
\end{aligned}$$

which for $\alpha > 1/2$ defines a continuous function of λ. Hence the integral (9.12) is a continuous function of (x,y) which coincides with the inverse Fourier transform (in $\mathscr{S}'(\mathbb{R}^2)$) of $\exp(it(\xi|\xi|^\alpha - \eta^2/\xi))$.

We next set for $t > 0$

$$A(t,x,y) \equiv (2\pi)^{-2}\int_{\mathbb{R}^2}\frac{1}{i\xi}e^{i(x\xi+y\eta)+it(\xi|\xi|^\alpha-\eta^2/\xi)}d\xi d\eta\,.$$

The last integral is clearly not absolutely convergent not only at infinity but also for ξ near zero. Nevertheless, the oscillations involved in its definition will allow us to show that $A(t,x,y)$ is in fact a continuous function. By the change of variables

$$\eta' = \frac{t^{1/2}}{|\xi|^{1/2}}\eta,$$

we obtain

$$\begin{aligned}
&A(t,x,y)\\
= \;& \frac{c}{t^{1/2}} \int_{\mathbb{R}_\xi} \frac{\operatorname{sgn}(\xi)}{|\xi|^{1/2}} \left(\int_{\mathbb{R}_\eta} e^{i(y/t^{1/2})|\xi|^{1/2}\eta - i\operatorname{sgn}(\xi)\eta^2} d\eta \right) e^{ix\xi + it\xi|\xi|^\alpha} d\xi\\
= \;& \frac{c}{t^{1/2}} \int_{\mathbb{R}} \frac{(\operatorname{sgn}(\xi))e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}}{|\xi|^{1/2}} e^{iy^2\xi/4t}\, e^{ix\xi + it\xi|\xi|^\alpha} d\xi\\
= \;& \frac{c}{t^{\frac{\alpha+2}{2(\alpha+1)}}} \int_{\mathbb{R}} \frac{(\operatorname{sgn}(\xi))e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}}{|\xi|^{1/2}} \exp\left(i\xi \left(\frac{x}{t^{\frac{1}{\alpha+1}}} + \frac{y^2}{4t^{\frac{\alpha+2}{\alpha+1}}} \right)\right) e^{i\xi|\xi|^\alpha} d\xi\,.
\end{aligned}$$

We now need the following lemma.

Lemma 9.2. *Let for $\alpha > 0$*

$$F(\lambda) = \int_{\mathbb{R}} \frac{(\operatorname{sgn}(\xi))e^{-i(\operatorname{sgn}(\xi))\frac{\pi}{4}}}{|\xi|^{1/2}} e^{i\lambda\xi + i\xi|\xi|^\alpha} d\xi\,.$$

Then F is a continuous function which tends to zero as $|\lambda| \to +\infty$.

Proof. Write F as

$$F(\lambda) = \int_{|\xi|\le 1} + \int_{|\xi|\ge 1} := F_1(\lambda) + F_2(\lambda)\,.$$

Since $|\xi|^{-1/2}$ in integrable near the origin, by the Riemann–Lebesgue lemma $F_1(\lambda)$ is continuous and

$$\lim_{|\lambda|\to\infty} F_1(\lambda) = 0\,.$$

We consider two cases in the analysis of $F_2(\lambda)$.

Case 1. $\lambda \ge -1$.

After an integration by parts, we obtain that

$$F_2(\lambda) = \frac{c\cos(\lambda + 1 - \frac{\pi}{4})}{\lambda + \alpha + 1} + c\int_1^\infty \cos\Big(\lambda\xi + \xi^{\alpha+1} - \frac{\pi}{4}\Big)\,\frac{\lambda + (\alpha+1)(2\alpha+1)\xi^\alpha}{\xi^{3/2}(\lambda + (\alpha+1)\xi^\alpha)^2}\,d\xi\,. \quad (9.13)$$

The first term is clearly a continuous function of λ which tends to zero as $\lambda \to \infty$. Observing that

$$0 \le \frac{\lambda + (\alpha+1)(2\alpha+1)\xi^\alpha}{\xi^{3/2}(\lambda + (\alpha+1)\xi^\alpha)^2} \le C_\alpha \xi^{-3/2}$$

uniformly with respect to $\xi \ge 1$ and $\lambda \ge -1$, we deduce from the dominated convergence theorem that the right-hand side of (9.13) is a continuous function of λ for $\lambda \ge -1$. On the other hand, for $\lambda \ge 1$,

$$\frac{\lambda + (\alpha+1)(2\alpha+1)\xi^\alpha}{\xi^{3/2}(\lambda + (\alpha+1)\xi^\alpha)^2} \le \frac{2\alpha+1}{\lambda\xi^{3/2}},$$

and thus the right-hand side of (9.13) tends to zero as $\lambda \to +\infty$.

Case 2. $\lambda \le -1$.

Set $\lambda = -\mu$ with $\mu \ge 1$. In the integral over $|\xi| \ge 1$ defining $F_2(\lambda)$, we consider only the integration over $[1, +\infty[$. The integration over $]-\infty, -1]$ can be treated in a completely analogous way. We perform the changes of variables

$$\xi \longrightarrow \xi^2$$

and

$$\xi \longrightarrow \mu^{\frac{1}{2\alpha}}\xi$$

to conclude that

$$\widetilde{F_2}(\lambda) := c\int_1^\infty \frac{1}{\xi^{1/2}} e^{i\lambda\xi + i\xi|\xi|^\alpha}\,d\xi = c\mu^{\frac{1}{2\alpha}} \int_{\mu^{-\frac{1}{2\alpha}}}^\infty e^{i\mu^{1+\frac{1}{\alpha}}\left[\xi^{2(\alpha+1)} - \xi^2\right]}\,d\xi.$$

Let us set

$$\varphi(\xi) = \xi^{2(\alpha+1)} - \xi^2.$$

Then

$$\varphi'(\xi) = 2\xi\,[(\alpha+1)\xi^{2\alpha} - 1]\,.$$

Let us split

$$\widetilde{F}_2(\lambda) = c\mu^{\frac{1}{2\alpha}} \int_{\mu^{-\frac{1}{2\alpha}}}^{1} + c\mu^{\frac{1}{2\alpha}} \int_{1}^{\infty} := I_1(\mu) + I_2(\mu)\,.$$

Since $\varphi'(\xi)$ does not vanish for $\xi \geq 1$, we can integrate by parts, which gives

$$I_2(\mu) = \frac{1}{2i\mu^{1+\frac{1}{2\alpha}}} \left(\frac{c}{\alpha} + c\int_1^{\infty} e^{i\mu^{1+\frac{1}{\alpha}}\left[\xi^{2(\alpha+1)}-\xi^2\right]} \frac{(\alpha+1)(2\alpha+1)\xi^{2\alpha}-1}{\xi^2((\alpha+1)\xi^{2\alpha}-1)^2} d\xi \right),$$

which is a continuous function of $\mu \geq 1$ thanks to the dominated convergence theorem. Moreover, it clearly tends to zero as $\mu \to +\infty$.

Let us next analyze $I_1(\mu)$. We first observe that thanks to the dominated convergence theorem, $I_1(\mu)$ is a continuous function of μ. It remains to prove that $I_1(\mu) \to 0$ as $\mu \to \infty$. For $\xi \in [\mu^{-\frac{1}{2\alpha}}, 1]$, the phase φ has a critical point, and a slightly more delicate argument is needed. Compute

$$\varphi''(\xi) = 2[(\alpha+1)(2\alpha+1)\xi^{2\alpha} - 1]\,.$$

Observe that $\varphi'(\xi)$ is vanishing only at zero and

$$\xi_1(\alpha) := \left(\frac{1}{\alpha+1}\right)^{\frac{1}{2\alpha}} .$$

Next, we notice that $\varphi''(\xi)$ is vanishing at

$$\xi_2(\alpha) := \left(\frac{1}{(\alpha+1)(2\alpha+1)}\right)^{\frac{1}{2\alpha}} .$$

Clearly $\xi_2(\alpha) < \xi_1(\alpha) < 1$, and we choose a real number δ such that

$$\xi_2(\alpha) < \delta < \xi_1(\alpha) < 1\,.$$

For $\mu \gg 1$, we can split

$$I_1(\mu) = c\mu^{\frac{1}{2\alpha}} \int_{\mu^{-\frac{1}{2\alpha}}}^{\delta} + c\mu^{\frac{1}{2\alpha}} \int_{\delta}^{1} := J_1(\mu) + J_2(\mu)\,.$$

For $\xi \in [\mu^{-\frac{1}{2\alpha}}, \delta]$, we have the lower bound

$$|\varphi'(\xi)| \geq c\mu^{-\frac{1}{2\alpha}} > 0,$$

and an integration by parts shows that

$$J_1(\mu) = \mu^{\frac{1}{\alpha}} \mathscr{O}(\mu^{-1-\frac{1}{\alpha}}) \leq C\mu^{-1},$$

which clearly tends to zero as $\mu \to \infty$. For $\xi \in [\delta, 1]$, we have the minoration

$$|\varphi''(\xi)| \geq c > 0,$$

and therefore we can apply the Van der Corput lemma to conclude that

$$J_2(\mu) = \mu^{\frac{1}{2\alpha}} \mathscr{O}(\mu^{-\frac{1}{2}-\frac{1}{2\alpha}}) \leq C\mu^{-\frac{1}{2}},$$

which tends to zero as $\mu \to \infty$. This completes the proof of Lemma 9.2. □

It is now easy to check that $\partial_x A = G$ in the sense of distributions. Since both A and G are continuous, we deduce that A has a classical derivative with respect to x which is equal to G. Finally, since $\varphi \in L^1(\mathbb{R}^2)$, applying Lemma 9.2 and the Lebesgue theorem completes the proof of Theorem 9.1.

The nonlinear case

After a change of frame we can eliminate the u_x term and reduce the Cauchy problem for (9.5) to

$$(u_t + uu_x - Lu_x)_x + u_{yy} = 0, \quad u(0,x,y) = \varphi(x,y). \tag{9.14}$$

In order to state our result concerning (9.14), for $k \in \mathbb{N}$, we denote by $H^{k,0}(\mathbb{R}^2)$ the Sobolev space of $L^2(\mathbb{R}^2)$ functions $u(x,y)$ such that $\partial_x^k u \in L^2(\mathbb{R}^2)$.

Theorem 9.3. *Assume that $\alpha > 1/2$. Let $\varphi \in L^1(\mathbb{R}^2) \cap H^{2,0}(\mathbb{R}^2)$ and*

$$u \in C([0,T]; H^{2,0}(\mathbb{R}^2)) \tag{9.15}$$

be a distributional solution of (9.14)*. Then, for every $t \in (0,T]$, $u(t,\cdot,\cdot)$ is a continuous function of x and y which satisfies*

$$\int_{-\infty}^{\infty} u(t,x,y)dx = 0 \quad \forall y \in \mathbb{R},\ \forall t \in (0,T]$$

in the sense of generalized Riemann integrals. Moreover, $u(t,x,y)$ is the derivative with respect to x of a C_x^1 continuous function which vanishes as $x \to \pm\infty$ for every fixed $y \in \mathbb{R}$ and $t \in [0,T]$.

Remark 9.3. The case $\alpha = 2$ corresponds to the classical KP-I, KP-II equations. In the case of the KP-II, we have global solutions for data in $L^1(\mathbb{R}^2) \cap H^{2,0}(\mathbb{R}^2)$ (see [45]). Thus Theorem 9.3 displays a striking smoothing effect of the KP-II equation: $u(t,\cdot,\cdot)$ becomes a continuous function of x and y (with zero mean in x) for $t \neq 0$ (note that $L^1(\mathbb{R}^2) \cap H^{2,0}(\mathbb{R}^2)$ is not included in $C^0(\mathbb{R}^2)$). A similar comment is valid for the local solutions of the KP-I equation in [187] and more especially in [188].

Remark 9.4. The numerical simulations in [147] display clearly the phenomena described in Theorem 9.3 in the case of the KP-I equation.

Proof of Theorem 9.3. Under our assumption on u, one has the Duhamel representation

$$u(t) = S(t)\varphi - \int_0^t S(t-s)\big(u(s)u_x(s)\big)ds, \tag{9.16}$$

where

$$\int_0^t S(t-s)\big(u(s)u_x(s)\big)ds = \int_0^t \partial_x \left(\int_{\mathbb{R}^2} A(x-x',y-y',t-s)(uu_x)(x',y',s)dx'dy' \right) ds\,.$$

From Theorem 9.1, it suffices to consider only the integral term in the right-hand side of (12.8). Using the notations of Lemma 9.2,

$$A(x-x',y-y',t-s) = \frac{c}{(t-s)^{\frac{\alpha+2}{2(\alpha+1)}}} F\left(\frac{x-x'}{(t-s)^{\frac{1}{\alpha+1}}} + \frac{(y-y')^2}{4(t-s)^{\frac{\alpha+2}{\alpha+1}}} \right).$$

Recall that F is a continuous and bounded function on $\mathbb{R}$. Next, we set

$$I(x,y,t-s,s) \equiv \partial_x \left(\int_{\mathbb{R}^2} A(x-x',y-y',t-s)(uu_x)(x',y',s)dx'dy' \right).$$

Using the Lebesgue differentiation theorem and the assumption (9.15), we can write

$$I(x,y,t-s,s) = \int_{\mathbb{R}^2} A(x-x',y-y',t-s)\partial_x(uu_x)(x',y',s)dx'dy'\,.$$

Moreover, for $\alpha > 0$,

$$\frac{\alpha+2}{2(\alpha+1)} < 1,$$

and therefore I is integrable in s on $[0,t]$. Therefore, by the Lebesgue differentiation theorem,

$$\int_0^t \int_{\mathbb{R}^2} A(x-x', y-y', t-s)(uu_x)(x', y', s)dx'dy'ds \tag{9.17}$$

is a C_x^1 function and

$$\int_0^t S(t-s)\big(u(s)u_x(s)\big)ds$$
$$= \partial_x \left(\int_0^t \int_{\mathbb{R}^2} A(x-x', y-y', t-s)(uu_x)(x', y', s)dx'dy'ds \right).$$

Let us finally show that for fixed y and t the function (9.17) tends to zero as x tends to $\pm\infty$. For that purpose, it suffices to apply the Lebesgue dominated convergence theorem to the integral in s, x', y'. Indeed, for fixed s, x', y', the function under the integral tends to zero as x tends to $\pm\infty$ thanks to the linear analysis. On the other hand, using Lemma 9.2, we can write

$$|A(x-x', y-y', t-s)(uu_x)(x', y', s)| \leq \frac{c}{(t-s)^{\frac{\alpha+2}{2(\alpha+1)}}} |(uu_x)(x', y', s)|.$$

Thanks to the assumptions on u, the right-hand side of the above inequality is integrable in s, x', y' and independent of x. Thus we can apply the Lebesgue dominated convergence theorem to conclude that the function (9.17) tends to zero as x tends to $\pm\infty$. This completes the proof of Theorem 9.3.

Remark 9.5. If $\alpha > 2$, the assumptions can be weakened to $\varphi \in L^1(\mathbb{R}^2) \cap H^{1,0}(\mathbb{R}^2)$ and $u \in C([0,T]; H^{1,0}(\mathbb{R}^2))$. This result follows from the fact that the fundamental solution G writes

$$G(t,x,y) = \frac{c}{t^{1/2+3/2(\alpha+1)}} B(t,x,y),$$

where $B \in L^\infty$ and $1/2 + 3/(2(\alpha+1)) < 1$ for $\alpha > 2$.

Remark 9.6. Similar results as those in the present Chapter are established in [186] for the KP-II/BBM equation

$$(u_t + u_x + uu_x - u_{xxt})_x + u_{yy} = 0$$

and for versions involving more general dispersion in x.

Chapter 10

Blow-up issues

We review here the three different type of finite time blow-up that may occur in the solutions of dispersive perturbations of hyperbolic nonlinear equations or systems. One has in fact three possible scenarios for blow-up.

10.1 Hyperbolic blow-up, or blow-up by shock formation

This phenomena is typical of quasilinear hyperbolic examples or systems, a paradigm being the Burgers equation

$$u_t + uu_x = 0, \quad u(\cdot, 0) = u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{10.1}$$

Setting $v = u_x$, (10.1) reduces to

$$v_t + uv_x + v^2 = 0,$$

that is

$$v' + v^2 = 0,$$

where $' = \frac{\partial}{\partial_t} + u\partial_x$ is the derivative along the characteristics $X_\xi = \{(x,t), x = X(t) = tu_0(\xi) + \xi\}$.

Integration gives

$$v(X(t),t) = \frac{v_0(\xi)}{1+tv_0(\xi)},$$

showing that, provided $\inf u_0' > -\infty$, $||u_x(\cdot,t)||_\infty$ blows up at $T^\star = -\frac{1}{\inf u_0'} > 0$, but u remains bounded since it is given implicitly by

$$u(x,t) = u_0(x - tu(x,t))$$

(shock formation).

To display the universal nature of the nonlinear hyperbolic blow-up we consider the case of a scalar conservation law in $\mathbb{R}^n$.

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{j=1}^n \partial_{x_j}(f_j(u)) = 0, \\ u(.,0) = u_0, \quad u_0 \in C^1(\mathbb{R}^N) \end{cases} \tag{10.2}$$

where $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ and where, for $i = 1,,,,,n$ the $f_i : \mathbb{R} \to \mathbb{R}$ are C^3 functions. We set $f(u) = (f_1(u), f_2(u), ..., f_n(u))$ and we denote $F = f' = (f_1', f_2', ..., f_n')$. The local C^1 theory results from the method of characteristics as it is shown now.

We define the characteristic curves $(X(t),t)$ by

$$\begin{cases} \dot{X}(t) = F(u(X(t),t), \\ X(0) = \xi \end{cases} \tag{10.3}$$

when u is a C^1 solution of (10.2) in $\mathbb{R}^n \times [0,T]$.

One checks at once that

$$\frac{du}{dt}(X(t),t) = \frac{\partial u}{\partial t}(X,t) + \sum_{j=1}^n f_j'(u)(u(X(t),t)\frac{\partial u}{\partial x_j}(X(t),t) = 0,$$

showing that u is constant along the characteristics, and that those are straight lines

$$X(t) = tF(u_0(\xi)) + \xi.$$

We introduce the scalar quantity $D = \operatorname{div} F(u_0) = F'(u_0) \cdot \nabla u_0 = \sum_{j=1}^n f_j''(u_0)\frac{\partial u}{\partial \xi_j}$.

Theorem 10.1. *Let $u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*

(i) Assume that $D \geq 0$ in $\mathbb{R}^n$. Then there exists a unique solution $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ of (10.2).

(ii) Assume that D takes strictly negative values. Let

$$T^* = (\sup\{-D(\xi), \xi \in \mathbb{R}^n\})^{-1} = (-\inf\{D(\xi), \xi \in \mathbb{R}^n\})^{-1}$$

If $T^* > 0$, (10.2) *admits a unique C^1 solution on $[0, T^*)$ that cannot be extended (as a C^1 solution) after T^*.*

Proof. (i) We first prove that under the assumption $D \geq 0$, F^t defined by

$$F^t(\xi) = tF(u_0(\xi)) + \xi$$

is a global diffeomorphism of $\mathbb{R}^n$, for any $t > 0$. Note first that $F \circ u_0$ is a bounded function since $u_0 \in L^\infty(\mathbb{R}^n)$ and moreover is a proper function (that is the inverse image of a compact set is compact). In fact there exist $R > 0$ such that $\xi \geq R$ implies $|F^t(\xi)| \geq \frac{1}{2}|\xi|$.

Lemma 10.2. *The jacobian determinant of F^t is*

$$1 + tD = 1 + tF'(u_0) \cdot \nabla u_0.$$

Proof. The coefficients of the jacobian matrix $J(t\xi)$ of F^t are

$$(\delta_{i,j} + tF_i'(u_0)(\xi))\frac{\partial u_0}{\partial \xi_j}(\xi))_{i,j}.$$

The determinant $P(t, \xi)$ of $J(t, \xi)$ is a polynomial in t of degree $\leq n$. One easily checks that its second derivative in t vanishes identically (apply the rule of differentiation of determinants and observe that all the subdeterminants occurring in the computation of $P"(t, \xi)$ have proportional columns). Since $J(0, \xi) = 0$, one has $P(t, \xi) = 1 + a(\xi)t = 1 + P'(0, \xi)t$. A simple computation leads to

$$P'(0, \xi) = \operatorname{Tr} J(t, \xi) = \sum_{j=1}^{n} F_j'(u_0(\xi))\frac{\partial u_0}{\partial \xi_j}(\xi) = D(\xi).$$

By Hadamard Theorem (see below), F^t is for any $t \geq 0$, a diffeomorphism of $\mathbb{R}^n$. For each $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, there exists therefore a unique $\xi = \xi(x, t) \in \mathbb{R}^n$ verifying

$$x = tF^t(u_0(\xi(x, t), t) + \xi(x, t).$$

Moreover, ξ is C^1 in (x,t) (the regularity in t results from the implicit function theorem). Defining then $u(x,t) = u_0(\xi(x,t))$, one checks as in the classical one-dimensional case that u is the unique C^1 solution on $\mathbb{R}^n \times \mathbb{R}_+$. □

(ii) We assume now that D has negative values and define T^* as in the Theorem. If $T^* > 0$, the formula $u(x,t) = u_0(\xi(x,t),t)$ where $\xi(x,t)$ is defined as above, proves that u is the unique solution in $C^1([0,T^*) \times \mathbb{R}^n$. Let us check that this solution cannot extends to a $C^1([0,T] \times \mathbb{R}^n)$ function with $T > T^*$. To this aim, we compute $\nabla_x u(x,t)$ from the implicit representation

$$u(x,t) = u_0(x - tF(u(x,t)),$$

which is valid as far as u is C^1.

One finds

$$\nabla_x u(x,t) = \frac{\nabla u_0(\xi)}{1 + t\nabla u_0(\xi) \cdot F'(u_0(\xi))} = \frac{\nabla u_0(\xi)}{1 + tD(\xi)}.$$

This proves that along a characteristic, (that is when ξ is fixed and t increases), $\nabla_x u$ does not change direction, only its length varies.

If for some ξ $D(\xi) < 0$, then necessarily $\nabla u_0(\xi) \neq 0$, and $|\nabla_x(u(x,t)| \to +\infty$ as $t \to -\frac{1}{D(\xi)}$, and thus a C^1 solution cannot exists beyond the time $T^* = (-\inf\{D(\xi), \xi \in \mathbb{R}^n\})^{-1}$.

□

As in the one dimensional case, one can prove that the blow-up of certain expression depending on u and its derivatives is governed by a Ricatti equation. More precisely,

Proposition 10.3. *If u is a C^2 solution of* (10.2)*, then the expression $q =$ div $F(u)$ satisfies the equation $q' + q^2 = 0$ along the characteristics.*

Proof. We differentiate (10.2) with respect to $x_i, i = 1,...,n$, we multiply the i^{th} by $F_i'(u)$, and sum with respect to i yielding

$$0 = \sum_{i=1}^{n} F_i'(u) \left[\frac{\partial u_{x_i}}{\partial t} + \sum_{i=1}^{n} F_j(u) \frac{\partial u_{x_i}}{\partial x_j} \right] + \sum_{i=1}^{n} F_i'(u) \sum_{i=1}^{n} F_j'(u) u_{x_i} u_{x_j},$$

that is

$$\begin{aligned} 0 &= \sum_{i=1}^{n} F_i'(u) \left[\frac{\partial u_{x_i}}{\partial t} + \sum_{i=1}^{n} F_j(u) \frac{\partial u_{x_i}}{\partial x_j} \right] + \left[\sum_{i=n}^{n} F_i'(u) u_{x_i} \right]^2 \\ &= \sum_{i=1}^{n} F_i'(u) \left[\frac{\partial u_{x_i}}{\partial t} + \sum_{i=1}^{n} F_j(u) \frac{\partial u_{x_i}}{\partial x_j} \right] + q^2(t). \end{aligned} \tag{10.4}$$

On the other hand, denoting here $\frac{d}{dt}q$ the derivative of q along the characteristics and observing that u, thus also $F_i'(u)$ is constant along the characteristics, one has

$$\frac{d}{dt} q = \frac{d}{dt} \left[\sum_{i=1}^{n} F_i'(u) u_{x_i} \right] = \sum_{i=1}^{n} F_i'(u) \frac{d}{dt} u_{x_i}. \tag{10.5}$$

Using (10.5) (10.4) writes also

$$q' + q^2 = 0.$$

Integrating this Ricatti equation, one finds that for a C^1 solution,

$$\operatorname{div} F(u(x,t)) = \frac{D(\xi)}{1 + tD(\xi)},$$

where $\xi = \xi(x,t)$ is the foot of the characteristic passing through (x,t). We find again the maximal existence time of the solution when $D(\xi)$ takes negative values. □

For the sake of completeness we present a proof of Hadamard inversion theorem we used in the proof of Theorem 10.1.

Theorem 10.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping. Then f is a global diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ if and only if f is proper (that is the inverse image of a compact set is compact) and the jacobian determinant $Jf(x) = \det\left(\frac{\partial f_i(x)}{\partial x_j}\right)$ of f never vanishes.*

Proof. The elegant proof that is sketched here is due to Gordon [99] but it needs the extra hypothesis that f is C^2.

One side of the equivalence being trivial, it suffices to prove that if f is proper and $Jf(x) \neq 0, \forall x \in \mathbb{R}^n$, then f is a diffeomorphism. The local inversion theorem reduces to proving that f is a bijection.

(i) f onto. Up to a translation, it is sufficient to prove that there exists $x \in \mathbb{R}^n$ such that $f(x) = 0 is$. Let $F(x) = \frac{1}{2}|f(x)|^2 = \frac{1}{2}\sum_{j=1}^n |f_j(x)|^2$, so that $\frac{\partial F}{\partial x_i} = \sum_{j=1}^n f_j(\frac{\partial f_j}{\partial x_i})$. Observe that F is proper since f is proper.

Let c strictly larger that an arbitrary value of F. Then $F^{-1}([0,c])$ is non empty and compact since F is proper. It results that F achieves its minimum at a point p of $F^{-1}[0,c])$, and this minimum is clearly an absolute minimum ($\geq 0!$) of F in $\mathbb{R}^n$. Thus $\nabla F(p) = 0$ and $f(p) = 0$.

(ii) f is one to one. Up to a translation this amounts to proving that $S = f^{-1}(0)$ reduces to one point. We proceed in several steps.

a) S is a finite set. If not, it would contains, by compactness, an accumulation point q contradicting the fact that f is a local diffeomorphism in the neighborhood of q. Thus $S = \{p_1, ..., p_m\}$.

We will use uses a descent method, considering the differential system

$$\frac{dx(t)}{dt} = -\nabla F(x(t)), \quad x(0) \in \mathbb{R}^n, (\text{arbitrary}). \tag{10.6}$$

Since ∇F is C^1, the Cauchy-Lipschitz theorem guarantees the local well-posedness of (10.6).

Observe that for any solution $x(t)$,

$$\frac{dF(x(t))}{dt} = -\nabla F(x(t)) \cdot \frac{dx(t)}{dt} = -|\nabla F(x,t))|^2.$$

b) For ay $i \in \{1,...,m\}$, there exists an open neighborhood U_i of p_i such that any solution of (10.6) entering in U_i remains in U_i, and in fact converges to p_i as $t \to +\infty$. In other words, every p_i is an asymptotically stable critical point of (10.6).

This follows from the fact that F is a Lyapunov function for (10.6) at every p_i : along any trajectory $x(t)$, one has $\frac{dF(x(t))}{dt} \leq 0$, with equality if and only if $x(t) = p_i$.

Let W_i be the set of $q's$ in $\mathbb{R}^n$ such that the solution of (10.6) with initial value $x(0) = q$ satisfies $x(t) \to p_i$ as $t \to +\infty$.

c) One has $\mathbb{R}^n = \cup W_i$ (finite union of non empty disjoint sets). This is a consequence of the surjectivity and b).

d) Each W_i is open and thus S reduces to a single element. This a consequence of the continuous dependence of solutions to (10.6) with respect to the initial values. Let $\varepsilon > 0$ such that the ball of center p_i and tradius 2ε is contained in U_i. Let $q \in W_i$ and $x = x(t)$ the solution emanating from $x(0) = q$. There exists consequently some $T > 0$ such that $|x(T) - p_i| \le \varepsilon$. Taking q' such that $|q - q'|$ is small enough, one can insures that the solution $y(t)$ starting from q' satisfies $|x(T) - y(T)| < \varepsilon$, so that $y(T) \in U_i$. But then b) implies that $q' \in W_i$. □

We now go back to the one-dimensional case. An interesting question is to investigate the influence of a dispersive perturbation on the shock formation.

As a paradigm (motivated by the fact that the quadratic nonlinearity is natural in most equations arising from Fluid Mechanics and that the dispersion is often “weak"’) we consider the so-called Whitham equation that Whitham [240] introduced exactly for that purpose.

$$u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)u_x(y,t)dy = 0. \tag{10.7}$$

This equation can also be written on the form

$$u_t + uu_x - Lu_x = 0, \tag{10.8}$$

where the Fourier multiplier operator L is defined by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where $p = \hat{k}$.

In the original Whitham equation, the kernel k was given by

$$k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi}\right)^{1/2} e^{ix\xi} d\xi, \tag{10.9}$$

that is $p(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^{1/2}$.

The dispersion is in this case that of the finite depth surface water waves without surface tension.

A typical result (see [190], [58]) suggest that for *not too dispersive Whitham type equations* that is for instance when $p(\xi) = |\xi|^{\alpha}$, $-1 < \alpha \leq 0$, (10.7) still presents a blow-up of Burgers type. This has been proved for Whitham type equations, with a regular kernel k satisfying

$$k \in C(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ symmetric and monotonically decreasing on } \mathbb{R}_+, \tag{10.10}$$

by Naumkin and Shishmarev [190] and by Constantin and Escher [58], without an unnecessary hypothesis made in [190]. The blow-up is obtained for initial data which are sufficiently asymmetric. More precisely :

Theorem 10.5. *[58] Let $u_0 \in H^{\infty}(\mathbb{R})$ be such that*

$$\inf_{x\in\mathbb{R}} |u_0'(x)| + \sup_{x\in\mathbb{R}} |u_0'(x)| \leq -2k(0).$$

Then the corresponding solution of (10.7) *undergoes a wave breaking phenomena, that is there exists $T = T(u_0) > 0$ with*

$$\sup_{(x,t)\in[0,T)\times\mathbb{R}} |u(x,t)| < \infty, \textit{while } \sup_{x\in\mathbb{R}} |u_x(t,x)| \to \infty \textit{ as } t \to T.$$

The previous result does not include the case of the Whitham equation (10.7) with kernel given by (10.9) since then $k(0) = \infty$, but it is claimed in [58] that the method of proof adapts to more general kernels.

This has been proven recently by Castro, Cordoba and Gancedo [48] for the equation

$$u_t + uu_x + D^{\beta}\mathscr{H}u = 0, \tag{10.11}$$

where $\mathscr{H}$ is the Hilbert transform and D^{β} is the Riesz potential of order $-\beta$, *i.e.* D^{β} is defined via Fourier transform by

$$\widehat{D^{\beta}f}(\xi) = |\xi|^{\beta}\hat{f}(\xi), \tag{10.12}$$

for any $\beta \in \mathbb{R}$. It is established in [48] (see also [112] for the case $\beta = \frac{1}{2}$) that for $0 \leq \beta < 1$, there exist initial data $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, $0 < \delta < 1$, and $T(u_0)$ such that the corresponding solution u of (10.11) satisfies

$$\lim_{t\to T} \|u(\cdot,t)\|_{C^{1+\delta}(\mathbb{R})} = +\infty.$$

This rules out the case $-1 < \alpha < 0$ in our notation. It would be interesting to extend this result to a non pure power dispersion, for instance (10.9).

The case $0 < \alpha < 1$ is much more delicate and is open. The numerical simulations in [145] suggest (as claimed in [152]) that no shock like blow-up occurs (but a blow-up in the sense of the next Section seems to occur when $0 < \alpha < \frac{1}{2}$).

10.2 "Nonlinear dispersive blow-up"'

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Both nonlinear and dispersive effects are crucial for this type of blow-up. A typical example is the L^2 critical and super critical focusing NLS, that is when $p \geq \frac{4}{n}$ in

$$i\psi_t + \Delta\psi + |\psi|^p\psi = 0, \quad \psi = \psi(x,t), x \in \mathbb{R}^n, t \in \mathbb{R}. \tag{10.13}$$

A formal proof of blow-up was given by Vlasov-Petrishev-Talanov 1971, Zakharov 1972; it was made rigorous by Glassey 1977, Ginibre-Velo 1979.

We refer to the book [223] for precise references and for a proof of this blow-up based on a virial identity and to [175] for a sharp analysis of the blow-up in the critical case $p = \frac{p}{2}$..

Note that both a blow-up of $|\nabla\psi|_{L^2}$ and of $|\psi|_\infty$ occur; actually the conservation of the L^2 norm and of the energy

$$\int_{\mathbb{R}^n} [\frac{1}{2}|\nabla\psi|^2 - \frac{1}{(p+1)(p+2)}|\psi|^{p+2}]$$

imply that a control on $|\psi|_\infty$ prevents the blow-up.

A similar result is expected for the L^2 critical and super critical KdV equation

$$u_t + u^p u_x + u_{xxx} = 0, \quad p \geq 4. \tag{10.14}$$

This was proved in [170] for $p = 4$, and conjectured for $p > 4$ according to the numerical simulations of Bona-Dougalis-Karakashian-McKinney [31]).

We consider

10.3 Dispersive blow-up

This type blow-up may occur for most *linear* dispersive equations and is due to the fact that monochromatic waves (simple waves) propagate at speeds that vary substantially with their wavelength. Indeed, what appears to be important is that the ratio of the phase speeds at different wave numbers is not suitably bounded. It can persist for *nonlinear* equations.

To explain the phenomenon, we consider the linear KdV equation (some times improperly called Airy equation since it was introduced by Stokes)

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{10.15}$$

Take

$$\phi(x) = \frac{Ai(-x)}{(1+x^2)^m},$$

with

$$\frac{1}{8} < m < \frac{1}{4},$$

where Ai is the Airy function defined by

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}\theta^3 + \theta z)d\theta.$$

We recall the classical asymptotics of the Airy function (see for instance [80]) :

- For $x > 0$,

$$0 < Ai(x) \leq \frac{1}{2\pi^{1/2}x^{1/4}}e^{-\xi} \quad \text{where} \quad \xi = \frac{2}{3}x^{3/2},$$

$$0 < -Ai'(x) \leq \frac{x^{1/4}}{2\pi^{1/2}}e^{-\xi}(1 + \frac{7}{72\xi}).$$

- For $x < 0$,

$$Ai(-x) = \frac{1}{2\pi^{1/2}x^{1/4}}\cos\left(\xi - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{\xi}\right)\right),$$

$$Ai'(-x) = \frac{x^{1/4}}{2\pi^{1/2}} \sin\left(\xi - \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{\xi}\right)\right).$$

Then $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The solution $u \in C(\mathbb{R}_+; L^2(\mathbb{R}))$ is given by

$$\frac{c}{t^{\frac{1}{3}}} \int_{\mathbb{R}} Ai(\frac{x-y}{t^{\frac{1}{3}}}) \frac{Ai(-y)}{(1+y^2)^m} dy.$$

When $(x,t) \to (0,1)$, $u(x,t) \to c \int_{\mathbb{R}} \frac{Ai^2(-y)}{(1+y^2)^m} dy = +\infty$.

Actually one can prove with some extra work using the previous asymptotics of the Airy function that (see [34]) u is continuous on $\mathbb{R} \times \mathbb{R}_+^*$ except at $(x,t) = (0,1)$. By a suitable change of variables one could have replace $(0,1)$ by any couple $(x*, t*) \in \mathbb{R} \times \mathbb{R}_+^*$.

We have thus proved

Theorem 10.6. *Let $(x*, t*) \in \mathbb{R} \times \mathbb{R}_+^*$. There exists $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{10.16}$$

has a unique solution $u \in C([0,\infty); L^2(\mathbb{R}) \cap L^2_{loc}(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$ which is continuous on $(\mathbb{R} \times \mathbb{R}_+^*) \setminus (x*, t*)$ and satisfies

$$\lim_{(x,t)\to(x*,t*)} |u(x,t)| = +\infty.$$

Remark 10.1. We will call this type of blow-up as the *dispersive blow-up*. It is a precise way to express that the Airy group $e^{t\partial_x^3}$ is not well-posed in $L^\infty(\mathbb{R})$ (and actually in any $L^p(\mathbb{R})$, $1 \le p \le +\infty$, $p \ne 2$.)

Remark 10.2. One establish by linearity (see [36]) that for any sequence $(x_n, t_n) \in \mathbb{R} \times (0, +\infty)$ without finite acumulation points and such that $\{t_n\}_{n=1}^\infty$ does not cluster at zero, thee exists an initial data $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that the corresponding solution $u \in C(\mathbb{R}_+ : L^2(\mathbb{R}))$ of

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$$

is coninuous everywhere in $\mathbb{R}\times\mathbb{R}_+^*$ except at the points $\{(x_n,t_n)\}_{n=1}^\infty$, with

$$\lim_{(x,t)\to(x_n,t_n)} u(x,t) = +\infty, \quad n = 1,2,..$$

The previous result can be extended to the generalized KdV equations

$$u_t + u^p u_x + u_{xxx} = 0,$$

but we will consider only the case of the usual KdV equation, $p = 1$. A natural idea is the following.

For a solution u of the Cauchy problem corresponding to the initial data $\phi \in L^2(\mathbb{R})$ given by $\phi(x) = \frac{Ai(-x)}{(1+x^2)^m}$, $\frac{3}{16} < m \le \frac{1}{4}$, we write the Duhamel representation

$$u(x,t) = S(t)\phi(x) + \int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{3}}} Ai\Big(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\Big) u u_x(y,s)\,ds\,dy$$

and integrating by parts in the integral,

$$u(x,t) = S(t)\phi(x) + C\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{2}{3}}} Ai'\Big(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\Big) u^2(y,s)\,ds\,dy. \quad (10.17)$$

This seems silly because Ai' grows as $(-x)^{\frac{1}{4}}$ as $x \to -\infty$, but actually the initial data ϕ belongs to some weighted L^2 space and this will be used to compensate the behavior of Ai' at $-\infty$.

For $\sigma > 0$, we consider a non-decreasing, smooth function w_σ such that $w_\sigma(x) = 1$ for $x < 0$ and $w_\sigma(x) = (1+x^2)^\sigma)$ for $x > 1$.

The class $L^2(\mathbb{R},\omega)$ is the class of measurable functions which are square integrable with respect to the measure $w^2(x)dx$.

Theorem 10.7. *Let $\sigma > 0$ and $\psi \in L^2(\mathbb{R},\omega)$. There exists a solution u to the KdV equation, with initial data ψ such that for any $T > 0$, $u \in L^\infty(0,T;L^2(\mathbb{R},\omega)) \cap L^2(0,T;H^1_{loc}(\mathbb{R})$.*

Proof. We merely sketch it. One can use a compactness method, smoothing the initial data by a sequence $\psi_j \in C_0^\infty(\mathbb{R})$, deriving appropriate bounds on the corresponding smooth solutions u_j, and passing to the limit. We only indicate how to derive the suitable energy estimates.

Setting $v_j = w_\sigma u_j$, we get (dropping the j's and the $\sigma' s$)

$$v_t + v_{xxx} + v\left(6\frac{w_x w_{xx}}{w^2} - 6\frac{w_x^3}{w^3} - \frac{w_{xxx}}{w}\right) + v_x\left(6\frac{w_x^2}{w^2} - 3\frac{w_{xx}}{w}\right) \\ -3\frac{w_x}{w}v_{xx} + \frac{1}{w}vv_x - \frac{w_x}{w^2}v^2 = 0 \tag{10.18}$$

By assumption, $v(\cdot,0) \in H^\infty(\mathbb{R})$ and moreover $||v(\cdot,0)||_k$ is bounded independently of j. Observe that the following functions are smooth and bounded

$$\frac{w_x}{w} \geq 0,\ 6\frac{w_x w_{xx}}{w^2} - 6\frac{w_x^3}{w^3} - \frac{w_{xxx}}{w},\ 6\frac{w_x^2}{w^2} - 3\frac{w_{xx}}{w},\ \frac{1}{w},\ \frac{w_x}{w^2}.$$

Classical results imply that there exists one (at least local) $H^\infty(\mathbb{R})$ solution of (10.18). By the uniqueness of $H^\infty(\mathbb{R})$ solutions of the KdV equation, one has $v/w = u$.

To get the a priori estimates we take the L^2 scalar product of (10.18) with v to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2 dx + 3\int_{-\infty}^{\infty}\frac{w_x}{w}v_x^2 dx = \int_{-\infty}^{\infty}\theta v^2 dx + \frac{2}{3}\int_{-\infty}^{\infty}\frac{w_x}{w^2}v^3 dx, \tag{10.19}$$

where

$$\theta = \frac{w_{xxx}}{w} + 6\frac{w_x^3}{w^3} - 6\frac{w_x w_{xx}}{w^2} - \frac{3}{2}\left(\frac{w_{xx}}{w}\right)_x + 3\left(\frac{w_x^2}{w^2}\right)_x + \frac{3}{2}\left(\frac{w_x}{w}\right)_{xx}.$$

The definition of θ implies that θ is a bounded and smooth function. On the other hand,

$$\int_{-\infty}^{\infty}\frac{w_x}{w^2}v^3 dx = \int_{-\infty}^{\infty}\frac{w_x}{w}uv^2 dx \leq ||u||_0||v||_0\left|\frac{w_x}{w}v\right|_\infty \leq c||v||_0\left|\frac{w_x}{w}v\right|_\infty.$$

Moreover, since w_x/w and $(w_x/w)_x$ are bounded, it follows that

$$\left|\frac{w_x}{w}v\right|_\infty \leq \left\|\frac{w_x}{w}v\right\|_0^{1/2}\left\|\left(\frac{w_x}{w}v\right)\right\|_0^{1/2} \\ \leq 2\left|\frac{w_x}{w}\right|_\infty ||v||_0^{1/2}\left\{\left\|\left(\frac{w_x}{w}\right)_x v\right\|_0^{1/2} + \left\|\frac{w_x}{w}v_x\right\|_0^{1/2}\right\} \\ \leq c\left\{\|v\|_0 + \|v\|_0^{1/2}\left\|\frac{w_x}{w}v_x\right\|_0^{1/2}\right\}, \tag{10.20}$$

where c denotes various constants depending on the weight w and on the norm of the initial data ψ but not on t or j. One also has

$$\left\|\frac{w_x}{w}v_x\right\|_0 \le c\left\|\left(\frac{w_x}{w}\right)^{1/2} v_x\right\|_0^{1/2}$$

and Young's inequality implies that

$$\int_{-\infty}^{\infty}\frac{w_x}{w^2}v^3dx \le c||v||_0^2+\frac{3}{2}\int_{-\infty}^{\infty}\frac{w_x}{w}v_x^2dx.$$

We deduce from those estimates and from the boundedness of θ that

$$\frac{1}{2}\int_{-\infty}^{\infty}v^2dx+2\int_{-\infty}^{\infty}\frac{w_x}{w}v_x^2dx \le c\int_{-\infty}^{\infty}v^2dx$$

from which one deduces the uniform bound on $||v(\cdot,t)||_0$ and the local L^2 one on v_x. □

Remark 10.3. With some extra technical details a similar result can be obtained for higher values of $p's$ (see [36]).

The following lemma will be used in the proof of the dispersive blow-up for the KdV equation.

Lemma 10.8. *Let* $\psi \in L^2(\mathbb{R},\omega)$ *were* $\sigma \ge \frac{1}{16}$. *Let* $u \in L^\infty(0,T;L^2(\mathbb{R},\omega)) \cap L^2(0,T;H^1_{loc}(\mathbb{R}))$ *constructed in Theorem 10.7. Then the integral*

$$\Lambda(x,t)=\int_0^t\int_{-\infty}^{\infty}\frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds$$

is continuous with respect to $(x,t)\in\mathbb{R}\times\mathbb{R}_+^*$.

Proof. We break the integral as follows

$$\begin{aligned}\Big|\Lambda(x,t)| \le |\int_0^t\int_{-\infty}^{m}\frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds\Big| \\ +|\int_0^t\int_{-m}^{m}\frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds| \\ +|\int_0^t\int_{m}^{\infty}\frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds|\end{aligned} \tag{10.21}$$

It results from the asymptotic of Ai' that

$$\left|\frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y)}{(t-s)^{1/3}}\right)\right| \leq \frac{C}{(t-s)^{2/3}}$$

for all $t \geq s$ and $x-y>0$.

Hence,

$$\left|\Lambda(x,t)\right| \leq \left|\int_0^t \int_{-\infty}^x \frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds\right|$$

$$\leq C\int_0^t \frac{1}{(t-s)^{2/3}}\int_{-\infty}^x u^2(y,s)dyds \leq CT^{1/3}||u||^2_{L^\infty(0,T;L^2(\mathbb{R}))^2} < +\infty. \tag{10.22}$$

In a similar way we estimate the second integral in the right hand side of (10.21) as

$$\begin{aligned}\left|\Lambda(x,t)\right| &\leq \left|\int_0^t \int_x^\infty \frac{1}{(t-s)^{2/3}}Ai'\left(\frac{x-y}{(t-s)^{1/3}}\right)u^2(y,s)dyds\right| \\ &\leq C\int_0^t \int_x^\infty \frac{1}{(t-s)^{3/4}}(y-x)^{1/4}u^2(y,s)dyds \\ &= C\int_0^t \frac{1}{(t-s)^{3/4}}\int_x^\infty \frac{(y-x)^{1/4}}{w_\sigma^2(y)}w_\omega^2(y)u^2(y,s)dyds \\ &\leq C\sup_{y\geq x}\left(\frac{(y-x)^{1/4}}{w_\sigma^2(y)}\right)\int_0^t \frac{1}{(t-s)^{3/4}}\int_{-\infty}^\infty w_\sigma^2(y)u^2(y,s)dyds \\ &\leq CT^{1/4}\sup_{y\geq x}\left(\frac{(y-x)^{1/4}}{w_\sigma^2(y)}\right)||u||^2_{L^\infty(0,T;L^2(\mathbb{R}))^2}.\end{aligned} \tag{10.23}$$

Notice that if $\sigma \geq 1/16$

$$\sup_{y\geq x}\left(\frac{(y-x)^{1/4}}{w_\sigma^2(y)}\right) \leq \begin{cases} C, & x \geq 0, \\ C|x|^{1/4}, & x \leq 0. \end{cases}$$

Combining (10.22), (10.23) imply that Λ is locally bounded. The continuity follows since its defining integral has been shown above to converge uniformly for (x,t) in bounded subsets of $\mathbb{R}\times\mathbb{R}_+^*$.

□

We now state the dispersive blow-up for the KdV equation.

Theorem 10.9. *Let* $(x*,t*) \in \mathbb{R} \times \mathbb{R}_+^*$. *There exists* $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0, \\ u(.,0) = \phi \end{cases} \tag{10.24}$$

has a unique solution $u \in C([0,\infty); L^2(\mathbb{R}) \cap L^2_{loc}(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$ *which is continuous on* $(\mathbb{R} \times \mathbb{R}_+^*) \setminus (x*,t*)$ *and satisfies*

$$\lim_{(x,t)\to(x*,t*)} |u(x,t)| = +\infty.$$

Proof. As in the linear case we may assume that $(x*,t*) = (0,1)$. Let us take as initial value $\psi(x) = \frac{Ai(-x)}{(1+x^2)^m}$ with $\frac{3}{16} < m < \frac{1}{4}$ and we consider the Duhamel formulation (10.17).

By Theorem 10.6, the free part blows up as stated in Theorem 10.9. On the other hand, the extra hypothesis on m implies that $\psi \in L^2(\mathbb{R}; w_\sigma)$ where $\sigma \geq 1/16$. In this case, Theorem 10.7 combined with Lemma 10.8 imply that the integral term in the Duhamel formula is bounded and continuous in $\mathbb{R} \times [0,T]$, for any $T > 0$ and the Theorem is proven.

□

Remark 10.4. As will be seen below, dispersive blow-up occurs for many equations relevant to water waves for instance for the linearized water waves equations (see [37]). The phase velocity in this later case is $g^{\frac{1}{2}}\left(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|}\right)^{\frac{1}{2}}\hat{\mathbf{k}}$ and thus a bounded function of $\mathbf{k}$. This is contrary to the case of the linear KdV-equations (Airy-equation) and the linear Schrödinger equation, where both the phase velocity and the group velocity become unbounded in the short wave limit.

The dispersive blow-up phenomenon is thus not linked to the unboundedness of the phase velocity, but simply to the fact that monochromatic waves (simple waves) propagate at speeds that vary substantially with their wavelength. Indeed, what appears to be important is that the ratio of the phase speeds at different wavenumbers is not suitably bounded.

Remark 10.5. Once the dispersive blow-up property is established, one can by a truncation process produce smooth and localized initial data with arbitrarily small amplitude leading to solutions having an arbitrary large amplitude at any given point in space-time.

Remark 10.6. *Weakly dispersive* equations do not display dispersive blow-up. For instance, the Cauchy problem for the BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad u(\cdot, 0) = \phi \tag{10.25}$$

can be written as the convolution equation

$$u_t = K \star \left(u + \frac{1}{2} u^2 \right) \tag{10.26}$$

where $K(x) = \frac{1}{2}(\operatorname{sign} x) e^{-|x|}$, from which one deduces immediately that the Cauchy problem (10.25) is locally well-posed in $L^\infty(\mathbb{R})$.

Dispersive blow-up for the linearized water waves equations

As aforementioned dispersive blow-up occurs for the linearized surface gravity waves as shown in [35] that we follow closely. We thus consider the linearized gravity waves

$$\begin{cases} \eta_{tt} + \omega^2(|D|)\eta = 0, \quad x \in \mathbb{R}^d, \quad d = 1,2 \quad t \in \mathbb{R}_+^* \\ \eta(x,0) = \eta_0(x) \\ \eta_t(x,0) = \eta_1(x) \end{cases} \tag{10.27}$$

for the elevation $\eta = \eta(x,y,t)$ (or $\eta(x,t)$ in case the motion does not vary much in the y direction) of the wave (see *e.g.* [240] and Chapter 4). Here, $\omega(|\mathbf{k}|)$ is the usual linearized dispersion relation for water waves given by

$$\omega^2(|\mathbf{k}|) = g|\mathbf{k}| \tanh(|\mathbf{k}| h_0), \tag{10.28}$$

where h_0 is the undisturbed depth, $\mathbf{k} = (k_1, k_2)$ and $|\mathbf{k}| = (k_1^2 + k_2^2)^{\frac{1}{2}}$ when $d = 2$. The phase velocity is therefore

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|} \hat{\mathbf{k}} = g^{\frac{1}{2}} \left(\frac{\tanh(|\mathbf{k}| h_0)}{|\mathbf{k}|} \right)^{\frac{1}{2}} \hat{\mathbf{k}}$$

where $\hat{\mathbf{k}}$ is the unit vector in the $\mathbf{k}$-direction. For waves of extreme length wherein $|\mathbf{k}| \to 0$, the phase velocity tends to $\sqrt{gh_0}\hat{\mathbf{k}}$. For water waves on an infinite layer (corresponding to $h_0 = +\infty$), the phase velocity is

$$\mathbf{c}(\mathbf{k}) = g^{\frac{1}{2}} \frac{1}{|\mathbf{k}|^{\frac{1}{2}}} \hat{\mathbf{k}}.$$

Thus, on deep water, plane waves travel faster and faster as the wavelength becomes large, contrary to the case of the Airy or Schrödinger equations where large phase velocities occur for short waves (large wavenumbers k).

Considered also will be the case of gravity-capillary waves whose linear dispersion relation is

$$\omega^2(|\mathbf{k}|) = g|\mathbf{k}|\tanh(|\mathbf{k}|h_0)\left(1+\frac{T}{\rho g}|\mathbf{k}|^2\right), \tag{10.29}$$

where ρ is the density and T the surface tension coefficient. In this case, the phase velocity is

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\hat{\mathbf{k}} = g^{\frac{1}{2}}\left(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|}\right)^{\frac{1}{2}}\left(1+\frac{T}{\rho g}|\mathbf{k}|^2\right)^{\frac{1}{2}}\hat{\mathbf{k}},$$

whose modulus tends to infinity as $|\mathbf{k}|$ tends to $+\infty$, that is in the limit of short wavelengths. In the infinite depth case, the phase velocity tends to infinity in the limit of both infinitely long and infinitely short waves.

In the sequel, the equations are scaled so that the gravity constant g and the mean depth h_0 are both equal to 1.

The solution of (10.27) with the dispersion law (10.28) is easily computed in Fourier transformed variables to be

$$\hat{\eta}(\mathbf{k},t) = \hat{\eta}_0(\mathbf{k})\cos\left[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}\right] + \frac{\sin\left[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}\right]}{(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}}\hat{\eta}_1(\mathbf{k}). \tag{10.30}$$

In consequence, the Cauchy problem is clearly well posed in L^2-based Sobolev classes. More precisely for any $(\eta_0,\eta_1)\in H^k(\mathbb{R}^d)\times H^{k-\frac{1}{2}}(\mathbb{R}^d)$, $k\geq 0$, (10.27) possesses a unique solution $\eta\in C(\mathbb{R},H^k(\mathbb{R}^d))$.

To establish ill-posedness in L^∞, it suffices to consider the situation wherein $\hat{\eta}_1 = 0$. Ill-posedness then amounts to proving that for each $t\neq 0$, the kernel

$$m_t(\mathbf{k}) = e^{it(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}}$$

is not a Fourier multiplier in L^∞, which is the same as showing that its Fourier transform is not a bounded Borel measure.

Let $t>0$ be fixed and focus on $m_t(\mathbf{k})$. The first point to note is that

$$(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}}\left(1-\frac{2}{1+e^{2|\mathbf{k}|}}\right)^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}} + r(|\mathbf{k}|) \tag{10.31}$$

where $r \in C(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ and $r(|\mathbf{k}|)$ behaves like $-\frac{|\mathbf{k}|^{\frac{1}{2}}}{1+e^{2|\mathbf{k}|}}$ as $|\mathbf{k}| \to +\infty$ and like $-|\mathbf{k}|^{\frac{1}{2}}(1-|\mathbf{k}|^{\frac{1}{2}})$ as $|\mathbf{k}| \to 0$. Note that $r \equiv 0$ when the depth h_0 is infinite.

Decompose the kernel m_t as follows:

$$e^{it(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}} = e^{ir(|\mathbf{k}|)t} e^{it|\mathbf{k}|^{\frac{1}{2}}} = (1+f_t(|\mathbf{k}|))e^{it|\mathbf{k}|^{\frac{1}{2}}} \tag{10.32}$$

where

$$f_t(|\mathbf{k}|) = 2i\sin\left(\frac{r(|\mathbf{k}|)t}{2}\right) e^{i\frac{r(|\mathbf{k}|)t}{2}}$$

is continuous, smooth on $\mathbb{R}^d \setminus \{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to +\infty$, uniformly on bounded temporal sets, since $r(\mathbf{k})$ does so. This decomposition leads to an associated splitting of the Fourier transform $I_t(\mathbf{x})$ of $m_t(\mathbf{k})$, namely

$$I_t(\mathbf{x}) =: \int_{\mathbb{R}^d} e^{it|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} + \int_{\mathbb{R}^d} f_t(|\mathbf{k}|) e^{it|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} = I_t^1(x) + I_t^2(x). \tag{10.33}$$

• **Study of $I_t^2(x)$.** Because f_t decays rapidly to 0 as $|\mathbf{k}|$ becomes large, the Riemann-Lebesgue lemma implies that I_t^2 is a bounded, continuous function of $\mathbf{x}$ and thus locally integrable, in both dimensions 1 and 2. In fact, when $d = 1$, it is actually a continuous L^1-function. To see this, restrict to $|x| \geq 1$ and integrate by parts to reach the equation

$$\begin{aligned} I_t^2(x) &= -\frac{1}{ix}\int_{-\infty}^{+\infty} \frac{d}{dk}\left(f_t(k)e^{it|k|^{\frac{1}{2}}}\right) e^{ikx} dk \\ &= -\frac{1}{ix}\int_{-\infty}^{+\infty} e^{ikx} e^{it|k|^{\frac{1}{2}}} \left[f_t'(k) + \frac{it\,\mathrm{sgn}\,k}{2|k|^{\frac{1}{2}}} f_t(k)\right] dk. \end{aligned}$$

The term in square brackets decays exponentially to 0 as $|k| \to \infty$ and has a singularity of order $|k|^{-\frac{1}{2}}$ at the origin, coming from $f_t'(k)$ (note that $f_t(k)|k|^{-\frac{1}{2}}$ is bounded at 0). It is therefore the Fourier transform of an L^p-function, where $1 \leq p < 2$, and so, by the Riesz-Thorin theorem, must itself be an L^q-function where $2 < q \leq +\infty$. Since $\frac{1}{x} \in L^s(|x| \geq 1)$ for any $s > 1$, the Hölder inequality thus insures that $I_t^2 \in L^1(\mathbb{R})$.

• **Study of** $I_t^1(x)$. The analysis of I_t^1 relies on detailed results about the Fourier transform of the kernel $\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a}$, for a in the range $0<a<1$, where $\psi \in C^\infty(\mathbb{R}), 0 \le \psi \le 1, \psi \equiv 0$ on $[0,1], \psi \equiv 1$ on $[2,+\infty)$. For $0<a<1$ and $\mathbf{k} \in \mathbb{R}^d$, let $F_a(x) = \mathscr{F}(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a})(x)$ be the Fourier transform of the kernel. Since $\mathbf{k} \mapsto \psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a} \in \mathscr{S}'(\mathbb{R}^d)$, $\psi(|\mathbf{k}|)e^{-\varepsilon|\mathbf{k}|}e^{i|\mathbf{k}|^a}$ converges to $\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a}$ as $\varepsilon \to 0$, at least in the sense of distributions. It follows that $\mathscr{F}(\psi(|\mathbf{k}|)e^{-\varepsilon|\mathbf{k}|}e^{i|\mathbf{k}|a}) \to \mathscr{F}(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|a})$ as $\varepsilon \to 0$.

For the readers' convenience, we recall the following general result (see Wainger [240], Theorem 9 , and Miyachi, [182] Proposition 5.1).

Theorem 10.10. *(Wainger, Miyachi) Let* $0<a<1$, $b \in \mathbb{R}$ *and define* $F^\varepsilon_{a,b}(\mathbf{x}) =: \mathscr{F}\left(\psi(|\mathbf{k}|)|\mathbf{k}|^{-b}\exp(-\varepsilon|\mathbf{k}|+i|\mathbf{k}|^a)\right)(\mathbf{x})$ *for* $\varepsilon>0$ *and* $\mathbf{x} \in \mathbb{R}^d$. *The following is true of the function* $F^\varepsilon_{a,b}$.

(i) $F^\varepsilon_{a,b}(\mathbf{x})$ *depends only on* $|\mathbf{x}|$.

(ii) $F_{a,b}(\mathbf{x}) = \lim_{\varepsilon\to 0^+} F^\varepsilon_{a,b}(\mathbf{x})$ *exists pointwise for* $\mathbf{x} \neq 0$ *and* $F_{a,b}$ *is smooth on* $\mathbb{R}^d \setminus \{0\}$.

(iii) For all $N \in \mathbb{N}$, *and* $\beta \in \mathbb{N}^d$, $|\left(\frac{\partial}{\partial x}\right)^\beta F_{a,b}(\mathbf{x})| = O(|\mathbf{x}|^{-N})$ *as* $|\mathbf{x}| \to +\infty$.

(iv) If $b > d(1-\frac{1}{2})$, $F_{a,b}$ *is continuous on* $\mathbb{R}^d$.

(v) If $b \le d(1-\frac{1}{2})$, *then for any* $m_0 \in \mathbb{N}$, *the function* $F_{a,b}$ *has the asymptotic expansion*

$$F_{a,b}(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{\frac{1}{1-a}(d-b-\frac{ad}{2})}} \exp\left(\frac{i\xi_a}{|\mathbf{x}|^{\frac{a}{1-a}}}\right) \sum_{m=0}^{m_0} \alpha_m |\mathbf{x}|^{\frac{ma}{1-a}} + 0(|\mathbf{x}|)^{\frac{(m_0+1)a}{(1-a)}} + g(\mathbf{x}) \tag{10.34}$$

as $\mathbf{x} \to 0$, *where* $\xi_a \in \mathbb{R}$, $\xi_a \neq 0$, *and* g *is a continuous function.*

(vi) When $a>1$ *and* $b \in \mathbb{R}^d$, $F_{a,b}$ *is smooth on* $\mathbb{R}^d$ *and has the asymptotic expansion*

$$F_{a,b}(\mathbf{x}) \sim C(a,b,d)|\mathbf{x}|^{\frac{b-d+da/2}{1-a}} \exp(iB(a)|\mathbf{x}|^{-\frac{a}{1-a}}) + o\left(|\mathbf{x}|^{\frac{b-d+da/2}{1-a}}\right) \tag{10.35}$$

as $|\mathbf{x}| \to +\infty$*, where* $C(a,b,d)$ *and* $B(a)$ *are explicit positive constants.*

(vii) For any $b \in \mathbb{R}$*,* $F_{1,b}$ *is smooth on* $\mathbb{R}^d \setminus \{|x| = 1\}$ *and for every* $\beta \in \mathbb{N}^d$ *and* $N \in \mathbb{N}$*,*

$$\left| \left(\frac{\partial}{\partial x} \right)^{\beta} F_{1,b}(x) \right| = O(|x|^{-N}) \quad as \quad |x| \to +\infty.$$

If $b < \frac{d+1}{2}$*, then*

$$F_{1,b}(x) = C(b)\,(1 - |x| + i0)^{b - \frac{d-1}{2}} \quad as \quad |x| \to 1.$$

.

Remark 10.7. The previous theorem will be used with $b = 0$. For notational convenience, set $F_{a,0} = F_a$. If $a = \frac{1}{2}$, the three first terms in the asymptotic expansion of F_a are, respectively, $\alpha_0 |x|^{\frac{-3d}{2}} \exp(\theta)$, $\alpha_1 |x|^{\frac{-3d+2}{2}} \exp(\theta)$ and $\alpha_2 |x|^{\frac{-3d+4}{2}} \exp(\theta)$, where $\theta = i\xi_a |x|^{\frac{-a}{1-a}}$. Notice that $F_a \notin L^1_{loc}(\mathbb{R}^d)$, but that it is defined as a distribution since, because of its oscillatory nature, it is locally integrable around 0 in the sense of generalized Riemann integration. For example, when $d = 1$, one has

$$F_{\frac{1}{2}}(x) = \frac{1}{|x|^{\frac{3}{2}}} \exp\left(i \frac{\xi}{|x|} \right) + G(x)$$

where $\xi \neq 0$ and $G \in L^1(\mathbb{R})$. For any $A > 0$, integration by parts reveals that

$$\int_{-A}^{A} |x|^{-\frac{3}{2}} \exp\left(i \frac{\xi}{|x|} \right) dx = 2 \int_0^A \frac{e^{i\frac{\xi}{r}}}{r^{\frac{3}{2}}} dr = 2 \frac{i}{\xi} \sqrt{A} e^{i\frac{\xi}{A}} - \frac{i}{\xi} \int_0^A \frac{1}{\sqrt{r}} e^{i\frac{\xi}{r}} dr.$$

The last integral exists in the $L^1(0,A)$ sense. Thus, $F_{\frac{1}{2}}$ can be defined as a distribution by writing it as $F_{\frac{1}{2}} = G + \mathfrak{F}$, where $G \in L^1_{loc}(\mathbb{R})$ and, for any test function $\phi \in \mathscr{C}_0^\infty(\mathbb{R})$,

$$\langle \mathfrak{F}, \phi \rangle = -\frac{i}{2\xi} \int_0^{+\infty} \frac{e^{i\frac{\xi}{x}}}{\sqrt{x}} \left[\phi(x) - \phi(-x) \right] dx - \frac{i}{\xi} \int_0^{+\infty} e^{i\frac{\xi}{x}} \sqrt{x} \left[\phi'(x) - \phi'(-x) \right] dx.$$

Returning to the study of $I_t^1(x)$, notice first that for dimensions $d = 1, 2$,

$$I_t^1(x) = \frac{1}{t^{2d}} \int_{\mathbb{R}^d} e^{i|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot\frac{x}{t^2}} d\mathbf{k} = \frac{1}{t^{2d}} J\left(\frac{x}{t^2}\right),$$

where

$$J(x) = \int_{\mathbb{R}^d} e^{i|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot x} d\mathbf{k}.$$

Introducing a truncation function ψ as above which is zero near the origin and one near infinity, the integral J can be broken down as

$$J(x) = \int_{\mathbb{R}^d} \psi(|\mathbf{k}|) e^{i|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot x} d\mathbf{k} + \int_{\mathbb{R}^d} (1 - \psi(|\mathbf{k}|)) e^{i|\mathbf{k}|^{\frac{1}{2}}} e^{i\mathbf{k}\cdot x} d\mathbf{k} =: J_1(x) + J_2(x). \tag{10.36}$$

Arguing as in the analysis of I_t^2, one checks that in dimension $d = 1$, $J_2 \in L^1(\mathbb{R})$. In dimension $d = 2, J_2$ is a bounded continuous function of x.

On the other hand, Theorem 10.10 implies that $(1 - \psi(|\mathbf{k}|)) e^{i|\mathbf{k}|^{\frac{1}{2}}}$ is not an $L^\infty(\mathbb{R}^d)$ multiplier. These considerations allow the following conclusion.

Proposition 10.11. *The linearized water-wave problem (10.27) is ill-posed in $L^\infty(\mathbb{R}^d)$, for both horizontal dimensions $d = 1, 2$.*

Proof. Take $\eta_1 \equiv 0$ in (10.27) and an appropriate choice of η_0 (see the proof of Theorem 10.12 for more details).

□

This proposition is reinforced by the following, more specific dispersive blow-up result.

Theorem 10.12. *Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$, $d = 1, 2$ be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (10.27) with $\eta_1 \equiv 0$ satisfies the three conditions*

(i) η is a continuous function of x and t on $\mathbb{R} \times \big((0, +\infty) \setminus \{t^*\}\big)$,

(ii) $\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R} \setminus \{x*\}$,

(iii) $\lim_{\substack{(x,t) \in \mathbb{R}^d \times (0,+\infty) \to (x^*, t^*) \\ (x,t) \neq (x^*, t^*)}} |\eta(x,t)| = +\infty.$

Proof. One may assume that $(x^*, t^*) = (0, 1)$. Again, take $\hat{\eta}_1 = 0$ in (10.27) so that the corresponding solution is

$$\eta(\cdot, t) = \eta_0 \star \mathscr{F}^{-1} \left(\exp i[t(|\mathbf{k}| \tanh |\mathbf{k}|)^{\frac{1}{2}}] \right).$$

Using the notation introduced in our earlier ruminations, write

$$\begin{aligned}\eta(\cdot,t) &= \eta_0 \star \mathscr{F}^{-1}\left(\psi(|\mathbf{k}|)e^{it|\mathbf{k}|^{\frac{1}{2}}} + (1-\psi(|\mathbf{k}|)e^{it|\mathbf{k}|^{\frac{1}{2}}} + f_t(|\mathbf{k}|)e^{it|\mathbf{k}|^{\frac{1}{2}}}\right)\\ &= \eta_0 \star \mathscr{F}^{-1}\big(f_1(t,\mathbf{k}) + f_2(t,\mathbf{k}) + f_3(t,\mathbf{k})\big)\\ &= N_1(\cdot,t) + N_2(\cdot,t) + N_3(\cdot,t).\end{aligned}$$

In spatial dimension $d = 1$, it has already been shown that for any fixed $t \in (0,+\infty)$, $\mathscr{F}^{-1}f_2(\cdot,t)$ and $\mathscr{F}^{-1}f_3(\cdot,t)$ are integrable functions of x, and, as is easily confirmed, uniformly so on compact subsets of $t \in (0,+\infty)$. Thus, the functions N_2 and N_3 are continuous on $\mathbb{R}\times(0,+\infty)$, for any $\eta_0 \in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})\cap C(\mathbb{R})$. On the other hand, in dimension $d = 2$, for a fixed $t \in (0,+\infty)$, $\mathscr{F}^{-1}f_2(\cdot,t)$ and $\mathscr{F}^{-1}f_3(\cdot,t)$ are bounded continuous functions of x, and uniformly so on compact subsets of $t \in (0,+\infty)$,

Choose the initial value η_0 to be $\eta_0(x) = |x|^\beta \bar{K}(x)$ for $x \in \mathbb{R}$, where $\frac{3d}{2} \le \beta \le 2d$, and

$$K = \mathscr{F}^{-1}\left(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}}\right).$$

In the notation arising in Theorem 10.10, this amounts to taking $b = 0$ and setting $K(x) = \bar{F}_{\frac{1}{2}}(x)$. Using the results of Theorem 10.10 along with the choice of β, it is easily seen that $\eta_0 \in C(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\cap\mathscr{C}^\infty(\mathbb{R}^d\setminus\{0\})\cap L^1(\mathbb{R}^d)$. In particular, η_0 is an L^2-function.

Note that although $\eta_0 \in L^1(\mathbb{R}^d)$, for $t \neq 0$, the solution $\eta(\cdot,t)$, does not necessarily belong to Ł$^\infty(\mathbb{R}^d)$ since $\mathscr{F}^{-1}\left(\exp i[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}]\right)$ is not an L^∞–function. Again this is in strong contrast with what obtains for the linear KdV-equation (10.15) or the linear Schrödinger equation.

The preceding analysis demonstrates that $N_2(\cdot,t)$ and $N_3(\cdot,t)$ are convolutions of an L^1–function with a bounded, continuous function of x. Hence, they are themselves bounded and continuous in x, and uniformly so on compact temporal subsets.

Theorem 10.10 applied to N_1 implies that as $(x,t)\to(0,1)$, the solution $\eta(x,t)$ tends to

$$C_1 + C_2\int_{\mathbb{R}^d}|K(y)|^2|y|^\beta dy = +\infty$$

since $\beta \le 2d$.

It is now demonstrated that η is continuous on $\mathbb{R}^d \times (0,+\infty) \setminus \{(0,1)\}$, which is to say, everywhere except at the dispersive blow-up point. Since N_2 and N_3 are continuous in x and t, it remains to consider $N_1(\cdot,t) = \eta_0 \star \mathscr{F}^{-1}(\psi(|\mathbf{k}|)e^{it|\mathbf{k}|^{\frac{1}{2}}})$.

We first show that $N_1(\cdot,1)$ is a continuous function of x on $\mathbb{R}^d \setminus \{0\}$. According to the definition of η_0,

$$N_1(x,1) = \int_{\mathbb{R}^d} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) F_{\frac{1}{2}}(y)dy. \tag{10.37}$$

Let $\delta > 0$ be fixed and suppose that $|x| > \delta$. Decompose the last integral in the form

$$N_1(x,1) = \int_{B_{\frac{\delta}{2}}(0)} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) F_{\frac{1}{2}}(y)dy + \int_{B_{\frac{\delta}{2}}(x)} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) F_{\frac{1}{2}}(y)dy$$

$$+ \int_{\mathbb{R}^d \setminus B_{\frac{\delta}{2}}(0) \cup B_{\frac{\delta}{2}}(x)} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) F_{\frac{1}{2}}(y)dy = N_1^1(x,1) + N_1^2(x,1) + N_1^3(x,1).$$

By Theorem 10.10, $N_1^3(\cdot,1)$ is a continuous function of x. By our choice of β, $N_1^2(\cdot,1)$ is a continuous function of x. The treatment of $N_1^1(\cdot,1)$ is a bit more delicate and makes use of the oscillatory nature of the integrand. By Theorem 10.10, $F_{\frac{1}{2}}(x)$ behaves like

$$\left[\frac{\alpha_1}{|x|^{\frac{3d}{2}}} + \frac{\alpha_2}{|x|^{\frac{3d}{2}-1}} + \frac{\alpha_3}{|x|^{\frac{3d}{2}-2}} + g(x) \right] e^{iC_3|x|^{-1}} \tag{10.38}$$

for x near 0, where g is continuous.

When $d = 1$, only the first term in (10.38) gives trouble as regards the continuity of $N_1^1(\cdot,1)$. Integration by parts reveals immediately that the integral

$$\int_{B_{\frac{\delta}{2}}(0)} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) \frac{e^{iC_3|y|^{-1}}}{|y|^{\frac{3}{2}}} dy$$

defines a continuous function of x.

When $d = 2$, the first two terms in (10.38) both lead to situations that are possibly singular. We are therefore lead to consider the two integrals

$$\int_{B_{\frac{\delta}{2}}(0)} |x-y|^\beta \bar{F}_{\frac{1}{2}}(x-y) \frac{e^{iC_3|y|^{-1}}}{|y|^3} dy \tag{10.39}$$

and

$$\int_{B_{\frac{\delta}{2}}(0)} |x-y|^{\beta} \bar{F}_{\frac{1}{2}}(x-y) \frac{e^{iC_3|y|^{-1}}}{|y|^2} dy. \tag{10.40}$$

Straightforward integration by parts shows that both these integrals define continuous functions of x.

Attention is now turned to the region $D_1 = \{(x,t); x \in \mathbb{R}^d, \; t > 0, \; t \neq 1\}$. It will be shown that N_1 is continuous throughout this domain. A first observation is that

$$\mathscr{F}^{-1}\left(\psi(|\mathbf{k}|e^{it|\mathbf{k}|^{\frac{1}{2}}}\right)(x) = \frac{1}{t^{\frac{d}{2}}}\mathscr{F}^{-1}\left(\psi\left(\frac{|\mathbf{k}|}{t^{\frac{1}{2}}}\right)e^{i|\mathbf{k}|^{\frac{1}{2}}}\right)\left(\frac{x}{t^{\frac{1}{2}}}\right) \tag{10.41}$$

On the other hand,

$$\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}} - \psi\left(\frac{|\mathbf{k}|}{t^{\frac{1}{2}}}\right)e^{i|\mathbf{k}|^{\frac{1}{2}}} = \tilde{\psi}_t(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}}$$

where $\tilde{\psi}_t$ is smooth, compactly supported and vanishes in a neighborhood of 0. Thus, the inverse Fourier transform of $\tilde{\psi}_t(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}}$ is smooth and decays rapidly to 0 as $|x| \to \infty$; it is certainly bounded and continuous on D_1. We may therefore write

$$N_1(\cdot,t) = \frac{1}{t^{\frac{d}{2}}}\eta_0 \star F_{\frac{1}{2}}\left(\frac{\cdot}{t^{\frac{1}{2}}}\right) + G(\cdot,t) =: \tilde{N}_1(\cdot,t) + G(\cdot,t), \tag{10.42}$$

where G is continuous in x and t. Split $\tilde{N}_1(x,t)$ as follows:

$$\begin{aligned}
\tilde{N}_1(x,t) &= \frac{1}{t^{\frac{d}{2}}}\int_{\mathbb{R}^d}\eta_0(x-y)F_{\frac{1}{2}}\left(\frac{y}{t^{\frac{1}{2}}}\right)dy \\
&= \frac{1}{t^{\frac{d}{2}}}\left(\int_{|y|\leq 1}\eta_0(x-y)F_{\frac{1}{2}}\left(\frac{y}{t^{\frac{1}{2}}}\right)dy + \int_{|y|\geq 1}\eta_0(x-y)F_{\frac{1}{2}}\left(\frac{y}{t^{\frac{1}{2}}}\right)dy\right) \\
&= \tilde{N}_1^1(x,t) + \tilde{N}_1^2(x,t).
\end{aligned}$$

Since $\eta_0(x-y)$ behaves like $C|x-y|^{-\frac{3d}{2}+\beta}$ when y is close to x, the choice of β and the properties of $F_{\frac{1}{2}}$ imply that $\tilde{N}_1^2$ is continuous in x and t.

The choice $\eta_0 = |x|^{\beta}\bar{K}(x)$ entails that

$$\tilde{N}_1(x,t) = \frac{1}{t^{\frac{d}{2}}}\int_{|y|\leq 1}|x-y|^{\beta}\bar{F}_{\frac{1}{2}}(x-y)F_{\frac{1}{2}}\left(\frac{y}{t^{\frac{1}{2}}}\right)dy. \tag{10.43}$$

When $x \neq 0$, the singularity at $y = 0$ can be dealt with as in the preceding analysis of N_1^1. Attention is thus restricted to $x = 0$ and the aim is to prove that the integral

$$\tilde{N}_1(0,t) = \frac{1}{t^{\frac{d}{2}}} \int_{|y|\leq 1} |y|^\beta \bar{F}_{\frac{1}{2}}(y) F_{\frac{1}{2}}\left(\frac{y}{t^{\frac{1}{2}}}\right) dy, \tag{10.44}$$

taken in the sense of generalized Riemann integration, is finite when $t \neq 1$. According to Theorem 10.10, the singular part of the integral defining $\tilde{N}_1(0,t)$ is

$$\Gamma(t) = t^{\frac{d}{4}} \int_{|y|\leq 1} |y|^{\beta - 3d} e^{i \frac{C_3}{|y|}(t^{\frac{1}{2}} - 1)} dy.$$

This integral is finite, as seen by integration by parts, provided $\beta > 2d - 1$, which is compatible with the restriction

$$\frac{3d}{2} \leq \beta \leq 2d$$

on β. The proof is complete.

□

Remark 10.8. It is proven in [35, 33] that dispersive blow-up holds true for a general class of nonlinear Schrödinger type equations in $\mathbb{R}^n$ (including the “hyperbolic" one and the Davey-Stewartson systems).

Chapter 11

Long time existence issues

11.1 Long time existence issues

As we have seen in the previous chapters, a basic issue for the rigorous justification of the asymptotic model is to establish long time existence results for the solutions, that is on time scales on which the models are meaningful.

This issue is of course irrelevant when, for *simple* models like KdV, KP, Benjamin-Ono,..., local well-posedness in large spaces combined with conservation laws insure the *global* well-posedness.

For most of asymptotic models however this strategy does not work. Either no conservation laws exist or they cannot control a useful norm. For instance, the Boussinesq systems (4.26) are hamiltonian only when $b = d$. In fact (4.26) can be then rewritten in the 2D caseon the form

$$\partial_t \mathbf{u} + A_\varepsilon \mathbf{u} + \varepsilon \mathscr{N}(\mathbf{u}) = 0, \tag{11.1}$$

where

$$\mathbf{u} = \begin{pmatrix} \zeta \\ v_1 \\ v_2 \end{pmatrix},$$

$$A_\varepsilon = (I - \varepsilon b \Delta)^{-1} \begin{pmatrix} 0 & (1 + a\varepsilon\Delta)\partial_{x_1} & (1 + a\varepsilon\Delta)\partial_{x_2} \\ (1 + c\varepsilon\Delta)\partial_{x_1} & 0 & 0 \\ (1 + c\varepsilon\Delta)\partial_{x_2} & 0 & 0 \end{pmatrix}, \tag{11.2}$$

and

$$\mathscr{N}(\mathbf{u}) = (I-\varepsilon b\Delta)^{-1}\begin{pmatrix} \partial_{x_1}(\zeta v_1)+\partial_{x_2}(\zeta v_2) \\ \frac{1}{2}\partial_{x_1}(v_1^2+v_2^2) \\ \frac{1}{2}\partial_{x_2}(v_1^2+v_2^2) \end{pmatrix}.$$

We will denote by $(\cdot,\cdot)$ the scalar product on $L^2(\mathbb{R}^2;\mathbb{R}^3)$, *i.e*

$$(\mathbf{u},\widetilde{\mathbf{u}}) = \int_{\mathbb{R}^2}\Big(\zeta\widetilde{\zeta}+v_1\widetilde{v}_1+v_2\widetilde{v}_2\Big)\,dx_1dx_2$$

and by J the skew adjoint matrix operator

$$J = (I-\varepsilon b\Delta)^{-1}\begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} \\ \partial_{x_1} & 0 & 0 \\ \partial_{x_2} & 0 & 0 \end{pmatrix}.$$

Then, the system (11.1) is equivalent to

$$\partial_t\mathbf{u} = -J\begin{pmatrix} (1+\varepsilon\Delta)\eta+\frac{\varepsilon}{2}|\mathbf{v}|^2 \\ (1+\varepsilon\Delta)v_1+\varepsilon\zeta v_1 \\ (1+\varepsilon\Delta)v_2+\varepsilon\zeta v_2 \end{pmatrix} = J(\operatorname{grad}H_\varepsilon)(\mathbf{u}),$$

where $H_\varepsilon(\mathbf{u})$ is the functional given by [1]

$$H_\varepsilon(\mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}^2}\big(-a\varepsilon|\nabla\zeta|^2-c\varepsilon|\nabla\mathbf{v}|^2-\zeta^2-|\mathbf{v}|^2-\varepsilon\zeta|\mathbf{v}|^2\big)dx_1dx_2.$$

Therefore, it follows that H_ε is a conserved quantity by the flow of (11.1), since

$$\begin{aligned}\frac{d}{dt}H_\varepsilon(\mathbf{u}) &= H_\varepsilon'(\mathbf{u})\,\partial_t\mathbf{u} = \big((\operatorname{grad}H_\varepsilon)(\mathbf{u}),\partial_t\mathbf{u}\big)\\ &= \big((\operatorname{grad}H_\varepsilon)(\mathbf{u}),J(\operatorname{grad}H_\varepsilon)(\mathbf{u})\big) = 0,\end{aligned}$$

where we used the fact that J is skew adjoint.

On the other hand one easily check that the Hamiltonian H does not control any Sobolev norm and one cannot use it to obtain useful a priori bounds.[2]

[1] Recall that the linear well-posedness of the Boussinesq systems implies that $a \le 0$ and $c \le 0$ or $a = c$.

[2] In 1D however, due to the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ one can nevertheless prove global existence results for some specific systems for small enough initial data, see [14, 209, 37].

Another example is the *dispersive Burgers equation* (10.9) when $\alpha < 1$. We have already seen that the associated blow-up issues are delicate. So are the long time existence problems. For the Burgers equation

$$u_t + \varepsilon u u_x = 0,$$

the maximal existence time is $O(1/\varepsilon)$ and one can ask whether and how this existence time is affected by a weak dispersive perturbation ($0<\alpha<1$).

This question is not a simple one, as shows the related example of the *Burgers-Hilbert equation*

$$u_t + \varepsilon u u_x + \mathscr{H} u = 0, \quad u(\cdot, 0) = u_0, \tag{11.3}$$

where $\mathscr{H}$ is the Hilbert transform.

In fact, Hunter and Ifrim [110] (see a different proof in [111]) have shown the rather unexpected result :

Theorem 11.1. *Suppose that $u_0 \in H^2(\mathbb{R})$. There are constants $k > 0$ and $\varepsilon_0 > 0$, depending only on $|u_0|_{H^2}$, such that for every ε with $|\varepsilon| \leq \varepsilon_0$, there exists a solution $u \in C(I_\varepsilon; H^2(\mathbb{R})) \cap C^1(I_\varepsilon; H^1(\mathbb{R}))$ of (11.3) defined on the time-interval $I_\varepsilon = [-k/\varepsilon^2, k/\varepsilon^2]$.*

As explained in [110], this enhanced lifespan is due to the fact that the quadratic nonlinear term of order ε in (11.3) is nonresonant for the linearized equation. To see this, note that the solution of the linearized equation

$$u_t = \mathscr{H}[u]]$$

is given by $u = e^{t\mathscr{H}} u_0$, or

$$u(x,t) = u_0(x) \cos t + h_0(x) \sin t, \quad h_0 = \mathscr{H}[u_0],$$

as may be verified by use of the identity $\mathscr{H}^2 = -I$. This solution oscillates with frequency one between the initial data and its Hilbert transform, and the effect of the nonlinear forcing term uu_x on the linearized equation averages to zero because it contains no Fourier component in time whose frequency is equal to one.

11.1.1 Long time existence for the Boussinesq systems

Since the Boussinesq systems (4.26) are not skew-adjoint perturbations of symmetric quasilinear systems, one cannot use the classical energy method (see Chapter) to get “for free” well-posedness on time scales of order $O(1/\varepsilon)$.

It has been nevertheless observed in [30] that, when $a = c$, one can find an *equivalent* (in the sense of consistency) family of *fully symmetric systems* that can be solved by the aforementioned method. More precisely, by performing the change of variables

$$\tilde{\mathbf{v}} = \mathbf{v}(1 + \frac{\varepsilon}{2}\zeta)$$

one gets (up to terms of order ε^2) systems having a *symmetric* nonlinear part. Those transformed systems can be viewed as skew-adjoint perturbations of *symmetric* first order hyperbolic systems (when $\mathbf{v}$ is curlfree up to $0(\varepsilon)$) and existence on time scales of order $1/\varepsilon$ follows classically. But of course the transformed systems are not members of the a,b,c,d class of Boussinesq systems and this does not solve the long time existence problem for this class.

This question is addressed in [206] (see also [177]) for all Boussinesq systems except the case $a = c = \frac{1}{6}, b = d = 0$. We first need a technical definition.

Definition 11.2. For any $s \in \mathbb{R}$, $k \in \mathbb{N}$, $\varepsilon \in (0,1)$, the Banach space $X^s_{\varepsilon^k}(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm:

$$|u|^2_{X^s_{\varepsilon^k}} = |u|^2_{H^s} + \varepsilon^k |u|^2_{H^{s+k}}. \tag{11.4}$$

k, and later k' are positive numbers which depend on (a,b,c,d).

For instance $(k,k') = (3,3)$ when $a,c < 0$, $b,d > 0, b \neq d$.

Theorem 11.3. *Let $t_0 > \frac{n}{2}$, $s \geq t_0 + 2$ if $b + d > 0$, $s \geq t_0 + 4$ if $b = d = 0$. Let a,b,c,d satisfy the condition* (7.7)*. Assume that $\zeta_0 \in X^s_{\varepsilon^k}(\mathbb{R}^n), \mathbf{v}_0 \in X^s_{\varepsilon^{k'}}(\mathbb{R}^n)$ satisfy the (non-cavitation) condition*

$$1 - \varepsilon\zeta_0 \geq H > 0, \quad H \in (0,1), \tag{11.5}$$

Then there exists a constant $\tilde{c}_0$ such that for any $\varepsilon \leq \varepsilon_0 = \frac{1-H}{\tilde{c}_0(|\zeta_0|_{X^s_{\varepsilon^k}} + |\mathbf{v}_0|_{X^s_{\varepsilon^{k'}}})}$, there exists $T > 0$ independent of ε and a unique solution $(\zeta, \mathbf{v})^T$ with

$\zeta \in C([0,T/\varepsilon];X^s_{\varepsilon^k}(\mathbb{R}^n))$ and $\mathbf{v} \in C([0,T/\varepsilon];X^s_{\varepsilon^{k'}}(\mathbb{R}^n))$. Moreover,

$$\max_{t\in[0,T/\varepsilon]}(|\zeta|_{X^s_{\varepsilon^k}} + |\mathbf{v}|_{X^s_{\varepsilon^{k'}}}) \leq \tilde{c}(|\zeta_0|_{X^s_{\varepsilon^k}} + |\mathbf{v}_0|_{X^s_{\varepsilon^{k'}}}). \tag{11.6}$$

Here $\tilde{c} = C(H^{-1})$ and $\tilde{c}_0 = C(H^{-1})$ are nondecreasing functions of their argument.

Proof. We only sketch it (see [206] for details).

- The idea of the proof is to perform a suitable symmetrization (up to lower order terms) of a linearized system and then to implement a energy method on an approximate system. The method is of "hyperbolic" spirit.

- This is why we need the non cavitation condition (the hyperbolicity condition for the Saint-Venant system). Note that no such condition is needed when one solves the*local* Cauchy problem by dispersive methods.

Setting $\mathbf{V} = (\zeta,\mathbf{v})^T$, $\mathbf{U} = (\eta,\mathbf{u})^T = \varepsilon\mathbf{V}$, we rewrite (7.5) as

$$\begin{cases} (1-b\varepsilon\Delta)\partial_t\eta + \nabla\cdot\mathbf{u} + \nabla\cdot(\eta\mathbf{u}) + a\varepsilon\nabla\cdot\Delta\mathbf{u} = 0, \\ (1-d\varepsilon\Delta)\partial_t\mathbf{u} + \nabla\eta + \dfrac{1}{2}\nabla(|\mathbf{u}|^2) + c\varepsilon\nabla\Delta\eta = 0. \end{cases} \tag{11.7}$$

with the initial data

$$(\eta,\mathbf{u})^T|_{t=0} = (\varepsilon\zeta_0, \varepsilon\mathbf{v}_0)^T \tag{11.8}$$

Let $g(D) = (1-b\varepsilon\Delta)(1-d\varepsilon\Delta)^{-1}$. Then (11.21) is equivalent after applying $g(D)$ to the second equation to the condensed system:

$$(1-b\varepsilon\Delta)\partial_t\mathbf{U} + M(\mathbf{U},D)\mathbf{U} = 0, \tag{11.9}$$

where

$$M(\mathbf{U},D) = \begin{pmatrix} \mathbf{u}\cdot\nabla & (1+\eta+a\varepsilon\Delta)\partial_{x_1} & (1+\eta+a\varepsilon\Delta)\partial_{x_2} \\ g(D)(1+c\varepsilon\Delta)\partial_{x_1} & g(D)(u_1\partial_{x_1}) & g(D)(u_2\partial_{x_1}) \\ g(D)(1+c\varepsilon\Delta)\partial_{x_2} & g(D)(u_1\partial_{x_2}) & g(D)(u_2\partial_{x_2}) \end{pmatrix}. \tag{11.10}$$

In order to solve the system (7.5)-(11.8), we consider the following linear system in $\mathbf{U}$

$$(1-b\varepsilon\Delta)\partial_t\mathbf{U} + M(\underline{\mathbf{U}},D)\mathbf{U} = \mathbf{F}, \tag{11.11}$$

together with the initial data

$$\mathbf{U}|_{t=0} = \mathbf{U}_0, \tag{11.12}$$

The key point to solve the linear system (11.24)-(11.12) is to search a "symmetrizer" $S_{\underline{\mathbf{U}}}(D)$ of $M(\underline{\mathbf{U}}, D)$ such that the principal part of $iS_{\underline{\mathbf{U}}}(\xi)M(\underline{\mathbf{U}},\xi)$ is self-adjoint and $((S_{\underline{\mathbf{U}}}(\xi)\mathbf{U},\mathbf{U}))^{\mathbf{1/2}}$ defines a norm under a smallness assumption on $\underline{\mathbf{U}}$.

It is not difficult to find that:

(i) if $b = d$, $g(D) = 1$, $S_{\underline{\mathbf{U}}}(D)$ is defined by

$$\begin{pmatrix} 1+c\varepsilon\Delta & \underline{u}_1 & \underline{u}_2 \\ \underline{u}_1 & 1+\underline{\eta}+a\varepsilon\Delta & 0 \\ \underline{u}_2 & 0 & 1+\underline{\eta}+a\varepsilon\Delta \end{pmatrix}; \tag{11.13}$$

(ii) if $b \neq d$, $S_{\underline{\mathbf{U}}}(D)$ is defined by

$$\begin{pmatrix} (1+c\varepsilon\Delta)^2 g(D) & g(D)(\underline{u}_1(1+c\varepsilon\Delta)) & g(D)(\underline{u}_2(1+c\varepsilon\Delta)) \\ g(D)(\underline{u}_1(1+c\varepsilon\Delta)) & (1+\underline{\eta}+a\varepsilon\Delta)(1+c\varepsilon\Delta) & 0 \\ g(D)(\underline{u}_2(1+c\varepsilon\Delta)) & 0 & (1+\underline{\eta}+a\varepsilon\Delta)(1+c\varepsilon\Delta) \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{u}_1\underline{u}_1 & \underline{u}_1\underline{u}_2 \\ 0 & \underline{u}_1\underline{u}_2 & \underline{u}_2\underline{u}_2 \end{pmatrix}(g(D)-1). \tag{11.14}$$

Note that $S_{\underline{\mathbf{U}}}(D)$ is not self-adjoint since at least its diagonal part is not.
Then we define the energy functional associated with (11.24) as

$$E_s(\mathbf{U}) = \big((1-b\varepsilon\Delta)\Lambda^s\mathbf{U}, S_{\underline{\mathbf{U}}}(D)\Lambda^s\mathbf{U}\big)_2. \tag{11.15}$$

One can show that $E_s(\mathbf{U})$ defined in (11.25) is trully a energy functional equivalent to (the square of) some $X^s_{\varepsilon^k}(\mathbb{R}^2)$ norm provided a smallness condition is imposed on $\underline{\mathbf{U}}$, which is satisfied (for ε small enough) if

$$1+\underline{\eta} \geq H > 0, \quad |\underline{\mathbf{U}}|_\infty \leq \kappa(H,a,b,c,d), \quad |\underline{\mathbf{U}}|_{H^s} \leq 1, \quad \text{for} \quad t \in [0,T']. \tag{11.16}$$

For the nonlinear system, if ε is small enough, this smallness condition holds for its solution $(\zeta,\mathbf{v})^T$, i.e., $(\eta,\mathbf{u})^T \equiv \varepsilon(\zeta,\mathbf{v})^T$ satisfies (14.20).

One has then (painful) task to derive a priori estimates on the linearized system (which use in particular the commutator estimates of D. Lannes [157], in the various cases.

- Construction of the nonlinear solution by an iterative scheme :

We construct the approximate solutions $\{\mathbf{V}^n\}_{n\geq 0} = \{(\eta^n, \mathbf{v}^n)^T\}_{n\geq 0}$ with $\mathbf{U}^{n+1} = \varepsilon \mathbf{V}^n$ solution to the linear system

$$(1 - b\varepsilon\Delta)\partial_t \mathbf{U}^{n+1} + M(\mathbf{U}^n, D)\mathbf{U}^{n+1} = 0, \quad \mathbf{U}^{n+1}|_{t=0} = \varepsilon \mathbf{V}_0 \equiv \mathbf{U}_0, \quad (11.17)$$

and with $\mathbf{U}^0 = \mathbf{U}_0$. Given $\mathbf{U}^n$ satisfying the above assumption, the linear system (11.36) is unique solvable.

Remark 11.1.
- Our proof for the (abcd) systems does not seem to work for the *"KdV-KdV Boussinesq system"* $b = d = 0$, $a = c = 1/6$ but it does when $b = d = 0$, $a < 0, c < 0$ (that may occurs for gravity-capillary waves).
- The proofs using dispersion (that is high frequencies) do not take into account the algebra (structure) of the nonlinear terms. They allows initial data in relatively large Sobolev spaces but seem to give only existence times of order $O(1/\sqrt{\varepsilon})$, see [72, 161].
- The existence proofs on existence times of order $1/\varepsilon$ are of "hyperbolic" nature.They do not take into account the dispersive effects (treated as perturbations).
- Is it possible to go till $O(1/\varepsilon^2)$, or to get global existence. This is plausible in one D (the Boussinesq systems should evolves into an uncoupled system of KdV equations [220] but not so clear in 2D... One should there take advantage of dispersion dispersion. A possible strategy would be to use of a normal form technique (*à la Germain-Masmoudi-Shatah*) [88, 89, 90, 91].

□

11.1.2 Long time existence for a class of Full dispersion systems

We recall that the Full dispersion systems for surface water waves write

$$\begin{cases} \partial_t \zeta - \dfrac{1}{\sqrt{\mu}\nu}\mathscr{H}_\mu v + \dfrac{\varepsilon}{\nu}\big(\mathscr{H}_\mu(\zeta\nabla\mathscr{H}_\mu v) + \nabla\cdot(\zeta v)\big) = 0, \\ \partial_t v + \nabla\zeta + \dfrac{\varepsilon}{\nu}\big(\dfrac{1}{2}\nabla(|v|^2) - \sqrt{\mu}\nu\nabla\zeta\mathscr{H}_\mu\nabla\zeta\big) = 0, \end{cases} \quad (11.18)$$

where $v = v(t,x)$ ($x \in \mathbb{R}^d, d = 1,2$) is an $O(\varepsilon^2)$ approximation of the horizontal velocity at the surface and $\zeta = \zeta(t,x)$ is the deviation of the free surface.

We recall that ε is here the steepness parameter defined as the ratio of a typical amplitude over a typical horizontal wave length.

Following Lannes ([155]), ν is a smooth function of μ such that $\nu \sim 1$ when $\mu \ll 1$ (shallow water) and $\nu \sim \frac{1}{\sqrt{\mu}}$ when $\mu = O(1)$ or $\mu \gg 1$ (deeper water), for instance $\nu = \frac{\tanh(\sqrt{\mu})}{\sqrt{\mu}}$. $\mathscr{H}_\mu$ is a Fourier multiplier defined as

$$\forall v \in \mathscr{S}(\mathbb{R}^d), \qquad \widehat{\mathscr{H}_\mu v}(\xi) = -\frac{\tanh(\sqrt{\mu}|\xi|)}{|\xi|}(i\xi)\cdot\widehat{v}(\xi). \tag{11.19}$$

Without loss of generality, we may assume that

$$\sqrt{\mu}\nu = 1, \quad \varepsilon = \varepsilon\sqrt{\mu} \leq \varepsilon_{\max} \leq 1, \quad \mu \geq \mu_{\min} > 0. \tag{11.20}$$

This model has been proven by D. Lannes ([155]) to be consistent with the full water wave system. To our knowledge, no (local or on large time) well-posedness result for the Cauchy problem associated to (11.18) seems to be available. Our goal is to derive an equivalent system, which is also consistent with the water wave system, and for which we can prove the large time well-posedness of the Cauchy problem. The new system, which has the same accuracy as the original one, is obtained after a (nonlinear and nonlocal) change of the two independent variables and turns out to be symmetrizable yielding the large time existence on the *hyperbolic* time scale $1/\varepsilon$. This method is inspired by [32] where it was used in the (simpler) case of Boussinesq systems having a skew-adjoint linear dispersive part (see the beginning of this Chapter).

One can now write the FD system as

$$\begin{cases} \partial_t\zeta - \mathscr{H}_\mu v + \varepsilon\big(\mathscr{H}_\mu(\zeta\nabla\mathscr{H}_\mu v) + \nabla\cdot(\zeta v)\big) = 0, \\ \partial_t v + \nabla\zeta + \varepsilon\big(\frac{1}{2}\nabla(|v|^2) - \nabla\zeta\mathscr{H}_\mu\nabla\zeta\big) = 0, \end{cases} \tag{11.21}$$

where $\mathscr{H}_\mu = \tanh(\sqrt{\mu}|D|)\mathscr{H}$ for $d=1$ while $\mathscr{H}_\mu = (\mathscr{H}_{\mu,1}, \mathscr{H}_{\mu,2})^T$ with $\mathscr{H}_{\mu,j} = \tanh(\sqrt{\mu}|D|)\mathscr{R}_j$ for $d=2$. Here $\mathscr{H} = -\frac{\partial_x}{|\partial_x|}$ is the Hilbert transform and $\mathscr{R} = (\mathscr{R}_1, \mathscr{R}_2)^T = -\frac{\nabla}{|D|}$ is the Riesz transform. In what follows, we shall use the notations

$$\mathscr{R}u = \sum_{j=1}^{2} \mathscr{R}_j u_j, \quad u \cdot \mathscr{R} f = \sum_{j=1}^{2} u_j \mathscr{R}_j f. \tag{11.22}$$

The same notations are valid for $\mathscr{H}_\mu$.

Note that the dispersive part (order zero part) is the linearized water waves system at zero velocity and flat surface.

We consider first the one-dimensional case. We consider the nonlinear changes of variables

$$\tilde{v} = v + \frac{\varepsilon}{2} v \mathscr{H}_\mu \partial_x \zeta, \qquad \tilde{\zeta} = \zeta - \frac{\varepsilon}{4}|v|^2. \tag{11.23}$$

Deleting the $O(\varepsilon^2)$ terms, we obtain the following system (omitting the tildes)

$$\begin{cases} \partial_t \zeta - \mathscr{H}_\mu v + \frac{\varepsilon}{2}\mathscr{H}_\mu (v \mathscr{H}_\mu \partial_x \zeta) + \frac{\varepsilon}{2} v \partial_x \zeta + \varepsilon\Big(\mathscr{H}_\mu(\zeta \mathscr{H}_\mu \partial_x v) + \zeta \partial_x v\Big) = 0, \\ \partial_t v + \partial_x \zeta - \frac{\varepsilon}{2}\partial_x \zeta \mathscr{H}_\mu \partial_x \zeta + \frac{3\varepsilon}{2} v \partial_x v - \frac{\varepsilon}{2} v \mathscr{H}_\mu^2 \partial_x v = 0. \end{cases} \tag{11.24}$$

In the two-dimensional case, we consider now the nonlinear changes of variables

$$\tilde{v} = v + \frac{\varepsilon}{2} v \mathscr{H}_\mu \nabla \zeta, \qquad \tilde{\zeta} = \zeta - \frac{\varepsilon}{4}|v|^2, \tag{11.25}$$

Discarding the $O(\varepsilon^2)$ terms, we finally get (omitting the tildes)

$$\begin{cases} \partial_t \zeta - \mathscr{H}_\mu v + \frac{\varepsilon}{2}\mathscr{H}_\mu (v \mathscr{H}_\mu \nabla \zeta) + \frac{\varepsilon}{2} v \cdot \nabla \zeta + \varepsilon\Big(\mathscr{H}_\mu(\zeta \nabla \mathscr{H}_\mu v) + \zeta \nabla \cdot v\Big) = 0, \\ \partial_t v + \nabla \zeta - \frac{\varepsilon}{2}\nabla \zeta \mathscr{H}_\mu \nabla \zeta - \frac{\varepsilon}{2} v(\mathscr{H}_\mu \nabla \mathscr{H}_\mu + \operatorname{div}) v \\ \qquad + \varepsilon \begin{pmatrix} 2v_1 \partial_1 v_1 + v_2 \partial_2 v_1 \\ 2v_2 \partial_2 v_2 + v_1 \partial_1 v_2 \end{pmatrix} + \frac{\varepsilon}{2} \begin{pmatrix} v_2 \partial_1 v_2 + v_1 \partial_2 v_2 \\ v_1 \partial_2 v_1 + v_2 \partial_1 v_1 \end{pmatrix} = 0. \end{cases} \tag{11.26}$$

In both cases the new systems are consistent with the original ones (assuming that curl $\mathbf{v} = O(\varepsilon)$ in $2D$, a condition which is satisfied when deriving the FD system). It turns out that the new systems are symmetrizable.

We need first to define some functional spaces

Definition 11.4. For any integer $N \geq 0$, $\mu \in (\mu_{\min}, \infty)$ with $\mu_{\min} > 0$, the Banach space $X^N_\mu(\mathbb{R}^d)$ $(d = 1,2)$ is defined as $H^{N+1/2}(\mathbb{R}^d) \times H^N(\mathbb{R}^d)$ equipped with the norm

$$\mathscr{E}^N(\mathbf{V}) = |\mathbf{V}|^2_{X^N_\mu} = \frac{1}{\sqrt{\mu}}|\zeta|_2^2 + \big||D|^{1/2}\zeta\big|^2_{H^N} + |\mathbf{v}|^2_{H^N}, \tag{11.27}$$

where $\mathbf{V} = (\zeta, \mathbf{v})^T$ with $\mathbf{v} = v$ for $d = 1$ and $\mathbf{v} = (v_1, v_2)^T$ for $d = 2$.

Remark 11.2. When $\mu = \infty$, the Banach space $X^N_\mu(\mathbb{R}^d)$ becomes

$$X^N(\mathbb{R}^d) = \{\mathbf{V} = (\zeta, \mathbf{v})^T \,|\, |D|^{1/2}\zeta \in H^N(\mathbb{R}^d), \quad \mathbf{v} \in H^N(\mathbb{R}^d)\}$$

equipped with the norm

$$\mathscr{E}^N(\mathbf{V}) = |\mathbf{V}|^2_{X^N} = \big||D|^{1/2}\zeta\big|^2_{H^N} + |\mathbf{V}|^2_{H^N}.$$

In the one-dimensional case we have the following theorem.

Theorem 11.5. *Given any $\mathbf{V}_0 = (\zeta_0, v_0)^T \in X^N_\mu(\mathbb{R})$ with $N \geq 2$, there exist T independent of ε and a unique solution $\mathbf{V} = (\zeta, v)^T \in C([0, T/\varepsilon]; X^N_\mu(\mathbb{R}))$ to* (11.24)-(11.12)*. Moreover, one has the energy estimate*

$$\max_{0 \leq t \leq T/\varepsilon} \mathscr{E}^N(\mathbf{V}) \leq C_0 \mathscr{E}^N(\mathbf{V}_0). \tag{11.28}$$

In the two-dimensional case we have the following theorem.

Theorem 11.6. *Given any $\mathbf{V}_0 = (\zeta_0, \mathbf{v}_0)^T \in X^N_\mu(\mathbb{R}^2)$ with $N \geq 3$, there exist T independent of ε and a unique solution $\mathbf{V} = (\zeta, \mathbf{v})^T \in C([0, T/\varepsilon]; X^N_\mu(\mathbb{R}^2))$ to* (11.26)*. Moreover, one has the energy estimate*

$$\max_{0 \leq t \leq T/\varepsilon} \mathscr{E}^N(\mathbf{V}) \leq C_0 \mathscr{E}^N(\mathbf{V}_0). \tag{11.29}$$

As in the case of the Boussinesq systems, the proof consists, in the spirit of the procedure used in hyperbolic quasilinear systems, to construct the suitable symmetrizers, and then to derive the corresponding energy estimates. In the final step one constructs approximate solutions and one passes to the limit thanks to the energy estimates.

Technically one has to derive some commutator estimates. For instance, in the 1D case :

Lemma 11.7. *Let $s \geq 0$, $t_0 > 1/2$. Then for any $r_1, r_2 \geq 0$, there hold*

(i) $|[\mathscr{H}, a(x)]f(x)|_{\dot{H}^s} \lesssim |\Lambda^{r_1} a|_{\dot{H}^{s+r_2}} |\Lambda^{t_0 - r_1} f|_{\dot{H}^{-r_2}}$;

(ii) $|[\mathscr{H}, a(x)]\partial_x f(x)|_{\dot{H}^s} \lesssim |\Lambda^{r_1} a|_{\dot{H}^{s+r_2+1}} |\Lambda^{t_0 - r_1} f|_{\dot{H}^{-r_2}}$,

Lemma 11.8. *Let $t_0 > 1/2$. Then for any $r \geq 0$, there hold*

(i) $|(\mathscr{H}_\mu - \mathscr{H})f(x)|_{\dot{H}^s} \lesssim \frac{1}{\sqrt{\mu}^r} |f|_{\dot{H}^{s-r}}$, *for* $s \in \mathbb{R}$;

(ii) $|(1 + \mathscr{H}_\mu^2)f(x)|_{\dot{H}^s} \lesssim \frac{1}{\sqrt{\mu}^r} |f|_{\dot{H}^{s-r}}$, *for* $s \in \mathbb{R}$;

(iii) $|[\mathscr{H}_\mu, a(x)]f(x)|_{\dot{H}^s} \lesssim \big(|a|_{\dot{H}^{s+r}} + \frac{1}{\sqrt{\mu}^r} |a|_{\dot{H}^s} + \frac{1}{\sqrt{\mu}^s} |a|_{\dot{H}^r} + \frac{1}{\sqrt{\mu}^{s+r}} |a|_2\big) |\Lambda^{t_0} f|_{\dot{H}^{-r}}$, *for* $s \geq 0$.

Introduce $U = (\eta, u)^T = (\varepsilon\zeta, \varepsilon v)^T$ and $U^{(\alpha)} = \partial_x^\alpha U$.
Write the modified FD system as

$$\begin{cases} \partial_t \eta - \mathscr{H}_\mu u + \dfrac{1}{2}\mathscr{H}_\mu(u \mathscr{H}_\mu \partial_x \eta) + \dfrac{1}{2} u \partial_x \eta = g_\mu(\eta, u), \\ \partial_t u + \partial_x \eta - \dfrac{1}{2}\partial_x \eta \mathscr{H}_\mu \partial_x \eta + \dfrac{3}{2} u \partial_x u - \dfrac{1}{2} u \mathscr{H}_\mu^2 \partial_x u = 0, \end{cases} \tag{11.30}$$

where

$$g_\mu(\eta, u) = -\mathscr{H}_\mu(\eta \mathscr{H}_\mu \partial_x u) - \eta \partial_x u = -[\mathscr{H}_\mu, \eta]\mathscr{H}_\mu \partial_x u - \eta(\mathscr{H}_\mu^2 + 1)\partial_x u.$$

(i) for $\alpha = 0$, $U^{(0)}$ satisfies

$$\partial_t U^{(0)} + \mathscr{A} U^{(0)} = G^{(0)}, \tag{11.31}$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & -\mathscr{H}_\mu \\ \partial_x & 0 \end{pmatrix},$$

and $G^{(0)} = (G_1^{(0)}, G_2^{(0)})^T$ with

$$\begin{aligned} G_1^{(0)} &= g_\mu(\eta, u) - \frac{1}{2}\mathscr{H}_\mu(u \mathscr{H}_\mu \partial_x \eta) - \frac{1}{2} u \partial_x \eta, \\ G_2^{(0)} &= \frac{1}{2}\partial_x \eta \mathscr{H}_\mu \partial_x \eta - \frac{3}{2} u \partial_x u + \frac{1}{2} u \mathscr{H}_\mu^2 \partial_x u. \end{aligned}$$

Note that $\mathscr{H}_\mu = -\tanh(\sqrt{\mu}|\partial_x|)\frac{\partial_x}{|\partial_x|}$. Then the symmetrizer of $\mathscr{A}$ is defined by

$$S = \begin{pmatrix} \frac{|\partial_x|}{\tanh(\sqrt{\mu}|\partial_x|)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{11.32}$$

(ii) for $1 \leq \alpha \leq N$, applying ∂_x^α to (11.30), we obtain that $U^{(\alpha)}$ satisfies

$$\partial_t U^{(\alpha)} + \big(\mathscr{A} + \mathscr{B}[U]\big)U^{(\alpha)} = G^{(\alpha)}, \tag{11.33}$$

where $\mathscr{A}$ is the same as above and $\mathscr{B}[U] = (b_{ij})_{i,j=1,2}$ is defined by

$$\begin{pmatrix} \frac{1}{2}\mathscr{H}_\mu(u\mathscr{H}_\mu\partial_x) + \frac{1}{2}u\partial_x & \frac{1}{2}\mathscr{H}_\mu(\mathscr{H}_\mu\partial_x\eta\,\cdot) - \frac{1}{2}\partial_x\eta\,\mathscr{H}_\mu^2 \\ -\frac{1}{2}\partial_x\eta\,\mathscr{H}_\mu\partial_x - \frac{1}{2}\mathscr{H}_\mu\partial_x\eta\,\partial_x & \frac{3}{2}u\partial_x - \frac{1}{2}u\mathscr{H}_\mu^2\partial_x \end{pmatrix},$$

and $G^{(\alpha)} = (G_1^{(\alpha)}, G_2^{(\alpha)})^T$ with

$$\begin{aligned} G_1^{(\alpha)} &= \partial_x^\alpha g_\mu(\eta,u) - \frac{1}{2}\sum_{\beta=1}^{\alpha-1} C_\alpha^\beta\Big(\mathscr{H}_\mu(u^{(\beta)}\mathscr{H}_\mu\partial_x\eta^{(\alpha-\beta)}) + u^{(\beta)}\partial_x\eta^{(\alpha-\beta)}\Big) \\ &- \frac{1}{2}\partial_x\eta(\mathscr{H}_\mu^2+1)u^{(\alpha)}, \\ G_2^{(\alpha)} &= \frac{1}{2}\sum_{\beta=1}^{\alpha-1} C_\alpha^\beta\partial_x\eta^{(\beta)}\mathscr{H}_\mu\partial_x\eta^{(\alpha-\beta)} \\ &\quad + \sum_{\beta=1}^{\alpha} C_\alpha^\beta\Big(-\frac{3}{2}u^{(\beta)}\partial_x u^{(\alpha-\beta)} + \frac{1}{2}u^{(\beta)}\mathscr{H}_\mu^2\partial_x u^{(\alpha-\beta)}\Big). \end{aligned}$$

The symmetrizer of $\mathscr{A} + \mathscr{B}[U]$ is also defined by S in (11.32).
Energy functionals associated to the quasilinear system (11.31)-(11.33):

$$\begin{aligned} E^{(\alpha)}(U) &= (U^{(\alpha)}, SU^{(\alpha)})_2 = \Big|\frac{|\partial_x|^{1/2}}{\tanh(\sqrt{\mu}|\partial_x|)^{1/2}}\eta^{(\alpha)}\Big|_2^2 + |u^{(\alpha)}|_2^2, \\ E^N(U) &= \sum_{\alpha=0}^{N} E^{(\alpha)}(U) = \Big|\frac{|\partial_x|^{1/2}}{\tanh(\sqrt{\mu}|\partial_x|)^{1/2}}\eta\Big|_{H^N}^2 + |u|_{H^N}^2. \end{aligned} \tag{11.34}$$

$$E^N(U) \sim \mathscr{E}^N(U) = \frac{1}{\sqrt{\mu}}|\eta|_2^2 + \big||\partial_x|^{1/2}\eta\big|_{H^N}^2 + |u|_{H^N}^2. \tag{11.35}$$

- The end of the proof consists in technical energy estimates.

11.1.3 Large time existence results for some internal wave models

We focus here on the Cauchy problem for some internal waves models in the Intermediate long wave and Benjamin-Ono regimes, following closely [245] (where the counterpart of the system below is derived in the case of an upper free surface). We consider the ILW system derived in Chapter 5.2

$$\begin{cases} (1+g(D))\partial_t\zeta - \frac{1}{\gamma}\nabla\cdot(\varepsilon\zeta\mathbf{v}) + \frac{1}{\gamma}(1+\frac{\beta-1}{\beta}g(D))\nabla\cdot\mathbf{v} = 0, \\ \partial_t\mathbf{v} + (1-g)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla(|\mathbf{v}|^2) = \mathbf{0}, \end{cases} \tag{11.36}$$

with the initial data

$$\mathbf{V}|_{t=0} = (\zeta,\mathbf{v})^T|_{t=0} = \mathbf{V}_0, \tag{11.37}$$

where

$$g(D) = \frac{\beta}{\gamma}\sqrt{\mu}|D|\coth(\sqrt{\mu_2}|D|). \tag{11.38}$$

From now on we will assume that

$$\beta > 1 \quad \text{and} \quad \gamma\in(0,1), \tag{11.39}$$

ensuring that (11.36) is linearly well-posed.

We first prove the local well-posedness of (11.36) with a small existence time $O(\sqrt{\mu})$ by using in a essential way the dispersion of the system.

Theorem 11.9. *Let* $\varepsilon\in(0,1)$ *and* $\mu\in(0,1)$, $\mu_2>0$. *For any* $\mathbf{V}_0\in H^{2+[\frac{d}{2}]}(\mathbb{R}^d)$, *there exists* $T>0$ *such that* (11.36)-(11.37) *has a unique solution* $\mathbf{V} =: (\zeta,\mathbf{v})^T \in C^1([0,\sqrt{\mu}T];H^{1+[\frac{d}{2}]}(\mathbb{R}^d))\cap C([0,\sqrt{\mu}T];H^{2+[\frac{d}{2}]}(\mathbb{R}^d))$. *The correspondence* $\mathbf{V}_0\to\mathbf{V}$ *from* $H^{2+[\frac{d}{2}]}(\mathbb{R}^d)$ *to* $C([0,\sqrt{\mu}T];H^{2+[\frac{d}{2}]}(\mathbb{R}^d))$ *is continuous. Here* T *is only depending on* $|\mathbf{V}_0|_{H^{2+[\frac{d}{2}]}}$ *and* γ,β, *independent of* μ *and* ε.

Remark 11.3. The same result holds in spaces $H^s(\mathbb{R}^d)$ for any $s\geq 2+[\frac{d}{2}]$.

Proof. For the sake of simplicity we restrict to the case $d=1$ and will only prove the suitable a priori estimates and the uniqueness.

Step 1. Setting $\eta = \varepsilon\zeta$, $u = \varepsilon v$, using the formula

$$(1+g(D))^{-1}(1+\frac{\beta-1}{\beta}g(D)) = \frac{\beta-1}{\beta} + \frac{1}{\beta}(1+g(D))^{-1}, \qquad (11.40)$$

(11.36) is rewritten in terms of (η, u) as

$$\begin{cases} \partial_t\eta - \frac{1}{\gamma}(1+g(D))^{-1}\partial_x(\eta u) + \frac{\beta-1}{\gamma\beta}\partial_x u + \frac{1}{\gamma\beta}(1+g(D))^{-1}\partial_x u = 0, \\ \partial_t u + (1-\gamma)\partial_x\eta - \frac{1}{\gamma}u\partial_x u = 0. \end{cases} \qquad (11.41)$$

Step 1.1. *A priori estimates.* Multiplying the first equation of (11.41) by $\frac{\gamma\beta}{\beta-1}\eta$, multiplying the second one by $\frac{1}{1-\gamma}u$, integrating over $\mathbb{R}$, integrating by parts, one gets

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\frac{\gamma\beta}{\beta-1}|\eta|_2^2 + \frac{1}{1-\gamma}|u|_2^2) &= \frac{\beta}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x(\eta u)\eta dx \\ &\quad - \frac{1}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x u\eta dx, \end{aligned} \qquad (11.42)$$

which with the fact that $\frac{|\xi|}{1+g(\xi)} \leq c(1+\frac{1}{\sqrt{\mu}})$ implies

$$\frac{1}{2}\frac{d}{dt}(\frac{\gamma\beta}{\beta-1}|\eta|_2^2 + \frac{1}{1-\gamma}|u|_2^2) \leq c(1+\frac{1}{\sqrt{\mu}})(|u|_{H^1}|\eta|_2^2 + |u|_2|\eta|_2). \qquad (11.43)$$

Multiplying the first equation of (11.41) by $-\frac{\gamma\beta}{\beta-1}\partial_x^2\eta$, multiplying the second one by $-\frac{1}{1-\gamma}\partial_x^2 u$, integrating over $\mathbb{R}$, integrating by parts, one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\frac{\gamma\beta}{\beta-1}|\partial_x\eta|_2^2 + \frac{1}{1-\gamma}|\partial_x u|_2^2) &= \frac{\beta}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x\partial_x(\eta u)\partial_x\eta dx \\ &\quad - \frac{1}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x\partial_x u\partial_x\eta dx - \frac{1}{(1-\gamma)\gamma}\int_{\mathbb{R}}u\partial_x u\partial_x^2 u dx, \end{aligned} \qquad (11.44)$$

which with the fact that $\frac{|\xi|}{1+g(\xi)} \leq c(1+\frac{1}{\sqrt{\mu}})$ gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\frac{\gamma\beta}{\beta-1}|\partial_x\eta|_2^2 + \frac{1}{1-\gamma}|\partial_x u|_2^2) \\ &\qquad \leq c(1+\frac{1}{\sqrt{\mu}})(|u|_{H^1}|\eta|_{H^1}^2 + |u|_{H^1}|\eta|_{H^1} + |u|_{H^1}^2|u|_{H^2}). \end{aligned} \qquad (11.45)$$

Multiplying the first equation of (11.41) by $\frac{\gamma\beta}{\beta-1}\partial_x^4\eta$, multiplying the second one by $\frac{1}{1-\gamma}\partial_x^4 u$, integrating over $\mathbb{R}$, integrating by parts, one has

$$\frac{1}{2}\frac{d}{dt}\Big(\frac{\gamma\beta}{\beta-1}|\partial_x^2\eta|_2^2+\frac{1}{1-\gamma}|\partial_x^2 u|_2^2\Big)=\frac{\beta}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x\partial_x^2(\eta u)\partial_x^2\eta dx$$
$$-\frac{1}{\beta-1}\int_{\mathbb{R}}(1+g(D))^{-1}\partial_x\partial_x^2 u\partial_x^2\eta dx+\frac{1}{(1-\gamma)\gamma}\int_{\mathbb{R}}\partial_x^2(u\partial_x u)\partial_x^2 u dx. \tag{11.46}$$

Integrating by parts, we get that

$$\int_{\mathbb{R}}\partial_x^2(u\partial_x u)\partial_x^2 u dx=3\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_x u dx+\int_{\mathbb{R}}u\partial_x\partial_x^2 u\partial_x^2 u dx=\frac{5}{2}\int_{\mathbb{R}}(\partial_x^2 u)^2\partial_x u dx, \tag{11.47}$$

from which, we deduce from (11.46) that

$$\frac{1}{2}\frac{d}{dt}\Big(\frac{\gamma\beta}{\beta-1}|\partial_x^2\eta|_2^2+\frac{1}{1-\gamma}|\partial_x^2 u|_2^2\Big)\le c\Big(1+\frac{1}{\sqrt{\mu}}\Big)\big(|u|_{H^2}|\eta|_{H^2}^2+|u|_{H^2}|\eta|_{H^2}+|u|_{H^2}^3\big) \tag{11.48}$$

Combining (11.43), (11.45) and (11.48), one obtains

$$\frac{1}{2}\frac{d}{dt}\Big(\frac{\gamma\beta}{\beta-1}|\eta|_{H^2}^2+\frac{1}{1-\gamma}|u|_{H^2}^2\Big)\le c\Big(1+\frac{1}{\sqrt{\mu}}\Big)\big(|u|_{H^2}|\eta|_{H^2}^2+|u|_{H^2}|\eta|_{H^2}+|u|_{H^2}^3\big), \tag{11.49}$$

which implies by denoting $Y(t)=:\frac{\gamma\beta}{\beta-1}|\zeta|_{H^2}^2+\frac{1}{1-\gamma}|v|_{H^2}^2$ that

$$Y(t)'\le c\Big(1+\frac{1}{\sqrt{\mu}}\Big)\big(Y(t)+\varepsilon Y(t)^{3/2}\big)\le c\Big(1+\frac{1}{\sqrt{\mu}}\Big)Y(t)(Y(t)+1), \tag{11.50}$$

that is

$$\frac{d}{dt}\ln\Big(\frac{Y(t)}{Y(t)+1}\Big)\le c\Big(1+\frac{1}{\sqrt{\mu}}\Big). \tag{11.51}$$

Then

$$Y(t)\le\frac{1}{(1+\frac{1}{Y(0)})e^{-c(1+\frac{1}{\sqrt{\mu}})t}-1}, \tag{11.52}$$

if $(1+\frac{1}{Y(0)})e^{-c(1+\frac{1}{\sqrt{\mu}})t}>1$. Taking $T=\frac{1}{4c}\ln(1+\frac{1}{Y(0)})$, by (11.52), there holds

$$Y(t)\le\Big(\big(1+\frac{1}{Y(0)}\big)^{1/2}+1\Big)Y(0),\quad \forall t\in(0,\sqrt{\mu}T). \tag{11.53}$$

The estimate on $(\partial_t \zeta, \partial_t v)$ is easy to be deduced from the equations (11.41) and (11.53).

Step 1.2. *Uniqueness.* Suppose that $(\zeta_1, v_1)^T$ and $(\zeta_2, v_2)^T$ are two solutions to (11.36)-(11.37). Denoting $\zeta =: \zeta_1 - \zeta_2$, $v =: v_1 - v_2$ and $\eta =: \varepsilon\zeta$, $u =: \varepsilon v$, $\eta_i =: \varepsilon\zeta_i$, $u_i =: \varepsilon v_i$ $(i = 1,2)$, then $(\eta, u)^T$ satisfies

$$\begin{cases} \partial_t \eta - \dfrac{1}{\gamma}(1+g(D))^{-1}\partial_x(\eta u_1 + \eta_2 u) + \dfrac{\beta-1}{\gamma\beta}\partial_x u + \dfrac{1}{\gamma\beta}(1+g(D))^{-1}\partial_x u = 0, \\ \partial_t u + (1-\gamma)\partial_x \eta - \dfrac{1}{\gamma}(u\partial_x u_1 + u_2 \partial_x u) = 0. \end{cases} \tag{11.54}$$

Multiplying the first equation of (11.54) by $\frac{\gamma\beta}{\beta-1}(\eta - \partial_x^2 \eta)$, multiplying the second one by $\frac{1}{1-\gamma}(u - \partial_x^2 u)$, integrating over $\mathbb{R}$, integrating by parts, after a similar argument as proving (11.43) and (11.45), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\frac{\gamma\beta}{\beta-1}|\eta|_{H^1}^2 + \frac{1}{1-\gamma}|u|_{H^1}^2\right) \\ &\leq c(|u_1|_{H^2} + |u_2|_{H^2})(|\eta|_{H^1}^2 + |u|_{H^1}^2) + c(|\eta_2|_{H^1}|u|_{H^1}|\eta|_{H^1} + |u|_{H^1}|\eta|_{H^1}). \end{aligned} \tag{11.55}$$

Since $(\zeta_1, v_1)^T$, $(\zeta_2, v_2)^T$ are bounded in $C([0, \sqrt{\mu}T]; H^2(\mathbb{R}))$ and satisfying inequality (11.53), we deduce from (11.55) that

$$\frac{d}{dt}\left(\frac{\gamma\beta}{\beta-1}|\eta|_{H^1}^2 + \frac{1}{1-\gamma}|u|_{H^1}^2\right) \leq C_T(|\eta|_{H^1}^2 + |u|_{H^1}^2), \tag{11.56}$$

where C_T is a constant depending on $\gamma, \beta, \mu, T, |\mathbf{V}_0|_{H^2}$. Since $(\zeta, v)^T|_{t=0} = \mathbf{0}$, we deduce from (11.56) that $(\zeta_1, v_1)^T = (\zeta_2, v_2)^T$. This proves the uniqueness of the solution to (11.36)-(11.37).

□

Still following [245] we now prove a large time existence of the Cauchy problem (11.36)-(11.37) that is needed for the fumll justification of the system More precisely we shall prove the solvability of (11.36)-(11.37) in Sobolev space with higher regularity with the existence time at order $O(\frac{1}{\varepsilon}) = O(\frac{1}{\sqrt{\mu}})$ when $\mu \sim \varepsilon^2 \ll 1$. To this purpose, we introduce the Sobolev space $X_\mu^s(\mathbb{R}^d)$ as being $H^{s+1}(\mathbb{R}^d)$ equipped with the norm $|u|_{X_\mu^s}^2 = |u|_{H^s}^2 + \mu|u|_{H^{s+1}}^2$.

Theorem 11.10. *Let* $t_0 > \frac{d}{2}$, $s \geq t_0 + 2$ *and* $\mu \in (0,1)$, $\mu_2 > 0$. *Assume that* $\mathbf{V}_0 = (\zeta_0, \mathbf{v}_0)^T \in X^s_\mu(\mathbb{R}^d)$ *satisfies that*

$$1 - \varepsilon\zeta_0 \geq H > 0, \tag{11.57}$$

with $\mathbf{v}_0 = v_0$ *for* $d = 1$ *and* $\mathbf{v}_0 = (v_{0,1}, v_{0,2})^T$ *for* $d = 2$. *Then there exists a constant* $\tilde{c}_0$ *such that for any* $\varepsilon \leq \zeta_0 = \frac{1-H}{\tilde{c}_0 |\mathbf{V}_0|_{X^s_\mu}}$, *there exists* $T > 0$ *independent of* μ *and* ζ, *such that* (11.36)-(11.37) *has a unique solution* $\mathbf{V} \in C^1([0,T/\varepsilon]; X^{s-1}_\mu(\mathbb{R}^d)) \cap C([0,T/\varepsilon]; X^s_\mu(\mathbb{R}^d))$. *Moreover,*

$$\max_{t\in[0,T/\varepsilon]} (|\mathbf{V}|_{X^s_\mu} + |\partial_t \mathbf{V}|_{X^{s-1}_\mu}) \leq \tilde{c} |\mathbf{V}_0|_{X^s_\mu}. \tag{11.58}$$

Here and in what follows, without confusion, we denote $\tilde{c} = C(\mu_2^{-1}, H^{-1})$ *a nondecreasing constant depending on* μ_2^{-1} *and* H^{-1}. *Otherwise, we denote* $\tilde{c}_i$ *(i=0,1,2,...) the distinguished constants with the same properties as* $\tilde{c}$.

Proof. The proof of Theorem 11.10 follows the line of that of Theorem 11.5 and we refer to [245] for details. □

Chapter 12

Focusing versus defocusing

Dispersive equations are sometimes classified into two categories, the *focusing* ones and the *defocusing* ones. Roughly speaking the long time dynamics of defocusing equations is expected to be dominated by dispersion and scattering, being thus in some sense "trivial". On the other hand, the long time dynamics of focusing equations is expected to be not trivial, due to the existence of special localized solutions ("solitons", "solitary waves",..) that are supposed to play a major role in the dynamics as for instance in the case of the KdV equation.

For the cubic nonlinear Schrödinger equation

$$iu_t + \Delta u \pm |u|^2 u = 0 \quad \in \mathbb{R}^2 \times \mathbb{R}, \tag{12.1}$$

the focusing case corresponds to the $+$ sign, the defocusing case to the $-$ sign. Besides the L^2 norm and the momentum (that we will note use here), (12.1) preserves the energy (Hamiltonian)

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx \mp \frac{1}{4}\int_{\mathbb{R}^2} |u|^4 dx. \tag{12.2}$$

As it is well known (see for instance [223]), (12.1) possesses *solitary waves solutions* $u(x,t) = e^{i\omega t}\phi(x)$, $\phi \in H^1(\mathbb{R}^2)$, $\omega > 0$ in the focusing case and no such solutions in the defocusing one.

Solitary waves in the focusing case satisfy the nonlinear elliptic equation

$$-\Delta u + \omega u = |u|^2 u. \tag{12.3}$$

can be obtained by minimizing in $H^1(\mathbb{R}^2)$ the action

$$\frac{1}{2}\int_{\mathbb{R}}^{2}[|\nabla u|^2 - \frac{1}{3}|u|^4]dx + \frac{\omega}{2}\int_{\mathbb{R}}^{2}|u|^2 dx.$$

It is well-known ([223]) that no such solutions exist for the defocusing NLS equation. In the next sections we will consider different aspects of a *defocusing* or *defocusing* dynamics, emphasizing the Davey-Stewartson and KP equations.

12.1 Non existence of solitary waves solutions for the nonelliptic nonlinear Schrödinger equation

We will consider first the *nonelliptic nonlinear Schrödinger equation* we had seen in Chapter 4 to be a model for very deep water waves in the modulation regime.

We will consider more generally equations of type

$$iu_t + Lu + f(|u|^2)u = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R},\ n \geq 2 \tag{12.4}$$

where L is the second order differential operator given by

$$Lu = \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial_k}.$$

We assume that f is a real valued continuous function such that

$$|f(s)| \leq c(1+|s|^p), \quad p \leq \frac{2}{n-2}, \quad (1 \leq p < \infty \text{ if } n = 2).$$

We are looking for travelling wave solutions of (12.4), that is solutions of the form(2.1), that is, solutions of the form

$$u(x,t) = e^{i\omega t}\phi(x+ct), \tag{12.5}$$

where $\omega \in \mathbb{R}, c \in \mathbb{R}^n, \phi \in H^1(\mathbb{R}^n) \cap H^2_{\text{loc}}(|R^n)$.

Theorem 12.1. *For any $\omega \in \mathbb{R}, c \in \mathbb{R}^n$,* (12.4) *has no nonzero solution of type* (12.5) *if the nonsingular matrix (a_{jk}) is not positive-definite (resp. negative definite).*

Proof. As usual in this kind of problem, we will derive and use Pohojaev type identities. One can assume without loss of regularity that $Lu = \sum_{i=1}^n \varepsilon_i \frac{\partial^2 u}{\partial x_j^2}, \varepsilon_i = \pm 1$, where $\varepsilon_k \neq \varepsilon_l$ for some (k,l). ϕ should therefore satisfy the equation

$$-\omega\phi + i\sum_{j=1}^n c_j + \frac{\partial \phi}{\partial x_j} + \sum_{j=1}^n \frac{\partial^2 \phi}{\partial x_j^2} + f(|\phi|^2)\phi = 0.$$

By setting $\phi(x) = e^{-i\frac{c}{2}\cdot x}\psi(x)$, we are reduced to the case $c = 0$, (with a different ω) and we consider therefore

$$-\omega\phi + \sum_{j=1}^n \varepsilon_j \frac{\partial^2 \phi}{\partial x_j^2} + f(|\phi|^2)\phi = 0. \tag{12.6}$$

Since $\phi \in H^1(\mathbb{R}^n) \cap H^2_{\text{loc}}(\mathbb{R}^n)$, the following formal computations can be justified by a standard truncation procedure, namely we replace x_k in the following argument by $\chi_j(x)x_k = \chi_0\left(\frac{|x|}{j}\right)x_k$, $\chi_0 \in C_0^\infty(\mathbb{R})$, $\chi_0 \geq 0$, $\chi_0(t) = 1$, $0 \leq t \leq 1$, $\chi_0(t) = 0$, $t \geq 2$ and will let $j \to +\infty$.

Let $k \in \{1,\cdots,n\}$ be fixed; we multiply (12.6) by $x_k \frac{\partial \bar{\phi}}{\partial x_k}$ and integrate the real part of the resulting equation. We obtain after several integrations by parts

$$\omega \int |\phi|^2 dx - \varepsilon_k \int \left|\frac{\partial \phi}{\partial x_k}\right|^2 dx + \sum_{j\neq k} \varepsilon_j \int \left|\frac{\partial \phi}{\partial x_j}\right|^2 dx - \int F(|\phi|^2)dx = 0, \tag{12.7}$$

where $F(x) = \int_0^s f(t)dt$.

Let $l \in \{1 \cdots, n\}$ such that $\varepsilon_k \neq \varepsilon_l$. One obtains similarly

$$\omega \int |\phi|^2 dx - \varepsilon_l \int \left|\frac{\partial \phi}{\partial x_l}\right|^2 dx + \sum_{j\neq l} \varepsilon_j \int \left|\frac{\partial \phi}{\partial x_j}\right|^2 dx - \int F(|\phi|^2)dx = 0. \tag{12.8}$$

One deduces from (12.7) and (12.8) that

$$-\varepsilon_l \int \left|\frac{\partial \phi}{\partial x_l}\right|^2 dx + \varepsilon_k \int \left|\frac{\partial \phi}{\partial x_k}\right|^2 dx = 0,$$

and therefore

$$\frac{\partial \phi}{\partial x_l} \equiv \frac{\partial \phi}{\partial x_k} \equiv 0.$$

Finally $\nabla\phi \equiv 0$ and $\phi \equiv 0$. □

Remark 12.1. Of course Theorem 12.1 does not exclude the existence of non trivial *non localized* traveling waves.

Let us consider for instance the "non elliptic" NLS :

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \tag{12.9}$$

Let (see [50]) $f \in H^1(\mathbb{R}^2)$ be the unique radial positive solution $(f(x,y) = R((x^2+y^2)^{1/2}))$ of

$$-f + \Delta f + f^3 = 0.$$

It is well-known that $f \in C^\infty(\mathbb{R}^2)$, and therefore that $R(r) = T(r^2)$. One easily checks that $\phi(x,y) = T(x^2 - y^2)$ is a solution to

$$-\phi + \phi_{xx} - \phi_{yy} + \phi^3 = 0,$$

and thus that $e^{it}\phi(x,y)$ is a nontrivial traveling wave of (12.9).

12.2 Non existence of solitary waves for the DSII Davey-Stewartson systems

We consider here the Davey-Stewartson system that appears in deep water which we write for convenience as

$$\begin{cases} iu_t + u_{xx} - u_{yy} = \alpha|u|^2u + \beta u\phi_x, \\ \Delta\phi = \frac{\partial}{\partial x}|u|^2. \end{cases} \tag{12.10}$$

Remark 12.2. We recall that the Davey-Stewartson system (12.10) is integrable by the Inverse Scattering method (see [17, 224, 225, 226, 194] for rigorous results) when

$$\alpha + \frac{\beta}{2} = 0.$$

It is then known as the Davey-Stewartson II (DS II) system. The case $\beta < 0$ corresponds to the *defocusing* DS II, $\beta > 0$ to the *focusing* DS II. We will keep this terminology in the non integrable case.

We consider localized traveling wave solutions of (12.10), namely solutions of the type

$$(e^{i\omega t+\psi(\mathbf{x}-\mathbf{c}t)}U(\mathbf{x}-\mathbf{c}t),\phi(\mathbf{x}-\mathbf{c}t)), \tag{12.11}$$

where $\mathbf{x}=(x,y)\in\mathbb{R}^2, \mathbf{c}=(c_1,c_2)\in\mathbb{R}^2$, $U\in H^1(\mathbb{R}^2)\cap H^2_{\text{loc}}(\mathbb{R}^2), \nabla\phi\in L^2(\mathbb{R}^2)$ and ψ is a linear function and $\omega=\frac{c_1^2}{4}-\frac{c_2^2}{4}$.

The following result is proven in [93].

Theorem 12.2. *The Davey-Stewartson system* (12.10) *can possess a non zero traveling wave solution only if*

$$\textit{(i)}\ \beta<0, \quad \alpha\in(0,-\beta).$$

Moreover, (12.10) *possesses a nontrivial* **radial** *traveling wave solution if and only if*

$$\textit{(ii)}\ \beta<0, \quad \alpha+\tfrac{\beta}{2}=0.$$

Remark 12.3. 1. The case (ii) corresponds to the *integrable* focusing DS II system (which can always be rescaled to $\alpha=1,\beta=-2$). Arkadiev et al. [15] have exhibited a family of explicit traveling waves (the lump solitons) having the profile ($c=0$):

$$u_{\text{lump}}(x,y,t)=\frac{2\bar{\nu}\exp\left(2i\,\text{Im}\,(\lambda z)+4i\,\text{Re}\,(\lambda^2)t\right)}{|z+4i\lambda t+\mu|^2+|\nu|^2},$$

where $z=x+iy$ and λ,ν,μ are arbitrary complex constants. Note that the traveling wave profile $|u(x,y,0)|$ is radial in appropriate variables and that our hypothesis concerning ψ and ω are satisfied by u_{lump}.

Ozawa [192] has independently exhibited a special case ($\lambda{=}\mu{=}0, \nu{=}1$) and used it to construct solutions to the corresponding Cauchy problem which blow up in the L^2 norm using a pseudo-conformal invariance derived in [95].

2. The fact that no traveling waves exist when $\beta>0$ justifies the terminology *defocusing* in this case and the long time dynamics should be governed by scattering. Actually in the defocusing integrable DSII, Sung [226] has proven, as we will see in more details in the next section, that the

Cauchy problem is globally well-posed for arbitrary large initial data in an appropriate functional space and that their sup norm in space decays as $\frac{C}{t}$, that is as the solution of the linear problem. Such a global well-posedness result is unknown for the defocusing nôt integrable DS II system, and also for the non elliptic Schrödinger equation (12.4).

3. In the focusing case $\beta < 0$, (i) and (ii) express that non trivial traveling waves can exist only "in the viciniity" of the integral case, and, for *radiial* ones, only in the integrable case. The computations in [25] suggest actually that *no* traveling waves exist in the focusing non integrable case, $\beta < 0, \alpha + \frac{\beta}{2} \neq 0$.

Proof. We first prove that no non zero traveling wave exists when (i) is violated. Inserting (12.11) into (12.10), we obtain

$$\left\{\begin{array}{l} -(\omega+\psi_x^2-\psi_y^2-c\psi_x-d\psi_y)U+U_{xx}-U_{yy}=\alpha U^3+\beta U\phi_x, \\ \Delta\phi=(U^2)_x, \\ (\psi_{xx}-\psi_{yy})U-(c-2\psi_x)U_x-(d+2\psi_y)U_y=0. \end{array}\right. \tag{12.12}$$

Since ψ is linear, the last equation reduces to

$$(c-2\psi_x)U_x+(d+2\psi_x)U_y=0,$$

that is U is constant on the characteristic lines

$$(d+2\psi_y)x-(c-2\psi_x)y=\text{constant}.$$

This can be satisfied by a H^1 function if and only if

$$\psi_x=c/2, \quad \psi_y=-d/2,$$

reducing the first equation in (12.12) to

$$\left\{\begin{array}{l} U_{xx}-U_{yy}=\alpha U^3+\beta U\phi_x, \\ \Delta\phi=(U^2)_x. \end{array}\right. \tag{12.13}$$

As in the proof of Theorem 12.1, the computations below can be justified by a truncation argument.

We multiply the first equation in (12.13) by xU_x. After several integrations by parts, we get

$$\int(U_x^2+U_y^2)dxdy=\int\left[\frac{\alpha}{2}-\beta x(U^2)_x\phi_x\right]dxdy,$$

that is, by using the second equation in (12.13)

$$\int \left[U_x^2 + U_y^2 - \frac{\alpha}{2}U^4 - \frac{\beta}{2}(\phi_x^2 - \phi_y^2)\right] dxdy = 0. \tag{12.14}$$

In a similar way, the first equation in (12.13) by xU_y and integrating by parts gives

$$\int \left[U_x^2 + U_y^2 + \frac{\alpha}{2}U^4 + \frac{\beta}{2}(\phi_x^2 + 3\phi_y^2)\right] dxdy = 0. \tag{12.15}$$

On the other hand, multiplying the first equation in (12.13) by U and integrating using the second one yields

$$\int [U_x^2 - U_y^2 - \alpha U^4 - \beta|\nabla\phi|^2] dxdy = 0. \tag{12.16}$$

Adding (12.14) and (12.15) gives

$$\int \left[|\nabla U|^2 + \beta\phi_y^2\right] dxdy = 0. \tag{12.17}$$

By adding (12.14) and (12.16) one obtains

$$\int \left[2U_y^2 - \frac{3\alpha}{2}U^4 - \frac{3\beta}{2}\phi_x^2 - \frac{\beta}{2}\phi_y^2\right] dxdy = 0. \tag{12.18}$$

Identities (12.17) and (12.18) imply that

$$\beta < 0, \quad \text{and} \quad \alpha > 0.$$

Substracting (12.14) from (12.15), we then obtain

$$\int \left[\alpha U^4 + \beta(\phi_x^2 + \phi_y^2)\right] dxdy = 0.$$

We now set $r = \widehat{|u|^2}$. By Plancherel theorem and the second equation in (12.13) we obtain

$$\int [\alpha U^4 + \beta(\phi_x^2 + \phi_y^2)] dxdy = \int \left(\alpha + \frac{|\xi_1|^2}{|\xi|^2}\beta\right) |r|^2 d\xi_1 d\xi_2 = 0,$$

where $\xi = (\xi_1, \xi_2)$ is the dual variable of (x,y) and hence

$$\int [\alpha U^4 + \beta(\phi_x^2 + \phi_y^2)]dxdy = \int \left(\alpha + \beta - \frac{\xi_2^2}{|\xi|^2}\beta\right)|r|^2 d\xi_1 d\xi_2 = 0. \tag{12.19}$$

Since $\beta < 0$, it follows that $\alpha + \beta < 0$, achieving to prove (i).

Let us now assume that U is radial. We then rewrite (12.19) as

$$\int \left[\alpha + \frac{\beta}{2} - \frac{\beta}{2}\left(\frac{\xi_2^2 - \xi_1^2}{|\xi|^2}\right)\right]|r|^2 d\xi_1 d\xi_2 = 0. \tag{12.20}$$

Since U is radial, the integral

$$\int \frac{\xi_2^2 - \xi_1^2}{|\xi|^2}|r|^2 d\xi_1 d\xi_2$$

vanishes and (12.20) reduces to

$$\alpha + \frac{\beta}{2} = 0$$

proving (ii).

□

12.3 Remarks on the Cauchy problem for the Davey-Stewartson systems

We will consider here some aspects of the Cauchy problem for the DS-II system (12.10). Inverting the equation for ϕ, (12.10) writes as a single NLS equation with a nonlocal cubic term:

$$iu_t + u_{xx} - u_{yy} = \alpha|u|^2 u + \beta u L(|u|^2), \tag{12.21}$$

where L is defined in Fourier variables by

$$\widehat{Lf}(\xi) = -\frac{\xi_1 \xi_2}{|\xi|^2}, \quad \text{(product of two Riesz transforms)}.$$

Noticing that the Strichartz estimates for the group $e^{i(\partial_x^2 - \partial_y^2)}$ are the same as the usual Schrödinger group $e^{\pm i\Delta}$ ones, this allows (see [92]) to obtain the same local Cauchy theory as for the cubic (focusing or defocusing) standard NLS equation, namely local well-posedness for initial data in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$ (global for small L^2 initial data).

No further results using PDE methods (global well posedness, finite time blow-up, dispersion,...) is known. However very nice rigorous results can be obtained for the *integrable* DS-II system ($\alpha + \frac{\beta}{2} = 0$). We refer to [17, 194, 224, 225, 225, 226, 227] and the references therein. In particular L.Y. Sung [224, 225, 226, 227] has obtained very nice global with decay results. To state his main result, we will follow his notations and write the integrable DS-II equation on the form

$$q_t = 2iq_{x_1x_2} + 16i[L(\pm|q|^2)|q|, \quad q(\cdot,0) = q_0, \tag{12.22}$$

where L is defined as above and the $+$ sign corresponds to the focusing case, the $-$ sign to the defocusing case. The articles [224, 225, 226] are devoted to present a rigorous theory of the Inverse Scattering method for the DS-II equation. In [227] L.Y Sung proves the following

Theorem 12.3. *Let $q_0 \in \mathscr{S}(\mathbb{R}^2)$. Then* (12.22) *possesses a unique global solution u such that the mapping $t \mapsto q(\cdot,t)$ belongs to $C^\infty(\mathbb{R}, \mathscr{S}(\mathbb{R}^2))$ in the two cases:*

(i) Defocusing.

(ii) Focusing and $|\widehat{q_0}|_1 |\widehat{q_0}|_\infty < \frac{\pi^3}{2}\left(\frac{\sqrt{5}-1}{2}\right)^2$.
Moreover, there exists $c_{q_0} > 0$ such that

$$|q(x,t)| \leq \frac{c_{q_0}}{|t|}, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^*.$$

We will not comment on the proof of theorem 12.3 whose techniques are outside the scope of this book. We recall that such a result is unknown for the general *non integrable* DS-II systems, and also for the nonelliptic cubic NLS.

Remark 12.4. 1. Sung obtains in fact the global well-posedness (without the decay rate) in the defocusing case under the assumption that $\hat{q}_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $q_0 \in L^p(\mathbb{R}^2)$ for some $p \in [1,2)$, see [226].

2. Recently, Perry [194] has precised the asymptotics in the defocusing case for initial data in $H^{1,1}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2)$ such that $\nabla f, (1+|\cdot|)f \in L^2(\mathbb{R}^2)\}$, proving that the solution obeys the asymptotic behavior in the $L^\infty(\mathbb{R}^2)$ norm :

$$q(x,t) = u(\cdot,t) + o(t^{-1}),$$

where u is the solution of the linear problem.

On the other hand we recall that Ozawa [192] has constructed a solution of the Cauchy problem in the *focusing integrable* case whose L^2 norm blows up in finite time (the solution converges to a Dirac measure having as mass the L^2 norm of the initial data). The solution persists after the blow-up time and disperses as $t \to \infty$.

This blow-up is carefully studied numerically in [144]. On the other hand, the numerical simulations of [25] suggest that this blow-up does not persist in the non integrable case.

12.4 Solitary waves for KP type equations

12.4.1 Non existence of traveling waves for the KP II type equations

We will illustrate here the defocusing nature of the KP II equation by proving, following [40] that it does not possess any non trivial localized solitary waves. We recall that the KP II equation writes

$$\partial_t u + u\partial_x u + \partial_x^3 u + \partial^{-1}\partial_y^2 u = 0. \tag{12.23}$$

We first recall the natural energy space associated to the KP equations. We set

$$Y = \{u \in L^2(\mathbb{R}^2); \partial_x u \in L^2(\mathbb{R}^2), \partial_x^{-1}\partial_y u \in L^2(\mathbb{R}^2)\},$$

equipped with its natural norm. A solitary wave for (12.23) is a solution of the form $u(x-ct,y)$ with $u \in Y$ and $c > 0$ and it should thus satisfy the equation

$$-cu_x + uu_x + u_{xxx} + \partial_x^{-1}u_{yy} = 0, \quad \text{or} \tag{12.24}$$

$$-cu_{xx} + \left(\frac{u^2}{2}\right)_{xx} + u_{xxxx} + u_{yy} = 0. \tag{12.25}$$

Observe that by the change of scale $\tilde{u}(x,y) = c^{-1}u\left(\frac{x}{c^{1/2}}, \frac{y}{c}\right)$, one can assume (that we will do from now on) that $c = 1$.

Theorem 12.4. *Equation* (12.23) *does not admit a nontrivial solitary wave u such that $u \in Y \cap H^1(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$ with $\partial_x^2 u \in L^2_{loc}(\mathbb{R}^2)$ and $\partial_x^{-1}\partial_y^2 u \in L^2_{loc}(\mathbb{R}^2)$.*

Proof. The extra regularity assumptions in the Theorem are needed to justify the following truncation argument (see also the previous Sections). Let $\chi_0 \in C_0^\infty(\mathbb{R})$, with $0 \le \chi_0 \le 1$, $\chi_0(t) = 1$ if $0 \le |t| \le 1$, $\chi_0(t) = 0$, $|t| \ge 2$. We set $\chi_j = \chi\left(\frac{|\cdot|}{j^2}\right)$, $j = 1, 2, \cdots$.

We multiply (12.24) by $x\chi_j u$ and integrate over $\mathbb{R}^2$ to get (the third integral has to be interpreted as a $H^1 - H^{-1}$ duality):

$$-\int x\chi_j\partial_x\left(\frac{u^2}{2}\right)dxdy + \frac{1}{3}\int x\chi_j\partial_x(u^3)dxdy + \int x\chi_j u\partial_x^3 u \\ + \int x\chi_j(\partial_x^{-1}\partial_y u)udxdy = 0. \tag{12.26}$$

After several integrations by parts, we obtain

$$\frac{1}{2}\int \chi_j u^2 dxdy - \frac{1}{3}\int \chi_j u^3 dxdy + \frac{3}{2}\chi_j u_x^2 dxdy \\ + \frac{1}{2}\int \chi_j(\partial_x^{-1}u_u - y)^2 + \frac{1}{j^2}\int x\chi_j'\left(\frac{r^2}{j^2}\right)u^2 dxdy \\ - \frac{2}{3j}\int x^2\chi_0'\left(\frac{r^2}{j^2}\right)u^2 dxdy - \frac{3}{j^2}\int \chi_0'\left(\frac{r^2}{j^2}\right)u^2 dxdy \\ - \frac{6}{j^4}\int \chi_0''\left(\frac{r^2}{j^2}\right)u^2 dxdy - \frac{6}{j^2}\int x\chi_0''\left(\frac{r^2}{j^4}\right)u^2 dxdy \\ - \frac{4}{j^6}\int x^3\chi_0'''\left(\frac{r^2}{j^2}\right)u^2 dxdy + \frac{3}{j^2}\int x\chi_0'\left(\frac{r^2}{j^2}\right)u_x^2 dxdy \\ - \frac{2}{j^2}\int xy\chi_0'\left(\frac{r^2}{j^2}\right)u(\partial_x^{-1}u_y)dxdy + \frac{1}{j^2}\int x\chi_0'\left(\frac{r^2}{j^2}\right)(\partial_x^{-1}u_y)^2 dxdy = 0, \tag{12.27}$$

where$r^2 = x^2 + y^2$. By Lebesgue dominated convergence theorem we deduce that

$$\int \left[-\frac{1}{2}u^2 + \frac{u^3}{3} - \frac{3}{2}u_x^2 - \frac{1}{2}(\partial_x^{-1}u_y)^2 \right] dxdy = 0. \qquad (12.28)$$

From now on we will proceed formally, he rigorous proof resulting from the same truncation argument as above. We multiply (12.24) by $y(\partial_x^{-1}u_y)$ and integrate (the last two integrals being understood as a $H^1 - H^{-1}$ duality). After several integrations by parts we obtain finally

$$\int \left[\frac{1}{2}u^2 - \frac{1}{6}u^3 + \frac{1}{2}u_x^2 + \frac{1}{2}(\partial_x^{-1}u_y)^2 \right] dxdy = 0. \qquad (12.29)$$

To get the third identity, we remark first that if $u \in Y \cap L^4(\mathbb{R}^2)$ satisfies (12.24) in $\mathfrak{D}'(\mathbb{R}^2)$, and if Y' is the dual of Y, then u satisfies

$$-u + u_{xx} + \frac{u^2}{2} + \partial_x^{-1}u_{yy} = 0 \quad \text{in } Y'$$

where $\partial_x^{-1}u_{yy} \in Y'$ is defined by $\langle \partial_x^{-1}u_{yy}, \psi \rangle_{Y,Y'} = (\partial_x^{-1}u_y, \partial_x^{-1}\psi_y)$ for any $\psi \in Y$. Taking then the $Y - Y'$ duality of the last equation with $u \in Y$, we obtain

$$\int \left[-u^2 + \frac{u^3}{3} - u_x^2 + (\partial_x^{-1}u_y)^2 \right] dxdy = 0. \qquad (12.30)$$

Substracting (12.28) from (12.29) we get

$$\int \left[u^2 - \frac{1}{2}u^3 + 2u_x^2 + (\partial_x^{-1}u_y)^2 \right] dxdy = 0. \qquad (12.31)$$

Adding (12.30) and (12.31) we get

$$\int [u_x^2 + (\partial_x^{-1}u_y)^2] dxdy = 0$$

which achieves the proof.

□

Remark 12.5. 1. A similar proof demonstrates that no non trivial solitary waves exist for the *generalized* KP-II equation, that is uu_x replaced by $u^p u_x, \forall p \in \mathbb{N}$.

2. Theorem 12.4 was reinforced by de Bouard and Martel [39] who proved that the KP-II equation does not possesses L^2- compact solutions, that is L^2 solutions u satisfying:

• there exist $x(t), y(t) : \mathbb{R} \to \mathbb{R}$ such that $\forall \varepsilon > 0$, there exists $R(\varepsilon > 0$ such that, $\forall t \in \mathbb{R}$,

$$\int_{|x|+|y| \geq R(\varepsilon} u^2(x+x(t), y+y(t), t)dxdy \leq \varepsilon.$$

Obviously, any solitary wave solution $\phi(x-ct, y)$ with $\phi \in L^2(\mathbb{R}^2)$ is L^2- compact. The result in [39] can be viewed as a first step for proving the expected conjecture that all bounded solutions of KP-II tend to zero in the sup-norm as $t \to \infty$.

12.4.2 Solitary waves for the KP-I equation

On the other hand, the *focusing nature of the KP-I equation* is revealed by the existence of non trivial localized solitary waves.

First, as a consequence of to its integrability properties, the KP I equation possesses a localized, finite energy, explicit solitary wave, the *lump* [169] :

$$\phi_c(x-ct, y) = \frac{8c(1 - \frac{c}{3}(x-ct)^2 + \frac{c^2}{3}y^2)}{[1 + \frac{c}{3}(x-ct)^2 + \frac{c^2}{3}y^2]^2}. \tag{12.32}$$

Another interesting explicit solitary wave of the KP I equation which is *localized in x and periodic in y* has been found by Zaitsev [246]. It reads

$$Z_c(x, y) = 12\alpha^2 \frac{1 - \beta \cosh(\alpha x) \cos(\delta y)}{[\cosh(\alpha x) - \beta \cos(\delta y)]^2}, \tag{12.33}$$

where

$$(\alpha, \beta) \in (0, \infty) \times (-1, +1),$$

and the propagation speed is given by

$$c = \alpha^2 \frac{4 - \beta^2}{1 - \beta^2}.$$

Let us observe that the transform $\alpha \to i\alpha$, $\delta \to i\delta$, $c \to ic$ produces solutions of the KP I equation which are periodic in x and localized in y.

Second, the existence of *ground states* solutions has been established in [40] for the generalized KP I equations

$$u_t + u_{xxx} - \partial_x^{-1} u_{yy} + u^p u_x = 0, \tag{12.34}$$

when $p < 4, p = \frac{m}{n}, m,n \in \mathbb{N}$, relatively prime, m odd.

In order to define this notion, we set

$$E_{KP}(\psi) = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_x \psi)^2 + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_x^{-1}\partial_y \psi)^2 - \frac{1}{2(p+2)}\int_{\mathbb{R}^2} \psi^{p+2},$$

and we define the action

$$S(N) = E_{KP}(N) + \frac{c}{2}\int_{\mathbb{R}^2} N^2.$$

We term *ground state*, a solitary wave N which minimizes the action S among all finite energy non-constant solitary waves of speed c (see [40] for more details).

It was proven in [40] that ground states exist if and only if $c > 0$ and $1 \le p < 4$. Moreover, when $1 \le p < \frac{4}{3}$, the ground states are minimizers of the Hamiltonian E_{KP} with prescribed mass (L^2 norm) and are thus orbitally stable;

An important related remark is that the anisotropic Sobolev embedding (see [24])

$$\int_{\mathbb{R}^2} |u|^{p+2} dxdy \le C||u||_{L^2}^{\frac{4-p}{2}} ||u_x||_{L^2}^{p} ||\partial_x^{-1} u_y||_{L^2}^{\frac{p}{2}},$$

which is valid for $0 \le p \le 4$, implies that the energy norm $||u||_Y$ is controlled in term of the L^2 norm and of the Hamiltonian

$$\int_{\mathbb{R}^2} \left(\frac{1}{2}u_x^2 + \frac{1}{2}(\partial_x^{-1} u_y)^2 - \frac{|u|^{p+2}}{(p+1)(p+2)} \right) dxdy,$$

if and only if $p < \frac{4}{3}$.

Remark 12.6. When $p = 1$ (the classical KP I equation), it is unknown (but conjectured) if the lump solution is a ground state. Observe however that ground states have the same space asymptotic as the lump ([100]).

More generally, the question of the (possible) multiplicity of solitary waves is largely open. One even does not know whether or not the ground states are unique (up to the obvious symmetries). Such a result is established for the focusing nonlinear Schrödinger equations but the proof uses in a crucial way that the ground states are radial in this case, allowing to use ODE's arguments.

12.5 Transverse stability of the KdV solitary wave

The KP I and KP II equations were introduced to study the *transverse* stability of the KdV 1-soliton (line-soliton). They behave quite differently in this aspect. Roughly speaking, the line-soliton is stable for the *defocusing* KP-II equation and unstable for the *focusing* KP-I equation. In the later case, a *breaking of symmetry* seems to occur.

Zakharov [249] has proven, by exhibiting an explicit perturbation using the integrability, that the KdV 1-soliton is *nonlinearly* unstable for the KP I flow. Rousset and Tzvetkov [199] have given an alternative proof of this result, which does not use the integrability, and which can thus be implemented on other problems (*eg* for nonlinear Schrödinger equations).

The *nature* of this instability is not known (rigorously) but the numerical simulations in [145] suggest a breaking of symmetry, the line-soliton evolving into two-dimensional localized structures.

On the other hand, Mizomachi and Tzvetkov [181] have recently proved the $L^2(\mathbb{R}\times\mathbb{T})$ orbital stability of the KdV 1-soliton for the KP II flow. The perturbation is thus localized in x and periodic in y. The precise result is as follows:

Theorem 12.5. *Let ϕ_c the KdV solitary wave of velocity c. For every $\varepsilon>0$, there exists a $\delta>0$ such that if the initial data of the KP-II Cauchy problem satisfies*
$||u_0-\phi_c]||_{L^2(\mathbb{R}_x\times\mathbb{T}_y)}<\delta$, *the corresponding solution u satisfies*

$$\inf_{\gamma\in\mathbb{R}}||u(x,y,t)-\phi_c(x+\gamma)||_{L^2(\mathbb{R}_x\times\mathbb{T}_y)}<\varepsilon,\quad t\in\mathbb{R}.$$

Moreover, there exists a constant $\tilde{c}$ satisfying $\tilde{c}-c=O(\delta)$ and a modulation parameter $x(t)$ satisfying

$$\lim_{t\to\infty}||u(x,y,t)-\phi_{\tilde{c}}(x-x(t))||_{L^2((x\geq ct)\times\mathbb{T}_y)}=0.$$

The proof involves in particular the global well-posedness of the Cauchy problem for the KP-II equation on a background of a non-localized solution (*eg* the KdV soliton) established in [189] and the Miura transform used in [135] to established the global well-posedness for a modified KP II equation.

Such a result is not known (but expected) for a perturbation which is localized in x and y.

One can similarly consider the transverse stability of the *periodic* KdV solitary wave. To our knowledge no *nonlinear* results are known. On the other hand spectral stability issues have been considered.

Johnson and Zumbrun [121] have derived a (spectral) instability criterion for x- periodic line solitary waves of a class of generalized KP equations with respect to perturbations which are periodic in the direction of propagation x and have long wavelengths in the transverse direction.

Haragus [105] has considered the spectral stability of *small amplitude* x-periodic line solitons of the Classical KP equations. In the KP I case, those solitary waves are spectrally transversally unstable with respect to perturbations which are either periodic in the direction x of propagation, with the same period as the one-dimensional solitary wave, or non-periodic (localized or bounded). In the KP II case, the periodic line solitons are spectrally stable with respect to transverse perturbations which are periodic in the direction of propagation and have long wavelengths in the transverse direction.

It is likely that the spectral instability results in those papers could be obtained by using the simple method in [200]. Also the methods developed in [199] to prove *nonlinear* instability of the localized line-soliton might be useful to treat the periodic case.

Chapter 13

Some models in the modulation regime

We present here results on the Cauchy problem for some systems arising in the modulation regime. The local well-posedness is obtained thanks to the smoothing properties due to the dispersive part.

13.1 The Cauchy problem for the Benney-Roskes system

The Benney-Roskes system has been introduced in Chapter 4, section 4.3. A similar system was derived by Zakharov and Rubenchik [252] as a "universal" model to describe the interaction of spectrally narrow high frequency wave packets with low-frequency acoustic type oscillations.

We present here, following [196] a proof of the local well-posedness of those systems. The case of the *full dispersion* Benney-Roskes systems is studied in [191].

Since we are not looking for existence on long time scales (and thus will not introduce any small parameter), we will write the Benney-Roskes, Zakharov-Kuznetsov in dimension $n = 2, 3$ (of course $n = 2$ for the Benney-Roskes system itself) in the nondimensional form displayed by [193], which uses a reference frame moving at the group velocity of the carrying wave, namely

$$\begin{cases} i\partial_t \psi = -\varepsilon\partial_z^2\psi - \sigma_1\Delta_\perp\psi + (\sigma_2|\psi|^2 + W(\rho + D\partial_z\phi))\psi, \\ \partial_t\rho + \sigma_3\partial_z\phi = \Delta\phi - D\partial_z|\psi|^2, \\ \partial_t\phi + \sigma_3\partial_z\phi = -\dfrac{1}{M}\rho - |\psi|^2. \end{cases} \tag{13.1}$$

Here $\Delta_\perp = \partial_x^2 + \partial_y^2$ or ∂_x^2, $\Delta = \Delta_\perp + \partial_z^2$, $\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the coupling with acoustic type waves, $M \in (0,1)$ is a Mach number, $D \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ is a nondimensional dispersion coefficient.

One first wants to decouple the two last equations in (13.1). We first apply the operator $\partial_t + \sigma_3\partial_z$ to them to obtain

$$\begin{cases} (\partial_t + \sigma_3\partial_z)^2\rho = -(\partial_t + \sigma_3)\Delta\phi - D(\partial_t + \sigma_3\partial_z)\partial_z|\psi|^2, \\ (\partial_t + \sigma_3\partial_z)^2\phi = -\dfrac{1}{M}(\partial_t + \sigma_3\partial_z)\rho - (\partial_t + \sigma_3\partial_z)|\psi|^2, \\ \Delta(\partial_t + \sigma_3\partial_z)\phi = -\dfrac{1}{M}\Delta\phi - \Delta|\psi|^2. \end{cases} \tag{13.2}$$

Therefore

$$\begin{cases} (\partial_t + \sigma_3\partial_z)^2\rho = -\dfrac{1}{M}\Delta\rho + \Delta|\psi|^2 - D(\partial_t + \sigma_3\partial_z)\partial_z|\psi|^2, \\ (\partial_t + \sigma_3\partial_z)^2\phi = \dfrac{1}{M}\Delta\phi + \dfrac{D}{M}\partial_z|\psi|^2 - (\partial_t + \sigma_3\partial_z)|\psi|^2. \end{cases} \tag{13.3}$$

Introducing the following notations

$$\begin{cases} \tilde\rho(x,y,z,t) = \tau_{3,\sigma_3 t}\rho(x,y,z,t) = \rho(x,y,z+\sigma_3 t,t), \\ \tilde\phi(x,y,z,t) = \tau_{3,\sigma_3 t}\phi(x,y,z,t) = \phi(x,y,z+\sigma_3 t,t), \\ \Box_M = \partial_t^2 - \dfrac{1}{M}\Delta, \\ F_1(\psi) = \Delta|\psi|^2 - D(\partial_t + \sigma_3\partial_z)\partial_z|\psi|^2, \\ F_2(\psi) = \dfrac{D}{M}\partial_z|\psi|^2 - (\partial_t + \sigma_3)|\psi|^2, \end{cases} \tag{13.4}$$

we rewrite (13.3) as

$$\begin{cases} \Box_M\tilde\rho = F_1(\psi), \\ \Box_M\tilde\phi = F_2(\psi), \\ \tilde\rho(\mathbf{x}),0) = \rho_0(\mathbf{x}), \\ \partial_t\tilde\rho(\mathbf{x},0) = (\partial_t + \sigma_3\partial_z)\rho(\mathbf{x},0) = -(\Delta\phi + D\partial_z|\psi|^2)(\mathbf{x},0), \\ \tilde\phi(\mathbf{x},0) = \phi_0(\mathbf{x}), \\ \partial_t\tilde\phi(\mathbf{x},0) = (\partial_t + \sigma_3\partial_z)\phi(\mathbf{x},0) = -\left(\dfrac{1}{M}\rho + |\psi|^2\right)(\mathbf{x},0). \end{cases} \tag{13.5}$$

where $\mathbf{x} = (x,y,z)$ if $n = 3$ and $\mathbf{x} = (x,z)$ if $n = 2$.

Next, defining

$$\mathscr{L} = \sigma_1 + \varepsilon\partial_z^2 \quad \text{not necessarily elliptic)}$$

the IVP associated to the system in (13.1) can be expressed as

$$\begin{cases} i\partial_t\psi + \mathscr{L}\psi = \{\sigma_2|\psi|^2 + W(\tau_{3,-\sigma_3}\tilde{\rho} + D\tau_{3,-\sigma_3}\partial_z\tilde{\phi})\}\psi, \\ \square_M\tilde{\rho} = F_1(\psi), \\ \square_M\tilde{\phi} = F_2(\psi), \\ \psi(\mathbf{x},0) = \psi_0(\mathbf{x}), \\ \tilde{\rho}(\mathbf{x}),0) = \rho_0(\mathbf{x}),\ \partial_t\tilde{\rho}(\mathbf{x}),0) = \rho_1(\mathbf{x}) = -(\Delta\phi_0 + D\partial_z|\psi_0|^2)(\mathbf{x}), \\ \tilde{\phi}(\mathbf{x},0) = \phi_0(\mathbf{x}),\ \partial_t\tilde{\phi}(\mathbf{x},0) = \tilde{\phi}_1(\mathbf{x}) = -\left(\dfrac{1}{M}\rho_0 + |\psi_0|^2\right)(\mathbf{x}). \end{cases} \tag{13.6}$$

Since the nonlinear terms F_1, F_2 depend only on ψ, we can express $(\tilde{\rho},\tilde{\phi})$ i terms of $(\rho_0,\rho_1,\phi_0,\phi_1)(\mathbf{x})$ and $\psi(\mathbf{x},t)$,

$$\begin{cases} \tilde{\rho}(t) = U'(t)\rho_0 + U(t)\rho_1 + \displaystyle\int_0^t U(t-t'F_1(\psi)(t')dt', \\ \tilde{\phi}(t) = U'(t)\phi_0 + U(t)\phi_1 + \displaystyle\int_0^t U(t-t')F_2(t')dt', \end{cases} \tag{13.7}$$

where $U(t)$ is the propagator of $\square_M$, that is

$$\begin{cases} U(t)f = M^{1/2}(-\Delta)^{1/2}\sin(M^{-1/2}(-\Delta)^{1/2}t)f, \\ U't()f = \cos(M^{-1/2}(-\Delta)^{1/2}t)f. \end{cases} \tag{13.8}$$

We have the classical estimates

$$\begin{cases} |U(t)f|_2 \le |t|\,|f|_2, \quad |\nabla_{\mathbf{x}}U(t)f|_2 \le M^{1/2}|f|_2, \\ |U'(t)f|_2 \le |f|_2. \end{cases} \tag{13.9}$$

Inserting (13.7) into the first equation of (13.6), we get a single, self-contained differential-integral equation for ψ. However, both $F_1(\psi)$ and $F_2(\psi)$ involve derivatives in the t- variable of ψ. It is thus convenient to remove them by using the following formula which results from integration by parts).

$$\int_0^t U(t-t')\partial_t G(t')dt' = -U(t)G(0) + \int_O^t U'(t-t')G(t')dt'.$$

More precisely, in (13.7) we shall use

$$\int_0^t U(t-t')F_1(\psi)(t')dt' = -U(t)G(0) + \int_0^t U(t-t')(\Delta - D\sigma_3\partial_z^2)|\psi|^2 dt' + DU(t)\partial_z|\psi_0|^2 - \int_0^t U'(t-t')D\partial_z|\psi|^2(t')dt', \tag{13.10}$$

and

$$\int_0^t U(t-t')F_2(\psi)(t')dt' = \int_0^t U(t-t')\left(\left(\frac{D}{M}-\sigma_3\right)\partial_z|\psi|^2(t')\right)dt' + U(t)|\psi_0|^2 - \int_0^t U'(t-t')|\psi|^2(t')dt'. \tag{13.11}$$

Collecting the informations above we see that the IVP associated to the system in (13.1) is formally equivalent to the following IVP for a scalar equation in ψ

$$\begin{cases} \partial_t\psi = i\mathscr{L}\psi - iH(\rho_0,\phi_0,\psi) \\ \psi(\cdot,0) = \psi_0. \end{cases} \tag{13.12}$$

and to its integral version

$$\psi(t) = e^{it\mathscr{L}}\psi_0 - i\int_0^t e^{i(t-t')\mathscr{L}} H(\rho_0,\phi_0,\psi)(t')dt', \tag{13.13}$$

where the nonlinear term $H(\cdot)$ is given by the lengthy expression

$$\begin{aligned} H(\rho_0,\phi_0,\psi)(t') &= \sigma_2|\psi|^2\psi(t') \\ &+ W\psi(t')\tau_{3,-\sigma_3 t'}\{U(t')\rho_0 + U(t')(\rho_1 + 2D\partial_z|\psi_0|^2)\} \\ &+ WD\psi(t')\tau_{3,-\sigma_3 t'}\{\partial_z U(t')\phi_0 + \partial_z U(t')\phi_1\} \\ &+ W\psi(t')\tau_{3,-\sigma_3 t'}\left\{\int_0^{t'} U(t'-t'')(\Delta - D\sigma_3\partial_z 2)\psi|^2(t'')dt''\right\} \\ &- 2W\psi(t')\tau_{3,-\sigma_3 t'}\left\{\int_0^{t'} U'(t'-t'''')D\partial_z|\psi|^2(t'')dt''''\right\} \\ & WD\psi(t')\tau_{3,-\sigma_3 t'}\left\{\int_0^{t'} U(t'-t'')\left(\frac{D}{M}-\sigma_3\right)\partial_z^2|\psi|^2(t'')dt''\right\} \\ &= H_1+H_2+H_3+H_4+H_5+H_6, \end{aligned} \tag{13.14}$$

with $H_j = H_j(\psi), j = 1,...,6$.

Thus equation (13.12) and its integral version (13.13) can be seen as a nonlinear Schrödinger equation where the dispersion is given by a non degenerate constant coefficients second order operator $\mathscr{L}$ which is not necessary elliptic, and with a nonlinear term H involving nonlocal terms and derivatives of the unknown ψ. The order of derivatives appearing in H is, roughly, one, since $\square^{-1}\partial_z^2$ is basically an operator of order one.

In the case where H depends only on ψ and $\nabla\psi$, the corresponding local well-posedness theory is due to Chihara [52] in the elliptic case $\mathscr{L} = \Delta$ and to Kenig-Ponce-Vega [142] for general $\mathscr{L}$.

Due to the specific nature of (13.12) (and (13.13)) we shall combine the structure of the nonlinearity H with estimates involving the smoothing effect, in its homogeneous and inhomegeneous versions, associated to the unitary group $\{e^{it\mathscr{L}}\}_{-\infty}^{\infty}$ as well as special properties of solutions to the wave equation to obtain the desired local existence theory.

We will first state the estimates for the group $\{e^{it\mathscr{L}}\}_{-\infty}^{\infty}$ that we will use in the proof of the main theorem. We introduce the following notations.

$\{Q_\mu\}_{\mu\in\mathbb{Z}^n}$ is a family of unit cubes paralel to the coordinates axis with disjoint interiors covering $\mathbb{R}^n$. We introduce the norms

$$||\cdot||_{l^1_\mu L^2_T L^2_x} \equiv \sup_{\mu\in\mathbb{Z}^n} ||\cdot||_{L^2(Q_\mu\times[0,T])} \equiv ||\cdot||_{l^\infty_\mu(L^2(Q_\mu\times[0,T]))},$$

and

$$||\cdot||_{l^1_\mu L^2_T L^2_x} \equiv \sum_{\mu\in\mathbb{Z}^n} ||\cdot||_{L^2(Q_\mu\times[0,T])} \equiv ||\cdot||_{l^1_\mu(L^2(Q_\mu\times[0,T]))}.$$

In general

$$||F||_{l^r_\mu L^p_T L^q_x} = \left(\sum_{\mu\in\mathbb{Z}^n}\left(\int_0^T\left(\int_{Q_\mu}|F(x,t)|^q dx\right)^{p/q} dt\right)^{r/p}\right)^{1/p}.$$

We now state the needed smoothing estimates.

Proposition 13.1.

$$||I_x^{1/2} e^{it\mathscr{L}} f||_{l^\infty_\mu L^2_T L^2_x} \leq c|f|_2 \tag{13.15}$$

$$\sup_{0\leq t\leq T} |I_x^{1/2}\int_0^t e^{i(t-t')\mathscr{L}} G(t')dt'|_2 \leq c||G||_{l^1_\mu L^2_T L^2_x}, \tag{13.16}$$

and

$$||\nabla_x \int_0^t e^{i(t-t')\mathscr{L}} G(t')dt'||_{l^\infty_\mu L^2_T L^2_x} \leq c||G||_{l^1\mu L^2_T L^2_x}, \tag{13.17}$$

where $I_x^{1/2} f = \mathscr{F}^{-1}(|\xi|^{1/2}\hat{f})$ *and* c *is a constant independent of* T.

Proof. The estimate sm1 was proved in [60, 216, 238]. The estimate (13.16) follows from the dual version of (13.15) and the group properties of $\{e^{it\mathscr{L}}\}_{-\infty}^{\infty}$. Finally, (13.17) was proved in [**?**].

□

We will also use the following version of the Sobolev inequality whose proof can be found in [196].

Let $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a smooth function such that for each $t \in [0,T]$, $f(\cdot,t) \in \mathscr{S}(\mathbb{R}^n)$. Let $J = (1-\Delta)^{1/2}$.

Lemma 13.2. . *Let* $p_1 \in [1,\infty)$ *and* $p_2 \in [p_1,\infty]$. *Let* $s \geq \frac{n}{p_1} - \frac{n}{p_2}$ *and* $s > \frac{n}{p_1}$ *if* $p_2 = \infty$. *Then for any* $r,q \in [1,\infty]$

$$||f||_{l^r_\mu L^q_T L^{p_2}_x} \leq c||J^s f||_{l^r_\mu L^q_T L^{p_1}_x}. \tag{13.18}$$

We remark that for $f : \mathbb{R}^n \to \mathbb{R}$, (13.18) tells us that

$$||f||_{l^r_\mu L^{p_2}_x} \leq c||J^s f||_{l^r_\mu L^{p_1}_x} \tag{13.19}$$

If s is a positive integer, this results from the Calderón extension theorem [47].

We will use in the proof of our main theorem the following inequalities on the wave propagator.

Lemma 13.3. *With* $U(t)$ *defined as above, we have in the* 3*-dimensional case*

$$|U(t')f||_{l^2\mu L^\infty_T L^2_x} \leq c(1+TM^{1/2})^3|f|_2, \tag{13.20}$$

$$||U(t)\nabla_x f||_{l^2_\mu L^\infty_T L^2_x} \leq cM^{1/2}(1+TM^{1/2})^3|f|_2 \tag{13.21}$$

$$||\nabla_x \int_0^t U(t-t')h(t')dt'||_{l^2_\mu L^\infty_T L^2_x} \leq cM^{1/2}(1+TM^{1/2})^3||h||_{l^2_\mu L^1_T L^2_x}, \tag{13.22}$$

and

$$||\int_0^t U'(t-t')h(t')dt'||_{l^2_\mu L^\infty_T L^2_x} \leq c(I+TM^{1/2})^3||h||_{l^2_\mu L^1_T L^2_x}. \tag{13.23}$$

Proof. To prove (13.3), we observe that $w(x,t) = U'(t)f(x)$ solves the IVP

$$\begin{cases} \Box_M w = \partial_t^2 w - \frac{1}{M}\Delta w = 0, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = 0. \end{cases} \tag{13.24}$$

The finite propagation speed of the solution implies that the values of w on $Q_\mu \times [0,T]$ depend on the values of f on $(1+M^{1/2})Q_\mu$, where λQ_μ denotes the cube with the same center asQ_μ and side λ. By a cutting off argument and (13.9) one has that

$$\sup_{[0,T]} ||w(\cdot,t)||_{L^2(Q_\mu)} \leq c||f||_{L^2((1+M^{1/2}T)Q_\mu}. \tag{13.25}$$

Hence, adding in μ and counting the cubes one obtains the desired result

$$||U'(t)f||_{l^2_\mu L^\infty_T L^2_x} \leq c(1+M^{1/2}T)^3|f|_2. \tag{13.26}$$

The proofs of (13.20)-(13.21) are similar and they will be omitted.

To prove (13.22) we observe that $v(x,t) = \int_0^t U(t-t')h(t')dt'$ solves the IVP

$$\begin{cases} \Box_M v = \partial_t^2 v - \frac{1}{M}\Delta w = h(x,t), \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ v(x,0) = 0, \\ \partial_t v(x,0) = 0. \end{cases} \tag{13.27}$$

The values of v on $Q_\mu \times [0,T]$ depend on those of h on $(1+M^{1/2}T)Q_\mu \times [0,T]$. By combining a cutting off argument and the standard energy estimate it follows that

$$\sup_{[0,T]} ||\nabla_x v(\cdot,t)||_{L^2(Q_\mu)} \leq M^{1/2} \int_0^T ||h(t)||_{L^2((1+M^{1/2}T)Q_\mu)} dt. \tag{13.28}$$

By adding in μ we get (13.22). A similar argument proves (13.23).

□

We now state our main result on the local well-posedness of Benney-Roskes type systems.

Theorem 13.4. *Let $s > n/2$, $n = 2,3$. Then given $(\psi - 0, \rho_1, \phi_0) \in H^s \times H^{s-1/2} \times H^{s+1/2}(\mathbb{R}^n)$, there exist $T = T(||\psi_0||_{H^s}; ||\rho_0||_{H^{s-1/2}}; ||\phi||_{H^{s-1/2}}) > 0$ and a unique solution $\psi(\cdot)$ of the integral equation* (13.13) *such that*

$$\psi \in C([0,T]; H^s(\mathbb{R}^n)) \tag{13.29}$$

with

$$||J^{s+1/2}\psi||_{l^\infty_\mu L^2_T L^2_x} < \infty. \tag{13.30}$$

Moreover, the map $(\psi_0, \rho_0, \phi_0) \mapsto \psi(t)$ from $H^s \times H^{s-1/2} \times H^{s+1/2}$ into the class (13.29)-(13.30) *is locally lipschitz.*

Finally, from (13.29)-(13.30) *one has*

$$(\rho, \phi) \in C([0,T]; H^{s-1/2}(\mathbb{R}^n) \times H^{s+1/2}(\mathbb{R}^n)). \tag{13.31}$$

Proof. To simplify the exposition we shall restrict to the case $n = 3$ and $s = 2 + 1/2$. The proof for general values of s follows by combining the argument below with the calculus of inequalities involving fractional derivatives deduced in [139].

For $(\psi_0, \rho_0, \phi_0) \in H^{5/2} \times H^2 \times H^3(\mathbb{R}^3)$ fixed we define the operator $\Phi(\omega)$ as

$$\Phi(\omega)(t) = e^{it\mathscr{L}}\psi_0 - i\int_0^t e^{i(t-t')\mathscr{L}} H(\rho_0, \phi_0, \omega)(t')dt', \tag{13.32}$$

with ω in the function space X^a_T, meaning that

$$\omega \in C([0,T]; H^s(\mathbb{R}^3)), \quad |||\omega|||_T = \sup_{0\le t\le T} ||\omega()||_{H^{5/2}} + \sum_{|\alpha|=3} ||\partial_x^\alpha \omega||_{l^\infty_\mu L^2_T L^2_x} \le a. \tag{13.33}$$

We remark that when $s' + 1/2 \in \mathbb{Z}^+$,

$$\sup_{0\le t\le T} ||\cdot||_{H^{s'}} + ||J^{s'+1/2}\cdot||_{l^\infty_\mu L^2_T L^2_x} \sim \sup_{0\le t\le T} ||\cdot||_{H^{s'}} + \sum_{|\alpha|=s'+1/2} ||\partial_x^\alpha \cdot||_{l^\infty_\mu L^2_T L^2_x}. \tag{13.34}$$

This is consequence of the results in [139].

From Proposition 13.1 one has

$$
\begin{aligned}
|||\Phi(\omega)|||_T &= \sup_{0\le t\le T} ||\Phi(\omega)(t)||_{H^{5/2}} + \sum_{|\alpha|=3} ||\partial_x^\alpha \Phi(\omega)||_{l^\infty_\mu L^2_T L^2_x} \\
&\le c||\psi_0||_{H^{5/2}} + \int_0^T ||H_1(\omega(t))||_{H^{5/2}} dt + \sum_{j=2}^{6} \int_0^T |H_j(\omega)(t))|_2 dt \\
&+ \sum_{j=2}^{6} \sum_{|\alpha|=2} ||\partial_x^\alpha H_j(\omega(t))||_{l^1_\mu L^2_T L^2_x}.
\end{aligned}
\tag{13.35}
$$

Since $H^s(\mathbb{R}^n), s > n/2$, is a muliplicative algebra we get

$$
\int_0^T ||H_1(\omega(t))||_{H^{5/2}} dt = \int_0^T || \, |\omega|^2 \omega||_{H^{5/2}}(t) dt \le T \sup_{0\le t\le T} ||\omega!t)||^3_{H^{5/2}}.
\tag{13.36}
$$

Next, we estimate $H_2(\omega)$ in the $L^\infty_T L^2_x$ norm. Thus

$$
\begin{aligned}
&\int_0^T |W\omega(t)\tau_{3,-\sigma_3 t}\{U'(t)\rho_0 + U(t)(\rho_1 + 2D\partial_z|\psi_0|^2)\}|_2 dt \\
&\le WT \sup_{0\le t\le T} [||\omega(t)||_{L^\infty_x} |\tau_{3,-\sigma_3 t}\{U'(t)\rho_0 + U(t)(\rho_1 + 2D\partial_z|\psi_0|^2 i\{|_2] \\
&\le WT \sup_{0\le t\le T} [||\omega(t)||_{L^\infty_x} |U'(t)\rho_0 + U(t)(\rho_1 + 2|D|\partial_z|\psi_0|^2)|_2] \\
&= WT \sup_{0\le t\le T} [||\omega(t)||_{L^\infty_x} (|\rho_0|_2 + T|\rho_1|_2 + 2|D|M^{1/2}| \, |\psi_0|^2 \, |_2)] \\
&\le cWT(|\rho_0|_2 + T|\rho_1|_2 + 2|D|M^{1/2}||\psi_0||^2_{H^{3/2+}}) \sup_{0\le t\le T} ||\omega(t)||_{H^{3/2+}}.
\end{aligned}
\tag{13.37}
$$

To estimate $H_2(\omega)$ in $\sum_{|\alpha|=2} ||\partial_x^\alpha \cdot ||_{l^1_\mu L^2_T L^2_x}$, we notice that an interpolation argument of Gagliardo-Nirenberg type allows to bound $||\nabla_x \omega||_{l^2 \mu L^2 T L^6_x}$ and $||\Delta\omega||_{l^6_\mu L^2_T L^3_x}$ in terms of $\sum_{|\beta|=3} ||\partial_x^\beta \omega||_{l^\infty L^2_T L^2_x} (\le |||\omega|||_T)$ and $||\omega||_{l^2_\mu L^2_T L^\infty_x} (\le cT^{1/2}|||\omega|||_T)$. Then, up to a multiplicative constant involving a power of T, all these terms can be bounded by $|||\omega|||_T$.

Now, we combine (13.17), Hölder inequality, Lemma 13.3 (inequalities

(13.20)-(13.21)) and the interpolation argument commented above to get

$$\sum_{|\alpha|=2} ||\partial_x^{\alpha} H_2(\omega)||_{l^1 \mu L_T^2 L_x^2}$$
$$= cW \sum_{|\alpha|=2} ||\partial_x^{\alpha}(\omega\tau_{\cdot,\cdot}\{U'(t)\rho_0 + U(t)(\rho_1 + 2D)\partial_z|\psi_0|^2)\}||_{l_\mu^1 L_T^2 L_x^2}$$
$$\leq cW \sum_{|\gamma|+|\beta|=2} ||\partial_x^{\gamma}\omega\tau\cdot,\cdot\partial_x^{\beta}(U'(t)\rho_0 + U(t)(\rho_1 + 2|D|\partial_z|\psi_0|^2)\}||_{l_\mu^1 L_T^2 L_x^2}$$
$$\leq cW \sum|\gamma| = 2||\partial_x^{\gamma}\omega||_{l_\mu^2 L_T^2 L_x^2}||\tau_{\cdot,\cdot}\{U'(t)\rho_0 + U(t)(\rho_1 + 2D\partial_z|\psi_0|^2)\}||_{l_\mu^2 L_T^\infty L_x^6}$$
$$+ cW \sum_{|\gamma|=1} ||\partial_x^{\gamma}\omega||_{l_\mu^2 L_T^2 L_x^6} \times$$
$$\sum_{|\beta|=2} ||\tau_{3,-\sigma_3 t}\partial_x^{\beta}\{U'(t)\rho_0 + U(t)(\rho_1 + 2|D|\partial_z|\psi_0|^2)\}||_{l_\mu^2 L_T^\infty L_x^2}$$
$$\leq cWT^{1/2} \sup_{0\leq t\leq T} ||\omega(t)||_{H^{5/2}} \times$$
$$(1 + |\sigma_3|T) \sum_{|\beta|\leq 2} ||\partial_x^{\beta}\{U'(t)\rho_0 + U(t)(\rho_1 + 2|D|\partial_z|\psi_0|^2)\}||_{l_\mu^2 L_T^\infty L_x^2}$$
$$\leq cWT^{1/2} \sup_{0\leq t\leq T} ||\omega(t)||_{H^{5/2}}(1 + |\sigma_3|T) \times$$
$$(1 + M^{1/2})(1 + TM^{1/2})^2\{||\rho_0||_{H^2} + ||\rho_1||_{H^1} + 2|D|||\psi_0||_{H^2}^2. \tag{13.38}$$

The bound for $H_3(\omega)$ is similar to that deduced above for $H_2(\omega)$. We turn thus to the estimate of $H_4(\omega)$. First from (13.9) we have

$$\int_0^T |H_4(\omega(t))|_2 dt$$
$$= \int_0^T \left| W\omega(t)\tau_{3,-\sigma_3}\left(\int_0^t U(t-t')(\Delta - D\sigma_3\partial_z^2)|\omega|^2(t')dt'\right)\right|_2 dt$$
$$\leq cWT \sup_{0\leq t\leq T} |\omega(t)|_\infty \int_0^T |U(t-t')(\Delta - D\sigma_3\partial_z^2)|\omega|^2|_2 dt' \tag{13.39}$$
$$\leq cWT \sup_{0\leq t\leq T} ||\omega(t)||_{H^{3/2+}} \times$$
$$(1 + |D|T) \sup_{0\leq t\leq T} |(-\Delta)^{-1/2}(\Delta - D\sigma - 3\partial_z^2)|\omega|^2(t)|_2$$
$$\leq cW(1 + |D|)T^2(1 + |\sigma_3|)M^{1/2} \sup_{0\leq t\leq T} ||\omega(t)||_{H^{3/2+}}^3.$$

For the other piece of $H_4(\omega)$ we write

$$
\begin{aligned}
&\sum_{|\alpha|=2} ||\partial_x^\alpha H_4(\omega)||_{l^1_\mu L^2_T L^2_x} \\
&= cW \sum_{|\alpha|=2} ||\partial_x^\alpha (\omega(t)\tau_{\cdot,\cdot} \left(\int_0^t U(t-t')(\Delta - D\sigma_3\partial_z^2)|\omega|^2(t')dt'\right) ||_{l^1\mu L^2_T L^2_x} \\
&\leq cW \sum_{|\gamma=2} ||\partial_x^\gamma \omega||_{l^2\mu L^2_T L^2_x} \times \\
&||\tau_{3,-\sigma_3 t} \left(\int_0^t U(t-t')(\Delta - D\sigma_3\partial_z^2)|\omega|^2(t')dt'\right) ||_{l^2_\mu L^\infty_T L^\infty_x} \\
&+ cW \sum_{|\gamma|=1} ||\partial_x^\gamma \omega||_{l^2_\mu L^2_T L^4_x} \times \\
&\sum_{|\beta|=1} ||\tau_{3,-\sigma_3 t} \left(\int_0^t U(t-t')\partial_x^\beta (\Delta - D\sigma_3\partial_z^2)|\omega|^2(t')dt'\right) ||_{l^2_\mu L^\infty_T L^4_x} \\
&+ cW ||\omega||_{l^2_\mu L^2_T L^\infty_x} \times \\
&\sum_{|\beta|=2} ||\tau 3, -3\sigma_3 t \left(\int_0^t U(t-t')\partial_x^\beta (\Delta - D\sigma_3\partial_z^2)|\omega|^2(t')dt'\right) ||_{l^2_\mu L^\infty_x L^2_x}.
\end{aligned}
\tag{13.40}
$$

Combining (13.18) (13.22), Hölder inequality and the interpolation argument used in (13.38) we can bound the last term in (13.40) by

$$
\begin{aligned}
&cWT^{1/2} \sup_{0\leq t\leq T} ||\omega(t)||_{H^2}(1+|\sigma_3|T)\times \\
&M^{1/2}(1+TM^{1/2})^3 \sum_{1\leq|\beta|\leq 3} ||\partial_x^\beta |\omega|^2||_{l^{2+}\mu L^1_T L^2_x} \\
&\leq cWT(1+T)(1+|\sigma_3|T))M^{1/2})^3 |||\omega|||_T.
\end{aligned}
\tag{13.41}
$$

The estimates for $H_5(\omega)$ and $H_6(\omega)$ are similar to those deduced above for $H_4(\omega)$. Thus, collecting the previous estimates it follows that

$$|||\Phi(\omega)|||_T \equiv \sup_{0\le t\le T} ||\Phi(\omega)(t)||_{H^{5/2}} + \sum_{|\alpha|=3} ||\partial_x^\alpha \Phi(\omega)||_{l_\mu^\infty L_T^2 L_x^2}$$
$$\le c||\psi_0||_{H^{5/2}} + cT|\sigma_2|\, |||\omega|||_T^3$$
$$+ c(|W|+|D|)T^{1/2}(1+M^{1/2})(1+|\sigma_3|T)(1+TM^{1/2})^3 \times$$
$$\{||\rho_0||_{H^2} + ||\rho_1||_{H^1} + ||\phi_0||_{H^3} + ||\phi_1||_{H^2} + ||\psi_0||_{H^2}^2\}|||\omega|||_T$$
$$+ c(1+|W|+|D|)T(1+T)(1+|\sigma_3|)(1+M^{1/2})(1+TM^{1/2})^3|||\omega|||_T. \tag{13.42}$$

The above inequality guarantees that for fixed values of the parameters $\varepsilon, \sigma_1, \sigma_2, \sigma_3, W, D$ and M, and for a given data $(\Psi_0, \rho_0, \phi_0) \in H^{5/2} \times H^2 \times H^3(\mathbb{R}^3))$, the operator $\Phi(\omega) = \Phi_{(\psi_0,\rho_0,\phi_0)}$ defined as the left-hand side of (13.12), with ω instead of ψ, define a contraction map in X_T^a with $a = 2c||\psi_0||_{H^{5/2}}$, T small enough depending on the value of the parameters and on $||\psi_0||_{H^{5/2}}, ||rho_0||_{H^2}$, and $||\phi_0||_{H^3}$. Therefore the map has a unique fixed point $\psi \in X_T^a$ that is $\Phi(\psi) = \psi$, which solves the integral equation (13.13). Moreover the contraction principle tells us that the map $(\psi_0, \rho_0, \phi_0) \mapsto \psi(t)$ from $H^{5/2} \times H^2 \times H^3$ into X_T^a is locally Lipschitz.

Finally we observe that the condition

$$(\psi_0, \rho_0, \phi_0) \in H^{5/2} \times H^2 \times H^3, \quad (H^s \times H^{s-1/2} \times H^{s+1/2},\ s > n/2)$$

implies that

$$\partial_t \tilde{\rho}(x,0) = \rho_1(x,0) = -\Delta\phi_0 + D\partial_z|\psi_0|^2 \in H^1, \quad (H^{s-1/2})$$

and

$$\partial_t \tilde{\phi}(x,0) = \phi_1(x,0) = -(\frac{1}{M}\rho_0 + |\psi_0|^2) \in H^2, \quad (H^{s+1/2}).$$

Since the arguments above show that

$$(F-1, F-2) \in L^1([0,T]; H^1 \times H^2), \quad (H^{s-3/2} \times H^{s-1/2})$$

we have that

$$(\rho, \phi) \in C([0,T]; H^2 \times H^3), \quad (H^{s-1/2} \times H^{s+1/2}).$$

No global well-posedness, nor finite time blow-up are known for Benney-Roskes type systems. However, it turns out that [193] has a Hamiltonian structure, as noticed in [251] where it is derived from a general Hamiltonian description of the interaction of short and long waves. Working directly on the system [193] one obtains the following conservation laws.

□

Proposition 13.5. *Let* (ψ,ρ,ϕ) *be a solution of* (13.1) *obtained in Theorem 13.4, defined in the time interval* $[0,T]$. *Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|\psi(x,t)|^2dx=0,\quad 0\le t\le T, \tag{13.43}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\{\varepsilon|\partial_z\psi|^2+\sigma-1|\nabla_\perp\psi|^2+\frac{W}{2M}\rho^2+\frac{W}{2}|\nabla\phi|^2+\sigma_3W\rho\partial_z\phi\\ +\frac{\sigma_2}{2}|\psi|^4+W\rho|\psi|^2+DW|\psi|^2\partial_z\phi\}dx=0,\quad 0\le t\le T.\end{aligned} \tag{13.44}$$

Proof. To obtain (13.43) we multiply the first equation in (13.1) by $\bar\psi$, integrate the result and take its imaginary part (if $s<2$ the integrals have to be understood as a H^s-H^{-s} duality).

To obtain (13.44) we proceed formally. A rigorous proof can be obtained by smoothing the initial data and passing to the appropriate limit.

We multiply the first equation in (13.1) by $\partial_t\bar\psi$, integrate the result and take its real part to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}t_{\mathbb{R}^n}[\varepsilon|\partial_z\psi|^2+\sigma_1|\nabla_\perp\psi|^2+\frac{\sigma_2}{2}|\psi|^4]dx+\operatorname{Re}W\int_{\mathbb{R}^n}\rho\psi\partial_t\bar\psi dx\\ +\operatorname{Re}W\int_{\mathbb{R}^n}\rho\psi\partial_t\bar\psi dx+\operatorname{Re}WD\int_{\mathbb{R}^n}\partial_z\phi\psi\partial_t\bar\psi dx=0.\end{aligned} \tag{13.45}$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}[\varepsilon|\partial_z\psi|^2+\sigma_1|\nabla_\perp\psi|^2+\frac{\sigma_2}{2}|\psi|^4+W\rho|\psi|^2+DW\partial_z\phi|\psi|^2]dx\\ -\frac{W}{2}\int_{\mathbb{R}^n}\partial_t\rho|\psi|^2dx+\frac{DW}{2}\int_{\mathbb{R}^n}\partial_t\phi\partial_z|\psi|^2dx=0.\end{aligned} \tag{13.46}$$

Using the third equation in (13.1) leads to

$$-\frac{W}{2}\int_{\mathbb{R}^n}\partial_t\rho|\psi|^2dx = \frac{W}{2}\int_{\mathbb{R}^n}\partial_t\phi\partial_t\rho dx + \frac{\sigma_3 W}{2}\int_{\mathbb{R}^n}\partial_z\rho\partial_t\phi dx + \frac{W}{4}\frac{d}{dt}\int_{\mathbb{R}^n}|\nabla\phi|^2dx. \tag{13.47}$$

Similarly, the second equation in [193] leads to

$$\frac{DW}{2}\int_{\mathbb{R}^n}\partial_t\phi\partial_z|\psi|^2dx = -\frac{W}{2}\int_{\mathbb{R}^n}\partial_t\phi\partial_t\phi dx - \frac{\sigma_3 W}{2}\int_{\mathbb{R}^n}\partial_z\rho\partial_t\phi dx + \frac{W}{4}\int_{\mathbb{R}^n}|\nabla\phi|^2dx. \tag{13.48}$$

Combining (13.45)-(13.48) we obtain (13.44). □

Proposition 13.5 implies the existence of *global* weak solutions of (13.1) for some range of the parameters. Namely

Theorem 13.6. *Assume that*

$$\varepsilon > 0, \quad \sigma_1 = \sigma_2 = 1,$$

and that the quadratic form

$$Q(x,y,z) = \frac{W}{2M}x^2 + \frac{W}{2}y^2 + \frac{1}{2}z^2 + \sigma_3 Wxy + DWyz + Wxz$$

is positive definite.

Then for any $(\psi_0,\phi_0,\rho_0) \in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, *there exists a global weak solution* (ψ,ϕ,ρ) *of* (13.1) *such that*

$$\begin{aligned}&\psi,\phi \in L^\infty((0,+\infty);\, H^1(\mathbb{R}^n)), \quad \rho \in L^\infty((0,+\infty;\, L^2(\mathbb{R}^n)),\\ &\partial_t\psi,\partial_t\rho \in L^\infty((0,+\infty;\, H^{-1}(\mathbb{R}^n)), \quad \partial_t\phi \in L^\infty((0,+\infty;\, L^2(\mathbb{R}^n)).\end{aligned} \tag{13.49}$$

Proof. We use a compactness method. We approximate (ψ_0,ϕ_0,ρ_0) by smooth functions $(\psi_{0,\varepsilon},\phi_{0,\varepsilon},\rho_{0,\varepsilon})$ and obtain local approximate solutions $(\psi_\varepsilon,\phi_\varepsilon,\rho_\varepsilon)(x,t)$ on some interval $[0,T_\varepsilon]$, thanks to Theorem 13.4. The assumptions in Theorem 13.6 and Proposition 13.5 imply that $(\psi_{\varepsilon,\phi_\varepsilon,\rho_(}\varepsilon)$ is bounded independently of ε in the space

$$L^\infty((0,+\infty;\, H^1)\times L^\infty((0,+\infty;\, H^1)\times L^\infty((0,+\infty;\, L^2).$$

One deduces that $(\partial_t \psi_\varepsilon, \partial_t \phi_\varepsilon, \partial_t \rho_\varepsilon)$ is bounded in $L^\infty((0,+\infty;\, H^{-1}) \times L^\infty((0,+\infty;\, L^2) \times L^\infty((0,+\infty;\, H^{-1})$.

Hence, up to a subsequence one can assume that

$$\begin{aligned} \psi_\varepsilon &\rightharpoonup \psi \quad \text{in} \quad L^\infty((0,+\infty;\, H^1(\mathbb{R}^n)) \quad \text{weak}^* \\ \phi_\varepsilon &\rightharpoonup \phi \quad \text{in} \quad L^\infty((0,+\infty;\, H^1(\mathbb{R}^n)) \quad \text{weak}^* \\ \rho_\varepsilon &\rightharpoonup \rho \quad \text{in} \quad L^\infty((0,+\infty;\, L^2(\mathbb{R}^n)) \quad \text{weak}^*. \end{aligned} \tag{13.50}$$

By Aubin-Lions lemma one can furthermore assume (up to a subsequence) that

$$\psi_\varepsilon \to \psi \quad \text{in} \quad L^p_{\text{loc}}(|0,+\infty);\, L^q_{\text{loc}}(\mathbb{R}^n)), \quad \text{and a.e. in} \quad [0,+\infty) \times \mathbb{R}^n,$$

for any p, $2 \le p < \infty$ and q, $2 \le q < 6$ if $n = 3$ (resp. q, with $2 \le q < \infty$ if $n = 2$). Similarly for $\phi_\varepsilon, \rho_\varepsilon$.

These convergences allow to pass to the limit in the distribution sense in (13.1) written for $(\psi_{\varepsilon,\phi_\varepsilon,\rho_(\varepsilon})$, proving that (ψ, ρ, ϕ) satisfies (13.1) in $\mathscr{D}(\mathbb{R}^+ \times \mathbb{R}^n)$, and actually in $L^\infty((0,+\infty;\, H^{-1}) \times L^\infty((0,+\infty;\, H^{-1}) \times L^\infty((0,+\infty;\, L^2))$.

The initial makes sense since by Strauss lemma,

$$(\psi, \phi, \rho) \in C_{\text{w}}([0,\infty);\, H^1) \times C_{\text{w}}([0,\infty);\, H^1) \times C_{\text{w}}([0,\infty);\, L^2).$$

□

13.2 The Cauchy problem for the Dysthe equation

Dysthe type equations [77, 109, 78] arise when one proceeds to the next order in the expansion leading to NLS type equations in the modulation regime. We are not aware of a rigorous derivation but the method sketched in Chapter 4 would lead to one.

Recall that the nonlinear Schrödinger equation is derived in the modulational regime when the wave steepness is small, $ka \ll 1$, and the bandwith is narrow, $|\Delta \mathbf{k}/k \ll 1$, [1] both of the same order of magnitude $O(\varepsilon)$, the nonlinear and dispersive terms being of order $O(\varepsilon^3)$. Recall that the deep water

[1] We have denoted by $\mathbf{k}$ the wave number of the carrying wave and $k = |\mathbf{k}|$.

case corresponds to $(kh)^{-1} = O(\varepsilon)$, the infinite depth to $(kh)^{-1} = 0$. The Dysthe type systems are obtained when the expansion is carried out one step further $O(\varepsilon^4)$.

To start with we consider the one-dimensional case. One gets (see for instance [168]) the following system coupling the complex envelope A of the wave and the potential ϕ of the induced mean current

$$
\begin{aligned}
&\frac{\partial A}{\partial t} + \frac{\omega}{2k}\frac{\partial A}{\partial x} + i\frac{\omega}{8k^2}\frac{\partial^2 A}{\partial x^2} + \frac{i}{2}\omega k^2 |A|^2 A \\
&\quad - \frac{1}{16}\frac{\omega}{k^3}\frac{\partial^3 A}{\partial x^3} - \frac{\omega k}{4}A^2\frac{\partial \bar{A}}{\partial x} + \frac{3}{2}\omega k|A|^2\frac{\partial A}{\partial x} + ikA\frac{\partial \phi}{\partial x}_{|z=0} = 0, \\
&\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (-h < z < 0), \\
&\frac{\partial \phi}{\partial z} = \frac{\omega}{2}\frac{\partial |A|^2}{\partial x} \quad (z = 0), \\
&\frac{\partial \phi}{\partial z} = 0 \quad (z = -h).
\end{aligned}
\tag{13.51}
$$

The equation for ϕ is easily solved by taking the Fourier transform in x and we find that

$$\frac{\partial \phi}{\partial x}_{|z=0} = \mathscr{L}(|A|^2)$$

where the nonlocal operator $\mathscr{L}$ is defined in Fourier variables by

$$\widehat{\mathscr{L}f}(\xi) = -\frac{\omega}{2}\xi \coth(h\xi)\hat{f}(\xi).$$

Note that for $h = +\infty$, $\mathscr{L}$ is given by

$\widehat{\mathscr{L}f}(\xi) = -\frac{\omega}{2}|\xi|\hat{f}(\xi$, that is $\mathscr{L} = \frac{\omega}{2}\mathscr{H}\partial_x$ where $\mathscr{H}$ is the Hilbert transform.

We thus can write (13.51) as a single equation:

$$
\begin{aligned}
&\frac{\partial A}{\partial t} + \frac{\omega}{2k}\frac{\partial A}{\partial x} + i\frac{\omega}{8k^2}\frac{\partial^2 A}{\partial x^2} + \frac{i}{2}\omega k^2|A|^2 A - \frac{1}{16}\frac{\omega}{k^3}\frac{\partial^3 A}{\partial x^3} - \frac{\omega k}{4}A^2\frac{\partial \bar{A}}{\partial x} \\
&\quad + \frac{3}{2}\omega k|A|^2\frac{\partial A}{\partial x} + ikA\mathscr{L}(|A|^2) = 0.
\end{aligned}
\tag{13.52}
$$

This equation is reminiscent of the cubic KdV equation

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{13.53}$$

In fact by eliminating the transport term in (13.52) by the change of variable $X = x - \frac{\omega}{2k}t$ (we will keep the notation x for the spatial variable), then writing, with $\alpha = -\frac{1}{16}\frac{\omega}{k^3}$ and $\beta = \frac{\omega}{8k^2}$

$$\left(\xi + \frac{\beta}{3\alpha}\right)^3 = \xi^3 + \frac{\beta}{\alpha}\xi^2 + \frac{\beta^2\xi}{3\alpha^2} + \frac{\beta^3}{27\alpha^3},$$

an easy computation shows that the fundamental solution of the linearization of (13.52) can be expressed as

$$\mathfrak{A}(x,t) = \frac{1}{(t\alpha)^{1/3}} \exp\left(\frac{2it\beta^3}{27\alpha^2}\right) \exp\left(\frac{-i\beta x}{2\alpha^2}\right) \int_{-\infty}^{\infty} e^{i\xi^3} e^{i\xi(\alpha t)^{-1/3}(x - \frac{t\beta^2}{3\alpha})} d\xi,$$

that is

$$\mathfrak{A}(x,t) = \frac{1}{(t\alpha)^{1/3}} \exp\left(\frac{2it\beta^3}{27\alpha^2}\right) \exp\left(\frac{-i\beta x}{2\alpha^2}\right) \mathrm{Ai}\left(\frac{1}{t^{1/3}\alpha^{1/3}}(x - \frac{\beta^2}{3\alpha}t)\right),$$

where we have used here as definition of the Airy function

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} e^{i(\xi^3 + iz\xi)} d\xi.$$

If follows that the dispersive estimates for $\mathfrak{A}$ are essentially the same as those of the linearized KdV equation and one obtains for (13.52) the same results as for the cubic KdV equation (13.53), that is (see [139], that is local well-posedness of the Cauchy problem in $H^s(\mathbb{R})$, $s \geq \frac{1}{4}$.

In the two-dimensional case, the mean flow potential is solution of the system (we follow the notations of [221]).

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (-h < z < 0),\\
&\frac{\partial\phi}{\partial z} = \frac{\omega}{2}\frac{\partial |A|^2}{\partial x} \quad (z = 0),\\
&\frac{\partial\phi}{\partial z} = 0 \quad (z = -h).
\end{aligned} \tag{13.54}$$

In the infinite depth case one obtains at the fourth order in ε the following *higher order nonlinear Schrödinger equation* for the slow variation of the wave envelope (where surface tension can be included) (see [109]) [2]

$$2i\left(\frac{\partial A}{\partial t}+c_g\frac{\partial A}{\partial x}\right)+p\frac{\partial^2 A}{\partial x^2}+q\frac{\partial^2 A}{\partial y^2}-\gamma|A|^2A=-is\frac{\partial^3 A}{\partial x\partial y^2}-ir\frac{\partial^3 A}{\partial x^3}$$
$$-iuA^2\frac{\partial \bar{A}}{\partial x}+iv|A|^2\frac{\partial A}{\partial x}+A\mathscr{R}_1\frac{\partial}{\partial x}|A|^2 \tag{13.55}$$

where $\mathscr{R}_1$ is the Riesz transform defined by $\widehat{\mathscr{R}_1\psi}(\xi)=i\frac{\xi_1}{|\xi|}\hat{\psi}(\xi)$.

The linear part is a third order dispersive equation of type

$$u_t+iP(D)u=0, \tag{13.56}$$

where p is a real polynomial of degree three in two variables.

A classification for all possible pointwise decay estimates leading to Strichartz estimates for the corresponding fundamental solution

$$G(x,t)=\int_{\mathbb{R}}^2 e^{itp(\xi)+ix\cdot\xi}d\xi$$

are given in [18].

We describe briefly below, following [148], how to obtain the local well-posedness of the Cauchy problem for a class of equations comprising (13.55). We consider thus the Cauchy problem

$$\begin{cases} iu_t+P(D)u=f(u,\partial_\nu u) & \text{in } \mathbb{R}^2\times\mathbb{R},\\ u((x,0)=u_0(x) & \text{for } x\in\mathbb{R}.\end{cases} \tag{13.57}$$

We make the following hypothesis on P,f. P is a differential operator with a real symbol of degree three $p(\xi)$ and there exist $c1,c_2>0$ such that

$$|\nabla p(\xi)|\geq c_1|\xi|^2-c_2, \quad \forall\xi\in\mathbb{R}^2. \tag{13.58}$$

Remark 13.1. It is straightforward to check that (13.58) is satisfied for the Dysthe type systems above.

[2] In the deep water case one obtains by solving (13.54) a system similar to (13.55) where $\mathscr{R}_1$ is a replaced by a zeroth order nonlocal operator with a inhomogeneous symbol (see the one-dimensional case above). It has essentially the same mathematical properties and one obtains the same local well-posedness result as in the infinite depth case .

Concerning the nonlinear part, ν is a unit vector in $\mathbb{R}^2$, to which we associates the Riesz transform $\mathscr{R}_\nu$ defined by

$$\widehat{\mathscr{R}_\nu f}(\xi) = i\frac{\xi\cdot\nu}{|\xi|}\hat{f}(\xi).$$

For real or complex constants $a_j, j = 0,1,2,3$, we define

$$f(u,\partial_\nu) = a_0|u|^2u + a_1u^2\partial_\nu\bar{u} + a_2|u|^2\partial_\nu u + a_3u\partial_\nu\mathscr{R}_\nu|u|^2. \tag{13.59}$$

Note that when

$$a_0, a_3 \in \mathbb{R}, \quad a_1, a_2 \in i\mathbb{R}, \tag{13.60}$$

the L^2 norm is formally conserved by (13.57). This condition is satisfied in the case of Dysthe type systems.

As previously mentioned we will rely on smoothing properties of the linear equation

$$\begin{cases} iu_t + P(D)u = g & \text{in } \mathbb{R}^2\times\mathbb{R}, \\ u((x,0) = u_0(x) & \text{for } x\in\mathbb{R}. \end{cases} \tag{13.61}$$

We now introduce related functional spaces. For $x\in\mathbb{R}^2$ we set

$$\tilde{Q}_x = \{y\in\mathbb{R}^2, \max_j |x_j - y_j| \le \frac{1}{2}\}$$

and for $T > 0$,

$$Q_{x,T} = \tilde{Q}_x \times [0,T].$$

The local smoothing estimate is as follows. [3]

Proposition 13.7. *Suppose that p satisfies* (13.58)*. Let* $T > 0$*. We have*

$$\begin{aligned} &\sup_x ||(1-\Delta)^{\frac{1}{4}}u||_{L^2(Q_{x,T})} + \sup_{0\le t\le T} ||u(t)||_{L^2} \\ &\qquad \le c\left(||u_0||_{L^2} + \sum_{k\in\mathbb{Z}^2} ||(1-\Delta)^{-\frac{1}{4}}f||_{L^2(Q_{k,T})}\right). \end{aligned} \tag{13.62}$$

[3]This result generalizes easily to polynomials of degree $\kappa \ge 3$, see [148]

Proof. We refer to [148] for the proof that consists in reducing to a one dimensional estimate. A key observation is the elementary lemma

□

Lemma 13.8. *Let $p(\sigma)$ be a real polynomial of degree κ. Then for all real $\varepsilon \neq 0$*

$$\left| \frac{1}{2i\pi} \int_{\mathbb{R}} \frac{e^{is\sigma} p'(\sigma)}{p(\sigma) - i\varepsilon} d\sigma \right| \leq \kappa.$$

The following spaces are natural ones in view of the smoothing property.

Definition 13.9. We fix a smooth compactly supported function ϕ which is identically zero on $\bar{Q}_0$.

The spaces $X_{t_0} \subset C([0,t_0]; H^{3/2}(\mathbb{R}^2))$ and Y_0 are defined through their norms

$$||u||_{X_{t_0}} = \sup_{0 \leq t \leq t_0} ||u||_{H^{3/2}(\mathbb{R}^2)} + \sup_{x \in \mathbb{R}^2} ||(1-\Delta)_x^{5/4} u||_{L^2(Q_{x,t_0})}$$
$$+ t_0^{-1/2} \left(\sum_{k \in \mathbb{Z}^2} \sup_{0 \leq t \leq t_0} (||\phi(\cdot - k)u(t)||^2_{H^{1/2}(\mathbb{R}^2)} - ||\phi(\cdot - k)u(0)||^2_{H^{1/2}(\mathbb{R}^2)}) \right)^{1/2} \tag{13.63}$$

and

$$||f||_{Y_{t_0}} = \inf_{f_1 + f_2 = f} \{ \sum_{k \in \mathbb{Z}^2} ||(1-\Delta)^{-1/4} f_1||_{L^2(Q_{k,t_0})} + \int_0^{t_0} ||f_2(\tau)||_{H^{3/2}(\mathbb{R}^2)} d\tau \}.$$

Before stating the main result we prove a consequence of the local smoothing estimates.

Lemma 13.10. *Suppose that $|\nabla p(\xi)| \sim |\xi|^2$ for large ξ. Let $u_0 \in H^{3/2}(\mathbb{R}^2)$ and u the solution of*

$$iu_t - P(D)u = f \text{ in } \mathbb{R}^2 \times (0,t_0), \quad u(\cdot,0) = u_0. \tag{13.64}$$

Then

$$||u||_{X_{t_0}} \lesssim ||u_0||_{H^{3/2}(\mathbb{R}^2)} + ||f||_{Y_{t_0}}.$$

Proof. Proposition 13.7 implies that

$$\begin{aligned}&\sup_{0\le t\le t_0}||u||_{H^1}+\sup_{x\in\mathbb{R}^2}||D_x^2u||_{L^2(Q_{x,t_0})}\\&\lesssim|D_xu_0|_2+\inf_{f_1+f_2=f}\left(\sum_{k\in\mathbb{Z}^2}||f_1||_{L^2(Q_{k,t_0})}+\int_0^{t_0}|f_2(t)|_2dt\right).\end{aligned}$$

We apply $(1-\Delta)^{1/4}$ to (13.64) and obtain

$$\sup_{0\le t\le t_0}||u||_{H^{3/2}}+\sup_{x\in\mathbb{R}^2}||(1-\Delta)^{5/4}u||_{L^2(Q_{x,t_0})}\lesssim||u_0||_{H^{3/2}(\mathbb{R}^2)}+||f||_{Y_{t_0}}. \tag{13.65}$$

We will use an elementary estimate (which actually is also used in the proof of the local smoothing). See [148] for the proof.

□

Lemma 13.11. *Let*

$$i\partial_t v-P(D)v=f$$

where p is a real polynomial of degree κ in n variables. Let $\phi\in C_0^\kappa(\mathbb{R}^n)$ be supported in $B_R(x)$. Then

$$\left||\phi v(t)|_2^2-|\phi v(0)|_2^2\right|\le 4t\int_0^t|\phi f(s)|_2^2ds+c\int_0^t||v(s)||_{H^{\frac{\kappa-1}{2}}(B_R(x))}ds$$

We now combine (13.65) with Lemma 13.11 where we choose $\phi\in C_0^\infty(\mathbb{R}^2)$, identically 1 in $[-1,1|]^2$ and supported in $[-2,2]^2$ to obtain

$$\sum_{k\in\mathbb{Z}^2}\left||\phi(x-k)u(t)|_2^2-|\phi(x-k)u(0)|_2^2\right|\lesssim\int_0^t|f(s)|_2^2ds+\int_0^t||u||^2_{H^1(\mathbb{R}^n)}ds.$$

We apply this estimate to $(1-\Delta)^{1/4}u$. The commutators $[(1-\Delta)^{1/4},\phi]$ are of order $-1/2$. Their kernels decay fast and the corresponding terms can be easily controlled. Hence

$$\begin{aligned}&t_0^{-1}\sum_{k\in\mathbb{Z}^2}\left|||\phi(x-k)u(t)||^2_{H^{1/2}(\mathbb{R}^2)}-||\phi(x-k)u(0)||^2_{H^{1/2}(\mathbb{R}^2)}\right|\\&\lesssim\int_0^t||f(s)||^2_{H^{1/2}(\mathbb{R}^2)}ds+\sup_{0\le t\le t_0}||u(t)||^2_{H^{1/2}(\mathbb{R}^2)}.\end{aligned} \tag{13.66}$$

The main result of this Section is the

Theorem 13.12. *Suppose that p satisfies* (13.58) *and that f is of the form* (13.59). *Given $c > 1$ there exists $t_0 \sim c^{-2}$ such that for $u_0 \in H^{3/2}(\mathbb{R}^2)$ with $||u_0||_{H^{3/2}} \leq c$ there exists a unique solution $u \in X_{t_0}$ to* (13.57). *Moreover, the map $u_0 \mapsto u$ is analytic from $H^{\frac{3}{2}}(\mathbb{R}^2)$ into X_{t_0}. When* (13.60) *is satisfied (and thus in the case of Dysthe systems), one has furthermore $|u(\cdot,t)|_2 = |u_0|_2$.*

Proof. We will merely sketch it, refering to [148] for details. Assuming f as in (13.59) wwe can write it on the form

$$f(u) = F[u,u,u]$$

with

$$F[u,v,w] = \frac{1}{6}(f(u+v+w)+f(u)+f(v) + f(w)-f(u+v)-f(u+w)-f(v+w)),$$

where F is symmetric and linear independently in each argument. A typical term is

$$F[u,v,w] = u\partial_y \mathscr{R}_y(vw) + v\mathscr{R}_y(wu) + w\partial_y \mathscr{R}_y(uv). \tag{13.67}$$

Let $w_0 \in H^4(\mathbb{R})$ to be precised below and let w be the solution to

$$iw_t - P(D)w = 0, \quad u(x,0) = w_0.$$

Instead of u we search by a fixed point argument the solution to

$$iv_t - P(D)v = f(v+w), \quad v(\cdot,0) = u_0 - w_0$$

Thanks to the unitarity of the group $e^{itP(D)}$ in L^2 based Sobolev spaces and to Sobolev embedding,

$$|D_x^2 w|_{L^\infty(\mathbb{R}\times\mathbb{R}^2)} \lesssim ||w_0||_{H^4(\mathbb{R}^2)}.$$

Let $c_0 = ||u_0||_{H^{3/2}}$. We choose ε small and $w_0 \in H^4(\mathbb{R}^2)$ with $||u_0 - w_0||_{H^{1/2}} \leq \varepsilon$, $||u_0 - w_0||_{H^{3/2}} \lesssim 2c_0$ and $||w_0||_{H^4(\mathbb{R}^2)} \lesssim c_0^{5/2}\varepsilon^{-3/2}$. The result will follow from Lemma 13.10 and from the following estimate on the non-linear term. We assume that f is of the form (13.59), that F is related to f as above and that $t_0 \leq 1$.

□

Lemma 13.13.

$$||F[u,v,w]||_{Y_{t_0}} \lesssim \left(|u(0)||_{H^{1/2}} + t_0^{1/2}||u||_{X_{t_0}}\right)\left(||v(0)||_{H^{1/2}} + t_0^{1/2}||v||_{X_{t_0}}\right)$$
$$\left(|v(0)||_{H^{1/2}} + t_0^{1/2}||v||_{X_{t_0}}\right)\left(||w(0)||_{H^{1/2}} + t_0^{1/2}||w||_{X_{t_0}}\right)$$
$$\left(|w(0)||_{H^{1/2}} + t_0^{1/2}||w||_{X_{t_0}}\right)\left(||u(0)||_{H^{1/2}} + t_0^{1/2}||u||_{X_{t_0}}\right). \tag{13.68}$$

Proof. It suffices to prove that

$$||f(u)||_{Y_{t_0}} \leq c||u(0)||^2_{H^{1/2}} + t_0||u||^2_{X_{t_0}}. \tag{13.69}$$

More precisely, by the Sobolev embedding

$$|h(s)||_{H^{1/2}(\tilde{Q}_x)} \lesssim ||h(s)||_{W^{1,4/3}(\tilde{Q}_x)},$$

we have to prove that

$$\sum_{k\in\mathbb{Z}^2} ||\,|g| + |\nabla g|\,||_{L^{4/3}(Q_k)} \leq c\left(||u(0)||_{H^{1/2}} + t_0^{1/2}||u||^2_{X_{t_0}}\right)^2 ||u||_{X_{t_0}}$$

where g is one of the following terms

$$u^3, u^2\partial_\nu u, u\partial_\nu \mathscr{R}_\nu |u|^2.$$

The treatment of $u\partial_\nu \mathscr{R}_\nu |u|^2$ is typical and contains all the difficulties. For clarity we consider first the local term $\nabla(u^2 Du)$. By Hölder inequality

$$||\nabla(u^2(s)Du(s)||_{L^{4/3}(\tilde{Q}_x)} \lesssim ||u(s)||_{L^8(\tilde{Q}_x)}||D^2u(s)||_{L^2(\tilde{Q}_x} + ||u(s)||_{L^8(\tilde{Q}_x)}||\nabla u||^2_{L^{16/5}(\tilde{Q}_x)}$$

and by interpolation,

$$||\nabla u||^2 L^{16/5}(\tilde{Q}_x) \lesssim ||u||_{L^8(\tilde{Q}_x)} + ||D^2u||_{L^2(\tilde{Q}_x)}.$$

Hence

$$\begin{aligned}
&\int_0^{t_0} ||\nabla(u^2(s)Du(s))||^2_{L^{4/3}(\tilde{Q}_x)} ds \\
&\lesssim \int_0^{t_0} ||u(s)||^4_{L^8(\tilde{Q}_x)} \left(||u(s)||^2_{L^2(\tilde{Q}_x)} + ||D^2u(s)||^2_{L^2(\tilde{Q}_x)}\right) ds \\
&\leq c_1 \int_0^{t_0} ||u(s)||^4_{H^{1/2}(\tilde{Q}_x)}||u(s)||^2_{H^{1/2}(\tilde{Q}_x)} ds \\
&\leq c_2 \sup_{0\leq t\leq t_0} ||(1-\Delta)^{1/4}u(t)||_{L^2(\tilde{Q}_x)}||(1-\Delta)^{5/4}u||_{L^2(Q_{x,t_0})}.
\end{aligned} \tag{13.70}$$

and by the definition of the spaces X_{t_0},

$$\sum_{k\in\mathbb{Z}^2}\left\{\int_0^t ||u^2(s)Du(s)||^2_{H^{1/2}(\tilde{Q}_x)}ds\right\}$$
$$\leq \left(||(1-\Delta)^{1/4}u(0)||_{L^2}+t_0^{1/2}||u||_{X_{t_0}}\right)^2||u||_{X_{t_0}}.$$

We now turn to the more difficult term

$$\sum_{k\in\mathbb{Z}^2}||\nabla(u\partial\nu\mathscr{R}_\nu|u|^2)||_{L^2([0,t_0];L^{4/3}(Q_k))}.$$

There are two terms to control

$$\sum_{k\in\mathbb{Z}^2}||(\nabla u)\partial_\nu\mathscr{R}_\nu|u|^2||_{L^2([0,t_0];L^{4/3}(Q_k))} \tag{13.71}$$

and

$$\sum_{k\in\mathbb{Z}^2}||u\nabla\partial_\nu\mathscr{R}_\nu|u|^2||_{L^2([0,t_0];L^{4/3}(Q_k))}. \tag{13.72}$$

By Hölder inequality for fixed t

$$||(\nabla u)\partial_\nu\mathscr{R}_\nu|u|^2||_{L^{4/3}(Q_k)}\lesssim||\nabla u||_{L^{16/5}(Q_k)}||\partial_\nu\mathscr{R}_\nu|u|^2||_{L^{16/7}(Q_k)}.$$

The kernel $K(x,y)$ of $\partial_\nu\mathscr{R}_\nu$ decays fast for $|x-y|\to\infty$, as

$$|K(x,y)|\leq c||x-y|^{-3}$$

and we estimate

$$||\partial_\nu\mathscr{R}_\nu|u|^2||_{L^{16/7}(Q_k)}\lesssim\sum_{\bar{k}}(1+|k-\bar{k}|)^{-3}\sup_k||u||^2_{L^2(Q_k)}$$
$$+\sum_{|k-\bar{k}|\leq 2}||\partial_\nu|u|^2||_{L^{16/7}(Q_{\bar{k}})}.$$

Proceeding as above, one can similarly bound the term

$$\sum_k||\nabla(u\partial_\nu\mathscr{R}_\nu|u|^2)||_{L^2([0,t_0],L^{4/3}(Q_k))}$$

and this complete the proof of Lemma 13.13.

□

We can now complete the proof of Theorem 13.12 using Lemmas 13.13 and 13.64.

Firstly we obtain

$$||w||_{X_{t_0}} \leq c(n)||w_0||_{H^{3/2}} \leq c(n)c_0.$$

Let $J : X_{t_0} \to X_{t_0}, \quad J(\tilde{v}) = v$ be defined as the solution to

$$v_t + P(D)v = f(w+\tilde{v}), \quad v(0) = u_0 - w_0.$$

Then by Lemmas 13.13 and 13.64

$$||v||_{X_{t_0}} \lesssim c_0 + \Big(||w_0||_{H^{1/2}} + ||u_0 - w_0||_{H^{1/2}} + t_0^{1/2}(||w_0||_{H^{3/2}} + ||\tilde{v}||_{X_{t_0}})\Big)^2 \\ \times (||w_0||_{H^{3/2}} + ||\tilde{v}||_{X_{t_0}} \lesssim c_0 + t_0(c_0 + ||\tilde{v}||_{X_{t_0}})^3. \tag{13.73}$$

We shall see that this map is a contraction, after posibly decreasing t_0. The lower bound on the life span follows from (13.73).

Let for $j = 1,2$

$$v_t^j + P(D)v^j = f(w+\tilde{v}^j), \quad v^j(0) = u_0 - w_0.$$

We expand the trilinear term

$$f(w+\tilde{v}^2) - f(w+\tilde{v}^1) = 3F[w,w,\tilde{v}^2 - \tilde{v}^1] + 3F[w,\tilde{v}^2 + \tilde{v}^1, \tilde{v}^2 - \tilde{v}^1] \\ + F[\tilde{v}^2 + \tilde{v}^1, \tilde{v}^2 + \tilde{v}^1, \tilde{v}^2 - \tilde{v}^1].$$

It is straightforward to estimate

$$||F[w(t),w(t),\tilde{v}^2 - \tilde{v}^1]||_{L^1(H^{3/2}} \lesssim t_0^{1/2}||w_0||_{H^4}^2 ||\tilde{v}^2 - \tilde{v}^1||_{X_{t_0}}$$

and hence

$$||v^2 - v^1||_{X_{t_0}} \leq \gamma ||\tilde{v}^2 - \tilde{v}^1||_{X_{t_0}}$$

where γ can be chosen such that

$$\gamma \sim t_0^{1/2}||w_0||_{H^4}^2 + (||w_0||_{H^{1/2}} + t_0^{1/2}||w||_{X_{t_0}} + \mu)\mu \lesssim t_0^{1/2}c_0^5\varepsilon^{-3} + (c_0 + \mu)\mu$$

with

$$\mu = \varepsilon + t_0^{1/2}(||\tilde{v}^2||_{X_{t_0}} + ||\tilde{v}^1||_{X_{t_0}}).$$

Suppose that $||\tilde{v}^j||_{X_{t_0}} \leq R$ with $R > c_0$. Then

$$||v^2 - v^1||_{X_{t_0}} \leq \frac{1}{2}||\tilde{v}^2 - \tilde{v}^1||_{X_{t_0}}$$

provided $\varepsilon R \ll 1$, $t_0^{1/2}R^2 \ll 1$, $t_O^{1/2}(c_0+1) \ll 1$ and $t_0 \ll c_0^{-10}\varepsilon^6$.

Given R we can satisfy all these inequalities. Let $R_0 = ||J(0)||_{X_{t_0}}$, $R \geq 2R_0$ and ε, t_0 as above. Then

$$||J(\tilde{v})||_{X_{t_0}} \leq R_0 + \frac{1}{2}R \leq R$$

if $||\tilde{v}||_{X_{t_0}} \leq ||\tilde{v}(0)||_{H^{1/2}(\mathbb{R}^2)} \leq \varepsilon$.

Now J maps this ball into itself and it is a contraction. The same argument gives uniqueness of solutions in that class. Finally we may reinterpret the considerations above as an application of the implicit function theorem with analytic nonlinearities, which implies analytic dependence on the initial data.

Remark 13.2. By similar arguments one could establish the local well-posedness of (13.57) for $n = 3$ in the cubic case for initial data in $H^2(\mathbb{R}^2)$ provided ε_1 and ε_2 are nonzero.

Remark 13.3. The result above was obtained by using only the local smoothing esffect or the associated linear propagator. It is likely that it could be improved by using other dispersive estimates (Strichartz, maximal function,...).

Chapter 14

Some internal waves system

14.1 The Shallow-Water/Shallow-Water system

We will consider in this Chapter some internal wave systems, mainly the shallow-water/shallow-water system. We will also make some comments of possible extensions of the systems introduced in Chapter 5 : surface tension effects; upper free surface; higher order systems.

The system that arises in the shallow-water/shallow-water regime for internal waves with a rigid lid (see Chapter 5) is studied here in some details. it can be seen as the counterpart of the Saint-Venant system for surface waves. It is thus of "hyperbolic" nature and does not possess a dispersive term. As for the Green-Naghdi system for surface waves, the dispersive effects appear when performing the expansion of the Dirichlet-Neumann operator to the next order (see below).

We will see that the rigid lid assumption introduces in the two-dimensional case a non locality which is not present in the case of a upper free surface.

We will study here the well-posedness of the Cauchy problem associated to (5.21) following [104]. The method of proof is essentially an hyperbolic one with some technical difficulties due to the nonlocal terms.

Since in the regime under study both ε and ε_2 are $O(1)$ quantities, we will take for simplicity $\varepsilon = 1$ (and thus $\varepsilon_2 = \delta$).

Introducing the operator

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{1}{\gamma+\delta}\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\delta\zeta](h_2\mathbf{u}),$$

one can rewrite (5.21) under the form

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(h_1\mathfrak{R}[\zeta]\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{1}{2}\nabla\Big(\big|\mathbf{v}-\gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) = 0. \end{cases} \tag{14.1}$$

Note also that when $d = 1$, one has $\Pi = 1$ and

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{h_2}{\delta h_1 + \gamma h_2}\mathbf{u},$$

so that (14.1) writes then

$$\begin{cases} \partial_t\zeta + \partial_x\big(\frac{h_1h_2}{\delta h_1+\gamma h_2}\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{1}{2}\partial_x\big(\frac{(\delta h_1)^2-\gamma h_2^2}{(\delta h_1+\gamma h_2)^2}\mathbf{v}^2\big) = 0, \end{cases} \tag{14.2}$$

Let the function $f(\zeta)$ be given by

$$f(\zeta) = \frac{h_1h_2}{\delta h_1+\gamma h_2}. \tag{14.3}$$

One easily checks that (14.2) can thus be recast under the conservative form

$$\begin{cases} \partial_t\zeta + \partial_x\big(f(\zeta)\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{1}{2}\partial_x\big(f'(\zeta)\mathbf{v}^2\big) = 0. \end{cases} \tag{14.4}$$

Remark 14.1. One derive from (14.4) the Rankine-Hugoniot condition for a piecewise C^1 solution

$$\begin{cases} (\zeta_+ - \zeta_-)n_t + [f(\zeta_+)\mathbf{v}_+ - f(\zeta_-)\mathbf{v}_-]n_x = 0, \\ (\mathbf{v}_+ - \mathbf{v}_-)n_t + (1-\gamma)(\zeta_+ - \zeta_-)n_x \\ \qquad +\frac{1}{2}[\big(f'(\zeta_+)\mathbf{v}_+^2\big) - \big(f'(\zeta_-)\mathbf{v}_-^2\big)]n_x = 0, \end{cases} \tag{14.5}$$

along the surfaces of discontinuity. A mathematical study of weak solution to (14.2) has not been performed yet, but Bouchut and Zeitlin recently managed to handle numerically shock waves for a similar type of system [43].

14.1.1 The one dimensional case

It is a tedious but simple computation to check that the SW/SW equations (14.2) can be put under the “quasilinear” form :

$$\partial_t U + A(U)\partial_x U = 0, \qquad U = (\zeta, \mathbf{v})^T, \tag{14.6}$$

with

$$A(U) = \begin{pmatrix} a(U) & b(\zeta) \\ c(U) & a(U) \end{pmatrix}$$

and

$$a(U) = f'(\zeta)\mathbf{v}, \quad b(\zeta) = f(\zeta), \quad c(U) = (1-\gamma) + \frac{1}{2} f''(\zeta)\mathbf{v}^2, \tag{14.7}$$

and where the function $f(\zeta)$ was defined in (14.3). Actually both formulations are equivalent.

A simple computation shows that (14.6) is strictly hyperbolic provided that

$$\begin{cases} \inf\limits_{\mathbb{R}}(1-\zeta) > 0, \\ \inf\limits_{\mathbb{R}}(1+\delta\zeta) > 0, \\ \inf\limits_{\mathbb{R}} \big[1 - \gamma(1 + \delta \dfrac{(1+\delta)^2}{(\delta+\gamma-\delta(1-\gamma)\zeta)^3}\mathbf{v}^2) \big] > 0. \end{cases} \tag{14.8}$$

The two first conditions (no cavitation) are the exact counterparts of the hyperbolicity condition for the classical Saint-Venant system. The third condition can be seen as a “trace” of the possible Kelvin-Helmholtz instabilities in the two-layer system.

The following theorem follows directly from standard results on hyperbolic systems but we will give the plan of the proof as a guideline for the much more difficult two-dimensional case.

Theorem 14.1. *Let $\delta > 0$ and $\gamma \in [0,1)$. Let also $t_0 > 1/2$, $s \geq t_0 + 1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R})^2$ be such that (14.8) is satisfied. Then*

- *There exists $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0,T_{max}); H^s(\mathbb{R})^2)$ to (14.2) satisfying (14.8) on $[0,T_{max})$ and with initial condition U^0;*

- *This solution satisfies the conservation of energy on $[0,T_{max})$:*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(1-\gamma)\zeta^2 + v^2 f(\zeta)]dx = 0,$$

  with $f(\zeta)$ as in (14.3).

- *If $T_{max} < \infty$ then $\lim_{t\to T_{max}} |U(t)|_{W^{1,\infty}} = \infty$ or one of the three conditions of (14.8) ceases to be true as $t \to T_{max}$.*

Proof. As previously mention we only indicate the steps (see [104] for more details).

Step 1. Construction of a regularized system of equations by truncating the high frequencies. Let χ be a smooth, even, compactly supported function defined over $\mathbb{R}$ and with values in $[0,\infty)$, and equal to 1 in a neighborhood of the origin. For all $\iota > 0$, we define the operator χ_ι as

$$\chi_\iota = \chi(\iota D);$$

the operator χ_ι is thus a smoothing operator mapping continuously H^s into H^r for all $s, r \in \mathbb{R}$. The regularization of (14.2) is then defined as

$$\partial_t U^\iota + \chi_\iota\big(A(U^\iota)\chi_\iota(\partial_x U^\iota)\big) = 0. \tag{14.9}$$

Since U^0 satisfies (14.8), the mapping $U \mapsto \chi_\iota(A(U)\chi_\iota(\partial_x U))$ is locally Lipschitz in a neighborhood of U^0 in H^s, for all $s \geq t_0 > 1/2$. Existence/uniqueness of a maximal solution $U^\iota \in C([0,T^\iota); H^s)$ (with $T^\iota > 0$) to (14.9) with initial condition U^0 and satisfying (14.8) is thus a direct consequence of the Cauchy-Lipschitz theorem for ODEs in Banach spaces.

Step 2. Choice of a symmetrizer. A natural choice is :

$$S(U) = \begin{pmatrix} b(\zeta)^{-1} & 0 \\ 0 & c(U)^{-1} \end{pmatrix}.$$

Step 3. Energy estimates. There are obtained by multiplying the equation for $\widetilde{U} = \Lambda^s U^\iota$ (where $\Lambda^s = (I-\Delta)^{\frac{s}{2}}$ by $S(U^\iota)$ and using then various commutator estimates.

Step 4. Convergence of U^ι to a solution U by standard methods (see *eg* [231], Chapter 16). The blow-up condition is also standard.

Step 5. The conservation of energy results from the Hamiltonian structure of (14.4). Actually, setting

$$H(\zeta, v) = \frac{1}{2}\int_{\mathbb{R}}[(1-\gamma)\zeta^2 + v^2 f(\zeta)]dx,$$

one can write (14.4) in Hamiltonian form (this corresponds to (5.24) in [62])

$$\partial_t U + \mathfrak{J}\nabla H(U) = 0, \tag{14.10}$$

where $\mathfrak{J}$ is the skew-adjoint operator $\mathfrak{J} = \partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

□

We refer to [104] for more precise blow-up conditions. In particular, one can prove the following result (which has been observed in the numerical simulations in [104] :

Corollary 14.2. *Under the assumptions of Theorem 14.1, if the maximal existence time T_{max} is finite and if $\gamma > 0$ then:*

- $U = (\zeta, \mathbf{v})$ *remains uniformly bounded on* $[0, T_{max}) \times \mathbb{R}$
- $\lim_{t\to T_{max}} |\partial_x U(t,\cdot)|_\infty = \infty$

It is possible that the height of one of the fluids vanishes as $t \to T_{max}$. In that case, additional information can be given on the blow up of $\partial_x U(t,\cdot)$:

- *If* $\lim_{t\to T_{max}} \inf_{\mathbb{R}}(1-\zeta(t,\cdot)) = 0$ *then* $\lim_{t\to T_{max}} \sup_{\mathbb{R}} \partial_x \mathbf{v}(t,\cdot) = \infty$.
- *If* $\lim_{t\to T_{max}} \inf_{\mathbb{R}}(1+\delta\zeta(t\cdot)) = 0$ *then* $\lim_{t\to T_{max}} \inf_{\mathbb{R}} \partial_x \mathbf{v}(t,\cdot) = -\infty$.

Remark 14.2. It is of course possible to have a shock on the velocity without vanishing of the fluid depth for the upper or lower fluid. This scenario can also be observed on our numerical computations.

Remark 14.3. We also refer to [104] for the proof that, under the assumptions of Theorem 14.1, one has always $T_{max} < \infty$ if $U^0 \neq (0,0)$ is suitably compactly supported. This results from the fact that the domain where the system is genuinely nonlinear is a "big" subset of the domain of strict hyperbolicity.

14.1.2 The two-dimensional case

We first summarize in the following lemma various useful properties of the operator $\mathfrak{R}[\zeta]$ (see [104] for a proof).

Lemma 14.3. *Let $\gamma \in [0,1)$, $\delta > 0$ and $t_0 > 1$. Assume also that $\zeta \in H^s(\mathbb{R}^2)$, with $s \geq t_0 + 1$, and satisfies*

$$\inf_{\mathbb{R}}(1 - |\zeta|_\infty) > 0 \quad \textit{and} \quad \inf_{\mathbb{R}}(1 - \delta|\zeta|_\infty) > 0.$$

Then, for all $\mathbf{v} \in L^2(\mathbb{R}^2)^2$, one has

$$\nabla \cdot \mathfrak{R}[\zeta]\mathbf{v} = \delta \frac{\mathfrak{S}[\zeta]\mathbf{v}}{\delta h_1 + \gamma h_2} \cdot \nabla \zeta + \frac{h_2}{\delta h_1 + \gamma h_2} \nabla \cdot \mathbf{v}$$

and, for $j = 1, 2$,

$$\partial_j \big(\mathfrak{R}[\zeta]\mathbf{v} \big) = \delta \mathfrak{R}[\zeta] \big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2} \partial_j \zeta \big) + \mathfrak{R}[\zeta] \partial_j \mathbf{v}.$$

Moreover, for all $\mathbf{v} \in L^2(\mathbb{R}^2)^2$,

$$|\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) - \frac{1}{\delta h_1 + \gamma h_2} \Pi \mathbf{v}|_2$$

$$\leq C\big(\frac{1}{\gamma + \delta - \delta(1-\gamma)|\zeta|_\infty}, \delta(1-\gamma)|\zeta|_{H^{t_0+1}} \big) |\Pi \mathbf{v}|_{H^{-1}}.$$

Remark 14.4. The first part of the Lemma shows how the divergence and differentiation operators act on $\mathfrak{R}[\zeta]$.

The second part proves that $\mathfrak{R}[\zeta]$ is a local operator on gradient vector fields up to a more regular term. We recall that in one dimension, $\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2}) = \frac{1}{\delta h_1 + \gamma h_2} \mathbf{v}$.

It is more tricky when $d = 1$ to put (14.1) under a quasilinear form because of the presence of the nonlocal term $\mathfrak{R}[\zeta]\mathbf{v}$. Nevertheless one can write (14.1) on the form :

$$\partial_t U + A^j[U]\partial_j U = 0, \qquad U = (\zeta, \mathbf{v})^T, \tag{14.11}$$

where

$$A^j[U] = \begin{pmatrix} a^j(U) & \mathbf{b}^j(U)^T \\ \mathbf{c}^j[U] & D^j[U] \end{pmatrix}, \qquad (j = 1, 2),$$

and

$$a^j(U) = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \frac{h_2}{\delta h_1 + \gamma h_2}, \tag{14.12}$$

$$\mathbf{b}^j(U) = \frac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{e}^j, \tag{14.13}$$

$$\mathbf{c}^j[U]\bullet = \mathbf{e}^j - \gamma\Big[\mathbf{e}^j + \delta(\mathfrak{S}[\zeta]\mathbf{v})_j \mathfrak{R}[\zeta]\big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\bullet\big)\Big], \tag{14.14}$$

$$D^j[U]\bullet = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j \mathrm{Id}_{2\times 2} - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \mathfrak{R}[\zeta]\bullet, \tag{14.15}$$

where

$$\mathfrak{S}[\zeta]\mathbf{v} = \mathbf{v} + (1-\gamma)\mathfrak{R}[\zeta]\mathbf{v} \tag{14.16}$$

One can prove, using in particular Lemma 14.3 :

Proposition 14.4 (The case $d = 2$). *Let $T > 0$, $t_0 > 1$ and $s \geq t_0 + 1$. Let also $U = (\zeta, v) \in C([0,T]; H^s(\mathbb{R}^2)^3)$ be such that for all $t \in [0,T]$,*

$$(1 - |\zeta(t,\cdot)|_\infty) > 0 \quad \textit{and} \quad (1 - \delta|\zeta(t,\cdot)|_\infty) > 0 \quad \text{and} \quad \operatorname{curl} \mathbf{v}(t,\cdot) = 0.$$

Then U solves (14.1) if and only U solves (14.11).

Remark 14.5. The system (14.11) is not *stricto sensu* a quasilinear system since $\mathbf{c}^j[U]$ (resp. $D^j[U]$) is not an $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) function but a linear operator defined over the space of $\mathbb{R}^2$-vector-valued (resp. 2×2-matrix-valued) functions. However, these operators are of order zero and, as shown below, (14.11) can be handled roughly as a quasilinear system.

One next proves that a solution of (14.11) which is initially curl-free remains curl-free on its existence time.

We now turn to the local well-posedness of the two-dimensional SW/SW system (14.1).

The following conditions generalize the hyperbolicity conditions of the one-dimensional system.

$$\begin{cases} 1-|\zeta|_\infty > 0, \\ 1-\delta|\zeta|_\infty > 0, \\ 1-\gamma-\gamma\delta\dfrac{|\mathfrak{S}[\zeta]\mathbf{v}|_\infty^2}{\gamma+\delta-\delta(1-\gamma)|\zeta|_\infty} > 0, \end{cases} \tag{14.17}$$

with $\mathfrak{S}[\zeta]\mathbf{v}$ as in (14.16).

The main result is the

Theorem 14.5. *Let $\delta > 0$ and $\gamma \in [0,1)$. Let also $t_0 > 1$, $s \geq t_0+1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R}^2)^3$ be such that (14.17) is satisfied and* curl $\mathbf{v}^0 = 0$. *Then there exists $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0, T_{max}); H^s(\mathbb{R}^2)^3)$ to (14.1) with initial condition U^0. Moreover, if $T_{max} < \infty$ then at least one of the following conditions holds:*

(i) $\quad \lim_{t\to T_{max}} |U(t)|_{H^{t_0+1}} = \infty$

(ii) *One of the three conditions of (14.17) is enforced as $t \to T_{max}$.*

Proof. As in the one-dimensional case we will only indicate the main steps, emphasizing the specific difficulties of the $2d$ case (see [104] for details).

Step 1. Regularized equations. This step is very similar to the $1d$ case after checking the smoothness of the coefficients of the matrices $A^j[U]$.

Step 2. Choice of a symmetrizer. Let us look for $S[U]$ under the form

$$S[U] = \begin{pmatrix} s_1(U) & 0 \\ 0 & S_2[U] \end{pmatrix}, \tag{14.18}$$

with $s_1(\cdot) : H^s(\mathbb{R}^2)^3 \mapsto H^s(\mathbb{R}^2)$ and $S_2[U]$ a linear operator mapping $L^2(\mathbb{R}^2)^2$ into itself. Defining $C[U]$ as

$$\forall \widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2)^T \in L^2(\mathbb{R}^2)^2, \qquad C[U]\widetilde{\mathbf{v}} = \mathbf{c}_1[U]\widetilde{\mathbf{v}}_1 + \mathbf{c}_2[U]\widetilde{\mathbf{v}}_2,$$

a straightforward generalization of the one dimensional case consists in taking $s_1(U) = b(U)^{-1}$ and $S_2[U] = C[U]^{-1}$; unfortunately, such a choice is not correct because the operator $C[U]$ is not self-adjoint. It turns out however that $C[U]$ is self-adjoint (up to a smoothing term) on the restriction of $L^2(\mathbb{R}^2)^2$ to gradient vector fields, as shown in the following lemma. We first need to define the operator $C_1[U]$ as

$$C_1[U] = (1-\gamma)\mathrm{Id} + \frac{1}{2}\delta\gamma \begin{pmatrix} c_1[U]+c_1[U]^* & 0 \\ 0 & c_1[U]+c_1[U]^* \end{pmatrix}, \quad (14.19)$$

with $c_1[U] : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$c_1[U] = \frac{1}{\delta h_1 + \gamma h_2}(2\mathfrak{S}_1\mathfrak{S}_2\Pi(\mathbf{e}^2\cdot)_1 + \mathfrak{S}_1^2\Pi(\mathbf{e}^1\cdot)_1 + \mathfrak{S}_2^2\Pi(\mathbf{e}^2\cdot)_2), \quad (14.20)$$

and where $\mathfrak{S}_j = (\mathfrak{S}[\zeta]\mathbf{v})_j$.

One can now state :

Lemma 14.6. *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v}) \in H^{t_0+1}(\mathbb{R}^2)^3$ be such that (14.17) is satisfied. Define also $C_1[U]$ as in (14.19) and let $C_2[U] = C[U] - C_1[U]$. For all $\widetilde{\zeta} \in L^2(\mathbb{R}^2)$, one has*

$$|C_2[U]\nabla\widetilde{\zeta}|_2 \leq \mathfrak{c}(U)|\widetilde{\zeta}|_2.$$

We now choose the coefficients $s_1[U]$ and $S_2[U]$ of the symmetrizer $S[U]$ given by (14.18) as follows

$$s_1(U) = b(U)^{-1}, \quad (14.21)$$
$$S_2[U] = C_1[U]^{-1}. \quad (14.22)$$

One then checks that $C_1[U]$ is invertible in $\mathscr{L}(L^2(\mathbb{R}^2)^2; L^2(\mathbb{R}^2))$.

The operator $S[U]$ would therefore be a symmetrizer in the sense given in Step 2 of the proof of Theorem 14.1 if $S[U]A^j[U]$ $(j = 1,2)$ were symmetric, which is unfortunately not the case. However, one can prove that $\Pi S[U]A^j[U]\Pi$, where Π denotes as before the projection onto gradient vector fields, is symmetric at leading order.

Step 3. Energy estimates. The idea is the same that in the $1d$ case but much more delicate. One uses various properties of the coefficients of the matrices $A^j[U]$ and commutator estimates (see for instance [157], Theorem 6).

Step 4. Convergence of the approximate solution. We first obtained the convergence to the unique solution of (14.11) which turns out to be also the unique solution of (14.1) since we assumed that curl $\mathbf{v}_0 = 0$.

Step 5. Blow-up condition. It results from a standard continuation argument. □

Remark 14.6. We do not know whether or not (14.1) possesses a conserved energy or a Hamiltonian structure. The existence of two-dimensional blowing-up solutions, though highly expected, is also unknown.

14.2 Extensions : free upper surface; surface tension effects

A natural extension, (very relevant in applications to ocean dynamics problems) of the previous results is to consider a free upper surface instead of a rigid lid and/or a non flat bottom (nontrivial bathymetry).

14.2.1 The case of an upper free surface

A formal derivation of asymptotic models in this situation has been carried out formally in [56] in the *weakly nonlinear* regime and in [16] in the *strongly nonlinear* regime (see also [174]). The *weakly nonlinear* regime was also considered in [62] by expanding the Hamiltonian of the full system.

The rigorous approach of [32] has been generalized to the case of an upper free surface by V. Duchêne ([73]) who also incorporates a non trivial bathymetry. This second aspect will be kept aside till the next subsection in order to focus here on the effect of an upper free surface.

Let us denote by Γ_1 and Γ_2 the upper and inner free surfaces respectively and ϕ_1 and ϕ_2 the velocity potentials of the upper and lower layers. The full system consists now in four equations for the elevations ζ_1 and ζ_2 of the upper free surface and of the inner free surface respectively and for the traces ψ_1 of ϕ_1 on Γ_1 and ψ_2 of ϕ_2 on Γ_2. It involves two Dirichlet-Neumann operators $G_1[\zeta_1,\zeta_2]$ and $G_2[\zeta_2]$ and a nonlocal interface operator $H[\zeta_1,\zeta_2]$. As in the rigid lid case the full system suffers from Kelvin-

Helmholtz instabilities in presence of discontinuous tangential velocities at the inner interface.

Performing the suitable expansions of the nonlocal operators, V. Duchêne obtains asymptotic models in the Boussinesq/Boussinesq and SW/SW regimes, but his approach applies as well to the other regimes. Let us consider for instance the SW/SW regime. We denote here by ε_1 (resp. ε_2) the ratio of the typical amplitude of the upper (resp inner) free surface over the depth of the upper layer at the rest position. Setting $\mathbf{v}_1 = \nabla\psi_1$, $\mathbf{v}_2 = \nabla\psi_2$ one gets the following system (compare with (5.21)):

$$\begin{cases} \alpha\partial_t\zeta_1 + \nabla.(h_1\mathbf{v}_1) + \nabla.(h_2\mathbf{v}_2) = 0, \\ \partial_t\zeta_2 + \nabla.(h_2\mathbf{v}_2) = 0 \\ \partial_t\mathbf{v}_1 + \alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_1|^2) = 0 \\ \partial_t\mathbf{v}_2 + (1-\gamma)\nabla\zeta_2 + \gamma\alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_2|^2) = 0, \end{cases} \tag{14.23}$$

where $\alpha = \frac{\varepsilon_1}{\varepsilon_2}$, $h_1 = 1 + \varepsilon_1\zeta_1 - \varepsilon_2\zeta_2$, $h_2 = \frac{1}{\delta} + \varepsilon_2\zeta_2$.

Remark 14.7. This system was derived in the one-dimensional case in [62] from the Hamiltonian formulation and in [57] (using the layer-mean formulation). In [73] V. Duchêne considers also the case of a non flat bottom.

Contrary to (5.21), the system (14.23) above is local. Actually it can be written as a symmetrizable hyperbolic system

$$\partial_t U + A_1(U)\partial_x U + A_2(U)\partial_y U = 0 \tag{14.24}$$

where $\mathbf{v}_1 = (u_1, u_2)^T$, $\mathbf{v}_2 = (v_1, v_2)^T$ and

$$U = (h_1, h_2, \varepsilon_2 u_1, \varepsilon_2 u_2, \varepsilon_2 v_1, \varepsilon_2 v_2),$$

$$A_1(U) = \begin{pmatrix} \varepsilon_2 u_1 & 0 & h_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 v_1 & 0 & 0 & h_2 & 0 \\ 1 & 1 & \varepsilon_2 u_1 & \varepsilon_2 u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 1 & 0 & 0 & \varepsilon_2 v_1 & \varepsilon_2 v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2(U) = \begin{pmatrix} \varepsilon_2 u_2 & 0 & 0 & h_1 & 0 & 0 \\ 0 & \varepsilon_2 v_2 & 0 & 0 & 0 & h_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \varepsilon_2 u_1 & \varepsilon_2 u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 1 & 0 & 0 & \varepsilon_2 v_1 & \varepsilon_2 v_2 \end{pmatrix}.$$

One can then prove the

Theorem 14.7. *([73]) Let $U_0 \in H^s(\mathbb{R}^2))^6$, $s > 2$, such that U_0 satisfies on $\mathbb{R}^2$, for some $h > 0$,*

$$h_1 \geq h, \;\; h_2 \geq h, \;\; \varepsilon_2^2|u_1^2 + u_2^2| \leq (1-\gamma)h_1, \;\; \varepsilon_2^2|v_1^2 + v_2^2| \leq h_2.$$

Then there exists $T > 0$ and a unique solution $U \in C([0,T); H^s(\mathbb{R}^2))^6$ of (14.24) *with initial data U_0.*

Remark 14.8. It is furthermore proven in [73] that the solutions of (14.23) approximate those of the full system, provided the later ones exist.

It thus appears that the nonlocal character of the SW/SW system (14.1) derived in the previous section is a rigid lid effect. As noticed in [73] one can actually derived it from (14.23). In fact, using the notations of the present subsection, the rigid lid assumption means that $\varepsilon_1 = 0$, while ε_2 remains > 0, so that $\alpha = 0$ and (14.23) writes now :

$$\begin{cases} \nabla.(h_1 \mathbf{v}_1) + \nabla.(h_2 \mathbf{v}_2) = 0, \\ \partial_t \zeta_2 + \nabla.(h_2 \mathbf{v}_2) = 0 \\ \partial_t \mathbf{v}_1 + \alpha \nabla \zeta_1 + \frac{\varepsilon_2}{2} \nabla(|\mathbf{v}_1|^2) = 0 \\ \partial_t \mathbf{v}_2 + (1-\gamma)\nabla \zeta_2 + \gamma\alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_2|^2), \end{cases} \tag{14.25}$$

Let $\mathbf{v} = \mathbf{v}_2 - \gamma \mathbf{v}_1$. We deduces from the first equation in (14.25) that

$$\nabla.(h_2 \mathbf{v}) = -\nabla.((h_1 + \gamma h_2)\mathbf{v}_1) = -\frac{\gamma+\delta}{\delta}\nabla.((1 + \frac{\gamma-1}{\gamma+\delta}\delta\varepsilon_2)\mathbf{v}_1).$$

Being given $\zeta \in L^\infty(\mathbb{R}^2)^2$, we define the nonlocal operator $\mathfrak{Q}[\zeta]$ (see **2.2.2**) as the mapping

$$\mathfrak{Q}[\zeta] : \quad \begin{array}{ccc} L^2(\mathbb{R}^2)^2 & \to & L^2(\mathbb{R}^2)^2 \\ W & \mapsto & V, \end{array}$$

where V is the unique gradient solution in $L^2(\mathbb{R}^2)^2$ of the equation

$$\nabla.((1+\zeta)V) = \nabla.W.$$

We have thus

$$\mathbf{v}_1 = \mathfrak{Q}[\frac{\gamma+1}{\gamma+\delta}\delta\varepsilon_2\zeta_2](-\frac{\delta}{\gamma+\delta}h_2\mathbf{v}).$$

Plugging this expression into (14.25) we obtain

$$\begin{cases} \partial_t\zeta_2 + \frac{\delta}{\gamma+\delta}\nabla\cdot\big(h_1\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta](h_2\mathbf{v})\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta_2 \\ +\frac{\varepsilon_2}{2}\nabla\Big(\big|\mathbf{v} - \frac{\gamma\delta}{\gamma+\delta}\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon_2\delta\zeta_2](h_2\mathbf{v})\big|^2 - \frac{\gamma^2}{(\gamma+\delta)^2}\big|\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon_2\delta\zeta_2](h_2\mathbf{v})\big|^2\Big) = 0, \end{cases} \tag{14.26}$$

which is (5.21).

14.2.2 Effect of surface tension and of a nontrivial bathymetry

The main effect of surface tension in the two-layer system is to prevent the formation of Kelvin-Helmholtz instabilities. For instance, in the related problem of horizontal shear flows (see [49] and the Introduction), if we denote by T the surface tension coefficient, the flat interface for horizontal shear flows with constant horizontal velocities U_1 and U_2 does not develop instabilities for perturbations in the direction of streaming having wave numbers $\mathbf{k}$ such that (compare to (5.16)):

$$\frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2 < gk\{\frac{\alpha_2-\alpha_1}{k} + \frac{kT}{g(\rho_1+\rho_2}\}. \tag{14.27}$$

In particular, no Kelvin-Helmholtz instabilities are present provided the surface tension is large enough to insure that

$$(U_1-U_2)^2 < \frac{2}{\alpha_1\alpha_2}\sqrt{\frac{Tg(\alpha_2-\alpha_1)}{\rho_1+\rho_2}}. \tag{14.28}$$

Coming back to the two-layer system (5.14), we have seen previously that the surface tension adds a term $-\frac{\sigma}{\rho_2}\nabla K(\widetilde{\zeta})$ to the LHS of the second equation of (5.14), where $K(\widetilde{\zeta}) = (\frac{\nabla\widetilde{\zeta}}{\sqrt{1+|\nabla\widetilde{\zeta}|^2}})$.

In fact, in oceanographic applications, the surface tension effects are very weak and can be ignored when deriving the aforementionned asymptotic models (they appear as a lower order effect). In situations where they are small but of the order of the "small" parameters involved in the asymptotic expansions (the $\varepsilon's$ or the $\mu's$) one has to add a cubic "capillary" term to the equation for $\mathbf{v}$ (see [66] where the various regimes are systematically investigated). For instance, in the SW/SW regime, (5.21) has to be replaced by

$$\begin{cases} \partial_t\zeta + \frac{1}{\gamma+\delta}\nabla\cdot\big(h_1\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta](h_2\mathbf{v})\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta \\ +\frac{\varepsilon}{2}\nabla\Big(\big|\mathbf{v} - \frac{\gamma}{\gamma+\delta}\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta](h_2\mathbf{v})\big|^2 - \frac{\gamma}{(\gamma+\delta)^2}\big|\mathfrak{Q}[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta](h_2\mathbf{v})\big|^2\Big) \\ -\varepsilon\sqrt{\mu}\nu\Delta\nabla\zeta = 0, \end{cases} \tag{14.29}$$

where $\nu = \frac{\sigma}{\rho_2\lambda^2}$.

Taking into account a varying bottom is more relevant for the oceanographic applications. For surface waves, F. Chazel [51] has incorporated those effects in the context of weakly nonlinear longwave systems of the Boussinesq type.

For internal waves, this issue has been settled by Cung The Anh [66] and V. Duchêne [73], who, as we said previously considers the case of a free upper surface. We refer to those papers for details.

14.2.3 Higher order systems

A natural question is to go one step further in the asymptotic expansion of the nonlocal operators $G^\mu[\varepsilon\zeta]$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in order to derive *higher order asymptotic systems*. This has already been carried out in the Boussinesq/Boussinesq and SW/SW regimes in the two-dimensional case and with a free upper surface by V. Duchêne [73] who proved that these systems are consistent with the full two-layer system.

In the SW/SW case (for the sake of simplicity we restrict to the flat bottom case) one obtains a internal waves generalization of the Green-Naghdi system which writes (compare with (14.23))

$$\begin{cases} \alpha\partial_t\zeta_1 + \nabla.(h_1\mathbf{v}_1) + \nabla.(h_2\mathbf{v}_2) = \mu(\nabla.\mathscr{T}_1 + \nabla.\mathscr{T}_2 - \frac{1}{2}\nabla.(h_1^2\nabla\mathscr{A}_2) \\ -\nabla.(h_1\varepsilon_1\nabla\zeta_1\mathscr{A}_2)), \\ \partial_t\zeta_2 + \nabla.(h_2\mathbf{v}_2) = \mu\nabla.\mathscr{T}_2, \\ \partial_t\mathbf{v}_1 + \alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_1|^2) = \mu\varepsilon_2\nabla\mathscr{N}_1, \\ \partial_t\mathbf{v}_2 + (1-\gamma)\nabla\zeta_2 + \gamma\alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_2|^2) = \mu(\gamma\partial_t\mathscr{H} + \gamma\varepsilon_2\nabla(\mathbf{v}_1.\nabla\mathscr{H}) \\ \qquad +\varepsilon_2\nabla\mathscr{N}_2 + \gamma\varepsilon_2\nabla\mathscr{N}_1), \end{cases} \tag{14.30}$$

where

$$\begin{cases} h_1 = 1 - \varepsilon_2\zeta_2, \quad h_2 = \frac{1}{\delta} + \varepsilon_2\zeta_2, \\ \mathscr{A}_1 = \nabla.(h_1\mathbf{v}_1), \qquad \mathscr{A}_2 = \nabla.(h_2\mathbf{v}_2), \\ \mathscr{T}[h,b]\mathbf{v} = -\frac{1}{3}\nabla(h^3\nabla.\mathbf{v}) + \frac{1}{2}(\nabla(h^2\nabla b.\mathbf{v}) - h^2\nabla b\nabla.\mathbf{v}) + h\nabla b\,\nabla b.\mathbf{v}, \\ \mathscr{T}_1 = \mathscr{T}[h_1,\varepsilon_2\zeta_2]\mathbf{v}_1, \quad \mathscr{T}_2 = -\frac{1}{3}\nabla(h_2^3\nabla.\mathbf{v}_2), \\ \mathscr{H} = h_1(\nabla.(h_1\mathbf{v}_1) + \nabla.(h_1\mathbf{v}_2) - \frac{1}{2}h_1\nabla.\mathbf{v}_1 - \varepsilon_1\nabla\zeta_1.\mathbf{v}_1), \\ \mathscr{N}_1 = \frac{1}{2}[\varepsilon_1\nabla\zeta_1.\mathbf{v}_1 - \nabla.(h_1\mathbf{v}_1) - \nabla.(h_1\mathbf{v}_2]^2, \\ \mathscr{N}_2 = \frac{1}{2}[(\varepsilon_2\nabla\zeta_1.\mathbf{v}_2 - \nabla.(h_1\mathbf{v}_2))^2 - \gamma(\varepsilon_2\nabla\zeta_2.\mathbf{v}_1 - \nabla.(h_1\nabla\mathbf{v}_2))^2]. \end{cases} \tag{14.31}$$

Note that this system is linearly ill-posed so that one should derived models with parameters to get linearly well-posed systems.

Previously, Choi and Camassa [57] derived formally a similar system (using the depth mean velocities) in one-dimension and with a rigid top. This system has been also derived in one dimension in [62] by expansion of the Hamiltonian.

Finally, Barros, Gavrilyuk and Teshukov [16] has considered the two-dimensional case with a free surface and obtained formally a version of the *generalized Green-Naghdi system* by expanding the Lagrangian of the full system with respect to the dispersion parameter.

We are not aware of any mathematical results on this type of systems, suitably modified to get linear well-posedness (see [13] for the rigorous complete justification of the *classical* Green-Naghdi system [101] for surface waves).

The asymptotic expansion in **2.2.2** could be of course carried out one order further to get the equivalent of (14.30) in case of a rigid lid.

Alternatively, following [73], one can try to obtain this system from (14.30). As in Subsection **5.1** this amounts to setting $\varepsilon_1 = 0$ and thus $\alpha = 0$ in (14.30). One gets

$$\begin{cases} \nabla.(h_1\mathbf{v}_1) + \nabla.(h_2\mathbf{v}_2) = \mu(\nabla.\mathscr{T}_1 + \nabla.\mathscr{T}_2 - \frac{1}{2}\nabla.(h_1^2\nabla\mathscr{A}_2)), \\ \partial_t\zeta_2 + \nabla.(h_2\mathbf{v}_2) = \mu\nabla.\mathscr{T}_2, \\ \partial_t\mathbf{v}_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_1|^2) = \mu\varepsilon_2\nabla\mathscr{N}_1, \\ \partial_t\mathbf{v}_2 + (1-\gamma)\nabla\zeta_2 + \gamma\alpha\nabla\zeta_1 + \frac{\varepsilon_2}{2}\nabla(|\mathbf{v}_2|^2) = \mu(\gamma\partial_t\mathscr{H} + \gamma\varepsilon_2\nabla(\mathbf{v}_1.\nabla\mathscr{H}) \\ \qquad\qquad + \varepsilon_2\nabla\mathscr{N}_2 + \gamma\varepsilon_2\nabla\mathscr{N}_1), \end{cases} \tag{14.32}$$

where now

$$\begin{cases} \mathscr{A}_1 = \nabla.(h_1\mathbf{v}_1), \qquad \mathscr{A}_2 = \nabla.(h_2\mathbf{v}_2), \\ \mathscr{T}[h,b]\mathbf{v} = -\frac{1}{3}\nabla(h^3\nabla.\mathbf{v}) + \frac{1}{2}(\nabla(h^2\nabla b.\mathbf{v}) - h^2\nabla b\nabla.\mathbf{v}) + h\nabla b\,\nabla b.\mathbf{v}, \\ \mathscr{T}_1 = \mathscr{T}[h_1,\varepsilon_2\zeta_2]\mathbf{v}_1, \quad \mathscr{T}_2 = -\frac{1}{3}\nabla(h_2^3\nabla.\mathbf{v}_2), \\ \mathscr{H} = h_1(\nabla.(h_1\mathbf{v}_1) + \nabla.(h_1\mathbf{v}_2) - \frac{1}{2}h_1\nabla.\mathbf{v}_1), \\ \mathscr{N}_1 = \frac{1}{2}[-\nabla.(h_1\mathbf{v}_1) - \nabla.(h_1\mathbf{v}_2]^2, \\ \mathscr{N}_2 = \frac{1}{2}[(\varepsilon_2\nabla\zeta_1.\mathbf{v}_2 - \nabla.(h_1\mathbf{v}_2))^2 - \gamma(\varepsilon_2\nabla\zeta_2.\mathbf{v}_1 - \nabla.(h_1\nabla\mathbf{v}_2))^2]. \end{cases} \tag{14.33}$$

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