Dynamics of Partial Actions

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Preface

This book arose when the second author asked in 2006, in a lecture to his graduated course at the Federal University of Rio de Janeiro, if it is possible to extend the result [47] to manifolds supporting codimension one Anosov actions of connected Lie groups. Fortunately the first author was present there and suddenly appeared with a partial solution which gave rise the paper [2]. At the same time all of us started to discuss in the coffee-shop *o anjinho* about a possible variation of the question: the one obtained by replacing the term *Anosov* by the term *sectional-Anosov* which eventually appeared in [40]. This issue carried us into other problems because the interesting examples of sectional-Anosov flows, apart of course from the Anosov's ones, come from *partial actions* rather than classical actions. We then devoted to investigate partial actions which is a rather general theory including not only partial semigroup or groupoid actions but also partial versions of the classical concepts in algebra as groupoids, semigroups, semirings and so on. Such investigations gave rise to a third question which is inside the cornerstone of this book: *Is it possible to extend Anosov group actions, which is a natural link between dynamics and group theory, to include partial actions of partial semigroups or partial semirings?* Behind this question relies also the (as far as we know) lack of an intrinsic definition of *Anosov foliation*, namely, one involving the holonomy pseudogroup only. The present book is nothing but an attempt to put together both the hyperbolic dynamical systems and the theory of partial semigroup action in a common context.

The first part of this book goes from Chapter 1 to Chapter 3. In Chapter 1 we expose some topics in the groupoid or semigroup including regular semigroups, monoids, inverse semigroups or completely regular semigroups. In Chapter 2 we present the definition of partial groupoids and partial semigroups which seems to be started in [13]. Some examples of these algebraic objects including direct sums, unions (also called amalgams), free partial semigroups, etc are given there. In Chapter 3 we present partial actions in order to define Anosov partial semigroup action at the end of the chapter.

In the second part (which is the longest one), we present some results on the dynamics of Anosov group actions. These results are motivated by the hyperbolic theory of diffeomorphisms and flows.

In Chapter 4, we study the ergodicity of volume-preserving central Anosov actions. Since the work of Hopf and Anosov, it was known that C^2 volume preserving diffeomorphisms and flows were ergodic, which means that any invariant subset must have zero or full measure. In this chapter we present a result by Pugh and Shub extending this result to the context of central Anosov actions. Centrality is used to guarantee that the invariant foliations are invariant not only by the Anosov element but also by the entire group. Hence, an analysis on the foliation, gives the fundamental property of absolute continuity. Being informal, absolute continuity says that the foliation satisfies a Fubini-type theorem, hence we can reintegrate a set of full measure on almost every leaf to obtain a set of full measure in the space. This is a key ingredient in Hopf's argument, which is used in many context, including the partially hyperbolic context, to obtain ergodicity.

In Chapter 5, we study the stability of Anosov action. More generally, we introduce the notion of Axiom A actions, based on the same notion for diffeomorphisms introduced by Smale, and show that these actions with an extra property (non existence of cycles) are stable. This means that every close action (in suitable topologies) is conjugated to the original one, by definition, this says that there exists a homeomorphism that sends orbits of one action onto orbits of the other one. For this purpose, we also show an spectral decomposition theorem, similar to the one obtained by Smale for diffeomorphisms. However, this decomposition theorem behaves better when the group is the Euclidian space, since in this setting there exists an Anosov-type closing lemma. Both the theorems of stability and spectral decomposition are due to Pugh and Shub.

In Chapter 6, we present some other results. First, a version of the well known closing lemma due to Pugh. This result deals with the problem of find a perturbation of the system such that if the first one presents a recurrent orbit, the same orbit for perturbation is closed. The extension of this result for action is due to Roussarie and Weyl, but in the setting of actions of $\mathbb{R}^2$ over 3-manifolds with no planar orbits. Second, we present a result due to Maquera and Tahzibi, which says that if an action of $\mathbb{R}^2$ on a 3-manifold is transitive with no planar orbits and every close action also is transitive then the action must be singular and hyperbolic, thus an Anosov flow. Third, we present a version of a Verjovsky‘s theorem due to Barbot and Maquera, which study the transitivity of codimension one Anosov actions of $\mathbb{R}^k$, the action will be transitive if the dimension of the ambient manifold is greater than $k + 2$. Finally, we pose some questions related to these topics.

We present some related basic concepts in the Appendix.

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Chapter 1

Semigroups

A *groupoid* is a pair (S, α) consisting of a set S and a binary operation $\alpha : S \times S \to S$ on S. We consistently use the notation gh instead of $\alpha(g, h)$ and, eventually, we write S instead of (S, α). A *subgroupoid* of a groupoid S is a subset $S' \subset S$ such that if $g, h \in S'$, then $gh \in S'$. Given the groupoids A, B a map $\phi : A \to B$ is a *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$. In such a case $\phi(A)$ is a subgroupoid of B.

Let S be a groupoid. An *identity* of S is an element 1 such that $1g = g1 = g$ for all $g \in S$ while a *zero* of S is an element 0 such that $g0 = 0g = 0$ for all $g \in S$. If S is a groupoid without identity (resp. zero), then the set $S \cup \{1\}$ (resp. $S \cup \{0\}$) equipped with the extended product $g1 = 1g = g$ (resp. $g0 = 0g = 0$) for all $g \in S \cup \{1\}$ (resp. $g \in S \cup \{0\}$) is a groupoid with identity 1 (resp. zero 0). For convenience we define $S^1 = S$ or $S \cup \{1\}$ depending on whether S has an identity or not. Similarly we define the groupoid with zero S^0.

An *idempotent* of S is an element $i \in S$ satisfying $i^2 = i$. We denote by $E = E(S)$ the set of idempotents of S. Clearly $\phi(E(A)) \subseteq E(B)$ for all homomorphism of groupoids $\phi : A \to B$.

Given $\Gamma \subset S$ we define its (inner) *centralizer*

$$Z(\Gamma) = \{g \in \Gamma : hg = gh, \forall h \in \Gamma\}.$$

The set $Z(S)$ is called the *center* of S. We say that S is *commutative* if it is equals to its own center. Equivalently, $hg = gh$ for all $g, h \in S$.

We say that S is *associative* if $(gh)f = g(hf)$ for all $g, h, f \in S$. In such a case we write ghf to mean either $g(hf)$ or $(gh)f$. Clearly if $\phi : A \to B$ is a homomorphism of groupoids with A associative, then $\phi(A)$ is an associative subgroupoid of B.

Let S be an associative groupoid. An element $g \in S$ is called *regular* if there is $x \in S$ such that $gxg = g$. An associative groupoid is *regular* if all its elements are regular.

An *inverse* of an element $g \in S$ is an element $g^* \in S$ such that $gg^*g = g$ and and $g^*gg^* = g^*$. It is customary to denote by

$$V(g) = \{g^* \in S : g^* \text{ is an inverse of } g\}$$

the set of inverses of g.

We say that g is *invertible* if $V(g)$ consists of a single element. Such an element is then denoted by g^{-1} and is called the *inverse* of g. An invertible element is clearly regular but not conversely. Every idempotent in an associative groupoid is regular and it is own inverse (if exists).

If A and B are associative groupoids and $\phi : A \to B$ is a homomorphism, then $\phi(a)$ is regular (in B) for all regular $a \in A$. Moreover, $\phi(V(a)) \subset V(\phi(a))$ for all $a \in A$.

Definition 1.1. *A groupoid S is*

- *a* (regular) semigroup *if it is (regular) associative;*
- *a* (regular) monoid *if it is a (regular) semigroup with an identity;*
- *an* inverse semigroup *if it is a semigroup where every element is invertible;*
- *an* inverse monoid *if it is an inverse semigroup with an identity;*
- *a* group *if it is an inverse monoid where $g^{-1}g = gg^{-1} = e$ for all $g \in S$.*

The substructures corresponding to the above definition are given as follows.

A *subsemigroup* of a semigroup S is a subgroupoid of S. A *submonoid* of a monoid is a subgroupoid containing the identity. An *inverse subsemigroup* of an inverse semigroup is a subsemigroup S' which is *symmetric*, i.e., $g^{-1} \in S'$ for all $g \in S'$. An *inverse submonoid* of an inverse monoid is an inverse subsemigroup containing the identity. A *subgroup* of a group G is an inverse submonoid of G.

To these substructures we can define their corresponding generated substructures by noting that, on all groupoids (resp. semigroups, monoids, inverse semigroups, inverse monoids, groups) S, the intersection of a non-empty family of subgroupoids (resp. subsemigroups, submonoids, inverse subsemigroups, inverse submonoids, subgroups) is either empty or a subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S. On the other hand, if Γ is an arbitrary non-empty subset of S, then the family of subgroupoids (resp. subsemigroups, submonoids, inverse subsemigroups, inverse submonoids, subgroups) of S containing Γ is non-empty for it contains S itself. Then, the intersection $< \Gamma >$ of this family is a subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S throughout called the *subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S generated by* Γ. If $< \Gamma >= S$ then we say that Γ *generates* S (or that S is *generated* by Γ). Eventually we write $< \Gamma >_{(\cdot)}$ in the cases when we need to emphasize the binary operation $\cdot$ of S.

In the sequel we present some useful properties of semigroups.

Lemma 1.2. *Let S be a semigroup. If $g \in S$ and $g^* \in V(g)$ satisfy $gg^*, g^*g \in Z(S)$, then $gg^* = g^*g$.*

Proof. Since $g^*g, gg^* \in Z(S)$ we have $g^*g^*g = g^*gg^* = g^*$ and $ggg^* = gg^*g = g$. Hence $gg^*g^*g = gg^*$ and $gg^*g^*g = g^*ggg^* = g^*g$ therefore $gg^* = g^*g$. □

Lemma 1.3. *An element g in a semigroup S is regular if and only if $V(g) \neq \emptyset$. Moreover, there is $x \in S$ such that $gxg = g$ and $gx = xg$ if and only if there is $g^* \in V(g)$ such that $gg^* = g^*g$.*

Proof. We only have to prove the direct implications for the reversed ones are trivial. If $g \in S$ is regular, then there is $x \in S$ such that $gxg = g$. Then $g^* = xgx$ satisfies $gg^*g = gxgxg = gxg = g$ and $g^*gg^* = xgxgxgx = x(gxg)xgx = x(gxg)x = xgx = g^*$ thus $g^* \in$

$V(g)$ so $V(g) \neq \emptyset$. The last assertion follows from the fact that if we assume in addition that $gx = xg$ then $gg^* = g^*g$. □

We use this lemma to prove the following well known equivalence [28].

Proposition 1.4. *A semigroup is an inverse semigroup if and only if it is regular and $Z(E) = E$.*

Proof. Since every inverse semigroup is regular we only need to prove that, in such semigroups, every pair of idempotents i, j commute. To see it we observe that $i^{-1} = i$ and $j^{-1} = j$ since they are idempotents, so, $(ij)^{-1} = ji$. On the other hand, $ij(ij)^{-1}ij = ij$ by the definition of inverse. But $ij(ij)^{-1}ij = ijjiij = ijij = (ij)^2$ hence ij is idempotent. Therefore $(ij)^{-1} = ij$ and then $ij = ij$ proving that i and j commute.

Conversely, suppose that S is regular and that $Z(E) = E$ (or, equivalently, that each pair of idempotents commute). If $g \in S$ then there exists by Lemma 1.3 an element $x \in S$ such that $gxg = g$ and $xgx = x$. We must prove that this x is unique. Indeed suppose that there is another $\overline{x} \in S$ such that $g\overline{x}g = g$ and $\overline{x}g\overline{x} = \overline{x}$. We have that $(gx)^2 = gxgx = gx$ hence $gx \in E$. Analogously $g\overline{x} \in E$ and also $xg, \overline{x}g \in E$. Since every pair of idempotents commute we have $gx = (g\overline{x}g)x = (g\overline{x})(gx) = gxg\overline{x} = g\overline{x}$. Analogously $xg = \overline{x}g$. Then, $\overline{x} = \overline{x}g\overline{x} = \overline{x}gx = xgx = x$ which proves the desired uniqueness. □

Notice that a semigroup S has a regular element if and only if $E \neq \emptyset$ (for every idempotent is regular and $gg^*g = g$ implies $g^*g, g^*g \in E$). On the other hand, a group is a regular semigroup with the identity as its unique idempotent. Conversely we have

Corollary 1.5. *A regular semigroup is a group if and only if it has a unique idempotent.*

Proof. As already noticed a group is a regular semigroup with the identity as its unique idempotent. Conversely, suppose that S is a regular semigroup with a single idempotent e. In such a case we have $Z(E) = E = \{e\}$ and so S is an inverse semigroup by Proposition 1.4. Now we fix $g \in S$. Since $gg^{-1}g = g$ where g^{-1} is the inverse of g we have that both gg^{-1} and $g^{-1}g$ are idempotents. Therefore

$gg^{-1} = g^{-1}g = e$ and so $ge = eg = e$ for all $g \in S$. This proves that S is an inverse monoid (with identity e) and also that $gg^{-1} = g^{-1}g = e$ hence S is a group. $\square$

1.1 Completely Regular Semigroups

A subgroup of a semigroup is a subsemigroup which is a group by itself and a semigroup is *completely regular* (or a *union of groups*) if each element belongs to a subgroup of it. In this section gives a simple characterization of completely regular semigroups due to Clifford [13]. We avoid the use of congruences or Green relations for the sake of brevity.

Clearly every completely regular semigroup S satisfies that for all $g \in S$ there is $x \in S$ such that $gxg = g$ and $gx = xg$ which by Lemma 1.3 is equivalent to the following property.

(P). For all $g \in S$ there is $g^* \in V(g)$ such that $gg^* = g^*g$.

Thus all such semigroups are regular. We shall see that this property characterizes completely regular semigroups, that is, a semigroup is completely regular if and only if it satisfies (P). The proof is based on the following lemma. For all semigroup S and $x \in S$ we define

$$S_x = \{g \in S : gg^* = g^*g = x \text{ for some } g^* \in V(g)\}.$$

Lemma 1.6. *For all semigroup S and all $x \in S$ one has $S_x \neq \emptyset$ if and only if $x \in E$ in whose case S_x is a subgroup of S. Moreover, $S_x \cap S_y \neq \emptyset$ if and only if $x, y \in E$ and $x \neq y$.*

Proof. Clearly $S_x \neq \emptyset$ if and only if $x \in E$ for $gg^* \in E$ for all $g^* \in V(g)$ and $x \in V(x)$ for all $x \in E$. It remains to prove that S_x is a subgroup.

For this suppose that $g, h \in S_x$ hence there are $g^* \in V(g)$ and $h^* \in V(h)$ such that $gg^* = g^*g = hh^* = h^*h = x$. Since $gh(h^*g^*)gh = g(hh^*)(g^*g)h = gxxh = gxh$ and $gx = gg^*g = g$ we have $gh(h^*g^*)gh = gh$. Analogously $(h^*g^*)gh(h^*g^*) = h^*g^*$ hence $h^*g^* \in V(gh)$.

On the other hand, since $gh(h^*g^*) = gxg^*$ and $gx = x$ we have $gh(h^*g^*) = x$. Analogously $(h^*g^*)gh = x$ and then $gh \in S_x$ for all $g, h \in S_x$. Therefore S_x is a subsemigroup of S. Moreover, since it is clear that $g^* \in V(g) \cap S_x$ whenever $g^* \in V(g)$ satisfies $gg^*g^*g = x$ we have that S_x is an regular semigroup by itself. Finally if $e \in E(S_x)$ then there is $e^* \in V(e)$ such that $ee^* = e^*e = x$ and then $e = ee^*e = e^*ee = e^*e = x$ which proves that x is the unique idempotent of S_x. We conclude from Corollary 1.5 that S_x is a subgroup.

Now we prove the last assertion of the lemma. Evidently $S_x \cap S_y \neq \emptyset$ if $x = y \in E$. Conversely, if $S_x \cap S_y \neq \emptyset$ then $x, y \in E$ and there are $g \in S$, $g^*, g^{**} \in V(g)$ such that $gg^* = g^*g = x$ and $gg^{**} = g^{**}g = y$. Then, $xy = (gg^*)(gg^{**}) = (gg^*g)g^{**} = gg^{**} = y$ and $xy = (g^*g)(g^{**}g) = g^*(gg^{**}g) = g^*g = x$ hence $x = y$. □

It follows from the above lemma that inside any semigroup S there is a disjoint collection of subgroups $\{S_e : e \in E\}$ indexed by E which is well-defined (i.e. $E \neq \emptyset$) at least when S is regular. We use it to prove the following equivalence due to Clifford [13] (see also [44]).

Proposition 1.7. *A semigroup S satisfies $S = \bigcup_{e \in E} S_e$ if and only if it satisfies (P). In particular, a semigroup is completely regular if and only if it satisfies (P). A semigroup is completely regular if and only if it is disjoint union of subgroups.*

Proof. If $S = \bigcup_{e \in E} S_e$ then S is a union of groups by Lemma 1.6 and so it satisfies (P). Conversely, suppose that S satisfies (P) and take $g \in S$. Hence there is $g^* \in V(g)$ with $gg^* = g^*g$ and then $g \in S_e$ with $e = gg^* \in E$ so $S = \bigcup_{e \in E} S_e$ as desired. The last two sentences of the proposition follows from the first and Lemma 1.6. □

Notice that from Proposition 1.7 we have that an inverse semigroup with a unique idempotent is a group. But this is consequence of Corollary 1.5 as well. The following is a direct consequence of Proposition 1.7.

Corollary 1.8. *A commutative semigroup is regular if and only if it is completely regular.*

Another application of Proposition 1.7 is the following well known result. A *Clifford semigroup* is a regular semigroup S satisfying $E \subset Z(S)$ (see [28] p. 93). A *Boolean semigroup* is a semigroup S such that $g^3 = g$ for all $g \in S$. Notice that every *idempotent semigroup* (i.e. a semigroup S satisfying $S = E$) is Boolean.

Proposition 1.9. *The class of completely regular semigroups contains both Clifford and Boolean semigroups.*

Proof. A Clifford semigroup satisfies (P) due to Lemma 1.2 while a Boolean semigroup satisfies (P) by taking $g^* = g$ in the definition of (P). Thus the assertion follows from Proposition 1.7. □

1.2 The symmetric inverse semigroup

In this section we prove that any inverse semigroup can be represented in the inverse semigroup of partial injective mappings of some set. This result is known as the *Vagner-Preston representation Theorem* plays fundamental role in semigroup theory. We start with an elementary lemma about semigroups.

Given a groupoid S, $H \subset S$ and $g \in S$ we denote $Hg = \{hg \in S : h \in H\}$.

Lemma 1.10. *If S is an inverse semigroup, then $Se \cap Sf = Sef$ for all $e, f \in E$ and $Sg^{-1}gh = Sh^{-1}g^{-1}gh$ for all $g, h \in S$.*

Proof. By Proposition 1.4 one has $ef = fe$ therefore $Sef \subset Sf \cap Sf$. Conversely, if $h = xe = yf \in Se \cap Sf$ then $h = xe = xee = he = yfe = yef \in Sef$ proving $Se \cap Sf = Sef$.

Now take $g, h \in S$. We clearly have $Sh^{-1}g^{-1}gh \subset Sg^{-1}gh$ for $Sh \subset S$. Conversely, $Sg^{-1}gh = Sg^{-1}ghh^{-1}h = Shh^{-1}g^{-1}gh \subset Sh^{-1}g^{-1}gh$ for $Sh \subset S$ and $hh^{-1}, g^{-1}g \in E = Z(E)$. Hence $Sg^{-1}gh \subset Sh^{-1}g^{-1}gh$ proving the result. □

Denote by $Dom(F)$ and $Rang(F)$ the domain and the range of a map F respectively. Given a set X we denote by $\mathcal{T}_X$ the set of all maps $g : Dom(g) \subset X \to X$. We allow the empty map $\emptyset : Dom(\emptyset) \subset X \to X$ where $Dom(\emptyset) = Rang(\emptyset) = \emptyset$ as an element of $\mathcal{T}_X$. On the other hand, for every subset $U \subset X$ we can associate the map $I_U \in \mathcal{T}_X$ by $Dom(I_U) = U$ and $I_U(x) = x$ for all $x \in U$. In particular, I_X is the identity map of X.

There is a natural operation $(g, h) \in \mathcal{T}_X \times \mathcal{T}_X \mapsto g \cdot h \in \mathcal{T}_X$ given by composition,

$$g \cdot h = \begin{cases} \emptyset, & \text{if} \quad Dom(g) \cap Rang(h) = \emptyset \\ g \circ h, & \text{if} \quad Dom(g) \cap Rang(h) \neq \emptyset. \end{cases}$$

Notice that $Dom(g \cdot h) = h^{-1}(Dom(g) \cap Rang(h))$ and $Rang(g \cdot h) = g(Dom(g) \cap Rang(h))$. Moreover, $(\mathcal{T}_X, \cdot)$ is a monoid with identity I_X, the identity map of X. Hereafter we write gh instead of $g \cdot h$.

Now define $\mathcal{S}_X$ as the set of all injective elements of $\mathcal{T}_X$. Notice that the pair $(\mathcal{S}_X, \cdot)$ is not only a submonoid of $(\mathcal{T}_X, \cdot)$ but also an inverse monoid where the inverse of $g \in \mathcal{S}_X$ is the set-theoretical inverse of g. This pair is usually called the *symmetric inverse semigroup* of X. The symmetric inverse semigroup plays important role in semigroup theory due to the following result by Vagner and Preston (see Theorem 1.10 p. 135 in [28]).

Theorem 1.11 (Vagner-Preston Representation Theorem)**.** *For all inverse semigroup S there is an injective homomorphism $\phi : S \to \mathcal{S}_S$.*

Proof. Define $\phi : S \to \mathcal{T}_S$ by $Dom(\phi(g)) = Sg^{-1}g$ and $\phi(g)(x) = xg^{-1}$ whenever $g \in S$. We have $Rang(\phi(g)) = Sgg^{-1}$ for if $y \in Rang(\phi(g))$ then $y = hg^{-1}gg^{-1}$ for some $h \in S$ thus $y = (hg^{-1})gg^{-1} \in Sgg^{-1}$ therefore $Rang(\phi(g)) \subset Sgg^{-1}$ and, conversely, if $y \in Sgg^{-1}$ then $y = hgg^{-1}$ for some $h \in S$ thus $y = xg^{-1}$ with $x = hgg^{-1}g \in Sg^{-1}g = Dom(\phi(g))$ hence $Sgg^{-1} \subset Rang(\phi(g))$. which proves $Rang(\phi(g)) = Sgg^{-1}$ for all $g \in S$.

It follows that $Rang(\phi(g)) = Dom(\phi(g^{-1}))$ thus the composition $\phi(g^{-1}) \circ \phi(g) : Dom(\phi(g) \to Dom(\phi(g))$ is well defined. Since for $x \in Dom(\phi(g)) = Sg^{-1}g$ one has $x = hg^{-1}g$ for some $h \in S$ we have $(\phi(g^{-1}) \circ \phi(g))(x) = xg^{-1}g = hg^{-1}gg^{-1}g = hg^{-1}g = x$ therefore $\phi(g^{-1}) \circ \phi(g) = I_{Dom(\phi(g))}$ for all $g \in S$. Hence $\phi(g) \in \mathcal{S}_S$ and $(\phi(g))^{-1} = \phi(g^{-1})$ for all $g \in S$.

To prove that ϕ is a homomorphism we must prove for all $g, h \in S$ that $Dom(\phi(gh)) = Dom(\phi(g) \cdot \phi(h))$ and $\phi(gh)(x) = (\phi(g) \cdot \phi(h))(x)$ for all $x \in Dom(\phi(gh))$. Now,

$$Dom(\phi(gh)) = S(gh)^{-1}(gh) = Sh^{-1}g^{-1}gh$$

and

$$\begin{aligned} Dom(\phi(g) \cdot \phi(h)) &= (\phi(h))^{-1}(Dom(\phi(g) \cap Rang(\phi(h))) \\ &= (Sg^{-1}g \cap Shh^{-1})h \end{aligned}$$

thus by Lemma 1.10 we get

$$\begin{aligned} Dom(\phi(g)\cdot\phi(h)) &= Sg^{-1}ghh^{-1}h \\ &= Sg^{-1}gh \\ &= Sh^{-1}g^{-1}gh \\ &= Dom(\phi(gh)). \end{aligned}$$

Evidently $(\phi(g)\cdot\phi(h))(x) = \phi(gh)(x) = x(gh)^{-1}$, $\forall x \in Dom(\phi(gh))$ therefore π is a homomorphism.

Finally, if $\phi(g) = \phi(h)$ then $Sg^{-1}g = Sh^{-1}h$ and $xg^{-1} = xh^{-1}$ for all $x \in Sg^{-1}g = Sh^{-1}h$. Now, $g = gg^{-1}g \in Sg^{-1}g$ hence $gg^{-1} = gh^{-1}$ by taking $x = g^{-1}$. Analogously $hg^{-1} = hh^{-1}$. On the other hand, we also have $\phi(g^{-1}) = \phi(h^{-1})$ so $Sgg^{-1} = Shh^{-1}$ and $xg = xh$ for all $x \in Sgg^{-1} = Shh^{-1}$. Since $h^{-1} = h^{-1}hh^{-1} \in Shh^{-1}$ we get $h^{-1}g = h^{-1}h$ and then

$$g = gg^{-1}g = gh^{-1}g = (hg^{-1})^{-1}g = hh^{-1}g = hh^{-1}h = h$$

so ϕ is injective and the result follows. □

1.3 Partial orders for semigroups

A relation on a set X is a subset ω of $X \times X$. It is customary to write $x\omega y$ instead of $(x, y) \in \omega$. We say that the relation ω of X is *reflexive* if $x\omega x$ for all $x \in X$; *symmetric* if $x\omega y$ implies $y\omega x$; *antisymmetric* if $x\omega y$ and $y\omega x$ imply $x = y$; and *transitive* if $x\omega y$ and $y\omega z$ imply $x\omega z$. A reflexive transitive relation is called an *equivalence* or a *partial order* depending on whether it is symmetric or antisymmetric. Partial orders are usually denoted by $\leq$.

Given a partial order $\leq$ on X and $x \in X$ we define $(-\infty, x] = \{y \in S : y \leq x\}$. If $H \subset X$ then we define the *closure* of H with respect to $\leq$ by

$$\overline{H} = \{y \in X : y \leq h \text{ for some } h \in H\}. \tag{1.1}$$

Clearly we have $\overline{\{x\}} = (-\infty, x]$ and further

$$\overline{H} = \bigcup_{h\in H} (-\infty, h]. \tag{1.2}$$

The following lemma is a direct consequence of the definitions.

Lemma 1.12. *If $\leq$ is a partial order on a set X, then the following properties hold for all subsets $\Gamma, \tilde{\Gamma} \subset X$ and all collection of subsets $\{\Gamma_r\}_{r\in I}$ of S:*

1. $\Gamma \subset \overline{\Gamma}$;
2. *If* $\Gamma \subset \tilde{\Gamma}$ *then* $\overline{\Gamma} \subset \overline{\tilde{\Gamma}}$;
3. $\overline{\bigcap_{r\in I} \Gamma_x} \subset \bigcap_{r\in I} \overline{\Gamma_r}$ *and* $\overline{\bigcup_{r\in I} \Gamma_x} = \bigcup_{r\in I} \overline{\Gamma_r}$;
4. $\overline{\overline{\Gamma}} = \overline{\Gamma}$.

For instance (1.2) implies

$$\overline{\bigcup_{r\in I} \Gamma_r} = \bigcup_{\gamma\in\bigcup_{r\in I}\Gamma_r} (-\infty, \gamma] = \bigcup_{r\in I} \bigcup_{\gamma\in\Gamma_r} (-\infty, \gamma] = \bigcup_{r\in I} \overline{\Gamma_r}.$$

A relation ω on a semigroup S is *left* or *right compatible* depending on whether $a\omega b$ implies $ca\omega cb$ for all $c \in S$ or $a\omega b$ implies $ac\omega bc$ for all $c \in S$. A *compatible relation* is one which is both left and right compatible.

There is a natural partial order in the set of idempotents E of S, the *Rees order*, defined by $e \leq f$ if and only $e = fe = ef$ ([28]). The problem as to whether this order can be extended to a (possibly compatible) order in the whole semigroup S has been investigated elsewhere (see the Introduction in [39]). For instance Vagner proved the following result in 1952.

Proposition 1.13. *If S is an inverse semigroup, then the relation $\leq$ on S defined by $h \leq g$ if and only if $h = ig$ for some $i \in E$ is a partial order which reduces to the Rees order when restricted to E.*

Proof. Since $gg^{-1} \in E$ and $g = gg^{-1}g$ for all $g \in S$ we have that $\leq$ is reflexive. Now suppose that $h \leq g$ and $g \leq h$, namely, there are idempotents e, f such that $h = eg$ and $g = fh$. Then, $fg = ffh = fh = g$ for $f \in E$ thus $h = eg = efg = feg = fh = g$ since $E = Z(E)$ by Proposition 1.4. This proves that $\leq$ is antisymmetric. Next suppose that $h \leq g \leq k$ thus $h = eg$ and $g = fk$ for some $e, f \in E$ hence $h = efk$ with $ef \in E$ proving that $\leq$ is transitive. Therefore $\leq$ is a partial order as claimed. The last statement of the proposition is evident. $\square$

The partial order in the previous proposition will be referred to as the *Vagner order* of an inverse semigroup. A remarkable property of this order is given below.

Lemma 1.14. *If $\leq$ is the Vagner order of an inverse semigroup S, then $a \leq b$ if and only if $a = bj$ for some $j \in E$. Hence $a \leq b$ and $c \leq d$ imply $a^{-1} \leq b^{-1}$ and $ac \leq bd$. Therefore $\leq$ is compatible.*

Proof. Suppose that $a \leq b$. Hence $a = ib$ for some $i \in E$. Now $bb^{-1} \in E$ is idempotent and $Z(E) = E$ by Proposition 1.4 therefore $ibb^{-1} = bb^{-1}i$. Then, $a = ib = ibb^{-1}b = b(b^{-1}ib) = bj$ where $j = b^{-1}ib$. But $j^2 = b^{-1}ibb^{-1}ib = b^{-1}bb^{-1}i^2b = b^{-1}ib = j$ hence $j \in E$ and we are done. The proof of the converse implication is similar.

Now suppose that $a \leq b$. By the first part of the lemma one has $a = bj$ for some idempotent j and then $a^{-1} = j^{-1}b^{-1} = jb^{-1}$ proving $a^{-1} \leq b^{-1}$. Finally suppose that $c \leq d$. Hence $a = bi$ and $c = jd$ for some idempotents i, j so $ac = bijd$. Now, $ijd \leq d$ by definition since ij is idempotent therefore, by the first part of the lemma, there is another idempotent k such that $ijd = dk$. Thus, $ac = bdk$ with k idempotent and the result follows. $\square$

In 1980 Hartwig and K. Nambooripad extended the partial order for inverse semigroups in Proposition 1.13 to regular semigroups by proving the following.

Proposition 1.15. *If S is a regular semigroup, then the relation $\leq$ defined $h \leq g$ if and only if $h = eg = gf$ for some $e, f \in E$ is a partial order which coincides with the partial order in Proposition 1.13 when S is an inverse semigroup.*

Proof. (See [45] p. 73). Notice that since S is regular we have by Lemma 1.3 that for all $h \in S$ there is $x \in S$ such that $hxh = h$ and $xhx = h$. Now, $(xh)(xh) = xh$ and $(hx)(hx) = hx$ hence $xh, hx \in E$ and $h = (hx)h = h(xh)$ yielding $h \leq h$ so $\leq$ is reflexive. On the other hand, if $h \leq g \leq h$ then $h = eg = gf$ and $g = xh = hy$ for some $e, f, x, y \in E$. Then, $xg = xh = g$ so $g = xh = xgf = gf = h$ proving that $\leq$ is antisymmetric.

Next suppose that $h \leq g \leq k$ namely $h = eg = gf$ and $g = xk = ky$ for some $e, f, x, y \in E$. Since S is regular we can fix $k^* \in V(k)$ so $h = eg = exk = (exk)k^*k = (hk^*)k$. But $(hk^*)(hk^*) = hk^*gfk^* =$

$(exk)k^*(ky)fk^* = (exk)yfk^* = hyfk^* = hfk^* = hk^*$, since $h = eg = egy = hy$ and $h = gf = gff = hf$ therefore $hk^* \in E$. A symmetric argument shows $h = k(k^*h)$ with $k^*h \in E$ therefore $h \leq k$ so $\leq$ is transitive. □

Nevertheless such an order is not necessarily compatible with the semigroup product (c.f. [39]).

An order for semigroups can be obtained from the following result due to Mitsch ([39]).

Proposition 1.16. *For any semigroup S the relation $\leq$ defined by $h \leq g$ if and only if $h = xg = gy$ and $h = xh$ for some $f, k \in S^1$ is a partial order on S.*

Proof. It is clear that $\leq$ is reflexive for $h = 1h = h1$ where 1 is the identity of S^1. Now suppose that $h \leq g$ and $g \leq h$. Then, there are $x, y, z, t \in S^1$ such that $h = xg = gy$, $h = xh$, $g = zh = ht$ and $g = zg$. Thus $h = xg = xht = ht = g$ so $\leq$ is antisymmetric. Finally suppose that $h \leq g \leq k$ then there are $x, y, z, t \in S^1$ such that $h = xg = gy$, $h = xh$, $g = zk = kt$ and $g = zg$. Then, $h = (xz)k = k(ty)$ and $(xz)h = x(zh) = x(zgy) = xgy = xh = h$ so $h \leq k$ thus $\leq$ is transitive. Therefore $\leq$ is a partial order. □

It can be also proved that the Mitsch order on a semigroup S in the above lemma both coincides with the Hartwig-Nambooripad order if S is regular and with the idempotent order when restricted to E.

Another order in a semigroup S comes from the lemma below.

Lemma 1.17. *For any semigroup S the relation $\leq$ defined by $h \leq g$ if and only if $h = ig$ for some $i \in Z(E(S^1))$ is a partial order on S.*

Proof. The proof is similar to the analogous proof for the Vagner's order on inverse semigroups (c.f. Lemma 3.1 p. 137 in [28]). Clearly $\leq$ is reflexive for $g = 1g$ with $1 \in Z(E(S^1))$. In addition, $\leq$ is antisymmetric for if $h = ig$ and $g = jh$ for some $i, j \in Z(E(S^1))$, then $jg = jjh = jh = g$ and so $g = jh = jig = ijg = ig = h$ hence $g = h$. Finally $\leq$ is transitive for if $h = ig$ and $g = jf$ for some $i, j \in Z(E(S^1))$, then $h = ijf$ and is clear that $ij \in Z(E(S^1))$. This proves the lemma. □

Since inverse monoids are inverse semigroups by definition we have for inverse monoids that $Z(E) = E$ by Proposition 1.4. Then, the Vagner order and the order in Lemma 1.17 coincide for inverse monoids.

Hereafter we assume that every inverse semigroup is equipped with the Vagner order and every monoid is equipped with the order in Lemma 1.17.

The following lemma present elementary properties of the closure operation for inverse semigroups. Denote by $\Gamma^{-1} = \{g^{-1} : g \in \Gamma\}$.

Lemma 1.18. *If S is an inverse semigroup, then $\overline{\Gamma^{-1}} = (\overline{\Gamma})^{-1}$ and $\overline{<\Gamma>} = <\overline{\Gamma}>$ for all $\Gamma \subset S$.*

Proof. Take $x \in \overline{\Gamma^{-1}}$ which is equivalent to $x \le g^{-1}$ for some $g \in \Gamma$. By well known properties of inverse semigroups [28] we have that $x^{-1} \le g$ thus $x^{-1} \in \overline{\Gamma}$ which is equivalent to $x \in (\overline{\Gamma})^{-1}$.

To prove the second property we appeal to the following explicit expression of $<\Gamma>$ which is valid for inverse semigroups:

$$<\Gamma> = \{g_1 \cdots g_k : k \in \mathbb{N} \text{ and } g_1, \cdots, g_k \in \Gamma \cup \Gamma^{-1}\}.$$

Hence if $x \in \overline{<\Gamma>}$ then $x = ig_1 \cdots g_k$ for some $g_1, \cdots, g_k \in \Gamma \cup \Gamma^{-1}$ and some idempotent i. Clearly $ig_1, g_2, \cdots, g_k \in \overline{\Gamma \cup \Gamma^{-1}} = \overline{\Gamma} \cup \overline{\Gamma^{-1}} = \overline{\Gamma} \cup (\overline{\Gamma})^{-1}$, by the previous properties, so $x \in <\overline{\Gamma}>$. Conversely if $x \in <\overline{\Gamma}>$ then $x = i_0 h_1 \cdots h_k$ for some idempotent i_0 and some $h_1, \cdots, h_k \in \overline{\Gamma \cup \Gamma^{-1}}$. However, by applying Lemma 1.14 we can write $x = (i_0 i_1 \cdots i_k) g_1 \cdots g_k$ for some idempotents $i_1, \cdots, i_k$ and some $g_1, \cdots, g_k \in \Gamma \cup \Gamma^{-1}$ which implies $x \in \overline{<\Gamma>}$ since the product of idempotents in an inverse semigroup is idempotent too. $\square$

We can use these natural orders to extend the concept of inverse subsemigroups or submonoids as follows. A subset S' of an inverse semigroup S is an *inverse pre-subsemigroup* if $g^{-1} \in S'$ and $gh \in \overline{S'}$ for all $g, h \in S'$. A subset S' of a monoid S is a *pre-submonoid* if $e \in S'$ and $gh \in \overline{S'}$ for all $g, h \in S'$. A subset of an inverse monoid is an *inverse pre-submonoid* if it is an inverse pre-subsemigroup containing the identity. Evidently, every inverse subsemigroup (resp. submonoid or inverse submonoid) is an inverse pre-subsemigroup (resp. pre-submonoid or inverse pre-submonoid).

A subset Γ of either an inverse semigroup or a monoid is *closed* if $\Gamma = \overline{\Gamma}$. The following lemma gives a characterization of closed inverse subsemigroups in terms of inverse pre-subsemigroups.

Lemma 1.19. *An inverse subsemigroup A of an inverse semigroup S is closed if and only if $A = \overline{B}$ for some inverse pre-subsemigroup B of S.*

Proof. Evidently if A is a closed inverse subsemigroup, then $B = A$ is an inverse pre-subsemigroup satisfying $A = \overline{B}$. To prove the converse we only need to prove that $\overline{B}$ is an inverse subsemigroup for all inverse pre-subsemigroups B of S. Indeed, if $\overline{g} \in \overline{B}$ then $\overline{g} \leq g$ for some $g \in B$ by the definition of closure. By Lemma 1.14 we have $\overline{g}^{-1} \leq g^{-1}$ and $g^{-1} \in B$ since B is an inverse pre-subsemigroup. Therefore $\overline{g}^{-1} \in \overline{B}$. Now take $\overline{g}, \overline{h} \in \overline{B}$. By the definition we have $\overline{g} \leq g$ and $\overline{h} \leq h$ for some $g, h \in B$, so, $\overline{g}\overline{h} \leq gh$ by Lemma 1.14. But B is an inverse pre-subsemigroup hence there is $k \in B$ such that $gh \leq k$. Then, $\overline{g}\overline{h} \leq k$ with $k \in B$ hence $\overline{g}\overline{h} \in \overline{B}$. □

The partial order on inverse semigroups or monoids allows us to extend the concept of groupoid homomorphism as follows.

Definition 1.20. *A map $\phi : A \to B$ from a groupoid A to an inverse semigroup or monoid B is a* premorphism *if $\phi(g)\phi(h) \leq \phi(gh)$ for all $g, h \in A$. If A is a semigroup we also require that $\phi(V(a)) \subset V(\phi(a))$ for all $a \in A$. If both A and B have an identity we require $\phi(1) = 1$.*

This definition is a slight generalization of the definition of ν-prehomomorphism from inverse semigroups in [38].

If A is a groupoid and B is an inverse semigroup or monoid, then any groupoid homomorphism from A to B is a premorphism. In addition, premorphic image of inverse semigroups (resp. monoids or inverse monoids) are inverse pre-subsemigroups (resp. pre-submonoids or inverse pre-submonoids).

We say that a map $\phi : A \to B$ from an inverse semigroup or monoid A to an inverse semigroup or monoid B is *closed* if $\phi(\Gamma)$ is closed for all closed subset $\Gamma \subset A$.

Lemma 1.21. *A map $\phi : A \to B$ from an inverse monoid or semigroup A to an inverse semigroup or monoid B is closed if and only if $\overline{\phi(\Gamma)} \subset \phi(\overline{\Gamma})$, $\forall \Gamma \subset A$.*

Proof. First suppose that ϕ is closed and take $\Gamma \subset A$. Hence $\overline{\phi(\overline{\Gamma})} = \phi(\overline{\Gamma})$ since $\overline{\Gamma}$ is closed. Because $\Gamma \subset \overline{\Gamma}$ we get $\phi(\Gamma) \subset \phi(\overline{\Gamma})$ so $\overline{\phi(\Gamma)} \subset \overline{\phi(\overline{\Gamma})} = \phi(\overline{\Gamma})$ which proves the direct implication. For the converse suppose that $\Gamma \subset A$ is closed hence $\overline{\Gamma} = \Gamma$. By the hypothesis we have $\overline{\phi(\Gamma)} \subset \phi(\overline{\Gamma}) = \phi(\Gamma)$ thus $\overline{\phi(\Gamma)} = \phi(\Gamma)$ and the result follows. $\square$

Chapter 2

Partial semigroups

In this section we extend some of the definitions in the previous chapter to partially defined operations.

A *partial groupoid* as a pair (S, α) where S is a set and α is a partially defined binary operation on S, i.e., a map $\alpha : Dom(\alpha) \subset S \times S \to S$. We consistently use the notation gh instead of $\alpha(g, h)$ and, eventually, the notations S and Dom instead of (S, α), and $Dom(\alpha)$ respectively. Partial groupoid are also referred to in the literature as *pargoids* [34] or *partial rings* [32].

Following [32] we have that any partial groupoid S is naturally equipped with a binary relation, the operation relation Dom, which is defined by $g \, Dom \, h$ if and only if $(g, h) \in Dom$. Conversely, a partial groupoid can be defined as a triple consisting of a set S, a binary relation R in S and a map $\alpha : R \to S$ (for this point of view see [56]). We then say that the partial groupoid S is *reflexive*, *symmetric*, *antisymmetric* or *transitive* depending on whether its operation relation is.

A *partial subgroupoid* of a partial groupoid S is a subset $S' \subset S$ such that if $g, h \in S'$ and $g \, Dom \, h$, then $gh \in S'$. A map $\phi : A \to B$ from a partial groupoid A to a partial groupoid B is a *homomorphism* whenever $a \, Dom \, b$ if and only if $\phi(a) \, Dom \, \phi(b)$ in whose case $\phi(ab) = \phi(a)\phi(b)$ (this is what is called strict homomorphism in [32]). It is clear that the homomorphic image of partial groupoids are partial subgroupoids.

Let S be a partial groupoid. An *identity* of S an element 1 such that $1\,Dom\,g$, $1\,Dom\,e$ and $1g = g1 = g$ for all $g \in S$. A *zero* is an element 0 such that $g\,Dom\,0$, $0\,Dom\,g$ and $g0 = 0g = 0$ for all $g \in S$. If S is a partial groupoid without identity (resp. zero), then the set $S \cup \{1\}$ (resp. $S \cup \{0\}$) equipped with the extended product $g1 = 1g = g$ (resp. $g0 = 0g = 0$) for all $g \in S \cup \{1\}$ (resp. $g \in S \cup \{0\}$) is a partial groupoid with identity 1 (resp. zero 0). For convenience we define $S^1 = S$ or $S \cup \{1\}$ depending on whether S has an identity or not. Similarly we define S^0.

An *idempotent* of S is an element i such that $i\,Dom\,i$ and $i^2 = i$. We denote by $E = E(S)$ the set of idempotents of S. As before we have $\phi(E(A)) \subset E(B)$ for all homomorphism of partial groupoids $\phi : A \to B$.

Given $\Gamma \subset S$ we define its *centralizer*

$$Z(\Gamma) = \{g \in \Gamma : g\,Dom\,h \text{ and } h \in \Gamma \Rightarrow h\,Dom\,g \text{ and } hg = gh\}.$$

The set $Z(S)$ is called the *center* of S and we say that S is *commutative* if it is equals to its own center (this is what is called symmetric partial ring in [32]). Equivalently, if $g\,Dom\,h$ implies $h\,Dom\,g$ and $hg = gh$. Evidently every commutative partial groupoid is symmetric but not conversely.

A partial groupoid S is *left associative* if

$$h\,Dom\,f \text{ and } gDom\,hf \Rightarrow g\,Dom\,h, gh\,Dom\,f \text{ and } (gh)f = g(hf)$$

and *right associative* if

$$g\,Dom\,h \text{ and } ghDom\,f \Rightarrow h\,Dom\,f, g\,Dom\,hf \text{ and } g(hf) = (gh)f.$$

An *associative partial groupoid* is a partial groupoid which is both left and right associative. See [32] p. 610 or [29] for another definition of associative partial groupoids. In such a case we write ghf to mean either $g(hf)$ or $(gh)f$ when appropriated. Homomorphic image of associative partial groupoids are associative partial subgroupoids. We shall be mostly interested on associative partial groupoids. The following elementary lemma mentioned in [32].

Lemma 2.1. *A commutative left associative partial groupoid is associative.*

Proof. Let S be a commutative left associative partial groupoid. Take $g \, Dom \, h$ and $x \in S$ with $gh \, Dom \, x$. As S is is symmetric we have $h \, Dom \, g$ and $hg = gh$ hence $x \, Dom \, hg$. But S is left associative so $x \, Dom \, h$ thus $h \, Dom \, x$, $g \, Dom \, hx$ and $g(hx) = (gh)x$. This finishes the proof. □

Let S be an associative partial groupoid. We say that $g \in S$ is *regular* if there is $x \in S$ satisfying $g \, Dom \, x$, $gx \, Dom \, g$ and $gxg = g$. An associative partial groupoid is *regular* if all its elements are.

An *inverse* of an element $g \in S$ is an element $g^* \in S$ satisfying $g \, Dom \, g^*$, $gg^* \, Dom \, g$, $g^* \, Dom \, g$, $g^*g \, Dom \, g^*$, $gg^*g = g$ and $g^*gg^* = g^*$. As before we denote $V(g) = \{g^* \in S : g^* \text{ is an inverse of } g\}$ and say that g is *invertible* if $V(g)$ consists of a single element g^{-1} which is called the *inverse* of g. An invertible element is clearly regular but not conversely. Every idempotent in an associative groupoid is regular and its own inverse (if exists).

If A and B are associative partial groupoids and $\phi : A \to B$ is a groupoid homomorphism, then $\phi(a)$ is regular for all $a \in A$ regular. In addition, $\phi(V(a)) \subseteq V(\phi(a))$ for all $a \in A$.

Definition 2.2. *A partial groupoid S is a*

- partial (regular) semigroup *if it is (regular) associative;*
- partial (regular) monoid *if it a partial (regular) semigroup with an identity;*
- partial inverse semigroup *if it is a partial semigroup where every element has an inverse;*
- partial inverse monoid *if it is a partial inverse semigroup with an identity;*
- partial group *if it is a partial inverse monoid where $g^{-1}g = gg^{-1} = e$ for all $g \in S$.*

The concept of partial semigroup can be found in [56] p. 46.

The substructures corresponding to the above definition are the followings. A *partial subsemigroup* of a partial semigroup S is a partial subgroupoid of S. A *partial submonoid* of a partial monoid is a partial subgroupoid containing the identity. A *partial inverse subsemigroup* of a partial inverse semigroup is a partial subsemigroup S' which is *symmetric*, i.e., $g^{-1} \in S'$ for all $g \in S'$. A *partial inverse submonoid* of a partial inverse monoid is a partial inverse subsemigroup containing the identity. A *partial subgroup* of a partial group G is a partial inverse submonoid of G.

To these substructures we can define their corresponding generated substructures by noting that, on all partial groupoids (resp. semigroups, monoids, inverse semigroups, inverse monoids, groups) S, the intersection of a non-empty family of partial subgroupoids (resp. subsemigroups, submonoids, inverse subsemigroups, inverse submonoids, subgroups) is either empty or a partial subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S. On the other hand, if Γ is an arbitrary non-empty subset of S, then the family of partial subgroupoids (resp. subsemigroups, submonoids, inverse subsemigroups, inverse submonoids, subgroups) of S containing Γ is non-empty for it contains S itself. Then, the intersection $< \Gamma >$ of this family is a partial subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S which is called the *partial subgroupoid (resp. subsemigroup, submonoid, inverse subsemigroup, inverse submonoid, subgroup) of S generated by* Γ. If $< \Gamma >= S$ then we say that Γ *generates* S (or that S is *generated* by Γ). Again we write $< \Gamma >_{(\cdot)}$ in the cases when we need to emphasize the binary operation $\cdot$ of S.

Evidently a transitive partial groupoid with an identity is a monoid while a transitive partial groupoid with a zero is a groupoid.

It is possible to generalize results in semigroup theory to partial semigroups (this idea was carried out in [56]). As a sample we state the following which is the partial version of Lemma 1.3. The proof is essentially the same as in that lemma but with some minor complications due to the domain relation.

Proposition 2.3. *If S is a partial semigroup, then $g \in S$ is regular if and only if $V(g) \neq \emptyset$.*

Proof. Obviously we only have to prove the direct implication. If g is regular, then there is $h \in S$ such that $g\, Dom\, h$, $gh\, Dom\, g$ and $ghg = g$. Associativity implies that $h\, Dom\, g$, $g\, Dom\, hg$ and $g(hg) = g$. Define $k = hg$. Then, $g\, Dom\, k$ and $gk = g$. As $g\, Dom\, h$ we have

$gk\, Dom\, h$ and then $k\, Dom\, h$, $g\, Dom\, kh$ and $g(kh) = (gk)h = gh$ by associativity.

Now set $g^* = kh$. Then, $g\, Dom\, g^*$ and $gg^* = gh$. As $gh\, Dom\, g$ we have $gg^*\, Dom\, g$ and so associativity implies $g^*\, Dom\, g$, $g\, Dom\, g^*g$ and $g(g^*g) = (gg^*)g = (gh)g = g$ therefore $gg^*g = g$.

On the other hand, since $g\, Dom\, g^*$ and $g(g^*g) = g$ we have $g(g^*g)\, Dom\, g^*$ and then $gg^*\, Dom\, g^*$ by associativity. To compute g^*gg^* we see that since $g^* = kh$ we have $kh\, Dom\, g$ therefore $k\, Dom\, hg$ and $k(hg) = (kh)g$. Thus, $gg^* = (kh)g = k(hg) = k^2$. But $k^2 = (hg)hg) = h(ghg) = hg = k$. Therefore $g^*g = k$ and then $g^*gg^* = k(kh) = k^2h = kh = g^*$. Hence $g^* \in V(g)$ so $V(g) \neq \emptyset$. □

We finish this section with the following definition.

Definition 2.4. *A map $\phi : A \to B$ from a partial groupoid A to an inverse semigroup or monoid B is a* premorphism *if $g\, Dom\, h$ if and only if $\phi(g)\, Dom\, \phi(h)$ in whose case $\phi(g)\phi(h) \leq \phi(gh)$. If A is a partial semigroup we require $\phi(V(a)) \subset V(\phi(a))$ for all $a \in A$ and if both A and B have an identity we require $\phi(1) = 1$.*

Clearly a homomorphism from a partial semigroup to an inverse semigroup or monoid is a premorphism.

2.1 Examples

In this section we collect some few examples of partial groupoids and partial semigroups.

Example 2.5. *Any groupoid S is a partial groupoid with $Dom = S \times S$. In addition S is a partial (regular) semigroup, partial (regular) monoid, partial inverse semigroup or a partial group depending on whether S is a (regular) semigroup or a (regular) monoid or an inverse semigroup or a group.*

Example 2.6 (Restriction)**.** *Let $(S, \cdot)$ be a partial groupoid with operation relation Dom and $A \subset S$. Define $Dom^A \subset A \times A$ by*

$$Dom^A = \{(a, b) \in A \times A : a\, Dom\, b \text{ and } a \cdot b \in A\}.$$

Then, the pair $(A, \cdot^A)$ *where* $\cdot^A$ *is the partial operation with domain* Dom^A *defined by* $a \cdot^A b = a \cdot b$ *whenever* $a\, Dom^A\, b$ *is a partial groupoid. Notice that such a partial groupoid may not be associative even if* S *is.*

Example 2.7 (Union of partial groupoids)**.** *Let* $\{S_\alpha : \alpha \in I\}$ *a disjoint family of partial groupoids. Denote by* $\cdot_\alpha$ *and* Dom_α *the partial operation and the operation relation of the partial groupoid* S_α *respectively.*

Define Dom *as the set of* $(g, h) \in \left(\bigcup_{\alpha \in I} S_\alpha\right) \times \left(\bigcup_{\alpha \in I} S_\alpha\right)$ *satisfying*

$$g, h \in S_\alpha,\ g \text{ and } Dom_\alpha\, h \text{ for some } \alpha \in I.$$

Then, the pair $\left(\bigcup_{\alpha \in I} S_\alpha, \cdot\right)$ *where* $\cdot$ *is the partial operation with domain* Dom *defined by* $g \cdot h = g \cdot_\alpha h$ *whenever* $g\, Dom_\alpha\, h$ *is a partial groupoid. Notice that such a partial groupoid is a partial (regular) semigroup if and only if every* S_α *is.*

Example 2.8 (Direct sums of partial groupoids)**.** *Again consider a family of partial groupoids* $\{S_\alpha : \alpha \in I\}$ *each one with partial operation* $\cdot_\alpha$ *and operation relation* Dom_α*. Define*

$$\bigoplus_{\alpha \in I} S_\alpha = \left\{ \eta : I \to \bigcup_{\alpha \in I} : \eta(\alpha) \in S_\alpha \right\}$$

and Dom *as the set of* $(\eta, \nu) \in \left(\bigoplus_{\alpha \in I} S_\alpha\right) \times \left(\bigoplus_{\alpha \in I} S_\alpha\right)$ *satisfying*

$$\eta(\alpha)\, Dom_\alpha\, \nu(\alpha), \quad \forall \alpha \in I.$$

Then, the pair $\left(\bigoplus_{\alpha \in I} S_\alpha, \cdot\right)$ *where* $\cdot$ *is the partial operation with domain* Dom *defined by*

$$(\eta \cdot \nu)(\alpha) = \eta(\alpha) \cdot_\alpha \nu(\alpha), \quad \forall \alpha \in I$$

whenever $\eta\, Dom\, \nu$ *is a partial groupoid called the* direct sum *of the partial groupoids* S_α*,* $\alpha \in I$*.*

Example 2.9. *If* S *is a partial groupoid with* $Dom = \{(g, g) : g \in S\}$*, then* S *is commutative and* S *is associative if and only if* S *is* idempotent *(i.e.* $S = E$*). Indeed, suppose that* S *is associative and take* $g \in S$*. Thus* $g\, Dom\, g$ *and* $gg\, Dom\, gg$*. Setting* $h = g$ *and* $k = gg$ *in the associativity law we get* $g\, Dom\, gg$ *so* $gg = g$ *which proves that* S *is idempotent. We left the converse implication to the reader.*

Example 2.10 (Free partial groupoids). *Consider a non-empty set A. Denote by F_A the set of non-empty words $a_1 a_2 \cdots a_n$ in the alphabet A. Define the maps $i, j : F_A \to A$ by $i(a) = a_n$ and $j(a) = 1$ whenever $a = a_1 a_2 \cdots a_n$. Given $B \subset A$ we define*

$$Dom^B = \{(a,b) \in F_A \times F_A : i(a), j(b) \notin B\}.$$

Then, the pair $(F_A, \cdot^B)$ where $\cdot^B$ is the partial operation with domain Dom defined by juxtaposition, i.e.,

$$a \cdot^B b = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$$

whenever $a = a_1 a_2 \cdots a_n$, $b = b_1 b_2 \cdots b_m$ and $a \, Dom \, b$ is a partial semigroup which is neither commutative nor regular. Notice that if $B = \emptyset$, then F_A is nothing but the free semigroup generated by A (e.g. [28] p. 29).

2.1.1 Partial groupoid Congruences

To present more examples we need the following definitions. A relation ω in a partial groupoid S is *left compatible* if for all $g, h, k \in S$ one has that $g \, \omega \, h$, $k \, Dom \, g$ and $k \, Dom \, h$ imply $kg \, Dom \, kh$. It is *right compatible* if $g \, \omega \, h$, $g \, Dom \, k$ and $h \, Dom \, k$ imply $gk \, \omega \, hk$. Finally it is *compatible* if for all $g, h, k, p \in S$ one has that $g \, \omega \, h$, $k \, \omega \, p$, $g \, Dom \, k$ and $h \, Dom \, p$ imply $gk \, \omega \, hp$.

A *congruence* (resp. *left congruence*, *right congruence*) of a partial groupoid S is an equivalence which is compatible (resp. left compatible, right compatible). As in [28] p. 21 we can prove that a relation in a partial groupoid is a congruence if and only if it is a left and right congruence simultaneously. (We left it to the reader.)

For any congruence ω of a partial groupoid S and $g \in S$ we define the equivalence class of $g \in S$, $g\omega = \{h \in S : g \, \omega \, h\}$, the set of equivalence classes $S/\omega = \{g\omega : g \in S\}$ and the projection $\omega^\# : S \to S/\omega$ by $\omega^\#(g) = g\omega$.

Example 2.11. *Let ω be a congruence of a partial groupoid S. Define*

$$Dom_\omega = \{(\rho, \mu) \in S/\omega \times S/\omega : g \, Dom \, h \text{ for all } (g,h) \in \rho \times \mu\}.$$

Define the partial operation $\cdot_\omega$ *with domain* Dom_ω *by* $\rho \cdot_\omega \mu = gh\omega$ *whenever* $(g,h) \in \rho \times \mu$ *with* $g\, Dom\, h$. *Since* ω *is a congruence we have that this operation is well defined and then the pair* $(S/\omega.\cdot_\omega)$ *is a partial groupoid and* $\omega^{\#} : S \to S/\omega$ *is a homomorphism of partial groupoids. Notice that* S/ω *is a partial semigroup if* S *is.*

Example 2.12. *Let the* kernel $Ker(\phi)$ *of a homomorphism of partial groupoids* $\phi : A \to B$ *be defined as the relation in* A *below*

$$Ker(\phi) = \{(g,h) \in A : \phi(g) = \phi(h)\}.$$

It is easy to prove that $Ker(\phi)$ *is in fact a congruence of* A. *Hence* $A/Ker(\phi)$ *is a partial groupoid and* $Ker(\phi)^{\#} : A \to A/Ker(\phi)$ *is a homomorphism of partial groupoids according to Example 2.11. Now the map* $\phi^{\#} : A/Ker(\pi) \to B$ *given by* $\phi^{\#}(\rho) = \phi(g)$ *whenever* $g \in \rho$ *is a well defined homomorphism of partial groupoids satisfying* $\phi^{\#} \circ (Ker(\phi))^{\#} = \phi$. *(Again we left it to the reader as an exercice.) This example corresponds to the First Isomorphism Theorem in group theory.*

2.1.2 K-theory for commutative semigroups

Now we introduce a classical construction of an abelian group from a commutative semigroup used by Alexander Grothendieck in his K-theory.

Let S be a partial semigroup. We say that S satisfies the *left* or *right cancellation property* depending on whether $c\, Dom\, a$, $c\, Dom\, b$ and $ca = cb$ implies $a = b$ or $a\, Dom\, c$, $b\, Dom\, c$ and $ac = bc$ implies $a = b$. We say that S satisfies the *cancellation property* if it satisfies the left and right cancellation property. Evidently a commutative partial semigroup satisfying either left or right cancellation property satisfies the cancellation property and conversely.

Now consider a commutative partial semigroup $S = (S, +)$ with the cancellation property and define the ω in $S \times S$ by

$$(h,g)\omega(h',g') \quad \Leftrightarrow \quad h\, Dom\, g',\quad h'\, Dom\, g \text{ and } h + g' = h' + g.$$

Then we have the following

Lemma 2.13. *If S is a commutative semigroup (i.e. Dom $= S \times S$) with the cancellation property, then the relation ω above is an equivalence in $S \times S$.*

Proof. Clearly ω is reflexive and symmetric. To prove that it is transitive suppose $(h,g)\,\omega\,(h',g')\,\omega\,(h'',g'')$ then $h+g' = h'+g$ and $h'+g'' = h''+g'$. Then, $h+g'+g'' = h'+g''+g = h''+g'+g$ and so $h+g'' = h''+g$ hence $(h,g)\omega(h'',g'')$ proving the transitivity. $\square$

It follows from this lemma that any commutative semigroup with the cancellation property S induces the set

$$K(S) = S \times S/\omega$$

which we shall call the *K-theory* of S.

Given $(h,g) \in S \times S$ we denote by $[(h,g)] \in K(S)$ the equivalence class of (h,g) with respect to ω. It is customary to write $h-g$ instead of (h,g) and so $[h-g]$ instead of $[(h,g)]$.

We define a binary operation $+$ in $K(S)$ by

$$[h-g]+[h'-g'] = [(h+h')-(g+g')]$$

Proposition 2.14. *If S is a commutative semigroup with the cancellation property, then the above operation is well defined and $K(S)$ with this operation is a commutative group.*

Proof. To prove that the operation is well defined we suppose that $(\overline{h}-\overline{g})\omega(h-g)$ and $(\overline{h}'-\overline{g}')\omega(h'-g')$. Then, $\overline{h}+g = h+\overline{g}$ and $\overline{h}'+g' = h'+\overline{g}'$ so $\overline{h}+\overline{h}'+g+g' = h+h'+\overline{g}+\overline{g}'$ hence $[(\overline{h}+\overline{h}')-(\overline{g}+\overline{g}')] = [(h+h')-(g-g')]$ proving that the operation is well defined.

It is clear that the operation is commutative since S is. Now suppose that $[h-g]$ is idempotent. Hence $[h-g]+[h-g] = [h-g]$ so $[(h+h)-(g+g)] = [h-g]$ thus $h+h+g = h+g+g$ which implies $h = g$ by cancellation. This proves that $K(S)$ has a unique idempotent which is $[h-h]$ for some fixed $h \in S$. Next we observe that a simple computation shows that $[h-g]+[g-h]+[h-g] = [h-g]$ for all $[h-g] \in K(S)$ which implies that $K(S)$ is regular. It then follows from Corollary 1.5 that $K(S)$ is a commutative group and we are done. $\square$

2.1.3 The overlap operation

This is another partial operation on $\mathcal{T}_X$, the set of all maps $g : Dom(g) \subset X \to X$. It is the operation $\cup : Dom(\cup) \subset \mathcal{T}_X \times \mathcal{T}_X \to \mathcal{T}_X$, defined by

$$Dom(\cup) = \{(g, h) \in \mathcal{T}_X \times \mathcal{T}_X : g(x) = h(x), \ \ \forall x \in Dom(g) \cap Dom(h)\},$$

and

$$(g \cup h)(x) = \begin{cases} g(x), & \text{if} \quad x \in Dom(g) \\ h(x), & \text{if} \quad x \in Dom(h). \end{cases}$$

Clearly $Dom(g \cup h) = Dom(g) \cup Dom(h)$ and $Rang(g \cup h) = Rang(g) \cup Rang(h)$ for all $(g, h) \in Dom(\cup)$.

With these notations we have the following.

Theorem 2.15. *The pair $(\mathcal{T}_X, \cup)$ is a commutative partial semigroup.*

Proof. The commutativity of $(\mathcal{T}_X, \cup)$ is clear from the definition. Hence, by Lemma 2.1, to prove that $(\mathcal{T}_X, \cup)$ is associative it suffices to prove that it is left associative. Suppose that (h, f) and $(g, h + f) \in Dom(\cup)$. Then $h = f$ in $Dom(h) \cap Dom(f)$ and $g = h \cup f$ in $(Dom(g) \cap Dom(h)) \cup (Dom(g) \cap Dom(f)$. If $x \in Dom(g) \cap dom(h)$ then $(h \cup f)(x) = h(x)$ hence $g(x) = h(x)$ and so $(g, h) \in Dom(\cup)$. Analogously $g = f$ in $Dom(g) \cap Dom(f)$. But $h = f$ in $Dom(h) \cap Dom(f)$ hence $g \cup h = f$ in $(Dom(g) \cup Dom(h)) \cap Dom(f)$ proving $(g \cup h, f) \in Dom(\cup)$. Finally $(g \cup h) \cup f$ and $g \cup (h \cup f)$ have the common domain $Dom(g) \cup Dom(h) \cup (f)$ and is clear that $(g \cup h) \cup f) = g \cup (h \cup f)$ in that domain. This proves that $(\mathcal{T}_X, \cup)$ is a partial semigroup.

Now take $(g, h) \in Dom(\cup)$ and $f \in \mathcal{T}_X$. Then, $g = h$ in $Dom(g) \cap Dom(h)$. On the other hand,

$$Dom(fg) = g^{-1}(Dom(f) \cap Rang(g)) \subset Dom(g)$$

and

$$Dom(fh) = h^{-1}(Dom(f) \cap Rang(h)) \subset Dom(h)$$

so $Dom(fg) \cap Dom(fh) \subset Dom(g) \cap Dom(h)$. If $x \in Dom(fg) \cap Dom(fh)$ then $x \in Dom(g) \cap Dom(h)$ and so $g(x) = h(x)$ yielding $f(g(x)) = f(h(x))$ hence $fg = fh$ in $Dom(fg) \cap Dom(fh)$ proving $(fg, fh) \in Dom(\cup)$.

Next observe that

$$Dom(gh) = f^{-1}(Dom(g) \cap Rang(f))$$

and

$$Dom(hf) = f^{-1}(Dom(h) \cap Rang(f)).$$

If $x \in Dom(gf) \cap Dom(hf)$ then $f(x) \in Dom(g) \cap dom(h)$ and so $g(f(x)) = h(f(x))$ hence $gf = hf$ in $Dom(gf) \cap Dom(hf)$ proving $(gf, hf) \in Dom(\cup)$.

If $x \in Dom(f(g \cup h))$ then $x \in Dom(g \cup h)$ and $(g \cup h)(x) \in Dom(f)$. When $x \in Dom(g)$ we have $(g \cup h)(x) = g(x)$ hence $g(x) \in Dom(f)$ so $x \in g^{-1}(Dom(f) \cap Rang(g))$. When $x \in Dom(h)$ we have $(g \cup h)(x) = h(x)$ hence $h(x) \in Dom(f)$ so $x \in h^{-1}(Dom(f) \cap Rang(h))$. All together imply $x \in Dom((fg) \cup (fh))$ therefore

$$Dom(f(g \cup h)) \subset Dom((fg) \cup (fh)).$$

Conversely, if $x \in Dom((fg) \cup (fh))$ then $x \in g^{-1}(Dom(f) \cap Rang(g))$ or $x \in h^{-1}(Dom(f) \cap Rang(h))$. In the first case we get $g(x) \in Dom(f)$ so $(g \cup h)(x) = g(x) \in Dom(f)$ thus $x \in (g \cup h)^{-1}(Don(f) \cap Rang(g \cup h))$. Analogous conclusion in the second case hence

$$Dom((fg) \cup (fh)) = Dom(f(g \cup h)).$$

Clearly both $f(g \cup h)$ and $(fg) \cup (fh)$ coincide in their common domain $Dom((fg) \cup (fh)) = Dom(f(g \cup h))$ therefore $f(g \cup h) = (fg) \cup (fh)$. The identity $(g \cup h)f = (gf) \cup (hf)$ is left to the reader. □

2.2 Partial inverse semirings

A *partial (inverse) semiring* is a set S with two binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative partial semigroup; $(S, \cdot)$ is an (inverse) monoid and the following distributive law holds: If $(g, h) \in Dom(+)$ and $f \in S$, then $(fg, fh) \in Dom(+)$, $(gf, hf) \in Dom(+)$, $f(g + h) = (fg) + (fh)$ and $(g + h)f = (gf) + (hf)$. This definition is a particular case of those given in [9], [21] or [22].

Frequently we say that the triple $(S, +, \cdot)$ is a partial (inverse) semiring in order to emphasize the operations $+$ and $\cdot$. We also denote the two operations in different partial (inverse) semirings with the same symbols $+$ and $\cdot$. Given two elements g, h of a partial semiring S we write $g \leq h$ to mean that $g \leq h$ with respect to the natural order of the (inverse) monoid $(S, \cdot)$. The *closure* of a subset $\Gamma \subset S$ is the closure $\overline{\Gamma}$ of Γ with respect to the monoid $(S, \cdot)$ and we say that Γ is *closed* if it does with respect to $(S, \cdot)$, namely, if $\overline{\Gamma} = \Gamma$. We say that a map $\phi : A \to B$ from a partial semiring A to a partial semiring B is *closed* if it does as a map from the monoid $(A, \cdot)$ to the monoid $(B, \cdot)$.

The corresponding substructure is as follows. A *partial (inverse) subsemiring* of a partial (inverse) semiring S is a subset S' of S which is both a partial subsemigroup of $(S, +)$ and an (inverse) submonoid of $(S, \cdot)$. We also define *partial (inverse) pre-subsemiring* of a partial (inverse) semiring S as a subset S' which is both a partial subsemigroup of $(S, +)$ and an (inverse) pre-submonoid of $(S, \cdot)$.

For the corresponding generating substructure we notice that, in all partial (inverse) semirings S, the intersection of a non-empty family of partial subsemirings is a partial subsemiring. On the other hand, if Γ is an arbitrary non-empty subset of S, then the family of partial (inverse) subsemirings of S containing Γ is non-empty for it contains S itself. Then, the intersection $[\Gamma]$ of this family is a partial (inverse) subsemiring of S which is called the *partial (inverse) subsemiring of S generated by* Γ. If $[\Gamma] = S$ then we say that Γ *generates* S (or that S is *generated* by Γ). We say that S is *finitely (countably) generated* if it is generated by a finite (countable) set.

For partial (inverse) semirings we can define an additional substructure in the following way.

Definition 2.16. *A* pseudogroup *of a partial (inverse) semigroup S is a closed partial (inverse) subsemiring of S.*

The following lemma, which is a direct consequence of Lemma 1.19, gives a characterization of pseudogroups on partial inverse semirings in terms of partial inverse pre-subsemirings.

Lemma 2.17. *A subset of a partial inverse semiring S is a pseudogroup if and only if it is the closure of some partial inverse pre-subsemiring of S.*

Once again we notice that the intersection of a non-empty family of pseudogroups is a pseudogroup and, furthermore, for any non-empty set Γ of S, the family of pseudogroups of S containing Γ is non-empty for it contains S itself. Then, the intersection S_Γ of this family is a pseudogroup of S which is called the *pseudogroup of S generated by* Γ.

We can obtain an equivalent expression for S_Γ. Define the subset's sequence S_Γ^n by $S_\Gamma^0 = \Gamma$, $S_\Gamma^{2k+1} = \overline{S_\Gamma^{2k}}$ (for $k \geq 0$) and $S_\Gamma^{2k} = [S_\Gamma^{2k-1}]$ (for $k \geq 1$). As $S_\Gamma^{2k-1} \subset S_\Gamma^{2k}$ (since $A \subset [A]$ for all $A \subset S$) and $S_\Gamma^{2k} \subset S_\Gamma^{2k+1}$ (by Lemma 1.18) one has that this sequence is increasing. With these notations we have the following.

Theorem 2.18. *If S is a partial (inverse) semiring and $\Gamma \subset S$ is non-empty, then*

$$S_\Gamma = \bigcup_{n=0}^{\infty} S_\Gamma^n.$$

Proof. For simplicity we denote by H the set in the the right-hand side of the above equation. To prove that $S_\Gamma = H$ we must prove that H is a pseudogroup of S containing Γ and that every pseudogroup of S containing Γ contains H too. To see the first part we notice that the following two equations

$$H = \bigcup_{n=0}^{\infty} S_\Gamma^{2n} \quad \text{and} \quad H = \bigcup_{n=0}^{\infty} S_\Gamma^{2n+1}$$

hold since S_Γ^n is increasing. Now, each S_Γ^{2n} is a partial (inverse) subsemiring of S by definition and S_Γ^{2n} is increasing hence H is a partial (inverse) subsemiring too by the first equation above. To see that H is closed we notice that each S_Γ^{2n+1} is closed by definition hence H also is by Lemma 1.18 and the second equation above. This proves that H is a pseudogroup of S which evidently contains Γ.

To finish let us consider a pseudogroup S' of S containing Γ. Then, $S_\Gamma^0 \subset S'$. Now suppose that $S_\Gamma^k \subset S'$ for some $k \geq 0$. If k is even we have $S_\Gamma^{k+1} = \overline{S_\gamma^k} \subset \overline{S'} = S'$ since S' is closed. If k is odd then $S_\Gamma^{k+1} = [S_\Gamma^k] \subset S'$ since S' is a partial (inverse) subsemiring of S. We conclude that $S_\Gamma^{k+1} \subset S'$ and then $S_\Gamma^n \subset S'$ for all n by induction. Therefore $H \subset S'$ and we are done. $\square$

Given a partial (inverse) subsemiring S we say that $\Gamma \subset S$ *pseudogenerates* S (or that S is *pseudogenerated* by Γ) if $S = S_\Gamma$. We say that S is *finitely (countably) pseudogenerated* if it is pseudogenerated by a finite (countable) set. Clearly every finitely (or countable) generated partial inverse semigroup is finitely (or countably) pseudogenerated but not conversely.

Now we extend the definition of homomorphisms and premorphisms to partial semirings.

Definition 2.19. *Let $\phi : A \to B$ be a map from a partial semiring A into another partial semiring B with identities e and f respectively. We say that ϕ is a* homomorphism *(resp.* premorphism*) if $\phi : (A, +) \to (B, +)$ is a homomorphism, $\phi : (A, \cdot) \to (B, \cdot)$ is a homomorphism (resp. premorphism) and $\phi(e) = f$.*

Clearly the homomorphic image of a partial semiring into a partial semiring is a partial subsemiring. Moreover, the homomorphic (resp. premorphic) image of a partial inverse semiring into a partial inverse semiring is a partial inverse subsemiring (resp. pre-subsemiring).

A map $\phi : A \to B$ from a partial semiring A to a partial semiring B is *closed* if $\phi(\Gamma)$ is closed for all $\Gamma \subset A$ closed. The following lemma is a direct consequence of Lemma 1.21.

Lemma 2.20. *A map $\phi : A \to B$ from a partial semiring A to a partial semiring B is closed if and only if $\overline{\phi(\Gamma)} \subset \phi(\overline{\Gamma})$, $\forall \Gamma \subset A$.*

2.3 The symmetric partial inverse semiring

This is an important example of a partial semiring. Recall that $\mathcal{T}_X$ denotes the set of all maps $g : Dom(g) \subset X \to X$ which is a monoid if equipped with the composition operation. We also defined in $\mathcal{T}_X$ a partial operation $\cup$ for which the pair $(\mathcal{T}_X, \cup)$ is a commutative partial semigroup by Theorem 2.15. It follows that the triple $(\mathcal{T}_X, \cup, \cdot)$ is a partial semiring.

Now recall that $\mathcal{S}_X$, the symmetric inverse semigroup of X, is the set of all injective elements of $\mathcal{T}_X$. As already noted the pair $(\mathcal{S}_X, \cdot)$ is not only a submonoid of $(\mathcal{T}_X, \cdot)$ but also an inverse monoid where the inverse of $g \in \mathcal{S}_X$ is the set-theoretical inverse of g. By the Vagner-Preston Representation Theorem there is an injective homomorphism $\phi : S \to \mathcal{S}_S$.

Since $(\mathcal{S}_X, \cdot)$ is an inverse submonoid of $(\mathcal{T}_X, \cdot)$ we have $(\mathcal{S}_X, \cdot)$ is itself an inverse monoid thus the triple $(\mathcal{S}_X, \cup, \cdot)$ is a partial inverse semiring. This suggests the following definition.

Definition 2.21. *The triple* $(\mathcal{S}_X, \cup, \cdot)$ *is called the* symmetric partial inverse semiring *of* X.

In light of the Vagner-Preston Represenation Theorem we left to the reader the question if for all partial inverse semirings $(S, +, \cdot)$ there is an injective homomorphism of partial inverse semirings from $(S, +, \cdot)$ to the symmetric inverse semiring of S.

We can further refine the symmetric partial inverse semiring of X in the case when X is a topological space. Indeed, if $Cont(X)$ denotes the set of all continuous elements in $\mathcal{T}_X$, then the triple $(Cont(X), \cup, \cdot)$ is a partial subsemiring.

Now denote by $Homeo(X) \subset Cont(X)$ the set of all $g \in S_X$ for which $g : Dom(g) \to Rang(g)$ is a homeomorphism. Clearly $(Homeo(X), \cdot)$ is a submonoid of $(Cont(X), \cdot)$, $g \cup h \in Cont(X)$ for all $g, h \in Homeo(X)$ with $(g, h) \in Dom(\cup)$ but it may happen that $g \cup h \notin Homeo(X)$ due to the following straightforward lemma.

Lemma 2.22. *If* $g, h \in Homeo(X)$ *and* $(g, h) \in Dom(\cup)$, *then* $g \cup h \in Homeo(X)$ *if and only if* $g(Dom(g) \cap Dom(h)) = Rang(g) \cup Rang(h)$.

Nevertheless, the partial operation $\cup$ in $Cont(X)$ induces one $\cup^*$ in $Homeo(X)$ defined by

$$Dom(\cup^*) = \{(g, h) \in Homeo(X) \times Homeo(X) : (g, h) \in Dom(\cup) \text{ and } g \cup h \in Homeo(X)\}$$

and $g \cup^* h = g \cup h$ whenever $(g, h) \in Dom(\cup^*)$. It is also clear that the triple $(Homeo(X), \cup^*, \cdot)$ is a partial inverse semiring.

One more refinement but now of $(Homeo(X), \cup^*, \cdot)$ can be obtained in the case when X is a differentiable manifold. Indeed, for all $r \geq 0$ we denote by $Diff^r(X)$ the subset of all $g \in Homeo(X)$ such that $Dom(g)$ is open and $g : Dom(g) \to Rang(g)$ is a C^r diffeomorphism. Again we have that the triple $(Diff^r(X), \cup^*, \cdot)$ is a partial inverse semiring.

Now the following definition is natural.

Definition 2.23. *A* pseudogroup of maps *of a set X is a pseudogroup of the partial inverse semiring $(\mathcal{T}_X, \cup, \cdot)$. A* pseudogroup of injective maps *of X is a pseudogroup of the symmetric partial inverse semiring of X. A* pseudogroup of continuous maps *of a topological space X is a pseudogroup of the partial semiring $(Cont(X), \cup, \cdot)$. A* pseudogroup of homeomorphisms *of a topological space X is a pseudogroup of the partial inverse semiring $(Homeo(X), \cup^*, \cdot)$. A* pseudogroup of C^r diffeomorphisms *of a manifold X is a pseudogroup of the partial inverse semiring $(Diff^r(X), \cup^*, \cdot)$, $r \geq 0$.*

The definition of pseudogroup of homeomorphisms above is equivalent to one given in [46].

Corresponding to this definition we have the following one.

Definition 2.24. *A pseudogroup of maps of a set X is* finitely *(or* countably*) generated if it is a finitely (or countably) pseudogenerated in $\mathcal{T}_X$. A pseudogroup of injective maps of X is* finitely *(or* countably*) generated if it is a finitely (or countably) pseudogenerated in the partial symmetric inverse semiring of X. A pseudogroup of continuous maps of a topological space X is* finitely *(or* countably*) generated if it is a finitely (or countably) pseudogenerated in $Cont(X)$. A pseudogroup of homeomorphisms of a topological space X is* finitely *(or* countably*) generated if it is a finitely (or countably) pseudogenerated in $Homeo(X)$. A pseudogroup of C^r diffeomorphisms of a manifold X, $r \geq 0$, is is* finitely *(or* countably*) generated if it is a finitely (or countably) pseudogenerated in $Diff^r(X)$.*

2.4 The holonomy pseudogroup

Let us present an important example of a pseudogroup of homeomorphisms. Recall that a *foliation of class C^r, $r \geq 0$, and codimension p* of a n-dimensional manifold M is a maximal atlas $\mathcal{F} = \{(\Psi_\alpha, U_\alpha)\}_{\alpha \in I}$ such that for all $\alpha, \beta \in I$ satisfying $U_\alpha \cap U_\beta \neq \emptyset$ the map $\Psi_\beta \circ \Psi_\alpha^{-1}$:

$\Psi_\alpha(U_\alpha \cap U_\beta) \to \Psi_\beta(U_\alpha \cap U_\beta)$ is C^r and has the form

$$(\Psi_\beta \circ \Psi_\alpha^{-1})(x, y) = (f_{\alpha\beta}(x, y), g_{\alpha\beta}(y))$$

for all $(x, y) \in \Psi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^{n-p} \times \mathbb{R}^p$. The integer $dim(\mathcal{F}) = n-p$ is called the dimension of $\mathcal{F}$. The pair (Ψ_α, U_α) (or (Ψ, U) for short) is called coordinate chart of $\mathcal{F}$. Without loss of generality we can assume $\Psi(U) = D^{n-p} \times D^p$ where D^k is the unitary disk in $\mathbb{R}^k$ (we shall write D^k_r to indicate the disk of radius r). The domain U is called *distinguished open set* and each set $P = \Psi^{-1}(D^{n-p} \times y)$ is called a *plaque* of $\mathcal{F}$. A *chain of plaques* is a finite sequence $(P_1, \cdots, P_r)$ for which $P_i \cap P_{i+1} \neq \emptyset$ for all $i = 1, \cdots r-1\}$. If $x \in M$ then it belongs to some plaque P and we define the *leaf of* $\mathcal{F}$ *through* x as the union $\mathcal{F}_x$ of all plaques contained in plaque chains starting with P. A *leaf* of $\mathcal{F}$ is a set $L = \mathcal{F}_x$ for some $x \in M$. Clearly the set of all leaves of $\mathcal{F}$ is a partition of M. It can be also proved that each leaf is a C^r submanifold of M and, moreover, that a leaf is compact if and only if it is finite union of plaques.

A distinguished open set U is called *regular* if $\Psi : U \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ can be extended to a chart $\Psi^{ext} : U^{ext} \to \mathbb{R}^{n-p} \times \mathbb{R}^p$ such that $Cl(U) \subset U^{ext}$ and $\Psi^{ext}(U^{ext}) = D_2^{n-p} \times D_2^p$. By a *regular plaque* we mean a plaque of a regular open set U while chain of plaques is regular if all its plaque components are. A *regular covering* of M will be a covering $\mathcal{U} = \{U_i : i \in I\}$ by regular distinguished sets such that every plaque in U_i^{ext} intersects at most one plaque in $U_j^{ext}\}$, $\forall i, j \in I$. Countably regular coverings always exist.

Now suppose that $\mathcal{U} = \{U_i : i \in I\}$ is a countably regular covering. Define $X_i = \Psi_i^{-1}(0 \times D^p)$ thus X_i is a p-dimensional submanifold of M transverse to the leaves of $\mathcal{F}$ (or transverse to $\mathcal{F}$ for short). Given $x \in X_i$ we denote by $P_i(x)$ the plaque of U_i containing x. Since $\mathcal{U}$ is regular we have that for all $i, j \in I$ and $x \in X_i$ there is at most one point $y \in X_j$ such that $P_i(x) \cap P_j(y) \neq \emptyset$. This allows us to define the so-called *transition functions* $\gamma_{ij} : Dom(\gamma_{ij}) \subset X_i \to X_j$ by setting $Dom(\gamma_{ij}) = \{x \in X_i : P_i(x) \cap U_j \neq \emptyset\}$ and $\gamma_{ij}(x) = y$ whenever $P_i(x) \cap P_j(y) \neq \emptyset$. Consider the disjoint union

$$X = \bigcup_{i \in I} X_i \tag{2.1}$$

which is also a submanifold transverse to $\mathcal{F}$. If $i, j \in I$ satisfy $U_i \cap U_j \neq \emptyset$, then $\gamma_{ij} \in Homeo(X)$, $Dom(\gamma_{ij}) \subset X_i$ and $Rang(\gamma_{ij}) \subset X_j$. Define the set

$$\Gamma = \{\gamma_{ij} : (i, j) \in I \times I \text{ and } U_i \cap U_j \neq \emptyset\} \tag{2.2}$$

which is clearly contained in $Diff^r(X)$.

Definition 2.25. *A* holonomy pseudogroup *of $\mathcal{F}$ is a pseudogroup of C^r diffeomorphisms of X as in (2.1) generated by Γ as in (2.2).*

Chapter 3

Partial actions

Given two sets S and X we define

$$\mathbb{A}(S,X) = \{\varphi : Dom(\varphi) \subset S \times X \to X\}$$

and

$$\mathbb{B}(S,X) = \{\phi : S \to \mathcal{T}_X\}.$$

It follows that there is a bijective map $\Psi : \mathbb{A}(S,X) \to \mathbb{B}(S,X)$ from $\mathbb{A}(S,X)$ to $\mathbb{B}(S,X)$ defined by $Dom(\Psi(\varphi)(g)) = \{x \in X : (g,x)) \in Dom(\varphi)\}$ and $\Psi(\varphi)(g)(x) = \varphi(g,x)$. The inverse $\Psi^{-1} : \mathbb{B}(S,X) \to \mathbb{A}(S,X)$ is defined by $Dom(\Psi^{-1}(\phi)) = \{(g,x) \in S \times X : x \in Dom(\phi(g))\}$ and $\Psi^{-1}(\phi)(g,x) = \phi(g)(x)$.

For every $\varphi \in \mathbb{A}(S,X)$ and $g \in S$ we define $D_g = \{x \in X : (g,x) \in Dom(\varphi)\}$ and $\varphi_g : Dom(\varphi_g) \subset X \to X$ by $Dom(\varphi_g) = D_g$ and $\varphi_g(x) = \varphi(g,x)$. We also denote $R_g = \varphi_g(D_g)$. With this notation we have $\Psi(\varphi)(g) = \varphi_g$ for all $\varphi \in \mathbb{A}(S,X)$ and $g \in S$. Given $x \in X$ we define its *orbit* and its *isotropy set* by

$$O_\varphi(x) = \{\varphi(g,x) : x \in D_g\}$$

and

$$S_x = \{g \in S : x \in D_g \text{ and } \varphi(g,x) = x\}$$

respectively. A subset $I \subset X$ is called φ-*invariant* if $O_\varphi(x) \in I$ for all $x \in I$.

If S is a partial groupoid we say that $\varphi \in \mathbb{A}(S,X)$ is *associative* if $(h,x) \in Dom(\varphi)$, $(g,\varphi(h,x)) \in Dom(\varphi)$ and $(g,h) \in Dom$ imply $(gh,x) \in Dom(\varphi)$ and $\varphi(gh,x) = \varphi(g,\varphi(h,x))$. We say that $\varphi \in \mathbb{A}(S,X)$ is *strong associative* if $(h,x) \in Dom(\varphi)$ and $(g,\varphi(h,x)) \in Dom(\varphi)$ if and only if $(g,h) \in Dom$ and $(gh,x) \in Dom(\varphi)$ in which case $\varphi(gh,x) = \varphi(g,\varphi(h,x))$. Strong associativity implies associativity but not conversely.

Definition 3.1. *A* (strong) partial action *(on the left) of a partial groupoid S on X is a (strong) associative map in $\mathbb{A}(S,X)$.*

The following are well known equivalences.

Proposition 3.2. *The equivalences below hold for a partial groupoid S and a set X.*

1. *$\varphi \in \mathbb{A}(S,X)$ is a partial action of S on X if and only if $\Psi(\varphi)$ is a premorphism from S to $\mathcal{T}_X$.*

2. *$\varphi \in \mathbb{A}(S,X)$ is a strong partial action if and only if $\Psi(\varphi)$ is a homomorphism from S to $\mathcal{T}_X$.*

Proof. Let us prove the first equivalence. For the direct implication take $g,h \in S$ with $(g,h) \in Dom$. If $x \in (\varphi_h)^{-1}(Dom(\varphi_g) \cap Rang(\varphi_h))$ then $(h,x) \in Dom(\varphi)$ and $(g,\varphi(h,x)) \in Dom(\varphi)$ so we have $(gh,x) \in Dom(\varphi)$ and $\varphi(gh,x) = \varphi(g,\varphi(h,x)))$. Hence $x \in Dom(\varphi_{gh})$ and $\varphi_{gh}(x) = (\varphi_g \circ \varphi_h)(x)$ therefore $Dom(\varphi_g \cdot \varphi_h)) \subset Dom(\varphi_{gh})$ and $\varphi_{gh}/Dom(\varphi_g \cdot \varphi_h) = \varphi_g \cdot \varphi_h$. Thus $\Psi(\varphi)(g)\Psi(\varphi)(h) \leq \Psi(\varphi)(gh)$ whenever $(g,h) \in Dom$ which proves that $\Psi(\varphi)$ is a premorphism. The reversed implication is left to the reader.

To prove the second equivalence we must prove $\varphi_g \cdot \varphi_h = \varphi_{gh}$ or, equivalently, $D_{gh} = Dom(\varphi_g \cdot \varphi_h)$ and $\varphi_{gh}(x) = (\varphi_g \cdot \varphi_h)(x)$ for all $x \in D_{gh}$ and all $g,h \in S$ with $(g,h) \in Dom$. Take $x \in D_{gh}$, i.e., $(gh,x) \in Dom(\varphi)$. Since $(g,h) \in Dom$ we have $(h,x) \in Dom(\varphi)$, $(g,\varphi(h,x)) \in Dom(\varphi)$ (or, equivalently, $x \in Dom(\varphi_g \cdot \varphi_h)$) and $\varphi_{gh}(x) = (\varphi_g \cdot \varphi_h)(x)$ since φ is strong. □

Elementary properties of partial actions are given below. If F is a map with $Dom(F) \subset X$ and $U \subset X$ we denote by F/U the restriction of F to U. Recall that if $D \subset X$, then $I_D : Dom(I_D) \subset X \to X$ is defined by $Dom(I_D) = D$ and $I_D(x) = x$ for all $x \in D$.

Lemma 3.3. *If φ is a partial action of a partial groupoid S on X and $i \in S$ is idempotent, then $\varphi_i/(R_i \cap D_i) = I_{R_i \cap D_i}$.*

Proof. If $y \in R_i$, then there is $x \in D_i$ such that $y = \varphi(i, x)$. If also $y \in D_i$ then $(i, y) \in Dom(\varphi)$ so $\varphi_i(y) = \varphi(i, y) = \varphi(i, \varphi(i, x)) = \varphi(i^2, x) = \varphi(i, x) = y$ proving the assertion. $\square$

Lemma 3.4. *If φ is a partial groupoid action of a partial groupoid S on X and $x \in X$, then S_x is a partial subgroupoid of S and $O_\varphi(x)$ is φ-invariant.*

Now we extend the definition of partial actions of partial groupoids to partial semigroups.

Definition 3.5. *A partial action of a partial semigroup S on a set X is a map $\varphi \in \mathbb{A}(S, X)$ with the following properties for all $g, h \in S$ and $x \in X$:*

1. *If $(h, x) \in Dom(\varphi)$, $(g, \varphi(h, x)) \in Dom(\varphi)$ and $g\, Dom\, h$, then $(gh, x) \in Dom(\varphi)$ and*
$$\varphi(gh, x) = \varphi(g, \varphi(h, x)).$$

2. *If $(g, x) \in Dom(\varphi)$ and $g^* \in V(g)$, then $(g^*, \varphi(g, x)) \in Dom(\varphi)$ and*
$$\varphi(g^*, \varphi(g, x)) = x.$$

Equivalently, a partial action of a partial semigroup S on a set X is a partial action φ of the groupoid S on X which satisfies the extra assumption (2) above. A partial action $\varphi \in \mathbb{A}(S, X)$ of a partial semigroup S on X is a *strong partial action* if it is strong associative.

Next we extend the definition of unital partial group action [33] to partial actions of partial semigroups.

Definition 3.6. *A partial action φ of a partial semigroup S on a set X is* unital *or* zerotal *depending on whether S has an identity 1 and $(1, x) \in Dom(\varphi)$ for all $x \in X$ or S has a zero 0 and $(0, x) \in Dom(\varphi)$ for all $x \in X$.*

We consistently drop the first or second "partial" in the last two definitions depending on whether $Dom(\varphi) = S \times X$ or $Dom = S \times S$ respectively.

The definition of (strong) partial action above corresponds in the monoid case to the one given by Hollings [27]. The definition of unital partial action of a group is equivalent to that of unital partial group action by Kellendonk and Lawson [33].

Frequently we say that φ *is a (strong) (partial) semigroup (resp. regular semigroup, (regular) monoid, inverse semigroup or inverse monoid or group) action of* S *on* X to mean that S is a semigroup (resp. regular semigroup, (regular) monoid, inverse semigroup or inverse monoid or group) and φ is a (strong) (partial) action of the semigroup S on X. Analogously, we say that φ is a partial groupoid action of S on X to mean that S is a groupoid and that φ is a partial action of the partial groupoid S on S.

Another related concept is the one due to R. Exel who used the Vagner-Preston Representation Theorem to define an *action* of an inverse semigroup S on a set X as a homomorphism $\varphi : S \to \mathcal{S}_X$ (c.f. [18]). Simultaneously he defined what we shall call here *Exel partial action* of a group G with identity e on a set X, that is, a couple $\Theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ where, for each $g \in G$, $D_g \subset X$ and $\theta_g : D_{g^{-1}} \to D_g$ is a bijective map satisfying the following properties for all $g, h \in G$: $D_e = X$ and $\theta_e = I_X$; $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$; and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$ for all $x \in D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$. But every Exel partial action is a partial group action by the following

Proposition 3.7. *If* $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ *is an Exel partial action of a group* G *on a set* X*, then the map* $\varphi : Dom(\varphi) \subset G \times X \to X$ *defined by* $Dom(\varphi) = \{(g, x) \in G \times X : x \in D_{g^{-1}}\}$ *and* $\varphi(g, x) = \theta_g(x)$ *is a partial group action.*

Proof. Evidently $(e, x) \in Dom(\varphi)$ and $\varphi(e, x) = x$ for all $x \in X$. Moreover, if $(g, x) \in Dom(\varphi)$ then $x \in D_{g^{-1}}$ and so $\varphi(g, x) = \theta_g(x) \in D_g$ hence $(g^{-1}, \varphi(g, x)) \in Dom(\varphi)$. Now suppose that (h, x) and $(g, \varphi(h, x))$ belong to $Dom(\varphi)$. Hence $x \in D_{h^{-1}}$ and $\theta_h(x) \in D_{g^{-1}}$ therefore $\theta_h(x) \in D_{g^{-1}} \cap D_h$ so $\theta_{h^{-1}}(\theta_h(x)) \in D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$. But we have $\theta_{h^{-1}}(\theta_h(x)) = \theta_{h^{-1}h}(x) = \theta_e(x) = x$ since $x \in D_{h^{-1}} = D_{h^{-1}} \cap X = D_{h^{-1}} \cap D_e = D_{h^{-1}} \cap D_{h^{-1}h}$. Therefore $x \in D_{h^{-1}g^{-1}}$ which implies $(gh, x) \in Dom(\varphi)$. In particular, $x \in D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$

and then $\varphi(gh,x) = \theta_{gh}(x) = \theta_g(\theta_h(x)) = \varphi(g,\varphi(h,x))$ so φ is a partial group action. $\square$

Conversely, any partial group action is given by an Exel partial action [33].

In the sequel we present some basic properties of partial inverse semigroup actions. Recall that $\mathcal{S}_X$ denotes the symmetric inverse semigroup of X.

Lemma 3.8. *Let φ be a partial action of a partial semigroup S on a set X. If $g \in S$ and $g^* \in V(g)$ then $D_{g^*} = R_g$ and $\varphi_{g^*} \circ \varphi_g = I_{D_g}$. In particular, $\varphi_g \in \mathcal{S}_X$ and $(\varphi_g)^{-1} = \varphi_{g^*}$.*

Proof. If $y \in R_g$ then $y = \varphi(g,x)$ for some $x \in D_g$. As $x \in D_g$ we have $(g,x) \in Dom(\varphi)$ then $(g^*,\varphi(g,x)) \in Dom(\varphi)$ so $y \in D_{g^*}$. Conversely, if $y \in D_{g^*}$ then $(g^*,y) \in Dom(\varphi)$ so $(g,\varphi(g^*,y)) \in Dom(\varphi)$ and $\varphi(g,\varphi(g^*,y)) = y$ since $g^* \in V(g)$ if and only if $g \in V(g^*)$. Then, $x = \varphi(g^*,y) \in D_g$ and $\varphi_g(x) = y$ hence $y \in R_g$. Therefore $D_{g^*} = R_g$ for all $g \in S$. Hence $D_g = R_{g^*}$ and so $D_g = R_{g^*} = \varphi_{g^*}(D_{g^*}) = \varphi_{g^*}(R_g)$, that is, $D_g = \varphi_{g^*}(R_g)$ for all $g \in S$ and all $g^* \in V(g)$. Consequently,

$$\begin{aligned} Dom(\varphi_{g^*} \circ \varphi_g) &= \varphi_{g^*}(D_{g^*} \cap R_g) \\ &= \varphi_{g^*}(R_g) \\ &= D_g \end{aligned}$$

and so $\varphi_{g^*} \circ \varphi_g$ and I_{D_g} has the common domain D_g. Finally, if $x \in D_g$ then $(g,x) \in Dom(\varphi)$ and then $(g^*,\varphi(g,x)) \in Dom(\varphi)$ and $\varphi(g^*,\varphi(g,x)) = x$ or, equivalently, $(\varphi_{g^*} \circ \varphi_g)(x) = x$. Reversing the roles of g^* and g above we get $\varphi_g \circ \varphi_{g^*}(y) = y$ for all $y \in R_g$ so $(\varphi_g)^{-1} = \varphi_{g^*}$. $\square$

The following is a direct corollary of the previous lemma.

Corollary 3.9. *If S is a partial semigroup, then $\varphi \in \mathbb{A}(S,X)$ is a partial action (resp. strong partial action) of the partial semigroup S on a set X if and only if $\Psi(\varphi)(S) \subset \mathcal{S}_X$ and $\Psi(\varphi) : S \to \mathcal{S}_X$ is a premorphism (resp. homomorphism).*

Lemma 3.10. *If φ is a partial action of a partial semigroup S on X and $i \in E$, then $\varphi_i = I_{D_i}$. Consequently if φ is unital (resp. zerotal), then $\varphi_1 = I_X$ (resp. $\varphi_0 = I_X$) where* 1 *(resp.* 0*) is the identity (resp. the zero) of S.*

Proof. If i is idempotent then $i \in V(i)$ so $R_i = D_i$ by Lemma 3.8 and then $\varphi_i = I_{D_i}$ by Lemma 3.3. The second part of the lemma follows from the first one applied to the idempotents 1 and 0 respectively. □

Now we extend the concept of partial action from partial groupoids to partial semirings. However, we do it in an indirect way using the map Ψ.

Definition 3.11. *A* (strong) partial action of a partial semiring S *on a set X is a map $\varphi \in \mathbb{A}(S, X)$ for which $\Psi(\varphi) \in \mathbb{B}(S, X)$ is a (homomorphism) premorphism from the partial semiring S to the partial semiring $\mathcal{T}_X$. A (strong) partial action of a partial inverse semiring S on X is a map $\varphi \in \mathbb{A}(S, X)$ such that $\Psi(\varphi)$ is a (homomorphism) premorphism from the partial inverse semiring S to the partial inverse semiring $\mathcal{S}_X$ (recall Corollary 3.9).*

We frequently write that φ is a *partial (inverse) semiring action* of S to X which means that φ is a partial action of the partial (inverse) semiring S on X.

For any set X and any subset $S \subset \mathcal{T}_X$ we define $\varphi = \varphi^{S,X} \in \mathbb{A}(S, X)$ by

$$Dom(\varphi) = \{(g, x) \in S \times X : x \in Dom(g)\} \quad \text{and} \quad \varphi(g, x) = g(x).$$

Clearly $\Psi(\varphi) \in \mathbb{B}(S, X)$ is the inclusion $g \in S \mapsto g$ so, if S is a partial semiring, then $\Psi(\varphi)$ is closed.

If S is a pseudogroup of maps (or continuous maps when X is a topological space) of X, then φ is a strong partial action of the partial semiring S on X. In the case when S is a pseudogroup of either injective maps or homeomorphism (if X is a topological space) or a pseudogroup of C^r diffeomorphisms (if X is a differentiable manifold), then φ is a strong partial action of the partial inverse semiring S on X.

In addition, if X is a topological space and S is a pseudogroup of continuous maps (or homeomorphisms) of X we have that $\Psi(\varphi)(S) \subset$

$Cont(X)$ and, if X is a differentiable manifold and S is a pseudogroup of C^r diffeomorphisms, we have that $\Psi(\varphi)(S) \subset Diff^r(X)$. These features suggest the following definition.

Definition 3.12. *We say that a partial action φ of a partial groupoid (resp. partial semiring) S on a set X is* closed, continuous *or* C^r *for $r \geq 0$ depending on whether $\Psi(\varphi)$ is closed, X is a topological space and $\Psi(\varphi)(S) \subset Cont(X)$ or X is a differentiable manifold and $\Psi(\varphi)(S) \subset Diff^r(X)$.*

3.1 Invariant measures for partial actions

In this section we consider the problem of existence of invariant measures for certain partial actions. The approach we shall use is due to J. Plante [46]. Before explain it we present the precise definition of invariant measure.

Definition 3.13. *Let φ be a partial action of a partial groupoid S on a set X. Given $\Gamma \subset S$ we say that a measure μ on X is* Γ-invariant *if for all $\gamma \in \Gamma$ and all measurable set $A \subset D_\gamma$ one has that $\varphi_\gamma(A)$ is mesurable and $\mu(\varphi_\gamma(A)) = \mu(A)$. If $g \in S$, then we shall say that μ is g-invariant instead of $\{g\}$-invariant and if $\Gamma = S$, then we say that μ is* φ-invariant.

We shall find invariant measures for certain continuous partial groupoid actions φ of S on a topological space X by using two classical approach. Firstly denote by $C(X)$ the space of all continuous maps $f : X \to \mathbb{R}$. If X is compact then $C(X)$ is a Banach space if endowed with the supremum norm

$$\|f\|_{C^0} = \sup_{y \in X} |f(y)|.$$

By a functional we mean a linear continuous map $I : C(X) \to \mathbb{R}$ which is non-negative (i.e. $I(f) \geq 0$ if $f \geq 0$) and normalized (i.e. $I(1) = 1$ where 1 is the constant map $x \in X \mapsto 1$). We say that a functional I is Γ-*invariant* if for all $\gamma \in \Gamma$ and all $f \in C(X)$ satisfying $\{x \in X : f(x) \neq 0\} \subset R_\gamma$ one has $I(f) = I(f_\gamma)$, where $f_\gamma \in C(X)$ is defined by

$$f_\gamma(y) = \begin{cases} f(\varphi_\gamma(y)), & \text{if} \quad y \in D\gamma \\ 0, & \text{if} \quad y \notin D_\gamma \end{cases} \tag{3.1}$$

Now it follows from the Riesz Representation Theorem that every Γ-invariant functional I has the form

$$I(f) = \int_X f d\mu,$$

for some Γ-invariant probability measure μ.

The second approach is based on growth-type theory. Let φ be a partial semigroup action of S on a set X. If $\Gamma \subset S$ is non-empty, $x \in X$ and $n \geq 1$ we define $\Gamma^n(x)$ as the set of those $y \in X$ such that $y = \varphi(\gamma_1 \cdots \gamma_k, x)$ for some integer $1 \leq k \leq n$ and some $\gamma_1, \cdots, \gamma_k \in \Gamma$ with $x \in D_{\gamma_1 \cdots \gamma_k}$.

Note that if Γ is finite, then so is $\Gamma^n(x)$ for all $n \geq 1$. Thus we can consider the cardinality $\#\Gamma^n(x)$. The following lemma present some elementary properties of $\Gamma^n(x)$. Denote by $A \Delta B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference between A and B.

Lemma 3.14. *If φ is a partial groupoid action of S on a set X, $x \in X$ and $\Gamma \subset X$ is finite non-empty, then $\Gamma^n(x) \subset \Gamma^{n+1}(x)$ and $\varphi_\gamma(\Gamma^n(x) \cap D_\gamma) \subset \Gamma^{n+1}(x)$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$. If additionally φ is a partial inverse semigroup action, Γ is symmetric and $n \geq 2$, then*

$$(\Gamma^n(x) \cap R_\gamma) \Delta \varphi_\gamma(\Gamma^n(x) \cap D_\gamma) \subset \Gamma^{n+1}(x) \setminus \Gamma^{n-1}(x), \quad \forall \gamma \in \Gamma. \quad (3.2)$$

Proof. The first part of the lemma is trivial. For the second part we have that if $z \in (\Gamma^n(x) \cap R_\gamma) \setminus \varphi_\gamma(\Gamma^n(x) \cap D_\gamma)$, then $z \in \Gamma^n(x)$, $z \in R_\gamma$ and $z \notin \varphi_\gamma(\Gamma^n(x) \cap D_\gamma)$. Thus $z \in \Gamma^{n+1}(x)$ and if $z \in \Gamma^{n-1}(x)$ then $\varphi_{\gamma^{-1}}(z) \in \Gamma^n(x) \cap D_\gamma$ which contradicts $z \notin \varphi_\gamma(\Gamma^n(x) \cap D_\gamma)$. Then, $z \notin \Gamma^{n-1}(x)$ and so $(\Gamma^n(x) \cap R_\gamma) \setminus \varphi_\gamma(\Gamma^n(x) \cap D_\gamma) \subset \Gamma^{n+1}(x) \setminus \Gamma^{n-1}(x)$ since z is arbitrary. On the other hand, if $z \in \varphi_\gamma(\Gamma^n(x) \cap D_\gamma)$ and $z \notin \Gamma^n(x) \cap R_\gamma$, then $z \in \Gamma^{n+1}(x)$, $z \in R_\gamma$ and if $z \in \Gamma^{n-1}(x)$, then $z \in \Gamma^n(x) \cap R_\gamma$ which is absurd. Thus $z \in \Gamma^{n+1}(x) \setminus \Gamma^{n-1}(x)$ and the lemma is proved. $\square$

Given a finite non-empty set $\Gamma \subset S$ and $x \in X$ we say that Γ has *exponential growth at x* if

$$\liminf_{n \to \infty} \frac{\log(\#\Gamma^n(x))}{n} > 0.$$

Otherwise we say that Γ has *subexponential growth at x*. The next theorem relates subexponential growth to the existence of invariant measures. Hereafter we denote by $Cl(A)$ the closure of a subset A.

Theorem 3.15. *Let φ be a continuous partial inverse semigroup action of S on a compact metric space X. If $\Gamma \subset S$ is finite, non-empty, symmetric and has subexponential growth at some point $x \in X$, then there is a Γ-invariant probability measure in X with support contained in $Cl(O_\varphi(x))$.*

Proof. Since Γ has subexponential growth at x we can find an integer sequence $n_i \to \infty$ for which

$$\lim_{i\to\infty} \frac{\#(\Gamma^{n_i+1} \setminus \Gamma^{n_i-1}(x))}{\Gamma^{n_i}(x)} = 0.$$

Next we define a sequence of functionals $I_i : C(X) \to \mathbb{R}$,

$$I_i(f) = \frac{1}{\#\Gamma^{n_i}(x)} \sum_{y\in\Gamma^{n_i}(x)} f(y).$$

Since X is compact we can assume that I_i converges to some functional I in the sense that $I_i(f) \to I(f)$ for all $f \in C(X)$. Let us prove that this limit functional I is Γ-invariant. Indeed, take $\gamma \in \Gamma$, $f \in C(X)$ with $\{x \in X : f(x) \neq 0\} \subset R_\gamma$ and consider f_γ as in (3.1). Then,

$$\begin{aligned} & I(f) - I(f_\gamma) = \\ = \;& \lim_{i\to\infty} \tfrac{1}{\#\Gamma^{n_i}(x)} \left(\textstyle\sum_{y\in\Gamma^{n_i}(x)} f(y) - \sum_{y\in\Gamma^{n_i}(x)} f_\gamma(y) \right) \\ = \;& \lim_{i\to\infty} \tfrac{1}{\#\Gamma^{n_i}(x)} \left(\textstyle\sum_{y\in\Gamma^{n_i}(x)\cap R_\gamma} f(y) - \sum_{y\in\Gamma^{n_i}(x)\cap D_\gamma} f(\varphi_\gamma(y)) \right) \\ = \;& \lim_{i\to\infty} \tfrac{1}{\#\Gamma^{n_i}(x)} \left(\textstyle\sum_{y\in\Gamma^{n_i}(x)\cap R_\gamma} f(y) - \sum_{y\in\varphi_\gamma(\Gamma^{n_i}(x)\cap D_\gamma)} f(y) \right) \\ = \;& \lim_{i\to\infty} \tfrac{1}{\#\Gamma^{n_i}(x)} \textstyle\sum_{y\in(\Gamma^{n_i}(x)\cap R_\gamma)\Delta\varphi_\gamma(\Gamma^{n_i}(x)\cap D_\gamma)} (\pm f(y)) \end{aligned}$$

so

$$|I(f) - I(f_\gamma)| \leq \|f\|_{C^0} \cdot \lim_{i\to\infty} \frac{\#\left((\Gamma^{n_i}(x)\cap R_\gamma)\Delta\varphi_\gamma(\Gamma^{n_i}(x)\cap D_\gamma)\right)}{\Gamma^{n_i}(x)}.$$

Now applying (3.2) in Lemma 3.14 to $n = n_i$ we get

$$|I(f) - I(f_\gamma)| \leq \|f\|_{C^0} \cdot \lim_{i\to\infty} \frac{\#\left(\Gamma^{n_i+1}(x) \setminus \Gamma^{n_i-1}(x)\right)}{\Gamma^{n_i}(x)} = 0.$$

Therefore I is Γ-invariant so there is a probability measure μ in X such that

$$I(f) = \int_X f d\mu,$$

for all $f \in C(X)$. Hence μ is Γ-invariant and from the above identity and the fact that $I_i \to I$ we obtain that the support of μ is contained in $Cl(O_\varphi(x))$. This finishes the proof. $\square$

For the next proposition we recall that if S is a partial semiring and $\Gamma \subset S$ then S_Γ denotes the pseudogroup of S generated by Γ.

Proposition 3.16. *If φ is a continuous strong partial inverse semiring action of S on a topological space X and $\Gamma \subset S$ is non-empty, then every Γ-invariant measure of φ is S_Γ-invariant.*

Proof. First remark that since φ is strong we have by Proposition 3.2 that every Γ-invariant measure of φ is both $[\Gamma]$-invariant and $\overline{\Gamma}$-invariant. On the other hand, by Theorem 2.18 we have that $S_\Gamma = \bigcup_{n=0}^{\infty} S_\Gamma^n$ where the sequence S_Γ^n is defined by $S_\Gamma^0 = \Gamma$, $S_\Gamma^{2k+1} = \overline{S_\Gamma^{2k}}$ (for $k \geq 0$) and $S_\Gamma^{2k} = [S_\Gamma^{2k-1}]$ (for $k \geq 1$). Applying the remark to the sequence S_Γ^n we have that every Γ-invariant measure is S_Γ^n-invariant for all n. Therefore it is also S_Γ-invariant and we are done. $\square$

3.2 Anosov partial semigroup actions

In this section we extend the classical definition of Anosov group action to include partial semigroups actions. Previously we recall the definition of Anosov group actions which consists of three steps (e.g. [25]). Firstly one defines C^r *group action* for $r \geq 0$ as a group action φ of G on a set X for which G is a Lie group, X is a differentiable manifold M (say) and $\varphi : G \times M \to M$ is a C^r map. In such a case the orbits of the action are known to be C^r submanifolds of M. Secondly one defines *foliated group action* as a C^r group action, for some $r \geq 1$, whose orbits are the leaves of a C^r foliation of M throughout we denote by $\mathcal{F}^\varphi$ (or $\mathcal{F}$ for short). As is well known, a C^r group action is a foliated group action if and only if its orbits have the same dimension on each connected component of M. We denote by $T\mathcal{F}$ the subbundle of TM tangent to $\mathcal{F}$, and by $m(L)$ the co-norm of a linear operator L. Given a C^1 map $f : Dom(f) \subset M \to M$ with open domain $Dom(f)$, we say that a tangent bundle splitting $TM = E^1 \oplus \cdots \oplus E^k$ is *f-invariant* if $Df(E^i_x) = E^i_{f(x)}$ for all $x \in Dom(f)$ and all $1 \leq i \leq k$. Recall that the *center of a groupoid* S is the set of all $z \in S$ such that $zg = gz$ for all $g \in S$.

Definition 3.17. *An* Anosov group action *is a foliated group action φ of a Lie group S on a manifold M for which there are a Riemannian metric $\|\cdot\|$ in M; $g \in S$ (called* Anosov element*); a continuous g-invariant splitting $TM = E^s \oplus T\mathcal{F} \oplus E^u$ and positive constants K, λ such that the following exponential expanding or contracting properties hold for all $x \in M$ and $n \in \mathbb{N}$:*

- $\|D\varphi_{g^n}(x)/E^s_x\|, \frac{\|D\varphi_{g^n}(x)/E^s_x\|}{m(D\varphi_{g^n}(x)/T_x\mathcal{F})}, \frac{\|D\varphi_{g^n}(x)/T_x\mathcal{F}\|}{m(D\varphi_{g^n}(x)/E^u_x)} \leq Ke^{-\lambda n}$,
- $m(D\varphi_{g^n}(x)/E^u_x) \geq K^{-1}e^{\lambda n}$.

A central Anosov group action *is an Anosov group action whose Anosov element belongs to the center of S.*

We would like to extend this definition in order to include partial actions of partial semigroups on manifolds, but we have to bypass some problems first. The first one is to define what a foliated partial action is. However we can mimic the definition of foliated group action and say that a *foliated partial groupoid action* as a C^r partial groupoid action of S on M, for some $r \geq 1$, whose orbits are the leaves of a C^r foliation $\mathcal{F}^\varphi$ (or $\mathcal{F}$) of M. The second problem is the domain of the maps φ_g which could be proper subsets of M. To handle this deficiency we note that the Anosov element g of an Anosov group action can be seem as an (obviously finite) sequence $\{g\}$ whose corresponding domain is the whole M. So, we can replace it in the desired definition by a sequence $\{g_1, \cdots, g_k, \cdots\}$ whose corresponding domains $\{D_{g_1}, \cdots, D_{g_k}, \cdots\}$ cover M in order to ensure that any $x \in M$ can be iterated by some element of the sequence. Now we present the detailed definition which is restricted to semigroups instead of groupoids for the sake of simplicity.

Definition 3.18. *An* Anosov partial semigroup action *is a foliated partial semigroup action φ of S on a manifold M for which there are a Riemannian metric $\|\cdot\|$ in M; a sequence $\{g_1, \cdots, g_k, \cdots\} \subset S$*

with $\bigcup_{k=1}^{\infty} D_{g_k} = M$ *(called* Anosov sequence*); a continuous tangent splitting* $TM = E^s \oplus T\mathcal{F} \oplus E^u$ *which is* g_k*-invariant* $(\forall k)$ *and positive constants* K, λ *such that the following properties hold for every integer* $n \geq 2$*, every set* $\{k_1, \cdots, k_n\} \subset \mathbb{N}$ *and every* $x \in D_{g_{k_1}}$ *satisfying* $\varphi(g_{k_i} g_{k_{i-1}} \cdots g_{k_2} g_{k_1}, x) \in D_{g_{k_{i+1}}}$ *for all* $i = 1, \cdots, n-1$*:*

- $\|D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/E^s_x\|, \frac{\|D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/E^s_x\|}{m(D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/T_x\mathcal{F})}, \frac{\|D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/T_x\mathcal{F}\|}{m(D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/E^u_x)}$ $\leq K e^{-\lambda n}$,
- $m(D\varphi_{g_{k_n} \cdots g_{k_1}}(x)/E^u_x) \geq K^{-1} e^{\lambda n}$.

A central Anosov partial semigroup action *is an Anosov partial semigroup action whose Anosov sequence belongs to the center of* $(S, \cdot)$*.*

It is clear that an Anosov group action is an Anosov partial semigroup action. An Anosov semigroup action which is not an Anosov group action can be obtained from an expanding map on a closed manifold.

Now we introduce the corresponding definition for partial semiring actions.

Definition 3.19. *A* (central) Anosov partial semiring action *is a partial semiring action of* $(S, +, \cdot)$ *on a manifold* M *whose corresponding partial semigroup action of* $(S, \cdot)$ *on* M *is (central) Anosov.*

To finish we present a definition for *Anosov foliation*. As a motivation we recall that the definition of the entropy of a foliation via its holonomy pseudogroup [20]. Another motivation comes from the concept of expansive foliation, also depending on the holonomy pseudogroup [30], [59]. In the same vein we introduce the following definition.

Definition 3.20. *An* Anosov foliation *is a* C^1 *foliation such that one of its holonomy pseudogroups is an Anosov partial semiring action.*

This definition does not depend on the holonomy pseudogroup. Furthermore, the stable or unstable foliations of an Anosov flow are both examples of Anosov foliations.

Chapter 4

Ergodicity of Anosov Group Actions

4.1 Introduction

There is a special class of dynamical systems which have the following property, it leaves a volume invariant along the evolution. It was Poincaré who noticed that this property implies that the structure of the orbits is rich, indeed, he noticed that almost every orbit must be recurrent.

In the other hand, advances in the theory of statistical mechanics leads to what is called Boltzman's ergodic hypothesis, which states that, over long periods, the time spent by orbits in a region of a energy level set is proportional to its volume. The celebrated Birkhoff's theorem says that this is true for almost every orbit, with respect to the volume, if every function which is almost everywhere invariant by the evolution is in fact a almost everywhere constant.

This last property can be extended to any other measure which is invariant by the evolution of the system. And, the question of whether an invariant measure is ergodic for some system is an important question. However, since the Lebesgue measure (volume) have a good relationship with the open sets (any open set has positive Lebesgue measure) then, by a simple argument, ergodicity of the

Lebesgue measure also gives a topological information of the structure of the orbits, it implies transitivity of the dynamical system, i.e. the existence of a dense orbit.

The works of Hopf and Anosov [4], sheds a light over this question when some hyperbolicity is guarantee. In fact, the main motivation was the study of the dynamics of geodesic flows. It was noticed by them that if the curvature of the manifold is negative then the geodesic flow presents exponential expansion and contraction on complementary directions in the complement of the flow direction, using the behaviour of the Jacobi fields by the derivative of the flow. With this property they showed that these flows are ergodic with respect to the Liouville measure.

In fact, the geometric counterpart is not used in their proof of the ergodicity. Any flow which presents this expansion/contraction feature of its derivative which preserves a Lebesgue measure is ergodic[1]. This type of flows now are called Anosov flows, and the same terminology holds for diffeomorphisms.

Nowadays, the hyperbolic theory evolutes to what is called partially hyperbolic theory. In this theory we also have expansion and contraction in some directions, but now, there exists central direction where we only know that if some hyperbolicity is present on it then it cannot be stronger that in the true hyperbolic directions. A program to show ergodicity for an open and dense set of these type of systems was started by Pugh and Shub, and there are many partial positive results on this program, specially if the central direction is one dimensional, we refer the reader to [10], [11] and [26] for more details.

The same problem can be posed to Anosov actions, asking about the ergodicity of the Lebesgue measure when all of the diffeomorphisms generated by the acting group preserve it. In fact, this is a kind of a special case of the problem in the partially hyperbolic theory. But we had a stronger property which aids the proof, the fact that the central direction is integrable, since by the locally free assumption it coincides with the orbit foliation of the action.

We say that that the action $A : G \to Diff(M)$:

[1]Here we need higher differentiability, C^2 for instance, the ergodicity of C^1 Anosov diffeomorphisms is an open question.

- Preserves the measure μ, if the action is measure preserving: For any $g \in G$ and B a measurable set we have $\mu(A(g^{-1}, B)) = \mu(B)$.
- Is ergodic if preserves μ and if $f : M \to \mathbb{R}$ μ-integrable which is invariant by the action, i.e. $\forall g \in G$, we have $f \circ A(g, .) = f$ μ-almost everywhere, then f is constant μ-almost everywhere.

The main theorem of this chapter is the following.

Theorem 4.1 (Pugh-Shub)**.** *Any central Anosov action $A : G \to Diff^2(M)$ which preserves the Lebesgue measure of M is ergodic.*

In the proof, we will follow [49] closely.

4.1.1 Absolute Continuity and an outline of the proof

One of the key notions for the proof of the theorem is the absolute continuity of the invariant foliations. For this purpose we define it for pre-foliations.

Definition 4.2. *Let G a pre-foliation by C^r discs of dimension k and $H_{p,q} : D_{p,q} \to R_{p,q}$ a holonomy map. We denote by μ_{D_p} and μ_{D_q} the restrictions of the measure μ to $D_{p,q}$ and $R_{p,q}$ respectively. If $H_{p,q}$ is measurable, we define the Jacobian $J : D_{p,q} \to \mathbb{R}$ of $H_{p,q}$ by:*

$$\mu_{D_q}(S) = \int_{H_{p,q}^{-1}(S)} J d\mu_{D_p} \text{ For every } S \subset R_{p,q}.$$

We say that a pre-foliation $\mathcal{G}$ is absolutely continuous if its Jacobians are continuous and positive for every holonomy map.

We make three remarks about the definition. First, if a pre-foliation $\mathcal{G}$ is absolutely continuous then if $A \subset D_{p,q}$ and $\mu_{D_p}(A) = 0$ then $\mu_{D_q}(H_{p,q}(A)) = 0$. Second, any holonomy map which is a C^1-embedding has a positive and continuous Jacobian. Third, we can speak about the absolute continuity of foliations, since we know that they generates pre-foliations naturally.

Now, we recall a useful property of absolutely continuous maps with respect to uniform convergence.

Proposition 4.3. *Let $g_n : D^k \to \mathbb{R}^k$ a sequence of C^1 embeddings converging to a topological embedding $h : D^k \to \mathbb{R}^k$. If their Jacobians $J(g_n)$ converge to a function J then h is absolutely continuous with Jacobian J.*

Proof. Denote by μ the Lebesgue measure of R^k and μ_D the restriction to D^k. Fix S a k-dimensional closed disc of D^k. Since $J(g_n)$ are continuous, by uniform convergence we have that J is also continuous. So given $\epsilon > 0$ there exists k-dimensional closed discs R and T such that:

$$R \subset int(S) \subset S \subset int(T) \text{ and } \int_{T-R} J d\mu_D \leq \frac{\epsilon}{2}.$$

Since h is a topological embedding, for n large enough we have that $g_n(R) \subset h(S) \subset g_n(T)$, in particular:

$$\mu(g_n(R)) \leq \mu(h(S)) \leq \mu(g_n(T)).$$

On the other hand, since g_n are absolutely continuous we have:

$$\mu(g_n(R)) = \int_R J(g_n) d\mu_D \leq \int_S J(g_n) d\mu_D \leq \int_T J(g_n) d\mu_D = \mu(g_n(T)).$$

So $|\mu(h(S)) - \int_S J(g_n) d\mu_D| < \epsilon$, taking $n \to \infty$ and observing that ϵ is arbitrarily, we have that:

$$\mu(h(S)) = \int_S J d\mu_D,$$

for any disc S. Since h is continuous, this holds for any measurable S. □

The main feature that absolutely continuous foliations have is that they have some Fubini-type properties. More precisely, if we denote by *Leb* the Lebesgue measure, then:

Proposition 4.4. *If $\mathcal{F}$ is an absolutely continuous foliation and $Z \subset M$ is a subset of M. Then $Leb(Z) = 0$ if, and only if, almost all leaves[2] of $\mathcal{F}$ have an intersection with Z with zero leaf-measure[3].*

[2] All leaves of $\mathcal{F}$ not lying in a set composed of whole $\mathcal{F}$-leaves having zero measure.

[3] The restriction of the measure μ to each leaf

Recall that the essential supremum of a function $\varphi : M \to \mathbb{R}$ is

$$\inf\{\sup \varphi|_{M-Z}; Leb(Z) = 0\}.$$

As a corollary we have that, if $\mathcal{F}$ is an absolutely continuous foliation and the essential supremum of $\varphi : M \to \mathbb{R}$ is bounded by c on almost every leaf then the essential supremum of φ is bounded by c.

Corollary 4.5. *Let $\mathcal{F}$ be an absolutely continuous foliation. If the essential supremum of $\psi : M \to \mathbb{R}$ on almost every leaf is bounded by c then the essential supremum of ψ is bounded by c.*

Proof. The hypothesis says that for every leaf $\mathcal{F}_p$ we have a subset $Z_p \subset \mathcal{F}_p$, such that $Leb(Z_p) = 0$ in a set D of $\mathcal{F}$-leaves and zero measure, and $\sup \psi|_{\mathcal{F}_p - Z_p} = 0$ outside D. Then $Z = D \cup \bigcup_p Z_p$ has zero measure by the proposition, and $\sup \psi|_{M-Z} \leq c$. □

Proposition 4.6. *If $\mathcal{F}$ and $\mathcal{G}$ are absolutely continuous and complementary foliations then any function $\varphi : M \to \mathbb{R}$ constant on almost every leaf of $\mathcal{F}$ and of $\mathcal{G}$ then φ is constant almost everywhere.*

We will prove theses propositions later. Also in the sequel, we will prove that stable, unstable, center-stable and center-unstable are absolutely continuous foliations, and so we can apply the propositions above.

We recall that, sometimes, if $g \in G$ we also see g as a diffeomorphism, using the action A, i.e. $g(x) := A(g, x)$.

Now, if for any $g \in G$ we define $Inv(g)$ as the set of integrable g-invariant functions, then the statement of the main theorem of this chapter is that $\bigcap\limits_{g \in G} Inv(g)$ is the set of constant functions.

We recall that, for any function $\varphi : M \to \mathbb{R}$, by Birkhoff's theorem we can define three projections over $Inv(g)$:

$$I_g\varphi(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \varphi(g^k(x))$$

$$I_g^+\varphi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(g^k(x))$$

$$I_g^-\varphi(x) = \lim_{n\to-\infty} \frac{1}{|n|} \sum_{k=0}^{n-1} \varphi(g^k(x))$$

And also we have that $I_g^+(x) = I_g^-(x) = I_g(x)$ almost everywhere. So all of these three maps are the same, if we consider as a continuous projection $L^1(M) \to Inv(g)$. Moreover the set of continuous functions is dense in $L^1(M)$, in particular their images by I_g are also dense in $Inv(g)$. The following lemma will be important:

Lemma 4.7. *If φ is a continuous function and f is the Anosov element then $I_f(\varphi)$ is constant on almost every leaf of $\mathcal{W}^u$ and $\mathcal{W}^s$.*

Proof. Let $x, y \in W_p^u$, since $d(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$ then $I_f^-\varphi(x)$ exists if, and only if, $I_f^-\varphi(y)$, and they are equal. Also, $I_f^-\varphi$ is defined almost everywhere and $\mathcal{W}^u$ is absolutely continuous, we have that $I_f^-\varphi$ is constant on almost all $\mathcal{W}^u$-leaf (and it is defined). Again, by absolute continuity and since $I_f^-\varphi = I_f\varphi$ we have that $I_f\varphi$ is constant on almost every leaf of $\mathcal{W}^u$. The same argument, shows that $I_f\varphi$ is constant on almost every leaf of $\mathcal{W}^s$. □

Now, any G-invariant function $\psi : M \to \mathbb{R}$ is approximated, almost everywhere, by functions $I_f\varphi$ where φ is continuous. In particular, on almost every leaf of $\mathcal{W}^u$, ψ is the pointwise limit, almost everywhere, of functions which are constant on these leaves, and the same holds for $\mathcal{W}^s$. Hence, ψ is constant on almost every leaf of $\mathcal{W}^s$ and $\mathcal{W}^u$ and since it is G-invariant, it is constant on almost every leaf of $\mathcal{F}$.

Lemma 4.8 (Dynamical Coherence)**.** *The foliations $\mathcal{F}$ and $\mathcal{W}^u$ are smooth when restricted to a leaf W_p^{cu}.*

Proof. The first assertion is true, since $\mathcal{F}$ is a smooth foliation on M. It is not true that $\mathcal{W}^u$ is smooth on M. But, W_p^u is a smooth manifold, and for any $q \in \mathcal{F}_p$ we have that $W_q^u = gW_p^u$ for some g in the connected component of $e \in G$. This implies the smoothness assertion. □

Now let Z be a set with zero measure, such that outside Z the function ψ is constant on almost every leaf of $\mathcal{W}^u$, $\mathcal{W}^s$ and $\mathcal{F}$. Since $\mathcal{W}^{cu}$ is absolutely continuous, we have that almost every leaf W_p^{cu} intersects Z in a set of zero leaf-measure. By proposition 4.4 almost every leaf $\mathcal{F}_y$ (resp. W_q^u) inside W_p^{cu} intersects $Z \cap W_p^{cu}$ in a set of zero leaf-measure of $\mathcal{F}$ (resp. $\mathcal{W}^u$). In particular, ψ is constant on almost every leaf of $\mathcal{F}$ and $\mathcal{W}^u$ inside $\mathcal{W}^{cu}$, then by proposition 4.6 ψ is constant in almost every point of W_p^{cu}.

Finally, we have that ψ is constant on almost every leaf of $\mathcal{W}^{cu}$ and on almost every leaf of $\mathcal{W}^s$, proposition 4.6 says that ψ is constant on almost every point of M. And this implies the ergodicity of the action, completing the proof of the theorem.

In the next sections we will show that the strong stable and strong unstable foliations are absolutely continuous, and prove the Fubini-type propositions.

4.2 Hölder Continuity and Angles

To obtain the absolute continuity of the invariant foliations, we need first to have some control on the continuity of their linear approximations given by the invariant subbundles. We will fix the Anosov element f and show that the invariant subbundles varies Hölder-continuously. This will imply some control on the angles between these subbundles, as a corollary of their distance in the Grassmanian bundle.

Theorem 4.9. *There exists $\theta > 0$ such that E^u and E^{cs} are θ-Hölder continuous.*

Proof. Let F^u and F^{cs} smooth bundles close to the invariant bundles and define the smooth disc bundle $\mathcal{D} = \bigcup \mathcal{D}_x$ over M formed by

$$\mathcal{D}_x = \{P_x \in L(F_x^{cs}, F_x^u); \|P\| \leq 1\}.$$

If the matrix of $D_x f^{-1}$ in the coordinates $F^{cs} \oplus F^u$ is given by:

$$\begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix}$$

Then the action of Df^{-1} on $\mathcal{D}$ is given by $F(P_x) = (C_x + K_x P)(A_x + B_x P)^{-1}$. We will denote, and use this notation until the end of the chapter,

$$\lambda = \inf_{x \in M} m(D_x f|_{E^u}) \text{ and } \mu = \sup_{x \in M} \|D_x f|_{E^{cs}}\|.$$

Using domination, we obtain $\frac{\mu}{\lambda} < 1$.

Moreover, F is a fiber contraction with Lipschitz constant k close to $\frac{\mu}{\lambda}$, if F^{cs} and F^u are close enough to E^{cs} and E^u. The invariant section theorem says that if F is C^1 and $kLip(f)^\theta < 1$ then the unique F-invariant section of $\mathcal{D}$ is θ-Hölder (see theorems 3.1 and 3.8 of [25] or p. 304 of [57]).

By Df-invariance the bundle E^{cs} is and F-invariant section when represented by a graph of a linear map from F^{cs} to F^u, also since f is C^2 there exists θ satisfying the hypothesis of the invariant section theorem. By uniqueness, we have that E^{cs} is θ-Hölder.

The same argument holds for E^u. □

Now we analyze the effect of the angle between the invariant subbundles and transverse discs on the holonomy maps. Recall the Hausdorff metric on the Grassmannian: If E and F are k-dimensional subbundles, for any $p \in M$ we define:

$$\angle(E_p, F_p) = \max\{\sup_{v \in E_p - \{0\}} \angle(v, F_p), \sup_{v \in F_p - \{0\}} \angle(E_p, v)\}.$$

Then $\angle(E, F) = \sup_{p \in M} \angle(E_p, F_P)$.

Proposition 4.10. *Suppose that $TM = N \oplus E^{cs}$ where N is a smooth distribution. Let $\mathcal{G}(\delta)$ the smooth pre-foliation given by $exp_p(N_p(\delta))$, and take $G_{p,q} : D_{p,q} \to R_{p,q}$ an holonomy map. Let $0 \le \beta < \frac{\pi}{2}$, if δ is small enough, $\angle(TD_p, (E^u)^\perp) \le \beta$ and $\angle(TD_q, (E^u)^\perp) \le \beta$ then the holonomy map $G_{p,q}$ is a smooth immersion.*

Proof. Observe that $G_{p,q}$ is a smooth map, so we need only to show that $D_y G_{p,q} : T_y D_p \to T_z D_q$ is a bijection where $z = G_{p,q}(y)$.

First, note that when y is close enough to p we have that $G_{p,q} = G_{y,z}$, so we need to show that only when $y = p$. Now observe that this is trivially true when $p = q = y$. The proposition follows now by the continuity of $DG_{p,q}$ and compactness of M and $\{A_p \subset T_p M; \angle(A_p, (E^u)^\perp) \le \beta\}$. □

4.3 Absolute Continuity of Foliations

In this section we show the following theorem, which applies immediately to the C^2-Anosov element of the Anosov action.[4]

Theorem 4.11. *Let f be a C^s diffeomorphism of M with $s \geq 2$ which leaves an invariant splitting $TM = E^u \oplus E^{cs}$ such that:*

$$\sup_{p \in M} \|Df|_{E^{cs}_p}\|^j < \inf_{p \in M} m(Df|_{E^u_p}) \text{ for } 0 \leq j \leq r \leq s.$$

If $r \geq 1$ then there exists W^u a strong stable foliation tangent to E^u, and it is absolute continuous.

We remark that the theorem holds for stable foliations (see chapter 4 of [25]), with the necessarily adaptations on the statement.

The existence of the strong stable foliation follows from the theory of partially hyperbolic dynamics. So, we will only to show the second part of the statement.

First, we take N a smooth distribution, fix $0 < \beta < \frac{\pi}{2}$ such that

$$\max\{\angle(E^{cs}, (E^u)^\perp), \angle(E^{cs}, N^\perp)\} < \beta.$$

Now, we choose δ as in Proposition 4.10. Also, we take the smooth pre-foliation $\{\mathcal{G}_x := exp_x(N_x(\delta))\}_{x \in M}$. Now we iterate the pre-foliation, obtaining pre-foliations $\mathcal{G}^n_x := f^n \mathcal{G}_{f^{-n}(x)}$, and we take the restriction, using the induced metric $d_{\mathcal{G}^n}$:

$$\mathcal{G}^n_x(\epsilon) = \{y \in \mathcal{G}^n_x; d_{\mathcal{G}^n}(x, y) \leq \epsilon\}$$

The uniform hyperbolicity implies that $\mathcal{G}^n(\epsilon)$ and $T\mathcal{G}^n(\epsilon)$ uniformly converge to $W^u(\epsilon)$ and E^u respectively, using a graph transform argument, see [25].

Fix $p \in M$ and take $q \in W^u_p$. For any discs D_p and D_q transversal to E^u we want to show the absolute continuity of the holonomy maps $H_{p,q} : D_{p,q} \to R_{p,q}$. Since $\mathcal{W}^u$ is a true foliation, we know that $H_{p,q}$ is a homeomorphism and $R_{p,q}$ is a neighborhood of q in D_q.

Since the foliation is f-invariant and f is a diffeomorphism, the holonomy map between $f^{-n}(p)$ and $f^{-n}(q)$ is given by conjugacy:

$$H_{f^{-n}(p), f^{-n}(q)} : f^{-n}(D_{p,q}) \to f^{-n}(R_{p,q})$$

[4]Some arguments in this section will be used in the next section.

$$H_{f^{-n}(p),f^{-n}(q)} = f^{-n} \circ H_{p,q} \circ f^n$$

Also, $H_{p,q}$ has a positive Jacobian if, and only if, $H_{f^{-n}(p),f^{-n}(q)}$ also has it too.

Now, since we are dealing with local objects and, again by hyperbolicity and a graph transform argument, $Tf^{-n}(D_p)$ and $Tf^{-n}(D_q)$ converges uniformly to E^{cs}, we can assume that $q \in W^u_p(\epsilon/2)$ and for any $n \geq 0$ we have:

$$\max\{\angle(Tf^{-n}(D_p), (E^u)^\perp), \angle(Tf^{-n}(D_q), (E^u)^\perp)\} \leq \beta.$$

Also we can suppose that $D_{p,q} = D_p$ and $R_{p,q} \subset int(D_q)$.

Observe that by the uniform convergence of $\mathcal{G}^n(\epsilon)$ to $W^u(\epsilon)$ we have that the holonomy maps $G^n_{p,q} : D_p \to D_q$ along $\mathcal{G}_n(\epsilon)$ are well defined and we will define $Q_n := \mathcal{G}^n_p(\epsilon) \cap D_q$, $g_n := G^n_{p,Q_n}|_{D_p}$ and $h := H_{p,q}$.

By definition of Q_n, we observe that if we call $p_n = f^{-n}(p)$ and $q_n = f^{-n}(Q_n)$ then $q_n \in \mathcal{G}_{p_n}$. In particular, the holonomy map $G^0_{p_n,q_n}$ along $\mathcal{G}$ is defined on $f^{-n}(D_p)$. Since, E^u expands, and $\mathcal{G}_{p_n}$ converges to E^u, we have that $q_n \in \mathcal{G}_{p_n}(\epsilon_n)$ with $\epsilon_n \to 0$.

So we can express $g_n : D_p \to D_q$ as:

$$g_n = f^n \circ G^0_{p_n,q_n} \circ f^{-n}$$

Lemma 4.12. *g_n are embeddings, for n sufficiently large.*

Lemma 4.13. *The Jacobians $J(g_n)$ uniformly converge to a function J, such that:*

$$J(x) = \lim_{n\to\infty} \frac{\det(Df^{-n}|_{T_x D_p})}{\det(Df^{-n}|_{T_{h(y)} D_q})}.$$

We will postpone the proof of the lemmas and finish the proof of the theorem. By proposition 4.3, we have that J is the Jacobian of h. J is continuous and finite since it is an uniform limit of continuous functions. It's not difficult to see, by the symmetry of the formula, that the Jacobian of $H_{q,p}$ will be $1/J$, and also it will be finite. So J must be positive, and this will complete the proof of the theorem.

Proof of lemma 4.12. The assumptions on the angle and proposition 4.10 shows that g_n are immersions. Also, by the local assumptions, both g_n and h are well defined on a large disc $D \supset D_p$ and g_n uniformly converge to h on D.

Let K a compact neighborhood of $R_{p,q}$, such that

$$K \subset int(h(D)) \subset h(D) \subset int(D_q).$$

Since h is a homeomorphism, its degree $deg(h, D, y)$ is equal to 1 for any $y \in K$. Now by uniform convergence, if n is large enough then $g_n(\partial D)$ is close to $h(\partial D)$ so they are homotopic:

$$g_n|_{\partial D} \sim h|_{\partial D} \text{ in } D_q - K.$$

Hence, for large n the degree $deg(g_n, D, y)$ is equal to 1 for any $y \in K$ and consequently g_n is an embedding on $g_n^{-1}(K)$. Finally, note that this set contains D_p for n large because $h^{-1}(K)$ does it. This completes the proof. $\square$

Proof of lemma 4.13. Using the formula of g_n and the Chain Rule, if we define:

$$\begin{aligned} A_n &= \det Df^n|_{T_{f^{-n}\circ g_n(y)} f^{-n}(D_q)} \\ B_n &= \det DG^0_{p_n,q_n}|_{T_{f^{-n}(y)f^{-n}(D_p)}} \\ C_n &= \det Df^{-n}|_{T_y D_p}. \end{aligned}$$

Actually, $B_n \to 1$ uniformly, since $T_f^{-n} D_q \to E^{cs}$, $T(f^{-n}(D_q) \to E^{cs}$ uniformly and $q_n \in \mathcal{G}_{p_n}(\epsilon_n)$ since $\epsilon_n \to 0$. Then, since:

$$J_y(g_n) = A_n B_n C_n$$

We only need to control A_n and C_n, which it means to prove that:

$$\frac{\det(Df^{-n}|_{T_y D_p})}{\det(Df^{-n}|_{T_{g_n(y)} D_q}} \text{ and } \frac{\det(Df^{-n}|_{T_y D_p})}{\det(Df^{-n}|_{T_{h(y)} D_q})}$$

have the same (uniform) limit.

Estimating $Df^{-n}|_{T_y D_p}$ and $Df^{-n}|_{T_y D_q}$.

Let $\pi : TM \to E^{cs}$ the projection along E^u, which commutes with Df, by invariance. So we have that:

$$Df^{-n}|_{T_y D_p} = (\pi|_{T_{f^{-n}(y)} f^{-n}(D_p)})^{-1} \circ Df^{-n}|_{E^{cs}} \circ (\pi|_{T_y D_p}).$$

Since $T(f^{-n}(D_p) \to E^{cs}$ uniformly when $n \to \infty$, then the determinant of the first term tends uniformly to 1. An analogous formula holds for $y \in D_q$.

Now, since $g_n \to h$ uniformly, and D_q is C^1 we have that:

$$\frac{\det(\pi|_{T_y D_p})}{\det(\pi|_{T_{g_n(y)} D_q})} \text{ and } \frac{\det(\pi|_{T_y D_p})}{\det(\pi|_{T_{h(y)} D_q})}$$

have the same limit.

So we only need to prove that:

$$\frac{\det(Df^{-n}|_{E^{cs}_y})}{\det(Df^{-n}|_{E^{cs}_{g_n(y)}})} \text{ and } \frac{\det(Df^{-n}|_{E^{cs}_y})}{\det(Df^{-n}|_{E^{cs}_{h(y)}})}$$

have the same limit.

We claim that the second limit exists uniformly and postpone the proof of the claim. So we only need to show that:

$$\lim_{n\to\infty} \frac{\det(Df^{-n}|_{E^{cs}_{h(y)}})}{\det(Df^{-n}|_{E^{cs}_{g_n(y)}})} = 1.$$

Taking logarithms, using the Chain Rule and recalling that f is C^2 and E^{cs} is θ-Hölder we have that:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} |\log\det(Df^{-1}|_{E^{cs}_{f^{-k}(h(y))}}) - \log\det(Df^{-1}|_{E^{cs}_{f^{-k}(g_n(y))}})|$$

is dominated by

$$C\sum_{k=0}^{n-1} d(f^{-k}(h(y)), f^{-k}(g_n(y)))^{\theta} \quad (*)$$

Estimating the distance between iterates.

Now, we fix $\max(\mu, 1) < \rho < \sigma < \lambda$. Recall that $\mathcal{G}$ is close to E^u, $f^{-n}(h(y)) \in W^u_{f^{-n}(y)}(\epsilon_n)$ and $f^{-n}(g_n(y)) \in \mathcal{G}_{f^{-n}(y)}(\epsilon_n)$, so we can take $\epsilon_n \leq \sigma^{-n}$ for large n. In particular:

$$d(f^{-n}(h(y)), f^{-n}(g_n(y))) \leq \sigma^{-n} \text{ for large } n.$$

Now, if k is large and $n > k$ we have that $f^{-k}(D_q), \dots, f^{-n}(D_q)$ are close to E^{cs}, and writing $f^{-k} = f^{n-k} \circ f^{-n}$, we obtain:

$$d(f^{-k}(h(y)), f^{-k}(g_n(y))) \leq C_1 \rho^{n-k} \sigma^{-n} \text{ for some } C_1 > 0.$$

And,

$$(*) \leq C(C_1)^\theta (\sum_{k=0}^{n-1} \rho)^{\theta(n-k)})(\sigma^{-\theta n}) = C_2 \rho^\theta \sigma^{-n\theta} \frac{1-\rho^{n\theta}}{1-\rho^\theta}$$

Which goes to zero uniformly.

The proof of the claim.

The proof of the claim is similar, taking logarithms, and using the Chain Rule we need to show the uniform convergence of

$$\sum_{k=0}^{\infty} |\log\det(Df^{-1}|_{E^{cs}_{f^{-k}(y)}}) - \log\det(Df^{-1}|_{E^{cs}_{f^{-k}(h(y))}})|$$

Again, using the Hölder continuity, we bound this sum by:

$$\sum_{k=0}^{\infty} C d(f^{-k}(y), f^{-k}(h(y)))^\theta$$

Now, since h is the holonomy of W^u we have that $h(y) \in W^u_y$ we have:

$$d(f^{-k}(p), f^{-k}(q)) \leq \lambda^{-k} d(p,q)$$

Again this implies that the sum converges uniformly. And the proof of the theorem is now complete. □

4.4 Center Foliations

Now we deal with the saturation along the orbits of the action of the hyperbolic foliation. Since the directions of the action are neutral, the proof is more involved, but the integrability of the central direction which, in fact, is the orbit foliation will be exploited to this purpose.

Since the action is locally free, the orbit foliation is C^1. Since the argument is more general, we will fix a C^2 diffeomorphism f and suppose that there exists a Df-invariant partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ and a C^1 foliation $\mathcal{F}$ tangent to E^c. As usual we define the *center-stable manifold* (resp. the *center-unstable manifold*) as:

$$W_p^{cs} = \bigcup_{q \in \mathcal{F}_p} W_q^s \text{ (resp. } W_p^{cu} = \bigcup_{q \in \mathcal{F}_p} W_q^u).$$

They form true foliations by the:

Theorem 4.14 ([25]). *If $\mathcal{F}$ is a C^1 foliation and f is normally hyperbolic at $\mathcal{F}$ then the center-stable manifolds are leaves of a foliation $\mathcal{W}^{cs}$ tangent to $E^s \oplus E^c$ called the center-stable foliation. The same holds for the center-unstable manifolds.*

The main theorem of this section is:

Theorem 4.15. *If f is a C^2 diffeomorphism normally hyperbolic at $\mathcal{F}$, a C^1 foliation, then $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are absolutely continuous foliations.*

As in the previous section, we will choose N a smooth distribution close to E^u. Then we take $\mathcal{G}$ as the pre-foliation by discs induced by N and by iteration by f we obtain the pre-foliation $\mathcal{G}_y^n(\delta)$. Now, we use the pre-foliation by submanifolds of the form:

$$\mathcal{H}_p^n = \bigcup_{y \in \mathcal{F}} \mathcal{G}_y^n(\delta).$$

Note that by domination, we have that $\mathcal{H}^n \to \mathcal{W}^{cu}$ and $T\mathcal{H}^n \to E^{cu}$ uniformly.

What we want to do is fix $p \in M$ and $q \in W_p^{cu}$, then take D_p and D_q s-discs transversal to E^{cu} and analyze the associated holonomy map $H_{p,q}$.

As we did in the previous section, we can suppose that $q \in W_p^u(\epsilon/2)$, also that D_p is the domain of $H_{p,q}$ with diameter less than $\epsilon/2$. And consider the holonomy maps $H_n := H_{p,q}^n$.

If we prove that H_n is an embedding for all n, $H_n \to H_{p,q}$ and $J(H_n) \to J$ uniformly and $J > 0$ then the same proof of the previous section will work here.

Since $\mathcal{H}^n \to \mathcal{W}^{cu}$ and $H_{p,q}$ is a homeomorphism the same proof of lemma 4.12 shows that H_n is an embedding and $H_n \to H_{p,q}$. So we only need to prove that $J(H_n) \to J$ and $J > 0$.

Definition 4.16. *Let $y \in D_p$ we define y_n as the unique point of $\mathcal{F}_y(\epsilon)$ such that $H_n(y) \in \mathcal{G}^n_{y_n}(\epsilon)$ and y^* as the unique point of $\mathcal{F}_y(\epsilon)$ such that $H(y) \in W^u_{y^*}(\epsilon)$. Observe that $y_n \to y^*$ uniformly.*

Let $\Sigma(y_n)$ and $\Sigma(y^*)$ smooth discs at y_n and y^*, inside D, transverse to E^c such that $\Sigma(y_n) \to \Sigma(y^*)$ and $T\Sigma(y_n) \to T\Sigma(Y^*)$ uniformly. We consider also $F_{y,y_n} : D_p \to \Sigma(y_n)$ the holonomy along $\mathcal{F}$ in D and $h_n : \Sigma(y_n) \to D$ the holonomy along $\mathcal{H}^n$ through D_p. This defines a factorization of the holonomy map H_n as $h_n \circ F_{y,y_n}$ (which depends on y_n).

Observe that $\det(DF_{y,y_n}(y)) \to \det(DF_{y,y^*}(y^*)) > 0$ uniformly, by continuity of the derivative. This implies that we only need to control the Jacobian of h_n. Indeed, this factorization together with the Chain Rule proves that:

Lemma 4.17. *We have the $J(H_n) \to J > 0$ uniformly if, and only if,*

$$\lim_{n\to\infty} J_{y_n}(h_n) = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|_{T_{f^{-k}(y^*)}f^{-k}\Sigma(y^*)})}{\det(Df^{-1}|_{T_{f^{-k}H(y)}f^{-k}(D_q)})} \text{ uniformly.}$$

Again, we fix $\max(\mu, 1) < \rho < \sigma < \lambda$, then as we did in the previous section, since $\mathcal{G}^{n-k}$ is close to E^u for $0 \le k \le n$ and n large enough we have that $f^{-k}(H_n(y)) \in \mathcal{G}^{n-k}_{f^{-k}(H_n(y_n))}(\epsilon\sigma^{-k})$ and $f^{-k}(H(y)) \in W^u_{f^{-k}(H(y))}(\epsilon\sigma^{-k})$. This will implies some backward contractions:

Lemma 4.18. *For n large enough and $0 \le k \le n$ we have:*

$$d(f^{-k}(y_n), f^{-k}(y^*)) \le \sigma^{-k}\epsilon \quad \text{and} \quad d(f^{-k}(H_n(y)), f^{-k}(H(y))) \le \sigma^{-k}\epsilon.$$

We will postpone the proof of this lemma. Let

$$R_n := H^0_{f^{-n}(y_n), f^{-n}(H_n(y))} : f^{-n}(\Sigma(y_n)) \to f^{-n}(D_q)$$

be the holonomy map along $\mathcal{H}^0$ through $f^{-n}(D_q)$, then by the Chain Rule we have:

$$J_{y_n}(h_n) = \frac{\det(DR_n(f^{-n}(y_n))) \det(Df^{-n}|_{T_{y_n}\Sigma(y_n)})}{\det(Df^{-n}|_{T_{H_n(y)}D_q})}.$$

Now, since $d(f^{-n}(y_n), f^{-n}(H_n(y))) \to 0$ and $T\Sigma(y_n) \to T\Sigma(y^*)$ uniformly we have that $\det(DR_n(f^{-n}(y_n)) \to 1$ uniformly. So, again by the Chain Rule, we only need to prove that:

$$\lim_{n\to\infty} \frac{\det(Df^{-n}|_{T_{y_n}\Sigma(y_n)})}{\det(Df^{-n}|_{T_{H_n(y)}(D_q)})} = \lim_{n\to\infty} \frac{\det(Df^{-n}|_{T_{y_n}\Sigma(y^*)})}{\det(Df^{-n}|_{T_{H(y)}D_q})} \text{ uniformly.}$$

The proof will be complete if we prove the following two lemmas:

Lemma 4.19.

$$\lim_{n\to\infty} \frac{\det(Df^{-n}|_{T_{y_n}\Sigma(y^*)})}{\det(Df^{-n}|_{T_{H(y)}D_q})} \textit{ converges uniformly.}$$

Lemma 4.20.

$$\begin{aligned} 0 \quad &= \lim_{n\to\infty} \textstyle\sum_{k=0}^{n-1} |\det(Df^{-1}|_{T_{f^{-k}(y_n)}f^{-k}(\Sigma(y_n))}) \\ &\qquad - \det(Df^{-1}|_{T_{f^{-k}(y^*)}f^{-k}(\Sigma(y^*))})| \\ 0 \quad &= \lim_{n\to\infty} \textstyle\sum_{k=0}^{n-1} |\det(Df^{-1}|_{T_{f^{-k}(H_n(y_n))}f^{-k}(D_q)}) \\ &\qquad - \det(Df^{-1}|_{T_{f^{-k}(H((y^*)))}f^{-k}(D_q)})| \end{aligned}$$

This completes the proof of the theorem, and now we give proofs of these lemmas.

First we prove lemma 4.19. We recall that as we did in the last section, since E^s is θ-Hölder and $d(f^{-k}(y^*), f^{-k}(H(y))) \le \sigma^{-k}$ we can prove that:

$$\prod_{k=0}^{\infty} \frac{\det(Df^{-1}|_{E^s_{f^{-k}(y^*)}})}{\det(Df^{-1}|_{E^s(f^{-k}(H(y)))})} \text{ converges uniformly. } (*)$$

Now, since the domination property implies projective hyperbolicity, we have that E^s attracts, under Df^{-1}, any complementary subspace to E^{cu}. More precisely, since $T\Sigma(y_n) \to T\Sigma(y^*)$ and $T\Sigma(y^*)$ is complementary to E^{cu} we have that if $k \le n$ are large enough then:

$$\angle(Df^{-k}\Sigma(y^*), E^s) \le (\rho/\sigma)^k \text{ , } \angle(Df^{-k}\Sigma(y_n), E^s) \le (\rho/\sigma)^k$$
$$\text{and } \angle(Df^{-k}D_q, E^s) \le (\rho/\sigma)^k.$$

Now the function $P \to \det(Df^{-1}|_P)$ is smooth, where P is a subspace of fixed dimension, so there exists a constant $C > 0$ such that:

$$|\det(Df^{-1}|_{T_{f^{-k}(y^*)}f^{-k}(\Sigma(y^*))}) - \det(Df^{-1}|_{E^s(f^{-k}(y^*))})| \leq C(\rho/\sigma)^k$$
$$|\det(Df^{-1}|_{T_{f^{-k}(H(y))}f^{-k}(D_q)}) - \det(Df^{-1}|_{E^s(f^{-k}(H(y)))})| \leq C(\rho/\sigma)^k.$$

Again, by the Chain Rule to prove the lemma we need to show that:

$$\prod_{k=0}^{\infty} \frac{\det(Df^{-1}|_{T_{f^{-k}(y^*)}f^{-k}(\Sigma(y^*))})}{\det(Df^{-1}|_{T_{f^{-k}(H(y))}f^{-k}(D_q)})} \text{ converges uniformly.}$$

But this occurs by comparison with (*).

Now we prove lemma 4.20. First, we will estimate each term of the sum, using the triangular inequality, by the sum of the following terms:

$$|\det(Df^{-1}|_{T_{f^{-k}(y_n)}f^{-k}\Sigma(y_n)}) - \det(Df^{-1}|_{E^s(f^{-k}(y_n))})| \text{ (I)}$$
$$|\det(Df^{-1}|_{E^s(f^{-k}(y_n))}) - \det(Df^{-1}|_{E^s(f^{-k}(y^*))})| \text{ (II)}$$
$$|\det(Df^{-1}|_{E^s(f^{-k}(y^*))}) - \det(Df^{-1}|_{T_{f^{-k}(y_n)}f^{-k}\Sigma(y^*)})| \text{ (III)}$$

We estimate (II) using that $Df^{-1}|_{E^s}$ is Hölder:

$$\begin{aligned} \text{(II)} \quad &\leq \quad Cd(f^{-k}(y_n), f^{-k}(y^*))^\theta \leq C\rho^{(n-k)\theta} d(f^{-n}(y_n), f^{-n}(y^*))^\theta \\ &\leq \quad C(\rho^{n-k}\sigma^{-n}\epsilon)^\theta \end{aligned}$$

Also (I) and (III) can be estimated using the angles estimates above, and we split the sum in two parts:

$$\begin{aligned} \sum_{k=0}^{n-1} \quad &\leq \quad \sum_{k=0}^{K} + \sum_{k=K+1}^{n-1} (I) + (II) + (III) \\ &\leq \quad \sum_{k=0}^{K} + 2C \sum_{k=K+1}^{\infty} (\sigma^{-1}\rho)^k + C\sigma^{-n\theta} \sum_{k=0}^{n-1} \rho^{(n-k)\theta}. \end{aligned}$$

Now, fix K large enough and then let n tend to infinity. The inequality for $H_n(y_n)$ is obtained analogously.

Finally, we prove lemma 4.18

Proof of lemma 4.18. We prove the lemma by induction on k. Let $k = 0$ then we want to show that:

$$d(y_n, y^*) \leq \epsilon \text{ and } d(H_n(y), H(y)) \leq \epsilon.$$

We observe that $d(y^*, H(y)) \leq \epsilon$ because of $H(y) \in W^u_{y^*}(\epsilon)$ and $d(y_n, H_n(y)) \leq \epsilon$ due to $H_n(y) \in \mathcal{G}^n_{y_n}(\epsilon)$. Also, observe that the twisted quadrilateral formed by $y_n, y^*, H(y)$ and $H_n(y)$ has diameter bounded by ϵ so this implies that the other edges also have length bounded by ϵ, and completes the proof in the case $k = 0$.

Now let's suppose that the lemma is true for $k - 1 < n$. If we define $\gamma = \sup \|Df^{-1}|_{E^c}\|$ then by domination and the induction hypothesis we have:

$$\begin{aligned}
d(f^{-k}(y_n), f^{-k}(y^*)) &\leq \gamma d(f^{-(k-1)}(y_n), f^{-(k-1)}(y^*)) \\
&\leq \gamma\epsilon\sigma^{-(k-1)} \\
d(f^{-k}(H_n(y)), f^{-k}(H(y))) &\leq \gamma d(f^{-(k-1)}(H_n(y)), f^{-(k-1)}(H(y))) \\
&\leq \gamma\epsilon\sigma^{-(k-1)}
\end{aligned}$$

Again, $f^{-k}(y^*), f^{-k}(H(y)), f^{-k}(H_n(y))$ and $f^{-k}(y_n)$ forms a twisted quadrilateral of diameter bounded by $\gamma\epsilon$.

Also, there are two edges of this quadrilateral in $\mathcal{G}^{n-k}_{f^{-k}(y^*)}$ and $\mathcal{G}^{n-k}_{f^{n-k}(y_n)}$ (which are nearly parallel), moreover their lengths are just bounded by $\epsilon\sigma^{-k}$. Now, since $k \leq n$ then $\mathcal{G}^{n-k}$ is close to E^u. Since $\mathcal{F}$, $f^{-k}(D_q)$ and $\mathcal{G}^{n-k}$ are almost perpendicular to each other, we have that the two edges in $\mathcal{F}$ and $f^{-k}(D_q)$ have lengths bounded by $\epsilon\sigma^{-k}$. And this proves the lemma. $\square$

4.5 Proof of Fubini-type propositions

In this section, we prove the Fubini-type propositions 4.4 and 4.6.

Proof of proposition 4.4. It is sufficiently to show the proposition in the local case. So we fix a neighborhood U of some point $p \in M$ which trivializes the foliation $\mathcal{F}$. In particular, there exists a map $\pi : D^k \times D^{m-k} \to U$ such that $\pi(D^k \times x)$ is a leaf of $\mathcal{F}$, and $\pi(x \times$

D^{m-k}) defines a transversal local smooth foliation $\mathcal{G}$. Also, we will suppose that the Lebesgue measures on the leaves are induced by the Riemannian metric by restriction, and then the respective Lebesgue measure on D^k and D^{m-k} will be given by pull-back using the map π, more precisely using the maps:

$$\pi : D^k \sim D^k \times \{0\} \to \mathcal{F}_p \text{ and } \pi : D^{m-k} \sim \{0\} \times D^{m-k} \to \mathcal{G}_p.$$

Also, we consider in $D^k \times D^{m-k}$ the product measure. We will denote all of this measures as Leb, when there are no confusion in what space we are calculating the Lebesgue measure.

Let $Z \subset D^k \times D^{m-k}$, and suppose the $Leb(Z) = 0$, since $\mathcal{G}$ is smooth, this occurs if, and only if, for almost everywhere $x \in \mathcal{F}_p$ we have $Leb(\mathcal{G}_x \cap Z) = 0$. Since $\mathcal{F}$ is absolutely continuous, this is equivalent to $Leb(\{x\} \times D^{m-k} \cap \pi^{-1}(Z)) = 0$ for almost everywhere $x \in D^k$. By, Fubini's theorem we have $0 = Leb(\pi^{-1}(Z)) = Leb(D^k \times \{y\} \cap \pi^{-1}(Z))$ for almost every $y \in D^{m-k}$.

Since $\mathcal{G}$ is smooth, in particular is absolutely continuous, we have that, the previous statement occurs if, and only if, $Leb(\mathcal{F}_y \cap Z) = 0$ for almost every $y \in \mathcal{G}_p$. And since $\mathcal{F}$ is absolutely continuous, this is equivalent to $Leb(\mathcal{F}_y \cap Z) = 0$ for every $x \in \mathcal{F}_p$ and almost every $y \in \mathcal{G}_x$.

Now let W be a $\mathcal{F}$-saturated set[5] such that every leaf which intersects Z has positive leaf-measure. In particular, we have that $Leb(S\mathcal{F}_y \cap Z = 0)$ if, and only if, $Leb(\mathcal{F}_y \cap W) = 0$. So, the statement of the previous paragraph is equivalent to $Leb(\mathcal{F}_y \cap W) = 0$ for any $x \in \mathcal{F}_p$ and almost every $y \in \mathcal{G}_x$. By definition of W, we have that this is equivalent to $Leb(\mathcal{G}_x \cap W) = 0$ for all $x \in \mathcal{F}_p$, and since $\mathcal{G}$ is smooth, this implies that $Leb(W) = 0$.

In the other hand, if $Leb(W) = 0$ then, again since $\mathcal{G}$ is smooth, we gave that $Leb(W \cap \mathcal{G}_x) = 0$ for almost every $x \in \mathcal{F}_p$. Again, by definition of W we have that $Leb(W \cap \mathcal{G}_x) = 0$ for all $x \in \mathcal{F}_p$. And this completes the proof. □

Now we proof the proposition 4.6

Proof. Let $c \in \mathbb{R}$ and consider $M^c = \{x \in M; \psi(x) \leq c\}$. We define W^c as the set of leaves of $\mathcal{F}$ contained in M^c with positive measure.

[5]A set which is the union of leaves of $\mathcal{F}$

Then $Z = M^c \Delta W^c$ has zero measure. By proposition 4.4 almost every leaf of $\mathcal{G}$ intersects Z only in sets of zero leaf measure. This means that almost every leaf of $\mathcal{G}$ intersects M^c in a set of positive leaf measure if, and only if, the same holds for W^c.

By compactness of M, there exists $\epsilon > 0$ such that for any $p \in M$ if $x \in \mathcal{F}_p(\epsilon)$ and $y \in \mathcal{G}_p(\epsilon)$ then $\mathcal{F}_y(2\epsilon) \cap \mathcal{G}_x(2\epsilon)$ is a unitary set. We define $M_p(\epsilon)$ as the neighborhood of p formed by the union of these intersections. In particular, we have that for any $x \in \mathcal{F}_p(\epsilon)$ and $y \in \mathcal{G}_p(\epsilon)$ we have that $\mathcal{F}_y(\epsilon/2)$ and $\mathcal{G}_x(\epsilon/2)$ are containing in $M_p(\epsilon)$.

Observe that if we prove that for any $c > 0$, we have that for any component $M_p(\epsilon)$, denoting by S_p (respectively s_p) the essential supremum (respectively the essential infimum) of φ we have that $S_p \geq c$ or $s_p < c$, then φ is constant almost everywhere on $M_p(\epsilon)$ and therefore since p is arbitrary, φ is constant almost everywhere. Based on this, we split the proof in two cases.

First, if $\mathcal{G}_p(\epsilon)$ intersects W^c with positive measure, then since $\mathcal{F}$ is absolutely continuous, we have that for any $q \in \mathcal{F}_p(\epsilon)$ we have that $\mathcal{G}_q(2\epsilon)$ intersects W^c with positive measure, therefore the essential supremum of φ on most of $\mathcal{G}_q$ for $q \in \mathcal{F}_p(\epsilon)$ is bounded by c from above, then by corollary 4.5 the essential supremum of φ on $M_p(\epsilon)$ is bounded by c.

Second, if $\mathcal{G}_p(\epsilon)$ intersects W^c in a zero measure set, again by absolute continuity of $\mathcal{F}$ we have that for any $q \in \mathcal{F}_p(\epsilon)$, we have that $\mathcal{G}_q(\epsilon)$ intersects W^c in a zero measure set. Therefore, essential infimum of φ on most of $\mathcal{G}_q$ is bounded below by c for every $q \in \mathcal{F}_p(\epsilon)$, and corollary 4.5 says again that the essential infimum of φ on $M_p(\epsilon)$ is bounded below by c. And this completes the proof. $\square$

Chapter 5

Stability of Anosov Group Actions

5.1 Introduction

One of the main applications of dynamical systems is to give models of phenomena which evolves with time. Unfortunately in practice, there are many truncations of the parameters important to describe the evolution. It is desirable then that the approximated system also has an orbit structure close to the ideal one. This is what we call structural stability of the system.

In its seminal work, Smale [54] introduces the concept of hyperbolicity and shows that, if the non-wandering set of a diffeomorphism is hyperbolic and if the periodic orbits are dense in the non-wandering set then the diffeomorphism restricted to the non-wandering set is stable, what he called Ω-stability. The reason to restrict to the non-wandering set, is that since the manifold is compact, the asymptotic behavior of the orbits are inside the non-wandering set. In fact, it is true that the stable manifold of the non-wandering set, in this case, is the whole manifold. Moreover, Smale also proved a Spectral Decomposition theorem, which gives a decomposition of the non-wandering set in a finite set of basic pieces, where the analysis of the system can be localized on them.

Since transitivity implies that the non-wandering set is the whole manifold, transitive Anosov diffeomorphisms with dense periodic orbits are stable. The hypothesis on the denseness of periodic orbits is superfluous, since it is implied by Anosov's closing lemma[1]. Actually, transitivity is not needed to prove the stability, see corollary 18.2.2 of [23].

In this chapter we will study both the existence of a spectral decomposition and the stability of Anosov actions. In fact, we will need to assume denseness of the compact orbits in the whole manifold, in the Anosov case, since for general actions we do not have a version of the Anosov closing lemma. However, for Anosov actions of $\mathbb{R}^k$, we can use the Anosov type closing lemma due to Katok and Spatzier to obtain the desired denseness.

In this chapter we will follow closely the works of [50] and [6].

In what follows, we will set the scenario and explain what we mean by stability. As usual, we denote by $A^r(G, M)$ the set of C^r-actions endowed with the C^r-topology. Also, we will assume that G is a Lie group with compact set of generators.

First, we will define several notions of stability trying to imitates the correspondent notions in the case of diffeomorphisms, flows and foliations.

Definition 5.1. *Two actions $A, B : G \to Diff(M)$ are:*

- parametrically *conjugate if there exists a homeomorphism $h : M \to M$ such that $A(g) = h \circ B(g) \circ h^{-1}$ for every $g \in G$.*
- orbit *conjugate if there exists a homeomorphism $h : M \to M$ which sends each A-orbit onto a B-orbit.*

Definition 5.2. *We say that an action $A : G \to Diff(M)$ is C^r-structurally stable if there exists a C^r-neighborhood of A such that any action in this neighborhood is orbit conjugated to A.*

Also, we will say that a point $x \in M$ is non-wandering if for every neighborhood U of x, and every compact subset $S \subset G$, there exists $g \in G - S$ with $g(U) \cap U \neq \emptyset$. We will denote by Ω_A the subset of non-wandering points of the action. Obviously this set is invariant by

[1]Here there is no perturbation.

the action. If there is no confusion, we will denote the non-wandering set Ω_A of an action A simply as Ω.

The main feature of the non-wandering set is that the boundary of any orbit (the asymptotic behavior of the dynamics) is inside Ω. Indeed, for any x in the boundary of the orbit of some p has a sequence $g_n \in G$ without cluster points such that $A(g_n, p) \to x$. Now, if $S \subset G$ is compact, U is a neighborhood of x and k is such that $y = A(g_k, p) \in U$ then $\lim_{n\to\infty} A(g_n, A(g_k^{-1}(y)) = x$, so there exists n large such that $g = g_n g_k^{-1} \in G - S$ and $A(g, U) \cap U \neq \emptyset$.

So it is natural to define:

Definition 5.3. *An action $A : G \to Diff(M)$ is Ω-stable if for any action $B : G \to Diff(M)$ sufficiently close to A, we have that $A|_{\Omega_A}$ and $B|_{\Omega_B}$ are orbit conjugated.*

Now we extend naturally the notion of Axiom A systems given by Smale to actions. But first, we recall that a hyperbolic element of an action over some compact set M is an element of the group which is normally hyperbolic to the orbit foliation of M.

Definition 5.4. *We say that an action is Axiom A if the orbit foliation laminates the non-wandering set, the action is central, the hyperbolic element is normally hyperbolic over the non-wandering set, and the compact orbits are dense in the non-wandering set.*

Notice that we are not assuming connectedness of the group. In this setting, we can obtain a spectral decomposition of the non-wandering set:

Proposition 5.5. *If A is a Axiom A action then there is a unique decomposition:*

$$\Omega = \Lambda_1 \cup \cdots \cup \Lambda_k$$

Where Λ_i are compact, invariant and disjoint sets. Also, the action is transitive on each Λ_i.

With this property, we can define the notion of cycles. We say that $\Lambda_i \prec \Lambda_j$ if there exists $x \in \Lambda_i$ and $y \in \Lambda_j$ such that $\emptyset \neq W^u_x \cap W^s_y \subsetneq \Omega$. A cycle is a sequence $\Lambda_{i_1} \prec \cdots \prec \Lambda_{i_j} = \Lambda_{i_1}$ such that $j \geq 2$.

The main theorems of this chapter are:

Theorem 5.6. *If $A : G \to Diff(M)$ is a transitive Anosov action such that the compact orbits are dense in M then A is structurally stable.*

In fact a stronger result holds:

Theorem 5.7. *If the action is Axiom A and there are no cycles then it is Ω-stable.*

5.2 Spectral Decomposition

We start with the following fact that is not too difficult to obtain (see [50] for more details): let $O(p)$ and $O(q)$ two orbits of an Axiom A action, such that there exists $x \in W^u_{O(p)} \pitchfork W^s_{O(q)}$, then $x \in W^{uu}_x \pitchfork W^s_{O(q)}$.

Lemma 5.8. *Let f be a central hyperbolic element of an action $A : G \to Diff(M)$ which acts over an invariant compact subset $\Lambda \subset M$. If $O(p)$ and $O(q)$ are two compact orbits inside Λ such that $W^u_{O(p)} \pitchfork W^s_{O(q)} \neq \emptyset$ then $\emptyset \neq W^s_{O(p)} \cap W^u_{O(q)} \subset \Omega_A$.*

Proof. The hypothesis and the previous fact says that there exists $x \in M$ and $p_1 \in O(p)$ such that $x \in W^{uu}_{p_1} \pitchfork W^s_{O(q)}$. Now let $y \in W^{ss}_{p_2} \cap W^{uu}_{q_2}$ for some $p_2 \in O(p)$ and $q_2 \in O(q)$. If U is a neighborhood of y and $S \subset G$ is compact, we must find some $g \in G - S$ such that $A(g, U) \cap U \neq \emptyset$.

If we denote $Qz := \{A(q, z); q \in Q\}$ then if $Q \subset G$ is a compact set large enough we have that for every $w \in O(p)$ and $t \in O(q)$, $Qw = O(p)$ and $Qt = O(q)$ holds. In particular, for every $n \in \mathbb{N}$ there exist $q_n \in Q$ such that $A(q_n f^n, p_2) = p_1$. By compactness of Q we can assume that $q_n f^n(y) \to p_1$. Using the λ-lemma, since the hyperbolic element is at the center, $q_n f^n(U)$ contains a disc D_n near $W^{uu}_{p_1}$, then it must intersect a compact fixed piece of $W^s_{O(q)}$ near x in a point x_n. In particular there exists y_n such that $x \in W^{ss}_{y_n}$ where $x_n \to x$ and $y_n \to p_2$.

In the same way, there exists $h_n \in Q$ such that $h_n f^n(y_n) = q_2$ and, again, the λ-lemma says that there exist discs D'_n in $h_n f^n q_n f^n(U)$ accumulating on $W^{uu}_{q_2}$. Observe that $g_n = h_n f^n q_n f^n = h_n q_n f^{2n}$, since f^n has no cluster points[2] and Q is compact, there exists n such that $g_n \notin S$ and $A(g_n, U) \cap U \neq \emptyset$. $\square$

As a consequence of this lemma, we obtain a local product structure for the non-wandering set.

Proposition 5.9. *If f is the hyperbolic element of an Axiom A action and Ω is the non-wandering set of the action, then $\mathcal{F}|_\Omega$, the restriction of the orbit foliation to the non-wandering set, has local product structure: There exists $\epsilon > 0$ such that $W^u_\Omega(\epsilon) \cap W^s_\Omega(\epsilon) = \Omega$.*

Proof. Since $\mathcal{W}^{uu}$ and $\mathcal{W}^s$ are transverse and the splitting is continuous, if $p, q \in \Omega$ are close enough we have that $W^{uu}_p(\epsilon) \pitchfork W^s_{O(q)}(\epsilon)$ and $W^{uu}_q(\epsilon) \pitchfork W^s_{O(p)}(\epsilon)$ are not empty.

Since the compact orbits are dense in the non-wandering set, there exists compact orbits $O(p_1)$ and $O(q_1)$ which pass close to p and q respectively. In particular, by transversality $W^{uu}_{p_1}(2\epsilon) \cap W^s_{O(q_1)}(2\epsilon)$ and $W^{uu}_{q_1}(2\epsilon) \cap W^s_{O(p_1)}(2\epsilon)$ are not empty and are close to the respective intersections for p and q. By the previous lemma, these intersections are inside the non-wandering set and, since the non-wandering set is closed, the respective intersections for p and q also are inside the non-wandering set. $\square$

Now we are ready to prove a spectral decomposition for the non-wandering set.

Theorem 5.10. *If $A : G \to Diff(M)$ is an Axiom A action then the non-wandering set Ω is a finite union $\Omega_1 \cup \cdots \cup \Omega_k$ of disjoint, compact and invariant sets Ω_i. Moreover, each Ω_i cannot be divide into two disjoint nonempty compact invariant subsets, and every two relatively nonempty open invariant subsets of Ω_i have nonempty intersection.*

Proof. Let f be the hyperbolic element of the action, and $\epsilon > 0$ very small. Let $p \in \Omega$, V and W neighborhoods of p such that $diam(V), diam(W) < \epsilon$.

[2] Indeed, for every $g \in G$, $\|DA(g)\|$ and $m(DA(g))$ are finite, but if $E^u \neq \{0\}$ (or $E^s \neq \{0\}$) then $m(DA(f^n)) \to \infty$ (or $\|DA(f^{-n})\| \to \infty$) as $n \to \infty$. In particular f^n cannot converge to some $g \in G$ since it must be a diffeomorphism. Also, if $E^u = E^s = \{0\}$ then the whole manifold M is an orbit.

If $z \in W \cap \Omega$ then there exists $p_1 \in V \cap \Omega$ and z_1 close to z with compact orbits. The local product structure says that $W^{uu}_{p_1}(2\epsilon) \cap W^s_{O(z_1)}(2\epsilon)$ and $W^{uu}_{q_1}(2\epsilon) \cap W^s_{O(p_1)}(2\epsilon)$ are not empty. In particular $z_1 \in Sat(V \cap \Omega)$ since the latter set is closed we have that $z \in Sat(V \cap \Omega)$. Interchanging the role of V and W, we obtain that:

$$Sat(V \cap \Omega) = Sat(W \cap \Omega).$$

In particular, if we set $\Omega(p) = Sat(V \cap \Omega)$, then the previous argument says that $\Omega(p)$ and $\Omega(q)$ are disjoint or equal. By compactness, there exists only a finite number of such sets $\Omega_1, \ldots, \Omega_k$. Since they are saturated sets, they are invariant nonempty compact subsets of Ω. Finally, if there exists i such that if V and W nonempty open invariant subsets with some $p \in V \cap \Omega_i$ and $q \in W \cap \Omega_i$, then $\Omega_i = \Omega(p) = \Omega(q)$. In particular there exists $g \in G$ such that $g(V \cap \Omega_i) \cap (W \cap \Omega_i)$ is nonempty. The proof is now complete. □

We observe that, in the special case where the group G is $\mathbb{R}^k$, we can obtain a spectral decomposition only assuming that the action is Anosov.

5.2.1 Spectral Decomposition for Anosov actions of $\mathbb{R}^k$

As we discuss above, in the $\mathbb{R}^k$ case we can find compact orbits near to recurrent orbits and obtain:

Theorem 5.11. *If $A : \mathbb{R}^k \to Diff(M)$ is an Anosov action then the non-wandering set Ω is a finite union $\Omega_1 \cup \cdots \cup \Omega_k$ of disjoint, compact and invariant sets Ω_i. Moreover, each Ω_i cannot be divide into two disjoint nonempty compact invariant subsets, and the action restricted to each Ω_i is transitive.*

The main tool used in the proof is the following Anosov type closing lemma due to Katok and Spatzier [31]:

Theorem 5.12 (Katok-Spatzier's closing lemma)**.** *Let $A : \mathbb{R}^k \to Diff(M)$ be an Anosov action and let f be the Anosov element. There exists $\epsilon > 0$, $C > 0$ and $\lambda > 0$ such that if $x \in M$ and $t \in \mathbb{R}$ satisfies $\delta := d(A(t.f, x), x) < \epsilon$ then there exists $y \in M$ and a curve $\gamma : [0, t] \to \mathbb{R}^k$ such that for every $s \in [0, t]$ we have:*

(i) $d(A(s.f,x), A(\gamma(s),y)) < Ce^{-\lambda \min\{s,t-s\}}\delta$.

(ii) *There exists* $g \in \mathbb{R}^k$ *satisfying* $\|g\| < C\delta$ *such that* $A(\gamma(t),y) = A(g,y)$.

(iii) $\|\gamma' - f\| < C\delta$.

Moreover, these constants depend continuously on the action in the C^1*-topology.*

This lemma will give denseness of compact orbits inside the non-wandering set generated by a chamber.

Corollary 5.13. *The union of compact orbits is dense in the non-wandering set* $\Omega(\mathcal{C})$ *of any chamber* $\mathcal{C}$ *of the Anosov action* $A : \mathbb{R}^k \to Diff(M)$.

Proof. Let $x \in \Omega(\mathcal{C})$, $\epsilon_0 > 0$ small enough and U be the ϵ_0-neighborhood of x. By definition, there exists $g \in \mathcal{C}$ such that $A(-g, U) \cap U$ contains some point y. Hence, $d(A(g,y), y) < 2\epsilon_0$ and so the closing lemma give us $g \in \mathcal{C}$ satisfying $(i), (ii)$ and (iii). Using (iii), if ϵ_0 is small enough, we have that the image of γ is contained in $\mathcal{C}$, and by (ii), again if ϵ_0 is small, we have that $\gamma(t) - \delta$ belongs to $\mathcal{C}$ and the isotropy group of some z such that $d(y,z) < 2C\epsilon_0$. As we saw before, the orbit of z must be compact and $d(x,z) \leq d(x,y) + d(y,z) < (2C+1)\epsilon_0$. Since ϵ_0 is as small as we want, the corollary follows. □

Now we are ready to prove the theorem 5.11.

Proof of theorem 5.11. The previous corollary says that the compact orbits are dense in the non-wandering set. As usual, we says that two compact orbits $O(x)$ and $O(y)$ are equivalent is they are homoclinically related, i.e. if $\mathcal{F}^s_x \pitchfork \mathcal{F}^u_y$ and $\mathcal{F}^s_y \pitchfork \mathcal{F}^u_x$ are nonempty.

Obviously this relation is reflexive and symmetric. Transitivity follows from the λ-lemma applied to $\{A(t.g,.)\}_{t \in \mathbb{R}}$ where g is an Anosov element. The local product structure given by proposition 5.9 says that if x and y are close enough then they are equivalent. In particular there are finitely many classes with disjoint closures.

Let Λ_i one of these classes. If p and q are equivalent compact orbits and $p \in \Lambda_i$ then there exists $z \in \mathcal{F}^u_p \cap \mathcal{F}^s_q$. In particular, there exists an Anosov element g in the chamber $\mathcal{C}$ which fixes p and such that if B is a small ball with center at z then $A(t.g, B)$ accumulates on $\mathcal{F}^u_q$. Hence, since the compact orbits are dense in the non-wandering set, $\mathcal{F}^u_p$ is dense in Λ_i.

Finally, let U and V two open sets of Λ_i. By denseness, there exists $p \in U$ with compact orbit and $g \in \mathcal{C}$ an Anosov element, such that p is fixed by g and $\mathcal{F}^{uu}_p(\delta)$ the δ-local strong unstable manifold of p is contained in U.

By compactness of the orbit, there exists a compact set K of $\mathbb{R}^k$ such that:

$$\mathcal{F}^u_p = \{A(g, \bigcup_{j=0}^{\infty} A(j.g, \mathcal{F}^{uu}_\delta(p))); g \in K\}.$$

As we saw $\mathcal{F}^u_p$ is dense in Λ_i then for every $m \in \mathbb{N}$ large there exists $g_m \in K$ such that:

$$V \cap A(g_m, \bigcup_{j=0}^{m} A(j.g, \mathcal{F}^{uu}_\delta(p))) \neq \emptyset.$$

Hence, $V \cap A(mg + g_m, U) \neq \emptyset$. Transitivity inside the chamber $\mathcal{C}$ follows observing that $m(g + \frac{g_m}{m}) \in \mathcal{C}$. $\square$

5.3 Proof of the Stability Theorem 5.6

We start recalling some notions that will be used in the proof.

Let $f : M \to M$ be a diffeomorphism. An ϵ-pseudo orbit of f is a sequence $\{x_n\}_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$ we have that $d(f(x_n), x_{n+1}) \leq \epsilon$. Moreover, if f preserves a foliation $\mathcal{L}$ with a fixed plaquation $\mathcal{P}$, we say that the pseudo orbit respects $\mathcal{P}$ if $f(x_n)$ and x_{n+1} are in the same plaque of $\mathcal{P}$. Also, if f preserves a foliation $\mathcal{L}$ with a fixed plaquation $\mathcal{P}$, we say that f is plaque expansive if there exists an $\epsilon > 0$ such that if $\{x_n\}$ and $\{y_n\}$ are ϵ-pseudo orbits which respect $\mathcal{P}$ and $d(x_n, y_m) \leq \epsilon$ for all $n \in \mathbb{Z}$ then x_n and y_n are in the same plaque for every $n \in \mathbb{Z}$.

The main property to prove the theorem 5.6 is the persistence of hyperbolic sets for hyperbolic actions. The hypothesis of plaque expansivity holds since the orbit foliation is C^1, as we shall prove in the appendix.

Theorem 5.14 (Persistence)**.** *If $A : G \to Diff(M)$ is a hyperbolic action over a compact and invariant set $\Lambda \subset M$ then there exists a neighborhood $\mathcal{U}$ of A in the C^1-topology such that for any $B \in \mathcal{U}$ there exists an orbit conjugacy $h : \Lambda \to \Lambda'$ between the two actions A and B, where Λ' is a compact B-invariant set canonically defined. Also, h is close to the inclusion $i : \Lambda \to M$ if B is sufficiently close to A.*

Proof. Let $g \in G$ be the hyperbolic element inside the group and define $f = A(g)$ and $f' = B(g)$. If $\mathcal{L}$ is the orbit foliation on Ω, the previous theorem says that there exists a canonical f'-invariant foliation $\mathcal{L}'$ close to $\mathcal{L}$ and a conjugacy $h : (f, \mathcal{L}) \to (f', \mathcal{L}')$ close to the inclusion. What we need to prove is that $\mathcal{L}'$ is the B-orbit foliation and that h sends orbits in orbits.

The main feature of the conjugacy h is that is given by shadowing (see the Appendix). More precisely, if η is a smooth bundle complementary to $T\mathcal{L}$, $\eta(\epsilon)$ is the subbundle of balls of radius ϵ of η, and a plaquation $\mathcal{P}$ is fixed then $h(x)$ is the unique point of $exp_x\eta(\epsilon)$ such that its orbit by f' is shadowed by an f-pseudo orbit which respects $\mathcal{P}$.

Fix $x \in M$ and set $y = h(x)$. If $\{x_n\}$ is a pseudo orbit through x which respects $\mathcal{P}$ and shadows $\{f'^n(y)\}$, i.e. $x_0 = x$ and there exists some g_n close to e such that $x_{n+1} = A(g_n, f(x_n))$.

Let W be a symmetric compact set of generators of G, and fix $g \in W$. If D is a disc in G transverse to the isotropy groups G_x at e, since the map $(d, z) \in D \times exp_x(\eta(\epsilon)) \to A(gd, z)$ is a local diffeomorphism which sends $D \times \{0\}$ to a neighborhood of $A(g, x)$ inside $\mathcal{L}_{A(g,x)}$, there exists some $p, q \in D$ such that the points $z' = B(gp, y)$ and $z'' = B(g, y)$ belongs to $exp_{A(g,x)}\eta(\epsilon)$ and $exp_{A(gq,x)}\eta(\epsilon)$ respectively. We will call $g' = gp$ and$g'' = gq$, and also postpone the proof of the following lemma.

Lemma 5.15. *The subsets $\{A(g, x_n)\}$ and $\{A(g'', x_n)\}$ are f-pseudo orbits which respect $\mathcal{P}$. Moreover, the first one shadows $\{f'^n(z')\}$ and the second one shadows $\{f'^n(z'')\}$. In particular, by the characterization of h, $h(A(g, x)) = z'$ and $h(A(g'', x)) = z''$.*

The previous arguments and the lemma says that for every $g \in W$ there exists some g' and g'' close to g such that:

$$h(A(g, x)) = B(g', y) \text{ and } h(A(g'', x)) = B(g, y).$$

The next lemma extends this fact to the entire group. We will also postpone the proof.

Lemma 5.16. *For any $g \in G$ there exists some g' and g'' in the same connected component of g in G such that:*

$$h(A(g,x)) = B(g',y) \text{ and } h(A(g'',x)) = B(g,y).$$

Now if g belongs to the connected component of e then g'' belongs to the same component and:

$$B(g,y) = B(g,h(x)) = h(A(g'',x)) \subset h(\mathcal{L}_x) = \mathcal{L}'_x.$$

In particular, $\mathcal{L}'_x$ is $B(g,.)$ invariant for any g in the connected component of e. But, since $\mathcal{L}_x = \bigcup_{g \in G(e)} A(g,x)$ where $G(e)$ is the connected component of e we have that:

$$\mathcal{L}'_x = h(\mathcal{L}_x) = \bigcup_{g \in G(e)} h(A(g,x)) \subset \bigcup_{g \in G(e)} B(g',y).$$

Thus, the connected component of the B-orbit through y is $\mathcal{L}'x$, this implies that the B-orbits foliate Λ' and $\mathcal{L}' = h(\mathcal{L})$. Finally, the previous lemma also says that:

$$h(\mathcal{L}_{A(g,x)}) = \mathcal{L}'_{h(A(g,x))} = \mathcal{L}'_{B(g,y)}.$$

And this completes the proof. □

The proof of the stability theorem follows in the transitive case since $\Lambda = \Omega = M$ and h is surjective (continuous and close to identity).

Now we give the proofs of the lemmas 5.15 and 5.16.

Proof of Lemma 5.15. By definition, $A(g,x_{n+1}) = A(g, A(g_n, f(x_n)))$, now since f is in the center of the group we have:

$$A(g,x_{n+1}) = A(gg_ng^{-1}fg, x_n) = A(gg_ng^{-1}, fA(g,x_n)).$$

Since gg_ng^{-1} is close to e, we have that $\{A(g,x_n)\}$ is an f-pseudo orbit which respects $\mathcal{P}$, the same holds for g'' since it is close to g, hence inside a compact set.

Now, we observe that $f'^n(z') = B(f^n, B(g', y)) = B(g', f'^n(y))$, since f is in the center. Moreover, x_n is close to $f'^n(y)$ because B is close to A, also g' is close to g which lives in W, thus $B(g', f'^n(y))$ is close to $A(g, x_n)$.

Analogously, $f'^n(z'') = B(f^n, B(g, y)) = B(g, f'^n(y))$ is close to $A(g'', x_n)$. This completes the proof. □

Proof of Lemma 5.16. Since W is a symmetric compact set of generators, we only need to prove the assertion for $g = g_1 g_2$, where the lemma holds for g_1 and g_2. In particular, we have that:

$$h(A(g_1 g_2, x)) = h(A(g_1, A(g_2, x))) = B(g_1', h(A(g_2, x))) = B(g_1' g_2', y).$$

Where g_1' and g_2' are in the same connected component of g_1 and g_2 respectively. Now, we observe that $G(e)$ is a normal subgroup of G this implies that $g_1' g_2'$ are in the same connected component of $g_1 g_2$. This implies the first equation.

The second equation follows from:

$$\begin{aligned} B(g_1 g_2, y) &= B(g_1, B(g_2, y)) = B(g_1, h(A(g_2'', x))) \\ &= h(A(g_1'', A(g_2'', x))) = h(A(g_1'' g_2'', x)). \end{aligned}$$

And, as above $g_1'' g_2''$ is in the same component of $g_1 g_2$. This completes the proof.

□

5.4 Stability of Axiom A actions

In this section we will give an outline of the proof of theorem 5.7, we refer the reader to [50] for details.

We are supposing that $\Omega \neq M$. As before, we denote by $\mathcal{L}$ the orbit lamination over Ω and f the hyperbolic element. As we saw, we have a local product structure in Ω given by the hyperbolic element. Hence, as we saw in the persistence theorem there exists $\mathcal{U}$ a C^1-neighborhood and U a neighborhood of Ω such that if $B \in \mathcal{U}$ and h is the map given by the persistence theorem then $h(\Omega)$ is the largest B-invariant set of U.

We fix then $B \in \mathcal{U}$, h the homeomorphism given by the persistence theorem and we denote by f' the hyperbolic element given by $B(f)$

if we saw f as an element of G. We call $\Omega' = h(\Omega)$ and notice that it is laminated by B-orbits, and h is an orbit conjugacy between Ω and Ω'.

Now, by hypothesis the compact A-orbits are dense in Ω, thus the compact B-orbits are dense in Ω', but since compact orbits are in the non-wandering set, then $\Omega' \subset \Omega_B$.

Now, we remark the effect of the hyperbolicity on the asymptotic structure of the group G. First, we fix K a symmetric compact neighborhood of e which generates the group G.

Definition 5.17. *A neighborhood of infinity in* G *is an unbounded set* $Q \subset G$ *such that* $\partial_K Q$ *is bounded, where* $\partial_K(Q) = \mathcal{H}_K(Q) \cap \mathcal{H}_K(Q^c)$ *and*

$$\mathcal{H}_K(Q) = KQ := \{kq \in G;\ k \in K \ \textit{and}\ q \in Q\}.$$

It can be proved that the definition of unbounded sets does not depends on such K.

Definition 5.18. *An end* E *of the group* G *is a class of subsets of* G *which are neighborhoods of infinity, closed by intersections and maximal with respects to these two properties. We say that the group* G *is* elliptic, parabolic or hyperbolic *if* G *has zero, one or two invariant ends respectively, i.e. ends which are fixed under multiplication of all elements of the groups (i.e.* $gE = Eg = E$ *for every* $g \in G$*).*

For instance, $\mathbb{R}$ has two ends, $\mathbb{R}^2$ has one end, compact groups has no ends, $\mathbb{Z} \times \mathbb{Z}$ has one end, $\mathbb{R} \times S^1$ has two ends and the free group has a continuum number of ends. In fact, a compactly generated locally compact group has 0,1,2 or a continuum number of ends, this was proved by Freudenthal (in fact, he defined the concept of end). The notion of an end translates, in the dynamics, in a way to go to infinity in time. For instance, since $\mathbb{R}$ has two ends, we can go to past and future and they are different ways to go to infinity.

We remark that if H is the subgroup generated by the hyperbolic element f, then since f^n has no cluster points in G, H is a non-compact closed subgroup isomorphic to $\mathbb{Z}$, in particular has two ends. Moreover, since f is central this subgroup is normal. The presence of this subgroup will give information on the number of ends of the group G can have.

Theorem 5.19. *Let G be a compactly generated locally compact group and $H < G$ be a compactly generated normal subgroup of G. If G/H is bounded then there exists a bijection between the set of ends of G and the set of ends of H. If G/H is unbounded then G has only one end.*

Hence, since the action is Axiom A, and we are supposing that it have more than one orbit, the group can have one or two ends. It turns out that in the non-Anosov case there exists two ends and they are invariant by a result of [50].

Theorem 5.20 (Theorem 4.12 of [50])**.** *If $A : G \to Diff(M)$ is an Axiom A action then if the non-wandering set is a proper subset of the manifold M then the group G is hyperbolic.*

The idea is that, in this case, the non-wandering set must have two basic pieces and it must have two ways to some orbits go to one piece and go to the other. This will give the two ends. The difficulty is to prove that these ends are invariant. Here we give an outline.

Let $\Lambda_+ = \Omega_i$ one piece of the spectral decomposition and set $\Lambda_- = \bigcup_{j \neq i} \Omega_j$. Let K as in the definition of an end, N_+ a neighborhood of Λ_+ and N_- a neighborhood of Λ_- such that $N_+ \cap N_- = \emptyset$. Then the group G is partitioned as $L_- \cup S \cup L_+$ where, $S = \{g; A(g,x) \in M - (N_- \cup N_+)\}$ and $L_\pm = \{g; A(g,x) \in N_\pm\}$.

By continuity, since $\Lambda_\pm$ are invariant, if N_+ and N_- are small enough then $A(K, N_-) \cap A(K, N_+) = \emptyset$. If $kg_\pm \in \mathcal{H}_K(L_\pm)$ then $A(kg_\pm, x) = A(k, A(g_\pm, x)) \in A(K, N_\pm)$, so we have that $\mathcal{H}_K(L_-) \cap \mathcal{H}_K(L_+) = \emptyset$. Therefore, $\partial_K((L_\pm) \subset \mathcal{H}_K S$, but since S is compact we have that $\partial_K(L_\pm)$ is bounded. But $L_\pm$ is unbounded since contains powers of f, then Zorn's Lemma says that there are ends $E^\pm$ containing $L_\pm$. Since $L_- \cap L_+ = \emptyset$, the ends are different.

If G were not hyperbolic, then let the isotropy group of the ends $H := \{s \in G; Es = E \,\forall\, E \text{ end of } G\}$, and take $g \in K \cap (G - H)$. Then $E^+ g = E^-$ and $E^- g = E^+$. In particular, if $f^{n_k} \in L_+$ then $f^{n_k} g \in L_-$. Centralness guarantees that:

$$A(f^{n_k} g, x) = A(g, A(f^{n_k}, x)) \in A(K, N_+).$$

But since $A(K, N_+) \cap N_- = \emptyset$ then such g cannot exist.

We will call E^+ and E^- the two ends of G. Let O be an orbit and $x \in O$, we define the ends of the orbit as:

$$\partial_\pm O = \bigcap_{Q \in E^\pm} \overline{\{gx, g \in Q\}}.$$

We remark that this definition does not depends on x, since the end is invariant by multiplication (this also implies that the ends of an orbit are A-invariant). Moreover, the ends of an orbit are compact and nonempty.

If $\Omega \subset \Lambda_1 \cup \cdots \cup \Lambda_m$, where Λ_i are compact, disjoint and A-invariant sets, it is an exercise to show that:

$$W^u_{\Lambda_i} = \{x; \partial_-(O(x)) \subset \Lambda_i\} \text{ and } W^s_{\Lambda_i} = \{x; \partial_+(O(x)) \subset \Lambda_i\}.$$

Lemma 5.21. *If $\partial_+(O(x)) \cap \Lambda_i \neq \emptyset$ then $\partial_+(O(x)) \subset \Lambda_i$. An analogous result holds for $\partial_-(O(x))$.*

Proof. If not, there exists $i \neq j$ such that $\partial_+(O(x)) \cap \Lambda_i \neq \emptyset$ and $\partial_+(O(x)) \cap \Lambda_j \neq \emptyset$. Then, if we take N_i and N_j disjoint neighborhoods of Λ_i and Λ_j respectively, by definition we obtain that $\{g \in G; A(g,x) \in N_1\}$ and $\{g \in G; A(g,x) \in N_2\}$ are in distinct ends. However they are inside E^+. A contradiction. $\square$

In particular, we have that:

$$\bigcup_{i=0}^{m} W^u_{\Lambda_i} = \bigcup_{i=0}^{m} W^s_{\Lambda_i} = M.$$

Theorem 5.22 (Theorem 5.1 of [50])**.** *If $V_1, \ldots, V_m$ are neighborhoods of $\Lambda_1, \ldots, \Lambda_m$ and there are no cycles then for any C^0-close action $B : G \to Diff(M)$ we have that $\Omega_B \subset V_1 \cup \cdots \cup V_m$.*

The stability follows from this theorem, since it implies that there are no Ω-explosions, then the local Ω-stability (i.e. over the basic pieces by the persistence theorem) implies global Ω-stability. We will give an outline of the proof, the details and proofs of the claims are in section 5 of [50].

We say K is a seed of G is it is a symmetric compact neighborhood of e such that its interior generates G. We also denote by K^n the set of products of at most n elements in K. If the group is hyperbolic then there exists an order on it:

Proposition 5.23 (Proposition 4.15 of [50]). *If G is a hyperbolic group with ends $E^{\pm}$ then there exists a seed K and a map $\tau : G \to \mathbb{Z}$ such that $\tau(e) = 0$, sequences c_k and C_k converging to ∞ and a constant K such that:*

- *If $k \geq 3$ then $g'g^{-1} \in K^k$ implies $|\tau(g) - \tau(g')| \leq C_k$, also $|\tau(g) - \tau(g')| \leq c_k$ implies $g'g^{-1} \in U^k$.*
- *For every $g \in G$ we have that $\tau(g') \geq K$ implies $\tau(g'g) > \tau(g)$ and $\tau(g') \leq -K$ implies $\tau(g'g) < \tau(g)$.*
- *Fix $n \in \mathbb{Z}$, for every $x \in \tau^{-1}(n)$ then $\mathcal{H}_K(x) \cap \mathcal{H}_K(\tau^{-1}(n+1)) \neq \emptyset$.*
- *$g_n \to E^{\pm} \Leftrightarrow \tau(g_n) \to \pm\infty$.*

Let W_i be a small neighborhood of $A(K^4, V_i)$, $W = \bigcup W_i$ and X_i a small neighborhood of $\overline{W_i}$. We need to prove that $\Omega_B \subset X_1 \cup \cdots \cup X_m$. If this is false then there exists actions A_n, such that $A_n \to A$ and $\Omega_{A_n} \cap (M - X) \neq \emptyset$. By a diagonal argument we can construct a sequence $g_n \in G$ with no cluster points, and points $x_n \in M - W$ such that $y_n := A_n(g_n, x_n) \to x$, $x_n \to x$ and $y_n, x \in M - W$.

Taking a subsequence we can suppose that $g_n \to E^+$. We also denote by N_i the set $W_i - V_i$.

Claim 5.24. *The set N_i acts as a fundamental neighborhood of Λ_i, i.e. if B is an action close to A and $x \in M$ such that $A(g, x) \in V_i$, $A(g', x) \in M - W_i$ and $\tau(g) < \tau(g')$ then there exists g'' such that $\tau(g) < \tau(g'') \leq \tau(g')$ and $B(g'', x) \in N_i$, moreover $B(a, x) \in A(K^3, V_i)$ for all a with $\tau(g) \leq \tau(a) < \tau(g'')$.*

The no cycles condition says that $\partial_- O(x) \subset \Lambda_i$ and $\partial_+ O(x) \subset \Lambda_j$ and $i \neq j$. There exists $g'_n \in G$ such that:

$$0 < \tau(g'_n) < \tau(g_n) \text{ and } A_n(g'_n, x_n) = x'_n \to \lambda_j \in \Lambda_j.$$

Hence, if n is large then $x'_n \in V_j$, recall that $y_n \in M - W$, then by the claim, there exists $g''_n \in G$ such that, $\tau(g'_n) \leq \tau(g''_n) \leq \tau(g_n)$ and $A_n(g''_n, x_n) \in N_j$. Moreover, if $g \in \tau^{-1}[\tau(g'_n), \tau(g''_n)]$ then $A_n(g, x_n) \in A_n(K^3, V_j)$.

By compactness of N_j we can suppose that $x''_n = A_n(g''_n, x_n) \to z \in N_j$.

Claim 5.25. *If we define $Q_- = \{g \in G; \tau(g) \leq -K\}$ then $Q_- \subset E^-$ and $A(Q_-, z) \subset W_j$.*

In particular, the claim implies that $\partial_- O(z) \subset \Lambda_j$. The no cycles condition then guarantees that $\partial_+ O(z) \subset \Lambda_l$ where if we denote i, j and l as i_1, i_2 and i_3, then this three numbers are distinct.

We can perform similar arguments to find $g_n''' \in G$, and $x_n''' = A_n(g_n''', x_n) \in V_{i_3}$ for n large, and again, using the previous proposition, find g_n'''' such that $\tau(g_n''') \leq \tau(g_n'''') \leq \tau(g_n)$ such that $x_n'''' = A(g_n'''', x_n) \to w \in N_{i_3}$ and, therefore, $\partial_- O(w) \subset \Lambda_{i_3}$. The no cycle conditions again says that $\partial_+ O(w) \subset \Lambda_{i_4}$ where i_1, i_2, i_3 and i_4 are distinct. Continuing this gives a contradiction since we only have a finite number of Λ_i's. And this will give the desired contradiction.

5.5 Appendix: Normal Hyperbolicity

In this appendix, we survey the theory of normal hyperbolicity used along the text and discuss the stability of normally hyperbolic manifolds. We will follow the book of Hirsch M., Pugh C. and M. Shub [25]. First, we recall the definition.

Definition 5.26. *Let $f : M \to M$ be a diffeomorphism of a compact Riemannian manifold, and $N \subset M$ an f-invariant smooth submanifold of M. We say that f is normally hyperbolic at N if there exists constants λ_i and μ_i for $i = 1, 2, 3$ and a Df-invariant splitting $TM = E^u \oplus TN \oplus E^s$ such that:*

$$\begin{aligned} \lambda_1 \leq \|Df|_{E_x^s}\| \leq \mu_1 \\ \lambda_2 \leq \|Df|_{T_x N}\| \leq \mu_2 \\ \lambda_3 \leq \|Df|_{E_x^u}\| \leq \mu_3. \end{aligned}$$

And,

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3 \ , \ \mu_1 < 1 \text{ and } \lambda_3 > 1.$$

The notion of normal hyperbolicity is robust in the sense that if f is normally hyperbolic at N and f' is C^1-close to f, such that N is still f'-invariant then f' is normally hyperbolic at N and the splitting $E'^u \oplus TN \oplus E'^s$ for f' is near that of f (see theorem 2.15 of [25]).

As in the chapter 4 and 5 of [25], it is possible to construct invariant manifolds using a graph transform for normally hyperbolic diffeomorphisms. In particular, for every $x \in N$ there exist local stable and unstable manifolds V_x^s and V_x^u which contains x and are tangent to E_x^s and E_x^u at x respectively. Also, there exists $\epsilon > 0$ small enough and $C > 0$, such that for every $n \geq 0$, we have:

$$\begin{aligned} d(f^n(x), f^n(y)) &\leq C(\mu_1 + \epsilon)^n d(x,y) \text{ for } y \in V_x^s \\ d(f^{-n}(x), f^{-n}(y)) &\leq C(\lambda_3 - \epsilon)^n d(x,y) \text{ for } y \in V_x^u. \end{aligned}$$

Also we have local center-stable and center-unstable manifolds given by $V_N^{cs} = \bigcup_{x \in N} V_x^s$ and $V_N^{cu} = \bigcup_{x \in N} V_x^u$. They are smooth submanifolds and have the property that:

$$N = V_N^{cs} \cap V_N^{cu}.$$

The main theorem about the invariant manifolds and their stability is the following (see [25] or [43]):

Theorem 5.27. *Let f be a C^q diffeomorphism normally hyperbolic at N. If $l_s, l_u \leq q$ are the biggest integers such that $\mu_1 < \lambda_2^{l_u}$ and $\mu^{l_s} < \lambda_3$ and $l = \min\{l_s, l_u\}$ then for every $\delta > 0$ there exists $r > 0$ and $\epsilon > 0$ such that:*

1. *There exists locally f-invariant submanifolds V_N^{cs} and V_N^{cu} tangent to $E^s \oplus TN$ and $E^u \oplus TN$ respectively.*

2. *If S is an f-invariant set contained in an ϵ-neighborhood $U_\epsilon(N)$ of N then $S \subset V_N^{cs} \cap V_N^{cu}$.*

3. *The center stable manifold W_N^{cs} is the set of points y such that $d(f^n(y), N) \leq r$ for every $n \geq 0$, and in fact, it converges exponentially fast to N. An analogous statement holds for W_N^{cu}.*

4. *V_N^{cs} is of class C^{l_s} and V_N^{cu} is of class C^{l_u}. In particular, N is a C^l submanifold.*

5. *V_N^{cs} and V_N^{cu} are subfoliated by V_x^s and V_x^u with $x \in N$ respectively.*

6. *If g is a diffeomorphism $\epsilon - C^1$ close to f then there exists a g-invariant smooth submanifold N_g, such that g is normally hyperbolic at N_g and N_g is contained in a r-neighborhood $U_r(N)$ of N.*

7. *$V_N^{cs}(g)$ is of class C^{l_s} and $V_N^{cu}(g)$ is of class C^{l_u}. In particular, N_g is a C^l submanifold and they depends continuously on g in the C^1-topology.*

8. *There exists a homeomorphism $h : U_r \to M$ δ-close to the identity in the C^0-topology such that $h(N) = N_g$.*

Now we discuss the properties of the map h. For this purpose we recall the concepts of expansiveness and shadowing.

Definition 5.28. *Let $f : M \to M$ be a diffeomorphism. An ϵ-pseudo orbit of f is a sequence $\{x_n\}_{n\in\mathbb{Z}}$ such that:*

$$d(f(x_n), x_{n+1}) \leq \epsilon \text{ for all } n \in \mathbb{Z}.$$

Moreover, if f preserves a foliation $\mathcal{L}$ with a fixed plaquation $\mathcal{P}$. We say that the pseudo orbit respects $\mathcal{P}$ if $f(x_n)$ and x_{n+1} are in the same plaque of $\mathcal{P}$.

Definition 5.29. *Let $f : M \to M$ be a diffeomorphism which preserves a foliation $\mathcal{L}$ with a fixed plaquation $\mathcal{P}$. We say that f is plaque expansive if there exists an $\epsilon > 0$ such that if $\{x_n\}$ and $\{y_n\}$ are ϵ-pseudo orbits which respect $\mathcal{P}$ and $d(x_n, y_m) \leq \epsilon$ for all $n \in \mathbb{Z}$ then x_n and y_n are in the same plaque for every $n \in \mathbb{Z}$.*

It is an exercise to show that the definition does not depends of the the metric and the plaquation (with small plaques).

Definition 5.30. *If $\{x_n\}$ is a g-pseudo orbit and $y \in M$. We say that the f-orbit of y ϵ-shadows $\{x_n\}$ if $d(x_n, f^n(y)) \leq \epsilon$ for all $n \in \mathbb{Z}$.*

One of the main features that the map h of theorem 5.27 is given by the theorem 6.8 of [25], which can be translated as follows. If $\mathcal{L}$ is a foliation such that f is normally hyperbolic to its leaves, $\mathcal{P}$ is a plaquation, η is a smooth complement of $T\mathcal{L}$, f' is C^1-close to f and $x \in M$ then $h(x)$ is the unique point in $exp_x\eta(\epsilon)$ such that the f'-orbit of $h(x)$ can be ϵ-shadowed by an f-pseudo orbit which respects $\mathcal{P}$.

Now we can analyze the stability of the foliation. We say that $(f, \mathcal{L})$ is structurally stable if, there exists a C^1-neighborhood $\mathcal{U}$ of f, such that any $g \in \mathcal{U}$ preserves some foliation $\mathcal{L}'$ and there exists a homeomorphism $h : M \to M$, which sends leaves of $\mathcal{L}$ on leaves of $\mathcal{L}'$ and $h \circ f(L) = g \circ h(L)$ for every leaf of $\mathcal{L}$.

Theorem 5.31. *If $f : M \to M$ is a diffeomorphism which is normally hyperbolic to a foliation $\mathcal{L}$ and plaque expansive then $(f, \mathcal{L})$ is structurally stable. The conjugacy h is a leaf conjugacy and every f' C^1-close to f is normally hyperbolic and plaque expansive at $\mathcal{L}' = h(\mathcal{L})$.*

Proof. Let V be the disjoint union of leaves of $\mathcal{L}$ with the leaf-topology and consider $i : V \to M$ the inclusion, which is a leaf immersion since f is normally hyperbolic. One of the consequences of theorem 6.8 of [25] is that if f' is C^1-close to f then there exists a (unique) leaf immersion $i' : V \to M$ such that $\mathcal{L}' = h(\mathcal{L})$ is a foliation if and only if i' is a bijection, and this is equivalent to the bijectivity of h.

Lemma 5.32. *$h : M \to M$ is injective.*

Proof. By hypothesis, there exists $\epsilon > 0$ such that f is ϵ-plaque expansive. Now, fix a C^1-neighborhood $\mathcal{U}$ of f such that if $f' \in \mathcal{U}$ then the map h is $\epsilon/2$-C^0-close to the identity. By the arguments above, there exists a unique f-pseudo orbit $\{x_n\}$ which $\epsilon/2$-shadows $\{f'^n(h(x))\}$ and each element of the orbit is contained in $exp_{x_n}(\eta_{x_n}(\frac{\epsilon}{2}))$. Actually, $h(x_n) = f'^n(h(x))$.

If $h(x) = h(y)$ then there are two pseudo-orbits $\{x_n\}$ and $\{y_n\}$ which $\epsilon/2$-shadows the same f'-orbit. Using the triangular inequality we have that $d(x_n, y_n) \leq \epsilon$. Plaque expansiveness says that x and y are in the same plaque, but h is an embedding in each plaque, which implies $x = y$. $\square$

Lemma 5.33. *$h : M \to M$ is continuous.*

Proof. Once again, theorem 6.8 of [25] says that for any $\delta > 0$ there exists some $N > 0$ and $\nu > 0$ such that if there exists a f-pseudo orbit

$\{p_n\}$, with $d(p_n, f'^n(y)) \leq 2\nu$ for $n = -N, \ldots, N$ then $d(y, h(x_0)) < \delta$.

If h is not continuous, then there exist some $x \in M$ and a sequence $\{z_k\}$ converging to x such that $d(h(z_k), h(x)) \geq \delta > 0$. Fix $k > 0$, by construction, there exists a unique f-pseudo orbit $\{x_n^k\}$ through z_k such that $h(x_n^k) = f'^n(h(z_k))$. Using a diagonal argument we can suppose that there exists a subsequence $x_{k_m}^n \to p_n$ as $m \to \infty$ for some $p_n \in M$. Also, p_n is a pseudo-orbit through x such that given $\nu > 0$ small and $N > 0$:

$$d(p_n, f'^n(h(z_n))) \leq d(p_n, x_n^n) + d(x_n^n, f'^n(h(z_n))) \leq 2\nu \text{ for all } |n| \leq N.$$

This gives that $d(h(z_k), h(x)) < \delta$ if k is large, a contradiction. □

As a corollary of this lemma, we have that h is surjective, since it is continuous and close to the identity in the C^0-topology.

Lemma 5.34. *f' is plaque expansive.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two f'-pseudo orbits. Then their preimages by h are f-pseudo orbits. In particular, if $d(x_n, y_n)$ is sufficiently small we have that $d(h^{-1}(x_n), h^{-1}(y_n)) < \epsilon$ and this implies that $h^{-1}(x_0)$ and $h^{-1}(y_0)$ are in the same $\mathcal{L}$-plaque. Hence, x_0 and y_0 are in the same $\mathcal{L}'$-plaque. □

The proof is now complete. □

To obtain plaque expansivity of the foliation the differentiability of the foliation plays a central role.

Proposition 5.35. *If f is normally hyperbolic at a C^1-foliation $\mathcal{F}$, then f is plaque expansive.*

To prove it we proceed as follows. Taking a iterate if necessary, we can assume that $\lambda_3 \geq 2$ and $\mu_1 \leq 1/2$. We will denote by $|.|$ the norm given by $|v| = \max\{\|v^s\|, \|v^c\|, \|v^u\|\}$ with respect to the splitting $E^s \oplus T\mathcal{F} \oplus E^u$ and fix ϵ small enough. Given a smooth path $\alpha : [0,1] \to M$ we will denote by $L(\alpha)$ its length and we define:

$$L_{us}(\alpha) := \int_0^1 \max\{\|\alpha'(t)^u\|, \|\alpha'(t)^s\|\} dt.$$

Observe that α belongs to a leaf of $\mathcal{F}$ if, and only if, $L_{us}(\alpha) = 0$. We also define a ρ-truncated distance as:

$$d_{us}^{\rho}(p, q) = \inf\{L_{us}(\alpha); \alpha \text{ joins } p \text{ and } q,\ L(\alpha) \leq \rho\}.$$

Now, we will enunciate some key lemmas and postpone the proof of them.

Lemma 5.36. *If $p' \in \mathcal{F}(p)$ and $q' \in \mathcal{F}(q)$ are points such that, if we have* $\max\{d(p, p'), d(q, q')\} \leq \min\{\rho, \rho'\}$ *then:*

$$\lim_{\rho,\rho' \to 0} \frac{d_{us}^{\rho}(p, q)}{d_{us}^{\rho'}(p', q')} = 1.$$

Lemma 5.37. *For any path α we have that:*

$$\max\{L_{us}(f^{-1} \circ \alpha), L_{us}(f \circ \alpha)\} \geq \frac{\lambda_3 - \epsilon}{2} L_{us}(\alpha).$$

Let $\{x_n\}$ and $\{y_n\}$ two ϵ-pseudo orbits such that $d(x_n, y_n) \leq \epsilon$ for all $n \in \mathbb{Z}$. Fix $\rho > 0$ and $\rho' > 0$ such that if $\min\{L(f \circ \alpha), L(f^{-1} \circ \alpha)\} \leq \rho'$ then $L(\alpha) \leq \rho$. Also we will set $\sigma = \sup_{n \in \mathbb{Z}} d_{us}^{\rho}(x_n, y_n)$.

The following claim implies the proposition.

Claim 5.38. $\sigma = 0$.

Proof. If not, then for any $\delta > 0$ there exists m such that:

$$1 \geq \frac{d_{us}^{\rho'}(x_m, y_m)}{\sigma} \geq 1 - \delta.$$

But,

$$\begin{aligned}
d_{us}^{\rho'}(f(x_m), f(y_m)) &= \inf\{L_{us}(\alpha); \alpha(0) = f(x_m), \\
&\quad \alpha(1) = f(y_m) \text{ and } L(\alpha) \leq \rho'\} \\
&\geq \inf\{L_{us}(f \circ \alpha); \alpha(0) = x_m, \alpha(1) = y_m, \\
&\quad \text{and } L(\alpha) \leq \rho\} \\
d_{us}^{\rho'}(f^{-1}(x_m), f^{-1}(y_m)) &= \inf\{L_{us}(\alpha); \alpha(0) = f^{-1}(x_m), \\
&\quad \alpha(1) = f^{-1}(y_m), \text{ and } L(\alpha) \leq \rho'\} \\
&\geq \inf\{L_{us}(f^{-1} \circ \alpha); \alpha(0) = x, \alpha(1) = y, \\
&\quad \text{and } L(\alpha) \leq \rho\}
\end{aligned}$$

Lemma 5.37 says that:

$$\max\{d^{\rho'}_{us}(f(x_m), f(y_m)), d^{\rho'}_{us}(f^{-1}(x_m), f^{-1}(y_m))\} \geq \frac{\lambda_3 - \epsilon}{2} d^{\rho}_{us}(x, y).$$

Using lemma 5.36, if ρ and ϵ are small enough, such that $\lambda_3 - \epsilon > 2 + \epsilon$, we have that:

$$\min\{\frac{d^{\rho}_{us}(x_{m+1}, y_{m+1})}{d^{\rho}_{us}(x_m, y_m)}, \frac{d^{\rho}_{us}(x_{m-1}, y_{m-1})}{d^{\rho}_{us}(x_m, y_m)}\} \geq \frac{2+\epsilon}{2}$$

And this gives a contradiction since $\frac{2+\epsilon}{2} > 1$ is fixed and δ is small. $\square$

Now we give the proof of lemmas 5.36 and 5.37.

Proof of lemma 5.36. If the foliation is trivial subordinated to $\mathbb{R}^s \times \mathbb{R}^c \times \mathbb{R}^u$, where $c = dim\mathcal{F}$ then the lemma is trivial, since the ratio will be identically 1.

Now, we observe that there exists a covering by foliated charts $\phi_i : B(p_i, r) \to \mathbb{R}^s \times \mathbb{R}^c \times \mathbb{R}^u$ which carries $T_{p_i}E^s \oplus T_{p_i}\mathcal{F} \oplus T_{p_i}E^u$ isometrically onto $R^s \oplus \mathbb{R}^x \oplus \mathbb{R}^u$, $\phi_i(p_i) = 0$, ϕ_i is close to an isometry and $\{B(p_i, r/2)\}$ still covers M. Now, the limit of the lemma relates only points p, p', q, q' and paths inside $B(p_i, r)$. Thus, since ϕ_i is close to an isometry we have that the ratio in question is nearly one. $\square$

Proof of lemma 5.37. Let $A(\alpha)$ be the set of $t \in \mathbb{R}$ in a way that $\|(\alpha'(t))^u\| \geq \|(\alpha'(t))^s\|$, and $B(\alpha) = [0,1] - A(\alpha)$. Since f contracts vectors in E^s and expands vectors in E^u wqe have $A(\alpha) \subset A(f \circ \alpha)$ and $B(\alpha) \subset B(f^{-1} \circ \alpha)$.

Thus:

$$\begin{aligned}
L_{us}(f \circ \alpha) &\geq \int_{A(f\circ\alpha)} \|(Df(\alpha')(t))^u\| dt \\
&\geq (\lambda_3 - \epsilon) \int_{A(\alpha)} \|(\alpha'(t))^u\| dt \\
L_{us}(f^{-1} \circ \alpha) &\geq \int_{B(f^{-1}\circ\alpha)} \|(Df^{-1}(\alpha')(t))^s\| dt \\
&\geq (\lambda_3 - \epsilon) \int_{B(\alpha)} \|(\alpha'(t))^s\| dt
\end{aligned}$$

Therefore:

$$\begin{aligned}
L_u s(f\circ\alpha) + L_{us}(f^{-1}\circ\alpha) &\geq (\lambda_3-\epsilon)(\int_{A(\alpha)} \|(\alpha'(t))^u\| dt \\
&\quad + \int_{B(\alpha)} \|(\alpha'(t)^s\| dt) \\
&= (\lambda_3-\epsilon) L_{us}(\alpha).
\end{aligned}$$

□

Chapter 6

Other Topics

6.1 Introduction

In this chapter we present some other topics and open questions on the theory of groups actions.

The first one is the closing lemma of Roussarie and Weyl, which is an extension of the well known Pugh's closing lemma [48]. The main problem is the following, given a system with a recurrent orbit, is it possible to find a nearby system such that this orbit is compact? In the case of flows or diffeomorphisms, compactness of the orbit means that the orbit is periodic. Pugh's closing lemma answers positively this question in this case in the C^1-topology, but its possible extension to actions is more difficult.

Nevertheless, Roussarie and Weyl obtained an extension in the case of locally free actions of $\mathbb{R}^2$ on 3-manifolds. One of the main tools is the theory of codimension one foliations on 3-manifolds, which allows to get a good structure of the possible orbits that can arise. This enable us to obtain by perturbation a torus near a recurrent orbit of such action.

The closing lemma is a basic tool in the dynamics of generic diffeomorphisms or flows. One challenge is to obtain generic results for actions on some open sets in the space of actions. This would require to obtain more perturbation techniques for actions.

The second one is about robustly transitive actions of $\mathbb{R}^2$ over

3-manifolds. One of the basic question in dynamics is to know what consequences have the presence of a dynamical property in a robust manner. Of course, the notion of robustness depends on the topology that we introduce in the space of systems, there is much work in the case of the C^1-topology for diffeomorphisms and flows, since perturbation techniques in this topology are available. For instance, transitivity were studied since the work of Mañe [35], where he shows that a robustly transitive diffeomorphism on a surface is Anosov, this was extended in the context of flows on 3-manifolds by Doering [17], an analogous result in the volume preserving setting was proved by Arbieto and Matheus in [1]. Moreover, these results were also extended in the higher dimensional setting, for instance Bonatti, Diaz and Pujals [7], shows that any robustly transitive diffeomorphisms admits a global dominated splitting.

One of the key tools to show some result of this type is the closing lemma, discussed in the first part of this chapter. The main theorem in the second section of this chapter is due to Maquera and Tahzibi [37] where they show that any robustly transitive action of $\mathbb{R}^2$ on a 3-manifold which does not have planar orbits is an Anosov flow. This technical hypothesis follows from Roussarie-Weyl's closing lemma which also has this hypothesis. One of the main problems here is that the action may be not locally free.

The third one deals with the question of whether a codimension one Anosov action is transitive. The main motivation is Verjovsky's theorem [58], where he shows that a codimension one Anosov flow on a manifold with dimension at least four is transitive. The hypothesis on the dimension is needed since there exists an example of an anomalous Anosov flow on a 3-manifold which is not transitive by Franks and Williams [19]. Hence, the natural dimensional hypothesis in the setting of an action of a group G on a manifold M would be $dim(M) \geq dim(G) + 3$.

Hence, in the third section we will give an outline of a result by Barbot and Maquera [6], where transitivity is obtained when the group G is $\mathbb{R}^k$. The proof deals with a criterion to obtain transitivity in this setting and an argument to pass this setting to the setting of irreducible Anosov actions, which shares many properties of Anosov flows which were studied in Barbot's thesis [5].

Finally, in the last section, we will state some open problems related to the issues mentioned along the book.

6.2 A closing lemma

As mentioned in the introduction, one main technical tool to study robust properties are perturbation tools, and among then, one of the most useful is the closing lemma. The first result on this topic is the remarkable:

Theorem 6.1 (Pugh's closing lemma [48])**.** *Let $f : M \to M$ be a diffeomorphism and $x \in M$ a recurrent point. Then any C^1 neighborhood of f contains a diffeomorphism g such that x is a periodic point of g.*

This theorem also holds for flows in the C^1-topology and it is an open question if it holds on higher topologies. In what follows, we will give an outline of the proof of an extension of the closing lemma for actions of $\mathbb{R}^2$ on 3-manifolds due to Roussarie and Weyl [52].

In this section M will be a 3-dimensional manifold. We recall that $A^1(\mathbb{R}^2, M)$ is the set of C^r (with $r > 2$) locally free actions of $\mathbb{R}^2$ on M, with the C^1-topology. As usual TM will denote the tangent bundle and $\mathcal{X}(M)$ the space of vector fields on M with the C^1-topology. We will also set T^2M as the set of 2-planes on TM and $\mathcal{X}^2(M)$ the space of sections over T^2M with the C^1-topology.

We observe that every locally free action of $\mathbb{R}^2$ on M gives rise to a element on $\mathcal{X}^2(M)$. Indeed, if A is such action and $p \in M$, we define $PA(p)$ as $\partial_v A(0,p).T_0\mathbb{R}^2$ where ∂_v is the partial derivative on $v \in \mathbb{R}^2$ (recall that the action can be viewed as $A : \mathbb{R}^\times M \to M$) then $PA \in \mathcal{X}^2(M)$.

We observe that both topologies on TM and T^2M are related in the following sense. If X and Y are two vector fields linearly independent generating a plane $P(X,Y)$ then for every $\epsilon > 0$ there exists some $\delta > 0$, such that if X' and Y' are two vector fields δ-C^1-close to X and Y respectively then X' and Y' are linearly independent and if $P(X',Y')$ is the plane field generated by X' and Y' then $P(X,Y)$ and $P(X',Y')$ are ϵ-close in T^2M.

The first closing lemma of Roussarie and Weyl is an adaptation of Pugh's closing lemma:

Theorem 6.2. *Let M be a compact 3-manifold and $A \in A^1(\mathbb{R}^2, M)$ be a locally free action such that its orbits are not planes. Then for every $\epsilon > 0$ there exists an action $B \in A^1(\mathbb{R}^2, M)$ such that PB is ϵ-C^1-close to PA and B has a compact orbit.*

Proof. If A has a compact orbit there is nothing to do. If not, then there exists a cylindrical orbit, say O. Hence, there is a minimal set $V \subset \overline{O}$ and, since we are assuming that there is no compact orbits, V is the closure of a recurrent cylindrical orbit. But V cannot be properly contained on M by Sacksteder's theorem[53]:

Theorem 6.3 (Theorem 8 of [53])**.** *If a foliation on M^n is defined by a locally free action of $\mathbb{R}^{n-1}$ then there are no exceptional leaves with minimal closure.*

Hence, V is the whole manifold and this implies that every orbit is cylindrical. Also, there exist local coordinates $(\theta, z, x) \in S^1 \times [-1, 1] \times [-2, 2]$ such that, if X and Y are the generator vector fields of the action the $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial \theta}$. Also, the orbit inside this chart are the cylinders $z = constant$. Using these coordinates the arguments of the proof of Pugh's closing lemma can be performed in the same way. $\square$

More difficult, is the problem of given a recurrent orbit, perturb the action and find an orbit with the topological type of a torus *near* to the recurrent orbit.

Theorem 6.4 (Theorem 2(1) of [52])**.** *Let $A \in A^1(\mathbb{R}^2, M)$, O a non-planar recurrent orbit of the action and $\epsilon > 0$. Then, there exists a submanifold O' diffeomorphic to $\mathbb{T}^2$, ϵ-close to O such that the plane field tangent to O' can be extended to a plane field C^1 close to the plane field PA.*

The proof of this theorem is more intricate and deals with the notion of linearization of return maps over a cylindrical orbit and the minimum lift. First we will define these notions and then give an outline of the proof. In the follows, A will be an action as in the statement of theorem 6.4

6.2.1 Returns and their linearizations over cylindrical orbits

Let $p \in M$ such that its orbit is a cylinder. Hence, its isotropy group has the form $u\mathbb{Z}$ for some $u \in \mathbb{R}^2$. Now, take $v \in \mathbb{R}^2$ such that $\{u, v\}$ is a basis of $\mathbb{R}^2$ and also consider u and v as constant vector fields on $\mathbb{R}^2$. We define two vector fields $X = \partial_t A(0,p).u$ and $Y = \partial_t A(0,p).v$ (where the partial derivative is related to the first coordinate) and by commutativity $[X, Y] = 0$. Also, these vector fields generates the action, since $A(su + tv, p) = x(t)(y(t)(p)) = y(s)(x(t)(p))$ where $s, t \in \mathbb{R}$ and $x(t)$, $y(t)$ are the flows generated by the vector fields.

Also the cylindrical orbit Γ of p has it circle component as closed orbits of period 1 of Y, and the non-compact part given by the orbits of X. The recurrence mentioned in the theorem follows from a lemma of Rosenberg:

Lemma 6.5. *If c is a closed orbit of Y, and the $\overline{\Gamma}$ is minimal then for any neighborhood U of c and t_0 there exists $t > t_0$ such that $x(t)(c) \subset U$.*

Let N be the vector field orthonormal to the plane field PA and $n(u)(p)$ the generated flow (starting at p) where $u \in \mathbb{R}$. We also define, $A_0 = \{n(u)(c)\}_{u \in [-\alpha, \alpha]}$ for some small α, and notice that A_0 is diffeomorphic to $S^1 \times [-\alpha, \alpha]$ and $c \sim S^1$ can be parametrized by $\theta \in [0, 1]$ such that $Y|_c = \frac{\partial}{\partial \theta}$.

Definition 6.6. *A return is an intersection of Γ with A_0 diffeomorphic to S^1. The set of returns of Γ is denoted by D.*

We can define a distance between returns and also a natural order as follows. If $p, q \in D$ then the distance between p and q is defined as $d(p,q) = \sup_{\theta \in [0,1]} |p(\theta) - q(\theta)|$. Also, we say that $p < q$ if for any $x_0 \in c$ and $t, t' \in \mathbb{R}$ we have $x(t)(x_0) \in p$ and $x(t')(x_0) \in q$ we have $t < t'$.

Now, if c' and c'' are returns distinct form c then they define a domain A of A_0 diffeomorphic to $S^1 \times I$ where I is an interval. We say that $C(p, l(\theta))$ is a *crown* if it is formed by points $(z, \theta) \in A$ such that $|z - p(\theta)| \leq l(\theta)$.

Definition 6.7. *Let c_i be the i-th return of Λ, starting in c and given by the order $<$. Then the orbits of X give a diffeomorphism between*

c and c_i:

$$(\theta, c(\theta) \to (\phi_i(\theta), c_i(\phi_i(\theta))).$$

We call $\phi_i(\theta)$ the angular component and denote by $k_i(\theta)$ the piece of an orbit of X between $(\theta, c(\theta))$ and $(\phi_i(\theta), c_i(\phi_i(\theta)))$.

Observe that the holonomy along $k_i(\theta)$ induces a local diffeomorphism between $R_\theta = \{p \in A; \theta(p) = \theta\}$ and $R_{\phi_i(\theta)} = \{p \in A; \theta(p) = \phi_i(\theta)\}$, given by $z \to c_i(\phi_i(\theta)) + \phi_i'(\theta, z)$ where $\phi_i'(\theta, 0) = 0$. Also, there exists a crown V_0 of c inside A and a crown V_i of c_i which are diffeomorphic by the map:

$$\Phi_i(\theta, z) = (\phi_i(\theta), c_i(\phi_i(\theta)) + \phi_i'(\theta, z)).$$

We call this diffeomorphism as the holonomy map in a neighborhood of the i-th return.

Definition 6.8. *We say that $\phi_i(\theta, z)$ is locally linearizable of V_0 on V_i in the coordinates U if, for any $(\theta, z) \in V_0$ we have:*

$$\phi_i'(\theta, z) = a_i(\theta) z.$$

Here, $a_i(\theta)$ will be periodic. Actually, $a_i(\theta) = \frac{\partial \phi_i'}{\partial z}(\theta, 0)$.

The main theorem about linearization is the following:

Theorem 6.9 (Theorem III.1 of [52]). *Let $A \in A^1(\mathbb{R}^2, M)$, for every $\epsilon > 0$ and $N > 0$ there exists a C^{r-1}-diffeomorphism $h : M \to M$ which is ϵ-C^1-close to the identity, such that, $h = id$ outside $\bigcup_{i=0}^{N} c_i \times [-2, 2]$ and there exist $L_0, \ldots, L_N$ neighborhoods of $c, c_1, \ldots, c_N$ such that the action $h \circ A$ is locally linearizable of L_0 to L_i for every $i = 1, \ldots, N$.*

6.2.2 The minimum lift

We denote the coordinates of $\mathbb{R}^3$ as (x_1, x_2, x_3) and the polar coordinates of $\{x_1 = 0\}$ as (ρ, θ). Now we define circles for $r > 1$, $c_r = \{(x_1, \rho, \theta); x_1 = 0, \rho = r\}$ and setting $z = \rho - r$ we obtain new coordinates (z, θ, x). We define $F_r = \{x = 0, z \in [-1, 1]\}$ and $G_r = \{x \in [0, 1]$ and $z \in [-1, +1]\}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $Q = [-1,1] \times [0,1]$ is its support, $f > 0$ on the interior of Q, $f < 1$ and $\max\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\} < 1$. If $b : [0,1] \to \mathbb{R}$ is such that $b(\theta) < r(\theta)$ then we say that the linear counterpart of f on $S = c(r(\theta), b(\theta)) \times [0,1]$ is the function $f_s(2, \theta, x) = b(\theta) f(\frac{2-r(\theta)}{b(\theta)}, \theta, x)$.

The main feature of the linear counterpart is the following lemma:

Lemma 6.10 (Minimum lift). *Given $\epsilon > 0$ and the flow of the vector field $(\epsilon f_s, 0, 1)$ (in the coordinates (z, θ, x)) given by $\psi(t, M)$ for $t \in [0,1]$ and $M \in S$. If we define $m(\theta) = \min\{\psi(1, (z, \theta, 0)) - z; z \in [r(\theta) + \frac{b(\theta)}{2}, r(\theta) - \frac{b(\theta)}{2}\}$, then there exists a constant K which does not depends on r such that $m(\theta) \geq \epsilon K b(\theta)$*

6.2.3 Proof of the theorem 6.4

The idea of the proof is the following. First, to perform the perturbations we linearize the action along returns for a sufficiently large piece of the orbit, this will make the notion of push the orbit more easily to be performed. Then we try to push the orbit along the flow generated by X, which is a non-compact direction of the orbit, and close it, forming a torus. We recall that the orbit of Y is already a circle. Finally, we perturb also Y such that the generated plane field extends to an action over the whole manifold, close to the original action.

Let Γ be the cylindrical orbit in the hypothesis of the theorem.

Linearizing

Using the theorem 6.9 there exists a C^1-close action B such that it is a local linearization of A for the N first returns. Let X_1 and Y_1 the respective generators of the action B, C^1-close to X and Y (the generators of A). Moreover, if Z is a vector field orthogonal to $P(X,Y)$ then it is still transversal to $P(X_1, Y_1)$.

This new action has the same N first returns of A and also has a recurrent orbit which does not pass by c. If δ is given by the relation between the topology of TM and T^2M, and K is given by the minimum lift, then we take $N > \frac{2}{K\delta}$. We also take $0 < \nu < \min\{\frac{\delta}{2}, \frac{d(c, \partial L_0)}{2}\}$, where $L_0, L_1, \ldots, L_N$ are given by theorem 6.9. Moreover, the linearizations satisfying $\Psi_i(L_0) = L_i$ for $i = 1, \ldots N$.

Finding a torus

Now, we invoke a version of a lemma of [48], with $\lambda = 1/2$ and $\nu > 0$ as defined above.

Lemma 6.11. *If Λ is a cylindrical recurrent orbit of an action A then for every $\nu > 0$ and $0 \leq \lambda < 1$ there exists returns p and q such that $\max\{d(p,c), d(q,c)\} < \nu$, $p < q$ and for every return r such that $p < r < q$ we have that:*

$$\min\{d(p,r), d(q,r)\} > \lambda d(p,q).$$

Moreover, if we define $W(p,q) := c(p, \frac{d(p(\theta),q(\theta))}{2}) \cup c(q, \frac{d(p(\theta),q(\theta))}{2})$, then $r \cap W(p,q) = \emptyset$.

We use this lemma for every i and we denote $q_i = \Psi_i(q)$, $p_i = \Psi_i(p)$, $b_i(\theta) = d(p_i(\theta), q_i(\theta))$ and $W_i = W(p_i, q_i)$. We notice that $W_0 \subset L_0$ and by linearity $W_i = \Psi_i(W_0)$. Recalling the coordinates (z, θ, x) we define the sets $S_i = \{(z,\theta,x) \in U; (z,\theta,0) \in W_i, x \in [0,1]\}$ and $W_i' = \{(z,\theta,x) \in U; (z,\theta,0) \in W_i, x = 1\}$.

Now, using the linear counterparts f_{s_i}, we define the function $\Delta_i(z,\theta,x)$ as:

$$\begin{array}{ll} \dfrac{\delta}{2} f_{s_i}(z,\theta,x)\dfrac{\partial}{\partial z} & \text{if} \quad q_i(\theta) < p_i(\theta) \\ -\dfrac{\delta}{2} f_{s_i}(z,\theta,x)\dfrac{\partial}{\partial z} & \text{if} \quad q_i(\theta) > p_i(\theta) \end{array}$$

Observe that the support of Δ_i is S_i, by construction. It is possible choose A such that any return p satisfies $\frac{\partial p(\theta)}{\partial \theta} < \frac{\delta}{5}$. For every $\sigma \in [0,1]$ we define $\Delta^\sigma = \sigma \sum\limits_{i=1}^{N} \Delta_i$ and observe that its support is $\bigcup_{i=1}^N S_i$ and the C^1-norm is bounded from above by $\frac{\delta}{2}$.

The perturbation on X_1 will be $X' = X_1 + \Delta^\mu$ for some $\mu \in [0,1]$ such that the orbit of the generated action will be a torus.

We denote by A' the projection of A over $\{x = 1\}$, $\Pi_i : W_i \to W_i'$ the projection given by $\Pi_i(z,\theta,0) = (z,\theta,1)$ and $\delta_i(\sigma) : W_i \to W_i'$ the diffeomorphism given by $X_1 + \Delta^\sigma$.

Now, for any $\sigma \in [0,1]$, by induction we define functions $\alpha_i(\sigma) : W_0 \to W_i$ as $\alpha_0(\sigma) = id|_{W_0}$ and for every $i \in [1,N]$, we set $\alpha_i(\sigma) = \psi_i \psi_{i-1}^{-1} \Pi_{i-1}^i \delta_i(\sigma) \alpha_{i-1}(\sigma)$.

Definition 6.12. *We say that a return γ is homothetic to p and q if there exists λ such that:*

$$\frac{d(\gamma(\theta), p(\theta)}{d(q(\theta), p(\theta))} = \lambda\gamma(\theta)$$

We observe that if γ is a homothety then its image by $\alpha_i(\sigma)$ is also homothetic to p_i and q_i. Now, we set $q_i(\theta,\sigma) = \alpha_i(\sigma)(q(\theta))$, $q_i'(\theta,\sigma) = \Pi_i(q_i(\theta,\sigma))$ and $\widetilde{q}_i(\theta,\sigma) = \delta_i(\sigma)(q_i(\theta,\sigma))$.

There are two possibilities:

$$\widetilde{q}_i(\theta,\sigma) \in [p_i'(\theta), q_i'(\theta)] \qquad \forall i \in [0,N] \text{ and } \forall\sigma \in [0,1] \qquad (a)$$
$$\exists\sigma \in [0,1] \text{ and } \exists i \in [0,N] \qquad \text{such that } p_i'(\theta) \in [q_i'(\theta), \widetilde{q}_i(\sigma,\theta)] \quad (b).$$

It is possible to show that (a) leads to a contradiction, using the constants defined above and the minimum lift lemma, also that since Δ_i points from q_i to p_i, then (b) is true for $i = N$ and $\sigma = 1$. For more details see section (VB) of [52].

In particular, since $\widetilde{q}_N(\theta,\sigma)$ is continuous, using the Intermediate value theorem, there exists some μ such that $\widetilde{q}_N(\theta,\mu) = p_N'(\theta)$. In particular the orbit of $X' = X_1 + \Delta^\mu$ which passes by q becomes a periodic orbit, and the action will have an orbit with the type of a torus.

Perturbing Y to obtain the properties of the extended plane field

If Γ_N is the piece of the orbit Γ between c and $c_N' = \Pi_N(c_N)$, then the orbits of X starting on L_0 gives rise to a tubular neighborhood, which we call T such that $T \cap U = \bigcup_{i=1}^{N} (L_i \times [-1,1])$. Using appropriated coordinates T is diffeomorphic to a submanifold of $S^1 \times I \times \mathbb{R}$, bounded by the two sections L_0 and L_N', where $I = [-\mu,\mu]$ obtained above.

Recalling the previous notation, we denote by $p_N'' = X(l)p$ and $L_N'' = X(l)(L_0)$ for some $l > \varphi_N(\theta,z)$ such that $T \subset T^0 = \{(z,\theta,x); x \in [0,l]\}$. We want to define a perturbation Y' of Y_1 such that $Y' = Y$ on $M - T$ and $\|Y' - Y\|_{C^1}$ is as small as we want.

If $\Gamma'(p,q)$ denotes the piece of orbit generated by X' until p_N'' and extended using X from p_N'' to q, then it is defined by an equation $z = \gamma(\theta,x)$ for some γ. The main properties of γ are the following:

- $\gamma(\theta, 0) = q(\theta)$ and $p''_N \in \Gamma'(p, q)$,
- $\frac{\partial \gamma}{\partial x}(m) = X'$ if $m \in \Gamma'(p, q) \cap (\bigcup_{i=1}^{N} L_i \times [-1, 1])$.
- Outside $\bigcup_{i=1}^{N} L_i \times [-1, 1]$ we have that $\frac{\partial \gamma}{\partial z} = 0$ and $\frac{\partial \gamma}{\partial \theta} = constant$, since γ does not depends of x.
- Inside $L_i \times [-1, 1]$ we have that $\frac{\partial \gamma}{\partial x} = X'$, $\frac{\partial \gamma}{\partial \theta} = \frac{\partial p_i}{\partial \theta}$ and $\frac{\partial \gamma}{\partial z} = \frac{\partial p_i}{\partial z}$.

If we take a C^∞ function $u : [-1, 1] \to [0, 1]$ such that $u(t) = u(-t)$, $u(0) = 1$, $u(1) = 0$, $u'(0) = u'(1) = 0$, $u'(t) > 0$ if $t \neq 0, \pm 1$ and $|u'(t)| < 2$ for every $t \in [-1, 1]$. Then, it can be showed that the following perturbation satisfies the properties that we want, as follows:

If $m \in \Gamma(p, q)$ we define $Y'(m)$ as the orthogonal projection of Y over $T_m \Gamma'(p, q)$. Then we extend Y' for $m = (z, \theta, x)$ as follows:

$$Y'(m) = Y(m) + u(\frac{z - \gamma(\theta, x)}{\mu - \gamma(\theta, x)})(Y'(\theta, z) - Y(\theta, z))$$

We refer the reader to section (VC) of [52] for the details.

6.3 Robustly transitive actions

In this section we consider M an orientable compact 3-manifold without boundary. We recall that an action $A : G \to Diff(M)$ is transitive if there exists some orbit $\{(A(g, x); g \in G\}$ which is dense on M. An important problem in dynamics is to study what consequences a dynamical property gives to the system if this property appears in a robust manner, this means that any action sufficiently close to the original also shares the property. Clearly, to make this notion precise we need to define what is the topology which will be used in the space of actions.

In what follows we will define a topology when the group is $\mathbb{R}^2$ which is a topology between the C^1 and C^2 well known topologies for differentiable maps. We recall that since $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is a basis of $\mathbb{R}^2$, we will denote the associated infinitesimal generators of A by X_{e_1} and X_{e_2}. We also denote by $\|.\|_1$ the C^1 norm in the space of vector fields. In this section, we will follows [37] closely:

Definition 6.13. *Given two C^2-actions A and B of $\mathbb{R}^2$ on M. The $(1,1)$-distance between A and B is defined as:*

$$d_{(1,1)}(A,B) = \max\{\|X_{e_1} - Y_{e_1}, X_{e_2} - Y_{e_1}\|\}.$$

Where X_{e_1}, X_{e_2} are the generators of A and Y_{e_1}, Y_{e_2} are the generators of B. With this distance, the set of C^2 actions $A : \mathbb{R}^2 \to Diff(M)$ becomes a complete metric space and we denote this space by $C^{(1,1)}(\mathbb{R}^2, Diff(M))$.

Using this topology we can introduce the notion of robust properties. In particular, we have the following definition:

Definition 6.14. *A C^2-action $A : \mathbb{R}^2 \to Diff(M)$ is robustly transitive if there exists $\epsilon > 0$ such that for any C^2-action $B : \mathbb{R}^2 \to Diff(M)$ such that $d_{(1,1)}(A,B) < \epsilon$, we have that B is transitive.*

Finally, we will say that an action $A : R^2 \to Diff(M)$ is a flow if their generators are linearly dependent. For example if it is trivial in one coordinate: $A(0,x) = id$.

The main theorem in this section is the following:

Theorem 6.15 (Maquera-Tahzibi [37])**.** *Let M is an orientable closed 3-manifold. If $A : \mathbb{R}^2 \to Diff(M)$ is a $C^{(1,1)}$ robustly transitive action with a dense orbit not homeomorphic to $\mathbb{R}^2$ then A is an Anosov flow.*

In fact, we need only prove the following theorem:

Theorem 6.16. *Let M is an orientable closed 3-manifold. If $A : \mathbb{R}^2 \to Diff(M)$ is a $C^{(1,1)}$ robustly transitive action with a dense orbit not homeomorphic to $\mathbb{R}^2$ then A is a transitive flow.*

Since, by the definition of the $C^{(1,1)}$-topology, we will have that the vector field X which generates A is C^1-robustly transitive and then we can apply a theorem due to Doering.

Theorem 6.17 (Doering [17])**.** *Any C^1-robustly transitive vector field on an orientable closed 3-manifold is an Anosov flow.*

6.3.1 The topological type of a two-dimensional orbit

First, we recall that a two-dimensional orbit of an action of $\mathbb{R}^2$ can have only three topological types: a plane, a cylinder or a torus.

Lemma 6.18. *All of the dense orbits of the action have the same topological type.*

Proof. It is enough to show that the isotropy groups of these orbits are the same, because this will imply that the orbits are homeomorphic. Let p and q be two dense orbits. If $g \in G_p$ and $h \in \mathbb{R}^2$ then $A(h, A(g,p)) = A(g,p)$, then by continuity for any $z \in O(q)$ we have that $A(h,z) = z$ and this implies that $h \in G_q$. $\square$

By the hypothesis of the main theorem, any dense orbit must be a cylinder or a line. If there exists a dense orbit with the type of a line then by continuity, both of the infinitesimal generators must be linearly dependent, and this will imply that the action is a transitive flow, and that will finish the proof of the theorem. So we need to rule out the existence of a dense orbit with the type of a cylinder.

We start doing the following remark: if there exists some dense orbit $O(p)$ with the type of a cylinder, let $u \in \mathbb{R}^2 - \{0\}$ such that the isotropy group of p is $\mathbb{Z}u$. Then p is a 1-periodic orbit for the flow $A(t.u)_{t\in\mathbb{R}}$, and let Y the vector field generated by this flow. Also, take $v \in \mathbb{R}^2$ linearly independent to u, and X the vector field generated by the flow $A(t.v)_{t\in\mathbb{R}}$. Since $\mathbb{R}^2$ is abelian, the two flows commute, so every point of $O(p)$ is a 1-periodic orbit for Y, and by denseness, every point of M is a 1-periodic orbit for Y. So, any two-dimensional orbit is a torus or a cylinder.

Now we recall a result from 3-manifold topology [51]:

Proposition 6.19. *Let M be a orientable closed 3-manifold. There exists k such that if $T_1, \dots T_k$ are submanifolds with the type of a torus then they form the boundary of a three dimensional submanifold of M.*

The proposition says that if there are at least k two-dimensional compact orbits, then the action cannot be transitive, since any dense two dimensional orbit should intersect one of this tori. Hence, by the previous remark this will say that most of orbits are cylinders. In other words, if we could find too many two dimensional compact orbits, for the action or some close perturbation of it, then this action cannot be robustly transitive.

One tool to obtain compact orbits, in the non-singular case, is the closing lemma by Roussarie-Weyl [52]:

Theorem 6.20. *Let $A : \mathbb{R}^2 \to Diff(M)$ be a locally free action on a orientable closed 3-manifold, if Λ is a non-planar, recurrent orbit, and for every $\epsilon > 0$ there exists a torus ϵ-close to Λ such that the plane field tangent to this torus can be extended to a plane field that is C^1 close to the plane field generated by the infinitesimal generators of A.*

Unfortunately, the action can be singular (in fact, we want to prove that!), so we need a version of this closing lemma in this scenario, where there are not orbits homeomorphic to a plane.

6.3.2 A singular version of the closing lemma in the non-planar case

We will assume, in the follows, that M has a Riemannian metrics and that there exists a point p with dense orbit with the type of a cylinder. We will show that there exits compact orbits near p for some action B close to A.

Proposition 6.21. *There exists a generator Y of A, a closed orbit of Y and an action B which is a $C^{(1,1)}$-perturbation of A supported on a neighborhood of this orbit such that B has a compact orbit.*

Now, we will give an sketch of the proof of the proposition. But first, we will introduce some local coordinates related to the cylindrical orbit as follows. First, we fix a basis $\{w_1, w_2\}$ of $\mathbb{R}^2$ such that the vector field Y generated by w_2 has a 1-periodic orbit c through p, and we call X the vector field generated by w_1, also we parametrize c with $\theta \in [0, 1]$ such that $\frac{\partial}{\partial \theta} = Y|_c$.

Now, we take then χ a unitary local vector field in a neighborhood of c orthogonal to the orbits of the action. Then we put a coordinated system (x, θ, z) in a small neighborhood of c diffeomorphic to

$S^1 \times (-\epsilon, \epsilon) \times (-1, 1)$, such that (X, Y, χ) corresponds to $(\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z})$. Observe that in this coordinates, the level sets $z = constant$ corresponds to pieces of orbits of the action.

This neighborhood acts like a flow box, where the annulus $S^1 \times (-\epsilon, \epsilon)$ is a transversal section (and foliated by orbits of Y).

Lemma 6.22. *Any neighborhood U of c has an unbounded sequence $t_i \in \mathbb{R}$ such that if X_t is the flow generated by X then $X_{t_n}(c) \subset U$.*

Proof. We will denote by Y_t the flow generated by Y. Let V an small neighborhood of c such that for any $z \in V$ and $t \in [0, 1]$ we have that $Y_t(z) \in U$. Since $O(p)$ is dense, there exists $z \in c$ and $t \in \mathbb{R}$ such that $X_t(z) \in V$, observe that t becomes large when we shrink V. Now for any $s \in [0, 1]$, we have that $Y_s(X_t(z)) \in U$. Finally, we have that $Y_s(z) \in c$ and $X_t(Y_s(z)) = Y_s(X_t(z)) \in U$. □

Now, we recall the Pugh's closing lemma for flows.

Theorem 6.23 (Pugh's closing lemma [48])**.** *Let X be a C^1 vector field with a recurrent orbit q, a neighborhood U of p and $\epsilon > 0$. Then there exists an ϵ-C^1-small perturbation Z of X supported on U. Such that q is a periodic orbit of Z.*

Using the transversal section, Maquera and Tahzibi can adapt the proof of the closing lemma for flows of Pugh to this particular case, in the setting of actions, obtaining the following theorem.

Theorem 6.24. *There exists a perturbation $B : \mathbb{R}^2 \to Diff(M)$ supported on a neighborhood of c which is $C^{(1,1)}$-close to A with a compact orbit.*

6.3.3 End of the proof

As in the previous section, we will fix p the cylindrical dense orbit, c the 1-periodic orbit associated to the vector field Y in the (x, θ, z)-coordinates and $S^1 \times (-\epsilon, \epsilon)$ the transversal section associated to these coordinates. Since there exists only a finite number of orbits with the type of a torus, we can choose ϵ small enough such that the orbits that intersects the transversal section are cylindrical.

Now, for any $0 \leq i < k$ we define $C_i = \{(x, \theta, z); \frac{\epsilon i}{k} < z < \frac{\epsilon(i+1)}{k}\}$. As we see above, there exists some s such that $X_s(c) \subset \{(x, \theta, z); |z| <$

$\frac{\epsilon}{k}\}$. Also, since the neighborhood acts like a flow box, there exists some $t \geq s$ such that $X_t(p) \in C_0$. But, since the flow box is foliated by Y-orbits, we have that $X_t(c) \subset C_0$.

By the adapted closing lemma of the previous section, there exists a perturbation A_1 of A supported on $\{(x, \theta, z); |z| < \frac{\epsilon}{k}\}$ with a compact orbit intersecting this neighborhood. But, since the support is inside C_0, the orbits of A_1 passing through C_i for $i > 0$ still are cylindrical. Then we can perform a series of perturbations on each C_i, obtaining an action A_k, sufficiently close to A, with k periodic tori. And this gives a contradiction.

6.4 A Verjovsky's theorem for actions of $\mathbb{R}^k$

In this section, following [6], we will give an outline of a proof of the following theorem:

Theorem 6.25 (Barbot-Maquera [6])**.** *Let M be a closed manifold such that $dim(M) \geq k+3$. Then every codimension one Anosov action $A : \mathbb{R}^k \to Diff(M)$ is transitive.*

First we will give a criterion for transitivity, essentially the absence of non-bi-homoclinic points guarantees the transitivity of the action. Then we show that every point is a bi-homoclinic point.

6.4.1 A criterion for transitivity

Using the codimension on hypothesis, we will assume that the codimension of the stable foliation $\mathcal{W}^s$ is one, then the strong unstable foliation $\mathcal{W}^{uu}$ has leaves diffeomorphic to $\mathbb{R}$. Also, considering the double covering if necessarily, we assume that $\mathcal{W}^{uu}$ is orientable. Now, considering the induced metric on $\mathcal{W}^{uu}$, if d is the signed distance, we can define a parametrization $u : \mathbb{R} \times M \to M$ of $\mathcal{W}^{uu}$ requiring that $d(u(t,x), x) = t$.

Lemma 6.26. *The application u is continuous, and C^∞ on the variable t. Furthermore, at $t = 0$ all of these derivatives are continuous on x.*

Now we define the following sets:

$$\begin{aligned}\mathcal{H}^+ &= \{x \in M; \{u(s,x)\}_{s>0} \cap \mathcal{W}^s_x = \emptyset\}\\ \mathcal{H}^- &= \{x \in M; \{u(s,x)\}_{s<0} \cap \mathcal{W}^s_x = \emptyset\}\end{aligned}$$

And we say that a point x is *bi-homoclinic* if $x \notin \mathcal{H}^+ \cup \mathcal{H}^-$.

The following two propositions gives a criterion for transitivity:

Proposition 6.27. *If every point x is bi-homoclinic then $\mathcal{W}^s$ is minimal.*

Proposition 6.28. *If $\mathcal{W}^s$ is minimal then the action is transitive.*

The first proposition follows from the fact that the strong unstable foliation is expanded by the Anosov element, and every leaf of the strong unstable manifold intersects every leaf of the stable manifold. This implies that the closure of the stable manifolds is open. For more details see [2].

The second one, follow from the existence of a Lyapunov function $L : M \to \mathbb{R}$ for the flow generated by the Anosov element, see [16]. Which is constant on the non-wandering set of the flow (which coincides with the non-wandering set of the action). Now observe that the spectral decomposition has only one basic set, since the stable foliation is minimal. This implies that the non-wandering set has a dense orbit. Now, since both of the invariant manifolds L is constant on the non-wandering set. And by compactness, since the ω-limit set and the α-limit set of any $x \in M$ are in the non-wandering set, it follows that L vanishes everywhere and then M is the non-wandering set.

6.4.2 Proof of the theorem

We can suppose that the action is irreducible. Also, as we saw above, the theorem will follows if we prove that every point is bi-homoclinic. And this will follows if we prove that both $\mathcal{H}^+$ and $\mathcal{H}^-$ are empty.

$\mathcal{H}^\pm$ are formed by compact orbits

First, we cover the manifold with a finite number of neighborhoods $\{U_i\}$ with local product structure. Take an orbit $y \in \mathcal{H}^+$ and W^s_y its stable leaf.

By definition, $O(y) \cap U_i$ is contained in a single plaque F_i, since $\bigcup F_i$ is compact, then $O(y)$ is pre-compact on $\mathcal{W}^s_y$. Since the canonical projection $p_{\mathcal{W}^s_y} : \mathcal{W}^s_y \to P$ restricted to $O(p)$ is a covering map then P is compact. Also the associated lattice contains an Anosov element v, since intersects a chamber. Hence $A(v, \mathcal{W}^{ss}_y)$ is a contracting map with a fixed point y_1.

In particular, by compactness on $\mathcal{W}^s_y$ of $O(y)$, we have that the distance $d_{\mathcal{W}^s_y}(A(-tv, y), A(-tv, y_1))$ is bounded, hence $y_1 = y$ is fixed by $A(v, .)$. Then $O(y)$ is compact.

Now, since $\mathcal{W}^s_y$ contains at most one compact orbit and this orbit intersects strong stable leaves at most one time. We have that intersection of $O(y)$ and U_i is connected for every i.

In the other hand, by local product structure, the volume of orbits in U_i is uniformly bounded from above and, since $\mathcal{H}^+ \cap U_i$ is connected. Hence, $\mathcal{H}^+$ is a union of compact orbits. Analogously for $\mathcal{H}^-$.

The meaning of $\mathcal{H}^+$ in the universal covering

Let us call Q the orbit space of $\overline{A}$ the induced action in the universal covering. Then, there exists foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ on Q induced by $\widetilde{\mathcal{W}^s}$ and $\widetilde{\mathcal{W}^u}$ respectively. Moreover $\mathcal{G}^u$ is one dimensional and orientable, hence, there exists a natural order in each leaf of $\mathcal{G}^u$. Using this order we denote by $(x, +\infty) \in \mathcal{G}^u_x$ the subset of points above x in this order.

Let $x_0 \in \mathcal{H}^+$ and consider $\Gamma = \pi_1(M, x_0)$, which acts as covering automorphisms on $\widetilde{M}$. In particular, induces an action on Q. Now, is x is a point in the fiber of x_0 then, since the orbit of x_0 is an incompressible torus $\mathbb{T}^k$, the isotropy group Γ_0 of x is isomorphic to $\mathbb{Z}^k$.

If $F = \mathcal{G}^s_x$ then $x_0 \in \mathcal{H}^+$ means that $(x, +\infty) \cap \Gamma.F = \emptyset$.

The map h

Take F' as $F - \{O(x)\}$, then it is not difficult to see that for every $y \in F'$, we have that $(y, +\infty) \cap \Gamma F \neq \emptyset$. We define $h(y)$ as the infimum of this intersection using the order on $(y, +\infty)$. Obviously, $h(y) > y$ since the leaves of $\mathcal{G}^s$ which passes through $(y, y + \epsilon)$ also passes through $(x, +\infty)$ and they are not in ΓF.

Lemma 6.29. *There exists a Γ_0-invariant leaf F_1 of $\mathcal{G}^s$ such that*

$h : F' \to F_1$ is an injective local homeomorphism onto its image and $h \circ \gamma = \gamma \circ h$ for every $\gamma \in \Gamma_0$.

Proof. We have that h is injective by the definition of h. The images are in the same leaf F_1 locally by local product structure. Let F'' be a leaf of $\mathcal{G}^s$ and $\Omega(F'')$ be the subset of points z in F' such that $h(z) \in F''$. Then $\{\Omega(F'')\}_{F''}$ form a partition of F' by open sets. But since $dim(M) \geq k+3$ we have that F' is connected! The lemma follows. $\square$

Since $h(x) > x$ we obtain that F' and F_1 are disjoint. Also, it is possible to show that the projection of F_1 is a bundle over $\mathbb{R}^k/\Gamma_1$ with contractible fiber for some lattice Γ_1 of $\mathbb{R}^k$. Hence, Γ_1 needs to meet a chamber. The presence of the Anosov element guarantees that there exists only one compact orbit in this projection. Take $O(\overline{x})$ the lift of this orbit which is the only orbit fixed by Γ_0 and define $F_1'' = F_1 - \{O(\overline{x})\}$.

Lemma 6.30 (Claim 4 of [6])**.** *The map $h : F'' \to F_1''$ is a homeomorphism.*

The contradiction

There exists $\gamma_0 \in \Gamma_0$ and an Anosov element f such that $A(f,x) = \gamma_0(x)$, in particular γ_0 contracts F and fixes $O(x)$.

Now, by contraction, there exists a ball B containing x inside F'' such that $\partial B \cup \gamma_0(\partial B)$ is the boundary of a domain W_0 diffeomorphic to $S^{p-1} \times [0,1]$, such that $\bigcup \gamma_0^n(W_0) = F''$. Since $h(\partial B)$ is a sphere in F_1, by Schöenflies's theorem, it must be the boundary of a closed ball B_1 in F_1. Let $\mathcal{R} = \bigcup_{x \in \partial B} [x, h(x)]$ and $\mathcal{S} = B \cup \mathcal{R} \cup B_1$. Hence, $\mathcal{S}$ is homeomorphic to a codimension one sphere on Q.

Since, by irreducibility, Q is homeomorphic to $\mathbb{R}^{n-k}$, $\mathcal{S}$ is the boundary of a ball $\mathcal{B}$.

The leaf $l = \mathcal{G}^u_{\overline{x}}$ crosses $\mathcal{B}$, hence intersects $\mathcal{S}$ in two points. One of them is $\overline{x}$. The other one cannot be in $\mathcal{R}$, since $\mathcal{R}$ is foliated by $\mathcal{G}^u$ leaves. Also, cannot be in B_1 since any leaf of $\mathcal{G}^u$ intersects one leaf of $\mathcal{G}^s$ in at most one point. Then l intersects B, and since this intersection is fixed by Γ_0 it must be x.

In particular, l contains two Γ_0-fixed points. A contradiction, since every unstable leaf contain at most one compact orbit. A similar argument holds for $\mathcal{H}^-$.

6.5 Open Questions

In this section we pose several questions on the subjects that were presented along the book.

6.5.1 Stable Actions

Is it true that Maquera-Tahzibi's theorem holds in general, i.e. without the assumption on the non-existence of planar orbits? Moreover, can we use stability instead robust transitivity and obtain an Anosov action? This is related to the following conjecture:

There are no stable actions of $\mathbb{R}^2$ on T^3 with all leaves homeomorphic to $\mathbb{R}^2$

In the context of diffeomorphism and flows, C^1-stability leads to hyperbolicity, as proved by Mañé [36] for diffeomorphisms and by Hayashi [24] for flows.

Theorem 6.31 (Mañé). *A C^1 diffeomorphism is C^1-structurally stable if, and only if, it is Axiom A and satisfies the strong transversality condition.*

In particular if it is stable and transitive, it must be Anosov. Is it true the same statement for stable actions of $\mathbb{R}^2$ over $\mathbb{T}^3$? If the answer is positive and leads to an Anosov action, then the previous conjecture is true by the following theorem:

Theorem 6.32 (Arbieto-Morales [2]). *There are no central Anosov actions of a connected 2-dimensional Lie group on a closed 3-manifold.*

6.5.2 Suspensions

In the study of codimension one Anosov flows, Verjovsky's conjecture is one of central open questions:

Every codimension one Anosov flow on a manifold with dimension at least four is conjugated to a suspension Anosov flow

A positive answer to this conjecture was announced joining the works of Simic and Asaoka. Nevertheless, we can make the same conjecture in the setting of Anosov actions with the appropriated dimensional hypothesis:

Every irreducible codimension one Anosov action of $\mathbb{R}^k$ *over a manifold with dimension at least* $k+3$ *is conjugated to a suspension of an Anosov action of* $\mathbb{Z}^k$

6.5.3 Equilibrium States and Physical measures

We saw that volume preserving C^2-Anosov actions are ergodic. What about its thermodynamical formalism? There exists a notion of topological entropy for foliations due to Ghys-Langevin-Walczak [20] and also the notion of topological and metric entropy for actions of $\mathbb{R}^k$ due to Tagi-Zade [55], moreover in the last work a variational principle is proved. Also, there are generalizations for actions of locally compact unimodular amenable groups.

Is it possible to extend the thermodynamical formalism to central Anosov actions?

This involves to show variational principles for pressure, and try to find equilibrium states for Hölder continuous potentials.

Can we define and find physical measures for central Anosov actions?

6.5.4 Partially Hyperbolic Actions

Based on the recent non-hyperbolic theory what happens if some complementary directions of the orbits are not hyperbolic. Is it possible to obtain splittings as $E^s \oplus T\mathcal{F} \oplus E^c \oplus E^u$, where $\mathcal{F}$ is the orbit foliation of an action? It will be needed to adapt the notion of normally hyperbolicity to normally partial hyperbolicity.

A previous step in the study of such objects could be suppose that the splitting $T\mathcal{F} \oplus E^c$ is dominated. The questions on these objects deals with topological-dynamical properties, or even ergodicity, in the volume preserving case, extending the Pugh-Shub's program to this setting.

What if we assume that there exists a non-uniformly contraction on E^c, for instance if we can impose negative Lyapunov exponents in the whole direction E^c. What if we impose positive Lyapunov exponents? These are inspired on the mostly contracting and mostly expanding diffeomorphisms introduced in Bonatti-Viana [8] and Alves-Bonatti-Viana [3]. Suit yourself to study your own definition on this issue.

Is there a notion of dominated actions? I.e. if there are no *a priory* hyperbolic directions, but there exists a dominated splitting involving the tangent direction of the orbit foliation.

6.5.5 The final question of the Book

To extend all the results of Anosov group actions to Anosov partial semigroup actions

Appendix A

In this Appendix we present some definitions and facts used in the book.

- Transitivity: An action $A : G \to Diff(M)$ is transitive if there exists a dense orbit $O(x)$.

- Incompressibility: We say that a submanifold F of a manifold M is incompressible if $\pi_1(F)$ injects on $\pi_1(M)$ by the inclusion $i : F \to M$.

- If $A : G \to Diff(M)$ is an action, then for every $g \in G$ we will denote $A(g)$ the diffeomorphism on M and by $A(g, x)$ as the point on M given by $A(g)(x)$. If there is no confusion on what action we are speaking, we will denote by g the diffeomorphism $A(g)$ and by $g(x)$ the point $A(g, x)$.

- The boundary $\partial O(p)$ of an orbit $O(p)$ of an action $A : G \to Diff(M)$ is the set of all limit points of sequences $A(g_n, p)$ where $g_n \in G$ is a sequence having no cluster points in G. It is an exercise to show that $O(p) = O(q)$ implies $\partial O(p) = \partial O(q)$.

- Laminations: A lamination is a topological space which can be covered by open charts of U_i, $\phi_i : U_i \to \mathbb{R}^n \times X$ in such a way that the manifold-like factor is preserved by the overlaps, i.e. for $U_i \cap U_j \neq \emptyset$ we have that $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \times X \to \mathbb{R}^n \times X$ is of the form $\phi_j \circ \phi_i^{-1}(t, x) = (f(t, x), g(x))$.

- If f is normally hyperbolic to a lamination $\mathcal{L}$ of an invariant set Λ, we say that (f, Λ) has local product structure if $W^u_\Lambda(\epsilon) \cap W^s_\Lambda(\epsilon) = \Lambda$, for some $\epsilon > 0$.

  Moreover, if in addition, the intersections of any leaf $\mathcal{L}_x$ with $W^u(O(y))$ and $W^s(O(y))$ are relative open sets of $\mathcal{L}_x$ then we say that $(f, \mathcal{L})$ has local product structure.

- The saturate of X is $Sat(X) = \overline{\bigcup_{g \in G} A(g, X)}$.

- Codimension one: We say that an Anosov action is a codimension one Anosov action, if E^s (or E^u) is one dimensional. Reversing the time, i.e. using the action $B(g,x) = A(g^{-1}, x)$ we can suppose that $dim(E^u) = 1$.

- Plaquations: A C^r plaque in an n-dimensional manifold M is a C^r embedding $\rho : B \to M$ of the closed unit n-ball B into M. If $w : W \to M$ is a C^r immersion then we say that a family of plaques $\mathcal{P} = \{\rho\}$ plaquates w if $W = \bigcup_{\mathcal{P}} \rho(int(B))$ and $\{w \circ \rho\}$ is precompact in $Emb^r(B, M)$.

  We say that a C^r-immersion $h : N \to M$ of a k-dimensional manifold is uniformly k-self tangent if, denoting by $T^k N$ the k-th order tangent space, we have that $\overline{T^k h(T^k N)}$ extends to a continuous subbundle of $T^k M$ over $\overline{h(N)}$. We say that $h : N \to M$ is leaf immersion if it is a uniformly k-self-tangent immersion, $\overline{h(N)}$ is compact, disjoint of ∂M and N is complete respecting the Finsler metric obtained by pull-back of the Finsler metric on M. If $\partial N \neq \emptyset$ then we say the leaf immersion is boundaryless.

  It is possible to show that every C^r-leaf immersion $i : V \to M$ has a C^r-plaquation.

A.1 Invariant foliations

Let $A : G \to Diff(M)$ be a central Anosov action and f its Anosov element. We denote by $E^s \oplus E^c \oplus E^u$ the invariant splitting, where $E^c = T\mathcal{F}$. We define the following constants:

$$\gamma = \sup \|Df|_{E^s}\|, \;\; \eta = \inf m(Df|_{E^c}), \;\; \mu = \inf \|Df|_{E^c}\|$$

and $\lambda = \inf m(Df|_{E^u})$.

The normally hyperbolic theory [25] says that there exists invariant stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, which are defined as:

$$\begin{aligned} W_p^s &= \{x \in M; d(f^n(x), f^n(p))\xi^{-n} \to 0; \text{ as } n \to \infty\} \\ W_p^u &= \{x \in M; d(f^{-n}(x), f^{-n}(p))\sigma^n \to 0; \text{ as } n \to \infty\}. \end{aligned}$$

In the case that the Anosov element f is in the center of G then the foliation is G-invariant. Indeed, if L is the Lipschitz constant of $g \in G$ then:

$$\begin{aligned} d(f^{-n}(g(x)), f^{-n}(g(p)))\sigma^n &= d(g(f^{-n}(x)), g(f^{-n}(p)))\sigma^n \\ &\leq Ld(f^{-n}(x), f^{-n}(p))\sigma^n \to 0. \end{aligned}$$

And this implies that $g(W_p^u) = W_{g(p)}^u$. Analogously for the stable foliation.

Also, we have the center-stable and center-unstable foliations defined by saturation:

$$\mathcal{W}_p^{cu} = \bigcup_{q \in \mathcal{F}_p} W_q^u \text{ and } \mathcal{W}_p^{cs} = \bigcup_{q \in \mathcal{F}_p} W_q^s.$$

Again these foliations are G-invariant.

A.2 Pre-Foliations and Pseudo-Foliations

Given a k-disc D^k, we denote by $Emb^r(D^k, 0, M, p)$ the space of C^r-embeddings $\varphi : D^k \to M$ such that $\varphi(0) = p$, with the C^r-distance it becomes a metric space. If there is no confusion on the choice of the point p we will denote this space simply as $Emb^r(D^k, M)$. Using the projection $\pi : Emb^r(D^k, M) \to M$ defined as $\pi(h) = h(0)$ this space becomes a fiber bundle over M.

Definition A.1. *A continuous map $p \in M \mapsto D_p \in Emb^r(D^k, M)$ is called a pre-foliation. More precisely, there exists a cover of M by charts U, such that the pre-foliation is given by a continuous section $\sigma : U \to Emb^r(D^k, U)$, so we have $D_p = \sigma_p(D^k)$ for $p \in U$. The pre-foliation is C^s with $s \leq r$ if the maps $(p, x) \mapsto \sigma_p(x)$ are C^s.*

Now, we list some examples of pre-foliations, the first one justifies the name:

- Let $\mathcal{F}$ a C^r-foliation of k-dimensional leaves. If $d_{\mathcal{F}}$ is the distance on the leaves induced by the Riemannian metric, the following map gives a C^r-pre-foliation:

$$p \mapsto \mathcal{F}_p(\delta) := \{x \in \mathcal{F}_p; d_{\mathcal{F}}(x,p) \leq \delta\}$$

- If N is a C^r-distribution of k-dimensional subspaces of TM then if $\delta > 0$ is small enough and $N_p(\delta)$ is the disc centered in the origin of N_p of radius δ then the following map defines a C^r-pre-foliation:

$$p \mapsto exp_p(N_p(\delta))$$

- If we denote $W^s_p(\delta)$ the local stable manifold of size δ of an Anosov diffeomorphism the following map

$$p \mapsto W^s_p(\delta)$$

  forms a continuous pre-foliation.[1]

Now we define the notion of holonomy for pre-foliations. Let $\mathcal{G}$ be a pre-foliation by k-dimensional discs and $p \in M$. Suppose that there exist $q \in int(\mathcal{G}_p)$ and embedded $(m-k)$-dimensional discs D_p and D_q transverse to $\mathcal{G}_p$ and $\mathcal{G}_q$ in p and q respectively. Then there exists a neighborhood $D_{p,q}$ of p inside D_p, a subset $R_{p,q}$ of D_q containing q, and a continuous surjection $H_{p,q} : D_{p,q} \to R_{p,q}$ such that $H_{p,q}(p) = q$ and $H_{p,q}(y) \in \mathcal{G}_y \cap D_q$.

It is not difficult to see that if the pre-foliation $\mathcal{G}$ is C^s then $H_{p,q}$ is C^s, also $H_{p,q}$ C^s-depends continuously on p, q, D_p and D_q. If q is sufficiently close to p then $H_{p,q}$ is a local diffeomorphism. This closeness assumption can be dropped if the pre-foliation arises from a true foliation and in this case $R_{p,q}$ is a neighborhood of q in D_q.

We can also consider pre-foliations by submanifolds, instead of discs, using unions of discs. For instance, if $\mathcal{F}$ is a C^1 foliation and N is a C^r distribution previously we considered the pre-foliations $\mathcal{G}_y = exp_y(N_y(\delta))$, now for any p we consider the leaf $\mathcal{F}_p$ which

[1]In general this pre-foliation is not C^1

contains p and take $\mathcal{H}_p = \bigcup_{y \in \mathcal{F}_p} \mathcal{G}_y(\delta)$, which is a immersed manifold. Similar to the construction of pre-foliations by discs, we can speak about pre-foliations by submanifolds.

A.3 Flows generated by $\mathbb{R}^k$ actions

Let $A : \mathbb{R}^k \to Diff(M)$ be a C^r-action. Then, for any $v \in \mathbb{R}^k - \{0\}$ we can define a C^r-flow given by:

$$\varphi_t(x) := A(tv, x) \text{ for every } x \in M.$$

The theory of ordinary differential equations says that this flow is generated by the C^{r-1}-vector field X_v given by $X_v(x) = D_1A(0, x).v$.

Also if we take $\{v_i\}_{i=1}^k$ a basis of $\mathbb{R}^k$, then the associated vector fields commutes: $[X_{v_i}, X_{v_j}] = 0$ for every $1 \leq i \leq j \leq k$ and it is an exercise to show that they generates the entire action A. We call these vector fields X_{v_i} as the infinitesimal generators of the action.

A.4 A remark

We recall that the leaves of the strong invariant foliations of a central Anosov action A are diffeomorphic to some euclidian space. For instance, any bounded domain in a leaf of $\mathcal{W}^{ss}$ is contracted to a point by the Anosov element.

Let W be a stable leaf and $W' \subset W$ a strong stable leaf, denote by $\Gamma_{W'} < \mathbb{R}^k$ the subgroup of elements v which leaves W' invariant, i.e. $A(v, W') = W'$. Then since the saturation $O_{W'}$ of W' by the action is open in W, by local structure product and W is connected then $O_{W'} = W$. Also, local product structure says that $\Gamma_{W'}$ is a discrete subgroup and also does not depend on W'. Moreover, $P = \mathbb{R}^k/\Gamma_{W'}$ is a cylinder, i.e. it is diffeomorphic to $\mathbb{R}^p \times \mathbb{T}^q$ for some p and q.

In particular, the canonical projection $p_W : W \to P$ given by $p_W(x) = v + \Gamma_{W'}$ such that $x \in A(v, W')$, is a locally trivial fibration, such that p_W restricted to an orbit is a covering map, and since the strong-stable leaves are planes, hence contractible, we have that $\pi_1(W) = \pi_1(P) = \Gamma_{W'}$ for every W'.

A.5 Irreducible Anosov Actions of $\mathbb{R}^k$

In this section, following [6], we study a class of codimension one Anosov actions which has many properties similar to the ones that codimension one Anosov flows possess. As usual we will assume that $dim(\mathcal{W}^{uu}) = 1$.

We recall that if $x \in M$ and $v \in \mathbb{R}^k - \{0\} \cap \mathcal{G}_x$ and $\gamma = \{A(sv, x); s \in [0, 1]\}$ is a curve inside the orbit of x then we denote by Hol_γ the holonomy along γ.

Definition A.2. *Let $A : \mathbb{R}^k \to Diff(M)$ be a codimension one Anosov action. We say that A is irreducible if any non-zero element Hol_γ is either a contraction or an expansion.*

We remark that codimension one Anosov flows ($k = 1$) are always irreducible.

If $A : G \to Diff(M)$ is an action and $\pi : \widetilde{M} \to M$ is the universal covering of M, then we can define an action $\widetilde{A} : G \to Diff(\widetilde{M})$ lifting A using π. As a consequence, we obtain foliations $\widetilde{\mathcal{W}^{ss}}, \widetilde{\mathcal{W}^{uu}}, \widetilde{\mathcal{W}^{s}}$ and $\widetilde{\mathcal{W}^{u}}$ which are, respectively, the lift of the foliations $\mathcal{W}^{ss}, \mathcal{W}^{uu}, \mathcal{W}^{s}$ and $\mathcal{W}^{u}$.

Proposition A.3. *If $A : \mathbb{R}^k \to Diff(M^n)$ is a codimension one Anosov action then the orbit space of $\widetilde{A}$ is homeomorphic to $\mathbb{R}^{n-k}$.*

By an argument using Haefliger's theorem [12], it is possible to obtain non-existence of vanishing cycles:

Lemma A.4 (Proposition 2 of [6])**.** *There are no homotopically trivial transversal loops to the stable foliation.*

This lemma give us information on the orbits of A and $\widetilde{A}$.

Corollary A.5. *If $A : \mathbb{R}^k \to Diff(M)$ is an irreducible action then:*

(i) The orbits of A are incompressible.

(ii) The leaves of the invariant foliations of $\widetilde{A}$ are closed planes.

(iii) If L is a leaf of $\widetilde{\mathcal{W}^{s}}$ and L' is a leaf of $\widetilde{\mathcal{W}^{u}}$ then $L \cap L'$ is at most an orbit of $\widetilde{A}$.

(iv) *Every orbit of $\widetilde{A}$ intersects a leaf of the strong invariant manifolds at most in one point.*

(v) *$\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$*

Proof. Since any loop in the orbit of $x \in M$ is homotopic to $c = \{A(t.g, x)\}_{t\in[0,1]}$ for some $g \in \mathbb{R}^k$ with $A(g, x) = x$, if the loop is homotopically non-trivial then the holonomy of $\mathcal{W}^s$ along c is non-trivial. In particular, we have that it is transverse to $\mathcal{W}^s$, so it must be homotopically non-trivial in M. This proves (i). This also says that $\widetilde{A}$ is free.

If $\widetilde{\mathcal{W}^{ss}_x}$ coincides with $\widetilde{\mathcal{W}^{ss}_{\widetilde{A}(g,x)}}$ for some $g \neq 0$ then irreducibility says that the holonomy generated by g on $\pi(x)$ is non-trivial. But this holonomy of $\mathcal{W}^s$ is constructed along a closed loop in $\mathcal{W}^s_{\pi(x)}$ which is homotopically trivial in M. A contradiction.

This implies that the orbits of $\widetilde{A}$ intersects the leaves of $\widetilde{\mathcal{W}^{ss}}$ at most once. By saturation, $\widetilde{\mathcal{W}^s}$ is an injective immersion of $\mathbb{R}^{p+k}$. Analogously the leaves of $\widetilde{\mathcal{W}^{uu}}$ are injective immersions of $\mathbb{R}^{2+k}$.

Now $\widetilde{\mathcal{W}^s}$ is a foliation by closed planes, since if one leaf is not close we can find a loop in $\widetilde{M}$ transverse to $\widetilde{\mathcal{W}^s}$. This implies (ii).

Observe that the leaves of $\widetilde{\mathcal{W}^s}$ disconnect $\widetilde{M}$ since they are closed hypersurfaces which are leaves of a oriented and transversely oriented foliation [2] Since the orientation must be preserved we obtain (iii) and also that the leaves are closed.

Since any orbit is the intersection between a stable leaf and a unstable leaf it must be closed, since both are. Again, this gives (iv). Finally, Palmeira's theorem [42] gives (v).

Theorem A.6 (Palmeira [42])**.** *If M^n admits a plane foliation, the universal covering of M^n is $\mathbb{R}^n$.*

□

Now we are ready to prove proposition A.3:

Proof of proposition A.3. We prove the proposition in two steps.

The orbit space of $\widetilde{A}$ is Hausdorff

[2] $\widetilde{M}$ is simply connected, see for instance [41]

Indeed, if not, then there exists two different orbits $\widetilde{O}(p)$ and $\widetilde{O}(q)$, which cannot be separable. Saturating $\widetilde{\mathcal{W}^s_p}$ and $\widetilde{\mathcal{W}^s_q}$ by $\widetilde{\mathcal{W}^{uu}}$ we obtain two neighborhoods V_p and V_q of $\widetilde{O}(p)$ and $\widetilde{O}(q)$ respectively and $V_p \cap V_q \neq \emptyset$.

If $\widetilde{O}(p)$ and $\widetilde{O}(q)$ are in the same leaf W of $\widetilde{\mathcal{W}^s}$, U_1 and U_2 are disjoint neighborhoods in W of p and q respectively then the previous corollary says that the $\widetilde{\mathcal{W}^{uu}}$-saturation of U_1 and U_2 are disjoint, a contradiction.

Hence, we have that $\widetilde{\mathcal{W}^s_p} \neq \widetilde{\mathcal{W}^s_q}$, and since $V_p \cap V_q \neq \emptyset$ then there exists $x \in \widetilde{\mathcal{W}^s_p}$ and $y \in \widetilde{\mathcal{W}^s_q}$ such that $\widetilde{\mathcal{W}^{uu}_x} = \widetilde{\mathcal{W}^{uu}_y}$ and $x \neq y$. In particular, there exists U_1 and U_2 disjoint neighborhoods inside $\widetilde{\mathcal{W}^{uu}_x}$ of x and y respectively.

Now if we saturate U_1 and U_2 by $\widetilde{\mathcal{W}^s}$ then we obtain two non-disjoint invariant neighborhoods of $\widetilde{O}(x)$ and $\widetilde{O}(y)$. In particular a leaf of $\widetilde{\mathcal{W}^s}$ through a point in this intersections meet $\widetilde{\mathcal{W}^{uu}_x}$ in two points. A contradiction with (iv) of the previous corollary. This shows that the orbit space of $\widetilde{A}$ is Hausdorff.

Using return maps

Let $x \in \widetilde{M}$, U be a neighborhood of x given by local product structure and $\Sigma \subset U$ a $(n-k)$ dimensional cross section.

Claim A.7. *Every orbit of A intersects Σ in at most one point.*

Thus the orbit space has a differentiable structure given by

$$\{(\Sigma_i, \pi|_{\Sigma_i})\}_{i \in I},$$

where $\{\Sigma_i\}_{i\in I}$ is a family of cross sections whose union meets all of the orbits of the action. In particular, if the action is C^r the orbit space is a C^r-manifold. Since π is a locally trivial bundle, the dimension of the orbit space is $n-k$ and simply connected. But the orbit space also has a codimension one foliation by planes induced by $\widetilde{\mathcal{W}^s}$ then Palmeira's theorem implies the statement of the proposition.

Now, to prove the claim, if an orbit O intersects Σ in two points, since by (iv) of the previous corollary O intersects a leaf of $\widetilde{\mathcal{W}^{ss}}$ at most in one point, so O intersects U along two different leaves of

$\widetilde{\mathcal{W}^s}|_U$. By local product structure there exist a leaf of $\widetilde{\mathcal{W}^{uu}}$ which intersects a leaf of $\widetilde{\mathcal{W}^{ss}}$ at two points, but this is impossible by (iii).

The proof of the proposition is now complete. □

A.5.1 Reducing codimension one Anosov actions

In this section, we show how to obtain a irreducible Anosov action from a general codimension one Anosov action. For this purpose we recall the definition of a principal bundle:

Definition A.8. *Let G be a Lie group and M and P be manifolds. We say that a fiber bundle $\pi : P \to M$ is a G-principal bundle if there exists a continuous group action $G \times P \to P$, such that G preserves the fibers of P and acts freely and transitively on each fiber.*

Theorem A.9. *If $A : \mathbb{R}^k \to Diff(M)$ is a codimension one Anosov action, then there exists a subgroup $H_0 < \mathbb{R}^k$ isomorphic to $\mathbb{R}^l$ for some l, a lattice $\Gamma_0 \subset H_0$, a $(n-l)$-dimensional smooth manifold N and a smooth $\mathbb{T}^l$-principal bundle $\pi : M \to N$ such that, every orbit of $A|_{H_0 \times M}$ is a fiber of π and if $\overline{H} = \mathbb{R}^k/H_0$ then A induces a irreducible codimension one Anosov action $\overline{A} : \overline{H} \to Diff(N)$.*

First, we recall the notion of the holonomy of a element of $v \in \mathbb{R}^k$ which fixes a strong stable manifold, i.e. there exists $x \in M$ such that $\mathcal{W}^{ss}_x = A(v, \mathcal{W}^{ss}_x)$. Let γ be a path joining $A(v, x)$ to x and $\alpha = \{A(tv, x)\}_{t\in[0,1]} * \gamma$ be a loop in $\mathcal{W}^s_x$ (we reparametrize to obtain a path $\alpha : [0, 1] \to \mathcal{W}^s_x$). Since $\mathcal{W}^{ss}_x$ is a plane, all of these loops are homotopic in $\mathcal{W}^s_x$, in particular the holonomy does not depend on such loop and we will denote by h^v_x.

Proof of theorem A.9. We choose Γ_0 as the kernel of A, which is isomorphic to $\mathbb{Z}^l$ for some l and discrete, since the action is locally free, we then set H_0 as the subspace generated by Γ_0, in particular H_0/Γ_0 is a torus $\mathbb{T}^l$. By the discussion on the previous paragraph, the holonomy h^v_x is trivial for every $v \in \Gamma_0$.

Now observe that the action of this torus is proper on M, by compactness. We also claim that this action is free, in fact if $v \in H_0$ fixes some x, then the holonomy h^v_x is trivial, now we invoke the following lemma:

Lemma A.10. *For such v, either x is a repelling or attracting fixed point of $A(v,.)|_{\mathcal{W}_x^{uu}}$ or the action of $A(v,.)$ is trivial on M.*

The lemma says that the action of $A(v,.)$ is trivial, in particular $v \in \Gamma_0$.

Hence, if we take N as the quotient space, it is a $(n-l)$-dimensional manifold, and the projection $\pi : M \to N$ is a $\mathbb{T}^l$-principal bundle. Moreover, taking the quotient action of A over N, we obtain a codimension one Anosov action.

Irreducibility follows, since if $h_{\overline{x}}^{\overline{v}}$ is trivial for some $\overline{v} \in \overline{H} - \{0\}$ fixing some $\overline{x} \in N$, then there exists $v \in \mathbb{R}^k$ which is projected over $\overline{v}$ and fixes some x such that $\pi(x) = \overline{x}$. In particular h_x^v is trivial and again by the previous lemma $v \in \Gamma_0$. A contradiction. $\square$

Now we prove the lemma used in the proof.

Proof of lemma A.10. Using the existence of a unique affine structure along the 1-dimensional leaves of $\mathcal{W}^{uu}$, we obtain that $A(v,.)|_{\mathcal{W}^{uu}}$ is conjugated to an affine transformation of the real line, then we can suppose that h_x^v is trivial (For more details see theorem 6 of [6]). We define,

$$\Theta_v = \{x \in M; A(v,x) \in \mathcal{W}_x^{ss} \text{ and } h_v^x \text{ is trivial}\}.$$

Then Θ_v is invariant, non-empty and since $\mathcal{W}_x^{ss}$ is a plane, the homotopy between the loops in the definition of h_x^v implies that Θ_v is $\mathcal{W}^s$-saturated. Also, since the holonomy h_x^v is trivial, every $y \in \mathcal{W}_x^{uu}$ near x, we have that $A(v,y) \in \mathcal{W}_{y,loc}^s$, then for some w near v we have that $y \in \Theta_w$. In particular, if $\Theta_U := \bigcup_{w \in U} \Theta_w$ then for some neighborhood of v then there exists $\delta > 0$ such that for every $y \in \mathcal{W}_x^{uu}(\delta)$ we have that $y \in \Theta_U$. Moreover, since the $\mathcal{W}^{uu}$-saturation of an $\mathcal{W}^s$-invariant set is the whole manifold. We have that $\Theta_U = M$, since U is arbitrarily we have that $\Theta_v = M$.

We claim that $A(v,.)$ is trivial over the closure of compact orbits. Indeed, there exists $\delta > 0$ such that if O is a compact orbit, then $O \cap \mathcal{W}^{ss}(\delta) = \{x\}$. If f is the Anosov element, then, by hyperbolicity, for every $y \in \mathcal{W}^{ss}(x)$ there exist t large enough such that $A(tf,y) \in \mathcal{W}_{A(tf,x)}^{ss}(\delta)$, this implies that $\mathcal{W}^{ss} \cap O = \{x\}$.

Fix $x \in M$, f an Anosov element and take $t_n \to \infty$ such that $x_n = A(-t_n f, x) \to x_\infty$. For n large, let c_n a path in $\mathcal{W}^{ss}(x_n)$ with arbitrarily small length joining x_n and $A(v, x_n)$. Then $A(t_n f, c_n)$ is a path joining x to $A(v,x)$ with arbitrarily small length, then $A(v,x) = x$. The proof of the lemma is now complete. $\square$

Remark A.11. *We observe that if $dim M > k+2$ the $dim N > dim\overline{H} + 2$. Also A is transitive, if and only if, $\overline{A}$ is transitive too. Finally, if the action is already irreducible then $\overline{H} = \mathbb{R}^k$.*

A.5.2 Non-compact orbits of Anosov $\mathbb{R}^k$ actions

Now, we focus on the topological structure of the non compact orbits of a codimension one Anosov action.

Theorem A.12. *If $A : \mathbb{R}^k \to Diff(M)$ is a codimension one Anosov action, then every non compact orbit is diffeomorphic to $\mathbb{T}^l \times \mathbb{R}^{k-l}$ for some integer l.*

First we deal with the irreducible case, and show that in this case $l = 0$.

Lemma A.13. *If the codimension one Anosov action is irreducible then $l = 0$.*

Proof. Let O be a non compact orbit which is not a plane. Then there exists $v \in \mathbb{R}^k - \{0\}$ such that $A(v,.)$ fixes O, $y \in \overline{O}$ and $x_n \in O$ such that $x_n \to y$. In particular, $A(v,y) = y$. Irreducibility says that the holonomy generated by v at y is non trivial, in particular there exists $\delta > 0$ such that if n is large then $x_n \in \mathcal{W}^s_y(\delta)$. In particular $\overline{O} \subset \mathcal{W}^s_y =: W$.

This implies that the space of strong stable leaves in W is compact. In particular, for any strong stable leaf W' of W, $\Gamma_{W'}$ is a lattice in $\mathbb{R}^k$. Hence, by lemma A.16 Γ_W intersects a chamber, in particular contains an Anosov element f.

Since f restricted to W' is a contraction, it must contain a fixed point z. Hence, the orbit of z is compact, with isotropy groups $\Gamma_{W'}$. Moreover, $W' \cap \overline{O}$ is compact, but for every $z' \neq z$ and every compact subset K of W' there exists some n such that $f^{-n}(z') \notin K$. Hence $\overline{O}$ is a compact orbit of z. This proves the lemma. $\square$

The proof of the theorem follows reducing the Anosov action, using theorem A.9.

A.6 Chambers

Let $A : \mathbb{R}^k \to Diff(M)$ be an Anosov action, and $\mathcal{A}$ the set of Anosov elements of A. Observe that $\mathcal{A}$ is an open subset of $\mathbb{R}^k$, since normally hyperbolic diffeomorphisms are robust, for any other element g near to one Anosov element there exists some foliation *a priori* close to the orbit foliation, such that g is normally hyperbolic to it, but then by invariance, this foliation must be the orbit foliation.

Also, since $\mathbb{R}^k$ is abelian every two Anosov elements that are close enough has the same stable/unstable bundle. Moreover the expansion/contraction property is invariant by multiplication of the generating vector field by a positive constant factor. This discuss permit us to define:

Definition A.14. *A chamber is a connected component of the set of Anosov elements. Any chamber is an open convex cone.*

Lemma A.15. *If the isotropy group of $x \in M$ intersects a chamber then the orbit of x is compact.*

Proof. Let g be an element of the intersection and $y \in \overline{O(x)}$. Since y is fixes by $A(g,.)$ there exists a local transversal section of the flow generating by g, which contains y which is locally invariant by $A(g,.)$. Since g is hyperbolic in this section and y is a fixed point of $A(g,.)$ of saddle type, y must be an isolated $A(g,.)$-fixed point. In particular $y \in O(x)$. □

We also define the non-wandering set $\Omega(\mathcal{C})$ of a chamber $\mathcal{C}$ as the set of points x such that for every neighborhood U of x, there exists $g \in C \subset \mathbb{R}^k$ with $\|g\| > 1$ such that $A(g,U) \cap U$ is nonempty.

Lemma A.16. *The isotropy group of a compact orbit intersects any chamber.*

Proof. Let Γ be the isotropy group and B a closed ball o radius $r > 0$ inside a chamber $\mathcal{C}$. If R is such that a ball of radius R intersects every orbit of the isotropy group then if $t > \frac{R}{r}$ the ball tB intersects the Γ-orbit of 0. □

Lemma A.17. *The set of compact orbits with volume bounded from above by C is finite for any $C > 0$.*

Proof. Let O_n be a sequence of distinct compact orbits with volume bounded from above by C, and G_{O_n} be their respective isotropy groups. Since the isotropy groups are discrete, the length of G_{O_n} are uniformly bounded by below. Now we use Mahler's criterion:

Let Θ be the set of all lattice in $\mathbb{R}^k$. Since the linear group $GL(\mathbb{R}^k)$ acts transitively, denoting by $GL(M)$ the stabilizer of some lattice M, we can put a topology on Θ such that the natural mapping of $GL(\mathbb{R}^k)/GL(M)$ onto Λ is a homeomorphism. Also we define $D(M) = \int_{\mathbb{R}^k/M} dx$, where the integral is over a fundamental domain of M.

Theorem A.18 (Mahler's criterion)**.** *If C is a closed subset of Θ then C is compact if, and only if, $D(M)$ is bounded on C and there exists a neighborhood of 0 such that $U \cap M = \{0\}$ for every $M \in C$.*

So, we can assume that G_{O_n} converges to some lattice G_∞ of $\mathbb{R}^k$. The previous lemma, says that there exist some $h \in G_\infty \cap \mathcal{C}$, where $\mathcal{C}$ is a chamber. In particular, there exists sequences $x_n \in M$ and $h_n \in G_{O_n}$ such that $A(h_n, x_n) = x_n$ and $h_n \to h$. Also, we can suppose that x_n converges to some $x \in M$. In particular, $A(h, x) = x$ and $O(x)$ is compact.

Now, since the action is Anosov, we can construct a local cross section Σ such that the first return map along $\{A(t.h)\}_{t\in\mathbb{R}}$ is hyperbolic, hence x is an isolated fixed point of this map. Since x_n are close to x, there exists y_n in the intersection of $O(x_n)$ and Σ that are close to x_n. Now, this gives a contradiction, since $\{A(t.h_n, y_n)\}_{t\in\mathbb{R}}$ converges to $\{A(t.h, x)\}_{t\in\mathbb{R}}$ and y_n are fixed points of the return map. □

A.7 Suspensions of Anosov $\mathbb{Z}^k$-actions

Let $A : \mathbb{Z}^k \to N$ be an action, and think $\mathbb{Z}^k$ as a lattice of $\mathbb{R}^k$. We produce the action $B : \mathbb{Z}^k \to \mathbb{R}^k \times N$ given by $B(z,(x,m)) = (x - z, A(z,m))$ and define the quotient manifold $M = (\mathbb{R}^k \times N)/Z^k$

given by the action B. Now observe that the action $C : \mathbb{R}^k \to Diff(\mathbb{R}^k \times N)$ given by $C(x, (y, n)) = (x + y, n)$ commutes with B, hence it descends to M. We call this action the *suspension* of the $\mathbb{Z}^k$ action. If $g \in \mathbb{Z}^k$ is an Anosov diffeomorphism on N then g can also be thought as an element of $\mathbb{R}^k$, hence the suspension is Anosov too.

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