Three Dimensional Flows

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Preface

In this book we present the elements of a general theory for flows on three-dimensional compact boundaryless manifolds, encompassing flows with equilibria accumulated by regular orbits.

The main motivation for the development of this theory was the Lorenz equations whose numerical solution suggested the existence of a robust chaotic attractor with a singularity coexisting with regular orbits accumulating on it.

More than three decades passed before the existence of the Lorenz attractor was rigorously established by Warwick Tucker in the year 2000.

The difficulty in treating this kind of systems is both conceptual and numerical. On the one hand, the presence of the singularity accumulated by regular orbits prevents this invariant set to be uniformly hyperbolic. On the other hand, solutions slow down as they pass near the saddle equilibria and so numerical integration errors accumulate without bound.

Trying to address this problem, a successful approach was developed by Afraimovich-Bykov-Shil'nikov and Guckenheimer-Williams independently, leading to the construction of a geometrical model displaying the main features of the behavior of the solutions of the Lorenz system of equations.

In the 1990's a breakthrough was obtained by Carlos Morales, Enrique Pujals and Maria José Pacifico following very original ideas developed by Ricardo Mañé during the proof of the C^1-stability conjecture, providing a characterization of robustly transitive attractors for three-dimensional flows, of which the Lorenz attractor is an example.

This characterization placed this class of attractors within the realm of a weak form of hyperbolicity: they are partially hyperbolic invariant sets with volume expanding central direction. Moreover robustly transitive attractors without singularities were proved to be uniformly hyperbolic. Thus these results extend the classical uniformly hyperbolic theory for flows with isolated singularities.

Once this was established it is natural to try and understand the dynamical consequences of partial hyperbolicity with central volume expansion. It is well known that uniform hyperbolicity has very precise implications on the dynamics, geometry and statistics of the invariant set. It is important to ascertain which properties are implied by this new weak form of hyperbolicity, known today as *singular-hyperbolicity*.

Significant advances at the topological and ergodic level where recently obtained through the work of many authors which deserve a systematic presentation.

This is the main motivation for writing these notes. We hope to provide a global perspective of this theory and make it easier for the reader to approach the growing literature on this subject.

Acknowledgments: we thank our co-authors Carlos Morales and Enrique Pujals who made definitive contributions and helped build the theory of singular-hyperbolicity. We also thank Ivan Aguilar for providing the figures of his MSc. thesis at UFRJ, and Serafin Bautista and Alfonso Artigue for having communicated to us some arguments which we include in this text.

We extend our acknowledgment to Marcelo Viana who, besides being our co-author, encouraged us to write this text.

We are thankful to DMP-FCUP, IM-UFRJ and IMPA for providing us with the necessary time and conditions to write this text. We also profited from partial financial support from CNPq, FAPERJ, PRONEX-Dynamical Systems (Brazil) and CMUP-FCT, POCI/MAT/61237/2004 (Portugal). Finally we thank the scientific committee of the XXVI Brazilian Mathematical Colloquium for selecting our proposal for this text.

Não sou nada.

Nunca serei nada.

Não posso querer ser nada.

À parte isso, tenho em mim todos os sonhos do mundo.

Álvaro de Campos. *Tabacaria.*

Maria José Pacifico dedica a Maria, Laura e Ricardo.

Contents

Chapter 1

Introduction

We start with an overview of the main results of uniformly hyperbolic dynamical systems to be used throughout the rest of the text, both from the geometrical viewpoint and the measure-theoretical or ergodic point-of-view. We also mention some by-now standard generic properties of flows in the C^1 topology, as the Kupka-Smale vector fields (which are in fact C^r generic for every $r \geq 1$), Pugh's Closing Lemma and Hayashi's Connecting Lemma. We restrict ourselves to the results which will be actually used in the course of the proofs of the main results of the text.

Then in Chapter 2 we describe the construction of the most simple non-trivial examples of singular-hyperbolic sets: the singular-horseshoe of Labarca-Pacifico, and the geometric Lorenz attractor of Afraimovich-Bykov-Shil'nikov and Guckenheimer-Williams.

Next in Chapter 3 we characterize robustly transitive sets with singularities as partially hyperbolic attractors with volume expanding central direction, either for the original flow, or for the time reversed flow. This naturally leads to the notion of *singular-hyperbolic set*: a compact partially hyperbolic invariant subset with volume expanding central direction.

We construct in Chapter 4 a physical measure for singular-hyperbolic attractors, i.e. for transitive attracting singular-hyperbolic sets.

We finish in Chapter 5 with a description of the Omega-limit set for C^1-generic flows: either the limit set contains an infinite collection of sinks or sources; or is a finite union of basic pieces, either uniformly hyperbolic transitive isolated sets, or singular-hyperbolic attractors or repellers.

In an attempt to provide a broader view of the dynamics of flows on three-dimensional manifolds, we close the text briefly mentioning in Chapter 6 many other related results: the *contracting Lorenz-attractor* introduced by Rovella, singular cycles exhibiting singular-hyperbolic and/or contracting Lorenz attractors in its unfolding, other attractor resembling the Lorenz attractor, decay of correlations for flows and global generic results for conservative flows on three-dimensional manifolds.

1.1 Notation, motivation and preliminary definitions

In this book we will consider a compact finite dimensional boundaryless manifold M of dimensions 1 to 3 and study the dynamics of the flow associated to a given smooth vector field X on M from the topological and measure-theoretic or ergodic point-of-view.

We fix on M some Riemannian metric which induces a distance dist on M and naturally defines an associated Riemannian volume form Leb which we call *Lebesgue measure* or simply *volume*, and always take Leb to be normalized: $\mathrm{Leb}(M) = 1$.

We always assume that a C^r vector field X on M is given, $r \geq 1$, and consider the associated global *flow* $(X^t)_{t\in\mathbb{R}}$ (since X is defined on the whole of M, which is compact, X is bounded and X^t is defined for every $t \in \mathbb{R}$.) Recall that the flow $(X^t)_{t\in\mathbb{R}}$ is a family of C^r diffeomorphisms satisfying the following properties:

1. $X^0 = Id : M \to M$ is the identity map of M;

2. $X^{t+s} = X^t \circ X^s$ for all $t, s \in \mathbb{R}$,

and it is *generated by the vector field X* if

(3) $\frac{d}{dt} X^t(q)\big|_{t=t_0} = X\big(X_{t_0}(q)\big)$ for all $q \in M$ and $t_0 \in \mathbb{R}$.

Note that reciprocally a given flow $(X^t)_{t\in\mathbb{R}}$ determines a unique vector field X whose associated flow is precisely $(X^t)_{t\in\mathbb{R}}$.

In what follows we denote by $\mathfrak{X}^r(M)$ the vector space of all C^r vector fields on M endowed with the C^r topology and by $\mathcal{F}^r(M)$ the space of all flows on M also with the C^r topology. Many times we usually denote the flow $(X^t)_{t\in\mathbb{R}}$ by simply X. For details on these topologies the reader is advised to consult standard references on Differential Equations [77] and/or Dynamical Systems [143].

Given $X \in \mathfrak{X}^r(M)$ and $q \in M$, an orbit segment $\{X^t(q); a \le t \le b\}$ is denoted by $X^{[a,b]}(q)$. We denote by DX^t *the derivative of* X^t *with respect to the ambient variable* q *and when convenient we set* $D_qX^t = DX^t(q)$. Analogously, DX *is the derivative of the vector field* X with respect to the ambient variable q, and when convenient we write D_qX for the derivative DX at q, $DY(q)$.

An *equilibrium* or *singularity* for X is a point $\sigma \in M$ such that $X^t(\sigma) = \sigma$ for all $t \in \mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field X: $X(\sigma) = 0$. We denote by $S(X)$ the set of singularities (zeroes) of the vector field X. Every point $p \in M \setminus S(X)$, that is p satisfies $X(p) \neq 0$, is a *regular* point for X.

An *orbit* of X is a set $\mathcal{O}(q) = \mathcal{O}_X(q) = \{X^t(q) : t \in \mathbb{R}\}$ for some $q \in M$. Hence $\sigma \in M$ is a singularity of X if, and only if, $\mathcal{O}_X(\sigma) = \{\sigma\}$. A *periodic orbit* of X is an orbit $\mathcal{O} = \mathcal{O}_X(p)$ such that $X^T(p) = p$ for some minimal $T > 0$ (equivalently $\mathcal{O}_X(p)$ is compact and $\mathcal{O}_X(p) \neq \{p\}$). We denote by $\mathrm{Per}(X)$ the set of all periodic orbits of X.

A *critical element* of a given vector field X is either a singularity or a periodic orbit. The set $C(X) = S(X) \cup \mathrm{Per}(X)$ is the set of *critical elements* of X.

We say that $p \in M$ is *non-wandering* for X if for every $T > 0$ and every neighborhood U of p there is $t > T$ such that $X^t(U) \cap U \neq \emptyset$. The set of non-wandering points of X is denoted by $\Omega(X)$. If $q \in M$, we define $\omega_X(q)$ as the set of accumulation points of the positive orbit $\{X^t(q) : t \ge 0\}$ of q. We also define $\alpha_X(q) = \omega_{-X}$, where $-X$ is the time reversed vector field X, corresponding to the set of accumulation points of the negative orbit of q. It is immediate that $\omega_X(q) \cup \alpha_X(q) \subset \Omega(X)$ for every $q \in M$.

A subset Λ of M is *invariant* for X (or X-invariant) if $X^t(\Lambda) = \Lambda$, $\forall t \in \mathbb{R}$. We note that $\omega_X(q)$, $\alpha_X(q)$ and $\Omega(X)$ are X-invariant. For every compact invariant set Λ of X we define the *stable set* of Λ

$$W^s_X(\Lambda) = \{q \in M : \omega_X(q) \subset \Lambda\},$$

and also its *unstable set*

$$W^u_X(\Lambda) = \{q \in M : \alpha_X(q) \subset \Lambda\}.$$

A compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \cap_{t \geq 0} X^t(U)$ for some neighborhood U of Λ satisfying $X^t(U) \subset U$, $\forall t > 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. We say that Λ is a *proper* attractor or repeller if $\emptyset \neq \Lambda \neq M$.

The *limit set* $L(X)$ is the closure of $\cup_{x \in M} \alpha_X(x) \cup \omega_X(x)$. Clearly $L(X) \subset \Omega(X)$. Using these notions we have the following simple and basic

Lemma 1.1. *For any flow X the limit set $L(X)$ can neither be a proper attractor nor a proper repeller.*

Proof. Suppose $L(X)$ is a proper attractor with isolating open neighborhood U (and $U \neq M$). Let $z \in U$. Then $\alpha(z) \in L(X) \subset U$ and so $X^{-t_n}(z) \in U$ for a sequence $t_n \to +\infty$, that is $z \in X^{t_n}(U)$ for all n. But since $X^{t_n - t}(U) \subset U$ for $0 < t < t_n$ by definition of U, we have that $z \in X^{t_n}(U) \subset X^t(U)$ (recall that each X^t is an invertible map) for all $0 < t < t_n$, and so $z \in X^t(U)$ for all $t > 0$. We conclude that $z \in L(X)$. Thus $L(X) \supset U$ and $L(X)$ is simultaneously open and closed, hence it cannot be a proper subset of the connected manifold M. The proper repeller case is similar. □

A *sink* of X is a singularity of X which is also an attractor of X, it is a trivial attractor of X. A *source* of X is a trivial repeller of X, i.e. a singularity which is a attractor for $-X$.

A *singularity* σ *is hyperbolic* if the eigenvalues of $DX(\sigma)$, the derivative of the vector field at σ, have a real part different from zero. In particular sinks and sources are hyperbolic singularities, where all the eigenvalues of the former have negative real part and those of the latter have positive real part.

A *periodic orbit* $\mathcal{O}_X(p)$ *of* X *is hyperbolic* if the eigenvalues of $DX^T(p) : T_pM \to T_pM$, the derivative of the diffeomorphism X^T, where $T > 0$ is the period of p, are all different from 1. In Section 1.2 we will define hyperbolicity in a geometric way.

When a critical element is hyperbolic, then its stable and unstable sets have the structure of an embedded manifold (a consequence of the Stable Manifold Theorem, see Section 1.2), and are called *stable* and *unstable manifolds*.

Given two vector fields $X, Y \in \mathfrak{X}^r(M), r \geq 1$ we say that X and Y are *topologically equivalent* if there exists a homeomorphism $h : M \to M$ taking orbits to orbits and preserving the time orientation, that is

- $h\big(\mathcal{O}_X(p)\big) = \mathcal{O}_Y\big(h(p)\big)$ for all $p \in M$, and
- for all $p \in M$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for $t \in (0,\delta)$ there is $s \in (0,\varepsilon)$ satisfying $h\big(X^t(p)\big) = Y^s\big(h(p)\big)$.

The map h is then said a *topological equivalence* between X and Y. This is an equivalence relation in $\mathfrak{X}^r(M)$.

We say that $X, Y \in \mathfrak{X}^r(M)$ are *conjugate* if there exists a topological equivalence h between X and Y which preserves the time, i.e. $X^t\big(h(p)\big) = h\big(Y^t(p)\big)$ for all $p \in M$ and $t \in \mathbb{R}$. This is also an equivalence relation on $\mathfrak{X}^r(M)$.

The qualitative behavior of two topologically equivalent vector fields are the same, as the following result shows.

Proposition 1.2. *Let h be a topological equivalence between $X, Y \in \mathfrak{X}^r(M)$. Then*

1. *$p \in S(X)$ if, and only if, $h(p) \in S(Y)$;*
2. *$\mathcal{O}_X(p)$ is closed if, and only if, $\mathcal{O}_Y\big(h(p)\big)$ is closed;*
3. *$h\big(\omega_X(p)\big) = \omega_Y\big(h(p)\big)$ and $h\big(\alpha_X(p)\big) = \alpha_Y\big(h(p)\big)$.*

We say that a vector field $X \in \mathfrak{X}^r(M), r \geq 1$ is *C^s-structurally stable*, $s \leq r$, if there exists a neighborhood $\mathcal{V}$ of X in $\mathfrak{X}^s(M)$ such that every $Y \in \mathcal{V}$ is topologically equivalent to X.

Roughly speaking, a vector field is structurally stable if its qualitative features are robust under small perturbations.

1.1.1 One-dimensional flows

The only connected one-dimensional compact boundaryless manifold M is the circle $\mathbb{S}^1$, which we represent by $\mathbb{R}/\mathbb{Z}$ or by the unit interval $I = [0,1]$ with its endpoints identified $0 \sim 1$.

Let X_0 be one of the two unit vector fields on $\mathbb{S}^1$, i.e. either $X_0 \equiv 1$ or $X_0 \equiv -1$. Then every $X \in \mathfrak{X}^r(\mathbb{S}^1)$ can be written in a unique way as $X(p) = f(p) \cdot X_0(p)$ for $p \in S_1$, where $f : \mathbb{S}^1 \to \mathbb{R}$ is a C^r-function.

It is well known (see for example [99, 136]) that given any compact set $K \subset \mathbb{S}^1$ and $r \geq 1$ there exists $f : \mathbb{S}^1 \to \mathbb{R}$ of class C^r with $f^{-1}(\{0\}) = K$.

Thus K is the set of singularities of $X = f \cdot X_0$. Since topological equivalence preserves singularities, we see that there exist as many topological equivalence classes of vector fields in $\mathbb{S}^1$ as there are homeomorphism classes of compact subsets of $\mathbb{S}^1$. Hence *the problem of classifying smooth vector fields on* $\mathbb{S}^1$ *up to topological equivalence is hopeless*, and we need to restrict our attention to a subset of $\mathfrak{X}^r(M)$ which is open and dense, or residual or, at least, dense.

Here by a *residual* subset of the space $\mathfrak{X}^r(M)$ we mean a set $\mathcal{R}$ which contains a countable intersection of open and dense subsets of $\mathfrak{X}^r(M)$: $\mathcal{R} \supset \cap_{n\geq 1}\mathcal{R}_n$ where each $\mathcal{R}_n$ is an open and dense subset of $\mathfrak{X}^r(M)$.

We say that *a generic vector field in* $\mathfrak{X}^r(M)$ *satisfies a property (P)* if there is a residual subset $\mathcal{R}$ of $\mathfrak{X}^r(M)$ such that (P) holds for every $X \in \mathcal{R}$.

A singularity $\sigma \in S(X)$ is *non-degenerate* if $DX(\sigma) \neq 0$ or $Df(\sigma) \neq 0$ where $X = f \cdot X_0$. It can be a sink ($Df(\sigma) < 0$) or a source ($Df(\sigma) > 0$) and in either case a non-degenerate singularity is *isolated*: there exists a neighborhood U of σ in M such that σ is the only zero of $f \mid U$.

Let $\mathcal{G} \subset \mathfrak{X}^r(\mathbb{S}^1)$ be the subset consisting of vector fields whose singularities are all non-degenerate. Since these are isolated there are only finitely many of them. It is not difficult to show that $\mathcal{G}$ is open and dense, that the number of singularities in even and that $X, Y \in \mathcal{G}$ are topologically conjugate if, and only if, the number of singularities is the same (see e.g. [143, 194]). Moreover the elements of $\mathcal{G}$ are precisely the structurally stable vector fields of $\mathbb{S}^1$, that is *generically a smooth vector field on the circle is structurally stable.*

1.1.2 Two-dimensional flows

Surfaces have a simple enough topology (albeit much more complex that the topology of the circle) to enable one to characterize the non-wandering set of the flow of a vector field. The most representative result in this respect is the Poincaré-Bendixson's Theorem on planar flows or flows on the two-dimensional sphere (essentially the result depends on the Jordan Curve Theorem: any closed simple curve splits the manifold in two connected components, see e.g. [136, 118, 66]).

Theorem 1.3 (Poincaré-Bendixson)**.** *Let* $X \in \mathfrak{X}^r(\mathbb{S}^2)$, $r \geq 1$ *be a smooth vector field with a finite number of singularities. Let* $p \in \mathbb{S}^2$ *be given. Then the omega-limit set* $\omega_X(p)$ *satisfies one of the following:*

1. *$\omega_X(p)$ is a singularity;*

2. *$\omega_X(p)$ is a periodic orbit;*

3. *$\omega_X(p)$ consists of singularities $\sigma_1, \dots, \sigma_n$ and regular orbits $\gamma \in \omega_X(p)$ such that $\alpha_X(\gamma) = \sigma_i$ and $\omega_X(\gamma) = \sigma_j$ for some $i, j = 1, \dots, n$.*

The proof of this basic result may be found e.g. in [77, 143]. This answers essentially all the questions concerning the asymptotic dynamics of the solutions of autonomous ordinary differential equations on the plane or on the sphere.

Observe that now hyperbolic singularities σ can be of three types: sink ($DX(\sigma)$ with two eigenvalues with negative real part), source ($DX(\sigma)$ whose eigenvalues have positive real part, see Figure 1.1) or a saddle ($DX(\sigma)$ with eigenvalues having negative and positive real parts, see Figure 1.3 on page 13).

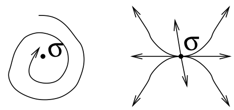

Figure 1.1: A sink and a source.

Historically the characterization of structurally stable vector fields on compact surfaces by Maurício Peixoto, based on previous work of Poincaré [155, 156, 157] and Andronov and Pontryagin [7], was the origin of the notion of structural stability for Dynamical Systems. In this setting structural stability is still synonym of a finite and hyperbolic non-wandering set. We now write S for any compact connected two-manifold without boundary.

Theorem 1.4 (Peixoto)**.** *A C^r vector field on a compact surface S is structurally stable if, and only if:*

1. *the number of critical elements is finite and each is hyperbolic;*

2. *there are no orbits connecting saddle points;*

3. *the non-wandering set consists of critical elements alone.*

Moreover if S is orientable, then the set of structurally stable vector fields is open and dense in $\mathfrak{X}^r(S)$.

The proof of this celebrated result can be found in [147, 148] and for an more detailed exposition of this results and sketch of the proof see [64]. The last part of the statement uses a version of Pugh's C^1-Closing Lemma [163, 164], which is a fundamental tool to be used repeatedly in many proofs in this book, see Section 1.3.7 for the statement of this result.

The extension of Peixoto's characterization of structural stability for C^r flows, $r \geq 1$, on non-orientable surfaces is known as *Peixoto's Conjecture*, and up until now it has been proved for the projective plane $\mathbb{P}^2$ [143], the Klein bottle $\mathbb{K}^2$ [113] and $\mathbb{L}^2$, the torus with one cross-cap [67].

In an attempt to extend this result to higher dimensions, Steve Smale considered in [190] the following type of vector field which preserves the main features of the structurally stable vector fields on surfaces.

We say that a vector field $X \in \mathfrak{X}^r(M), r \geq 1$ is *Morse-Smale* (where now M is a compact manifold of any dimension) if

1. the number of critical elements of X is finite and each one of them is hyperbolic;
2. every stable and unstable manifold of each critical element intersects transversely the unstable or stable manifold of any other critical element;
3. the non-wandering set consists only of the critical elements of X: $\Omega(X) = C(X)$.

Hence *structurally stable vector fields in two-dimensions are Morse-Smale and they are open and dense on the set of all smooth vector fields of an orientable surface.*

There exists a similar notion of Morse-Smale diffeomorphisms on any compact manifold. *Smale's Horseshoe*, presented in [190], showed that Morse-Smale diffeomorphisms are neither dense on the space of all diffeomorphisms, nor the only structurally stable type of diffeomorphisms.

Moreover the singular horseshoe, which we present in Section 2.1, is a compact invariant set for a flow similar to a Smale Horseshoe which is structurally stable but non-hyperbolic, defined on manifolds with boundary.

It is well known that Morse-Smale vector fields are structurally stable in any dimension, see e.g. [144, 143]. However early hopes that they might form an open and dense subset of the space of all smooth vector fields or that they are the representatives of structurally stable vectors fields where shattered in higher dimensions, as the following section explains.

1.1.3 Three dimensional chaotic attractors

In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [102] an example of a parametrized polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y-x) & a &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{1.1}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast. Later Lorenz [103] together with other experimental researches showed that the equations of motions of a certain laboratory water wheel are given by (1.1). Hence equations (1.1) can be deduced directly in order to model a physical phenomenon instead of as an approximation to a partial differential equation.

Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a stranger attractor, called the *Lorenz attractor*. However Lorenz's equations proved to be very resistant to rigorous mathematical analysis, and also presented very serious difficulties to rigorous numerical study.

A very successful approach was taken by Afraimovich, Bykov and Shil'nikov [1], and Guckenheimer, Williams [65], independently: they constructed the so-called *geometric Lorenz models* (see Section 2.3) for the behavior observed by Lorenz. These models are flows in 3-dimensions for which one can rigorously prove the existence of an attractor that contains an equilibrium point of the flow, together with regular solutions. The accumulation of regular orbits near a singularity prevents such sets to be hyperbolic (see Section 1.2). Moreover, for almost every pair of nearby initial conditions, the corresponding solutions move away from each other exponentially fast as they converge to the attractor, that is, the attractor is *sensitive to initial conditions*: this unpredictability is a characteristic of *chaos*. Most remarkably, this attractor is robust: it can not be destroyed by any small perturbation of the original flow.

Another approach was through rigorous numerical analysis. In this way, it could be proved, by [71, 72, 119, 120], that the equations (1.1) exhibit a suspended Smale Horseshoe. In particular, they have infinitely many closed solutions, that is, the attractor contains infinitely many periodic orbits. However, proving the existence of a strange attractor as in

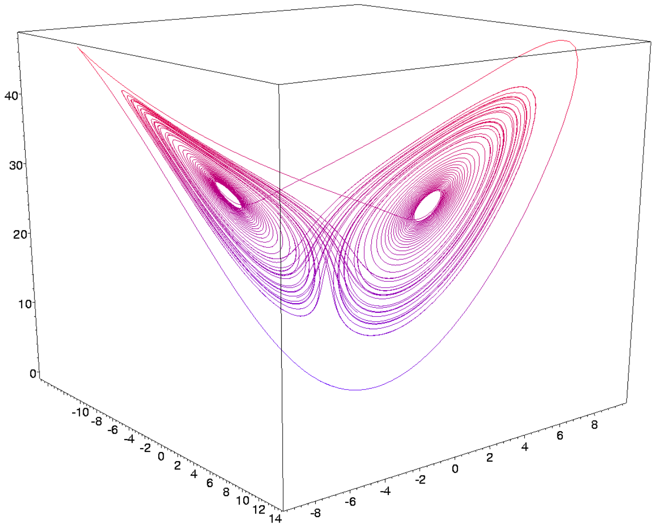

Figure 1.2: Lorenz strange attractor

the geometric models is an even harder task, because one cannot avoid the main numerical difficulty posed by Lorenz's equations, which arises from the very presence of an equilibrium point: solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors.

As a matter-of-fact, proving that equations (1.1) support a strange attractor was listed by Steve Smale in [191] as one of the several challenging problems for the twenty-first century. In the year 2000 this was finally settled by Warwick Tucker who gave a mathematical proof of the existence of the Lorenz attractor, see [196, 197, 198].

The algorithm developed by Tucker incorporates two kinds of ingredients: a numerical integrator, used to compute good approximations of trajectories of the flow far from the equilibrium point sitting at the origin, together with quantitative results from normal form theory, that make it possible to handle trajectories close to the origin.

The consequences of the sensitiveness to initial conditions on a (albeit simplified) model of the atmosphere were far-reaching: assuming that the weather behaves according to this model, then long-range weather forecasting is impossible. Indeed the unavoidable errors in determining the present state of the weather system are magnified as time goes by casting off any reliability of the values obtained by numerical integration within a small time period.

This observation was certainly not new. Since the development of the kinetic theory of gases and thermodynamics in the end of the nineteenth century it was known that gas environments, specifically the Earth atmosphere, are very complex systems whose dynamics involves the interaction of a huge number of particles, so it is not surprising that the evolution of such systems be hard to predict. What bewildered mathematicians was the simplicity of the Lorenz system, the fact that it arises naturally as a model of a physical phenomenon (convection) and, notwithstanding, its solutions exhibit sensitiveness with respect to the initial conditions. This suggests that sensitiveness is the rule rather than the exception in the natural sciences.

For an historical account of the impact of the Lorenz paper [102] on Dynamical Systems and an overview of the proof by Tucker see [202].

The robustness of this example provides an open set of flows which are not Morse-Smale, nor hyperbolic, and also non-structurally stable, as we will see in Section 2.3.

1.2 Hyperbolic flows

In an attempt to identify what properties were common among stable systems, Stephen Smale introduced in [190] the notion of *Hyperbolic Dynamical System*. Remarkably it turned out that stable systems are essentially the hyperbolic ones, plus certain transversality conditions. In the decades of 1960 and 1970 an elegant and rather complete mathematical theory of hyperbolic systems was developed, culminating with the proof of the Stability Conjecture, by Mañé in the 1990's in the setting of C^1 diffeomorphisms, followed by Hayashi for C^1 flows.

In what follows we present some results of this theory which will be used throughout the text.

Let $X \in \mathfrak{X}^r(M)$ be a flow on a compact manifold M. Denote by $m(T) = \inf_{\|v\|=1} \|T(v)\|$ the *minimum norm* of a linear operator T. A compact invariant set $\Lambda \subset M$ of X is *hyperbolic* if

1. admits a continuous DX-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$, that is we can write the tangent space TxM as a direct sum $E^s_x \oplus E^X_x \oplus E^u_x$, where E^X_x is the subspace in T_xM generated by $X(x)$, satisfying
   - $DX^t(x) \cdot E^i_x = E^i_{X^t(x)}$ for all $t \in \mathbb{R}$, $x \in \Lambda$ and $i = s, X, u$;
2. there are constants $\lambda, K > 0$ such that
   - E^s_Λ *is* (K,λ)-*contracting*, i.e. for all $x \in \Lambda$ and every $t \geq 0$
     $$\|DX^t(x) \mid E^s_x\| \leq K^{-1}e^{-\lambda t},$$
   - E^u_Λ *is* (K,λ)-*expanding*, i.e. for all $x \in \Lambda$ and every $t \geq 0$
     $$m(DX^t \mid E^u) \geq Ke^{\lambda t},$$

By the Invariant Manifold Theory [76] it follows that for every $p \in \Lambda$ the sets

$$W^{ss}_X(p) = \{q \in M : \text{dist}(X_t(q), X_t(p)) \xrightarrow[t\to\infty]{} 0\}$$

and

$$W^{uu}_X(p) = \{q \in M : \text{dist}(X_t(q), X_t(p)) \xrightarrow[t\to-\infty]{} 0\}$$

are invariant C^r-manifolds tangent to E^s_p and E^u_p respectively at p. Here dist is the *distance on M induced by some Riemannian norm*.

If $\mathcal{O} = \mathcal{O}_X(p) \subset \Lambda$ is an orbit of X one has that

$$W^s_X(\mathcal{O}) = \cup_{t\in\mathbb{R}} W^{ss}_X(X^t(p)) \quad \text{and} \quad W^u_X(\mathcal{O}) = \cup_{t\in\mathbb{R}} W^{uu}_X(X^t(p))$$

are invariant C^r-manifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ at p, respectively. We shall denote $W^s_X(p) = W^s_X(\mathcal{O}_X(p))$ and $W^u_X(p) = W^u_X(\mathcal{O}_X(p))$ for the sake of simplicity.

A *singularity* (respectively *periodic orbit*) of X is *hyperbolic* if its orbit is a hyperbolic set of X. Note that $W^{ss}_X(\sigma) = W^s_X(\sigma)$ and $W^{uu}_X(\sigma) = W^u_X(\sigma)$ for every hyperbolic singularity σ of X. A sink and a source are both hyperbolic singularities. A *hyperbolic* singularity which is *neither* a sink *nor* a source is called a *saddle*.

A hyperbolic set Λ of X is called *basic* if it is transitive and *isolated*, that is $\Lambda = \cap_{t\in\mathbb{R}} \overline{X^t(U)}$ for some neighborhood U of H. It follows from

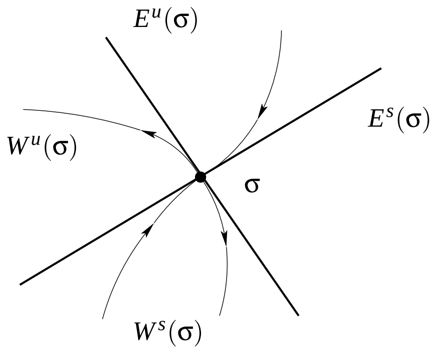

Figure 1.3: A saddle singularity σ for bi-dimensional flow.

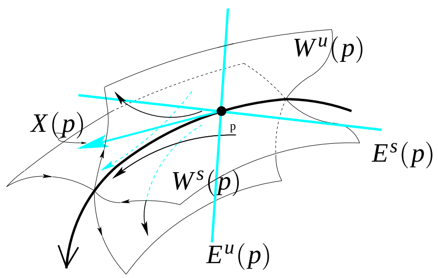

Figure 1.4: The flow near a hyperbolic saddle periodic orbit through p.

the Shadowing Lemma [137] that every hyperbolic basic set of X either reduces to a singularity or else has no singularities and it is the closure of its periodic orbits.

We say that X is *Axiom A* if the non-wandering set $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The *Spectral Decomposition Theorem* asserts that if X is Axiom A, then there is a disjoint decomposition $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$, where each Λ_i is a hyperbolic basic set of X, $i = 1, \cdots, k$.

A *cycle* of a Axiom A vector field X is a sub-collection $\{\Lambda_{i_0}, \cdots, \Lambda_{i_k}\}$ of $\{\Lambda_1, \cdots, \Lambda_n\}$ such that $i_0 = i_k$ and $W^u_X(\Lambda_{i_j}) \cap W^s_X(\Lambda_{i_{j+1}}) \neq \emptyset$, $\forall 0 \leq j \leq k-1$.

Hyperbolic sets and singularities

The continuity of the DX-invariant splitting on the tangent space of a uniformly hyperbolic set Λ is a consequence of the uniform expansion and contraction estimates (see e.g. [143]). This means that if $x_n \in \Lambda$ is a sequence of points converging to $x \in \Lambda$, and we consider orthonormal basis $\{e_i^n\}_{i=1,\dots,\dim E^s(x_n)}$ of $E^s(x_n)$, $\{f_i^n\}_{i=1,\dots,\dim E^u(x_n)}$ of $E^u(x_n)$ and $X(x_n)$ of $E^X(x_n)$, then these vectors converge to a basis of $E^s(x), E^u(x)$ and $E^X(x)$ respectively. In particular the dimension of the subspaces in the hyperbolic splitting is constant if Λ is transitive.

This shows that a uniformly hyperbolic basic set Λ cannot contain singularities, except if Λ is itself a singularity. Indeed, if $\sigma \in \Lambda$ is a singularity then it is hyperbolic but the dimension of the central sub-bundle is zero since the flow is zero at σ. Therefore the dimensions of either the stable or the unstable direction at σ and those of a transitive regular orbit in Λ do not match.

In other words *an invariant subset* Λ *containing a singularity accumulated by regular orbits cannot be uniformly hyperbolic*.

1.2.1 Examples of hyperbolic sets and Axiom A flows

Any hyperbolic singularity or hyperbolic periodic orbit is a hyperbolic invariant set. Also any finite collection of hyperbolic critical elements is a hyperbolic set. We refer to these sets as *trivial hyperbolic sets*.

The first examples of a non-trivial (different from a singularity or a periodic orbit) hyperbolic basic set (on the whole manifold) was the *geodesic flow on any Riemannian manifold with negative curvature*, studied by Dmitri Victorovich Anosov [8], whose name is attached to this type of systems today, and the *Smale Horseshoe*, presented in [190] in the setting of diffeomorphisms.

We use a global construction of a (linear) Anosov diffeomorphism (hyperbolic with dense orbit) on the 2-torus and then consider its suspension on the solid (3-)torus to obtain an example of a transitive Axiom A flow.

A linear Anosov diffeomorphism on the 2-torus

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ with the following matrix in the canonical base

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Consider the 2-torus $\mathbb{T}^2$ as the quotient $\mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2/\sim$, where $(x,0) \sim (x,1)$ and $(y,0) \sim (y,1)$ for all $x,y \in [0,1]$, that is the square $[0,1]^2$ whose parallel sides are identified. We denote by $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ the quotient map or projection from $\mathbb{R}^2$ to $\mathbb{T}^2$. Since A preserves $\mathbb{Z}^2$, i.e. $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$, then there exists a well defined quotient map $F_A : \mathbb{T}^2 \to \mathbb{T}^2$. This is a linear automorphism of $\mathbb{T}^2$, see e.g. [204, 107].

The matrix A is hyperbolic: its eigenvalues are $\lambda_1, \lambda_2 = (3 \pm \sqrt{5})/2$ and the corresponding eigenvectors $v_1, v_2 = (1, (-1 \mp \sqrt{5})/2)$, with irrational slope. Given any point $p \in \mathbb{T}^2$, if we take the projection $W_i(p)$ of the line L_i through p parallel to v_i, $W_i(p) = \pi(L_i)$, then distances along $W_i(p)$ are multiplied by λ_i under the action of F_A, for $i = 1,2$. These are the stable and unstable manifolds of p. Due to the irrationality of the slope every such "line" is dense in the torus. Moreover there is a transitive orbit and a dense set of periodic orbits for the map F_A (see e.g. [52]). The entire torus is then a uniformly hyperbolic set.

General definition of suspension flow over a roof function

Let (X,d) be a metric space with distance d and $r : X \to \mathbb{R}$ be a strictly positive function. The *phase space* X_r of the suspension flow is defined as

$$X_r = \{(x,y) \in X \times [0,+\infty) : 0 \le y < r(x)\}.$$

Let $f : X \to X$ be a map on X. The *suspension semi-flow over f with roof r* is the following family of maps $X_f^t : X_r \to X_r$ for $t \ge 0$: X^0 is the identity and for each $x = x_0 \in X$ denote by x_n the nth iterate $f^n(x_0)$ for $n \ge 0$. Denote also $S_n r(x_0) = \sum_{j=0}^{n-1} r(x_j)$ for $n \ge 1$. Then for each pair $(x_0, y_0) \in X_r$ and $t > 0$ there exists a unique $n \ge 1$ such that $S_n r(x_0) \le y_0 + t < S_{n+1} r(x_0)$ and we define (see Figure 1.5 on the next page).

$$X_f^t(x_0,y_0) = \big(x_n, y_0 + t - S_n r(x_0)\big)$$

This construction is the basis of many examples and also of many techniques to pass from a flow with a transverse section to a suspension flow and viceversa, enabling us to transfer results which are easy to prove for suspension flows, due to their "almost product structure", to more general flows. In this text we will see several examples of this.

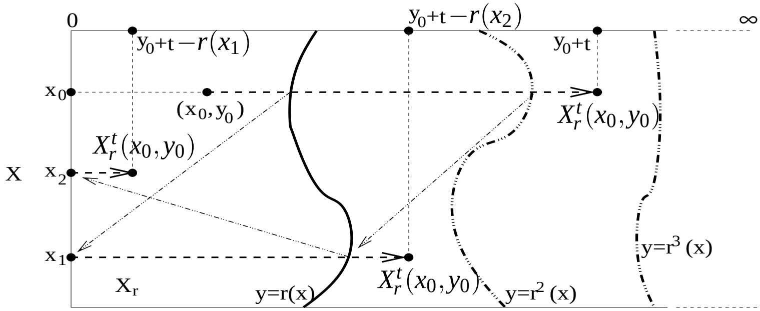

Figure 1.5: The equivalence relation defining the suspension flow of f over the roof function r.

An Anosov flow on $\mathbb{T}^3$ though the suspension of an Anosov diffeomorphism

Consider the suspended flow X_r over $F_A : \mathbb{T}^2 \to \mathbb{T}^2$ defined in Section 1.2.1 with a constant roof function $r \equiv 1$. Then X_r is the 3-cube $[0,1]^3$ with parallel sides identified, that is, we obtain a flow on the 3-torus such that the *first return map* R_z from any section $\mathbb{T}^2 \times \{z\}$ to itself can be naturally identified with F_A, see Figure 1.6 on the facing page.

This flow $X^t_{F_A}$ is uniformly hyperbolic since the hyperbolic structure exhibited by the map F_A is naturally carried by the flow to $\mathbb{T}^3$, e.g. it has a dense orbits and a dense set of periodic orbits, each of which are the suspension of the corresponding dense orbit and periodic orbits for F_A. The invariant manifolds of a point (x,y,s) are simply the translate of the corresponding invariant manifolds of (x,y) for F_A: $W^k_{X_r}(x,y,z) = W^k(x,y) \times \{z\}$ for $k = uu, ss$ and any $z \in [0,1]$.

We will see in Section 6.4 that this Anosov flow *is not topologically mixing*.

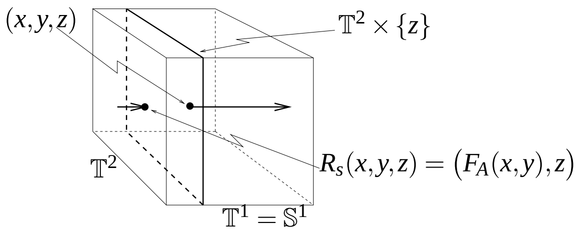

Figure 1.6: Suspension flow over Anosov diffeomorphism with constant roof

The solenoid attractor

Consider now the solid 2-torus $\mathbb{S}^1 \times \mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk in $\mathbb{C}$, together with the map $f : \mathbb{S}^1 \times \mathbb{D} \to \mathbb{S}^1 \times \mathbb{D}$ given by

$$(\theta, z) \mapsto (2\theta, \alpha z + \beta e^{i\theta/2}),$$

$\theta \in \mathbb{R}/\mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta < 1$. This transformation maps $\mathbb{S}^1 \times \mathbb{D}$ strictly inside itself, that is $\overline{f(\mathbb{S}^1 \times \mathbb{D})} \subset \mathbb{S}^1 \times \mathbb{D}$. The maximal positively invariant set $\Lambda = \cap_{n \geq 0} f^n(\mathbb{S}^1 \times \mathbb{D})$ is a uniformly hyperbolic basic set: the $\mathbb{S}^1$ direction is uniformly expanding and the $\mathbb{D}$ direction is uniformly contracting, see Figure 1.7. This set is transitive and has a dense subset of periodic orbits [52, 177].

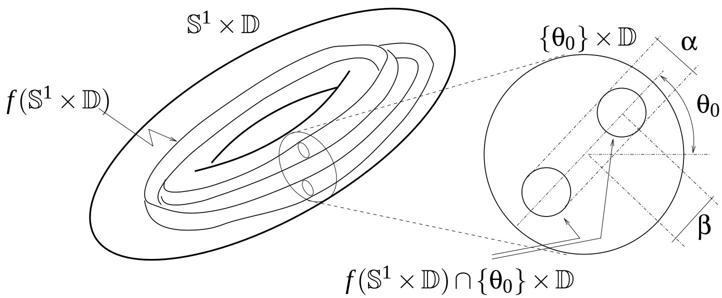

Figure 1.7: The solenoid attractor

Uniformly hyperbolic basic set for a flow

Consider the suspension of the solenoid map f of the previous subsection over the constant roof function $r \equiv 1$ to get a flow with an attractor $\Lambda_f = \cap_{t\geq 0} X_f^t\big((\mathbb{S}^1 \times \mathbb{D})_r\big)$ which is a uniformly hyperbolic basic set for the flow X_f.

This is an example of an Axiom A attractor for a flow. As before X_f^t on Λ is *not* topologically mixing.

1.2.2 Expansiveness and sensitive dependence on initial conditions

The development of the theory of dynamical systems has shown that models involving expressions as simple as quadratic polynomials (as the *logistic family* or *Hénon attractor*), or autonomous ordinary differential equations with a hyperbolic singularity of saddle-type accumulated by regular orbits, as the *Lorenz flow*, exhibit *sensitive dependence on initial conditions*, a common feature of *chaotic dynamics*: small initial differences are rapidly augmented as time passes, causing two trajectories originally coming from practically indistinguishable points to behave in a completely different manner after a short while. Long term predictions based on such models are unfeasible since it is not possible to both specify initial conditions with arbitrary accuracy and numerically calculate with arbitrary precision.

Formally the definition of sensitivity is as follows for a flow X^t: a X^t-invariant subset Λ is *sensitive to initial conditions* or has *sensitive dependence on initial conditions* if, for every small enough $r > 0$ and $x \in \Lambda$, and for any neighborhood U of x, there exists $y \in U$ and $t \neq 0$ such that $X^t(y)$ and $X^t(x)$ are r-apart from each other: $\operatorname{dist}\big(X^t(y), X^t(x)\big) \geq r$. See Figure 1.8.

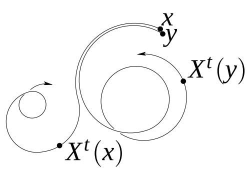

Figure 1.8: Sensitive dependence on initial conditions.

A related concept is that of expansiveness, which roughly means that points whose orbits are always close for all time must coincide. The concept of expansiveness for homeomorphisms plays an important role in the study of transformations. Bowen and Walters [40] gave a definition of expansiveness for flows which is now called *C-expansiveness* [87]. The basic idea of their definition is that two points which are not close in the orbit topology induced by $\mathbb{R}$ can be separated at the same time even if one allows a continuous time lag — see below for the technical definitions. The equilibria of C-expansive flows must be isolated [40, Proposition 1] which implies that the Lorenz attractors and geometric Lorenz models are not C-expansive.

Keynes and Sears introduced [87] the idea of restriction of the time lag and gave several definitions of expansiveness weaker than C-expansiveness. The notion of *K-expansiveness* is defined allowing only the time lag given by an increasing surjective homeomorphism of $\mathbb{R}$. Komuro [90] showed that the Lorenz attractor (presented in Section 1.1.3) and the geometric Lorenz models (to be presented in Section 2.3) are not K-expansive. The reason for this is not that the restriction of the time lag is insufficient but that the topology induced by $\mathbb{R}$ is unsuited to measure the closeness of two points in the same orbit.

Taking this fact into consideration, Komuro [90] gave a definition of *expansiveness* suitable for flows presenting equilibria accumulated by regular orbits. This concept is enough to show that two points which do not lie on a same orbit can be separated.

Let $C\big(\mathbb{R},\mathbb{R}\big)$ be the set of all continuous functions $h:\mathbb{R}\to\mathbb{R}$ and set $C_0\big((\mathbb{R},0),(\mathbb{R},0)\big)$ for the subset of all $h\in C\big(\mathbb{R},\mathbb{R}\big)$ such that $h(0)=0$. Define

$$\mathcal{K}_0=\{h\in C\big(\mathbb{R},0),(\mathbb{R},0)\big):h(\mathbb{R})=\mathbb{R},\,h(s)>h(t)\,,\forall s>t\},$$

and

$$\mathcal{K}=\{h\in C\big(\mathbb{R},\mathbb{R}\big):h(\mathbb{R})=\mathbb{R},\,h(s)>h(t)\,,\forall s>t\},$$

A flow X is *C-expansive* (*K-expansive* respectively) on an invariant subset $\Lambda\subset M$ if for every $\varepsilon>0$ there exists $\delta>0$ such that if $x,y\in\Lambda$ and for some $h\in\mathcal{K}_0$ (respectively $h\in\mathcal{K}$) we have

$$\operatorname{dist}\big(X^t(x),X^{h(t)}(y)\big)\le\delta\quad\text{for all}\quad t\in\mathbb{R}, \tag{1.2}$$

then $y \in X^{[\varepsilon,\varepsilon]}(x) = \{X^t(x) : -\varepsilon \le t \le \varepsilon\}$.

We say that the flow X is *expansive* on Λ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in \Lambda$ and $h \in \mathcal{K}$ (note that now we do not demand that 0 be fixed by h) satisfying (1.2), then we can find $t_0 \in \mathbb{R}$ such that $X^{h(t_0)}(y) \in X^{[t_0-\varepsilon, t_0+\varepsilon]}(x)$.

Observe that expansiveness on M implies sensitive dependence on initial conditions for any flow on a manifold with dimension at least 2. Indeed if ε, δ satisfy the expansiveness condition above with h equal to the identity and we are given a point $x \in M$ and a neighborhood U of x, then taking $y \in U \setminus X^{[-\varepsilon,\varepsilon]}(x)$ (which always exists since we assume that M is not one-dimensional) there exists $t \in \mathbb{R}$ such that $\operatorname{dist}\big(X^t(y), X^t(x)\big) \ge \delta$. The same argument applies whenever we have expansiveness on an X-invariant subset Λ of M containing a dense regular orbit of the flow.

Clearly C-expansive $\implies$ K-expansive $\implies$ expansive by definition. When a flow has no fixed point then C-expansiveness is equivalent to K-expansiveness [138, Theorem A]. In [40] it is shown that on a connected manifold a C-expansive flow has no fixed points. The following was kindly communicated to us by Alfonso Artigue from IMERL.

Proposition 1.5. *A flow is C-expansive on a manifold M if, and only if, it is K-expansive.*

Proof. From the results of Bowen [40], a C-expansive flow admits only finitely many isolated fixed points on M. We assume now that X^t has non-isolated fixed points in M, that is, there exists at least a singularity σ which is accumulated by other points of M (this always holds on a connected manifold). Then X is not C-expansive. We now show that it is not K-expansive either, proving the proposition.

Using the continuity of X^t we have that for all $R > 0$ and $T > 0$ there exists $x \in M \setminus \{\sigma\}$ such that $\operatorname{dist}\big(X^t(x), \sigma\big) \le R$ whenever $|t| < T$.

Let $\varepsilon, \delta > 0$ be given and let us set $T = 3\varepsilon$ and $R = \delta/2$. Define $y = X^{\varepsilon}(x)$ and

$$h(t) = \begin{cases} t+\varepsilon & \text{if } t \notin (-2\varepsilon, \varepsilon) \\ 2t & \text{if } 0 \le t < \varepsilon \\ t/2 & \text{if } t \in (-2\varepsilon, 0) \end{cases},$$

which is a monotonously increasing homeomorphism of $\mathbb{R}$ with $h(0) = 0$.

Next we verify that $\operatorname{dist}(X^t(y), X^{h(t)}(x)) \le \delta/2$ for all $t \in \mathbb{R}$.

- if $t \notin (-2\varepsilon, \varepsilon)$ then $X^{h(t)}(x) = X^{t+\varepsilon}(x) = X^t(y)$ and so we are done,
- if $t \in (0, \varepsilon)$ then $h(t) = 2t < T$ and so $\mathrm{dist}(X^{h(t)}(x), \sigma) \le \delta/2$ which implies $\mathrm{dist}(X^t(y), X^{h(t)}(x)) \le \mathrm{dist}(X^t(y), \sigma) + \mathrm{dist}(X^{h(t)}(x), \sigma)$. As $X^t(y) = X^{t+\varepsilon}(x)$ and for $t < \varepsilon$ we have $t + \varepsilon < 3\varepsilon = T$ we obtain $\mathrm{dist}(X^t(y), \sigma) < \delta/2$. Hence $\mathrm{dist}(X^t(y), X^{h(t)}(x)) < \delta$.
- if $t \in (-2\varepsilon, 0)$ then $|h(t)| = |t/2| < 3\varepsilon$ and so $\mathrm{dist}(X^{h(t)}(x), \sigma) \le \delta/2$. Now, $\mathrm{dist}(X^t(y), X^{h(t)}(x)) \le \mathrm{dist}(X^t(y), \sigma) + \mathrm{dist}(X^{h(t)}(x), \sigma) \le \mathrm{dist}(X^t(y), \sigma) + \delta/2$. But $t \in (-2\varepsilon, 0)$, $t + \varepsilon \in (-\varepsilon, \varepsilon)$ and so $|t + \varepsilon| < \varepsilon$ implying that $\mathrm{dist}(X^{t+\varepsilon}(x), \sigma) < \delta/2$ and hence, as $X^\varepsilon(x) = y$, we get $\mathrm{dist}(X^t(y), \sigma) < \delta/2$ and replacing this in the inequality above we obtain $\mathrm{dist}(X^t(y), X^{h(t)}(x)) < \delta$.

All together we have proved $\mathrm{dist}(X^t(y), X^{h(t)}(x)) \le \delta/2$ for all $t \in \mathbb{R}$. Now there are two possibilities.

1. either $X^t(x) \neq y$ for all $|t| < \varepsilon$, and we are done, or
2. or there exists $s \in \mathbb{R}$ such that $X^s(x) = y$, and in this case x is a periodic orbit with period $\tau \le s - \varepsilon < 2\varepsilon$. Thus $\mathrm{dist}(X^t(x), X^{h(t)}(\sigma) < \delta$.

Either way we found a pair of points (x and y in case (1), x and σ in case (2)) which remain δ-close even when time is reparametrized through h in one of the orbits, and both points are not connected through any X-orbit in a time less than ε. Since we may take $\delta > 0$ arbitrarily close to zero for a fixed $\varepsilon > 0$ in this construction, we have shown that X is not K-expansive. □

We will prove in Section 4.1 that singular-hyperbolic attractors are expansive so, in particular, the Lorenz attractor and the geometric Lorenz examples are all expansive and sensitive to initial conditions. Since these families of flows exhibit equilibria accumulated by regular orbits, we see that expansiveness is compatible with the existence of fixed points by the flow.

1.3 Basic tools

Here we state two basic classical results which enable us to understand in many cases the local dynamics near many flow orbits. Then we state the powerful *closing* and *connecting lemmas* which will be used in a fundamental way in several key points in the following chapters.

1.3.1 The tubular flow theorem

The following result shows that the local behavior of orbits near a regular point of any flow is very simple.

Theorem 1.6 (Tubular Flow)**.** *Let $X \in \mathfrak{X}^r(M)$ and let $p \in M^n$ be a regular point of X where $n \geq 1$ is the dimension of M. Let $V = \{(x_1,\dots,x_n) \in \mathbb{R}^n : \|x_i\| < 1\}$ and Y be the vector field on V given by $Y = (1,0,\dots,0)$. Then there is a C^r diffeomorphism $h : U \to V$ for some neighborhood U of p in M, which takes trajectories of X to trajectories of Y, that is $X \mid U$ is topologically equivalent to $Y \mid V$.*

This shows that near a regular point p every smooth flow can be smoothly linearised: under a change of coordinates orbits near p look like the orbits of a constant flow, see Figure 1.9.

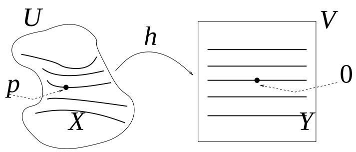

Figure 1.9: Linearization of orbits near a regular point of a flow.

1.3.2 Transverse sections and the Poincaré return map

Now we describe a standard and extremely useful consequence of the tubular flow theorem, which provides a converse to the construction of suspensions semiflows (presented in Section 1.2.1).

Let $X \in \mathfrak{X}^1(M^3)$ be a flow on a three-dimensional manifold and let S be an embedded surface in M which is transverse to the vector field X at all points, i.e. for every $x \in S$ we have $T_xS + E^X_x = T_xM$ or equivalently $X(x) \notin T_xS$. We say in what follows that such S is a *cross-section* to the flow X^t or to the vector field X.

Let S_0 and S_1 be a pair of cross-sections to X and $x_0 \in S_0$ be a regular point of X and suppose that there exists $T > 0$ such that $x_1 = X^T(x_0) \in S_1$. Applying the Tubular Flow Theorem 1.6 to a finite open covering of the compact arc $\gamma = X^{[0,T]}(x_0)$ we obtain a tubular flow in a neighborhood of γ. This shows that there exists a smooth map R from a neighborhood V_0 of x_0

in S_0 to a neighborhood V_1 of x_1 in S_1, with the same degree of smoothness of the flow, such that $R(x) = X^T(x)(x)$ for all $x \in V_0$ with $R(x_0) = x_1$ and $T : V_0 \to \mathbb{R}$ also smooth with $T(x_0) = T$. Moreover R is a bijection and thus a diffeomorphism.

We can reapply the Tubular Flow Theorem and extend the domain of definition of R to its maximal domain relative to S_0 and S_1 and to the connection time T. Notice that x_1 need not be the first entry to S_1, that is T might be bigger than $\inf\{t > 0 : X^t(x_0) \in S_1\}$.

Note that if x_0 is a periodic orbit of X then taking $S_1 = S_0$ we see that x_0 is a fixed point of R and the local behavior of the flow near x_0 can be studied through the map R acting on a space with less dimension than M. This is an important example where we can reduce the study of a flow to a lower dimensional transformation. The power and applicability of this method should be clear after Chapters 2 and 4.

1.3.3 The Linear Poincaré Flow

If x is a regular point of X (i.e. $X(x) \neq 0$), denote by

$$N_x = \{v \in T_xM : v \cdot X(x) = 0\}$$

the orthogonal complement of $X(x)$ in T_xM. Denote by $O_x : T_xM \to N_x$ the orthogonal projection of T_xM onto N_x. For every $t \in \mathbb{R}$ define

$$P_x^t : N_x \to N_{X^t(x)} \quad \text{by} \quad P_x^t = O_{X^t(x)} \circ DX^t(x).$$

It is easy to see that $P = \{P_x^t : t \in \mathbb{R}, X(x) \neq 0\}$ satisfies the cocycle relation

$$P_x^{s+t} = P_{X^t(x)}^s \circ P_x^s \quad \text{for every} \quad t, s \in \mathbb{R}.$$

The family P is called the *Linear Poincaré Flow* of X.

Hyperbolic splitting for the Linear Poicaré Flow

Let a compact subset Λ invariant under the flow of $X \in \mathfrak{X}^1$ be given. Assume that Λ is *(uniformly) hyperbolic*, as defined in Section 1.2. Then the normal space N_x is defined for all $x \in \Lambda$, since Λ does not contain singularities. Hence Linear Poincaré Flow is defined everywhere on the family of normal spaces $N_\Lambda = \{N_x\}_{x \in \Lambda}$. Compactness and absence of singularities enables us to obtain the following characterization of (uniformly) hyperbolic subsets for flows.

Theorem 1.7. *Let Λ be a compact invariant subset for $X \in \mathfrak{X}^1(M)$. Then Λ is (uniformly) hyperbolic if, and only if, the Linear Poincaré Flow is everywhere defined over Λ and P_Λ admits a (uniformly) hyperbolic splitting of N_Λ.*

The relation between the hyperbolic splitting $E^s \oplus E^X \oplus E^u$ over $T_\Lambda M$ and the splitting $N^s \oplus N^u$ over N_Λ is the obvious one: $N^s_x = O_x(E^s_x)$ and $N^u_x = O_x(E^u_x)$ for all $x \in \Lambda$.

Dominated splitting for the Linear Poicaré Flow

Assume that a C^1 flow X admits a proper attractor with an isolating neighborhood U, that is $\Lambda = \Lambda_X(U) = \cap_{t\in\mathbb{R}} X^t(U)$. Hence there exists a C^1-neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$, $x \in \mathrm{Per}(Y)$ and $\mathcal{O}_Y(x) \cap U \neq \emptyset$, then

$$\mathcal{O}_Y(x) \subset \Lambda_Y(U). \tag{1.3}$$

Given $Y \in \mathcal{U}$ define $\Lambda^*_Y(U) = \Lambda_Y(U) \setminus S(Y)$. In what follows, E^X stands for the bundle spanned by the flow direction, and P^t stands for the linear Poincaré flow of X over $\Lambda^*_X(U)$.

Using (1.3) and the same arguments as in [53, Theorem 3.2] (see also [205, Theorem 3.8]) we obtain

Theorem 1.8 (Dominated splitting for the Linear Poincarï¿½ Flow)**.** *Assume that there exists a C^1 open set in $\mathfrak{X}^1(M)$ such that for all $X \in \mathcal{U}$ there are no sinks nor sources in U and every critical element of X in $\Lambda_X(U)$ is hyperbolic. Then there exists an invariant, continuous and dominated splitting $N_{\Lambda^*_X(U)} = N^{s,X} \oplus N^{u,X}$ for the Linear Poincaré Flow P^t. Moreover*

1. *for all hyperbolic sets $\Gamma \subset \Lambda^*_X(U)$ with splitting $E^{s,X} \oplus E^X \oplus E^{u,X}$ and for every $x \in \Gamma$*

$$E^{s,X}_x \subset N^{s,X}_x \oplus E^X_x \quad \text{and} \quad E^{u,X}_x \subset N^{u,X}_x \oplus E^X_x.$$

2. *If $Y_n \to X$ in $\mathfrak{X}^1(M)$ and $x_n \to x$ in M, with $x_n \in \Lambda^*_{Y_n}(U), x \in \Lambda^*_X(U)$, then $N^{s,Y_n}_{x_n} \xrightarrow[n\to\infty]{} N^{s,X}_x$ and $N^{u,Y_n}_{x_n} \xrightarrow[n\to\infty]{} N^{u,X}_x$.*

3. *If $\sigma \in S(X) \cap \Lambda_X(U)$ is Lorenz-like and $x \in W^s(\sigma) \setminus \{\sigma\}$, then on N_x the invariant splitting for the Linear Poincaré Flow is given by $N^s_x = N_x \cap T_q W^s(x)$ and $N^u_x = N_x \cap T_q W^u(x)$.*

1.3.4 The Hartman-Grobman Theorem on local linearization

The following result due to Hartman and Grobman [63, 70] shows that a flow of a vector field X is locally equivalent to its linear part at a hyperbolic singularity. Since linear hyperbolic flows can be completely classified by topological equivalence, this result enables us to classify the local behavior of the flow of any smooth vector field near a hyperbolic singularity. See [145] for generalizations and more references on this subject.

Theorem 1.9 (Hartman-Grobman)**.** *Let $X \in \mathfrak{X}^r(M)$ and let $p \in M$ be a hyperbolic singularity of X. Let $Y = DX^0 : T_pM \to T_pM$ be the linear vector field on T_pM given by the linear transformation DX^0. Then there exists a neighborhood U of p in M, a neighborhood V of 0 in T_pM and a homeomorphism $h : U \to V$ which takes trajectories of X to trajectories of Y, that is $X \mid U$ is topologically equivalent to $Y \mid V$.*

1.3.5 The (strong) Inclination Lemma (or λ-Lemma)

This are basic results of dynamics near a hyperbolic singularity which are extremely useful to obtain intersections between stable and unstable manifold through simple geometric arguments.

The Inclination Lemma

Let $\sigma \in M$ be a hyperbolic singularity of $X \in \mathfrak{X}^r(M)$ for some $r \geq 1$, with its local stable and unstable manifolds $W^s_{\text{loc}}(\sigma), W^u_{\text{loc}}(\sigma)$. Fix an embedded disk B in $W^u_{\text{loc}}(\sigma)$ which is a neighborhood of σ in $W^u_{\text{loc}}(\sigma)$, and a neighborhood V of this disk in M. Then let D be a transverse disk to $W^s_{\text{loc}}(\sigma)$ at z with the same dimension as B, and write D^t for the connected component of $X^t(D) \cap V$ which contains $X^t(z)$, for $t \geq 0$, see Figure 1.10.

Lemma 1.10 (Inclination Lemma [142, 143])**.** *Given $\varepsilon > 0$ there exists $T > 0$ such that for all $t > T$ the disk D^t is ε-close to B in the C^r-topology.*

This means that the embeddings whose images are the disks B and D^t are close in the C^r topology.

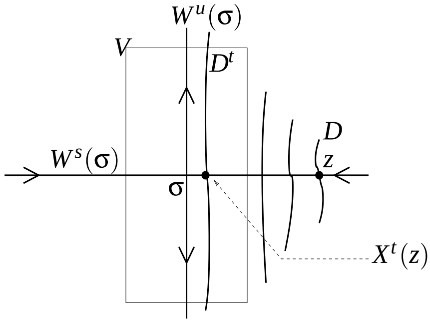

Figure 1.10: The inclination lemma

The strong Inclination Lemma

In the same setting as above but imposing that the eigenvalues of $DX(\sigma)$ closest to the imaginary axis be real and simple it is possible to improve the convergence estimates. This condition on $DX(\sigma)$ is satisfied in particular by all hyperbolic singularities with distinct real eigenvalues, and so also by the so-called Lorenz-like singularities, see Definition 2.1. These are the only kind of singularities allowed on singular-hyperbolic sets, see Chapter 3.

Lemma 1.11 (Strong Inclination Lemma [51]). *There are $c,\lambda,T>0$ such that for all $t>T$ the C^r distance between the embeddings of B and of D^t is bounded by $c\cdot e^{-\lambda t}$.*

Homoclinic classes, transitiveness and denseness of periodic orbits

Given a hyperbolic period orbit p of saddle-type for a flow $X\in\mathfrak{X}^1$ we can define its associated *homoclinic class* $H_X(p)$ by the *closure of the set of transverse intersections between the stable and unstable manifolds of p*

$$H_X(p)=\overline{W^u_X(p)\pitchfork W^s_X(p)}.$$

Note that there are cases where $W^u_X(p)$ coincides with $W^s_X(p)$, a saddle-connection, and then $H_X(p)=\emptyset$. Observe that a nonempty homoclinic class is always an invariant subset of the flow.

Otherwise we have the following important classical result from the early works of Poincarï¿½[154] (who showed that transverse homoclinic orbits are accumulation points of other homoclinic orbits) and developed by Birkhoff [27] (transverse homoclinic orbits are accumulation points of periodic orbits) and by Smale [189].

Theorem 1.12 (Birkhoff-Smale)**.** *Any non-empty homoclinic class has a dense orbit and contains a dense set of periodic orbits.*

See [146] for a general modern presentation of this result including motivation, proofs and other non-trivial dynamical consequences.

The transitiveness part of this theorem is a consequence of the Inclination Lemma and we present a short proof here.

Lemma 1.13. *Every homoclinic class H of a flow X is topologically transitive.*

Proof. Let $q, r \in H = \text{closure}[W^s_X(p) \pitchfork W^u_X(p)]$ be distinct points and U, V two disjoint neighborhoods of q, r in H, respectively. Let q_1, r_1 be points of intersection between the stable and unstable manifolds of p in U and V, respectively. Then for some future time $t > 0$ very large and some $s > 0$ close to the period of p we have that $X^{t+s}(q_1)$ is on $W^s(p)$ very close to p and $X^{-t}(r_1)$ is on $W^u(p)$ very close to p also.

The invariance of the stable and unstable manifolds and the Inclination Lemma imply that there exists a point w in the intersection between $W^{uu}\big(X^{t_1}(q_1)\big)$ and $W^{ss}\big(X^{-t_2}(r_1)\big)$ for some $t_1, t_2 > t$. Hence $X^{-t_1}(w)$ is inside U near q_1 and $X^{t_2}(w)$ is inside V near r_1. Then $X^{t_1+t_2}(U) \cap V \neq \emptyset$. □

1.3.6 Generic vector fields and Lyapunov stability

Recall that a compact set $L \subset M$ is called *Lyapunov stable* for $X \in \mathfrak{X}^1(M)$ if for every neighborhood U of L there is a neighborhood $V \subset U$ of L such that $X^t(V) \subset U$, $\forall t \geq 0$. Every attractor is a transitive Lyapunov stable set but not conversely.

The following lemmas summarize some classical properties of Lyapunov stable sets, see Chapter V in [25] for proofs.

Lemma 1.14. *Let Λ be a Lyapunov stable set of X. Then,*

1. *If $x_n \in M$ and $t_n \geq 0$ satisfy $x_n \to x \in \Lambda$ and $X^{t_n}(x_n) \to y$, then $y \in \Lambda$;*

2. *$W^u_X(\Lambda) \subset \Lambda$;*

3. *if* Γ *is a transitive set of* X *and* $\Gamma \cap \Lambda \neq \emptyset$*, then* $\Gamma \subset \Lambda$*.*

The following provides a necessary and sufficient conditions for a Lyapunov stable set to be an attractor.

Lemma 1.15. *A Lyapunov stable set* Λ *of a vector field* X *is an attractor of* X *if and only if there is a neighborhood* U *of* Λ *such that* $\omega_X(x) \subset \Lambda$*, for all* $x \in U$*.*

Let us collect some properties for generic vector fields $X \in \mathfrak{X}^1(M)$ for future reference.

L1. X is *Kupka-Smale*, i.e. every periodic orbit and singularity of X is hyperbolic and the corresponding invariant manifolds intersect transversely, see [143]. In particular, $S(X)$ is a finite set.

L2. $\Omega(X) = \overline{\mathrm{Per}(X) \cup S(X)}$, see [163].

L3. $\overline{W^u_X(\sigma)}$ is Lyapunov stable for X for each $\sigma \in S(X)$.

L4. $\overline{W^s_X(\sigma)}$ is Lyapunov stable for $-X$, for every $\sigma \in S(X)$.

L5. If $\sigma \in S(X)$ and $\dim(W^u_X(\sigma)) = 1$ then $\omega_X(q)$ is Lyapunov stable for X, for every $q \in W^u_X(\sigma) \setminus \{\sigma\}$.

L6. If $\sigma \in S(X)$ and $\dim(W^s_X(\sigma)) = 1$ then $\alpha_X(q)$ is Lyapunov stable for $-X$, for all $q \in W^s_X(\sigma) \setminus \{\sigma\}$.

The proofs of items L3 to L6 can be found in [41].

1.3.7 The Closing Lemma

This celebrated result, proved by Charles Pugh [163, 164, 165], says that every regular orbit which accumulates on itself can be closed by an arbitrarily small C^1 perturbation of the vector field. The question whether a vector field with a recurrent trajectory through a point p can be perturbed so that the solution through p for the new vector field is closed, albeit trivial in class C^0, is a deep problem in class C^r for $r \geq 1$, as first remarked by Peixoto [148].

In [163, 164] Pugh proved the C^1 Closing Lemma for compact manifolds of dimensions two and three and generalized the result for arbitrary dimensions and to the case of closing a non-wandering trajectory, rather than a recurrent one. In [161] he proved that for a weaker type of recurrent point, for which $\alpha_X(p) \cap \omega_X(p) \neq \emptyset$, the C^2 double-closing is not always possible on the 2-torus $\mathbb{T}^2$. Later Pugh and Robinson [165] established the Closing Lemma when M is non-compact, provided the point q to be closed satisfies $\alpha_X(q) \cap \omega_X(q) \neq \emptyset$.

We remark that the C^r Closing Lemma, for $r \geq 1$, in the case of M being the 2-torus and the vector field has no singularities, was proved earlier by Peixoto [148] and later by Gutierrez [68] for the "constant type" vector fields on the 2-torus with finitely many singularities. In [69] Gutierrez gave a counter-example to the C^2 Closing Lemma for the punctured torus. There exists also the "ergodic closing lemma" from Ricardo Mañé, see below.

Theorem 1.16 (C^1-Closing Lemma)**.** *Let $X \in \mathfrak{X}^1(M)$ be a C^1-flow on a compact boundaryless finite dimensional manifold M and $p \in M$ be a non-wandering point of X. Given a C^1-neighborhood $\mathcal{U}$ of X and a neighborhood V of p, then there exists $Y \in \mathcal{U}$ and $q \in V$ such that q belongs to a periodic orbit of Y.*

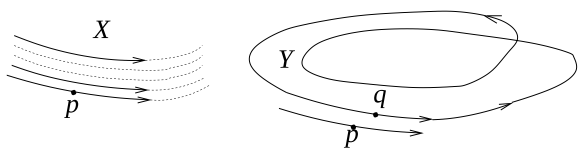

Figure 1.11: Closing a recurrent orbit

Observe that in the Closing Lemma above the point whose orbit is closed is not necessarily the initial non-wandering point, but only a point arbitrarily close to it. The same situation appears in the "Ergodic Closing Lemma" of Mañé, see Section 1.4. Later this was improved in the Connecting Lemma by Hayashi, see the next subsection.

1.3.8 The Connecting Lemma

The connecting lemma is motivated by the following situation often faced when studying dynamical systems. Suppose the unstable manifold of a hyperbolic periodic orbit accumulates on the stable manifold of another hyperbolic periodic orbit. We would like to find a vector field close to the given one such that the continuation of the invariant manifolds of the periodic orbits above really intersect.

Observe that although very similar to the closing lemma, now we are demanding that the orbits whose manifolds intersect are continuations of the original ones, so by a change of coordinates we can assume they are the same! The closing lemma only provides a point arbitrarily close to the initially given recurrent point.

The result below is the flow version of [206, Theorem E, p. 5214] first proved by Hayashi [74, 75] (see also [12]). This shows that if two distinct points p,q have orbits which visit a given neighborhood a point x and the points p,q are far way from a piece of the negative orbit of x, then we can find a C^1-close vector field such that p,q are in the same orbit, see Figure 1.12.

Theorem 1.17 (Connecting Lemma (Hayashi))**.** *Let $X \in \mathfrak{X}^1(M)$ and $x \notin S(X)$. For any C^1 neighborhood $\mathcal{U}$ of X there are $\rho > 1$, $L > 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and any two points $p,q \in M$ satisfying*

1. $p,q \notin B_\varepsilon(X^{[-L,0]}(x))$;
2. $\mathcal{O}_X^+(p) \cap B_{\varepsilon/\rho}(x) \neq \emptyset$;
3. $\mathcal{O}_X^-(q) \cap B_{\varepsilon/\rho}(x) \neq \emptyset$,

there is $Y \in \mathcal{U}$ such that $Y = X$ outside of $B_\varepsilon(X^{[-L,0]}(x))$ and such that $q \in \mathcal{O}_Y^+(p)$.

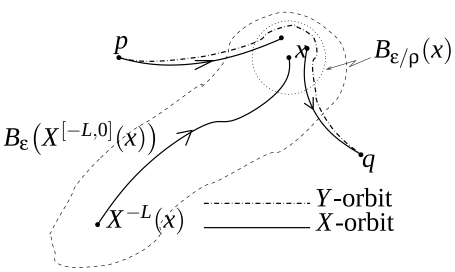

Figure 1.12: The Connecting Lemma for C^1 flows

There is an extension of this result [33] showing that it is possible to connect pseudo-orbits in the C^1 setting.

Theorem 1.17 above gives a solution to the problem of connecting stable and unstable manifolds of periodic orbits. In fact this result can be stated in a slightly different way, more adapted to our needs in Chapter 3.

Theorem 1.18. *Let $X \in \mathfrak{X}^1(M)$ and $\sigma \in S(X)$ be hyperbolic. Suppose that there are $p \in W^u_X(\sigma) \setminus \{\sigma\}$ and $q \in M \setminus C(X)$ such that:*

(H1) For all neighborhoods U, V of p, q (respectively) there is $x \in U$ such that $X^t(x) \in V$ for some $t \geq 0$.

Then there are Y arbitrarily C^1 close to X and $T > 0$ such that $p \in W^u_Y(\sigma(Y))$ and $Y^T(p) = q$. If in addition $q \in W^s_X(x) \setminus \mathcal{O}_X(x)$ for some $x \in C(X)$ hyperbolic, then Y can be chosen so that $q \in W^s_Y(x(Y)) \setminus \mathcal{O}_Y(x(Y))$.

Moreover we can use it to connect orbits of two distinct points which accumulate a third point, but with one of the points in the unstable manifold of a hyperbolic singularity. This singularity persists under perturbation and the connecting orbits will still be in its unstable manifold.

Theorem 1.19. *Let $X \in \mathfrak{X}^1(M)$ and $\sigma \in S(X)$ be hyperbolic. Suppose that there are $p \in W^u_X(\sigma) \setminus \{\sigma\}$ and $q, x \in M \setminus C(X)$ such that:*

(H2) For all neighborhoods U, V, W of p, q, x (respectively) there are $x_p \in U$ and $x_q \in V$ such that $X^{t_p}(x_p) \in W$ and $X^{t_q}(x_q) \in W$ for some $t_p > 0$, $t_q < 0$.

Then there are Y arbitrarily C^1 close to X and $T > 0$ such that $p \in W^u_Y(\sigma(Y))$ and $Y^T(p) = q$.

1.3.9 A perturbation lemma for 3-flows

A very useful result of Franks [60, Lemma 1.1] shows that it is possible to modify a diffeomorphism to achieve a desired derivative at a finite number of points, as long as the modification is made in the C^1 topology. Here we state a version for vector fields of this result: under some mild conditions, any C^2 perturbation of the derivative of the vector field along a compact orbit segment is realized by the derivative of a C^1 nearby vector field. Hence this result allows one to locally change the derivative of the flow along a compact trajectory, while the original result of Franks allows only perturbations on a finite number of points of the orbit of a diffeomorphism.

The version we present here is very useful and it was already used in several published works [53], [124], and [125] but a proof was never provided.

To simplify notations we shall state it for flows defined on compact sets of $\mathbb{R}^n$. Using local charts it is straightforward to obtain the result for flows on compact boundaryless n-manifolds. Let M be an open subset of $\mathbb{R}^n$.

Theorem 1.20. *Let us fix $Y \in \mathfrak{X}^2(M)$, $p \in M$ and $\varepsilon > 0$. Given an orbit segment $Y^{[a,b]}(p)$, a neighborhood U of $Y^{[a,b]}(p)$, and a C^2 parametrized family of invertible linear maps $A_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $t \in [a,b]$ (i.e. the coefficients of the matrices A_t with respect to a fixed basis are C^2 functions of t), such that for all s,t with $t+s \leq b$ we have*

1. *$A_0 = Id : \mathbb{R}^n \to \mathbb{R}^n$ and $A_t\big(Y(Y^s(q))\big) = Y(Y^{t+s}(q))$,*
2. *$\|\partial_s A_{t+s} A_t^{-1}|_{s=0} - D_{Y^t(p)} Y\| < \varepsilon$,*

then there is $Z \in \mathfrak{X}^1(M)$ such that $\|Y - Z\|_1 \leq \varepsilon$ and Z coincides with Y in $M \setminus U$. Moreover $Z^s(p) = Y^s(p)$ for every $a \leq s \leq b$ and $DZ^t(p) = A_t$ for every $t \in [a,b]$.

A proof of this result is presented in Appendix A.

Assume that there is such Z as in Theorem 1.20. On one hand A_t must preserve the direction of the vector field along the orbit segment $Y^{[a,b]}(p)$ for all $t \in [a,b]$ by item 1 above. On the other hand since

$$\begin{aligned}\partial_s A_{t+s} A_t^{-1}|_{s=0} &= \frac{\partial}{\partial s} D_p Z^{t+s} \big(D_p Z^t\big)^{-1}\Big|_{s=0} = \frac{\partial}{\partial s} D_p Z^{t+s} D_{Z^t(p)} Z^{-t}\Big|_{s=0} \\ &= \frac{\partial}{\partial s} DZ\big(Z^{t+s}(p)\big)|_{s=0} = \frac{\partial}{\partial s} D_{Z^t(p)} Z^s\Big|_{s=0} = DZ(Z^t(p))\end{aligned}$$

we see that item 2 above ensures that Z is C^1 near Y along the orbit segment $Y^{[a,b]}(p)$.

We observe that although we start with a C^2 vector field we obtain at the end a C^1 vector field nearby the original one. If we increase the class of differentiability of the initial vector field Y and of A_t with respect to the parameter t, then we obtain Z of higher order of differentiability. But even in this setting we can only control the distance between Y and the final vector field in the C^1 topology, by results of Pujals and Sambarino in [166] which we now explain.

There is an example of a homoclinic class H (recall Section 1.3.5 for the definition of homoclinic class) of a C^2 diffeomorphism f on a compact surface with a unique fixed point which is a saddle-node, i.e. one of its eigenvalues if equal to one, corresponding to an indifferent direction, and the other is smaller than one in modulus, corresponding to a contracting direction. Hence there are periodic orbits x_n with arbitrarily large period p_n whose a normalized Lyapunov exponent λ_n^{1/p_n} tends to 1 when $n \to +\infty$, where λ_n is an eigenvalue of $Df^{p_n}(x_n)$.

Therefore if it were possible to have a C^2 perturbation lemma analogous to Theorem 1.20, then we would obtain a C^2 diffeomorphisms arbitrarily close to f in the C^2 topology exhibiting a non-hyperbolic periodic orbit.

However in [166] the authors show that for homoclinic classes H of C^2 diffeomorphisms, if k is the maximum period of non-hyperbolic periodic orbits in H, then every periodic point with period $2k$ must be hyperbolic for every C^2 close diffeomorphisms (a kind of C^2 rigidity result). This shows that a straightforward extension of Theorem 1.20 for C^2 diffeomorphisms is impossible.

1.4 Ergodic Theory

The ergodic theory of uniformly hyperbolic systems was initiated by Sinai's Theory of Gibbs States for Anosov flows [9, 188] and was extended to Axiom A flows and diffeomorphisms by Bowen and Ruelle [37, 39]. The special measures studied by these authors are commonly referred to by their combined name *Sinai-Ruelle-Bowen* or just *SRB* in the literature since.

Recall that an *invariant probability measure* μ for a flow $X \in \mathfrak{X}^r(M)$ is a probability measure such that $\mu\big((X^t)^{-1}A\big) = \mu(A)$ for all measurable subsets A and any $t > 0$ or, equivalently, $\int \varphi \circ X^t d\mu = \int \varphi\, d\mu$ for all continuous functions $\varphi : M \to \mathbb{R}$ and any $t > 0$.

Recall also that an invariant measure μ is *ergodic* if the only X-invariant subsets have either measure 0 or 1 with respect to μ. Equivalently, any X-invariant function $\varphi \in L^1(\mu)$, i.e. $\varphi \circ X^t = \varphi$ $\mu-$ almost everywhere for all $t > 0$ is constant μ-almost everywhere. The cornerstone of Ergodic Theory is the following celebrated result of George David Birkhoff (see [26] or for a recent presentation [204]).

Theorem 1.21 (Ergodic Theorem)**.** *Let $f : M \to M$ be a measurable transformation, μ a f-invariant probability measure and $\varphi : M \to \mathbb{R}$ a bounded* *measurable function. Then the time average* $\tilde{\varphi}(x) = \lim \frac{1}{n}\sum_{j=0}^{n-1}\varphi\big(f^n(x)\big)$ *exists for* μ*-almost every point* $x \in M$. *Moreover* $\tilde{\varphi}$ *is* f*-invariant and* $\int \tilde{\varphi}\, d\mu = \int \varphi\, d\mu$. *In addition, if* μ *is ergodic, then* $\tilde{\varphi} = \int \varphi\, d\mu$ *almost everywhere with respect to* μ.

For a flow X^t just replace in the statement of Theorem 1.21 on the previous page above the discrete time average with $\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi\big(X^t(z)\big)\, dt$ and f-invariance by X-invariance. For invertible transformations or flows forward and backward (i.e. with $T \to -\infty$) time averages are equal μ-almost everywhere.

Every invariant probability measure μ is a generalized convex linear combination of ergodic measures in the following sense: for μ-a.e. x there exists an ergodic measure μ_x satisfying for every continuous function φ

$$\int \varphi\, d\mu_x = \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T \varphi\big(X^t(z)\big)\, dt$$

and for every bounded measurable function ψ we have

$$\int \psi\, d\mu = \int \left(\int \psi\, d\mu_x\right) d\mu(x).$$

1.4.1 Physical or SRB measures

The chaotic nature of hyperbolic phenomena prevents accurate long term predictions for many models of physical, biological or economic origin. Inspired by an analogous situation of unpredictability faced in the field of Statistical Mechanics/Thermodynamics — although due to the large number of particles involved, whereas dynamical systems exhibit unpredictability even for models expressed with few variables and simple mathematical formulas, e.g. the Lorenz flow in Section 1.1.3 — researchers focused on the statistics of the data provided by the time averages of some observable (a continuous function on the manifold) of the system. Time averages are guaranteed to exist for a positive volume subset of initial states (also called an *observable subset*) on the mathematical model if the transformation, or the flow associated to the ordinary differential equation, admits a smooth invariant measure (a density) or a *physical* measure.

Indeed, if μ_0 is an ergodic invariant measure for the transformation T_0, then the Ergodic Theorem ensures that for every μ-integrable function

$\varphi : M \to \mathbb{R}$ and for μ-almost every point x in the manifold M the time average $\tilde{\varphi}(x) = \lim_{n\to+\infty} n^{-1}\sum_{j=0}^{n-1}\varphi(T_0^j(x))$ exists and equals the space average $\int \varphi\, d\mu_0$. A *physical measure* μ is an invariant probability measure for which it is *required* that *time averages of every continuous function* φ *exist for a positive Lebesgue measure (volume) subset of the space and be equal to the space average* $\mu(\varphi)$.

We note that if μ is a density, that is, is absolutely continuous with respect to the volume measure, then the Ergodic Theorem ensures that μ is physical. However not every physical measure is absolutely continuous. To see why in a simple example we just have to consider a singularity p of a vector field which is an attracting fixed point (a sink), then the Dirac mass δ_p concentrated on p is a physical probability measure, since every orbit in the basin of attraction of p will have asymptotic time averages for any continuous observable φ given by $\varphi(p) = \delta_p(\varphi) = \int \varphi\, d\delta_p$.

Physical measures need not be unique or even exist in general, but when they do exist it is desirable that *the set of points whose asymptotic time averages are described by physical measures* (such set is called the *basin* of the physical measures) *be of full Lebesgue measure* — only an exceptional set of points with zero volume would not have a well defined asymptotic behavior. This is yet far from being proved for most dynamical systems, in spite of much recent progress in this direction.

There are robust examples of systems admitting several physical measures whose basins together are of full Lebesgue measure, where *robust* means that there are whole open sets of maps of a manifold in the C^2 topology exhibiting these features. For typical parametrized families of one-dimensional unimodal maps (maps of the circle or of the interval with a unique critical point) it is known that the above scenario holds true for Lebesgue almost every parameter [106]. It is known that there are systems admitting no physical measure [85], but the only known cases are not robust, i.e. there are systems arbitrarily close which admit physical measures.

Physical probability measures for a flow

Given an invariant probability measure μ for a flow X^t, let $B(\mu)$ be the the *(ergodic) basin* of μ, i.e., the set of points $z \in M$ satisfying for all continuous functions $\varphi : M \to \mathbb{R}$

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \varphi\big(X^t(z)\big)\,dt = \int \varphi\,d\mu.$$

We say that μ is a *physical* (or *SRB*) measure for X if $B(\mu)$ has positive Lebesgue measure: $\operatorname{Leb}\big(B(\mu)\big) > 0$.

The notion of *SRB* measure captures the intuitive idea that the natural measure for a dynamical system should be one which gives probabilistic information on the asymptotic behavior of trajectories starting from a "big" set of initial states. Here the notion of "big" can arguably be taken to mean "positive volume". In this sense an *SRB* measure provides information on the behavior of trajectories starting from a set of initial states which is in principle "physically observable" in practice, say when modeling some physical experiment. That is why the name *physical measure* is also attached to them

This kind of measures was first constructed for C^2 Anosov flows by Anosov and Sinai [9] and later for every Axiom A attractor for C^2 flows and for C^2 diffeomorphisms by Bowen and Ruelle [37, 39]. Moreover if the attractor is transitive (i.e. a basic piece in the spectral decomposition of an Axiom A flow), then there is a unique such measure supported in the attractor whose basin covers a full neighborhood of the attractor except for a volume zero subset. In addition, in the setting of diffeomorphisms these measures are ergodic and mixing (see Section 6.4 for the definition of mixing for an ergodic probability measure).

The existence of physical measures shows that uniformly hyperbolic attractors have well defined asymptotic behavior in a probabilistic sense for Lebesgue almost all points in a neighborhood.

1.4.2 Gibbs measures versus SRB measures

The concept of SRB measure is closely related to the concept of Gibbs measure introduced in the setting of uniformly hyperbolic flows and transformation by Sinai [9, 188] and by Bowen and Ruelle [37, 39].

Recall that for a given flow X the *Lyapunov exponent of x in the direction of $v \in T_xM \setminus \{0\}$* is the number

$$L(x,v) = \liminf_{t\to+\infty} \frac{1}{t} \log \big\|DX^t(x)v\big\|. \tag{1.4}$$

Given an invariant ergodic probability measure μ for the flow X the Multiplicative Ergodic Theorem of Oseledets [139, 204] ensures that for Lebesgue almost every x there exists a DX^t-invariant splitting (for all $t > 0$) $T_xM = E_1 \oplus \cdots \oplus E_k$ and numbers $\lambda_1 < \cdots < \lambda_k$ such that for all $i = 1, \dots, k$ and $v_i \in E_i \setminus \{0\}$

$$\lambda_i = \lim_{t\to+\infty} \frac{1}{t} \log \left\| DX^t(x)v_i \right\|. \tag{1.5}$$

Observe that since M is compact and X is smooth, then we have that the invariant direction given by E^X_z cannot have positive Lyapunov exponent, since for all $t > 0$ and $z \in M$

$$\frac{1}{t} \log \left\| DX^t(z) \cdot X(z) \right\| = \frac{1}{t} \log \left\| X\left(X^t(z)\right) \right\| \leq \frac{1}{t} \log \|X\|_0, \tag{1.6}$$

where $\|X\|_0 = \sup\{\|X(z)\| : z \in M\}$ is a constant. Analogously this direction cannot have positive exponent for negative values of time, thus the Lyapunov exponent along the flow direction must be zero.

Consequently *the flow direction is never tangent to a direction along which all Lyapunov exponents are non-zero*. In particular E^X is never tangent either to a strong-stable or strong-unstable direction.

Absolutely continuous disintegration

In the uniformly hyperbolic setting it is well known that physical measures for uniformly hyperbolic attractors admit a disintegration into conditional measures along the unstable manifolds of almost every point which are absolutely continuous with respect to the induced Lebesgue measure on these sub-manifolds, see [37, 39, 151, 201]. We explain the meaning of this technical notion in what follows.

Assume that an ergodic invariant probability measure μ for the flow X has a positive Lyapunov exponent. In this setting the existence of unstable manifolds through μ-almost every point x and tangent at x to $F_x = \bigoplus_{\lambda_i>0} E_i(x)$ is guaranteed by the non-uniform hyperbolic theory of Pesin [152]: the strong-unstable manifolds $W^{uu}(x)$ are the "integral manifolds" in the direction of the (measurable) sub-bundle F, tangent to F_x at almost every x. The sets $W^{uu}(x)$ are embedded sub-manifolds in a neighborhood of x which, in general, depend only measurably (including its size) on the base point x. Let $W^u(x)$ be the unstable manifold through x whenever the strong-unstable manifold $W^{uu}(x)$ is defined (see Section 1.2). These manifolds are tangent at x to the center-unstable direction $E^X_x \oplus F_x$. Assume that $n = \dim(M)$ and $l = \dim(F)$.

Given $x \in M$ let S be a co-dimension one submanifold of M everywhere transverse to the vector field X and $x \in S$, which we call a *cross-section of the flow at x*. Let ξ_0 be the connected component of $W^u(x) \cap S$ containing x. Then ξ_0 is a smooth submanifold of S and we take a parametrization $\psi : [-\varepsilon, \varepsilon]^l \times [-\varepsilon, \varepsilon]^{n-l-1} \to S$ of a compact neighborhood S_0 of x in S, for some $\varepsilon > 0$, such that

- $\psi(0,0) = x$ and $\psi\big((-\varepsilon, \varepsilon)^l \times \{0^{n-l-1}\}\big) \subset \xi_0$;
- $\xi_1 = \psi\big(\{0^l\} \times (-\varepsilon, \varepsilon)^{n-l-1}\big)$ is transverse to ξ_0 at x: $\xi_0 \pitchfork \xi_1 = \{x\}$.

Consider the family $\Pi(S_0)$ of connected components ζ of $W^u(z) \cap S_0$ which *cross* S_0. We say that a *submanifold* ζ *crosses* S_0 if it can be written as the graph of a map $\xi_0 \to \xi_1$.

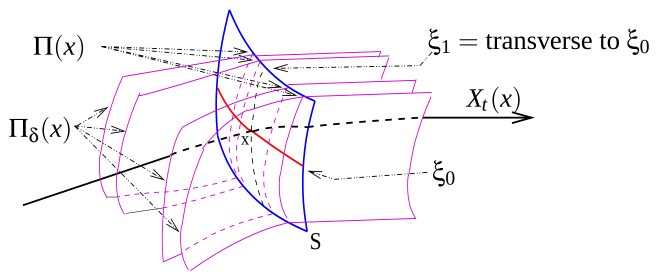

Figure 1.13: Disintegration.

Given $\delta > 0$ we let $\Pi_\delta(x) = \{X_{(\delta,\delta)}(\zeta) : \zeta \in \Pi(S_0)\}$ be a family of co-dimension one submanifolds inside unstable leaves in a neighborhood of x crossing S_0, see Figure 1.13. The volume form Leb induces a volume form Leb_γ on each $\gamma \in \Pi_\delta(x)$ naturally. Moreover, since $\gamma \in \Pi_\delta(x)$ is a measurable family of submanifolds (S_0 is compact and each curve is tangent to a measurable sub-bundle E^{cu}), it forms a measurable partition of $\hat{\Pi}_\delta(x) = \cup\{\gamma : \gamma \in \Pi_\delta(x)\}$. We say that $\Pi_\delta(x)$ is a δ-*adapted foliated neighborhood of x*.

Hence (see [178]) $\mu \mid \hat{\Pi}_\delta(x)$ can be disintegrated along the partition

$\Pi_\delta(x)$ into a family of *conditional measures* $\{\mu_\gamma\}_{\gamma\in\Pi_\delta(x)}$ such that

$$\mu \mid \hat{\Pi}_\delta(x) = \int \mu_\gamma \, d\hat{\mu}(\gamma),$$

where $\hat{\mu}$ is a measure on $\Pi_\delta(x)$ defined by

$$\hat{\mu}(A) = \mu\left(\cup_{\gamma\in A}\gamma\right) \quad \text{for all Borel sets } A \subset \Pi_\delta(x).$$

In this setting we say that μ *has an absolutely continuous disintegration along the center-unstable direction* or a *Gibbs state* if *for μ-almost every $x \in M$, each δ-adapted foliated neighborhood $\Pi_\delta(x)$ of x induces a disintegration $\{\mu_\gamma\}_{\gamma\in\Pi_\delta(x)}$ of $\mu \mid \hat{\Pi}_\delta(x)$, for all small enough $\delta > 0$, such that $\mu_\gamma \ll \mathrm{Leb}_\gamma$ for $\hat{\mu}$-almost all $\gamma \in \Pi_\delta(x)$*. In this setting we also say that μ *is a Gibbs measure* for the flow X.

Note that completely dual properties and definitions can be stated for the strong-stable $W^{ss}(x)$ and stable leaves $W^s(x)$ of μ-almost every point x for a system with an invariant probability measure μ having a negative Lyapunov exponent.

Absolute continuity of foliations

In the same setting above, assume that x has a unstable leaf $W^u(x)$ and let D_1, D_2 be embedded disk in M transverse to $W^u(x)$ at x_1, x_2, that is $T_{x_i}D_i \oplus T_{x_i}W^u(x) = T_{x_i}M$, $i = 1,2$. Then the strong-unstable leaves through the points of D_1 which cross D_2 define a map h between a subset of D_1 to D_2: $h(y_1) = y_2 = W^{uu}(y_1) \cap D_2$, called the *holonomy* map of the strong-unstable foliation between the transverse disks D_1, D_2. The holonomy is injective if D_1, D_2 are close enough due to uniqueness of the strong-unstable leaves through μ-a.e. point.

We say that h is *absolutely continuous* if there is a measurable map $J_h : D_1 \to [0, +\infty]$, called the *Jacobian of* h, such that

$$\mathrm{Leb}_2\left(h(A)\right) = \int_A J_h \, d\,\mathrm{Leb}_1 \ \text{ for all Borel sets } A \subset D_1,$$

and J_h is integrable with respect to Leb_1 on D_1, where Leb_i denotes the Lebesgue measure induced on D_i by the Riemannian metric, $i = 1,2$.

The foliation $\{W^{uu}(x)\}$ is *absolutely continuous* (Hölder continuous) if every holonomy map is absolutely continuous (or J_h is Hölder continuous, respectively).

Since the pioneering work of Anosov and Sinai [8, 9] it became clear that for C^2 transformations or flows (in fact it is enough to have transformations or flows which are C^1 with α-Hölder derivative for some $0 < \alpha < 1$) the strong-unstable foliation is absolutely continuous and Hölder continuous. See also [107]. *When the leaves are of co-dimension one, then the Jacobian J_h of the holonomy map h coincides with the derivative h' since h is a map between curves in M.* In this case the holonomy map can be seen as a $C^{1+\alpha}$ transformation between subsets of the real line.

Going back to the case of the unstable foliation for a flow, see Figure 1.14, we have that for any pair of disks γ_1, γ_2 inside S_0 transverse to $W^u(x) \cap S_0$ at distinct points y_1, y_2, the holonomy H between γ_1 and γ_2 along the leaves $W^u(z) \cap S_0$ crossing S_0 is also Hölder continuous *if the flow is C^2*.

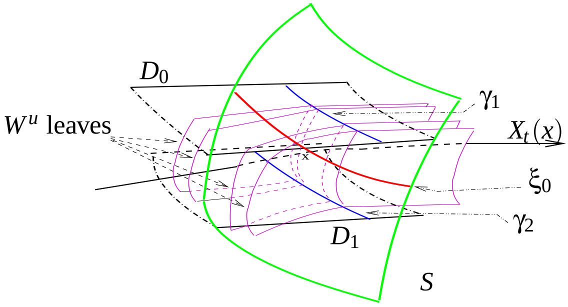

Figure 1.14: The holonomy maps.

Indeed note that this holonomy map H can be obtained as a composition of the holonomy map h between two disks D_1, D_2 transverse to the strong-unstable leaves which cross S_0, and the “projection along the flow” sending $w \in X^{(-\delta,\delta)}(S_0)$ to a point $X^t(w) \in S_0$ uniquely defined, with $t \in (-\delta, \delta)$. The disks are defined simply as $D_i = X^{(-\varepsilon,\varepsilon)}(\gamma_i)$ for $0 < \varepsilon < \delta$ and satisfy $D_i \cap S_0 = \gamma_1$, $i = 1, 2$. Since the the holonomy h is Hölder continuous and the projection along the flow has the same differentiability class of the flow (due to the Tubular Flow Theorem 1.6), we see that the holonomy H is also Hölder continuous.

A very important consequence of absolute continuity is the following.

Lemma 1.22. *Assume that for some given submanifold W of M one knows that through* Leb_W*-almost every point $x \in W$ (*Leb_W *is the induced volume form on W by the volume form* Leb *of M) there passes a strong-stable manifold $W^{ss}(x)$ transverse to W. Then the union of the points of all these strong-stable manifolds has positive volume in M.*

Proof. In a neighborhood of one of its points W can be written as $\mathbb{R}^k \times \{0^{n-k}\}$ and by the transversely assumption on $W^{ss}(x)$ these submanifolds can be written as graphs $\mathbb{R}^{n-k} \to \mathbb{R}^k$ on a neighborhood of 0^{n-k} which depends measurably on $x \in \mathbb{R}^k$. This change of coordinates through some local chart of M affects the derivatives of maps and holonomies at most by multiplication by bounded smooth functions.

The measurability ensures that given $\varepsilon > 0$ we can find $\delta, \alpha > 0$ small enough such that there exists $\Lambda \subset \mathbb{R}^k$ satisfying:

1. $W^{ss}(x)$ is the graph of a map $\gamma_x : B_\delta(0^{n-k}) \to \mathbb{R}^k$ defined on a δ-ball around the origin;

2. the slope of the tangent space to $W^{ss}(x)$ at every point is smaller than α (meaning that $\|D\gamma_x(w)\| \le \alpha$ for all w);

3. $\mathrm{Leb}_W(\Lambda) > 1 - \varepsilon$.

Then the submanifold $W_t = \mathbb{R}^k \times \{t\}$ for $t \in \mathbb{R}^{n-k}$ near 0^{n-k} is transverse to $W^{ss}(x)$ for all $x \in \Lambda$. Thus the holonomy map h_t from a subset of W_t to $W = W_0$ contains Λ in its image, which has positive volume in W_0. By absolute continuity of h_t, the intersection $W_t \cap \cup_x W^s(x)$ has positive volume in W_t. Hence $\mathrm{Leb}(\cup_x W^s(x)) = \int \mathrm{Leb}_{W_t}\left(W_t \cap \cup_x W^s(x)\right) d\,\mathrm{Leb}^{n-k}(t) > 0$, and this concludes the proof. □

Hyperbolic measures, Gibbs property and construction of physical measures

These technical notions have crucial applications in the construction of physical measures for a dynamical system. Indeed, if the measure μ is ergodic and *hyperbolic*, meaning that all Lyapunov exponents are non-zero except the one corresponding to the flow direction, and also a Gibbs measure, then transverse to a center-unstable manifold $W^u(z)$ there exist strong-stable manifolds through μ_γ-almost every point and also through Leb_γ-almost all points $w \in W^u(z)$. Along strong-stable manifolds forward time averages of continuous functions are constant and along center-unstable manifolds backward time averages of continuous functions are constant. Moreover forward and backward time averages are equal μ-almost everywhere and through disintegration and ergodic decomposition, we deduce that μ-almost every z has a strong-unstable manifold $W^{uu}(z)$ where Leb-a.e. point has the same forward and backward time averages.

We are in the setting of Lemma 1.22 thus the absolute continuity of the strong-stable foliation implies that the family of all the strong-stable manifolds through $W^u(z)$ covers a positive Lebesgue measure subset of M *if the flow is of class* C^2. By the previous observations this set is inside the (ergodic) basin of μ. Hence *a hyperbolic ergodic invariant probability measure for a* C^2 *flow which is a Gibbs measure is also a physical measure*.

1.4.3 The Ergodic Closing Lemma

In several proofs in this text we shall use the Ergodic Closing Lemma for flows which shows that any given invariant measure can be approximated by an invariant measure supported on critical elements. The Ergodic Closing Lemma was first proved by Mañé [110] for diffeomorphisms and for flows by Wen [205].

We need the following definition. A point $x \in M \setminus S(X)$ is δ-*strongly closed* if for any C^1 neighborhood $\mathcal{U} \subset \mathfrak{X}^1(M)$ of X, there are $Z \in \mathcal{U}$, $z \in M$ and $T > 0$ such that $Z^T(z) = z$, $X = Z$ on $M \setminus B_\delta(X^{[0,T]}(x))$ and $\operatorname{dist}(Z^t(z), X^t(x)) < \delta$, for all $0 \le t \le T$.

Denote by $\Sigma(X)$ the set of points of M which are δ-strongly closed for any δ sufficiently small.

Theorem 1.23 (Ergodic Closing Lemma, flow version)**.** *Let μ be any X-invariant Borel probability measure. Then* $\mu\big(S(X) \cup \Sigma(X)\big) = 1$.

1.5 Stability conjectures

The search for a characterization of stable systems, from Smale's seminal work in the sixties [190], led to several conjectures some of which are still open.

The famous stability conjecture, by Palis and Smale [144], states that a vector field X is structurally stable if, and only if, the non-wandering set is hyperbolic, coincides with the closure of the set of critical elements, there are no cycles between the stable and unstable manifolds of the critical elements and the intersection between the stable and unstable manifolds of points at the non-wandering set is transverse. In short terms, this conjecture states that *a system is structurally stable if, and only if, its non-wandering set is uniformly hyperbolic, the periodic orbits are dense and it satisfies the strong transversality condition.*

This conjecture was proved in the setting of C^1 diffeomorphisms by the combined work of several people along the years. First Robbin [170] showed that if a diffeomorphisms f is C^2, $\Omega(f)$ is Axiom A and satisfies the strong transversality condition, then f is C^1-structurally stable. Then Wellington de Melo [50] obtained the same result for C^1 diffeomorphisms on surfaces and finally Robinson [173] showed that for C^1 diffeomorphisms on any compact manifold the strong transversality condition plus Axiom A is sufficient for C^1 structural stability. The proof of this conjecture, in the C^1 topology, was completed by Mañé [109, 110, 108] (see also Liao [97] for a proof for surface diffeomorphisms) who showed that C^1-structural stability implies that the non-wandering set is uniformly hyperbolic and satisfies the strong transversality condition.

For flows the proof that uniform hyperbolicity together with strong transversality is sufficient for C^1 structural stability was given by Robinson [171, 172]. Finally, that these conditions are also necessary for structural stability was proved much later by Hayashi [74] using the Connecting Lemma.

Developments in the last decades led Palis to conjecture [141] that the set of dynamical systems exhibiting finitely many attractors is dense in the set of all dynamical systems (in a suitable topology) and, moreover, each attractor supports a physical/SRB measure and the union of the (ergodic) basin of all physical measures covers Lebesgue almost every point of the ambient manifold. This conjecture admits a version for parametrized families where denseness is to be taken in the set of parameters corresponding to finitely many attractors whose basins cover the ambient manifold Lebesgue almost everywhere.

In the context of three-dimensional flows, one has to consider another homoclinic phenomenon involving singularities of the vector field: the situation in which the stable and unstable manifolds of a singularity have intersections other than the singularity itself. In this case, it is said that the vector field has a singular cycle.

In the setting of C^1 surface diffeomorphisms this conjecture was proved true by E. Pujals and M. Sambarino [167]. In the setting of real analytic families of unimodal maps of the interval or the circle, this was obtained by M. Lyubich [106]. In higher dimensions this conjecture is still wide open in spite of much recent progress, see e.g [34] and references therein.

Chapter 2

Singular cycles and robust singular attractors

A cycle Γ for a flow X^t is a finite sequence $\{\sigma_i, 0 \le i \le n\} \subset C(X)$ of hyperbolic critical elements of X^t, with $\sigma_0 = \sigma_n$, such that $W^u(\sigma_j) \cap W^s(\sigma_{j+1}) \neq \emptyset$ for $0 \le j \le n$, that is the unstable manifold of one element intersects the stable manifold of the next element. A cycle is *singular* if at least one of its critical elements is a fixed point of X^t.

Cycles play an important role in the bifurcating theory of Dynamical Systems. A singular cycle is one of the mechanisms to go from a Morse-Smale flow (whose non-wandering set is a finite collection of hyperbolic critical elements) to a hyperbolic flow (whose non-wandering set is a finite collection of basic sets) through a one parameter family of flows.

In this chapter we shall describe three types of singular cycles, that will be used in the sequel. Nowadays the first one, presented in Section 2.1, is denominated *singular-horseshoe*. It was introduced by Labarca and Pacifico in [92] as a model for stable non hyperbolic flows in the context of boundary manifolds. We show that this set satisfies some properties which, in Chapter 3, will be defined as singular-hyperbolicity. This generalization of (uniform) hyperbolicity will characterize a much broader class of invariant sets for flows.

The second cycle is a homoclinic connection associated to a hyperbolic singularity of saddle-type. There are several possibilities for these cycles which are used in the proofs presented in the following chapters. We provide a brief description the dynamics of perturbations of these cycles here. One of them is a inclination flip cycle. This was studied by many authors, see e.g. [80, 44] among others. The study of this type of cycle is crucial for the proof, in Chapter 3, that a robust transitive set with singularities for a 3-flow is either an attractor or a repeller, together with the Shil'nikov bifurcation, first considered in [185]. These are presented in Section 2.2.

Finally the third one is the Lorenz geometrical model, introduced by Guckenheimer and Williams [65] and presented in Section 2.3. This is a model for a robust attractor with singularities for a 3-flow, as we will see in Chapter 3.

2.1 Singular horseshoe

We start in Section 2.1.1 with the description of a map defined on a rectangle into itself which resembles the Smale horseshoe map [190]. For this reason this type of map is nowadays denominated singular horseshoe.

Afterward, in Section 2.1.2, we exhibit a singular cycle presenting a singular horseshoe map as a first return map. Then, in Section 2.1.3, we show in several stages that the singular horseshoe is a transitive partially hyperbolic set with volume expanding central direction.

2.1.1 A singular horseshoe map

Given $\delta > 0$ small enough, $\lambda < 1/2$ and $\mu > 1$, let $Q = [-1,0] \times [0,1+\delta]$ and define

$$R_\delta = Q \setminus \big((\mu^{-1}(1+\delta), 1/2-\delta) \times (1/2,1)\big).$$

Let $F : R_\delta \to Q$, $(x,y) \mapsto (g(x,y), f(y))$ be a smooth map satisfying:

(a) $|\partial_x g(x,y)| < 1/2$ for all $(x,y) \in R_\delta$ and

$$g(x,y) = \lambda \cdot x \quad \text{for} \quad 0 \le y \le \mu^{-1}(1+2\delta).$$

(b) $f : I \setminus (J \cup K) \to I$ where $I = [0,1]$, $J = (\mu^{-1}(1+2\delta), 1/2-\delta)$ and $K = (1/2,1)$ satisfying

(i) $f(y) = \mu \cdot y$ for $0 \le y \le \mu^{-1}(1+2\delta)$,

(ii) $f'(y) >> \mu$ for $y \in [1/2-\delta, 1/2] \cup [1, 1+\delta]$.

(c) $F(x,1) = F(x,1/2) = (\alpha, 0)$ for $-1 \le x \le 0$ with a fixed $-1 < \alpha < \lambda$.

(d) the following sets

$$\gamma_{-1} = F(\{-1\} \times (1, 1+\delta]), \qquad \gamma_0 = F(\{0\} \times (1, 1+\delta]),$$
$$\beta_0 = F(\{0\} \times [1/2-\delta, 1/2)), \quad \beta_{-1} = F(\{-1\} \times [1/2-\delta, 1/2))$$

are disjoint C^1 curves, except for the point $(\alpha, 0)$ where all are tangent. These curves are contained in $(-1, -\lambda) \times [0, 1+\delta]$ and are transverse to the horizontal lines. Moreover, if $d(A,B)$ denotes the distance between the sets A and B, and $L = \{-1\} \times [0, 1+\delta]$ then

$$d(\gamma_{-1}, L) < d(\gamma_0, L) < d(\beta_0, L) < d(\beta_{-1}, L).$$

Figure 2.1 displays the main features of the map F.

Observe that, by construction, the horizontal lines $\{x\} \times [0, 1+\delta]$, for $x \in [-1, 0]$, are invariants by F. They are also uniformly contracted by a factor $0 < c_0 < 1/2$. This guaranties that Q has a uniformly contracted (strong-)stable foliation invariant by F that we denote by $\mathcal{F}^{ss}(Q)$.

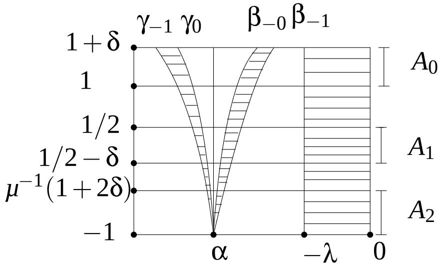

Figure 2.1: A singular-horseshoe map

Define the following rectangles

$$A_0 = [-1, 0] \times [1, 1+\delta], \qquad A_1 = [-1, 0] \times [1/2-\delta, 1/2],$$
$$A_2 = [-1, 0] \times [0, \mu^{-1}(1+2\delta)].$$

Note that

$$R_\delta = \bigcup_{i=0}^{i=1} A_i \quad \text{and define} \quad \Omega_F = \bigcap_{n\in\mathbb{Z}} F^n(R_\delta)\,.$$

It is clear that $F^{-1}(\Omega_F) = \Omega_F$.

Singular symbolic dynamics

We now associate a symbolic dynamics to the restriction $F \mid \Omega_F$. For this, consider a map $\widetilde{F} : R_\delta \to Q$ such that $\widetilde{F}$ has the same properties described for F, except that $\widetilde{F}([-1,1]\times\{1\})$ and $\widetilde{F}([-1,1]\times\{1/2\})$ are disjoint intervals I and J contained in the interior of $[-1,\lambda]\times\{0\}$ as in Figure 2.2. Define $\Omega = \cap_{n\in\mathbb{Z}}\widetilde{F}^n(R_\delta)$.

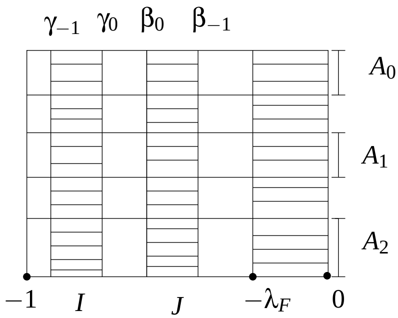

Figure 2.2: A Smale horseshoe map

Clearly $\widetilde{F}$ is a Smale horseshoe map. Roughly speaking, F is obtained from $\widetilde{F}$ pinching the intervals I and J into a unique point in such a way that the resulting boundary lines $\widetilde{\gamma}_{-1}, \widetilde{\gamma}_0, \widetilde{\beta}_0$, and $\widetilde{\beta}_{-1}$ are tangent at this point.

Let Σ^3 be the set of doubly infinite sequences of symbols in $\{0,1,2\}$ endowed with the topology given by the distance

$$d(x,y) = \sum_{i\in\mathbb{Z}} \frac{|x_i - y_i|}{3^{|i|}}$$

and $\sigma : \Sigma^3 \to \Sigma^3$ be the left shift map $\sigma(x)_i = x_{i+1}$.

It is well known (see e.g. [190] but also the textbooks of e.g. Devaney [52] or Robinson [177]) that there exists a homeomorphism $\widetilde{H} : \Omega \to$

Σ^3 which conjugates $\widetilde{F}$ and σ, i.e. $\widetilde{H}\circ\widetilde{F}=\sigma\circ\widetilde{H}$. The image $\widetilde{H}(x)$ of $x\in\Omega$ is the sequence $(\widetilde{H}(x)_i)\in\Sigma^3$ defined by

$$\widetilde{H}(x)_i = j\in\{0,1,2\} \iff \widetilde{F}^i(x)\in A_j, \quad i\in\mathbb{Z}. \tag{2.1}$$

Recall that the set of periodic orbits for σ is dense in Σ^3 and that there exists a dense orbit.

We now describe the sequences associated, in a similar way, to points Ω_F.

Observe that the tangency point $(\alpha,0)$ is the unique point of Ω_F outside of $[-\lambda,0]\times[0,1+\delta]$ which remains forever in the bottom boundary of Q. This line corresponds to the local stable manifold of the fixed point $(0,0)$ of F.

- Since $[-1,0]\times\{0\}=\cap_{n\le 0}\widetilde{F}^n(A_2)$ we have $z\in[-1,0]\times\{0\}\cap\Omega$ if, and only if, $\theta_i(z)=2$ for all $i\ge 0$, i.e. $\widetilde{H}(z)=(\dots,x_{-1},2,2,2,\dots)$.

The points belonging to this line which are outside of $[-\lambda,0]\times[0,1+\delta]$ are the points of the local stable manifold of $(0,0)$ which are different from $(0,0)$, i.e. their corresponding codes differ from the constant sequence $x_i\equiv 2$ at some coordinate with negative index. Defining Σ^3_* the subset of Σ^3 of those sequences $(x_i)_{i\in\mathbb{Z}}$ with $x_0\in\{0,1\}$ and $x_i=2$ for all $i\ge 1$, then

$$W^s_{\widetilde{F}}\big(\widetilde{H}(0,0)\big)\setminus\widetilde{H}\big([-\lambda,0]\times[0,1+\delta]\big)=\bigcup_{k\ge 1}\sigma^k\Sigma^3_*=\widetilde{\Sigma}^3_*$$

Note that $\sigma^{-1}\widetilde{\Sigma}^3_*\subseteq\widetilde{\Sigma}^3_*$. Defining an equivalence relation on Σ^3 by $\theta\sim\widetilde{\theta}$ if and only if $\theta,\widetilde{\theta}\in\widetilde{\Sigma}^3_*$, then this relation is preserved by the shift.

Let $\widetilde{\Sigma}^3$ be the corresponding quotient space and $\widetilde{\sigma}$ the associated quotient shift map. This map can be seen as the original full shift map on three symbols after identifying the sequences on $\widetilde{\Sigma}^3_*$, which correspond to the points which are taken to $(\alpha,0)$ by F.

By the above considerations and the dynamics of F we get

Lemma 2.1. *There exists a homeomorphism $H_F:\Omega_F\to\widetilde{\Sigma}^3$ which conjugates $F\mid\Omega_F$ and H_F, that is $H_F\circ(F\mid\Omega_F)=\widetilde{\sigma}\circ H_F$.*

The homeomorphism H_F is defined just as in (2.1) replacing $\widetilde{F}$ by F.

Observe that the set of periodic orbits for $\widetilde{\sigma}$ is the same set of periodic orbits for σ. Note also that the dense orbit for σ is not contained in $\widetilde{\Sigma}^3_*$.

Therefore the set of periodic orbits for $\widetilde{\sigma}$ is dense in $\widetilde{\Sigma}^3$ and this space contains a dense orbit. The existence of the conjugation above ensures that Ω_F has a dense subset of periodic orbits and a dense orbit for the dynamics of F.

2.1.2 A singular cycle with a singular horseshoe first return map

We start by giving a definition of a special type of singularity of a vector field X in a 3-manifold.

Definition 2.1. We say that a singularity σ of a 3-flow X^t is Lorenz-like if the eigenvalues λ_i, $1 \leq i \leq 3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1. \tag{2.2}$$

Next we shall exhibit a singular cycle $\mathcal{C}$ having a Lorenz-like singularity p and a hyperbolic saddle-type closed orbit σ, connected through a branch of the unstable manifold associated to p: this branch is contained in the stable manifold associated to σ. Moreover there are two orbits of transverse intersection between $W^s(p)$ and $W^u(\sigma)$. The cycle will be constructed in such away that it is contained in the maximal invariant set $\Lambda(X)$ of a vector field X in a neighborhood U of $\mathcal{C}$, and the first return map associated to $\mathcal{C}$ is a singular horseshoe map, see Figure 2.3.

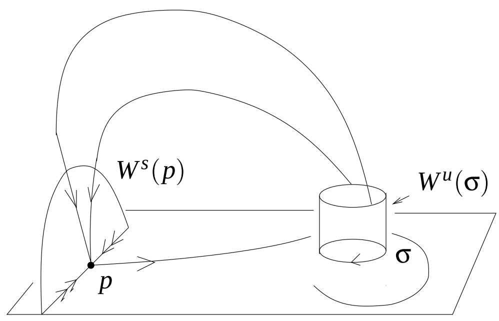

Figure 2.3: A singular cycle

We start with a vector field $X_0 \in \mathfrak{X}^r(\mathbb{D}^3)$ on the 3-disk $\mathbb{D}^3$ in $\mathbb{R}^3$. This vector field has one repeller singularity r_1 at the north pole. Outside a neighborhood of r_1, X_0 has four singularities which we denote by p, p_1, p_2, r_2, plus a hyperbolic closed orbit σ. These satisfy the following:

1. p is a Lorenz-like singularity.

2. (p,σ) is a saddle connection with a branch $\gamma^u(p)$ of $W^u(p) \setminus \{p\}$ whose ω-limit set is σ. By the Hartman-Grobman Theorem there exists a neighborhood $p \in \mathcal{N} \subset \mathbb{R}$ such that the restriction of X_0 to $\mathcal{N}$ is equivalent to the linear vector field $L(x_1,x_2,x_3) = (\lambda_2 x_1, \lambda_1 x_2, \lambda_3 x_3)$.

3. p_1 is an attractor and is also the ω-limit set of the other branch of $W^u(p) \setminus \{p\}$.

4. p_2 is an attractor and is the ω-limit of $W^u(\sigma) \setminus \{\sigma\}$.

5. r_2 is a repeller contained in the interior of the 2-disk $\mathbb{D}^2$ bounded by σ in $\mathbb{S}^2$.

6. We assume that

   (a) $p_1, p, \gamma^u(p), \sigma$ and $\mathbb{D}^2$ are contained in the boundary $\partial(\mathbb{D}^3) = \mathbb{S}^2$ of the 3-disk;

   (b) the eigenvalues of $DX_0(r_2)$ corresponding to eigenvectors in $T\mathbb{S}^2$ are complex conjugates. Therefore the part of $W^u(r_2) \setminus \{r_2\}$ in $\mathbb{S}^2$ is a spiral whose ω-limit set is σ.

   (c) the strong unstable manifold $W^{uu}(r_2) \setminus \{r_2\}$ is contained in the interior of $\mathbb{D}^3$ and its ω-limit set is the attractor p_2.

7. The α-limit set of $W^s(p) \setminus \{p\}$ is the repeller r_1 and $W^s(p)$ separates the two attractors.

Figure 2.4 shows the essential features of the vector field X_0 outside a neighborhood of r_1. Observe that X_0 constructed in this way is a Morse-Smale vector field.

Now we can modify the vector field X_0 away from its critical elements, in particular away from the neighborhood $\mathcal{N}$ of p, in order to produce a unique tangency between $W^s(p)$ and $W^u(\sigma)$, see Figure 2.5.

By another slight perturbation of the above vector field we get a vector field X such that $W^u(\sigma)$ is transverse to $W^s(p)$ at two orbits, see Figure 2.6.

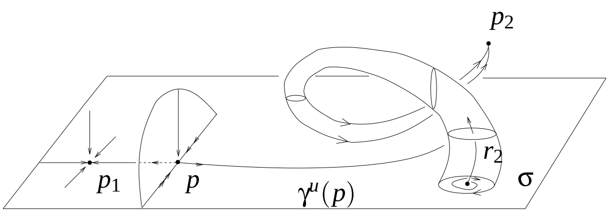

Figure 2.4: The vector field X_0.

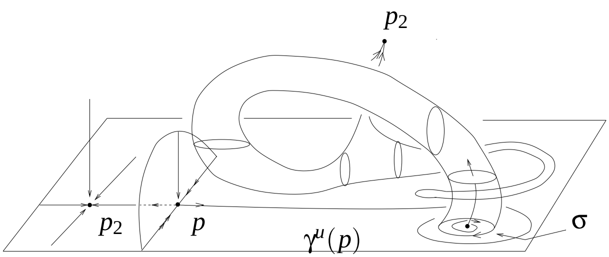

Figure 2.5: Producing a unique tangency.

The first return map associated to $\mathcal{C}$ is a singular horseshoe map

Now we study the first return map associated to $\mathcal{C}$ and show that it is a singular horseshoe map.

Let S be a cross section to X at $q \in \sigma$. Reparametrizing X, if necessary, we can assume that the period of σ is equal to one and that S is invariant by X^1: there exists a small neighborhood $U \subset S$ of q such that $X^1(S \cap U) \subset S$.

Since there are two orbits of transverse intersection of $W^u(\sigma)$ with $W^s(p)$ and the branch $\gamma^u(p)$ has σ as ω-limit set, there exists a first return map F defined on subsets of S, taking points of S back to S under the action of the flow. The goal now is to describe F.

From now we assume mild non-resonant conditions on the eigenvalues of p to ensure that there are C^1 linearizing coordinates (x_1, x_2, x_3) in a neighborhood U_0 containing p.

Let $D^s(p) \subset U_0$ and $D^u(p) \subset U_0$ be fundamental domains for the action of the flow inside $W^s(p)$ and $W^u(p)$ respectively. That is $D^s(p)$ is a circle in $W^s(p) \setminus \{p\}$ containing p in its interior and transverse to X, and $D^u(p)$

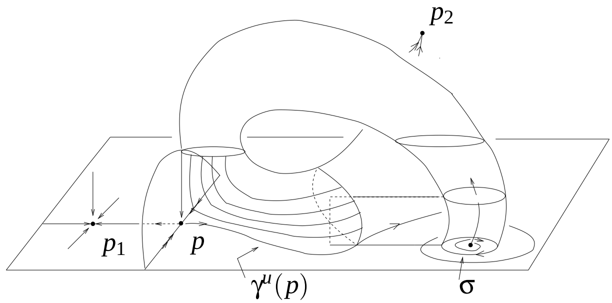

Figure 2.6: One point of tangency

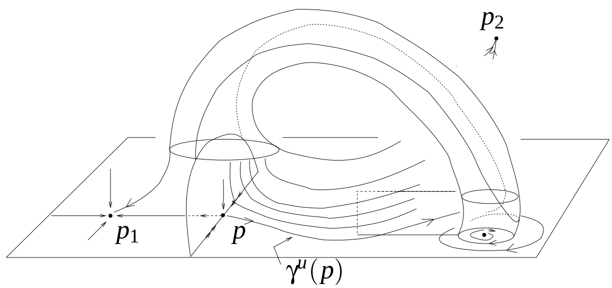

Figure 2.7: The final vector field.

is a pair of points, one in each branch of $W^u(p)\setminus\{p\}$.

Let $C^s(p)\subset U_0$ be a cross section to X, as in Figure 2.8, with several components: $C^s = C^s(p) = C^+(p)\cup D^s(p)\cup C^-(p)$. We assume that $C^-(p)$ is contained in the stable manifold of the attractor p_1. We also assume that the plane $\{x_1=0\}$ is a center-unstable manifold for p and we denote it by $W^{cu}(p)$. Let $C^u(p)$ be a cross section to X formed by a 2-disk through the point of $\gamma^(p)\cap D^u(p)$.

Observe that if γ is a C^1 curve transverse to $D^s(p)$ and $\gamma\cap W^{ss}(p)=\emptyset$, then

$$C^u(p)\cap\Big(\bigcup_{t\geq 0}X^t(\gamma)\Big)$$

is a C^1 curve tangent to $W^{cu}(p)\cap C^u(p)$ at $D^u(p)\cap\gamma^u(p)$.

Let $D^s(p_2)\subset\mathbb{D}^3$ be a fundamental domain for the dynamics on $W^s(p_2)$,

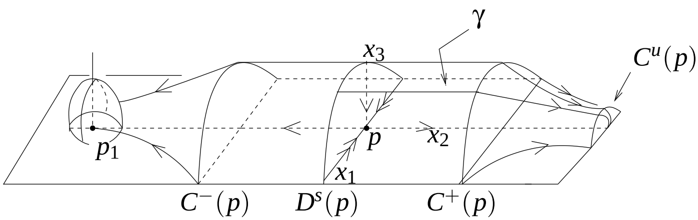

Figure 2.8: The cross section C^s at p.

i.e. the boundary of a 3-ball containing p_2. Let $V \subset S$ be a small neighborhood of $q \in \sigma$, where we have C^1 linearizing coordinates (x,y) for the Poincaré first return map F associated to σ. The eigenvalues of $DF(q)$ are λ, μ both bigger than 1.

Let $Q = [-1,1] \times [0,1]$ be a rectangle contained in the interior of V. Assume that

$$[-1,1] \times \{\frac{1}{2}, 1\} \subset W^s(p) \quad \text{and} \quad [-1,1] \times \{0\} \subset \mathbb{S}^2.$$

There are only two orbits of transverse intersection between $W^u(\sigma)$ and $W^s(p)$, and the points in $\{1\} \times (1/2,1)$ will fall in the stable set of p_1, by construction of the vector field X. Since $W^s(p_1)$ is open we can assume that $[-1,1] \times (1/2,1) \subset W^s(p_1)$ (taking V small enough) and also

$$X^1([-1,1] \times (1/2,1)) \subset C^-(p)$$

through a reparametrization of time if necessary. Assume further that there exists $\delta > 0$ such that $(1+2\delta)\mu^{-1} < 1/2 - \delta$ and

(a) for $A_0 = [-1,1] \times (1, 1+\delta]$ we have $X^1(A_0) \subset C^+(p)$;

(b) for $A_1 = [-1,1] \times [1/2-\delta, 1]$ we have $X^1(A_1) \subset C^+(p)$;

(c) $X^2\big([-1,1] \times [1+\delta, 1+2\delta]\big) \subset D^s(p_2)$;

(d) $X^2\big([-1,1] \times [1/2-2\delta, 1/2-\delta]\big) \subset D^s(p_2)$;

(e) for $A_2 = [-1,1] \times [0,(1+2\delta)\mu^{-1}]$ we have

$$X^1(A_2) = [-\lambda,0] \times [0,1+2\delta] \subset Q.$$

Now define

$$H_1(X) = \bigcup_{t\geq 0} X^t(X^1(A_0)) \cap C^u(p), \quad H_2(X) = \bigcup_{t\geq 0} X^t(X^1(A_1)) \cap C^u(p)\,.$$

Clearly $H_i(X)$ are cones tangent to $W^{cu}(p) \cap C^u(p)$ at $D^u(p)$ for $i = 1,2$, see Figures 2.9 and 2.10.

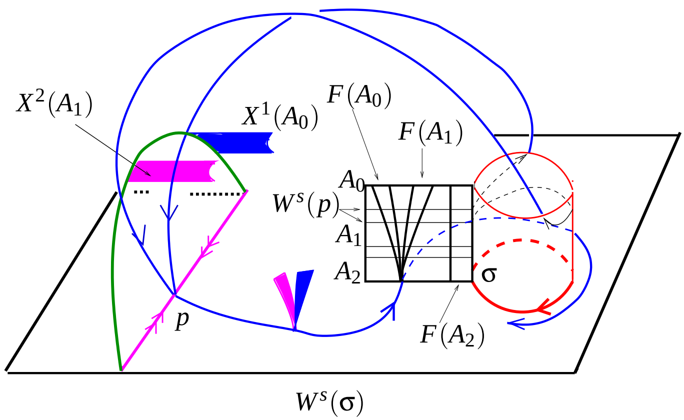

Figure 2.9: The first return map to Q.

Let α be the first intersection point between $W^u(p)$ and Q. We can assume that $\tau_i(X) = X^1(H_i(X))$ is contained in Q and that these sets are cones tangent to $W^{cu}(p) \cap Q$ at α, for $i = 1,2$.

Clearly we can also assume that

$$X^3\big([-1,1] \times \{1+\delta\}\big) \subset \tau_1(X) \quad \text{and} \quad X^3\big([-1,1] \times \{\tfrac{1}{2} - \delta\}\big) \subset \tau_2(X).$$

If necessary, we modify the vector field X in order to have (see Figure 2.11):

(a) horizontal lines $\{y = constant\}$ going to horizontal lines in $\tau_i(X)$;

(b) writing π_y for the projection on the y-axis in V

$$\pi_y\Big(X^3\big([-1,1]\times\{1+\delta\}\big)\Big)=\{1+2\delta\} \quad \text{and}$$
$$\pi_y\Big(X^3\big([-1,1]\times\{\frac{1}{2}-\delta\}\big)\Big)=\{1+2\delta\};$$

(c) for $D^s_\sigma=[-1,\lambda]\times[0,1+\delta]$ we have $\tau_i(X)\subset \operatorname{int} D^s_\sigma$ for $i=1,2$.

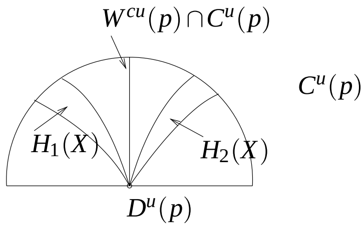

Figure 2.10: The first return map at $D^u(p)$

Now we describe the first return map F.

- If we take a point (x,y) with $1+\delta<y\le 1+2\delta$, then (x,y) is contained in the stable manifold of the attractor p_2 and F is not defined at these points.
- For either a point $(x,1)\in Q$ or $(x,1/2)\in Q$ we define $F(x,1)=\alpha=F(x,1/2)$.
- For points $(x,y)\in Q$ with $0\le y\le \mu^{-1}(1+2\delta)$ we define $F(x,y)=(\lambda x,\mu y)$.
- For points (x,y) such that either $1<y\le 1+\delta$ or $1/2-\delta\le y<1/2$, we define $F(x,y)$ as the first intersection of the positive orbit through (x,y) with the rectangle $Q_\delta=[-1,1]\times[0,1+2\delta]$.
- For points (x,y) with $1/2<y<1$ the first return F is not defined, since these points are in the stable manifold of the attractor p_1.
- F is also not defined for points (x,y) with $\mu^{-1}(1+2\delta)<y<1/2-\delta$. Indeed, these points are such that the projection on the y-axis of their first return to S is larger than $1+2\delta$. So these points return once to S and then they are taken to the attractor p_2.

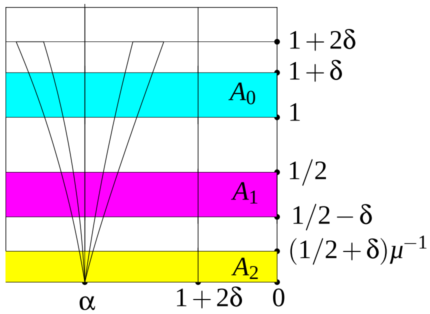

Figure 2.11: The singular horseshoe return map.

Then the first return map F has the expression:

$$F(x,y)=\begin{cases}(\lambda x,\mu y) & \text{if} \quad 0\le y\le \mu^{-1}(1+2\delta)\\ (g_1(x,y),f_1(y)) & \text{if} \quad 1\le y\le 1+\delta\\ (g_2(x,y),f_2(y)) & \text{if} \quad 1/2-\delta\le y\le 1/2\end{cases}$$

with

- $g_i(x,y)$ is some smooth function with $|\partial_x g_i|<c<\frac{1}{2}$, and
- f_i is a smooth function satisfying $f_i'(y)>\mu$ and $0\le f_i(y)\le 1+2\delta$, for $i=1,2$.

We assume that the image $F(\{0\}\times[0,1+\delta])$ is transverse to the horizontal lines in Q_δ.

The non-trivial dynamics of F is concentrated in the square Q_δ.

Let $\Omega_F=\cap_{n\ge0}\overline{F^n(Q_\delta)}$. Observe that the non-wandering set $\Omega(X)$ is the disjoint union of the critical elements $\{r_1,r_2,p_1,p_2\}$ and Λ, where Λ is the closure of the saturation by the flow X^t of the non-wandering set of the first return map F described above, i.e. $\Lambda=\overline{\cup_{t\ge0}X^t(\Omega_F)}$.

The set Λ is the maximal invariant set containing the singular cycle $\mathcal{C}$ in the neighborhood U chosen at the beginning of the construction. This invariant set is the so called *singular horseshoe*.

Remark 2.2. On the boundary of the manifold $\mathbb{D}^3$, which is preserved by the flow, we have a Morse-Smale system. Hence any vector field Y close to X preserving the boundary will have the same features as X on the boundary.

Moreover the features of X depend on the transverse intersection of certain invariant manifolds of the hyperbolic critical elements, all of which lie on the boundary of the ambient manifold. Hence every vector field close to X preserving the boundary will exhibit the same critical elements and the same transversality relations between them, so *the singular-horseshoe is robust among the vector fields which preserve the boundary of* $\mathbb{D}^3$.

2.1.3 The singular horseshoe is a partially hyperbolic set with volume expanding central direction

We start by constructing local stable and unstable manifolds through points of Ω_F with respect to F. The stable and unstable foliation of the singular horseshoe Λ is the obtained as the saturation by the flow of these manifolds. Then we explain how to obtain the strong-stable foliation. Having these foliations we can define a splitting of the tangent space at Λ which will behave much like a hyperbolic splitting.

Stable manifold for points in Ω_F

Let $F : Q_\delta \to Q$ be the singular horseshoe map defined in the previous subsection.

It is easy to see that any horizontal line crossing Q is uniformly contracted by a factor of $c \in (0, 1/2)$ by the definition of F. Then, given any pair of points x, y of Ω_F in the same horizontal line one has

$$\operatorname{dist}\big(F^k(x), F^k(y)\big) \le c^k \xrightarrow[k\to+\infty]{} 0.$$

Hence these curves are the local stable manifolds through points of Ω_F with respect to F. Saturating these curves by the flow we obtain the foliation of stable manifolds $\mathcal{F}^s$ through the points of the singular horseshoe.

For the particular case of the saddle singularity p and the periodic orbits σ the stable leaves are given by the stable manifolds of these hyperbolic critical points.

Unstable manifolds for points of Ω_F

Define $R_0 = Q \cap F(A_0)$, $R_1 = Q \cap F(A_1)$ and $R_2 = Q \cap F(A_2)$. Then R_0 and R_1 are, except for their vertexes, disjoint cones. R_2 is a rectangle, crossing Q from bottom to top.

For each $i, j \in \{0,1,2\}$, let $R_{ij} = R_i \cap F(R_j)$. Then $F(R_j) = \cup_{i=0}^{2} R_{ij}$. Since $F(x,y) = (g(x,y), f(y))$ with $|g_x(x,y)| < c < 1/2$, we have that the horizontal lines are contracted by a factor of c when iterated by F. Thus, except for R_{22} (which is a rectangle strictly contained in R_2), R_{ij} is a cone strictly contained in R_i.

Inductively, given any sequence of n-symbols $x_1, x_2 \cdots, x_n$ with $x_i \in \{0,1,2\}$ and $n \geq 2$ define $R_{ix_1x_2\cdots x_n} = R_i \cap F(R_{x_1\cdots x_n})$ for $i = 0,1,2$. Then

$$F(R_{x_1\cdots x_n}) = \bigcup_{i=0}^{2} R_{ix_1\cdots x_n}\,.$$

Note that

- If all the x_i are equal to 2, then

$$R_2, \quad R_2 \cap F(R_2), \quad \cdots, \quad R_2 \cap F(R_2) \cap \cdots \cap F^n(R_2)$$

  is a strictly decreasing sequence of rectangles converging, in the C^1 topology, to the vertical line $\{0\} \times [0, 1+\delta]$.

- If there is any $x_{i_0} \in \{0,1\}$, then the sequence

$$R_{x_0}, \quad R_{x_0} \cap F(R_{x_1}), \quad \cdots, \quad R_{x_0} \cap F(R_{x_1}) \cap F^2(R_{x_2}) \cap \cdots \cap F^n(R_{x_n})$$

  is a strictly decreasing sequence of C^1-cones. Hence this sequence converges to a C^1 curve, denoted by $\gamma(x_0, x_1, \cdots)$, which crosses Q from top to bottom, that is, γ intersects each horizontal line of Q in a unique point, see Figure 2.12.

Note that every point $x \in \Omega_F \setminus \{(\alpha, 0)\}$ has a corresponding code $H_F(x)$ in $\tilde{\Sigma}^3$ whose coordinates with positive index define a unique regular curve $\gamma = \gamma(x_1, x_2, \ldots)$ as above. This curve γ is the same for every $y \in \Omega_F$ having a code $H_F(y)$ with the same coordinates as $H_F(x)$ at positive indexes. Such points y form the unstable manifold of x with respect to F, since $d\big(\sigma^{-k}H_F(x), \sigma^{-k}H_F(y)\big) \xrightarrow[k\to+\infty]{} 0$.

Indeed, from the description of the map F, it is clear that γ is expanded by all iterates of F whenever its image is defined. Or, reversing time, by the construction of γ, the pre-image of any pair of points $y, z \in \gamma$ by F^k is well

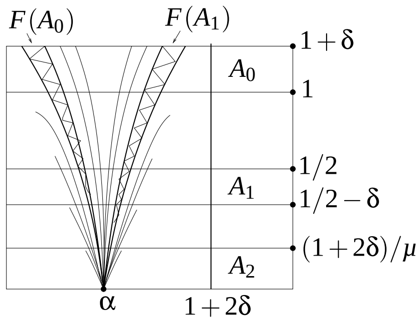

Figure 2.12: The unstable curves of Ω_F tangent at $(\alpha,0)$.

defined for all $k \geq 1$ and, moreover, for any pair y_{-k}, z_{-k} of such pre-images under the same sequence of inverse branches of F satisfies

$$\operatorname{dist}\left(y_{-k}, z_{-k}\right) \leq c^k \xrightarrow[k\to+\infty]{} 0.$$

Saturating these curves by the flow we obtain the central-unstable foliation $\mathcal{F}^u$ through the points of Λ.

The point $(\alpha,0)$ has already a well defined unstable manifold: the vertical line crossing Q through $(\alpha,0)$, corresponding to the intersection of the unstable manifold of the orbit of $W^u(p)$ connecting the saddle singularity p to the periodic orbit σ, see Figure 2.9.

In addition, the saddle singularity p and the periodic orbit σ also have a well defined unstable foliation compatible with the leaves defined above.

Strong-stable foliation for the singular-horseshoe

The previous observations show that *every periodic orbit of F on Ω_F is hyperbolic of saddle-type*. Since F is the Poincaré first return map to Q of the flow X, we deduce that *every periodic orbit of X in Λ is hyperbolic of saddle-type*. Moreover the density of periodic orbits for $F \mid \Omega_F$ implies that *the family of periodic orbits of X in Λ is dense in Λ*.

In addition the stable foliation of the periodic orbits coincides with the stable foliation defined above for all points, including the singularity p and the periodic orbit σ. Hence the strong-stable leaves $\widetilde{\mathcal{F}}^{ss}$ defined on the periodic orbits extend (by continuity and coherence) to a strong-stable foliation $\mathcal{F}^{ss}$ defined throughout Λ. Notice that at the singularity p the strong-stable foliation coincides with its strong-stable manifold corresponding to the weakest contracting eigenvalue.

Partial hyperbolicity

The flow invariance of the stable $\mathcal{F}^s$, strong-stable $\mathcal{F}^{ss}$ and unstable $\mathcal{F}^u$ foliations through points of Λ and the smoothness of their leaves enables us to define the following DX invariant sub-bundles: for every point $z \in \Lambda$

$$E_z = T_z\mathcal{F}^{ss}(z) \quad \text{and} \quad F_z = T_z\mathcal{F}^u(z)$$

satisfy $DX^t \cdot E_z = E_{X^t(z)}$ and $DX^t \cdot F_z = F_{X^t(z)}$, for all $t \in \mathbb{R}$.

Now we show that the flow X contracts E uniformly, and contracts strongly than any contraction along F. Then we conclude by showing that X expands volume along F.

Let V be a neighborhood of p where linearizing coordinated are defined. Assume without loss of generality that $X^1(Q) \subset V$. In V the solutions of the linear flow can be given explicitly as in (2.3).

Write $J_t^c(z)$ for the absolute value of the determinant of the linear map $DX^t \mid F_z : F_z \to F_{X^t(z)}$ where z is any point of Λ and $t \in \mathbb{R}$.

For points z in $X^1(Q)$ and for $s > 0$ such that $X^t(z)$ remains in V for $0 \le t \le s$ we have

- $\left\|DX^t \mid E_z\right\| = e^{\lambda_2 t}$;
- $\left\|DX^t \mid E_z\right\| = e^{(\lambda_2-\lambda_3)t} \cdot m(DX^t \mid F_z)$;
- $|\det DX^t \mid F_z| = e^{(\lambda_1+\lambda_3)t}$,

where $m(\cdot)$ denotes the minimum norm of the linear map. Note that because $\lambda_1 + \lambda_3 > 0$ the flow in V expands volume along the F direction. Moreover since $\lambda_2 < \lambda_3$ the flow contracts along the E direction strongly than it expands along the F direction, by the second item above. We say that F *dominates* E, see Chapter 3 for more on dominated splitting. Observe that the above properties are also valid for the singularity p and the periodic orbit σ.

In what follows we extend these properties for the action of X on points of Λ for all times.

Notice that the flow takes a finite amount of time, bounded from above and from below, to take points in Q to $X^1(Q)$, and from $D^u(p)$ to Q (these times are constant and equal to 1 in our construction).

Hence if we are given a point $z \in \Lambda \setminus \{p, \sigma\}$, then its negative orbit $X^{-t}(z)$ for $t > 0$ will have consecutive and alternate hits on $D^u(p)$ and Q, at times $t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < \dots$ respectively, with $t_0 = s_0 = 0$ and $r_n = |t_{n+1} - s_n|$ bounded from below by T_0 independently of $n \geq 1$.

Let $B > 0$ be a upper bound on $\|DX^{-t}(z)\|$ from 0 to T_1 and for all $z \in \Lambda$. Then from the volume expansion on V we have for $t_n < t \leq s_n$

$$
\begin{aligned}
|\det DX^{-t} \mid F_z| &\leq \exp\Big(B \cdot n - (\lambda_1 + \lambda_3) \cdot \big(t - \sum_{i=1}^{n-1} r_i\big)\Big) \\
&= \exp\Big(t \cdot (\lambda_1 + \lambda_3) \cdot \big(\frac{Bn}{t(\lambda_1 + \lambda_3)} - 1 + \frac{\sum_{i=1}^{n-1} r_i}{t}\big)\Big).
\end{aligned}
$$

Since $t > T_0 n$ and $\sum_{i=1}^{n-1} r_i < t$ we see that there exists $K > 0$ such that $|\det DX^{-t} \mid F_z| \leq K^{-1} \cdot e^{(\lambda_1 + \lambda_3)t}$, which is equivalent to volume expansion.

The uniform contraction along E and the domination of F over E are obtained by similar arguments, see also Section 2.3.3.

2.2 Bifurcations of saddle-connections

An homoclinic orbit associated to a singularity σ of $X \in \mathfrak{X}^1(M)$ is a regular orbit $\mathcal{O}(q)$ satisfying $\lim_{t \to +\infty} X^t(q) = \sigma$ and $\lim_{t \to -\infty} X^t(q) = \sigma$. Here we focus on the dynamics close to $\mathcal{O}(q)$ for small perturbations of the flow.

2.2.1 Saddle-connection with real eigenvalues

Consider the following one-parameter system of ordinary differential equations in $\mathbb{R}^3$

$$
\begin{cases}
\dot{x} &= \lambda_1 x + f_1(x,y,z;\mu) \\
\dot{y} &= \lambda_2 y + f_2(x,y,z;\mu) \\
\dot{z} &= \lambda_3 z + f_3(x,y,z;\mu)
\end{cases}
\qquad (x,y,z,\mu) \in \mathbb{R}^4
$$

where f_i are C^2 functions which vanish together with Df_i at the origin of $\mathbb{R}^4$. So $\sigma = (0,0,0)$ is a singularity. We assume that the eigenvalues λ_i, $i = 1,2,3$ of σ are real and $\lambda_2 \leq \lambda_3 < 0 < \lambda_1$.

Note that any other case of a hyperbolic saddle singularity *with only real eigenvalues* for a three-dimensional flow can be reduced to the present case by considering the time reversed flow.

The hyperbolicity of σ ensures the existence of C^1 stable $W^s(\sigma)$ and unstable manifolds $W^u(\sigma)$. The manifold $W^s(\sigma)$ is tangent at σ to the eigenspace $\{0\} \times \mathbb{R}^2$ associated to the eigenvalues λ_2, λ_3, and $W^u(\sigma)$ is tangent at σ to the eigenspace $\{(0,0)\} \times \mathbb{R}$ associated to λ_1.

In this setting an homoclinic orbit associated to σ is any orbit $\Gamma = \mathcal{O}_X(q)$ of a point $q \in W^s(\sigma) \cap W^u(\sigma) \setminus \{\sigma\}$. We assume that there exists such an homoclinic orbit. Moreover we make the supposition that the saddle-connection brakes as in Figure 2.13.

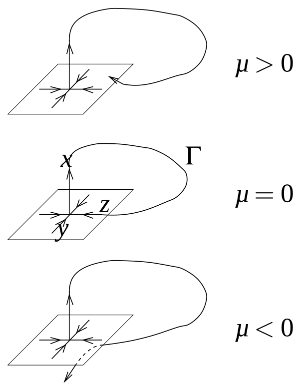

Figure 2.13: Breaking the saddle-connection.

Using linearizing coordinates and an analysis of the return maps to convenient cross-sections near σ one can prove the following.

Theorem 2.3. *For $\mu \neq 0$ small enough a periodic orbit bifurcates from Γ. This periodic orbit is*

1. *a sink for $\lambda_1 < -\lambda_3 \leq -\lambda_2$;*

2. *a saddle for $\lambda_1 < -(\lambda_2 + \lambda_3)$, $-\lambda_2 < \lambda_1$ and/or $-\lambda_3 < \lambda_1$;*

3. *a source for $-(\lambda_2 + \lambda_3) < \lambda_1$.*

A proof of this result can be found in [207, pp. 207-219].

Observe that if σ is Lorenz-like (recall Definition 2.1), then only item 2 above is possible. That is, *a Lorenz-like saddle singularity is the only one which persists in the unfolding of a saddle-connection with real eigenvalues*. It is natural that these are the only allowed singularities for robustly transitive attractors, see Section 3.

2.2.2 Inclination flip and orbit flip

Here we consider degenerated homoclinic orbits. We assume that σ satisfies some generic conditions: the eigenvalues $\lambda_i, i = 1,2,3$ of σ are real and distinct and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$, that is, σ is a Lorenz-like singularity, as in Definition 2.1.

The condition $\lambda_2 < \lambda_3 < 0$ ensures that there is an invariant C^1 manifold $W^{ss}(\sigma)$, the *strong-stable manifold*, tangent at σ to the eigendirection of the eigenvalue λ_2. There are also invariant manifolds $W^{cu}(\sigma)$ containing σ, called *center-unstable* manifolds, tangent at σ to the eigendirection generated by the eigenvectors associated to λ_3, λ_1. There are several of these center-unstable manifolds but all of them are tangent along $W^u(\sigma)$ at σ (the reader should consult Hirsch, Pugh and Shub [76] for a proof of these facts).

Let Γ be a homoclinic orbit associated to σ. The following conditions are generic, that is, both are true for a residual subset of flows in $\mathfrak{X}^1(M)$ exhibiting a homoclinic orbit associated to a Lorenz-like singularity:

(G1) $W^{cu}(\sigma)$ intersects $W^s(\sigma)$ transversely along Γ, i.e.

$$\Gamma = W^{cu}(\sigma) \pitchfork W^s(\sigma); \quad \text{and}$$

(G2) $\Gamma \cap W^{ss}(\sigma) = \emptyset$.

We are going to study what happens when such generic conditions fail.

Definition 2.2. Let $X \in \mathfrak{X}^r(M)$, $r \geq 1$, be a smooth vector field and Γ be a homoclinic orbit associated to a Lorenz-like singularity $\sigma \in S(X)$. We say that Γ is of *inclination-flip* type if (G1) fails and of *orbit-flip* type if (G2) fails.

Generically inclination-flip homoclinic orbits are not orbit-flip and conversely.

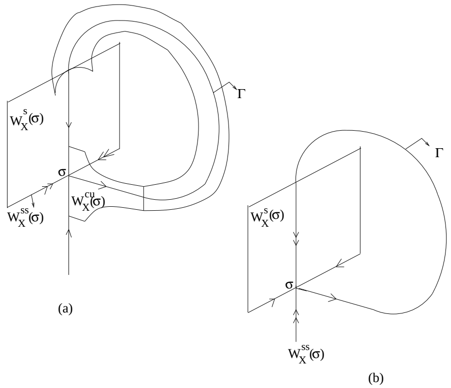

Figure 2.14: (a) Inclination-flip; (b) orbit-flip.

Every C^r vector fields ($r \geq 1$) exhibiting orbit-flip homoclinic orbits can be C^r approximated by smooth vector field exhibiting inclination-flip homoclinic orbits, as stated in the following

Theorem 2.4. *Let X be a C^1 vector field in M exhibiting an orbit-flip homoclinic orbit associated to a singularity σ of X. Suppose that σ has real eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ satisfying $-\lambda_3 < \lambda_1$. Then X can be C^1 approximated by C^1 vector fields exhibiting an inclination-flip homoclinic orbit.*

The proof of this theorem can be found in [129] and follows from standard perturbation techniques (see e.g. [143]). Observe that a vector field exhibiting a inclination-flip type homoclinic orbit cannot have a dominated splitting for the linear Poincaré flow. Indeed, the definition of inclination-flip implies the existence of a tangency between the strong-stable and center-unstable manifolds along a regular orbit of the flow.

As a consequence, by Theorem 2.4, for vector fields having every critical element hyperbolic and no sinks nor sources inside an isolating neighborhood U in a robust way, there cannot be either orbit-flip or inclination-flip type homoclinic orbits because of Theorem 1.8, since this would contradict the existence of a dominated splitting for the linear Poincaré flow.

2.2.3 Saddle-focus connection and Shil'nikov bifurcations

Consider the following one-parameter system of ordinary differential equations in $\mathbb{R}^3$

$$\begin{cases} \dot{x} &= -\rho x + \omega y + f_1(x,y,z;\mu) \\ \dot{y} &= \omega x - \rho y + f_2(x,y,z;\mu) \qquad (x,y,z,\mu) \in \mathbb{R}^4 \\ \dot{z} &= \lambda z + f_3(x,y,z;\mu) \end{cases}$$

where $\lambda, \omega, \rho > 0$ and f_i are C^2 functions which vanish together with Df_i at the origin of $\mathbb{R}^4$. Then $\sigma = (0,0,0)$ is a saddle-focus with eigenvalues λ and $-\rho \pm \omega i$.

These families exhibit very interesting dynamics when there exists a homoclinic orbit Γ associated to σ, see Figure 2.15.

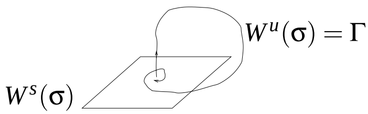

Figure 2.15: Saddle-focus connection

Again by the use of linearizing coordinates and an analysis of the return maps to convenient cross-sections near σ one can prove the following.

Theorem 2.5. *For $\mu \neq 0$ small enough we can find near Γ:*

1. *either an attracting periodic orbit (a sink), for $\rho > \lambda$;*
2. *or infinitely many generic unfoldings of homoclinic tangencies when $\mu \to 0$, inducing in particular the appearance of attracting or repelling periodic orbits near Γ, for $\rho < \lambda$.*

The setting of the second item above is known as *Shil'nikov bifurcation*. The proof of these results can be found in Shil'nikov's work [185] and also in [195, 13, 207, 168].

2.3 Lorenz attractor and geometric models

Here we present a study of the Lorenz system of equations (1.1) and then explain the construction of the geometric Lorenz models, which initially where intended to mimic the behavior of the solutions of the system (1.1), but actually give an accurate description of this flow. Recall the relation between the Lorenz flow, and the associated geometrical model, with sensitive dependence on initial conditions and its historical impact, briefly touched upon in Section 1.1.3.

2.3.1 Properties of the Lorenz system of equations

Here we list analytical properties directly obtained from the Lorenz equations, which can be found with much more details in the books of Sparrow [62] and Guckenheimer-Holmes [64].

Let $X:\mathbb{R}^3\to\mathbb{R}^3$ be the flow defined by the equations (1.1).

1. *Singularities of* X. The origin $\sigma_0=(0,0,0)$ is a singularity of the field X which does not depend on the parameters of X. The others are
$$\sigma_1=(-\sqrt{b(r-1)},-\sqrt{b(r-1)},r-1) \quad \text{and}$$
$$\sigma_2=(\sqrt{b(r-1)},\sqrt{b(r-1)},r-1).$$

2. *Symmetry of* X. The map $(x,y,z)\mapsto(-x,-y,z)$ preserves the Lorenz system of equations, that is if $\big(x(t),y(t),z(t)\big)$ is a solution of the system of equations, then $\big(-x(t),-y(t),z(t)\big)$ will also be a solution.

3. *Divergence of* X. We have
$$DX(x,y,z)=\begin{pmatrix}\partial_x(\dot{x}) & \partial_y(\dot{x}) & \partial_z(\dot{x})\\ \partial_x(\dot{y}) & \partial_y(\dot{y}) & \partial_z(\dot{y})\\ \partial_x(\dot{z}) & \partial_y(\dot{z}) & \partial_z(\dot{z})\end{pmatrix}=\begin{pmatrix}-a & a & 0\\ r-z & -1 & -x\\ y & x & -b\end{pmatrix}$$
hence
$$\operatorname{div}X(x,y,z)=\nabla\cdot X=\operatorname{trace}\big(DX(x,y,z)\big)=-(a+1+b)<0.$$
This shows the strongly dissipative character of this flow and implies that the flow contracts volume: if V_0 is the initial volume of a subset B of $\mathbb{R}^3$ we have by Liouville's Formula that the volume $V(t)$ of the image $X^t(B)$ is $V(t)=V_0e^{-(\sigma+b+1)t}$. For the parameters in (1.1) we have $V(t)=V_0e^{-\frac{41}{3}t}$.

   In particular any maximally positively invariant subset under X^t has zero volume: $\operatorname{Leb}\big(\cap_{t>0}X^t(U)\big)=0$ for any open subset U of $\mathbb{R}^3$.

4. *Eigenvalues of the singularities.* For the parameters in (1.1) the singularities are, besides σ_0

$$\sigma_1 = (-6\sqrt{2}, -6\sqrt{2}, 27) \quad \text{and} \quad \sigma_2 = (6\sqrt{2}, 6\sqrt{2}, 27).$$

For $DX(\sigma_0)$ we have the eigenvalues

$$\lambda_1 = -11/2 + \sqrt{1201}/2 \approx 11.83;$$
$$\lambda_2 = -11/2 - \sqrt{1201}/2 \approx -22.83;$$
$$\lambda_3 = -8/3 \approx -2.67.$$

Note that $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$ which corresponds to a *Lorenz-like* singularity (Definition (2.2)).
For σ_1 the characteristic polynomial of $DX(\sigma_1)$ is of odd degree $p(x) = x^3 + \frac{41}{3}x^2 + \frac{304}{3}x + 1440$ and its derivative $p'(x) = 3x^2 + \frac{41}{4}x + \frac{304}{3}$ is strictly positive for all $x \in \mathbb{R}$, hence there exists a single real root λ of p. Since $p(0) > 0 > p(-15)$ the root is negative and simple numerical calculations show that $\lambda \approx -13.85457791$. Factoring p we get

$$p(x) = \big(x-\lambda\big)\big(x^2 - 0.187911244x + 103.9367643\big)$$
$$= \big(x-\lambda\big)\big(x-z\big)\big(x-\bar{z}\big)$$

and thus $z \approx 0.093955622 + 10.19450522i$.
For σ_2 the eigenvalues are the same by the symmetry of X.

Using this we obtain the following Figure 2.16 of the local invariant manifolds and thus the local dynamics near the singularities

(5) *The trapping ellipsoid.* There exists an ellipsoid E where eventually every positive orbit of the flow enters. Moreover E is transverse to the flow X. Therefore the open region V bounded by E is a *trapping region* for X, that is $\overline{X^t(U)} \subset U$ for all $t > 0$.
This is obtained by finding an appropriate Lyapunov function. We follow Sparrow [192, Appendix C] (see also the original work of Lorenz [102]). Consider

$$L(x,y,z) = rx^2 + ay^2 + a(z-2r)^2.$$

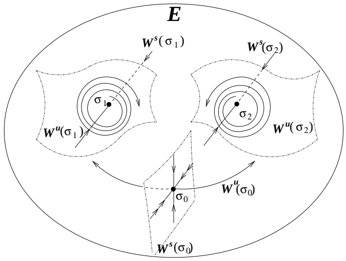

Figure 2.16: Local stable and unstable manifolds near σ_0, σ_1 and σ_2, and the ellipsoid E.

Then along solutions of the system (1.1) we have

$$\frac{dL}{dt} = -2a\big(rx^2 + y^2 + bz^2 - 2brz\big).$$

Let D be domain where $dL/dt \geq 0$ and let M be the maximum of L in D. Now define E to be the set of points such that $L \leq M + \varepsilon$ for some $\varepsilon > 0$ small. Since $D \subset E$ for x outside E we have $dL/dt = \nabla L \cdot X < -\delta < 0$ where $\delta = \delta(\varepsilon) > 0$ and X is the vector field defined by the equations (1.1). Then after a finite time the solution of the Lorenz system through x will enter the set E. Moreover for the values $(a, r, b) = (10, 28, 8/3)$ it is routine to check that ∇L points to the exterior of V over $\partial V = E$ and so all trajectories through E move towards the interior of V. Once in V any trajectory will remain there forever in the future.

Since $\overline{V}$ is compact the maximal positively invariant set $A = \cap_{t>0}\overline{X^t(V)}$ is an attracting set where trajectories of the flow accumulate when t grows without limit.

In fact numerical simulations show that there exists a subset B homeomorphic to a bi-torus such that every positive trajectory crosses B transversely and never leaves it. Hence the open set U bounded by B (see Figure 2.17) is a better candidate for the trapping region of the set with interesting limit dynamics for X, since σ_1 and σ_2 are isolated points in the ω-limit set of X. Hence $\Lambda = \cap_{t>0}\overline{X^t(U)}$ is also an attracting set and the origin is the only singularity contained in U.

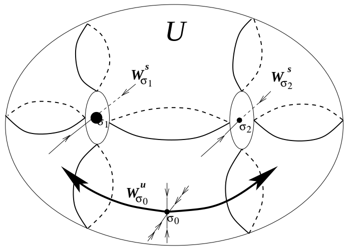

Figure 2.17: The trapping bi-torus.

The evolution of a regular orbit inside the attracting basin

Lorenz observed numerically what today is known as *sensitive dependence on initial conditions*, see Section 1.2.2. Due to this the actual path of any given orbit is impossible to calculate for all large values of integration time.

The “butterfly” which appear on the computer screens can be explained heuristically through the analytical properties already determined and by some numerical results. In fact the set of points whose orbits will draw the butterfly is the complement $\mathbb{R}^3 \setminus N$ of the union $N = W^s(\sigma_0) \cup W^s(\sigma_1) \cup W^s(\sigma_2)$ of the stable manifolds of the three singularities. Note that N is a bi-dimensional immersed surface in $\mathbb{R}^2$ and so has zero volume.

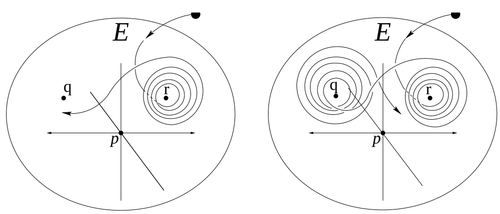

Figure 2.18: The evolution of a generic orbit inside U.

Figure 2.18 provides a very general view of how the orbit of a generic point in the trapping region U evolves in time. The trajectory starts spiraling around one of the singularities, σ_2 say, and suddenly “jumps” to the other singularity and then starts spiraling around σ_2. This process repeats endlessly. The number of turns around each singularity is essentially random. The ω-limit of a generic orbit is the following "butterfly" in Figure 2.19.

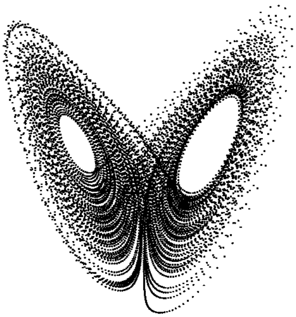

Figure 2.19: Another view of the Lorenz attractor.

2.3.2 The geometric model

The work of Lorenz on the famous flow was published in 1963 [102] but more than 10 years passed before new works on the subject appeared. Williams [208] wrote (in 1977):

> *...Several years ago Jim Yorke figured out some things about the Lorenz equation and got other mathematicians interested. He gave some talks on the subject, including one here at Berkeley. Ruelle, Lanford and Guckenheimer became interested and did some work on these equations. Unfortunately, except for the preprint of Ruelle, Guckhenheimer's paper, is the only thing these four people ever wrote on the subject as far as I know.*

Lorenz had already conjectured the existence of a strange attractor according to the available numerical simulations. The rigorous proof of this fact took many years due to the presence of a singularity accumulated by regular orbits of the flow, which prevents this set from being uniformly hyperbolic — see e.g. Section 1.2.

An important breakthrough in the understanding of the dynamics of the solutions of the Lorenz system of equations was achieved through the introduction of geometric models independently by Afraimovich, Bykov, Shil'nikov [1] in 1977 and by Guckenheimer, Williams [65] in 1979. These models were based on the properties suggested by the numerical simulations. In fact they were able to show the existence of a strange attractor for the geometric model.

This model inspired many others. Today there are different extensions and there are singular attractors which are not of the "Lorenz type": neither conjugated nor equivalent to the Lorenz geometrical model, see e.g. [125].

As explained in Section 1.1.3 on page 9, in 1998 a positive answer to the existence of a strange attractor in the original Lorenz system of equations was given by Tucker [196] in his PhD thesis, through the theory of normal forms together with rigorous numerical algorithms.

Construction of the geometric model

To present the detailed construction of the geometric Lorenz model we first analyze the dynamics in a neighborhood of the singularity at the origin, and then we imitate the effect of the pair of saddle singularities in the original Lorenz flow, as in Figure 2.16.

Near the singularity

By the Hartman-Grobman Theorem or by the results of Sternberg [193], in a neighborhood of the origin the Lorenz equations are equivalent to the linear system $(\dot{x},\dot{y},\dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ through conjugation, thus

$$X^t(x_0,y_0,z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}), \tag{2.3}$$

where $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$ and $(x_0,y_0,z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $(0,0,0)$.

Consider $S = \{(x,y,1) : |x| \le 1/2, \quad |y| \le 1/2\}$ and

$$S^- = \{(x,y,1) \in S : x < 0\}, \qquad S^+ = \{(x,y,1) \in S : x > 0\} \quad \text{and}$$
$$S^* = S^- \cup S^+ = S \setminus \Gamma \qquad \text{, where} \quad \Gamma = \{(x,y,1) \in S : x = 0\}.$$

Assume that S a transverse section to the flow so that every trajectory eventually crosses S in the direction of the negative z axis as in Figure 2.20.

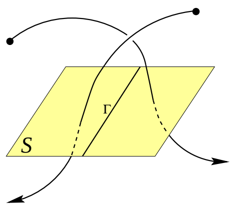

Figure 2.20: S is a cross-section of the flow.

Consider also $\Sigma = \{(x,y,z) : |x| = 1\} = \Sigma^- \cup \Sigma^+$ with $\Sigma^\pm = \{(x,y,z) : x = \pm 1\}$. For each $(x_0, y_0, 1) \in S^*$ the time τ such that $X^\tau(x_0, y_0, 1) \in \Sigma$ is given by $\tau(x_0) = -\frac{1}{\lambda_1} \log |x_0|$, which depends on $x_0 \in S^*$ only and is such that $\tau(x_0) \to +\infty$ when $x_0 \to 0$. This is one of the reasons many standard numerical algorithms were unsuited to tackle the Lorenz system of equations. Hence we get (where $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$ as usual)

$$X^\tau(x_0, y_0, 1) = \left(\operatorname{sgn}(x_0), y_0 e^{\lambda_2 \tau}, e^{\lambda_3 \tau}\right) = \left(\operatorname{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}\right).$$

Since $0 < -\lambda_3 < \lambda_1 < -\lambda_2$, we have $0 < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$. Let $L : S^* \to \Sigma$ be such that $L(x,y) = \left(y|x|^\beta, |x|^\alpha\right)$ with the convention that $L(x,y) \in \Sigma^+$ if $x > 0$ and $L(x,y) \in \Sigma^-$ if $x < 0$. It is easy to see that $L(S^\pm)$ has the

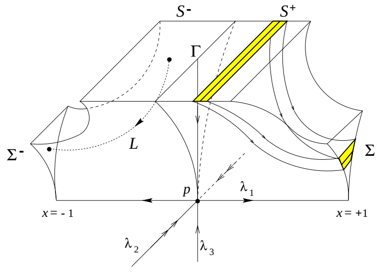

Figure 2.21: Behavior near the origin.

shape of a triangle without the vertex $(\pm 1, 0, 0)$. In fact these are cusp points of the boundary of each of these sets.

From now on we denote by $\Sigma^{\pm}$ the closure of $L(S^{\pm})$. Clearly each line segment $S^{*} \cap \{x = x_0\}$ is taken to another line segment $\Sigma \cap \{z = z_0\}$ as sketched in Figure 2.21.

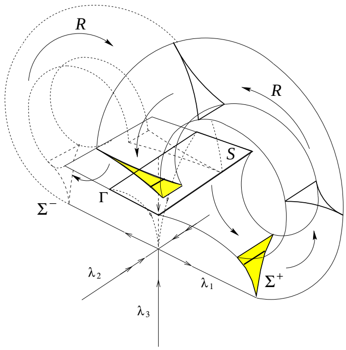

Figure 2.22: R takes $\Sigma^{\pm}$ to S.

The effect of the saddles

The sets $\Sigma^{\pm}$ should return to the cross section S through a rotation around $W^{s}(\sigma_1)$ and $W^{s}(\sigma_2)$. We assume that this rotation takes line segments $\Sigma \cap \{z = z_0\}$ into line segments $S \cap \{x = x_1\}$ as sketched in Figure 2.22.

We are assuming that the "triangles" $\Sigma^{\pm}$ are compressed in the y-direction and stretched on the other transverse direction. This is related to the eigenvalues of σ_1, σ_2 of the original Lorenz flow as explained below.

The rotation R mentioned above is assume to be such that for each $(y,z) \in \Sigma^{\pm}$

$$DR(y,z) = \begin{pmatrix} 0 & \pm M \\ \sigma & 0 \end{pmatrix} \quad \text{for some} \quad 0 < \sigma < 1 \quad \text{and} \quad M > 1,$$

and we define the Poincaré first return map $P : S^{*} \to S$ as $P = R \circ L$.

The combined effects of R and L on lines implies that the foliation of S given by the lines $S \cap \{x = x_0\}$ is invariant under the return map, meaning that for any given leaf γ of this foliation, its image $P(\gamma)$ is contained in a leaf of the same foliation. Hence P must have the form $P(x,y) = \big(f(x), g(x,y)\big)$

for some functions $f : I \setminus \{0\} \to I$ and $g : (I \setminus \{0\}) \times I \to I$, where $I = [-1/2, 1/2]$.

A consequence of all this is that every $x \in S$ has a positive orbit disjoint from $W^{ss}(\sigma)$. Since every point $x \in \Lambda \setminus \{\sigma\}$ has a positive orbit that will eventually cross S by construction, we see that

$$W^{ss}(\sigma) \cap \Lambda = \{\sigma\}. \tag{2.4}$$

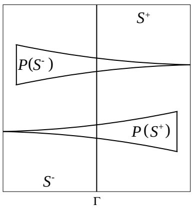

Figure 2.23: $P(S^*)$.

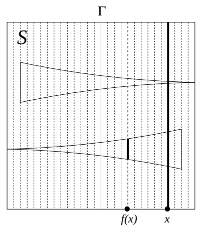

Figure 2.24: Projection on I.

Properties of the one-dimensional map f

Now we specify the properties which we impose on f:

(f1) the symmetry of the Lorenz equations implies $f(-x) = -f(x)$.

(f2) f is discontinuous at $x = 0$ with lateral limits $f(0^-) = +\frac{1}{2}$ and $f(0^+) = -\frac{1}{2}$, since P is not defined at Γ because $\Gamma \subset W^s(0,0,0)$.

(f3) f is differentiable on $I \setminus \{0\}$ and $f'(x) > \sqrt{2}$, since the real part of the (complex) eigenvalues of the saddles σ_1, σ_2 is positive (see the previous Section 2.3.1).

(f4) the lateral limits of f' at $x = 0$ are $f'(0^-) = +\infty$ and $f'(0^+) = -\infty$.

On the other hand $g : S^* \to I$ is defined in such a way that it contracts the second coordinate: we assume $g'_y(w) \le \mu < 1$ for all $w \in S^*$. This is suggested by the eigenvalues $\lambda_2 \approx -22.83$ of σ_0 and $\lambda \approx -13.8545$ of the saddles σ_1, σ_2 (see Section 2.3.1). Moreover the rate of contraction of g on the second coordinate should be much higher than the expansion rate of f.

Figure 2.23 sketches $P(S^*)$. In addition the expansion rate is big enough to obtain a strong mixing property for f (it is locally eventually onto, see Section 2.3.5).

The foliation is contracting in the following sense: for a given leaf γ of the foliation and for $x, y \in \gamma$ then dist $\left(P^n(x), P^n(y)\right) \to 0$ when $n \to \infty$.

Thus the study of the 3-flow can be reduced to the study of a bi-dimensional map and, moreover, the dynamics of this map can be further reduced to a one-dimensional map, since the invariant contracting foliation enables us to identify two points on the same leaf, since their orbits remains forever on the same leaf and the distance of their images tends to zero under iteration, see Figure 2.24 for a sketch of this identification.

The quotient map obtained through this identification will be called *the Lorenz map*. Figure 2.25 shows the graph of this one-dimensional transformation.

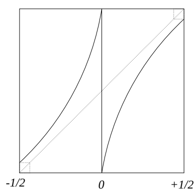

Figure 2.25: The Lorenz map f.

2.3.3 The geometric Lorenz attractor is a partially hyperbolic set with volume expanding central direction

Observe that the time $t(w)$ it takes a point $w \in \Sigma$ to go to S, that is $X^{t(w)}(w) \in S$ and $X^t(w) \in \mathbb{R}^3 \setminus (S \cup \Sigma)$ for $0 < t < t(w)$, is bounded by some constant independently of the point: $t(w) \le t_0$. This ensures that the behavior of the flow on the maximal positively invariant subset of the trapping region is prescribed by the behavior from the cross-section S to the cross-sections Σ^+, Σ^-, as we now explain.

Figure 2.21 makes it clear that the linear flow (2.3) preserves lines in the direction of the y-axis when taking points from S to Σ. Moreover it is not difficult to check that *its derivative DX^t* also preserves planes orthogonal to the y-axis.

In addition, by the choice of the flow from Σ to S and as Figure 2.22 suggests, horizontal lines at Σ, i.e. parallel to the y-axis, are taken to lines parallel to same axis in S, that is *the flow preserves lines parallel to the y-axis from Σ to S. Since the flow from Σ to S is essentially a rotation, we can assume that its derivative also preserves planes orthogonal to the y-axis.*

¿From this we deduce that the following splitting of $\mathbb{R}^3$: $E = \mathbb{R} \times \{(0,0\}$ and $F = \{0\} \times \mathbb{R}^2$, is preserved by the flows, that is $DX^t_w \cdot E = E$ and $DX^t_w \cdot F = F$ for all t and every point w in an orbit inside the trapping ellipsoid.

Moreover we can check that for w on the linearised part of the flow, from S to Σ, we have for $t > 0$ such that $X^{[0,t]}(w)$ is contained in the domain of the linearised coordinates:

- $\|DX^t_w \mid E\| = e^{\lambda_2 t}$;
- $\|DX^t_w \mid E\| = e^{(\lambda_2-\lambda_3)t} \cdot m\big(DX^t \mid F\big)$,

where $m\big(DX^t \mid F\big)$ is the minimum norm of the linear map. Since $\lambda_2 < 0$ we see that E is uniformly contracting, this a stable direction. But $\lambda_2 - \lambda_3 < 0$ and so the contraction along the direction of F is weaker than the contraction along E. This kind of splitting $E \oplus F$ of $\mathbb{R}^3$ is called a *partially hyperbolic splitting*.

Observe also that since $\lambda_1 + \lambda_3 > 0$ we have that $|\det DX^t \mid F| = e^{(\lambda_1+\lambda_3)t}$ and so the flow *expands volume* along the F direction.

We will see in Chapter 3 that these properties characterize compact invariant sets which are robustly transitive.

However we have only checked these properties in the linearised region. But if the orbit of a point w passes outside the linear region k times from Σ to S lasting $s_1 + \cdots + s_k$ from time 0 to time t, then $t > s_1 + \cdots + s_k$ and for some constant $b > 0$ bounding the derivatives of DX^t from 0 to t_0 we have

$$\|DX^t_w \mid E\| \le e^{bk+\lambda_2(t-s_1-\cdots-s_k)} = \exp\Big\{\lambda_2 t\big(1 - \frac{bk}{\lambda_2 t} - \frac{s_1+\cdots+s_k}{t}\big)\Big\},$$

so the last expression in brackets is bounded. We see that E is (K,λ_2)-contracting for some $K > 0$.

An entirely analogous reasoning shows that the direction E dominates F uniformly for all t and that DX^t expands volume along F also uniformly.

Thus the maximal positively invariant set in the trapping ellipsoid is partially hyperbolic and the flow expands volume along a bi-dimensional invariant direction.

2.3.4 Existence and robustness of invariant stable foliation

Now we prove, partially following the work of [65], that the geometric Lorenz attractor constructed in the previous subsection is *robust*, that is, it persists for all nearby vector fields.

More precisely: there exists a neighborhood U in $\mathbb{R}^3$ containing the attracting set Λ such that for all vector fields Y which are C^1-close to X the maximal invariant subset in U, that is $\Lambda_Y = \cap_{t\geq 0} Y^t(U)$, is still a transitive Y-invariant set.

This is a striking property of these flows since the Lorenz flow exhibits sensitive dependence on initial conditions. The robustness will be a consequence of the persistence of the invariant contracting foliation on the cross-section S to the flow.

Numerically this is expected since in spite of the huge integration errors involved and the various integration algorithms used the solutions obtained always have a shape similar to the one in Figure 2.19, independently of the initial point chosen to start the integration.

We start by obtaining the persistence of the stable foliation for points in the attractor, then explain why these attractors, although robust, are *not structurally stable*, in Section 2.3.5.

We note that C^1-robustly transitive attractors in 3-manifolds were completely described from the geometrical point-of-view in [133] and the proof of this result is presented in Chapter 3.

Geometric idea of the proof

Theorem 2.6 (Persistence of contracting foliation)**.** *Let X be the vector field obtained in the construction of the geometric Lorenz model and $\mathcal{F}_X$ the invariant contracting foliation of the cross-section S. Then any vector field Y which is sufficiently C^1-close to X admits an invariant contracting foliation $\mathcal{F}_Y$ on the cross-section S.*

We first present a geometric idea of the proof and then proceed to the details in the following Section 2.3.4.

Observe first that the cross-section S remains transverse to any flow C^1-close to X and that the singularities $\sigma_0, \sigma_1, \sigma_2$ persist with eigenvalues satisfying the same relations as before since they are hyperbolic. In addition, since $W^u_X(\sigma_0)$ intersects S transversely, then just by the C^1 continuous variation of compact parts of the unstable manifolds of a hyperbolic singularity we have that $W^u_Y(\sigma_0(Y))$ still intersects S transversely for all Y close to X in the C^1 norm.

Without loss we can assume, after a C^1 change of coordinates, that the Lorenz-like singularity $\sigma_0(Y)$ remains at the origin and that the eigenvectors of $DY(\sigma_0(Y))$ have the directions of the coordinate axis as before, with the plane $x = 0$ containing the stable manifold of $\sigma_0(Y)$.

Thus for a neighborhood $\mathcal{U}$ of X in the C^1 topology and for each $Y \in \mathcal{U}$ we can define the Poincaré first return map $P_Y : S^* \to S$ as $P_Y = R_Y \circ L_Y$ where $L_Y : S^* \to \Sigma$ is such that $L_Y(x,y) = \big(y|x|^\beta, |x|^\alpha\big)$ with $\alpha = -\frac{\lambda_3(Y)}{\lambda_1(Y)}$ and $\beta = -\frac{\lambda_2(Y)}{\lambda_1(Y)}$ (note that $\beta - \alpha > 1$).

On the other hand $R_Y : \Sigma \to S$ is a C^1-diffeomorphism which can be expressed by the composition $R_Y = J_Y \circ R_0$, where J_Y is a C^1-perturbation of the identity and R_0 is the diffeomorphism associated to X_0.

Now let $\mathcal{A}$ be the space of continuous maps $\phi : \mathcal{U} \times S \to [-1,+1]$. For each $Y \in \mathcal{U}$ and $\phi \in \mathcal{A}$ we define $\phi_Y : S \to [-1,1]$ by $\phi_Y(q) = \phi(Y,q)$ for all $q \in S$. We associate to ϕ_Y a vector field $\eta^\phi_Y : S \to [-1,1] \times \{1\}$ given by $\eta^\phi_Y(q) = \big(\phi_Y(q), 1\big)$ which we view as a vector on $T_qS = \mathbb{R}^2$. Integrating

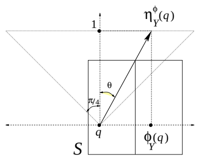

Figure 2.26: The field η^ϕ_Y.

the field η^ϕ_Y we get a family of curves which induces a foliation on S. We must show that there exists $\phi \in \mathcal{A}$ such that η^ϕ_Y induces an invariant foliation under P_Y. Before explaining the proof of this fact we state a necessary and sufficient condition for the invariance of this foliation.

Let F be a continuous vector field defined on S and P the map defined above. Integrating F we get a foliation of S. Let $q \in S^*$ have image $P(q)$ and consider the vectors $F(q)$ and $F\big(P(q)\big)$. Observe that the foliation induced by F is invariant under P if

$$DP(q)\big(F(q)\big) \quad \text{and} \quad F\big(P(q)\big) \quad \text{are parallel,} \quad \text{or}$$
$$F(q) \quad \text{and} \quad [DP(q)]^{-1}F\big(P(q)\big) \quad \text{are parallel.}$$

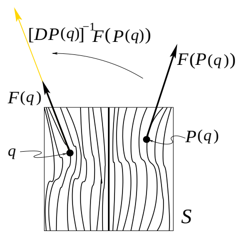

Figure 2.27: The field F and the parallel condition.

On the other hand if we consider the slope of vectors with respect to the vertical direction $(0,1)$, two vectors are parallel if, and only if, their slope is the same. For $(a,b) \in \mathbb{R}^2$ we set slope$(a,b) = a/b$ and hence to check that the foliation defined by F is invariant under P amounts to obtain

$$\text{slope}\big(F(q)\big) = \text{slope}\Big([DP(q)]^{-1}F\big(P(q)\big)\Big).$$

Translating this for η_Y^ϕ we obtain the condition

$$\phi_Y(q) = \text{slope}\Big(\eta_Y^\phi(q)\Big) = \text{slope}\Big(\big[DP_Y(q)\big]^{-1}\eta_Y^\phi\big(P_Y(q)\big)\Big).$$

The last term above depends on ϕ, X and q and if we define $T : \mathcal{A} \to \mathcal{A}$ as

$$T\big(\phi_Y\big)(q) = \text{slope}\Big(\big[DP_Y(q)\big]^{-1}\eta_Y^\phi\big(P_Y(q)\big)\Big),$$

then the condition of invariance becomes $T\big(\phi_Y\big)(q) = \phi_Y(q)$, that is $T(\phi) = \phi$. Hence the element $\phi \in \mathcal{A}$ for which η_X^ϕ induces and invariant foliation on S is a fixed point of T. Thus we are left to prove that the operator T has a fixed point.

For this, we first show that T is well defined and then that T is a contraction in an appropriate space, which concludes the proof of Theorem 2.6.

Proof of existence of invariant stable foliation

The Poincaré map P_Y associated to $Y \in \mathcal{U}$ can be written

$$P_Y(q) = \big(f_Y(q), g_Y(q)\big)$$

for $q \in S^*$. We rewrite T as a function of f and g. First we calculate

$$\big(DP_Y(q)\big)^{-1} = \frac{1}{\Delta}\begin{pmatrix} \partial_y g_Y(q) & -\partial_y f_Y(q) \\ -\partial_x g_Y(q) & \partial_x f_Y(q) \end{pmatrix} \quad \text{with} \quad \Delta = \det DP_Y(q).$$

Then it is not difficult to see that the slope of

$$\big(DP_Y(q)\big)^{-1}\eta_Y^{\phi}\big(P_Y(q)\big) = \frac{1}{\Delta}\begin{pmatrix} \partial_y g_Y(q) & -\partial_y f_Y(q) \\ -\partial_x g_Y(q) & \partial_x f_Y(q) \end{pmatrix}\begin{pmatrix} \phi_Y\big(P_Y(q)\big) \\ 1 \end{pmatrix}$$

is

$$\text{slope}\Big(\big(DP_Y(q)\big)^{-1}\eta_Y^{\phi}\big(P_Y(q)\big)\Big) = \frac{[\phi_Y\big(P_Y(q)\big)]\partial_y g_Y(q) - \partial_y f_Y(q)}{-[\phi_Y\big(P_Y(q)\big)]\partial_x g_Y(q) + \partial_x f_Y(q)}$$

Writing $\widehat{P}(Y,q) = \big(Y, P_Y(q)\big)$we get

$$T(\phi_Y)(q) = \frac{(\phi \circ \widehat{P})\partial_y g - \partial_y f}{\partial_x f - (\phi \circ \widehat{P})\partial_x g}(Y,q)$$

Lemma 2.7. *Let $a_0 \in (0, 1/5)$ and Y a vector field C^1-close to X. If $P_Y(q) = P(Y,q) = \big(f(Y,q), g(Y,q)\big)$, then there are positive constants k_i, $i = 1,2,3$ such that for all $q \in S^*$*

1. $\left|\frac{\partial_x g(Y,q)}{\partial_x f(Y,q)}\right| \le a_0$; $\quad \frac{|\partial_y g(Y,q)|}{|\partial_x f(Y,q)|} \le k_1 |x|^{(\beta-\alpha+1)}$; *and* $\frac{|\partial_y f(Y,q)|}{|\partial_x f(Y,q)|} \le k_2 |x|^{(\beta-\alpha+1)}$;

2. $\|D_q P(Y,q)\| \le k_3 |x|^{(\alpha-1)}$ *and* $\quad |\det D_q P(Y,q)| \le a_0 |x|^{(\beta+\alpha-1)}$;

3. $\sup_{S^*}\left\{\frac{|\partial_y g|}{|\partial_x f|}, \frac{|\partial_y f|}{|\partial_x f|}, \frac{|\partial_x g|}{|\partial_x f|}, |\det DP|\right\} < a_0$.

Proof. We provide the calculations for $x > 0$, the other case being analogous. Since $P_Y = R_Y \circ L_Y$ we have $DP_Y(q) = DR_Y\big(L_Y(q)\big)DL_Y(q)$ and

$$DL_Y(x,y) = \begin{pmatrix} \beta y x^{\beta-1} & x^{\beta} \\ \alpha x^{\alpha-1} & 0 \end{pmatrix}.$$

Recall that $R_Y = J_Y \circ R_0$ and so we have that $DR_Y\big(L_Y(x,y)\big)$ can be written as $DJ_Y\big(R_0(L_Y(x,y))\big) \cdot DR_0\big(L_Y(x,y)\big)$. Since J_Y is close to the identity we may write $J_Y(x,y) = (x+\varepsilon_1, y+\varepsilon_2)$ with $(\varepsilon_1,\varepsilon_2) = \varepsilon(X,y,z)$ small in the C^1-norm. Thus we have

$$DJ_Y = \begin{pmatrix} 1+\partial_x\varepsilon_1 & \partial_y\varepsilon_1 \\ \partial_x\varepsilon_1 & 1+\partial_y\varepsilon_2 \end{pmatrix} \quad \text{and} \quad DR_0 = \begin{pmatrix} 0 & M \\ \sigma & 0 \end{pmatrix}$$

and so

$$DR_Y = \begin{pmatrix} \sigma\cdot\partial_y\varepsilon_1 & M + M\cdot\partial_x\varepsilon_1 \\ \sigma+\sigma\cdot\partial_y\varepsilon_2 & M\cdot\partial_x\varepsilon_1 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & M+\varepsilon_2 \\ \sigma+\varepsilon_3 & \varepsilon_4 \end{pmatrix}.$$

Finally we multiply the last pair of matrices to get

$$\begin{aligned} DP_Y(x,y) &= \begin{pmatrix} \varepsilon_1\beta y x^{\beta-1} + M\alpha x^{\alpha-1} + \varepsilon_2\alpha x^{\alpha-1} & \varepsilon_1 x^{\beta} \\ \sigma\beta y x^{\beta-1} + \varepsilon_3\beta y x^{\beta-1} + \varepsilon_4\alpha x^{\alpha-1} & \sigma x^{\beta} + \varepsilon_3 x^{\beta} \end{pmatrix} \\ &= \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix}. \end{aligned}$$

We may now assume without loss in what follows that $\varepsilon = \varepsilon_i$ since $\varepsilon_i \to 0$ when $Y \to X$. Hence

$$\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} \big[\varepsilon\beta y x^{(\beta-\alpha)} + (M+\varepsilon)\alpha\big] x^{(\alpha-1)} & \varepsilon x^{\beta} \\ \big[(\sigma+\varepsilon)\beta y x^{(\beta-\alpha)} + \varepsilon\alpha\big] x^{(\alpha-1)} & (\sigma+\varepsilon) x^{\beta} \end{pmatrix}.$$

Now we may find the stated bounds as follows.

1. When $\varepsilon \to 0$ we have both

$$\frac{\sigma+\varepsilon}{\varepsilon\beta y x^{(\beta-\alpha)} + (M+\varepsilon)\alpha} \to \frac{\sigma}{M\alpha} \quad \text{and} \quad \frac{\varepsilon}{\varepsilon\beta y x^{(\beta-\alpha)} + (M+\varepsilon)\alpha} \to 0,$$

hence these quotients are bounded: there are k_1 and k_2 so that $\frac{|\partial_y g|}{|\partial_x f|} \le k_1 |x|^{(\beta-\alpha+1)}$ and $\frac{|\partial_y f|}{|\partial_x f|} \le k_2 |x|^{(\beta-\alpha+1)}$.

On the other hand, again when $\varepsilon \to 0$ we get

$$\frac{|\partial_x g|}{|\partial_x f|} = \frac{(\sigma+\varepsilon)\beta y x^{\beta-\alpha} + \varepsilon\alpha}{\varepsilon\beta y x^{\beta-\alpha} + (M+\varepsilon)\alpha} \to \frac{\sigma\beta y x^{\beta-\alpha}}{M\alpha} \leq \frac{\sigma\beta}{2\cdot 2^{\beta-\alpha} M\alpha}$$

where the bound above follows because $\beta - \alpha > 0$. Since $0 < \sigma < 1$ and $M > 1$ where chosen arbitrarily in the construction, we may assume that σ is very small and M big enough so that $\frac{|\partial_x g|}{|\partial_x f|} \leq a_0$.

2. It is clear that since $0 < \alpha < 1 < \beta$ and

$$\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{pmatrix} M\alpha x^{(\alpha-1)} & 0 \\ \sigma\beta y x^{(\alpha-1)} & \sigma x^{\beta} \end{pmatrix},$$

the norm of the matrix is dominated by the value of $|x|^{\alpha-1}$ for $x \approx 0$, thus there exists k_3 such that $\|D_q P(Y,q)\| \leq k_3 |x|^{\alpha-1}$. On the other hand

$$\begin{aligned} |\det D_q P(Y,q)| &= |\partial_x f \partial_y g - \partial_y f \partial_x g| \leq |\partial_x f||\partial_y g| + |\partial_y f||\partial_x g| \\ &\leq r_1 |x|^{\beta+\alpha-1} + r_2 |x|^{\beta+\alpha-1} \leq K|x|^{\beta+\alpha-1} \end{aligned}$$

where the existence of $r_1, r_2 > 0$ as above is a consequence of both $|\partial_x f|\cdot|\partial_y g| \xrightarrow[\varepsilon\to 0]{} M\alpha\sigma x^{(\beta+\alpha-1)}$ and $|\partial_y f|\cdot|\partial_x g| \xrightarrow[\varepsilon\to 0]{} 0$ and also of $\beta+\alpha > 1$. Note that we may assume $K \leq a_0$ by setting $M\sigma$ small (that is, we assume that the volume is contracted).

3. Finally for the quotients of the entries of DP_Y note that we can again use the bounds already obtained and get then smaller than a_0 by letting σ close to 0 and M big enough.

□

Proposition 2.8. *Let T be defined as before depending on f and g. Then*

1. *$T(\mathcal{A}) \subset \mathcal{A}$, that is, $T : \mathcal{A} \to \mathcal{A}$ is well defined;*

2. *$T : \mathcal{A} \to \mathcal{A}$ is a contraction.*

Proof. First we show that for $\phi \in \mathcal{A}$ then $T(\phi)$ is continuous and $|T(\phi)| \leq 1$, which would prove the first item of the statement. According to the definition of T we have

$$|T(\phi)(Y,q)| = \frac{|(\phi \circ \widehat{P})\partial_y g - \partial_y f|}{|\partial_x f - (\phi \circ \widehat{P})\partial_x g|}(Y,q) = \frac{|(\phi \circ \widehat{P})\frac{\partial_y g}{\partial_x f} - \frac{\partial_y f}{\partial_x f}|}{|1 - (\phi \circ \widehat{P})\frac{\partial_x g}{\partial_x f}|}(Y,q)$$
$$\leq \frac{|\frac{\partial_y g}{\partial_x f}| + |\frac{\partial_y f}{\partial_x f}|}{1 - |\frac{\partial_x g}{\partial_x f}|}(Y,q) \leq \frac{k_2|x|^{\beta-\alpha+1} + k_3|x|^{\beta-\alpha+1}}{1 - k_1} \leq K|x|^{\beta-\alpha+1}.$$

Thus $|T(\phi)(Y,q)| \to 0$ if $|x| \to 0$, which shows that $T(\phi)$ is continuous at 0. Then ϕ is continuous since the expression is continuous away from $x = 0$. Moreover

$$|T(\phi)(Y,q)| \leq \frac{|\frac{\partial_y g}{\partial_x f}| + |\frac{\partial_y f}{\partial_x f}|}{1 - |\frac{\partial_x g}{\partial_x f}|}(Y,q) \leq \frac{2a_0}{1 - a_0} < 1,$$

by Lemma 2.7(3) and because $a_0 \in (0, 1/5)$ implies $\frac{2a_0}{1-a_0} < 1/2$.

Now the contraction is easy, since for $\phi_1, \phi_2 \in \mathcal{A}$ and for fixed $(Y,q) \in \mathcal{U} \times S$

$$|T(\phi_1) - T(\phi_2)| = \frac{|\det DP(q)| \cdot |\phi_1 \circ \widehat{P} - \phi_2 \circ \widehat{P}|}{\left|\partial_x f - (\phi_1 \circ \widehat{P})\partial_x g\right| \cdot \left|\partial_x f - (\phi_2 \circ \widehat{P})\partial_x g\right|}$$
$$\leq \frac{a_0}{(1 - a_0)^2}|\phi_1 - \phi_2|$$

and again $\frac{a_0}{(1-a_0)^2} < 1/2$, as long as $\mathcal{U}$ is taken small enough around X so that Lemma 2.7 remains valid. □

We have shown that there exists a unique fixed point for T on $\mathcal{A}$ as we wanted and so we have an invariant foliation on S.

Differentiability of the foliation

Now we prove that the fixed point $\phi(Y,q)$ depends on Y, q continuously on the C^1 topology. We do this by showing that $D\phi_Y$ depends continuously on (Y,q) and that the operator T is also a contraction on the C^1 norm.

Again using the definition of T at a point (Y,q) we obtain the following expression

$$DT(\phi) = \frac{D\big[(\phi\circ\widehat{P})\partial_y g - \partial_y f\big]}{\partial_x f - (\phi\circ\widehat{P})\partial_x g} - \frac{(\phi\circ\widehat{P})\partial_y g - \partial_y f}{\big(\partial_x f - (\phi\circ\widehat{P})\partial_x g\big)^2}\cdot D\big[\partial_x f - (\phi\circ\widehat{P})\partial_x g\big]$$
$$= V_1(\phi) + T(\phi)V_2(\phi) + N(\phi)D\phi\big(\widehat{P}(X,q)\big),$$

where we have used

$$V_1(\phi) = \frac{\phi\circ\widehat{P}}{\partial_x f - (\phi\circ\widehat{P})\partial_x g}\cdot D\partial_y g - \frac{1}{\partial_x f - (\phi\circ\widehat{P})\partial_x g} D\partial_y f;$$
$$V_2(\phi) = \frac{\phi\circ\widehat{P}}{\partial_x f - (\phi\circ\widehat{P})\partial_x g}\cdot D\partial_x f - \frac{1}{\partial_x f - (\phi\circ\widehat{P})\partial_x g} D\partial_x g;$$
$$N(\phi) = \frac{\det DP(X,q)}{\big(\partial_x f - (\phi\circ\widehat{P})\partial_x g\big)^2}.$$

Now define the space $\mathcal{A}_1$ of continuous maps $A : \mathcal{U}\times S \to \mathcal{L}(\mathcal{X}\times\mathbb{R}^2,\mathbb{R})$ such that

$$\sup_{(X,q)} |A(X,q)| < 1 \quad \text{and} \quad A\big(X,(0,y)\big) = 0 \quad \text{for all} \quad y \in \left[-\frac{1}{2},\frac{1}{2}\right]$$

and consider the operator $\widetilde{T} : \mathcal{A}\times\mathcal{A}_1 \to \mathcal{A}\times\mathcal{A}_1$ such that for $\phi\in C^1$ we have $\widetilde{T}(\phi,D\phi) = \big(T(\phi),DT(\phi)\big)$, defined as $\widetilde{T}\big(\phi,A) = \big(T\phi,S(\phi,A)\big)$ where $S(\phi,A)$ is given by

$$S(\phi,A)(Y,q) = \big[V_1(\phi) - T(\phi)V_2(\phi) + N(\phi)(A\circ\widehat{P})\cdot D\widehat{P}\big](Y,q),$$

where $V_1(\phi),V_2(\phi)$ and $N(\phi)$ were defined previously during the calculation of $DT(\phi)$.

Again we need to show that $\widetilde{T}$ is well defined and a contraction.

Lemma 2.9. *Take Y C^1-close to X such that the estimates of Lemma 2.7 are valid. If $P_Y(q) = P(Y,q) = \big(f(Y,q),g(Y,q)\big)$ then there are positive constants $k_i = 4,\dots,8$ such that for all $q\in S^*$*

1. $\frac{|D\partial_y g|}{|\partial_x f|} \le k_4|x|^{\beta-\alpha}$, $\frac{|D\partial_x g|}{|\partial_x f|} \le k_5|x|^{-1}$;

2. $\frac{|D\partial_y f|}{|\partial_x f|} \le k_6|x|^{\beta-\alpha}, \frac{|D\partial_x f|}{|\partial_x f|} \le k_7|x|^{-1}$;

3. $\frac{|\det D_q P|}{|\partial_x f|^2}|D\widehat{P}| \le k_8|x|^{\beta}, |N(\phi)| \cdot |D\widehat{P}| < 1/2$;

4. $\sup_{S^*}\left\{\frac{|D\partial_y g|}{|\partial_x f|}, \frac{|D\partial_y f|}{|\partial_x f|}\right\} < a_0$.

Proof. Using Lemma 2.7, since $\partial_x f \underset{\varepsilon\to 0}{\longrightarrow} M\alpha x^{\alpha-1}$ we see there are K_1, K_2 satisfying

$$K_1|x|^{\alpha-1} \le |\partial_x f| \le K_2|x|^{\alpha-1}. \tag{2.5}$$

On the other hand, taking derivatives we see that

$$\begin{aligned}
\partial_X(\partial_y g) &= x\beta\partial_X\varepsilon + (\sigma+\varepsilon)x^{\beta}\log(\beta)\partial_X\beta) \\
\partial_x(\partial_y g) &= \partial_x\varepsilon x^{\beta} + (\sigma+\varepsilon)\beta x^{\beta-1} \\
\partial_y(\partial_y g) &= \partial\varepsilon x^{\beta}.
\end{aligned}$$

Then $|D\partial_y g| \le K_3|x|^{\beta-1}$ and by (2.5) we see there exists k_4 such that $\frac{|D\partial_y g|}{|\partial_x f|} \le k_4|x|^{\beta-\alpha}$. Analogously we may estimate the derivatives $\partial_X(\partial_x g), \partial_x^2, \partial_y(\partial_x g)$ obtaining

$$|\partial_X(\partial_x g)| \le K|x|^{\alpha-1}, \quad |\partial_x^2 g| \le K|x|^{\alpha-2}, \quad |\partial_y(\partial_x g)| \le K|x|^{\beta-1}$$

and thus $|D\partial_x g| \le K|x|^{\alpha-2}$ and by (2.5) we get k_5 so that $\frac{|D\partial_x g|}{|\partial_x f|} \le k_5|x|^{-1}$. This proves the first item of the statement.

Again analogously we obtain $|D\partial_y f| \le K|x|^{\beta-1}$ and by (2.5) also $\frac{|D\partial_y f|}{|\partial_x f|} \le k_6|x|^{\beta-\alpha}$ for a constant k_6.

¿From the explicit expression of $\partial_x f$ we get $|\partial_X(\partial_x f)| \le K|x|^{\alpha-1}$ and also

$$\partial_x(\partial_x f) = \beta y x^{\beta-1}\partial_x\varepsilon + \varepsilon\beta(\beta-1)yx^{\beta-2} + \alpha x^{\alpha-1}\partial_x\varepsilon + \alpha(M+\varepsilon)(\alpha-1)x^{\alpha-2}$$

implying that $|\partial_x(\partial_x f)| \le K|x|^{\alpha-2}$. We also have

$$\partial_y(\partial_x f) = \beta y x^{\beta-1}\partial_y\varepsilon + \varepsilon\beta x^{\beta-1} + \alpha x^{\alpha-1}\partial_y\varepsilon$$

which implies $\partial_y(\partial_x f) \le K|x|^{\beta-1}$, and so $|D\partial_x f| \le K|x|^{\alpha-2}$ showing the existence of k_7 such that $\frac{|D\partial_x f|}{\partial_x f} \le k_7|x|^{-1}$, and proving the second item of the statement.

Now recall the definition of $N(\phi)$ and use Lemma 2.7 to deduce

$$
\begin{aligned}
|N(\phi)||D\widehat{P}| &= \frac{|\det D_q P|}{(\partial_x f - (\phi\circ\widehat{P})\partial_x g)^2}|D\widehat{P}| \\
&\le \frac{a_0|x|^{\beta+\alpha-1}}{|\partial_x f|^2(1-\frac{\partial_x g}{\partial_x f})^2}|D\widehat{P}| \le \frac{a_0}{(1-a_0)^2}|x|^\beta \le \frac{a_0}{2^\beta(1-a_0)^2}
\end{aligned}
$$

which concludes the proof of the third item since $\beta > 1$. □

Now using the estimates of Lemmas 2.7 and 2.9 we are to prove the following.

Proposition 2.10. *The map $S : \mathcal{A}\times\mathcal{A}_1 \to \mathcal{A}_1$ is well defined, continuous and $S(\phi,\cdot) : \mathcal{A}_1 \to \mathcal{A}_1$ is a contraction whose contraction rate is independent of ϕ.*

Note that this shows that for every $\phi \in \mathcal{A}$ which is derivable there exits $A \in \mathcal{A}_1$ such that $S(\phi,A) = A$.

Proof. We can estimate

$$
\begin{aligned}
|V_1(\phi)| &\le \frac{|\phi\circ\widehat{P}|}{|\partial_x f - (\phi\circ\widehat{P})\partial_x g|}\cdot|D\partial_y g| + \frac{1}{|\partial_x f - (\phi\circ\widehat{P})\partial_x g|}|D\partial_y f| \\
&\le \left(\frac{|\partial_x f|^{-1}}{1-|\frac{\partial_x g}{\partial_x f}|}\right)\cdot|D\partial_y g| + \left(\frac{|\partial_x f|^{-1}}{1-|\frac{\partial_x g}{\partial_x f}|}\right)\cdot|D\partial_y f| \\
&\le \frac{1}{|1-a_0|}\cdot\Big(|\partial_x f|^{-1}|D\partial_y g| + |\partial_x f|^{-1}|D\partial_y f|\Big) \le K|x|^{\beta-\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
|T(\phi)V_2(\phi)| &\le K|x|^{\beta-\alpha+1}\frac{1}{1-a_0}\left\{\frac{|D\partial_x g|}{|\partial_x f|} + \frac{|D\partial_x f|}{|\partial_x f|}\right\} \\
&\le K|x|^{\beta-\alpha+1}|x|^{-1} \le K|x|^{\beta-\alpha}
\end{aligned}
$$

and also

$$|N(\phi)| \cdot |(A \circ \widehat{P}) \cdot D\widehat{P}| \le \frac{1}{(1-a_0)^2} |\det D_q P| \cdot |A| \cdot |D\widehat{P}| \cdot |\partial_x f|^{-2} \le K|x|^{\beta}.$$

Therefore we arrive at

$$|S(\phi,A)| \le |V_1(\phi)| + |T(\phi)| \cdot |V_2(\phi)| + |N(\phi)| \cdot |(A \circ \widehat{P}) D\widehat{P}| \le K|x|^{\beta-\alpha}.$$

Since $\beta - \alpha > 0$ we see that S is continuous at $x = 0$. Moreover

$$|V_1(\phi)| < \frac{a_0}{1-a_0} \quad \text{and} \quad |T(\phi)V_2(\phi)| < \frac{a_0^2}{(1-a_0)^2}$$

and for $a_0 \in (0, 1/5)$ we get $\frac{a_0^2}{(1-a_0)^2} < 1/2$ so $|S(\phi,A)| \le 1$ and thus S is well defined.

Finally taking $A_1, A_2 \in \mathcal{A}_1$ and fixing $\phi \in \mathcal{A}$ we get

$$S(\phi,A_1) - S(\phi,A_2) = N(\phi) \cdot \left[A_1 \circ \widehat{P} - A_2 \circ \widehat{P}\right] \cdot D\widehat{P}$$

hence

$$|S(\phi,A_1) - S(\phi,A_2)| \le |N(\phi)| \cdot |A_1 - A_2| \cdot |D\widehat{P}| < \frac{1}{2}|A_1 - A_2|$$

and we conclude that $S(\phi,\cdot)$ is a contraction as stated. □

This shows that $\widehat{T}$ has a fixed point (ϕ_0, A_0) where ϕ_0 is a fixed point of T. Clearly (ϕ_0, A_0) is a global attractor inside $\mathcal{A} \times \mathcal{A}_1$. In particular taking ϕ of class C^1 we obtain

$$\widehat{T}^n(\phi, D\phi) = \left(T^n(\phi), D(T^n(\phi))\right) \xrightarrow[n \to +\infty]{} (\phi_0, D\phi_0)$$

then $A_0 = D\phi_0$ and hence ϕ_0 is continuously differentiable.

2.3.5 Robustness of the geometric Lorenz attractors

Here we conclude the proof that the geometric Lorenz attractor is a robustly transitive attractor and show that it is not structurally stable. Here we drop condition (f1) on the symmetry of the one-dimensional map f.

Robust properties of the one-dimensional map f

We start by showing that the properties of the one-dimensional map f are robust for small C^1 perturbations of X.

Indeed, note that since the stable foliation is robust, we can define the one-dimensional map f_Y as the quotient map of the corresponding Poincaré map P_Y over the leaves of the foliation $\mathcal{F}_Y$, for all flows Y close to X in the C^1 topology.

Moreover since the leaves of $\mathcal{F}_Y$ are C^1 close to those of $\mathcal{F}$, hence f_Y is C^1 close to f and thus there exists $c \in [-1/2, 1/2]$ which play for f_Y the same role of 0 so that properties (f2)-(f4) from Section 2.3.2 are still valid for f_Y on a subinterval $[-b, b]$ for some $0 < b < 1/2$ close to $1/2$.

This implies that every f_Y is *locally eventually onto* for all Y close to X, that is for any interval $J \subset (-b, b)$ there exists an iterate $n \geq 1$ such that $f_Y^n(J) = (-b, b)$.

Lemma 2.11. *Let* $f : [-1/2, 1/2] \setminus \{0\} \to [-1/2, 1/2]$ *be given satisfying the properties (f1)-(f4) on Section 2.3.2 on page 75. Then f is locally eventually onto: for any open interval J not containing* 0 *there exists n such that* $f^n \mid J$ *is a diffeomorphism between J and* $(f(-1/2), f(1/2))$.

This implies in particular the maps f_Y are (robustly) transitive and periodic points are dense. Moreover this also implies that the pre-orbit set $\cup_{n\geq 0} f^{-n}\{x\}$ is dense for every $x \neq 0$.

Proof. Let $J_0 \subset (-1/2, 1/2)$ be an open interval with $0 \notin J_0$ and let $\eta = \inf |f'| > \sqrt{2}$.

Since $0 \notin J_0$ then $f(J_0)$ is such that $\ell\big(f(J_0)\big) \geq \eta \ell(J_0)$, where $\ell(\cdot)$ denotes length, and $f(J_0)$ is connected.

1. If $0 \notin f(J_0)$, set $J_1 = f^2(J_0)$ and then $\ell(J_1) \geq \eta^2 \ell(J_0)$.
2. If $0 \in f(J_0)$, then $f^2(J_0) = I^- \cup I^+$, where I^+ is the biggest connected component. Thus

$$\ell(I^+) \geq \frac{\ell\big(f^2(J_0)\big)}{2} \geq \frac{\eta^2}{2} \ell(J_0).$$

Now replace J_0 by I^+ in case (2) or by J_1 in case (1). Since $\min\{\eta, \eta^2/2\} > 1$ we obtain after finitely many steps one of the intervals $(-1/2, 0)$ or $(0, 1/2)$. One more interate then covers the interval $(f(-1/2), f(1/2))$.

□

Transitivity and denseness of periodic orbits

We deduce these features from a stronger property: we show that the geometric Lorenz attractor is a homoclinic class (see Section 1.3.5).

Proposition 2.12. *There exists a periodic orbit $\mathcal{O}_X(p)$ in the geometric Lorenz attractor Λ such that $\Lambda = H_X(p) = \overline{W^s_X(p) \pitchfork W^u_X(p)}$.*

We prove this in Section 2.3.6. Observe that *every periodic orbit $\mathcal{O}(p)$ in Λ must be hyperbolic* since

- the uniformly contracting foliation obtained in Section 2.3.4 provides a uniformly contracting direction and a stable manifold for $\mathcal{O}(p)$: if $\mathcal{F}(p)$ is the leaf of $\mathcal{F}$ through $p = \mathcal{O}(p) \cap S$, then

$$W^s\big(\mathcal{O}(p)\big) = \bigcup_{t \geq 0} X^{-t}\big(\mathcal{F}(p)\big);$$

- the expansion of the one-dimensional projection map f (property (f3) from Section 2.3.2 on page 75) ensures that there exists a forward DP-invariant expanding cone field around the horizontal direction, which in turn ensures the existence of a DP-invariant expanding direction at p.

Proposition 2.12 implies after the Birkhoff-Smale Theorem 1.12 that the geometric Lorenz attractor Λ has a dense orbit and a dense subset of periodic orbits.

Since the arguments we use to prove Proposition 2.12 depend only on the properties of f and these properties are robust, we conclude that the geometric Lorenz attractors are robustly transitive.

The geometric Lorenz models are not structurally stable

The dynamics of two nearby geometric Lorenz models are in general not topologically equivalent. In fact Guckenheimer and Williams [65, 209] show that the conjugacy classes are completely described by two parameters: the *kneading sequences* of the two singular values

$$f(0^+) = \lim_{x \to 0^+} f(x) \quad \text{and} \quad f(0^-) = \lim_{x \to 0^-} f(x)$$

with respect to the singular point 0 — *a pair of one-dimensional Lorenz-like maps are conjugate if, and only if, they have the same pair of kneading sequences* and, moreover, the corresponding flows are topologically equivalent if, and only if, the one-dimensional maps are conjugated (recall that we have dropped condition (f1)).

The kneading sequence of $x^+ = f(0^+)$ with respect to 0 is a sequence defined by

$$a_n = \begin{cases} 0 & \text{if } f^n(x^+) < 0 \\ 1 & \text{otherwise} \end{cases} ; \quad \text{for} \quad n \geq 0,$$

and analogously we define the kneading sequence $(b_n)_{n\geq 0}$ for x^-.

It is easy to see that *if two nearby geometric Lorenz flows are topologically conjugated* (see Section 1.1 for definitions and basic properties) *then the kneading sequences must be equal*, since the equivalence relation preserves the orbit structure and in particular preserves also the first return iterates to the cross-section S.

Now given a geometric Lorenz flow X with corresponding kneading sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, we can through a small perturbation find a C^1 close vector field Y whose corresponding one-dimensional map has kneading sequences $(a'_n)_{n\geq 0}$ and $(b'_n)_{n\geq 0}$ distinct from the pair $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$.

Indeed, if one of the orbits of $x^\pm$ is dense in $(-1,1)$, then one of its iterate is arbitrarily close to 0. Thus a small perturbation of the map will flip one of the elements of the kneading sequence from 0 to 1 or viceversa. Otherwise there exists $\varepsilon > 0$ such that the orbits of $x^\pm$ do not enter $(-\varepsilon, \varepsilon)$. As we have already proved, the one-dimensional map f is locally eventually onto and in particular topologically transitive. Hence there exists a point $0 < y < \delta \ll \varepsilon$ with $0 < f(y) - x^+ < \delta$ whose orbit is dense. Let $n > 0$ be the smallest integer such that $|f^n(y)| < \delta$. Consider $\tilde{f}$ a small perturbation of f such that

- $\tilde{f}$ satisfies all the properties (f1) through (f4);
- $\tilde{f} \mid [-1,1] \setminus (0,\delta) \equiv f$;
- $\tilde{f}(0^+) = y$.

Then $\tilde{f}^k\big(\tilde{f}(0^+)\big) = f^k(y)$ for $k = 0, \ldots, n$ and so $\tilde{f}^n(y) \in (-\delta, \delta)$. Now we can perturb $\tilde{f}$ so that $\tilde{f}^n(y)$ changes sign and this would change one of the kneading sequences of $\tilde{f}$. Since δ can be taken arbitrarily small, then we obtain a very small perturbation of f whose kneading sequences are distinct. Since we can build a geometric Lorenz flow from $\tilde{f}$ and from any of its small perturbations, we have shown that we can always find a nearby geometric Lorenz flow Y not topologically conjugated to the given X.

2.3.6 The geometric Lorenz attractor is a homoclinic class

Here we prove Proposition 2.12 following Bautista [21].

Observe first that the *geometric Lorenz attractor* Λ *must contain a hyperbolic periodic orbit.* Indeed since the associated Lorenz transformation f is locally eventually onto, the periodic orbits of f are dense. Let $x_0,\dots,x_k$ be a periodic orbit of f. Then the leaves $\ell_0,\dots,\ell_k$ of $\mathcal{F}$ in S which project on these points form a invariant set under the map P. Since P preserves the leaves of the foliation $\mathcal{F}$ and is a contraction along $\mathcal{F}$, then P^k must send each ℓ_i into itself with a uniform contraction rate. Hence there exists a point p_i which is fixed by P^k on each leaf ℓ_i, i.e. $p_0,\dots,p_k$ is a periodic orbit of P.

The definition of P shows that the orbit of p_0 by the flow X is periodic and $\mathcal{O}_X(p_0)\cap S=\{p_0,\dots,p_k\}$.

As already observed every periodic orbit in Λ must be hyperbolic of saddle-type: the expanding and contracting directions can be easily read from the discussion in Section 2.3.3. Hence the unstable manifold of p is a disk transverse to S which intersects S in a one-dimensional manifold. The connected component of $W^u(p)\cap S$ which contains p is then a small line transverse to the foliation $\mathcal{F}$.

Now observe that since Λ is an attracting set, that is $\Lambda=\cap_{t>0}\overline{X^t(U)}$, where U is the trapping ellipsoid, then the unstable manifold $W^u(p)$ of the orbit of $p=p_0$ must be contained in Λ. Indeed if $z\in W^{uu}(p)$ then $\operatorname{dist}(X^{-t}(z),X^{-t}(p))\xrightarrow[t\to+\infty]{}0$ and hence $X^{-t}(z)\in U$ for big $t>0$, thus $z\in X^t(U)$. This shows that $W^{uu}(p)\subset\Lambda$ and since Λ is X-invariant we also get $W^u(p)\subset\Lambda$.

The definition of homoclinic class and the fact that Λ is closed imply that $H_X(p)\subset\Lambda$. For the converse we need a stronger fact.

Lemma 2.13. *If Λ is the geometric Lorenz attractor and $p\in\Lambda$ is the point of some periodic orbit, then $\Lambda=\overline{W^u(p)}$.*

Proof. Let $w \in \Lambda \setminus \{\sigma\}$ be given. Then there exists $t \geq 0$ such that $y = X^t(w) \in S$. Let $\ell = \mathcal{F}(y)$ be the corresponding leaf of $\mathcal{F}$ through y. Then ℓ is not the leaf $S \setminus S^*$. Therefore it projects to a point $x \in (-1/2, 0) \cup (0, 1/2)$. Since the pre-orbit set of every point is dense (because f is locally eventually onto), by definition of f this implies that $\Lambda \cap S = \Lambda \cap \overline{\cup_{n \geq 0} P^{-n} \ell}$.

Hence we have that $P^{-n}\ell \cap W^u(p) \neq \emptyset$ for some $n \geq 0$. But this means that $W^s(y) \cap W^u(p) \neq \emptyset$ and so $w, y \in \overline{W^u(p)}$. Thus $\Lambda \setminus \{\sigma\} \subset \overline{W^u(p)}$. □

Finally to prove that $\Lambda \subset H_X(p)$ it is enough to show that $W^u(p) \subset H_X(p)$. Every point $w \in W^u(p)$ admits $t < 0$ such that $q = X^t(w) \in S$. Take a small neighborhood J of q in $W^u(p) \cap S$, which is a small line transverse to $\mathcal{F}$.

Let l be the leaf of $\mathcal{F}$ containing p and let I be the interval inside $(-1/2, 1/2)$ corresponding to J by the projection $S \to S/\mathcal{F} = (-1/2, 1/2)$. Recall that $l \subset W^s(p) \cap S$. Write x for the point corresponding to p under this projection.

Again by Lemma 2.11 there exists $n \geq 0$ such that $f^{-n}\{x\} \cap I \neq \emptyset$. This means that $J \pitchfork P^{-n}(l) \neq \emptyset$, hence in J there exists a point of the homoclinic class of p. Since J can be taken arbitrarily small near q, we conclude that $q \in H_X(p)$. This concludes the proof that $\Lambda = H_X(p)$.

Chapter 3

Robust transitivity and singular-hyperbolicity

In the theory of differentiable dynamics for flows, i.e., in the study of the asymptotic behavior of orbits $\{X^t(x)\}_{t\in\mathbb{R}}$ for $X \in \mathfrak{X}^r(M)$, $r \geq 1$ a fundamental problem is to understand how the behavior of the tangent map DX controls or determines the dynamics of the flow X^t. Since the 1970's there is a complete description of the dynamics of a system under the assumption that the tangent map has a hyperbolic structure.

The spectral decomposition theorem, presented in Section 1.2 and first proved in [190], provides a description of the non-wandering set of a structural stable system as a finite number of disjoint *compact maximal invariant and transitive sets*, each of these pieces being well understood, both from the deterministic and from statistical viewpoints. Moreover such a decomposition persists under small C^1 perturbations. This naturally leads to the study of isolated transitive sets that remain transitive for all nearby systems (robustness).

The Lorenz equations (1.1) provide an example of a robust attractor containing an equilibrium point at the origin and periodic points accumulating on it. This is a non-uniformly hyperbolic attractor which cannot be destroyed by any small perturbation of the parameters. See Section 1.1.3 for more on this.

The existence of robust non-hyperbolic attractors for flows was first proved rigorously through the study of *geometric models for Lorenz attractors*, see Section 2.3. In particular, they exhibit in a robust way an attracting transitive set with an equilibrium (singularity) whose eigenvalues $\lambda_i, 1 \leq i \leq 3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. In the definition of geometrical models, another key requirement was the existence of an invariant foliation whose leaves are forward contracted by the flow. These features enable us to extract very complete topological, dynamical and ergodic information about these geometrical Lorenz models, as explained in Section 2.3. We prove now that these features are present for any robustly transitive set.

Hence the main properties of the Lorenz attractor and geometric Lorenz models are consequences of their robust transitivity. Building on this characterization, in Chapter 4 we elaborate on the ergodic properties of singular-hyperbolic attractors.

Definition 3.1. An isolated set Λ of a C^1 vector field X is robustly transitive if it has an open neighborhood U such that

$$\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y^t(U)$$

is both transitive and non-trivial (i.e. it is neither a singularity nor a periodic orbit) for any vector field Y C^1-close to X.

First we state the following simpler result for global transitive flows on 3-manifolds which was first proved by Doering in [53].

Theorem 3.1. *Assume $\Lambda = M$ is a robustly transitive set (on a three dimensional manifold). Then the flow is Anosov. In particular the flow has no singularities.*

In the general case, when Λ is a proper subset of M and contains singularities, we have the following characterization.

Theorem 3.2. *A robustly transitive set containing singularities of a flow on a closed* 3*-manifold is either a proper attractor or a proper repeller.*

Note that Theorem 3.2 is false for dimensions bigger than three. Indeed consider vector field $Y : (z,w) \in \mathbb{S}^3 \times \mathbb{S}^1 \mapsto \big(X(z), N(w)\big)$ in $\mathbb{S}^3 \times \mathbb{S}^1$, where

- X is the vector field given by the Lorenz equations (1.1) or the vector field obtained after the construction of any geometric Lorenz attractor (see Section 2.3.2) suitably embedded in $\mathbb{S}^3$, for example with a hyperbolic unstable equilibria at infinity;

- N is the "North-South" vector field on the circle $\mathbb{S}^1 = \{(x,y) : x^2 + y^2 = 1\}$ given by $-k \cdot \nabla(\pi \mid \mathbb{S}^1)$ where π is the projection on the second coordinate and $k > 0$ is big enough so that the expansion rate e^k at the "North" $(0,1)$ and the contraction rate e^{-k} at the "South" $(0,-1)$ dominate every eventual expansion or contraction along the directions of $T\mathbb{S}^3 \times \{0\}$.

Then $\Lambda_Y = \cap_{t>0} Y^t\big(U \times V\big)$, which is the maximal invariant subset of U with respect to Y, equals $\Lambda_X \times \{(0,1)\}$ and is transitive, where

- U is an isolating neighborhood of the (geometric) Lorenz attractor for the X-flow;

- V is a small neighborhood of $(0,1)$ in $\mathbb{S}^1$; and

- Λ_X is the (geometric) Lorenz attractor.

Notice that $\mathbb{S}^3 \times \{(0,1)\}$ is an invariant and normally hyperbolic compact submanifold of $\mathbb{S}^3 \times \mathbb{S}^1$, see [76]. It follows that for all vector fields Z C^1-close to Y, there exists an "analytic continuation" $\tilde{M}$ of the submanifold $\mathbb{S}^3 \times \{(0,1)\}$ such that

- $\tilde{M}$ is Z^t-invariant, compact and normally hyperbolic submanifold of $\mathbb{S}^3 \times \mathbb{S}^1$, in particular any smooth curve transverse to $\tilde{M}$ inside $U \times V$ is expanded by $Z^t, t > 0$.

- $\tilde{M}$ is C^1-close to $\mathbb{S}^3 \times \{(0,1)\}$ as embeddings in $\mathbb{S}^3 \times \mathbb{S}^1$.

Hence there exists a diffeomorphism $\phi : \tilde{M} \to \mathbb{S}^3$, close to the identity, and the restriction of the vector field Z to $\tilde{M}$ can be seen as a vector field C^1-close to X under a global change of coordinates extending ϕ. Therefore the maximal invariant subset of $U \times V$ for Z is $\cap_{t>0} Z^t\big(\phi^{-1} U\big) \subset \tilde{M}$, which is transitive by the robustness of the (geometric) Lorenz attractor. In this way we get a robustly transitive set Λ_Y which is neither an attractor nor a repeller.

In the setting of boundary-preserving vector fields, on 3-manifolds with boundary, the singular-horseshoe provides another counter-example (see

Remark 2.2) since it is robustly transitive in the space of vector fields preserving the boundary, but it is not an attractor nor a repeller.

The converse to Theorem 3.2 is also not true: proper attractors (or repellers) with singularities are not necessarily robustly transitive, even if their periodic points and singularities are hyperbolic in a robust way. For examples see e.g. Morales and Pujals [126].

Theorem 3.2 follows from a general result on n-manifolds, $n \geq 3$, which shows that the next conditions are sufficient for an isolated set to be an attracting set:

1. all its periodic points and singularities are hyperbolic, and

2. it robustly contains the unstable manifold of either a periodic point or a singularity.

Before the proofs let us describe a global consequence of Theorem 3.2 which improves Theorem 3.1.

Theorem 3.3. *A C^1 vector field on a 3-manifold having a robustly transitive non-wandering set is Anosov.*

Proof. Let X be a C^1 vector field satisfying the conditions of the statement above, that is: $\Omega(X)$ is (an isolated set and) robustly transitive.

If $\Omega(X)$ has singularities, then $\Omega(X)$ is either a proper attractor or a proper repeller of X by Theorem 3.2, which is impossible by Lemma 1.1 from Section 1.1. Then $\Omega(X)$ is a robustly transitive set without singularities. By [53, 205] we conclude that $\Omega(X)$ is hyperbolic and so X is Axiom A with a unique basic set in its spectral decomposition. Since Axiom A vector fields always exhibit at least one attractor and $\Omega(X)$ is the unique basic set of X, it follows that $\Omega(X)$ is an attractor. By Lemma 1.1 again this implies that $\Omega(X)$ is the whole manifold.

Hence we are in the setting of Theorem 3.1 and we conclude that X is Anosov as desired. □

Remark 3.4. As observed after the proof of Lemma 1.1 in Section 1.1 the same argument shows that Theorem 3.3 remains true if one exchanges non-wandering set by limit set in its statement.

The singularities of robust attractors are Lorenz-like

We say that an isolated set $\Lambda \subset M$ is *robustly singular* for $X \in \mathfrak{X}^1(M)$ if there is a neighborhood U of Λ in M and a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ such that $\Lambda_Y(U)$ contains a singularity for all $Y \in \mathcal{U}$.

Theorem 3.5. *Let Λ be a robustly singular transitive set of $X \in \mathfrak{X}^1(M)$. Then, either for $Y = X$ or $Y = -X$, every $\sigma \in S(Y) \cap \Lambda$ is Lorenz-like and satisfies $W_Y^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.*

As a consequence, considering *robust attractors*, that is attractors which persist for all C^1-nearby vector fields and remain transitive, we get

Theorem 3.6. *Every singularity of a robust attractor on a closed 3-manifold is Lorenz-like.*

Robust attractors are singular-hyperbolic

A compact invariant set Λ of X is *partial hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and constants $\lambda, K > 0$ such that

- E^s_Λ (K,λ)-*dominates* E^c_Λ, i.e. for all $x \in \Lambda$ and for all $t \geq 0$

$$\|DX^t(x) \mid E^s_x\| \leq \frac{e^{-\lambda t}}{K} \cdot m(DX^t(x) \mid E^c_x);$$

- E^s_Λ is (K,λ)-contracting (see Section 1.2).

We shall say that $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is a (K,λ)-*splitting* for short. For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J^c_t(x)$ be the absolute value of the determinant of the linear map $DX^t(x) \mid E^c_x : E^c_x \to E^c_{X^t(x)}$. We say that the sub-bundle E^c_Λ of the partial hyperbolic set Λ is *volume expanding* if

$$J^c_t(x) = \left|\det(DX^t \mid E^c_x)\right| \geq K e^{\lambda t},$$

for every $x \in \Lambda$ and $t \geq 0$ (in this case we say that E^c_Λ is (K,λ)-*volume expanding* to indicate the dependence on (K,λ).

It is known (see e.g. [132]) that a non-singular partially hyperbolic set for a three-dimensional flow, with volume expanding central direction, is uniformly hyperbolic.

Definition 3.2. A partial hyperbolic set is *singular-hyperbolic* if its singularities are hyperbolic and it has volume expanding central direction.

A *singular-hyperbolic attractor* is a singular-hyperbolic set which is an attractor as well: an example is the (geometric) Lorenz attractor presented in Section 2.3. A *singular-hyperbolic repeller* of X is a singular-hyperbolic attractor of $-X$. An example of a singular-hyperbolic set which is neither an attractor or a repeller is the singular horseshoe presented in Section 2.1.

The following result characterizes robust attractors for three-dimensional flows.

Theorem 3.7. *Robust attractors of $X \in \mathfrak{X}^1(M^3)$ containing singularities are singular-hyperbolic sets for X.*

Note that robust attractors cannot be C^1 approximated by vector fields presenting either attracting or repelling periodic points. This implies that, on 3-manifolds, any periodic point lying in a robust attractor is hyperbolic of saddle-type. Thus, as in Liao [98, Theorem A], we conclude that robust attractors *without singularities* on closed 3-manifolds are hyperbolic. Therefore we obtain a dichotomy as follows.

Theorem 3.8. *Let Λ be a robust attractor of $X \in \mathfrak{X}^1(M)$. Then Λ is either hyperbolic or singular-hyperbolic.*

Brief sketch of the proofs

To prove Theorem 3.2 we first obtain a sufficient condition for a transitive isolated set with hyperbolic critical elements of a C^1 vector field on a n-manifold, $n \geq 3$, to be an attractor (Theorem 3.15). We use this to prove that a robustly transitive set whose critical elements are hyperbolic is an attractor if it contains a singularity whose unstable manifold has dimension one (Theorem 3.16). This implies that C^1 robustly transitive sets with singularities on closed 3-manifolds are either proper attractors or proper repellers (Theorem 3.2).

The characterization of singularities in a robust transitive set (Theorem 3.5) is obtained by contradiction. Using the Connecting Lemma (see Section 1.3.8), we can produce special types of cycles (inclination-flip or Shil'nikov, see Chapter 2) associated to a singularity leading to nearby vector fields which exhibit attracting or repelling periodic points. This contradicts the robustness of the transitivity condition.

Theorem 3.7 is proved in Section 3.3. We start by proposing an invariant splitting over the periodic points lying in Λ and prove uniform estimates on angles between stable, unstable, and central unstable bundles for periodic points. Roughly speaking, if such angles are not uniformly bounded away from zero, we construct a new vector field near the original one exhibiting either a sink or a repeller, yielding a contradiction. Such a perturbation is obtained using the extension for flows of a perturbation Lemma of Franks, given by Theorem 1.20. This allows us to prove that the splitting proposed for the periodic points is partially hyperbolic with volume expanding central direction. We then extend this splitting to the closure of the periodic points. We show that the splitting proposed for the periodic points is compatible with the local partial hyperbolic splitting at the singularities (Proposition 3.37) using that the linear Poincaré flow has a dominated splitting outside the singularities ([205, Theorem 3.8] stated as Theorem 1.8 in Section 1.3.3); and that the non-wandering set outside a neighborhood of the singularities is hyperbolic (Lemma 3.39). We next extend this splitting to all of Λ, obtaining Theorem 3.7.

3.1 Consequences of singular-hyperbolicity

Under the sole assumption of singular-hyperbolicity one can show that at each point there exists a strong stable manifold; more precisely, the attractor is a subset of a lamination by strong stable manifolds. It is also possible to show the existence of local central manifolds tangent to the central unstable direction, see [76] and Section 3.4.1. Although these central manifolds do not behave as unstable manifolds, in the sense that their points are not necessarily asymptotic in the past, the fact that the flow expands volume along the central unstable direction implies rather strong properties.

We list some of these properties that give us a nice description of the dynamics of robustly transitive sets with singularities and, in particular, for robust attractors, or of singular-hyperbolic attracting sets with a dense subset of periodic orbits.

The first two properties do not depend either on the fact that the set is robust transitive or an attractor, but only on the fact that the set has a dominated splitting and that the flow expands volume in the central-unstable direction.

Proposition 3.9. *Let Λ be a singular-hyperbolic compact set of $X \in \mathfrak{X}^1(M)$. Then any invariant compact set $\Gamma \subset \Lambda$ without singularities is a uniformly hyperbolic set.*

For singular hyperbolic attracting sets having only one singularity we can obtain a partial converse to the results in the previous section. As commented below, Arroyo and Pujals have recently obtained in [15] a condition on a singular-hyperbolic attractor implying robust transitivity.

We start by stating a corollary of the arguments used to prove Theorems 3.5 and 3.6 (see Remarks 3.23 and 3.29 in the following sections). Observe that we assume partial hyperbolicity with volume expanding central direction but do not assume transitivity.

Theorem 3.10. *Let Λ be a nonempty compact invariant isolated set for a three-dimensional flow $X \in \mathfrak{X}^1$. Assume that Λ is partially hyperbolic with volume expanding central direction. If σ is a singularity accumulated by regular orbits in Λ, then*

- *either σ is Lorenz-like for X and $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$;*
- *or σ is Lorenz-like for $-X$ and $W^{uu}_X(\sigma) \cap \Lambda = \{\sigma\}$.*

This shows that partial hyperbolicity and volume expansion on an isolated set alone imply that the possible singularities are Lorenz-like, either for the positive or for the negative time flow.

Definition 3.3. Let Λ be an isolated set of $X \in \mathfrak{X}^r(M)$. We say that Λ is C^r *robustly periodic* if there are an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^r(M)$ such that $\Lambda_Y(U) = \overline{\mathrm{Per}(Y) \cap \Lambda_Y(U)}$ for all $Y \in \mathcal{U}$.

Examples of C^1 robustly periodic sets are the hyperbolic attractors and the geometric Lorenz attractor (see Sections 1.2 and 2.3). These examples are also C^1 robustly transitive. On the other hand, the singular horseshoe (from Section 2.1) and the example by Morales and Pujals in [126] are neither C^1 robust transitive nor C^1 robustly periodic. These examples motivate the question whether *all* C^1 robust transitive sets for vector fields are C^1 robustly periodic. Arroyo and Pujals have recently obtained a positive answer to this question in [15], see below.

Nevertheless, on compact 3-manifolds, C^r robustly periodic sets are C^r robust *among singular-hyperbolic attractors with only one singularity*.

Theorem 3.11. *A C^r robustly periodic singular-hyperbolic attractor, with only one singularity, on a compact* 3*-manifold is C^r robust.*

This result, first proved in [131], gives explicit sufficient conditions for robustness of attractors *depending on the perturbed flow*. One should aim to obtain sufficient conditions *depending only on the unperturbed flow*. This was recently achieved by Arroyo and Pujals in [15], where they obtain a criteria for a singular-hyperbolic attractor to be C^1 robustly transitive, depending only on the attractor and with no restriction on the number of singularities. As a consequence in this setting the attractor is automatically robustly periodic.

Theorem 3.12. *Let $\Lambda = \Lambda_X(U)$ be a singular-hyperbolic attractor of $X \in \mathfrak{X}^1(X)$ with isolating neighborhood U. Then the set of periodic orbits is dense in Λ and Λ is the homoclinic class of at least one of those orbits.*

Moreover, assume there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ the positive maximal invariant subset of $U \setminus B_\delta\big(S(X)\big)$ is transitive. Then Λ is C^1 robustly transitive.

We present a proof of Theorem 3.11 following [131] in Section 3.4.

The next two results show that important features of hyperbolic attractors and of the geometric Lorenz attractor are present for singular-hyperbolic attractors, and so, for robust attractors with singularities.

Proposition 3.13. *A singular-hyperbolic attractor Λ of $X \in \mathfrak{X}^1(M)$ has positive Lyapunov exponent uniformly bounded away from zero at every orbit.*

The following generic property in the space $\mathfrak{X}^1(M)$ can also be deduced from Theorems 3.2 and 3.5.

Proposition 3.14. *For X in a residual subset (a set containing an intersection of a enumerable family of open dense subsets) of $\mathfrak{X}^1(M)$, each robust transitive set with singularities is the closure of the stable or unstable manifold of one of its hyperbolic periodic points.*

Now we present the proofs of these propositions assuming the results stated in the previous section.

Proof of Proposition 3.9: The argument relies on the fact that the intersection of the dominated splitting $E^s \oplus E^{cu}$ with the normal bundle N_Γ over Γ

induces a hyperbolic splitting for the linear Poincaré flow defined over Γ (recall the definition of the Linear Poincaré Flow in Section 1.3.3). Thus by Theorem 1.7 we conclude that Γ is uniformly hyperbolic, finishing the proof.

From the fact that Γ does not contain singularities, there exists $K > 0$ such that $1/K < \|X(x)\| < K$ for every $x \in \Gamma$. Consider the following splitting on the normal bundle N_Γ: define $N^u_x = E^{cu}_x \cap N_x$ and $E^{cs}_x \cap N_x$ for $x \in \Gamma$, where $E^{cs}_x = E^X_x \oplus E^s_x$.

Now we show that this splitting is hyperbolic for the linear Poincaré flow P_t over Γ. Note that for any $t \in \mathbb{R}$ the Jacobian of DX^t along the sub-bundle E^{cu}_x) can be given by

$$\sin\angle\big(DX^t(x)\cdot n^u_x, X(X^t(x))\big)\cdot\|DX^t(x)\cdot n^u_x\|\cdot\frac{\|X(X^t(x))\|}{\|X(x)\|},$$

where $n^u_x \in N^u_x$ is any choice of a unit vector. The last expression is the same as

$$\big\|O_{X^t(x)}\big(DX^t(x)\cdot n^u_x\big)\|\cdot\frac{\|X(X^t(x))\|}{\|X(x)\|},$$

where $O_{X^t(x)}$ denotes the orthogonal projection from $T_{X^t(x)}M$ onto $N_{X^t(x)}$, recall Section 1.3.3. Thus

$$\big|\det(DX^t \mid E^{cu}_x)\big| = \|O_{X^t(x)}(DX^t(x)\cdot n^u_x)\|\cdot\frac{\|X(X^t(x))\|}{\|X(x)\|}. \tag{3.1}$$

Since the central direction is (c,λ)-volume expanding, we know that the value of the expression in (3.1) is bigger than $c\cdot e^{\lambda t}$. Hence we get

$$\big\|O_{X^t(x)}(DX^t(x)\cdot n^u_x)\big\| > \frac{c}{K}\cdot e^{\lambda t} \text{ for all } t \geq 0.$$

This proves that N^u is uniformly expanded by P^t.

To see that N^s is uniformly contracted by the linear Poincaré flow, first note that the splitting $E^s \oplus E^{cu}$ is partially hyperbolic along Γ. Thus there exists $A > 0$ such that $\angle(E^s_x, X(x)) \geq A$ for every $x \in \Gamma$. Hence we can find a_0 such that for all $x \in \Gamma$ and $v \in N^s_x$ with $\|v\| = 1$, there is $w \in E^s_x$ with

$\|w\| = 1$ such that $v = aw + b \cdot \frac{X(x)}{\|X(x)\|}$ with $|a| < a_0$. Therefore we have

$$\begin{aligned}\left\|O_{X^t(x)}(DX^t(x)\cdot v)\right\| &= \left\|O_{X^t(x)}\Big(DX^t(x)\cdot\big(aw + b\cdot\frac{X(x)}{\|X(x)\|}\big)\Big)\right\| \\ &= \left\|O_{X^t(x)}(DX^t(x)\cdot(aw))\right\| \\ &\le \left\|DX^t(x)\cdot(aw)\right\| \le a_0\cdot K\cdot e^{-\lambda t}\end{aligned}$$

for some $K, \lambda > 00$ (recall that E^s is (K,λ) contracting). Thus N^s is uniformly contracted by P_t. Proposition 3.9 is proved. □

Proof of Proposition 3.13: Let Λ be as in the statement of Proposition 3.13. Given $x \in \Lambda$, if x is a singularity then the result follows from the fact that x is Lorenz-like for X. Now assume $X(x) \neq 0$ and take $v \in E^{cu}_x$ with $\|v\| = 1$ and orthogonal to $X(x)$. We have for some $c, \lambda > 0$

$$\begin{aligned}c\cdot e^{\lambda t} \le |\det(DX^t \mid E^{cu}_x)| &\le \left\|DX^t(x)\cdot v\right\|\cdot\frac{\|DX^t(x)\cdot X(x)\|}{\|X(x)\|} \\ &= \left\|DX^t(x)\cdot v\right\|\cdot\frac{\|X(X^t(x))\|}{\|X(x)\|}\end{aligned}$$

and then for $t > 0$ we get

$$\frac{1}{t}\log\left\|DX^t(x)\cdot v\right\| \ge \lambda + \frac{1}{t}\log c - \frac{1}{t}\log\frac{\|X(X^t(x))\|}{\|X(x)\|}.$$

Since $\|X(X^t(x))\|$ is uniformly bounded for all $t > 0$ by compactness of Λ, we see that $\limsup_{t\to+\infty} t^{-1}\log\left\|DX^t(x)\cdot v\right\| > 0$. □

Proof of Proposition 3.14: Let $\Lambda = \Lambda_X(U)$ be a robustly transitive set with singularities for $X \in \mathfrak{X}^1(M)$ with isolating neighborhood U. By Theorems 3.2 and 3.7 we can assume that Λ is a partially hyperbolic attractor for X. By the genericity properties from Section 1.3.6 (the Kupka-Smale property, item 1) for a generic subset $\mathcal{G}$ in a C^1 neighborhood $\mathcal{V}$ of X we have that $\Lambda = \Lambda_Y(U)$ has a hyperbolic period orbit p, for all $Y \in \mathcal{G}$.

As Λ is an attractor, the unstable manifold $W^u(p)$ of any periodic point p of Λ is contained in Λ. In particular its the closure $\overline{W^u(p)}$ is contained in Λ. We show that Λ is contained in $\overline{W^u(p)}$.

Let $q \in \Lambda$ be such that $\Lambda = \omega_Y(q)$ (recall that an attractor is transitive by definition). Let V be a small neighborhood of p. On the one hand, by transitivity, we can assume without loss of generality that $q \in V$. On the other hand, since Λ is partially hyperbolic, projecting q into $W^u(p)$ through the stable manifold of q, we can assume that q is actually contained in $W^u(p)$. Indeed, being in the same stable manifold, q and its projection have the same ω-limit sets.

Finally observe that $\omega_Y(q) \subset \overline{W^u(p)}$ because $W^u(p)$ is invariant by the flow. Thus $\Lambda = \omega_Y(q) \subset \overline{W^u(p)}$ finishing the proof. □

3.2 Attractors and isolated sets for C^1 flows

Here prove Theorems 3.2 and 3.5. We start by focusing on isolated sets, obtaining the following sufficient conditions for an isolated set of a C^1 flow on an n-manifold, $n \geq 3$, to be an attractor:

- all its periodic points and singularities are hyperbolic, and
- it contains, in a robust way, the unstable manifold of some critical element.

Using this we prove that isolated sets Λ satisfying the following conditions are attractors:

- the critical elements $C(X) \cap \Lambda$ are hyperbolic;
- Λ contains a singularity with one-dimensional unstable manifold, and
- Λ is
  - either robustly non-trivial and transitive (robustly transitive),
  - or $\Lambda = \overline{C(X) \cap \Lambda}$ is robustly the closure of its periodic points (C^1 robustly periodic).

In particular robustly transitive sets with singularities on closed 3-manifolds are either proper attractors or proper repellers, proving Theorem 3.2. Then we characterize the singularities on robustly transitive sets on 3-manifolds, obtaining Theorem 3.5.

Elementary topological dynamics ensures that an attractor containing a hyperbolic critical element contains the unstable manifold of this critical element. The converse, although false in general, is true for a residual subset of C^1 vector fields, as shown in [42]. We derive a sufficient condition for the converse to hold inspired by the following property of uniformly hyperbolic attractors (see e.g. [146]): if Λ is a uniformly hyperbolic attractor of a vector field X, then there is an isolating block U of Λ and $x_0 \in C(X) \cap \Lambda$ such that $W^u_Y\big(x_0(Y)\big) \subset U$ for every Y close to X, where $x_0(Y)$ is the hyperbolic continuation of x_0 for Y. This property motivates the following definition.

Definition 3.4. Let Λ be an isolated set of $X \in \mathfrak{X}^r(M)$, $r \geq 1$. We say that Λ *robustly contains the unstable manifold of a critical element* if there are $x_0 \in C(X) \cap \Lambda$ hyperbolic, an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^r(M)$ such that $W^u_Y(x_0(Y)) \subset U$, for all $Y \in \mathcal{U}$.

With this definition in mind we are able to prove

Theorem 3.15. *Let Λ be a transitive isolated set of $X \in \mathfrak{X}^1(M)$ where M is a compact n-manifold, $n \geq 3$, and suppose that every $x \in C(X) \cap \Lambda$ is hyperbolic. If Λ robustly contains the unstable manifold of a critical element, then Λ is an attractor.*

Now we derive an application of Theorem 3.15. Recall Definition 3.3 of C^r *robustly periodic set*.

The geometric Lorenz attractor is a robustly transitive (periodic) set, and it is an attractor satisfying (see Section 2.3)

- all its periodic points are hyperbolic, and

- it contains a singularity whose unstable manifold has dimension one.

The result below shows that such conditions are enough for a robustly transitive (periodic) set to be an attractor.

Theorem 3.16. *Let Λ be either a robustly transitive or a transitive C^1 robustly periodic set of $X \in \mathfrak{X}^1(M)$, where M is a n-dimensional compact manifold, $n \geq 3$. If*

1. *every $x \in C(X) \cap \Lambda$ is hyperbolic and*

2. *Λ has a singularity whose unstable manifold is one-dimensional,*

then Λ is an attractor of X.

Theorem 3.16 follows from Theorem 3.15 showing that Λ robustly contains the unstable manifold of the singularity provided by condition 2 above.

The following lemma is well known, see for instance [48, p.3].

Lemma 3.17. *Let Λ be an isolated set of $X \in \mathfrak{X}^r(M)$, $r \geq 0$. Then for every isolating block U of Λ and every open set V containing Λ, there is a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^r(M)$ such that $\Lambda_Y(U) \subset V$ for all $Y \in \mathcal{U}$.*

Proof. We have by assumption $\Lambda = \Lambda_X(U) = \cap_{t\in\mathbb{R}} X^t(U) = \cap_{t\in\mathbb{R}} X^t(\overline{U})$. For any neighborhood V of Λ there is a big enough $L > 0$ such that

$$\bigcap_{-L\leq t\leq L} X^t(\overline{U}) \subset V.$$

Then using the continuous dependence of the flow with the vector field and the compactness of $\overline{U}$, there exists a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^r(M)$ such that

$$\bigcap_{-L\leq t\leq L} Y^t(\overline{U}) \subset V \quad \text{for all} \quad Y \in \mathcal{U}.$$

Thus $\Lambda_Y(U) \subset V$ for all $Y \in \mathcal{U}$. □

Lemma 3.18. *If Λ is an attracting set and a repelling set of $X \in \mathfrak{X}^1(M)$, then $\Lambda = M$.*

Proof. Suppose that Λ is an attracting set and a repelling set of X. Then there are neighborhoods V_1 and V_2 of Λ satisfying $X^t(V_1) \subset V_1$, $X^{-t}(V_2) \subset V_2$ for every $t \geq 0$,

$$\Lambda = \bigcap_{t\geq 0} X^t(V_1) \quad \text{and} \quad \Lambda = \bigcap_{t\geq 0} X^{-t}(V_2).$$

Define $U_1 = \mathrm{int}(V_1)$ and $U_2 = \mathrm{int}(V_2)$. Clearly $X^t(U_1) \subset U_1$ and $X^{-t}(U_2) \subset U_2$ for all $t \geq 0$, since X^t is a diffeomorphism. As U_2 is open and contains Λ, the first equality implies that there is $t_2 > 0$ such that $X^{t_2}(V_1) \subset U_2$ (see for instance [83, Lemma 1.6]). As $X^{t_2}(U_1) \subset X^{t_2}(V_1)$ it follows that $U_1 \subset X^{-t_2}(U_2) \subset U_2$ proving

$$U_1 \subset U_2.$$

Similarly, as U_2 is open and contains Λ, the second equality implies that there is $t_1 > 0$ such that $X^{-t_1}(V_2) \subset U_1$. As $X^{-t_1}(U_2) \subset X^{-t_1}(V_2)$ it follows that $U_2 \subset X^{t_1}(U_1) \subset U_1$ proving

$$U_2 \subset U_1.$$

Thus, $U_1 = U_2$. From this we obtain $X^t(U_1) = U_1$ for all $t \geq 0$ proving $\Lambda = U_1$. As Λ is compact by assumption we conclude that Λ is open and closed. As M is connected and Λ is not empty we obtain that $\Lambda = M$ as desired. □

The lemma below gives a sufficient condition for an isolated set to be attracting.

Lemma 3.19. *Let Λ be an isolated set of $X \in \mathfrak{X}^1(M)$. If there are an isolating block U of Λ and an open set W containing Λ such that $X^t(W) \subset U$ for every $t \geq 0$, then Λ is an attracting set of X.*

Proof. Let Λ and X be as in the statement. To prove that Λ is attracting we have to find a neighborhood V of Λ such that $X^t(V) \subset V$ for all $t > 0$ and

$$\Lambda = \cap_{t \geq 0} X^t(V). \tag{3.2}$$

To construct V we let W be the open set in the statement of the lemma and define $V = \cup_{t>0} X^t(W)$. Clearly V is a neighborhood of Λ satisfying $X^t(V) \subset V$ for each $t > 0$.

We claim that V satisfies (3.2). Indeed, as $X^t(W) \subset U$ for every $t > 0$ we have that $V \subset U$ and so $\cap_{t \in \mathbb{R}} X^t(V) \subset \Lambda$ because U is an isolating block of Λ. But $V \subset X^{-t}(V)$ for every $t \geq 0$ since V is forward invariant. So $V \subset \cap_{t \leq 0} X^t(V)$ and from this we have

$$\begin{aligned} \bigcap_{t \geq 0} X^t(V) &\subset V \cap \bigcap_{t>0} X^t(V) \\ &\subset \bigcap_{t \leq 0} X^t(V) \cap \bigcap_{t>0} X^t(V) = \bigcap_{t \in \mathbb{R}} X^t(V). \end{aligned}$$

Thus, $\cap_{t \geq 0} X^t(V) \subset \Lambda$. Now, as $\Lambda \subset V$ and Λ is invariant, we have $\Lambda \subset X^t(V)$ for every $t \geq 0$. Then $\Lambda \subset \cap_{t \geq 0} X^t(V)$, proving (3.2). □

3.2.1 Proof of sufficient conditions to obtain attractors

The proof of Theorem 3.15 is based on the following lemma.

Lemma 3.20. *Let Λ be a transitive isolated set of $X \in \mathfrak{X}^1(M)$ such that every $x \in C(X) \cap \Lambda$ is hyperbolic. Suppose that the following condition holds:*

(H3) *There are $x_0 \in C(X) \cap \Lambda$, an isolating block U of Λ and a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ such that*

$$W^u_Y(x_0(Y)) \subset U, \quad \forall \quad Y \in \mathcal{U}.$$

Then $W^u_X(x) \subset \Lambda$ for every $x \in C(X) \cap \Lambda$.

Proof. Let x_0, U and $\mathcal{U}$ as in (H3). By assumption $\mathcal{O}_X(x_0)$ is hyperbolic. If $\mathcal{O}_X(x_0)$ is attracting then $\Lambda = \mathcal{O}_X(x_0)$ since Λ is transitive and we are done. We can then assume that $\mathcal{O}_X(x_0)$ is not attracting. Thus, $W^u_X(x_0) \setminus \mathcal{O}_X(x_0) \neq \emptyset$.

By contradiction, suppose that there is $x \in C(X) \cap \Lambda$ such that $W^u_X(x)$ is not contained in Λ. Then $W^u_X(x)$ is not contained in $\overline{U}$. As $M \setminus \overline{U}$ is open there is a cross-section $\Sigma \subset M \setminus \overline{U}$ of X such that $W^u_X(x) \cap \Sigma \neq \emptyset$ is transverse. Shrinking $\mathcal{U}$ if necessary we can assume that $W^u_Z(x(Z)) \cap \Sigma \neq \emptyset$ is transverse for every $Z \in \mathcal{U}$.

Now $W^u_X(x_0) \subset \Lambda$ by (H3) applied to $Y = X$. Choose $p \in W^u_X(x_0) \setminus \mathcal{O}_X(x_0)$. As Λ is transitive and $p, x \in \Lambda$, there is $q \in W^s_X(x) \setminus \mathcal{O}_X(x)$ such that p, q satisfy (H1) in Theorem 1.18 on page 31. Indeed, the dense orbit of Λ accumulates both p and x. Then, by Theorem 1.18, there are $Z \in \mathcal{U}$ and $T > 0$ such that $p \in W^u_Z(x(Z))$, $q \in W^s_Z(x(Z))$ and $Z_T(p) = q$. In other words, $\mathcal{O}_Z(q)$ is a saddle connection between $x_0(Z)$ and $x(Z)$. On the other hand, as $Z \in \mathcal{U}$, we have that $W^u_Z(x(Z)) \cap \Sigma \neq \emptyset$ is transverse. It follows from the λ-Lemma (see Section 1.3.5 of Chapter 1) that $Z^t(\Sigma)$ accumulates on q as $t \to \infty$. This allows us to break the saddle-connection $\mathcal{O}_Z(q)$ in the standard way in order to find $Z' \in \mathcal{U}$ such that $W^u_{Z'}(x_0(Z')) \cap \Sigma \neq \emptyset$ (see e.g. [143] or the proof of Theorem 2.4 in [129]). In particular, $W^u_{Z'}(x_0(Z'))$ is not contained in U. This contradicts (H3) and the lemma follows. □

Proof of Theorem 3.15. Let Λ and X be as in the statement of Theorem 3.15. It follows that there are $x_0 \in C(X) \cap \Lambda$, U and $\mathcal{U}$ such that (H3) holds.

Next we prove that Λ satisfies the hypothesis of Lemma 3.19, that is, there is an open set W containing Λ such that $X^t(W) \subset U$ for every $t \geq 0$.

Indeed, suppose that such a W does not exist. Then, there are sequences $x_n \to x \in \Lambda$ and $t_n > 0$ such that $X^{t_n}(x_n) \in M \setminus U$. By compactness we can assume that $X^{t_n}(x_n) \to q$ for some $q \in \overline{M \setminus U}$.

Fix an open set $V \subset \overline{V} \subset U$ containing Λ. As $q \in \overline{M \setminus U}$,

$$\overline{M \setminus U} \subset M \setminus \mathrm{int}(U), \quad \text{and} \quad M \setminus \mathrm{int}(U) \subset M \setminus \overline{V}$$

we have that $q \notin \overline{V}$. By Lemma 3.17 there is a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of X such that

$$\Lambda_Y(U) \subset V, \quad \text{for all} \quad Y \in \mathcal{U}_0. \tag{3.3}$$

Then condition (H3), the invariance of $W^u_Y(x_0(Y))$ and the relation (3.3) imply

$$W^u_Y(x_0(Y)) \subset V \subset \overline{V}, \quad \text{for every} \quad Y \in \mathcal{U}_0. \tag{3.4}$$

Now we have two cases:

1. either $x \notin C(X)$;

2. or $x \in C(X)$.

In Case 1 we obtain a contradiction as follows. Let $\mathcal{O}_X(z)$ be the dense orbit of Λ, i.e. $\Lambda = \omega_X(z)$. Fix $p \in W^u_X(x_0) \setminus \mathcal{O}_X(x_0)$. Then $p \in \Lambda$ by (H3) applied to $Y = X$. As $x \in \Lambda$ we can choose sequences $z_n \in \mathcal{O}_X(z)$ and $t'_n > 0$ such that

$$z_n \to p \quad \text{and} \quad X^{t'_n}(z_n) \to x.$$

It follows that p, q, x satisfy (H2) of Theorem 1.19 for $Y = X$. Then from Theorem 1.19 there is $Z \in \mathcal{U}_0$ such that $q \in W^u_Z(x_0(Z))$. As $q \notin \overline{V}$ we have that $W^u_Z(x_0(Z))$ is not contained in U. And this is a contradiction by (3.4) since $Z \in \mathcal{U}_0$.

In Case 2 we use (H3) to obtain a contradiction as follows. By assumption $\mathcal{O}_X(x)$ is a hyperbolic closed orbit. Clearly $\mathcal{O}_X(x)$ is neither attracting nor repelling. In particular $W^u_X(x) \setminus \mathcal{O}_X(x) \neq \emptyset$. But $x_n \notin W^s_X(x)$ since $x_n \to x$ and $X^{t_n}(x_n) \notin U$. Then, using linearizing coordinates given by the Grobman-Hartman Theorem around $\mathcal{O}_X(x)$ (see Section 1.3.4), we can find x'_n in the positive orbit of x_n such that $x'_n \to r \in W^u_X(x) \setminus \mathcal{O}_X(x)$. Note that $r \notin C(X)$ and that there are $t'_n > 0$ such that $X^{t'_n}(x'_n) \to q$.

Since (H3) holds, by Lemma 3.20 we have $W^u_X(x) \subset \Lambda$. This implies that $r \in \Lambda$. Then we have Case 1 replacing x by r, t_n by t'_n and x_n by x'_n. As Case 1 results in a contradiction, we conclude that Case 2 also results in a contradiction.

Hence Λ satisfies the hypothesis of Lemma 3.19, and Theorem 3.15 follows. □

Proof of Theorem 3.16. Let Λ be either a robust transitive set or a transitive C^1 robust periodic set of $X \in \mathfrak{X}^1(M)$ satisfying the following conditions:

1. Every critical element of X in Λ is hyperbolic.

2. Λ contains a singularity σ with $\dim(W^u_X(\sigma)) = 1$.

On the one hand, if Λ is robustly transitive, we can fix by Definition 3.1 a neighborhood $\mathcal{U}$ of X and an isolating block U of Λ such that $\Lambda_Y(U)$ is a non-trivial transitive set of Y, for every $Y \in \mathcal{U}$. Clearly we can assume that the continuation $\sigma(Y)$ is well defined for all $Y \in \mathcal{U}$. Since transitive sets are connected sets, we have:

(C) $\Lambda_Y(U)$ is connected for each $Y \in \mathcal{U}$.

On the other hand, if Λ is C^1 robustly periodic, we can fix by Definition 3.3 a neighborhood $\mathcal{U}$ of X and an isolating block U of Λ such that for each $Y \in \mathcal{U}$ we have $\Lambda_Y(U) = \overline{\mathrm{Per}(Y) \cap \Lambda_Y(U)}$. Assuming that $\sigma(Y)$ is well defined for $Y \in \mathcal{U}$ we have

(C') $\sigma(Y) \in \overline{\mathrm{Per}(Y) \cap \Lambda_Y(U)}$, for every $Y \in \mathcal{U}$.

Claim 3.1. Λ *robustly contains the unstable manifold of a critical element.*

By Definition 3.4, if $\mathcal{U}$ is the neighborhood of X described in either Property (C) or (C'), then it suffices to prove $W^u_Y(\sigma(Y)) \subset \overline{U}$ for all $Y \in \mathcal{U}$.

Arguing by contradiction, suppose that there exists $Y \in \mathcal{U}$ such that $W^u_Y(\sigma(Y))$ *is not contained in* U.

¿From Condition 2 above it follows that $W^u_X(\sigma) \setminus \{\sigma\}$ has two branches which we denote by w^+ and w^- respectively. Fix $q^+ \in w^+$ and $q^- \in w^-$. Denote by $q^\pm(Y)$ the continuation of $q^\pm$ for Y close to X. We can assume that the $q^\pm(Y)$ are well defined for all $Y \in \mathcal{U}$.

As $q^\pm(Y) \in W^u_Y(\sigma(Y))$, the negative orbit of $q^\pm(Y)$ converges to $\sigma(Y) \in \mathrm{int}(U) \subset U$. If the positive orbit of $q^\pm(Y)$ is in U, then $W^u_Y(\sigma(Y)) \subset U$, which is a contradiction. Consequently the positive orbit of either $q^+(Y)$ or $q^-(Y)$ leaves U. It follows that there is $t>0$ such that either $Y^t(q^+(Y))$ or $Y^t(q^-(Y)) \notin U$. Assume the first case. The other case is analogous. As $M \setminus U$ is open, the continuous dependence of the unstable manifolds implies that there is a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of Y such that

$$Z^t(q^+(Z)) \notin U, \quad \text{for every} \quad Z \in \mathcal{U}'. \tag{3.5}$$

Now we split the proof into two cases.

Case I: Λ is robustly transitive.

In this case $\Lambda_Y(U)$ is a non-trivial transitive set of Y. Fix $z \in \Lambda_Y(U)$ such that $\omega_Y(z) = \Lambda_Y(U)$. As $\sigma(Y) \in \Lambda_Y(U)$ it follows that either $q^+(Y)$ or $q^-(Y) \in \omega_Y(z)$. As $Y \in \mathcal{U}'$, the relation (3.5) implies $q^-(Y) \in \omega_Y(z)$. Thus, there is a sequence $z_n \in \mathcal{O}_Y(z)$ converging to $q^-(Y)$. Similarly there is a sequence $t_n > 0$ such that $Y^{t_n}(z_n) \to q$ for some $q \in W^s_Y(\sigma(Y) \setminus \{\sigma(Y)\}$. Define $p = q^-(Y)$.

It follows that p, q, Y satisfy (H1) in Theorem 1.18, and so, there is $Z \in \mathcal{U}'$ such that $q^-(Z) \in W^s_Z(\sigma(Z))$. This gives a homoclinic connection associated to $\sigma(Z)$. Breaking this connection as in the proof of Lemma 3.20, we can find $Z' \in \mathcal{U}'$ close to Z and $t' > 0$ such that

$$Z'_{t'}(q^-(Z')) \notin U. \tag{3.6}$$

Now, (3.5), (3.6) together with the Grobman-Harman Theorem 1.9, imply that the set $\{\sigma(Z')\}$ is isolated in $\Lambda_Z(U)$. But $\Lambda_{Z'}(U)$ is connected by Property (C) since $Z' \in \mathcal{U}' \subset \mathcal{U}$. Then $\Lambda_{Z'}(U) = \{\sigma(Z')\}$, a contradiction since $\Lambda_{Z'}(U)$ is non-trivial. This proves Claim 3.1 in this case.

Case II: Λ is C^1 robustly periodic.

The proof is similar to the previous one. In this case $\Lambda_Y(U)$ is the closure of its periodic orbits and $\dim(W^u_Y(\sigma(Y)) = 1$. As the periodic points of $\Lambda_Y(U)$ do accumulate either $q^+(Y)$ or $q^-(Y)$, relation (3.5) implies that there is a sequence $p_n \in \mathrm{Per}(Y) \cap \Lambda_Y(U)$ such that $p_n \to q^-(Y)$. Clearly there is another sequence $p'_n \in \mathcal{O}_Y(p_n)$ now converging to some $q \in W^s_Y(\sigma(Y) \setminus \{\sigma(Y)\}$. Set $p = q^-(Y)$.

Again p,q,Y satisfy (H1) in Theorem 1.18, and so, there is $Z \in \mathcal{U}'$ such that $q^-(Z) \in W^s_Z(\sigma(Z))$. As before we have a homoclinic connection associated to $\sigma(Z)$. Breaking this connection we can find $Z' \in \mathcal{U}'$ close to Z and $t' > 0$ such that

$$Z'_{t'}(q^-(Z')) \notin U.$$

Again this relation together with the Grobman-Harman Theorem 1.9 and the relation (3.5) would imply that every periodic point of Z' passing close to $\sigma(Z')$ is not contained in $\Lambda_{Z'}(U)$. But this contradicts Property (C') since $Z' \in \mathcal{U}' \subset \mathcal{U}$. This completes the proof of Claim 3.1 in this case.

It follows that Λ is an attractor by condition (1) above, Theorem 3.15 and Claim 3.1. This completes the proof of Theorem 3.16. □

3.2.2 Robust singular transitivity implies attractors or repellers

In this section M is a closed 3-manifold and Λ is a robustly transitive set of $X \in \mathfrak{X}^1(M)$.

According to Definition 3.1 we can fix an isolating block U of Λ and a neighborhood $\mathcal{U}_U$ of X such that $\Lambda_Y(U) = \cap_{t\in\mathbb{R}} Y^t(U)$ is a non-trivial transitive set of Y, for every $Y \in \mathcal{U}_U$. Robustness of transitivity implies that $X \in \mathcal{U}_U$ cannot be C^1-approximated by vector fields exhibiting either sinks or sources in U. And since $\dim(M) = 3$ this easily implies the following.

Lemma 3.21. *Let $X \in \mathcal{U}_U$. Then X has neither sinks nor sources in U, and any $p \in \mathrm{Per}(X) \cap \Lambda_X(U)$ is hyperbolic.*

The following result shows that singularities in this setting are Lorenz-like, either for the given flow X or for the reversed flow $-X$.

Lemma 3.22. *Let $Y \in \mathcal{U}_U$ and $\sigma \in S(Y) \cap \Lambda_Y(U)$ be such that*

(i) every C^1-nearby flow admits neither sinks nor sources in U;

(ii) every critical element in U is hyperbolic, and

(iii) σ is accumulated by regular orbits in Λ.

Then

1. *the eigenvalues of* σ *are real.*
2. *if* $\lambda_2 \le \lambda_3 \le \lambda_1$ *are the eigenvalues of* σ*, then* $\lambda_2 < 0 < \lambda_1$.
3. *for* λ_i *as above we have*
   (a) $\lambda_3(\sigma) < 0 \Longrightarrow -\lambda_3(\sigma) < \lambda_1(\sigma)$*;*
   (b) $\lambda_3(\sigma) > 0 \Longrightarrow -\lambda_3(\sigma) > \lambda_2(\sigma)$.

Remark 3.23. Assume that we are given a nonempty compact invariant isolated set $\Lambda = \Lambda_X(U)$ under a flow on a 3-manifold, which is also partially hyperbolic with volume expanding central direction. Since partial hyperbolicity is a robust property, then for every close flow Y we have that $\Lambda_Y(U)$ is also partially hyperbolic. This implies that there are no sources in $\Lambda_Y(U)$. The uniform volume expansion along the central direction of $T_\Lambda M$ for X implies that there are no sinks in $\Lambda_Y(U)$, for otherwise we would get volume contraction along the central direction for points and flows arbitrary close to Λ and X. This is a contradiction since dominated splittings depend continuously on the base point and on the dynamics, thus taking limits we obtain a point in Λ with central direction whose volume is contracted by the X flow.

Hence the conclusion of Lemma 3.21 is also valid in this setting and we conclude that *every singularity accumulated by regular orbits of a singular-hyperbolic isolated set of a flow* X *is either Lorenz-like for* X*, or Lorenz-like for* $-X$.

Remark 3.24. For an example of a singular-hyperbolic isolated set of a flow X with non-Lorenz-like singularities, consider the maximal invariant set inside the ellipsoid E of the flow described in Figure 2.16.

Proof. Let us prove the first item by contradiction. Suppose that there is $Y \in \mathcal{U}_U$ and $\sigma \in S(Y) \cap \Lambda_Y(U)$ with a complex eigenvalue ω. We can assume that σ is hyperbolic by Lemma 3.21. As $\dim(M) = 3$ the remaining eigenvalue λ of σ is real. We have either $Re(\omega) < 0 < \lambda$ or $\lambda < 0 < Re(\omega)$. Reversing the flow direction if necessary we can assume that we are in the first case. We can further assume, by a small perturbation keeping the vector field inside $\mathcal{U}_U$, that Y is C^∞ and

$$\lambda \neq -Re(\omega). \tag{3.7}$$

According to a form of the Connecting Lemma stated in Theorem 1.18 on page 31, we can assume that there is a homoclinic loop $\Gamma \subset \Lambda_Y(U)$ associated to σ. Then Γ is a *Shil'nikov bifurcation*, see Section 2.2, and thus there is a vector field Z arbitrarily C^1 close to Y exhibiting a sink or a source in $\Lambda_Z(U)$. This contradicts Lemma 3.21 and concludes the proof of the first item.

Thus we can arrange the eigenvalues λ_1, λ_2, λ_3 of σ in such a way that $\lambda_2 \leq \lambda_3 \leq \lambda_1$. By Lemma 3.21 we have that $\lambda_2 < 0$ and $\lambda_1 > 0$. This proves the second item in the statement.

To prove the third item we can apply Theorem 2.3 from Section 2.2. This shows that there is Z arbitrarily C^1 close to Y exhibiting either a sink in $\Lambda_Z(U)$ (if item (a) fails) or a source in $\Lambda_Z(U)$ (if item (b) fails). This is a contradiction as before, concluding the proof of the lemma. □

Lemma 3.25. *There is no $Y \in \mathcal{U}_U$ exhibiting two hyperbolic singularities in $\Lambda_Y(U)$ with different unstable manifold dimensions.*

Proof. Suppose by contradiction that there is $Y \in \mathcal{U}_U$ exhibiting two hyperbolic singularities with different unstable manifold dimensions in $\Lambda_Y(U)$. Note that $\Lambda' = \Lambda_Y(U)$ is a robust transitive set of Y and $-Y$ respectively. Since Kupka-Smale vector fields are generic (by the results in Section 1.3.6) we can assume that all the critical elements of Y in Λ' are hyperbolic.

As $\dim(M) = 3$ and Y has two hyperbolic singularities with different unstable manifold dimensions, it follows that both Y and $-Y$ have a singularity in Λ' whose unstable manifold has dimension one. Then, by Theorem 3.16 applied to Y and $-Y$ respectively, Λ' is a proper attractor and a proper repeller of Y. In particular, Λ' is an attracting set and a repelling set of Y. It would follow from Lemma 3.18 that $\Lambda' = M$. But this is a contradiction since Λ' is proper. □

Corollary 3.26. *If $Y \in \mathcal{U}_U$, then every critical element of Y in $\Lambda_Y(U)$ is hyperbolic.*

Proof. By Lemma 3.21 every periodic point of Y in $\Lambda_Y(U)$ is hyperbolic, for all $Y \in \mathcal{U}$. It remains to prove that every $\sigma \in S(Y) \cap \Lambda_Y(U)$ is hyperbolic, for all $Y \in \mathcal{U}_U$. By Lemma 3.22 the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of σ are real and satisfy $\lambda_2 < 0 < \lambda_1$. Then, to prove that σ is hyperbolic, we only have to prove that $\lambda_3 \neq 0$. If $\lambda_3 = 0$, then σ is a generic saddle-node singularity (after a small perturbation if necessary). Unfolding this saddle-node we obtain $Y' \in \mathcal{U}_U$ close to Y having two hyperbolic singularities with different unstable manifold dimensions in $\Lambda_{Y'}(U)$. This contradicts Lemma 3.25 and the proof follows. □

Proof of Theorem 3.2. Let Λ be a robustly transitive set with singularities of $X \in \mathcal{X}^1(M)$ with $\dim(M) = 3$. By Corollary 3.26 applied to $Y = X$ we have that every critical element of X in Λ is hyperbolic. So Λ satisfies condition (1) of Theorem 3.16. As $\dim(M) = 3$ and Λ is non-trivial, if Λ has a singularity, then this singularity has unstable manifold dimension equal to one, either for X or $-X$. So Λ also satisfies condition (2) of Theorem 3.16, either for X or $-X$. Applying Theorem 3.16 we have that Λ is an attractor (in the first case) or a repeller (in the second case).

We shall prove that Λ is proper in the first case. The proof is similar in the second case. If $\Lambda = M$ then we would have $U = M$. From this it would follow that $\Omega(X) = M$ and, moreover, that X cannot be C^1 approximated by vector fields exhibiting attracting or repelling critical elements. It would follow from the the work of Doering [53, p. 60] that X is Anosov. But this is a contradiction since Λ (and so X) has a singularity and Anosov vector fields do not. This finishes the proof of Theorem 3.2. □

Now we prove Theorem 3.5. We start with the following corollary.

Corollary 3.27. *If $Y \in \mathcal{U}_U$ then, either for $Z = Y$ or $Z = -Y$, every singularity of Z in $\Lambda_Z(U)$ is Lorenz-like.*

Proof. Apply Lemmas 3.22, 3.25 and Corollary 3.26. □

Now we use the existence of dominated splitting for the linear Poincaré flow with respect to $X \in \mathcal{U}_U$, see Section 1.3.3 for the relevant results and definitions.

Given $X \in \mathcal{U}_U$ define $\Lambda^*_X(U) = \Lambda_X(U) \setminus S(X)$. According to Theorem 3.2 we can assume that $\Lambda_X(U)$ is a proper and isolated attractor of X. Using Lemma 3.21 and the fact that $\Lambda^*_X(U) \subset \Omega(X)$, we see that we are in the setting of Theorem 1.8. Then we conclude that the Linear Poincaré Flow P^t on $\Lambda^*_X(U)$ admits a partially hyperbolic splitting: $N_{\Lambda^*_X(U)} = N^{s,X} \oplus N^{u,X}$.

The following consequence of this is used in a crucial way for the proof of expansiveness in Chapter 4.

Lemma 3.28. *Let Λ be a compact isolated invariant set for X, with isolating neighborhood U such that every C^1-close vector field admits a dominated splitting for the corresponding Linear Poincaré Flow on U away from singularities. Fix $\sigma \in S(X) \cap \Lambda$ and write $\lambda_2 < \lambda_3 < \lambda_1$ for its eigenvalues.*

1. *If $\lambda_2 < \lambda_3 < 0$, then σ is Lorenz-like for X and $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$.*
2. *If $0 < \lambda_3 < \lambda_1$, then σ is Lorenz-like for $-X$ and $W^{uu}_X(\sigma) \cap \Lambda = \{\sigma\}$.*

Remark 3.29. If we are given a singular-hyperbolic isolated set Λ for a flow X with isolating neighborhood U then, by Remark 3.23, the singularities of Λ, which are accumulated by regular orbits in Λ, are Lorenz-like either for X or $-X$. Moreover the Linear Poincaré Flow on $\Lambda^* = \Lambda \setminus S(X)$ admits a partially hyperbolic splitting naturally. Indeed the Linear Poincaré Flow is dominated by Theorem 1.8 (since singular-hyperbolicity prevents sinks and sources for nearby flows on U and guarantees hyperbolicity of all critical elements in U) and its central-stable bundle is uniformly contracted by the same argument in the proof of Proposition 3.9.

In addition, for all close enough vector fields Y the corresponding locally maximal invariant subset $\Lambda_Y(U)$ is also partially hyperbolic with volume expanding central direction, and so the domination property of the splitting for the Linear Poincaré Flow of X on Λ is robust.

Hence we have the same properties used in the proof of Lemma 3.28. We conclude that *every singularity accumulated by regular orbits in a singular-hyperbolic isolated set satisfies either item 1 or item 2 of Lemma 3.28 above.*

Proof. To prove the first item we assume that $\lambda_2 < \lambda_3 < 0$. Then σ is Lorenz-like for X by Corollary 3.27. Assume by contradiction that $W^{ss}_X(\sigma) \cap \Lambda \neq \{\sigma\}$.

Since Λ is transitive, by Theorem 1.18 there is $Z \in \mathcal{U}_U$ exhibiting a homoclinic connection $\Gamma \subset W^u_Z(\sigma(Z)) \cap W^{ss}_Z(\sigma(Z))$. This connection is called *orbit-flip*, see Section 2.2.2. By using Theorem 2.4 we can approximate Z by $Y \in \mathcal{U}_U$ wit ha homoclinic connection

$$\Gamma' \subset W^u_Y(\sigma(Y)) \cap (W^s_Y(\sigma(Y)) \setminus W^{ss}_Y(\sigma(Y))).$$

Hence there exists a center-unstable manifold $W^{cu}_Y(\sigma(Y))$ containing Γ' and tangent to $W^s_Y(\sigma(Y))$ along Γ'. This connection is called *inclination-flip*.

The existence of inclination-flip connections contradicts the existence of the dominated splitting for the Linear Poincaré Flow from Theorem 1.8, as a direct consequence of Theorem 2.4 in Section 2.2.2. This contradiction proves the first item.

The proof of the second item follows from the above argument applied to $-X$. $\square$

Proof of Theorem 3.5. Let Λ be a robust transitive set of $X \in \mathfrak{X}^1(M)$ with $\dim(M) = 3$. By Corollary 3.27, if $\sigma \in \sigma_X(\Lambda)$, then σ is Lorenz-like either for X or $-X$. If σ is Lorenz-like for X we have that $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$ by Lemma 3.28-(1) applied to $Y = X$. If σ is Lorenz-like for $-X$ we have that $W^{uu}_X(\sigma) \cap \Lambda = \{\sigma\}$ by Lemma 3.28-(2) again applied to $Y = X$. As $W^{ss}_{-X}(\sigma) = W^{uu}_X(\sigma)$ the proof is complete. $\square$

3.3 Attractors and singular-hyperbolicity

The main goal here is the proof of Theorem 3.7.

Let Λ be a robust attractor of $X \in \mathfrak{X}^1(M)$ with $\dim(M) = 3$, U an isolating block of Λ, and $\mathcal{U}_U$ a neighborhood of X such that for all $Y \in \mathcal{U}_U$, $\Lambda_Y(U) = \cap_{t\in\mathbb{R}} Y^t(U)$ is transitive. By definition $\Lambda = \Lambda_X(U)$.

As we already proved (in Lemma 3.21 and Corollary 3.26), for all $Y \in \mathcal{U}_U$ all the singularities of $\Lambda_Y(U)$ are Lorenz-like and all the critical elements in $\Lambda_Y(U)$ are hyperbolic of saddle type.

For future reference we state precisely the technical conditions for the arguments that follow.

Theorem 3.30. *Let Λ be a compact transitive Lyapunov stable invariant subset of $X \in \mathfrak{X}^1(M)$ such that for every vector field close to X all critical elements nearby Λ are hyperbolic, and there are no sinks nor sources. Suppose further that for all close vector fields every singularity nearby Λ is Lorenz-like. Then Λ is singular-hyperbolic.*

The strategy to prove Theorem 3.7 is the following: given $X \in \mathcal{U}_U$ we show that there exists a neighborhood $\mathcal{V}$ of X, $c > 0$, $0 < \lambda < 1$ and $T_0 > 0$ such that for all $Y \in \mathcal{V}$, the set

$$\mathrm{Per}^{T_0}_Y(\Lambda_Y(U)) = \{y \in \mathrm{Per}_Y(\Lambda_Y(U)) : (\text{minimal period of } y) \geq T_0\}$$

has a continuous invariant (c,λ)-dominated splitting $E^s \oplus E^{cu}$, with the dimension of E^s equal to 1.

Using the Closing Lemma of Pugh (Theorem 1.16) and the robust transitivity, we induce a dominated splitting over $\Lambda_X(U)$. The natural difficulty is to obtain the splitting around the singularities. By Theorem 3.6 the singularities are Lorenz-like and, consequently, they carry the local hyperbolic bundle $\hat{E}^{ss}$ associated to the strongest contracting eigenvalue of $DX(\sigma)$, and the central bundle $\hat{E}^{cu}$ associated to the remaining eigenvalues of $DX(\sigma)$. These bundles induce a local partial hyperbolic splitting $\hat{E}^{ss} \oplus \hat{E}^{cu}$ around the singularities.

The main step now is to prove that the splitting proposed for the periodic points is compatible with the local partial hyperbolic splitting at the singularities. Proposition 3.37 expresses this fact. Finally we prove that E^s is contracting and that the central direction E^{cu} is volume expanding, concluding the proof of Theorem 3.7.

We point out that the splitting for the Linear Poincaré Flow obtained in Theorem 1.8 is not invariant by DX^t. When $\Lambda_X^*(U) = \Lambda_X(U) \setminus S(X)$ is closed, this splitting induces a hyperbolic one for X, see [53, Proposition 1.1] and [98, Theorem A]. However the arguments used there do not apply here, since $\Lambda_X^*(U)$ is not closed. We also note that a hyperbolic splitting for X over $\Lambda_X^*(U)$ cannot be automatically extended to a hyperbolic one over $\overline{(}\Lambda_X^*(U))$: the presence of a singularity is a natural obstruction for it. On the other hand, Theorem 3.7 shows that this can be circumvented to get a partially hyperbolic structure for X over $\overline{(\Lambda_X^*(U))}$.

Let us establishing some notations, definitions and preliminary results.

Uniformly dominated splitting over $\mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$

Let $\Lambda_Y(U)$ be a robust attractor of $Y \in \mathcal{U}_U$, where U and $\mathcal{U}_U$ are as in the previous section.

Since every $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ is hyperbolic of saddle type, we have that the tangent bundle of M over p can be written as

$$T_pM = E_p^s \oplus E_p^Y \oplus E_p^u,$$

where E_p^s is the eigenspace associated to the contracting eigenvalue of $DY^{t_p}(p)$, E_p^u is the eigenspace associated to the expanding eigenvalue of $DY^{t_p}(p)$, and we write t_p for the (minimal) period of p.

Note that $E^s_p \subset N^s_p \oplus E^Y_p$ and $E^u_p \subset N^u_p \oplus E^Y_p$, where $N^s \oplus N^u$ is the splitting for the linear Poincaré flow over regular orbits.

Observe that, if we consider the previous splitting over all $\mathrm{Per}_Y(\Lambda_Y(U))$, the presence of a singularity in $\overline{\mathrm{Per}_Y(\Lambda_Y(U))}$ is an obstruction for the extension of the stable and unstable bundles E^s and E^u to $\overline{\mathrm{Per}_Y(\Lambda_Y(U))}$. Indeed, near a singularity, the angle between either E^u and E^X, or E^s and E^X, goes to zero. To bypass this difficulty, we introduce the following.

Definition 3.5. Given $Y \in \mathcal{U}_U$ define for any $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ the splitting

$$T_pM = E^{s,Y}_p \oplus E^{cu,Y}_p, \quad \text{where} \quad E^{cu,Y}_p = E^Y_p \oplus E^u_p.$$

In addition we define a splitting over $\mathrm{Per}_Y(\Lambda_Y(U))$ by

$$T_{\mathrm{Per}_Y(\Lambda_Y(U))}M = \bigcup_{p \in \mathrm{Per}_Y(\Lambda_Y(U))} (E^{s,Y}_p \oplus E^{cu,Y}_p).$$

When no confusion arises we drop the Y-dependence on the notation just defined. To simplify notation we denote the restriction of $DY^t(p)$ to $E^{s,Y}_p$ (respectively $E^{cu,Y}_p$) simply by $DY^t \mid E^s_p$ (respectively $DY^t \mid E^{cu}_p$) for $t \in \mathbb{R}$ and $p \in \mathrm{Per}(\Lambda_Y(U))$.

We now prove that the splitting over $\mathrm{Per}_Y(\Lambda_Y(U))$ given by Definition 3.5 is a DY^t-invariant and uniformly dominated splitting along periodic points with large period.

Theorem 3.31. *Given $X \in \mathcal{U}_U$ there are a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ and constants $0 < \lambda < 1$, $c > 0$, and $T_0 > 0$ such that, for every $Y \in \mathcal{V}$, if $p \in \mathrm{Per}^{T_0}_Y(\Lambda_Y(U))$ and $T > 0$, then*

$$\|DY^T \mid E^s_p\| \cdot \|DY^{-T} \mid E^{cu}_{Y^T(p)}\| < c \cdot \lambda^T.$$

Theorem 3.31 will be proved in Section 3.3.2, with the help of Theorems 3.32 and 3.33 below. The proofs of these theorems are in Section 3.3.2.

Theorem 3.32 establishes, first, that the periodic points are uniformly hyperbolic, i.e., the periodic points are of saddle-type and the Lyapunov exponents are uniformly bounded away from zero. Secondly the angle between the stable and the unstable eigenspaces at periodic points are uniformly bounded away from zero.

Before the statement we need the following definition: given two subspaces $A \subset T_xM$ and $B \subset T_xM$ the angle $\angle(A,B)$ between A and B is defined as $\angle(A,B) = \inf\{\angle(v,w) : v \in A, w \in B\}$.

Theorem 3.32. *Given $X \in \mathcal{U}_U$, there are a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and constants $0 < \lambda < 1$ and $c > 0$, such that for every $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ and t_p is the period of p then*

a) (a1) $\|DY^{t_p} \mid E^s_p\| < \lambda^{t_p}$ (uniform contraction on the period)
(a2) $\|DY^{-t_p} \mid E^u_p\| < \lambda^{t_p}$ (uniform expansion on the period).

b) $\angle(E^s_p, E^u_p) > c$ (angle uniformly bounded away from zero).

Theorem 3.33 is a strong version Theorem 3.32-b). It establishes that, at periodic points, the angle between the stable and the central unstable bundles is uniformly bounded away from zero.

Theorem 3.33. *Given $X \in \mathcal{U}_U$ there are a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and a positive constant C such that for every $Y \in \mathcal{V}$ and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ we have angles uniformly bounded away from zero: $\angle(E^s_p, E^{cu}_p) > C$.*

We shall prove that, if Theorem 3.31 fails, then we can create a periodic point for a nearby flow with the angle between the stable and the central unstable bundles arbitrarily small. This yields a contradiction with Theorem 3.33. In proving the existence of such a periodic point for a nearby flow we use Theorem 3.32.

Assuming Theorem 3.31, we establish in the following section the extension of the splitting given in Definition 3.5 to all of $\Lambda_X(U)$. Afterward, with the help of Theorem 3.32, we show that E^s is uniformly contracting and that E^{cu} is volume expanding.

In the proof that E^s is uniformly contracted (respectively E^{cu} is volume expanding) we show that the opposite assumption leads to the creation of periodic points for flows nearby the original one with contraction (respectively expansion) along the stable (respectively unstable) bundle arbitrarily small, contradicting the first part of Theorem 3.32.

All of these facts together imply Theorem 3.7.

3.3.1 Dominated splitting over a robust attractor

Here we induce a dominated splitting over $\Lambda_X(U)$ using the dominated splitting over $\mathrm{Per}^{T_0}_Y(\Lambda_Y(U))$ for flows near X, given by Definition 3.5.

On the one hand, since $\Lambda_Y(U)$ is a proper attractor for every Y close to X in $\mathfrak{X}^1$, we can assume without loss of generality that for all $Y \in \mathcal{V}$, and

$x \in \mathrm{Per}(Y)$ with $\mathcal{O}_Y(x) \cap U \neq \emptyset$, we have

$$\mathcal{O}_Y(x) \subset \Lambda_Y(U). \tag{3.8}$$

On the other hand, since $\Lambda_X(U)$ is a non-trivial transitive set, we get that $\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$ is dense in $\Lambda_X(U)$. So, to induce an invariant splitting over $\Lambda_X(U)$ it is enough to do so over $\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$ (see [109] and references therein). For this we proceed as follows.

Given $X \in \mathcal{U}_U$, let $K(X) \subset \Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}$ be such that $X^t(x) \notin K(X)$ for all $x \in K(X)$ if $t \neq 0$. In other words, $K(X)$ is a set of representatives of the quotient $\Lambda_X(U) \setminus \{p \in \mathrm{Per}_X(\Lambda_X(U)) : t_p < T_0\}/\sim$, where $\sim$ is the equivalence relation given by $x \sim y \iff x \in \mathcal{O}_X(y)$. Since $\Lambda_X(U) = \omega(z)$ for some $z \in M$, we have that for any $x \in K(X)$ there exists $t_n > 0$ such that $X^{t_n}(z) \to x$. Then by the Closing Lemma (Theorem 1.16) there exist $Y_n \to X$ in $\mathfrak{X}^1$ and $y_n \to x$ such that $y_n \in \mathrm{Per}(Y_n)$. We can assume that $Y_n \in \mathcal{U}_U$ for all n. In particular, inclusion (3.8) holds for all $Y = Y_n$, that is $\mathcal{O}_{Y_n}(y_n) \subset \Lambda_{Y_n}(U)$.

Moreover, since the period of the periodic points in $K(X)$ are larger than T_0, we can also assume that the periods of y_n are $t_{y_n} > T_0$ for all n. Thus the (c,λ)-dominated splitting $E^{s,Y_n} \oplus E^{cu,Y_n}$ over $\mathrm{Per}_{Y_n}^{T_0}(\Lambda_{Y_n}(U))$, provided by Theorem 3.31, is well defined.

Take a converging subsequence $E^{s,Y^{n_k}}_{y_{n_k}} \oplus E^{cu,Y^{n_k}}_{y_{n_k}}$ and set

$$E^{s,X}_x = \lim_{k\to\infty} E^{s,Y^{n_k}}_{y_{n_k}}, \qquad E^{cu,X}_x = \lim_{k\to\infty} E^{cu,Y^{n_k}}_{y_{n_k}}.$$

Since $E^{s,Y_n} \oplus E^{cu,Y_n}$ is a (c,λ)-dominated splitting for all n, then this property is also true for the limit $E^{s,X}_x \oplus E^{cu,X}_x$. Moreover $\dim(E^{s,X}_x) = 1$ and $\dim(E^{cu,X}_x) = 2$ for all $x \in K(X)$.

Define the following eigenspaces along $X^t(x)$ for $t \in \mathbb{R}$

$$E^{s,X}_{X^t(x)} = DX^t(E^{s,X}_x) \quad \text{and} \quad E^{cu,X}_{X^t(x)} = DX^t(E^{cu,X}_x).$$

Since for every n the splitting over $\mathrm{Per}_{Y_n}^{T_0}(\Lambda_{Y_n}(U))$ is (c,λ)-dominated, it follows that the splitting defined above along X-orbits of points in $K(X)$ is also (c,λ)-dominated. Moreover we also have that $E^{s,X}_{X^t(x)}$ is unidimensional and $E^{cu,X}_{X^t(x)}$ is bi-dimensional, for all $t \in \mathbb{R}$. This provides the desired extension of a dominated splitting to $\Lambda_X(U)$.

We denote by $E^s \oplus E^{cu}$ the splitting over $\Lambda_X(U)$ obtained in this way. Since this splitting is uniformly dominated we deduce that $E^s \oplus E^{cu}$ depends continuously on the points of $\Lambda_X(U)$ and also on the vector field X, see [76] or [34].

When necessary we denote by $E^{s,Y} \oplus E^{cu,Y}$ the above splitting for Y near X.

Remark 3.34. If $\sigma \in S(X) \cap \Lambda_X(U)$ then E^s_σ is the eigenspace E^{ss}_σ associated to the strongest contracting eigenvalue of $DX(\sigma)$, and E^{cu}_σ is the bidimensional eigenspace associated to the remaining eigenvalues of $DX(\sigma)$. This follows from the uniqueness of dominated splittings, [53, 108] or [34].

3.3.2 Robust attractors are singular-hyperbolic

Next we prove that the splitting $E^s \oplus E^{cu}$ over $\Lambda_X(U)$ is partially hyperbolic with volume expanding central-unstable direction.

E^s is uniformly contracting

We start by proving the following elementary lemmas.

Lemma 3.35. *If* $\liminf_{t\to\infty} \|DX^t \mid E^s_x\| = 0$ *for all* $x \in \Lambda_X(U)$*, then there is* $T_0 > 0$ *such that* $\|DX_{T_0} \mid E^s_x\| < \frac{1}{2}$ *for every* $x \in \Lambda_X(U)$*.*

Proof. For each $x \in \Lambda_X(U)$ there is t_x such that $\|DX^{t_x} \mid E^s_x\| < 1/3$. Hence for each x there is a neighborhood $B(x)$ such that for all $y \in B(x)$ we have $\|DX^{t_x} \mid E^s_y\| < 1/2$. Since $\Lambda_X(U)$ is compact, there are $B(x_i)$, $i = 1,\dots,n$, such that $\Lambda_X(U) \subset B(x_1) \cup \dots \cup B(x_n)$.

Let $K_0 = \sup\{\|DX^t \mid E^s_y\|, y \in B(x_i), 0 \le t \le t_{x_i}, i = 1,\dots,n\}$, let j_0 be such that $2^{-j_0} \cdot K_0 < 1/2$ and fix $T_0 > j_0 \cdot \sup\{t_{x_i}, i = 1,\dots,n\}$. We claim that T_0 satisfies the statement of the lemma.

Indeed, given $y \in \Lambda_X(U)$ we have $y \in B(x_{i_1})$ for some $1 \le i_1 \le n$. Let $t_{i_1},\dots,t_{i_k},t_{i_{k+1}}$ satisfy

- $X^{t_{i_1}+\cdots+t_{i_j}}(y) \in B(x_{i_{j+1}})$, $1 \le j \le k$, and
- $t_{i_1} + \cdots + t_{i_k} \le T_0 \le t_{i_1} + \cdots + t_{i_{k+1}}$.

Observe that $k \geq j_0$. Then for $\ell_j = t_{i_1} + \cdots + t_{i_j}$, $j = 1, \ldots, k+1$, we have

$$\|DX_{T_0} \mid E^s_y\| \leq \|DX_{T_0 - \ell_k} \mid E^s_{X^{\ell_k}(y)}\| \cdot \prod_{j=1}^{k} \|DX^{t_{i_j}} \mid E^s_{X^{\ell_{j-1}}(y)}\| < \frac{K_0}{2^k} < \frac{1}{2}.$$

The proof is complete. $\square$

Lemma 3.36. *If there exists $T_0 > 0$ such that $\|DX_{T_0} \mid E^s_x\| < 1/2$ for all $x \in \Lambda_X(U)$, then there are $c > 0$ and $0 < \lambda < 1$ such that $\|DX_T / E^s_x\| < c\lambda^T$ for all $x \in \Lambda_X(U)$ and $T > 0$.*

Proof. Let $K_1 = \sup\{\|DX^t\|, 0 \leq t \leq T_0\}$. Choose $0 < \lambda < 1$ such that $1/2 < \lambda^{T_0}$ and $c > 0$ such that $K_1 < c \cdot \lambda^r$ for all $0 \leq r \leq T_0$. Then for any $x \in \Lambda_X(U)$ and all $T > 0$ we have $T = nT_0 + r$ with $n = [T/T_0] = \max\{k \in \mathbb{Z} : k \leq T/T_0\}$ and $0 \leq r = T - nT_0 < T_0$. Consequently

$$\begin{aligned}\|DX^T \mid E^s_x\| &= \|DX^r \mid E^s_{X^{nT_0}(x)}\| \cdot \prod_{j=0}^{n-1} \|DX^{T_0} \mid E^s_{X^{jT_0}(x)}\| \\ &< \frac{K_1}{2^n} < c \cdot \lambda^r \cdot (\lambda^{T_0})^n < c \cdot \lambda^T,\end{aligned}$$

concluding the proof. $\square$

By Lemmas 3.35 and 3.36, in order to prove that the bundle E^s is uniformly contracting, it is enough to prove that $\liminf_{t\to\infty} \|DX^t \mid E^s_x\| = 0$ for every $x \in \Lambda_X(U)$.

Arguing by contradiction, assume that there exists $x \in \Lambda_X(U)$ satisfying $\liminf_{t\to\infty} \|DX^t \mid E^s_x\| > 0$. Then there exists $s_n \xrightarrow[n\to\infty]{} \infty$ such that

$$\lim_{n\to\infty} \frac{1}{s_n} \log \|DX^{s_n} \mid E^s_x\| \geq 0. \tag{3.9}$$

Let $C^0(\Lambda_X(U))$ be the set of real continuous functions defined on $\Lambda_X(U)$ with the C^0 topology, and define the sequence of continuous operators

$$\Psi_n : C^0(\Lambda_X(U)) \to \mathbb{R}, \qquad \varphi \in C^0(\Lambda_X(U)) \mapsto \frac{1}{s_n} \int_0^{s_n} \varphi(X^s(x))\, ds.$$

Since in the C^0 norm this sequence is bounded, $\|\Psi_n\| \leq 1$, and the unit ball of the dual $C^0(\Lambda_X(U))^*$ is weak*-compact (see any standard reference on

Functional Analysis e.g. [181]), there exists a subsequence of Ψ_n, which we still denote by Ψ_n, converging to a continuous map $\Psi \in C^0(\Lambda_X(U))^*$ in the weak* topology. Let $\mathcal{M}(\Lambda_X(U))$ be the space of measures with support on $\Lambda_X(U)$. By the Riez's Representation Theorem (see e.g. [180]) there exists a probability measure $\mu \in \mathcal{M}(\Lambda_X(U))$ such that

$$\int_{\Lambda_X(U)} \varphi\, d\mu = \lim_{n\to+\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(X^s(x))\, ds = \Psi(\varphi), \tag{3.10}$$

for every continuous function $\varphi : \Lambda_X(U) \to \mathbb{R}$. Note that such μ is invariant by the flow since for all $t \in \mathbb{R}$

$$\begin{aligned}\Psi(\varphi\circ X^t) &= \lim_{n\to+\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(X^{s+t}(x))\, ds\\ &= \lim_{n\to+\infty} \frac{s_n+t}{s_n}\cdot\frac{1}{s_n+t}\left(\int_0^{s_n+t}\varphi(X^s(x))\,ds - \int_0^t \varphi(X^s(x))\,ds\right) = \Psi(\varphi).\end{aligned}$$

Define $\varphi_X : C^0(\Lambda_X(U)) \to \mathbb{R}$ by

$$\varphi_X(p) = \partial_h(\log\|DX^h \mid E^s_p\|)_{h=0} = \lim_{h\to 0}\frac{1}{h}\log\|DX^h \mid E^s_p\|,$$

which is continuous and so satisfies (3.10). Observe that for $T \in \mathbb{R}$,

$$\begin{aligned}\int_0^T \varphi_X(X^s(p))\,ds &= \int_0^T \partial_h(\log\|DX^h \mid E^s_{X^s(p)}\|)_{h=0}\,ds\\ &= \log\|DX^T \mid E^s_p\|.\end{aligned} \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) we get

$$\int_{\Lambda_X(U)} \varphi_X\, d\mu \geq 0. \tag{3.12}$$

By The Ergodic Theorem 1.21 we deduce

$$\int_{\Lambda_X(U)} \varphi_X\, d\mu = \int_{\Lambda_X(U)} d\mu(y) \lim_{T\to\infty}\frac{1}{T}\int_0^T ds\, \varphi_X(X^s(y)). \tag{3.13}$$

Let $\Sigma(X)$ be the set of strongly closed points, see Section 1.4.3. Since μ is X-invariant and $\operatorname{supp}(\mu) \subset \Lambda_X(U)$, the Ergodic Closing Lemma (Theorem 1.23) ensures that $\mu\big(\Lambda_X(U)\cap(S(X)\cup\Sigma(X))\big) = 1$.

We claim that $\mu\big(\Lambda_X(U)\cap\Sigma(X)\big)>0$. For otherwise $\mu\big(\Lambda_X(U)\cap S(X)\big)=1$ and since $S(X)$ is X-invariant and discrete, we would get that μ is a finite convex linear combination of point masses in $S(X)$: $\mu=\sum_{\sigma\in S(X)}a_i\delta_\sigma$. But E^s_σ coincides with the strong-stable eigenspace E^{ss}_σ at every $\sigma\in S(X)$ (recall Remark 3.34) and the corresponding eigenvalues λ_{ss} are negative, thus

$$\int_{\Lambda_X(U)}\varphi_X\,d\mu=\int_{S(X)}\varphi_X\,d\mu=\sum_{\sigma\in S(X)}a_i\varphi_X(\sigma)<0$$

contradicting (3.12). This contradiction proves the claim.

The Ergodic Decomposition Theorem (see Section 1.4) enables us to assume without loss of generality that μ is ergodic. Hence $\mu(\Lambda_X(U)\cap\Sigma(X))=1$. Therefore by (3.12) and (3.13) there exists $y\in\Lambda_X(U)\cap\Sigma(X)$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi_X(X^s(y))ds\geq 0. \tag{3.14}$$

Because $y\in\Sigma(X)$, there are $\delta_n\xrightarrow[n\to+\infty]{}0$, $Y_n\in\mathcal{U}_U$, $p_n\in\mathrm{Per}_{Y_n}(\Lambda_{Y_n}(U))$ with period t_n such that

$$\|Y_n-X\|<\delta_n\quad\text{and}\quad\sup_{0\le s\le t_n}\mathrm{dist}(Y_n^s(p_n),X^s(y))<\delta_n.$$

Observe that we must have $t_n\xrightarrow[n\to\infty]{}\infty$. For otherwise y would be periodic for X and, if t_y is its period, then (3.11) with (3.14) imply that $DX^{t_y}\mid E^s_y$ is not a contraction. Combining this with Theorem 3.32-a2) and Lemma 3.21 we see that y can belong neither to a hyperbolic periodic orbit of saddle type nor to a repelling periodic orbit, a contradiction.

Let $\gamma<0$ be arbitrarily small. By (3.14) again, there is T_γ such that for $t\geq T_\gamma$ we have

$$\frac{1}{t}\int_0^t\varphi_X(X^s(y))ds\geq\gamma. \tag{3.15}$$

Since $t_n\xrightarrow[n\to\infty]{}\infty$, we can assume that $t_n>T_\gamma$ for every n. The continuity of the splitting $E^s\oplus E^{cu}$ over $T_{\Lambda_X(U)}M$ with the flow together with (3.15) implies that for n big enough

$$\frac{1}{t_n}\log\|DY_n^{t_n}/E^{s,Y_n}_{p_n}\|\geq\gamma\quad\text{or}\quad\|DY_n^{t_n}/E^{s,Y_n}_{p_n}\|\geq e^{\gamma t_n}.$$

Taking n sufficiently large and $\gamma<0$ sufficiently small, this last inequality contradicts item (a1) in Theorem 3.32.

This completes the proof that E^s is a uniformly contracting bundle.

E^{cu} is uniformly volume expanding

Using results analogous to Lemmas 3.35 and 3.36, one can see that to prove that E^{cu} is uniformly volume expanding it is enough to prove

$$\liminf_{t\to\infty} |\det(DX^{-t} \mid E^{cu}_x)| = 0 \quad \text{for every} \quad x \in \Lambda_X(U).$$

Arguing again by contradiction, assume that there exists $x \in \Lambda_X(U)$ such that $\liminf_{t\to\infty} |\det(DX^{-t} \mid E^{cu}_x)| > 0$. Then there is $s_n \xrightarrow[n\to\infty]{} \infty$ such that

$$\lim_{n\to\infty} \frac{1}{s_n} \log|\det(DX^{-s_n} \mid E^{cu}_x)| \geq 0. \tag{3.16}$$

Again define the sequence of continuous operators

$$\Psi_n : C^0(\Lambda_X(U)) \to \mathbb{R}, \quad \varphi \mapsto \frac{1}{s_n}\int_0^{s_n} \varphi(X^{-s}(x))\, ds.$$

Analogously to the previous arguments for E^s, there exists a convergent subsequence $\Psi_n \to \Psi \in C^0(\Lambda_X(U))^*$ and there exists a probability measure $\mu \in \mathcal{M}(\Lambda_X(U))$ such that

$$\int_{\Lambda_X(U)} \varphi\, d\mu = \lim_{n\to\infty} \frac{1}{s_n}\int_0^{s_n} \varphi(X^{-s}(x))\, ds = \Psi(\varphi),$$

for every continuous function $\varphi : \Lambda_X(U) \to \mathbb{R}$. As before such μ is invariant by the flow. Likewise define $\varphi_X : C^0(\Lambda_X(U)) \to \mathbb{R}$ by

$$\varphi_X(p) = \partial_h\big(\log|\det(DX^{-h} \mid E^{cu}_p)|\big)_{h=0} = \lim_{h\to 0} \frac{1}{h} \log|\det(DX^{-h}(p) \mid E^{cu}_p)|.$$

Hence we obtain

$$\int_{\Lambda_X(U)} \varphi_X\, d\mu = \lim_{n\to\infty} \frac{1}{s_n} \log|\det(DX^{-s_n} \mid E^{cu}_p)| \tag{3.17}$$

and using (3.16) we get

$$\int_{\Lambda_X(U)} \varphi_X\, d\mu \geq 0. \tag{3.18}$$

By The Ergodic Theorem

$$\int_{\Lambda_X(U)} \varphi_X\, d\mu = \int_{\Lambda_X(U)} d\mu(x) \lim_{T\to\infty} \frac{1}{T}\int_0^T ds\, \varphi_X(X^{-s}(x)). \tag{3.19}$$

Arguing as in the previous section, we have $\mu(\Lambda_X(U)\cap\Sigma(X)) = 1$. From (3.18) and (3.19) there exists $y \in \Lambda_X(U)\cap\Sigma(X)$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi_X(X^{-s}(y))\,ds \geq 0. \tag{3.20}$$

Hence there are $\delta_n \xrightarrow[m\to\infty]{} 0$, $Y_n \in \mathcal{U}_U$, $p_n \in \mathrm{Per}_{Y_n}(\Lambda_{Y_n}(U))$ with period t_n such that

$$\|Y_n - X\| < \delta_n \quad \text{and} \quad \sup_{0\leq s\leq t_n} \mathrm{dist}(Y_n^{-s}(p_n), X^{-s}(y)) < \delta_n.$$

Again we must have $t_n \xrightarrow[n\to\infty]{} \infty$. Fix $\gamma < 0$ be arbitrarily small. By (3.20) we can find $T_\gamma > 0$ such that for $t \geq T_\gamma$

$$\frac{1}{t}\int_0^t \varphi_X(X^{-s}(y))\,ds \geq \gamma \tag{3.21}$$

and we can assume that $t_n > T_\gamma$ for every n. The continuity of the splitting $E^s \oplus E^{cu}$ with the flow together with (3.17) and (3.21) imply that for n big enough we have

$$\frac{1}{t_n}\log|\det(DY_n^{-t_n} \mid E^{cu,Y_n}_{p_n})| \geq \gamma \quad \text{or} \quad |\det(DY_n^{-t_n} \mid E^{cu,Y_n}_{p_n})| \geq e^{\gamma t_n}.$$

This implies

$$|\det(DY_n^{t_n} \mid E^{cu,Y_n}_{p_n})| \leq e^{-\gamma t_n} = (e^{-\gamma})^{t_n}. \tag{3.22}$$

We can make $e^{-\gamma}$ arbitrarily close to one and taking n sufficiently big, from (3.22) we obtain a contradiction with item (a2) in Theorem 3.32, since for periodic orbits we have

$$|\det(DY_n^{t_n} \mid E^{cu,Y_n}_{p_n})| = |\det(DY_n^{t_n} \mid E^{u,Y_n}_{p_n})|.$$

This completes the proof that E^{cu} is volume expanding.

Uniform dominated splitting on periodic orbits

Let us assume Theorems 3.32 and 3.33 on page 121 and show how we obtain Theorem 3.31. The central idea of the proof is to show that if the Theorem 3.31 fails then we can obtain a flow near X exhibiting a periodic point with angle between the stable and the central bundles arbitrarily small, leading to a contradiction with Theorem 3.33.

As in the proof of Lemma 3.36, to obtain Theorem 3.31 it is enough to show that there exist a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and $T_0 > 0$, such that for every vector field $Y \in \mathcal{V}$, if $p \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ then

$$\|DY^{T_0} \mid E^s_p\| \cdot \|DY^{-T_0} \mid E^{cu}_{Y^{T_0}(p)}\| \leq \frac{1}{2}. \tag{3.23}$$

We prove (3.23) arguing by contradiction. If (3.23) fails then given $X \in \mathcal{U}_U$ and $T_0 > 0$, we can find $Y \in \mathcal{U}_U$ arbitrarily close to X and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ satisfying

$$\|DY^{T_0} \mid E^s_y\| \cdot \|DY^{-T_0} \mid E^{cu}_{Y^{T_0}(y)}\| > \frac{1}{2}. \tag{3.24}$$

Claim 3.2. *For any positive number T_0, there are Y arbitrarily close to X, $T > T_0$ and $y \in \mathrm{Per}_Y(\Lambda_Y(U))$, with period t_y larger than T, admitting a direction $v \in E^{cu}_{Y^T(y)}$ not collinear to $Y(Y^T(y))$ such that*

$$\|DY^T \mid E^s_y\| \cdot \|DY^{-T}(Y^T(y))(v)\| = 1.$$

Proof. First we show that there exist Y arbitrarily close to X, $T > T_0$ and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ with period t_y larger than T admitting $v \in E^{cu}_{Y^T(y)}$ such that

$$\|DY^T \mid E^s_y\| \cdot \|DY^{-T}(Y^T(y))(v)\| = \frac{1}{2}. \tag{3.25}$$

Take Y close to X, $T_0 > 0$ large enough and $y \in \mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ satisfying (3.24). If there exists $T \in (T_0, t_y)$ such that $\|DY^T \mid E^s_y\| \cdot \|DY^{-T} \mid E^{cu}_{Y^T(y)}\| <$ $1/2$, then by continuity of the norms and the flow we can find some other T in the same interval satisfying $\|DY^T \mid E^s_y\| \cdot \|DY^{-T} \mid E^{cu}_{Y^T(y)}\| = 1/2$. Since the unit ball in any tangent space is compact, there exists $v \in E^{cu}_{Y^T(y)}$ satisfying (3.25). Otherwise we have

$$\|DY^T \mid E^s_y\| \cdot \|DY^{-T} \mid E^{cu}_{Y^T(y)}\| \geq \frac{1}{2} \text{ for all } T_0 \leq T < t_y. \tag{3.26}$$

In this case we observe that Theorem 3.32 implies

$$\|DY^{t_y} \mid E^s_y\| \cdot \|DY^{-t_y} \mid E^u_{Y^{t_y}(y)}\| < \lambda^{2t_y} < \frac{1}{2}.$$

Hence the inequality above is still true for some $T' < t_y$ and close enough to t_y. Since $E^u_{Y^t(y)} \subset E^{cu}_{Y^t(y)}$ for all $t \in \mathbb{R}$, we deduce from (3.26) that there exist $T \in (T', t_y)$ and $v \in E^{cu}_{Y^T(y)}$ satisfying (3.25).

Define now the following one-parameter family of linear maps which deform DY^t

$$A_t \mid E^s_y = 2^{t/T} \cdot DY^t \mid E^s_y \quad \text{and} \quad A_t \mid E^{cu}_y = DY^t \mid E^{cu}_y \quad \text{for} \quad 0 \le t \le T.$$

By a straightforward computation we obtain

$$\begin{aligned} \partial_s (A_{t+s} \cdot A_t^{-1})\big|_{s=0} \mid E^s_{Y^t(y)} &= \partial_s \big(2^{s/T} \cdot DY^{t+s}(y) \cdot DY^{-t} \mid E^s_{Y^t(y)}\big)\big|_{s=0} \\ &= DY \mid E^s_{Y^t(y)} + \frac{\log(2)}{T}, \end{aligned}$$

which implies that $\big\|\partial_s (A_{t+s} \cdot A_t^{-1})\big|_{s=0} \mid E^s_{Y^t(y)} - DY \mid E^s_{Y^t(y)}\big\| = \log(2)/T$ or, since $A_t \mid E^{cu}_y = DY^t \mid E^{cu}_y$ by definition

$$\big\|\partial_s (A_{t+s} \cdot A_t^{-1})\big|_{s=0} - DY\big\| = \frac{\log(2)}{T}.$$

Note that this can be made arbitrarily small by taking T big enough and, since the flow direction is contained in E^{cu}, the family A_t preserves the flow direction. Thus A_t satisfies the conditions of the extension to the flow setting of Frank's Perturbation Lemma 1.20 (see Section 1.3.9 on page 31).

Hence on the one hand, by at most a small C^1 perturbation, we can assume that $DY^t(y) = A_t$ for $0 \le t \le T$. On the other hand, by (3.25) and by definition of A_t we get $\|A_T \mid E^s_y\| \cdot \|A_T^{-1}(v)\| = 1$ which ensures that

$$\big\|DY^T \mid E^s_y\big\| \cdot \big\|DY^{-T}(Y^T(y))(v)\big\| = 1.$$

If v is not collinear to $Y(Y^T(y))$, then we are done. Otherwise let w be near v inside $E^{cu}_{Y^T(y)}$, not collinear to $Y(Y^T(y))$, so that

$$b_T = \|DY^T \mid E^s_y\| \cdot \|DY^{-T}(Y^T(y))(w)\| \approx 1.$$

Now perturb the vector field as before, keeping the flow direction: define the one-parameter family of linear maps B_t by

$$B_t \mid E^s_y = b_T^{-t/T} \cdot DY^t \mid E^s_y \quad \text{and} \quad B_t \mid E^{cu}_y = DY^t \mid E^{cu}_y \quad \text{for} \quad 0 \le t \le T.$$

Again this family is in the setting of Frank's Lemma 1.20 and by the same arguments we can assume that, by at most a small C^1 perturbation, we get Y near X and $w \in E^{cu}_{Y^T(y)}$ not collinear to $Y(Y^T(y))$ with $T_0 < T < t_y$ arbitrarily large, such that $\left\|DY^T \mid E^s_y\right\| \cdot \left\|DY^{-T}(Y^T(y))(w)\right\| = 1$. This completes the proof of Claim 3.2. □

Claim 3.3. *There are a vector field Z which is C^1 near Y and a periodic point $y \in \mathrm{Per}_Z(\Lambda_Z(U))$ such that $\angle(E_y^{s,Z}, E_y^{cu,Z})$ is arbitrarily small.*

This contradicts Theorem 3.33 and hence this proves Theorem 3.31 assuming Theorems 3.32 and 3.33

Proof of Claim 3.3. Fix $T > 0$ large and let $Y \in \mathcal{U}_U$ be C^1 close to X. Take $y \in \mathrm{Per}_Y(\Lambda_Y(U))$ with period $t_y > T$ and $v \in E^{cu}_{Y^T(y)}$ not collinear to $Y(Y^T(y))$ given by Claim 3.2. Let w be the unit vector

$$DY^{-T}(Y^T(y))(v)/\|DY^{-T}(Y^T(y))(v)\|,$$

which belongs to E^{cu}_y and is not collinear to $Y(y)$. Let e^t_y be the unit vector in the direction of $Y(Y^t(y))$ and for each $r \in [0,T]$ define a basis of $T_{Y^r(y)}M$ by $\mathcal{B}_r = \{f^r_y, w_r, e^s_{Y^r(y)}\}$, where $w_r = DY^r(y)(w)/\|DY^r(y)(w)\|$ and $e^s_{Y^r(y)}$ is a unit vector in $E^s_{Y^r(y)}$.

By Theorem 3.33 there exists $C > 0$ such that $\angle(E^s_{Y^r(y)}, E^{cu}_{Y^r(y)}) \geq C$ for all $y \in \mathrm{Per}_Y(\Lambda_Y(U))$ and every $r \in [0,T]$. Then we can find $K = K(C) > 0$ and a neighborhood $\mathcal{V}$ of X such that for all $Y \in \mathcal{V}$ and all $y \in \mathrm{Per}(\Lambda_Y(U))$, there is a inner product $< \cdot, \cdot >_{(Y^r(y))}$ inducing a norm $\|\cdot\|^*_{(Y^r(y))}$ such that for all $r \in [0,T]$

$$E^s_{Y^r(y)} \text{ and } E^{cu}_{Y^r(y)} \text{ are orthogonal, and } \frac{1}{K}\|\cdot\| \leq \|\cdot\|^*_{(Y^r(y))} \leq K\|\cdot\|.$$

The matrix of $DY^r(y)$ with respect to the basis $\mathcal{B}_r$ in the new metric is

$$DY^r(y) = \begin{bmatrix} Y(r) & * & 0 \\ 0 & a(r) & 0 \\ 0 & 0 & b(r) \end{bmatrix} \text{ where } \begin{cases} Y(r) = \|Y(Y^r(y))\|^*_{Y^r(y)} \\ a(r) = \|DY^r(y)(w)\|^*_{Y^r(y)} \\ b(r) = \|DY^r(y)(e^s_y)\|^*_{Y^r(y)} \end{cases}.$$

Note that in this basis $Y(Y^r(y)) = (1,0,0)$ and $a(T) = b(T) \cdot \|v\|^*_{Y^T(y)}$ by the choices of v and w.

Let $A_{s,y}$ be the restriction of $DY^s(y)$ to the subspace $[w, e^s_y]$ spanned by the vectors w and e^s_y. Observe that any perturbation on $A_{s,y}$ does not affect the direction of Y.

For each $\delta > 0$ define

$$A^+_{s,y} = \begin{bmatrix} a(s) & \delta a(s) \int_0^s \frac{b(r)}{a(r)} dr \\ 0 & b(s) \end{bmatrix} \text{ and } A^-_{s,y} = \begin{bmatrix} a(s) & 0 \\ \delta b(s) \int_0^s \frac{a(r)}{b(r)} dr & b(s) \end{bmatrix}.$$

Note that for $h \geq 0$ we have

$$A^+_{s+h,y}(A^+_{s,y})^{-1} = \begin{bmatrix} a(s,h) & c(s,s+h) \\ 0 & b(s,h) \end{bmatrix},$$

where $a(s,h) = a(s+h)/a(s)$, $b(s,h) = b(s+h)/b(s)$ and

$$c(s,s+h) = \delta \cdot \frac{a(s+h)}{b(s)} \int_s^{s+h} \frac{b(r)}{a(r)} dr.$$

An analogous formula holds for $A^-_{s+h}(A^-_s)^{-1}$. We claim that

$$\|\partial_h A^+_{s+h,y}(A^+_{s,y})^{-1}|_{h=0} - DY(Y^s(y))\|^*_{Y^s(y)} \leq \delta. \tag{3.27}$$

Indeed since

$$\partial_h A^+_{s+h,y}(A^+_{s,y})^{-1}|_{h=0} = \begin{bmatrix} \frac{a'(s)}{a(s)} & \partial_h c(s,s+h)|_{h=0} \\ 0 & \frac{b'(s)}{b(s)} \end{bmatrix}$$

and

$$DY(Y^s(y)) = \begin{bmatrix} \frac{a'(s)}{a(s)} & 0 \\ 0 & \frac{b'(s)}{b(s)} \end{bmatrix},$$

all we need to show is $\|\partial_h c(s,s+h)|_{h=0}\|_{(Y,y)} \leq \delta$. But for some $\eta \in [s, s+h]$ we have

$$c(s,s+h) = \delta \cdot \frac{a(s+h)}{b(s)} \int_s^{s+h} \frac{b(r)}{a(r)} dr = \delta \cdot \frac{a(s+h)}{b(s)} \cdot \frac{b(\eta)}{a(\eta)} h,$$

and from this we deduce

$$\partial_h c(s,s+h)|_{h=0} = \lim_{h \to 0} \frac{c(s,s+h)}{h} = \delta$$

which implies (3.27). A similar result holds for $A^-_{s+h}(A^-_s)^{-1}$.

Observe that

$$A^+_{T,y}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\delta a(T)\int_0^T\frac{b(r)}{a(r)}dr\\ b(T)\end{bmatrix},\ A^-_{T,y}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a(T)\\ \delta b(T)\int_0^T\frac{a(r)}{b(r)}dr\end{bmatrix}.$$

We shall prove next that shrinking δ we obtain either

$$\frac{b(T)}{\delta a(T)\int_0^T\frac{b(r)}{a(r)}dr}\ \text{vanishes or} \tag{3.28}$$

$$\delta\frac{b(T)}{a(T)}\int_0^T\frac{a(r)}{b(r)}dr \quad \text{is arbitrarily large.} \tag{3.29}$$

This implies

$$\text{either } A^+_{T,y}\begin{bmatrix}0\\1\end{bmatrix}\text{ is nearly horizontal, or } A^-_{T,y}\begin{bmatrix}1\\0\end{bmatrix}\text{ is nearly vertical.}$$

If (3.28) holds we consider the family

$$B^+_{s,y}=\begin{bmatrix}Y(s) & * & 0\\ 0 & a(s) & \delta a(s)\int_0^s\frac{b(r)}{a(r)}dr\\ 0 & 0 & b(s)\end{bmatrix}$$

and if (3.29) holds we take

$$B^-_{s,y}=\begin{bmatrix}Y(s) & * & 0\\ 0 & a(s) & 0\\ 0 & \delta b(s)\int_0^s\frac{a(r)}{b(r)}dr & b(s)\end{bmatrix}.$$

Suppose (3.28) is true. Observe that (3.27) implies that $B^+_{s,y}$ satisfies Frank's Lemma 1.20. So there is $Z\in\mathcal{U}_U$ C^1 near Y (and C^1 near X) such that y is a periodic point of Z with period T, $Z^t(y)=Y^t(y)$ for every t, and $DZ^s(y)=B^+_{s,y}$. In particular, the restriction of $DZ^s(y)$ to $[w,e^s_y]$ equals $A^+_{s,y}$ for all s. Hence we have $E^{cu,Z}_{Z^s(y)}=E^{cu,Y}_{Y^s(y)}$ for all s.

Moreover, Theorem 3.33 combined with the fact that $\|\cdot\|^*_{Y^s(y)}$ is equivalent to $\|\cdot\|$, ensures that $\angle(E^{cu,Z}_y,E^{s,Z}_y)>C'$ for some positive constant

$C' = C'(C)$ in the adapted metric. Thus, in the basis $\mathcal{B}_{Y^s(y)}$ fixed above, we obtain $E^{s,Z}_y = (g,d,1)$ with $|g|$ and $|d|$ bounded above by a constant depending only on C. Hence

$$E^{s,Z}_{Z_T(y)} = DZ_{Z_T(y)} \begin{bmatrix} g \\ d \\ 1 \end{bmatrix} = \begin{bmatrix} gY(T) + *d \\ da(T) + \delta a(T) \int_0^T \frac{b(r)}{a(r)} dr \\ b(T) \end{bmatrix}.$$

As $a(T) = b(T) \cdot \|v\|^*_{Y^T(y)}$, we obtain that the ratio between the third and the second coordinate of $E^{s,Z}_{Z_T(y)}$ is equal to

$$\left(d + \delta \int_0^T \frac{b(r)}{a(r)} dr \right)^{-1}.$$

and (3.28) implies $\delta \int_0^T \frac{b(r)}{a(r)} dr > K_1$, with K_1 arbitrarily large. Therefore

$$d + \delta \int_0^T \frac{b(r)}{a(r)} dr > K_1 + d \quad \text{and so} \quad \left(d + \delta \int_0^T \frac{b(r)}{a(r)} dr \right)^{-1} < \frac{1}{K_1 + d}.$$

Since $(K_1 + d)^{-1}$ is arbitrarily small, we see that $\angle(E^{s,Z}_{Z_T(y)}, E^{cu,Z}_{Z_T(y)})$ vanishes in the metric $\|\cdot\|^*_{(Y,y)}$, and so $\angle(E^{s,Z}_{Z_T(y)}, E^{cu,Z}_{Z_T(y)})$ is also arbitrarily small in the original metric, contradicting Theorem 3.33.

We also obtain that $\angle(E^{cu,Z}_{Z_T(y)}, E^{s,Z}_{Z_T(y)})$ is arbitrarily small in the original metric, if we assume (3.29) and use similar arguments.

Finally to prove Claim 3.3 we need to show that either (3.28) or (3.29) are true. For this, set $\delta = T^{-1/2}$, so that $\delta \xrightarrow[T\to+\infty]{} 0$. Since $a(s), b(s) > 0$ for all $s \in [0,T]$, we can write

$$T = \int_0^T dY = \int_0^T \sqrt{\frac{a(Y)}{b(Y)}} \sqrt{\frac{b(Y)}{a(Y)}} dY \le \sqrt{\int_0^T \frac{a(Y)}{b(Y)} dY} \sqrt{\int_0^T \frac{b(Y)}{a(Y)} dY}$$

and so

$$T^2 \le \int_0^T \frac{a(Y)}{b(Y)} dY \cdot \int_0^T \frac{b(Y)}{a(Y)} dY,$$

implying that

$$\frac{T}{\delta \int_0^T \frac{b(Y)}{a(Y)} dY} = \frac{T^2 \delta^2}{\delta \int_0^T \frac{b(Y)}{a(Y)} dY} \le \delta \int_0^T \frac{a(Y)}{b(Y)} dY.$$

Thus if $(\delta\int_0^T \frac{b(Y)}{a(Y)}dY)^{-1} > T^{-1/2}$, then $T\,(\delta\int_0^T \frac{b(Y)}{a(Y)}dY)^{-1} > T\,T^{-1/2} = \sqrt{T}$, which implies

$$\delta\int_0^T \frac{a(Y)}{b(Y)}dr \geq \sqrt{T}\,.$$

Since T can be taken arbitrarily large, we see that either (3.28) or (3.29) are true. The proof of Theorem 3.31 is complete, and we conclude that the splitting $E^s \oplus E^{cu}$ over $\mathrm{Per}_Y^{T_0}(\Lambda_Y(U))$ given by Definition 3.5 is invariant and uniformly dominated. □

Uniformly bounded angles between stable and unstable directions at periodic orbits

Here we prove Theorem 3.32 and 3.33, used in the proofs of the results in the previous section.

Proof of Theorem 3.32: Let us start with item (a). Suppose, by contradiction, that given $\delta > 0$ small, there is $Y \in \mathfrak{X}^\infty(M)$ arbitrarily C^1 close to X, and a periodic orbit y of Y with period t_y, such that $\|DY^{t_y} \mid E^s_y\| \geq (1-\delta)^{t_y}$.

Let A_t be the following one-parameter family of linear maps

$$A_t = (1-2\delta)^{-t} \cdot DY^t(y), \quad 0 \leq t \leq t_y.$$

By construction A_t preserves the direction of the flow and the eigenspaces of DY^{t_y}. Moreover

$$\|\partial_h A_{t+h} A_t^{-1}|_{h=0} - DY(Y^t(y))\| < -\log(1-\delta)\,.$$

Since we can take δ as close to 0 as needed, the inequality above together with $Y \in C^\infty$ imply that A_t satisfies Frank's Lemma 1.20. Hence there exists $Z \in C^1$ C^1 near Y such that y is a periodic point of Z with period t_y, and $DZ^t(Z^t(y)) = A_t$ for $0 \leq t \leq t_y$. By definition of A_t we get $\|DZ^{t_y} \mid E^s_y\| > 1$, implying that y is a source for Z, which contradicts Lemma 3.21. This proves subitem (a1).

By the same argument we prove subitem (a2). This finishes the proof of the first item.

Now let us prove item (b). By contradiction, assume that for every $\gamma > 0$ there exist $Y \in \mathfrak{X}^\infty$ C^1 close to X and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ such that $\angle(E^s_p, E^u_p) < \gamma$.

Let t_p be the period of p and λ_s, λ_u be the stable and unstable eigenvalues of $DY^{t_p}(p)$. Then $\lambda_s < \lambda^{t_p}$ and $\lambda_u > \lambda^{-t_p}$, where λ is given by item (a) already proved. Observe that there is t_0 such that $t_p > t_0$ and thus, $|1 - \frac{\lambda_s}{\lambda_u}|$ is uniformly bounded away from 0. In addition, if $\lambda_s \cdot \lambda_u > 0$, then there is $D_1 > 0$ such that

$$D_1^{-1} < \left| \frac{2\sqrt{\lambda_s \lambda_u} - \lambda_s - \lambda_u}{\lambda_u - \lambda_s} \right| < D_1, \tag{3.30}$$

or else, if $\lambda_s \cdot \lambda_u < 0$ then there is $D_2 > 0$ such that

$$D_2^{-1} < \left| \frac{-(\lambda_s + \lambda_u)}{(\lambda_u - \lambda_s)} \right| < D_2. \tag{3.31}$$

Let $\hat{\gamma}$ be the slope between E_p^s and E_p^u. Observe that $\hat{\gamma}$ is small if the angle $\angle(E_p^s, E_p^u)$ is small. In the case $\lambda_s \cdot \lambda_u > 0$, we set $\delta = \left| \frac{2\sqrt{\lambda_s \lambda_u} - \lambda_s - \lambda_u}{\lambda_u - \lambda_s} \right| \hat{\gamma}$. Otherwise set $\delta = \left| \frac{-(\lambda_s + \lambda_u)}{(\lambda_u - \lambda_s)} \right| \hat{\gamma}$. By hypothesis, $\hat{\gamma}$ can be taken arbitrarily small. Thus (3.30) and (3.31) imply that δ also can be taken arbitrarily small.

Now, let $\mathcal{B}_{t\,0 \le t \le t_p}$ be a continuous family of positively oriented basis in $T_{Y^t(p)}M$, defined by

$$\mathcal{B}_t(p) = \left\{ \frac{Y^t(p)}{\|Y^t(p)\|}, v_2(t), v_3(t) \right\},$$

with $v_2(t) \in E^{cs}_{Y^t(p)}$ orthonormal to $Y(Y^t(p))$, and $v_3(t)$ is orthonormal to $E^{cs}_{Y^t(p)}$. In this basis we have

$$DY^{t_p}(p) = \begin{bmatrix} 1 & * & * \\ 0 & \lambda_s & \frac{\lambda_u - \lambda_s}{\hat{\gamma}} \\ 0 & 0 & \lambda_u \end{bmatrix}.$$

For each δ as above, let

$$A(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \delta & 1 \end{bmatrix},$$

and consider $B(\delta) = A(\delta)\cdot DY^{t_p}(p)$. Since δ is arbitrarily small, $B(\delta)$ is arbitrarily near $DY^{t_p}(p)$. Moreover a straightforward calculation shows that $B(\delta)$ has one eigenvalue equal to 1, and the product of the other two eigenvalues have modulus equal to $\sqrt{|\lambda_s \cdot \lambda_u|}$ (δ was chosen so that the other eigenvalues besides 1 are equal), which is either bigger than 1 or smaller or equal to 1.

Taking δ small enough, there is a non-negative C^2 real function $\delta(t)$ such that $\delta(0) = 0$, $\delta(t_p) = \delta$, $|\delta'(t)| < 2\delta$ and $|\delta(t)| < 2\delta$. Define the one-parameter family of linear maps whose matrix in the basis $\mathcal{B}_t$ is

$$A_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \delta(t) & 1 \end{bmatrix}, \quad 0 \le t \le t_p.$$

Let $C_t = A_t \cdot DY^t(p)$ for $0 \le t \le t_p$. By construction the transformation C_t preserves the flow direction along the Y-orbit of p. The choice of $\delta(t)$ implies that A_t is a small perturbation of the identity map $I_t : T_{Y^t(p)}M \to T_{Y^t(p)}M$ for $0 \le t \le t_p$ and so C_t is in the setting of Frank's Lemma 1.20 again.

Hence we can find a vector field Z which is C^1 near Y, and a periodic point $p \in \mathrm{Per}_Z(\Lambda_Z(U))$ such that $DZ^t(p) = C_t = A_t \cdot DY^t(p)$, for $0 \le t \le t_p$. Moreover $DZ^{t_p} = A_{t_p} \cdot DY^{t_p}(p) = B(\delta)$ (recall $B(\delta)$ was defined above). Thus, taking δ small enough, we get a C^1 vector field Z nearby Y exhibiting a periodic point p which is either a sink or a source. This contradicts Lemma 3.21. This completes the proof of Theorem 3.32. $\square$

Proof of Theorem 3.33: Arguing by contradiction, we show that if Theorem 3.33 fails, then we can create periodic points with angle between the stable and unstable direction arbitrarily small, leading to a contradiction with the second part of Theorem 3.32, already proved.

Theorem 3.33 is a consequence of Propositions 3.37 and 3.38 below. The first one establishes that for periodic points close to a singularity, the stable direction remains close to the strong stable direction of the singularity, and the central unstable direction is close to the central unstable direction of the singularity. This result gives the compatibility between the splitting proposed for the periodic points in Definition 3.5 and the local partially hyperbolic splitting at the singularities.

Proposition 3.37. *Given $X \in \mathcal{U}_U$, $\varepsilon > 0$ and $\sigma \in S(X) \cap \Lambda_X(U)$ there exist a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X and $\delta > 0$ such that for all $Y \in \mathcal{V}$ and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ satisfying $\mathrm{dist}\big(p, \sigma_Y\big) < \delta$ we have*

(a) $\angle\big(E_p^{s,Y}, \hat{E}_{\sigma_Y}^{ss,Y}\big) < \varepsilon$, *and*

(b) $\angle\big(E_p^{cu,Y}, \hat{E}_{\sigma_Y}^{cu,Y}\big) < \varepsilon$.

The second one says that, *far from singularities*, the angle between the stable direction and the central unstable direction of any periodic point inside the maximal invariant set is uniformly bounded away from zero.

Proposition 3.38. *Given $X \in \mathcal{U}_U$ and $\delta > 0$, there are a neighborhood $\mathcal{V} \subset \mathcal{U}$ of X and a positive constant $C = C(\delta)$ such that if $Y \in \mathcal{V}$ and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ satisfies $\mathrm{dist}\big(p, S(Y) \cap \Lambda_Y(U)\big) > \delta$ then*

$$\angle\big(E_p^{s,Y}, E_p^{cu,Y}\big) > C.$$

Theorem 3.33 follows from these propositions since

- far away from singularities the uniform domination between the stable and center-unstable directions at periodic orbits is a consequence of the uniform growth rates provided by Theorem 3.32 together with the angle estimate of Proposition 3.38;

- for orbits passing close to the singularities, Proposition 3.37 ensures that the stable and center-unstable directions are essentially the same as the strong-stable and center-unstable direction at the singularity. The angle between these is bounded away from zero since every singularity is Lorenz-like, by Theorem 3.5, and the set $S(X) \cap \Lambda_Y(U)$ is finite because each singularity is hyperbolic. This together with the uniform growth rates provided by Theorem 3.32 ensure the uniform domination between the stable and center-unstable directions.

The proof of Theorem 3.33 is complete depending only on Propositions 3.37 and 3.38. □

3.3.3 Flow boxes near singularities

Since the singularities σ in our setting are all Lorenz-like, the unstable manifold $W^u(\sigma)$ is one-dimensional, and there is a one-dimensional strong-stable manifold $W^{ss}(\sigma)$ contained in the two-dimensional stable manifold

$W^s(\sigma)$. Using the linearization given by the Hartman-Grobman Theorem 1.9 or, in the absence of resonances, the smooth linearization results provided by Sternberg [193], orbits of the flow in a small neighborhood U_0 of the singularity are solutions of the linear system (2.3), modulo a continuous change of coordinates.

Then for some $\delta > 0$ we may choose cross-sections contained in U_0

- $\Sigma^{o,\pm}$ at points $y^\pm$ in different components of $W^u_{loc}(\sigma) \setminus \{\sigma\}$
- $\Sigma^{i,\pm}$ at points $x^\pm$ in different components of $W^s_{loc}(\sigma) \setminus W^{ss}_{loc}(\sigma)$

and Poincaré first hitting time maps $R^\pm : \Sigma^{i,\pm} \setminus \ell^\pm \to \Sigma^{o,-} \cup \Sigma^{o,+}$, where $\ell^\pm = \Sigma^{i,\pm} \cap W^s_{loc}(\sigma)$, satisfying (see Figure 3.1)

1. every orbit in the attractor passing through a small neighborhood of the singularity σ intersects some of the incoming cross-sections $\Sigma^{i,\pm}$;
2. $R^\pm$ maps each connected component of $\Sigma^{i,\pm} \setminus \ell^\pm$ diffeomorphically inside a different outgoing cross-section $\Sigma^{o,\pm}$, preserving the corresponding stable foliations.

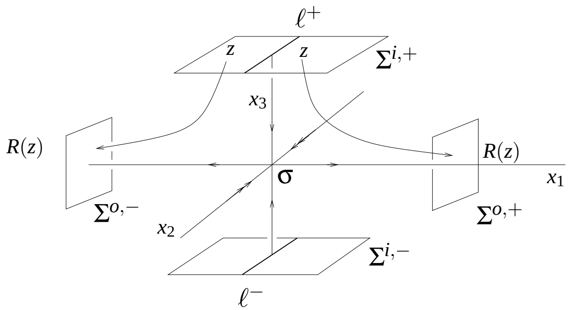

Figure 3.1: Cross-sections near a Lorenz-like singularity.

These cross-sections may be chosen to be planar relative to some linearizing system of coordinates near σ e.g. for a small $\delta > 0$

$$\Sigma^{i,\pm} = \{(x_1, x_2, \pm 1) : |x_1| \le \delta, |x_2| \le \delta\} \quad \text{and}$$
$$\Sigma^{o,\pm} = \{(\pm 1, x_2, x_3) : |x_2| \le \delta, |x_3| \le \delta\},$$

where the x_1-axis corresponds to the unstable manifold near σ, the x_2-axis to the strong-stable manifold and the x_3-axis to the weak-stable manifold of the singularity which, in turn, is at the origin, see Figure 3.1.

The singularity is hyperbolic for the vector field X. Hence for every C^1 nearby vector field Y there exists a unique Lorenz-like singularity σ_Y in U_0. Moreover the submanifolds $\Sigma^{i,\pm}$ and $\Sigma^{o,\pm}$ remain transverse to Y. So all local properties of these cross-sections are robust under small C^1 perturbations of the flow.

3.3.4 Uniformly bounded angle between stable and center-unstable directions on periodic orbits

Let us recall some facts and notation before starting the proof of Propositions 3.37 and 3.38.

Given a singularity σ of $X \in \mathcal{U}_U$, we know that σ is hyperbolic. So for Y close to X there exists a unique continuation of σ, which we write σ_Y. Since every singularity of X is hyperbolic, we conclude that the singularities of Y nearby X are the continuations of the singularities of X. Hence we can assume that, for any Y close to X, the singularities of Y in $\Lambda_Y(U)$ coincide with the ones of X in $\Lambda_X(U)$.

According to Theorem 3.5, for all $Y \in \mathcal{U}_U$ the eigenvalues $\lambda_i = \lambda_i(Y)$, $i = 1,2,3$ of $DY(\sigma_Y)$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. We write $\hat{E}^{ss,Y}_{\sigma_Y}$ for the eigenspace associated to the strongest contracting eigenvalue λ_2 and $\hat{E}^{cu,Y}_{\sigma_Y}$ for the bidimensional eigenspace associated to $\{\lambda_3, \lambda_1\}$. Without loss of generality we can assume that, for Y close to X, the eigenvalues of of $DY(\sigma_Y)$ are the same as the ones of of $DX(\sigma)$.

Since M is a Riemannian manifold, for any $x \in M$ and every neighborhood U of x there exists a *normal neighborhood* $V \subset U$ of x, i.e., for any pair of points in V there is a unique geodesic contained in V connecting them. Thus using parallel transport in V we can define angles between any pair of tangent vectors at points of V. We will use this in what follows to compare angles of tangent vectors at nearby points.

We reduce the proof of Propositions 3.37 and 3.38 to the following results.

The first one is the next lemma establishing that any compact invariant set $\Gamma \subset \Lambda_X(U)$ *containing no singularities* is uniformly hyperbolic.

Lemma 3.39. *Let $X \in \mathcal{U}_U$ and $\Gamma \subset \Lambda_X(U)$ be a compact invariant set without singularities. Then Γ is uniformly hyperbolic.*

Note that we cannot use Proposition 3.9 at this point since we still have not completed the proof of uniform domination for the splitting on periodic orbits, so we cannot use that the splitting $E^s \oplus E^{cu}$ is dominated over Λ.

We will not provide the detailed proof of Lemma 3.39 here, since the arguments are very similar to the one proving that E^s is uniformly contracting and E^{cu} is uniformly volume expanding over $\Lambda_X(U)$. The main tool is the Ergodic Closing Lemma and the main point of the argument is to show that, if the splitting of the Linear Poincaré Flow is not dominated or not hyperbolic, then we can find a C^1 close flow exhibiting either a sink or a source, contradicting Lemma 3.21. The detailed arguments can be found in [133, pages 421-424].

Given $X \in \mathcal{U}_U$ and $\delta > 0$ we define

$$C_\delta = \bigcup_{\sigma \in S(X) \cap \Lambda_X(U)} B_\delta(\sigma)$$

the δ-neighborhood around the singularities of X in Λ. Write $U_\delta = \overline{U \setminus C_\delta}$ for the closure of the complement of C_δ in U and define

$$\Omega_X(U_\delta) = \{x \in \Omega(X) : \mathcal{O}_X(x) \subset U_\delta\}.$$

We use the following application of Lemma 3.39.

Corollary 3.40. *For any $\delta > 0$, $\Omega_X(U_\delta)$ is hyperbolic.*

Recall that given a regular point $x \in M$ we define N_x^Y as the orthogonal complement of E_x^Y in T_xM, $\Lambda_Y^*(U) = \Lambda_Y(U) \setminus (S(X) \cap \Lambda_Y(U))$ and

$$N_{\Lambda_Y^*(U)} = \left\{N_x^{s,Y} \oplus N_x^{u,Y}\right\}_{x \in \Lambda_Y^*(U)}$$

denotes the splitting for the linear Poincaré flow P_t^Y of Y, see Theorem 1.8 in Section 1.3.3. For $x \in \Lambda_Y^*(U)$ we define the bundles $E_x^{cs,Y} = N_x^{s,Y} \oplus E_x^Y$ and $E_x^{cu,Y} = N_x^{u,Y} \oplus E_z^Y$.

Recall also that for Y near X and $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ we denote by $E_p^{s,Y} \oplus E_p^{cu,Y}$ the splitting induced by the hyperbolic splitting along the periodic orbit as in Definition 3.5. In this case, we have that $E_p^{cu,Y} = N_p^{u,Y} \oplus E_p^Y$ and $E_p^{s,Y} \subset E_p^{cs,Y} = N_p^{s,Y} \oplus E_p^Y$.

Using that a uniformly hyperbolic set has a unique locally defined continuation for flows close to the initial one, we obtain that for every point whose orbit does not go away from Γ for any nearby flow, any tangent vector in E^{cs} close to the flow direction remains close to the flow direction under the action of the flow.

Lemma 3.41. *Let $X \in \mathcal{U}_U$ and Γ be a compact invariant set without singularities. Then, there are neighborhoods $\mathcal{V}$ of X, V of Γ and $\gamma > 0$ such that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ satisfying: if $Y \in \mathcal{V}$, $y \in V \cap \Lambda_Y(U)$ with $Y^s(y) \in V$ for $0 \le s \le t$ and some $t \ge T$, and also $v \in N_y^{s,Y} \oplus E_y^Y$ with $\angle(v, Y(y)) < \gamma$, then $\angle\big(DY^t(y) \cdot v, Y(Y^t(y))\big) < \varepsilon$.*

The next result provides angle estimates for orbits passing nearby a singularity. For a point y in $\Lambda_Y(U)$ and vectors v with angle bounded away from zero with the strong-stable bundle at the singularity, after passing near the singularity, $DY^t(v)$ lands in the direction of the central unstable bundle at $Y^t(y)$.

Given $\sigma_Y \in S(Y) \cap \Lambda_Y(U)$, $W^s_{loc}(\sigma_Y)$ ($W^u_{loc}(\sigma_Y)$ respectively) stands for the local stable (unstable) manifold at σ_Y. We set $\hat{W}^s_{loc}(\sigma_Y) = W^s_{loc}(\sigma_Y) \setminus \{\sigma_Y\}$, and $\hat{W}^u_{loc}(\sigma_Y) = W^u_{loc}(\sigma_Y) \setminus \{\sigma_Y\}$. Since σ_Y is Lorenz-like, there is a unique bundle $\hat{E}^{ss,Y}$ in $TW^s_{loc}(\sigma_Y)$ which is strongly contracted by the derivative of the flow. For each $y \in W^s_{loc}(\sigma_Y)$, $\hat{E}^{ss,Y}_y$ is the fiber of $\hat{E}^{ss,Y}$ at y.

In the following we use the notation from Section 3.3.3 for cross-sections near a δ-neighborhood of the singularities.

Definition 3.6. If $y \in B_\delta(\sigma_Y)$ we write y_* for the point in $\hat{W}^s_{loc}(\sigma_Y)$ satisfying $\operatorname{dist}\big(y, \hat{W}^s_{loc}(\sigma_Y)\big) = \operatorname{dist}(y, y_*)$.

Now we can state the result precisely.

Lemma 3.42. *Let $X \in \mathcal{U}_U$, $\sigma \in S(X) \cap \Lambda_X(U)$ and $\delta > 0$. There exists a neighborhood $\mathcal{V}$ of X such that given $\gamma > 0$ and $\varepsilon > 0$ there is $r = r(\varepsilon, \gamma) > 0$ such that for $Y \in \mathcal{V}$, $y \in B_\delta(\sigma) \cap \Lambda_Y(U)$ satisfying $\operatorname{dist}\big(y, \hat{W}^s_{loc}(\sigma_Y)\big) < r$ and for $v \in T_yM$ with $\angle\big(v, \hat{E}^{ss,Y}_{y_*}\big) > \gamma$, then $\angle\big(DY^{s_y}(y)(v), E^{cu,Y}_{Y^{s_y}(y)}\big) < \varepsilon$, where s_y is the smallest positive time such that $Y^{s_y}(y) \in \Sigma^{o,\pm}$.*

Given $\delta' \in (0, \delta)$ we define a neighborhood of the local stable manifold of σ in $\Sigma^{i,\pm}$ by

$$\Sigma^{i,\pm}_{\delta,\delta'} = \{x \in \Sigma^{i,\pm} : \operatorname{dist}(x, \hat{W}^s_{loc}(\sigma) \cap \Sigma^{i,\pm}) \le \delta'\}. \qquad (3.32)$$

Finally next result gives also estimates for the angles after passing near a singularity: for vectors v in the central direction with angle bounded away from zero with the flow direction, then after passing near σ, $DX^t(v)$ becomes closer to the direction of the flow.

Lemma 3.43. *Let $X \in \mathcal{U}_U$, $\sigma \in S(X) \cap \Lambda_X(U)$ and $\delta > 0$. There is a neighborhood $\mathcal{V}$ of X such that, given $\varepsilon > 0$, $\kappa > 0$, $\delta > 0$ and cross sections $\Sigma^{i,\pm}$, $\Sigma^{o,\pm}$ as above, there exists $\delta' > 0$ satisfying: for all $Y \in \mathcal{V}$, $p \in \Sigma^{i,\pm}_{\delta,\delta'}$ and $v \in N^{u,Y}_p \oplus E^Y_p$, if $\angle(v, Y(p)) > \kappa$, then $\angle\big(DY^{s_p}(p) \cdot v, Y(Y^{s_p}(p))\big) < \varepsilon$, where s_p is the first positive time such that $Y^{s_p}(p) \in \Sigma^{o,\pm}$.*

We postpone the proof of Lemmas 3.41, 3.42 and 3.43 to the end of this section, and continue with the proof of Propositions 3.37 and 3.38 assuming these results.

Since we have only a finite number of singularities, we can assume that the estimates given by the previous lemmas are simultaneously valid for all singularities of Y in $\Lambda_Y(U)$ and for all $Y \in \mathcal{V}$.

Proof of Proposition 3.37-(a): We argue by contradiction. Using that hyperbolic singularities depend continuously on the vector field, we have that if item (a) of Proposition 3.37 fails then there are a singularity σ of X, $\gamma > 0$, a sequence of vector fields Y_n converging to X and a sequence of periodic points $p_n \in \mathrm{Per}_{Y_n}(\Lambda_{Y_n}(U))$ with $p_n \to \sigma$ such that

$$\angle\big(E^{s,Y_n}_{p_n}, \hat{E}^{ss,Y_n}_{\sigma_{Y_n}}\big) > \gamma. \tag{3.33}$$

We prove using (3.33) that after a first passage through a neighborhood of a singularity, the stable direction and the flow direction become close. This property persists up to the next return to that neighborhood. After a second passage through it, we show that the stable direction and the flow direction are close, and that the unstable direction and the flow direction are also close. This implies the stable and the unstable direction are close to each other, leading to a contradiction with Theorem 3.32(b).

Fix a neighborhood $B_\delta(\sigma)$ and cross sections $\Sigma^{o,\pm}, \Sigma^{i,\pm}$ contained in $B_\delta(\sigma)$ as in Section 3.3.3. Since $p_n \to \sigma$, we have that for all sufficiently large n there exists the smallest $t_n > 0$ such that $q_n = Y_n^{t_n}(p_n) \in \Sigma^{o,\pm}$.

Note that there is $q \in \hat{W}^u_{loc}(\sigma) \cap \Lambda_X(U)$ such that $q_n \to q$.

Claim 3.4. *The bound* (3.33) *implies that* $\angle\big(E^{s,Y_n}_{q_n}, Y_n(q_n)\big) \xrightarrow[n\to+\infty]{} 0$.

Proof. We prove first that, as a consequence of (3.33), the stable direction at q_n is close to the central-unstable direction at q_n. Using some properties of the splitting given by the Poincaré flow, we deduce then the stable direction at q_n is necessarily close to the flow direction at q_n, proving the claim.

By (3.33) and since $p_n \to \sigma$, by Lemma 3.42 we get

$$\angle\big(E^{s,Y_n}_{q_n}, N^{u,Y_n}_{q_n} \oplus E^{Y_n}_{q_n}\big) \xrightarrow[n\to+\infty]{} 0. \tag{3.34}$$

Now we deduce from (3.34) that $E^{s,Y_n}_{q_n}$ is leaning in the direction of the flow. Indeed, since $q_n \to q \in \Lambda^*_X(U)$, Theorem 1.8 for the Linear Poincaré Flow ensures $\angle\big(N^{s,Y_n}_{q_n}, N^{u,Y_n}_{q_n}\big) > \frac{9}{10} \cdot \angle\big(N^{s,X}_q, N^{u,X}_q\big)$ for every n big enough. Because $N^{s(u)}_{q_n}$ is orthogonal to $Y_n(q_n)$, we deduce

$$\angle(N^{s,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}, N^{u,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}\big) = \angle\big(N^{s,Y_n}_{q_n}, N^{u,Y_n}_{q_n}\big).$$

Hence $\angle\big(N^{s,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}, N^{u,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}\big)$ is uniformly bounded away from zero. Since $E^{s,Y_n}_{q_n} \subset N^{s,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}$ and $Y_n(q_n) = \big(N^{s,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}\big) \cap \big(N^{u,Y_n}_{q_n} \oplus E^{Y^n}_{q_n}\big)$, by (3.34) we obtain

$$\angle\big(E^{s,Y_n}_{q_n}, Y_n(q_n)\big) \xrightarrow[n\to\infty]{} 0. \tag{3.35}$$

This completes the proof of Claim 3.4. □

Now we apply Lemma 3.43. For this, let δ be as above, $\kappa = c$ with c given by Theorem 3.32(b) and $\varepsilon < c/2$. Let δ' be given by Lemma 3.43.

Fix $\delta^* < \max\{\delta, \delta'\}$ and consider $U_{\delta^*} = \overline{U \setminus C_{\delta^*}}$. Since the singularities of $Y \in \mathcal{V}$ are continuations of the singularities of X, we can assume that $U_{\delta^*} \cap S(Y) \cap \Lambda_Y(U) = \emptyset$ for all $Y \in \mathcal{V}$.

Since σ is an accumulation point of $\{\mathcal{O}_{Y_n}(q_n)\}_{n\geq 1}$ we have that, for n large enough, there is a first positive time s_n such that $\tilde{q}_n = Y_n^{s_n}(q_n)$ belongs to C_{δ^*}. We can take s_n in such a way that $\tilde{q}_n \in \Sigma^{i,\pm}_{\delta,\delta'}$ (defined in (3.32)).

We assume, without loss of generality, that every $\tilde{q}_n$ belongs to the same cross section $\Sigma^{i,\pm}_{\delta,\delta'}$ associated to the same singularity of Y_n and of X. Note that from the choice of δ^* we have $Y^s_n(q_n) \in U_{\delta^*}$ for all $0 \leq s \leq s_n$.

Next we prove that (3.35) is also true replacing q_n by $\tilde{q}_n$, that is

$$\angle\big(E^{s,Y_n}_{\tilde{q}_n}, Y_n(\tilde{q}_n)\big) \xrightarrow[n\to+\infty]{} 0. \tag{3.36}$$

Indeed, if there exists $S > 0$ such that for infinitely many n we have $s_n < S$, then (3.35) immediately implies (3.36).

Otherwise, let q be such that $Y_n^{s_n/2}(q_n) \xrightarrow[s_n\to\infty]{} q$. Then $\overline{\mathcal{O}_X(q)} \subset U_{\delta^*}$ which implies $\omega_X(q) \subset \Omega_X(U_{\delta^*})$. By Corollary 3.40 we know that $\Omega_X(U_{\delta^*})$ is uniformly hyperbolic. Let V be a neighborhood of $\Omega_X(U_{\delta^*})$ given by Lemma 3.41. Now we establish that the time spent by the Y_n-orbit segment $\{Y_n^t(q_n), 0 \le t \le s_n\}$ outside V is uniformly bounded.

Claim 3.5. *There is $S > 0$ such that for all n there are $0 \le s_n^1 < s_n^2 \le s_n$ with $s_n^1 < S$ and $s_n - s_n^2 < S$ satisfying $Y_n^s(q_n) \in V$ for all $s_n^1 \le s \le s_n^2$.*

Proof. It is enough to prove that there exists S' such that given q_n and $0 < s_n' < s_n$ with $Y_n^{s_n'}(q_n) \notin V$, then either $s_n' < S'$, or $s_n - s_n' < S'$.

If this were not the case, there would exist s_n' such that $Y_n^{s_n'}(q_n) \notin V$ and both $s_n - s_n' \to +\infty$ and $s_n' \to +\infty$. Then we can take a sequence $Y_n^{s_n'}(q_n) \to q'$ with $q' \notin V$. This implies that $\overline{\mathcal{O}_X(q')} \subset U_{\delta^*}$. So $\omega_X(q') \subset \Omega_X(U_{\delta^*})$ and hence $\omega_X(q') \subset V$. Thus for large n we would get $Y_n^{s_n'}(q_n) \in V$, contradicting the assumption. This finishes the proof of Claim 3.5. $\square$

Returning to the proof of (3.36), recall that $\angle\big(E_{q_n}^{s,Y_n}, Y_n(q_n)\big)$ is arbitrarily small for n large enough, by relation (3.35). Now Lemma 3.41 combined with Claim 3.5 imply (3.36), since we know that the time spent by $Y_n^s(q_n)$ in V for $s \in [0, s_n]$ is arbitrarily big.

Now since $\tilde{q}_n \in \Sigma_{\delta,\delta'}^{i,\pm}$ there is a first time $r_n > 0$ such that $\hat{q}_n = Y_n^{r_n}(\tilde{q}_n) \in \Sigma^{o,\pm}$, by the choice of the cross-sections near the singularities. We prove that also $\angle\big(E_{\hat{q}_n}^{s,Y_n}, Y_n(\hat{q}_n)\big) \xrightarrow[n\to\infty]{} 0$.

If there is $S > 0$ such that $0 < r_n < S$ for infinitely many n, then taking a subsequence we obtain the desired conclusion. Otherwise, taking a subsequence if necessary, we have $\tilde{q}_n \to \hat{W}_{loc}^s(\sigma) \cap \Sigma_{\delta,\delta'}^{i,\pm}$ and there exists $\hat{q} \in \hat{W}_{loc}^u(\sigma) \cap \Sigma^{o,\pm}$ such that $\hat{q}_n \to q$. Observe that there is $d > 0$ satisfying: for any $y \in \hat{W}_{loc}^s(\sigma) \cap \Sigma_{\delta,\delta'}^{i,\pm}$ we have $\angle\big(X(y), \hat{E}_y^{ss}\big) > d$. So provided n is large enough we obtain

$$\angle\big(Y_n(\tilde{q}_n), \hat{E}_{\tilde{q}_n}^{ss,Y_n}\big) > d. \tag{3.37}$$

Combining (3.36) and (3.37) we obtain $\angle\big(E_{\tilde{q}_n}^{s,Y_n}, \hat{E}_{\tilde{q}_n}^{ss,Y_n}\big) > d$ for n large. Arguing as in the proof of Claim 3.4, replacing q_n by $\tilde{q}_n$ for $n \ge 0$, we obtain

$$\lim_{n\to\infty} \angle\big(E_{\tilde{\tilde{q}}_n}^{s,Y_n}, Y_n(\tilde{\tilde{q}}_n)\big) = 0. \tag{3.38}$$

Moreover from (3.36) Theorem 3.32(b) ensures that

$$\angle\big(E^{u,Y_n}_{\tilde{q}_n}, Y_n(\tilde{q}_n)\big) > c \text{ for } n \text{ big enough.} \tag{3.39}$$

Since $E^{u,Y_n}_{\tilde{q}_n} \subset N^{u,Y_n}_{\tilde{q}_n} \oplus E^{Y_n}_{\tilde{q}_n}$ from (3.39) Lemma 3.43 implies that

$$\angle\big(DY_n^{r_n}(E^{u,Y_n}_{\tilde{q}_n}), Y_n(\hat{q}_n)\big) < \varepsilon < c/2 \tag{3.40}$$

by the choice of ε.

Finally (3.38) and (3.40) combined with $E^{u,Y_n}_{\hat{q}_n} = DY_n^{r_n}(E^{u,Y_n}_{\tilde{q}_n})$ give

$$\angle\big(E^{u,Y_n}_{\hat{q}_n}, E^{s,Y_n}_{\hat{q}_n}\big) < c/2 \text{ for } n \text{ big enough.}$$

This contradicts Theorem 3.32(b). This contradiction proves Proposition 3.37(a). □

Proof of Proposition 3.37(b): We show that given Y nearby X and a periodic point p of Y close to σ_Y then $E^{cu,Y}_p$ is close to $\hat{E}^{cu,Y}_{\sigma_Y}$. We split the argument into the following claims.

Given $\delta, \delta' > 0$ we consider the cross sections $\Sigma^{i,\pm}$ and $\Sigma^{i,\pm}_{\delta,\delta'}$ as in Section 3.3.3 and definition (3.32).

Claim 3.6. *Let $X \in \mathcal{U}_U$, $\sigma \in S(X) \cap \Lambda_X(U)$ and $\delta > 0$. There are a neighborhood $\mathcal{V}$ of X such that given $\gamma > 0$ and $\varepsilon > 0$, there is $r = r(\varepsilon, \gamma) > 0$ such that if $y \in \Sigma^{i,\pm}$ and $L_y \subset T_yM$ is a plane with $\angle\big(L_y, \hat{E}^{ss}_y\big) > \gamma$, then $\angle\big(DY^{s_y}(y) \cdot L_y, \hat{E}^{cu}_{\sigma_Y}\big) < \varepsilon$, where s_y is such that $Y^{s_y}(y) \in B_r(\sigma_Y)$ and $Y^s(y) \in B_\delta(\sigma_Y)$ for all $0 \le s \le s_y$.*

The proof of this claim is analogous to the proof of Lemma 3.42, which is presented at the end of this section.

Given $y \in \Sigma^{i,\pm}_{\delta,\delta'}$ let y_* be as in Definition 3.6.

Claim 3.7. *Let $X \in \mathcal{U}_U$, $\sigma \in S(X) \cap \Lambda_X(U)$ and $\delta > 0$. There are a neighborhood $\mathcal{V}$ of X, $\gamma > 0$ and $\delta' > 0$ such that for all $Y \in \mathcal{V}$ and all $y \in \Lambda_Y(U) \cap \Sigma^{i,\pm}_{\delta,\delta'}$ we have $\angle\big(E^{cu,Y}_y, \hat{E}^{ss,Y}_{y_*}\big) > \gamma$.*

Assuming the claims, let us complete the proof of the proposition.

Observe that for p close to σ_Y there is $s_p > 0$ such that $\tilde{p} = Y_{-s_p}(p) \in \Sigma^{i,\pm}_{\delta,\delta'}$, where δ and δ' are as in Claim 3.7. Let $\tilde{p}_*$ be as in Definition 3.6. By Claim 3.7 we have $\angle\big(E^{cu,Y}_{\tilde{p}}, \hat{E}^{ss,Y}_{\tilde{p}_*}\big) > \gamma$. Hence by Claim 3.6 we get

$\angle\big(DY^t(\tilde{p})(E^{cu,Y}_{\tilde{p}}), \hat{E}^{cu,Y}_{\sigma_Y}\big)$ arbitrarily small, provided p is close enough to σ_Y, concluding the proof of Proposition 3.37(b). □

Proof of Claim 3.7: First we consider points $q \in \Sigma^{i,\pm}_{\delta,\delta'} \cap \Lambda_X(U) \cap \hat{W}^s_{loc}(\sigma)$.

In this case, observe that $\angle\big(E^{cu,X}_q, \hat{E}^{ss}_q\big) \geq \angle\big(E^{cu,X}_q, T_q W^s_{loc}(\sigma)\big)$. By item 3 of Theorem 1.8 we have $N^{s,X}_q = T_q W^s_{loc}(\sigma) \cap N_q$ and since $X(q) \in T_q W^s_{loc}(\sigma)$ we get that $T_q W^s_{loc}(\sigma) = N^{s,X}_q \oplus E^X_q$. We conclude

$$\angle\big(E^{cu,X}_q, T_q W^s_{loc}(\sigma)\big) = \angle\big(E^{cu,X}_q, N^{s,X}_q \oplus E^X_q\big) = \angle\big(N^{u,X}_q, N^{s,X}_q\big). \quad (3.41)$$

Since $\Sigma^{i,\pm}_{\delta,\delta'}$ is compact and does not contain singularities by construction, Theorem 1.8 ensures that there is $\gamma = \gamma(\delta, \delta')$ such that $\angle\big(N^{u,X}_q, N^{s,X}_q\big) > \gamma$ for all $q \in \Sigma^{i,\pm}_{\delta,\delta'}$. Replacing this inequality in (3.41) we conclude the proof of the claim in this case.

For p close enough to σ_Y, we have $\operatorname{dist}(\tilde{p}, \Sigma^{i,\pm}_{\delta,\delta'} \cap \hat{W}^s_{loc}(\sigma_Y))$ arbitrarily small. Using the continuous dependence of the splitting $N^{s,X} \oplus N^{u,X}$ with the flow together with Theorem 1.8, we get that the estimate (3.41) above still holds replacing q by $\tilde{p}$ and X by Y, concluding the proof of Claim 3.7. □

Proof of Proposition 3.38: Assume, by contradiction, that there exists a sequence of periodic points $p_n \notin C_\delta$ for flows $Y_n \to X$ such that

$$\angle\big(E^{cu,Y_n}_{p_n}, E^{s,Y_n}_{p_n}\big) \xrightarrow[n\to+\infty]{} 0. \quad (3.42)$$

We claim that $\overline{\cup_n \mathcal{O}_{Y_n}(p_n)} \cap S(X) \cap \Lambda_X(U) \neq \emptyset$. Indeed, if this were not the case, we would get $\delta^* > 0$ such that $\overline{\cup_n \mathcal{O}_{Y_n}(p_n)} \subset \Omega_X(U_{\delta^*})$. By Corollary 3.40 the set $\Omega_X(U_{\delta^*})$ is hyperbolic and so there are neighborhoods V and $\mathcal{V}$ of $\Omega_X(U_{\delta^*})$ and Y, respectively, and $c > 0$ satisfying $\angle\big(E^{s,Y}_p, E^{cu,Y}_p\big) > c$ for all $p \in \mathrm{Per}_Y(\Lambda_Y(U))$ such that $\mathcal{O}_Y(p) \subset V$. Since $Y_n \xrightarrow[n\to+\infty]{} X$ we have $\mathcal{O}_{Y_n}(p_n) \subset V$ for n sufficiently large. We conclude that $\angle\big(E^{s,Y_n}_{p_n}, E^{cu,Y_n}_{p_n}\big) > c$, leading to a contradiction. Thus $\overline{\cup_n \mathcal{O}_{Y_n}(p_n)} \cap S(X) \cap \Lambda_X(U) \neq \emptyset$ as claimed.

Fix $\delta > 0$ and take cross sections $\Sigma^{i,\pm}$ and $\Sigma^{o,\pm}$ as in Section 3.3.3.

Since $\overline{\cup_n \mathcal{O}_{Y_n}(p_n)} \cap S(X) \cap \Lambda_X(U) \neq \emptyset$, we get for each n a positive s_n such that $\tilde{p}_n = Y^{s_n}_n(p_n) \in \Sigma^{i,\pm}$.

Now take $\kappa = c$ with c given by Theorem 3.32(b), $\varepsilon < c/2$ and δ' as in Lemma 3.43. Fix $\delta^* < \min\{\delta, \delta'\}$ and consider $U_{\delta^*} = \overline{U \setminus C_{\delta^*}}$. By Corollary 3.40 the subset $\Omega_X(U_{\delta^*})$ is hyperbolic.

From the choice of δ^*, we get that $Y_n^s(p_n) \in U_{\delta^*}$ for any $0 \le s \le s_n$. We assume, without loss of generality, that every $\tilde{p}_n$ is in a neighborhood of the same singularity σ. Reasoning as in Claim 3.4 we prove that (3.42) implies

$$\angle\big(E^{s,Y_n}_{\tilde{p}_n}, Y_n(\tilde{p}_n)\big) \xrightarrow[n\to+\infty]{} 0. \tag{3.43}$$

Once (3.43) is settled, the proof follows analogously to the one of the previous proposition. □

We finally present the proofs Lemmas 3.41, 3.42 and 3.43.

Proof of Lemma 3.41: Since Γ is hyperbolic, there are $0 < \lambda_\Gamma < 1$ and $c > 0$ such that $N^{s,X}_\Gamma = E^{s,X}_\Gamma \oplus E^X$ with $\|DX^t \mid E^{s,X}\| < c \cdot \lambda^t_\Gamma$, and $c^{-1} < \|X \mid \Gamma\| < c$. Changing the metric in a neighborhood of Γ, we can assume without loss of generality that $E^{s,X}_x$ is orthogonal to E^X_x and $\|X(x)\| = 1$ for all $x \in \Gamma$. In other words, in the new metric $E^{s,X}_\Gamma$ coincides with the stable bundle $N^{s,X}_\Gamma$ of the linear Poincaré flow restricted to Γ.

For each $x \in \Gamma$, let $n^{s,X}_x \in N^{s,X}_x$ be a unit vector and consider the orthogonal basis $\mathcal{B}_x = \{X(x), n^{s,X}_x\}$ of $E^X_x \oplus N^{s,X}_x$. In this basis the matrix of $DX^t(x)$ restricted to $E^X_x \oplus N^{s,X}_x$ is

$$DX^t \mid \big(E^X_x \oplus N^{s,X}_x\big) = \begin{bmatrix} 1 & 0 \\ 0 & n^{s,X}_{x,t} \end{bmatrix},$$

where $\|n^{s,X}_{x,t}\| < c \cdot \lambda^t_\Gamma$.

Fix $t_0 > 0$ such that $\|n^{s,X}_{x,t_0}\| < 1/2$ for all $x \in \Gamma$. There exists $c' > 0$ such that $\|n^{s,X}_{x,t_0}\| > c'$ for all $x \in \Gamma$ by continuity of the flow and compactness of Γ. Taking a neighborhood V of Γ and a neighborhood $\mathcal{V} \subset \mathcal{U}_U$ of X, both sufficiently small, and a change of metric varying continuously with the flow, we can get $\|Y(y)\| = 1$ for all $Y \in \mathcal{V}$ and all $y \in \Lambda_Y(U)$. Thus the matrix of $DY^{t_0}(y)$ restricted to $E^Y_y \oplus N^{s,Y}_y$ with respect to the basis $\mathcal{B}_y = \{Y(y), n^{s,Y}_y\}$ is

$$DY^{t_0} \mid \big(E^Y_y \oplus N^{s,Y}_y\big) = \begin{bmatrix} 1 & \delta^Y_y \\ 0 & n^{s,Y}_{y,t_0} \end{bmatrix},$$

where $\delta_y^Y < \delta_0$ and δ_0 is small for Y sufficiently close to X. Moreover $\|n_{y,t_0}^{s,Y}\| < 1/2$. Hence

$$DY_{n\cdot t_0} \mid \left(E_y^Y \oplus N_y^{s,Y}\right) = \begin{bmatrix} 1 & \delta_{y,n}^Y \\ 0 & n_{y,n\cdot t_0}^{s,Y} \end{bmatrix},$$

with $\delta_{y,n}^Y \ll 2\delta_0$. Let $\varepsilon > 0$ and n_0 be such that $2^{-n} < \varepsilon$ for all $n \geq n_0$. Given $v \in E_y^Y \oplus N_y^{s,Y}$ we can write $v = (1, \gamma_0)$ in the basis B_y. Then for any positive integer m we get

$$\angle\left(DY_{n_0\cdot m}(y)\cdot v, (1,0)\right) \leq \frac{\gamma_0 \cdot n_{y,n_0\cdot m}^{s,Y}}{1-\delta_{y,n_0}^Y \cdot n_{y,n_0\cdot m}^{s,Y}} < \frac{(1/2)^{n_0\cdot m}}{1-2\delta_0}.$$

For $t > n_0$ we write $t = m\cdot n_0 + s$ with $0 \leq s \leq n_0$ and then

$$\angle\left(DY^t(y)\cdot v, Y(Y^t(y)\right) < K\cdot\varepsilon$$

for some positive constant K, proving Lemma 3.41. □

Proof of Lemma 3.42: We prove the lemma introducing linearizing coordinates in a normal neighborhood V of σ. For this we assume that there is a neighborhood V of σ where all Y near enough X is linearisable. This is no restriction since we can always get rid of resonances between the eigenvalues by small C^∞ perturbations of the flow. Fix $\delta > 0$ small so that $B_\delta(\sigma) \subset V$. Assume also that $\sigma_Y = \sigma$ and the eigenvalues of $DY(\sigma_Y)$ are the same as the ones of $DX(\sigma)$. Let $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ be the eigenvalues of $DX(\sigma)$. So, in local coordinates $\overline{x}, \overline{y}, \overline{z}$, we have that $Y \mid V$ can be written as

$$Y(\overline{x}, \overline{w}, \overline{z}) = \begin{cases} \dot{\overline{x}} = \lambda_1 \overline{x} \\ \dot{\overline{y}} = \lambda_2 \overline{y} \\ \dot{\overline{z}} = \lambda_3 \overline{z}. \end{cases} \tag{3.44}$$

Note that in this case for $y \in W_{loc}^s(\sigma)$

$$W_{loc}^s(\sigma) = V \cap \left(\{0\} \times \mathbb{R}^2\right), \qquad W_{loc}^u(\sigma) = V \cap \left(\mathbb{R} \times \{(0,0)\}\right),$$
$$W_{loc}^{cu}(\sigma) = V \cap \left(\mathbb{R} \times \{0\} \times \mathbb{R}\right), \qquad \hat{E}_y^{ss,Y} = V \cap \left(\{0\} \times \mathbb{R} \times \{0\}\right).$$

For $y \in W_{loc}^u(\sigma)$ we have $\hat{E}_y^{cu,Y} = \mathbb{R} \times \{0\} \times \mathbb{R}$ and $\Sigma^{o,\pm} \cap W_{loc}^u(\sigma) = \{(\pm 1, 0, 0)\}$.

For $y \in V$ and for $v = (v_1, v_2, v_3)$, if $t > 0$ is such that $Y^s(y) \in V$ for all $0 \le s \le t$, then

$$DY^t(y) \cdot v = (e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2, e^{\lambda_3} v_3). \tag{3.45}$$

Given two vectors v and w we set slope(v,w) for the slope between v and w.

Let $r > 0$ and $y \in B_\delta(\sigma)$ be such that $\text{dist}(y, \hat{W}^s_{loc}(\sigma)) < r$, $v = (v_1, v_2, v_3) \in T_yM$ and $t > 0$ satisfying $Y^s(y) \in V$ for all $0 \le s \le t$. Then

$$\text{slope}\left(DY^t(y) \cdot v, \hat{E}^{cu,X}_\sigma\right) = \frac{|e^{\lambda_2 t} \cdot v_2|}{\sqrt{(e^{\lambda_1 t} \cdot v_1)^2 + (e^{\lambda_3 t} \cdot v_3)^2}}.$$

On the other hand, assuming that $\angle\big(\hat{E}^{ss,X}_\sigma, v\big) = \angle\big((0,1,0), v\big) > \gamma$ we get that there is $0 < \hat{\gamma} < 1$ such that $0 \le |v_2| < \hat{\gamma}$. Hence $v_1^2 + v_3^2 > 1 - \hat{\gamma}^2$. This implies that either $v_1 > \sqrt{(1-\hat{\gamma}^2)/2}$, or $v_3 > \sqrt{(1-\hat{\gamma}^2)/2}$. Thus

$$\text{slope}\left(DY^t(y) \cdot v, \hat{E}^{cu,X}_\sigma\right) \le \frac{|e^{\lambda_2 t} \cdot v_2|}{|e^{\lambda_i t} \cdot v_i|} \le \frac{\hat{\gamma}}{\sqrt{(1-\hat{\gamma}^2)/2)}} \cdot e^{(\lambda_2 - \lambda_i)t}, \tag{3.46}$$

where $i \in \{1,3\}$ is chosen so that v_i satisfies $v_i^2 > \sqrt{(1-\gamma^2)/2}$. As both $\lambda_2 - \lambda_3$ and $\lambda_2 - \lambda_1$ are strictly negative, there is $T = T(\varepsilon, \gamma) > 0$ such the bound given by (3.46) is smaller than ε for all $t > T$.

Now taking r sufficiently small, for $y \in (B_\delta(\sigma) \setminus \hat{W}^s_{loc}(\sigma))$, we can ensure that if $Y^t(y) \in \Sigma^{o,\pm}$, then $t > T$. These last two facts combined complete the proof. □

Proof of Lemma 3.43: For the proof of this lemma we use local linearisable coordinates in a neighborhood of σ as in the proof of Lemma 3.42.

Let $\delta > 0$ be small enough so that $B_\delta(\sigma) \subset V$ and consider $\Sigma^{i,\pm}, \Sigma^{o,\pm}$ as in Section 3.3.3. Take $\delta' > 0$ and consider $\Sigma^{i,\pm}_{\delta,\delta'}$ as in (3.32). Let $p \in \Sigma^{i,\pm}_{\delta,\delta'} \cap \Lambda_Y(U)$ and $v \in N^{u,Y}_p \oplus E^Y_p$ with $\angle\big(v, Y(p)\big) > \kappa > 0$. Write $v = a \cdot (1,0,0) + b \cdot (0,1,0) + c \cdot (0,0,1)$ with $a^2 + b^2 + c^2 = 1$.

Claim 3.8. *There are $R > 0$ and δ' such that, if p and v are as above then, $|a| > R$.*

Proof. By the continuity of the flow direction and the normal bundle splitting far from singularities, it suffices to verify the claim for $p \in W^s_{loc}(\sigma) \setminus \{\sigma\}$. In this case $E^{cs,Y}_p = \{0\} \times \mathbb{R}^2$. Thus all we need to prove is that $\angle\big(v, E^{cs,Y}_p\big) > \kappa$ for some $\kappa > 0$. For this, observe that since $\operatorname{dist}\big(p,\sigma\big) > \delta$, by Theorem 1.8 there is $k' = k'(\delta)$ such that $\angle\big(N^{s,Y}_p, N^{u,Y}_p\big) > k'$. Since $\angle\big(E^{cu,Y}_p, E^{cs,Y}_p\big) = \angle\big(N^{s,Y}_p, N^{u,Y}_p\big)$, we conclude that

$$\angle\big(E^{cu,Y}_p, E^{cs,Y}_p\big) > k'. \tag{3.47}$$

On the other hand, $v \in E^Y_p \oplus N^{u,Y}_p = E^{cu,Y}_p$ and $\angle\big(v, Y(p)\big) = \angle\big(v, E^{cs,Y}_p \cap E^{cu,Y}_p\big) > \kappa$ by hypothesis. This fact combined with (3.47) give the proof of the claim. $\square$

Returning to the proof of Lemma 3.43, let $t_p > 0$ be such that $Y^{t_p}(p) \in \Sigma^u_\delta$. Next we prove that for δ' small we have

1. $\angle\big(Y(Y^{t_p}(p)), (1,0,0)\big)$ is small, and
2. $\angle\big(DY^{t_p}(p)(v), (1,0,0)\big)$ is small.

Observe that if $\delta' \to 0$, then $t_p \to \infty$ and $Y^{t_p}(p)$ converges to a point in $\hat{W}^u_{loc}(\sigma)$, where the flow direction is $(1,0,0)$. Hence the continuity of the flow direction implies the first item above.

To prove the second item, recall (3.45). Then by Claim 3.8

$$\left|\frac{b \cdot e^{\lambda_2 \cdot t_p}}{a \cdot e^{\lambda_1 \cdot t_p}}\right| < e^{(\lambda_2 - \lambda_1)\cdot t_p} \cdot \frac{|b|}{R}.$$

Similarly, we have $\left|\frac{c \cdot e^{\lambda_3 \cdot t_p}}{a \cdot e^{\lambda_1 \cdot t_p}}\right| < e^{(\lambda_3 - \lambda_1)\cdot t_p} \cdot |c|/R$. Since $t_p \to \infty$ as $\delta' \to 0$, $R > 0$ and both $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ are negative numbers, we deduce that the bounds on both inequalities above tend to 0 when $\delta' \to 0$, concluding the proof of Lemma 3.43. $\square$

3.4 Sufficient conditions for robustness of singular-hyperbolic attractors

Here we present a proof of Theorem 3.11. This is based on the following result whose proof we postpone to Section 3.4.2.

Theorem 3.44. *Let Λ be a singular-hyperbolic attracting set of $X \in \mathfrak{X}^r$ for some $r \geq 1$. Suppose that Λ is connected and contains a dense subset of periodic orbits. Moreover assume that Λ contains only one singularity and is not transitive.*

Then for every neighborhood U of Λ there exists a flow Y close to X in the C^r topology such that $\Lambda_Y(U) \not\subset \Omega(Y)$.

Let Λ be a singular-hyperbolic attractor of a C^r flow X on a compact 3-manifold M. Assume that Λ is C^r robustly periodic and has a unique singularity σ.

Denote by U a neighborhood of Λ such that $\Lambda_Y(U) \cap \mathrm{Per}(Y)$ is dense in $\Lambda_Y(U)$ for every flow Y which is C^r close to X. Clearly $\Lambda_Y(U)$ is a singular-hyperbolic set of Y for all Y close to X.

Because Λ has a unique singularity, which is Lorenz-like, then $\Lambda_Y(U)$ has a unique singularity as well. Indeed, by Remark 3.23, every singularity of $\Lambda_Y(U)$ must be either singular-hyperbolic for Y or for $-Y$ (we can show that every singularity in this setting is Lorenz-like, see Lemma 3.45 in the next section). In both cases the singularities are hyperbolic and bifurcations are not allowed for every Y close to X. Hence if Λ_Y had more than one singularity there would exist at least two distinct singularities in the original set Λ, by the property of analytic continuation of any hyperbolic critical element.

Recalling that Λ is an attractor by assumption, thus transitive in particular, we see that Λ is connected, and so we can assume that the neighborhood U above is connected. Then $\Lambda_Y(U)$ is connected as well.

Summarizing: $\Lambda_Y(U)$ is a connected singular-hyperbolic attracting set of Y containing only one singularity.

Were Λ not C^r robust, then it would exist Y close to X such that $\Lambda_Y(U)$ is not transitive. In this case $\Lambda_Y(U)$ would satisfy all the conditions of Theorem 3.44. Hence there would exist Z close to Y satisfying $\Lambda_Z(U) \not\subset \Omega(Z)$. But we are assuming that $\Lambda_Z(U) \cap C(Z)$ is dense in $\Lambda_Z(U)$ and $C(Z)$ is always contained in $\Omega(Z)$.

This contradiction completes the proof of Theorem 3.11, assuming Theorem 3.44.

3.4.1 Cross-sections and Poincaré maps

For future reference we give here a few properties of *Poincaré maps*, that is, continuous maps $R:\Sigma \to \Sigma'$ of the form $R(x) = X^{t(x)}(x)$ between cross-sections Σ and Σ' of the flow near a singular-hyperbolic set. We always assume that the Poincaré time $t(\cdot)$ is large enough as explained in what follows.

There properties will be often used in the following chapters to obtain many dynamical and ergodic consequences of singular-hyperbolicity. In particular they will be used in Section 3.4.2 to prove Theorem 3.44.

We assume that Λ is a compact invariant subset for a flow $X \in \mathfrak{X}^1(M)$ such that

- either Λ is a singular-hyperbolic attractor,
- or Λ is a singular-hyperbolic attracting set with a dense subset of periodic orbits.

In the former case it has already been proved that every singularity in Λ is Lorenz-like. Next result shows that the same is true in the latter case.

Lemma 3.45. *Let Λ be a singular hyperbolic attracting set for a flow X. Then every singularity σ accumulated by regular orbits in Λ is Lorenz-like and $W^{ss}(\sigma)\cap\Lambda = \{\sigma\}$.*

Proof. Let σ be a singularity of X in Λ. According to Theorem 3.10 we have that σ is Lorenz-like for X or for $-X$. Arguing by contradiction, assume that σ is Lorenz-like for $-X$.

Again by Theorem 3.10 we have $W^{ss}_Y(\sigma)\cap\Lambda = \{\sigma\}$ either for $Y = -X$ or for $Y = X$. Since we assume that σ is Lorenz-like for $-X$, this means that either $W^{ss}(\sigma)\cap\Lambda = \{\sigma\}$ or $W^{uu}(\sigma)\cap\Lambda = \{\sigma\}$ for X. Since Λ is attracting, then $W^{uu}(\sigma) \subset \Lambda$ and so the latter condition is impossible. We now show that the former condition is also violated, concluding the proof.

By assumption we can find points p_n in regular orbits inside Λ such that $p_n \xrightarrow[n\to+\infty]{} \sigma$.

Consider the linearised flow (3.44) and its solutions (2.3) given on a neighborhood V of σ by the Hartman-Grobman Theorem 1.9. It is easy to see that the accumulation of p_n on σ implies that the orbit of p_n through V must also pass nearby points on the connected components containing σ of of $W^{ss}(\sigma)\cap V$. Since Λ is closed we obtain $W^{ss}(\sigma)\cap\Lambda\setminus\{\sigma\} \neq \emptyset$. ∎

We start by observing that cross-sections have co-dimension one foliations which are dynamically defined: the leaves $W^s(x,\Sigma) = W^s_{loc}(x) \cap \Sigma$ correspond to the intersections with the stable manifolds of the flow. These leaves are uniformly contracted and, assuming the cross-section is *adapted*, then the foliation is invariant:

$$R(W^s(x,\Sigma)) \subset W^s(R(x),\Sigma') \quad \text{for all } x \in \Lambda \cap \Sigma.$$

Moreover we show that R is uniformly expanding in the transverse direction. Then we analyze the flow close to singularities, again by means of cross-sections.

Stable foliations on cross-sections

We recall classical facts about partially hyperbolic systems, especially existence of strong-stable and center-unstable foliations. The standard reference is [76].

We have that Λ is a singular-hyperbolic isolated set of $X \in \mathfrak{X}^1(M)$ with invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ with $\dim E^{cu} = 2$. Let $\tilde{E}^s \oplus \tilde{E}^{cu}$ be a continuous extension of this splitting to a small neighborhood U_0 of Λ. For convenience we take U_0 to be forward invariant. Then $\tilde{E}^s$ may be chosen invariant under the derivative: just consider at each point the direction formed by those vectors which are strongly contracted by DX_t for positive t. In general $\tilde{E}^{cu}$ is not invariant. However we can consider a cone field around it on U_0

$$C^{cu}_a(x) = \{v = v^s + v^{cu} : v^s \in \tilde{E}^s_x \text{ and } v^u \in \tilde{E}^{cu}_x \text{ with } \|v^s\| \le a \cdot \|v^{cu}\|\}$$

which is forward invariant for $a > 0$:

$$DX^t(C^{cu}_a(x)) \subset C^{cu}_a(X^t(x)) \quad \text{for all large } t > 0. \tag{3.48}$$

Moreover we may take $a > 0$ arbitrarily small, reducing U_0 if necessary. For notational simplicity we write E^s and E^{cu} for $\tilde{E}^s$ and $\tilde{E}^{cu}$ in all that follows.

The next result says that there are locally strong-stable and center-unstable manifolds, defined at every regular point $x \in U_0$ and which are embedded disks tangent to $E^s(x)$ and $E^{cu}(x)$, respectively. The strong-stable manifolds are locally invariant. Given any $x \in U_0$ define the strong-stable manifold $W^{ss}(x)$ and the stable-manifold $W^s(x)$ as in Section 1.2.

Given $\varepsilon > 0$ denote $I_\varepsilon = (-\varepsilon, \varepsilon)$ and let $\mathcal{E}^1(I_1, M)$ be the set of C^1 embedding maps $f : I_1 \to M$ endowed with the C^1 topology.

Proposition 3.46 (stable and center-unstable manifolds)**.** *There are continuous maps $\phi^{ss} : U_0 \to \mathcal{E}^1(I_1, M)$ and $\phi^{cu} : U_0 \to \mathcal{E}^1(I_1 \times I_1, M)$ such that given any $0 < \varepsilon < 1$ and $x \in U_0$, if we denote $W^{ss}_\varepsilon(x) = \phi^{ss}(x)(I_\varepsilon)$ and $W^{cu}_\varepsilon(x) = \phi^{cu}(x)(I_\varepsilon \times I_\varepsilon)$,*

(a) $T_x W^{ss}_\varepsilon(x) = E^s(x)$*;*

(b) $T_x W^{cu}_\varepsilon(x) = E^{cu}(x)$*;*

(c) $W^{ss}_\varepsilon(x)$ *is a neighborhood of* x *inside* $W^{ss}(x)$*;*

(d) if $y \in W^{ss}(x)$ *then there is* $T \geq 0$ *such that* $X^T(y) \in W^{ss}_\varepsilon(X^T(x))$ *(local invariance);*

(e) $d(X^t(x), X^t(y)) \leq K \cdot e^{-\lambda t} \cdot d(x,y)$ *for all* $t > 0$ *and all* $y \in W^{ss}_\varepsilon(x)$*.*

The constants $K, \lambda > 0$ are taken as in the definition of (K, λ)-splitting in the beginning of Chapter 3, and the distance $d(x,y)$ is the intrinsic distance between two points on the manifold $W^{ss}_\varepsilon(x)$, given by the length of the shortest smooth curve contained in $W^{ss}_\varepsilon(x)$ connecting x to y.

Denoting $E^{cs}_x = E^s_x \oplus E^X_x$, where E^X_x is the direction of the flow at x, it follows that

$$T_x W^{ss}(x) = E^s_x \quad \text{and} \quad T_x W^s(x) = E^{cs}_x.$$

We fix ε once and for all. Then we call $W^{ss}_\varepsilon(x)$ the local *strong-stable manifold* and $W^{cu}_\varepsilon(x)$ the local *center-unstable manifold* of x.

Now let Σ be a *cross-section* to the flow, that is, a C^2 embedded compact disk transverse to X: at every point $z \in \Sigma$ we have $T_z\Sigma \oplus E^X_z = T_z M$ (recall that E^X_z is the one-dimensional subspace $\{s \cdot X(z) : s \in \mathbb{R}\}$) . For every $x \in \Sigma$ we define $W^s(x, \Sigma)$ to be the connected component of $W^s(x) \cap \Sigma$ that contains x. This defines a foliation $\mathcal{F}^s_\Sigma$ of Σ into co-dimension 1 sub-manifolds of class C^1.

Remark 3.47. Given any cross-section Σ and a point x in its interior, we may always find a smaller cross-section also with x in its interior and which is the image of the square $[0,1] \times [0,1]$ by a C^2 diffeomorphism h that sends horizontal lines inside leaves of $\mathcal{F}^s_\Sigma$. In what follows we always assume that cross-sections are of this kind, see Figure 3.2. We denote by $\text{int}(\Sigma)$ the image of $(0,1) \times (0,1)$ under the above-mentioned diffeomorphism, which we call the *interior* of Σ.

We also assume that each cross-section Σ is contained in U_0, so that every $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$.

Remark 3.48. In general, we can not choose the cross-section such that $W^s(x,\Sigma) \subset W^{ss}_\varepsilon(x)$. The reason is that we want cross-sections to be C^2. Cross-section of class C^1 are enough for the proof of expansiveness in Section 4.1.1 but C^2 is needed for the construction of the physical measure in Section 4.2.1 and for the absolute continuity results in Section 4.2.8. See Section 1.4.2 for the technical definitions.

On the one hand $x \mapsto W^{ss}_\varepsilon(x)$ is usually not differentiable if we assume that X is only of class C^1, see e.g. [146]. On the other hand, assuming that the cross-section is small with respect to ε, and choosing any curve $\gamma \subset \Sigma$ crossing transversely every leaf of $\mathcal{F}^s_\Sigma$, we may consider a Poincaré map

$$R_\Sigma : \Sigma \to \Sigma(\gamma) = \bigcup_{z\in\gamma} W^{ss}_\varepsilon(z)$$

with Poincaré time close to zero, see Figure 3.2. This is a homeomorphism onto its image, close to the identity, such that $R_\Sigma(W^s(x,\Sigma)) \subset W^{ss}_\varepsilon(R_\Sigma(x))$. So, identifying the points of Σ with their images under this homeomorphism, we may pretend that indeed $W^s(x,\Sigma) \subset W^{ss}_\varepsilon(x)$. We shall often do this in the sequel, to avoid cumbersome technicalities.

Hyperbolicity of Poincaré maps

Let Σ be a small cross-section to X and let $R : \Sigma \to \Sigma'$ be a Poincaré map $R(y) = X^{t(y)}(y)$ to another cross-section Σ' (possibly $\Sigma = \Sigma'$). Note that R needs not correspond to the first time the orbits of Σ encounter Σ'.

The splitting $E^s \oplus E^{cu}$ over U_0 induces a continuous splitting $E^s_\Sigma \oplus E^{cu}_\Sigma$ of the tangent bundle $T\Sigma$ to Σ (and analogously for Σ'), defined by

$$E^s_\Sigma(y) = E^{cs}_y \cap T_y\Sigma \quad \text{and} \quad E^{cu}_\Sigma(y) = E^{cu}_y \cap T_y\Sigma. \tag{3.49}$$

We now show that if the Poincaré time $t(x)$ is sufficiently large then (3.49) defines a hyperbolic splitting for the transformation R on the cross-sections, at least restricted to Λ, in the following sense.

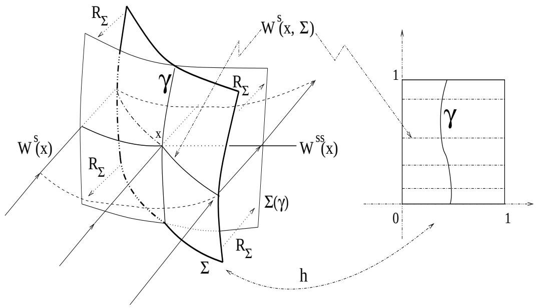

Figure 3.2: The sections Σ, $\Sigma(\gamma)$, the manifolds $W^s(x), W^{ss}(x)$, $W^s(x,\Sigma)$ and the projection R_Σ, on the right. On the left, the square $[0,1]\times[0,1]$ is identified with Σ through the map h, where $\mathcal{F}^s_\Sigma$ becomes the horizontal foliation and the curve γ is transverse to the horizontal direction. Solid lines with arrows indicate the flow direction.

Proposition 3.49. *Let $R:\Sigma\to\Sigma'$ be a Poincaré map as before with Poincaré time $t(\cdot)$. Then $DR_x(E^s_\Sigma(x)) = E^s_\Sigma(R(x))$ at every $x\in\Sigma$ and $DR_x(E^{cu}_\Sigma(x)) = E^{cu}_\Sigma(R(x))$ at every $x\in\Lambda\cap\Sigma$.*

Moreover for every given $0<\lambda<1$ there exists $t_1 = t_1(\Sigma,\Sigma',\lambda)>0$ such that if $t(\cdot)>t_1$ at every point, then

$$\|DR \mid E^s_\Sigma(x)\| < \lambda \quad \text{and} \quad \|DR \mid E^{cu}_\Sigma(x)\| > 1/\lambda \quad \text{at every } x\in\Sigma.$$

Remark 3.50. In what follows we use K as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction, and on upper and lower bounds for the norm of the vector field on the cross-sections. The conditions on t_1 in the proof of the proposition depend only on these bounds as well. In all our applications, all these angles and norms will be uniformly bounded from zero and infinity, and so both K and t_1 may be chosen uniformly.

Proof. The differential of the Poincaré map at any point $x\in\Sigma$ is given by

$$DR(x) = P_{R(x)}\circ DX^{t(x)} \mid T_x\Sigma,$$

where $P_{R(x)}$ is the projection onto $T_{R(x)}\Sigma'$ along the direction of $X(R(x))$.

Note that $E^s_\Sigma(x)$ is tangent to $\Sigma \cap W^s(x) \supset W^s(x,\Sigma)$. Since the stable manifold $W^s(x)$ is invariant, we have invariance of the stable bundle:

$$DR(x)\big(E^s_\Sigma(x)\big) = E^s_{\Sigma'}\big(R(x)\big).$$

Moreover for all $x \in \Lambda$ we have

$$DX^{t(x)}\big(E^{cu}_\Sigma(x)\big) \subset DX^{t(x)}\big(E^{cu}_x\big) = E^{cu}_{R(x)}\,.$$

As $P_{R(x)}$ is the projection along the vector field, it sends $E^{cu}_{R(x)}$ to $E^{cu}_{\Sigma'}(R(x))$. This proves that the center-unstable bundle is invariant restricted to Λ, i.e. $DR(x)\big(E^{cu}_\Sigma(x)\big) = E^{cu}_{\Sigma'}(R(x))$.

Next we prove the expansion and contraction statements. We start by noting that $\|P_{R(x)}\| \le K$. Then we consider the basis $\{\frac{X(x)}{\|X(x)\|}, e^u_x\}$ of E^{cu}_x, where e^u_x is a unit vector in the direction of $E^{cu}_\Sigma(x)$. Since the flow direction is invariant, the matrix of $DX^t \mid E^{cu}_x$ relative to this basis is upper triangular:

$$DX^{t(x)} \mid E^{cu}_x = \begin{bmatrix} \frac{\|X(R(x))\|}{\|X(x)\|} & \star \\ 0 & \Delta \end{bmatrix}.$$

Moreover

$$\frac{1}{K}\cdot \det\big(DX^{t(x)} \mid E^{cu}_x\big) \le \frac{\|X(R(x))\|}{\|X(x)\|}\Delta \le K\cdot \det\big(DX^{t(x)} \mid E^{cu}_x\big).$$

Then

$$\begin{aligned}\|DR(x)\,e^u_x\| &= \|P_{R(x)}\big(DX^{t(x)}(x)\cdot e^u_x\big)\| = \|\Delta\cdot e^u_{R(x)}\| = |\Delta| \\ &\ge K^{-3}\,|\det(DX^{t(x)} \mid E^{cu}_x)| \ge K^{-3}\lambda^{-t(x)} \ge K^{-3}\,\lambda^{-t_1}.\end{aligned}$$

Taking t_1 large enough we ensure that the latter expression is larger than $1/\lambda$.

To prove $\|DR \mid E^s_\Sigma(x)\| < \lambda$, let us consider unit vectors $e^s_x \in E^s_x$ and $\hat{e}^s_x \in E^s_\Sigma(x)$, and write

$$e^s_x = a_x\cdot \hat{e}^s_x + b_x \cdot \frac{X(x)}{\|X(x)\|}\,.$$

Since $\angle(E^s_x, X(x)) \ge \angle(E^s_x, E^{cu}_x)$ and the latter is uniformly bounded from zero, we have $|a_x| \ge \kappa$ for some $\kappa > 0$ which depends only on the flow.

Then

$$\begin{aligned} \|DR(x)\,e_x^s\| &= \|P_{R(x)} \circ \big(DX^{t(x)}(x)\cdot e_x^s\big)\| \\ &= \frac{1}{|a_x|}\left\|P_{R(x)} \circ \Big(DX^{t(x)}(x)\big(e_x^s - b_x\frac{X(x)}{\|X(x)\|}\big)\Big)\right\| \\ &= \frac{1}{|a_x|}\left\|P_{R(x)} \circ \big(DX^{t(x)}(x)\cdot \hat{e}_x^s\big)\right\| \le \frac{K}{\kappa}\lambda^{t(x)} \le \frac{K}{\kappa}\lambda^{t_1}. \end{aligned} \tag{3.50}$$

Once more it suffices to take t_1 large to ensure that the right hand side is less than λ. □

Given a cross-section Σ, a positive number ρ, and a point $x \in \Sigma$, we define the unstable cone of width ρ at x by

$$C_\rho^u(x) = \{v = v^s + v^u : v^s \in E_\Sigma^s(x), v^u \in E_\Sigma^{cu}(x) \text{ and } \|v^s\| \le \rho\|v^u\|\} \tag{3.51}$$

(we omit the dependence on the cross-section in our notations).

Let $\rho > 0$ be any small constant. In the following consequence of Proposition 3.49 we assume the neighborhood U_0 has been chose sufficiently small, depending on ρ and on a bound on the angles between the flow and the cross-sections.

Corollary 3.51. *For any $R : \Sigma \to \Sigma'$ as in Proposition 3.49, with $t(\cdot) > t_1$, and any $x \in \Sigma$, we have $DR(x)(C_\rho^u(x)) \subset C_{\rho/2}^u(R(x))$ and*

$$\|DR_x(v)\| \ge \frac{5}{6}\lambda^{-1}\cdot\|v\| \quad \textit{for all} \quad v \in C_\rho^u(x).$$

Proof. Proposition 3.49 immediately implies that $DR_x(C_\rho^u(x))$ is contained in the cone of width $\rho/4$ around $DR(x)\big(E_\Sigma^{cu}(x)\big)$ relative to the splitting

$$T_{R(x)}\Sigma' = E_{\Sigma'}^s(R(x)) \oplus DR(x)\big(E_\Sigma^{cu}(x)\big).$$

(We recall that E_Σ^s is always mapped to $E_{\Sigma'}^s$.) The same is true for E_Σ^{cu} and $E_{\Sigma'}^{cu}$, restricted to Λ. So the previous observation already gives the conclusion of the first part of the corollary in the special case of points in the attractor. Moreover to prove the general case we only have to show that $DR(x)\big(E_\Sigma^{cu}(x)\big)$ belongs to a cone of width less than $\rho/4$ around $E_{\Sigma'}^{cu}(R(x))$. This is easily done with the aid of the flow invariant cone field C_a^{cu} in (3.48), as follows. On the one hand,

$$DX^{t(x)}\big(E_\Sigma^{cu}(x)\big) \subset DX^{t(x)}\big(E_x^{cu}\big) \subset DX^{t(x)}\big(C_a^{cu}(x)\big) \subset C_a^{cu}(R(x))\,.$$

We note that $DR(x)\big(E^{cu}_\Sigma(x)\big) = P_{R(x)} \circ DX^{t(x)}\big(E^{cu}_\Sigma(x)\big)$. Since $P_{R(x)}$ maps $E^{cu}_{R(x)}$ to $E^{cu}_{\Sigma'}(R(x))$ and the norms of both $P_{R(x)}$ and its inverse are bounded by some constant K (see Remark 3.50), we conclude that $DR(x)\big(E^{cu}_\Sigma(x)\big)$ is contained in a cone of width b around $E^{cu}_{\Sigma'}(R(x))$, where $b = b(a,K)$ can be made arbitrarily small by reducing a. We keep K bounded, by assuming the angles between the cross-sections and the flow are bounded from zero and then, reducing U_0 if necessary, we can make a small so that $b < \rho/4$. This concludes the proof since the expansion estimate is a trivial consequence of Proposition 3.49. □

As usual a *curve* is the image of a compact interval $[a,b]$ by a C^1 map. We use $\ell(\gamma)$ to denote its length. By a *cu-curve* in Σ we mean a curve contained in the cross-section Σ and whose tangent direction is contained in the unstable cone $T_z\gamma \subset C^u_\rho(z)$ for all $z \in \gamma$. The next lemma says that *the length of cu-curves linking the stable leaves of nearby points x,y must be bounded by the distance between x and y.*

Lemma 3.52. *Let us we assume that ρ has been fixed, sufficiently small. Then there exists a constant κ such that, for any pair of points $x,y \in \Sigma$, and any cu-curve γ joining x to some point of $W^s(y,\Sigma)$, we have $\ell(\gamma) \le \kappa \cdot d(x,y)$.*

Here d is the intrinsic distance in the C^2 surface Σ, that is, the length of the shortest smooth curve inside Σ connecting two given points in Σ.

Proof. We consider coordinates on Σ for which x corresponds to the origin, $E^{cu}_\Sigma(x)$ corresponds to the vertical axis, and $E^s_\Sigma(x)$ corresponds to the horizontal axis; through these coordinates we identify Σ with a subset of its tangent space at x, endowed with the Euclidean metric. In general this identification is not an isometry, but the distortion is uniformly bounded, which is taken care of by the constants C_1 to C_4 in what follows.

The hypothesis that γ is a cu-curve implies that its velocity vector $\dot\gamma(s)$ is contained in the cone of width $C_1 \cdot \rho$ centered at $\gamma(s)$ for all values of the parameter s. In the coordinates described above this means that we may write $\gamma(s) = (\xi(s), s)$ for some C^1 function $\xi : [0, s_0] \to [0, +\infty)$ with $\xi(0) = 0$, $\xi(s) > 0$ for all $s > 0$ and $|\dot\xi| \le C_1\rho$.

On the other hand, stable leaves are close to being horizontal, that is, fixing some stable leaf through $y \in \Sigma$ we may write it as a graph $(u, \eta(u))$ for a C^1 function $\eta : (-u_0, u_0) \to \mathbb{R}$ with $\eta(0) = d > 0$ and $|\dot\eta| \le C_2\rho$ (see Figure 3.3).

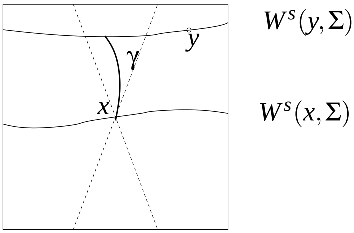

Figure 3.3: The stable manifolds on the cross-section and the *cu*-curve γ connecting them.

Observe now that $h = \eta \circ \xi$ satisfies $|h'| \le \delta = C_1C_2\rho^2$ and $h(0) = d$, thus $|h(s) - d| \le \delta \cdot s$ and hence $h(s_*) = 0$ for some $0 < s_* < d/(1-\delta) < d(1+\delta)$. But this means that

$$\begin{cases} u = \xi(s)) \\ s = \eta(\xi(s)) = \eta(u) \end{cases} \quad \text{or} \quad \gamma(s) = (\xi(s), s) = (u, \eta(u)),$$

that is, we have an intersection between γ and the stable leaf at a distance from x along γ bounded by $d(1+\delta)\sqrt{1+(C_1\rho)^2} < d(1+C_3\rho)$, where C_3 is a constant depending on C_1, C_2 only.

Finally y has coordinates $(\eta(u_1), u_1)$ for some $|u_1| < u_0$ and since $u_0 < \rho$ we get that $\eta(u_1) \ge d - \delta u_1 > d - \delta\rho$ so in Euclidean coordinates $\|x - y\| > d - \delta\rho = d(1 - \delta\rho/d)$ and hence $d(x,y) > C_4 d$ for some $C_4 > 0$ depending on all the previous constants (remember that $d < \rho$ also) including the distortion due to the change of metric.

It follows that the length of γ is bounded by $\kappa \cdot d(x,y)$ where $\kappa = (1+\delta)\sqrt{1+(C_1\rho)^2}/C_4$. □

In what follows we take t_1 in Proposition 3.49 for $\lambda = 1/3$. From Section 4.2.1 onwards we will need to decrease λ once taking a bigger t_1.

Adapted cross-sections

Now we exhibit stable manifolds for Poincaré transformations $R : \Sigma \to \Sigma'$. The natural candidates are the intersections $W^s(x,\Sigma) = W^s_\varepsilon(x) \cap \Sigma$ we introduced previously. These intersections are tangent to the corresponding sub-bundle E^s_Σ and so, by Proposition 3.49, they are contracted by the transformation. For our purposes it is also important that the stable foliation be invariant:

$$R(W^s(x,\Sigma)) \subset W^s(R(x),\Sigma') \qquad \text{for every } x \in \Lambda \cap \Sigma. \tag{3.52}$$

In order to have this we restrict our class of cross-sections whose center-unstable boundary is disjoint from Λ. Recall (Remark 3.47) that we are considering cross-sections Σ that are diffeomorphic to the square $[0,1] \times [0,1]$, with the horizontal lines $[0,1] \times \{\eta\}$ being mapped to stable sets $W^s(y,\Sigma)$. The *stable boundary* $\partial^s\Sigma$ is the image of $[0,1] \times \{0,1\}$. The *center-unstable boundary* $\partial^{cu}\Sigma$ is the image of $\{0,1\} \times [0,1]$. The cross-section is δ-*adapted* if

$$d(\Lambda \cap \Sigma, \partial^{cu}\Sigma) > \delta,$$

where d is the intrinsic distance in Σ, see Figure 3.4. We call *horizontal strip* of Σ the image $h([0,1] \times I)$ for any compact subinterval I, where $h : [0,1] \times [0,1] \to \Sigma$ is the coordinate system on Σ as in Remark 3.47. Notice that every horizontal strip is a δ-adapted cross-section.

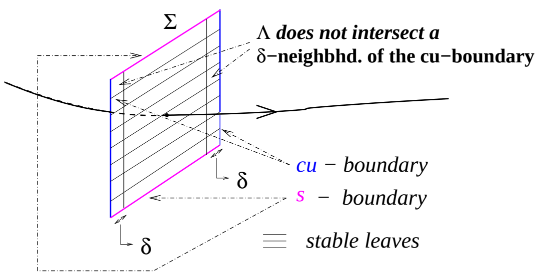

Figure 3.4: An adapted cross-section for Λ.

In order to prove that adapted cross-sections do exist, we need the following result.

Lemma 3.53. *Let Λ be either a transitive singular-hyperbolic Lyapunov stable set, or a connected singular-hyperbolic attracting set admitting a dense subset of periodic orbits. Then every point $x \in \Lambda$ is in the closure of $W^{ss}(x) \setminus \Lambda$.*

Note that a singular-hyperbolic attractor satisfies the first condition of the statement of Lemma 3.53.

Proof. The proof is by contradiction. Let us suppose that there exists $x \in \Lambda$ such that x is in the interior of $W^{ss}(x) \cap \Lambda$. Let $\alpha(x) \subset \Lambda$ be its α-limit set. Then

$$W^{ss}(z) \subset \Lambda \quad \text{for every } z \in \alpha(x), \tag{3.53}$$

since any compact part of the strong-stable manifold of z is accumulated by backward iterates of any small neighborhood of x inside $W^{ss}(x)$. It follows that $\alpha(x)$ does not contain any singularity: indeed, Theorem 3.5 or Lemma 3.45 proves that the strong-stable manifold of each singularity meets Λ only at the singularity (observe that every singularity of Λ is accumulated by regular orbits in Λ).

Therefore by Proposition 3.9 the invariant set $\alpha(x) \subset \Lambda$ is hyperbolic. It also follows from (3.53) that the set

$$H = \overline{\cup\{W^{ss}(y) : y \in \alpha(x) \cap \Lambda\}}$$

is contained in Λ. Also by the same argument as before, this set contains the strong-stable manifolds of all its points. Hence H does not contain any singularity, that is H is uniformly hyperbolic.

We claim that $\overline{W^u(H)}$, the closure of the union of the unstable manifolds of the points of H, is an open set (it is clearly a closed set).

First we show that $W^u(H)$ is open. Note that H contains the whole stable manifold $W^s(z)$ of every $z \in H$: this is because H is invariant and contains the strong-stable manifold of z. Note that the union of the strong-unstable manifolds through the points of $W^s(z)$ contains a neighborhood of z. This proves that $W^u(H)$ is a neighborhood of H. Thus the backward orbit of any point in $W^u(H)$ must enter the interior of $W^u(H)$. Since the interior is, clearly, an invariant set, this proves that $W^u(H)$ is open, as claimed.

Now observe that because $W^u(H)$ is open and invariant, the strong-stable manifold of any $z \in W^u(H)$ is contained in $W^u(H)$, which is contained in Λ since we are assuming that Λ is either Lyapunov stable or attracting. Therefore taking limits we see that $W^{ss}(w) \subset W^u(H)$ for all $w \in \overline{W^u(H)}$. This implies that $\overline{W^u(H)}$ does not contain singularities and is hyperbolic. Finally the unstable manifolds of points in $\overline{W^u(H)}$ are well defined by hyperbolicity and are contained in $\overline{W^u(H)}$, just by taking limits of points in $W^u(H)$. Hence $\overline{W^u(H)}$ contains its stable and unstable manifolds, so it is an open set inside Λ.

Since Λ is also a connected set (which is always the case if Λ is transitive) we obtain $\Lambda = \overline{W^u(H)}$. This means that any singularity $\sigma \in \Lambda$ must be in $\overline{W^u(H)}$, a contradiction. The proof of the lemma is complete. $\square$

Corollary 3.54. *For any $x \in \Lambda$ there exist points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in distinct connected components of $W^{ss}(x) \setminus \{x\}$.*

Proof. Otherwise there would exist a whole segment of the strong-stable manifold entirely contained in Λ. Considering any point in the interior of this segment, we would get a contradiction to Lemma 3.53. $\square$

Lemma 3.55. *Let $x \in \Lambda$ be a regular point, that is, such that $X(x) \neq 0$. Then there exists $\delta > 0$ for which there exists a δ-adapted cross-section Σ at x.*

Proof. Fix $\varepsilon > 0$ as in the stable manifold theorem. Any cross-section Σ_0 at x sufficiently small with respect to $\varepsilon > 0$ is foliated by the intersections $W^s_\varepsilon(x) \cap \Sigma_0$. By Corollary 3.54, we may find points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in each of the connected components of $W^s_\varepsilon(x) \cap \Sigma_0$. Since Λ is closed, there are neighborhoods $V^\pm$ of $x^\pm$ disjoint from Λ. Let $\gamma \subset \Sigma_0$ be some small curve through x, transverse to $W^s_\varepsilon(x) \cap \Sigma_0$. Then we may find a continuous family of segments inside $W^s_\varepsilon(y) \cap \Sigma_0$, for $y \in \gamma$ with endpoints contained in $V^\pm$. The union Σ of these segments is a δ-adapted cross-section, for some $\delta > 0$, see Figure 3.5. $\square$

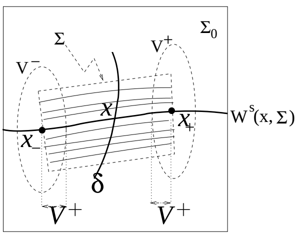

Figure 3.5: The construction of a δ-adapted cross-section for a regular $x \in \Lambda$.

We are going to show that if the cross-sections are adapted, then we have the invariance property (3.52). Given $\Sigma, \Sigma' \in \Xi$ we set $\Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\}$ the domain of the return map from Σ to Σ'.

Lemma 3.56. *Given $\delta > 0$ and δ-adapted cross-sections Σ and Σ', there exists $T_2 = T_2(\Sigma, \Sigma') > 0$ such that if $R : \Sigma(\Sigma') \to \Sigma'$ defined by $R(z) = R_{t(z)}(z)$ is a Poincaré map with time $t(\cdot) > T_2$, then*

1. $R\big(W^s(x,\Sigma)\big) \subset W^s(R(x),\Sigma')$ *for every* $x \in \Sigma(\Sigma')$, *and also*
2. $d(R(y),R(z)) \le \frac{1}{2}\, d(y,z)$ *for every* y, $z \in W^s(x,\Sigma)$ *and* $x \in \Sigma(\Sigma')$.

Proof. This is a simple consequence of the relation (3.50) from the proof of Proposition 3.49: the tangent direction to each $W^s(x,\Sigma)$ is contracted at an exponential rate $\|DR(x)\, e^s_x\| \le Ce^{-\lambda t(x)}$. Choosing T_2 sufficiently large we ensure that

$$Ce^{-\lambda T_2} \cdot \sup\{\ell(W^s(x,\Sigma)) : x \in \Sigma\} < \delta.$$

In view of the definition of δ-adapted cross-section this gives part (1) of the lemma. Part (2) is entirely analogous: it suffices that $Ce^{-\lambda T_2} < 1/2$. $\square$

Remark 3.57. Clearly we may choose $T_2 > T_1$. Remark 3.50 applies to T_2 as well.

The following is a technical consequence of the uniform contraction and the way cross-sections where chosen near real stable leaves.

Lemma 3.58. *Let Σ be a δ-adapted cross-section. Then given any $r > 0$ there exists ρ such that for all $s > 0$, every y, $z \in W^s(x,\Sigma)$, and every $x \in \Lambda \cap \Sigma$ we have* $\operatorname{dist}\big(X^s(y), X^s(z)\big) < r$ *if* $d(y,z) < \rho$.

Proof. Let y and z be as in the statement. As in Remark 3.48, we may find $z' = X^\tau(z)$ in the intersection of the orbit of z with the strong-stable manifold of y satisfying

$$\frac{1}{K} \le \frac{\operatorname{dist}(y,z')}{d(y,z)} \le K \quad \text{and} \quad |\tau| \le K \cdot d(y,z).$$

Then, given any $s > 0$,

$$\begin{aligned} \operatorname{dist}(X^s(y), X^s(z)) &\le \operatorname{dist}(X^s(y), X^s(z')) + \operatorname{dist}(X^s(z'), X^s(z)) \\ &\le C \cdot e^{-\lambda s} \cdot \operatorname{dist}(y,z') + \operatorname{dist}(X^{s+\tau}(z), X^s(z)) \\ &\le KC \cdot e^{\gamma s} \cdot d(y,z) + K|\tau| \le \big(KC + K^2\big) \cdot d(y,z). \end{aligned}$$

Taking $\rho < r/(KC+K^2)$ we get the statement of the lemma. $\square$

A very useful consequence of the hyperbolicity of Poincaré maps is the following criterion for the existence of a periodic orbit.

Lemma 3.59. *Let $x \in \Lambda$ be a regular point and suppose there exists another regular point $z \in W^{ss}(x) \cap \Lambda$ such that $x \in \omega(z)$. Then x belongs to a periodic orbit.*

Proof. Take an adapted cross-section Σ through x. The conditions on z imply that there exists a Poincaré return map R defined on some substrip $\Sigma(\Sigma)$ containing $W = W^s(x,\Sigma)$, and that this line W is forward invariant $R(W) \subset W$. The contracting property given by Lemma 3.56 ensures there exists a periodic point p for R. Therefore p belongs to a periodic orbit for the flow and to the line W. Hence $z \in W^s(p)$ and so $\omega(z) = \mathcal{O}(p)$, thus $x = p$ since there can be only one intersection $\mathcal{O}(p)$ with Σ on the same stable manifold. □

From Proposition 3.9 any compact invariant subset H of a singular-hyperbolic set Λ is uniformly hyperbolic, and of saddle-type. Using adapter cross sections we can say a bit more.

Lemma 3.60. *Let Λ be a singular-hyperbolic set. Suppose that one of the following conditions is true:*

1. *Λ is Lyapunov stable and transitive;*
2. *Λ is an attractor and H is a compact proper invariant subset of Λ;*
3. *Λ is a attracting set with a dense subset of periodic orbits, H is the set of accumulation points of a branch of the unstable manifold of some singularity σ of Λ, and H does not contain σ.*

Then either $H \subset S(X)$ or, for any adapted cross section Σ through some regular point of H, the intersection $H \cap \Sigma$ is totally disconnected.

Note that the compact invariant set H is covered by a finite number of tubular flow boxes or flow boxes near singularities $U_{\Sigma_i} = X^{(-\varepsilon,\varepsilon)}(\Sigma_i)$, for $\varepsilon > 0$ small and $i = 1, \dots, k$. From Lemma 3.60 we conclude that each $U_{\Sigma_i} \cap H$ has topological dimension one. Hence H in each case of the statement above is a one-dimensional set. For the definition and main properties of topological dimension see e.g. [82].

Proof. We follow the arguments in Morales [127]. If H is not contained in the set of singularities, fix a regular point $x \in H \cap \Sigma$. From Lemma 3.53 together with Remark 3.48 we have that the connected component C of $H \cap \Sigma$ containing x cannot contain intervals inside $W^s(x,\Sigma)$. Then either $C = \{x\}$ or C contains some point y in $\Sigma \setminus W^s(x,\Sigma)$. We show that the latter cannot happen in each case according to the assumption in the statement.

Observe first that since Σ is adapted there are no points of $H \cap \Sigma$ near the center-unstable boundary $\partial^{cu}\Sigma$. Hence there must be some point $h_0 \in H$ in the interior of the substrip Σ' of Σ formed by the points of Σ between the two horizontal lines $W^s(x,\Sigma)$ and $W^s(y,\Sigma)$. For otherwise $y \in C$ would be disconnected from x.

1. If Λ is transitive, then there exists $w \in \Sigma'$ close to h_0 with $\omega(w) = \Lambda$. Arguing as above, there must exist a point $\zeta \in H \cap W^s(w,\Sigma)$, for otherwise y and x would be in different connected components of $H \cap \Sigma \setminus W^s(w,\Sigma)$. Then $\Lambda = \omega(w) = \omega(\zeta) \subset H$. This is not possible because H is a proper subset of Λ.

2. Let $H = \omega(z)$ for some $z \in W^u(\sigma) \setminus \{\sigma\}$ and some singularity σ, as in item 2 of the statement, and suppose H is not a singularity. Let Σ be some cross section through some regular point h of H. Since $\mathrm{Per}(X)$ is dense in Λ, we can find a sequence p_n of points in periodic orbits such that $p_n \xrightarrow[n\to+\infty]{} \sigma$. By assumption we can find a point w in the positive orbit of z such that $w \in \Sigma$ close to h.

   Observe that since $W^u(\sigma)$ is one-dimensional, we can assume without loss of generality that $z \in \Sigma_\sigma^{o,\pm}$ for some outgoing cross section near σ. Then there are points $p'_n \in \mathcal{O}(p_n)$ satisfying $p'_n \xrightarrow[n\to+\infty]{} z$. So we can also find points $\tilde{p}_n \in \mathcal{O}(p_n)$ such that $\tilde{p}_n \xrightarrow[n\to+\infty]{} w$.

   As before, there exists a point $\zeta \in H \cap W^s(w,\Sigma)$. Hence we can find a sequence ζ_n in the positive orbit of ζ arbitrarily close to σ. But then $\sigma \in H$, which is a contradiction.

We conclude that either $H \subset S(X)$ (and H is a singularity different from σ in the scenario of item 3), or the connected component of $H \cap \Sigma$ containing x is formed by x itself. $\square$

Poincaré times near singularities

Recall that since singularities are Lorenz-like, we have that the unstable manifold $W^u(\sigma_k)$ is one-dimensional, and there is a one-dimensional strong-stable manifold $W^{ss}(\sigma_k)$ contained in the two-dimensional stable manifold $W^s(\sigma_k)$. Most important for what follows, the attractor intersects the strong-stable manifold at the singularity only, by Theorem 3.5 on page 98.

Hence for some $\delta > 0$ we may take δ-adapted cross-sections contained $\Sigma^{o,\pm}$ and $\Sigma^{i,\pm}$ in U_0 as in Section 3.3.3. Reducing the cross-sections if necessary, i.e. taking $\delta > 0$ small enough, we ensure that the Poincaré times are larger than T_2, so that the same conclusions as in the previous subsections apply here. Indeed using linearizing coordinates it is easy to see that for points $z = (x_1, x_2, \pm 1) \in \Sigma^{i,\pm}$ the time $\tau^\pm$ it takes the flow starting at z to reach one of $\Sigma^{o,\pm}$ depends on x_1 only and is given by

$$\tau^\pm(x_1) = -\frac{\log x_1}{\lambda_1}.$$

We then fix these cross-sections once and for all and define for small $\varepsilon > 0$ the *flow-box*

$$U_{\sigma_k} = \bigcup_{x \in \Sigma^{i,\pm} \setminus \ell^\pm} X_{(-\varepsilon, \tau^\pm(x)+\varepsilon)}(x) \cup (-\delta, \delta) \times (-\delta, \delta) \times (-1, 1)$$

which is an open neighborhood of σ_k with σ_k the unique zero of $X \mid U_{\sigma_k}$. We note that the function $\tau^\pm : \Sigma^{i,\pm} \to \mathbb{R}$ is integrable with respect to the Lebesgue (area) measure over $\Sigma^{i,\pm}$: we say that *the exit time function in a flow box near each singularity is Lebesgue integrable*.

More precisely, we can determine the expression of the Poincaré maps between ingoing and outgoing cross-sections easily thought linearised coordinates

$$\Sigma^{i,+} \cap \{x_1 > 0\} \to \Sigma^{0,+}, \quad (x_1, x_2, 1) \mapsto \left(1, x_2 \cdot x_1^{-\lambda_3/\lambda_1}, x_1^{-\lambda_2/\lambda_1}\right). \quad (3.54)$$

This shows that the map obtained identifying points with the same x_2 coordinate, i.e. points in the same stable leaf, is simply $x_1 \mapsto x_1^\beta$ where $\beta = -\lambda_2/\lambda_1 \in (0,1)$. For the other possible combinations of ingoing and outgoing cross-sections the Poincaré maps have a similar expression. This will be useful to construct physical measures for the flow, in Chapter 4.

3.4.2 Denseness of periodic orbits and transitivity with a unique singularity

Here we start the proof of Theorem 3.44.

We present the proof as a sequence of several simpler results which will be proved in the sequel.

Let $X \in \mathfrak{X}^r$ and Λ be a singular-hyperbolic set of X satisfying the conditions in the statement of Theorem 3.44: it contains a unique singularity σ, it has a dense subset of periodic orbits and it is a singular hyperbolic *non-connected* attracting set. The singularity is Lorenz-like by Lemma 3.45. Then $W^{ss}(\sigma)$ divides $W^s(\sigma)$ in two connected components, which we denote by $W^{s,+}$ and $W^{s,-}$.

Note that $\Lambda \neq \{\sigma\}$, for otherwise we would get an attracting set consisting of a singularity with an expanding eigenvalue which is impossible. Therefore the set of periodic orbits in Λ is non-empty.

A crucial result in this setting is that *the unstable manifold of every periodic orbit in Λ crosses the stable manifold of the singularity transversely.* We present a proof in Section 3.4.3 following the arguments in [130].

Theorem 3.61. *Let Λ be either a singular-hyperbolic attractor, or a connected singular-hyperbolic attracting set with a dense subset of periodic orbits. Then for every $p \in \mathrm{Per}(X) \cap \Lambda$ there exists a singularity σ of Λ such that $W^u(p)$ and $W^s(\sigma)$ intersect transversely.*

The intersections provided by this results together with the uniqueness assumption on $S(X)$ enables us to relate two distinct periodic orbits of Λ or to deduce non-trivial consequences if Λ is not transitive or a disconnected set, using the two connected components $W^{s,+}$ and $W^{s,-}$ of $W^s(\sigma)$.

For that we consider the following invariant subsets of Λ:

$$P^{\pm} = \{p \in \mathrm{Per}(X) \cap \Lambda : W^u_X(p) \pitchfork W^{s,\pm}(\sigma) \neq \emptyset\} \quad \text{and} \quad H^{\pm} = \overline{P^{\pm}}.$$

The rest of this section is devoted to prove the following result. Then we use it to prove Theorem 3.44.

Theorem 3.62. *Let Λ be a connected singular-hyperbolic attracting set of a flow $X \in \mathfrak{X}^r, r \geq 1$, on a closed three-manifold M. Suppose that Λ contains a dense subset of periodic orbits and a unique singularity. Moreover assume that Λ is not transitive. Then H^+ and H^- are homoclinic classes of X.*

From Theorem 3.61 and the assumption that $S(X)\cap\Lambda$ is a singleton together with denseness of periodic orbits in Λ, we easily deduce that $P^{\pm}$ cover the whole attractor.

Lemma 3.63. *Let Λ be a connected singular-hyperbolic attracting set with dense periodic orbits and only one singularity σ. Then $\Lambda = H^+\cup H^-$.*

In this setting we can state Theorem 3.62 in the following useful way: *a singular hyperbolic attracting set having dense periodic orbits with only one singularity is either transitive or the union of two homoclinic classes.*

Since each element $\mathcal{O}\in \mathrm{Per}(X)\cap\Lambda$ is hyperbolic of saddle-type, then $W^u(\mathcal{O})\setminus\{\mathcal{O}\}$ has two connected components. For $\mathcal{O}\in P^{\pm}$ one of those components intersects $W^{s,\pm}(\sigma)$. We write that component $W^{u,\pm}(\mathcal{O})$.

Now we show that both H^+ and H^- are transitive sets.

Lemma 3.64. *Let Λ be a singular-hyperbolic attracting set with dense periodic orbits and only one singularity σ. Then H^+ and H^- are transitive. Moreover $H^{\pm}\subset\overline{W^{u,\pm}(q)}=\overline{W^{u,\pm}(q)\pitchfork W^{s,\pm}(\sigma)}$ for all $q\in P^{\pm}$.*

Proof. Let p,q be two points in distinct orbits inside H^+ (the argument for H^- is analogous). Then their unstable manifolds intersect transversely the same side of the stable manifold of the unique singularity. Hence through the local behavior of the flow near a singularity, flowing to an incoming cross section $\Sigma=\Sigma^{i,+}$ we obtain two small curves $\gamma\subset W^u(p)\cap\Sigma$ and $\zeta\subset W^u(q)\cap\Sigma$ crossing ℓ^+ transversely. See Figure 3.1.

Fix neighborhoods U of p and V of q. Since periodic orbits are dense in Λ and $\gamma\subset\Lambda$ (because Λ is an attracting set), then we can find a periodic orbit r so close to w such that

- $W^s(r,\Sigma)$ intersects both γ and ζ transversely;
- the orbit of r intersects U.

Hence taking $z\in\zeta\pitchfork W^s(r,\Sigma)\subset W^u(q)\cap W^s(r)$ we have that the positive orbit of z visits U infinitely many times, and the negative orbit of z converges to $\mathcal{O}(q)$, thus visits V infinitely many times. This means that there exists some $t>0$ such that $X^t\big(V\cap\Lambda\big)\cap\big(U\cap\Lambda\big)\neq\emptyset$. Since U and V where arbitrarily chosen, this proves transitivity.

Recall the convention $W^{u,+}(q)$ for the branch of $W^u(q)\setminus\mathcal{O}(q)$ which intersects $W^{s,+}(\sigma)$. The above argument also shows that the $W^{u,+}(q)$ is arbitrarily near p, that is $P^+\subset\overline{W^{u,+}(q)}$ for every $q\in P^+$, thus $H^+\subset\overline{W^{u,+}(q)}$.

Since Λ is attracting we have $W^{u,+}(q) \subset \Lambda$. Therefore given any $y \in W^{u,+}(q)$ there is a sequence $p_n \in \mathrm{Per}(X) \cap \Lambda$ such that $p_n \xrightarrow[n\to+\infty]{} y$. By Theorem 3.61 together with the Inclination Lemma, we get that $W^s(\sigma)$ crosses $W^{u,+}(q)$ very near y. This shows that $\overline{W^s(\sigma) \cap W^{u,+}(q)} \supset W^{u,+}(q)$.

Analogously with $-$ instead of $+$. Note that the intersections above are always transverse. The lemma is proved. $\square$

From this we deduce the following condition for transitivity.

Lemma 3.65. *In the same setting as the previous lemma, suppose there exists a sequence $\{p_n\}_{n\geq 1} \subset P^-$ converging to some point in $W^{s,+}(\sigma)$ (or similarly interchanging $+$ with $-$). Then Λ is transitive.*

Proof. Fix $p \in P^+$ and let $p_n \in P^-$ be as in the statement. From the construction of flow boxes near singularities in Section 3.3.3 we can fix an adapted cross-section $\Sigma = \Sigma^{i,+}$ through $W^{s,+}(\sigma)$ and a open arc $J \subset \Sigma \cap W^u(p)$ intersecting $W^{s,+}(\sigma)$ transversely.

Again by the behavior of the flow near singularities we can assume that $p_n \in \Sigma$ for every n. By the choice of adapted cross-sections, we know that the local stable manifolds $W^s(p_n, \Sigma)$ of p_n inside Σ intersect J transversely, for every big enough n.

The Inclination Lemma 1.10 applied to the positive orbit of $J \subset W^u_X(p)$ together with the assumption $p_n \in P^-$ imply that $W^u(p) \cap W^{s,-}(\sigma) \neq \emptyset$. Hence $p \in P^-$.

This shows that $H^+ \subset H^-$. Thus $\Lambda = H^-$ by Lemma 3.63, and from Lemma 3.64 we conclude that Λ is transitive. $\square$

Proposition 3.66. *In the same setting as above, if there is $z \in W^u(\sigma) \setminus \{\sigma\}$ such that $\sigma \in \omega(z)$, then Λ is transitive.*

Proof. Let z be as in the statement. By the local dynamics in flow boxes near σ we can assume there are points $z_n \in \Sigma^{i,+}$ in the positive orbit of z such that $z_n \to z_0 \in \ell$, where $\ell = W^{s,+}(\sigma) \cap \Sigma$. (The argument for the $-$ case above is analogous.)

If $\mathrm{Per}(X) \cap P^- = \emptyset$, then we would have $\mathrm{Per}(X) \cap \Lambda \subset P^+$ by Theorem 3.61. In this case Λ would be transitive by Lemma 3.64. Hence we can assume that there exists $q \in P^-$.

This allows us to choose a sequence of points $w_n \in W^u(q)$ in the same side where z is, such that $w_n \xrightarrow[n\to+\infty]{} w \in W^{s,-}(\sigma)$. Since Λ is attracting, it contains the unstable manifolds of its points and so $w \in \Lambda$. Thus we can find a sequence $p_n \in \mathrm{Per}(X) \cap \Lambda$ tending to w, whose orbit passes very close to z. Consequently there are $p'_n \in \mathcal{O}(p_n)$ converging to z_0.

We have found a sequence of periodic orbits accumulating simultaneously $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$. Arguing by contradiction, suppose that Λ is not transitive. Then Lemma 3.65 would imply that $p'_n \notin P^+$ and $p'_n \notin P^-$ for all n large enough. This contradicts Theorem 3.61 and concludes the proof. □

Now we assume that Λ is not transitive and use the previous results to disconnect Λ.

Lemma 3.67. *If Λ is not transitive, then for all $q \in P^\pm$ we have $H^\pm = \overline{W^{u,\pm}(q) \pitchfork W^{s,\pm}(\sigma)} = \overline{W^{u,\pm}(q)}$.*

Proof. Fix $q \in P^+$ (for P^- the argument is the same). From Lemma 3.64 it is enough to show that every point $y \in W^{u,+}(q)$ is an accumulation point of elements of P^+. This implies that y is accumulated by points in $W^{u,+}(q) \cap W^{s,+}(\sigma)$ by the Inclination Lemma and, in addition, also ensures that $y \in H^+$.

By denseness of periodic orbits there exists a sequence $p_n \in \mathrm{Per}(X) \cap \Lambda$ such that $p_n \xrightarrow[n\to+\infty]{} y$. Then $p_n \in P^+$ for all n big enough, for otherwise we would get $y \in H^-$ and thus $H^+ \subset \overline{W^{u,+}(q)} = \overline{\mathcal{O}(y)} \subset H^-$, since H^- is invariant. Hence $\Lambda = H^+$. This contradicts the assumption that Λ is not transitive. □

Theorem 3.68. *If Λ is not transitive, then for all $a \in W^u(\sigma) \setminus \{\sigma\}$ there exists a periodic orbit $O \subset \Lambda$ such that $a \in W^s(O)$, that is $\omega(a) = O$.*

Note that by Theorem 3.61 the periodic orbits given by Theorem 3.68 are homoclinically related to σ.

Proof. Fix $a \in W^u(\sigma) \setminus \{\sigma\}$ and assume that $\omega(a)$ is *not* a periodic orbit.

Since Λ is not transitive and periodic orbits are dense by assumption, we have $P^+ \neq P^-$ and both are non-empty. Take $p \in P^+$ and $q \in P^-$.

Using the flow we can assume that a belongs to some outgoing cross section $\Sigma = \Sigma^{o,\pm}$ of a flow box near σ. Since the unstable manifolds of p and q cross $W^s(\sigma)$ on sides opposite to $W^{ss}(\sigma)$, both their intersections with $W^s(\sigma)$ contain a curve having σ as an accumulation point and tangent to the eigendirection corresponding to the weak contracting eigenvalue of σ, see Figure 3.6. Using the flow box near σ we can find a curve $I = I_a$ contained

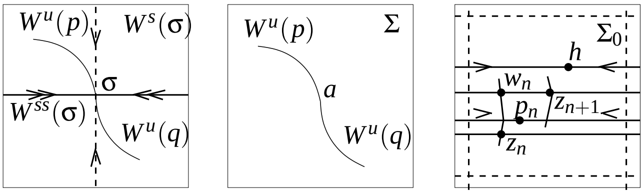

Figure 3.6: The stable manifold of σ, the unstable manifolds of p,q and the points in Σ_0.

in Σ through a such that $I \setminus \{a\}$ is formed by two arcs $I^+ \subset W^u(p)$ and $I^- \subset W^u(q)$, see Figure 3.6. Observe that the tangent space of I is contained in $E^c \cap T\Sigma$ by construction.

Proposition 3.66 ensures that $\sigma \notin \omega(a)$ since Λ is not transitive. Therefore from Proposition 3.9 we see that $H = \omega(a)$ is a uniformly hyperbolic saddle-type set. Moreover $I \subset \Lambda$ because Λ is a closed attracting set.

Consider an adapted cross section Σ_0 through some point of H. Then by item 2 of Lemma 3.60 and shrinking Σ_0 if necessary, we can assume that *the stable boundary $\partial^s\Sigma_0$ of Σ_0 does not touch H*. Moreover since $\Sigma_0 \setminus H$ is open we can in addition assume that $d(\Sigma_0 \cap H, \partial^s\Sigma) > \delta$ for some $\delta > 0$, just as in the definition of δ-adapted cross section, but now in the center-unstable direction.

Using a tubular flow construction we can linearise X in an *open* tube-like set $U_{\Sigma_0} = X^{(-\varepsilon,\varepsilon)}\big(\operatorname{int}(\Sigma_0)\big)$ for a small $\varepsilon > 0$. We can cover H by a finite number $\Xi = \{U_{\Sigma_0}, \dots, U_{\Sigma_l}\}$ of this type of *open tubular flow boxes,* since $H \cap S(X) = \emptyset$, H is compact and H satisfies item 2 of Lemma 3.60.

Consider the Poincaré map $R : \Xi \cap H \to \Xi$ defined by $z \in \Xi \cap H \mapsto X^{T_2+t(z)}(z)$ where T_2 is defined in Section 3.4.1 and $\tau(z)$ is the first return time of $X^{T_2}(z)$ to Ξ. The map is defined on entire strips of Ξ by the construction of adapted cross sections.

Fix now $h_0 \in H$ and let z_n be points in the positive orbit of z such that $d\big(z_n, R^n(h_0)\big) \xrightarrow[n\to+\infty]{} 0$. Note that $h_n = R^n(h_0)$ always belongs to the interior of Ξ and the same is true of z_n. Observe that there exists a corresponding sequence of images I_n of I such that $z_n \in I_n \subset \Xi$. Since I is transverse to the flow direction, we have that z_n belongs to the interior of I_n. In addition, the expansion of R in the central-unstable direction and the fact that z_n is δ-away from the boundary of Ξ ensures that there is an arc J_n with length bounded away from zero such that $z_n \in J_n \subset I_n$.

Let h be a limit point of h_n. Hence J_n converges in the C^1 topology to an interval in $W^u(h)$ (recall that $h \in H$ and H is uniformly hyperbolic). Since Ξ has finitely many components, we can assume that h_n, J_n and h all belong to the same component Σ_0 of Ξ.

Notice that we cannot have $z_n \in W^s(h,\Sigma_0)$ for infinitely many n, for otherwise by Lemma 3.59 we conclude that h is periodic and $z \in W^s(h)$, thus $H = \omega(z) = \mathcal{O}(h)$ contradicting the assumption. Hence $z_n \notin W^s(h,\Sigma_0)$ for all big enough n.

Therefore the intersection of $J_n \setminus \{z_n\}$ with $W^s(z_{n+1},\Xi)$ is non-empty for big enough n. If w_n belongs to this intersection, then it is either in the image of I^+ or in the image of I^- inside J_n. We write $J_n^{\pm}$ for the corresponding components.

Now we use that periodic orbits are dense. Assume that $w_n \in J_n^+$ and take $p_n \in \mathrm{Per}(X) \cap \Sigma_0$ close to a point in in J_n^- near z_{n+1}, see the rightmost rectangle in Figure 3.6. Then we ensure that

$$W^s(p_n) \pitchfork J_n^+ \neq \emptyset \neq J_{n+1}^- \pitchfork W^s(p_n)$$

which implies

$$W^s(p_n) \pitchfork W^u(p) \neq \emptyset \neq W^u(q) \pitchfork W^s(p_n).$$

By the choice of p_n we have that $\mathcal{O}(p_n)$ goes very close to $W^{s,-}(\sigma)$. We can find a sequence of such orbits converging to a point in $W^{s,-}(\sigma)$. Since Λ is not transitive, by Lemma 3.65 we must have that $p_n \in P^-$. But then p must be in P^- by the Inclination Lemma 1.10. Since p was an arbitrary point in P^+, we conclude that $P^+ \subset P^-$ and so $\Lambda = H^-$ is transitive, a contradiction.

Otherwise we have $w_n \in J_n^-$ and by the same arguments we deduce that $q \in P^+$, implying that $\Lambda = H^+$ is transitive as before.

Hence $\omega(z)$ must be a periodic orbit, as claimed. □

The orbit $\mathcal{O}$ provided by Theorem 3.68 is hyperbolic of saddle-type (because it carries a dominated splitting with volume expanding central direction). Hence there are two connected components $W^{u,\pm}$ of the unstable manifold of O such that $W^{u,+} \cup W^{u,-} = W^u(O) \setminus O$. The labels $\pm$ on each component are chosen according to whether the corresponding component is accumulated by the unstable manifold of a periodic point in P^+ or P^-, as in the proof of Theorem 3.68, see Figure 3.7. The above convention does not depend on $p \in P^+, q \in P^-$ nor on I^+, I^- (this is easily proved using the Inclination Lemma).

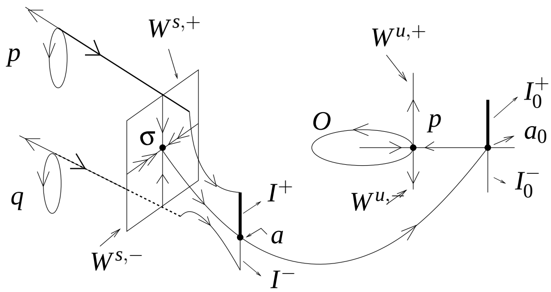

Figure 3.7: Definition of $W^{u,+}$ and $W^{u,-}$.

Next results shows that the choice of signs for the branches of $W^u(\mathcal{O})$ coincides with the previous convention for the unstable manifolds of periodic orbits in Λ.

Lemma 3.69. *We have $W^{u,+} \cap W^{s,-}(\sigma) = \emptyset$ and $W^{u,+} \cap W^{s,+}(\sigma) \neq \emptyset$, and the similar facts interchanging $+$ and $-$. In particular $\mathcal{O} \in P^+ \cap P^-$.*

Proof. Arguing by contradiction, note that if $W^{u,+} \cap W^{s,-}(\sigma) \neq \emptyset$, then because this intersection is transverse and every $p \in P^+$ has an unstable manifold accumulating on $W^{u,+}$, we deduce that $p \in P^-$, and again $P^+ \subset P^-$ thus $\Lambda = H^-$ is transitive, a contradiction. Similarly exchanging $+$ with $-$ in the above argument.

For the other part, if $W^{u,+} \cap W^{s,+}(\sigma) = \emptyset$, then $W^u(\mathcal{O}) \cap W^s(\sigma) = \emptyset$ since $\Lambda \cap W^{ss}(\sigma) = \{\sigma\}$ by Theorem 3.5, contradicting Theorem 3.61. $\square$

Lemma 3.70. *Assume that $W^s(p) \pitchfork W^{u,+} \neq \emptyset$ for some $p \in \mathrm{Per}(X) \cap \Lambda$. Then $\overline{W^s(p) \cap W^{u,+}} = \overline{W^{u,+}}$. Similarly replacing $+$ by $-$.*

Proof. Choose a neighborhood U of $x \in W^{u,+}$. By Lemma 3.67 we have in particular $\overline{W^{s,+}(\sigma) \cap W^{u,+}} = \overline{W^{u,+}}$. Then we can find a point $y \in W^{s,+}(\sigma) \cap W^{u,+} \cap U$. Let γ be a curve through y inside $U \cap W^{u,+}$ transverse to $W^{s,+}(\sigma)$. Then the positive orbit of γ contains open arcs which converge in the C^1 topology to any compact neighborhood of $\mathcal{O}$ inside $W^{u,+}$, by the Inclination

Lemma. Hence the positive orbit of γ intersects $W^s(p)$ by the assumption on p. Therefore there exists a point of $W^s(p)$ in U, proving that $W^{u,+} \subset \overline{W^s(p) \cap W^{u,+}}$. □

Now we are ready to consider homoclinic classes inside Λ (see Section 1.3.5 for the definition and basic properties).

Lemma 3.71. *For $p \in P^{\pm}$ such that $W^s(p) \pitchfork W^{u,\pm} \neq \emptyset$ we have that its homoclinic class $H(p)$ equals $\overline{W^{u,\pm}}$.*

Observe that since periodic orbits are dense we can choose $p \in \mathrm{Per}(X) \cap \Lambda$ very close to $W^{u,+}$ to obtain the condition on p in Lemma 3.71. Then by Lemma 3.67 we have that $H^+ = \overline{W^{u,\pm}} = H(p)$ is a homoclinic class. This completes the proof of Theorem 3.62.

Proof of Lemma 3.71: Fix $z \in P^+$, $y \in W^{u,\pm}$ and a neighborhood U of y. By definition there exists an arc $I \subset W^u(p)$ such that its forward orbit crosses $W^{s,+}(\sigma)$. Lemma 3.70 ensures that we can find a disk D transverse to $W^{u,\pm}$ inside $W^s(p) \cap U$.

The Inclination Lemma implies that the positive orbit of a sub-arc $J \subset I$ accumulates $W^{u,+}$. Then there exists $t > 0$ such that $X^t(J) \pitchfork D \neq \emptyset$. This means that $H(p) \cap U \neq \emptyset$. Since U was arbitrarily chosen and $H(p)$ is closed by definition, we have that $y \in H(p)$. Hence $W^{u,+} \subset H(p)$ and $\overline{W^{u,\pm}} \subset H(p)$.

For the opposite inclusion note that by the assumption $W^s(p) \pitchfork W^{u,\pm} \neq \emptyset$ and the Inclination Lemma we have that $\overline{W^{u,\pm}} \supset W^u(p) \supset H(p)$. □

Proof of Theorem 3.44: Note first that by Lemma 3.71 we must have $\overline{W^{u,\pm}} \cap W^{s,-}(\sigma) = \emptyset$. For otherwise we can find a sequence $p_n \in P^+$ converging to a point in $W^{s,-}(\sigma)$. By Lemma 3.65 this implies that Λ is transitive, a contradiction.

Therefore there exists a neighborhood B of $\overline{W^{u,\pm}}$ disjoint from $W^{s,-}(\sigma)$. Let $J = [a,b]$ be a fundamental neighborhood of $W^{ss}(p_0)$ for some $p_0 \in \mathcal{O}$, where $\mathcal{O}$ is the periodic orbit given by Theorem 3.68. That is, J is an arc with $b = X^t(a)$ for some $t > 0$ such that $X^s(a) \notin W^{ss}(p_0)$ for all $0 < s < t$. Take $V \subset B$ a small neighborhood of J such that every point of V belongs to a stable manifold of a point in $V \cap W^{u,\pm}$. The forward orbits of points in V never leave B, since $\overline{W^{u,\pm}}$ is invariant.

We are going to describe a perturbation of the flow X close to the point $a \in W^u(\sigma) \setminus \{\sigma\}$ (which defines the orbit $\mathcal{O} = \omega(a)$). Consider the following cross sections of X (recall the definition of flow box near a singularity in Section 3.3.3):

- $\Sigma^{o,+}$ containing a in its interior and $\Sigma' = X^1(\Sigma^{o,+})$.
- Σ_0 intersecting O in a single point in the center-unstable boundary.
- Σ^- a substrip of $\Sigma^{i,-}$ which is a one-sided neighborhood of ℓ^- not touching B on the same side of a.
- Σ^+ a substrip of $\Sigma^{i,+}$ which is a one-sided neighborhood of ℓ^+ also on the same side of a.

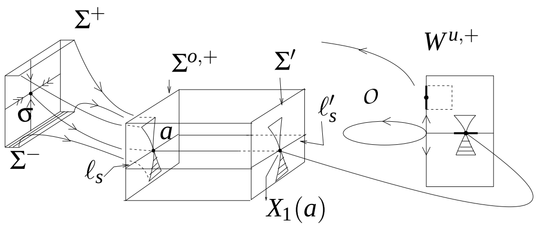

Figure 3.8: The unperturbed flow X.

Observe that the positive orbit of any point in $\Sigma^+ \cup \Sigma^-$ by X will cross $\Sigma^{o,+}$. Define $W = X^{[0,1]}(\Sigma^{o,+})$. The support of the perturbation from X to Y sketched in Figures 3.8 and 3.9 is contained in W. This perturbation is standard, see e.g. [143], amounting to "push a' upwards so that its image under the flow of Y lands in Σ_O above the stable manifold of $\mathcal{O}$".

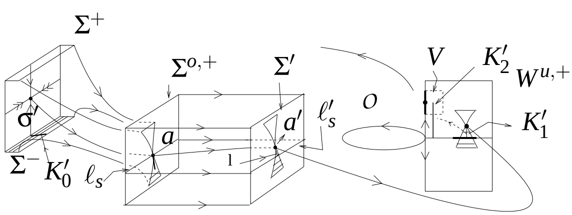

Figure 3.9: The perturbed flow Y.

Recall that $\Lambda = \Lambda_X(U) = \cap_{t>0} X^t(U)$. Since Λ is not transitive there exists $q \in P^-$ and so there is an interval K_0 in $\Sigma^- \cap W^u(q)$ crossing Σ^- as in Figure 3.9.

Denote by $q', W^u(q'), \sigma', K_0'$ the continuation of these objects for the perturbed flow Y. The Y-flow carries K_0' to an interval K_1' as in Figure 3.9.

Note that $K_0' \subset \Lambda_Y(U)$ since $\Lambda_Y(U)$ is an attracting set, $q' \in \Lambda_Y(U)$ and $K_0' \subset W^u(q')$.

We claim that $K_0' \not\subset \Omega(Y)$.

Arguing by contradiction, assume that $K_0' \subset \Omega(Y)$ and choose $x \in \operatorname{int}(K_0')$.

On the one hand, the flow of Y carries points nearby x to V as sketched in 3.9, close to the line K_2'. By assumption on K_0' we have that x is non-wandering for Y. In particular there exists $x' \in K_0'$ close to x such that *the positive Y-orbit of x' returns to Σ^-.*

On the other hand, by construction, the positive orbit of every point in V by the flow of X does not intersect Σ^-.

Since $Y = X$ outside W we conclude that the positive orbit of x' by Y intersects Σ^+ by the definition of $W^{u,+}$. The positive orbit of such an intersection passes through the flow box W and arrives to V again. Then we conclude that *the positive Y-orbit of x' never returns to Σ^-*. This contradiction proves that $K_0' \not\subset \Omega(Y)$, as claimed.

This implies $\Lambda_Y(U) \not\in \Omega(X)$ and finishes the proof of Theorem 3.44. □

3.4.3 Unstable manifolds of periodic orbits inside singular-hyperbolic attractors

Here we present a proof of Theorem 3.61 following the proof presented in [122].

Let Λ be either a singular-hyperbolic attractor, or a connected singular-hyperbolic attracting set having a dense subset of periodic orbits.

We start by showing that the closure of the unstable manifold of any periodic orbit in Λ must contain some singularity of the flow.

Lemma 3.72. *Let Λ be a connected singular-hyperbolic attracting set containing either a dense subset of periodic orbits, or a dense regular orbit. Fix a periodic point $p_0 \in \operatorname{Per}(X) \cap \Lambda$ (necessarily hyperbolic of saddle-type). Let $J = [a,b]$ be an arc on a connected component of $W^{uu}(p_0) \setminus \{p_0\}$ with $a \neq b$. Then $H = \overline{\cup_{t>0} X^t(J)}$ contains some singularity of Λ.*

Proof. Observe that $H = \overline{W^u_0(p_0)} \subset \Lambda$ by construction, where $W^u_0(p_0)$ is the connected component of $W^u(p_0) \setminus \mathcal{O}(p_0)$ containing J. In addition H contains the unstable manifolds through any of its points, since every point in H is accumulated by forward iterates of the arc J.

Consider the set $W^{ss}(H) = \cup\{W^{ss}(y) : y \in H\}$. Note that $W^u(y) \subset H$ for $y \in H$ and the family $\{W^{ss}(z)\}_{y \in W^u(y)}$ covers an open neighborhood of y, so $W^{ss}(H)$ is a neighborhood of H in M.

Let x be a point in $W^{ss}(H)$. Then by forward iteration this point is sent close to H. This shows that x is in the interior of $W^{ss}(H)$ and hence $W^{ss}(H)$ is open in M. Thus $H^s = W^{ss}(H) \cap \Lambda$ is an open neighborhood of H in Λ. If Λ is transitive, we can take $z \in H^s$ such that $\omega(z) = \Lambda$ and since $\omega(z) = \omega(x)$ for some $x \in H^s$, we conclude that $\Lambda \subset H$ and so $H \cap S(X) \neq \emptyset$.

If Λ is not transitive, we claim that either $H^s \cap S(X) \neq \emptyset$, or the closure of H^s is an open subset of Λ (besides being clearly a closed set).

First note that if $\sigma \in H^s \cap S(X)$, then $\sigma \in W^{ss}(y)$ for some $y \in H$ implying $\sigma \in H$. For otherwise we would get $y \in W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\}$, a contradiction with Lemma 3.45.

Suppose that $H^s \cap S(X) = \emptyset$. From Proposition 3.9 we known that H^s is a uniformly hyperbolic compact subset of Λ. Then every $w \in H^s$ has a well defined strong-unstable manifold. Moreover $W^{uu}_\varepsilon(w) \subset \Lambda \cap H^s$ for some $\varepsilon > 0$, because Λ is attracting and H^s is open. We conclude that H^s contains the unstable manifold of all its points. Hence taking limits we obtain that the closure $\overline{H^s}$ also contains every unstable manifold. Analogously we see that $\overline{H^s}$ contains the strong-stable manifold $W^{ss}(z) \cap \Lambda$ relative to Λ for all $z \in \overline{H^s}$. The union of the unstable manifolds through all points in the strong-stable manifolds provides a neighborhood of $\overline{H^s}$ in Λ.

Since Λ is connected we obtain $\overline{H^s} = \Lambda$. Hence there exists some singularity σ of Λ in the closure of the stable manifolds of H inside Λ. Let h_n be points in H^s converging to σ. Hence the orbits of h_n contain points h'_n very close to $W^u(p_0)$ by definition of H^s. Using the assumption of dense periodic orbits, consider a periodic orbit p_n very close to h_n. Then the orbit of p_n will be close to $W^u_0(p_0)$ and so $W^s(p_n) \pitchfork W^u_0(p_0) \neq \emptyset$ (to see this, consider an adapted cross section Σ through h'_n, a small tubular flow box through Σ and recall that stable manifolds cross Σ horizontally). The Inclination Lemma now ensures that $\overline{W^u_0(p_0)} = H$ contains p_n. Thus H is arbitrarily close to σ. Therefore the closed set H contains some singularity of $S(X) \cap \Lambda$. $\square$

Fix p_0 and $\sigma \in S(X) \cap H$ as in the statement of Lemma 3.72. We can assume that J is a fundamental domain for $W^u(p_0)$, that is $b = X^T(a)$ with $T > 0$ the first return time of the orbit of a to $W^{uu}(p_0)$, i.e. $X^t(a) \notin W^{uu}(p_0)$ for all $0 < t < T$.

Fix ingoing adapted cross sections $\hat{\Sigma}_\sigma^{i,\pm}$ of every $\sigma \in S(X) \cap \Lambda$ and horizontal substrips $\Sigma_\sigma^{i,\pm}$ around $\ell_\sigma^\pm$ of small width so that $\mathcal{O}(p_0)$ does not touch $\Sigma_\sigma^{i,\pm}$. We assume that $\hat{\Sigma}_\sigma^{i,\pm} \setminus \Sigma_\sigma^{i,\pm}$ have nonempty interior.

Consider also a cross section Σ_p containing p_0. We can then take $J = [a,b]$ so close to p_0 that $J \subset \mathrm{int}(\Sigma_p)$ and $X^{-t}(J)$ never intersects $\Sigma_\sigma^{i,\pm}$ for all $t > 0$ and every $\sigma \in S(X) \cap \Lambda$.

Since $S(X) \cap \overline{W^u(p_0)} \neq \emptyset$ there exists a Poincaré map R from a subset D of Σ_p to $\mathrm{int}(\cup_\sigma \Sigma_\sigma^{i,\pm})$ given by the first return time $\tau(x)$ of $x \in D$. Without loss of generality assume that $R(b) \in \mathrm{int}(\Sigma_\sigma^{i,+})$ for some singularity σ fixed from now on. We drop the σ from the notation of the cross sections in what follows.

Note that $R(a)$ must equal $R(b)$. Using this with some tubular flow boxes together with the fact that $\Sigma^{i,+}$ is an adapted cross section, we show that the image of J under R must cross $\Sigma^{i,+}$ from one stable boundary to the other, thus intersecting ℓ^+. Since $\ell^+ = \Sigma^{i,+} \cap W_{loc}^s(\sigma)$, this argument proves Theorem 3.61.

Observe that because both J and $R(a) = R(b)$ belong to the interior of the respective cross sections to X, then there exists a tubular flow box, given by Theorem 1.6, and open neighborhoods $V \subset \Sigma_p$ of b and $W \subset \Sigma^{i,+}$ of $R(b)$, such that $V \subset D$, that is $R \mid V : V \to W$ is well defined and a diffeomorphism. Moreover since J is transverse to the stable foliation in Σ_p, then the image $R(V \cap J)$ is also transverse to the stable foliation of $\Sigma^{i,+}$. In addition, since Λ is attracting, we have that J and $R(J \cap V)$ are contained in Λ. Because $\Sigma^{i,+}$ is adapted, the image of J is δ-away the center-unstable boundary. Identifying the arc $[a,b]$ with some interval $[a,b] \subset \mathbb{R}$ we define, see Figure 3.10

$$q = \sup\{s \in [a,b] : R([a,s]) \subset \mathrm{int}(\Sigma^{i,+})\}.$$

By the existence of the pair V,W we have $q > a$. Moreover given $s \in (a,q)$ and covering the compact arcs $[a,s]$ and $R([a,s])$ by a finite number of open tubular flow boxes $U_1, \ldots, U_k$ we easily see that $R([a,s])$ is connected. Indeed, $R([a,s])$ is the union of a sequence $R([s_i, s_{i+1}])$ of arcs inside $U_i \cap \mathrm{int}(\Sigma^{i,+})$, where $a = s_0 < s_1 < \cdots < s_k = s$ and $R \mid U_i \cap \mathrm{int}(\Sigma_p) : U_i \cap \mathrm{int}(\Sigma_p) \to U_i \cap \mathrm{int}(\Sigma^{i,+})$ is a diffeomorphism, $i = 1, \ldots, k$.

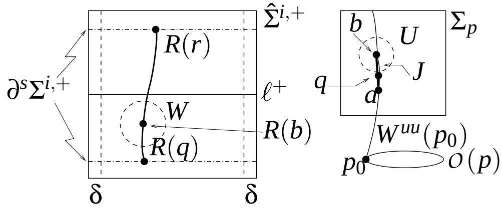

Figure 3.10: The arc J and cross-sections $\Sigma_p, \Sigma_0^{i,+}$.

Note that by the choice of $\Sigma^{i,+}$ strictly inside $\hat{\Sigma}^{i,+}$, *if q belongs to the domain D of R*, then there exists a tubular flow box U_0 taking q to $R(q)$, so that $R(q)$ is well defined. Hence $R(q) = \lim_{s \nearrow q} R(s)$ is not on the the center-unstable boundary $\partial^{cu}\Sigma^{i,+}$ by construction. Moreover using the tubular flow box U_0 we see that $R(q) \in \partial^s\Sigma^{i,+}$. For otherwise, in case $R(q) \in \mathrm{int}(\Sigma^{i,+})$, we would be able to extend the definition of R along J through the flow box U_0.

Now apply the same arguments to

$$r = \inf\{s \in [a,b] : R([s,b]) \subset \mathrm{int}(\Sigma^{i,+})\}.$$

We obtain $R(a) = R(b)$ and $R(q), R(r) \in \partial^s\Sigma^{i,+}$ *if r belongs to the domain of D*. We obtain in this way $\gamma = R([a,q] \cup [r,b])$, a connected smooth arc joining two points in the stable boundary.

If $R(q), R(r)$ belong to the same stable-manifold on $\partial^s\Sigma^{i,+}$, then by smoothness and connectedness there must be a tangency between γ and the stable foliation on $\Sigma^{i,+}$. This is a contradiction.

Hence $R(q), R(r)$ are on different stable leaves on the boundary of $\Sigma^{i,+}$, thus γ crosses ℓ^+ transversely. This means that $W^u(p_0) \pitchfork W^s(\sigma) \neq \emptyset$. The proof of Theorem 3.61 now rests on the claim that *both q and r belong to the domain of R*. To prove this claim we need the following result, whose proof we postpone.

Lemma 3.73. *Let $\tilde{\Sigma}$ be a cross section of X containing a compact cu-curve ζ, which is the image of a regular parametrization $\zeta : [0,1] \to \tilde{\Sigma}$, and assume that ζ is contained in Λ. Let Σ be another cross section of X. Suppose that ζ* falls off Σ*, that is*

- *the positive orbit of* $\zeta(t)$ *visits* $\mathrm{int}(\Sigma)$ *for all* $t \in [0,1)$*;*

- *and the* ω*-limit of* $\zeta(1)$ *is disjoint from* Σ*.*

Then $\zeta(1)$ *belongs either to the stable manifold of some periodic orbit* p *in* Λ*, or to the stable manifold of some singularity.*

Observe that $[a,q]$ (and $[r,b]$) *fall off* $\Sigma^{i,+}$*, if* q *(and* r*) does not belong to* D. Then $\omega(q)$ (and $\omega(r)$) is either a periodic orbit in Λ, or a singularity. In the first case the arc $J \supset [a,q]$ is transverse at q to the stable manifold of a periodic orbit p. The Inclination Lemma ensures that there exists a fundamental domain L of $W^{uu}(p)$ accumulated by iterates of the open arc (a,q), see Figure 3.11. Hence the flow takes *every point of* L through $\Sigma^{i,+}$.

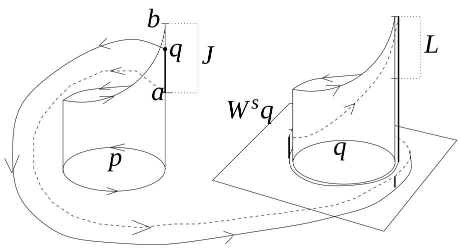

Figure 3.11: How I accumulates $D^u(x^*)$

As before the image of L by the corresponding first return map must be a *cu*-curve C in $\Sigma^{i,+}$. Moreover since the endpoints of L are on the same orbit of the flow, C must be a *closed cu*-curve. This is impossible.

This contradiction shows that q (and r) either does not fall off $\Sigma^{i,+}$, so that q (and r) is in the domain of R, or q is in the stable manifold of some singularity. In the former case, we are done. In the latter case, since the stable manifold is transverse to $W^{uu}(p_0)$ by the assumption of singular-hyperbolicity, we obtain the statement of the theorem as well.

Now to finish the proof of Theorem 3.61 we prove the remaining lemma.

Proof of Lemma 3.73: Define $H = \omega(\zeta(1))$ and suppose H is not a singularity. By an argument similar to the proof of Lemma 3.60 we have that H has totally disconnected intersection with any cross-section.

Indeed, consider an adapted cross-section Σ_x of X through $x \in H$ and consider C the connected component of $H \cap \Sigma_x$ containing x. As in the proof of Lemma 3.60, we have that $C \cap W^s(x, \Sigma_x) = \{x\}$.

If there exists $y \in C \setminus W^s(x, \Sigma_x)$ consider the horizontal strip S of Σ_x between the stable leaves $W^s(x, \Sigma_x)$ and $W^s(y, \Sigma_x)$. Then there exists a point w of $H \cap \mathrm{int}(S)$ for otherwise $W^s(w, \Sigma_x)$ would disconnect y from x. From this we find ξ in the positive orbit of $\zeta(1)$ inside $\mathrm{int}(S)$ and close to w. But ζ is a *cu*-curve. Hence considering the tubular flow on a neighborhood around the piece of orbit from $\zeta(1)$ to ξ, we find in the image of ζ under the tubular flow a *cu*-curve ζ', a connected image of a neighborhood of $\zeta(1)$ in ζ, with ξ as a boundary point. (Here we use the hyperbolicity of the Poincaré maps between cross sections assuming that the time from $\zeta(1)$ to ξ is big enough.)

So we have a positive iterate of a point $\zeta(s)$ in $\mathrm{int}(S)$ for some $s \in [0,1)$. Use the density of periodic orbits to find a point of a periodic orbit p' very close to $\zeta(s)$ in $\mathrm{int}(S)$. Then the orbit of p' crosses $\mathrm{int}(\Sigma)$ by the assumption on the curve ζ. Again there exists $h \in H \cap W^s(p', S)$. This means that the orbit of h will cross $\mathrm{int}(\Sigma)$. Since $h \in \omega(\zeta(1))$, then the orbit of $\zeta(1)$ must cross $\mathrm{int}(\Sigma)$ also. We have reached a contradiction.

We conclude that $\Sigma_x \cap H$ is totally disconnected.

Hence we can cover the set H with a finite number of flow boxes around the singularities contained in H together with finitely many tubular neighborhoods through adapted cross sections, i.e. sets of the form $X^{(-\varepsilon,\varepsilon)}(\Sigma_x)$. Let Ξ be the collection of adapted cross sections used in this cover, some of them ingoing or outgoing cross sections around singularities.

Since $\Sigma_x \cap H$, if non-empty, is totally disconnected, then H is contained in the interior of these flow boxes. Thus $\Sigma_x \cap H$ is not only δ-away from the center-unstable boundary of Σ, but δ-away from the stable boundary of Σ as well, for some uniform $\delta > 0$ valid for every cross section of Ξ.

The definition of H ensures that $\zeta^t(1) = X^t\big(\zeta(1)\big)$ for big enough $t > 0$ is contained in a small closed neighborhood W around H, which can be taken disjoint from the reference section Σ.

Let $t_n \to +\infty$ be such that $\zeta_n(1) = \zeta^{t_n}(1) \in \mathrm{int}(\Xi)$ for all $n \geq 1$. Since Ξ is a finite collection of sections, we can assume without loss of generality that $\zeta_n(1)$ always belongs to the same section $S \in \Xi$.

Observe that the positive orbit of $\zeta(s)$, with $s < 1$ and close to 1, enters W by continuity of the flow, but does not stay in W, since it must cross $\mathrm{int}(\Sigma)$. Then the first return of $\zeta(s)$ to S, which we write $\zeta_n(s)$, is well defined for $s < 1$ and close to 1.

For infinitely many values of n there exists some $s_n \in [0,1)$ such that $\zeta_n([s_n,1])$ is contained in S, the orbit segment from $\zeta(s)$ to $\zeta_n(s)$ is disjoint from Σ for all $s_n \leq s \leq 1$, and $\zeta_n(s_n)$ is in the boundary of S. For otherwise we would get $\zeta_n([0,1]) \subset \mathrm{int}(S) \subset W$ and so $\zeta(s)$ would never reach Σ.

This means that the *cu*-curve $\gamma_n = \zeta_n([s_n,1])$ has length at least δ inside S and

- either the end point $\zeta_n(1)$ of γ_n has a subsequence contained in the same stable manifold inside S, which by Lemma 3.59 implies that $\zeta_n(1)$ is in the stable manifold of a periodic orbits, and thus H is a periodic orbit;

- or γ_n has an accumulation curve inside S in the C^1 topology (using the Ascoli-Arzela Theorem, since γ_n have bounded derivative by definition of *cu*-curve and length bounded away from zero, and S is compact), so that we can find a point $\zeta_n(s)$ in the stable manifold of $\zeta_m(1)$, for m,n very big. This is impossible because the positive orbit of $\zeta_n(s)$ would stay forever close to the orbit of $\zeta_m(1)$, inside W, and would never reach Σ.

We conclude that H is a periodic orbit if it is not a singularity. The proof of Lemma 3.73 is complete. □

Chapter 4

Singular-hyperbolicity, sensitiveness and physical measure

Here we obtain another consequence of singular-hyperbolicity: *a singular-hyperbolic attractor is sensitive to initial conditions*.

Theorem 4.1. *Let* Λ *be a singular-hyperbolic attractor of* $X \in \mathfrak{X}^1(M)$*. Then* Λ *is expansive.*

From the comments of Section 1.2.2 from Chapter 1, we have the following.

Corollary 4.2. *A singular-hyperbolic attractor of a* 3*-flow is sensitive to initial data.*

The proof of Theorem 4.1 is the content of Section 4.1. The argument is based on analyzing Poincaré return maps of the flow to a convenient cross-section.

We show first that there exists a family of *Poincaré maps*, that is, continuous maps $R : \Sigma \to \Sigma'$ of the form $R(x) = X_{t(x)}(x)$ between cross-sections Σ and Σ' to X. Assuming that the Poincaré time $t(\cdot)$ is large and that the attractor Λ is singular-hyperbolic, we show that cross-sections have co-dimension 1 foliations which are dynamically defined: the leaves correspond to the intersections of the cross-sections with the stable manifolds of the flow. These leaves are uniformly contracted and choosing *adapted cross-sections* the foliation is also invariant:

$$R(W^s(x,\Sigma)) \subset W^s(R(x),\Sigma') \quad \text{for all } x \in \Lambda \cap \Sigma.$$

In addition R is uniformly expanding in the transverse direction. By means of cross-sections we can likewise analyze the flow close to the singularities.

From here we argue by contradiction: if the flow is not expansive on Λ, then we can find a pair of orbits hitting the cross-sections infinitely often on pairs of points uniformly close. We derive a contradiction by showing that the uniform expansion in the transverse direction to the stable foliation must take the pairs of points apart, unless one orbit is on the stable manifold of the other.

Existence and uniqueness of a physical measure

It was proved by Colmenarez in [47] that if Λ is a singular-hyperbolic attractor of a C^2 flow X with a dense set of periodic orbits, then the central direction $E^{cu}_{\widetilde{\Lambda}}$ can be continuously decomposed into $E^u \oplus E^X$ along each orbit of $\widetilde{\Lambda}$, where the E^u direction is non-uniformly hyperbolic, that is, has a positive Lyapunov exponent, and $\widetilde{\Lambda} = \Lambda \setminus \cup_{\sigma \in S(X) \cap \Lambda} W^u(\sigma)$. In [46] again under the assumption of a dense set of periodic orbits Colmenarez showed that every C^2 singular-hyperbolic attractor supports a physical probability measure — see Section 1.4.1 of Chapter 1 for the relevant definitions.

However in another recent work, Arroyo and Pujals [15] show that every singular-hyperbolic attractor has a dense set of periodic orbits, so the denseness assumption is no restriction. Here we give an independent proof of the existence of SRB measures which does not use denseness of periodic orbits and that enables us to obtain the hyperbolicity of the SRB measure.

Theorem 4.3. *Let Λ be a singular-hyperbolic* attractor *of a flow $X \in \mathfrak{X}^2(M)$ on a three-dimensional manifold. Then Λ supports a unique physical probability measure μ which is ergodic, hyperbolic and its ergodic basin covers a full Lebesgue measure subset of the topological basin of attraction, i.e.* $B(\mu) = W^s(\Lambda)$, Leb − mod 0. *Moreover the support of μ is the whole attractor* $\operatorname{supp}(\mu) = \Lambda$.

Here we need to assume that $(X^t)_{t\in\mathbb{R}}$ is a flow of class C^2 since for the construction of physical measures a bounded distortion property for one-dimensional maps is needed. These maps are naturally obtained as quotient maps over the set of stable leaves, which form a $C^{1+\alpha}$ foliation of a finite number of cross-sections associated to the flow if the flow is C^2, see Section 1.4.2. This will be detailed in Section 4.2.

Recall from Section 1.4 of Chapter 1 that hyperbolicity here means *non-uniform hyperbolicity*: the tangent bundle over Λ splits into a sum $T_zM = E^s_z \oplus E^X_z \oplus F_z$ of three one-dimensional invariant subspaces defined for μ-a.e. $z \in \Lambda$ and depending measurably on the base point z, where μ is the physical measure in the statement of Theorem 4.3, E^X_z is the flow direction (with zero Lyapunov exponent) and F_z is the direction with positive Lyapunov exponent.

Theorem 4.3 is another statement of sensitiveness, this time applying to the whole open set $B(\Lambda)$. Indeed, since non-zero Lyapunov exponents express that the orbits of infinitesimally close-by points tend to move apart from each other, this theorem means that most orbits in the basin of attraction separate under forward iteration. See Kifer [88], and Metzger [115], and references therein, for previous results about invariant measures and stochastic stability of the geometric Lorenz models.

In the uniformly hyperbolic setting it is well known that physical measures for hyperbolic attractors admit a disintegration into conditional measures along the unstable manifolds of almost every point which are absolutely continuous with respect to the induced Lebesgue measure on these sub-manifolds, see [37, 39, 151, 201].

Here the existence of unstable manifolds is guaranteed by the hyperbolicity of the physical measure: the strong-unstable manifolds $W^{uu}(z)$ are the “integral manifolds” in the direction of the one-dimensional sub-bundle F, tangent to F_z at almost every $z \in \Lambda$. The tools developed to prove Theorem 4.3 enable us to prove that the physical measure obtained there has absolutely continuous disintegration along the center-unstable direction, see Section 1.4 of Chapter 1 for the definition of conditional measures and the notion of adapted foliated neighborhoods of a point.

Theorem 4.4. *Let Λ be a singular-hyperbolic attractor for a C^2 three-dimensional flow. Then the physical measure μ supported in Λ has a disintegration into absolutely continuous conditional measures μ_γ along center-unstable surfaces $\gamma \in \Pi_\delta(x)$ such that $\frac{d\mu_\gamma}{dm_\gamma}$ is uniformly bounded from above, for all δ-adapted foliated neighborhoods $\Pi_\delta(x)$ and every $\delta > 0$. Moreover* $\mathrm{supp}(\mu) = \Lambda$.

Remark 4.5. The proof that $\text{supp}(\mu) = \Lambda$ that we present depends on the absolutely continuous disintegration property of the physical measure μ and the transitivity of X on Λ. However most singular-hyperbolic attractors are topologically mixing in the C^1 topology [122] and the Lorenz geometric models are always topologically mixing [105], so we should expect a more general argument proving $\text{supp}(\mu) = \Lambda$ without the need to obtain *first* that μ is a *cu*-Gibbs measure or *SRB*-measure.

Remark 4.6. It follows from the proof that the densities of the conditional measures μ_γ are bounded from below away from zero on $\Lambda \setminus B$, where B is any neighborhood of the singularities $\sigma(X \mid \Lambda)$. In particular the densities tend to zero as we get closer to the singularities of Λ.

The absolute continuity property along the center-unstable sub-bundle given by Theorem 4.4 ensures that

$$h_\mu(X^1) = \int \log \left| \det(DX^1 \mid E^{cu}) \right| d\mu,$$

by the characterization of probability measures satisfying the Entropy Formula [93]. The above integral is the sum of the positive Lyapunov exponents along the sub-bundle E^{cu} by Oseledets Theorem [107, 204]. Since in the direction E^{cu} there is only one positive Lyapunov exponent along the one-dimensional direction F_z, μ-a.e. z, the ergodicity of μ then shows that the following is true.

Corollary 4.7. *If Λ is a singular-hyperbolic attractor for a C^2 three-dimensional flow X^t, then the physical measure μ supported in Λ satisfies the Entropy Formula*

$$h_\mu(X^1) = \int \log \|DX^1 \mid F_z\| \, d\mu(z).$$

Again by the characterization of measures satisfying the Entropy Formula we get that μ *has absolutely continuous disintegration along the strong-unstable direction*, along which the Lyapunov exponent is positive, thus μ *is a u-Gibbs state* [151]. This also shows that μ *is an equilibrium state for the potential* $-\log \|DX^1 \mid F_z\|$ with respect to the diffeomorphism X^1. We note that the entropy $h_\mu(X^1)$ of X^1 is the entropy of the flow X^t with respect to the measure μ [204].

Hence we are able to extend most of the basic results on the ergodic theory of hyperbolic attractors to the setting of singular-hyperbolic attractors.

4.1 Expansiveness

For the proof of Theorem 4.1 we need the construction of cross-sections and Poincaré return maps, which is the subject of Section 3.4.1. We use the construction and notations defined there in what follows.

4.1.1 Proof of expansiveness

Here we prove Theorem 4.1. The proof is by contradiction: let us suppose that there exist $\varepsilon > 0$, a sequence $\delta_n \to 0$, a sequence of functions $h_n \in \mathcal{K}$ (see Section 1.2.2 of Chapter 1 for the definition of expansiveness), and sequences of points $x_n, y_n \in \Lambda$ such that

$$d\big(X^t(x_n), X^{h_n(t)}(y_n)\big) \le \delta_n \quad \text{for all } t \in \mathbb{R}, \tag{4.1}$$

but

$$X^{h_n(t)}(y_n) \notin X^{[t-\varepsilon,t+\varepsilon]}(x_n) \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

The main step in the proof is a reduction to a forward expansiveness statement about Poincaré maps which we state in Theorem 4.8 below.

We are going to use the following observation: there exists some regular (i.e. non-equilibrium) point $z \in \Lambda$ which is accumulated by the sequence of ω-limit sets $\omega(x_n)$. To see that this is so, start by observing that accumulation points do exist, since M is compact. Moreover, if the ω-limit sets accumulate on a singularity then they also accumulate on at least one of the corresponding unstable branches which, of course, consists of regular points. We fix such a z once and for all. Replacing our sequences by subsequences, if necessary, we may suppose that for every n there exists $z_n \in \omega(x_n)$ such that $z_n \to z$.

Let Σ be a δ-adapted cross-section at z, for some small δ. Reducing δ (but keeping the same cross-section) we may ensure that z is in the interior of the subset

$$\Sigma_\delta = \{y \in \Sigma : d(y, \partial\Sigma) > \delta\}.$$

By definition, x_n returns infinitely often to the neighborhood of z_n which, on its turn, is close to z. Thus dropping a finite number of terms in our sequences if necessary, we have that the orbit of x_n intersects Σ_δ infinitely many times. Let t_n be the time corresponding to the nth intersection.

Replacing x_n, y_n, t, and h_n by $x^{(n)} = X_{t_n}(x_n)$, $y^{(n)} = X_{h_n(t_n)}(y_n)$, $t' = t - t_n$, and $h_n'(t') = h_n(t' + t_n) - h_n(t_n)$, we may suppose that $x^{(n)} \in \Sigma_\delta$, while preserving both relations (4.1) and (4.2). Moreover there exists a sequence $\tau_{n,j}$, $j \geq 0$ with $\tau_{n,0} = 0$ such that

$$x^{(n)}(j) = X_{\tau_{n,j}}(x^{(n)}) \in \Sigma_\delta \quad \text{and} \quad \tau_{n,j} - \tau_{n,j-1} > \max\{t_1, t_2\} \tag{4.3}$$

for all $j \geq 1$, where t_1 is given by Proposition 3.49 and t_2 is given by Lemma 3.56.

Theorem 4.8. *Given $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $x \in \Sigma_\delta$ and $y \in \Lambda$ satisfy*

(a) there exist τ_j such that

$$x_j = X^{\tau_j}(x) \in \Sigma_\delta \quad \textit{and} \quad \tau_j - \tau_{j-1} > \max\{T_1, T_2\} \quad \textit{for all } j \geq 1;$$

(b) $\operatorname{dist}\left(X^t(x), X^{h(t)}(y)\right) < \delta_0$*, for all $t > 0$ and some $h \in \mathcal{K}$;*

then there exists $s \in \mathbb{R}$ such that $X^{h(s)}(y) \in W^{ss}_{\varepsilon_0}(X^{[s-\varepsilon_0, s+\varepsilon_0]}(x))$.

We postpone the proof of Theorem 4.8 until the next section and explain first why it implies Theorem 4.1. We are going to use the following observation.

Lemma 4.9. *There exist $\rho > 0$ small and $c > 0$, depending only on the flow, such that if z_1, z_2, z_3 are points in Λ satisfying $z_3 \in X^{[-\rho,\rho]}(z_2)$ and $z_2 \in W^{ss}_\rho(z_1)$, then*

$$\operatorname{dist}(z_1, z_3) \geq c \cdot \max\{\operatorname{dist}(z_1, z_2), \operatorname{dist}(z_2, z_3)\}.$$

Proof. This is a direct consequence of the fact that the angle between E^{ss} and the flow direction is bounded from zero which, on its turn, follows from the fact that the latter is contained in the center-unstable sub-bundle E^{cu}. Indeed consider for small enough $\rho > 0$ the C^1 surface $X^{[-\rho,\rho]}\left(W^{ss}_\rho(z_1)\right)$. The Riemannian metric here is uniformly close to the Euclidean one and we may choose coordinates on $[-\rho, \rho]^2$ putting z_1 at the origin, sending $W^{ss}_\rho(z_1)$ to the segment $[-\rho, \rho] \times \{0\}$ and $X^{[-\rho,\rho]}(z_1)$ to $\{0\} \times [-\rho, \rho]$, see Figure 4.1. Then the angle α between $X^{[-\rho,\rho]}(z_2)$ and the horizontal is bounded from below away from zero and the existence of c follows by standard arguments using the Euclidean metric. □

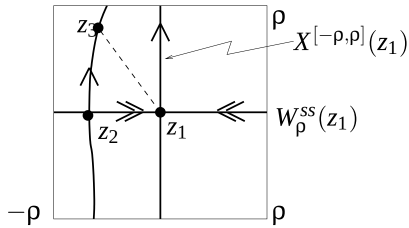

Figure 4.1: Distances near a point in the stable-manifold.

We fix $\varepsilon_0 = \varepsilon$ as in (4.2) and then consider δ_0 as given by Theorem 4.8. Next, we fix n such that $\delta_n < \delta_0$ and $\delta_n < c\rho$, and apply Theorem 4.8 to $x = x^{(n)}$ and $y = y^{(n)}$ and $h = h_n$. Hypothesis (a) in the theorem corresponds to (4.3) and, with these choices, hypothesis (b) follows from (4.1). Therefore we obtain that $X^{h(s)}(y) \in W^{ss}_\varepsilon(X^{[s-\varepsilon,s+\varepsilon]}(x))$. In other words, there exists $|\tau| \le \varepsilon$ such that $X^{h(s)}(y) \in W^{ss}_\varepsilon(X^{s+\tau}(x))$. Hypothesis (4.2) implies that $X^{h(s)}(y) \neq X^{s+\tau}(x)$. Hence since strong-stable manifolds are expanded under backward iteration, there exists $\theta > 0$ maximum such that

$$X^{h(s)-t}(y) \in W^{ss}_\rho(X^{s+\tau-t}(x)) \quad \text{and} \quad X^{h(s+\tau-t)}(y) \in X^{[-\rho,\rho]}(X^{h(s)-t}(y))$$

for all $0 \le t \le \theta$, see Figure 4.2. Since θ is maximum

$$\begin{aligned} &\text{either } \operatorname{dist}\left(X^{h(s)-t}(y), X^{s+\tau-t}(x)\right) = \rho \\ &\text{or } \operatorname{dist}\left(X^{h(s+\tau-t)}(y), X^{h(s)-t}(y)\right) = \rho \text{ for } t = \theta. \end{aligned}$$

Using Lemma 4.9, we conclude that $d(X^{s+\tau-t}(x), X^{h(s+\tau-t)}(y)) \ge c\rho > \delta_n$ which contradicts (4.1). This contradiction reduces the proof of Theorem 4.1 to that of Theorem 4.8.

4.1.2 Infinitely many coupled returns

We start by outlining the proof of Theorem 4.8. There are three steps.

- The first one, which we carry out in the present section, is to show that to each return x_j of the orbit of x to Σ there corresponds a nearby return y_j of the orbit of y to Σ. The precise statement is in Lemma 4.10 below.

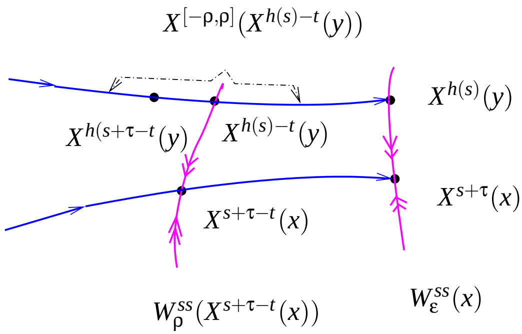

Figure 4.2: Relative positions of the strong-stable manifolds and orbits.

- The second, and most crucial step, is to show that there exists a smooth Poincaré map, with large return time, defined on the whole strip of Σ in between the stable manifolds of x_j and y_j. This is done in Section 4.1.3.

- The last step, Section 4.1.7, is to show that these Poincaré maps are uniformly hyperbolic, in particular, they expand *cu*-curves uniformly (recall the definition of *cu*-curve in Section 3.4.1).

The theorem is then easily deduced: to prove that $X^{h(s)}(y)$ is in the orbit of $W^{ss}_{\varepsilon}(x)$ it suffices to show that $y_j \in W^s(x_j,\Sigma)$, by Remark 3.48. The latter must be true, for otherwise, by hyperbolicity of the Poincaré maps, the stable manifolds of x_j and y_j would move apart as $j \to \infty$, and this would contradict condition (b) of Theorem 4.8. See Section 4.1.7 for more details.

Lemma 4.10. *There exists $K > 0$ such that, in the setting of Theorem 4.8, there exists a sequence $(\upsilon_j)_{j\geq 0}$ such that*

1. $y_j = X^{\upsilon_j}(y)$ *is in* Σ *for all* $j \geq 0$.

2. $|\upsilon_j - h(\tau_j)| < K \cdot \delta_0$ *and* $d(x_j, y_j) < K \cdot \delta_0$.

Proof. By assumption $d(x_j, X^{h(\tau_j)}(y)) < K \cdot \delta_0$ for all $j \geq 0$. In particular $y'_j = X^{h(\tau_j)}(y)$ is close to Σ. Using a flow box in a neighborhood of Σ we obtain $X^{\varepsilon_j}(y'_j) \in \Sigma$ for some $\varepsilon_j \in (-K \cdot \delta_0, K \cdot \delta_0)$. The constant K

depends only on the vector field X and the cross-section Σ (more precisely, on the angle between Σ and the flow direction). Taking $\upsilon_j = h(\tau_j) + \varepsilon_j$ we get the first two claims in the lemma. The third one follows from the triangle inequality; it may be necessary to replace K by a larger constant, still depending on X and Σ only. □

4.1.3 Semi-global Poincaré map

Since we took the cross-section Σ to be adapted, we may use Lemma 3.56 to conclude that there exist Poincaré maps R_j with $R_j(x_j) = x_{j+1}$ and $R_j(y_j) = y_{j+1}$ and sending $W^s_\varepsilon(x_j, \Sigma)$ and $W^s_\varepsilon(y_j, \Sigma)$ inside the lines $W^s_\varepsilon(x_{j+1}, \Sigma)$ and $W^s_\varepsilon(y_{j+1}, \Sigma)$, respectively. The goal of this section is to prove that R_j *extends to a smooth Poincaré map on the whole strip* Σ_j *of* Σ *bounded by the stable manifolds of* x_j *and* y_j.

We first outline the proof. For each j we choose a curve γ_j transverse to the stable foliation of Σ, connecting x_j to y_j and such that γ_j is disjoint from the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$. Using Lemma 3.56 in the same way as in the last paragraph, we see that it suffices to prove that R_j extends smoothly to γ_j. For this purpose we consider a tube-like domain $\mathcal{T}_j$ consisting of local stable manifolds through an *immersed surface* S_j whose boundary is formed by γ_j and γ_{j+1} and the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$, see Figure 4.3. We will prove that the orbit of any point in γ_j must leave the tube through γ_{j+1} in finite time. We begin by showing that the tube contains no singularities. This uses hypothesis (b) together with the local dynamics near Lorenz-like singularities. Next, using hypothesis (b) together with a Poincaré-Bendixson argument on S_j, we conclude that the forward orbit of any point in $\mathcal{T}_j$ must leave the tube. Another argument, using hyperbolicity properties of the Poincaré map, shows that orbits through γ_j must leave $\mathcal{T}_j$ through γ_{j+1}. In the sequel we detail these arguments.

4.1.4 A tube-like domain without singularities

Since we took γ_j and γ_{j+1} disjoint from the orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$, the union of these four curves is an embedded circle. We recall that the two orbit segments are close to each other, by hypothesis (b)

$$d(X^t(x), X^{h(t)}(y)) < \delta_0 \quad \text{for all } t \in [t_j, t_{j+1}].$$

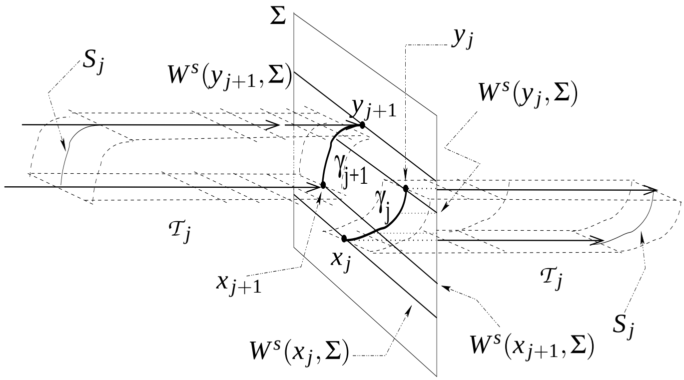

Figure 4.3: A tube-like domain.

Assuming that δ_0 is smaller than the radius of injectiveness of the exponential map of the ambient manifold (i.e. $\exp_x : T_xM \to M$ is locally invertible in a δ_0-neighborhood of x in M for any $x \in M$), there exists a unique geodesic linking each $X^t(x)$ to $X^{h(t)}(y)$, and it varies continuously (even smoothly) with t. Using these geodesics we easily see that the union of $[y_j, y_{j+1}]$ with γ_j and γ_{j+1} is homotopic to a curve inside the orbit of x, with endpoints x_j and x_{j+1}, and so it is also homotopic to the segment $[x_j, x_{j+1}]$. This means that the previously mentioned embedded circle is homotopic to zero. It follows that there is a *smooth immersion* $\phi : [0,1] \times [0,1] \to M$ such that

- $\phi(\{0\} \times [0,1]) = \gamma_j$ and $\phi(\{1\} \times [0,1]) = \gamma_{j+1}$
- $\phi([0,1]\} \times \{0\}) = [y_j, y_{j+1}]$ and $\phi([0,1]\} \times \{1\}) = [x_j, x_{j+1}]$.

Moreover $S_j = \phi([0,1] \times [0,1])$ may be chosen such that, see Figure 4.4

- all the points of S_j are at distance less than δ_1 from the orbit segment $[x_j, x_{j+1}]$, for some uniform constant $\delta_1 > \delta_0$ which can be taken arbitrarily close to zero, reducing δ_0 if necessary;
- the intersection of S_j with an incoming cross-section of any singularity (Section 3.4.1) is transverse to the corresponding stable foliation.

Then we define $\mathcal{T}_j$ to be the union of the local stable manifolds through the points of that disk.

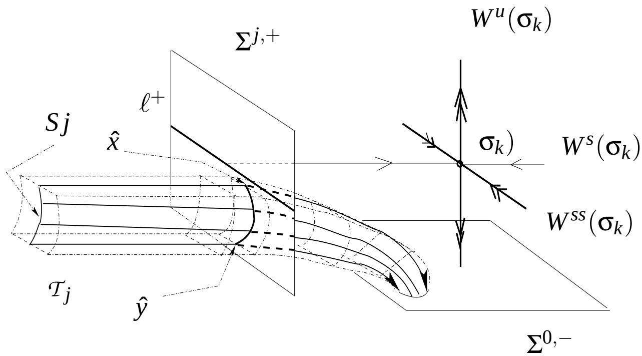

Figure 4.4: Entering the flow box of a singularity.

Proposition 4.11. *The domain $\mathcal{T}_j$ contains no singularities of the flow.*

Proof. By construction, every point of $\mathcal{T}_j$ is at distance $\leq \varepsilon$ from S_j and, consequently, at distance $\leq \varepsilon + \delta_1$ from $[x_j, x_{j+1}]$. So, taking ε and δ_0 much smaller than the sizes of the cross-sections associated to the singularities (Section 3.4.1), we immediately get the conclusion of the proposition in the case when $[x_j, x_{j+1}]$ is disjoint from the incoming cross-sections of all singularities. In the general case we must analyze the intersections of the tube with the flow boxes at the singularities. The key observation is in the following statement whose proof we postpone.

Lemma 4.12. *Suppose $[x_j, x_{j+1}]$ intersects an incoming cross-section Σ_k^i of some singularity σ_k at some point $\hat{x}$ with $d(\hat{x}, \partial\Sigma_k^i) > \delta$. Then $[y_j, y_{j+1}]$ intersects Σ_k^i at some point $\hat{y}$ with $d(\hat{x}, \hat{y}) < K \cdot \delta_0$ and, moreover $\hat{x}$ and $\hat{y}$ are in the same connected component of $\Sigma_k^i \setminus W^s_{loc}(\sigma_k)$.*

Let us recall that by construction the intersection of S_j with the incoming cross-section Σ_k^i is transverse to the corresponding stable foliation, see Figure 4.4. By the previous lemma this intersection is entirely contained in one of the connected components of $\Sigma_k^i \setminus W^s_{loc}(\sigma_k)$. Since $\mathcal{T}_j$ consists of local stable manifolds through the points of S_j its intersection with

Σ_k^i is contained in the region bounded by the stable manifolds $W^s(\hat{x},\Sigma_k^i)$ and $W^s(\hat{y},\Sigma_k^i)$, and so it is entirely contained in a connected component of $\Sigma_k^i \setminus W^s_{loc}(\sigma_k)$. In other words, the crossing of the tube $\mathcal{T}_j$ through the flow box is disjoint from $W^s_{loc}(\sigma_k)$, in particular, it does not contain the singularity. Repeating this argument for every intersection of the tube with a neighborhood of some singularity, we get the conclusion of the proposition. □

Proof of Lemma 4.12. The first part is proved in exactly the same way as Lemma 4.10. We have

$$\hat{x} = X^{r_0}(x) \quad \text{and} \quad \hat{y} = X^{s_0}(y)$$

with $|s_0 - h(r_0)| < K\delta_0$. The proof of the second part is by contradiction and relies, fundamentally, on the local description of the dynamics near the singularity. Associated to $\hat{x}$ and $\hat{y}$ we have the points $\tilde{x} = X^{r_1}(x)$ and $\tilde{y} = X^{s_1}(y)$, where the two orbits leave the flow box associated to the singularity. If $\hat{x}$ and $\hat{y}$ are in opposite sides of the local stable manifold of σ_k, then $\tilde{x}$ and $\tilde{y}$ belong to different outgoing cross-sections of σ_k. Our goal is to find some $t \in \mathbb{R}$ such that

$$\operatorname{dist}\left(X^t(x), X^{h(t)}(y)\right) > \delta_0,$$

thus contradicting hypothesis (b).

We assume by contradiction that $\hat{x},\hat{y}$ are in different connected components of $\Sigma_k^{i,\pm} \setminus \ell^{\pm}$. There are two cases to consider. We suppose first that $h(r_1) > s_1$ and note that $s_1 \gg s_0 \approx h(r_0)$, so that $s_1 > h(r_0)$. It follows that there exists $t \in (r_0, r_1)$ such that $h(t) = s_1$ since h is non-decreasing and continuous. Then $X^t(x)$ is on one side of the flow box of σ_k, whereas $X^{h(t)}(y)$ belongs to the outgoing cross-section at the other side of the flow box. Thus $\operatorname{dist}\left(X^t(x), X^{h(t)}(y)\right)$ has the order of magnitude of the diameter of the flow box, which we may assume to be much larger than δ_0.

Now we suppose that $s_1 \geq h(r_1)$ and observe that $h(r_1) > h(r_0)$, since h is increasing. We recall also that $X^{h(r_0)}(y)$ is close to $\hat{y}$, near the incoming cross-section, so that the whole orbit segment from $X^{h(r_0)}(y)$ to $X^{s_1}(y)$ is contained in (a small neighborhood of) the flow box, to one side of the local stable manifold of σ_j. The previous observation means that this orbit segment contains $X^{h(r_1)}(y)$. However $X^{r_1}(x)$ belongs to the outgoing cross-section at the opposite side of the flow box, and so $\operatorname{dist}\left(X^{r_1}(x), X^{h(r_1)}(y)\right)$ has the order of magnitude of the diameter of the flow box, which is much larger than δ_0. □

4.1.5 Every orbit leaves the tube

Our goal in this subsection is to show that the forward orbit of every point $z \in \mathcal{T}_j$ leaves the tube in finite time: The proof is based on a Poincaré-Bendixson argument applied to the flow induced by X^t on the disk S_j .

We begin by defining this induced flow. For the time being, we make the following simplifying assumption:

(H) $S_j = \phi([0,1] \times [0,1])$ is an embedded disk and the stable manifolds $W^s_\varepsilon(\xi)$ through the points $\xi \in S_j$ are pairwise disjoint.

This condition provides a well-defined continuous projection $\pi : \mathcal{T}_j \to S_j$ by assigning to each point $z \in \mathcal{T}_j$ the unique $\xi \in S_j$ whose local stable manifold contains z. The (not necessarily complete) flow Y^t induced by X^t on S_j is given by $Y^t(\xi) = \pi(X^t(\xi))$ for the largest interval of values of t for which this is defined. It is clear, just by continuity, that given any subset E of S_j at a positive distance from ∂S_j , there exists $\varepsilon > 0$ such that $Y^t(\xi)$ is defined for all $\xi \in E$ and $t \in [0,\varepsilon]$. In fact this remains true even if E approaches the curve γ_j (since Σ is a cross-section for X^t , the flow at γ_j points inward S_j) or the X^t-orbit segments $[x_j, x_{j+1}]$ and $[y_j, y_{j+1}]$ on the boundary of S_j (because they are also Y^t-orbit segments). Thus we only have to worry with the distance to the remaining boundary segment:

(U) given any subset E of S_j at positive distance from γ_{j+1} , there exists $\varepsilon > 0$ such that $Y^t(\xi)$ is defined for all $\xi \in E$ and $t \in [0,\varepsilon]$.

We observe also that for points ξ close to γ_{j+1} the flow $Y^t(\xi)$ must intersect γ_{j+1} , after which it is no longer defined.

Now we explain how to remove condition (H). In this case, the induced flow is naturally defined on $[0,1] \times [0,1]$ rather than S_j , as we now explain. *Recall that* $\phi : [0,1] \times [0,1] \to M$ *is an immersion.* So given any $w \in [0,1] \times [0,1]$ there exist neighborhoods U of w and V of $\phi(w)$ such that $\phi : U \to V$ is a diffeomorphism. Moreover, just by continuity of the stable foliation, choosing V sufficiently small we may ensure that each stable manifold $W^s_\varepsilon(\xi)$, $\xi \in V$, intersects V only at the point ξ. This means that we have a well-defined projection π from $\cup_{\xi \in V} W^s_\varepsilon(\xi)$ to V associating to each point z in the domain the unique element of V whose stable manifold contains z. Then we may define $Y^t(w)$ for small t, by

$$Y^t(w) = \phi^{-1}(\pi(X^t(\phi(w)))).$$

As before, we extend Y^t to a maximal domain. This defines a (partial) flow on the square $[0,1] \times [0,1]$, such that both $[0,1] \times \{i\}$, $i \in \{0,1\}$ are trajectories.

Remark 4.13. A singularity ζ for the flow Y^t corresponds to a singularity of X in the local strong-stable manifold of ζ in M by the definition of Y^t through the projection π.

Notice also that forward trajectories of points in $\{0\} \times [0,1]$ enter the square. Hence, the only way trajectories may exit is through $\{1\} \times [0,1]$. So, we have the following reformulation of property (U):

(U) given any subset E of $[0,1] \times [0,1]$ at positive distance from $\{1\} \times [0,1]$, there exists $\varepsilon > 0$ such that $Y^t(w)$ is defined for all $w \in E$ and $t \in [0,\varepsilon]$.

Moreover for points w close to $\{1\} \times [0,1]$ the flow $Y^t(\xi)$ must intersect $\{1\} \times [0,1]$, after which it is no longer defined.

Proposition 4.14. *Given any point $z \in \mathcal{T}_j$ there exists $t > 0$ such that $X^t(z) \notin \mathcal{T}_j$.*

Proof. The proof is by contradiction. First, we assume condition (H). Suppose there exists $z \in \mathcal{T}_j$ whose forward orbit remains in the tube for all times. Let $z_0 = \pi(z)$. Then $Y^t(z_0)$ is defined for all $t > 0$, and so it makes sense to speak of the ω-limit set $\omega(z_0)$. The orbit $Y^t(z_0)$ can not accumulate on γ_{j+1} for otherwise it would leave S_j. Therefore $\omega(z_0)$ is a compact subset of S_j at positive distance from γ_{j+1}. Using property (U) we can find a uniform constant $\varepsilon > 0$ such that $Y^t(w)$ is defined for every $t \in [0,\varepsilon]$ and every $w \in \omega(z_0)$. Since $\omega(z_0)$ is an invariant set, we can extend Y^t to a complete flow on it.

In particular we may fix $w_0 \in \omega(z_0)$, $w \in \omega(w_0)$ and apply the arguments in the proof of the Poincaré-Bendixson Theorem. On the one hand, if we consider a cross-section S to the flow at w, the forward orbits of z_0 and w_0 must intersect it on monotone sequences; on the other hand, every intersection of the orbit of w_0 with S is accumulated by points in the orbit of z_0. This implies that w is in the orbit of w_0 and, in fact, that the later is periodic.

We consider the disk $D \subset S_j$ bounded by the orbit of w_0. The flow Y^t is complete restricted to D and so we may apply Poincaré-Bendixson's

Theorem (see [143]) once more, and conclude that Y^t has some singularity ζ inside D. This implies by Remark 4.13 that X^t has a singularity in the local stable manifold of ζ, which contradicts Proposition 4.11. This contradiction completes the proof of the proposition, under assumption (H). The general case is treated in the same way, just dealing with the flow induced on $[0,1]\times[0,1]$ instead of on S_j. □

4.1.6 The Poincaré map is well-defined on Σ_j

We have shown that for the induced flow Y^t on S_j (or, more generally, on $[0,1]\times[0,1]$) every orbit must eventually cross γ_{j+1} (respectively, $\{1\}\times[0,1]$). Hence there exists a continuous Poincaré map

$$r:\gamma_j\to\gamma_{j+1},\quad r(\xi)=Y_{\theta(\xi)}(\xi).$$

By compactness the Poincaré time $\theta(\cdot)$ is bounded. We are going to deduce that every forward X^t-orbit eventually leaves the tube $\mathcal{T}_j$ through Σ_{j+1}, which proves that R_j is defined on the whole strip of Σ_j between the manifolds $W^s(x_j,\Sigma_j)$ and $W^s(y_j,\Sigma_j)$, as claimed in Section 4.1.2.

To this end, let γ be a *central-unstable curve in* Σ_δ *connecting the stable manifolds* $W^s(x_j,\Sigma)$ and $W^s(y_j,\Sigma)$. Observe that γ is inside $\mathcal{T}_j$. For each $z\in\gamma$, let $t(z)$ be the smallest positive time for which $X^{t(z)}$ is on the boundary of $\mathcal{T}_j$.

The crucial observation is that, in view of the construction of Y^t, *each* $X^{t(z)}(z)$ *belongs to the (global) stable manifold of* $Y_{t(z)}(\pi(z))$. We observe also that for $\{\xi\}=\gamma\cap W^s(x_j,\Sigma)$ we have $Y^t(\xi)=X^t(\xi)$ and so $t(\xi)=\theta(\xi)$.

Now we take $z\in\gamma$ close to ξ. Just by continuity, the X^t-trajectory of ξ and z remain close, and by the forward contraction along stable manifolds, the X^t-trajectory of ξ remains close to the segment $[x_j,x_{j+1}]$. Moreover orbit of z cannot leave the tube through the union of the local strong stable manifolds passing through $[x_j,x_{j+1}]$, for otherwise it would contradict the definition of Y^t. Hence the trajectory of z must leave the tube through Σ_{j+1}. In other words $X^{t(z)}(z)$ is a point of Σ_{j+1}, close to $\tilde{\xi}=X^{t(\xi)}(\xi)$.

Let $\hat{\gamma}\subset\gamma_j$ be the *largest connected subset* which contains x_j such that $X^{t(x)}(z)\in\Sigma_{j+1}$ for all $z\in\hat{\gamma}$. We want to prove that $\hat{\gamma}=\gamma$ since this implies that R_j extends to the whole γ and so, using Lemma 3.56, to the whole Σ_j.

The proof is by contradiction. We assume $\hat{\gamma}$ is not the whole γ, and let $\hat{x}$ be the endpoint different from ξ. Then by definition of $\mathcal{F}^s_\Sigma$ and of Y_t (from

Section 4.1.5) $\tilde{x} = X^{t(\hat{x})}(\hat{x})$ is on the center-unstable boundary $\partial^{cu}\Sigma_{j+1}$ of the cross-section Σ_{j+1}, between the stable manifolds $W^s(x_{j+1},\Sigma_{j+1})$ and $W^s(y_{j+1},\Sigma_{j+1})$, see Figure 4.5. By the choice of γ and by Corollary 3.51, $\tilde{\gamma} = \{X_{t(z)}(z) : z \in \hat{\gamma}\}$ is a *cu*-curve.

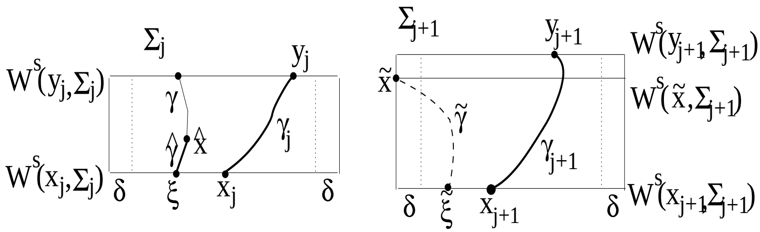

Figure 4.5: Exiting the tube at Σ_{j+1}.

On the one hand, by Lemma 3.52, the distance between $\tilde{x}$ and $\tilde{\xi} = X_{t(\xi)}(\xi)$ dominates the distance between their stable manifolds and $\ell(\tilde{\gamma})$

$$\ell(\tilde{\gamma}) \leq \kappa \cdot d(\xi,\tilde{x}) \leq \kappa \cdot d\big(W^s(x_{j+1},\Sigma), W^s(\tilde{x},\Sigma)\big).$$

We note that $\ell(\tilde{\gamma})$ is larger than δ, since ξ is in Λ and the section Σ_{j+1} is adapted. On the other hand, the distance between the two stable manifolds is smaller than the distance between the stable manifold of x_{j+1} and the stable manifold of y_{j+1}, and this is smaller than $K \cdot \delta_0$. Since δ_0 is much smaller than δ, this is a contradiction. This proves the claim that $X^{t(z)}(z) \in \Sigma$ for all $z \in \gamma$.

4.1.7 Expansiveness of the Poincaré map

We have shown that there exists a well defined Poincaré return map R_j on the whole strip between the stable manifolds of x_j and y_j inside Σ. By Proposition 3.49 and Corollary 3.51 we know that the map R_j is hyperbolic where defined and, moreover, that the length of each *cu*-curve is expanded by a factor of 3 by R_j (since we chose $\lambda = 1/3$ in Section 3.4.1). Hence the distance between the stable manifolds $R_j\big(W^s(x_j,\Sigma)\big)$ and $R_j\big(W^s(y_j,\Sigma)\big)$ is increased by a factor strictly larger than one, see Figure 4.6 on the next page. This contradicts item (2) of Lemma 4.10 since this distance will eventually become larger than $K \cdot \delta_0$. Thus y_j must be in the stable manifold $W^s(x_j, \Sigma)$. Since the strong-stable manifold is locally flow-invariant and $X^{h(\tau_j)}(y)$ is in the orbit of $y_j = X^{\upsilon_j}(y)$, then $X^{h(\tau_j)}(y) \in W^s(x_j) = W^s\big(X^{\tau_j}(x)\big)$, see Lemma 4.10 on page 192.

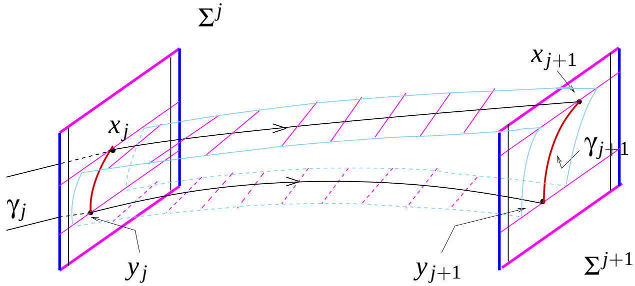

Figure 4.6: Expansion within the tube.

According to Lemma 4.10 we have $|\upsilon_j - h(\tau_j)| < K \cdot \delta_0$ and, by Remark 3.48, there exits a small $\varepsilon_1 > 0$ such that

$$R_\Sigma(y_j) = X^t(y_j) \in W^{ss}_\varepsilon(x_j) \quad \text{with} \quad |t| < \varepsilon_1.$$

Therefore the piece of orbit $\mathcal{O}_y = X^{[\upsilon_j - K\cdot\delta_0 - \varepsilon_1, \upsilon_j + K\cdot\delta_0 + \varepsilon_1]}(y)$ must contain $X^{h(\tau_j)}(y)$. We note that this holds for all sufficiently small values of $\delta_0 > 0$ fixed from the beginning.

Now let $\varepsilon_0 > 0$ be given and let us consider the piece of orbit $\mathcal{O}_x = X^{[\tau_j - \varepsilon_0, \tau_j + \varepsilon_0]}(x)$ and the piece of orbit of x whose strong-stable manifolds intersect $\mathcal{O}_y$, i.e.

$$\mathcal{O}_{xy} = \{X^s(x) : \exists \tau \in [\upsilon_j - K\cdot\delta_0 - t, \upsilon_j + K\cdot\delta_0 + t] \text{ s. t. } X^\tau(y) \in W^{ss}_\varepsilon\big(X^s(x)\big)\}.$$

Since $y_j \in W^s(x_j)$ we conclude that $\mathcal{O}_{xy}$ is a neighborhood of $x_j = X^{\tau_j}(x)$ which can be made as small as we want taking δ_0 and ε_1 small enough. In particular this ensures $\mathcal{O}_{xy} \subset \mathcal{O}_x$ and so $X^{h(\tau_j)}(y) \in W^{ss}_\varepsilon\big(X^{[\tau_j - \varepsilon_0, \tau_j + \varepsilon_0]}(x)\big)$. This finishes the proof of Theorem 4.8.

4.2 Singular-hyperbolic attractors are non-uniformly hyperbolic

Here we start the proof of Theorem 4.3.

The starting point

We show in Section 4.2.1 that choosing a *global Poincaré section* Ξ (with several connected components) for X on Λ, we can reduce the transformation R to the quotient over the stable leaves. We can do this using Lemma 3.56 with the exception of finitely many leaves Γ, corresponding to the points whose orbit falls into the local stable manifold of some singularity or are sent into the stable boundary $\partial^s\Sigma$ of some $\Sigma \in \Xi$ by R, where the return time function τ is discontinuous.

As will be explained in Section 4.2.1, the global Poincaré map $R : \Xi \to \Xi$ induces in this way a map $f : \mathcal{F} \setminus \Gamma \to \mathcal{F}$ on the leaf space, diffeomorphic to a finite union of open intervals I. which is piecewise expanding and admits finitely many $\upsilon_1, \dots, \upsilon_l$ ergodic absolutely continuous (with respect to Lebesgue measure on I) invariant probability measures (acim) whose basins cover Lebesgue almost all points of I.

Moreover the Radon-Nikodym derivatives (densities) $\frac{d\upsilon_k}{d\lambda}$ are *bounded from above* and *the support of each* υ_k *contains nonempty open intervals*, so the basin $B(\upsilon_k)$ contains nonempty open intervals Lebesgue modulo zero, $k = 1, \dots, l$.

Description of the construction

Afterward we unwind the reductions made in Section 4.2.1 and obtain a physical measure for the original flow at the end.

We divide the construction of the physical measure for Λ in the following steps.

1. The compact metric space Ξ is endowed with a partition $\mathcal{F}$ and map $R : \Xi \setminus \Gamma \to \Xi$, where Γ is a finite set of elements of $\mathcal{F}$ (see Section 4.2.1). The map R preserves the partition $\mathcal{F}$ and contracts its elements by Lemma 3.56. We have a finite family $\upsilon_1, \dots, \upsilon_l$ of absolutely continuous invariant probability measures for the induced quotient map $f : \mathcal{F} \setminus \Gamma \to \mathcal{F}$.

   We show in Section 4.2.2 that each υ_i defines a R-invariant ergodic probability measure η_i. In Section 4.2.3 we show that the basin $B(\eta_i)$ is a union of strips of Ξ, and η_i are therefore physical measures for R. Moreover these basins cover Ξ:

$$\lambda^2\big(\Xi \setminus (B(\eta_1) \cup \dots \cup B(\eta_l))\big) = 0,$$

where λ^2 is the area measure on Ξ.

2. We then pass from R-invariant physical measures $\eta_1, \dots, \eta_l$ to invariant probability measures $\nu_1, \dots, \nu_l$ for the suspension semiflow over R with roof function τ. In the process we keep the ergodicity (Section 4.2.5) and the basin property (Section 4.2.5) of the measures: the whole space $\Xi \times [0, +\infty)/\sim$ where the semiflow is defined equals the union of the ergodic basins of the ν_i Lebesgue modulo zero.

3. Finally in Section 4.2.6 we convert each physical measure ν_i for the semiflow into a physical measure μ_i for the original flow. We use that the semiflow is semi-conjugated to X_t on a neighborhood of Λ by a local diffeomorphism. Uniqueness of the physical measure μ is then deduced in Section 4.2.6 through the existence of a dense regular orbit in Λ (recall that our definition of attractor *demands* transitivity) and by the observation that the basin of μ contains open sets Lebesgue modulo zero. In Section 4.2.7 we show that μ is (non-uniformly) hyperbolic.

The details are exposed in the following sections.

4.2.1 Global Poincaré maps & reduction to 1-dimensional map

Here we construct a global Poincaré map for the flow near the singular-hyperbolic attractor Λ. We then use the hyperbolicity properties of this map to reduce the dynamics to a one-dimensional piecewise expanding map through a quotient map over the stable leaves.

Cross-sections and invariant foliations

We observe first that by Lemma 3.55 we can take a δ-adapted cross-section at each non-singular point $x \in \Lambda$. We know also that near each singularity σ_k there is a flow-box U_{σ_k} as in Section 3.4.1, see Figure 3.1

Using a tubular neighborhood construction near any given adapted cross-section Σ, we linearise the flow in an open set $U_\Sigma = X^{(-\varepsilon,\varepsilon)}(\text{int}(\Sigma))$ for a small $\varepsilon > 0$, containing the interior of the cross-section. This provides an open cover of the compact set Λ by flow-boxes near the singularities and tubular neighborhoods around regular points.

We let $\{U_{\Sigma_i}, U_{\sigma_k} : i = 1,\dots,l; k = 1,\dots,s\}$ be a finite cover of Λ, where $s \geq 1$ is the number of singularities in Λ, and we set $T_3 > 0$ to be an upper bound for the time it takes any point $z \in U_{\Sigma_i}$ to leave this tubular neighborhood under the flow, for any $i = 1,\dots,l$. We assume without loss that $T_2 > T_3$.

To define the Poincaré map R, for any point z in one of the cross-sections in

$$\Xi = \{\Sigma_j, \Sigma_{\sigma_k}^{i,\pm}, \Sigma_{\sigma_k}^{o,\pm} : j = 1,\dots,l; k = 1,\dots,s\},$$

we consider $\hat{z} = X^{T_2}(z)$ and wait for the next time $t(z)$ the orbit of $\hat{z}$ hits again one of the cross-sections. Then we define $R(z) = X^{T_2+t(z)}(z)$ and say that $\tau(z) = T_2 + t(z)$ is the *Poincaré time* of z. If the point z never returns to one of the cross-sections, then the map R is not defined at z (e.g. at the lines $\ell^{\pm}$ in the flow-boxes near a singularity). Moreover by Lemma 3.56, if R is defined for $x \in \Sigma$ on some $\Sigma \in \Xi$, then R is defined for every point in $W^s(x,\Sigma)$. Hence *the domain of $R \mid \Sigma$ consists of strips of Σ*. The smoothness of $(t,x) \mapsto X^t(x)$ ensures that the strips

$$\Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\} \tag{4.4}$$

have non-empty interior in Σ for every $\Sigma, \Sigma' \in \Xi$.When R maps to an outgoing strip near a singularity σ_k, there might be a boundary of the strip corresponding to the line $\ell_k^{\pm}$ of points which fall in the stable manifold of σ_k.

Remark 4.15. Consider the Poincaré map given by the *first return map* $R_0 : \Xi \to \Xi$ defined simply as $R_0(z) = X^{T(z)}(z)$, where

$$T(z) = \inf\{t > 0 : X^t(z) \in \Xi\}$$

is the time the X-orbit of $z \in \Xi$ takes to arrive again at Ξ. This map R_0 is not defined on those points z which do not return and, moreover, R_0 might not satisfy the lemmas of Section 3.4.1, since we do not know whether the flow from z to $R_0(z)$ has enough time to gain expansion. However the stable manifolds are still well defined. By the definitions of R_0 and of R we see that *R is induced by R_0*, i.e. *if R is defined for $z \in \Xi$, then there exists an integer $r(x)$ such that*

$$R(z) = R_0^{r(z)}(z).$$

We note that since the number of cross-sections in Ξ is finite and the time t_2 is a constant, then the function $r : \Xi \to \mathbb{N}$ is bounded: there exists $r_0 \in \mathbb{N}$ such that $r(x) \leq r_0$ for all $x \in \Xi$.

Finite number of strips

We show that fixing a cross-section $\Sigma \in \Xi$ the family of all possible strips as in (4.4) covers Σ except for finitely many stable leaves $W^s(x_i, \Sigma), i = 1, \dots, m = m(\Sigma)$. Moreover we also show that each strip given by (4.4) has finitely many connected components. Thus the number of strips in each cross-section is finite.

We first recall that each $\Sigma \in \Xi$ is contained in U_0, so $x \in \Sigma$ is such that $\omega(x) \subset \Lambda$. Note that R is locally smooth for all points $x \in \mathrm{int}(\Sigma)$ such that $R(x) \in \mathrm{int}(\Xi)$ by the flow box theorem and the smoothness of the flow, where $\mathrm{int}(\Xi)$ is the union of the interiors of each cross-section of Ξ. Let $\partial^s\Xi$ denote the union of all the leaves forming the stable boundary of every cross-section in Ξ.

Lemma 4.16. *The set of discontinuities of R in $\Xi \setminus \partial^s\Xi$ is contained in the set of points $x \in \Xi \setminus \partial^s\Xi$ such that:*

1. *either $R(x)$ is defined and belongs to $\partial^s\Xi$;*
2. *or there is some time $0 < t \leq T_2$ such that $X^t(x) \in W^s_{loc}(\sigma)$ for some singularity σ of Λ.*

Moreover this set is contained in a finite number of stable leaves of the cross-sections $\Sigma \in \Xi$.

Proof. We divide the proof into several steps.

Step 1 Cases (1) and (2) in the statement of the lemma correspond to all possible discontinuities of R in $\Xi \setminus \partial^s\Xi$.

Let x be a point in $\Sigma \setminus \partial^s\Sigma$ for some $\Sigma \in \Xi$, *not* satisfying any of the conditions in items (1) and (2). Then $R(x)$ is defined and $R(x)$ belongs to the interior of some cross-section Σ'. By the smoothness of the flow and by the flow box theorem we have that R is smooth in a neighborhood of x in Σ. Hence any discontinuity point for R must be in one the situations (1) or (2).

Step 2 Points satisfying item (2) are contained in finitely many stable leaves in each $\Sigma \in \Xi$.

Indeed if we set $W = X^{[-T_2,0]}\big(\cup_\sigma W^s_{loc}(\sigma)\big)$, where the union above is taken over all singularities σ of Λ, then W is a compact sub-manifold of M with boundary, tangent to the center-stable sub-bundle $E^s \oplus E^X$. This means that W is transverse to any cross-section of Ξ.

Hence the intersection of W with any $\Sigma \in \Xi$ is a one-dimensional sub-manifold of Σ. Thus the number of connected components of the intersection is finite in each Σ. This means that there are finitely many points $x_1, \dots, x_k \in \Sigma$ such that

$$W \cap \Sigma \subset W^s(x_1, \Sigma) \cup \cdots \cup W^s(x_k, \Sigma).$$

Step 3 Points satisfying item (1) are contained in a finite number of stable leaves of each $\Sigma \in \Xi$.

We argue by contradiction. Assume that the set of points D of Σ sent by R into stable boundary points of some cross-section of Ξ is such that

$$L = \{W^s(x, \Sigma) : x \in D\}$$

has *infinitely many lines*. Note that D in fact equals L by Lemma 3.56. Then there exists an accumulation line $W^s(x_0, \Sigma)$. Since the number of cross-sections in Ξ is finite we may assume that $W^s(x_0, \Sigma)$ is accumulated by *distinct* $W^s(x_i, \Sigma)$ with $x_i \in D$ satisfying $R(x_i) \in W^s(z, \Sigma') \subset \partial^s \Sigma'$ for a fixed $\Sigma' \in \Xi$, $i \geq 1$. We may assume that x_i tends to x_0 when $i \to \infty$, that x_0 is in the interior of $W^s(x_0, \Sigma)$ and that the x_i are all distinct — in particular the points x_i do not belong to any periodic orbit of the flow since we can choose the x_i anywhere in the stable set $W^s(x_i, \Sigma)$.

As a preliminary result we show that $R(x_i) = X^{s_i}(x_i)$ is such that s_i is a bounded sequence in the real line. For otherwise $s_i \to \infty$ and this means, by definition of R, that the orbit of $X^{T_2}(x_i)$ is very close to the local stable manifold of some singularity σ of Λ and that $R(x_i)$ belongs to the outgoing cross-section near this singularity: $R(x_i) \in \Sigma^{o,\pm}_\sigma$. Hence we must have that $X^{s_i}(x_i)$ tends to the stable manifold of σ when $i \to \infty$ and that $R(x_i)$ tends to the stable boundary of $\Sigma^{o,\pm}_\sigma$. Since no point in any cross-section in Ξ is sent by R into this boundary line, we get a contradiction.

Now the smoothness of the flow and the fact that $W^s(z,\Sigma')$ is closed imply that $R(x_0) \in W^s(z,\Sigma')$ also since we have the following

$$R(x_0) = \lim_{i\to\infty} R(x_i) = \lim_{i\to\infty} X^{s_i}(x_i) = X^{s_0}(x_0) \quad \text{and} \quad \lim_{i\to\infty} s_i = s_0.$$

Moreover $R(W^s(x_0,\Sigma)) \subset W^s(z,\Sigma')$ and $R(x_0)$ is in the interior of the image $R(W^s(x_0,\Sigma))$, then $R(x_i) \in R(W^s(x_0,\Sigma))$ for all i big enough. This means that there exists a sequence $y_i \in W^s(x_0,\Sigma)$ and a sequence of real numbers τ_i such that $X^{\tau_i}(y_i) = R(y_i) = R(x_i)$ for all sufficiently big integers i. By construction we have that $x_i \neq y_i$ and both belong to the same orbit. Since x_i, y_i are in the same cross-section we get that $x_i = X^{\alpha_i}(y_i)$ with $|\alpha_i| \geq T_3$ for all big i.

However we also have that $\tau_i \to s_0$ because $R(y_i) = R(x_i) \to R(x_0)$, $y_i \in W^s(x_0,\Sigma)$ and $R \mid W^s(x_0,\Sigma)$ is smooth. Thus $|s_i - \tau_i| \to 0$. But $|s_i - \tau_i| = |\alpha_i| \geq T_3 > 0$. This is a contradiction.

This proves that D is contained in finitely many stable leaves.

Combining the three steps above we conclude the proof of the lemma.

□

Let Γ be the finite set of stable leaves of Ξ provided by Lemma 4.16 together with $\partial^s\Xi$. Then the complement $\Xi \setminus \Gamma$ of this set is formed by finitely many open strips where R is smooth. Each of these strips is then a connected component of the sets $\Sigma(\Sigma')$ for $\Sigma, \Sigma' \in \Xi$.

Integrability of the global Poincaré return time

We claim that *the Poincaré time τ is integrable with respect to the Lebesgue area measure on* Ξ. Indeed given $z \in \Xi$, the point $\hat{z} = X^{t_2}(z)$ either is inside a flow-box U_{σ_k} of a singularity σ_k, or not. In the former case, the time $\hat{z}$ takes to reach an outgoing cross-section $\Sigma^{o,\pm}_{\sigma_k}$ is bounded by the exit time function $\tau^{\pm}_{\sigma_k}$ of the corresponding flow-box, which is integrable, see Section 3.4.1. In the latter case, $\hat{z}$ takes a time of at most $2 \cdot T_3$ to reach another cross-section, by definition of T_3. Thus the Poincaré time on Ξ is bounded by $t_2 + 2 \cdot T_3$ plus a sum of finitely many integrable functions, one for each flow-box near a singularity, by finiteness of the number of singularities, of the number of cross-sections in Ξ and of the number of strips at each cross-section. This proves the claim.

Remark 4.17. Given $z \in \Sigma \in \Xi$ we write $\tau^k(z) = \tau(R^{k-1}(z)) + \cdots + \tau(z)$ for $k \geq 1$ and so $\tau = \tau^1$. Since

$$R^k\big(W^s(z,\Sigma)\big) \subset X^{\tau^k(z)}\big(W^s(z,\Sigma)\big) \subset X^{\tau^k(z)}(U),$$

the length $\ell\Big(R^k\big(W^s(z,\Sigma)\big)\Big)$ is uniformly contracted and $\tau^k(z) \to +\infty$ when $k \to +\infty$, we get that $R^k\big(W^s(z,\Sigma)\big) \subset \Sigma'$ for some $\Sigma' \in \Xi$ and

$$d\Big(R^k\big(W^s(z,\Sigma)\big), \partial^{cu}\Sigma'\Big) > \delta/2$$

for all big enough k, by the definition of U and of δ-adapted cross-section. (The distance $d(A,B)$ between two sets A,B means $\inf\{d(a,b) : a \in A, b \in B\}$.) We may assume that this property holds for all stable leaves $W^s(z,\Sigma)$, all $z \in \Sigma$ and every $\Sigma \in \Xi$ for all $k \geq k_0$, for some fixed big $k_0 \in \mathbb{N}$, by the uniform contraction property of R in the stable direction.

The Hölder property of the projection

From now on we assume that the flow $(X^t)_{t\in\mathbb{R}}$ is C^2. Under this condition it is well known [107, 146] that the stable leaf $W^s(x,\Sigma)$ for every $x \in \Sigma \in \Xi$ is a C^2 embedded disk and these leaves define a C^1 foliation $\mathcal{F}^s_\Sigma$ of each $\Sigma \in \Xi$ with a Hölder-C^1 holonomy (since the leaves are one-dimensional).

From Section 1.4.2 we know that in this setting the holonomy (projection) along transverse curves to $\mathcal{F}^s_\Sigma$ are $C^{1+\alpha}$ for some $0 < \alpha < 1$ which depends on X only, since they can be seen as maps between subsets of the real line.

Recall also Remark 3.48: the projections we are dealing with consist really on the composition of two projections. The first along the strong-stable leaves and the second along the flow to Σ. Since the flow is assumed to be C^2, the end result is a holonomy map in Σ which is Hölder-C^1.

Reduction to the quotient leaf space

We choose once and for all a C^2 *cu*-curve γ_Σ transverse to $\mathcal{F}^s_\Sigma$ in each $\Sigma \in \Xi$. Then the projection p_Σ along leaves of $\mathcal{F}^s_\Sigma$ onto γ_Σ is a $C^{1+\alpha}$ map. We set

$$I = \bigcup_{\Sigma,\Sigma' \in \Xi} \operatorname{int}\big(\Sigma(\Sigma')\big) \cap \gamma_\Sigma$$

and observe that by the properties of $\Sigma(\Sigma')$ obtained earlier in the beginning of this Section 4.2.1, the set I is diffeomorphic to a finite union of non-degenerate open intervals $I_1,\dots,I_m$ by a C^2 diffeomorphism and $p_\Sigma \mid p_\Sigma^{-1}(I)$ becomes a $C^{1+\alpha}$ submersion. Note that since Ξ is finite we can choose γ_Σ so that p_Σ has bounded derivative: there exists $\beta_0 > 1$ such that

$$\frac{1}{\beta_0} \le \left|Dp_\Sigma \mid \gamma\right| \le \beta_0 \textit{ for every cu-curve } \gamma \text{ inside any } \Sigma \in \Xi.$$

In particular, denoting the Lebesgue area measure over Ξ by λ^2 and the Lebesgue length measure on I by λ, we have $(p_\Sigma)_*\lambda^2 \ll \lambda$.

According to Lemma 3.56, Proposition 3.49 and Corollary 3.51 the Poincaré map $R : \Xi \to \Xi$ takes stable leaves of $\mathcal{F}^s_\Sigma$ inside stable leaves of the same foliation and is hyperbolic. In addition a *cu*-curve $\gamma \subset \Sigma$ is taken by R into a *cu*-curve $R(\gamma)$ in the image cross-section. Hence the map

$$f : I \to I \quad \text{given by} \quad I \ni z \mapsto p_{\Sigma'}\Big(R\big(W^s(z,\Sigma) \cap \Sigma(\Sigma')\big)\Big)$$

for $\Sigma,\Sigma' \in \Xi$ is a $C^{1+\alpha}$ map and for points in the interior of I_i, $i = 1,\dots,m$

$$\left|Df\right| = \left|D\big(p_{\Sigma'} \circ R \circ \gamma_\Sigma\big)\right| \ge \frac{1}{\beta_0}\cdot\sigma. \tag{4.5}$$

Thus choosing t_1 (and consequently t_2) big enough so that $\sigma/\beta_0 > 3/2 > 1$ in Proposition 3.49, we obtain that f is piecewise expanding. Moreover $|f'|^{-1} \mid I_j$ is a α-Hölder function since for all $x,y \in I_j$ we have

$$\frac{1}{|f'(x)|} - \frac{1}{|f'(y)|} \le \frac{|f'(x)-f'(y)|}{|f'(x)f'(y)|} \le \frac{C}{(3/2)^2}\cdot|x-y|^\alpha,$$

for some $0 < \alpha < 1$. Thus $f : I \to I$ is a $C^{1+\alpha}$ *piecewise expanding map.*

Remark 4.18. By Lemma 3.56 the Poincaré time τ is constant on stable leaves $W^s(x,\Sigma)$ for all $x \in \Sigma \in \Xi$. Thus there exists a return time function τ_I on I such that $\tau = \tau_I \circ p$, where $p : \Xi \to \gamma_\Xi$ is the joining of all p_Σ, $\Sigma \in \Xi$ and $\gamma_\Xi = \{\gamma_\Sigma : \Sigma \in \Xi\}$. The integrability of τ with respect to λ^2 (see Section 4.2.1) implies the λ-integrability of τ_I naturally since $(p_\Sigma)_*\lambda^2 \ll \lambda$ and $\tau_I \circ p = \tau$.

Existence and finiteness of acim's

It is well known [201, 210, 78] that C^1 piecewise expanding maps f of the interval such that $1/|f'|$ is of bounded variation, have finitely many absolutely continuous invariant probability measures whose basins cover Lebesgue almost all points of I.

Using an extension of the notion of bounded variation (defined below) it was shown in [86] that the results of existence and finiteness of absolutely continuous ergodic invariant measures can be extended to C^1 piecewise expanding maps f such that $g = 1/|f'|$ is α-Hölder for some $\alpha \in (0,1)$. These functions are of universally bounded variation, i.e.

$$\sup_{a=a_0<a_1<\cdots<a_n=b} \left(\sum_{j=1}^{n} \left|\varphi(a_i) - \varphi(a_{i-1})\right|^{1/\alpha} \right)^{\alpha} < \infty,$$

where the supremum is taken over all finite partition of the interval $I = [a,b]$. Moreover from [86, Theorem 3.2] the densities φ of the absolutely continuous invariant probability measures for f satisfy the following: there exists constants $A, C > 0$ such that

$$\int \operatorname{osc}(\varphi, \varepsilon, x)\, dx \le C \cdot \varepsilon^{\alpha} \quad \text{for all} \quad 0 < \varepsilon \le A,$$

where $\operatorname{osc}(\varphi, \varepsilon, x) = \operatorname{ess\,sup}_{y,z \in B(x,\varepsilon)} \left|\varphi(y) - \varphi(z)\right|$ and the essential supremo is taken with respect to Lebesgue measure. From this we can find a sequence $\varepsilon_n \to 0$ such that $\operatorname{osc}(\varphi, \varepsilon_n, \cdot) \xrightarrow[n\to\infty]{} 0$ (with respect to Lebesgue measure). This implies that $\operatorname{supp}(\varphi)$ *contains non-empty open intervals*.

Indeed, for a given small $\delta > 0$ let $\alpha > 0$ be so small and n so big that $W = \{\varphi > \alpha\}$ and $V = \{\operatorname{osc}(\varphi, \varepsilon_n, \cdot) > \alpha/2\}$ satisfy $\lambda(I \setminus W) < \delta$ and $\lambda(V) < \delta$. Then $\lambda(W \cap I \setminus V) > 1 - 2\delta > 0$. Let x be a Lebesgue density point of $W \cap I \setminus V$. Then there exists a positive Lebesgue measure subset of $B(x, \varepsilon_n)$ where $\varphi > \alpha$. By definition of $\operatorname{osc}(\varphi, \varepsilon_n, x)$ this implies that for Lebesgue almost every $y \in B(x, \varepsilon_n)$ we have $\varphi(y) > \alpha/2 > 0$, thus $B(x, \varepsilon_n) \subset \operatorname{supp}(\varphi)$.

In addition from [86, Theorem 3.3] there are finitely many ergodic absolutely continuous invariant probability measures $\upsilon_1, \ldots, \upsilon_l$ of f and every absolutely continuous invariant probability measure υ decomposes into a convex linear combination $\upsilon = \sum_{i=1}^{l} a_i \upsilon_i$. From [86, Theorem 3.2] considering any subinterval $J \subset I$ and the normalized Lebesgue measure $\lambda_J = (\lambda \mid$

$J)/\lambda(J)$ on J, then every weak* accumulation point of $n^{-1}\sum_{j=0}^{n-1} f_*^j(\lambda_J)$ is an absolutely continuous invariant probability measure υ for f (since the indicator function of J is of generalized $1/\alpha$-bounded variation). Hence the basin of the $\upsilon_1,\ldots,\upsilon_l$ cover I Lebesgue modulo zero: $\lambda\big(I\setminus(B(\upsilon_1)\cup\cdots\cup B(\upsilon_l))\big)=0$.

Note that from [86, Lemma 1.4] we also know that *the density* φ *of any absolutely continuous* f*-invariant probability measure is bounded from above*. In what follows we show how to use these properties to build physical measures for the flow.

4.2.2 Suspending Invariant Measures

Here we show how to construct an invariant measure for a transformation from an invariant measure for the quotient map obtained from a partition of the space. We show also that if the measure is ergodic on the quotient, then we also obtain ergodicity on the starting space.

In Section 4.2.3 we apply these results to the global Poincaré map R of a singular-hyperbolic attractor and its corresponding one-dimensional quotient map f.

Later we extend the transformation to a semi-flow through a suspension construction and show that each invariant and ergodic measure for the transformation corresponds to a unique measure for the semi-flow with the same properties.

In Section 4.2.5 we again apply these results to the transformation R to obtain physical measures for the suspension semiflow over R with roof function τ.

Reduction to the quotient map

Let Ξ be a compact metric space, $\Gamma\subset\Xi$ and $F:(\Xi\setminus\Gamma)\to\Xi$ be a measurable map. We assume that there exists a partition $\mathcal{F}$ of Ξ into measurable subsets, having Γ as an element, which is

- *invariant:* the image of any $\xi\in\mathcal{F}$ distinct from Γ is contained in some element η of $\mathcal{F}$;

- *contracting:* the diameter of $F^n(\xi)$ goes to zero when $n\to\infty$, uniformly over all the $\xi\in\mathcal{F}$ for which $F^n(\xi)$ is defined.

We denote $p : \Xi \to \mathcal{F}$ the canonical projection, i.e. p assigns to each point $x \in \Xi$ the atom $\xi \in \mathcal{F}$ that contains it. By definition, $A \subset \mathcal{F}$ is measurable if and only if $p^{-1}(A)$ is a measurable subset of Ξ and likewise A is open if, and only if, $p_\Sigma^{-1}(A)$ is open in Ξ. The invariance condition means that there is a uniquely defined map

$$f : (\mathcal{F} \setminus \{\Gamma\}) \to \mathcal{F} \quad \text{such that} \quad f \circ p = p \circ F.$$

Clearly, f is measurable with respect to the measurable structure we introduced in $\mathcal{F}$. We assume from now on that the leaves are sufficiently regular so that $\Xi/\mathcal{F}$ is a metric space with the topology induced by p.

Let μ_f be any probability measure on $\mathcal{F}$ invariant under the transformation f. For any bounded function $\psi : \Xi \to \mathbb{R}$, let $\psi_- : \mathcal{F} \to \mathbb{R}$ and $\psi_+ : \mathcal{F} \to \mathbb{R}$ be defined by

$$\psi_-(\xi) = \inf_{x\in\xi} \psi(x) \qquad \text{and} \qquad \psi_+(\xi) = \sup_{x\in\xi} \psi(x).$$

Lemma 4.19. *Given any continuous function $\psi : \Xi \to \mathbb{R}$, both limits*

$$\lim_n \int (\psi \circ F^n)_- \, d\mu_f \quad \textit{and} \quad \lim_n \int (\psi \circ F^n)_+ \, d\mu_f \tag{4.6}$$

exist, and they coincide.

Proof. Let ψ be fixed as in the statement. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|\psi(x_1) - \psi(x_2)| \le \varepsilon$ for all x_1, x_2 with $d(x_1, x_2) \le \delta$. Since the partition $\mathcal{F}$ is assumed to be contracting, there exists $n_0 \ge 0$ such that $\operatorname{diam}(F^n(\xi)) \le \delta$ for every $\xi \in \mathcal{F}$ and any $n \ge n_0$. Let $n + k \ge n \ge n_0$. By definition,

$$(\psi \circ F^{n+k})_-(\xi) - (\psi \circ F^n)_-(f^k(\xi)) = \inf(\psi \mid F^{n+k}(\xi)) - \inf(\psi \mid F^n(f^k(\xi))).$$

Observe that $F^{n+k}(\xi) \subset F^n(f^k(\xi))$. So the difference on the right hand side is bounded by

$$\sup\big(\psi \mid F^n(f^k(\xi))\big) - \inf\big(\psi \mid F^n(f^k(\xi))\big) \le \varepsilon.$$

Therefore

$$\left| \int (\psi \circ F^{n+k})_- \, d\mu_f - \int (\psi \circ F^n)_- \circ f^k \, d\mu_f \right| \le \varepsilon.$$

Moreover, one may replace the second integral by $\int(\psi\circ F^n)_-\,d\mu_f$, because μ_f is f-invariant.

At this point we have shown that $\{\int(\psi\circ F^n)_-\,d\mu_F\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. In particular, it converges. The same argument proves that $\{\int(\psi\circ F^n)_+\,d\mu_F\}_{n\geq 1}$ is also convergent. Moreover, keeping the previous notations,

$$0\leq(\psi\circ F^n)_+(\xi)-(\psi\circ F^n)_-(\xi)=\sup\big(\psi\mid F^n(\xi)\big)-\inf\big(\psi\mid F^n(\xi)\big)\leq\varepsilon$$

for every $n\geq n_0$. So the two sequences in (4.6) must have the same limit. The lemma is proved. □

Corollary 4.20. *There exists a unique probability measure μ_F on Ξ such that*

$$\int\psi\,d\mu_F=\lim\int(\psi\circ F^n)_-\,d\mu_f=\lim\int(\psi\circ F^n)_+\,d\mu_f.$$

for every continuous function $\psi:\Xi\to\mathbb{R}$. Besides, μ_F is invariant under F. Moreover the correspondence $\mu_f\mapsto\mu_F$ is injective.

Proof. Let $\hat\mu(\psi)$ denote the value of the two limits. Using the expression for $\hat\mu(\psi)$ in terms of $(\psi\circ F^n)_-$ we immediately get that

$$\hat\mu(\psi_1+\psi_2)\geq\hat\mu(\psi_1)+\hat\mu(\psi_2).$$

Analogously, the expression of $\hat\mu(\psi)$ in terms of $(\psi\circ F^n)_+$ gives the opposite inequality. So, the function $\hat\mu(\cdot)$ is additive. Moreover, $\hat\mu(c\psi)=c\hat\mu(\psi)$ for every $c\in\mathbb{R}$ and every continuous function ψ. Therefore, $\hat\mu(\cdot)$ is a linear real operator in the space of continuous functions $\psi:\Xi\to\mathbb{R}$.

Clearly, $\hat\mu(1)=1$ and the operator $\hat\mu$ is non-negative: $\hat\mu(\psi)\geq 0$ if $\psi\geq 0$. By the Riesz-Markov theorem, there exists a unique measure μ_F on Ξ such that $\hat\mu(\psi)=\int\psi\,d\mu_F$ for every continuous ψ. To conclude that μ_F is invariant under F it suffices to note that

$$\hat\mu(\psi\circ F)=\lim_n\int(\psi\circ F^{n+1})_-\,d\mu_f=\hat\mu(\psi)$$

for every ψ.

To prove that the map $\mu_f\mapsto\mu_F$ is injective, we note that if $\mu_F=\mu'_F$ are obtained from μ_f and μ'_f respectively, then for any continuous function $\varphi:\mathcal{F}\to\mathbb{R}$ we have that $\psi=\varphi\circ p:\Xi\to\mathbb{R}$ is continuous. But

$$\mu_f\big((\psi\circ F^n)_\pm\big)=\mu_f\big((\varphi\circ p\circ F^n)_\pm\big)=\mu_f\big((\varphi\circ f^n\circ p)_\pm\big)=\mu_f(\varphi\circ f^n)=\mu_f(\varphi)$$

for all $n \geq 1$ by the f-invariance of μ_f. Hence by definition

$$\mu_f(\varphi) = \mu_F(\psi) = \mu'_F(\psi) = \mu'_f(\varphi)$$

and so $\mu_f = \mu'_f$. This finishes the proof of the corollary. □

Remark 4.21. We note that $\int \psi \, d\mu_F = \lim_n \int (\psi \circ F^n)_\# \, d\mu_F$ for every continuous $\psi : \Xi \to \mathbb{R}$ and any choice of a sequence $(\psi \circ F^n)_\# : \mathcal{F} \to \mathbb{R}$ with

$$\inf(\psi \mid F^n(\xi)) \leq (\psi \circ F^n)_\#(\xi) \leq \sup(\psi \mid F^n(\xi)).$$

Moreover we can define $\int \psi \, d\mu_F$ for any measurable $\psi : \Xi \to \mathbb{R}$ such that

$$\lim_{n \to +\infty} \big(\sup(\psi \mid F^n(\xi)) - \inf(\psi \mid F^n(\xi)) \big) = 0$$

uniformly in $n \in \mathbb{N}$ and in $\xi \in \mathcal{F}$. This will be useful in what follows.

Lemma 4.22. *Let $\psi : \Xi \to \mathbb{R}$ be a continuous function and $\xi \in \mathcal{F}$ be such that*

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)_-(f^j(\xi)) = \int (\psi \circ F^k)_- \, d\mu_f$$

for every $k \geq 1$. Then $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \psi(F^j(x)) = \int \psi \, d\mu_F$ for every $x \in \xi$.

Proof. Let us fix ψ and ξ as in the statement. Then by definition of $(\psi \circ F^k)_\pm$ and by the properties of $\mathcal{F}$ we have

$$(\psi \circ F^k)_-\big(f^j(\xi)\big) \leq (\psi \circ F^k)\big(F^j(x)\big) \leq (\psi \circ F^k)_+\big(f^j(\xi)\big)$$

for all $x \in \xi$ and $j, k \geq 1$. Given $\varepsilon > 0$, by Corollary 4.20 there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\mu_F(\psi) - \frac{\varepsilon}{2} \leq \mu_f\big((\psi \circ F^k)_-\big) \leq \mu_f\big((\psi \circ F^k)_+\big) \leq \mu_F(\psi) + \frac{\varepsilon}{2}$$

and there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 = n_0(k)$

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} (\psi \circ F^k)_-\big(f^j(\xi)\big) - \mu_f\big((\psi \circ F^k)_-\big) \right| < \frac{\varepsilon}{2}.$$

Hence we have that for all $n \geq n_0(k)$

$$\mu_F(\psi) - \varepsilon \leq \frac{1}{n}\sum_{j=0}^{n-1}(\psi \circ F^k)(F^j(x))$$
$$= \frac{n+k}{n} \cdot \frac{1}{n+k}\sum_{j=0}^{n+k-1}(\psi \circ F^j)(x) - \frac{1}{n}\sum_{i=0}^{k-1}(\psi \circ F^j)(x) \leq \mu_F(\psi) + \varepsilon.$$

Since n can be made arbitrarily big and $\varepsilon > 0$ can be taken as small as we want, we have concluded the proof of the lemma. □

Corollary 4.23. *If μ_f is f-ergodic, then μ_F is ergodic for F.*

Proof. Since $\Xi/\mathcal{F}$ is a metric space with the topology induced by p we have that $C^0(\mathcal{F}, \mathbb{R})$ is dense in $L^1(\mathcal{F}, \mathbb{R})$ for the L^1-topology and $p : \Xi \to \mathcal{F}$ is continuous. Hence there exists a subset $\mathcal{E}$ of $\mathcal{F}$ with $\mu_f(\mathcal{E}) = 1$ such that the conclusion of Lemma 4.22 holds for a subset $E = p^{-1}(\mathcal{E})$ of Ξ. To prove the corollary it is enough to show that $\mu_F(E) = 1$.

Let $\varphi = \chi_E = \chi_{\mathcal{E}} \circ p$ and take $\psi_n : \mathcal{F} \to \mathbb{R}$ a sequence of continuous functions such that $\psi_n \to \chi_{\mathcal{E}}$ when $n \to +\infty$ in the L^1 topology with respect to μ_f. Then $\varphi_n = \psi_n \circ p$ is a sequence of continuous functions on Ξ such that $\psi_n \to \psi$ when $n \to +\infty$ in the L^1 norm with respect to μ_F.

Then it is straightforward to check that

$$\mu_F(\psi_n) = \lim_{k\to+\infty} \mu_f\Big((\psi_n \circ F^k)_-\Big) = \lim_{k\to+\infty} \mu_f(\varphi_n \circ f^k) = \mu_f(\varphi_n)$$

which converges to $\mu_f(\mathcal{E}) = 1$. Since $\mu_F(\psi_n)$ tends to $\mu_F(E)$ when $n \to +\infty$, we conclude that $\mu_F(E) = 1$, as we wanted. □

4.2.3 Physical measure for the global Poincaré map

Let us now apply these results (with R replacing F) to the case of the global Poincaré map for a singular-hyperbolic attractor.

From the previous results in Sections 4.2.1 and 4.2.2 the finitely many acim's $\upsilon_1, \ldots, \upsilon_l$ for the one-dimensional quotient map f uniquely induce R-invariant ergodic probability measures $\eta_1, \ldots, \eta_l$ on Ξ.

We claim that the basins of each $\eta_1, \ldots, \eta_l$ have positive Lebesgue area λ^2 on Ξ and cover λ^2 almost every point of $p^{-1}(I)$. Indeed the uniform contraction of the leaves $\mathcal{F}^s_\Sigma \setminus \Gamma$ provided by Lemma 3.56, implies that the forward time averages of any pair x,y of points in $\xi \in \mathcal{F} \setminus p(\Gamma)$ on continuous functions $\varphi : \Xi \to \mathbb{R}$ are equal

$$\lim_{n\to+\infty}\left[\frac{1}{n}\sum_{j=0}^{n-1}\varphi\big(R^j(x)\big)-\frac{1}{n}\sum_{j=0}^{n-1}\varphi\big(R^j(y)\big)\right]=0.$$

Hence $B(\eta_i) \supset p^{-1}\big(B(\upsilon_i)\big), i = 1,\dots,l$. This shows that $B(\eta_i)$ contains an entire strip except for a subset of λ^2-null measure, because $B(\upsilon_i)$ contains some open interval λ modulo zero. Since $p_*(\lambda^2) \ll \lambda$ we get in particular

$$\lambda^2\big(B(\eta_i)\big) > 0 \quad \text{and} \quad \lambda^2\Big(p^{-1}(I)\setminus\bigcup_{i=1}^{l}B(\eta_i)\Big) = p_*(\lambda^2)\Big(I\setminus\bigcup_{i=1}^{l}B(\upsilon_i)\Big) = 0,$$

showing that $\eta_1,\dots,\eta_l$ are physical measures whose basins cover $p^{-1}(I)$ Lebesgue almost everywhere. We observe that $p^{-1}(I) \subset \Xi$ is forward invariant under R, thus it contains $\Lambda \cap \Xi$.

4.2.4 Suspension flow from the Poincaré map

Let Ξ be a measurable space, Γ be some measurable subset of Ξ, and $F : (\Xi\setminus\Gamma) \to \Xi$ be a measurable map. Let $\tau : \Xi \to (0,+\infty]$ be a measurable function such that $\inf\tau > 0$ and $\tau \equiv +\infty$ on Γ.

Let $\sim$ be the equivalence relation on $\Xi\times[0,+\infty)$ generated by $(x,\tau(x)) \sim (F(x),0)$, that is, $(x,s) \sim (\tilde{x},\tilde{s})$ if and only if there exist

$$(x,s) = (x_0,s_0),\ (x_1,s_1),\ \dots,\ (x_N,s_N) = (\tilde{x},\tilde{s})$$

in $\Xi\times(0,+\infty)$ such that, for every $1 \le i \le N$

$$\begin{aligned} &\text{either} \quad x_i = F(x_{i-1}) \quad \text{and} \quad s_i = s_{i-1} - \tau(x_{i-1});\\ &\text{or} \quad x_{i-1} = F(x_i) \quad \text{and} \quad s_{i-1} = s_i - \tau(x_i). \end{aligned}$$

We denote by $V = \Xi\times[0,+\infty)/\sim$ the corresponding quotient space and by $\pi : \Xi \to V$ the canonical projection which induces on V a topology and a Borel σ-algebra of measurable subsets of V.

Definition 4.1. The *suspension of F with roof function (or return-time)* τ is the semi-flow $(X_\tau^t)_{t\ge0}$ defined on V by

$$X_\tau^t(\pi(x,s)) = \pi(x,s+t) \quad \text{for every } (x,s)\in\Xi\times[0,+\infty) \text{ and } t > 0.$$

It is easy to see that this is indeed well defined as in Section 1.2.1 on page 15. In what follows we write X^t instead of X^t_τ since τ is fixed and no ambiguity can arise.

Remark 4.24. If F is injective then we can also define

$$X^{-t}\big(\pi(x,s)\big) = \pi\big(F^{-n}(x), s+\tau(F^{-n}(x))+\cdots+\tau(F^{-1}(x))-t\big)$$

for every $x \in F^n(\Xi)$ and $0 < t \leq s+\tau(F^{-n}(x))+\cdots+\tau(F^{-1}(x))$. The expression on the right does not depend on the choice of $n \geq 1$. In particular, the restriction of the semi-flow $(X^t)_{t\geq 0}$ to the maximal invariant set

$$\Lambda = \left\{(x,t) : x \in \bigcap_{n\geq 0} F^n(\Xi) \text{ and } t \geq 0\right\}$$

extends, in this way, to a flow $(X^t)_{t\in\mathbb{R}}$ on Λ.

Let μ_F be any probability measure on Ξ that is invariant under F. Then the product $\mu_F \times dt$ of μ_F by Lebesgue measure on $[0,+\infty)$ is an infinite measure, invariant under the trivial flow $(x,s) \mapsto (x,s+t)$ in $\Xi \times [0,+\infty)$. In what follows we assume that the return time is integrable with respect to μ_F, i.e.

$$\mu_F(\tau) = \int \tau \, d\mu_F < \infty. \tag{4.7}$$

In particular $\mu_F(\Gamma) = 0$. Then we introduce the probability measure μ_X on V defined by

$$\int \varphi \, d\mu_X = \frac{1}{\mu_F(\tau)} \int \int_0^{\tau(x)} \varphi(\pi(x,t)) \, dt \, d\mu_F(x)$$

for each bounded measurable $\varphi : V \to \mathbb{R}$.

We observe that the correspondence $\mu_F \mapsto \mu_X$ defined above is injective. Indeed for any bounded measurable $\psi : \Xi \to \mathbb{R}$, defining φ on $\{x\} \times [0,\tau(x))$ to equal $\mu_F(\tau) \cdot \psi(x)/\tau(x)$ gives a bounded measurable map $\varphi : V \to \mathbb{R}$ (since $\inf \tau > 0$) such that $\mu_X(\varphi) = \mu_F(\psi)$. Hence if $\mu_X = \mu_X'$ then $\mu_F = \mu_F'$.

Lemma 4.25. *The measure μ_X is invariant under the semi-flow $(X^t)_{t\geq 0}$.*

Proof. It is enough to show that $\mu_X\big((X^t)^{-1}(B)\big) = \mu_X(B)$ for every measurable set $B \subset V$ and any $0 < t < \inf \tau$. Moreover, we may suppose that B is of the form $B = \pi(A \times J)$ for some $A \subset \Xi$ and J a bounded interval in

$[0, \inf(\tau \mid A))$. This is because these sets form a basis for the σ-algebra of measurable subsets of V.

Let B be of this form and (x,s) be any point in Ξ with $0 \le s < \tau(x)$. Then $X^t(x,s) \in B$ if and only if $\pi(x, s+t) = \pi(\tilde{x}, \tilde{s})$ for some $(\tilde{x}, \tilde{s}) \in A \times J$. In other words, $(x,s) \in (X^t)^{-1}(B)$ if and only if there exists some $n \ge 0$ such that

$$\tilde{x} = F^n(x) \quad \text{and} \quad \tilde{s} = s + t - \tau(x) - \cdots - \tau(F^{n-1}(x)).$$

Since $s < \tau(x)$, $t < \inf \tau$, and $\tilde{s} \ge 0$, it is impossible to have $n \ge 2$. So,

- either $\tilde{x} = x$ and $\tilde{s} = s + t$ (corresponding to $n = 0$),
- or $\tilde{x} = F(x)$ and $\tilde{s} = s + t - \tau(x)$ (corresponding to $n = 1$)

The two possibilities are mutually exclusive: for the first one (x,s) must be such that $s + t < \tau(x)$, whereas in the second case $s + t \ge \tau(x)$. This shows that we can write $(X^t)^-(B)$ as a disjoint union $(X^t)^-(B) = B_1 \cup B_2$, with

$$B_1 = \pi\big\{(x,s) : x \in A \text{ and } s \in (J-t) \cap [0, \tau(x))\big\}$$
$$B_2 = \pi\big\{(x,s) : F(x) \in A \text{ and } s \in (J + \tau(x) - t) \cap [0, \tau(x))\big\}.$$

Since $t > 0$ and $\sup J < \tau(x)$, we have $(J-t) \cap [0, \tau(x)) = (J-t) \cap [0, +\infty)$ for every $x \in A$. So, by definition, $\mu_X(B_1)$ equals

$$\int_A \ell\Big((J-t) \cap [0, \tau(x))\Big)\, d\mu_F(x) = \mu_F(A) \cdot \ell\Big((J-t) \cap [0, +\infty)\Big).$$

Similarly $\inf J \ge 0$ and $t < \tau(x)$ imply that

$$(J + \tau(x) - t) \cap [0, \tau(x)) = \tau(x) + (J-t) \cap (-\infty, 0).$$

Hence $\mu_X(B_2)$ is given by

$$\int_{F^{-1}(A)} \ell\Big((J-t) \cap (-\infty, 0)\Big)\, d\mu_F(x) = \mu_F(F^{-1}(A)) \cdot \ell\Big((J-t) \cap (-\infty, 0)\Big).$$

Since μ_F is invariant under F, we may replace $\mu_F(F^{-1}(A))$ by $\mu_F(A)$ in the last expression. It follows that

$$\mu_X\big((X^t)^{-1}(B)\big) = \mu_X(B_1) + \mu_X(B_2) = \mu_F(A) \cdot \ell\big((J-t)\big).$$

Clearly, the last term may be written as $\mu_F(A) \cdot \ell(J)$ which, by definition, is the same as $\mu_X(B)$. This proves that μ_X is invariant under the semi-flow and ends the proof. $\square$

Given a bounded measurable function $\varphi : V \to \mathbb{R}$, let $\hat{\varphi} : \Xi \to \mathbb{R}$ be defined by

$$\hat{\varphi}(x) = \int_0^{\tau(x)} \varphi(\pi(x,t))\,dt\,. \tag{4.8}$$

Observe that $\hat{\varphi}$ is integrable with respect to μ_F and by the definition of μ_X

$$\int \hat{\varphi}\,d\mu_F = \mu_F(\tau) \cdot \int \varphi\,d\mu_X\,.$$

Lemma 4.26. *Let $\varphi : V \to \mathbb{R}$ be a bounded function, and $\hat{\varphi}$ be as above. We assume that $x \in \Xi$ is such that $\tau(F^j(x))$ and $\hat{\varphi}(F^j(x))$ are finite for every $j \geq 0$, and also*

(a) $\displaystyle\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \tau(F^j(x)) = \int \tau\,d\mu_F$, *and*

(b) $\displaystyle\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \hat{\varphi}(F^j(x)) = \int \hat{\varphi}\,d\mu_F$.

Then $\displaystyle\lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(\pi(x,s+t))\,dt = \int \varphi\,d\mu_X$ *for every* $\pi(x,s) \in V$.

Proof. Let x be fixed, satisfying (a) and (b). Given any $T > 0$ we define $n = n(T)$ by

$$T_{n-1} \leq T < T_n \quad \text{where} \quad T_j = \tau(x) + \cdots \tau(F^j(x)) \text{ for } j \geq 0$$

Then using $(y,\tau(y)) \sim (F(y),0)$ we get

$$\begin{aligned}\frac{1}{T}\int_0^T \varphi(\pi(x,s+t))\,dt = \frac{1}{T}\Bigg[\sum_{j=0}^{n-1}\int_0^{\tau(F^j(x))} \varphi(\pi(F^j(x),t))\,dt \\ + \int_0^{T-T_{n-1}} \varphi(\pi(F^n(x),t))\,dt - \int_0^s \varphi(\pi(x,t))\,dt\Bigg]\,.\end{aligned} \tag{4.9}$$

Using the definition of $\hat{\varphi}$, we may rewrite the first term on the right hand side as

$$\frac{n}{T}\cdot\frac{1}{n}\sum_{j=0}^{n-1}\hat{\varphi}(F^j(x)). \tag{4.10}$$

Now we fix $\varepsilon > 0$. Assumption (a) and the definition of n imply that,

$$n \cdot \Big(\int \tau \, d\mu_F - \varepsilon \Big) \le T_{n-1} \le T \le T_n \le (n+1) \cdot \Big(\int \tau \, d\mu_F + \varepsilon \Big),$$

for every large enough n. Observe also that n goes to infinity as $T \to +\infty$, since $\tau(F^j(x)) < \infty$ for every j. So, for every large T,

$$\mu_F(\tau) - \varepsilon \le \frac{T}{n} \le \frac{n+1}{n} \mu_F(\tau) + \varepsilon \le \mu_F(\tau) + 2\varepsilon.$$

This proves that T/n converges to $\mu_F(\tau)$ when $T \to +\infty$. Consequently, assumption (b) implies that (4.10) converges to

$$\frac{1}{\mu_F(\tau)} \int \hat{\varphi} \, d\mu_F = \int \varphi \, d\mu_X \, .$$

Now we prove that the remaining terms in (4.9) converge to zero when T goes to infinity. Since φ is bounded

$$\left| \frac{1}{T} \int_0^{T-T_{n-1}} \varphi(\pi(F^n(x), t)) \, dt \right| \le \frac{T - T_{n-1}}{T} \sup |\varphi|. \tag{4.11}$$

Using the definition of n once more,

$$T - T_{n-1} \le T_n - T_{n-1} \le (n+1)\Big(\int \tau \, d\mu_F + \varepsilon \Big) - n\Big(\int \tau \, d\mu_F - \varepsilon \Big)$$

whenever n is large enough. Then

$$\frac{T - T_{n-1}}{T} \le \frac{\int \tau \, d\mu_F + (2n+1)\varepsilon}{n\big(\int \tau \, d\mu_F - \varepsilon \big)} \le \frac{4\varepsilon}{\int \tau \, d\mu_F - \varepsilon}$$

for all large enough T. This proves that $(T - T_{n-1})/T$ converges to zero, and then so does (4.11). Finally, it is clear that

$$\frac{1}{T} \int_0^s \varphi(\pi(x,t)) \, dt \to 0 \quad \text{when} \quad T \to +\infty.$$

This completes the proof of the lemma. □

Corollary 4.27. *If μ_F is ergodic then μ_X is ergodic.*

Proof. Let $\varphi : V \to \mathbb{R}$ be any bounded measurable function, and $\hat{\varphi}$ be as in (4.8). As already noted, $\hat{\varphi}$ is μ_F-integrable. It follows that $\hat{\varphi}(F^j(x)) < \infty$ for every $j \geq 0$, at μ_F-almost every point $x \in \Xi$. Moreover, by the Ergodic Theorem, condition (b) in Lemma 4.26 holds μ_F-almost everywhere. For the same reasons, $\tau(F^j(x))$ is finite for all $j \geq 0$, and condition (a) in the lemma is satisfied, for μ_F-almost all $x \in \Xi$.

This shows that Lemma 4.26 applies to every point x in a subset $A \subset \Xi$ with $\mu_F(A) = 1$. It follows that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(X^t(z))\, dt = \int \varphi\, d\mu_X$$

for every point z in $B = \pi(A \times [0, +\infty))$. Since the latter has $\mu_X(B) = 1$, we have shown that the Birkhoff average of φ is constant μ_X-almost everywhere. Then the same is true for any integrable function, as bounded functions are dense in $L^1(\mu_X)$. Thus μ_X is ergodic and the corollary is proved. □

4.2.5 Physical measures for the suspension

Using the results from Sections 4.2.3 and 4.2.4 it is straightforward to obtain ergodic probability measures $\nu_1, \ldots, \nu_l$ invariant under the suspension $(X_\tau^t)_{t \geq 0}$ of R with return time τ, corresponding to the R-physical probability measures $\eta_1, \ldots, \eta_l$ respectively.

Now we use Lemma 4.26 to show that each ν_i is a physical measure for $(X_\tau^t)_{t \geq 0}$, $i = 1, \ldots, l$. Let $x \in \Sigma \cap B(\nu_i)$ for a fixed $\Sigma \in \Xi$ and $i \in \{1, \ldots, l\}$. According to Remark 4.18 the return time τ_I on I is Lebesgue integrable, thus υ_i-integrable also since $\frac{d\upsilon_i}{d\lambda}$ is bounded. Hence τ is η_i-integrable by the construction of η_i from υ_i (see Section 4.2.2).

Lemma 4.26 together with the fact that η_i is physical for R, ensures that $B(\nu_i)$ contains the positive X_τ^t orbit of almost every point $(x, 0), x \in B(\nu_i)$, with respect to λ^2 on $B(\eta_i)$. If we denote by $\lambda^3 = \pi_*(\lambda^2 \times dt)$ a natural volume measure on V, then we get $\lambda^3\big(B(\nu_i)\big) > 0$.

This also shows that the basins $B(\nu_1), \ldots, B(\nu_l)$ cover λ^3-almost every point in $V_0 = \pi\big(p^{-1}(I) \times [0, +\infty)\big)$. Notice that this subset is a neighborhood of the suspension $\pi\big((\Lambda \cap \Xi \setminus \Gamma) \times [0, +\infty)\big)$ of $\Lambda \cap \Xi \setminus \Gamma$.

4.2.6 Physical measure for the flow

Here we extend the previous conclusions to the original flow, completing the proof of Theorem 4.3.

We relate the suspension $(X_\tau^t)_{t\geq 0}$ of R with return time τ to $(X_t)_{t\geq 0}$ in U as follows. We define

$$\Phi : \Xi \times [0,+\infty) \to U \quad \text{by} \quad (x,t) \mapsto X^t(x)$$

and since $\Phi\big(x,\tau(x)\big) = \big(R(x),0\big) \in \Xi \times \{0\}$, this map naturally defines a quotient map

$$\phi : V \to U \quad \text{such that} \quad \phi \circ X_\tau^t = X^t \circ \phi, \quad \text{for all} \quad t \geq 0, \tag{4.12}$$

through the identification $\sim$ from Section 4.2.4.

Let $\Xi_\tau = \{(x,t) \in (\Xi \setminus \Gamma) \times [0,+\infty) : 0 < t < \tau(x)\}$. Note that Ξ_τ is a open set in V and that $\pi \mid \Xi_\tau : \Xi_\tau \to \Xi_\tau$ is a homeomorphism (the identity). Then the map $\phi \mid \Xi_\tau$ is a local diffeomorphism into $V_0 = \phi\big(\Xi \times [0,+\infty)\big) \subset U$ by the natural identification given by π and by the Tubular Flow Theorem, since points in Ξ_τ are not sent into singularities of X. Notice that Ξ_τ is a full Lebesgue (λ^3) measure subset of V. Thus ϕ is a semi-conjugation modulo zero. Note also that the number of pre-images of ϕ is globally bounded by r_0 from Remark 4.15.

Therefore the measures ν_i constructed for the semiflow X_τ^t in the previous Section 4.2.5 define physical measures $\mu_i = \phi_*(\nu_i)$, $i = 1,\ldots,l$, whose basins cover a full Lebesgue (m) measure subset of V_0, which is a neighborhood of Λ. Indeed the semi-conjugacy (4.12) ensures that $\phi(B(\nu_i)) \subset B(\mu_i)$ and since ϕ is a local diffeomorphisms on a full Lebesgue measure subset, then

$$m\Big(\phi\big(B(\nu_1) \cup \cdots \cup B(\nu_l)\big)\Big) = 0.$$

Since $V_0 \subset U$ we have

$$W^s(\Lambda) = \bigcup_{t<0} X^t(V_0).$$

Moreover X^t is a diffeomorphism for all $t \in \mathbb{R}$, thus preserves subsets of zero m measure. Hence $\cup_{t<0} X^t\big(B(\mu_1) \cup \cdots \cup B(\mu_l)\big)$ has full Lebesgue measure in $W^s(\Lambda)$. In other words, Lebesgue (m) almost every point x in the basin $W^s(\Lambda)$ of Λ is such that $X^t(x) \in B(\mu_i)$ for some $t > 0$ and $i = 1,\ldots,l$.

Uniqueness of the physical measure

The set Λ is an attractor. According to our definition of attractor there exists $z_0 \in \Lambda$ such that $\{X^t(z_0) : t > 0\}$ is a dense regular orbit in Λ.

We prove uniqueness of the physical measure by contradiction, assuming that the number l of distinct physical measures is bigger than one. Then we can take distinct physical measures η_1, η_2 for R on Ξ associated to distinct physical measures μ_1, μ_2 for $X \mid \Lambda$. Then there are open sets $U_1, U_2 \subset \Xi$ such that

$$U_1 \cap U_2 = \emptyset \quad \text{and} \quad \lambda^2\big(B(\eta_i) \setminus U_i\big) = 0, \quad i = 1, 2.$$

For a very small $\zeta > 0$ we consider the open subsets $V_i = X^{(-\zeta,\zeta)}(U_i)$, $i = 1, 2$ of U such that $V_1 \cap V_2 = \emptyset$. According to the construction of μ_i we have $\mu_i(B(\mu_i) \setminus V_i) = 0$, $i = 1, 2$.

The transitivity assumption ensures that there are positive times $T_1 < T_2$ (exchanging V_1 and V_2 if needed) such that $X^{T_i}(z_0) \in V_i$, $i = 1, 2$. Since V_1, V_2 are open sets and $g = X^{T_2 - T_1}$ is a diffeomorphism, there exists a small open set $W_1 \subset V_1$ such that $g \mid W_1 : W_1 \to V_2$ is a C^1 diffeomorphism into its image $W_2 = g(W_1) \subset V_2$.

Now the C^1 smoothness of $g \mid W_1$ ensures that a full Lebesgue (m) measure subset of W_1 is sent into a full Lebesgue measure subset of W_2. By the definition of g and the choice of V_1, V_2, there exists a point in $B(\mu_1) \cap W_1$ whose positive orbit contains a point in $B(\mu_2) \cap W_2$, thus $\mu_1 = \mu_2$. Hence *singular-hyperbolic attractors have a unique physical probability measure* μ.

4.2.7 Hyperbolicity of the physical measure

For the hyperbolicity of the measure μ we note that

- the sub-bundle E^s is one-dimensional and uniformly contracting, thus on the E^s-direction the Lyapunov exponent is negative for every point in U;

- the sub-bundle E^{cu} is two-dimensional, dominates E^s, contains the flow direction and is volume expanding, thus by Oseledets Theorem [107, 204] the sum of the Lyapunov exponents on the direction of E^{cu} is given by $\mu_i(\log|\det DX_1 \mid E^{cu}|) > 0$. Hence there is a positive

Lyapunov exponent for μ_i-almost every point on the direction of E^{cu}, $i = 1,\dots,l$.

We already know from Section 1.4 that expanding direction in E^{cu} does not coincide with the flow direction $E^X_z = \{s\cdot X(z) : s\in\mathbb{R}\}$, $z\in\Lambda$, since E^X_z always has zero Lyapunov exponent for regular points for a smooth flow on a compact manifold.

This shows that at μ-almost every point z the Oseledets splitting of the tangent bundle has the form

$$T_zM = E^s_z \oplus E^X_z \oplus F_z,$$

where F_z is the one-dimensional measurable sub-bundle of vectors with positive Lyapunov exponent. The proof of Theorem 4.3 is complete.

4.2.8 Absolutely continuous disintegration of the physical measure

Here we prove Theorem 4.4. We let μ be a physical ergodic probability measure for a singular-hyperbolic attractor Λ of a C^2-flow in an open subset $U\subset M^3$, obtained through the sequence of reductions of the dynamics of the flow X^t to the suspension flow X^t_τ of the Poincaré map R and return time function τ, with corresponding X^t_τ-invariant measure ν obtained from the R-invariant measure η. In addition η is obtained through the ergodic invariant measure υ of the one-dimensional map $f : I\to I$. This is explained in Section 4.2.3. We know that μ is hyperbolic as explained in Section 4.2.6.

Let us fix $\delta_0 > 0$ small. Then by Pesin's non-uniformly hyperbolic theory [149, 58, 162] we know that there exists a compact subset $K\subset\Lambda$ such that $\mu(\Lambda\setminus K) < \delta_0$ and there exists $\delta_1 > 0$ for which every $z\in K$ admits a strong-unstable manifold $W^{uu}_{\delta_1}(z)$ with inner radius δ_1. We refer to this kind of sets as *Pesin's sets*. The *inner radius* of $W^{uu}_{\delta_1}(z)$ is defined as the length of the shortest smooth curve in this manifold from z to its boundary. Moreover $K\ni z\mapsto W^{uu}_{\delta_1}(z)$ is a continuous map $K\to\mathcal{E}^1(I_1,M)$ (recall the notations in Section 3.4.1).

The suspension flow X^t_τ defined on V in Section 4.2.4 is semi-conjugated to the X^t-flow on an open subset of U through a finite-to-1 local homeomorphism ϕ, defined in Section 4.2.6, which takes orbits to orbits and preserves time as in (4.12). Hence there exists a corresponding set $K' = \phi^{-1}(K)$ satisfying the same properties of K with respect to X^t_τ, where the constants δ_0,δ_1

are changed by at most a constant factor due to ϕ^{-1} by the compactness of K. In what follows we use the measure $\nu = (\phi^{-1})_*\mu$ instead of μ and write K for K'.

We fix a density point $x_0 \in K$ of $\nu \mid K$. We may assume that $x_0 \in \Sigma$ for some $\Sigma \in \Xi$. Otherwise if $x_0 \notin \Xi$, since $x_0 = (x,t)$ for some $x \in \Sigma$, $\Sigma \in \Xi$ and $0 < t < T(x)$, then we use $(x,0)$ instead of x_0 in the following arguments, but we still write x_0. Clearly the length of the unstable manifold through $(x,0)$ is unchanged due to the form of the suspension flow, at least for small values of δ_1. Since ν is given as a product measure on the quotient space V (see Section 4.2.5), we may assume without loss of generality that x_0 is a density point of η on $\Sigma \cap K$.

We set $W^u(x,\Sigma)$ to be the connected component of $W^u(x) \cap \Sigma$, the unstable manifold of x that contains x, for $x \in K \cap \Sigma$. Recall that $W^u(x) \subset \Lambda$ because Λ is an attracting set. Then $W^u(x,\Sigma)$ has inner radius bigger than some positive value $\delta_2 > 0$ for $x \in K \cap \Sigma$, which depends only on δ_1 and the angle between $W^{uu}_{\delta_1}(x)$ and $T_x\Sigma$.

Let us define $\mathcal{F}^s(x_0,\delta_2) = \{W^s(x,\Sigma) : x \in W^u(x_0,\Sigma)\}$ and the corresponding horizontal strip $F^s(x_0,\delta_2) = \cup_{\gamma \in \mathcal{F}^s(x_0,\delta_2)} \gamma$. Points $z \in F^s(x_0,\delta_2)$ can be specified using coordinates $(x,y) \in W^u(x_0,\Sigma) \times \mathbb{R}$, where x is given by $W^u(x_0,\Sigma) \cap W^s(z,\Sigma)$ and y is the length of the shortest smooth curve connecting x to z in $W^s(z,\Sigma)$. Let us consider

$$\mathcal{F}^u(x_0,\delta_2) = \{W^u(z,\Sigma) : z \in \Sigma \quad \text{and} \quad W^u(z,\Sigma) \quad \text{crosses} \quad F^s(x_0,\delta_2)\},$$

where we say that a curve γ *crosses* $F^s(x_0,\delta_2)$ if the trace of γ can be written as the graph of a map $W^u(x_0,\Sigma) \to W^s(x_0,\Sigma)$ using the coordinates outlined above. We stress that $\mathcal{F}^u(x_0,\delta_2)$ is not restricted to leaves through points of K.

We may assume that $F^u(x_0,\delta_2) = \cup \mathcal{F}^u(x_0,\delta_2)$ satisfies $\eta(F^u(x_0,\delta_2)) > 0$ up to taking a smaller $\delta_2 > 0$, since x_0 is a density point of $\eta \mid K \cap \Sigma$. Let $\hat{\eta}$ be the measure on $\mathcal{F}^u(x_0,\delta_2)$ given by

$$\hat{\eta}(A) = \eta\Big(\bigcup_{\gamma \in A} \gamma\Big) \quad \text{for every measurable set} \quad A \subset \mathcal{F}^u(x_0,\delta_2).$$

Proposition 4.28. *The measure $\eta \mid F^u(x_0,\delta_2)$ admits a disintegration into conditional measures η_γ along $\hat{\eta}$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$ such that $\eta_\gamma \ll \lambda_\gamma$, where λ_γ is the measure (length) induced on γ by the natural Riemannian* *measure* λ^2 *(area) on* Σ*. Moreover there exists* $D_0 > 0$ *such that*

$$\frac{1}{D_0} \le \frac{d\eta_\gamma}{d\lambda_\gamma} \le D_0, \quad \eta_\gamma\text{-almost everywhere for } \hat{\eta}\text{-almost every } \gamma.$$

This is enough to conclude the proof of Theorem 4.4 since both δ_0 and δ_2 can be taken arbitrarily close to zero, so that all unstable leaves $W^u(x,\Sigma)$ through almost every point with respect to η will support a conditional measure of η.

Indeed, to obtain the disintegration of ν along the center-unstable leaves that cross any small ball around a density point x_0 of K, we project that neighborhood of x_0 along the flow in negative time on a cross section Σ. Then we obtain the family $\{\eta_\gamma\}$, the disintegration of η along the unstable leaves $\gamma \in \mathcal{F}^u$ on a strip F^s of Σ, and consider the family $\{\eta_\gamma \times dt\}$ of measures on $\mathcal{F}^u \times [0,T]$ to obtain a disintegration of ν, where $T > 0$ is a fixed time slightly smaller than the return time of the points in the strip F^s, see Figure 4.7.

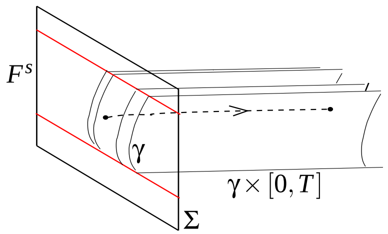

Figure 4.7: Center-unstable leaves on the suspension flow.

In fact, $\eta_\gamma \times dt \ll \lambda_\gamma \times dt$ and $\lambda_\gamma \times dt$ is the induced (area) measure on the center-unstable leaves by the volume measure λ^3 on V, and it can be given by restricting the volume form λ^3 to the surface $\gamma \times [0,T]$ which we write λ^3_γ, for $\gamma \in \mathcal{F}^u$. Thus by Proposition 4.28 and by the definition of ν, we have

$$\nu_\gamma = \eta_\gamma \times dt = \frac{d\eta_\gamma}{d\lambda_\gamma} \cdot \lambda^3_\gamma, \quad \gamma \in \mathcal{F}^u$$

and the densities of the conditional measures $\eta_\gamma \times dt$ with respect to λ^3_γ are also uniformly bounded from above and from below away from zero – we have left out the constant factor $1/\mu(\tau)$ to simplify the notation.

Since $\mu = \phi_* \nu$ and ϕ is a finite-to-1 local diffeomorphism when restricted to Ξ_τ, then μ also has an absolutely continuous disintegration along the center-unstable leaves. The densities on unstable leaves γ are related by the expression (where m_γ denotes the area measure on the center-unstable leaves induced by the volume form m)

$$\mu_\gamma = \phi_*(\nu_\gamma) = \phi_*\Big(\frac{d\eta_\gamma}{\lambda_\gamma}\cdot\lambda_\gamma^3\Big) = \Big(\frac{1}{\det D(\phi \mid \gamma\times[0,T])}\cdot\frac{d\eta_\gamma}{\lambda_\gamma}\Big)\circ\phi^{-1}\cdot m_\gamma,$$

for $\gamma \in \mathcal{F}^u$, which implies that the densities along the center-unstable leaves are uniformly bounded from above.

Indeed observe first that the number of pre-images of x under ϕ is uniformly bounded by r_0 from Remark 4.15, i.e. by the number of cross-sections of Ξ hit by the orbit of x from time 0 to time t_2. Moreover the tangent bundle of $\gamma\times[0,T]$ is sent by $D\phi$ into the bundle E^{cu} by construction and recalling that $\phi(x,t) = X^t(x)$ then, if e_1 is a unit tangent vector at $x\in\gamma$, $\hat{e}_1$ is the unit tangent vector at $\phi(x,0)\in W^u(x,\Sigma)$ and e_2 is the flow direction at (x,t) we get

$$\begin{aligned} &D\phi(x,t)(e_1) = DX_t\big(X^t(x)\big)(\hat{e}_1) \quad \text{and}\\ &D\phi(x,t)(e_2) = DX^t\big(X^t(x)\big)\big(X(x,0)\big) = X\big(X^t(x)\big). \end{aligned}$$

Hence $D\big(\phi \mid \gamma\times[0,T]\big)(x,t) = DX^t \mid E^{cu}_{\phi(x,t)}$ for $(x,t)\in\gamma\times[0,T]$ and so

$$|\det D\big(\phi \mid \gamma\times[0,T]\big)(x,t)| = J^c_t(x).$$

Now the volume expanding property of X^t along the center-unstable sub-bundle, together with the fact that the return time function τ is not bounded from above near the singularities, show that the densities of μ_γ are uniformly bounded from above throughout Λ but not from below. In fact, this shows that these densities will tend to zero close to the singularities of X in Λ.

This finishes the proof of Theorem 4.4 except for the proof of Proposition 4.28 and of $\operatorname{supp}(\mu) = \Lambda$, which we present in what follows.

4.2.9 Constructing the disintegration

Here we prove Proposition 4.28. We split the proof into several lemmas keeping the notations of the previous sections.

Let λ^2, $R : p^{-1}(I) \to \Xi$, $\mathcal{F}^u(x_0,\delta_2)$, $F^u(x_0,\delta_2)$ and η be as before, where $x_0 \in K \cap \Sigma$ is a density point of $\eta \mid K$ and K is a compact Pesin set. We write $\{\eta_\gamma\}$ and $\{\lambda^2_\gamma\}$ for the disintegrations of $\eta \mid F^u(x_0,\delta_2)$ and λ^2 along $\gamma \in \mathcal{F}^u(x_0,\delta_2)$.

Lemma 4.29. *Either $\eta_\gamma \ll \lambda^2_\gamma$ for $\hat\eta$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$, or $\eta_\gamma \perp \lambda^2_\gamma$ for $\hat\eta$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$.*

Proof. We start by assuming that the first item in the statement does not hold and proceed to show that this implies the second item. We write η for $\eta(F^u(x_0,\delta_2))^{-1} \cdot \eta \mid F^u(x_0,\delta_2)$ to simplify the notation in this proof.

Let us suppose that there exists $A \subset F^u(x_0,\delta_2)$ such that $\eta(A) > 0$ and $\lambda^2_\gamma(A) = 0$ for $\hat\eta$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$. Let $B = \cup_{k\geq 0} R^k(A)$. We claim that $\eta(B) = 1$.

Indeed, we have $R(B) \subset B$, then $B \subset R^{-1}(B)$ and $\big(R^{-k}(B)\big)_{k\geq 0}$ is a nested increasing family of sets. Since η is R-ergodic we have for any measurable set $C \subset \Xi$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \eta\big(C \cap R^{-j}(B)\big) = \eta(C)\cdot\eta(B). \tag{4.13}$$

But $\eta\big(\cup_{k\geq 0} R^{-k}(B)\big) = 1$ because this union is R-invariant and $\eta(B) = \eta\big(R^{-k}(B)\big) > 0$ by assumption, for any $k \geq 0$. Because the sequence is increasing and nested we have $\eta\big(R^{-k}(B)\big) \nearrow 1$. Hence from (4.13) we get that $\eta(C) = \eta(C)\cdot\eta(B)$ for all sets $C \subset X$. Thus $\eta(B) = 1$ as claimed.

Therefore $1 = \eta(B) = \int \eta_\gamma(B)\, d\hat\eta(\gamma)$ and so $\eta_\gamma(B) = 1$ for $\hat\eta$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$ since every measure involved is a probability measure.

We now claim that $\lambda^2_\gamma(B) = 0$ for $\hat\mu$-a.e. $\gamma \in \mathcal{F}^u(x_0,\delta_2)$. For if $R(A) \cap \gamma \neq \emptyset$ for some $\gamma \in \mathcal{F}^u(x_0,\delta_2)$, then $A \cap R^{-1}(\gamma) \cap F^u(x_0,\delta_2) \neq \emptyset$ and so it is enough to consider only $A \cap F^u_1$, where $F^u_1 = R^{-1}(F^u(x_0,\delta_2)) \cap F^u(x_0,\delta_2)$. But $\lambda^2_\gamma(A \cap F^u_1) \leq \lambda^2_\gamma(A) = 0$ thus

$$0 = \lambda^2_\gamma\big(R_0(A \cap F^u_1)\big) \geq \lambda^2_\gamma\big(R_0(A) \cap F^u(x_0,\delta_2)\big) = \lambda^2_\gamma(R_0(A))$$

for $\hat\eta$-a.e. γ since R_0 is piecewise smooth, hence a regular map. Therefore we get $\lambda^2_\gamma(R^k(A)) = 0$ for all $k \geq 1$ implying that $\lambda^2_\gamma(B) = 0$ for $\hat\eta$-a.e. γ.

This shows that η_γ is singular with respect to λ^2_γ for $\hat\eta$-a.e. γ. The proof is finished. □

Existence of hyperbolic times for f and consequences to R

Now we show that a positive measure subset of $\mathcal{F}^u(x_0,\delta_2)$ has absolutely continuous disintegrations, which is enough to conclude the proof of Proposition 4.28 by Lemma 4.29, except for the bounds on the densities.

We need the notion of *hyperbolic time* for the one-dimensional map f [5]. We know that this map is piecewise $C^{1+\alpha}$ and the boundaries Γ_0 of the intervals $I_1,\dots,I_n$ can be taken as a *singular set* for f (where the map is not defined or is not differentiable) which behaves like a *power of the distance to* Γ_0, as follows. Denoting by d the usual distance on the intervals I, there exist $B>0$ and $\beta>0$ such that

- $\frac{1}{B}\cdot d(x,\Gamma_0)^{\beta} \le \left|f'\right| \le B\cdot d(x,\Gamma_0)^{-\beta}$;
- $\left|\log|f'(x)|-\log|f'(y)|\right| \le B\cdot d(x,y)\cdot d(x,\Gamma_0)^{-\beta}$,

for all $x,y\in I$ with $d(x,y)<d(x,\Gamma_0)/2$. This is true of f since in Section 4.2.1 it was shown that $f' \mid I_j$ either is bounded from above and below away from zero, or else is of the form x^{β} with $\beta\in(0,1)$.

Given $\delta>0$ we define $d_\delta(x,\Gamma_0)=d(x,\Gamma_0)$ if $d(x,\Gamma_0)<\delta$ and 1 otherwise.

Definition 4.2. Given $b,c,\delta>0$ we say that $n\ge 1$ is a (b,c,δ)-hyperbolic time for $x\in I$ if

$$\prod_{j=n-k}^{n-1}\left|f'\big(f^j(x)\big)\right|^{-1}\le e^{-ck} \quad \text{and} \quad \prod_{j=n-k}^{n-1} d_\delta\big(f^j(x),\Gamma_0\big)\ge e^{-bk} \tag{4.14}$$

for all $k=0,\dots,n-1$.

Since f has positive Lyapunov exponent υ-almost everywhere, i.e.

$$\lim_{n\to+\infty}\frac{1}{n}\log\left|(f^n)'(x)\right|>0 \quad \text{for } \upsilon\text{-almost all } x\in I,$$

and $\frac{d\upsilon}{d\lambda}$ is bounded from above (where λ is the Lebesgue length measure on I), thus $|\log d(x,\Gamma_0)|$ is υ-integrable and for any given $\varepsilon>0$ we can find $\delta>0$ such that for υ-a.e. $x\in I$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}-\log d_\delta(f^j(x),\Gamma_0)=\int -\log d_\delta(x,\Gamma_0)\,d\upsilon(x)<\varepsilon.$$

This means that f is *non-uniformly expanding* and has *slow recurrence to the singular set*. Hence we are in the setting of the following result.

Theorem 4.30 (Existence of a positive frequency of hyperbolic times)**.** *Let $f : I \to I$ be a $C^{1+\alpha}$ map, behaving like a power of the distance to a singular set Γ_0, non-uniformly expanding and with slow recurrence to Γ_0 with respect to an absolutely continuous invariant probability measure υ. Then for $b, c, \delta > 0$ small enough there exists $\theta = \theta(b, c, \delta) > 0$ such that υ-a.e. $x \in I$ has infinitely many (b, c, δ)-hyperbolic times. Moreover if we write $0 < n_1 < n_2 < n_2 < \dots$ for the hyperbolic times of x then their asymptotic frequency satisfies*

$$\liminf_{N\to\infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \theta \quad \textit{for} \quad \upsilon\textit{-a.e. } x \in I.$$

Proof. A complete proof can be found in [5, Section 5] with weaker assumptions corresponding to Theorem C in that paper.

□

From now on we fix values of (b, c, δ) so that the conclusions of Theorem 4.30 are true.

We now outline the properties of these special times. For detailed proofs see [5, Proposition 2.8] and [3, Proposition 2.6, Corollary 2.7, Proposition 5.2].

Proposition 4.31. *There are constants $\beta_1, \beta_2 > 0$ depending on (b, c, δ) and f only such that, if n is (b, c, δ)-hyperbolic time for $x \in I$, then there are neighborhoods $W_k(x) \subset I$ of $f^{n-k}(x)$, $k = 1, \dots, n$, such that*

1. *$f^k \mid W_k(x)$ maps $W_k(x)$ diffeomorphically to the ball of radius β_1 around $f^n(x)$;*

2. *for every $1 \leq k \leq n$ and $y, z \in W_k(x)$*

$$d\big(f^{n-k}(y), f^{n-k}(z)\big) \leq e^{-ck/2} \cdot d\big(f^n(y), f^n(z)\big);$$

3. *for $y, z \in W_n(x)$*

$$\frac{1}{\beta_2} \leq \frac{\big|(f^n)'(y)\big|}{\big|(f^n)'(z)\big|} \leq \beta_2.$$

The conjugacy $p \circ R = f \circ p$ between the actions of the Poincaré map and the one-dimensional map on the space of leaves, together with the bounds on the derivative (4.5), enables us to extend the properties given by Proposition 4.31 to any *cu*-curve inside $B(\eta)$, as follows.

Let $\gamma : J \to \Xi$ be a *cu*-curve in $\Xi \setminus \Gamma$ such that $\gamma(s) \in B(\eta)$ for Lebesgue almost every $s \in J$, J a non-empty interval — such a curve exists since the basin $B(\eta)$ contains entire strips of some section $\Sigma \in \Xi$ except for a subset of zero area. Note that we have the following limit in the weak* topology

$$\lim_{n\to+\infty} \lambda_\gamma^n = \eta \quad \text{where} \quad \lambda_\gamma^n = \frac{1}{n}\sum_{j=0}^{n-1} R_*^j(\lambda_\gamma),$$

by the choice of γ and by an easy application of the Dominated Convergence Theorem.

Proposition 4.32. *There are constants $\kappa_0, \kappa_1 > 0$ depending on (b,c,δ) and $R_0, \beta_0, \beta_1, \beta_2$ only such that, if $x \in \gamma$ and n is big enough and a (b,c,δ)-hyperbolic time for $p(x) \in I$, then there are neighborhoods $V_k(x)$ of $R^{n-k}(x)$ on $R^{n-k}(x)(\gamma)$, $k = 1,\dots,n$, such that*

1. *$R^k \mid V_k(x)$ maps $V_k(x)$ diffeomorphically to the ball of radius κ_0 around $R^n(x)$ on $R^n(\gamma)$;*
2. *for every $1 \le k \le n$ and $y, z \in V_k(x)$*

$$d_{R^{n-k}(\gamma)}\big(R^{n-k}(y), R^{n-k}(z)\big) \le \beta_0 \cdot e^{-ck/2} \cdot d_{R^n(\gamma)}\big(R^n(y), R^n(z)\big);$$

3. *for $y, z \in V_n(x)$*

$$\frac{1}{\kappa_1} \le \frac{\big|D(R^n \mid \gamma)(y)\big|}{\big|D(R^n \mid \gamma)(z)\big|} \le \kappa_1;$$

4. *the inducing time of R^k on $V_k(x)$ is constant, i.e. $r^{n-k} \mid V_k(x) \equiv const.$.*

Here d_γ denotes the distance along γ given by the shortest smooth curve in γ joining two given points and λ_γ denotes the normalized Lebesgue length measure induced on γ by the area form λ^2 on Ξ.

Proof of Proposition 4.32. Let $x_0 = p(x)$ and $W_k(x_0)$ be given by Proposition 4.31, $k = 1,\dots n$. We have that $p(\gamma)$ is an interval in I and that

$p \mid \gamma : \gamma \to p(\gamma)$ is a diffeomorphism — we may take γ with smaller length if needed.

If n is big enough, then $W_n(x_0) \subset p(\gamma)$. Moreover the conjugacy implies that the following maps are all diffeomorphisms

$$\begin{array}{ccc} V_k(x) & \xrightarrow{R^k} & R^k(V_k(x)) \\ p \downarrow & & \downarrow p \\ W_k(x_0) & \xrightarrow{f^k} & B\big(f^k(x_0), \kappa_0\big) \end{array} ,$$

and the diagram commutes, where $V_k(x) = \big(p \mid R^k(\gamma)\big)^{-1}\big(W_k(x_0)\big)$, $k = 1, \ldots, n$, see Figure 4.8. Using the bounds (4.5) to compare derivatives we get $\kappa_0 = \beta_1/\beta_0$ and $\kappa_1 = \beta_0 \cdot \beta_2$.

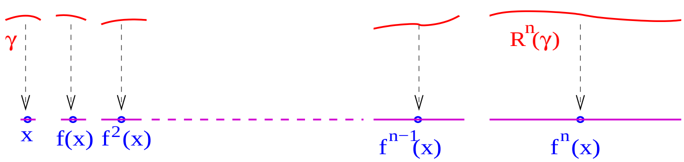

Figure 4.8: Hyperbolic times and projections.

To get item (4) we just note that by definition of (b, c, δ)-hyperbolic time none of the sets $W_k(x_0)$ may intersect Γ_0. According to the definition of Γ_0, this means that orbits through $x, y \in V_k(x)$ cannot cut different cross-sections in Ξ before the next return in time $\tau(x), \tau(y)$ respectively. Hence every orbit through $W_k(x_0)$ cuts the same cross-sections in its way to the next return cross-section. In particular the number of cross-section cuts is the same, i.e. $r \mid V_k(x)$ is constant, $k = 1, \ldots, n$. Hence by definition of r^k we obtain the statement of item (4) since $R(V_k(x)) = V_{k-1}(x)$ by definition. This completes the proof of the proposition. $\square$

Approximating η by push forwards of Lebesgue measure at hyperbolic times

We define for $n \geq 1$

$$H_n = \{x \in \gamma : n \text{ is a } (b, c/2, \delta)\text{-hyperbolic time for } p(x)\}.$$

As a consequence of items (1-2) of Proposition 4.32, we have that H_n is an open subset of γ and for any $x \in \gamma \cap H_n$ we can find a connected component γ^n of $R^n(\gamma) \cap B(R^n(x), \kappa_0)$ containing x such that $R^n \mid V_n(x) : V_n(x) \to \gamma^n$ is a diffeomorphism. In addition γ^n is a *cu*-curve according to Corollary 3.51, and by item (3) of Proposition 4.32 we deduce that

$$\frac{1}{\kappa_1} \leq \frac{d\big(R^n_*(\lambda_\gamma) \mid B(R^n(x), \kappa_0)\big)}{d\lambda_{\gamma^n}} \leq \kappa_1, \quad \lambda_{\gamma^n} - \text{a.e. on } \gamma^n, \tag{4.15}$$

where λ_{γ^n} is the Lebesgue induced measure on γ^n for any $n \geq 1$, if we normalize both measures so that $\big((R^n)_*(\lambda_\gamma) \mid B(R^n(x), \kappa_0)\big)(\gamma^n) = \lambda_{\gamma^n}(\gamma^n)$, i.e. their masses on γ^n are the same.

Moreover the set $R^n(\gamma \cap H_n)$ has an at most countable number of connected components which are diffeomorphic to open intervals. Each of these components is a *cu*-curve with diameter bigger than κ_0 and hence we can find a *pairwise disjoint family* γ_i^n *of* κ_0*-neighborhoods around* $R^n(x_i)$ *in* $R^n(\gamma)$*, for some* $x_i \in H_n$*, with maximum cardinality*, such that

$$\Delta_n = \bigcup_i \gamma_i^n \subset R^n\big(\gamma \cap H_n\big) \quad \text{and} \quad \big((R)^n_*(\lambda_\gamma) \mid \Delta_n\big)(\Delta_n) \geq \frac{1}{2\kappa_1} \cdot \lambda_\gamma(H_n). \tag{4.16}$$

Indeed since $R^n(\gamma \cap H_n)$ *is one-dimensional*, for each connected component the family Δ_n may miss a set of points of length at most equal to the length of one γ_i^n, for otherwise we would manage to include an extra κ_0-neighborhood in Δ_n. Hence we have in the worst case (assuming that there is only one set γ_i^n for each connected component)

$$\lambda_{\gamma^n}\big(R^n(\gamma \cap H_n) \setminus \Delta_n\big) \leq \lambda_{\gamma^n}\Big(\bigcup_i \gamma_i^n\Big) = \lambda_{\gamma^n}(\Delta_n)$$

so that

$$\lambda_{\gamma^n}(\Delta_n) \geq \frac{1}{2} \cdot \lambda_{\gamma^n}\big(R^n(\gamma \cap H_n)\big)$$

and the constant κ_1 comes from (4.15).

For a fixed small $\rho > 0$ we consider $\Delta_{n,\rho}$ given by the balls γ_i^n with the same center $x_{n,i}$ but a reduced radius of $\kappa_0 - \rho$. Then the same bound in (4.16) still holds with $2\kappa_1$ replaced by $3\kappa_1$.

We write D_n for the family of disks from $\cup_{j \geq 1} \Delta_j$ with the same expanding iterate (the disks with the same centers as the ones from $D_{n,\rho}$ but with their original size).

We define the following sequences of measures

$$\omega_\rho^n = \frac{1}{n}\sum_{j=0}^{n-1} R_*^j(\lambda_\gamma) \mid \Delta_{j,\rho} \quad \text{and} \quad \overline{\lambda}_\gamma^n = \lambda_\gamma^n - \omega_\rho^n, \quad n \geq 1.$$

Then any weak* limit point $\tilde{\eta} = \lim_k \omega_\rho^{n_k}$ for some subsequence $n_1 < n_2 < \ldots$ and $\overline{\eta} = \lim_k \overline{\lambda}_\gamma^{n_k'}$ (where n_k' may be taken as a subsequence of n_k), are R-invariant measures which satisfy $\eta = \tilde{\eta} + \overline{\eta}$.

We claim that $\tilde{\eta} \not\equiv 0$, thus $\eta = \tilde{\eta}$ as a consequence of the ergodicity of η. In fact, we can bound the mass of ω_ρ^n from below using the density of hyperbolic times from Theorem 4.30 and the bound from (4.16) through the following Fubini-Toneli-type argument. Write $\#_n(J) = \#J/n$ for any $J \subset \{0,\ldots,n-1\}$, the uniform discrete measure on the first n integers. Also set $\chi_i(x) = 1$ if $x \in H_i$ and zero otherwise, $i = 0,\ldots,n-1$. Then

$$\begin{aligned} \omega_\rho^n(M) &\geq \frac{1}{3\kappa_1 \cdot n}\sum_{j=0}^{n-1}\lambda_\gamma(H_j) = \frac{n}{3\kappa_1 n}\int\int \chi_i(x)\,d\lambda_\gamma(x)\,d\#_n(i) \\ &= \frac{1}{3\kappa_1}\int\int \chi_i(x)\,d\#_n(i)\,d\lambda_\gamma(x) \geq \frac{\theta}{6\kappa_1} > 0, \end{aligned}$$

for every n big enough by the choice of γ.

Approximating unstable curves by images of curves at hyperbolic times

We now observe that since $\eta(F^u(x_0,\delta_2)) > 0$ and x_0 is a density point of $\eta \mid F^u(x_0,\delta_2)$, then $\omega_\rho^n(F^u(x_0,\delta_2)) \geq c$ for some constant $c > 0$ for all big enough n. If we assume that $\delta_2 < \rho$, which poses no restriction, then we see that the cu-curves from $D_{j,\rho}$ intersecting $F^u(x_0,\delta_2)$ will cross this horizontal strip when we restore their original size. Thus the leaves $\cup_{j=0}^{n-1} D_j$ in the support of ω_0^n which intersect $F^u(x_0,\delta_2)$ cross this strip. Given any sequence γ^{n_k} of leaves in D_{n_k} crossing $F^u(x_0,\delta_2)$ with $n_1 < n_2 < n_3 < \ldots$, then there exists a C^1-limit leaf γ^∞ also crossing $F^u(x_0,\delta_2)$, by the Ascoli-Arzela Theorem. We claim that this leaf coincides with the unstable manifold of its points, i.e. $\gamma^\infty = W^u(x,\Sigma)$ for all $x \in \gamma^\infty$. This shows that the accumulation curves γ^∞ are defined independently of the chosen sequence γ^{n_k} of curves in Σ.

To prove the claim let us fix $l > 0$ and take a big k so that $n_k \gg l$. We note that for any distinct $x,y \in \gamma^\infty$ there are $x_k,y_k \in \gamma^{n_k}$ such that $(x_k,y_k) \to$

(x,y) when $k \to \infty$. Then for x_k, y_k there exists a neighborhood V_{n_k} of a point γ such that $\gamma^{n_k} = R^{n_k}(V_{n_k})$.

We take $j = n_k - l$. We can now write for some $w_k, z_k \in V_{n_k}$

$$\begin{aligned} d(x_k,y_k) &= d\Big(R^{n_k-j}\big(R^j(w_k)\big), R^{n_k-j}\big(R^j(z_k)\big)\Big) \\ &\geq \frac{e^{lc/4}}{\beta_0} \cdot d\big(R^{n_k-l}(w_k), R^{n_k-l}(z_k)\big). \end{aligned}$$

Note that each pair $R^{n_k-l}(w_k), R^{n_k-l}(z_k)$ belongs to a section $\Sigma_k \in \Xi$ and that $R^l\big(R^{n_k-l}(w_k)\big) = x_k$ and $R^l\big(R^{n_k-l}(z_k)\big) = y_k$. Letting $k \to \infty$ we obtain limit points $\big(R^{n_k-l}(w_k), R^{n_k-l}(z_k)\big) \to (w_l, z_l)$ in some section $\Sigma \in \Xi$ (recall that Ξ is a finite family of compact adapted cross-sections) satisfying

$$R^l(w_l) = x, \quad R^l(z_l) = y \quad \text{and} \quad d(w_l, z_l) \leq \beta_0 e^{-lc/4} \cdot d(x,y).$$

Since this is true for any $l > 0$ we conclude that y is in the unstable manifold of x with respect to R, i.e. $y \in W^u_R(x)$, thus $y \in W^u(x,\Sigma)$ by the following lemma. This proves the claim.

Lemma 4.33. *In the same setting as above, we have* $W^u_R(x) \subseteq W^u(x,\Sigma)$.

Notice that since both sets $W^u_R(x)$ and $W^u(x,\Sigma)$ are one-dimensional manifolds embedded in a neighborhood of x in Σ, then they coincide in a (perhaps smaller) neighborhood of x.

Proof. Let $y_0 \in W^u(x,\Sigma)$. Then there exists ε so that $z_0 = X_\varepsilon(y_0) \in W^{uu}(x)$, with $|\varepsilon|$ small by Remark 3.48 and tending to 0 when we take $y_0 \to x$. Let $t_l > 0$ be such that $X^{-t_l}(x) = w_l \in \Sigma$ for $l \geq 1$. Then we have

$$\operatorname{dist}\big(X^{-t_l}(z_0), X^{-t_l}(x)\big) \xrightarrow[l\to\infty]{} 0 \tag{4.17}$$

and so there exists ε_l such that $X^{\varepsilon_l - t_l}(z_0) = z_l = X^{\varepsilon_l + \varepsilon - t_l}(y_0) \in \Sigma$ with $|\varepsilon_l|$ small. Notice that (4.17) ensures that $|\varepsilon_l| \to 0$ also.

Hence there exists $\delta = \delta(\varepsilon, \varepsilon_l)$ satisfying $\delta \to 0$ when $(\varepsilon + \varepsilon_l) \to 0$ and also $d(z_l, w_l) < \delta$ for all $l \geq 1$. Since $R^l(z_l) = y_0$ we conclude that $y_0 \in W^u_R(x)$, finishing the proof. □

Upper and lower bounds for densities through approximation

We define $\mathcal{F}_\infty^u$ to be the family of all leaves γ^∞ obtained as C^1 accumulation points of leaves in

$$\mathcal{F}_n^u = \{\xi \in \cup_{j=0}^{n-1} D_j : \xi \quad \text{crosses} \quad F^s(x_0,\delta_2)\}.$$

We note that $\mathcal{F}_\infty^u \subset \mathcal{F}^u(x_0,\delta_2)$. Since for all n we have $\omega_0^n \geq \omega_\rho^n$ and so $\omega_0^n(\cup \mathcal{F}_n^u) > c$, we get that $\eta\big(\cup \mathcal{F}_\infty^u\big) \geq c$. By definition of $\mathcal{F}_n^u$ and by (4.15) we see that $\omega_0^n \mid F_n^u$ disintegrates along the partition $\mathcal{F}_n^u$ of $F_n^u = \cup \mathcal{F}_n^u$ into measures ω_ξ^n having density with respect to λ_ξ uniformly bounded from above and below, for almost every $\xi \in \mathcal{F}_n^u$.

To take advantage of this in order to prove Proposition 4.28 we consider a sequence of increasing partitions $(\mathcal{V}_k)_{k\geq 1}$ of $W^s(x_0,\Sigma)$ whose diameter tends to zero. This defines a sequence $\mathcal{P}_k$ of partitions of $\tilde{\mathcal{F}} = \cup_{0\leq n\leq\infty} \mathcal{F}_n^u$ as follows: we fix $k \geq 1$ and say that two elements $\xi \in \mathcal{F}_i^u, \xi' \in \mathcal{F}_j^u, 0 \leq i,j \leq \infty$ are in the same atom of $\mathcal{P}_k$ when both intersect $W^s(x,\Sigma)$ in the same atom of $\mathcal{V}_k$ and either $i,j \geq k$ or $i = j < k$.

If q is the projection $q : \tilde{\mathcal{F}} \to W^s(x_0,\Sigma)$ given by the transverse intersection $\xi \cap W^s(x_0,\Sigma)$ for all $\xi \in \tilde{\mathcal{F}}$, then $\tilde{\mathcal{F}}$ can be identified with a subset of the real line. Thus we may assume without loss that the union $\partial\mathcal{P}_k$ of the boundaries of $\mathcal{P}_k$ satisfies $\eta(\partial\mathcal{P}_k) = \hat{\eta}(\partial\mathcal{P}_k) = 0$ for all $k \geq 1$, by suitably choosing the sequence $\mathcal{V}_k$.

Upper and lower bounds for densities

Given $\zeta \in \tilde{\mathcal{F}}$ we write $p : F^u(x_0,\delta_2) \to \zeta$ the projection along stable leaves and ω for ω_0. Writing $\mathcal{P}_k(\zeta)$ for the atom of $\mathcal{P}_k$ which contains ζ, then since $\mathcal{P}_k(\zeta)$ is a union of leaves, for any given Borel set $B \subset \zeta$ and $n \geq 1$

$$\omega^n\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big) = \int \omega_\xi^n\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big)\, d\hat{\omega}^n(\xi) \tag{4.18}$$

through disintegration, where $\hat{\omega}^n$ is the measure on $\tilde{\mathcal{F}}$ induced by ω^n. Moreover by (4.15) and because each curve in $\tilde{\mathcal{F}}$ crosses $F^u(x_0,\delta_2)$

$$\frac{1}{\kappa_1\kappa_2} \cdot \lambda_\zeta(B) \leq \frac{1}{\kappa_1} \cdot \lambda_\xi\big(p^{-1}(B)\big) \leq \omega_\xi^n\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big) \tag{4.19}$$

$$\omega_\xi^n\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big) \leq \kappa_1 \cdot \lambda_\xi\big(p^{-1}(B)\big) \leq \kappa_1\kappa_2 \cdot \lambda_\zeta(B) \tag{4.20}$$

for all $n, k \geq 1$ and $\hat{\omega}^n$-a.e. $\xi \in \tilde{\mathcal{F}}$, where $\kappa_2 > 0$ is a constant such that

$$\frac{1}{\kappa_2} \cdot \lambda_\zeta \leq \lambda_\xi \leq \kappa_2 \cdot \lambda_\zeta \quad \text{for all} \quad \xi \in \tilde{\mathcal{F}},$$

which exists since the angle between the stable leaves in any $\Sigma \in \Xi$ and any *cu*-curve is bounded from below, see Figure 4.9.

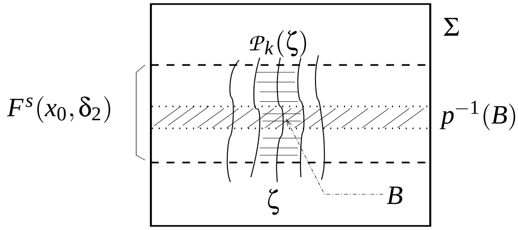

Figure 4.9: Leaves crossing $F^s(x_0, \delta_2)$ and the projection p.

Finally letting $\zeta \in \mathcal{F}_\infty^u$ and choosing B such that $\eta\big(\partial p^{-1}(B)\big) = 0$ (which poses no restriction), assuming that $\eta\Big(\partial\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big)\Big) = 0$ we get from (4.18), (4.19) and (4.20) for all $k \geq 1$

$$\frac{1}{\kappa_1\kappa_2} \cdot \lambda_\zeta(B) \cdot \hat{\eta}\big(\mathcal{P}_k(\zeta)\big) \leq \eta\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big) \leq \kappa_1\kappa_2 \cdot \lambda_\zeta(B) \cdot \hat{\eta}\big(\mathcal{P}_k(\zeta)\big) \tag{4.21}$$

by the weak* convergence of ω^n to η. Thus to conclude the proof we are left to check that $\eta\big(\partial\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big)\big) = 0$. For this we observe that $\mathcal{P}_k(\zeta) \cap p^{-1}(B)$ can be written as the product $q(\mathcal{P}_k(\zeta)) \times B$. Hence the boundary is equal to

$$\big(\partial q(\mathcal{P}_k(\zeta)) \times B\big) \cup \big(q(\mathcal{P}_k(\zeta)) \times \partial B\big) \subset q^{-1}\big(\partial q(\mathcal{P}_k(\zeta))\big) \cup p^{-1}(B)$$

and the right hand side has η-zero measure by construction.

This completes the proof of Proposition 4.28 since we have $\{\zeta\} = \cap_{k \geq 1}\mathcal{P}_k(\zeta)$ for all $\zeta \in \tilde{\mathcal{F}}$ and, by the Theorem of Radon-Nikodym, the bounds in (4.21) imply that the disintegration of $\eta \mid \cup \mathcal{F}_\infty^u$ along the curves $\zeta \in \mathcal{F}_\infty^u$ is absolutely continuous with respect to Lebesgue measure along these curves and with uniformly bounded densities from above and from below.

4.2.10 The support covers the whole attractor

Finally to conclude that $\mathrm{supp}(\mu) = \Lambda$ it is enough to show that $\mathrm{supp}(\mu)$ contains some *cu*-curve $\gamma : (a,b) \to \Sigma$ in some subsection $\Sigma \in \Xi$. Indeed, see Figure 4.10, letting $x_0 \in \Lambda \cap \Sigma$ be a point of a forward dense regular X-orbit and fixing $c \in (a,b)$ and $\varepsilon > 0$ such that $a < c - \varepsilon < c + \varepsilon < b$, then for any $\rho > 0$ there exists $t > 0$ satisfying $\mathrm{dist}\big(\gamma(c), X^t(x_0)\big) < \rho$. Since $W^s\big(X^t(x_0), \Sigma\big) \pitchfork \big(\gamma \mid (c-\varepsilon, c+\varepsilon)\big) = \{z\}$ (because γ is a *cu*-curve in Σ and $\rho > 0$ can be made arbitrarily small, where $\pitchfork$ means transverse intersection), then, by the construction of the adapted cross-section Σ (see Section 3.4.1), this means that $z \in W^s\big(X^t(x_0)\big)$. Hence the ω-limit sets of z and x_0 are equal to Λ. Thus $\mathrm{supp}(\mu) \supseteq \Lambda$ because $\mathrm{supp}(\mu)$ is X-invariant and closed, and $\Lambda \supseteq \mathrm{supp}(\mu)$ because Λ is an attracting set.

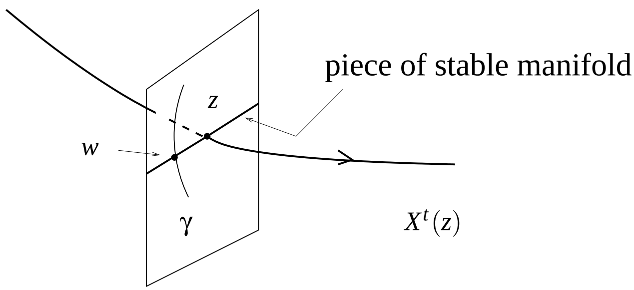

Figure 4.10: Transitiveness and support of the physical measure.

We now use (4.21) to show that $\hat\eta$-almost every $\gamma \in \tilde{\mathcal{F}}$ is contained in $\mathrm{supp}(\eta)$, which is contained in $\mathrm{supp}(\mu)$ by the construction of μ from η in Section 4.2.3. In fact, $\hat\eta$-almost every $\zeta \in \tilde{\mathcal{F}}$ is a density point of $\hat\eta \mid \tilde{\mathcal{F}}$ and so for any one ζ of these curves we have $\hat\eta\big(\mathcal{P}_k(\zeta)\big) > 0$ for all $k \geq 1$. Fixing $z \in \zeta$ and choosing $\varepsilon > 0$ we may find $k \geq 1$ big enough and a small enough open neighborhood B of z in ζ such that

$$\mathcal{P}_k(\zeta) \cap p^{-1}(B) \subset B(z,\varepsilon) \cap \Sigma \quad \text{and} \quad \eta\big(\mathcal{P}_k(\zeta) \cap p^{-1}(B)\big) > 0,$$

by the left hand side inequality in (4.21). Since $\varepsilon > 0$ and $z \in \zeta$ where arbitrarily chosen, this shows that $\zeta \in \mathrm{supp}(\eta) \subset \mathrm{supp}(\mu)$ and completes the proof of Theorem 4.4.

Chapter 5

Global dynamics of generic 3-flows

The results in Chapter 3 form the basis of a theory of flows on three-dimensional manifolds and paved the way for a global understanding of the dynamics of C^1-generic flows in dimension 3.

In this chapter we show that a generic C^1 vector field on a closed 3-manifold either has infinitely many sinks or sources or else is singular Axiom A without cycles. These results are contained in [130].

Theorem 5.1. *A generic vector field $X \in \mathfrak{X}^1(M)$ satisfies (only) one of the following properties:*

1. *X has infinitely many sinks or sources.*
2. *X is singular Axiom A without cycles.*

Singular Axiom A means that the non-wandering set of the vector field has a decomposition into finitely many compact invariant sets $\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_k$, each one being either a (uniformly) hyperbolic basic set (i.e. transitive, isolated and with a dense subset of periodic orbits) or a singular-hyperbolic attractor, or a singular-hyperbolic repeller with dense subset of periodic orbits (these are defined in Chapter 3, note that in this decomposition the singular-hyperbolic sets are transitive by definition).

An analogous result was proved by Mañé in [110] for C^1-generic diffeomorphisms on surfaces. For *conservative* flows on three-dimensional manifolds a related result was obtained recently by Bessa in [24], see Section 6.5 for more details.

It is known that a generic *non-singular* vector field $X \in \mathfrak{X}^1(M)$ either has infinitely many sinks or sources, or else is Axiom A without cycles, see Mañé [110] or Liao [97]. The robustness of the geometric Lorenz attractor obtained in Section 2.3 shows that this is not true in general if singularities are allowed. Allowing singularities we can improve this as follows. Let $\mathfrak{Y}^1(M) \subset \mathfrak{X}^1(M)$ be the set of vector fields that *cannot be* C^1 *approximated by homoclinic loops*. The Connecting Lemma 1.17 implies that any singularity of every $X \in \mathfrak{Y}^1(M)$ is separated from the non-wandering set. Using the arguments of Wen [205] and Hayashi [74] we conclude that a generic vector field in $\mathfrak{Y}^1(M)$ either has infinitely many sinks or sources or else it is Axiom A without cycles.

Recently Arroyo and Hertz [16] proved that every vector field in $\mathfrak{Y}^1(M)$ can be either approximated by one that is Axiom A without cycles, or exhibits a homoclinic tangency associated to a periodic orbit.

Let us describe some consequences Theorem 5.1. The first one is related with the abundance of three-dimensional vector fields exhibiting either attractors or repellers. As noted by Mañé in [110], a generic C^1 diffeomorphism in the 2-sphere S^2 does exhibit either sinks or sources. It is then natural to ask whether such a result is valid for C^1 vector fields in the 3-sphere S^3 instead of C^1 diffeomorphisms in S^2. The answer is negative as the following example shows.

Write $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider in $\mathbb{R}^3$ an unknotted two-torus T^2. Then the closure in S^3 of each connected component of $S^3 \setminus T^2$ is a solid two-torus. Consider a Lorenz attractor in one of the solid two-torus and a Lorenz repeller in the other. Since a fundamental domain for the Lorenz attractor (respectively repeller) is an unknotted solid two-torus, we can glue the two solid two-torus through the unknotted torus, obtaining a flow in S^3 whose non-wandering set equals the disjoint union of one Lorenz attractor and one Lorenz repeller. Such a flow is singular Axiom A, and it can not be approximated by vector fields with either sinks or sources. However from Theorem 5.1 we deduce

Corollary 5.2. *A generic vector field in* $\mathfrak{X}^1(M)$ *does exhibit either attractors or repellers.*

The second one is related with a conjecture by Palis in [141], see also Section 1.5, asserting the denseness of vector fields exhibiting a finite number of attractors whose basin of attraction forms a full Lebesgue measure subset. Theorem 5.1 gives an approach to this conjecture in the (open) set $\mathfrak{N}^1(M)$ of C^1 vector fields on a closed 3-manifold M which cannot be C^1 approximated by ones exhibiting infinitely many sinks or sources.

Corollary 5.3. *A generic vector field in $\mathfrak{N}^1(M)$ exhibits a finite number of attractors whose basins of attraction form an open and dense subset of M.*

This corollary follows from the no-cycle condition by the classical construction of filtrations adapted to the decomposition of the positive limit set of the flow, as the reader can easily see in [186, Chapter 2 & 3].

Using the filtration to isolate the dynamics around each basic piece of the singular Axiom A decomposition, since the critical elements are robustly hyperbolic nearby each basic piece (recall that singular-hyperbolicity is a robust property of the action of the flow on the tangent bundle), we obtain

Corollary 5.4. *A C^r singular Axiom A flow without cycles is in $\mathcal{G}^r(M)$, the interior of the set of C^r vector fields whose critical elements are hyperbolic, for any $r \geq 1$.*

We note that there exists a classification by Hayashi [73] of the C^1 interior of the set of diffeomorphisms whose periodic points are hyperbolic: they are Axiom A without cycles.

The corresponding result for vector fields is false since the Lorenz attractor is not uniformly hyperbolic. Indeed observe that we can easily construct a singular Axiom A vector field without cycles and with a singular basic set equivalent to the Lorenz attractor: just take the geometric Lorenz attractor constructed in Section 2.3, and embed and extend this flow to S^3 with a repelling singularity at the north pole and a sink at the south pole.

Proof of Theorem 5.1: The argument is based on the following result whose proof we postpone to Section 5.2. Denote by $\mathfrak{H}^r(M)$ the interior of the set of vector fields $X \in \mathfrak{X}^r(M)$ such that every periodic orbit and singularity of X is hyperbolic, for any $r \geq 1$.

Theorem 5.5. *Generic vector fields in $\mathfrak{H}^1(M)$ are singular Axiom A without cycles.*

Following the arguments of Mañé in [110], we can obtain Theorem 5.1 from Theorem 5.5. Indeed, let $\mathfrak{S}^1(M) \subset \mathfrak{X}^1(M)$ be the subset of C^1 vector fields such that every singularity of X is hyperbolic. Then $\mathfrak{G}^1(M)$ is open and dense in $\mathfrak{X}^1(M)$ by the local stability of hyperbolic critical elements. For $X \in \mathfrak{G}^1(M)$ define $A(X)$ to be the set of periodic orbits and singularities of X that are sinks or sources.

The set valued function $\mathfrak{X}^1(M) \ni X \mapsto \overline{A(X)} \in \mathcal{P}(M)$ is lower semicontinuous, again by the local stability of hyperbolic critical elements, where $\mathcal{P}(M)$ denotes the family of compact subsets of M endowed with the Hausdorff distance. Well known topological properties imply that the continuity points $\mathcal{O}$ of this map form a residual subset of $\mathfrak{G}^1(M)$.

This ensures that *every* $X \in \mathcal{O}$ *not satisfying the first item of Theorem 5.1 is in* $\mathfrak{H}^1(M)$.

Indeed for $X_0 \in \mathcal{O}$ with finitely many sinks and sources the set $A(X_0)$ is a finite collection of critical elements of X_0. Assume by contradiction that $X_0 \notin \mathfrak{H}^1(M)$. Then we can find a C^1-near vector field Y with a non-hyperbolic critical element ξ. Hence ξ is away from a neighborhood of $A(X_0)$. However $\mathcal{O} \subset \mathfrak{G}^1(M)$ and $\mathfrak{G}^1(M)$ is open, thus we can take $Y \in \mathfrak{G}^1(M)$. This guarantees that ξ is *not a singularity* of Y. Then the return map to a Poincaré section of the periodic orbit ξ has two eigenvalues, one of which has modulus 1. Perturbing Y we can find $Z \in \mathfrak{G}^1(M)$ arbitrarily C^1-close to Y (and to X_0) having either an attracting or repelling periodic orbit close to ξ. This contradicts the continuity of the set map $A(X)$ at X_0.

Now from Theorem 5.5 there exists a residual set $\mathfrak{R} \subset \mathfrak{H}^1(M)$ such that every vector field in $\mathfrak{R}$ is singular Axiom A without cycles. The class

$$\mathfrak{Y} = \left(\mathcal{O} \setminus \overline{\mathfrak{H}^1(M)}\right) \cup \left(\mathcal{O} \cap \mathfrak{R}\right)$$

is residual in $\mathfrak{X}^1(M)$ by construction (recall that $\mathfrak{G}^1(M)$ is open and dense in $\mathfrak{X}^1(M)$). Note that if $X_0 \in \mathfrak{Y}$ does not satisfy the first item of Theorem 5.1, then $X_0 \in \mathcal{O} \cap \mathfrak{R}$, since X_0 cannot belong to $\mathcal{O} \setminus \overline{\mathfrak{H}^1(M)}$ be the previous claim. This means that X_0 satisfies the second item of the statement of Theorem 5.1. □

5.1 Spectral decomposition

The *Spectral Decomposition Theorem for hyperbolic systems* plays a central role in dynamics. It ensures that an attracting hyperbolic set having dense periodic orbits must be a finite disjoint union of homoclinic classes.

Here we provide a version of this result in the setting of *singular-hyperbolic systems*, presented in Section 3. More precisely, we prove that an attracting singular-hyperbolic set with dense periodic orbits and a *unique singularity* is a finite union of transitive sets. Moreover, either the union is disjoint or the set contains finitely many distinct homoclinic classes. *For C^r-generic flows the union is in fact disjoint.* We shall follow [123].

The straightforward extension of the result on finite *disjoint* union of homoclinic classes from uniformly hyperbolic to singular hyperbolic attracting set with a dense subset of periodic orbits is false, as the next counterexample shows.

Consider a modification of the construction of the geometric Lorenz attractor given in Section 2.3, obtained by adding two singularities to the flow located at $W^u(\sigma)$ as indicated in Figure 5.1. This modification can be done in such a way that the new flow restricted to the cross section S has a C^∞ invariant stable foliation and the quotient map in the leaf space is piecewise expanding with a single discontinuity c as in the Lorenz case. The resulting attracting set can be proved to be a homoclinic class just as in the geometrical Lorenz case (see Section 2.3.6. In particular, such a set is transitive with *dense periodic orbits* and is also singular-hyperbolic by construction. Now glue two copies of this flow along the unstable manifold of the singularity σ obtaining the flow depicted in Figure 5.2. The resulting flow can be made C^∞ easily.

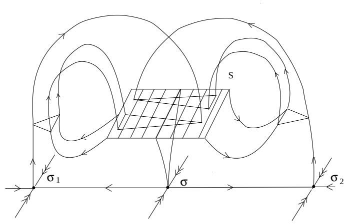

Figure 5.1: A modified geometric Lorenz attractor.

In this way we construct an attracting singular-hyperbolic set with dense periodic orbits and three equilibria which is not the *disjoint* union of homoclinic classes (although it is the union of two transitive sets). It is possible to construct a similar counter-example with a *unique* singularity, while this counterexample has *three* equilibria. However the construction in this case is more involved, see [20].

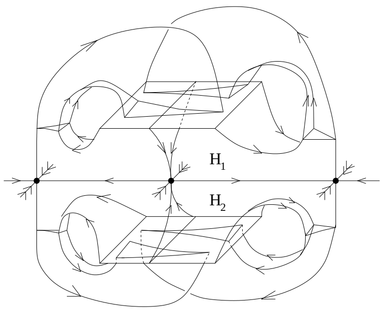

Figure 5.2: The counterexample.

The above counterexample shows that dealing with the spectral decomposition for singular-hyperbolic sets it is possible to obtain a *finite union of transitive sets* rather than a finite disjoint union of homoclinic classes. Next result shows that the former situation always occurs if the attracting set has only one singularity.

Theorem 5.6. *An attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite union of transitive sets.*

Proof. Split Λ into finitely many connected components. On the one hand such components are clearly attracting with dense periodic orbits and the non-singular ones are hyperbolic, hence transitive, by the Spectral Theorem for uniformly hyperbolic sets, see e.g. [190]. On the other hand, the singular component satisfies the conditions of Theorem 3.62. Hence this component is either transitive of the union of two homoclinic classes, which are transitive sets. Therefore Λ, which is the union of its connected components, is a finite union of transitive sets. □

Note that by a result of Morales [128] *every transitive set of a flow* Y, close to X, *contained in the isolating neighborhood* U *of a singular-hyperbolic attractor* Λ *of* X *must contain a singularity.* Therefore, since compact invariant subsets in Λ not containing singularities are hyperbolic and admit a spectral decomposition, and the number of singularities in U

is finite, the ω-limit set in U for Y has finitely many transitive pieces only, all of which are singular. *Hence near a singular-hyperbolic attractor the number of transitive pieces is robustly finite.*

It is natural to ask whether the union in Theorem 5.6 is disjoint. Recall that a vector field is *Kupka-Smale* if all its closed orbits are hyperbolic and their associated invariant manifolds are in general position, see Section 1.3.6.

Theorem 5.7. *An attracting singular-hyperbolic set, with dense periodic orbits and a unique singularity, of a Kupka-Smale vector field is a finite* disjoint *union of transitive sets.*

Proof. Let X be a Kupka-Smale vector field in a compact 3-manifold and Λ be an attracting singular-hyperbolic set of X with dense periodic orbits and a unique singularity σ. It suffices to prove that the connected component of Λ containing the singularity σ is transitive. By contradiction, suppose that this is not so.

On the one hand, by Theorem 3.68, we obtain a regular point a in the unstable manifold $W^u(\sigma)$ of σ such that $\omega(a)$ is a periodic orbit $\mathcal{O}(p)$. On the other hand, the unstable manifold $W^u(\sigma)$ is one-dimensional, so the vector field exhibits a non-transverse intersection between $W^u(\sigma)$ and $W^s(p)$, contradicting the choice of X in the Kupka-Smale class. □

Theorem 5.7 implies that the union in Theorem 5.6 is disjoint for *most* vector fields on closed 3-manifolds. Denote by $\mathfrak{R}^r(M)$ the subset of all vector fields $X \in \mathfrak{X}^r(M)$ for which every attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of X is a finite *disjoint* union of transitive sets. Standard C^1-generic arguments (see e.g. [34]) imply that $\mathfrak{R}^r(M)$ is residual in $\mathfrak{X}^r(M)$ when $r = 1$. The following corollary proves this assertion for all $r \geq 1$. The proof combines Theorem 5.7 with the classical Kupka-Smale Theorem (see e.g. [143]).

Corollary 5.8. *The class $\mathfrak{R}^r(M)$ is residual in $\mathfrak{X}^r(M)$ for every $r \geq 1$.*

Now consider the complement of $\mathfrak{R}^r(M)$. For a compact invariant subset Λ of a vector field X define the family $\mathcal{C}(\Lambda)$ of homoclinic classes contained in Λ. Note that if Λ is hyperbolic then $\mathcal{C}(\Lambda)$ is finite. We are able to give sufficient conditions for finiteness of $\mathcal{C}(\Lambda)$ when Λ is a singular-hyperbolic set.

Theorem 5.9. *Let Λ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of $X \in \mathfrak{X}^r(M)$. If Λ is* not *a disjoint union of transitive sets, then $\mathcal{C}(\Lambda)$ contains finitely many homoclinic classes only.*

Theorem 5.6 applies to the class of singular-hyperbolic vector fields introduced by Bautista in [22]. A vector field X is *singular-hyperbolic* if its non-wandering set $\Omega(X)$ has dense critical elements and if $A(X)$ denotes the union of the attracting and repelling closed orbits, then there is a *disjoint* union

$$\Omega(X) \setminus A(X) = \Omega_1(X) \cup \Omega_2(X),$$

where $\Omega_1(X)$ is a singular-hyperbolic set for X and $\Omega_2(X)$ is a singular-hyperbolic set for $-X$.

The class of singular-hyperbolic vector fields contains the Axiom A vector fields (uniformly hyperbolic) and the singular Axiom A example resembling the geometric Lorenz attractor, described after Corollary 5.4. An example of a singular-hyperbolic vector field in S^3 which is not Kupka-Smale can be derived from the example described in Figure 5.1, just weaken the contraction any one of the pair of saddle singularities which are accumulated by regular orbits only on one side. The following is a direct consequence of Theorems 5.6 and 5.7.

Corollary 5.10. *Let X be a singular-hyperbolic vector field* with a unique singularity *on a compact* 3*-manifold. If $\Omega_1(X)$ is attracting and $\Omega_2(X)$ is repelling, then $\Omega(X)$ is a finite union of transitive sets. If X is Kupka-Smale, then such an union is disjoint. In particular, the union is disjoint for a residual subset of vector fields in $\mathfrak{X}^r(M)$, $r \geq 1$.*

An example of a singular-hyperbolic vector field in S^3 satisfying the conditions of Corollary 5.10, without sinks nor sources, was described just before the statement of Corollary 5.2.

The extension of these results to general singular-hyperbolic attracting sets, with several singularities, is still work in progress.

Proof of Theorem 5.9. Suppose that Λ is *not* a disjoint union of transitive sets. Split Λ into finitely many connected components as before. It suffices to prove that the $\mathcal{C}(\Lambda')$ contains finitely many homoclinic classes for all connected components Λ' of Λ. On the one hand, for non-singular Λ' we have nothing to prove, since Λ' is uniformly hyperbolic by Proposition 3.9.

On the other hand, the singular connected component Λ_0 must contain $W^u(\sigma)$ (since it is connected), and $W^u(\sigma)$ has two connected components. Choose points a, a' in each one. Observe that Λ_0 must not be transitive by the assumptions on Λ. Then by Theorem 3.68 there are periodic orbits such that $\omega(a) = \mathcal{O}(a)$ and $\omega(a') = \mathcal{O}(a')$. By contradiction assume that there are infinitely many distinct homoclinic classes in Λ_0.

Then there exists an infinite sequence of pairwise distinct periodic orbits $O_n \subset \Lambda_0$ and an infinite sequence $z_n \in O_n$, so the set $A = \overline{\cup_n H(z_n)}$ must contain σ. For otherwise $A \subset \Lambda_0 \setminus \{\sigma\}$ is uniformly hyperbolic and the number of homoclinic classes would be finite.

Consider then $x_n \in \mathcal{O}_n$ such that $x_n \xrightarrow[n\to+\infty]{} \sigma$. Since x_n is not σ the accumulation on σ and the flow boxes near σ show that the orbit $\mathcal{O}_n$ accumulates also either a or a'. Without loss of generality, assume the former case is true.

Since $\omega_X(a) = \mathcal{O}(p)$ and $\mathcal{O}_n$ accumulates at a, we can find $z'_n \in \mathcal{O}_n$ passing close to O as indicated in Figure 5.3.

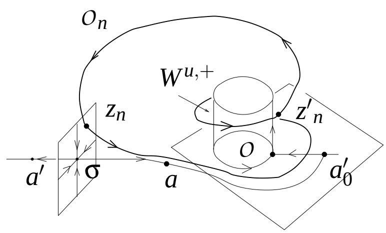

Figure 5.3: The accumulation on one of the components of $W^s(\sigma)$.

By the Inclination-Lemma we can assume that z'_n converges to a point either in one component $W^{s,+}$ of $W^s(\mathcal{O}) \setminus \mathcal{O}$, or in the other component $W^{s,-}$. Again suppose we are in the former case. By Lemma 3.67 and the Inclination Lemma we get that $z'_n \in \overline{W^{u,+}} = H^+$. But then $H(z'_n) = H(x) = H(z_n) = H^+$ for infinitely many n (since Theorem 1.12 ensures that every homoclinic class contains a dense subset of periodic orbits, all of which homoclinically related).

This contradicts the choice of z_n. □

5.2 A dichotomy for C^1-generic 3-flows

Here we present a proof of Theorem 5.5. We use the auxiliary Theorems 5.11 and 5.12 below. Recall the definition and properties of Lyapunov stable sets in Section 1.3.6.

The first theorem ensures that transitive Lyapunov stable sets containing singularities, if not equal to a critical element, are C^1-generically singular-hyperbolic sets.

Theorem 5.11. *For generic vector fields $X \in \mathfrak{X}^1(M)$, every nontrivial transitive Lyapunov stable set with singularities of X is singular-hyperbolic.*

The second result provides a way to obtain a singular-hyperbolic attractor from a singularity belonging to a Lyapunov stable set of a generic three-dimensional vector field. Together with the previous results, it asserts that the unstable manifold of a singularity accumulates on a singular-hyperbolic set containing the singularity.

Theorem 5.12. *Every Lyapunov stable singular-hyperbolic set with dense singular unstable branches of $X \in \mathfrak{X}^1(M)$ is an attractor of X.*

Here we say that a singular-hyperbolic set Λ has *dense singular unstable branches* if $\Lambda = \omega(x)$ for all $x \in W^u(\sigma) \setminus \{\sigma\}$ and for every singularity $\sigma \in \Lambda$.

Now we explain how Theorem 5.5 is a consequence of Theorems 5.11 and 5.12. For that we need some preliminary results. The first one gives a sufficient condition for a transitive Lyapunov stable set with singularities to have singular unstable branches.

Lemma 5.13. *For generic vector fields $X \in \mathfrak{X}^1(M)$, a transitive Lyapunov stable set with singularities Λ of X, such that the unstable manifold of every singularity in Λ is one-dimensional, has dense singular unstable branches.*

Proof. Generically we can assume that $X \in \mathfrak{X}^1(M)$ satisfies the properties presented in Section 1.3.6 (in particular X is Kupka-Smale). Let Λ be a transitive Lyapunov stable set of X, σ a singularity of Λ and $q \in W^u(\sigma) \setminus \{\sigma\}$.

On the one hand, since Λ is Lyapunov stable we have $W^u(\sigma) \subset \Lambda$ and in particular $\omega(q) \subset \Lambda$. On the other hand, we have that $\dim\left(W^u(\sigma)\right) = 1$ by assumption. Then $\omega(q)$ is Lyapunov stable by Property L5 in Section 1.3.6.

But Λ is transitive by construction and intersects $\omega(q)$, so by Lemma 1.14 we have $\Lambda \subset \omega(q)$. Then $\omega(q) = \Lambda$ and Λ has dense singular unstable branches as desired. □

The next one shows that the closure of the unstable manifold of a singularity accumulated by periodic orbits is transitive, provided that the unstable manifold is one-dimensional and its closure is Lyapunov stable.

Lemma 5.14. *Let $X \in \mathfrak{X}^1(M)$ and $\sigma \in S(X) \cap \overline{\mathrm{Per}(X)}$ be such that $W^u(\sigma)$ is one-dimensional and $\omega(q)$ is Lyapunov stable for every q in any of the branches of $W^u_X(\sigma) \setminus \{\sigma\}$. Then $\overline{W^u(\sigma)}$ is transitive.*

Proof. We have that $W^u_X(\sigma) \setminus \{\sigma\} = \mathcal{O}(q_1) \cup \mathcal{O}(q_2)$ for every q_1, q_2 belonging to different connected components of $W^u(\sigma) \setminus \{\sigma\}$.

On the one hand, since $\sigma \in \overline{\mathrm{Per}(X)}$ we can assume that $q_1 \in \overline{\mathrm{Per}(X)}$ without loss of generality. Then $\omega(q_1) \subset \overline{\mathrm{Per}(X)}$ by invariance. On the other hand, $\omega(q_1)$ is Lyapunov stable for X by assumption. These two properties imply that $\sigma \in \omega(q_1)$, since for $p_n \in \mathrm{Per}(X)$ with $p_n \xrightarrow[n \to +\infty]{} q_1$ we also have $X^{t_n}(p_n) \to \sigma$ for some sequence $t_n > 0$, and we can apply Lemma 1.14.

Therefore $\overline{W^u(\sigma)} \subset \omega(q_1)$ by the Lyapunov stability of $\omega(q_1)$ once more. But $\overline{W^u(\sigma)} \supset \omega(q_1)$ by construction, so we conclude that $\overline{W^u(\sigma)} = \omega(q_1)$. This shows that $\overline{W^u(\sigma)}$ is transitive. □

Using this we now show that any hyperbolic singularity accumulated by regular orbits of X is in a singular-hyperbolic attractor or repeller of the flow induced by X.

Theorem 5.15. *For generic $X \in \mathfrak{X}^1(M)$ every $\sigma \in S(X) \cap \overline{\mathrm{Per}(X)}$ belongs to a singular-hyperbolic attractor or a singular hyperbolic repeller.*

Proof. Let $X \in \mathfrak{X}^1(M)$ and σ be as in the statement. Since X is generic we can assume that σ is hyperbolic. Note that σ must be of saddle-type, for otherwise σ is either a sink or a source, and in any case no periodic orbit would approach σ. Hence either $W^u(\sigma_0)$ or $W^s(\sigma_0)$ is one-dimensional.

Suppose the former case is true. The latter case is the same for $-X$. Define $\Lambda = \overline{W^u(\sigma)}$. Property L3 in Section 1.3.6 implies that Λ is Lyapunov stable for X because X is generic. Property L5 then guarantees we are in the setting of Lemma 5.14 and so Λ is transitive.

Therefore Λ is a nontrivial transitive Lyapunov stable set of X. As X is generic, Theorem 5.11 ensures that Λ is singular-hyperbolic. By Theorem 3.5 we have that every singularity in Λ has one-dimensional unstable manifold. We conclude that Λ has dense singular unstable branches by Lemma 5.13, since X is generic. Then Λ is an attractor by Theorem 5.12. $\square$

Now we have the tools to complete the proof of Theorem 5.5 using all the previous results which assume Theorems 5.11 and 5.12.

Proof of Theorem 5.5: For $X \in \mathfrak{X}^1(M)$ denote by $S^*(X) = S(X) \cap \overline{\mathrm{Per}(X)}$ the (finite) set $\{\sigma_1, \ldots, \sigma_k\}$ of singularities accumulated by periodic orbits of X.

Theorem 5.15 ensures that for generic $X \in \mathfrak{X}^1(M)$ and for every $i = 1, \cdots, k$ there is a compact invariant set Λ_i of X such that $\sigma_i \in \Lambda_i$ and Λ_i is either a singular-hyperbolic attractor or a singular-hyperbolic repeller of X.

We claim that $H^* = \Omega(X) \setminus \cup_{i=1}^{k} \Lambda_i$ is a finite disjoint union of uniformly hyperbolic basic sets. Indeed $H^* \setminus S(X)$ is closed in M, for otherwise we can find a sequence of regular points x_n in H^* converging to some singularity $\sigma \in S(X) \setminus S^*(X)$. But Property L2 gives that $\Omega(X) = \overline{\mathrm{Per}(X) \cup S(X)}$, so σ is accumulated by periodic orbits because $S(X)$ is finite. Hence $H^* \setminus S(X)$ is a closed invariant subset of X without singularities. It is known, after Wen [205], that C^1 generically such sets are uniformly hyperbolic. Property L2 again ensures that $H^* = \overline{\mathrm{Per}(X) \cap H^*} \cup S(X) \setminus S^*(X)$. The Spectral Decomposition Theorem for uniformly hyperbolic sets now guarantees that H^* decomposes in finitely many basic pieces, together with finitely many singularities.

From this we have that $\Omega(X)$ splits into a disjoint union of compact invariant sets each one being either a hyperbolic basic set or a singular-hyperbolic attractor, or a singular-hyperbolic repeller. Hence X is a singular Axiom A vector field. For generic X we can also assume that the vector field is Kupka-Smale, thus there are no cycles between the transitive pieces in the above decomposition. The proof of Theorem 5.5 is complete depending on Theorems 5.11 and 5.12. $\square$

Proof of Theorem 5.11: Recall that there exists a residual subset $\mathcal{O}$ of the family $\mathfrak{G}^1(M)$ of vector fields whose singularities are hyperbolic, such that the map $X \in \mathfrak{S}^1(M) \mapsto A(X)$ restricted to $\mathcal{O}$ is continuous (see the arguments after the statement of Theorem 5.5 on page 241). Define $\mathcal{R} = \mathcal{O} \cap \mathfrak{H}^1(M)$ which is residual in $\mathfrak{H}^1(M)$.

Given $X \in \mathcal{R}$ and $\sigma \in S(X) \cap \Lambda$ for a non-trivial attractor Λ, observe that every vector field Y sufficiently C^1-close to X has no sources nor sinks nearby Λ, for otherwise we deduce a contradiction with the choice of X in the continuity set $\mathcal{O}$. All the critical elements of Y are also hyperbolic. Then Y is in the setting of Theorem 1.8, thus the Linear Poincaré Flow over $\Lambda \setminus S(X)$ is robustly dominated. This means that Λ is in the setting of Lemmas 3.22 and 3.28. Thus we have that *for $X \in \mathcal{R}$, if $\sigma \in S(X)$ belongs to a non-trivial attractor Λ of X, then σ is Lorenz-like for X and $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.*

Now let $X \in \mathcal{R}$ have a non-trivial transitive Lyapunov stable set Λ containing a singularity. The previous arguments ensure that Λ is in the setting of Theorem 3.30, hence Λ is a singular-hyperbolic attractor. □

Proof of Theorem 5.12: We need the following sufficient condition for a Lyapunov stable singular-hyperbolic set, with dense singular unstable branches, to be an attractor.

Lemma 5.16. *Let Λ be a Lyapunov stable singular-hyperbolic set with dense singular unstable branches of $X \in \mathfrak{X}^r(M)$, $r \geq 1$. If Λ admits an adapted cross-section Σ such that every point in the interior of Σ belongs to the stable leaf of some point of $\Lambda \cap \Sigma$, then Λ is an attractor.*

Proof. From Lemma 1.15 it is enough to prove that if x_n is a sequence converging to some point $p \in \Lambda$, then $\omega(x_n)$ is contained in Λ for every big enough n. Now $\omega(p)$ satisfies one of the following alternatives.

1. $\omega(p)$ contains a singularity σ of Λ.

   The orbits of x_n will have σ has an accumulation point. Hence the orbit of x_n also accumulates on some regular point q of the unstable manifold of σ. Since $\omega(q) = \Lambda$ by assumption, we have that for every big enough n the orbit of x_n crosses the interior of Σ. Then by the assumption on Σ we get $y \in \Lambda$ such that $\mathcal{O}(x_n) \subset W^s(y)$, that is $\omega(x_n) \subset \Lambda$ for all sufficiently big n.

2. $\omega(p)$ is far from singularities.

Take S an adapted cross-section to a point q of $\omega(p)$. Then for all big enough n the orbit of x_n crosses the interior of S at some point x_n' very close to q. Since $\omega(p)$ is uniformly hyperbolic by Proposition 3.9, the unstable manifold of q is well defined and $W^u(q)\cap S$ is a line in S crossing all stable manifolds of S in a neighborhood of q. Then x_n' belongs to some of these stable lines. Since $W^u(q)$ is inside Λ by Lyapunov stability, we see that x_n belongs to the stable manifold of some point of Λ. Again $\omega(x_n)\subset\Lambda$ for all sufficiently big n.

□

Now suppose that Λ *is not an attractor*. Then by Lemma 5.16 given any regular point $x\in\Lambda$ we can find an adapted-cross section Σ' such that the intersection $\Lambda\cap\Sigma'$ is contained in the interior of Σ. Indeed, $\Sigma\cap\Lambda$ contains z_0 such that $W^s(z_0,\Sigma)$ does not touch Λ, and then one of the connected components of $\Sigma\setminus W^s(z_0,\Sigma)$, which is also an adapted cross-section containing x, contains z_1 such that $W^s(z_1,\Sigma)\cap\Lambda=\emptyset$. The substrip Σ' between $W^s(z_0,\Sigma)$ and $W^s(z_1,\Sigma)$ only intersects Λ in its interior.

Cover Λ by finitely many flow boxes near singularities and tubular flow boxes through adapted cross-sections, around regular pieces of Λ, just as in Chapter 4, but with the family Ξ of adapted-cross sections chosen so that $\Lambda\cap\Xi\subset\mathrm{int}(\Xi)$.

Observe that since Λ is Lyapunov stable, we can find a neighborhood U of Λ such that $U\cap\Xi\subset\mathrm{int}(\Xi)$ and then another neighborhood $V\subset U$ of Λ satisfying $X^t(V)\subset U$ for all $t>0$. Then the Poincaré map R defined as in Section 3.4.1 between the sections of Ξ admits only finitely many discontinuity points, at the intersection of Ξ with a compact part of the stable manifolds of the singularities of Λ, since its image cannot touch the boundary of Ξ. We can choose the "waiting time t_2" of R so that the expansion rate on center-unstable cones is at least 4.

Let Ξ^* be the the subset of ingoing cross-sections near singularities of Ξ. Fix a point $x_0\in\Lambda\cap\Xi^*\setminus\cup\{W^s(\sigma):\sigma\in S(X)\cap\Lambda\}$ and a connected *cu*-curve γ_0 inside Ξ^* through x_0 not touching the lines of intersection of Ξ^* with the local stable manifold of the singularities. The image curve $R^i(\gamma_1)$, for $i>0$, is well defined until it returns to Ξ^*, because the image of R does not fall outside of $\mathrm{int}(\Xi)$. Let γ_2 be the next return to Ξ^*. Then its length $\ell(\gamma_2)$ is at least $4\cdot\ell(\gamma_1)$.

The image of γ_2 is well defined except perhaps at $\gamma_2\cap W^s_{loc}(\sigma)$ for some singularity σ of Λ. In this case we replace γ_2 by the lengthiest connected component of $\gamma_2 \setminus W^s_{loc}(\sigma)$. Then $\ell(\gamma_2) \geq 2 \cdot \ell(\gamma_1)$.

Inductively we obtain a sequence γ_n, $n \geq 1$ of larger and larger *cu*-curves in the interior Ξ^*, which is a finite collection of bounded cross-sections. Since the *cu*-curves cannot be tangent to the stable foliation, and so cannot curl inside Ξ, this is impossible.

This contradiction shows that Λ must be an attractor and concludes the proof of Theorem 5.12. $\square$

Chapter 6

Related results and recent developments

Here we present other related results about three-dimensional flows and some recent developments.

6.1 Lorenz-like attractors through the unfolding of singular cycles

It is natural to investigate whether an attractor resembling the Lorenz attractor can be obtained as a result of a bifurcation of a singular cycle of a given vector field.

Rychlik and Robinson studied the existence of Lorenz-like attractors in generic unfoldings of resonant double homoclinic loops, for flows in dimension three, in a series of works [184, 174, 175, 176]. Rychlik starts with a vector field with a Lorenz-like singularity σ with a connection between both branches of the unstable manifold of σ and the bidimensional stable manifold of σ, such that the singular cycle obtained is of *inclination-flip type*, see Section 2.2.2. Robinson considers a resonant connection, that is, the eigenvalues at σ are $\lambda_2 < \lambda_3 < 0 < \lambda_1$ but $\lambda_3 + \lambda_1 = 0$, that is, the singularity neither expands nor contracts volume in the central-unstable direction. In the setting of axially-symmetric vector fields, both cases are co-dimension two bifurcations.

Similarly Ushiki, Oka, Kokubu [200] and Dumortier, Kokubu, Oka [57] show that Lorenz-like attractor occur in the unfolding of local bifurcation of certain degenerate singularities. Analogously Bamón in [19] obtains attractor resembling the Lorenz attractor in higher dimensions unfolding cycles associated to degenerate singularities.

An extension of the results of Robinson, in dimension 3, was obtained in [134].

6.2 Contracting Lorenz-like attractors

Rovella [140, 179] presented a parametrized model similar to the geometric Lorenz model described in Section 2.3 which exhibits an attractor for a positive Lebesgue measure subset of the parameter space. This attractor contains a singularity with three real eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ but, unlike a Lorenz-like singularity, we have $\lambda_1 < -\lambda_3$, that is, the central-unstable direction at the singularity is *volume contracting*.

This construction is very similar to the geometric Lorenz model, amounting essentially to replace the one-dimensional map f, whose graph is presented in Figure 2.25 on page 76 and obtained through projecting along the contracting foliation, by the map g whose graph can be any of the ones sketched in Figure 6.1.

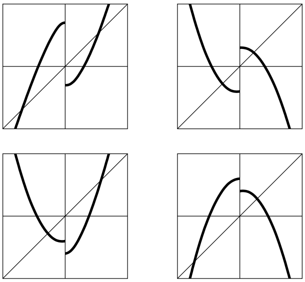

Figure 6.1: The one-dimensional map for the contracting Lorenz model

The parameters of these maps describe the vertical coordinates of the critical points of each branch of continuity of the maps in Figure 6.1.

Metzger in [116, 115] proved the existence of a physical measure and its stability for the contracting Lorenz model. More recently Metzger and Morales, in [117], showed that the contracting Lorenz attractor is also a homoclinic class.

6.2.1 Contracting Lorenz-like attractors through the unfolding of singular cycles

Recently in [135] the authors proved that, similarly to the (expanding) geometric Lorenz attractors, contracting Lorenz-like attractors can be obtained unfolding a resonant double homoclinic connection with a *contracting Lorenz-like singularity* σ, i.e. the eigenvalues are $\lambda_2 < \lambda_3 < 0 < \lambda_1$ with $\lambda_3 + \lambda_1 < 0$, see Figure 6.2.

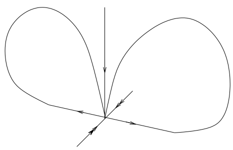

Figure 6.2: A double homoclinic connection

Note that contracting Lorenz-like attractors persist only in a measure theoretical sense. In this setting the authors prove the existence of non-degenerate two-parameter family of vector fields generically unfolding the singular cycle described above, which admits a positive Lebesgue measure subset of parameters such that the corresponding flow exhibits a contracting Lorenz-like attractor.

6.3 More on singular-hyperbolicity

Vivier, in [203], extended the results of Doering [53] to higher dimensions, showing that a C^1 robustly transitive vector field on a compact boundaryless n-manifold, with $n \geq 3$, does not have any singularity. Note that Doering was able to prove, for 3-manifolds, that such vector fields are Anosov. Similarly Vivier showed that, for n-dimensional manifolds, robustly transitive vector fields admit a global dominated splitting.

6.3.1 Attractors that resemble the Lorenz attractor

In Chapter 3 we presented several results showing that every robust attractor of a 3-flow containing equilibria looks like a geometric Lorenz attractor.

Shil'nikov and Turaev, in [199], present an example of a 4-dimensional quasi-attractor and study its perturbations. The quasi-attractor is pseudo-hyperbolic, contains a singularity with a complex eigenvalue and can not be destroyed by small perturbations of the system.

Lorenz, in [104], reports a careful numerical study of what seems to be a strange (chaotic) attractor in four dimensions for a system of 2-degree polynomial equations. Rovella in [179] proves existence and persistence of *contracting Lorenz attractors*, that is, with the contracting eigenvalue condition $-\lambda_3 > \lambda_1$, see the following Section 6.2.

In [84] the authors prove that certain parametrized families of one-dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way. Later in [10] it was proved that these maps have a unique physical (absolutely continuous) measure which varies continuously in the space of parameters with very nice statistical properties. An application of the methods developed in these works yields a proof of existence and even persistence of global spiral attractors for smooth flows in three dimensions, to be given in [49].

In [150] Pesin proposed abstract models for attractors with singularities, called generalized hyperbolic attractors, and studied their properties.

Bonatti, Pumariño and Viana, in [35], construct a multidimensional Lorenz-like attractor that is C^1-robust and contains a singularity with at least two positive eigenvalues. Their construction works in dimensions greater or equal to 5. They also obtain a physical measure for these attractors for an open set of flows in the C^∞ topology.

More recently Metzger and Morales [114] introduced the class of *sectionally hyperbolic vector fields* on n-manifolds containing the singular-hyperbolic systems on 3-manifolds, the multidimensional Lorenz attractors of [35] and the C^1 robustly transitive sets in Li, Gan and Wen [96].

An attractor Λ of a vector field X is *sectionally hyperbolic*, if there exists a splitting $E^s_\Lambda \oplus E^{cu}_\Lambda$ of the tangent bundle of Λ which is partial hyperbolic and the central-unstable bundle is 2-*sectionally volume expanding*, i.e. there are $K, \lambda > 0$ such that for every $x \in \Lambda$ and for every bidimensional plane L contained in E^{cu}_x one has $\left|\det(DX^t \mid L)\right| \geq Ke^{\lambda t}$ for all $t > 0$.

Moreover in [114] the authors show that if an attractor Λ is C^1 robustly transitive and

- *strongly homogeneous*, i.e. every hyperbolic periodic orbit of every C^r nearby vector field has the same index, that is, the dimension of their stable manifold is constant on a neighborhood of the vector field;

then all singularities of Λ are hyperbolic and Λ is sectionally-hyperbolic.

6.3.2 Topological dynamics

Some aspects of the topological dynamics of the geometric model were studied by Komuro in [90, 91], where it was proved that most geometrical Lorenz attractors do not have the shadowing property, and their expansive properties are investigated. In [89] the author finds a topological invariant for the Lorenz attractor allowing him to exhibit an uncountable number of non-homeomorphic Lorenz attractors in the unfolding of a certain homoclinic loop. In [28] the knot type of the geometric model is analyzed, and in [61] the Lorenz attractor is used to investigate the existence of flows realizing all links and knots as periodic orbits in 3-manifolds and an explicit ordinary differential equation with such properties is exhibited. The reader is advised to consult the survey [153].

Morales in [128] shows that a vector field Y which is C^1 close to a given one X in a 3-manifold exhibiting a singular-hyperbolic attractor Λ must have at least one singularity, and the number of attractors of Y near Λ is bounded above by the number of singularities of X in Λ.

Bautista showed in [21] that the geometric Lorenz model is a homoclinic class and, together with Morales, proved in [23] that every singular-hyperbolic attractor admits a (hyperbolic) periodic orbit.

Arroyo and Hertz, in [16], have advanced a significant step towards an affirmative answer to the Palis Conjecture, see Section 1.5. They show that any C^1 vector field on a compact 3-manifold can be approximated by another one showing one of the following phenomena:

- uniform hyperbolicity with the non-cycle condition,
- a homoclinic tangency, or
- a singular cycle.

Arroyo and Pujals, in [15], show that a singular-hyperbolic attractor has a dense set of periodic orbits and is the homoclinic class associated to one of this orbits. These results show that singular-hyperbolic attractors do play the same role as the basic pieces of Smale's Spectral Decomposition. They also provide a criterion for C^1 robustness of singular-hyperbolic attractors which depends only on the attractor.

6.3.3 Dimension theory, ergodic and statistical properties

Afraimovich and Pesin in [2] investigate the dimensional properties of "triangular maps" which are a class of maps generalizing the Poicaré first return map P of the geometric Lorenz model.

Concerning fractal dimensions of Lorenz attractors we mention the results of Leonov [94, 95] together with Bouichenko [32]. The first contains explicit formulas for the Lyapunov dimension of the Lorenz attractor and in the second a simple upper bound on the Hausdorff dimension of Lorenz attractors is given in terms of the parameters of the Lorenz systems of equations (1.1). In [127] Morales shows that every (nontrivial) compact invariant subset of a transitive singular set containing a singularity is one-dimensional, extending a similar result of Bowen in [36] in the setting of uniform hyperbolic flows.

Statistical and ergodic properties of the geometrical model were investigated, among others, by the authors together with Pujals and Viana [11] and Colmenarez [47], which are contained in Section 4.2.

In [211] Young shows that the geometrical Lorenz attractor can be approximated by horseshoes with entropy close to that of the Lorenz attractor.

The construction of the geometric Lorenz models forces the divergence of the vector field to be strictly negative in a isolating neighborhood of the attractor. This feature is also present in the Lorenz system of equations (1.1) for the classical parameters. It is then trivial to show that the corresponding attractor has zero volume. Recently it was proved [4] that singular-hyperbolic attractors always have zero volume for flows which are Hölder-C^1, although there is no volume dissipative condition on the definition of singular-hyperbolicity.

6.4 Decay of correlations

After obtaining an interesting invariant probability measure for a dynamical system the next thing to do is to study the properties of this measure. Besides ergodicity there are various degrees of mixing (see e.g.[204, 107]).

Given a flow X and an invariant ergodic probability measure μ, we say that the system (X,μ) is *mixing* if for any two measurable sets A,B

$$\mu\big(A\cap X^{-t}B\big) \xrightarrow[t\to\infty]{} \mu(A)\cdot\mu(B) \tag{6.1}$$

or equivalently

$$\int \varphi\cdot\big(\psi\circ X^t\big)\,d\mu \xrightarrow[t\to\infty]{} \int \varphi\,d\mu\int\psi\,d\mu$$

for any pair $\varphi,\psi : M\to\mathbb{R}$ of continuous functions.

Considering $\varphi,\psi\circ X^t : M\to\mathbb{R}$ as random variables over the probability space (M,μ), this definition just says that “the random variables $\varphi,\psi\circ X^t$ are asymptotically independent” since the expected value $\mathbb{E}\big(\varphi\cdot(\psi\circ X^t)\big)$ tends to the product $\mathbb{E}(\varphi)\cdot\mathbb{E}(\psi)$ when t goes to infinity. The *correlation function*

$$\begin{aligned} C_t(\varphi,\psi) &= \big|\mathbb{E}\big(\varphi\cdot(\psi\circ X^t)\big)-\mathbb{E}(\varphi)\cdot\mathbb{E}(\psi)\big| \\ &= \Big|\int\varphi\cdot\big(\psi\circ X^t\big)\,d\mu-\int\varphi\,d\mu\int\psi\,d\mu\Big| \end{aligned} \tag{6.2}$$

satisfies in this case $C_t(\varphi,\psi)\xrightarrow[t\to\infty]{} 0$. The *rate of approach to zero of the correlation function* is called *the rate of decay of correlations* for the observables φ,ψ of the system (X,μ).

The study of decay of correlations for hyperbolic systems goes back to the work of Sinai [188] and Ruelle [182]. Many results were obtained for transformations. For a diffeomorphism f the notion of decay of correlations is the same as above exchanging X^t by f^n and letting n go to infinity. Since [37, 182] it is known that the *physical* (SRB) measures for Axiom A *diffeomorphisms* are mixing and have *exponential decay of correlations*, that is there exists a constant $\alpha\in(0,1)$ such that given φ,ψ there is $C=C(\varphi,\psi)>0$ such that

$$C_n(\varphi,\psi)\le C\cdot e^{-\alpha n}\quad\text{for all}\quad n\ge 1, \tag{6.3}$$

for a suitable class of continuous functions $M \to \mathbb{R}$, in this case the Hölder continuous functions.

In more general cases for smooth endomorphisms (see e.g. [79, 6] and references therein) where the inverse in (6.1) is to be taken as the inverse image of f^n, it is possible to have slower rates of decay.

In contrast to the results available in the case of discrete dynamical systems, obtaining the rate of decay of correlations for flows seems to be much more complex and some results have been established for Anosov flows only recently. Ergodicity and mixing for geodesic flows on manifolds of negative curvature are known since the early half of the XXth century [81, 9, 187].

The proof of exponential decay of correlations for geodesic flows on manifolds of constant negative curvature was first obtained in two [45, 121, 169] and three dimensions [159] through group theoretical arguments.

6.4.1 Non-mixing flows and slow decay of correlations

Let $f : M \to M$ be a diffeomorphism with an invariant probability measure μ and consider the suspension flow X_f over f with constant roof function $r \equiv 1$. Then the probability measure $\nu = \mu \times \text{Leb}$ on $M \times [0,1)$ defines in a straightforward way a X_f-invariant probability measure on X_r *which is NOT mixing*, whatever f may be.

Indeed consider $A = \pi\big(M \times [0,1/2)\big)$ and $B = M_r \setminus A$ (recall that $\pi : M \times \mathbb{R} \to X_r$ is the projection defined in Section 1.2.1 on page 15). Then the function $t \mapsto \nu\big(A \cap X^{-t}B\big)$ for $t > 0$ has the graph in Figure 6.3 (here X^{-t} is a shorthand for $(X^t)^{-1}$, the inverse image of the map X^t).

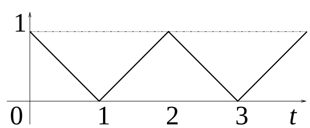

Figure 6.3: A correlation function for a non-mixing flow

This system is clearly *not* mixing since the sawtooth pattern in Figure 6.3 goes on for all positive t. Moreover this shows in particular that this suspension flow is not even topologically mixing (see below for the definition).

However if (X, f, μ) is ergodic, then ν is X_f-ergodic also: indeed, given $A \subset X_r$ such that $(X_f^t)^{-1}(A) = A$ for all $t > 0$ (an X_f-invariant set), then A is saturated, i.e. $p \in A$ if, and only if, $\mathcal{O}_{X_f}(p) \subset A$; thus we may find $\hat{A} \subset X$ such that $A \cap \pi(X \times \{0\}) = \pi(\hat{A})$ is X_f^1-invariant by construction (because $r \equiv 1$), $\hat{A}$ is f-invariant and $\nu(A) = \mu(\hat{A}) \cdot \mathrm{Leb}([0,1))$. Hence $\mu(\hat{A}) \cdot \mu(X \setminus \hat{A}) = 0$ by the ergodicity of (f, μ) which implies $\nu(A) \cdot \nu(X_r \setminus A) = 0$.

In addition to the examples of non-mixing suspension flows, which arguably can be characterized as very particular cases, not all Axiom A mixing flows have exponential decay of correlations: Ruelle [183] and Pollicott [158] exhibited suspensions with piecewise constant ceiling functions with arbitrarily slow decay rates.

Anosov [8] showed that geodesic flows for negatively curved compact Riemannian manifolds are mixing and obtained the *Anosov alternative*: given a transitive volume preserving Anosov flow, either it is mixing (with respect to the volume measure), or a suspension of an Anosov diffeomorphisms by a constant roof function. Note that Bowen [38] showed that if a mixing Anosov flow is the suspension of an Anosov diffeomorphism then it is *stably mixing*, that is, the mixing property remains true for all nearby flows (which are Anosov also by the structural stability of Axiom A flows).

Bowen also showed [38] that the class of C^r Axiom A flows, $r \geq 1$, admits a residual subset $\mathcal{R}$ such that for every $X \in \mathcal{R}$ the spectral decomposition of $\Omega(X)$ is formed by pairwise disjoint pieces $\Omega_1 \cup \cdots \cup \Omega_k$ each of which is *topologically mixing*. That is, given any pair of open sets U, V in Ω_i, there exists $T_0 = T_0(U,V) > 0$ such that $U \cap X^t(V) \neq \emptyset$ for all $t > T_0$.

6.4.2 Decay of correlations for flows

Chernov [43] provided a dynamical proof showing sub-exponential decay of correlations for geodesic flows on surfaces of variable negative curvature through a suitable stochastic approximation of the flow (see also [100] for a generalization and previous results [45]).

Much more recently a breakthrough was obtained by Dolgopyat [54, 55, 56]: smooth (C^r with $r \geq 7$) geodesic flows on manifolds of negative curvature, under a non-integrability condition exhibit exponential decay of correlations. Also Liverani [101] building on the work [54] obtained exponential decay of correlations for C^4 contact Anosov flows.

Using these ideas applied to the particular case of a suspension over uniformly expanding base dynamics, a conjecture of Ruelle was proved by Pollicott [160]: on a mild (cohomological) condition on the ceiling function, the decay of correlations for this type of suspension flows is exponential for observables not supported on the base. This was extended by Baladi-Vallée [18] clarifying the assumptions on the base and on the ceiling function which suffice to obtain exponential decay of correlations for suspension of one-dimensional expanding maps and all these ideas were used in a more abstract setting by Avila-Gouezel-Yoccoz [17] to obtain exponential decay of correlations for the Teichmüller flow on flat surfaces.

Recently Field-Melbourne-Török obtained [59] what they call *stability of rapid mixing* among Axiom A flows, meaning that the correlation function $C_t(\varphi,\psi)$ decays to zero faster than t^{-k} for all $k \in \mathbb{N}$ when $t \to \infty$, for a C^2-open and C^r-dense set of flows among the family of C^r Axiom A flows with $r \geq 2$.

Luzzatto, Melbourne and Paccaut [105] showed that the physical measure for the geometric Lorenz flow is mixing. The speed of mixing for the Lorenz flow is still an open problem.

6.5 Generic conservative flows in dimension 3

Conservative flows are a traditional object of study from Classical Mechanics, see e.g. [14]. These flows preserve a volume form on the ambient manifold and thus come equipped with a natural invariant measure. On compact manifolds this provides an invariant probability giving positive measure (volume) to all open subsets. Therefore for vector fields in this class we have $\Omega(X) = M$ by the Recurrence Theorem. In particular such flows cannot have Lyapunov stable sets, either for X or for $-X$.

Let $\mathfrak{X}^r_\omega(M)$ be the space of C^r vector fields defining flows which preserve the volume form ω on M, for any $r \geq 1$. It is natural to study these flows under the measure theoretic point of view, besides the geometrical one.

The device of Poincaré sections has been used extensively in the previous chapters to reduce several problems arising naturally in the setting of flows to lower dimensional questions about the behavior of a transformation. In the opposite direction, recent breakthroughs on the understanding of generic volume preserving diffeomorphisms on surfaces have non-trivial consequences for the dynamics of generic conservative flows on three-dimensional manifolds.

The Bochi-Mañé Theorem [29] asserts that, for a C^1 residual subset of area preserving diffeomorphisms, either we the transformation is Anosov, or the Lyapunov exponents are zero Lebesgue almost everywhere. This was announced by Mañé in [111] but only a sketch of a proof was available [112]. The complete proof presented by Jairo Bochi admits extensions to higher dimensions, obtained by Bochi and Viana in [31], stating in particular that either the Lyapunov exponents of a C^1 generic conservative diffeomorphism are zero Lebesgue almost everywhere, or the systems admits a dominated splitting for the tangent bundle dynamics. A survey of this theory can be found in [30].

Recently Mário Bessa was able to use adapt and extend the ideas of the original proof by Bochi to the setting of generic conservative flows on three-dimensional compact boundaryless manifolds. Again the presence of singularities imposes some differences between the discrete and continuous systems. Denote by $\mathfrak{X}^r_\omega(M)^*$ the subset of $\mathfrak{X}^r_\omega(M)$ of C^r flows with zero divergence but *without singularities*.

Theorem 6.1. *There exists a residual set $\mathcal{R} \subset \mathfrak{X}^1_\omega(M)^*$ such that, for $X \in \mathcal{R}$, either X is Anosov or else for Lebesgue almost every $p \in M$ all the Lyapunov exponents of X^t are zero.*

Developing the ideas of the proof of this result Bessa also obtains the following statement on denseness of dominated splitting, now admitting singularities.

Recall the definition of Linear Poincaré Flow in Section 1.3.3. Given an invariant subset Λ for $X \in \mathfrak{X}^1(M)$, an invariant splitting $N^1 \oplus N^2$ of the normal bundle N_Λ for the Linear Poincaré Flow P^t is said to be *n-dominated*, if there exists an integer n such that for *every* $p \in \Lambda$ we have the domination relation

$$\frac{\left\|P^n \mid N^1(p)\right\|}{\left\|P^n \mid N^2(p)\right\|} \leq \frac{1}{2}.$$

Theorem 6.2. *There exists a dense set $\mathfrak{D} \subset \mathfrak{X}^1_\omega(M)$ such that for $X \in \mathfrak{D}$, there exist invariant subsets D and Z whose union has full measures, such that*

- *for $p \in Z$ the flow has only zero Lyapunov exponents;*

- *D is a countable increasing union* Λ_n *of compact invariant sets admitting a n-dominated splitting for the Linear Poincaré Flow.*

Appendix A

Perturbation lemma for flows

Here we present a proof of Theorem 1.20 on page 32. This is an unpublished joint work of M. J. Pacifico and E. R. Pujals. Let Y be a vector field in the setting of the statement of the theorem.

Given $v,w \in \mathbb{R}^n$, $v \cdot w$ stands for the inner product of v and w. Given $v \in \mathbb{R}^n$ we set $[v]^\perp = \{w \in \mathbb{R}^n, w \cdot v = 0\}$. Given p, let $\Sigma \subset [Y(p)]^\perp$ be a cross section to Y at p whose size will be fixed later.

Define, for $q \in \Sigma$, the following parametrized family of maps

$$\hat{A}_t(q) = Y^t(p) + A_t(q).$$

Observe first that $\hat{A}_t$ is C^2 and if Σ is taken small enough then

$$\mathcal{T} = \{\hat{A}_t(q) : q \in \Sigma, t \in [a,b]\}$$

gives a neighborhood of $Y^{[a,b]}(p)$.

Lemma A.1. *There exists $r > 0$ such that the following is true for* $\operatorname{diam}(\Sigma) < r$*: if $\hat{A}_{t_1}(q_1) = \hat{A}_{t_2}(q_2)$ with $q_i \in \Sigma$ and $t_i \in [a,b]$, then $t_1 = t_2$ and $q_1 = q_2$.*

Proof. Assume $\hat{A}_{t_1}(q_1) = \hat{A}_{t_2}(q_2)$ with $t_1 < t_2$ and $q_1 \neq q_2$. Then

$$Y^{t_1}(p) - Y^{t_2}(p) = A_{t_2}(q_2) - A_{t_1}(q_1). \tag{A.1}$$

On one hand there is $t_0 \in (t_1, t_2)$ such that

$$\begin{aligned} Y^{t_1}(p) - Y^{t_2}(p) &= (t_1 - t_2)\frac{\partial}{\partial s} Y^{t_0+s}(p)|_{s=0} \\ &= \frac{\partial}{\partial s} Y^s\Big|_{s=0}(p) = (t_1 - t_2) Y(Y^{t_0}(p)). \end{aligned} \tag{A.2}$$

On the other hand there is $l \in (t_1, t_2)$ such that

$$\begin{aligned} A_{t_2}(q_2) - A_{t_1}(q_1) &= (A_{t_2} - A_{t_1})(q_2) + A_{t_1}(q_2 - q_1) \\ &= \frac{\partial}{\partial t} A_t|_{t=l}(q_2)(t_2 - t_1) + A_{t_1}(q_2 - q_1). \end{aligned} \tag{A.3}$$

Replacing (A.2) and (A.3) in (A.1) we get

$$Y(Y^{t_0}(p)) = -\frac{\partial}{\partial t} A_t|_{t=l}(q_2) + A_{t_1} \frac{(q_2 - q_1)}{(t_1 - t_2)}. \tag{A.4}$$

Observe that $A_{t_0}(Y(p)) = Y(Y^{t_0}(p))$. Since the family A_t depends continuously on t there are $h > 0$ and $\gamma > 0$ such that, if $P_{Y(p)}$ denotes the projection on the direction of $Y(p)$, then

$$\|P_{Y(p)}(A_t^{-1}(Y(Y^{t_0}(p))\| > \gamma \tag{A.5}$$

for all t with $|t - t_0| < h$. Define the numbers

$$K_1 = \sup\{\|\frac{\partial}{\partial t} A_t\|, t \in [a,b]\}, \quad K_2 = \sup\{\|A_t\|, t \in [a,b]\}$$

and $\gamma_0 = \inf\{\|Y(Y^t(p))\|, t \in [a,b]\}$. Observe that γ_0 is positive since p is a regular point. Let $r > 0$ be such that $r < \gamma/K_1$, $K_1 r + K_2 r/h < \gamma_0$ and take Σ with $\mathrm{diam}(\Sigma) < r$. We split the arguments in a pair of cases.

First case $|t_1 - t_2| \geq h$. Taking norms in (A.4) leads to $\gamma_0 < K_1 r + K_2 r/h < \gamma_0$, which is a contradiction.

Second case $|t_1 - t_2| < h$. Observe that (A.1) and (A.2) imply

$$(t_1 - t_2) Y(Y^{t_0}(p)) = A_{t_2}(q_2) - A_{t_1}(q_1),$$

which is the same as $(t_1-t_2)A_{t_1}^{-1}Y(Y^{t_0}(p)) = A_{t_1}^{-1}A_{t_2}(q_2)-q_1$. Thus

$$\begin{aligned} P_{Y(p)}[(t_1-t_2)A_{t_1}^{-1}Y(Y^{t_0}(p))] &= P_{Y(p)}[A_{t_1}^{-1}A_{t_2}(q_2)-q_1] \\ &= P_{Y(p)}[A_{t_1}^{-1}A_{t_2}(q_2)-q_2] \\ &= P_{Y(p)}[(A_{t_1}^{-1}A_{t_2}-Id)(q_2)]. \end{aligned} \tag{A.6}$$

Observe that we used above $q_2 \in \Sigma = [Y(p)]^{\perp}$.

But there is $l \in (t_1,t_2)$ such that

$$(A_{t_1}^{-1}A_{t_2}-Id)(q_2) = (t_1-t_2)\frac{\partial}{\partial s}A_{l+s}A_l^{-1}|_{s=0}(q_2).$$

Replacing this in (A.6) we get

$$P_{Y(p)}[A_{t_1}^{-1}Y(Y^{t_0}(p))] = P_{Y(p)}[\frac{\partial}{\partial s}A_{l+s}A_l^{-1}|_{s=0}(q_2)]. \tag{A.7}$$

Taking norms in (A.7) and using (A.5) we obtain $\gamma < K_1\|q_2\|$. Since diam$(\Sigma) < r$ and $r < \gamma(K_1)^{-1}$, this is a contradiction. All together this shows that $t_1 = t_2$ and from (A.4) we see that $q_1 = q_2$. The proof of Lemma A.1 is complete.

□

Now define $\widetilde{A} : \mathcal{T} \subset [a,b]\times\Sigma \to \mathcal{T}$ as follows. For $w \in \mathcal{T}$ there exists, by Lemma A.1, a unique pair $(q_w,t_w) \in \Sigma\times[a,b]$ such that $\hat{A}_{t_w}(q_w) = w$. We define

$$\widetilde{A}(w) = \hat{A}_{t_w+s}(q_w). \tag{A.8}$$

In other words, we have $\widetilde{A}_s(\hat{A}_t(q)) = Y^{t+s}(p) + A_{t+s}A_t^{-1}(A_t(q))$,,for $q \in \Sigma$ and $t+s<b$.

Lemma A.2. *The family $\widetilde{A}_s$ defines a C^2 flow in $\mathcal{T}$. Moreover*

$$\frac{\partial}{\partial s}D_w\widetilde{A}_s = D_w\frac{\partial}{\partial s}\widetilde{A}_s. \tag{A.9}$$

Proof. Clearly $\widetilde{A}_s$ is C^2. Let us prove that $\widetilde{A}_{s+t} = \widetilde{A}_s\widetilde{A}_t$.

Let $w \in \mathcal{T}$. Then $\hat{A}_{t_w}(q_w) = w$, for a unique $(q_w,t_w) \in \Sigma\times[a,b]$. By definition (A.8)

$$\widetilde{A}_s\widetilde{A}_t(w) = \widetilde{A}_s(\widetilde{A}_t(\hat{A}_{t_w}(q_w)) = \widetilde{A}_s(\hat{A}_{t+t_w}(q_w)). \tag{A.10}$$

Define now $\hat{A}_{t+t_w}(q_w) = \hat{w}$. Note that $\hat{w} = \hat{A}_{t_{\hat{w}}}(q_{\hat{w}})$. By the uniqueness of $t_{\hat{w}}$ and $q_{\hat{w}}$ given by Lemma A.1, we get $t + t_w = t_{\hat{w}}$ and $q_w = q_{\hat{w}}$. Thus

$$\begin{aligned}\widetilde{A}_s(\hat{A}_{t+t_w}(q_w)) = \widetilde{A}_s(\hat{A}_{t_{\hat{w}}}(q_{\hat{w}})) = \hat{A}_{s+t_{\hat{w}}}(q_{\hat{w}}) \\ = \hat{A}_{s+t+t_w}(q_w) = \widetilde{A}_{s+t}(A_{t_w}(q_w)) = \widetilde{A}_{s+t}(w).\end{aligned} \tag{A.11}$$

Combining (A.10) and (A.11) we deduce $\widetilde{A}_{s+t} = \widetilde{A}_s \widetilde{A}_t$.

Now we prove (A.9). Define $\widehat{A}(t,q) = \hat{A}_t(q)$. We have that $\widehat{A}$ is C^2 and

a) $\frac{\partial}{\partial t}\widehat{A} = \frac{\partial}{\partial t}Y^t(p) + \frac{\partial}{\partial t}A_t(q)$ is C^1,

b) $D_q\widehat{A} = A_t$ is C^1.

Note that a) and b) imply

$$\frac{\partial}{\partial t}A_t = \frac{\partial}{\partial t}D_q\widehat{A}_t = D_q\frac{\partial}{\partial t}\widehat{A}_t = \frac{\partial}{\partial t}A_t. \tag{A.12}$$

Note also that $\frac{\partial}{\partial t}D_q\widehat{A}_t$ and $D_q\frac{\partial}{\partial t}\widehat{A}_t$ are C^1 maps since A_t is a family of invertible linear maps depending C^2 in the parameter .

Now Lemma A.1 gives that $\widehat{A}$ has an inverse map R defined in the image $\widehat{\mathcal{T}} = \widehat{A}(\mathcal{T})$. Moreover R is C^1 since $\widehat{A}$ is C^1.

Again for $s \in [a,b]$ and $w \in \mathcal{T}$ define $\widetilde{A}(s,w) = \widetilde{A}_s(w)$.

Let π_1, π_2 be the projections on the first and second coordinates:

$$\pi_1 : [a,b] \times \mathcal{T} \to [a,b], \quad (s,w) \mapsto s \qquad \pi_2 : [a,b] \times \mathcal{T} \to \mathcal{T}, \quad (s,w) \mapsto w.$$

Clearly π_i is C^∞, $i = 1,2$. Since

$$\widetilde{A}(s,w) = \widetilde{A}_s(w) = \hat{A}_{s+t_w}(q_w) = \hat{A}(s+t_w, q_w) = \hat{A}(s+\pi_1 \circ R(w), \pi_2 \circ R(w))$$

we get that $\widetilde{A}$ is C^1, which implies that $\widetilde{A}_s$ induces a C^1 flow in $\mathcal{T}$.

Finally let us verify (A.9). For this, let $\widehat{R}(s,w) = (s+\pi_1 \circ R(w), \pi_2 \circ R(w))$, where R was defined above. Clearly $\widetilde{A} = \widehat{A} \circ \widehat{R}$. Observe that (A.12) and the fact that $\widehat{R}$ and $\widehat{A}$ are C^2 imply that

$$\begin{aligned} D_w\widetilde{A} &= D_{\widehat{R}(s,w)}\widehat{A} \cdot D_w\widehat{R}, \\ \frac{\partial}{\partial s}\widetilde{A} &= D_{\widehat{R}(s,w)}\widehat{A} \cdot \frac{\partial}{\partial s}\widehat{R}, \\ \frac{\partial}{\partial s}D_w\widetilde{A} &= \frac{\partial}{\partial s}D_{\widehat{R}(s,w)}\widehat{A} \cdot \partial_s\widehat{R} \cdot D_w\widehat{R} + D_{\widehat{R}(s,w)}\widehat{A} \cdot \frac{\partial}{\partial s}D_w\widehat{R} \end{aligned}$$

and $D_w \frac{\partial}{\partial s}\widetilde{A}$ all exist and are continuous. Thus by Schwartz Lemma we obtain (A.9). □

Let Z_A be the vector field induced by $\widetilde{A}_s$, that is, $Z_A(w) = \frac{\partial}{\partial s}\widetilde{A}_s(w)\big|_{s=0}$.

Lemma A.3. *The vector field Z_A is C^1. Moreover*

$$D_w Z_A = \frac{\partial}{\partial s} A_{t_w+s} A_{t_w}^{-1}\big|_{s=0}. \tag{A.13}$$

Proof. Since $\widetilde{A}_s$ is a C^2 flow we have that Z_A is C^1.

Let us calculate $D_w Z_A$. We first calculate $D_w\widetilde{A}_s\big|_{s=0}$. For this recall that $w = \hat{A}_{t_w}(q_w)$ with $t_w \in [a,b]$ and $q_w \in \Sigma$. To simplify notation we set $t_w = t$ and $q_w = q$. Then, $\widetilde{A}_s(w) = \widetilde{A}_s(\hat{A}_t(q)) = \hat{A}_{t+s}(q)$ and so $D_q\hat{A}_{t+s} = D_{\hat{A}_t(q)}\widetilde{A}_s \,.\, D_q\hat{A}_t$. This implies

$$D_{\hat{A}_t(q)}\widetilde{A}_s = D_q\hat{A}_{t+s}(D_q\hat{A}_t)^{-1}. \tag{A.14}$$

On the other hand $\hat{A}_{t+s}(q) = Y^{t+s}(p) + A_{t+s}(q)$ implies that $D_q\hat{A}_{t+s} = A_{t+s}$. Replacing this in (A.14) and using the fact that $(D_q\hat{A}_t)^{-1} = A_t^{-1}$ we get

$$D_{\hat{A}_t(q)}\widetilde{A}_s = A_{t+s}A_t^{-1}.$$

Thus

$$D_w Z_A = \frac{\partial}{\partial s} D_w\widetilde{A}_s\big|_{s=0} = \frac{\partial}{\partial s} D_{\hat{A}_t(q)}\widetilde{A}\big|_{s=0} = \frac{\partial}{\partial s} A_{t+s}A_t^{-1}\big|_{s=0}$$

proving (A.13). The proof of Lemma A.3 is completed. □

If $U \subset \mathbb{R}^n$, then U^c stands for the complement of U.

Fix $\varepsilon > 0$ and take $0 < r < \varepsilon$. For each $t \in [a,b]$ let Σ_r be a cross section to $Y^t(p)$ satisfying $\operatorname{diam}(\Sigma_r) < r$ and $\Sigma_r \subset [Y(Y^t(p))]^\perp$.

Let $\Omega = \bigcup_{t\in[a,b]} \Sigma_r$. Note that Ω is a neighborhood of $Y^{[a,b]}(p)$. Thus there are neighborhoods $U_1 \subset \overline{U_2} \subset \overline{\Omega}$ of $Y^{[a,b]}(p)$ and a C^1 function $f : \mathcal{T} \to \mathbb{R}$ satisfying:

- $f \mid U = 1$, $f \mid U^c = 0$ and $|f| \leq 1$; and
- given $w \in U_2$, let t_w be such that $\operatorname{dist}(w, Y^{[a,b]}(p)) = \operatorname{dist}(w, Y^{t_w}(p))$. We require

$$\|D_w f\|.\|w - Y^{t_w}(p)\| < \varepsilon.$$

Define the C^1 vector field in $\mathbb{R}^n$

$$Z(w) = f(w)\cdot Z_A(w) + (1-f(w))\cdot Y(w).$$

Lemma A.4. *Z is C^0-near Y.*

Proof. Indeed,

$$Z(w) - Y(w) = f(w)\cdot (Z_A(w) - Y(w)). \tag{A.15}$$

On the other hand, given w, there are t_w and q_w such that $w = \hat{A}_{t_w}(q_w)$. Taking into account (A.8) and the definition of Z_A we get

$$\begin{aligned} Z_A(w) &= \frac{\partial}{\partial s}\widetilde{A}_s(w)\big|_{s=0} = \frac{\partial}{\partial s}\big(Y^s(Y^{t_w}(p))\big|_{s=0} + \frac{\partial}{\partial s}\big(A_{t_w+s}\big)\big|_{s=0}(q_w) \\ &= Y(Y^{t_w}(p)) + \frac{\partial}{\partial s}(A_{t_w+s})\big|_{s=0}(q_w). \end{aligned}$$

Replacing this last inequality in (A.15) we obtain

$$Z_A(w) - Y(w) = f(w)\cdot\Big(Y(Y^{t_w}(p)) - Y(w) + \frac{\partial}{\partial s}A_{t_w+s}\big|_{s=0}(q_w)\Big)$$

and then

$$\|Z_A(w) - Y(w)\| \le \|Y(Y^{t_w}(p)) - Y(w)\| + \|\frac{\partial}{\partial s}A_{t_w+s}\big|_{s=0}\|\cdot\|(q_w)\|. \tag{A.16}$$

Now we can assume that Σ is sufficiently small so that $\|A_t(q)\| \le \|A_t\|\cdot\|q\|$ is small for all t and q. We can estimate the first term in the right hand side of (A.16)

$$\begin{aligned} \|Y(Y^{t_w}(p)) - Y(w)\| &\le \|Y\|\cdot\|Y^{t_w}(p) - w\| = \|Y\|\cdot\|Y^{t_w}(p) - \hat{A}_{t_w}(q_w)\| \\ &= \|Y\|\cdot\|Y^{t_w}(p) - Y^{t_w}(p) + A_{t_w}(q_w)\| \\ &= \|Y\|\cdot\|A_{t_w}(q_w)\| \le \|Y\|\cdot\|A_{t_w}\|\cdot\|q_w\| \le \varepsilon. \end{aligned} \tag{A.17}$$

The second term on the right hand side of (A.16) can be bounded by

$$\|\partial s A_{t_w+s}\big|_{s=0}\|\cdot\|(q_w)\| \le \varepsilon, \tag{A.18}$$

if Σ is small. Replacing (A.17) and (A.18) in (A.16) we conclude the proof. □

To finish we need one last lemma.

Lemma A.5. *The vector field Z is C^1-near Y.*

Proof. We have

$$D_w Z - D_w Y = D_w f \cdot (Z_A(w) - Y(w)) + f(w) \cdot (D_w Z_A - D_w Y) + D_w Y. \tag{A.19}$$

The norm of the first term above is bounded by

$$\begin{aligned} &\|D_w f\| \cdot \|Z_A(w) - Y(w)\| \\ &\qquad \le \|D_w f\| \cdot \|Z_A(w) - Z_A(Y^{t_w}(p)\| + \|Y(Y^{t_w}(p)) - Y(w)\| \\ &\qquad \le \|D_w f\| \cdot \|Z_A\| \cdot \|w - Y^{t_w}(p)\| + \|Y\| \cdot \|w - Y^{t_w}(p)\| \cdot \|D_w f\| \end{aligned}$$

and the condition on the bump function f imply that both terms in the last expression are small is Σ is small.

To estimate the second hand term in the right hand side of (A.19) we recall that Lemma A.3 gives

$$D_w Z_A = \frac{\partial}{\partial s} A_{t+s} A_t^{-1}\Big|_{s=0}.$$

On the one hand this is, by hypothesis, near $D_{Y^t(p)} Y$.

On the other hand, since $w = \hat{A}_t(q) = Y^t(p) + A_t(q)$, we also get that w is near $Y^t(p)$ and so $D_w Y$ is near $D_{Y^t(p)} Y$. Combining these last two observations we obtain that $D_w Z_A$ is near $D_w Y$, concluding the proof of Lemma A.5. □

The proof of Theorem 1.20 is completed.

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