Schubert Calculus:
An Algebraic Introduction

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A Sheila e Giuseppe

Preface

The goal of these notes is to let the Reader get familiar with a natural algebraic formalism equivalent to *Schubert Calculus on Complex Grassmannians*, basing on the following

Theorem. *The Chow intersection (or integral cohomology) ring of the complex grassmannian variety $G(k,n)$, parametrizing k-dimensional vector subspaces of $\mathbb{C}^n$, can be realized as a commutative ring of endomorphisms on the k^{th} exterior power of a free $\mathbb{Z}$-module of rank n (Cf. [23]).*

The explanation of the above result, as well as a list of consequences, which seem relevant at least from a "pedagogical" point of view, will be postponed to the Introduction. However, it is worth to anticipate that, from a purely algebraic point of view, Schubert calculus for grassmannians $G(k,n)$, for all $0 \leq k \leq n$ and all $n \geq 0$ at once, amounts to study the formal properties of a suitable algebra homomorphism

$$D_t : \bigwedge M \longrightarrow \bigwedge M[[t]] \qquad (1)$$

from the exterior algebra $\bigwedge M$ of a free $\mathbb{Z}$-module M, of infinite countable rank, to the algebra of formal power series in one indeterminate t over it. In a sense, *Schubert calculus* can be proclaimed with the slogan:

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta, \qquad \forall \alpha, \beta \in \bigwedge M,$$

which is the explicit way to phrase that the map (1) is an algebra homomorphism.

These notes have been written as a ready-for-use reference for the minicourse *Schubert Calculus: an Algebraic Introduction* given at IMPA during the 25° *Colóquio Brasileiro de Matemática.* By no means they can be intended as substitutes of the many good textbooks on either the geometry or the intersection theory of complex grassmannian varieties. For these subjects there is a plenty of canonical references, rich of geometrical insight, starting from the beautiful [39] up to [27] and [16], or [22]. For a hot update of the state of arts, the Reader should also consult the very recent volume edited by P. Pragacz (including his paper [65] therein). These notes are just a very personal (and, therefore, not necessarily the best) point of view which the author look at those subjects with: nothing more than a pair of coloured glasses to read the references quoted in the bibliography, whom the Reader is really referred.

Although the intersection theory of grassmannian varieties is a fairly delicate subject, certainly requiring some mathematical maturity, this formulation shows that one can learn Schubert calculus with minimal pre-requisites (basic linear algebra and calculus) without even knowing what is it. In particular, any reader, even the less experienced one, can read Chapter 4, the core of the notes, independently from the rest of the exposition.

Acknowledgements.[1] My first debt of gratitude is with my distinguished friend *Israel Vainsencher*, who last year suggested me to submit a proposal for a minicourse on this subject, to be given during the 25° *Colóquio Brasileiro de Matemática*, at IMPA. I could profit of his corrections and remarks because he was the only person I could send my very prelyminary informals drafts without feeling (too) ashamed. The friendly support of Israel was strengthened by the convinced one of *Abramo Hefez*, whom I want to adress a feeling of gratitude, too. My original proposal to *Comissão Organizadora do Colóquio* was somewhat unbalanced and I could improve it thanks to the friendly suggestions and remarks by *Eduardo Esteves*, who

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influenced the organization of the contents, whom I am grateful too. I am also strongly indebted with *Aron Simis* (who gave me valuable hints) and *Francesco Russo* that, together with *Israel* and *Eduardo*, have made possible my visit to Brazil.

The subject of these notes, based on [23], has been widely discussed with Dan Laksov whose wise advises, sharp criticism and friendly support are priceless. Also I want to thank *Gary Kennedy* for discussions on the quantum cohomology side, and *Giorgio Ottaviani* for discussions on the representation theory one, as well as *Piotr Pragacz*, who suggested me new difficult but exciting directions to push forward my current research interests.

I got a precious help from my collaborator *Taíse Santiago Costa Oliveira* (obrigado!), Ph.D. student at Politecnico di Torino, and *Caterina Cumino*. Not only the latter shared (as still does) an office with me, but also many delightful mathematical conversations, including the subject of these notes, which helped me a lot to improve the exposition.

Last, but not the least, I warmly thank all the readers, without blaming the others, who will forgive the several misprints, errors and/or omissions, certainly left, here and there, along the text.

These notes have two hidden co- authors: my wife *Sheila* and my son *Giuseppe*. The former solved all the practical problems I should have dealt myself in the daily life, letting me spending, with lovely comprehension, almost all my home time playing with Mathematics. The latter put abundant grains of joy with his wide smile, brighting even during cloudy days. Without them these notes could have not been written and for this reason, and many others, to them are dedicated.

♡

Sangano, 30 de Abril, 2005

Contents

Introduction

Schubert Calculus for *grassmannians* [1] has an ancient and venerable history. It is related with the deep investigations in enumerative geometry pursued by Schubert[2] in his celebrated treatise [68], which dates back to 1879. The first systematic study of what, in modern language, would be called intersection theory on the grassmannian variety of lines in the 3-dimensional projective space, is due to him. Schubert's work was somewhat revolutionary for his time and provided new powerful techniques to deal with many enumerative questions. The most popular is perhaps *how many lines do meet* 4 *others in general position in the projective* 3*-space?* Schubert's methods, based on heuristics, although ingenious, degeneration arguments, cried however for more mathematical rigour, explicitly claimed by Hilbert, who so entitled his 15^{th} Problem:

Rigorous Foundation of Schubert's Enumerative Calculus.

In Hilbert's words (borrowing the english translation from [38], p. 327), the problem consisted in establishing

> "... rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert ([68]) especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him."

[1] After Hermann Günther Grassmann (1809-1877).
[2] Hermann Cäsar Hannibal Schubert (1848–1911).

Schubert's work was continued and generalized, at the beginning of last century, by two italian mathematicians, Pieri and Giambelli ([28], [29], [30] and [58]), whose formulas allow in principle, speaking a modern language, the explicit computation of two arbitrary generators, said to be *Schubert cycles*, of the intersection ring of a grassmannian (see Section 3.4). They were far, however, from giving answers to Hilbert's problem, which had to wait for the modern developments of intersection theory and/or of the singular cohomology of complex projective algebraic varieties. Readers interested to follow this nice historical path, should not hesitate to look at the surveys [38] and [39].

Nowadays, talking about *Schubert Calculus*, without any further adjective, is too vague to indicate any specific mathematical subject. Schubert calculus for grassmannians has been generalized to flag varieties (see [18] and [8]) and, more generally, to homogeneous spaces (quotients of an algebraic group modulo a parabolic subgroup; see [6] or [76] for general vocabulary). There are also versions of Schubert calculus for other types of grassmannians (orthogonal, symplectic, see [62], [66]) of vector spaces equipped with an additional structure (metric, symplectic, etc.), without talking about the rich flourishing of the literature regarding combinatorial aspects of small quantum Schubert calculus initiated with Bertram in [4] and continued in a series of papers (e.g. [3], [5], [9], and references therein). More than that, the structural constants of the cohomology ring of the grassmannians appear in the representation theory of the general linear group $Gl_n(\mathbb{C})$ (see [20], [75], and the beautiful paper [77]), as well as in the more classical and fascinating theory of the symmetric functions (see [52] or the more recent [53]), where the $k \times k$ Vandermonde determinant plays the same role as the fundamental class of a grassmannian $G(k, n)$. People interested to the marriage of grassmannians with physics should have a look at [26] and, of course, to [81].

Regarding more specifically these notes, they aim to be a walk, back and forth, between the land of *Schubert's algebra* (governing the intersection theory of the grassmannians) and that of *Grassmann's algebra* (the exterior algebra of a module) through the bridge of *wronskians*[3]. Metaphores aside, the goal is to introduce the Reader to an

[3] After the polish mathematician J. M. Hoene-Wroński (1778-1853).

abstract algebraic model equivalent to Schubert calculus for grassmannian varieties, whose corner stones, *Pieri*'s and *Giambelli*'s formulas, can be rephrased in terms of *derivatives of wronskians* determinants of a system of linearly independent (C^∞ or holomorphic) functions of one (real or complex) variable. It is easily checked that the i^{th}-derivative ($i \geq 1$) of a wronskian turns out to be an integral linear combination of certain determinantal expressions, said to be, imitating a terminology used in [59], *generalized wronskians* (Cf. Section 1.2.1). Pieri's formula then corresponds to *differentiate* wronskians, while Giambelli's formula amounts to *integrate* the generalized ones, as explained with more details in Chapter 1, aimed to advertise the contents of Chapter 4, the core of the notes. The algebraic formalism there developed turns out to be particularly flexible, due to the natural identification between the Chow (or homology) group $A_*(G_k(V))$ of the grassmannian of k-planes of V, and the k^{th} *exterior power* $\bigwedge^k M_n$ of a free $\mathbb{Z}$-module M_n of rank $n = \dim(V)$. Schubert calculus is then described through a family $D = (D_1, D_2, \ldots)$ of *derivations* of the *Grassmann algebra* $\bigwedge M_n$ that, when restricted to $\bigwedge^k M_n$, generate a commutative subalgebra $\mathcal{A}^*(\bigwedge^k M_n, D)$ of $End_{\mathbb{Z}}(\bigwedge^k M_n)$, which is isomorphic to the intersection ring $A^*(G_k(V))$ of the grassmannian of k-planes of a n dimensional vector space. Moreover there is the isomorphism

$$\mathcal{A}^*(\bigwedge^k M_n, D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k \cong \bigwedge^k M_n, \tag{2}$$

translating *Poincaré duality* for grassmannians (Cf. Section 3.4.2). Within this framework, Pieri's formula is nothing else than a consequence of Leibniz's rule enjoyed by the operators D_i's (which correspond to the *special Schubert cycles* σ_i), while Giambelli's formula, which formula (2) is based on, turns out to be a consequence of the following *integration by parts* (Lemma 4.5.3)

$$D_i\alpha \wedge \epsilon^j = D_i(\alpha \wedge \epsilon^j) - D_{i-1}(\alpha \wedge \epsilon^{j+1}),$$

holding for each $\alpha \in \bigwedge M_n$.

Rather than working with the k^{th} exterior power of a finitely generated free $\mathbb{Z}$-module M_n, in Chapter 4 one works directly on the exterior algebra $\bigwedge M$ of a free module M of infinite countable rank. This corresponds to deal with all the grassmannians at once, rather than a single one, making things easier and more elegant. Everything is described in terms of a particular *Hasse-Schmidt Derivation* on $\bigwedge M$ (said to be *Schubert* for obvious reasons; see Sections 4.1 and 4.3), permitting to summarize Schubert Calculus formalism (and hence Pieri's and Giambelli's formulas) via the holding-by-definition equality:

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta, \quad \forall \alpha, \beta \in \bigwedge M.$$

One should dutifully remark that the picture of Chapter 4 has been rephrased, and considerably refined, by Laksov and Thorup ([48]). Using suitable symmetrizing operators (analogous to taking residues of formal Laurent series) and the notion of universal splitting algebra, they are able to give very transparent proofs of the analogous of Giambelli's formula, discussed in Section 4.8, even for grassmannian bundles. More than that, those authors are able to show ([49]) how it is possible, using the theory of splitting algebras, to prove that the picture exposed in Section 4 is indeed the intersection theory of the grassmannian variety without relying on any previous knowledge of Schubert Calculus!

These book-shaped notes are organized as follows. Chapter 1 is aimed to convince the Reader that *Schubert Calculus for grassmnannians* can be phrased in terms of everybody's first mathematical experiences and that, to learn it, is not necessary to know what is it. Most of the examples of Chapters 5 could be in fact computed by first year's undergraduate students. Chapter 2 should be skipped by most readers, as it is a list of quickly collected basic definitions and preliminary results (without proof), to be used mainly as a vocabulary. Chapter 3 is about an introduction to the notion of the grassmannian as a holomorphic variety. This also may be avoided byt experienced readers. However the presentation aims to prepare the way to the, although easy, abstract formalism developed in Chapter 4. What in the latter is a basis element $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ of $\bigwedge^k M$, in Chapter 3 is a Plücker coordinate or a section of the top exterior power of the dual of the tautological bundle over the grassmannian. Finally, Chapter 5 is a miscellanea of fragmented examples, intended to show that the formalism offers indeed some computational advantages. Section (5.1) fully works out the computation of the intersection theory of the grassmannian of lines $G_1(\mathbb{P}^3)$, while Section 5.5 is devoted to algebraically analyze what in the literature is known as *quantum Schubert calculus* of $G(k,n)$. It turns out that the latter can be seen as Schubert calculus in a grassmannian $G(k,N)$, with N sufficiently large. The examples about degree of grassmannians, shown in Section 5.3, cry for the search of a more general pattern. The degrees of Schubert varieties in the Plücker embedding of the grassmannians are very well known (see e.g. [36]). However, the Hasse-Schmidt presentation suggests a search for degrees through recursive formulas, as illustrated by the shown experiments. Using the algebraic yoga introduced in this book, one may easily show that the degree of the *Plücker image* (Section 3.2.5) of the grassmnannian variety $G(3,n)$ is

$$d_{3,n} = \binom{3(n-3)}{n-3} d_{2,n-1} + \left(\binom{3(n-3)}{n-5} - \binom{3(n-3)}{n-4} \right) d_{2,n},$$

having denoted by $d_{k,n}$ the degree of $G(k,n)$. A similar, although slightly more complicated formula expressing $d_{4,n}$ as a suitable linear combination of $d_{3,n}$, $d_{3,n-1}$, $d_{2,n}$ and $d_{2,n-1}$, is displayed in Example 5.3.4. Computations get trickier for $d_{5,n}$ and one wonders if it is possible to find a unified description via some nice generating function, possibly encoding top intersection degrees of others Schubert cycles, as being investigated in [67].

The notes end with a list of references related with Schubert calculus, certainly not complete, but probably sufficient to let the Reader finding his own path to study this beautiful subject.

Convention

• The *grassmannian of k-planes* of a vector space V shall be denoted by $G_k(V)$. If $V = \mathbb{F}^n$, $\mathbb{F}$ any field, one will write $G_k(\mathbb{F}^n)$. When $\mathbb{F} = \mathbb{C}$, the notation $G(k,n)$ shall be used instead of $G_k(\mathbb{C}^n)$.

• Einstein's convention summation will be used. If $(v^1, \ldots, v^n)$ are the components of an element $\mathbf{v} \in M$ with respect to the basis $(e_1, \ldots, e_n)$, then one will write

$$\mathbf{v} = v^i e_i \quad \text{instead of} \quad \mathbf{v} = \sum_{i=1}^{n} v^i e_i.$$

Similarly, if $\underline{\alpha} \in M^\vee$, one will write $\underline{\alpha} = \alpha_i \epsilon^i$ instead of $\sum_{i=1}^n \alpha_i \epsilon^i$. Also one will write $A^i_j \epsilon^j$ instead of $\sum_{j=1}^n A^i_j \epsilon^j$.

Chapter 1

Propaganda

The "official" mathematical language, which the author tried to write these notes with, may cause people, who never looked at this subject before, to think that things are more difficult than they really are. To avoid such an undesired feeling, this section aims, thus, to draw an informal and friendly path to show that whatever will be going on, it will in the most natural way.

Instead of starting by explaining what a grassmannian is, which is not strictly necessary at this stage, one chooses, for the Reader's convenience, to enter soon into the core of the subject, by describing the algebraic model which is the main character of this play.

1.1 Revisiting (some) Calculus

1.1.1 Derivatives and Leibniz's Rules. Believe you or not, the basic manipulation devices proper of Schubert Calculus should already be in the toolkit of any reader who took a basic course in calculus and principles of linear algebra. To start with, and to see why, let $C^\infty(\mathbb{R})$ be the algebra of all real C^∞-differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of one real variable x. The *derivative*

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}),$$

is a linear map $f \mapsto df/dx$, which operates on the product of two functions according to the well known *Leibniz's rule*:

$$\frac{d}{dx}(fg) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}. \tag{1.1}$$

The k^{th}-iterated of d/dx is a linear operator, too, denoted by d^k/dx^k, which satisfies a Leibniz-like rule, induced by that enjoyed by d/dx. For instance:

$$\begin{aligned}\left(\frac{d}{dx}\right)^2(fg) &= \frac{d}{dx}\left(\frac{d}{dx}(fg)\right) = \frac{d}{dx}\left(\frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}\right) = \\ &= \frac{d^2f}{dx^2} \cdot g + \frac{df}{dx} \cdot \frac{dg}{dx} + \frac{df}{dx} \cdot \frac{dg}{dx} + f \cdot \frac{d^2g}{dx^2} = \\ &= \frac{d^2f}{dx^2} \cdot g + 2\frac{df}{dx} \cdot \frac{dg}{dx} + f \cdot \frac{d^2g}{dx^2}. \end{aligned} \tag{1.2}$$

For sake of notational brevity, let

$$D_0 := 1 \quad \text{and} \quad D_n := \frac{1}{n!}\frac{d^n}{dx^n} = \frac{1}{n!}D_1^n.$$

In particular, D_1 is just the first derivative d/dx, while (1.2) can be rewritten more compactly in the form:

$$D_2(fg) = D_2f \cdot g + D_1f \cdot D_1g + f \cdot D_2g.$$

In general, by simple induction:

$$D_n(fg) = \sum_{i=0}^{n} D_if \cdot D_{n-i}g. \tag{1.3}$$

Again, by induction on the number of factors:

$$D_n(f_1 \dots f_k) = \sum_{n_1+\dots+n_k=n} D_{n_1}(f_1) \cdot \dots \cdot D_{n_k}(f_k). \tag{1.4}$$

The Reader will have no difficulty in proving that the following Newton binomial formula holds:

$$D_1^n(fg) = \sum_{h=0}^{n} \binom{n}{h} D_1^h f \cdot D_1^{n-h} g. \tag{1.5}$$

1.1.2 Example. The coefficient of $D_1^i f\cdot D_1^{n-i}g$ in the expansion (1.5) is the binomial coefficient $\binom{n}{i}$.

Formula (1.3) shall be referred to as Leibniz's rule for D_n. If t is an indeterminate over $C^\infty(\mathbb{R})$, then

$$D_t := \sum_{i\geq 0} t^i D_i : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})[[t]]$$

is the *Taylor formal power series.* The terminology is motivated by the fact that $\{(D_i f)(x_0), i\geq 0\}$ is precisely the set of coefficients of the Taylor series associated to the function f in a neighbourhood of the point x_0. Using the formal power series D_t, all the Leibniz's rules holding for D_i, $i\geq 0$, can be summarized by the following elegant equality:

$$D_t(f\cdot g) = D_t(f)\cdot D_t(g),$$

i.e. D_t is a $\mathbb{R}$-algebra homomorphism from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})[[t]]$. In general, if A is a commutative B-algebra, any B-algebra homomorphism $A\longrightarrow A[[t]]$ is said to be a *Hasse-Schmidt derivation* (Cf. [54], p. 208). To say it with a slogan, Hasse-Schmidt derivations realize everybody's student dream:

the "derivative" of the product is the product of "derivatives".

1.1.3 Determinants in $C^\infty(\mathbb{R})$. For $1\leq i\leq k$, let

$$\mathbf{u}_i = (u_{i1},\ldots,u_{ik}),$$

be a k-tuple of C^∞ real functions. For sake of brevity, denote by $\mathbf{u}_1\wedge\ldots\wedge\mathbf{u}_k$ the determinant:

$$\begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1k}\\ u_{21} & u_{22} & \ldots & u_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ u_{k1} & u_{k2} & \ldots & u_{kk}\end{vmatrix}. \tag{1.6}$$

The function

$$\mathbf{u}_1\wedge\ldots\wedge\mathbf{u}_k : \mathbb{R}\longrightarrow\mathbb{R}$$

is C^∞, being sums of products of C^∞ functions, and is skew-symmetric with respect to the arguments, namely:

$$\mathbf{u}_{\tau(1)} \wedge \ldots \wedge \mathbf{u}_{\tau(k)} = (-1)^{|\tau|}\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k,$$

where $\tau \in S_k$ is a permutation on k elements and $|\tau|$ is the number of simple transpositions modulo 2: this follows from the known properties of determinants. Moreover, if $\mathbf{u}_i = \mathbf{u}_j$, for some $i < j$, then $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k = 0$. We assume known the following formula:

$$D_1(\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k) = \sum_{i=1}^{k} \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_i' \wedge \ldots \wedge \mathbf{u}_k, \qquad (1.7)$$

where $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_i' \wedge \ldots \wedge \mathbf{u}_k$ denotes the determinant gotten from (1.6) by differentiating the i^{th} row and keeping untouched all the others. In other words:

the operator D_1 satisfies Leibniz's rule with respect to the symbol $\wedge$.

The proof of (1.7) merely consists in expanding the determinant as sum of products of the u_{ij}'s and then exploiting the linearity of D_1 together with the necessary iterations of Leibniz's rule.

Fantasy now suggests to embed the derivative D_1, with respect to $\wedge$, into a full family of differential operators

$$D := (D_1, D_2, \ldots),$$

by imitating formula (1.4), in a perfectly formal way:

$$D_h(\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k) = \sum_{\substack{h_1+\ldots+h_k=h \\ h_i \geq 0}} \mathbf{u}_1^{(h_1)} \wedge \ldots \wedge \mathbf{u}_k^{(h_k)}, \qquad (1.8)$$

where $\mathbf{u}_i^{(h)}$ means $D_1^h \mathbf{u}_i = (D_1^h u_{i1}, \ldots, D_1^h u_{ik})$.

1.2 Wronskians and their Derivatives

1.2.1 Let $\mathbf{f} := (f_1, \ldots, f_k)$ be a k-tuple of C^∞ real functions on the real line. The *wronskian determinant* of $\mathbf{f}$ is:

$$W(\mathbf{f}) := W(f_1, \ldots, f_k) = \begin{vmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{vmatrix}, \qquad (1.9)$$

where $f^{(i)} = D_1^i f$. As in Sect. 1.1.3, wronskians shall be shortly denoted as

$$\mathbf{f} \wedge \mathbf{f}' \wedge \ldots \wedge \mathbf{f}^{k-1}.$$

Since D_1 behaves obeying Leibniz's rule with respect to $\wedge$, one has, e.g.:

$$D_1(\mathbf{f} \wedge \mathbf{f}' \wedge \ldots \wedge \mathbf{f}^{k-1}) = \mathbf{f} \wedge \mathbf{f}' \wedge \ldots \wedge \mathbf{f}^{(k-2)} \wedge \mathbf{f}^{(k)}.$$

A *generalized wronskian* is an expression of the form:

$$\mathbf{f}^{(i_1)} \wedge \mathbf{f}^{(i_2)} \wedge \ldots \wedge \mathbf{f}^{(i_{k-1})}. \qquad (1.10)$$

The Reader may easily convince himself that iterated derivatives of "standard" wronskians like (1.9) are integral linear combinations of generalized wronskians.

1.2.2 Remark. What we called *generalized wronskian* is a particular case of a *Schmidt wronskian*, used by Schmidt ([72]) to study Weierstrass points in positive characteristic. If A is a B-algebra and $D_t := \sum_{i\geq 0} D_i t^i : A \longrightarrow A[[t]]$ is a Hasse-Schmidt derivation, he defines:

$$\Delta_{(i_1,\ldots,i_k)}(\mathbf{a}) = \Delta_{(i_1,\ldots,i_k)}(a_1, \ldots, a_k) = \begin{vmatrix} D_{i_1}a_1 & D_{i_1}a_2 & \ldots & D_{i_1}a_k \\ D_{i_2}a_1 & D_{i_2}a_2 & \ldots & D_{i_2}a_k \\ \vdots & \vdots & \ddots & \vdots \\ D_{i_k}a_1 & D_{i_k}a_2 & \ldots & D_{i_k}a_k \end{vmatrix}, \qquad (1.11)$$

for each k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $a_i \in A$, and any $1 < i_1 < i_2 < \ldots < i_k$. What we call, somewhat unproperly, a *generalized wronskian* is a Schmidth wronskian, where $D_i = D_1^i$.

1.2.3 Example. Let $\mathbf{f} := (f_1, f_2) \in C^\infty(\mathbb{R})$. Then

$$\mathbf{f}\wedge\mathbf{f}' = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1',$$

from which:

$$\begin{aligned} D_1(\mathbf{f}\wedge\mathbf{f}') &= D_1(f_1f_2' - f_2f_1') = \\ &= f_1'f_2' + f_1f_2'' - f_2'f_1' - f_2f_1'' = \\ &= f_1f_2'' - f_2f_1'' = \begin{vmatrix} f_1 & f_2 \\ f_1'' & f_2'' \end{vmatrix} = \mathbf{f}\wedge\mathbf{f}''. \end{aligned}$$

Similarly,

$$\begin{aligned} D_1(\mathbf{f}\wedge\mathbf{f}'') &= D_1(f_1f_2'' - f_2f_1'') = \\ &= f_1'f_2'' + f_1f_2''' - f_2'f_1'' - f_2f_1''' = \\ &= f_1'f_2'' - f_2'f_1'' + f_1f_2''' - f_2f_1''' = \\ &= \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 \\ f_1''' & f_2''' \end{vmatrix} = \mathbf{f}'\wedge\mathbf{f}'' + \mathbf{f}\wedge\mathbf{f}'''. \end{aligned}$$

Differentiating once more, one has:

$$D_1^3(\mathbf{f}\wedge\mathbf{f}') = D_1(\mathbf{f}'\wedge\mathbf{f}'' + \mathbf{f}\wedge\mathbf{f}''') = 2\mathbf{f}'\wedge\mathbf{f}''' + \mathbf{f}\wedge\mathbf{f}^{(iv)},$$

and then[1]

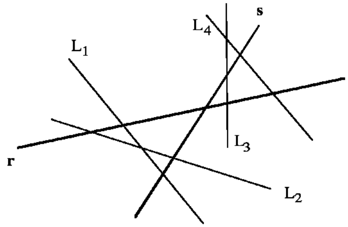
$$D_1^4(\mathbf{f}\wedge\mathbf{f}') = 2\cdot\mathbf{f}''\wedge\mathbf{f}''' + 3\mathbf{f}'\wedge\mathbf{f}^{(iv)} + \mathbf{f}\wedge\mathbf{f}^{(v)}. \qquad (1.12)$$

1.2.4 Claim. *Any Reader able to follow computations of Example 1.2.3 (who did not?) is also able to perform computations in the Chow (or cohomology) intersection ring of the grassmannians $G(2,n)$ $(n \geq 2)$ (See Sect. 1.4 below), whatever that means.*

Indeed, all the coefficients of the determinants occurring in the expansion of a derivative of a wronskian, are positive integers having beautiful interpretations in terms of enumerative geometry of projective spaces. For instance, the coefficient 2 multiplying $\mathbf{f}''\wedge\mathbf{f}'''$ in formula (1.12), *has computed*

[1]It is being implicitly assumed that no generalized wronskian vanishes identically as a function: this is the case, e.g., if f_1, f_2 form a basis of the space of solutions of a linear differential equation like $y'' - (a+b)y' + aby = 0$, with $a \neq b$.

the number of lines of the projective 3*-space intersecting* 4 *others*



in general position.

In general, the coefficient of $\mathbf{f}^{n-k} \wedge \mathbf{f}^{n-k+1} \wedge \ldots \wedge \mathbf{f}^{(n-1)}$ occurring in the expansion of $D_1^{k(n-k)}(\mathbf{f} \wedge \ldots \wedge \mathbf{f}^{(k-1)})$ is

the degree of the Plücker image of the complex grassmannian $G(k,n)$ *in* $\mathbb{P}^{\binom{n}{k}-1}$.

Explaining why the emphasized sentence holds true[2], is among the purposes of these notes. It basically amounts to prove Theorem 4.4.1 which implies that differentiating a wronskian is the same as applying a Pieri-like formula in the intersection ring of the grassmannian.

1.2.5 The operators D_h defined by formula (1.8) deserve a few more words. Indeed one has:

$$D_h(\mathbf{u}^{(i_1)} \wedge \mathbf{u}^{(i_2)} \wedge \ldots \wedge \mathbf{u}^{(i_k)}) =$$

$$= \sum_{\substack{h_1+\ldots+h_k=h \\ h_i \geq 0}} \mathbf{u}^{(i_1+h_1)} \wedge \mathbf{u}^{(i_2+h_2)} \wedge \ldots \wedge \mathbf{u}^{(i_k+h_k)}.$$

So, for example,

$$D_h(\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(k-1)}) = \mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(k-2)} \wedge \mathbf{u}^{(k+h-1)}.$$

[2] The explanation given in [48] is that the k^{th} exterior power of the polynomial ring in one indeterminate is a free module of rank 1 over the ring of symmetric functions on k-variables, which is in turn isomorphic to the intersection ring of the grassmannian of k-planes.

1.2.6 The other formal tool, that the Reader should be aware of, is *integration by parts* to find primitives of functions. For instance, wanting to look for a function $f \in C^\infty(\mathbb{R})$ such that $D_1 f = x\cos x$, one argues as follows:

$$\begin{aligned} x\cos\, x &= D_1(x\sin\, x) - \sin\, x = \\ &= D_1(x\sin\, x) - D_1(-\cos\, x) = \\ &= D_1(x\sin\, x + \cos\, x + c), \end{aligned}$$

having applied the formula

$$f \cdot D_1 g = D_1(fg) - D_1 f \cdot g,$$

often written in the calculus textbooks as:

$$\int f dg = fg - \int df \cdot g.$$

One wishes to practise a similar *integration by parts* on generalized wronskians, with the purpose to climb up to reach the wronskian itself.

$$\begin{aligned} &\mathbf{f}' \wedge \mathbf{f}'' = D_1(\mathbf{f} \wedge \mathbf{f}'') - \mathbf{f} \wedge \mathbf{f}''' = \\ = \;& D_1^2(\mathbf{f} \wedge \mathbf{f}') - D_2(\mathbf{f} \wedge \mathbf{f}') = (D_1^2 - D_2)(\mathbf{f} \wedge \mathbf{f}'). \end{aligned} \quad (1.13)$$

Equation (1.13) can also be written in the form:

$$\mathbf{f}' \wedge \mathbf{f}'' = \begin{vmatrix} D_1 & D_2 \\ 1 & D_1 \end{vmatrix} \mathbf{f} \wedge \mathbf{f}'.$$

Some healthy experimental mathematics, based on explicitly working out examples, may easily convince the Reader that suitable integrations by parts yield, in general:

$$\mathbf{u}^{(l_1)} \wedge \mathbf{u}^{(1+l_2)} \wedge \ldots \wedge \mathbf{u}^{(k-1+l_k)} = \det(D_{l_j - j + i}) \cdot \mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(k-1)} \quad (1.14)$$

where, by convention, $D_0 = 1$ and $D_h = 0$ if $h < 0$. Relation (1.14) is *Giambelli's formula* for wronskians or, possibly in spite of the most skeptical Reader, Giambelli's formula of Schubert calculus!

1.3 Algebraic Schubert Calculus

Now that the Reader believes to possess the main tools for playing with Schubert calculus, here is an attempt to define it in a purely abstract algebraic setting, with no reference at all to the cohomology of grassmannians. It is just an abstract formalization of the yoga of differentiating wronskians.

1.3.1 Let M_n be a free $\mathbb{Z}$-module of rank n spanned by $(\epsilon^1, \ldots, \epsilon^n)$[3] and, for each $k \geq 2$, let $\bigwedge^k M_n$ be the k^{th} exterior power of M_n (see Section 2.2). The latter is a $\mathbb{Z}$-module spanned by the symbols $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ with $1 \leq i_j \leq n$, $1 \leq j \leq k$ modulo the relations:

$$\epsilon^{i_{\tau(1)}} \wedge \ldots \wedge \epsilon^{i_{\tau(k)}} = (-1)^{|\tau|} \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k},$$

where $\tau \in S_k$ is any permutation on k-elements and $|\tau|$ is the number of its transpositions modulo 2. In particular $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = 0$ whenever $\epsilon^{i_j} = \epsilon^{i_l}$ for some $j < l$. It follows that $\bigwedge^k M_n$ is freely generated by $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ with $1 \leq i_1 < \ldots < i_k \leq n$. One sets, by convention, $\bigwedge^0 M_n = \mathbb{Z}$ and $\bigwedge^1 M_n = M_n$.

1.3.2 Let $D_1 : M_n \longrightarrow M_n$ be, now, the endomorphism sending $D_1 \epsilon^i$ onto ϵ^{i+1}, if $1 \leq i \leq n-1$, and to 0 otherwise. Denote by D_i the i^{th} iteration D_1^i of D_1, and set $D_1^0 = id_{M_n}$. Clearly $D_i(\epsilon^j) = 0$, if $i + j < n$ and $D_j = 0$, if $j \geq n$. The goal is now to extend the family of endomorphisms

$$D := \{D_i : M_n \longrightarrow M_n\}_{i \geq 0}$$

to endomorphisms of $\bigwedge^k M_n$, for all $2 \leq k \leq n$. Abusing notation, they will be still indicated by D_i. The extension will be achieved by imposing Leibniz's rule with respect to $\wedge$. Suppose that D_i has been extended to $\bigwedge^{k-1} M_n$. Since any $\eta \in \bigwedge^k M_n$ can be written as a finite sum of elements of the form $\epsilon^i \wedge \alpha_i$, $\alpha_i \in \bigwedge^{k-1} M_n$, it

[3] The reason why one puts the upper indices is twofold: on one side because it reminds us the "primes" used to denote derivatives, and on the other hand because it will be interpreted, later on, as a basis of a dual vector space, and in these notes we use the Einstein notation, to put lower indices to the element of a basis and the upper indices to the elements of its dual basis.

is sufficient to define the extension on elements of this form via the equality:

$$D_h(\epsilon^i \wedge \alpha_i) = \sum_{j=0}^{h} D_j\epsilon^i \wedge D_{h-j}\alpha_i.$$

Indeed, this makes D_h $(h \geq 0)$ into an endomorphism of the *exterior algebra of* M_n :

$$\bigwedge M_n := \bigoplus_{k\geq 0} \bigwedge^k M_n$$

where, by definition, $\bigwedge^0 M_n \cong \mathbb{Z}$ and $\bigwedge^1 M_n \cong M_n$. The sequence $D = \{D_i\}_{i\geq 0}$, as well as the formal power series $D_t = \sum_{i\geq 0} D_i t^i$ associated to it, will be named *Schubert derivation* ($\mathcal{S}$-derivation) (Cf. Sect. 4.3).

1.3.3 Example. Compute:

$$D_2(\epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6)$$

One has:

$$D_2(\epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6) = D_2(\epsilon^2) \wedge \epsilon^4 \wedge \epsilon^6 +$$
$$+D_1(\epsilon^2) \wedge D_1(\epsilon^4 \wedge \epsilon^6) + \epsilon^2 \wedge D_2(\epsilon^4 \wedge \epsilon^6) =$$

The first summand vanishes, since $D_2(\epsilon^2) = \epsilon^4$ and $\epsilon^4 \wedge \epsilon^4 = 0$, hence

$$D_2(\epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6) = \epsilon^3 \wedge (D_1\epsilon^4 \wedge \epsilon^6 + \epsilon^4 \wedge D_1\epsilon^6) + \epsilon^2 \wedge D_2(\epsilon^4 \wedge \epsilon^6) =$$
$$\epsilon^3 \wedge \epsilon^5 \wedge \epsilon^6 + \epsilon^3 \wedge \epsilon^4 \wedge \epsilon^7 + \epsilon^2 \wedge \epsilon^5 \wedge \epsilon^7 + \epsilon^2 \wedge \epsilon^4 \wedge \epsilon^8.$$

1.3.4 All the matter consists, then, in differentiating and cancelling terms with opposit sign. However, the practice of many examples suggests how to predict the terms surviving after vanishing and cancelations. This is the content of *Pieri's formula for $\mathcal{S}$-derivations* (Theorem 4.4.1):

$$D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{(h_i)} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k},$$

the sum over all $(h_1, \ldots, h_k)$ such that $\sum h_i = h$ and

$$i_1 \leq i_1 + h_1 < i_2 \leq \ldots \leq i_{k-1} + h_{k-1} < i_k.$$

To each strictly increasing sequence $I := (i_1, \ldots, i_k)$ of k positive integers, said to be a *k-schindex* in Section 2.1.5 , one may associate a *partition* $I(\underline{\lambda}) = (r_k, \ldots, r_1)$, where $r_j = i_j - j$. The *weight* of $\underline{\lambda}$ is $|\underline{\lambda}| = \sum r_i$. If one writes $\epsilon^{\underline{\lambda}}$ instead of $\epsilon^{1+r_1} \wedge \ldots \wedge \epsilon^{k+r_k}$, then Pieri's formula translates into:

$$D_h \epsilon^{\underline{\lambda}} = \sum_{\underline{\mu}} \epsilon^{\underline{\mu}},$$

where the sum is over all partitions $\underline{\mu} = (s_k \geq \ldots \geq s_1 \geq 0)$ such that $\sum s_i = \sum r_i + h$ and:

$$0 \leq r_1 \leq s_1 \leq r_2 \leq s_2 \leq \ldots \leq s_{h-1} \leq r_h \leq s_h, \tag{1.15}$$

coinciding with the *combinatorial Pieri's formula*: indeed, the *Young diagram* (Sect. 3.4) $Y(\underline{\mu})$ of $\underline{\mu}$ satisfying (1.15) is gotten by *adding i boxes to $Y(\underline{\lambda})$ in all possible ways, not two on the same column.*

Furthermore, everybody's basic calculus experience (Cf. Sect. 1.2.6) proves that, for any $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^k M_n$, there exists a $\mathbb{Z}$-polynomial expression $G_{i_1 \ldots i_k}(D)$ in the D_i's such that

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = G_{i_1 \ldots i_k}(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.$$

The above fact shall be formally proven in Section 4.5, Theorem 4.5.9. Useless to say, the proof consists in using *integration by parts* and it is nothing more than a mere generalization and formalization of the following example:

$$\begin{aligned}
\epsilon^2 \wedge \epsilon^3 &= D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge D_1 \epsilon^3 = \\
&= D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 = \\
&= D_1^2(\epsilon^1 \wedge \epsilon^2) - D_2(\epsilon^1 \wedge \epsilon^2) = \\
&= (D_1^2 - D_2)(\epsilon^1 \wedge \epsilon^2)
\end{aligned}$$

already encountered in "wronskian shape".

1.3.5 Exercise. Let $\epsilon^2 \wedge \epsilon^5 \wedge \epsilon^6 \in \bigwedge^3 M_n$, $n \geq 6$. Find a polynomial expression $G_{256}(D)$ in D_1, D_2, D_3 such that

$$\epsilon^2 \wedge \epsilon^5 \wedge \epsilon^6 = G_{256}(D) \cdot \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3.$$

(**Hint:** *I would start so:* $\epsilon^2 \wedge \epsilon^5 \wedge \epsilon^6 = D_1(\epsilon^1 \wedge \epsilon^5 \wedge \epsilon^6) - \epsilon^2 \wedge D_1(\epsilon^5 \wedge \epsilon^6) = \ldots$)

The general philosophy is that one does not need to know the explicit form of $G_{i_1\ldots i_k}$: everybody just computes its own expression[4]. The practise of many computations then develops the feeling for what should be the most rapid way to get the answer. It is however natural to apply the same "integration" procedure to figure out a canonical shape for the polynomial $G_{i_1\ldots i_k}$. This will be done in Sect. 4.8, where Giambelli's formula will be proven as a corollary of a more general expression holding in the exterior algebra of the module M (Theorem (4.8.1)).

1.3.6 To believe in the existence of $G_{i_1\ldots i_k}(D)$ such that

$$G_{i_1\ldots i_k}(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k = \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

means to believe that the map

$$P(D) \mapsto P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k$$

is surjective. This allows to define a product structure on $\bigwedge^k M$:

$$(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) \cdot (\epsilon^{j_1} \wedge \ldots \wedge \epsilon^{j_k}) = (G_{i_1\ldots i_k}(D) \cdot G_{j_1\ldots j_k}(D))\epsilon^1 \wedge \ldots \wedge \epsilon^k$$

which is indeed isomorphic to the intersection product in the Chow group of the grassmannian $G(k, n)$.

1.3.7 To end this section it seems worth to remark that formula (1.5) holds verbatim up to replacing $\cdot$ with $\wedge$:

$$D_1^n(\alpha \wedge \beta) = \sum_{i=0}^{n} \binom{n}{i} D_1^i \alpha \wedge D_1^{n-i}\beta.$$

The easy (and probably not necessary) proof is given in Sect. 5.3.1. Observe now that

$$D_1^h(\epsilon^1 \wedge \epsilon^2) = \sum_{\substack{0 \le i < j \\ i+j=h+3}} \alpha_{ij} \cdot \epsilon^i \wedge \epsilon^j,$$

with $\alpha_{ij} \in \mathbb{Z}$ to be determined. This is obvious by the definition of D_1. It follows that $\epsilon^{n-1} \wedge \epsilon^n$ can only occur in the expression of $D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2)$.

[4] There are more than one and all equivalent modulo the ideal of relations of the cohomology ring of the grassmannian (See Sect. 4.5)

1.3.8 Example (Analogous to 1.1.2). One aims to compute the coefficient of $\epsilon^{n-1}\wedge\epsilon^n$ in the expansion of $D_1^{2(n-2)}\epsilon^1\wedge\epsilon^2$. Notice that such a coefficient gets the contribution of that multiplying $\epsilon^{n-1}\wedge\epsilon^n$ as well as that multiplying $\epsilon^n\wedge\epsilon^{n-1}$. As a matter of fact one has:

$$D_1^{2(n-2)}(\epsilon^1\wedge\epsilon^2)) = \binom{2(n-2)}{n-2}\epsilon^{n-1}\wedge\epsilon^n + \binom{2(n-2)}{n-1}\epsilon^n\wedge\epsilon^{n-1} + o.t.$$

where o.t. means all the remaining summands. Therefore, the sought for coefficient is:

$$d_{2,n} = \binom{2(n-2)}{n-2} - \binom{2(n-2)}{n-1} = \frac{(2(n-2))!}{(n-2)!(n-1)!}.$$

See example 5.3.3.

Later on, it will be shown that this example is just computing

the degree of the Plücker image of the grassmannian $G(2,n)$

Or, alternatively,

the number of lines of $\mathbb{P}^{n-1}$ meeting $2(n-2)$ linear subspaces of dimension $n-3$ in general position.

1.4 Grassmannians

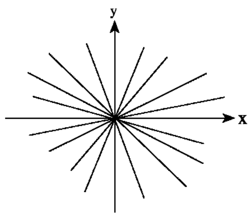
1.4.1 To deal with enumerative problems such as that of Example 1.3.8, one needs to introduce *grassmannians*. A grassmannian is primarily a linear algebraic object: if V is a vector space over any field $\mathbb{F}$, denote by $G_k(V)$ the set of all vector subspaces of V of dimension k. What can be said about $G_k(V)$? If $k=0$, then $G_0(V)=\{\mathbf{0}_V\}$ and if $\dim_{\mathbb{F}} V = n$, then $G_n(V)=V$. Since any inclusion $W_1\subseteq W_2$ of vector spaces implies the inequality $\dim(W_1)\le\dim(W_2)$, one sees that $G_k(V)=\emptyset$ whenever k is bigger than the dimension of V, when finite. If $\dim(V)=n\le 1$, there is not much to say: $G_0(V)$ and $G_1(V)$ consist of at most 1 element. One may think, at first, that even for higher dimensional vector spaces, the situation is not that interesting. Indeed, all the k-dimensional subspaces of V look the same in at least two different senses. The former is that any two vector spaces over the same field and of the same dimension are abstractly isomorphic. The second is that, given any two k-planes of V, there always exists an element of the general linear group $Gl(V)$ sending one into the other, as it will be proved in Proposition (3.2.1).

1.4.2 Another way to look at grassmannians is to think of them as the set of all k-dimensional linear subspaces of a projective space $\mathbb{P}^n$. A linear subspace of a projective space $\mathbb{P}(V)$ is the projective space associated to a vector subspace $W \subseteq V$. So, for instance, the grassmannian $G_2(\mathbb{R}^4)$ of 2-planes of $\mathbb{R}^4$ can be regarded as the grassmannian of projective lines of $\mathbb{P}^3(\mathbb{R})$. There is an obvious bijection of sets:

$$G_{1+k}(V) = G(k, \mathbb{P}(V)).$$

In particular, $\mathbb{P}(V) = G_1(V) = G(0, \mathbb{P}(V))$, i.e. *the set of lines of* V is the same as *the set of points of* $\mathbb{P}(V)$.

1.4.3 Example. The set $G_1(\mathbb{R}^2)$ can be thought of as the set of all the lines passing through the origin of the standard (x, y)-plane of high school analytic geometry.

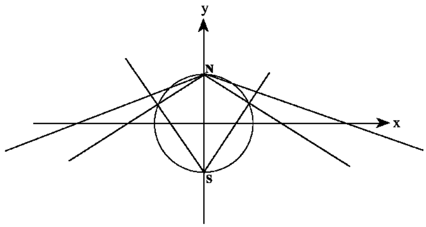


A pencil of lines is a model for the projective line $\mathbb{P}^1$

Any such line can be written in the form $ax + by = 0$, where (a, b) is a non-zero pair of $\mathbb{R}^2$, uniquely determined up to a non-zero multiple. The set $G_1(\mathbb{R}^2)$ is called the *projective* $\mathbb{R}$*-line*. It turns out that $\mathbb{P}^1(\mathbb{R}) := G_1(\mathbb{R}^2)$ is a compact connected differentiable manifold of dimension 1, diffeomorphic to the circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

with the differentiable structure induced by the *stereographic projections*.



Stereographic projections: from the "north" and "south" poles

Similarly, if $\mathbb{F} = \mathbb{C}$, then $\mathbb{P}^1(\mathbb{C}) := G_1(\mathbb{C}^2)$ is a compact connected *Riemann Surface* bi-holomorphically equivalent to the Riemann Sphere S^2 (see e.g. [10]).

1.4.4 For the limited purposes of these notes, even the field $\mathbb{R}$ is not enough to keep the exposition as simple as possible, without loosing elegance. Readers are invited to feel the reason through the following example, whose main character is the quadric hyperfurface

$$x_0x_5 - x_1x_4 + x_2x_3 = 0$$

of $\mathbb{P}^5(\mathbb{R})$, the real 5-dimensional projective space with its homogeneous coordinates $[x_0, x_1, \ldots, x_5]$. Consider the line

$$[v, 0, u, u, au + bv, v],$$

where $[u, v] \in \mathbb{P}^1$ and $(0, 0) \neq (a, b) \in \mathbb{R}^2$. Where do they intersect? Trying to solve the system one finds:

$$u^2 + v^2 = 0$$

which has no solution over the reals, while the line

$$[v, 0, u, -u, au + bv, v]$$

intersects the given quadric at two points. To avoid such a dichotomy, one wishes that any *general* line intersects a quadric surface at two distinct points. Allowing the coefficients to run over an algebraically closed field, and ours will be $\mathbb{C}$, one is led to work with *complex grassmannians*, parametrizing k-dimensional subspaces of a complex vector space V. The reason is that the intersection of a line in $\mathbb{P}^5$

with a quadric hypersurface, counts indeed the number of *lines in* $\mathbb{P}^3$ *intersecting four others in general position.* For this reason, in the rest of such introduction, all the vector spaces will be thought over the complex field.

1.4.5 The fact that any two k-planes are indistinguishable in the sense of Proposition 3.2.1 cries for the introduction of some kind of "landmark". For instance, picking an h-plane Π of V, a *general* k-plane will intersect it in a subspace of dimension $n-h-k$: general means, here, out of the space of solutions of some linear system of equations. However, they may be k-planes in special position intersecting Π in a subspace of dimension bigger than expected : if $h<k$, this is the case, for instance, of any k-plane containing Π! In this sense Π discriminates planes having an *expected* behaviour from those that are in special position. One may refine the *landamrk* : for instance one can pick two vector subspaces W_1 and W_2, of dimension h_1 and h_2 respectively, and then distinguish the k-planes into:

i) those which are in general position with respect to W_1 and W_2,

ii) those which are in general position with respect to one but not with respect to the other;

iii) and those which are in special position with respect to both.

One soon realizes that the finest description is achieved by refining such kinds of "reference systems" as much as possible. This naturally leads to the important notion of a *flag of subspaces of* V, introduced in Section 3.3

1.4.6 It turns out that the complex grassmannian $G_k(V)$ (or her projective sister $G_{k-1}(\mathbb{P}(V))$, is a complete smooth connected algebraic variety. As a geometrical object it is interesting in its own, either from the point of view of differential geometry (see e.g. [41], [11]) or from the point of view of algebraic geometry.

For the purposes of this introductory chapter it is enough for us to recall that very natural bundles (i.e. algebraic varieties which locally look like the product of a variety and a vector space) live on $G_k(V)$. Indeed, there is a *tautological exact sequence*:

$$0 \longrightarrow \mathcal{T}_k \longrightarrow G_k(V)\times V \longrightarrow \mathcal{Q}_k \longrightarrow 0$$

where

$$\mathcal{T}_k := \{([\Lambda],\mathbf{v}) \in G_k(V)\times V \mid \mathbf{v}\in[\Lambda]\}$$

is said to be the *tautological bundle*: the fiber of the projection $\pi : \mathcal{T}_k \longrightarrow G_k(V)$ over $[\Lambda] \in G_k(V)$ is the k-plane $[\Lambda]$ itself[5]. As for the *universal quotient bundle* $\mathcal{Q}_k$, its fiber over $[\Lambda]$ is the quotient vector space $V/[\Lambda]$. A linear form ϕ on V induces a bundle homomorphism:

$$\phi : \mathcal{T}_k \longrightarrow G_k(V) \times \mathbb{C},$$

defined by $\phi([\Lambda], \mathbf{v}) = ([\Lambda], \phi(\mathbf{v}))$, which will be very important for us. In fact the operators D_i quickly introduced in Section 1.3.2 can be identified with the Chern classes of the universal quotient bundle: they are the so calles *special Schubert cycles* on $G_k(V)$.

[5] The reason for brackets around Λ will be clear in Chapter 3.

Chapter 2

Preliminaries

The purpose of this section is to render the notes as self contained as possible. Most of the readers can skip this part or use it as a list of terms he may get better acquainted with in many good available references.

2.1 Combinatorics

2.1.1 Partitions. A *partition* is a non-increasing sequence

$$\underline{\lambda} = (\lambda_1, \lambda_2, \ldots)$$

of non-negative integers such that all but finitely many terms are 0. Partitions will be denoted with underlined greek letters (e.g. $\underline{\lambda}, \underline{\mu}, \underline{\nu}$).

Let $\mathcal{P}$ be the set of all partitions. The *length* of $\underline{\lambda} \in \mathcal{P}$ is the number of its *parts*:

$$\ell(\underline{\lambda}) = \sharp\{i \geq 1 \mid \lambda_i \neq 0\}.$$

while its *weight* is

$$|\underline{\lambda}| = \sum_{i\geq 1} \lambda_i.$$

One denotes by $\mathcal{P}_n$ the set of all partitions of the integer n, i.e.:

$$\mathcal{P}_n := \{\underline{\lambda} \in \mathcal{P} \mid |\underline{\lambda}| = n\}.$$

If $\underline{\lambda} \in \mathcal{P}_n$, then $1 \leq \ell(\underline{\lambda}) \leq n$. The partition of minimum length is (n) while that of maximum length is $\underbrace{(1, 1, \ldots, 1)}_{n \text{ times}}$. A partition of length k will be usually identified with the finite non-increasing sequence of its non-zero terms $(\lambda_1, \ldots, \lambda_k)$. Let:

$$\ell(k) := \{\underline{\lambda} \in \mathbb{N}^k \mid \underline{\lambda} \text{ is a partition of length } \leq \text{k}\}.$$

Then $\underline{\lambda} \in \ell(k)$ is a k-tuple of non increasing integers (some of them may be zero).

2.1.2 Example. The elements of the set $\mathcal{P}_5$ of the partitions of the integer 5 are:

$$(5), \quad (41), \quad (32), \quad (311), \quad (221), \quad (2111), \quad (11111),$$

where it has been omitted the infinite sequence of final zeroes.

2.1.3 Historical Remark. Let $p(n)$ be the cardinality of the set $\mathcal{P}_n$ of the partitions of the integer n. By convention $p(0) = 1$. Below is a table of the values of $p(n)$ for $n \leq 10$.

n	0	1	2	3	4	5	6	7	8	9	10
p(n)	1	1	2	3	5	7	11	15	22	30	42

A nice formula by Hardy and Ramanujan, see [55], shows that the function $p(n)$ is asymptotic to

$$\frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}$$

as $n \to \infty$. So, for instance, $p(100) \sim 1.9 \cdot 10^8$.

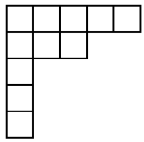
2.1.4 There are several way to represent a partition.

a) **Reverse Order.** For each $1 \leq i \leq k$ define:

$$r_i : \ell(k) \longrightarrow \mathbb{N},$$

by $r_i(\underline{\lambda}) = \lambda_{k+1-i}$. The k-tuple ${}^r\underline{\lambda} = (r_1(\underline{\lambda}), \ldots, r_k(\underline{\lambda}))$ is $\underline{\lambda}$ in the *reverse order.* Most of the times, $\underline{\lambda}$ will be skipped from the notation and one will shortly write r_i instead of $r_i(\underline{\lambda})$. The partition $\underline{\lambda}$ will be simply written as $(r_k, \ldots, r_1)$.

b) **Young Diagrams.** To each partition $\underline{\lambda} \in \mathcal{P}$, one may associate its *Young diagram* $Y(\underline{\lambda})$. The diagram $Y(\underline{\lambda})$ is an array with k-rows, the rows are left-aligned boxes such that the i-th rows has λ_i boxes. The Young diagram of the partition (53111) is depicted below.



More formally:

$$Y(\underline{\lambda}) := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i\}.$$

The length of a partition is the number of rows of its Young diagram while its weight is the number of boxes.

c) **Multiplicities.** Define

$$m_i(\underline{\lambda}) = \sharp\{\lambda_j \mid \lambda_j = i\}.$$

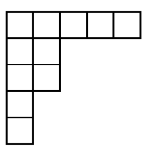
The integer $m_i(\underline{\lambda})$ is the *multiplicity* which the integer i occurs with in the partition $\underline{\lambda}$. Then a partition can be also written as

$$\underline{\lambda} = (1^{m_1} 2^{m_2} \dots h^{m_h}).$$

In this case one has:

$$\ell(\underline{\lambda}) = m_1 + \ldots + m_h \qquad \text{and} \qquad |\underline{\lambda}| = m_1 + 2m_2 + \ldots + hm_h.$$

The *conjugated* of a partition $\underline{\lambda}$ is the partition $\underline{\lambda}'$ whose Young diagram $Y(\underline{\lambda}')$ is the *transpose* of $Y(\underline{\lambda})$. For example, if $\underline{\lambda} = (53111)$, then $\underline{\lambda} = (52211)$ with Young diagram depicted below



2.1.5 Schindices. Let

$$\mathcal{I}^k = \{(i_1, \ldots, i_k) \in \mathbb{N}^k \,|\, 1 \le i_1 < i_2 < \ldots < i_k\}$$

and

$$\mathcal{I}_n^k = \{I \in \mathcal{I}^k \mid i_k \le n\}.$$

In other words, $\mathcal{I}^k$ is the set of all increasing ordered k-tuples of positive integers. Any such k-tuple will be called, for comodity, a k-*schindex*, as a contraction of *Schubert* and *index*. The reason for this name is that for each k-schindex I there is a *grassmannian* $G_k(V)$ with a *Schubert variety* having precisely that *Schubert index*. Let $\mathcal{I} = \cup_{k \ge 0} \mathcal{I}^k$. The set $\mathcal{I}$ is said to be the set of *schindices*. Any $I \in \mathcal{I}$ is a k-schindex for some k. Schindices will be denoted by capital roman letters $I, J, K, \ldots$.

2.1.6 Schindices and Partitions are strictly related. To each $I \in \mathcal{I}^k$, one may associate the partition $\underline{\lambda}(I)$ defined by:

$$\underline{\lambda}(I) = (i_k - k, \ldots, i_1 - 1).$$

Conversely, if $\underline{\lambda} = (r_k, \ldots, r_1) \in \ell(k)$, one sets:

$$I(\underline{\lambda}) = (1 + r_1, \ldots, k + r_k).$$

Clearly $\underline{\lambda} : \mathcal{I}^k \longrightarrow \mathcal{P}$ and $I : \mathcal{P} \longrightarrow \mathcal{I}^k$ are bijections inverse one of each other. The notation for such bijections is clearly abused. The *length* of a k-schindex is k, while its *weight* is, by definition,

$$wt(I) = |\underline{\lambda}(I)| = \sum_{i=1}^{k} r_i = (i_1 - 1) + \ldots + (i_k - k).$$

On the set $\mathcal{I}^k$ we shall put two different orders:

a) The *Bruhat-Chevalley* order:

$$(i_1, \ldots, i_k) \preceq (j_1, \ldots, j_k) \iff i_p \le j_p, \quad \forall\, 1 \le p \le k,$$

writing $(i_1, \ldots, i_k) \prec (j_1, \ldots, j_k)$ if $(i_1, \ldots, i_k) \preceq (j_1, \ldots, j_k)$ and $(i_1, \ldots, i_k) \ne (j_1, \ldots, j_k)$.

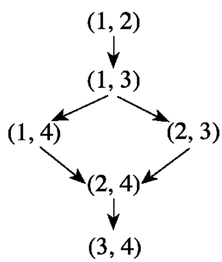
b) The *lexicographic order*:

$$(i_1, \ldots, i_k) \preceq (j_1, \ldots, j_k)$$

if and only if either $i_p = j_p$ for all $1 \leq p \leq k$ or if

$$q := \min\{p \mid i_p \neq j_p\} \Rightarrow i_q < j_q.$$

2.1.7 Example. The set $\mathcal{I}_4^2 = \{(i_1, i_2) \in \mathcal{I}^2 \mid i_2 \leq 4\}$ can be ordered be ordered, according to the Bruhat order, as follows:



while in the lexicographic order:

$$(1,2) \prec (1,3) \prec (1,4) \prec (2,3) \prec (2,4) \prec (3,4).$$

2.1.8 The ring $\mathbb{Z}[\mathbf{T}]$. Let $\mathbf{T} = (T_1, T_2, \ldots)$ be an infinite sequence of indeterminates over $\mathbb{Z}$ and denote by $\mathbb{Z}[\mathbf{T}]$ the ring $\mathbb{Z}[T_1, T_2, \ldots]$ of polynomials in the indeterminates $\mathbf{T}$ with $\mathbb{Z}$-coefficients. For each partition

$$\underline{\lambda} = (1^{m_1} \ldots h^{m_h}),$$

define the monomial $\mathbf{T}^{\underline{\lambda}}$ by:

$$\mathbf{T}^{\underline{\lambda}} = T_1^{m_1} \cdot \ldots \cdot T_h^{m_h}.$$

Then $\mathbb{Z}[\mathbf{T}]$ admits a direct sum decomposition of submodules:

$$\mathbb{Z}[\mathbf{T}] = \bigoplus_{h \geq 0} \mathbb{Z}[\mathbf{T}]_h,$$

where $\mathbb{Z}[\mathbf{T}]_h = \bigoplus_{|\underline{\lambda}|=h} \mathbb{Z} \cdot \mathbf{T}^{\underline{\lambda}}$. For example:

$$\mathbb{Z}[\mathbf{T}]_0 = \mathbb{Z}; \quad \mathbb{Z}[\mathbf{T}]_1 = \mathbb{Z} \cdot T_1; \quad \mathbb{Z}[\mathbf{T}]_2 = \mathbb{Z} \cdot T_1^2 \oplus \mathbb{Z} \cdot T_2.$$

One defines the *degree of a monomial* $\mathbf{T}^{\underline{\lambda}}$ as the *weight* of the partition $\underline{\lambda}$. Then $\deg(T_i) = i$, for all $i \geq 0$ (where by convention $T_0 = 1$). The ring $\mathbb{Z}[T_1, \ldots, T_n]$ of polynomials in n-indeterminates is a quotient of $\mathbb{Z}[\mathbf{T}]$.

2.1.9 Let A be a commutative ring with unit and let $c : \mathbb{N} \longrightarrow A$ be a sequence. Set $c_i = c(i)$. To any partition $\underline{\lambda} = (r_k, r_{k-1}, \ldots, r_1)$ of length k, let $\Delta_{\underline{\lambda}}(c) \in A$ be defined as:

$$\Delta_{\underline{\lambda}}(c) = \begin{vmatrix} c_{r_1} & c_{r_2+1} & \cdots & c_{r_k+k-1} \\ c_{r_1-1} & c_{r_2} & \cdots & c_{r_k+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r_1-k} & c_{r_2-k+1} & \cdots & c_{r_k} \end{vmatrix} = \det(c_{r_j+j-i}). \qquad (2.1)$$

setting $c_i = 0$ if $i < 0$. Let $I(\underline{\lambda}) = I = (i_1, \ldots, i_k)$ be the k-schindex associated to the partition $\underline{\lambda}$. Then (2.1) can be equivalently written as:

$$\Delta_I(c) := \Delta_{(i_1 \ldots i_k)}(c) = \begin{vmatrix} c_{i_1-1} & c_{i_2-1} & \cdots & c_{i_k-1} \\ c_{i_1-2} & c_{i_2-2} & \cdots & c_{i_k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1-k} & c_{i_2-k} & \cdots & c_{i_k-k} \end{vmatrix}. \qquad (2.2)$$

Both equations (2.1) and (2.2) are said to be *Giambelli's determinant* of c associated to the partition $\underline{\lambda}$ or to the k-schindex I. If $A := \mathbb{Z}[\mathbf{T}]$ as above, one writes $\Delta_{\underline{\lambda}}(\mathbf{T}) \in \mathbb{Z}[\mathbf{T}]$ (or $\Delta_I(\mathbf{T})$) for the corresponding Giambelli's determinants associated to the sequence $\mathbf{T} = (T_1, T_2, \ldots)$, i.e. $\Delta_{\underline{\lambda}}(\mathbf{T}) = \det(T_{r_j+j-i})$.

2.2 Exterior Algebra of a Free Module

2.2.1 To any module M over a commutative ring A, one may associate an algebra $(T(M), \otimes)$, said to be *tensor algebra* (see e.g. [2]). The *exterior (or Grassmann) algebra* $\bigwedge M$ of M is a suitable quotient of $T(M)$. For the limited purposes of this exposition, it is sufficient to look at free modules over an integral domain A of characteristic 0[1]

[1] Vector spaces, among them.

and, in this case, we shall give a practical *ad hoc* definition of exterior algebra, ready for use.

Let M be a free A-module of infinite countable rank, generated by $\mathcal{E} = (\epsilon^1, \epsilon^2, \ldots)$.

2.2.2 Definition. *Define* $\bigwedge^0 M = A$*,* $\bigwedge^1 M = M$ *and, for all* $k \geq 2$*, let* $\bigwedge^k M$ *be the* A*-module generated by all the expressions of the form* $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ *modulo the relations:*

$$\epsilon^{i_{\tau(1)}} \wedge \ldots \wedge \epsilon^{i_{\tau(k)}} = (-1)^{|\tau|} \cdot \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}. \tag{2.3}$$

for all $\tau \in S_k$*. For all* $k \geq 0$*, the* A*-module* $\bigwedge^k M$ *is said to be the* k^{th} exterior power *of* M*.*

Since the characteristic of the ring is 0 (hence $\neq 2$), relation (2.3) implies that $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = 0$ whenever $i_{j_1} = i_{j_2}$ for some $j_1 < j_2$. It follows that a basis of $\bigwedge^k M$ is given by

$$\bigwedge^k \mathcal{E} = \{\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} : 1 \leq i_1 < \ldots < i_k\},$$

the elements being ordered lexicographically. One has clearly a map

$$\bigwedge^h M \times \bigwedge^k M \longrightarrow \bigwedge^{h+k} M.$$

which is the unique A-bilinear extension of that sending

$$(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_h}, \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) \mapsto \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_h} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}. \tag{2.4}$$

2.2.3 Definition. *The* exterior algebra *of* M *is the pair* $(\bigwedge M, \wedge)$*, where*

$$\bigwedge M := \bigoplus_{k \geq 0} \bigwedge^k M = A \oplus M \oplus \bigwedge^2 M \oplus \ldots$$

and the product $\wedge$ *is defined according to formula (2.4).*

Any $\bigwedge^k M$ will be thought of as an A-submodule of $\bigwedge M$. If $\alpha \in \bigwedge^k M \subset \bigwedge M$, then α is said to be homogeneous of degree k. Any $\alpha \in \bigwedge^k M$ is a finite A-linear combination of elements of $\bigwedge^k \mathcal{E}$. Any

$\beta \in \bigwedge M$ is a finite sum of homogeneous elements. There is an obvious map $\underbrace{M \times \ldots \times M}_{k\ times} \longrightarrow \bigwedge^k M$ defined by

$$(m_1, \ldots, m_k) \mapsto m_1 \wedge \ldots \wedge m_k.$$

Such a map is neither injective (any $\mathbf{m} \in M^k$ with at least two equal components is mapped to 0) nor surjective, as it can be shown on the basis of easy examples: the image is said to be the set of *decomposable elements.*

2.2.4 Exterior Algebras of Finite Modules. The same construction holds verbatim for the free submodule M_n of M spanned by $\mathcal{E}_n = (\epsilon^1, \ldots, \epsilon^n)$. A basis for $\bigwedge^k M_n$ is then

$$\bigwedge^k \mathcal{E}_n = \{\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^k \mathcal{E} \mid i_k \leq n\}.$$

Clearly one has $\bigwedge^k M_n = 0$ if $k > n$, because in this case any $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ must have two equal indices. One has:

$$rk_A(\bigwedge^k M_n) = \sharp(\mathcal{I}_n^k) = \binom{n}{k}$$

Therefore

$$\bigwedge M_n = \bigoplus \bigwedge^k M_n = A \oplus M_n \oplus \bigwedge^2 M_n \oplus \ldots \oplus \bigwedge^n M_n$$

is itself a free module of finite rank equal to

$$\sum_{k \geq 0} rk_A \left(\bigwedge^k M_n \right) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

2.2.5 Example. Let $M_4 = \mathbb{Z}\epsilon^1 \oplus \mathbb{Z}\epsilon^2 \oplus \mathbb{Z}\epsilon^3 \oplus \mathbb{Z}\epsilon^4$. Then a basis of $\bigwedge^2 M$, with lexicographic ordering, is:

$$(\epsilon^1 \wedge \epsilon^2, \epsilon^1 \wedge \epsilon^3, \epsilon^1 \wedge \epsilon^4, \epsilon^2 \wedge \epsilon^3, \epsilon^2 \wedge \epsilon^4, \epsilon^3 \wedge \epsilon^4).$$

Then $rk_{\mathbb{Z}}(M) = 6$. Similarly, a basis of $\bigwedge^3 M_4$ is

$$(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3, \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^4, \epsilon^1 \wedge \epsilon^3 \wedge \epsilon^4, \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4).$$

Then the rank of $\bigwedge^3 M_4$ is 4.

2.2.6 Example. Let $\phi : V \longrightarrow W$ be a homomorphism of $\mathbb{F}$-vector spaces. The map ϕ induces a map $\bigwedge^k \phi : \bigwedge^k V \longrightarrow \bigwedge^k W$ defined as

$$(\bigwedge^k \phi)(\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k) = \phi(\mathbf{v}_1) \wedge \ldots \wedge \phi(\mathbf{v}_k).$$

One checks that $rk(\phi) = k$ if $\bigwedge^k \phi \neq 0$ and $\bigwedge^{k+1} \phi = 0$. If $\phi \in End_{\mathbb{F}}(V)$ and $\dim(V) = n$, then $\bigwedge^n \phi(e_1 \wedge \ldots \wedge e_n) = a \cdot e_1 \wedge \ldots \wedge e_n$, where $E_n = (e_1, \ldots, e_n)$ is a basis of V. The scalar $a \in \mathbb{F}$ is said to be the *determinant* of the map ϕ.

2.3 Review of Intersection Theory

Schubert calculus for grassmannians is a formalism enabling to perform explicit computations in the Chow intersection ring (or integral Cohomology Ring) of a complex grassmannian variety – a smooth algebraic projective variety. The aim of this section is then to quickly collect a few notions intended to describe the idea of Chow ring of a smooth complex projective variety. Universal references for the section below are [16] and [17]. The brazilian reader should also look at [78].

2.3.1 Order Along a Subvariety. Let X be a smooth complex projective variety of dimension n and let O_X be its structure sheaf. The function field $k(X)$ of X is the quotient field of the integral domain $O_X(U)$, for some affine open subset U of X. The definition makes sense because if V is another affine open set of X, then the quotients fields of $O_X(U)$ and $O_X(V)$ are isomorphic. If $r \in k(X)$, it can be written as a/b, with $a \in O_X(U)$ and $b \in O_X(U)^* := O_X(U) \setminus \{0\}$, for some affine open set U. If $a \in O_X(U)$ and Y is a $(n-1)$-dimensional subvariety of X such that $Y \cap U \neq \emptyset$, write:

$$ord_Y(a) = l_{A_Y(U)}\left(\frac{A_Y(U)}{\mathbf{p}A_Y(U)}\right),$$

where $A_Y(U)$ denotes the localization of $O_X(U)$ at the prime $\mathbf{p}$ defining $Y \cap U$ in U and $l_A(M)$ denotes the *lenght* of an A-module M (Cf. [16], p. 8 and p. 407). If $r \in k(X)^*$, one sets:

$$ord_Y(r) = ord_Y(a) - ord_Y(b).$$

2.3.2 Cycles. The group $Z_k(X)$ of k-cycles of the variety X is the free $\mathbb{Z}$-module generated by all the irreducible subvarieties of X of dimension k. The group of cycles of X is

$$Z_*(X) := Z_0(X) \oplus Z_1(X) \oplus \ldots \oplus Z_n(X).$$

If V is a subvariety of X of dimension k one shall denote by $[V]$ the corresponding k-cycle. To each $r \in k(X)^*$ one may associate the cycle

$$div(r) := \sum_{[Y] \in Z_{n-1}(X)} ord_Y(r)[Y].$$

One can prove that the above sum is finite (Cf. [16], p. 432).

2.3.3 Rational Equivalence. One says that $V \in Z_k(X)$ is *rationally equivalent* to 0 ($V \sim 0$) if there are $(k+1)$-dimensional subvarieties $W_1, \ldots, W_j$ and $r_i \in k(W_i)^*$ such that

$$[V] = \sum div(r_i).$$

The k-cycles rationally equivalent to zero form a $\mathbb{Z}$-submodule $Rat_k(X)$ of $Z_k(X)$. The quotient:

$$A_k(X) := \frac{Z_k(X)}{Rat_k(X)}$$

is said to be the Chow group of k-dimensional cycles modulo rational equivalence. The *Chow group* (of cycles modulo rational equivalence) of X is:

$$A_*(X) := \bigoplus_{i \geq 0} A_i(X).$$

In particular, the group $A_n(X)$ is a free $\mathbb{Z}$-module of rank 1 generated by $[X] \in Z_n(X) = A_n(X)$, said to be the *fundamental class* of X.

Each proper morphism $f : X \longrightarrow Y$ of $\mathbb{C}$-schemes induces a $\mathbb{Z}$-module homomorphism:

$$f_* : Z_k(X) \longrightarrow Z_k(Y),$$

defined as follows. If $[V]$ is a generator of $Z_k(X)$, then:

$$f_*[V] = \begin{cases} deg(f_{|V}) \cdot [f(V)] & \text{if } dim(V) = dim(f(V) \\ 0 & \text{if } dim(f(V)) < dim(V) \end{cases},$$

where by $deg(f_{|_V})$ we mean the degree of the algebraic field extension $[K(V) : K(f(V))]$. Moreover, if $f : X \longrightarrow Y$ is a flat morphism of relative dimension m, one may define a *pull-back* map f^* at the level of the cycles. It is defined as:

$$\begin{aligned} f^* \quad : \quad & Z_k(Y) \longrightarrow Z_{k+m}(X) \\ & [V] \mapsto f^*([V]) = [f^{-1}(V)]. \end{aligned}$$

Now, let $f : X \longrightarrow Y$ be a proper morphism. If $\alpha \in Rat_k(X)$, then $f_*(\alpha) \in Rat_k(Y)$, while if f is flat of relative dimension m and $\alpha \in Rat_k(Y)$, then $f^*\alpha \in Rat_{k+m}(X)$. This implies that f_* and f^* define two homomorphisms of graded $\mathbb{Z}$-modules:

$$f_* \quad : \quad A_*(X) \longrightarrow A_*(Y) \quad \text{and} \tag{2.5}$$

$$f^* \quad : \quad A_*(Y) \longrightarrow A_*(X). \tag{2.6}$$

Such a fact is proven in [16], Ch. 1. The homomorphisms (2.5) and (2.6) shall be called respectively the *proper push-forward* (under the assumption that f is proper) and the *flat pull-back* (under the assumption that f is flat). From now on, if V is a k-dimensional subvariety, the symbol $[V]$ will denote its Chow class, i.e. the k-cycle determined by it, *modulo rational equivalence.*

Let now W be a k-dimensional scheme. Let W_i be its irreducible components. Then the local rings O_{W,W_i} are *artinian local rings* ([2]). Define the *multiplicity* as the O_{W,W_i}-lenght of the module O_{W,W_i} itself. Set:

$$m_i = \ell_{O_{W,W_i}}(O_{W,W_i}).$$

2.3.4 Definition. *The* fundamental class *of a k-dimensional subscheme is defined to be the Chow class:*

$$[W] = \sum_i m_i[W_i] \in A_k(X).$$

Notice that if X is irreducible and reduced then the fundamental class of W is simply the rational equivalence class of the cycle $[W]$.

Another very important definition, which we already used more than once, without setting it, is that of the *degree of a cycle.*

2.3.5 Definition. *Let X be a proper scheme of finite type over $Spec(\mathbb{F})$, $\mathbb{F}$ being any algebraically closed field. The* degree homomorphism*:*

$$\int_X : A_*(X) \longrightarrow \mathbb{Z},$$

is defined as:

$$\int_X \alpha = \begin{cases} \sum_P n_P[k(P):\mathbb{F}] & if \quad \alpha = n_P[P] \in A_0(X) \\ 0 & if \quad \alpha \in A_i(X), \quad i > 0. \end{cases}$$

In the above formula $[k(P):\mathbb{F}]$ denotes the algebraic degree extension of $\mathbb{F}$ by the field of the point P, $k(P) = O_P/m_P$, m_P being the maximal ideal of the local ring O_P.

Clearly $[k(P):\mathbb{F}] = 1$ when $\mathbb{F}$ (as in our case) is algebraically closed. If $f : X \longrightarrow Y$ is a morphism of schemes proper over $Spec(\mathbb{F})$, then:

$$\int_X \alpha = \int_Y f_*(\alpha).$$

2.3.6 Example. Let $\mathbb{A}^n$ be the affine n-space. Then $A^n(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$. We claim that $A_0(\mathbb{A}^n) = 0$: in fact any point in the support of a 0-cycle $c = \sum_P n_P P$ belongs to a line, and on a line any 0-cycle is the set of poles and zeros of a rational function. Therefore $\sum n_p P = div(r_P)$ where r_P is some rational function on some line passing through P, and then equivalent to 0. From the fact that the obvious projection maps $p : \mathbb{A}^n \longrightarrow \mathbb{A}^{n-k}$ forgetting the first k coordinates induces a surjective map $p^* : A_i(\mathbb{A}^{n-k}) \longrightarrow A_{i+k}(\mathbb{A}^{n-k})$ (Cf. [16], p. 22), it follows that:

$$A_i(\mathbb{A}^n) = p^*(A_0(\mathbb{A}^{n-i})) = 0.$$

2.3.7 The Chow Ring. One defines:

$$A^k(X) := A_{n-k}(X),$$

the *Chow group of cycles of X of codimension k*. There is an obvious $\mathbb{Z}$-module isomorphism between $A^*(X) := \oplus A^i(X)$ and $A_*(X)$. If X is smooth (as in our hypothesis), one can put on $A^*(X) \cong A_*(X)$ an *intersection product*

$$\begin{cases} \cdot \quad : & A^i(X) \times A^j(X) & \longrightarrow & A^{i+j}(X) \\ & (\alpha, \beta) & \longmapsto & \alpha \cdot \beta \end{cases},$$

making it into a ring. Such a product has a geometrical interepretation. Recall that two subvarieties V_1 and V_2 of X intersect *properly* if and only if

$$codim(V_1 \cap V_2) = codim(V_1) + codim(V_2).$$

Moreover they intersect *transversally* along W, if $\exists w \in W$ such that $T_w V_1 + T_w V_2 = T_w X$: the point w is then non singular for W and one has $T_w W = T_w V_1 \cap T_w V_2$. If V_1 and V_2 intersect properly,

$$[V_1] \cdot [V_2] = [V_1 \cap V_2] = \sum_W m_W [W] \tag{2.7}$$

where the sum is over all the irreducible subvarieties W of the scheme theoretical intersection $V_1 \cap V_2$ and m_W is the *intersection multiplicity* of V_1 and V_2 along W. The intersection multiplicity is defined in such a way that if V_1 and V_2 intersect transversally along W, then $m_W = 1$. This is the reason why the product above is also called *intersection product.*

2.3.8 Example. If $A_0(X) \cong \mathbb{Z}$, generated by the class $[pt]$ of a point (like, e.g., in the case of grassmannians) $codim(V_1)+codim(V_2) = \dim(X)$, and V_1 and V_2 intersect transversally, then

$$[V_1] \cup [V_2] = \sharp(V_1 \cap V_2)[pt].$$

However, the number on the right hand side does not depend on V_1 and V_2 but only on their classes modulo rational equivalence.

2.3.9 Intersection product on homogeneous varieties. If X is transitively acted (say) on the left by a connected algebraic group G (i.e. X is a G-homogeneous variety), an important theorem of Kleiman ([37]) ensures that if $[Y_1]$ and $[Y_2]$ are any two cycles, then there exists a Zariski dense open set $U \subset G$ such that for each $g \in U$, Y_1 and $L_g(Y_2)$ meet properly, where $L_g : X \longrightarrow X$ denotes the left translation. In this case, according to (2.7),

$$[Y_1] \cdot [gY_2] = [Y_1 \cap L_g(Y_2)].$$

Moreover, if the group is rational, e.g. $G = Gl_n(\mathbb{C})$ as in the case of grassmannians, then the map (L_g) is proper and

$$(L_g)_* : A_*(X) \longrightarrow A_*(X)$$

is the identity (Cf. [16], p. 207). Then, in particular, the *self intersection* $[Y]^2$ is represented by the intersection of Y with a general translate of it. The conditions above are met for grassmannians varieties (See Proposition 3.2.1).

2.3.10 The Chow group as a module over the Chow ring. The Chow group $A_*(X)$ inherits a structure of module over $A^*(X)$ with respect to the *cap product*:

$$\begin{cases} \cap \ : \ A^*(X) \times A_*(X) & \longrightarrow \quad A_*(X) \\ \qquad\qquad (\alpha, [V]) & \longmapsto \quad \alpha \cap [V]. \end{cases}$$

Poincaré duality says that the map

$$\alpha \mapsto \alpha \cap [X]$$

induces an isomorphism between $A^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $A_{n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The variety X being smooth, the above isomorphism holds indeed over the integers. It is not easy, in general, to compute the Chow ring even of a nice smooth complex variety.

2.3.11 Suppose that $U \subseteq X$ is a Zariski open set and let $Y := X \setminus U$ be its (closed) complement in X. Let $i : U \longrightarrow X$ and $j : Y \longrightarrow X$ be the inclusions. Then the following exact sequence holds

$$A_i(Y) \overset{j_*}{\longrightarrow} A_i(X) \overset{i^*}{\longrightarrow} A_i(U) \longrightarrow 0. \tag{2.8}$$

The proof consists first in showing that the exact sequence holds at the cycle level:

$$Z_i(Y) \longrightarrow Z_i(X) \longrightarrow Z_i(U) \longrightarrow 0,$$

and then that the above maps pass to rational equivalence. See [16].

2.3.12 Example. One has that $A_n(\mathbb{P}^n) = \mathbb{Z}[\mathbb{P}^n]$ and $A^i(\mathbb{P}^n) = 0$ for $i > n$. If $0 < r < n$, $A_r(\mathbb{P}^n)$ is generated by the class of a linear subspace $H^{(r)} \cong \mathbb{P}^r$ of dimension r. In fact exact sequence (2.8) gives:

$$A_{n-1}(H^{(1)}) \longrightarrow A_{n-1}(\mathbb{P}^n) \longrightarrow A_{n-1}(\mathbb{P}^n \setminus H^{(1)}) \longrightarrow 0$$

and this proves that $A_{n-1}(\mathbb{P}^n)$ is generated by $H^{(1)}$. Similarly one sees that $A_{n-i}(\mathbb{P}^n)$ is generated by $A_{n-i}(\mathbb{P}^{n-1})$ which, by induction, is generated by a linear space of dimension $n - i$.

2.3.13 Vector Bundles. A *holomorphic vector bundle* of rank r over X is a complex algebraic variety V together with a surjective morphism $\pi : V \longrightarrow X$, such that the map π is locally trivial (in the holomorphic or algebraic category). The latter means that there is a family (U, ψ_U), where $\{U\}$ is an open covering of X, and such that the map $\psi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$ is a biholomorphic bijection such that $pr_2 \circ \psi_U = \pi$. It follows that the transition functions $g_{UV} := \psi_U \circ \psi_V^{-1} : \psi_V(V \cap U) \longrightarrow \psi_U(V \cap U)$ are biholomorphic. A meromorphic, (resp. holomorphic) section of a vector bundle is a meromorphic (resp. holomorphic) map $s : X \longrightarrow V$ such that $\pi \circ \sigma = id_X$ and such that $s_U : U \longrightarrow \mathbb{C}^r$ defined as $\psi_U \circ s_{|U}$ is a meromorphic (resp. holomorphic). Then, locally, a holomorphic section s is represented by r holomorphic functions on U:

$$s_U = (s_1, \ldots, s_r).$$

If U is an affine open set of X, the section is said to be *regular* if $(s_1, \ldots, s_r)$ is a regular sequence in the ring $O_X(U)$. Denote by $O_X^{\oplus n}$ the trivial vector bundle $pr_1 : X \times \mathbb{C}^r \longrightarrow X$ of rank r, pr_1 being the projection onto the 1^{st} factor. Its sections can be seen as functions from $X \longrightarrow \mathbb{C}^r$. The structural sheaf O_X can be seen as the trivial vector bundle of rank 1 over X. If V_1 and V_2 are two vector bundles of rank r_1 and r_2, respectively, a *bundle morphism* is a morphism $\phi : V_1 \longrightarrow V_2$ such that $\pi_2 \circ \phi = \pi_1$ and $\phi_x : (V_1)_x \longrightarrow (V_2)_{\phi(x)}$ is a vector space homomorphism.

2.3.14 Chern Classes. Any vector bundle over X of rank 1 will be said to be a *line bundle*. If s is a regular non-zero holomorphic section of a line bundle L, the zero scheme $Z(s) \subseteq X$ is a codimension 1 subvariety said to be a *Cartier divisor*. By definition $c_1(L)$ is the class in $A^1(X)$ of $Z(s)$. The set of all Cartier divisors is a group called $Pic(X)$: analytically is the group $H^1(X, O_X^*)$ parametrizing isomorphism classes of line bundles over X. On a smooth variety, the natural map $Pic(X) \longrightarrow A^1(X)$ is a group isomorphism (this is not true in general if X is not smooth).

To each holomorphic vector bundle $V \longrightarrow X$ of rank r, one can attach some distinguished classes:

$$c_1(V), \ldots, c_r(V) \in A^*(X),$$

said to be the *Chern classes* of V. The r^{th} Chern class has the following interpretation: if $s : O_X \longrightarrow V$ is a regular holomorphic section of V, then the cycle class associated to the zero scheme $Z(s)$ of s is precisely $c_r(V) \cap [X]$. The *Chern polynomial* is

$$c_t(V) := \sum_{i \geq 0} c_i(V)t^i.$$

where $c_0(V) = 1$. The Chern polynomial is certainly a finite sum. In fact $c_k(V) = 0$ if k is bigger of the dimension of X, because $A^k(X) = 0$ whenever $k > n$. Moreover one can prove (see [16], p. 50) that $c_k(V) = 0$ if k exceeds the rank of the bundle. All the Chern classes c_i of the trivial bundle $pr_1 : X \times \mathbb{C}^r \longrightarrow X$ are zero for $i > 0$. If

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

is an exact sequence of vector bundles, one has $c_t(V') \cdot c_t(V'') = c_t(V)$, the equality holding in the polynomial ring $A^*(X)[t]$.

2.3.15 Example. Let V be a vector space of dimension n and let $V^\vee$ be its dual. The *tautological* (or *universal*) exact sequence of vector bundles on $\mathbb{P}(V)$ is:

$$0 \longrightarrow \mathcal{T} \longrightarrow O_{\mathbb{P}(V)} \otimes V \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $O_{\mathbb{P}(V)} \otimes V$ is the trivial vector bundle $\mathbb{P}(V) \times V$, $\mathcal{T}$ is the bundle

$$\mathcal{T} = \{([\mathbf{u}], \mathbf{v}) \in \mathbb{P}(V) \times V \mid \mathbf{v} \in [\mathbf{u}]\},$$

whose fiber over $[\mathbf{u}] \in \mathbb{P}(V)$ is $[\mathbf{u}] := \mathbb{C} \cdot \mathbf{u}$ itself, thought of as a 1-dimensional subspace of V, and $\mathcal{Q}$ is the *universal quotient bundle*, whose fiber over $[\mathbf{u}] \in \mathbb{P}(V)$ is the quotient $V/[\mathbf{u}]$. Let $E_{1+n} := (e_0, e_1, \ldots, e_n)$ be a basis of V, and let $\mathcal{E}_{1+n} := (\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$ be the basis of $V^\vee$, dual of E_{1+n}. Any $\alpha \in V^\vee$ induces a bundle map

$$\begin{cases} \alpha \quad : & \mathcal{T} & \longrightarrow & \mathbb{P}(V) \times \mathbb{C} \\ & ([\mathbf{v}], \mathbf{u}) & \longmapsto & ([\mathbf{v}], \alpha(\mathbf{u})) \end{cases}$$

so that $V^\vee$ can be seen as a global holomorphic section of the dual bundle $\mathcal{T}^\vee$, usually denoted by $O_{\mathbb{P}(V)}(1)$, and said to be the *hyperplane bundle* or also the *twisting sheaf of Serre* (Cf. [34], p. 117). Conversely, any bundle map

$$\alpha^\vee : \mathcal{T} \longrightarrow \mathbb{P}(V) \times \mathbb{C}$$

induces a homomorphism $\alpha : V \longrightarrow \mathbb{C}$. In fact if $0 \neq \beta \in V^\vee$, the set $U_\beta = \{\mathbf{v} \in V \mid \beta(\mathbf{v}) \neq 0\}$ is an open set of V. Let U'_β be the image of U_β via the map $\mathbf{w} \mapsto \frac{\mathbf{w}}{\beta(\mathbf{w})}$. Then $\alpha^\vee$ is represented by a family of holomorphic functions $\alpha_\beta : U'_\beta \longrightarrow \mathbb{C}$ such that

$$\beta\alpha_\beta|_{U'_\beta \cap U'_\gamma} = \gamma\alpha_\gamma|_{U'_\beta \cap U'_\gamma}$$

(one used the hypothesis that $\alpha^\vee$ is a bundle morphism). In particular α_β admits a Taylor series expansion around any point of U'_β converging at α_β at each point of U'_β (which will turn out to be, see below, a polynomial function of degree 1). Let $\mathbf{w} \in V$ be arbitrarily chosen and let $[\mathbf{v}]$ be any subspace containing it (if $\mathbf{w} = 0$ any $[\mathbf{v}]$ does the job). If $\mathbf{v} \in U_\beta$, then $\mathbf{w} = t\frac{\mathbf{v}}{\beta(\mathbf{v})}$ for some $t \in \mathbb{C}$ (possibly 0). Define

$$\alpha(\mathbf{w}) = \beta(\mathbf{v})\alpha_\beta\left(t\frac{\mathbf{v}}{\beta(\mathbf{v})}\right) = t \cdot \beta(\mathbf{v})\alpha_\beta\left(\frac{\mathbf{v}}{\beta(\mathbf{v})}\right) \in \mathbb{C}.$$

Then α is a well defined holomorphic map $V \longrightarrow \mathbb{C}$ (i.e., if $\mathbf{w} \in U_\beta \cap U_\gamma$, the result does not depend on the representation of α). Moreover, since by construction $\alpha(\lambda\mathbf{w}) = \lambda\alpha(\mathbf{w})$, Euler's theorem on homogeneous functions says that α is linear.

We have hence proven that $H^0(O_{\mathbb{P}(V)}(1))$, the space of the global holomorphic sections of $O_{\mathbb{P}(V)}(1)$ is isomorphic to $V^\vee$.

If $0 \neq \alpha \in H^0(O_{\mathbb{P}(V)}(1))$, then

$$Z(\alpha) = H_\alpha := \mathbb{P}(\ker(\alpha)),$$

is a *projective hyperplane*. Since for any $\alpha, \beta \in H^0(O_{\mathbb{P}(V)}(1))$ the ratio α/β defines a rational function on $\mathbb{P}(V)$, any two hyperplanes H_α and H_β determine the same class in $A^{n-1}(X)$ modulo rational equivalence. The 1^{st} Chern class $c_1(O_{\mathbb{P}(V)}(1)) = h \in A^1(X)$ is defined as:

$$h \cap [\mathbb{P}(V)] = [H] \in A_{n-1}(\mathbb{P}(V)),$$

where $[H]$ is the hyperplane class. Hence *capping* with the 1^{st} Chern class corresponds to *cutting with a hyperplane* (modulo rational equivalence).

2.3.16 Example. Example 2.3.15 above can be generalized. On the grassmannian variety $G_k(V)$, parametrizing k dimensional vector subspaces of an n-dimensional vector space V, the *tautological exact sequence* is:

$$0 \longrightarrow \mathcal{T}_k \longrightarrow O_{G_k(V)} \otimes V \longrightarrow \mathcal{Q}_k \longrightarrow 0. \tag{2.9}$$

where the *tautological bundle* $\mathcal{T}_k$ is:

$$\mathcal{T}_k := \{([\Lambda], \mathbf{v}) \in G_k(V) \times V \mid \mathbf{v} \in [\Lambda]\}.$$

Any element of $V^\vee$ can be seen as an element of $\mathcal{T}_k^\vee := Hom(\mathcal{T}_k, O_{G_k(V)})$, by defining:

$$\alpha([\Lambda], \mathbf{v}) = ([\Lambda], \alpha(\mathbf{v})).$$

The quotient $\mathcal{Q}_k$ is a bundle of rank $n-k$ whose fiber over $[\Lambda] \in G_k(V)$ is $V/[\Lambda]$.

2.3.17 Example. The intersection ring of the complex projective space $\mathbb{P}^n$ is:

$$A^*(\mathbb{P}^n) := \frac{\mathbb{Z}[h]}{(h^{1+n})},$$

where $h \cap [\mathbb{P}(V)]$ is the class $[H] \in A_{n-1}(X)$ of any hyperplane. One already knows (Cf. Example 2.3.12) that $A^i(X)$ is generated by h^i. Moreover $h^i \cdot h^j = h^{i+j}$, because one may find a $(n-i)$-dimensional plane $H^{(i)}$ and $(n-j)$-dimensional plane $H^{(j)}$ intersecting tranversally, and hence along a $n-(i+j)$-dimensional linear variety, representing the class h^{i+j}. Since $A_0(\mathbb{P}^n)$ is cyclic, it has no torsion (because any rational function on a curve has as many zeros as poles) and therefore $A^i(\mathbb{P}^n)$ is not torsion as well, and hence isomorphic to $\mathbb{Z} \cdot h^i$. In fact, were $ah^i \in A^i(\mathbb{P}^n)$ such that $ah^i \cap [\mathbb{P}^n] = 0$ in $A_{n-i}(\mathbb{P}^n)$, then $ah^i h^{n-i} \cap [\mathbb{P}^n] = ah^n \cap [\mathbb{P}^n]$, i.e. $a = 0$. If $Y \subseteq \mathbb{P}^n$ is any projective subvariety of dimension i, its degree is defined as

$$\deg(Y)[pt] = h^i \cap [Y],$$

where $[pt] := h^n \cap [X]$ is the class of the point. This corresponds with the intuitive geometric idea that the degree of Y can be gotten by intersecting it with sufficiently many hyperplanes in sufficiently general position. The relation $h^{1+n} = 0$ says that $(1+n)$ hyperplanes in general position do not intersect at all. Therefore, by the discussion above, the Chow ring of $\mathbb{P}(V)$ is generated by the first Chern class of the hyperplane bundle $O_{\mathbb{P}(V)}(1)$.

Chapter 3

Frames and Grassmannians

The purpose of this chapter is to provide a smooth introduction to grassmannians for beginners. Many important topics have been omitted (one for all, the discussion of the tangent space to a grassmannian at a k-plane, see e.g. [32] or also [1]), but what is in here is more than needed to follow the rest of the exposition. At the beginning we work over an arbitrary field $\mathbb{F}$, to stress that most of the features of a grassmannian are of linear algebraic nature. We shall finally switch to the complex field when geometry will ask for it.

3.1 Warming Up

3.1.1 The Set of k-Frames. Let V be an $\mathbb{F}$-vector space. If $\Lambda := (\mathbf{u}_1, \ldots, \mathbf{u}_k) \in V^k$ is any ordered k-tuple of V, one agrees to denote by

$$[\Lambda] := [\mathbf{u}_1, \ldots, \mathbf{u}_k]$$

the vector subspace $\mathbb{F}\mathbf{u}_1 + \ldots + \mathbb{F}\mathbf{u}_k$ of V generated by Λ.

If $\dim_{\mathbb{F}}[\Lambda] = h$, one briefly says that $[\Lambda]$ is a *h-plane* of V. Any

k-tuple $\Lambda \in V^k$ can be seen as a linear map:

$$\begin{cases} \Lambda & : & \mathbb{F}^k & \longrightarrow & V \\ & & \mathbf{v} & \longmapsto & \Lambda \cdot \mathbf{v}, \end{cases}$$

where:

$$\Lambda \cdot \mathbf{v} = \Lambda \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^k \end{pmatrix} = v^1 \mathbf{u}_1 + \ldots + v^k \mathbf{u}_k,$$

and therefore every $\mathbf{v} \in [\Lambda]$ is of the form $\Lambda \cdot \mathbf{u}$ for some $\mathbf{u} \in \mathbb{F}^k$.

3.1.2 Definition. *A k-tuple $\Lambda := (\mathbf{v}_1, \ldots, \mathbf{v}_k) \in V^k$ is said to be a* k-frame of V *if $[\Lambda]$ is a k-plane, i.e. if $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ are linearly independent.*

Let:

$$F_k(V) := \{\Lambda \in V^k \mid \dim_{\mathbb{F}}[\Lambda] = k\},$$

be the set of all k-frames of V. If $\Lambda \in F_k(V)$, each $\mathbf{u} \in [\Lambda]$ is of the form $\mathbf{u} = \Lambda \cdot \mathbf{v}$ for a unique $\mathbf{v} \in \mathbb{F}^k$.

3.1.3 Example.

i) $[\mathbf{0}_V] = \{\mathbf{0}_V\}$;

ii) a 1-frame of V is just a non zero vector of V. Hence $F_1(V) = V^*$, where $V^* := V \setminus \{\mathbf{0}_V\}$;

iii) the set $F_k(\mathbb{F}^n)$ may be identified with the set of all $n \times k$ matrices with linearly independent columns, i.e. matrices of rank k;

iv) if $\dim V = n$ and $\Lambda \in F_n(V)$, then $[\Lambda] = [V]$.

3.1.4 Let $V^\vee$ be the dual vector space $Hom_{\mathbb{F}}(V, \mathbb{F})$ of V, and suppose that $\dim_{\mathbb{F}}(V) = n$. Let us fix once and for all a basis

$$E_n := (e_1, \ldots, e_n),$$

of V and the basis

$$\mathcal{E}_n := (\epsilon^1, \ldots, \epsilon^n)$$

of $V^\vee$ dual of E_n (i.e. $\epsilon^i(e_j) = \delta^i_j$). Each $\mathbf{v} \in V$ can be uniquely written as:

$$\mathbf{v} = \sum_{i=1}^{n} v^i e_i,$$

where $v^i = \epsilon^i(\mathbf{v})$. If $\Lambda = (\mathbf{u}_1, \ldots, \mathbf{u}_k) \in V^k$, one sets:

$$\epsilon^i(\Lambda) := \epsilon^i(\mathbf{u}_1, \ldots, \mathbf{u}_k) = (u_1^i, \ldots, u_k^i) := (\epsilon^i(\mathbf{u}_1), \ldots, \epsilon^i(\mathbf{u}_k)).$$

One also writes:

$$\begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^n \end{pmatrix} (\Lambda) = \begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^n \end{pmatrix} (\mathbf{u}_1, \ldots, \mathbf{u}_k) = \begin{pmatrix} \epsilon^1(\mathbf{u}_1) & \ldots & \epsilon^1(\mathbf{u}_k) \\ \vdots & \ddots & \vdots \\ \epsilon^n(\mathbf{u}_1) & \ldots & \epsilon^n(\mathbf{u}_k) \end{pmatrix}, \quad (3.1)$$

so that the basis $\mathcal{E}_n$ gives an identification:

$$\begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^n \end{pmatrix} : V^k \longrightarrow M_{n\times k}(\mathbb{F}),$$

where $M_{n\times k}(\mathbb{F})$ is the $\mathbb{F}$-vector space of all the $n \times k$ matrices with entries in $\mathbb{F}$. Each $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^k V$ (Section 2.2) induces a function:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} : V^k \longrightarrow \mathbb{F}$$

defined by:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) := \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \det \begin{pmatrix} \epsilon^{i_1}(\Lambda) \\ \vdots \\ \epsilon^{i_k}(\Lambda) \end{pmatrix} =$$

$$= \begin{vmatrix} \epsilon^1(\mathbf{u}_1) & \ldots & \epsilon^1(\mathbf{u}_k) \\ \vdots & \ddots & \vdots \\ \epsilon^n(\mathbf{u}_1) & \ldots & \epsilon^n(\mathbf{u}_k) \end{vmatrix} = \begin{vmatrix} \mathbf{u}_1^1 & \ldots & \mathbf{u}_k^1 \\ \vdots & \ddots & \vdots \\ \mathbf{u}_1^n & \ldots & \mathbf{u}_k^n \end{vmatrix}.$$

Notice that:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(e_{j_1}, \ldots, e_{j_k}) = \begin{vmatrix} \delta^{i_1}_{j_1} & \ldots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \ldots & \delta^{i_k}_{j_k} \end{vmatrix} = \sum_{\tau \in S_k} (-1)^{|\tau|} \delta^{i_1}_{j_{\tau(1)}} \cdot \ldots \cdot \delta^{i_k}_{j_{\tau(k)}}.$$

In particular

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(e_{i_1}, \ldots, e_{i_k}) = 1, \quad (3.2)$$

whenever the indices i_j are pairwise distincts and

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(e_{j_1}, \ldots, e_{j_k}) = 0, \tag{3.3}$$

whenever there is at least a $j_p \notin \{i_1, \ldots, i_k\}$ $(1 \leq p \leq k)$.

We have hence got a map

$$\begin{cases} V^k & \longrightarrow & \bigwedge^k V \\ (\mathbf{u}_1, \ldots, \mathbf{u}_k) & \longmapsto & \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k \end{cases}$$

Using the basis $(e_{i_1} \wedge \ldots \wedge e_{i_k})$ of $\bigwedge^k V$ one has:

$$\wedge^k \Lambda := \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k = \sum_{1 \leq i_j \leq n} \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) e_{i_1} \wedge \ldots \wedge e_{i_k}. \tag{3.4}$$

Because of formula (2.3), the sum occurring on the r.h.s. of (3.4) is only over k-schindices (Cf. Sect. 2.1.5) such that $i_k \leq n$.

3.1.5 Proposition. *A k-tuple $\Lambda := (\mathbf{u}_1, \ldots, \mathbf{u}_k) \in V^k$ is a k-frame if and only if*

$$\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k \neq 0 \tag{3.5}$$

as a vector of $\wedge^k V$.

Proof. Indeed $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k$ is a vector of $\bigwedge^k V$ whose components, with respect to the basis $\wedge^k E_n$ of $\bigwedge^k V$, are precisely the $k \times k$ minors of the matrix of formula (3.1). The latter has rank k if and only if there is at least one non vanishing minor among them. One must check, however, that condition (3.5) does not depend on the choice of a basis of V. But were $(f_1, \ldots, f_n)$ another basis of V, then $f_i = e_j A^j_i$ for some invertible matrix $A := (A^i_j)$ and $\epsilon^j = A^j_i \phi^i$. Thus:

$$\begin{aligned} &\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k = \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) e_{i_1} \wedge \ldots \wedge e_{i_k} = \\ &A^{i_1}_{j_1} \cdot \ldots \cdot A^{i_k}_{j_k} \cdot \phi^{j_1} \wedge \ldots \wedge \phi^{j_k}(\Lambda) e_{i_1} \wedge \ldots \wedge e_{i_k} = \\ &\phi^{j_1} \wedge \ldots \wedge \phi^{j_k}(\Lambda) A^{i_1}_{j_1} e_{i_1} \wedge \ldots \wedge A^{i_k}_{j_k} e_{i_k} = \\ &\phi^{j_1} \wedge \ldots \wedge \phi^{j_k}(\Lambda) \cdot f_{j_1} \wedge \ldots \wedge f_{j_k}, \end{aligned}$$

proving that $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k \neq \mathbf{0}$ is an intrinsic condition independent from the choice of a basis of V. ∎

3.1.6 Corollary. *Let $\Lambda_u := (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ and $\Lambda_v := (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be two k-frames of V. They span the same k-plane of V if and only if*

$$\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k = a \cdot \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k, \tag{3.6}$$

holds for some non zero $a \in \mathbb{F}$.
Proof. Suppose that $[\Lambda_u] = [\Lambda_v]$. Then Λ_u and Λ_v are both bases of the same k-plane. Therefore[1]:

$$\mathbf{v}_i = A_i^j \mathbf{u}_j,$$

for some invertible $k \times k$ matrix $A = (A_j^i)$. Then:

$$\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k = \det(A) \cdot \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k.$$

Conversely, suppose that (3.6) holds for some non-zero $a \in \mathbb{F}$. Then we claim that $[\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_i]$ is not a $(k+1)$-plane for each $1 \leq i \leq k$. In fact

$$\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k \wedge \mathbf{u}_i = a \cdot \mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k \wedge \mathbf{u}_i = 0,$$

because two equal indices occur on the r.h.s. ∎

As a consequence a k-plane is uniquely determined by the data $(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda))$ up to a scalar multiple.

3.1.7 Towards Grassmannians. Grassmannians are among the first examples of parameter spaces contructed with the purpose to solve concrete enumerative problems (see e.g. [79]). The simplest one such is surely *how many lines do pass through two distinct points in the projective $\mathbb{R}$-plane*? This is an easy problem: everybody knows that the answer is 1 and can be gotten by writing down the equation of a general line and imposing the passing through two distinct points. One so gets a linear homogeneous system of two equations in three unknowns (the coefficients of the equation of the line) having a 1-dimensional subspace of solutions. Since a line is determined by the coefficients of its equation up to a non-zero scalar multiple, one sees that there is only one such a line. However it is worth to present here a way to solve the same problem that can be generalized to less easy cases.

[1] We use Einstein's convention on sums.

Picking two distinct points of $\mathbb{P}^2$ is the same as picking two distinct 1-planes $[\mathbf{u}_1]$ and $[\mathbf{u}_2]$ in $\mathbb{R}^3$, which is the same as picking a 2-plane $[\mathbf{v},\mathbf{w}] \in G_2(\mathbb{R}^3)$. The problem is then to find how many 2-planes contain two distinct 1-lines. But $\mathbf{u}_i \in [\mathbf{v},\mathbf{w}]$ iff $[\mathbf{v},\mathbf{w},\mathbf{u}_i] \neq \mathbb{R}^3$, i.e. if and only if:

$$\begin{cases} \mathbf{v}\wedge\mathbf{w}\wedge\mathbf{u}_1 &= 0 \\ \mathbf{v}\wedge\mathbf{w}\wedge\mathbf{u}_2 &= 0 \end{cases}.$$

The reader may easily check that such a system can be written in the form:

$$(\epsilon^2\wedge\epsilon^3(\mathbf{v},\mathbf{w}), \epsilon^3\wedge\epsilon^1(\mathbf{v},\mathbf{w}), \epsilon^1\wedge\epsilon^2(\mathbf{v},\mathbf{w}))\cdot\begin{pmatrix} \epsilon^1(\mathbf{u}_1) & \epsilon^1(\mathbf{u}_2) \\ \epsilon^2(\mathbf{u}_1) & \epsilon^2(\mathbf{u}_2) \\ \epsilon^3(\mathbf{u}_1) & \epsilon^3(\mathbf{u}_2) \end{pmatrix} = 0.$$

Since the rank of the matrix (notation as in formula (3.1))

$$\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \\ \epsilon^3 \end{pmatrix}(\mathbf{u}_1,\mathbf{u}_2)$$

is 2, because $\mathbf{u}_1\wedge\mathbf{u}_2 \neq 0$, one finds a unique 1-dimensional space of solutions, i.e. a unique 2-plane containing $\mathbf{u}_1$ and $\mathbf{u}_2$.

3.1.8 Another exercise. In the same vein, let us try to look for the number of lines of the projective space $\mathbb{P}^3(\mathbb{F})$, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, meeting 4 others in general position (i.e. any two of them do not lie in a same 2-plane). This is the same as looking for the 2-planes in $\mathbb{F}^4$ intersecting four others in general position along a positive dimensional subspace. Let $[\Lambda_i] \in G_2(\mathbb{F}^4)$ $(1 \le i \le 4)$ be given. Then $\Lambda \in G_2(\mathbb{F}^4)$ intersects Λ_i in a positive dimensional subspace if and only if $[\Lambda,\Lambda_i] \neq \mathbb{F}^4$. This gives the following system (notation as in (3.1)):

$$(X^{34},X^{24},X^{23},X^{14},X^{13},X^{12})(\Lambda)\cdot\begin{pmatrix} \epsilon^1\wedge\epsilon^2 \\ \epsilon^1\wedge\epsilon^3 \\ \epsilon^1\wedge\epsilon^4 \\ \epsilon^2\wedge\epsilon^3 \\ \epsilon^2\wedge\epsilon^4 \\ \epsilon^3\wedge\epsilon^4 \end{pmatrix}\cdot(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.7}$$

where

$$(X^{34},X^{24},X^{23},X^{14},X^{13},X^{12})(\Lambda) =$$
$$= (X^{34}(\Lambda),X^{24}(\Lambda),X^{23}(\Lambda),X^{14}(\Lambda),X^{13}(\Lambda),X^{12}(\Lambda))$$

and

$$X^{ij}(\Lambda) = \epsilon^i \wedge \epsilon^j(\Lambda).$$

Notice that equation (3.7) can be seen as the intersection of four hyperplanes in $\mathbb{P}^5(\mathbb{F})$, with homogeneous coordinates (X^{ij}). Since the system has rank 4, this is precisely the equation of a line in such a $\mathbb{P}^5$. Last remark consists in observing that not any point of such a $\mathbb{P}^5$ can be seen as the set of coordinates of some 2-plane. In fact, the check being left to the Reader's care, any $[\Lambda] \in G_2(\mathbb{F}^4)$ satisfies the following quadratic equation:

$$X^{12}X^{34} - X^{13}X^{24} + X^{14}X^{23} = 0. \tag{3.8}$$

Therefore, the solution of our enumerative problem can be seen as the number of intersection points between a line and the quadric hypersurface (3.8) $\mathbb{P}^5(\mathbb{F})$, called the *Klein's quadric.* If $\mathbb{F} = \mathbb{R}$ one may have either 0 or 2 solutions, since $\mathbb{R}$ is not algebraically closed. If $\mathbb{F} = \mathbb{C}$, which is algebraically closed, instead, one would always find 2 solutions (according to the multiplicity). For sake of uniformity, then, the field $\mathbb{R}$ will be from now on ignored and it will be assumed that

the ground field $\mathbb{F}$ is $\mathbb{C}$.

The letter $\mathbb{F}$, used to denote a general field, exits now from our play.

3.2 Complex Grasmannian Varieties

The aim of this section is to imitate the construction of projective spaces to put a structure of complex manifold on the Grassmann Set of k-planes of a finite dimensional complex vector space. This will give us the *grassmannian variety* or, briefly, the *Grassmannian.* First of all, let us mention the following remarkable property.

3.2.1 Proposition. *The group $Gl(V)$ acts transitively on $F_k(V)$ and $G_k(V)$.*

Proof. Consider the map:

$$Gl(V) \times F_k(V) \longrightarrow F_k(V),$$

defined by $(A, (\mathbf{u}_1, \ldots, \mathbf{u}_k)) \mapsto (A \cdot \mathbf{u}_1, \ldots, A \cdot \mathbf{u}_k)$. It is clearly an action on $F_k(V)$, inducing an action on $G_k(V)$ as well, by setting:

$$A \cdot [\Lambda] = [A \cdot \Lambda].$$

Let now $\Lambda_u := (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ and $\Lambda_v := (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be any two k-frames. Let W be a finite dimensional subspace of V containing both $[\mathbf{u}_1, \ldots, \mathbf{u}_k]$ and $[\mathbf{v}_1, \ldots, \mathbf{v}_k]$. If $n := \dim(W)$, let:

$$(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n) \quad \text{and} \quad (\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n),$$

be any two bases of W gotten by completing Λ_u and Λ_v. Let $\phi \in Gl(W)$ be the unique automorphism of W sending $\mathbf{u}_i$ onto $\mathbf{v}_i$. If $V = W \oplus W'$, where W' is the linear complement of W in V, denote by $id_{W'}$ the identity of W'. Then the the automorphism

$$\phi \oplus id_{W'} : V \longrightarrow V,$$

maps Λ_u onto Λ_v, as well as $[\Lambda_u]$ onto $[\Lambda_v]$. This proves that $Gl(V)$ acts transitively on $F_k(V)$ and $G_k(V)$. ∎

3.2.2 According to Proposition 3.1.5, $F_k(V)$ is the complement in V^k of the zero locus of finitely many polynomial equations $(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) = 0, \ \forall (i_1, \ldots, i_k) \in \mathcal{I}_n^k)$. These are continuous in both the Zariski or the usual topology. Then $F_k(V)$ is a (Zariski-)open set in V^k, because $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a k-frame if and only if $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k \neq 0$ (i.e. at least one component of it, with respect to the basis of $\wedge^k V$, is different from zero). However, as already implicitly remarked, the same k-plane can be determined by several different k-frames. The reason is that there is a natural right action of the group $Gl_k(\mathbb{C})$ of the $k \times k$ $\mathbb{C}$-valued invertible matrices on the set of k-frames, given by:

$$\begin{cases} F_k(V) \times Gl_k(\mathbb{C}) & \longrightarrow & F_k(V) \\ \quad (\Lambda \ \ , \ \ A) & \longmapsto & \Lambda \cdot A \end{cases},$$

where if $\Lambda = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and $A = (A_j^i)$ then

$$\Lambda \cdot A = (\mathbf{v}_i A_1^i, \ldots, \mathbf{v}_i A_k^i).$$

Straightforward computations show that

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda \cdot A) = \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) \cdot \det(A).$$

Given that any two k-frames Λ, Λ' defines the same k-plane if there is $A \in Gl_k(\mathbb{C})$ such that $\Lambda' = \Lambda \cdot A$, it follows that $G_k(V) = F_k(V)/Gl_k(V)$. Moreover Λ and Λ' live in the same $F_k(V)$-orbit if the $\dim[\Lambda, \Lambda'] = k$. This amounts to impose that the rank of any matrix associated to the subspace $[\Lambda, \Lambda']$ is smaller or equal than k, i.e. all its $(k+1) \times (k+1)$ minors must vanish. Therefore the graph in $F_k(V) \times F_k(V)$ of the relation $\Lambda \sim \Lambda'$ if $[\Lambda] = [\Lambda']$ is (Zariski) closed, showing that the quotient topology of $G_k(V)$ is Hausdorff (or separated in the Zariski topology).

Let

$$\rho : F_k(V) \longrightarrow G_k(V),$$

be the canonical projection and write $\rho(\Lambda) = [\Lambda]$. The fiber over $[\Lambda] \in G_k(V)$ is the set $\rho^{-1}([\Lambda]) = \Lambda \cdot Gl_k(\mathbb{C})$.

For each k-schindex $I = (1 \leq i_1 < \ldots < i_k \leq n)$, let

$$U_I = \{[\Lambda] \in G_k(V) \,|\, \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) \neq 0\}.$$

As remarked, the condition $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) \neq 0$ depends only on the k-plane $[\Lambda]$ and not on its particular representative.

3.2.3 Proposition. *For each k-schindex $I = (i_1, \ldots, i_k)$, the set U_I is open in the quotient topology of $G_k(V)$.*

Proof. Let $F_k(V)_I$ be the set of all $\Lambda \in F_k(V)$ such that

$$\begin{pmatrix} \epsilon^{i_1} \\ \vdots \\ \epsilon^{i_k} \end{pmatrix} (\Lambda) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} = \mathbf{1}_{Gl_k(\mathbb{C})}.$$

One has $\rho(F_k(V)_I) = U_I$. Therefore

$$\rho^{-1}(U_I) = F_k(V)_I \cdot Gl_k(V)$$

which is the Zarisky open set of $F_k(V)$ complement of

$$Z(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \{\Lambda \in F_k(V) \mid \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) = 0\}. \quad \blacksquare$$

It follows that the collection $\mathcal{U}_{\mathcal{E}} := \{U_I \,|I \in \mathcal{I}^k,\, i_k \leq n\}$ forms an affine open covering of $G_k(V)$ and each U_I is isomorphic to $\mathbb{C}^{k(n-k)}$. Moreover, a subset $U \subseteq G_k(V)$ is open if and only if $U \cap U_I$ is (Zarisky) open in $U_{i_1 \dots i_k}$. In fact if $U \cap U_I$ is (Zariski) open in U_I, then $\rho^{-1}(U) \cap \rho^{-1}(U_I)$ is (Zariski) open in $\rho^{-1}(U_I)$. But $\rho^{-1}(U_I)$ is an open covering of $F_k(V)$ and, therefore, $\rho^{-1}(U)$ is open, i.e. U is open in $G_k(V)$.

Let $\psi_I : U_I \longrightarrow F_k(V)_I$ be the map defined by

$$\psi_I([\Lambda]) = \Lambda_I := \Lambda \cdot \begin{pmatrix} \epsilon^{i_1}(\Lambda) \\ \vdots \\ \epsilon^{i_k}(\Lambda) \end{pmatrix}^{-1} .$$

Then, if $[\Lambda] \in U_I \cap U_J$, one has:

$$\Lambda_J = \Lambda_I \cdot g_{IJ},$$

where $g_{IJ} \in Gl_k(\mathbb{C})$ is given by:

$$g_{IJ} = \begin{pmatrix} \epsilon^{i_1}(\Lambda) \\ \vdots \\ \epsilon^{i_k}(\Lambda) \end{pmatrix} \cdot \begin{pmatrix} \epsilon^{j_1}(\Lambda) \\ \vdots \\ \epsilon^{j_k}(\Lambda) \end{pmatrix}^{-1} .$$

It is straightforward to check that $g_{IJ} \cdot g_{JK} = g_{IK}$. This cocycle relation shows two things: first, that one can see the grassmann set $G_k(V)$ as an algebraic variety gotten by glueing $\binom{n}{k}$ affine spaces (the U_I's) via the linear transition function(s) g_{IJ}, which are polynomials (being product of matrices in the components of the frame Λ with respect to the basis E_n). Secondly, the maps $g_{IJ} : U_I \cap U_J \longrightarrow Gl_k(\mathbb{C})$ are matrices whose entries are quotient of polynomials with non zero denominator. Then such functions exhibit $F_k(V)$ as an algebraic principal $GL_k(\mathbb{C})$-bundle over $G_k(V)$.

3.2.4 Theorem. *The Grassmannian $G_k(V)$ is a smooth irreducible algebraic variety.*

Proof. Since $F_k(V)$ is an open set of V^k, which is irreducible in the Zariski topology, then it is irreducible. The irreducibility of $G_k(V)$ then follows from the the fact that $G_k(V)$ is the continuous image of $F_k(V)$ via the projection map ρ. It is smooth because it has certainly smooth points, being covered by affine sets (the U_I's) which are smooth. If it has a smooth points than all the points are smooth because it is a homogeneous variety (the group $Gl(V)$ acts transitively as an algebraic group of isomorphisms, see Proposition 3.2.1). ∎

3.2.5 Let now

$$l_{\mathcal{E}} : F_k(V) \longrightarrow F_1(\bigwedge^k V),$$

be defined by $(\mathbf{v}_1, \ldots, \mathbf{v}_k) \mapsto \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k$; here $F_1(\bigwedge^k V)$ stands for $\bigwedge^k V \setminus \{\mathbf{0}\}$, the set of 1-frames of $\bigwedge^k V$. Clearly $F_1(\wedge^k V)$ is acted on by the multiplicative group $\mathbb{G}_m(\mathbb{C}) := Gl_1(\mathbb{C}) := \mathbb{C}^*$. The map $l_{\mathcal{E}}$ is *equivariant*, in the sense that $l_{\mathcal{E}}(\Lambda \cdot A) = l_{\mathcal{E}}(\Lambda) \cdot \det(A)$. Hence $l_{\mathcal{E}}$ factors through a map

$$Pl_{\mathcal{E}} : G_k(V) \longrightarrow \mathbb{P}(\bigwedge^k V) := G_1(\bigwedge^k V).$$

said to be the *Plücker map*. In other words, two k-frames span the same plane if their *Plücker images* are the same. Equip $Gl_k(V)$ with the quotient topology of $F_k(V)/Gl_k(V)$.

3.2.6 Proposition. *The Plücker map $Pl_{\mathcal{E}}$ is an embedding.*

Proof. By Corollary 3.1.6, the Plücker map is injective. We are left to show that it has injective tangent map. Let

$$B_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\},$$

be a disc in the complex plane. Then, any tangent vector to $[\Lambda] \in G_k(V)$ is of the form

$$\left.\frac{d\gamma}{dz}\right|_{z=0},$$

where $\gamma : B_\epsilon \longrightarrow G_k(V)$ such that $\gamma(0) := [\Lambda]$. Therefore the tangent of the Plücker map is:

$$T_{[\Lambda]}Pl_{\mathcal{E}}\left(\left.\frac{d\gamma}{dz}\right|_{z=0}\right) = \left(\left.\frac{d}{dz}(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(z))\right|_{z=0}\right)_{(i_1,\ldots,i_k)\in\mathcal{I}_n^k}.$$

It is then sufficient to show that

$$\left.\frac{d\gamma}{dz}\right|_{z=0} \neq 0 \Leftarrow T_{[\Lambda]}Pl_{\mathcal{E}}\left(\left.\frac{d}{dz}\right|_{z=0}\gamma(z)\right) \neq 0.$$

The property being local, it suffices to check it on an affine open set of the Grassmannian of the form U_I containing $[\Lambda]$. Up to a linear transformation permuting the elements of the basis E_n, one may assume that $I = (12\ldots k)$. Any k-plane in U_I can be represented by a maximal rank matrix of the form:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ x_{1,1} & x_{1,2} & \ldots & x_{1,k-1} & x_{1,k} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,k-1} & x_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-k,1} & x_{n-k,2} & \ldots & x_{n-k,k-1} & x_{n-k,k} \end{pmatrix},$$

where a tangent vector can be represented in the form $(dx_{ij}/dz)_{|z=0}$. Since for each pair (i,j), such that $1 \leq i \leq k$ and $k+1 \leq j \leq n$,

$$(\epsilon^1 \wedge \ldots \wedge \widehat{\epsilon^j} \wedge \ldots \wedge \epsilon^k \wedge \epsilon^{k+i})(\Lambda) = (-1)^{k-j+i-1}x_{ij},$$

the tangent map can be written as

$$\left((-1)^{k-j+i-1}\left.\frac{dx_{ij}}{dz}\right|_{z=0}, \left.\frac{d}{dz}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\gamma(z))\right|_{z=0}\right)_{(i_1,\ldots,i_k)\in B},$$

where B is the set of all k-schindices such that $\sharp(B \cap \{1,2,\ldots,k\}) \leq k-2$. Then the tangent map to the Plücker embedding is injective as claimed (the null tangent vector is the unique pre-image of null vector through the tangent map). ∎

3.3 Schubert Varieties

3.3.1 A *complete flag* of a n-dimensional vector space is a filtration $E^\bullet$ of V:

$$E^\bullet: \qquad E^0 = V \supset E^1 \supset E^2 \supset \ldots \supset E^n = (0),$$

where E^i is a vector subspace of V of codimension i. Suppose that $E_n := (e_1, \ldots, e_n)$ is an adapted basis to E in the sense that $E^i = [e_{i+1}, \ldots, e_n]$, $1 \le i \le n-1$.

3.3.2 Definition (Cf. Example 5.4.3). *The i^{th} $\mathcal{E}$-Schubert matrix of $[\Lambda] \in G_k(V)$ is the $i \times k$-matrix:*

$$Sch([\Lambda] \cap E^i) := \begin{pmatrix} \epsilon^1(\Lambda) \\ \vdots \\ \epsilon^i(\Lambda) \end{pmatrix}.$$

.

Let $\rho_i(E, \Lambda) := rk(Sch([\Lambda] \cap E^i))$. Then one has:

3.3.3 Proposition. *The following equality holds:*

$$\dim(E^i \cap [\Lambda]) := k - \rho_i(E, \Lambda).$$

Proof. In fact, the vectors $\mathbf{v} \in [\Lambda]$ belonging to E^i must satisfy the linear system of equations:

$$\begin{cases} \epsilon^1(\mathbf{v}) = 0 \\ \quad\vdots \\ \epsilon^i(\mathbf{v}) = 0 \end{cases}.$$

The dimension of the space of solutions is precisely k minus the rank of the system, which is exactly $\rho_i(E, \Lambda)$. ∎

Because of the obvious inequalities:

$$\rho_i(E, \Lambda) \le \rho_{i+1}(E, \Lambda) \le \rho_i(E, \Lambda) + 1$$

(adding a row to a matrix the rank increases at most of 1), one deduces that

$$\dim(E^{i+1} \cap [\Lambda]) \le \dim(E^i \cap [\Lambda]) \le \dim(E^{i+1} \cap [\Lambda]) + 1,$$

i.e., any possible dimension “jump” is not bigger than 1. The upshot is that in the sequence:

$$k := \dim([\Lambda]\cap E^0) \geq \dim([\Lambda]\cap E^1) \geq \ldots \geq \dim([\Lambda]\cap E^n) = 0,$$

there are exactly k *dimension jumps.*

3.3.4 Definition. *An integer* $1 \leq i \leq n$ *is a* Schubert jump *if* $\dim([\Lambda]\cap E^{i-1}) > \dim([\Lambda]\cap E^i)$.

3.3.5 Definition. *The* Schubert index *of the k-plane* $[\Lambda]$ *is the k-schindex*

$$I([\Lambda], E^\bullet) = (1 \leq i_1 < \ldots < i_k \leq n)$$

of the Schubert jumps at $[\Lambda]$.

To any k-plane one can obviously attach one and only one Schubert index. The general k-plane has Schubert index $(1, 2, \ldots, k)$. In fact, a k-plane is general with respect to the flag $E^\bullet$ if it intersects E^k (the subspace of codimension k) in the null vector, because the general homogeneous linear system of k equations in k unknowns has no solution but the trivial one. As a matter of fact, the $E^\bullet$-general k-plane lives in the complement of a Zariski closed set. Let us see that. If $[\Lambda]$ is not general, then $\dim([\Lambda]\cap E^k) > 0$. Hence there exists $0 \neq \mathbf{v} \in [\Lambda]\cap E^k$, i.e. there exists $\mathbf{u} \in \mathbb{C}^k \setminus \{\mathbf{0}\}$, such that $\Lambda \cdot \mathbf{u}$ satisfies the linear system:

$$\epsilon^i(\Lambda)\cdot \mathbf{u} = 0, \qquad 1 \leq i \leq k.$$

This is possible if $\det(\epsilon^i(\Lambda)) = 0$, i.e if $\epsilon^1 \wedge \ldots \wedge \epsilon^k(\Lambda) = 0$. If $\Lambda = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$, the condition can be recast as:

$$\epsilon^1 \wedge \ldots \wedge \epsilon^k([\Lambda]) = \begin{vmatrix} \epsilon^1(\mathbf{v}_1) & \ldots & \epsilon^1(\mathbf{v}_k) \\ \vdots & \ddots & \vdots \\ \epsilon^k(\mathbf{v}_1) & \ldots & \epsilon^k(\mathbf{v}_k) \end{vmatrix} = 0.$$

3.3.6 Definition. *The expression* $\epsilon^1 \wedge \ldots \wedge \epsilon^k([\Lambda])$ *is said to be the* $E^\bullet$-Schubert Wronskian *at* $[\Lambda]$.

Hence the k-planes in $E^\bullet$-special position live in the zero-scheme $Z(\epsilon^1 \wedge \ldots \wedge \epsilon^k)$ of $\epsilon^1 \wedge \ldots \wedge \epsilon^k$. It is worth to remark that the latter depends only on the flag $E^\bullet$ and not on the adapted basis to the flag itself. For, were $(\phi^1, \ldots, \phi^n)$ another basis such that $E^i = [\phi^{i+1}, \ldots, \phi^n]$, then the transformation matrix T from (ϵ^j) to the (ϕ^j) would be triangular, so that

$$\phi^1 \wedge \phi^2 \wedge \ldots \wedge \phi^k = \det(T) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k.$$

Since $(\epsilon^1, \ldots, \epsilon^n)$ are sections of $\mathcal{T}^\vee$ (see Example 2.3.16), it follows that $\epsilon^1 \wedge \ldots \wedge \epsilon^k$ is a section of the line bundle $\wedge^k \mathcal{T}^\vee$ and thus that the class of the $E^\bullet$-special k-planes in the Chow group $A_*(G_k(V)$ is precisely:

$$[Z(\epsilon^1 \wedge \ldots \wedge \epsilon^k)] = c_1(\wedge^\vee \mathcal{T}^\vee) \cap [G_k(V)].$$

One easily sees that this class does not depend on the flag chosen: any two Schubert-wronskians are sections of the same line bundle. Set $\sigma_1 = c_1(\bigwedge^k \mathcal{T}_k^\vee)$. Because of the tautological exact 2.9 and the fact that $c_t(\mathcal{T}_k)c_t(\mathcal{Q}_k) = 1$, it also follows that:

$$\sigma_1 = c_1(\bigwedge^k \mathcal{T}_k^\vee) = -c_1(\bigwedge^k \mathcal{T}_k) = -c_1(\mathcal{T}_k) = c_1(\mathcal{Q}_k).$$

3.3.7 Example. Let $V := \mathbb{C}[X]/(X^n)$ be the $\mathbb{C}$-vector space of polynomials of degree at most $n-1$. It is a n-dimensional $\mathbb{C}$-vector space spanned by the classes of $1, X, X^2, \ldots, X^n \in \mathbb{C}[X]$ modulo (X^n). Let $z_0 \in \mathbb{C}$ and consider the flag:

$$V \supset V(-z_0) \supset V(-2z_0) \supset \ldots \supset V(-nz_0) = 0$$

where

$$V(-iz_0) := \frac{(X - z_0)^i + (X^n)}{X^n}$$

is the vector subspace of polynomials of degree $\leq n$ contained in the i^{th} power of the maximal ideal $(X - z_0)$ of $\mathbb{C}[X]$, or, spelled in a more friendly way, the subspace of polynomials of degree $\leq n$ vanishing at z_0 with multiplicity at least i (or divisible by $(X - z_0)^i$).

Let $[\Lambda]$ be a subspace of dimension k of V and let $(p_1(X), \ldots, p_k(X))$ be a basis of it. Then $[\Lambda]$ is special with respect to the given flag, if there exists $0 \neq p(X) \in [\Lambda]$ vanishing at z_0 with multiplicity at least k. Write:

$$p(X) = a^1 p_1(X) + a^2 p_2(X) + \ldots + a^k p_k(X).$$

Then $p^{(i)}(z_0) = 0$ for all $1 \leq i \leq k$, where $p^{(i)}(X)$ is the i^{th} derivative of the polynomial $p(X)$. Hence $[\Lambda]$ is special if the following determinant:

$$W(p_1, \ldots, p_k)(z_0) := \begin{vmatrix} p_1(z_0) & \ldots & p_k(z_0) \\ p_1'(z_0) & \ldots & p_k'(z_0) \\ \vdots & \ddots & \vdots \\ p_1^{(k-1)}(z_0) & \ldots & p_k^{(k-1)}(z_0) \end{vmatrix} = 0.$$

This is a true wronskian and hence motivates the terminology of Definition 3.3.6.

Let $W_{i_1 \ldots i_k}(E^\bullet) := \{[\Lambda] \in G_k(V) | I([\Lambda], E^\bullet) = (i_1, \ldots, i_k)\}$.

3.3.8 Proposition. *The set $W_{i_1 \ldots i_k}(E^\bullet)$ is an affine cell of codimension $(i_1 - 1) + \ldots + (i_k - k)$.*

Proof. If $[\Lambda] \in W_{i_1 \ldots i_k}$ then, in particular, $[\Lambda] \in U_I$ (Sect. 3.2.2). Therefore $W_{i_1 \ldots i_k}$ is entirely contained in U_I, which is affine of dimension $k(n-k)$. If $[\Lambda] \in W_{i_1 \ldots i_k}$, then $[\Lambda] = [\Lambda_I]$, where $\Lambda_I =$

$(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a k-frame whose matrix ${}^t(\epsilon^1, \ldots, \epsilon^n) \cdot \Lambda_I$ is given by:

$$\begin{pmatrix} \epsilon^1 \\ \vdots \\ \epsilon^{i_1-1} \\ \epsilon^{i_1} \\ \epsilon^{i_1+1} \\ \vdots \\ \epsilon^{i_2-1} \\ \epsilon^{i_2} \\ \epsilon^{i_2+1} \\ \vdots \\ \epsilon^{i_3-1} \\ \epsilon^{i_3} \\ \epsilon^{i_3+1} \\ \vdots \\ \epsilon^{i_k-1} \\ \epsilon^{i_k} \\ \epsilon^{i_k+1} \\ \vdots \\ \epsilon^n \end{pmatrix} (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ * & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ * & * & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ * & * & * & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ * & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & * \end{pmatrix} \tag{3.9}$$

Then $W_{i_1,\ldots,i_k}$ is an irreducible affine subvariety of U_I, isomorphic to $\mathbb{A}^N$, where N is the numbers of stars occurring in the huge matrix (3.9). The stars correspond to those components of the vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ not constrained by the condition $\Lambda_I \in W_I$. The constraints in the first column are: the first $i_1 - 1$ components are 0, while the i_1^{th} is 1. Moreover one must put a zero in the $i_2^{th}, \ldots, i_k^{th}$ because $(\epsilon^{i_1}, \ldots, \epsilon^{i_k})(\Lambda_I)$ must be the identity matrix. Hence, in the first column one has $n - (i_1 - k + 1)$ stars. Similarly, in the j^{th} column one has the first $i_j - 1$ components equal to 0, the i_j^{th} equal to 1 and then one still put $k - j - 1$ zero corresponding to the $i_{j+1}^{th}, \ldots i_k^{th}$ rows. It follows that in the i_j^{th} column one has $n - (i_j - (k - (j-1)))$

stars. Then one has:

$$\begin{aligned} & n-i_1-(k-1)+n-i_2-(k-2)+\ldots+n-i_k-(k-k)= \\ = \quad & kn-nk-(i_1-1)-(i_2-2)+\ldots+(i_k-k)= \\ = \quad & k(n-k)-wt(I) \end{aligned}$$

where $wt(I)=\sum_{j=1}^{k}(i_j-j)$ is then precisely the codimension of W_I.

■

3.4 Intersection Theory on $G_k(V)$

This section is just a quick review intended to let the Reader doing the necessaries comparisons with the material of Chapter 4. For proofs see [16], [27], [53]. For a geometrically stimulating exposition, see [39].

3.4.1 Definition. *Let*

$$\Omega_I(E^\bullet)=\Omega_{i_1\ldots i_k}(E^\bullet)=\overline{W_{i_1\ldots i_k}(E^\bullet)},$$

the closure being taken in $G_k(V)$. It is called the Schubert variety *associated to the flag E.*

3.4.2 Suppose $F^\bullet$ is another flag of V. Because of the transitive action of $Gl(V)$ on $G_k(V)$, there exists an automorphism g of V sending the flag $F^\bullet$ onto the flag $E^\bullet$ and, consequently, the Schubert variety $\Omega_I(F^\bullet)$ isomorphically onto $\Omega_I(E^\bullet)$. Since $Gl(V)$ is rational (and connected), their classes modulo rational equivalence in $A_*(G_k(V))$ are equal (Cf. Section 2.3.9). Then one let:

$$\Omega_I=[\Omega_I(E^\bullet)]\in A_*(G_k(V)),$$

for some complete flag $E^\bullet$ of V. The class $\Omega_I\in A_*(G_k(V))$ will be said to be the *Schubert cycle* associated to the k-schindex I. One may also denote the same Schubert cycle as $\Omega_{\underline{\lambda}}$, where $\underline{\lambda}=\underline{\lambda}(I)$ (Cf. Sect. 2.1.6). Clearly $\Omega_{12\ldots k}=\Omega_{(0\ldots 0)}=[G_k(V)]$, the fundamental class of $G_k(V)$. The class of $\Omega_I(E^\bullet)$ corresponds to a class in $A^*(G_k(V))$ classically denoted by $\sigma_{\underline{\lambda}}$, related to it via the equality:

$$\sigma_{\underline{\lambda}}\cap[G_k(V)]=\Omega_{\underline{\lambda}},$$

expressing *Poincaré duality* for grassmannians. The equality

$$\sigma_{\underline{\lambda}} \cap \Omega_{\underline{\mu}} = \sigma_{\underline{\lambda}} \cap (\sigma_{\underline{\mu}} \cap [G_k(V)]) = (\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\mu}}) \cap [G_k(V)].$$

expresses instead the fact that $A_*(G_k(V))$ is a module over $A^*(G_k(V))$. One also has:

3.4.3 Proposition (Chow basis theorem). *The classes $\Omega_{\underline{\lambda}} := \sigma_{\underline{\lambda}} \cap [G_k(V)]$, of Schubert varieties modulo rational equivalence, freely generate the Chow group $A_*(G_k(V))$.*

Proof (idea of). This is a particular case of a general result regarding cellular complexes. In our specific example, one may argue as follows. Any k-plane of V belongs to at least one open $E^\bullet$-Schubert cell, namely that indexed by its Schubert cycle. Moreover, two distinct open Schubert cells are disjont, because otherwise there would be k-planes having more than 1 Schubert index with respect to the same flag, which is absurd. Then, using exact sequence (2.8), one sees that the Chow group of $G_k(V)$ (or its integral homology), is generated by the classes modulo rational equivalence of the Schubert Varieties (the closure of the open Schubert cells). To show that the classes $\sigma_{\underline{\lambda}} \cap [G_k(V)]$ freely generated the Chow group, one uses duality: for each $\underline{\lambda} \in \ell(k)$,

$$\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\lambda}^\vee} = \sigma_{((n-k)^k)} = \sigma_{(n-k,\ldots,n-k)}$$

and the latter freely generates $A_{k(n-k)}(G_k(V))$. For the proof of this fact see, e.g., [16], p. 268 or [27]. ∎

The following example serves as illustration of how Schubert's devices should work.

3.4.4 Example. In the projective space $\mathbb{P}^5$ fix:

a) a 3-dimensional projective linear subspace H and a point $P \in H$;

b) a set of 2-codimensional projective linear subspaces $\Pi_1, \ldots, \Pi_5$ in general position. Suppose one wants to look for the class in $A_*(G(2, \mathbb{P}^6))$ of all the planes intersecting H along a line passing through P and incident to Π_i, for $1 \le i \le 5$.

Equivalently, one is asking to look for the class in $A_*(G(3, 6))$ of all the (affine) 3-dimensional vector subspaces of $\mathbb{C}^6$ such that

a) intersect a 4-dimensional subspace $\overline{H}$ of $\mathbb{C}^6$ along a 2-plane containing a given 1-plane (corresponding to the point P) and

b) intersecting 5 subspaces of dimension 3 along a positive dimensional subspace. The first step to solve such an exercise is to identify the involved Schubert varieties . Let $E^\bullet$ be the complete flag:

$$\mathbb{C}^7 = E^0 \supset E^1 \supset E^2 \supset E^3 \supset E^4 \supset E^5 \supset E^6 = (0)$$

If $[\Lambda] \in G(3,6)$ satisfy a), then one has that $E^5 \subset [\Lambda] \cap E^2$, and that $\dim([\Lambda] \cap E^3) \geq 2$. The general such plane is contained in the Schubert cell $W_{136}(E^{\cdot})$. In fact if $[\Lambda]$ verifies $a)$ one has $dim(E^2 \cap [\Lambda]) \geq 2$ and $\dim(E^5 \cap [\Lambda]) = 1$. The most general such plane is when the equality holds, which corresponds indeed to the k-planes having $1,3,5$ as a Schubert index. If a 3-plane $[\Lambda]$, instead, meet E^3 (which has (affine) dimension 3) along a positive dimensional vector subspace, then it belongs to the Schubert cycle $\Omega_{124}(E^\bullet)$, since for the most general such 3-plane one has $\dim([\Lambda] \cap E^4) = 0$. By Kleiman's theorem (Section 2.3.9), one knows that it is possible to choose sufficiently general flags $F_1^\bullet \ldots, F_5^\bullet, F_0^\bullet$ such that the intersection:

$$Y := \Omega_{124}(F_1^\bullet) \cap \ldots \cap \Omega_{124}(F_5^\bullet) \cap \Omega_{136}(F_0^\bullet)$$

is proper, i.e. such that the codimension of the intersection scheme coincides with the sum of the codimensions of those one is intersecting. The class of Y in $A_*(G(3,7)$ is then:

$$[Y] = \sigma_1^5 \cdot \sigma_{31} \cap [G(3,6)]$$

and the problem now amounts to compute explicitly the product $\sigma_1^5\sigma_{31} \in A^*(G(3,6)) = A^*(G(2,\mathbb{P}^5)$. Notice that Y has codimension $9 = \dim(G(3,6))$. To continue computations see Example 3.4.9

3.4.5 Since any Schubert variety is characterized by the Schubert index of a general point of it, one sees, almost by definition, that $\Omega_I(E^\bullet)$ is the Zariski closure $\overline{W_I(E^\bullet)}$ of the Schubert cell

$$W_I(E^\bullet) = \left\{ [\Lambda] \in G_k(V) \;\middle|\; \begin{cases} \epsilon^{j_1} \wedge \ldots \wedge \epsilon^{j_k}(\Lambda) = 0, & \forall (j_1,\ldots,j_k) \prec I \\ \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\Lambda) \neq 0 \end{cases} \right\} \tag{3.10}$$

In consideration of this remark, i.e. that the Schubert varieties are defined, even scheme theoretically, by equations (3.10), one shall also use (see e.g. Example 5.1) the notation:

$$[\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}], \tag{3.11}$$

to denote the *Schubert cycle* $\Omega_{i_1 \ldots i_k} \in A_*(G_k(V))$.

So, imitating Lascoux (see [61], p. 12) who said

"The Schubert variety is a Schur function",

we may say as well:

The Schubert variety is a generalized wronskian.

Indeed the two sentences are equivalent, as it is implicitly shown in the recent work [48]. However we shall not insist on that.

3.4.6 Example. Let us fix a flag $E^\bullet$ in $\mathbb{C}^4$. The homogeneous coordinate ring of the Plücker embedding of the grassmannian is the homogeneous polynomial ring

$$S(G(2,4)) = \frac{\mathbb{C}[\epsilon^i \wedge \epsilon^j]}{(Q)},$$

where $Q = 0$ is the Klein quadric relation. The ideal of the Schubert cycle $[\epsilon^2 \wedge \epsilon^3]$ in $S(G(2,4))$, represented by the Schubert variety $\Omega_{23}(E^\bullet)$, is $(\epsilon^1 \wedge \epsilon^2, \epsilon^1 \wedge \epsilon^3, \epsilon^1 \wedge \epsilon^4)$, while that of $[\epsilon^1 \wedge \epsilon^4]$, represented by the Schubert variety $\Omega_{14}(E^\bullet)$, is $(\epsilon^1 \wedge \epsilon^2, \epsilon^1 \wedge \epsilon^3, \epsilon^2 \wedge \epsilon^3)$. The Reader should check, for sake of exercise, that the sum of the two ideals cut out precisely the Schubert variety $[\epsilon^2 \wedge \epsilon^4]$.

3.4.7 Via the Poincaré isomorphism:

$$A^*(G_k(V)) \longrightarrow A_*(G_k(V)),$$

sending $\sigma_{\underline{\lambda}} \mapsto \sigma_{\underline{\lambda}} \cap [G_k(V)]$, and by Proposition 3.4.3, it follows that $A^*(G_k(V))$ is generated as a $\mathbb{Z}$-module by the Schubert cycles $\sigma_{\underline{\lambda}}$. It turns out that $\sigma_i = c_i(\mathcal{Q}_k)$ (see [16], p. 271). Doing intersection theory on the grassmannian amounts to knowing how to multiply any two Schubert classes $\sigma_{\underline{\lambda}}$ and $\sigma_{\underline{\mu}}$, i.e. to know $\sigma_{\underline{\lambda}} \cdot \sigma_{\mu}$ in $A^*(G_k(V))$ or, equivalently, $\sigma_{\underline{\lambda}} \cap \Omega_{\underline{\mu}} \in A_*(G_k(V))$. Using the combinatorial language of Young diagrams (see Section 2.1 or, better, [20]), one may also say that the Chow ring $A^*(G_k(V))$ is freely generated, as a module over the integers, by the Schubert (co)cycles:

$$\{\sigma_{\underline{\lambda}} \mid \underline{\lambda} \text{ is a partition contained in a } k(n-k) \text{ rectangle}\},$$

where $\sigma_{\underline{\lambda}} \cap [G_k(V)]$ is the class of a Schubert variety $\Omega_{\underline{\lambda}}(E^\bullet)$ associated to any flag $E^\bullet$ of V. Schubert Calculus allows to write the product $\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\mu}}$ as an explicit linear combination of elements of the given basis of $A^*(G_k(V))$. It consists, indeed, in an explicit algorithm to determine the structure constants $\{C^{\underline{\nu}}_{\underline{\lambda\mu}}\}$ defined by:

$$\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\mu}} = \sum_{|\underline{\nu}|=|\underline{\lambda}|+|\underline{\mu}|} C^{\underline{\nu}}_{\underline{\lambda\mu}} \sigma_{\underline{\nu}}.$$

The coefficients $C^{\underline{\nu}}_{\underline{\lambda\nu}}$ can be determined combinatorially via the *Littlewood-Richardson rule* ([52], p. 68).

The other recipe consists in determining any product via reduction to known cases. To this purpose, one first establishes a rule to multiply any Schubert cycle with a *special one*. A *special Schubert cycle* is a cycle indexed by a partition of length 1. Such a product is ruled by

3.4.8 Theorem (Pieri's Formula). *The following multiplication rule holds:*

$$\sigma_h \cdot \sigma_{\underline{\lambda}} = \sum_{\underline{\mu}} \sigma_{\underline{\mu}} \tag{3.12}$$

the sum over all partitions such that $|\mu| = |\underline{\lambda}| + h$ *and*

$$n - k \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_k \geq \lambda_k.$$

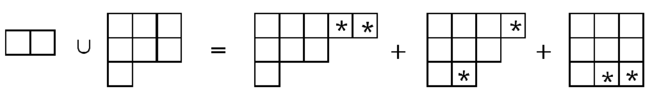
where $n = \dim(V)$.

Proof. (see e.g. [27], p. 203).

Pieri's formula can be also phrased by saying that sum (3.12) is over all the partitions $\underline{\mu}$ whose Young diagram $Y(\underline{\mu})$ is gotten by adding h boxes to $Y(\underline{\lambda})$, in all possible ways, not two on the same column. For instance, in $G(3, n)$, with $n \geq 9$, one has:

$$\sigma_2 \cdot \sigma_{331} = \sigma_{531} + \sigma_{432} + \sigma_{333}.$$

The "graphical Pieri's formula", in terms of Young diagrams is depicted below:



3.4.9 Example (3.4.4 continued). With Pieri's formula at our disposal one may compute the intersection product $\sigma_1^5 \cdot \sigma_{31} \in A^*(G(3,6))$. One applies Pieri's formula 5-times, getting:

$$\sigma_1^4(\sigma_{32} + \sigma_{311}) = \ldots (\textit{four more times}) \ldots = 5 \cdot \sigma_{333}.$$

Therefore the answer to the question proposed in Example 3.4.4 is 5 (see Example 5.2.3 for computational details in a different formalism).

It is not difficult to prove, and in fact we shall do it in Chapter 4, that *Pieri's* formula determines, indeed, the ring structure of $A^*(G)$. In particular, one can see that $A^*(G)$ is generated, as a ring, by the first k special Schubert cycles $\sigma_1, \ldots, \sigma_k$. This is a consequence of another explicit consequence of Pieri's formula, i.e. the determinantal *Giambelli's formula*, expressing the Schubert cycle $\sigma_{\underline{\lambda}}$ as a polynomial in the special ones:

3.4.10 Proposition (Giambelli's Formula). *The Schubert cycle associated to a partition $\underline{\lambda} = (r_k, \ldots, r_1)$ is a (determinantal) polynomial expression in the special Schubert cycle σ_i's:*

$$\sigma_{\underline{\lambda}} = \Delta_{\underline{\lambda}}(\sigma) = \begin{vmatrix} \sigma_{r_1} & \sigma_{r_2+1} & \cdots & \sigma_{r_k+k-1} \\ \sigma_{r_1-1} & \sigma_{r_2} & \cdots & \sigma_{r_k+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{r_1-k+1} & \sigma_{r_2-k+2} & \cdots & \sigma_{r_k} \end{vmatrix} = \det(\sigma_{r_j+j-i}).$$

Proof. It will be given in Section 4.8, within the formalism of $\mathcal{S}$-derivations as a consequence of a suitable *"integration by parts"*. ■

Therefore, the computation of an arbitrary product $\sigma_{\underline{\lambda}} \cdot \sigma_{\underline{\mu}}$ is reduced to a sequence of applications of Giambelli's and Pieri's formula: one first writes $\sigma_{\underline{\lambda}}$ as a polynomial in the σ_i's, and then applies Pieri's formula as many times as necessary and then again Giambelli's and so on. Computations become intricate in big grassmannians and for lengthy partitions, but products are computable in principle.

3.4.11 Example. To compute $\sigma_{11} \cdot \sigma_{21}$ in $A^*(G(3,6))$, one first use Giambelli's formula to write, e.g., σ_{11} as $\sigma_1^2 - \sigma_2$ and then expand the product:

$$(\sigma_1^2 - \sigma_2)\sigma_{21},$$

by applying thrice Pieri's formula. The result, whose checking is left to the reader, is $\sigma_{221} + \sigma_{311} + \sigma_{32}$.

3.4.12 A few comments. As already said, Giambelli's formula is a formal consequence of Pieri's formula. However, the way to show it may be rather tricky. The reader may look at the proof of [27], p. 204-206, which, in spite of being conceptually easy, is based on a formula whose proof consists in a case by case combinatorial analysis. Another equivalent proof is that presented in [53], p. 13. There. the Author proves Giambelli's formula in the realm of symmetric functions, where a *Schur polynomial* associated to a given partition is expressed as a determinant of *complete symmetric functions*: that is the so-called *Jacobi-Trudy formula* (see [52], pp. 23 ff. and also [16], p. 422). The proof works by induction and the details of a preliminary lemma (analogous to that used in [27], p. 204) are left to the reader's care. The proof we propose in Section 4.8, within the formalism of $\mathcal{S}$-derivations, instead, is based on the definition of determinant of a square matrix!

The shortest elegant proof the author knows is the very general one, by Laksov and Thorup, in a recent preprint ([48]), within the framework of symmetric functions. The philosophy is however quite different from that of these notes: there is there more mathematics, more generality and hence more naturality.

3.4.13 Giambelli's formula proves that $A^*(G_k(V))$ is generated, as a ring, by the *special Schubert cycles* only. Its presentation can be gotten combining Giambelli's and Pieri's formula and is given by

$$A^*(G_k(V)) = \frac{\mathbb{Z}[\sigma_1, \ldots, \sigma_k]}{(\Sigma_{n-k+1}(\sigma), \ldots, \Sigma_n(\sigma))},$$

where each $\Sigma_{n-k+i}(\sigma)$ is an explicit weighted homogeneous polynomial in $\sigma_1, \ldots, \sigma_k$ of degree $n-k+i$. Such a presentation shall be explicitly computed in Section 4.7. Notice that since $\sigma_i = c_i(\mathcal{Q})$ (as recalled in Section 3.4.7), the Chow ring of the Grassmannian is the generated by the first k Chern classes of the universal quotient bundle $\mathcal{Q}_k$. In particular $\sigma_{n-k+1}, \ldots, \sigma_n$ vanish if $k < n-k$, by properties of the Chern Classes.

Chapter 4

Schubert's Algebra

This chapter is devoted to construct natural endomorphisms $(D_1, D_2, \ldots)$ of the k^{th} exterior power $\bigwedge^k M_n$ $(k \geq 1)$ of a free $\mathbb{Z}$-module M_n of rank n. The image via D_h $(h \geq 1)$ of a basis element of $\bigwedge^k M_n$ can be computed through a Pieri-like formula (Theorem 4.4.1 and Proposition 4.7.4). This will prove that $\bigwedge^k M_n$ is a free module of rank 1 over the subring of $End_{\mathbb{Z}}(\bigwedge^k M_n)$ generated by the D_i's, isomorphic to the Chow group of the grassmannian variety $G_k(V)$ ($\dim(V) = n$), seen as a free module of rank 1 over the Chow ring $A^*(G_k(V))$ (Theorem 4.7.6). Since applying D_h to a basis element "simulates" the operation of taking the h^{th} derivative of a (generalized) wronskian (see Chapter 1), the quoted theorem proves indeed that such a differentiation formalism coincides with Schubert calculus. It will be then developed in a purely formal way. The proofs look natural and particularly transparents: for instance, the relations occurring in the presentation of the Chow ring of the grassmannian, are phrased in terms of "derivatives" vanishing on $\bigwedge^k M_n$, while Giambelli's determinantal formula (necessary to explicitly multiply two arbitrary Schubert cycles in $A^*(G_k(V))$) is seen as a consequence of a more general "integration by parts" formula.

4.1 Hasse-Schmidt Derivations on Exterior Algebras

For the time being, let M be any A-module, t an indeterminate over A and $A[[t]]$ the ring of formal power series in t. Let $\bigwedge M[[t]] := \bigwedge M \otimes_A A[[t]]$: an element of $\bigwedge M[[t]]$ is a formal power series with "coefficients" in $\bigwedge M$[1].

4.1.1 Definition. *A* Hasse-Schmidt derivation $\mathfrak{D}_t$ *on* $\bigwedge M$ *is an* A*-algebra homomorphism:*

$$\mathfrak{D}_t : \bigwedge M \longrightarrow \left(\bigwedge M\right)[[t]].$$

A Hasse-Schmidt derivation on $\bigwedge M$ determines, and is determined by, its *coefficients* $\mathfrak{D}_i : \bigwedge M \longrightarrow \bigwedge M$, defined via the equality

$$\mathfrak{D}_t\alpha = \sum_{i\geq 0}\mathfrak{D}_i(\alpha)t^i, \qquad \forall\alpha\in\bigwedge M.$$

4.1.2 Proposition. *For each* $i \geq 0$*,* $\mathfrak{D}_i$ *satisfies (the generalized) Leibniz's rule:*

$$\mathfrak{D}_h(\alpha\wedge\beta) = \sum_{\substack{h_1+h_2=h\\ h_1,h_2\geq 0}} \mathfrak{D}_{h_1}\alpha\wedge\mathfrak{D}_{h_2}\beta, \tag{4.1}$$

Proof. By explicitly writing that $\mathfrak{D}_t$ is an A-algebra homomorphism:

$$\mathfrak{D}_t(\alpha\wedge\beta) = \mathfrak{D}_t\alpha\wedge\mathfrak{D}_t\beta, \quad \forall\alpha,\beta\in\bigwedge M.$$

one sees tha $\mathfrak{D}_h(\alpha\wedge\beta)$ is the coefficient of t^h in the expansion of $\mathfrak{D}_t(\alpha\wedge\beta)$, which is also the coefficient of t^h in the expansion of the wedge product

$$(\mathfrak{D}_0\alpha + \mathfrak{D}_1\alpha\, t + \mathfrak{D}_2\alpha\, t^2 + \ldots)\wedge(\mathfrak{D}_0\beta + \mathfrak{D}_1\beta\, t + \mathfrak{D}_2\beta\, t^2 + \ldots),$$

[1] Since the exterior algebra has been defined only for free modules, the reader not familiar with the general definition, may assume that M is free, as will be needed later on. However the construction below holds in general.

i.e. exactly the right hand side of eq. (4.1). ∎

In particular, $\mathfrak{D}_0$ is an algebra homomorphism:

$$\mathfrak{D}_0(\alpha \wedge \beta) = \mathfrak{D}_0(\alpha) \wedge \mathfrak{D}_0(\beta),$$

while $\mathfrak{D}_1$ is a "usual" derivation:

$$\mathfrak{D}_1(\alpha \wedge \beta) = \mathfrak{D}_1\alpha \wedge \beta + \alpha \wedge \mathfrak{D}_1\beta.$$

If $\mathfrak{D}_t = \sum_{i\geq 0} \mathfrak{D}_i t^i$, the sequence of the "coefficients"

$$\mathfrak{D} := (\mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2, \ldots)$$

of $\mathfrak{D}_t$ will be also said a HS-derivation and the $\mathfrak{D}_i$'s will be also said to be the *components* of $\mathfrak{D}$. To denote a HS-derivation the symbols $\mathfrak{D}$ and $\mathfrak{D}_t$ shall be used interchangeably.

4.1.3 Proposition. *If $\mathfrak{d}_t : M \longrightarrow M[[t]]$ is an A-module homomorphism there is one and only one HS-derivation $\mathfrak{D}_t : \bigwedge M \longrightarrow \bigwedge M[[t]]$ such that $\mathfrak{D}_{t|_M} = \mathfrak{d}_t$.*

Proof. Since any $\mu \in \bigwedge M$ is a finite sum of homogeneous elements $\mu_{k_1} + \ldots + \mu_{k_h}$, we may assume that μ is homogeneous of degree k, i.e. $\mu \in \bigwedge^k M$. Therefore μ is a finite sum products of the form $m_{i_1} \wedge \ldots \wedge m_{i_k}$. Then one sets:

$$\mathfrak{D}_t(\mu) = \mathfrak{D}_t(m_{i_1} \wedge \ldots \wedge m_{i_k}) = \mathfrak{d}_t(m_{i_1}) \wedge \ldots \wedge \mathfrak{d}_t(m_{i_k})$$

getting an extension to the exterior algebra. Suppose now that $\mathfrak{C}_t$ is another extension such that $\mathfrak{C}_{t|_M} = \mathfrak{d}_t$. Then:

$$\begin{aligned} & \mathfrak{C}_t(m_{i_1} \wedge \ldots \wedge m_{i_k}) = \mathfrak{C}_t(m_{i_1}) \wedge \ldots \wedge \mathfrak{C}_t(m_{i_k}) = \\ = \quad & \mathfrak{d}_t(m_{i_1}) \wedge \ldots \wedge \mathfrak{d}_t(m_{i_k}) = \mathfrak{D}_t(m_{i_1} \wedge \ldots \wedge m_{i_k}), \end{aligned}$$

for each $k \geq 0$ and each $m_{i_1} \wedge \ldots \wedge m_{i_k} \in \bigwedge^k M$. The extension is hence unique. ∎

4.1.4 Definition. *A HS-derivation $\mathfrak{D}$ on $\bigwedge M$ is said to be* regular *if $\mathfrak{D}_0 \in End_A(\bigwedge M)$ is an A-automorphism; a regular HS-derivation is* normalized *if $\mathfrak{D}_0 = id_{\bigwedge M}$.*

The distinction between regular and normalized HS-derivation is immaterial, because to each regular HS-derivation $\mathfrak{D}_t$ corresponds the normalized one:

$$\mathfrak{D}_0^{-1}\mathfrak{D}_t = id_{\bigwedge M} + \sum_{i\geq 0} {\mathfrak{D}_0}^{-1}\mathfrak{D}_i t^i.$$

If $\mathfrak{D}_t$ is a HS-derivation, denote by $\overline{\mathfrak{D}_t}$ the map

$$\overline{\mathfrak{D}_t} : (\bigwedge M)[[t]] \longrightarrow (\bigwedge M)[[t]],$$

defined as:

$$\overline{\mathfrak{D}_t}(\sum_{i\geq 0} \alpha_i t^i) = \sum_{i\geq 0} \mathfrak{D}_t \otimes_A \mathbf{1}_{A[[t]]}(\alpha_i t^i) = \sum_{i\geq 0} \mathfrak{D}_t(\alpha_i)t^i. \tag{4.2}$$

This is an $A[[t]]$-algebra endomorphism of $\bigwedge M[[t]]$. In fact:

$$\begin{aligned}
\overline{\mathfrak{D}_t}(\sum_i \alpha_i t^i \wedge \sum_j \beta_j t^j) &= \overline{\mathfrak{D}_t} \sum_{k\geq 0} \left(\sum_{i+j=k} \alpha_i \wedge \beta_j \right) t^k = \\
= \sum_{k\geq 0} \sum_{i+j=k} \mathfrak{D}_t(\alpha_i \wedge \beta_j)t^k &= \sum_{k\geq 0} \sum_{i+j=k} (\mathfrak{D}_t\alpha_i \wedge \mathfrak{D}_t\beta_j)t^k = \\
\overline{\mathfrak{D}_t}(\sum_i \alpha_i t^i) \wedge \overline{\mathfrak{D}_t}(\sum_j \beta_j t^j).
\end{aligned} \tag{4.3}$$

Let $HS_t(\bigwedge M)$ be the set of all regular HS- derivations on $\bigwedge M$, and let $\mathfrak{C}_t, \mathfrak{D}_t \in HS_t(\bigwedge M)$. Define their product in $End_A(\bigwedge M)[[t]]$ as being:

$$\mathfrak{C}_t * \mathfrak{D}_t = \sum_{h\geq 0} \sum_{i+j=h} (\mathfrak{C}_i \circ \mathfrak{D}_j)t^h \tag{4.4}$$

and notice that

$$\mathfrak{C}_t * \mathfrak{D}_t(\alpha) = \overline{\mathfrak{C}_t} \circ \overline{\mathfrak{D}_t}(\alpha), \quad \forall \alpha \in \bigwedge M,$$

by just expanding the two sides of the above equation according to (4.2) and (4.4)

4.1.5 Proposition. *The pair $(HS_t(\bigwedge M), *)$ is a group.*

Proof. Let $\mathfrak{C}_t, \mathfrak{D}_t \in HS_t(\bigwedge M)$. Let us prove that $\mathfrak{C}_t * \mathfrak{D}_t \in HS_t(M)$. In fact

$$\begin{aligned} \mathfrak{C}_t * \mathfrak{D}_t(\alpha \wedge \beta) &= (\overline{\mathfrak{C}_t} \circ \overline{\mathfrak{D}_t})(\alpha \wedge \beta)) = \\ &= \overline{\mathfrak{C}_t}(\overline{\mathfrak{D}_t}(\alpha) \wedge \overline{\mathfrak{C}_t}(\beta)) = \\ &= \overline{\mathfrak{C}_t}(\overline{\mathfrak{D}_t}(\alpha)) \wedge \overline{\mathfrak{C}_t}(\overline{\mathfrak{C}_t}(\beta)) = \\ &= \overline{\mathfrak{C}_t} \circ \overline{\mathfrak{D}_t}(\alpha) \wedge \overline{\mathfrak{C}_t} \circ \overline{\mathfrak{D}_t}(\beta) = \\ &= \mathfrak{C}_t * \mathfrak{D}_t(\alpha) \wedge \mathfrak{C}_t * \mathfrak{D}_t(\beta) \end{aligned}$$

so that $\mathfrak{C}_t \circ \mathfrak{D}_t \in HS_t(\bigwedge M)$. The composition $\circ$ is obviously associative. The identity $1 := 1_M$ of M belongs to $HS_t(\bigwedge M)$. Let $\mathfrak{D}_t^{-1}$ be the formal inverse of $\mathfrak{D}_t$ in $End(\bigwedge M)[[t]]$, i.e.

$$\mathfrak{D}_t \circ \mathfrak{D}_t^{-1} = \mathfrak{D}_t^{-1} \circ \mathfrak{D}_t = id_{\bigwedge M},$$

existing by the regularity hypothesis on $\mathfrak{D}_t$. One has:

$$\begin{aligned} \mathfrak{D}_t^{-1}(\alpha \wedge \beta) &= \mathfrak{D}_t^{-1}(\mathfrak{D}_t(\mathfrak{D}_t^{-1})(\alpha) \wedge \mathfrak{D}_t(\mathfrak{D}_t^{-1})(\beta)) = \\ &= \mathfrak{D}_t^{-1}(\mathfrak{D}_t((\mathfrak{D}_t^{-1})\alpha \wedge (\mathfrak{D}_t^{-1})\beta)) = \\ &= (\mathfrak{D}_t^{-1} \circ \mathfrak{D}_t)(\mathfrak{D}_t^{-1}\alpha \wedge \mathfrak{D}_t^{-1}\beta) = (\mathfrak{D}_t^{-1}\alpha \wedge \mathfrak{D}_t^{-1}\beta), \end{aligned}$$

proving that $\mathfrak{D}_t^{-1} \in HS_t(\bigwedge M)$, which is obviously regular. ∎

4.2 Shift Endomorphisms

4.2.1 From now on M will be a $\mathbb{Z}$-module freely generated by $\mathcal{E} := (\epsilon^1, \epsilon^2, \ldots)$:

$$M := \bigoplus_{i \geq 0} \mathbb{Z} \cdot \epsilon^i.$$

Any $m \in M$ is a unique finite linear combination $\sum_{i \geq 1} m_i \epsilon^i$ of elements of $\mathcal{E}$. Regard M as a graded $\mathbb{Z}$-module:

$$M = \bigoplus_{w \geq 0} M_w, \tag{4.5}$$

where $M_w = \mathbb{Z} \cdot \epsilon^{1+w}$. If $m \in M_j$, the integer $wt(m) = j - 1$ is said to be the *weight* of m. For instance, ϵ^1 has weight 0, ϵ^2 has weight 1 and so on.

4.2.2 Remarks and Examples.

a) The module M will be interpreted, later on, as being the dual of a $\mathbb{Z}$-module N, freely generated by a basis $E := (e_1, e_2, \ldots)$. In this case $\mathcal{E}$ will be taken to be the basis of $N^\vee$ dual of E, i.e. such that $\epsilon^i(e_j) = \delta^i_j$. This partly motivates the use of superscripts to index the elements of the basis of M.

b) Let $\overline{M} := \mathbb{Z}\oplus M$. It is the $\mathbb{Z}$-module freely generated by $\overline{\mathcal{E}} = (\epsilon^0, \epsilon^1, \ldots,)$, where $\epsilon^0 := 1\oplus \mathbf{0}_M$. A model for $\overline{M}$ is $\mathbb{Z}^\infty$, the $\mathbb{Z}$-module of all the functions $P : \mathbb{N} \longrightarrow \mathbb{Z}$ such that $\{n \mid P(n) \neq 0\}$ is a finite set. The canonical basis of $\mathbb{Z}^\infty$ is $\{\delta^i | i \geq 0\}$, where $\delta^i : \mathbb{N} \longrightarrow \mathbb{Z}$ is defined by $\delta^i(j) = \delta^i_j$. Any $P \in \mathbb{Z}^\infty$ can be uniquely written as $P = \sum_{i\geq 0} P(i)\delta^i$. All the free $\mathbb{Z}$-modules of infinite countable rank are isomorphic to $\mathbb{Z}^\infty$, via a unique isomorphism $\overline{\mathcal{E}} : \mathbb{Z}^\infty \longrightarrow \overline{M}$, defined by $\overline{\mathcal{E}}(\delta^i) = \epsilon^i$ $(i \geq 0)$. The isomorphism $\mathbb{Z}^\infty \longrightarrow \overline{M}$ and the basis $(\epsilon^i)_{i\geq 0}$ have been indicated with the same letter $\overline{\mathcal{E}}$, by abuse of notation.

c) Take $\overline{M} = \mathbb{Z}[X]$, the $\mathbb{Z}$-module of polynomials with $\mathbb{Z}$-coefficients in one indeterminate. A basis is given by the monic monomials of degree $i \geq 0$:

$$(X^n)_{n\geq 0} := \{1, X, X^2, X^3, \ldots\}.$$

This is another piece of motivation for using superscripts to index the elements of the basis $\overline{\mathcal{E}}$ of $\overline{M}$. In this case the module $M \subseteq \overline{M}$ can be interpreted as the prime ideal (X).

4.2.3 Definition. *Let $i \geq 0$ be an integer. The i^{th} $\mathcal{E}$-*shift endomorphism *$D_i : M \longrightarrow M$ is the unique $\mathbb{Z}$-endomorphism such that $D_i(\epsilon^j) = \epsilon^{i+j}$. The* shift endomorphism sequence *is:*

$$D := (D_0, D_1, \ldots).$$

Clearly $D_0 = id_M$ and, for each $i > 0$, D_i is a monomorphism: moreover $D_i = D_1^i$ (agreeing that $D_1^0 = D_0$).

With respect to the grading (4.5) of M, D_i is a homogeneous endomorphism of M of *weight* (=degree) i, because $D_iM_j = M_{i+j}$. Let D_iM denote the free submodule of M generated by

$$\mathcal{E}^i := D_i\mathcal{E} = (\epsilon^{i+1}, \epsilon^{i+2}, \ldots).$$

One has a filtration of M by means of free submodules:

$$DM : \quad M = D_0M \supset D_1M \supset D_2M \supset \ldots.$$

Let $M^\vee := Hom_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual module of M and denote by $E := (e_1, e_2, \ldots)$ the basis of $M^\vee$ dual of $\mathcal{E}$, i.e. $e_i(\epsilon^j) = \delta_i^j$.

4.2.4 Examples.

a) If $\overline{M} = \mathbb{Z}[X]$, the i^{th}-shift endomorphism of $\mathbb{Z}[X]$ (with respect to the basis $(1, X, \ldots)$) is the multiplication by X^i. In fact $X^i \cdot X^j = X^{i+j}$. Also in this case one has $X^i = X_1^i$.

b) Let $I_\delta = (-\delta, \delta) \subset \mathbb{R}$, with $0 < \delta < 1$ and $f : I_\delta \longrightarrow \mathbb{R}$ be the function

$$f(x) = \frac{1}{1-x}.$$

Then f is a function of class C^∞ on I_δ. Let $f^{(i)}(x)$ be the i-th derivative of f. Since

$$f^{(i)}(x) = \frac{\mathfrak{C}_i}{(1-x)^{i+1}}, \quad \mathfrak{C}_i \in \mathbb{R}$$

then $(f, f', f'', \ldots)$ are linearly independent in the $\mathbb{R}$-vector space $C^\infty(I_\delta)$, and hence also linearly independent over $\mathbb{Z}$. Define M as $\bigoplus_{i \geq 0} \mathbb{Z} \cdot f^{(i)}$. Then, the shift operator is precisely d^i/dx^i sending $f^{(j)}(x)$ onto $f^{(i+j)}(x)$. This kind of examples inspired the choice of the symbol D_i (as for *derivative*) for the i^{th}-shift operator and again the superscripts to denote the elements of the basis $\overline{\mathcal{E}}$ of $\overline{M}$.

4.2.5 The ring $\mathcal{A}^*(M, D)$. If $\underline{\lambda} = (1^{m_1} 2^{m_2} \ldots)$ is a partition, define:

$$D^{\underline{\lambda}} = D_1^{m_1} \circ \ldots \circ D_k^{m_k} \in End_{\mathbb{Z}}(M),$$

where $\circ$ is the composition in $End_{\mathbb{Z}}(M)$ and

$$D_i^j = \underbrace{D_i \circ \ldots \circ D_i}_{j \ times},$$

is the j^{th} iterated of D_i.

Let $\mathrm{ev}_D : \mathbb{Z}[\mathbf{T}] \longrightarrow End_{\mathbb{Z}}(M)$ be the module homomorphism defined by:

$$\mathbf{T}^{\underline{\lambda}} \longmapsto \mathrm{ev}_D(\mathbf{T}^{\underline{\lambda}}) := D^{\underline{\lambda}}.$$

The image is the $\mathbb{Z}$-submodule of $End_{\mathbb{Z}}(M)$:

$$\mathcal{A}^*(M, D) := \left\{ \sum_{\underline{\lambda} \in L} a_{\underline{\lambda}} D^{\underline{\lambda}} \mid L \text{ is a finite subset of } \mathcal{P} \right\}.$$

If $P(\mathbf{T}) \in \mathbb{Z}[\mathbf{T}]$, we shall simply write $P(D)$ instead of $\mathrm{ev}_D(P(\mathbf{T}))$.

4.2.6 Proposition. *The set $\mathcal{A}^*(M, D)$ is the minimum $\mathbb{Z}$-subalgebra of $End_{\mathbb{Z}}(M)$ containing all the D_i's. Moreover it is commutative.*

Proof. The fact that $\mathcal{A}^*(M, D)$, with respect to the composition of endomorphisms, is a $\mathbb{Z}$-subalgebra of $End_{\mathbb{Z}}(M)$ is obvious; it is commutative because $D_i D_j = D_j D_i$, for all $i, j \geq 0$. Suppose now that A is any $\mathbb{Z}$-subalgebra containing all the D_i's $(i \geq 0)$. Then it contains $D^{\underline{\lambda}}$, too, for each $\underline{\lambda} \in \mathcal{P}$, and therefore any finite $\mathbb{Z}$-linear combination of them. Then $A \supset \mathcal{A}^*(M, D)$. ∎

Because of the commutativity of $\mathcal{A}^*(M, D)$, the natural evaluation epimorphism:

$$\mathrm{ev}_D : \mathbb{Z}[\mathbf{T}] \longrightarrow \mathcal{A}^*(M, D),$$

defined by $\mathrm{ev}_D(\mathbf{T}^{\underline{\lambda}}) = D^{\underline{\lambda}}$, extends to a ring homomorphism, again denoted by the same symbol ev_D. It can be defined on the algebra generators by $\mathrm{ev}_D(T_i) = D_i$. Then $\mathcal{A}^*(M, D)$ is itself a graded ring:

$$\mathcal{A}^*(M, D) = \bigoplus_{h \geq 0} \mathcal{A}^h(M, D),$$

where $\mathcal{A}^h(M, D) = \mathrm{ev}_D(\mathbb{Z}[\mathbf{T}]_h)$. Let

$$\begin{cases} \mathrm{ev}_{\epsilon^1} & : & \mathcal{A}^*(M, D) & \longrightarrow & M \\ & & P(D) & \longmapsto & P(D) \cdot \epsilon^1 \end{cases}.$$

Clearly ev_{ϵ^1} is surjective. In fact for any $m = a_1\epsilon^1 + a_2\epsilon^2 + \ldots + a_n\epsilon^n$, there is $G_m(D) \in \mathcal{A}^*(M, D)$ such that $G_m(D)\epsilon^1 = m$: it suffices to choose

$$G_m(D) = a_1 id_M + a_2 D_1 + \ldots + a_{n-1} D_n.$$

Furthermore

$$\mathrm{ev}_{D,\epsilon^1} := \mathrm{ev}_{\epsilon^1} \circ \mathrm{ev}_D : \mathbb{Z}[\mathbf{T}] \longrightarrow M,$$

is an epimorphism and the equality $\ker(\mathrm{ev}_{D,\epsilon^1}) = \ker(\mathrm{ev}_D)$ holds. In fact $\ker(\mathrm{ev}_D) \subseteq \ker(\mathrm{ev}_{D,\epsilon^1})$, by the very definition of $\mathrm{ev}_{D,\epsilon^1}$.

Conversely, if $P(\mathbf{T}) \in \ker(\mathrm{ev}_{D,\epsilon^1})$, then $\mathrm{ev}_D(P(\mathbf{T}))\epsilon^1 = \mathbf{0}_M$, i.e. for each $m \in M$:

$$\mathrm{ev}_D(P(\mathbf{T}))m = \mathrm{ev}_D(P(\mathbf{T})) \cdot G_m(D)\epsilon^1 = G_m(D)\mathrm{ev}_D(P(\mathbf{T}))\epsilon^1 = 0,$$

meaning that $\mathrm{ev}_D(P(\mathbf{T})) = \mathbf{0}_{\mathcal{A}^*(M,D)}$, i.e. $P(\mathbf{T}) \in \ker(\mathrm{ev}_D)$. This proves that:

4.2.7 Proposition. *There are $\mathbb{Z}$-module isomorphisms:*

$$\frac{\mathbb{Z}[\mathbf{T}]}{\ker(\mathrm{ev}_D)} \stackrel{\mathrm{ev}_D}{\longrightarrow} \mathcal{A}^*(M,D) \stackrel{\mathrm{ev}_{\epsilon^1}}{\longrightarrow} M. \qquad \blacksquare$$

Next task consists in determining explicitly $\ker(\mathrm{ev}_D)$, but this is easy. Since $D_i = D_1^i$, the ideal $(T_i - T_1^i)_{i\geq 1}$ of $\mathbb{Z}[\mathbf{T}]$ is contained in the kernel, whence the surjection:

$$\mathbb{Z}[T_1] \cong \frac{\mathbb{Z}[\mathbf{T}]}{(T_2 - T_1^2, T_3 - T_1^3, \ldots)} \longrightarrow \frac{\mathbb{Z}[\mathbf{T}]}{\ker(\mathrm{ev}_D)} \cong M,$$

given by $T_1^{i-1} \mapsto \epsilon^i$. This latter map is clearly injective (if $\mathrm{ev}_D(a_0 + a_1T^{i_1} + \ldots + a_nT^{i_n}) = 0$, then $\sum_{j=1}^n a_{i_j}\epsilon^{i_j+1} = 0$, i.e. all $a_{i_j} = 0$) and then $\ker(\mathrm{ev}_D) = (T_i - T_1^i)_{i\geq 0}$ and $\mathbb{Z}[T_1] \cong \mathcal{A}^*(M,D) \cong M$.

4.2.8 A few words about the "extended" module $\overline{M} = \mathbb{Z} \oplus M$. First of all the shift endomorphisms $D_i : M \longrightarrow M$ clearly extend to shift endomorphisms $D_i : \overline{M} \longrightarrow \overline{M}$ (by abuse denoted by the same letter), by defining $D_i\epsilon^0 = \epsilon^i$. The corresponding filtration is then:

$$D\overline{M} : \quad \overline{M} = D_0\overline{M} \supset D_1\overline{M} \supset D_2\overline{M} \supset \ldots$$

extending (4.2.1). In particular one has $D_1\overline{M} = M$. In this case one has again $\mathcal{A}^*(\overline{M}, D) \cong \mathbb{Z}[T_1]$ and the isomorphism

$$\mathcal{A}^*(\overline{M}, D) \longrightarrow \overline{M}$$

is given by sending $D_1^n \mapsto \epsilon^n = \mathrm{ev}_{D,\epsilon^0}(T_1^n)$.

4.3 Schubert Derivations

4.3.1 If $D := (D_0, D_1, \ldots)$ is the $\mathcal{E}$-shift sequence of M, denote by:

$$D_t := \sum_{i\geq 0} D_it^i = id_M + D_1t + D_2t^2 + \ldots \in End_{\mathbb{Z}}(M)[[t]],$$

its corresponding *shift formal power-series.* Let $\bigwedge M$ be the exterior algebra of M.

4.3.2 Definition. *The Hasse-Schmidt extension of D_t to $\bigwedge M$ is said to be* Schubert derivation *($\mathcal{S}$-derivation, for short).*

The $\mathcal{S}$-derivation is then the unique $\mathbb{Z}$-algebra homomorphism

$$D_t : \bigwedge M \longrightarrow \bigwedge M[[t]],$$

such that $D_t(\epsilon^i) = \sum_{j\geq 0} \epsilon^{i+j}t^j$. The algebra homomorphism condition means that $D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta$, for each $\alpha, \beta \in \bigwedge M$. If $\alpha \in \bigwedge M$ and $h \geq 0$, $D_h\alpha$ will be said to be the h^{th} *$\mathcal{S}$-derivative* of α. In particular one has that D_0 is the identity and that, see 4.1.2, $D_h \in End_{\mathbb{Z}}(\bigwedge M)$ satisfies Leibniz's rule (4.1).

For each $h \geq 0$, D_h is a homogeneous endomorphism of $\bigwedge M$ of degree 0 with respect to the grading

$$\bigwedge M := \bigoplus_{k\geq 0} \bigwedge^k M,$$

i.e. $D_h(\bigwedge^k M) \subseteq \bigwedge^k M$. By abuse, the induced endomorphism

$$D_i|_{\bigwedge^k M} : \bigwedge^k M \longrightarrow \bigwedge^k M$$

will be still denoted by D_i.

4.3.3 Proposition. *The endomorphisms $D_i : \bigwedge^k M \longrightarrow \bigwedge^k M$ $(i \geq 0)$, are pairwise commuting.*

Proof. By induction on k. For $k = 1$ the claim is true by construction. Assume that the property holds for $k-1$. Since any $m \in \bigwedge^k M$ is a finite sum of k-vectors of the form $\alpha \wedge \beta$, with $\alpha \in M$ and $\beta \in \bigwedge^{k-1} M$, without loss of generality one may check the property for any m of this form. Then:

$$D_iD_j(\alpha \wedge \beta) = D_i(\sum_{\substack{j_1+j_2=j \\ j_1,j_2\geq 0}} D_{j_1}\alpha \wedge D_{j_2}\beta) =$$

$$= \sum_{\substack{i_1+i_2=i \\ i_1,i_2\geq 0}} \sum_{\substack{j_1+j_2=j \\ j_1,j_2\geq 0}} D_{i_1}D_{j_1}\alpha \wedge D_{i_2}D_{j_2}\beta. \tag{4.6}$$

By the inductive hypothesis, last member of equality (4.6) is equal to:

$$\sum_{\substack{j_1+j_2=j\\ j_1,j_2\geq 0}}\sum_{\substack{i_1+i_2=i\\ i_1,i_2\geq 0}} D_{j_1}D_{i_1}\alpha\wedge D_{j_2}D_{i_2}\beta = D_j(\sum_{\substack{i_1+i_2=i\\ i_1,i_2\geq 0}} D_{i_1}\alpha\wedge D_{i_2}\beta) = $$
$$= D_jD_i(\alpha\wedge\beta).$$

■

As a consequence, the natural evaluation $\mathbb{Z}$-module homomorphism

$$\begin{cases} \mathrm{ev}_D & : & \mathbb{Z}[\mathbf{T}] & \longrightarrow & End_{\mathbb{Z}}(\bigwedge^k M) \\ & & \mathbf{T}^{\underline{\lambda}} & \longmapsto & D^{\underline{\lambda}} \end{cases}$$

maps $\mathbb{Z}[\mathbf{T}]$ onto a commutative subring $\mathcal{A}^*(\bigwedge^k M, D)$ of $End_{\mathbf{Z}}(\bigwedge^k M)$.

It turns out that for $k>1$, as already for $k=1$, $\bigwedge^k M$ is itself a graded $\mathbb{Z}$-module with respect to the *weight* grading defined below.

4.3.4 Definition. *The* weight *of a non zero element* $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}\in\bigwedge^k M$ *is the weight of the* k*-schindex* I*:*

$$wt(\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}) = (i_1-1)+\ldots+(i_k-k) = \sum_{j=1}^{k} i_j - \frac{k(k+1)}{2}.$$

If $I=(1\leq i_1<\ldots<i_k)$, the weight of $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$ coincides with the weight of the associated partition $\underline{\lambda}(I)=(i_k-k,\ldots,i_1-1)$. Notice also that

$$wt(\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}) = wt(\epsilon^{i_{\tau(1)}}\wedge\ldots\wedge\epsilon^{i_{\tau(k)}}),$$

for all $\tau\in S_k$. Let $(\bigwedge^k M)_w$ be the submodule of $\bigwedge^k M$ spanned by all $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$ of weight w. Then, clearly:

$$\bigwedge^k M = \left(\bigwedge^k M\right)_0 \oplus \left(\bigwedge^k M\right)_1 \oplus \left(\bigwedge^k M\right)_2 \oplus \ldots = \bigoplus_{w\geq 0}\left(\bigwedge^k M\right)_w,$$

and this makes $\bigwedge^k M$ into a graded $\mathbb{Z}$-module.

4.3.5 Remark on the terminology. The terminology *weight* has been suggested by the theory of Weierstrass points. In fact the *Weierstrass gap sequence* of a point P of a smooth curve C is a g-schindex $1 = n_1 < \ldots < n_g$, and its weight is precisely $(n_1 - 1) + \ldots + (n_g - g)$. In his thesis [59], Ponza shows that if $1 < n_2 < \ldots < n_g$ is the Weierstrass gap sequence at P, then the *generalized wronskian section*

$$\underline{\omega} \wedge D^{n_1-1}\underline{\omega} \wedge \ldots \wedge D^{n_g-1}\underline{\omega}.$$

does not vanishes at P.

Proposition 4.3.6 below shows that $D_h : \bigwedge^k M \longrightarrow \bigwedge^k M$ is indeed a homogeneous endomorphism of degree (=weight) h of $\bigwedge^k M$.

4.3.6 Proposition. *The "generalized" Leibniz's rule holds:*

$$D_h(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{\substack{h_1+\ldots+h_k=h \\ h_i \geq 0}} \epsilon^{i_1+h_1} \wedge \epsilon^{i_2+h_2} \wedge \ldots \wedge \epsilon^{i_k+h_k},$$

for each $h \geq 0$.

Proof. By induction on k. If $k = 1$ the property is obviously true. Assume it holds for $k - 1$. Then

$$D_h(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{h_1=0}^{h} \epsilon^{i_1+h_1} \wedge D_{h-h_1}(\epsilon^{i_2} \wedge \ldots \wedge \epsilon^{i_k}). \quad (4.7)$$

By the inductive hypothesis:

$$D_{h-h_1}(\epsilon^{i_2} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{h_2+\ldots+h_k=h-h_1} \epsilon^{i_2+h_2} \wedge \ldots \wedge \epsilon^{i_k+h_k},$$

and the r.h.s. of formula (4.7) turns into:

$$D_h(\epsilon^{i_1} \wedge \epsilon^{i_2} \wedge \ldots \wedge \epsilon^{i_k}) = \textstyle\sum_{h_1+\ldots+h_k=h} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k}. \quad \blacksquare$$

In particular $\mathrm{ev}_D(\mathbf{T}^{\underline{\lambda}}) := D^{\underline{\lambda}}$ is a homogeneous endomorphism of $\bigwedge^k M$ of weight $|\underline{\lambda}|$, i.e.:

$$D^{\underline{\lambda}}(\bigwedge^k M)_w \subseteq (\bigwedge^k M)_{w+|\underline{\lambda}|}.$$

4.3.7 Example. It is of course not necessary to remember formula (4.7) by head, since it is just the formalization of a mechanical practical rule. For instance, let us compute $D_2(\epsilon^2\wedge\epsilon^3\wedge\epsilon^5)\in\bigwedge^3 M$. One has:

$$\begin{aligned}
&D_2(\epsilon^2\wedge\epsilon^3\wedge\epsilon^5)=D_2\epsilon^2\wedge\epsilon^3\wedge\epsilon^5+D_1\epsilon^2\wedge D_1(\epsilon^3\wedge\epsilon^5)+\epsilon^2\wedge D_2(\epsilon^3\wedge\epsilon^5)=\\
&=\epsilon^4\wedge\epsilon^3\wedge\epsilon^5+\epsilon^3\wedge(\epsilon^4\wedge\epsilon^5+\epsilon^3\wedge\epsilon^6)+\epsilon^2\wedge(\epsilon^5\wedge\epsilon^5+\epsilon^4\wedge\epsilon^6+\epsilon^3\wedge\epsilon^7)=\\
&=\epsilon^4\wedge\epsilon^3\wedge\epsilon^5+\epsilon^3\wedge\epsilon^4\wedge\epsilon^5+\epsilon^3\wedge\epsilon^3\wedge\epsilon^6+\epsilon^2\wedge\epsilon^5\wedge\epsilon^5+\epsilon^2\wedge\epsilon^4\wedge\epsilon^6+\\
&\qquad+\epsilon^2\wedge\epsilon^3\wedge\epsilon^7=\epsilon^2\wedge\epsilon^4\wedge\epsilon^6+\epsilon^2\wedge\epsilon^3\wedge\epsilon^7,
\end{aligned}$$

where last equality results from the vanishing of the terms having equal $\wedge$-factors and to the cancellations due to skew-symmetry.

4.4 Pieri's Formula for $\mathcal{S}$-Derivations

Example 4.3.7 shows that computing $\mathcal{S}$-derivatives of k-vectors is a straightforward matter and that one has to care only about possible vanishing and cancellations.

However, the practice of many examples naturally suggests the following theorem which says that our algebraic model is isomorphic to *Schubert Calculus* (Section 3.4).

4.4.1 Theorem. *Let I be a k-schindex. Then* Pieri's formula *for $\mathcal{S}$-derivatives holds:*

$$D_h(\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k})=\sum_{\substack{(h_i)\in H(I,h)\\ h_1+\ldots+h_k=h}}\epsilon^{i_1+h_1}\wedge\ldots\wedge\epsilon^{i_k+h_k},\qquad(4.8)$$

where $H(I,h)$ is the set of all k-tuples $(h_i)\in\mathbb{N}^k$ such that:

$$1\le i_1\le i_1+h_1<i_2\le i_2+h_2<\ldots\le i_{k-1}+h_{k-1}<i_k\qquad(4.9)$$

and $h_1+h_2+\ldots+h_k=h$.

Proof. (It is copied from [23]) By induction on the integer k. For $k=1$, formula (4.8) is trivially true. Let us prove it directly for $k=2$. For each $h\ge 0$, let us split sum (4.7) as:

$$D_h(\epsilon^{i_1}\wedge\epsilon^{i_2})=\sum_{\substack{h_1+h_2=h\\ h_1,h_2\ge 0}}\epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h_2}=\mathcal{P}+\overline{\mathcal{P}},\qquad(4.10)$$

where:

$$\mathcal{P} = \sum_{\substack{i_1+h_1<i_2\\ h_1+h_2=h}} \epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h_2} \quad \text{and} \quad \overline{\mathcal{P}} = \sum_{\substack{i_1+h_1\geq i_2\\ h_1+h_2=h}} \epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h_2}.$$

One contends that $\overline{\mathcal{P}}$ vanishes. In fact, on the finite set

$$A := \{a\in\mathbb{Z}\,|\,i_2-i_1\leq a\leq i_2-i_1+h\}$$

define the bijection $\varrho(a) = i_2-i_1+h-a$. Then:

$$2\overline{\mathcal{P}} = \sum_{h_1=i_2-i_1}^{h} \epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h-h_1} + \sum_{h_1=i_2-i_1}^{h} \epsilon^{i_1+\rho(h_1)}\wedge\epsilon^{i_2+h-\rho(h_1)} =$$

$$= \sum_{h_1=i_2-i_1}^{h} \epsilon^{i_2+h-h_1}\wedge\epsilon^{i_1+h_1} - \sum_{h_1=i_2-i_1}^{h} \epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h_2} = 0,$$

hence $\overline{\mathcal{P}} = 0$ and (4.8) holds for $k=2$. Suppose now that (4.8) holds for all $1\leq k'\leq k-1$. Then, for each $h\geq 0$:

$$D_h(\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}) = \sum_{h'_k+h_k=h} D_{h'_k}(\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_{k-1}})\wedge D_{h_k}\epsilon^{i_k}$$

and, by the inductive hypothesis:

$$\sum_{(h_i)}(\epsilon^{i_1+h_1}\wedge\ldots\wedge\epsilon^{i_{k-2}+h_{k-2}}\wedge\epsilon^{i_{k-1}+h_{k-1}})\wedge\epsilon^{i_k+h_k}, \tag{4.11}$$

summed over all (h_i) such that

$$\begin{cases} h_1+\ldots+h_k = h, \quad h_i\geq 0\\ 1\leq i_1+h_1<i_2\leq\ldots\ldots\leq i_{k-2}+h_{k-2}<i_{k-1} \end{cases}. \tag{4.12}$$

But now (4.11) can be equivalently written as:

$$\sum_{(h_i)}\epsilon^{i_1+h_1}\wedge\ldots\wedge\epsilon^{i_{k-2}+h_{k-2}}\wedge D_{h''}(\epsilon^{i_{k-1}}\wedge\epsilon^{i_k}), \tag{4.13}$$

where the sum is over all $(h_1, \ldots, h_{k-2}, h'')$ such that $h_1 + \ldots + h_{k-2} + h'' = h$ and satisfying (4.12). Since

$$D_{h''}(\epsilon^{i_{k-1}} \wedge \epsilon^{i_k}) = \sum_{\substack{i_{k-1}+h_{k-1}<i_k \\ h_{k-1}+h_k=h''}} \epsilon^{i_{k-1}+h_{k-1}} \wedge \epsilon^{i_k+h_k},$$

by the inductive hypothesis, substituting into (4.13), one gets exactly sum (4.8). ∎

4.4.2 Corollary.

$$D_h(\epsilon^s \wedge \ldots \wedge \epsilon^{s+j-1} \wedge \epsilon^{s+j} \wedge \alpha) = \epsilon^s \wedge \ldots \wedge \epsilon^{s+j-1} \wedge D_h(\epsilon^{s+j} \wedge \alpha).$$

for each $\alpha \in \bigwedge M$.
Proof. It is a straightforward application of Pieri's formula (4.8). ∎

The consequences of Theorem 4.4.1 and its relationship with Section 1.2.4 shall be discussed in Section 4.7.

4.5 Giambelli's Problem

As in the case for $k = 1$, denote by $\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}$ the composition:

$$\mathbb{Z}[\mathbf{T}] \stackrel{\mathrm{ev}_D}{\longrightarrow} \mathcal{A}^*(\bigwedge^k M, D) \stackrel{\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}}{\longrightarrow} \bigwedge^k M.$$

One wonders if $\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}$ or, equivalently, $\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}$, is surjective. Let us fix some terminology.

4.5.1 Definition. *Let* $\mathbf{m} \in \bigwedge^k M$ *and* $G_{\mathbf{m}} \in \mathbb{Z}[\mathbf{T}]$ *be such that*

$$G_{\mathbf{m}}(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k = \mathbf{m}.$$

Then $G_{\mathbf{m}}$ *is said to be a* Giambelli's polynomial *for* $\mathbf{m}$.

Notice that a Giambelli's polynomial for $\mathbf{m}$ is not unique. For instance $T_1^2T_2 - T_1T_3$ and $T_2^2 - T_1T_3$ are distinct Giambelli's polynomials for $\epsilon^3 \wedge \epsilon^4$.

From now on, and for sake of brevity, the notation

$$G_{i_1\ldots i_k}\in\mathbb{Z}[\mathbf{T}],$$

will stand for a Giambelli's polinomial of $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$.

Proving the surjectivity of $\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}$ amounts to solve *Giambelli's problem* for $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$:

find a polynomial $G_{i_1\ldots i_k}\in\mathbb{Z}[\mathbf{T}]$, *such that*:

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=G_{i_1\ldots i_k}(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k \tag{4.14}$$

for each $1\le i_1<\ldots<i_k$.

This is in fact the case, as we shall see in a moment. However it is not difficult to guess this fact. Our formalism suggests that it can be achieved via a suitable "integration by parts", as shown in the following:

4.5.2 Example. To look for a Giambelli's polynomial for $\epsilon^2\wedge\epsilon^4\wedge\epsilon^5$, one first perform a first integration:

$$\begin{aligned}\epsilon^2\wedge\epsilon^4\wedge\epsilon^5 &= D_1(\epsilon^2\wedge\epsilon^3\wedge\epsilon^5)-\epsilon^2\wedge\epsilon^3\wedge\epsilon^6=\\ &= (D_1^2-D_2)\epsilon^2\wedge\epsilon^3\wedge\epsilon^4.\end{aligned} \tag{4.15}$$

Our problem now amounts to solve Giambelli's problem for $\epsilon^2\wedge\epsilon^3\wedge\epsilon^4$. Further integrations give:

$$\begin{aligned}&\epsilon^2\wedge\epsilon^3\wedge\epsilon^4=D_1(\epsilon^1\wedge\epsilon^3\wedge\epsilon^4)-\epsilon^1\wedge D_1(\epsilon^3\wedge\epsilon^4)=\\ =\;& D_1(D_1(\epsilon^1\wedge\epsilon^2\wedge\epsilon^4)-\epsilon^1\wedge\epsilon^2\wedge\epsilon^5))-\epsilon^1\wedge\epsilon^3\wedge\epsilon^5=\\ =\;&(D_1^3-D_1D_2)\epsilon^1\wedge\epsilon^2\wedge\epsilon^3-D_1(\epsilon^1\wedge\epsilon^2\wedge\epsilon^5)+\epsilon^1\wedge\epsilon^2\wedge\epsilon^6=\\ =\;&(D_1^3-2D_1D_2+D_3)\epsilon^1\wedge\epsilon^2\wedge\epsilon^3.\end{aligned}$$

Substituting into (4.15) one gets, finally:

$$\epsilon^2\wedge\epsilon^4\wedge\epsilon^5=(D_1^2-D_2)(D_1^3-2D_1D_2+D_3)\cdot\epsilon^1\wedge\epsilon^2\wedge\epsilon^3.$$

To prove surjectivity, in general one needs a couple of lemmas. Let us start with the most important.

4.5.3 Lemma. *The following identity holds:*

$$D_h(\epsilon^j\wedge\alpha)=\epsilon^j\wedge D_h\alpha+D_{h-1}(\epsilon^{j+1}\wedge\alpha) \tag{4.16}$$

for each $h\ge 0$, $j\ge 1$ *and each* $\alpha\in\bigwedge M$.

Proof.

$$D_h(\epsilon^j\wedge\alpha)=\sum_{i=0}^{h}\epsilon^{j+h-i}\wedge D_i\alpha=\epsilon^j\wedge D_h\alpha+\sum_{i=0}^{h-1}\epsilon^{j+h-i}\wedge D_i\alpha=$$

$$\epsilon^j\wedge D_h\alpha+\sum_{i=0}^{h-1}\epsilon^{j+1+(h-1-i)}\wedge D_i\alpha=\epsilon^j\wedge D_h\alpha+D_{h-1}(\epsilon^{j+1}\wedge\alpha).\quad\blacksquare$$

For any pair of positive integers (l,n), let $L_{l,n}$ be the set of all n-tuples $(l_1,\ldots,l_n)$ such that $0\le l_i\le 1$ and $l_1+\ldots+l_n=l$. If $n<l$ the set $L_{l,n}$ is clearly empty.

4.5.4 Lemma. *The following identity holds:*

$$(D_{h_1}\ldots D_{h_{p-1}}D_{h_p}\alpha)\wedge\epsilon^i=$$

$$=\sum_{l=0}^{p}(-1)^l\sum_{(l_j)\in L_{l,p}}D_{h_1-l_1}\ldots D_{h_{p-1}-l_{p-1}}D_{h_p-l_p}(\alpha\wedge\epsilon^{i+l}).\tag{4.17}$$

Proof. The proof is by induction on the integer p. For $p=1$, formula (4.17) is nothing else than formula (4.16). Suppose that (4.17) holds for the integer $p-1>1$ and any $\alpha\in\bigwedge M$. One may then write

$$(D_{h_1}\ldots D_{h_{p-1}}D_{h_p}\alpha)\wedge\epsilon^i=(D_{h_1}\ldots D_{h_{p-1}}(D_{h_p}\alpha))\wedge\epsilon^i=$$

$$=\sum_{l=0}^{p-1}(-1)^l\sum_{(l_j)\in L_{l,p-1}}D_{h_1-l_1}\ldots D_{h_{p-1}-l_{p-1}}(D_{h_p}\alpha\wedge\epsilon^{i+l})\tag{4.18}$$

Using (4.16), last side of formula (4.18) becomes:

$$\begin{aligned}
&= \sum_{l=0}^{p-1}(-1)^l\sum_{(l_j)\in L_{l,p-1}}D_{h_1-l_1}\ldots D_{h_{p-1}-l_{p-1}}D_{h_p}(\alpha\wedge\epsilon^{i+l})+\\
&+ \sum_{l=0}^{p-1}(-1)^{l+1}\sum_{(l_j)\in L_{l,p-1}}D_{h_1-l_1}\ldots D_{h_{p-1}-l_{p-1}}D_{h_p-1}(\alpha\wedge\epsilon^{i+l+1})=\\
&= \sum_{l=0}^{p}(-1)^l\sum_{(l_j)\in L_{l,p}}D_{h_1-l_1}\ldots D_{h_{p-1}-l_{p-1}}D_{h_p-l_p}(\alpha\wedge\epsilon^{i+l}).\quad\blacksquare
\end{aligned}$$

4.5.5 Corollary. *For each* $1 \leq i_1 < \ldots < i_k$*, there is a polynomial* $P(\mathbf{T}) \in \mathbb{Z}[\mathbf{T}]$ *such that:*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-1}.$$

Proof. By induction on the integer k. For $k = 1$ it is obvious. Suppose now that the property holds for $k-1$. Then one may write:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = Q(D) \cdot (\epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-2}) \wedge \epsilon^{i_k} \tag{4.19}$$

for some $Q(\mathbf{T}) \in \mathbb{Z}[\mathbf{T}]$. Write

$$Q(D) = \sum a_{j_1 \ldots j_p} D_{j_1} \ldots D_{j_p},$$

the sum over a finite subset of $\mathbb{N}^p$, where some of the j_i's may possibly coincide. The r.h.s. of (4.19) is then an integral linear combination of terms like:

$$D_{j_1} \ldots D_{j_p} (\epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-2}) \wedge \epsilon^{i_k}.$$

Using formula (4.17), any such term can be written as:

$$\sum_{l=0}^{p} (-1)^l \sum_{(l_j) \in L_{l,k-1}} D_{j_1-l_1} \ldots D_{j_p-l_p} (\epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-2} \wedge \epsilon^{i_k+l})$$

which, because of Corollary (4.4.2), can be also written as:

$$\sum_{l=0}^{p} (-1)^l \sum_{(l_j) \in L_{l,k-1}} D_{j_1-l_1} \ldots D_{j_p-l_p} D_{i_k+l-i_1-k+1} (\alpha \wedge \epsilon^{i_1+k-1}),$$

where $\alpha = \epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-2}$. This proves the claim. ∎

4.5.6 To continue with, recall that the formal inverse of D_t:

$$D_t^{-1} = \sum_{i \geq 0} (-1)^i \Delta_i t^i,$$

is a HS-derivation of $\bigwedge M$ (see Proposition 4.1.5). Therefore, for each $h \geq 0$, Δ_h satisfies Leibniz's rule:

$$\Delta_h (\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{h_1 + \ldots + h_k} \Delta_{h_1} \epsilon^{i_1} \wedge \ldots \wedge \Delta_{h_k} \epsilon^{i_k}. \tag{4.20}$$

To know how Δ operates on $\bigwedge^k M$, it is sufficient to know how it operates on $M := \bigwedge^1 M$. One knows that $D_{t|_M} = \sum_{i\geq 0} D_1^i t^i$. It follows that:

$$D_{t|_M}{}^{-1} = 1 + D_1 t.$$

In other words, $\Delta_h : M \longrightarrow M$ is such that $\Delta_h \epsilon^i = \delta_1^h \epsilon^{i+h}$, i.e. $\Delta_{0|_M} = id_M$, $\Delta_{1|_M} = D_1$ and $\Delta_{h|_M} = 0$, for each $h > 1$. Then one proves the following:

4.5.7 Lemma. *For all $k > 0$ and all $h > k$, $\Delta_h \alpha = 0$ for all $\alpha \in \bigwedge^k M$, while:*

$$\Delta_k(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_k+1}. \qquad (4.21)$$

Proof. If $h > k$, each summand of the r.h.s. of (4.20) possesses at least an index $1 \leq i \leq k$, such that $h_i > 1$. On the other hand, if in the sum (4.20) $h = k$, only the term for which $h_1 = \ldots = h_k = 1$ survives, i.e. precisely the r.h.s. of (4.21) ∎

4.5.8 Corollary. *Giambelli's problem has a solution for*

$$\epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n.$$

Proof. Let $\Delta_k \in \mathbb{Z}[\mathbf{T}]$ be the polynomial such that $\mathrm{ev}_D(\Delta_h(\mathbf{T})) = \Delta_k(D)$. It is sufficient to take $G_{n-k+1,\ldots,n} = (\Delta_k)^{k(n-k)}$. In fact:

$$\mathrm{ev}_D(\Delta_k(\mathbf{T})^{k(n-k)})\epsilon^1 \wedge \ldots \wedge \epsilon^k = \mathrm{ev}_D(\Delta_k(\mathbf{T}))^{k(n-k)} \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k =$$
$$= \underbrace{\Delta_k(D) \circ \ldots \circ \Delta_k(D)}_{k(n-k)\ times} \epsilon^1 \wedge \ldots \wedge \epsilon^k = \epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n.$$

∎

We can hence prove the most important result of this section:

4.5.9 Theorem. *The map*

$$\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k} : \mathbb{Z}[\mathbf{T}] \longrightarrow \bigwedge^k M$$

is surjective for all k.

Proof. It is sufficient to prove that Giambelli's problem has solution for each k-schindex $I = (i_1, \ldots, i_k)$. By Corollary 4.5.5, there is a polynomial $P \in \mathbb{Z}[\mathbf{T}]$ such that:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D)\epsilon^{i_1} \wedge \epsilon^{i_1+1} \wedge \ldots \wedge \epsilon^{i_1+k-1}.$$

Then, applying 4.5.8:

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = P(D) \cdot \Delta_k(D)^{k(i_1-1)} \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k,$$

proving our claim. ∎

4.5.10 Corollary. *The following maps are isomorphisms:*

$$\frac{\mathbb{Z}[\mathbf{T}]}{(\ker(\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}))} \stackrel{\mathrm{ev}_D}{\longrightarrow} \mathcal{A}^*(\wedge^k M, D) \stackrel{\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}}{\longrightarrow} \bigwedge^k M,$$

where by abuse of notation one denotes by ev_D *the induced map on the quotient.*

Proof. The surjectivity of $\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}$ and of ev_D implies the surjectivity of $\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}$. One must only show that

$$\ker(\mathrm{ev}_D) = \ker(\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}),$$

and for this one argues exactly as in the proof of Proposition 4.2.6. ∎

Next section shall be devoted to find an explicit presentation in terms of generators and relations of $\mathcal{A}^*(\bigwedge^k M, D)$.

4.6 A Presentation for $\mathcal{A}^*(\bigwedge^k M, D)$

Up to translations, most of the preliminary propositions of this Section can be found in [52], there proven within the language of symmetric functions.

4.6.1 Definition. *A Giambelli function*

$$G : \mathcal{I}^k \longrightarrow \mathbb{Z}[\mathbf{T}]$$

is a function which associates to each k*-schindex* $I = (i_1, \ldots, i_k)$ *a Giambelli polynomial for* $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$*. Its image* $\mathcal{G}$ *will be said a* Giambelli's set.

4.6.2 Corollary. *Any Giambelli's set is a $\mathbb{Z}$-basis for $\mathbb{Z}[\mathbf{T}]$.*

Proof. First of all we prove that $\mathcal{G}$ is a linearly independent subset of elements of $\mathbb{Z}[\mathbf{T}]$. Let $\sum_{\underline{\lambda}\in\mathcal{I}'} a_I[G_I(\mathbf{T})] = 0$ be any non trivial linear dependence relation, where $\mathcal{I}'$ is a finite subset of $\mathcal{I}$. There is then $k \geq 1$, such that $\mathcal{I}' \subseteq \mathcal{I}^k$ and

$$\sum_{I\in\mathcal{I}'} a_I G_I(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k = 0,$$

which is impossible, because $G_{i_1\ldots i_k}(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k = \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ are all linearly independents, being part of a basis of $\bigwedge^k M$.

Let us now prove that they are $\mathbb{Z}$-generators of $\mathbb{Z}[\mathbf{T}]$. Let $\mathcal{G}^h = \{G_I \in \mathcal{G} \mid wt(I) = h\}$. It is sufficient to show that $\mathcal{G}^h$ is in fact a $\mathbb{Z}$-basis for $\mathbb{Z}[\mathbf{T}]_h$. To this purpose, notice that $G_I(\mathbf{T}) \in \mathcal{G}_h$ is a unique $\mathbb{Z}$-linear combination of $T^{\underline{\lambda}}$'s, ($\underline{\lambda} \in \mathcal{P}_h$):

$$G_I(\mathbf{T}) = \sum_{\underline{\lambda}_I\in\mathcal{P}_h} a_{\underline{\lambda}_I} D^{\underline{\lambda}_I}.$$

Then, since $\sharp\{\mathcal{G}^h(\mathbf{T})\} = \sharp\{T^{\underline{\mu}} : \underline{\mu} \in \mathcal{P}^h\}$ (because of the bijection between k-schindices and partitions of length at most k, see Sect. 2.1.6), and all the $G_I \in \mathcal{G}^h$ are linearly independent, they freely span $\mathbb{Q}[\mathbf{T}]_h = \mathbb{Z}[\mathbf{T}]_h \otimes_{\mathbf{Z}} \mathbb{Q}$. In particular, for each $\underline{\lambda} \in \mathcal{P}_k$, $T^{\underline{\lambda}}$ is a unique $\mathbb{Q}$-linear combination of $G_I(\mathbf{T}) \in \mathcal{G}^h(\mathbf{T})$:

$$T^{\underline{\lambda}} = \sum_{I_{\underline{\lambda}}} b_{I_{\underline{\lambda}}} G_{I_{\underline{\lambda}}}(\mathbf{T}). \tag{4.22}$$

We only need to show that the coefficients $b_{I_{\underline{\lambda}}} \in \mathbb{Z}$. But equality (4.22) implies:

$$\mathrm{ev}_D(T^{\underline{\lambda}})\epsilon^1 \wedge \ldots \wedge \epsilon^k = \sum_{I_{\underline{\lambda}}\in\mathcal{I}_h} b_{I_{\underline{\lambda}}} \mathrm{ev}_D(G_{I_{\underline{\lambda}}}(\mathbf{T}))\epsilon^1 \wedge \ldots \wedge \epsilon^k \tag{4.23}$$

The l.h.s. is a $\mathbb{Z}$-linear combination of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$:

$$\sum_{I_{\underline{\lambda}}\in\mathcal{P}_h} a_{I_{\underline{\lambda}}} \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}.$$

By unicity, the coefficients $b_{I_{\underline{\lambda}}}$'s must be the same as those occurring in the expansion of the l.h.s. of (4.23) as linear combination of of the $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$, i.e. they must be integers. ∎

4.6.3 Proposition. *Let $\phi \in \mathbb{Z}[T_1, \ldots, T_k] \subseteq \mathbb{Z}[\mathbf{T}]$ such that*

$$\phi(D_1, \ldots, D_k) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k = 0,$$

then $\phi = 0$.

Proof. Let $\phi \in \mathbb{Z}[T_1, \ldots, T_k]$ of degree h. Writing ϕ as the sum $\phi_0 + \phi_1 + \ldots + \phi_h$, where ϕ_i is homogeneous of degree i, one has

$$\phi_i(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k \in (\bigwedge^k M)_i.$$

Therefore $\phi(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k = 0$ if and only if $\phi_i(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k = 0$, for each homogeneous summand of ϕ. One may then assume that $\phi \in \mathbb{Z}[\mathbf{T}]_h$ (i.e. it is homogeneous of weight h). Since $\mathcal{G}_h$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\mathbf{T}]_h$, one may write:

$$\phi(T_1, \ldots, T_k) = \sum_{I \in \mathcal{I}^k} a_I G_I(\mathbf{T}).$$

Hence:

$$0 = \phi(D_1, \ldots, D_k)\epsilon^1 \wedge \ldots \wedge \epsilon^k = \sum_{I \in \mathcal{I}^k} a_I G_I(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k,$$

implying that all $a_I = 0$, because $\{G_I(D) \cdot \epsilon^1 \wedge \ldots \wedge \epsilon^k\}_{I \in \mathcal{I}^k}$ are linearly independent. ∎

We may now prove the following result computing explicitly the presentation of $\mathcal{A}^*(M, D)$.

4.6.4 Theorem. *The ring $\mathcal{A}^*(M, D)$ is the polynomial ring $\mathbb{Z}[D_1, \ldots, D_k]$. (Cf. [7])*

Proof. By Proposition 4.6.3, $D_1, \ldots, D_k$ are algebraically independents. Moreover D_{k+i} $(i \geq 1)$ is a $\mathbb{Z}$-polynomial in $D_1, \ldots, D_k$. In fact, for each $j \geq 0$, the relation $(1/D_t) \cdot D_t = D_t \cdot (1/D_t) = 1$ implies:

$$D_{k+i} - \Delta_1 D_{k+i-1} + \ldots + (-1)^{k+i-1}\Delta_{k+i-1} D_1 + (-1)^{k+i}\Delta_{k+i} = 0. \tag{4.24}$$

Since Δ_p is a polynomial expression in $D_1, \ldots, D_p$ only, it follows that (for $i = 1$) D_{k+1} is a polynomial in $D_1, \ldots, D_k$, because $\Delta_{k+1} = 0$ by Lemma 4.5.7. Suppose now that the property holds for all $1 \leq j \leq i-1$. Then formula (4.24) reads as:

$$D_{k+i} = -\Delta_1 D_{k+i-1} + \ldots + (-1)^{k+1}\Delta_k D_i, \tag{4.25}$$

because $\Delta_{k+i} = 0$ for all $i \geq 1$; it follows that D_{k+i} is a polynomial in $D_1, \ldots, D_k$, by induction. As a conclusion:

$$\mathcal{A}^*(\bigwedge^k M, D) = \frac{\mathbb{Z}[D_1, D_2, \ldots]}{(D_{k+i} - D_{k+i}(D_1, \ldots, D_k))_{i\geq 1}} \cong \mathbb{Z}[D_1, \ldots, D_k].$$

■

4.7 The Finite Case

Let $D_n M$ be the submodule of M spanned by $\mathcal{E}^n = (\epsilon^{n+1}, \epsilon^{n+2}, \ldots)$ and denote by M_n the free submodule of M generated by $\mathcal{E}_n = (\epsilon^1, \ldots, \epsilon^n)$. Let p_n be the projection map:

$$\begin{cases} p_n & : & M & \longrightarrow & M_n \\ & & m & \longmapsto & \sum_{i=1}^n e_i(m)\epsilon^i \end{cases}$$

and $\iota_n : M_n \longrightarrow M$ be the section associated to p_n, defined by $\iota_n(m) = m$, for each $m \in M_n$. One has the sequence:

$$0 \longrightarrow D_n M \longrightarrow M \longrightarrow M_n \longrightarrow 0,$$

which can easily be checked to be exact.

4.7.1 Proposition. *The natural epimorphism:*

$$p_n \circ \mathrm{ev}_{\epsilon^1} : \mathbb{Z}[T_1] \longrightarrow M_n$$

induces the isomorphism:

$$\frac{\mathbb{Z}[T_1]}{(T_1^n)} \cong M_n \tag{4.26}$$

Proof. From the epimorphism:

$$p_n \circ \mathrm{ev}_{D,\epsilon^1} : \mathbb{Z}[T_1] \longrightarrow M_n$$

one gets

$$\mathbb{Z}[T_1]/\ker(p_n \circ \mathrm{ev}_{D,\epsilon^1}) \cong M_n.$$

Now, $p_n(D_1^i\epsilon^1) = 0$ if and only if $i \geq n$, proving that the kernel of $p_n \circ \mathrm{ev}_{D,\epsilon^1}$ is precisely (T_1^n). This proves the claim. ∎

4.7.2 Remark. Notice that the l.h.s. of (4.26) is isomorphic to the Chow ring of $\mathbb{P}^{n-1}$. Denote by $\mathcal{A}^*(M_n, D)$ the image of $\mathbb{Z}[T_1]/(T_1^n)$ in $End_{\mathbb{Z}}(M_n)$. Then $\mathcal{A}^*(M_n, D)$ is isomorphic to $\mathbb{Z}[D_1]$ where D_1 may be identified with the class of T_1 mod (T_1^n).

Let now

$$\bigwedge M \wedge D_n M$$

be the ideal of the exterior algebra $\bigwedge M$ generated by $D_n M$: a typical element of it is of the form $\alpha \wedge \epsilon^j$, with $j > n$, $\alpha \in \bigwedge M$. Let

$$\bigwedge^{k-1} M \wedge D_n M := \bigwedge^k M \cap (\bigwedge M \wedge D_n M).$$

It is the submodule of $\bigwedge^k M$ generated by all $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ such that $i_k > n$. We claim that:

$$\bigwedge^k M_n := \frac{\bigwedge^k M}{\bigwedge^{k-1} M \wedge D_n M}$$

In fact, one has a surjective module homomorphism:

$$\wedge^k p_n : \bigwedge^k M \longrightarrow \bigwedge^k M_n \tag{4.27}$$

defined by

$$\wedge^k p_n(\sum_{1 \leq i_1 < \ldots < i_k} a_{i_1\ldots i_k}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1\ldots i_k}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k},$$

whose kernel is precisely the submodule $\bigwedge^{k-1} M \wedge D_n M$. Therefore:

$$\bigwedge M_n = \bigoplus_{k\geq 0} \bigwedge^k M_n = \bigoplus_{k\geq 0} \frac{\bigwedge^k M}{\bigwedge^{k-1} M \wedge D_n M} = \frac{\bigwedge M}{\bigwedge M \wedge D_n M}.$$

Clearly $\bigwedge^k M_n = 0$ if $k > n$. Consider now the map:

$$\wedge p_n \circ D_t \circ \wedge \iota_n : \bigwedge M_n \longrightarrow \bigwedge M_n[[t]].$$

4.7.3 Proposition/Definition. *The map $p_n \circ D_t \circ \iota_n$ is a Hasse-Schmidt derivation on $\bigwedge M_n$ said to be the $\mathcal{S}$-derivation on $\bigwedge M_n$.*

Proof. It is a straightforward application of the definition:

$$\begin{aligned} \wedge p_n \circ D_t \circ \wedge \iota_n(\alpha \wedge \beta) &= \wedge p_n \circ D_t(\iota_n(\alpha) \wedge \iota_n(\beta)) = \\ &= \wedge p_n(D_t(\iota(\alpha)) \wedge D_t(\iota(\beta))) \\ &= (\wedge p_n \circ D_t \circ \iota_n)(\alpha) \wedge (\wedge p_n \circ D_t \circ \iota_n)(\beta). \end{aligned}$$

∎

It is worth to emphasize the following corollary of Theorem 4.4.1.

4.7.4 Proposition (Pieri's formula for $\bigwedge^k M_n$).
Let $I := (1 \leq i_1 < i_2 \ldots < i_k \leq n)$ and $0 \leq h \leq n$. Then:

$$\wedge^k p_n \circ D_h \circ \wedge^k \iota_n(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{\substack{(h_i)\in H(I,h) \\ i_k + h_k \leq n}} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k}, \quad (4.28)$$

Proof. Equation (4.28) is obvious: one writes down expansion (4.8) and then projects via p_n, cancelling all the terms such that $i_k > n$. ∎

4.7.5 The $\mathbb{Z}$-module isomorphism:

$$\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} : \mathcal{A}^*(\bigwedge^k M, D) \longrightarrow \bigwedge^k M,$$

induces a $\mathbb{Z}$-module epimorphism $\mathcal{A}^*(\bigwedge^k M, D) \longrightarrow \bigwedge^k M_n$, by composition with $\wedge^k p_n$. Let:

$$\mathcal{A}^*(\bigwedge^k M_n, D) := \frac{\mathcal{A}^*(\bigwedge^k M, D)}{\ker(\wedge p_n \circ \mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k})} \cong \bigwedge^k M_n, \quad (4.29)$$

where last isomorphism holds by construction. We have here, as will be done from now on, abused notation deciding to write simply D_t instead of the heavier (though more precise) $\wedge p_n \circ D_t \circ \wedge \iota_n$.

Patching together formula (4.29) with Proposition 4.7.4 one has esplicitly proven (as implicitly done by Theorem 4.4.1) that our formalism is indeed Schubert Calculus. More precisely:

4.7.6 Theorem. *The $A^*(G_k(V))$-module $A_*(G_k(V))$ is isomorphic to the $\mathcal{A}^*(\bigwedge^k M_n, D)$-module $\bigwedge^k M_n$, i.e. the following diagram*

$$\begin{array}{ccccc} \mathcal{A}^*(\bigwedge^k M_n, D) & \times & \bigwedge^k M_n & \stackrel{\mathsf{d}}{\longrightarrow} & \mathcal{A}^*(\bigwedge^k M_n, D) \\ \Big\downarrow \sigma & & \Big\downarrow \Omega & & \Big\downarrow \sigma \\ A^*(G_k(V)) & \times & A_*(G_k(V) & \stackrel{\mathsf{c}}{\longrightarrow} & A_*(G_k(V)) \end{array} \tag{4.30}$$

is commutative, where σ and Ω, defined by

$$\sigma(D_i) = \sigma_i$$

and

$$\Omega(\epsilon^{1+r_1} \wedge \ldots \wedge \epsilon^{k+r_k}) = \Omega_{(r_k,\ldots,r_1)} = [\epsilon^{1+r_1} \wedge \ldots \wedge \epsilon^{k+r_k}]$$

are isomorphisms (Cf. Section 1.3.4 and formula (3.11)*)* ■

Of course σ and Ω are isomorphisms because Chow basis theorem ensures that not only the cycles $\sigma_{\underline{\lambda}}$ generate $A^*(G_k(V))$, but they are indeed a basis of it. The inverse isomorphism Ω^{-1} may be also written as, according to formula (3.11):

$$\begin{aligned} &\Omega^{-1}\Big(\sum_{(i_1,\ldots,i_k)\in\mathcal{I}_n^k} a_{i_1,\ldots,i_k}\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}\Big) = \\ &= \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_n^k} a_{i_1,\ldots,i_k}[\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}]. \end{aligned} \tag{4.31}$$

As for the map $\mathtt{c}$, it is induced by the *cap* product $\cap$:

$$\mathtt{c}(\sigma_{\underline{\lambda}}, \Omega_{\underline{\mu}}) = \sigma_{\underline{\lambda}} \cap \Omega_{\underline{\mu}}$$

while the map $\mathtt{d}$ is induced by *differentiation*:

$$\mathtt{d}(P(D), \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = P(D) \cdot \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}.$$

We are hence in condition to explain the emphasized sentences in Section (1.2.4), through the following

4.7.7 Example. The Schubert cycle σ_1 is a very ample divisor class embedding the Grassmannian $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ à la Plücker. In such an embedding, its degree $d_{k,n}$ is the intersection of its Plücker image with $k(n-k)$ general hyperplanes and the hyperplane section is cut out precisely by the divisor $\sigma_1 \cap [G(k,n)]$. Therefore one has:

$$\sigma_1^{k(n-k)} \cap [G(k,n)] = d_{k,n} \cdot \sigma_{(n-k,\ldots,n-k)},$$

which has also the meaning of the numbers of $k-1$ planes intersecting $k(n-k)$ linear spaces of dimension $n-k-1$ in $\mathbb{P}^{n-1}$. By virtue of Theorem (4.7.6), this is the same as computing:

$$D_1^{k(n-k)} \epsilon^1 \wedge \ldots \wedge \epsilon^k \in \bigwedge^k M_n \tag{4.32}$$

which is formally the same as differentiating a wronskian like in Section (1.2.4). Since $(\bigwedge^k M_n)_{k(n-k)}$ is generated by $\epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n$, i.e. a multiple of the unique generator of $(\bigwedge^k M_n)_{k(n-k)}$, the degree of $G(k,n)$ is its coefficient in the expansion (4.32). The expansion of expression (4.32) as an element of $\bigwedge^k M$ would instead be a $\mathbb{Z}$-linear combination of $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$, with $wt(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = k(n-k)$, which can be written as:

$$a \cdot \epsilon^{n-k+1} \wedge \ldots \wedge \epsilon^n + \beta$$

for some $a \in \mathbb{Z}$ and where $\beta \in \bigwedge^{k-1} M \wedge D_n M$. Therefore (applying $\wedge^k p_n$ to both sides of (4.32), one concludes that $a = d_{k,n}$, as claimed in the last emphasized sentence of Section 1.2.4.

4.7.8 Example. The *number of lines of* $\mathbb{P}^3$ *meeting four others in general position* coincides with the degree of the grassmannian $G(2,4)$ in the Plücker embedding. To compute it, one just expands

$$D_1^4(\epsilon^1 \wedge \epsilon^2),$$

This is the same computation we did with the wronskian $\mathbf{f} \wedge \mathbf{f}'$ in Example 1.2.3. Exactly the same formal steps give

$$D_1^4(\epsilon^1 \wedge \epsilon^2) = 2 \cdot \epsilon^3 \wedge \epsilon^4,$$

i.e. there are 2 lines answering our question. For two different generalization of this problem look at example (5.2.1).

4.7.9 Next target is to find an explicit presentation of $\mathcal{A}^*(\bigwedge^k M_n, D)$ in terms of generators and relations. To this purpose, if

$$P_1(D), \ldots, P_r(D)$$

are elements of $\mathcal{A}^*(\bigwedge^k M, D)$, denote by $(P_1(D), \ldots, P_r(D))$ both the ideal of $\mathcal{A}^*(\bigwedge^k M, D)$ generated by them and the submodule of $\bigwedge^k M$ generated by $(P_1(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k, \ldots, P_r(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k)$. Notice that the shift operators D_i are elements of $\mathcal{A}^*(\bigwedge^k M, D)$ and that, because of the isomorphism of the latter with the polynomial ring $\mathbb{Z}[D_1, \ldots, D_k]$, any D_i is an explicit polynomial expression of $D_1, \ldots, D_k$. Because of the algebraic independence of such generators, such polynomial expression is indeed unique. We first prove the following

4.7.10 Lemma. *Let $\alpha \in \bigwedge M$ and n, k integers such that $n-k \geq 0$. Then*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \wedge D_n\alpha \in (D_{n-k}, \ldots, D_n) \bigwedge M \subseteq \bigwedge M.$$

Proof.

Induction on the integer k. The property is true for $k = 1$, by virtue of identity (4.16). Suppose the property holds for $k-1$ and for all $n \geq 0$. Then:

$$\begin{aligned}
\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \wedge D_n\alpha &= \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge (\epsilon^{i_k} \wedge D_n\alpha) = \\
&= \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge (D_n(\epsilon^{i_k} \wedge \alpha) - D_{n-1}(\epsilon^{i_k+1)} \wedge \alpha)) = \\
&= \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge (D_n(\epsilon^{i_k} \wedge \alpha) - D_{n-1}(\epsilon^{i_k+1)} \wedge \alpha)) = \\
= \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge D_n(\epsilon^{i_k} \wedge \alpha) &- \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge D_{n-1}(\epsilon^{i_k+1} \wedge \alpha).
\end{aligned}$$

The former summand belongs, by induction, to $(D_n, \ldots, D_{n-k+1})$ while the latter belongs to $(D_{n-1}, \ldots, D_{n-1-(k-1)})$. Then, the sum belongs to $(D_n, \ldots, D_{n-k})$ as claimed. ∎

4.7.11 Corollary. *If* $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^{k-1} M \wedge D_n M$, *then*

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in (D_{n-k+1}, \ldots, D_n).$$

Proof. Apply Lemma 4.7.10 to

$$\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k - 1} \wedge D_n \epsilon^{i_k - n}.$$

■

We can finally prove the following

4.7.12 Theorem. *There is the following isomorphism:*

$$\frac{\mathbb{Z}[D_1, \ldots, D_k]}{(D_{n-k+1}, \ldots, D_n)} = \mathcal{A}^*(\bigwedge^k M_n, D) \cong \bigwedge^k M_n. \tag{4.33}$$

Proof. Last isomorphism is that of formula (4.29). We are only left to prove the presentation of $\mathcal{A}^*(\bigwedge^k M_n, D)$. First of all

$$(D_{n-k+1}, \ldots, D_n) \in \ker(\wedge^k p_n \circ \mathrm{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k}).$$

In fact:

$$D_h \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = D_h G_{i_1 \ldots i_k}(D) \epsilon^1 \wedge \ldots \wedge \epsilon^k =$$
$$= G_{i_1 \ldots i_k}(D) D_h \epsilon^1 \wedge \ldots \wedge \epsilon^k = G_{i_1 \ldots i_k}(D) \epsilon^1 \wedge \ldots \epsilon^{k-1} \wedge \epsilon^{k+h}$$

and last term is zero modulo $D_n M$ precisely when $k + h \geq n + 1$, i.e. precisely when $h \geq n - k + 1$. Conversely, suppose that for some homogeneous element $P(D) \in \mathcal{A}^*(\bigwedge^k M, D)$

$$p_n \circ \mathrm{ev}_{\epsilon^1 \wedge \ldots \wedge \epsilon^k}(P(D)) = 0.$$

Since $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ is a basis of $\bigwedge^k M$, then $P(D)\epsilon^1 \wedge \ldots \wedge \epsilon^k$ is a unique linear combination of k-vectors $\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ belonging to $\wedge^{k-1} M \wedge D_n M$. But Corollary (4.7.11) ensures that each such term belongs to the submodule of $\bigwedge^k M_n$ generated by $(D_{n-k+1}, \ldots, D_n)$, and this concludes the proof. ■

4.8 Giambelli's Formula

In this section we want to offer another explicit proof of the surjectivity of $\mathrm{ev}_{D,\epsilon^1\wedge\ldots\wedge\epsilon^k}$ based on a determinantal formula which is natural to call *Giambelli's Formula* for $\mathcal{S}$-derivations. By formula (2.2), Giambelli's determinant $\Delta_I(\mathbf{T})\in\mathbb{Z}[\mathbf{T}]$ associated to the k-schindex $I=(i_1,\ldots,i_k)$ is:

$$\Delta_I(\mathbf{T})=\sum_{\sigma\in S_k}(-1)^{|\sigma|}T_{i_{\sigma(1)}-1}\cdot T_{i_{\sigma(2)}-2}\cdot\ldots\cdot T_{i_{\sigma(k)}-k}.$$

Let us denote by $\Delta_{\underline{\lambda}}(D)$ (resp. $\Delta_I(D)$) the elements of $End_{\mathbb{Z}}(\bigwedge M)$ defined by $\mathrm{ev}_D(\Delta_{\underline{\lambda}}(\mathbf{T}))$ (resp. $\mathrm{ev}_D(\Delta_I(\mathbf{T}))$. Our target is to show that Giambelli's determinant $\Delta_I(D)$ is an explicit Giambelli's polynomial for $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$.

Let $\underline{\lambda}=(r_k,\ldots,r_1)$ and denote by $\Delta^{ij}_{\underline{\lambda}}(D)$ the determinant of the matrix one gets by erasing the i^{th} row and the j^{th} column.

4.8.1 Theorem. *Giambelli's formula on $\bigwedge M$ holds:*

$$\Delta^{k,k}_{\underline{\lambda}}(D)(\alpha)\wedge\epsilon^{k+r_k}=\sum_{l=0}^{k-1}(-1)^l\Delta^{k-l,k}_{\underline{\lambda}}(D)(\alpha\wedge\epsilon^{k+r_k+l}).\qquad(4.34)$$

Proof. Since $\Delta^{k,k}_{\underline{\lambda}}(D)=\Delta_{(r_{k-1}\ldots r_1)}(D)$, one has:

$$\Delta^{k,k}_{\underline{\lambda}}(D)(\alpha)\wedge\epsilon^{k+r_k}=\sum_{\sigma\in S_{k-1}}(-1)^{|\sigma|}D_{i_{\sigma(1)}-1}\circ\ldots\circ D_{i_{\sigma(k-1)}-(k-1)}(\alpha)\wedge\epsilon^{k+r_k}$$

Now one applies formula (4.17) to the r.h.s. of the above equation, getting:

$$\sum_{\sigma\in S_{k-1}}(-1)^{|\sigma|}\sum_{l=0}^{k-1}(-1)^l\sum_{(l_a)\in L_{l,k-1}}D_{i_{\sigma(1)}-1-l_1}\ldots D_{i_{\sigma(k-1)}-(k-1)-l_{k-1}}(\alpha\wedge\epsilon^{k+r_k+l})=$$

$$=\sum_{l=0}^{k-1}(-1)^l\sum_{(l_a)\in L_{l,k-1}}\sum_{\sigma\in S_{k-1}}(-1)^{|\sigma|}D_{i_{\sigma(1)}-1-l_1}\ldots D_{i_{\sigma(k-1)}-(k-1)-l_{k-1}}(\alpha\wedge\epsilon^{k+r_k+l})=$$

$$=\sum_{l=0}^{k-1}(-1)^l\sum_{(l_a)\in L_{l,k-1}}\begin{vmatrix}D_{i_1-1-l_1}&\ldots&D_{i_k-1-l_1}\\D_{i_1-2-l_2}&\ldots&D_{i_k-2-l_2}\\\vdots&\ddots&\vdots\\D_{i_1-(k-1)-l_{k-1}}&\ldots&D_{i_k-(k-1)-l_{k-1}}\end{vmatrix}(\alpha\wedge\epsilon^{k+r_k+l}).$$

But

$$\sum_{(l_a)\in L_{l,k-1}} \begin{vmatrix} D_{i_1-1-l_1} & \dots & D_{i_k-1-l_1} \\ D_{i_1-2-l_2} & \dots & D_{i_k-2-l_2} \\ \vdots & \ddots & \vdots \\ D_{i_1-(k-1)-l_{k-1}} & \dots & D_{i_k-(k-1)-l_{k-1}} \end{vmatrix} (\alpha\wedge\epsilon^{k+r_k+l}) =$$

$$= \Delta_{\underline{\lambda}}^{k-l,k}(D)(\alpha\wedge\epsilon^{k+r_k+l})$$

proving Giambelli's formula (4.34) ∎

4.8.2 Corollary. *Giambelli's formula on $\bigwedge^k M$ holds:*

$$\epsilon^{1+r_1}\wedge\ldots\wedge\epsilon^{k+r_k} = \Delta_{(r_k\ldots r_1)}(D)\cdot\epsilon^1\wedge\ldots\wedge\epsilon^k. \tag{4.35}$$

Proof. The proof is by induction on the integer k. For $k=1$ one has $\epsilon^{1+r_1} = D_{r_1}\epsilon^1$ and the property holds. Suppose it holds for $k-1$. Then one has, using induction:

$$\begin{aligned} & \epsilon^{1+r_1}\wedge\ldots\wedge\epsilon^{k-1+r_{k-1}}\wedge\epsilon^{k+r_k} = \\ = \; & \Delta_{(r_{k-1}\ldots r_1)}(\epsilon^1\wedge\ldots\wedge\epsilon^{k-1})\wedge\epsilon^{k+r_k} = \\ = \; & \Delta_{\underline{\lambda}}^{k,k}(\alpha)\wedge\epsilon^{k+r_k}, \end{aligned}$$

where one set $\alpha = \epsilon^1\wedge\ldots\wedge\epsilon^{k-1}$. Since for such an α one has

$$\alpha\wedge\epsilon^{k+r_k+l} = D_{r_k+l}(\alpha\wedge\epsilon^k),$$

by applying Corollary 4.4.2, formula (4.34) can be written as:

$$\Delta_{\underline{\lambda}}^{k,k}(D)(\alpha)\wedge\epsilon^{k+r_k} = \sum_{l=0}^{k-1}(-1)^j D_{r_k+l}\Delta_{\underline{\lambda}}^{k-l,k}(D)(\alpha\wedge\epsilon^k) =$$

$$= \Delta_{(r_k\ldots r_1)}(D)\epsilon^{1+r_1}\wedge\ldots\wedge\epsilon^{k+r_k},$$

proving the claim. ∎

Chapter 5

Miscellanea

This Chapter shall be mainly devoted to discussions and examples. As usual M will be a free $\mathbb{Z}$-module spanned by $\mathcal{E} = (\epsilon^1, \epsilon^2, \ldots)$. By M_n, instead, we shall mean the submodule of M generated by $\mathcal{E}_n = (\epsilon^1, \ldots, \epsilon^n)$. We shall assume $V = \mathbb{C}^n$ and we shall write $G(k, n)$ instead of $G_k(\mathbb{C}^n)$.

5.1 The Intersection Ring of $G(2, 4)$

To exemplify the methods developed in Chapter 4, we study in details the case of the Grassmannian $G(2, 4)$, thought of as the variety parametrizing either 2-planes in $\mathbb{C}^4$ or lines in $\mathbb{P}^3(\mathbb{C})$. The model for its intersection theory is provided by the pair $(\bigwedge^2 M_4, D)$, where M_4 is a free $\mathbb{Z}$-module of rank 4 spanned by, say, $(\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4)$ and D_t an $\mathcal{S}$-derivation on its exterior algebra $\bigwedge M_4$. The intersection ring of $G(2, 4)$ will be then isomorphic to $\mathcal{A}^*(\wedge^2 M_4, D)$.

5.1.1 The model. The $\mathcal{S}$-derivation D_t is given by

$$D_t = 1 + D_1 t + D_2 t^2 + D_3 t^3$$

where $D_j = D_1^j$ and $D_1^4 = 0$. Then $D_t : \bigwedge^2 M_4 \longrightarrow \bigwedge^2 M_4[[t]]$, evaluated on $\epsilon^1 \wedge \epsilon^2$, gives:

$$\begin{aligned} &\sum_{i\geq 0} D_i(\epsilon^1 \wedge \epsilon^2)t^i = D_t(\epsilon^1 \wedge \epsilon^2) = D_t(\epsilon^1) \wedge D_t(\epsilon^2) = \\ = &\quad (D_0 + D_1 t + D_2 t^2 + D_3 t^3)\epsilon^1 \wedge (D_0 + D_1 t + D_2 t^2 + D_3 t^3)\epsilon^2 = \\ = &\quad (\epsilon^1 + \epsilon^2 t + \epsilon^3 t^2 + \epsilon^4 t^3) \wedge (\epsilon^2 + \epsilon^3 t + \epsilon^4 t^2) = \\ = &\quad \epsilon^1 \wedge \epsilon^2 + \epsilon^1 \wedge \epsilon^3 \cdot t + \epsilon^1 \wedge \epsilon^4 \cdot t^2. \end{aligned}$$

These computations show that D restricted to $\bigwedge^2 M_4$ has only three components, namely $D_0 = id_{\wedge^2 M}$, D_1 and D_2. Moreover:

$$D_1(\epsilon^1 \wedge \epsilon^2) = \epsilon^1 \wedge \epsilon^3 \quad \text{and} \quad D_2(\epsilon^1 \wedge \epsilon^2) = \epsilon^1 \wedge \epsilon^4.$$

Let us now check, by hand, that $\bigwedge^2 M_4$ is a principal $\mathbb{Z}[D] \cong \mathbb{Z}[D_1, D_2]$-module generated by $\epsilon^1 \wedge \epsilon^2$. We have:

$$\begin{aligned} \epsilon^2 \wedge \epsilon^3 &= D_1(\epsilon^1 \wedge \epsilon^3) - \epsilon^1 \wedge \epsilon^4 = D_1^2(\epsilon^1 \wedge \epsilon^2) - D_2(\epsilon^1 \wedge \epsilon^2) = \\ &= (D_1^2 - D_2)(\epsilon^1 \wedge \epsilon^2); \end{aligned} \tag{5.1}$$

$$\epsilon^2 \wedge \epsilon^4 = D_1(\epsilon^1 \wedge \epsilon^4) = D_1 D_2(\epsilon^1 \wedge \epsilon^2); \tag{5.2}$$

$$\epsilon^3 \wedge \epsilon^4 = D_1(\epsilon^2 \wedge \epsilon^4) = D_1^2 D_2(\epsilon^1 \wedge \epsilon^2). \tag{5.3}$$

Hence, by definition,

$$D_0, \quad D_1, \quad D_2, \quad D_1^2 - D_2, \quad D_1 D_2, \quad D_1^2 D_2,$$

are Giambelli's polynomials (see Section 4.5) of $\epsilon^1 \wedge \epsilon^2$, $\epsilon^1 \wedge \epsilon^3$, $\epsilon^1 \wedge \epsilon^4$, $\epsilon^2 \wedge \epsilon^3$, $\epsilon^2 \wedge \epsilon^4$, $\epsilon^3 \wedge \epsilon^4$ respectively.

However, it is worth of remarking that a different "integration by parts" gives another Giambelli's polynomial for $\epsilon^3 \wedge \epsilon^4$:

$$\epsilon^3 \wedge \epsilon^4 = D_2(\epsilon^1 \wedge \epsilon^4) - D_1\epsilon^1 \wedge D_1\epsilon^4 - \epsilon^1 \wedge D_2\epsilon^4 = D_2^2(\epsilon^1 \wedge \epsilon^2).$$

In addition, notice that:

$$D_1^2 - D_2 = \begin{vmatrix} D_1 & D_2 \\ D_0 & D_1 \end{vmatrix}, \quad D_1 D_2 = \begin{vmatrix} D_2 & D_3 \\ D_0 & D_1 \end{vmatrix}, \quad D_2^2 = \begin{vmatrix} D_2 & D_3 \\ D_1 & D_2 \end{vmatrix},$$

where we used the fact that $D_3 = 0$ on $\bigwedge^2 M_4$. We have already found a relation in $\mathcal{A}^*(\wedge^2 M_4, D)$, namely the equality of the two Giambelli's polynomials associated to $\epsilon^3 \wedge \epsilon^4$, which can be expressed as $D_1^2 D_2 - D_2^2 = 0$. Let us find all the relations. Clearly there cannot be any relation in degree 2. In fact the only monomials of degree 2 are D_1^2 and D_2. But $D_1^2(\epsilon^1 \wedge \epsilon^2) = \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^4$ and $\epsilon^1 \wedge \epsilon^4$ and $\epsilon^2 \wedge \epsilon^3$ are linearly independent. All the monomials of degree 5, D_1^5, $D_1^3 D_2$ and $D_1 D_2^2$ vanish on $\bigwedge^2 M_4$. This can be seen via direct computation using Leibniz's rule! Moreover one can conclude that there must be exactly one relation in degree 3 and exactly one relation in degree 4. In fact all the monomials in D_1 and D_2 in degree 3 are D_1^3 and $D_1 D_2$, while those of degree 4 are D_2^2 and D_1^4. But the part of degree 3 of $\bigwedge^2 M_4$ is generated by $\epsilon^2 \wedge \epsilon^4$, while the part of degree 4 is generated by $\epsilon^3 \wedge \epsilon^4$ only. It is immediate to see that $2D_1 D_2 - D_1^3 = 0$ and that (already observed) $2D_2^2 - D_1^4 = 0$. Hence the presentation of the *intersection ring* $\mathcal{A}^*(\bigwedge^2 M_4, D)$ is given by:

$$\mathcal{A}^*(\bigwedge^2 M_4, D) = \frac{\mathbb{Z}[D_1, D_2]}{(D_1^3 - 2D_1 D_2, D_1^4 - 2D_2^2)}.$$

5.1.2 Interpretation. To do intersection theory on $G(2,4)$, one chooses a flag $E^\bullet$ of V and represents the Schubert cycle σ_{r_2,r_1} in the form $[\epsilon^{1+r_1} \wedge \epsilon^{2+r_2}]$, where $(\epsilon^i)_{1\le i\le 4}$ is the basis of $V^\vee$ adapted to the polar flag $\mathcal{E}^\bullet$. Then we have:

1. $[\epsilon^1 \wedge \epsilon^2]$ (the fundamental class $[G(2,4)]$),

2. $[\epsilon^1 \wedge \epsilon^3]$, the class of the 2-planes of V intersecting a given two plane along a line, or the class of lines of $\mathbb{P}^3 := \mathbb{P}^3(\mathbb{C})$ intersecting a given line;

3. $[\epsilon^1 \wedge \epsilon^4]$, the class of 2-planes of V containing a line, or of lines of $\mathbb{P}^3$ passing through a point;

4. $[\epsilon^2 \wedge \epsilon^3]$, the class of 2-planes contained in a given 3-space, or of the lines of $\mathbb{P}^3$ contained in a plane;

5. $[\epsilon^2 \wedge \epsilon^4]$, the class of the 2-planes intersecting two fixed 2 planes along a line and intersecting a fixed line, or the class of the lines meeting two lines and passing through a point;

6. $[\epsilon^3 \wedge \epsilon^4]$, the class of a point, i.e the class of all planes coinciding with a given 2-plane, or of the lines of $\mathbb{P}^3$ coinciding with a given line.

5.1.3 Example. As explained, the grassmannian $G(2,4)$ embeds in $\mathbb{P}^5$ as a quadric hypersurface, the Klein quadric. Therefore, using (the classical) notation as in Section 3.4.2, it follows that:

$$\sigma_0\sigma_{22} - \sigma_1\sigma_{21} + \sigma_2\sigma_{11} = 0$$

i.e., the generators $\{\sigma_0, \sigma_1, \sigma_{11}, \sigma_2, \sigma_{21}, \sigma_{22}\}$ of $A^*(G(2,4))$ satisfy the equation of the Klein quadric. This is due to the fact that $\sigma_{\underline{\lambda}}$ can be identified with $\Delta_{\underline{\lambda}}(D)$ and that

$$\epsilon^{1+r_1} \wedge \epsilon^{2+r_2} = \Delta_{(r_2,r_1)}(D)\epsilon^1 \wedge \epsilon^2.$$

Since $\epsilon^i \wedge \epsilon^j$ are Plücker coordinates on $G(2,4)$, they satisfy the equation of the Klein quadric: hence the same holds for the elements $\Delta_{(r_2,r_1)}(D)$, and therefore for the corresponding $\sigma_{(r_2 r_1)}$. A similar statement holds for all grassmannians, i.e.

The Schubert cycles $\sigma_{\underline{\lambda}} \in A^*(G_k(V))$ satisfy the equations defining the Plücker embedding of $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$.

5.2 Playing with $\mathcal{S}$-Derivations

In this section we shall offer some examples of computational applications of the formalism of Chapter 4.

5.2.1 Example. Example 4.7.8, about computing the number of lines incident 4 others in $\mathbb{P}^3$ can be generalized in two different ways. The first is to look for all the lines of $G(1,\mathbb{P}^{n-1})$ intersecting $2(n-2)$ linear subspaces of codimension 2 in general position in $\mathbb{P}^{n-1}$. This amounts to compute the degree of the grassmannian $G(2,n)$, shown in Example 5.3.3 below. The other is to compute the number $\mathcal{L}_{1+n}$ of lines incident 4 linear subspaces of codimension n in general position in $\mathbb{P}^{2n+1}$. If $n = 1$ we already found $\mathcal{L}_2 = 2$. This example is proposed in [27], p. 206, and computed by direct application of Pieri's formula. Here we use a somewhat inductive argument. The problem lives in the grassmannian $G(2, 2n+2)$. Let us work on $\bigwedge^2 M_{2n+2}$, which is a model for its Chow group. The solution of our problem is the coefficient multiplying $\epsilon^{1+2n} \wedge \epsilon^{2+2n}$ in the expansion of $D_4^n(\epsilon^1 \wedge \epsilon^2)$.

We shall indeed prove that

$$D_n^4(\epsilon^1 \wedge \epsilon^2) = \epsilon^{1+2n} \wedge \epsilon^{2+2n} + D_{n-1}^4(\epsilon^3 \wedge \epsilon^4). \tag{5.4}$$

Since $D_{n-1}^4(\epsilon^3 \wedge \epsilon^4)$ in $\bigwedge^2 M_{2+2n}$ is formally the same as computing $D_{n-1}^4(\epsilon^1 \wedge \epsilon^2)$ in $\bigwedge^2 M_{2n}$ (prove it!), it follows that

$$\mathcal{L}_{1+n} = 1 + \mathcal{L}_n.$$

Therefore $\mathcal{L}_{1+n} = n + \mathcal{L}_1$, and $\mathcal{L}_1 = 1$: there is only one line in $\mathbb{P}^1$ meeting 4 points of $\mathbb{P}^1$! One is left, then, to prove formula (5.4). One first applies twice formula 4.16:

$$\begin{aligned} D_n^2(\epsilon^1 \wedge \epsilon^2) &= D_n(\epsilon^1 \wedge \epsilon^{2+n}) = \\ &= \epsilon^1 \wedge \epsilon^{2+2n} + D_{n-1}(\epsilon^2 \wedge \epsilon^{2+n}) = \\ &= \epsilon^1 \wedge \epsilon^{2+2n} + D_{n-1}^2(\epsilon^2 \wedge \epsilon^3) \end{aligned} \tag{5.5}$$

where in the last equality one used Corollary 4.4.2. Hence:

$$D_n^4(\epsilon^1 \wedge \epsilon^2) = D_n^2(\epsilon^1 \wedge \epsilon^{2+2n}) + D_{n-1} D_n^2(\epsilon^2 \wedge \epsilon^{2+n})$$

The first summand is simply $\epsilon^{1+2n} \wedge \epsilon^{2+2n}$. One can argue, e.g., by applying again formula (4.16) with $\alpha = \epsilon^{2+2n}$ and observing that $D_i \epsilon^{2+2n} = 0$ in $\bigwedge^2 M_{2+2n}$, for each $i > 0$. On the other hand:

$$D_n^2 D_{n-1}^2(\epsilon^2 \wedge \epsilon^3) = D_{n-1}^2 D_n^2(\epsilon^2 \wedge \epsilon^3) = D_{n-1}^2 D_n^2 \Delta_2(\epsilon^1 \wedge \epsilon^2)$$

where Δ_2 is defined as in Lemma 4.5.7. Again:

$$\begin{aligned} D_{n-1}^2 D_n^2 \Delta_2(\epsilon^1 \wedge \epsilon^2) &= \\ &= D_{n-1}^2 \Delta_2 D_n^2(\epsilon^1 \wedge \epsilon^2) = \\ &= D_{n-1}^2 \Delta_2(\epsilon^{1+2n} \wedge \epsilon^{2+2n} + D_{n-1}^2(\epsilon^2 \wedge \epsilon^3)). \end{aligned}$$

But $\Delta_2(\epsilon^{1+2n} \wedge \epsilon^{2+2n}) = 0$, so that

$$D_n^2 D_{n-1}^2(\epsilon^2 \wedge \epsilon^3) = D_{n-1}^4 \Delta_2(\epsilon^2 \wedge \epsilon^{2+n})) = D_{n-1}^4(\epsilon^3 \wedge \epsilon^4)$$

Substitution in (5.5) gives exactly formula (5.4).

5.2.2 Example/Exercise. Formula (**) at p. 205 of [27], used to prove Giambelli's formula at p. 206, can be re-written in the following transparent form:

$$\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} = D_{i_1-1}(\epsilon^1\wedge\epsilon^{i_2}\wedge\ldots\wedge\epsilon^{i_k}) + D_{i_2-1}(\epsilon^{i_1}\wedge\epsilon^1\wedge\ldots\wedge\epsilon^{i_k}) + \ldots$$
$$\ldots + D_{i_k-1}(\epsilon^{i_1}\wedge\epsilon^{i_2}\wedge\ldots\wedge\epsilon^1).$$

The Reader is invited to prove it as an exercise.

5.2.3 Example. Here we shall end some computations of Example 3.4.9, which were then left to the Reader. In the Grassmannian $G(3,6)$, $\sigma_1 = D_1$ and $\sigma_{31}\cap[G(3,6)] = [\epsilon^1\wedge\epsilon^3\wedge\epsilon^6]$. Therefore:

$$\sigma_1^5\cdot\sigma_{31}\cap[G_k(V)] = D_1^5(\epsilon^1\wedge\epsilon^3\wedge\epsilon^6)$$

(the brackets have been omitted). One has (using the fact that $D_i(\epsilon^6)=0$ in M_6, for $i>0$):

$$D_1^5(\epsilon^1\wedge\epsilon^3\wedge\epsilon^6) = D_1^5(\epsilon^1\wedge\epsilon^3)\wedge\epsilon^6 = D_1^4(\epsilon^2\wedge\epsilon^3+\epsilon^1\wedge\epsilon^4)\wedge\epsilon^6 =$$
$$= D_1^3(2\epsilon^2\wedge\epsilon^4+\epsilon^1\wedge\epsilon^5)\wedge\epsilon^6 = D_1^2(\epsilon^3\wedge\epsilon^4+3\epsilon^2\wedge\epsilon^5)\wedge\epsilon^6 =$$
$$= D_1(5\epsilon^3\wedge\epsilon^5)\wedge\epsilon^6 = 5\cdot\epsilon^4\wedge\epsilon^5\wedge\epsilon^6.$$

Since $[\epsilon^4\wedge\epsilon^5\wedge\epsilon^6]$ is the class of the point in $A_*(G(3,6))$, the solution of our enumerative exercise is 5 as claimed in Example 3.4.9.

5.2.4 Example. Here we concludes the computation of the product

$$(\sigma_1^2-\sigma_2)\cdot\sigma_{21}$$

proposed in Example (3.4.11). It is the same as computing:

$$(D_1^2-D_2)(\epsilon^1\wedge\epsilon^3\wedge\epsilon^5).$$

We left to the Reader the task of computing directly the above expression. Instead, we use the fact that, indeed, $D_1^2-D_2=\Delta_2$ (see Lemma 4.5.7), and that Δ_2 enjoy Leibniz's rule for second derivatives. One has:

$$\Delta_2(\epsilon^1\wedge\epsilon^3\wedge\epsilon^5) = \Delta_2(\epsilon^1\wedge\epsilon^3)\wedge\epsilon^5 + \Delta_1(\epsilon^1\wedge\epsilon^3)\wedge\Delta_1\epsilon^5 =$$

where we used again Lemma 4.5.7 for concluding that $\Delta_2(\epsilon^5)=0$. Thus, continuing computations and remembering that $\Delta_1 = D_1$:

$$= \epsilon^2\wedge\epsilon^4\wedge\epsilon^5+\epsilon^2\wedge\epsilon^3\wedge\epsilon^6+\epsilon^1\wedge\epsilon^4\wedge\epsilon^6 = (\sigma_{221}+\sigma_{311}+\sigma_{32})\cap[G(3,6)],$$

having used, in the last equality, the dictionary translating the expression $\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$ into the corresponding $\sigma_{\underline{\lambda}(I)}$.

5.3 Degree of Grassmannians

5.3.1 Proposition. *Newton's binomial formula holds:*

$$D_1^n(\alpha\wedge\beta)=\sum_{k=0}^{n}\binom{n}{k}D_1^k\alpha\wedge D_1^{n-k}\beta, \qquad (5.6)$$

for each $\alpha,\beta\in\bigwedge M$.

Proof. For $n=1$ the formula is obvious. Suppose it holds for $n-1$. Then

$$D_1^n(\alpha\wedge\beta)=D_1(D_1^{n-1}(\alpha\wedge\beta))=D_1\sum_{k=0}^{n-1}\binom{n-1}{k}D_1^k\alpha\wedge D_1^{n-1-k}\beta=$$

$$=\sum_{k=0}^{n-1}\binom{n-1}{k}D_1\left(D_1^k\alpha\wedge D_1^{n-1-k}\beta\right)=$$

$$=\sum_{k=0}^{n-1}\binom{n-1}{k}D_1^{k+1}\alpha\wedge D_1^{n-1-k}\beta+\sum_{k=0}^{n-1}\binom{n-1}{k}D_1^k\alpha\wedge D_1^{n-k}\beta=$$

$$=\sum_{k=1}^{n}\binom{n-1}{k-1}D_1^k\alpha\wedge D_1^{n-k}\beta+\sum_{k=0}^{n-1}\binom{n-1}{k}D_1^k\alpha\wedge D_1^{n-k}\beta=$$

$$=\sum_{k=1}^{n-1}\left(\binom{n-1}{k-1}+\binom{n-1}{k}\right)D_1^k\alpha\wedge D_1^{n-k}\beta+D_1^n\alpha\wedge\beta+\alpha\wedge D_1^n\beta=$$

$$=\sum_{k=0}^{n}\binom{n}{k}D_1^k\alpha\wedge D_1^{n-k}\beta.$$

where in the first row we used induction in the last equality. The rest are straightforward manipulations. ∎

5.3.2 The degree of the grassmannian $G(k,n)$ in the Plücker embedding is, by definition:

$$d_{k,n}=\int\sigma_1^{k(n-k)}\cap[G(k,n)],$$

i.e. it is the coefficient of the generator of $\wedge^k M_n$ of maximum weight:

$$D_1^{k(n-k)}\epsilon^1\wedge\ldots\wedge\epsilon^k=d_{k,n}\cdot\epsilon^{n-k+1}\wedge\ldots\wedge\epsilon^n \qquad (5.7)$$

The reason is that $\sigma_1 \cap [G(k,n)]$ is a very ample divisor class embedding the Grassmannian in $\mathbb{P}^{\binom{n}{k}-1}$ à la Plücker. The degree of the Plücker image of $G(k,n)$ is therefore the degree of its intersection with as many hyperplanes as the dimension of the grassmannian. The pullback of the hyperplane class of $Pl(G(k,n))$ is precisely σ_1 and this explains formula (5.7).

5.3.3 Example (Degree of $G(2,n)$). To compute the degree $d_{2,n}$ of $G(2,n)$ is then sufficient to expand expression:

$$D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2)$$

which is certainly an integral multiple of $\epsilon^{n-1} \wedge \epsilon^n$. First we apply the binomial formula:

$$D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2) = \sum_{k=0}^{2(n-2)} \binom{2(n-2)}{k} D_1^k \epsilon^1 \wedge D_1^{2(n-2)-k}\epsilon^2.$$

Now $D_1^{2(n-1)}\epsilon^1 \wedge \epsilon^2 \in (\bigwedge^k M_n)_{2(n-2)}$. The latter is a free $\mathbb{Z}$-module spanned by $\epsilon^{n-1} \wedge \epsilon^n$ only. Therefore one knows a priori that all the summands cancel and that only a $\mathbb{Z}$-linear combination of $\epsilon^{n-1} \wedge \epsilon^n$ and $\epsilon^n \wedge \epsilon^{n-1}$ will survive. More precisely one is interested in the sum:

$$\begin{aligned} D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2) &= \binom{2(n-2)}{n-2} D_1^{n-2}\epsilon^1 \wedge D_1^{2(n-2)-n+2}\epsilon^2 + \\ + &\binom{2(n-2)}{n-1} D_1^{n-1}\epsilon^1 \wedge D_1^{2(n-2)-n+1}\epsilon^2 = \\ = &\left(\binom{2(n-2)}{n-2} - \binom{2(n-2)}{n-1}\right)\epsilon^{n-1} \wedge \epsilon^n. \end{aligned}$$

Hence:

$$d_{2,n} = \binom{2(n-2)}{n-2} - \binom{2(n-2)}{n-1} = \frac{(2n-4)!}{(n-1)!(n-2)!}.$$

5.3.4 Example. Working on the full exterior algebra, instead of on a single exterior power, suggests that Schubert calculus for grassmannians $G(k,n)$ must be linked recursively with Schubert calculus of $G(k', n')$ with $k' < k$ and $n' \leq n$. This is also geometrically observed in [27]. Some nice exercises can be done to this purpose and, as a matter of example, we look for a formula expressing $d_{3,n}$ in functions of degrees of $G(2, n-1)$ and $G(2, n)$. As before, one aims to determine the integer $d_{3,n}$ defined by the equality below:

$$D_1^{3(n-3)}\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 = d_{3,n}\epsilon^{n-2} \wedge \epsilon^{n-1} \wedge \epsilon^n$$

To this purpose one first uses Newton bynomial formula (5.6):

$$D_1^{3(n-3)}\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 = \sum_{k=0}^{3(n-3)} \binom{3(n-3)}{k} D_1^{3(n-3)-k}(\epsilon^1 \wedge \epsilon^2) \wedge D_1^k\epsilon^3$$

In the above sum will only survive multiples of the three terms below:

$$\epsilon^{n-2} \wedge \epsilon^{n-1} \wedge \epsilon^n, \quad \epsilon^{n-1} \wedge \epsilon^n \wedge \epsilon^{n-2}, \quad \epsilon^{n-2} \wedge \epsilon^n \wedge \epsilon^{n-1}.$$

One then selects in the above sum those summands for which the values of k are $n-3$, $n-2$ and $n-1$. In other words:

$$\begin{aligned} D_1^{3(n-3)}\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 &= \binom{3(n-3)}{n-3} D_1^{2(n-3)}(\epsilon^1 \wedge \epsilon^2) \wedge \epsilon^n + \\ &+ \binom{3(n-3)}{n-4} D_1^{2n-5}(\epsilon^1 \wedge \epsilon^2) \wedge \epsilon^{n-1} + \\ &+ \binom{3(n-3)}{n-5} D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2) \wedge \epsilon^{n-2}. \end{aligned}$$

To end computations we only need to know the coefficients of

$$\begin{array}{lll} \epsilon^{n-2} \wedge \epsilon^{n-1} & \text{in the expansion of} & D_1^{2(n-3)}(\epsilon^1 \wedge \epsilon^2) \\ \epsilon^n \wedge \epsilon^{n-2} & \text{in the expansion of} & D_1^{2n-5}(\epsilon^1 \wedge \epsilon^2) \\ \epsilon^{n-1} \wedge \epsilon^n & \text{in the expansion of} & D_1^{2(n-2)}(\epsilon^1 \wedge \epsilon^2) \end{array}$$

By the previous discussion, the first and the third coefficient are nothing else than $d_{2,n-1}$ and $-d_{2,n}$ respectively. As for the second coefficient, remark that the coefficient of $\epsilon^n \wedge \epsilon^{n-2}$ in the expansion of $D_1^{2n-5}(\epsilon^1 \wedge \epsilon^2)$ is the same as the coefficient of $\epsilon^n \wedge \epsilon^{n-1}$ in the expansion of $D_1^{2n-4}(\epsilon^1 \wedge \epsilon^2)$ and this latter is precisely the degree $d_{2,n}$ of the Grassmannian $G(2,n)$. Therefore we have proven the following formula:

$$d_{3,n} = \binom{3(n-3)}{n-3} d_{2,n-1} + \left(\binom{3(n-3)}{n-5} - \binom{3(n-3)}{n-4}\right) d_{2,n}.$$

In an attempt to recognize a general pattern, Taise Santiago computed a similar recursive formula for $d_{4,n}$:

$$\begin{aligned} d_{4,n} &= \binom{4(n-4)}{n-4} d_{3,n-1} + \left[\binom{4(n-4)}{n-6} - \binom{4(n-4)}{n-7}\right] d_{3,n} + \\ &+ \binom{4(n-4)}{n-5} \left\{ \left[\binom{3n-11}{n-5} - \binom{3n-11}{n-6}\right] d_{2,n} + \right. \\ &- \left. \binom{3n-11}{n-5} d_{2,n-1} - \binom{3n-11}{n-3} d_{2,n-2} \right\}. \end{aligned}$$

5.4 Wronskians Correspondences

A *complete flag* of V is a filtration:

$$E^\bullet : E^0 \supset E^1 \supset E^2 \supset \ldots \supset E^n = (0) \tag{5.8}$$

into subspaces E^i of V of codimension i (hence $E^0 = V$). Let $Fl(V)$ be the set of all the complete flags of the complex vector space V. It is a honest smooth connected complex projective variety of dimension $\binom{n}{2}$, intimately related with any grassmannian $G_k(V)$, $1 \le k \le n$. It is easy to guess that a k-plane $[\Lambda]$ is in general position with respect to a flag $E^\bullet$ if and only if $E^k \cap [\Lambda] = (0)$. In fact one has the following chain of inequalities:

$$k = \dim(E^0 \cap [\Lambda]) \ge \dim(E^1 \cap [\Lambda]) \ge \ldots \ge \dim(E^k \cap [\Lambda]) = 0,$$

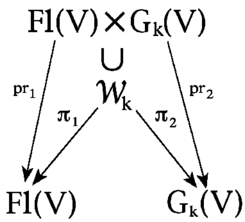
implying that $\dim(E^i \cap [\Lambda]) = k - i$ for $i \le k$ and $E^i \cap [\Lambda] = 0$ for all $i > k$. That is, $[\Lambda]$ is in general position with respect to the flag (5.8) if it is in general position with respect to all subspaces defining it. Such a regular behaviour gets suddenly lost for all k-planes such that:

$$\dim([\Lambda] \cap E^k) > 0,$$

i.e. those we call $E^\bullet$-special k-planes – k-planes in special position with respect to the flag $E^\bullet$. Similarly, pick a k-plane $[\Lambda]$ in V. Then the general complete flag of V

$$F^\bullet : F^0 \supset F^1 \supset F^2 \supset \ldots \supset F^n = (0)$$

has the property that $[\Lambda]$ is not $F^\bullet$-special: it suffices to pick a general $(n-k)$-plane, intersecting $[\Lambda]$ only at the zero vector, and then completing it in a flag around it. Hence the geometry of the flag varieties and that of grassmannians someway interacts as suggested by the following diagram



where

$$\mathcal{W}_k := \{([\Lambda], F^\bullet) \in Fl(V) \times G_k(V) \mid [\Lambda] \text{ is } \mathrm{F}^\bullet - \text{special}\}.$$

Indeed, *Schubert calculus for Grassmannian* studies the cohomology (or the intersection theory) of fibers of the map π_1, while the so called *Schubert Calculus for complete Flag Varieties* (see e.g. [22], [75]) studies the cohomology of the fibers of π_2.

5.4.1 The smooth irreducible algebraic variety $\mathcal{W}_k$ is what it would be reasonable to call a *Wronskian Correspondence*. In fact the fiber of $\mathcal{W}_k$ over $E^\bullet$ is the zero locus of a section $\mathbb{W}_E$ of the line bundle $\wedge^k \mathcal{T}_k^\vee$. Moreover this section looks like a true wronskian (as shown in Example 3.3.7) and one has that $[\Lambda] \in \pi_1^{-1}(E^\bullet)$ if and only if $\epsilon^1 \wedge \ldots \wedge \epsilon^k(\Lambda) = 0$, where $(\epsilon^1, \epsilon^2, \ldots, \epsilon^n)$ is a basis of V adapted to the flag $E^\bullet$.

5.4.2 Example (Weierstrass Points on Curves). This example has been the main inspiration of these notes. The reader not familiar with elementary theory of algebraic curves may skip it with no essential loss or consult [24] for a friendly introduction. Let C be a smooth projective complex curve and let K be its canonical bundle. The space $H^0(C,K)$ of holomorphic differentials has dimension g over the complex field. Denote by J^iK the i^{th} jet extension of K: it is a rank $i+1$ bundle whose sections behave as do i-th derivatives of local sections of K. If $\omega \in H^0(K)$, then $D^i\omega$ is a section of J^iK which locally is the local representation of ω together with its first i-derivatives. A point P is said to be a *Weierstrass point* if there exists a non zero $\omega \in H^0(K)$ such that $D^{g-1}(\omega)(P) = 0$, i.e. if there exists a non zero holomorphic differential vanishing at P with multiplicity at least g. Hence a Weierstrass point is in the locus where the map

$$D^{g-1} : H^0(K) \otimes O_C \longrightarrow J^{g-1}K$$

defined by $D^{g-1}(P,\omega) = (D^{g-1}\omega)(P)$ drops rank. This is equivalent to look at the zero locus of the determinant map:

$$\bigwedge^k D^{g-1} : O_C \longrightarrow \bigwedge^g J^{g-1}K \cong K^{\otimes \frac{g(g+1)}{2}},$$

said to be *wronskian*. If $\underline{\omega} = (\omega_1, \ldots, \omega_g)$ is a basis of $H^0(K)$ and U is an open set of C (in the Zariski or usual topology) trivializing K , where $\omega_{i|_U} = u_i(z) \cdot dz$ (z being a coordinate on U), then

$$\begin{aligned}
(\wedge^k D^{g-1})_{|_U} &= (\mathbf{u} \wedge \mathbf{u}' \wedge \ldots \wedge \mathbf{u}^{(g-1)}) \cdot dz^{\otimes \frac{g(g+1)}{2}} = \\
&= \begin{vmatrix} u_1 & u_2 & \ldots & u_g \\ u_1' & u_2' & \ldots & u_g' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(g-1)} & u_2^{(g-1)} & \ldots & u_g^{(g-1)} \end{vmatrix} \cdot dz^{\otimes \frac{g(g+1)}{2}}.
\end{aligned}$$

Abusing notation the wronskian section is also written as

$$\underline{\omega} \wedge D\underline{\omega} \wedge \ldots \wedge D^{g-1}\underline{\omega}.$$

Let (U,z) be a trivialization around a point P. Then the points of the fiber J^i_PK of J^iK at P are $(i+1)$-tuples of complex numbers $(u_0, u_1, \ldots, u_i)$. Consider the map $J^{2g-2}_PK \longrightarrow J^i_PK$ defined by

$$(u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_{2g-2}) \mapsto (u_0, u_1, \ldots, u_i)$$

and let $\mathcal{K}^i_P \subset J^{2g-2}_PK$ be the kernel, which has codimension $i+1$. Then one has a filtration $\mathcal{K}^\bullet_P$:

$$\mathcal{K}^0_P := J^{2g-2}_PK \supset \mathcal{K}^1_P \supset \ldots \supset \mathcal{K}^{2g-3}_P \supset \mathcal{K}^{2g-2}_P = (0).$$

The space $H^0(K)$ can be identified with a g-dimensional subspace of $J_P^{2g-2}K$ via the map:

$$\begin{cases} D_P^{2g-2} \quad : \quad H^0(K) & \longrightarrow \quad J_P^{2g-2}K \\ \qquad\qquad\quad \omega & \longmapsto \quad D^{2g-2}\omega(P). \end{cases}$$

This map is in fact injective, because $(D^{2g-2}\omega)(P) = 0$ implies $\omega = 0$: there is no non zero canonical divisor having degree bigger than $2g-2$. Then, the point P is a Weierstrass point if $H^0(K)$ is in special position with respect to the flag $\mathcal{K}_P^\bullet$, i.e. if $D_P^{2g-2}(H^0(K)) \cap \mathcal{K}_P^g \neq 0$, i.e. if there exists $0 \neq \omega \in H^0(K)$ such that $(D^{g-1}\omega)(P) = 0$, i.e. if

$$(u, u', \ldots, u^{(g-1)})$$

vanishes at P, where $\omega_{|_U} = u(z) \cdot dz$. In the Grassmannian $G_g(J_P^{2g-2}K)$ the g-plane $H^0(K)$ belongs to one and only one Schubert cell $\Omega_{i_1 \ldots i_g}(\mathcal{K}_P^\bullet)$. The k-schindex $(i_1, \ldots, i_g)$ is precisely the *Weierstrass Gap Sequence* of the point P (see [24]).

5.4.3 Example (The Brill-Nöther Matrix). (Cf. [1], p. 154). Let C be a smooth complex projective curve of genus g, K its canonical bundle and $\underline{\omega} = (\omega_1, \ldots, \omega_g)$ a basis of the holomorphic differentials. Let $\mathtt{D} = n_1P_1 + \ldots + n_kP_k$ be an effective divisor of degree $d = n_1 + \ldots + n_k$. The *Brill Nöther matrix* associated to D and to the basis $\underline{\omega}$ is by definition:

$$\mathcal{BN}(\mathtt{D}, \underline{\omega}) = \begin{pmatrix} D^{n_1-1}\underline{\omega}(P_1) \\ D^{n_2-1}\underline{\omega}(P_1) \\ \vdots \\ D^{n_k-1}\underline{\omega}(P_k) \end{pmatrix}.$$

Its rank is defined as follows. Let (U_i, z_i) be a local holomorphic chart around P_i, such that $(\omega_j)_{|_{U_i}} = f_{ij}(z_i)dz_i$, for some $f_{ij} \in O(U_i)$. Then the *Brill-Nöther matrix* admits the following local representation with respect to the local parameters $\mathbf{z} = (z_1, \ldots, z_k)$:

$$\mathcal{BN}(\mathtt{D}, \mathbf{z}, \underline{\omega}) = \begin{pmatrix} f_{11}(z_1) & f_{12}(z_1) & \ldots & f_{1g}(z_1) \\ f_{11}'(z_1) & f_{12}'(z_1) & \ldots & f_{1g}'(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_{11}^{(n_1-1)}(z_1) & f_{12}^{(n_1-1)}(z_1) & \ldots & f_{1g}^{(n_1-1)}(z_1) \\ & & & \\ f_{21}(z_2) & f_{22}(z_2) & \ldots & f_{2g}(z_2) \\ f_{21}'(z_2) & f_{22}'(z_2) & \ldots & f_{2g}'(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_{21}^{(n_2-1)}(z_2) & f_{22}^{(n_2-1)}(z_2) & \ldots & f_{2g}^{(n_2-1)}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ & & & \\ \vdots & \vdots & \ddots & \vdots \\ f_{k1}(z_k) & f_{k2}(z_k) & \ldots & f_{kg}(z_k) \\ f_{k1}'(z_k) & f_{k2}'(z_k) & \ldots & f_{kg}'(z_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_{k1}^{(n_k-1)}(z_k) & f_{k2}^{(n_k-1)}(z_k) & \ldots & f_{kg}^{(n_k-1)}(z_k) \end{pmatrix}.$$

Its rank does not depend neither on $\mathbf{z}$, nor on $\underline{\omega}$, but only on the divisor $\mathtt{D}$.

If $h^0(K(-\mathtt{D}))$ is the dimension of the space $H^0(K(-\mathtt{D}))$ of the holomorphic differentials vanishing at the P_i's with multiplicity at least n_i, then one has:

$$h^0(K(-\mathtt{D})) = g - rk(\mathcal{BN}(\mathtt{D})).$$

Using Riemann-Roch formula:

$$h^0(O(\mathtt{D})) - h^0(K(-\mathtt{D})) = 1 - g + \deg(\mathtt{D})$$

one sees that n is a *Weierstrass gap* at P if and only if

$$rk\ \mathcal{BN}((n-1)P) < rk\ \mathcal{BN}(nP).$$

In an unpublished part of his doctoral thesis ([59]), Ponza constructed some *generalized wronskian sections* which are sections of some *generalized wronskian bundles*, and show that the Weierstrass gap sequence at P is $(1, n_2, \ldots, n_g)$ if and only if the *generalized wronskian section*

$$\underline{\omega} \wedge D^{n_2-1}\underline{\omega} \wedge \ldots \wedge D^{n_g-1}\underline{\omega},$$

does not vanish at P. Hence the Weierstrass gap sequence can be seen as a Schubert index attached to $H^0(C, K)$ embedded in $J_P^{2g-2}K$ as shown in Example 5.4.2.

5.5 (Small) Quantum Cohomology of the Grassmannian

This section is devoted to extract the algebraic content of the so-called *small quantum cohomology* of the Grassmannian ([4]). For a nice geometrical introduction to the ideas of quantum cohomology, the reader (especially the brazilian one) is advised to have a look at [42] and to the references therein. For a nice concise account on quantum cohomology of grassmannians look at [57].

As for small quantum Schubert Calculus of the grassmannian $G(k,n)$, the idea is that it is the same as classical Schubert calculus in a grassmannian $G(k,N)$, with N sufficiently big. To be more precise, let q be an indeterminate over $\mathbb{Z}$ and let $M_n[q] := M_n \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ – the free $\mathbb{Z}[q]$-module spanned by $\mathcal{E}_n$. As a $\mathbb{Z}$-module, $M_n[q]$ is an infinite free $\mathbb{Z}$-module spanned by $(\epsilon^i)_{i\geq 1}$, with $\epsilon^{nj+r} = q^j\epsilon^r$ $(0 \leq r < n)$. Then one writes Pieri's and Giambelli's formulas for such a basis keeping into account that q belongs to the ring of coefficients and may be factorized. This is because in Chapter 4 one preferred to work with infinite free modules rather than finite one.

5.5.1 As a $\mathbb{Z}$-module, $M_n[q]$ is isomorphic to M via the isomorphism

$$\begin{cases} \mathcal{Q}_n : & M & \longrightarrow & M_n[q] \\ & \epsilon^{\alpha\cdot n+i} & \longmapsto & q^\alpha\epsilon^i \end{cases}, \quad (\forall \alpha \geq 0, \quad 1 \leq i \leq n-1).$$

Let $\bigwedge^k M_n$ and $\bigwedge^k M_n[q] \cong \bigwedge^k M_n \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ be the k-th exterior power of M_n and $M_n[q]$ (the latter thought as a $\mathbb{Z}[q]$-module) respectively. Both are freely generated, over $\mathbb{Z}$ and $\mathbb{Z}[q]$ respectively, by

$$\{(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) : 1 \leq i_1 < \ldots < i_n \leq n\}.$$

Let $\wedge^k p_n : \bigwedge^k M \longrightarrow \bigwedge^k M_n$ be the natural projection (4.27).

$$\sum_{1\leq i_1<\ldots<i_k} a_{i_1\ldots i_k}\cdot\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k} \longmapsto \sum_{1\leq i_1<\ldots<i_k\leq n} a_{i_1\ldots i_k}\cdot\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$$

and $\wedge^k\mathcal{Q}_n : \bigwedge^k M \longrightarrow \bigwedge^k M_n[q]$ the $\mathbb{Z}$-module isomorphism induced by $\mathcal{Q}_n$. It is easy to see that $\wedge^k p_n \circ D_h : \bigwedge^k M \longrightarrow \bigwedge^k M_n$ is the 0 homomorphism, for all $h \geq n+1$. The proposition below, together with Proposition 4.7.4, rules the case $h \leq n$.

5.5.2 Proposition. *Let* $I := (1 \leq i_1 < i_2 < \ldots < i_k \leq n)$ *and* $0 \leq h \leq n$. *Then:*

$$\wedge^k \mathcal{Q}_n \circ D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = p_n D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) + \\ +(-1)^{k-1} q \cdot \sum_{\substack{(h_i)\in H(I,h) \\ i_k+h_k-n<i_1}} \epsilon^{i_k+h_k-n} \wedge \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_{k-1}+h_{k-1}}. \quad (5.9)$$

Proof. (As in [23]) One first uses (4.8) to expand $D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})$ and then splits the sum as:

$$D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum_{\substack{(h_i)\in H(I,h) \\ i_k+h_k\leq n}} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k} + \\ + \sum_{\substack{(h_i)\in H(I,h) \\ i_k+h_k>n}} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k}.$$

The first summand occurring on the r.h.s. is precisely $p_n D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})$. Applying $\wedge^k \mathcal{Q}_n$ to both sides:

$$\wedge^k \mathcal{Q}_n(D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})) = p_n D_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) + \\ \sum_{(h_i)\in H(I,h)} \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_{k-1}+h_{k-1}} \wedge q\epsilon^{i_k+h_k-n}. \quad (5.10)$$

Using the $\mathbb{Z}_2$-symmetry of $\wedge$, last term of (5.10) can be written as $(-1)^{k-1} q(C + \overline{C})$, where:

$$C := \sum_{\substack{(h_i)\in H(I,h) \\ i_k+h_k-n<i_1}} \epsilon^{i_k+h_k-n} \wedge \epsilon^{i_1+h_1} \wedge \epsilon^{i_2+h_2} \wedge \ldots \wedge \epsilon^{i_{k-1}+h_{k-1}},$$

so that $(-1)^{k-1} qC$ is exactly the second summand of the r.h.s. of formula (5.9). As for $\overline{C}$:

$$\overline{C} := \sum_{\substack{(h_i)\in H(I,h)\\ i_k+h_k-n\geq i_1}} \epsilon^{i_k+h_k-n}\wedge\epsilon^{i_1+h_1}\wedge\epsilon^{i_2+h_2}\wedge\ldots\wedge\epsilon^{i_{k-1}+h_{k-1}} =$$

$$= \sum_{h'=0}^{h}\sum_{h_k=i_1+n-i_k}^{h'} \epsilon^{i_k+h_k-n}\wedge\epsilon^{i_1+h'-h_k}\wedge D_{h-h'}(\epsilon^{i_2}\wedge\ldots\wedge\epsilon^{i_k-1}). \tag{5.11}$$

For each $0\leq h'\leq h$, let $\rho_{h'}$ be the bijection of the set

$$\{a\in\mathbb{N} : i_1+n-i_k\leq a\leq h'\}$$

onto itself, defined by $\rho_{h'}(a) = i_1+n+h'-i_k-a$. Then expression (5.11) can also be written as:

$$\overline{C} = \sum_{h'=0}^{h}\sum_{h_k=\rho_0}^{h'} \epsilon^{i_k+\rho_{h'}(h_k)-n}\wedge\epsilon^{i_1+h'-\rho_{h'}(h_k)}\wedge D_{h-h'}(\epsilon^{i_2}\wedge\ldots\wedge\epsilon^{i_k-1}) =$$

$$= \sum_{h'=0}^{h}\sum_{h_k=\rho_0}^{h'} \epsilon^{i_1+h_1}\wedge\epsilon^{i_k+h_k-n}\wedge D_{h-h'}(\epsilon^{i_2}\wedge\ldots\wedge\epsilon^{i_k-1}) = -\overline{C},$$

where, for sake of not breaking the formula, one set $\rho_0 = \rho_0(0) = i_1+n-i_k$. Thus $\overline{C}=0$ and the proof of (5.9) is complete. ∎

This proves that

5.5.3 Theorem. *The small quantum intersection ring ([81],[4]) can be realized as a commutative ring of linear operators $\mathcal{A}^*(M_n, D, q)$ of $\bigwedge^k M[q]$ via the map $\sigma_i\mapsto D_i$ and $q\mapsto(-1)^{k-1}q$).*

Notice that Giambelli's formula (4.35) still holds without any q-correction. In particular the natural evaluation morphism $\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k}$: $\mathcal{A}^*(M_n, D, q)\longrightarrow\bigwedge^k M_n[q]$ is onto. Furthermore:

5.5.4 Corollary. *The epimorphism*

$$\mathrm{ev}_{\epsilon^1\wedge\ldots\wedge\epsilon^k} : \mathbb{Z}[q][D]\longrightarrow\bigwedge^k M_n[q]$$

induces an isomorphism:

$$\frac{\mathbb{Z}[q][D_1,\ldots,D_k]}{(D_{n-k+1},\ldots,D_{n-1},D_n-q)} \longrightarrow \bigwedge^k M_n[q]. \qquad (5.12)$$

If one replaces D_i by σ_i, keeping into account the relations prescribed by Theorem 4.6.4, formula (5.12) gives precisely the presentation of the *small quantum cohomology ring of the grassmannian* given by Siebert and Tian ([73]) and Witten ([81]).

Proof. Notice that if $0 \neq \psi(T_1,\ldots,T_k) \in \mathbb{Z}[\mathbf{T}]$ is homogeneous of degree $\leq n-k$ (remind that $\deg T_i = i$, see Sect. 2.1.8), then $E_D(\psi)\epsilon^1 \wedge \ldots \wedge \epsilon^k \notin \bigwedge^{k-1} M \wedge D_n M$, and is hence not zero in $\bigwedge^k M_n$. Now, if $1 \leq i \leq k-1$, then

$$\begin{aligned} D_{n-k+i}(\epsilon^1 \wedge \ldots \wedge \epsilon^k) &= \epsilon^1 \wedge \ldots \wedge \epsilon^{k-1} \wedge \epsilon^{n+i} = \\ &= \epsilon^1 \wedge \ldots \wedge \epsilon^{k-1} \wedge q\epsilon^i = q(\epsilon^1 \wedge \ldots \wedge \epsilon^{k-1} \wedge \epsilon^i) = 0, \end{aligned}$$

and this shows that $(D_{n-k+1},\ldots,D_{n-1})$ is in the ideal of relations. Suppose now that $0 \neq \phi \in \mathbb{Z}[q][T_1,\ldots,T_k]$ is such that

$$E_D(\phi)\cdot\epsilon^1 \wedge \ldots \wedge \epsilon^k = 0 \in \bigwedge^k M_n[q].$$

Let $\widetilde{\phi}$ be ϕ thought of as polynomial in $\mathbb{Z}[T_1,\ldots,T_k,q]$. It follows that $E_D(\widetilde{\phi})\epsilon^1 \wedge \ldots \wedge \epsilon^k \in \bigwedge^{k-1} M \wedge D_n M$, otherwise

$$E_D(\widetilde{\phi})\cdot\epsilon^1 \wedge \ldots \wedge \epsilon^k \neq 0,$$

in $\bigwedge^k M_n$ and this would be a contradiction since

$$\bigwedge^k M_n \cong \bigwedge^k M_n[q]/(q).$$

It follows that $\widetilde{\phi} \in (T_{n-k+1},\ldots,T_{n-1},q)$, thought of as polynomial with $\mathbb{Z}$-coefficients, and one may write $\widetilde{\phi} = q\phi_1 + \phi_1'$, with $\phi_1' \in (T_{n-k+1},\ldots,T_{n-1})$. Since $q\epsilon^1 \wedge \ldots \wedge \epsilon^k \neq 0$, ϕ_1 must be a $\mathbb{Z}$-polynomial in $T_1,\ldots,T_k,q$ of degree $\deg(\widetilde{\phi})-n$. If $\deg(\phi_1) \leq n-k$, then it must be 0, by the initial remark; if not one repeats for ϕ_1 the same argument used for ϕ, up to reach such a situation. ∎

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