An Introduction to Gauge Theory and its Applications

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Preface

For almost a century, classical and quantum field theory have occupied center stage in theoretical physics, whose main object of study are the so-called gauge fields. Examples are Weyl's description of classical electromagnetism as a $U(1)$ gauge theory, Yang & Mills' model of isotopic spin and Weinberg & Salam's standard model of particle physics.

The theory of fiber bundles, connections and characteristic classes was being developed at around the same time, with little relation to the physicists' work. It was only in the early 70's that physicists and mathematicians realized that the gauge fields were best described as the curvature of a connection on some principal or vector bundle, and that characteristic classes were topological "charges", akin to the usual electric charge.

Once made, this observation led to a hurricane of activity in mathematical physics during the 70's and 80's. I believe it is no exaggeration to say that the interplay between pure mathematics and theoretical physics during those two decades have revolutionized both fields.

The goal of these notes is to provide the reader with one aspect of this revolution. We will study anti-self-dual connections on 4-dimensional Riemannian manifolds and some closely related objects in lower dimensions. More precisely, we will discuss the two most fundamental results of the subject, in my opinion: Donaldson's non-existence theorems in differential topology and the Hitchin-Kobayashi correspondence between stable holomorphic bundles and anti-self-dual connections. Rather than focusing on the (very hard) techniques involved, we will concentrate on the ideas and basic definitions, trying to give a flavor of how these results are proved.

While it is probably true that these (and other) results could have been proved without any input from physics, such input provided important motivation and inspiration. It is hard to imagine how one could even dream of these two fundamental results without the physics input.

Another important feature of these two Theorems is that they provide true examples of the unity within mathematics. To prove a result in differential topology, Donaldson uses the geometry of connections and deep analytical tools together with the more traditional methods of algebraic topology; and it is worth noting that most of the examples that have been studied actually come from algebraic geometry. The Hitchin-Kobayashi correspondence is a bridge from differential geometry to algebraic geometry through the analysis of partial differential equations.

These notes are organized as follows. In Chapter 1 we revise the basic notions of vector bundles, connections and their curvature and characteristic classes. The anti-self-duality equations are introduced in chapter 2, where we also explain the construction of the instanton moduli space as a Riemannian manifold. This manifold is then used in Chapter 3 to sketch the proof of Donaldson's non-existence theorems and the existence of fake R^4's. Chapter 4 is dedicated to complex geometry and the second main result, the Hitchin-Kobayashi correspondence. We conclude in Chapter 5 by examining the simplest dimensional reductions of the anti-self-duality equations to dimension 3 (Bogomolny equations), dimension 2 (Hitchin's equations) and dimension 1 (Nahm's equations). We focus on Hitchin's equations to emphasize how gauge theory provides a bridge between algebraic geometry and the theory of integrable systems.

Donaldson's non-existence theorems links the differential geometry of vector bundles to the topology of smooth 4-dimensional manifolds, while the Hitchin-Kobayashi correspondence links the differential geometry of vector bundles to algebraic geometry. Gauge theory has also found equally profound links and applications to other traditional fields of mathematics, like representation theory, symplectic geometry and differential geometry itself, which unfortunately are outside the scope of these notes. Moreover, the infiltration of gauge theory within other mathematical subjects is by no means over; in fact, gauge theory has evolved into a set of tools available for use in a wide variety of problems. It is our hope that these notes will motivate the reader to further explore such links, which place gauge theory as a keystone of 21$^{\text{st}}$ century mathematics.

Contents

Chapter 1

Vector bundles with connections

We start with a brief review of the central objects of study in gauge theory: vector bundles with connections. A more detailed exposition can be found at [14, 16, 17].

1.1 Vector bundles

Let X be a smooth manifold of dimension d. Recall that a rank r complex vector bundle over X is a smooth manifold E together with a smooth map $\pi : E \to X$ satisfying the following conditions:

1. π is surjective;
2. for each point $x \in X$, $E_x = \pi^{-1}(x)$ is a complex vector space of dimension r;
3. for each point $x \in X$, there is a neighborhood U of x and a smooth map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\phi_U|_{E_x} : E_x \to \mathbb{C}^r$ is an isomorphism.

The vector space E_x is called the *fiber* of E over x, while the map ϕ_U is called a *local trivialization* of E.

In other words, a complex vector bundle $E \to X$ is a family of complex vector spaces parameterized by X which is locally a product of the parameter space with a fixed complex vector space. Clearly, one can define real vector bundles in the same manner simply by substituting $\mathbb{C}^r$ for $\mathbb{R}^r$ in the definition.

Every complex vector bundle defines a set of *transition functions*. More precisely, consider the complex vector bundle $\pi : E \to X$, and let $\{U_\alpha\}$ be an open covering of X such that $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ are local trivializations. If $U_\alpha \cap U_\beta \neq \emptyset$, we can consider the maps

$$g_{\alpha\beta} = \phi_\beta \phi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^r \to U_\alpha \cap U_\beta \times \mathbb{C}^r$$

which are called the transition functions of E with respect to the covering $\{U_\alpha\}$. It is easy to see that they must satisfy the following relation:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbf{1} \quad , \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \quad .$$

Conversely, given the set of transition functions, one can reconstruct the bundle E.

Proposition 1. *Let X be a smooth manifold, let $\{U_\alpha\}$ be an open covering of X, and let $\tau_{\alpha\beta}$ be a collection of smooth functions*

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^r) \quad ,$$

satisfying

$$\tau_{\alpha\beta} \tau_{\beta\gamma} \tau_{\gamma\alpha} = \mathbf{1} \quad , \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \quad .$$

Then there exist a rank r vector bundle $E \to X$ with local trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$$

satisfying $\phi_\beta \phi_\alpha^{-1}(x, v) = (x, \tau_{\alpha\beta}(x) v)$.

The proposition above helps us to form new vector bundles out of two vector bundles E and F over the same base space X. Assume that E and F are given by transition functions $g_{\alpha\beta} \in \mathrm{GL}(\mathbb{C}^r)$ and $h_{\alpha\beta} \in \mathrm{GL}(\mathbb{C}^s)$, respectively. Then one can define the bundles:

- $E \oplus F$ is the bundle given by transition functions

$$k_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^{r+s}) \quad ;$$

- $E \otimes F$ is the bundle given by transition functions

$$k_{\alpha\beta} = g_{\alpha\beta} \otimes h_{\alpha\beta} \in \mathrm{GL}(\mathbb{C}^r \otimes \mathbb{C}^s) \quad ;$$

- $\Lambda^p E$ is the bundle given by transition functions

$$k_{\alpha\beta} = \Lambda^p g_{\alpha\beta} \in \mathrm{GL}(\Lambda^p \mathbb{C}^r) \quad ;$$

- E^* is the bundle given by transition functions

$$k_{\alpha\beta} = \overline{g_{\alpha\beta}}^{\mathrm{t}} \in \mathrm{GL}(\mathbb{C}^r) \quad ;$$

and so on. It is easy to see that $(E \oplus F)_x = E_x \oplus F_x$, $(E \otimes F)_x = E_x \otimes F_x$, etc.

The simplest example of a (real) vector bundle over X is the tangent bundle TM. The bundle of p-forms on X is defined as $\Lambda^p(TM)$.

A map between vector bundles E and F is a smooth map $f : E \to F$ such that $f(E_x) \subset F_x$ and $f|_{E_x} : E_x \to F_x$ is linear for each $x \in M$. An isomorphism $g : E \to E$ of the vector bundle E is called an automorphism or a gauge transformation of E; note that the set of gauge transformations of a given vector bundle form a group, denoted by $\mathcal{G}(E)$.

A section σ of a complex vector bundle $\pi : E \to X$ over $U \subset M$ is a smooth map $s : U \to E$ such that $\pi\sigma = \mathbf{1}_X$, i.e. $\sigma(x) \in E_x$. The section σ is said to be global if it is defined over all of M. The set of all sections of E, denoted by $\Gamma(E)$, is a complex vector space, and it has the structure of a $C^\infty(M)$-module. For instance, the space of smooth p-forms on X being its space of sections of $\Lambda^p(TM)$, i.e. $\Omega^p_X = \Gamma(\Lambda^p(TM))$.

Proposition 2. *Given two vector bundles $E, F \to M$, there exist an isomorphism of $C^\infty(M)$-modules*

$$\Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F) \quad .$$

1.2 Connections and curvature

A connection ∇ on E is a $\mathbb{C}$-linear map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1_X$ satisfying the Leibnitz rule:

$$\nabla(f\sigma) = f \cdot \nabla\sigma + \sigma \otimes df \quad \text{for} \quad f \in C^\infty(M) \quad \text{and} \quad \sigma \in \Gamma(E) \quad .$$

On a local trivialization,

Two connections ∇ and ∇' on E are said to be gauge equivalent is there exists a gauge transformation $g : E \to E$ such that $\nabla' = g^{-1}\nabla g$.

Any connection ∇ on $E \to X$ can be extended to an operator:

$$\nabla^{(p)} : \Gamma(E) \otimes \Omega^p_X \to \Gamma(E) \otimes \Omega^{p+1}_X$$

satisfying the following Leibnitz rule:

$$\nabla^{(p)}(f\sigma) = (-1)^{\deg f} f \wedge \nabla^{(p)}\sigma + df \wedge \sigma \quad .$$

Given two bundles with connection (E_1, ∇_1) and (E_2, ∇_2), one can form connections on all of the bundles derived from E_1 and E_2, as in the previous section. For instance,

$$\nabla_1 \oplus \nabla_2 : \Gamma(E_1 \oplus E_2) \to \Gamma(E_1 \oplus E_2) \otimes \Omega^1_X$$

is a connection on the direct sum bundle $E_1 \oplus E_2$, while

$$\nabla_1 \otimes \mathbf{1}_{E_2} + \mathbf{1}_{E_1} \otimes \nabla_2 : \Gamma(E_1 \otimes E_2) \to \Gamma(E_1 \otimes E_2) \otimes \Omega^1_X$$

is a connection on the tensor bundle $E_1 \otimes E_2$.

A connection ∇ on E is said to be reducible if there are bundles with connection (E_1, ∇_1) and (E_2, ∇_2) such that $(E, \nabla) \simeq (E_1 \oplus E_2, \nabla_1 \oplus \nabla_2)$. ∇ is said to be irreducible if it is not reducible.

The curvature F_∇ of the connection ∇ is defined as the composition:

$$F_\nabla : \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes \Omega^1_X) \xrightarrow{\nabla^{(1)}} \Gamma(E \otimes \Omega^2_X)$$

Notice that F_∇ is linear as a map of $C^\infty(M)$-modules; indeed:

$$\begin{aligned} F_\nabla(f\sigma) &= \nabla^{(1)}(f \cdot \nabla\sigma + \sigma \otimes df) = \nabla^{(1)}(f \cdot \nabla\sigma) + \nabla^{(1)}(df \otimes \sigma) = \\ &= f \cdot \nabla^{(1)}\nabla\sigma + df \wedge \nabla\sigma - df \wedge \nabla\sigma + d^2 f \otimes \sigma = f \cdot F_\nabla\sigma \quad . \end{aligned}$$

Thus we can think of F_∇ as a section of $\mathrm{End}(E) \otimes \Omega^2_M$, i.e. a 2-form with values in the endomorphisms of E. It satisfies the *Bianchi identity*:

$$\nabla F_\nabla = 0 \tag{1.1}$$

Finally, note that if ∇ and ∇' are gauge equivalent, then $F_{\nabla'} = g^{-1}F_\nabla g$:

$$F_{\nabla'} = \nabla'\nabla' = g^{-1}\nabla\nabla g = g^{-1}F_\nabla g \quad .$$

1.3 Chern classes

Characteristic classes are topological invariants of vector bundles. In this course, we will only need Chern classes, which we now briefly introduce. An excellent, more general approach to characteristic classes can be found at [14].

Let $(E, \nabla) \to X$ be a rank r complex vector bundle with connection over X; as above, F_∇ denotes the curvature of ∇. We define the *total Chern class* by:

$$c(E) = \det\left(\mathbf{1} + \frac{i}{2\pi} F_\nabla\right) \tag{1.2}$$

Since F_∇ is a 2-form, $c(E)$ is a sum of even degree forms

$$c(E, \nabla) = 1 + c_1(E) + c_2(E) + \cdots$$

where $c_j(E, \nabla) \in \Omega^{2j}_X$ is called the j^{th} *Chern class* of E.

Proposition 3. *Each $c_j(E, \nabla)$ is a closed $2j$-form, and the cohomology class it defines does not depend on the connection ∇.*

We will therefore denote $c_j(E) = c_j(E, \nabla)$. The Chern character of E is defined by:

$$\begin{aligned} ch(E) &= \sum_{j=0}^{[n/2]} \frac{1}{j!} \left(\frac{i}{2\pi}\right)^j \operatorname{tr}(F_\nabla^j) \\ &= \operatorname{rk}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \cdots \end{aligned}$$

Given two vector bundles E and F, the Chern character satisfies the following summation and product formulas:

$$\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F) \tag{1.3}$$

$$\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \cdot \operatorname{ch}(F) \tag{1.4}$$

Since we will only be working mostly in dimension ≤ 4, it is enough for us to keep in mind the following two formulas:

$$c_1(E) = \frac{i}{2\pi} \operatorname{tr}(F_\nabla) \tag{1.5}$$

$$c_2(E) = \frac{1}{8\pi^2} \left(-\operatorname{tr}(F_\nabla \wedge F_\nabla) + (\operatorname{tr}(F_\nabla))^2\right) \tag{1.6}$$

In particular, if E is a hermitian vector bundle then $\mathrm{tr}(F_\nabla) = 0$, and $c_1(E) = 0$.

1.4 What is Gauge Theory?

Roughly speaking, gauge theory is the study of vector bundles over manifolds, provided with connections satisfying some gauge invariant curvature condition.

For instance, the simplest such condition is that of flatness. More precisely, a connection ∇ on a vector bundle E is said to be flat if its curvature vanishes, i.e. $F_\nabla = 0$. Clearly, if ∇ and ∇' are gauge equivalent then ∇ is flat if and only if ∇' is, i.e. flatness is a gauge invariant condition. Note that if E admits a flat metric, then $c(E) = 1$ and E is topologically trivial. In other words, there are topological obstructions to the existence of solutions to the equation $F_\nabla = 0$.

As another example, let X be a closed 2-dimensional manifold and let ω denote its (normalized) volume 2-form. Using $\mathbf{1}_E : E \to E$ to denote the identity map and $\lambda \in \mathbb{R}$, the equation:

$$F_\nabla = \lambda \mathbf{1}_E \cdot \omega \tag{1.7}$$

is clearly gauge invariant. Note that if ∇ satisfies the above equation, then $c_1(E) = \lambda \cdot \mathrm{rk}(E)$; this means that (1.7) admits a solution only if $\lambda = c_1(E)/\mathrm{rk}(E)$.

In these lectures, we will focus on examples of gauge-theoretic equations over manifolds of dimension four, three and two. Higher dimensional gauge theory is also extremely interesting, though, see for instance [10].

Chapter 2

Yang-Mills equation in dimension 4

In this chapter, X will denote a closed (compact without boundary) smooth manifold of dimension 4, provided with a Riemannian metric denoted by g. Recall that the associated Hodge operator:

$$* : \Omega_X^p \to \Omega_X^{4-p}$$

satisfies $*^2 = (-1)^p$. In particular, $*$ splits Ω_X^2 into two sub-bundles $\Omega_X^{2,\pm}$ with eigenvalues ± 1:

$$\Omega_X^2 = \Omega_X^{2,+} \oplus \Omega_X^{2,-} \quad . \tag{2.1}$$

Note also that this decomposition is an orthogonal one, with respect to the inner product:

$$\langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge *\omega_2 \quad .$$

A 2-form ω is said to be self-dual if $*\omega = \omega$ and it is said to be anti-self-dual if $*\omega = -\omega$. Any 2-form ω can be written as the sum

$$\omega = \omega^+ + \omega^-$$

of if its self-dual ω^+ and anti-self-dual ω^- components.

2.1 The Yang-Mills and anti-self-duality equations

Let E be a complex vector bundle over X as above, provided with a connection ∇. As we have seen, the curvature F_∇ is a 2-form with values in $\mathrm{End}(E)$ satisfying the Bianchi identity $\nabla F_\nabla = 0$.

The *Yang-Mills equation* is:

$$\nabla * F_\nabla = 0 \tag{2.2}$$

It is a 2^{nd}-order non-linear equation on the connection ∇. Note that if F_∇ is self-dual or anti-self-dual as a 2-form, then the Yang-Mills equation is automatically satisfied:

$$*F_\nabla = \pm F_\nabla \quad \Rightarrow \quad \nabla * F_\nabla = \pm \nabla F_\nabla = 0$$

by the Bianchi identity.

An *instanton* on E is a smooth connection ∇ whose curvature F_∇ is anti-self-dual as a 2-form, i.e. it satisfies:

$$F_\nabla^+ = 0 \quad . \tag{2.3}$$

The instanton equation is still non-linear (it is linear only if E is a line bundle), but it is only 1^{st}-order on the connection. It is easy to see that it is gauge invariant; it is also *conformally invariant*: a conformal change in the metric g does not change the decomposition (2.1), so it preserves self-dual and anti-self-dual 2-forms. The topological charge k of the instanton ∇ is defined by the integral:

$$k = \frac{1}{8\pi^2} \int_X \mathrm{tr}(F_\nabla \wedge F_\nabla) \quad . \tag{2.4}$$

Note that $k = c_2(E) - \frac{1}{2} c_1(E)^2$, and it is always an integer.

If X is a smooth, non-compact, complete Riemmanian manifold, an instanton on X with be an anti-self-dual connection for which the quantity (2.4) is finite. Note that in this case, k as above need not be an integer; however it is expected to always be quantized, i.e. always a multiple of some fixed (rational) number.

Variational aspects of Yang-Mills equation. Given a fixed smooth vector bundle $E \to X$, let $\mathcal{A}(E)$ be the set of all (smooth) connections on E. The Yang-Mills functional is defined by

$$\mathrm{YM} : \mathcal{A}(E) \to \mathbb{R} \tag{2.5}$$

$$\mathrm{YM}(\nabla) = ||F_\nabla||^2_{L^2} = \int_M \mathrm{tr}(F_\nabla \wedge *F_\nabla)$$

The Euler-Lagrange equation for this functional is exactly the Yang-Mills equation (2.2). In particular, self-dual and anti-self-dual connections yield critical points of the Yang-Mills functional.

Splitting the curvature into its self-dual and anti-self-dual parts, we have

$$\mathrm{YM}(\nabla) = ||F^+_\nabla||^2_{L^2} + ||F^-_\nabla||^2_{L^2}$$

It is then easy to see that every anti-self-dual connection ∇ is an absolute minimum for the Yang-Mills functional, and that $\mathrm{YM}(\nabla)$ coincides with the topological charge (2.4) of the instanton ∇ times $8\pi^2$.

One can construct, for various 4-manifolds but most interestingly for $X = S^4$, solutions of the Yang-Mills equations which are neither self-dual nor anti-self-dual. Such solutions do not minimize (2.5). Indeed, at least for gauge group $SU(2)$ or $SU(3)$, it can be shown that there are no local minima: any critical point which is neither self-dual nor anti-self-dual is unstable and must be a “saddle point” [2].

2.2 The moduli space of instantons

Now fix a rank r complex vector bundle $E \to X$. Observe that the difference between any two connection is a linear operator:

$$(\nabla - \nabla')(f\sigma) = f\nabla\sigma + \sigma \cdot df - f\nabla'\sigma - \sigma \cdot df = f(\nabla - \nabla')\sigma \quad .$$

In other words, any two connections on E differ by an endomorphism valued 1-form. Therefore, the set of all smooth connections on E, denoted by $\mathcal{A}(E)$, has the structure of an affine space over $\Gamma(\mathrm{End}(E)) \otimes \Omega^1_M$.

The gauge group $\mathcal{G}(E)$ acts on $\mathcal{A}(E)$ via conjugation:

$$g \cdot \nabla := g^{-1} \nabla g \quad .$$

We can form the quotient set $\mathcal{B}(E) = \mathcal{A}(E)/\mathcal{G}(E)$, which is the set of gauge equivalence classes of connections on E.

The set of gauge equivalence classes of anti-self-dual connections on E is a subset of $\mathcal{B}(E)$, and it is called the *moduli space of instantons* on $E \to X$. These sets are labeled by the topological invariants of the bundle E; for the sake of simplicity, we take $c_1(E) = 0$ (i.e. E is equipped with a hermitian metric), and denote by $\mathcal{M}_X(r,k)$ the moduli space of instantons on a rank r complex vector bundle $E \to X$ with $c_1(E) = 0$ and $c_2(E) = k > 0$. The subset of $\mathcal{M}_X(r,k)$ consisting of irreducible anti-self-dual connections is denoted $\mathcal{M}^*_X(r,k)$.

It turns out that $\mathcal{M}_X(r,k)$ can be given the structure of a Hausdorff topological space. In general, $\mathcal{M}_X(r,k)$ will be singular as a differentiable manifold, but there is an open dense subset which can always be given the structure of a smooth Riemannian manifold.

We start by explaining the notion of a L^2_p vector bundle. Recall that $L^2_p(\mathbb{R}^n)$ denotes the completion of the space of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ with respect to the norm:

$$||f||^2_{L^2_p} = \int_X \left(|f|^2 + |df|^2 + \cdots |d^{(p)} f|^2 \right) \quad .$$

In dimension $n = 4$ and for $p > 2$, by virtue of the Sobolev embedding theorem, L^2_p consists of continuous functions, i.e. $L^2_p(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$. So we define the notion of a L^2_p vector bundle as a topological vector bundle whose transition functions are in L^2_p, where $p > 2$.

Now fixed a L^2_p vector bundle E over X, we can consider the metric space $\mathcal{A}_p(E)$ of all connections on E which can be represented locally on an open subset $U \subset X$ as a $L^2_p(U)$ 1-form. In this topology, $\mathcal{A}^*_p(E)$ becomes an open dense subset of $\mathcal{A}_p(E)$. Since any topological vector bundle admits a compatible smooth structure, we may regard L^2_p connections as those that differ from a smooth connection by a L^2_p 1-form. In other words, $\mathcal{A}_p(E)$ becomes an affine space modeled over the Hilbert space of L^2_p 1-forms with values in the endomorphisms of E. The curvature of a connection in $\mathcal{A}_p(E)$ then becomes a L^2_{p-1} 2-form with values in the endomorphism bundle $\mathrm{End}(E)$.

Moreover, let $\mathcal{G}_{p+1}(E)$ be defined as the topological group of all L^2_{p+1} bundle automorphisms. By virtue of the Sobolev multiplication theorem, $\mathcal{G}_{p+1}(E)$ has the structure of an infinite dimensional Lie group modeled on a Hilbert space; its Lie algebra is the space of L^2_{p+1} sections of $\mathrm{End}(E)$.

The Sobolev multiplication theorem is once again invoked to guarantee that the action $\mathcal{G}_{p+1}(E) \times \mathcal{A}_p(E) \to \mathcal{A}_p(E)$ is a smooth map of Hilbert manifolds. The quotient space $\mathcal{B}_p(E) = \mathcal{A}_p(E)/\mathcal{G}_{p+1}(E)$ inherits a topological structure; it is a metric (hence Hausdorff) topological space. Therefore, the subspace $\mathcal{M}_X(r,k)$ of $\mathcal{B}_p(E)$ is also a Hausdorff topological space; moreover, the topology of $\mathcal{M}_X(r,k)$ does not depend on p.

The quotient space $\mathcal{B}_p(E)$ fails to be a Hilbert manifold because in general the action of $\mathcal{G}_{p+1}(E)$ on $\mathcal{A}_p(E)$ is not free.

Proposition 4. *Let A be any connection on a rank r complex vector bundle E over a connected base manifold X, which is associated with a principal G-bundle. Then the isotropy group of A within the gauge group:*

$$\Gamma_A = \{g \in \mathcal{G}_{p+1}(E) \mid g(A) = A\}$$

is isomorphic to the centralizer of the holonomy group of A within G.

This means that the subspace of irreducible connections $\mathcal{A}^*_p(E)$ can be equivalently defined as the open dense subset of $\mathcal{A}_p(E)$ consisting of those connections whose isotropy group is minimal, that is:

$$\mathcal{A}^*_p(E) = \{A \in \mathcal{A}_p(E) \mid \Gamma_A = \mathrm{center}(G)\} \quad .$$

Now $\mathcal{G}_{p+1}(E)$ acts with constant isotropy on $\mathcal{A}^*_p(E)$, hence the quotient $\mathcal{B}^*_p(E) = \mathcal{A}^*_p(E)/\mathcal{G}_{p+1}(E)$ acquires the structure of a smooth Hilbert manifold.

Remark. The analysis of neighborhoods of points in $\mathcal{B}_p(E) \setminus \mathcal{B}^*_p(E)$ is very relevant for applications of the instanton moduli spaces to differential topology. The simplest situation occurs when A is an $SU(2)$-connection on a rank 2 complex vector bundle E which reduces to a pair of $U(1)$ and such $[A]$ occurs as an isolated point in $\mathcal{B}_p(E) \setminus \mathcal{B}^*_p(E)$. Then a neighborhood of $[A]$ in $\mathcal{B}_p(E)$ looks like a cone on an infinite dimensional complex projective space.

Alternatively, the instanton moduli space $\mathcal{M}_X(r,k)$ can also be described by first taking the subset of all anti-self-dual connections and then taking the quotient under the action of the gauge group. More precisely, consider the map:

$$\rho : \mathcal{A}_p(E) \to L^2_p(\mathrm{End}(E) \otimes \Omega^{2,+}_X) \tag{2.6}$$

$$\rho(A) = F^+_A$$

Thus $\rho^{-1}(0)$ is exactly the set of all anti-self-dual connections. It is $\mathcal{G}_{p+1}(E)$-invariant, so we can take the quotient to get:

$$\mathcal{M}_X(r,k) = \rho^{-1}(0)/\mathcal{G}_{p+1}(E) \quad .$$

It follows that subspace $\mathcal{M}^*_X(r,k) = \mathcal{B}^*_p(E) \cap \mathcal{M}_X(r,k)$ has the structure of a smooth Hilbert manifold. Index theory comes into play to show that $\mathcal{M}^*_X(r,k)$ is finite-dimensional. Recall that if D is an elliptic operator on a vector bundle over a compact manifold, then D is Fredholm (i.e. $\ker D$ and coker D are finite dimensional) and its index

$$\text{ind } D = \dim\ker D - \dim \text{coker } D$$

can be computed in terms of topological invariants, as prescribed by the Atiyah-Singer index theorem. The goal here is to identify the tangent space of $\mathcal{M}^*_X(r,k)$ with the kernel of an elliptic operator.

It is clear that for each $A \in \mathcal{A}_p(E)$, the tangent space $T_A\mathcal{A}_p(E)$ is just $L^2_p(\mathrm{End}(E) \otimes \Omega^1_X)$. We define the pairing:

$$\langle a, b\rangle = \int_X a \wedge *b \tag{2.7}$$

and it is easy to see that this pairing defines a Riemannian metric (so-called L^2-*metric*) on $\mathcal{A}_p(E)$.

The derivative of the map ρ in (2.6) at the point A is given by:

$$d^+_A \ : \ L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) \to L^2_{p-1}(\mathrm{End}(E) \otimes \Omega^2_X)$$

$$a \mapsto (d_Aa)^+ \quad ,$$

so that for each $A \in \rho^{-1}(0)$ we have:

$$T_A\rho^{-1}(0) = \left\{ a \in L^2_p(\mathrm{End}(E)) \otimes \Omega^1_X \mid d^+_A a = 0 \right\} \quad .$$

Now for a gauge equivalence class $[A] \in \mathcal{B}_p^*(E)$, the tangent space $T_{[A]}\mathcal{B}_p^*(E)$ consists of those 1-forms which are orthogonal to the fibers of the principal $\mathcal{G}_{p+1}(E)$ bundle $\mathcal{A}_p^*(E) \to \mathcal{B}_p^*(E)$. At a point $A \in \mathcal{A}_p(E)$, the derivative of the action by some $g \in \mathcal{G}_{p+1}(E)$ is

$$-d_A \ : \ L^2_{p+1}(\mathrm{End}(E)) \to L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) \ \ .$$

Usual Hodge decomposition gives us that there is an orthogonal decomposition:

$$L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) = \mathrm{im}\ d_A \oplus \ker d_A^* \ \ ,$$

which means that:

$$T_{[A]}\mathcal{B}_p^*(E) = \left\{ a \in L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) \mid d_A^* a = 0 \right\} \ \ .$$

Thus the pairing (2.7) also defines a Riemannian metric on $\mathcal{B}_p^*(E)$.

Putting these together, we conclude that the space $T_{[A]}\mathcal{M}_X^*(r,k)$ tangent to $\mathcal{M}_X^*(r,k)$ at an equivalence class $[A]$ of anti-self-dual connections can be described as follows:

$$T_{[A]}\mathcal{M}_X^*(r,k) = \left\{ a \in L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) \mid d_A^* a = d_A^+ a = 0 \right\} \qquad (2.8)$$

It turns out that the so-called *deformation operator* $\delta_A = d_A^* \oplus d_A$:

$$\delta_A \ : \ L^2_p(\mathrm{End}(E) \otimes \Omega^1_X) \to L^2_{p+1}(\mathrm{End}(E)) \oplus L^2_{p-1}(\mathrm{End}(E) \otimes \Omega^2_X)$$

is elliptic. Moreover, if A is anti-self-dual then coker δ_A is empty, so that $T_{[A]}\mathcal{M}_X^*(r,k) = \ker \delta_A$. The dimension of the tangent space $T_{[A]}\mathcal{M}_X^*(r,k)$ is then simply given by the index of the deformation operator δ_A. Using the Atiyah-Singer index theorem, we have for $SU(r)$-bundles with $c_2(E) = k$:

$$\dim \mathcal{M}_X^*(r,k) = 4rk - (r^2-1)(1 - b_1(X) + b_+(X)) \ \ .$$

It is interesting to note that $\mathcal{M}_X^*(r,k)$ inherits many of the geometrical properties of the original manifold X. Most notably, if X is a Kähler manifold, then $\mathcal{M}_X^*(r,k)$ is also Kähler; if X is a hyperkähler manifold, then $\mathcal{M}_X^*(r,k)$ is also hyperkähler. Other geometric structures on X can also be transfered to the instanton moduli spaces $\mathcal{M}_X^*(r,k)$.

2.3 Instantons on Euclidean space

Let $X = \mathbb{R}^4$ with the flat Euclidean metric, and consider a hermitian vector bundle $E \to \mathbb{R}^4$. Any connection ∇ on E is of the form $d + A$, where $A \in \Gamma(\mathrm{End}(E)) \otimes \Omega^1_{\mathbb{R}^4}$ is a 1-form with values in the endomorphisms of E; this can be written as follows:

$$A = \sum_{k=1}^{4} A_k dx^k \quad , \quad A_k : \mathbb{R}^4 \to \mathfrak{u}(r) \quad .$$

In the Euclidean coordinates x_1, x_2, x_3, x_4, the anti-self-duality equation (2.3) is given by:

$$F_{12} = F_{34} \quad , \quad F_{13} = -F_{24} \quad , \quad F_{14} = F_{23}$$

where

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] \quad .$$

The simplest explicit solution is the charge 1 $SU(2)$-instanton on $\mathbb{R}^4$. The connection 1-form is given by:

$$A_0 = \frac{1}{1 + |x|^2} \cdot \mathrm{Im}(q d\overline{q}) \tag{2.9}$$

where q is the quaternion $q = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, while Im denotes the imaginary part of the product quaternion; we are regarding $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as a basis of the Lie algebra $\mathfrak{su}(2)$. From this, one can compute the curvature:

$$F_{A_0} = \left(\frac{1}{1 + |x|^2} \right)^2 \cdot \mathrm{Im}(dq \wedge d\overline{q}) \tag{2.10}$$

We see that the action density function

$$|F_A|^2 = \left(\frac{1}{1 + |x|^2} \right)^2$$

has a bell-shaped profile centered at the origin and decaying like r^{-4}.

Let $t_{\lambda,y} : \mathbb{R}^4 \to \mathbb{R}^4$ be the isometry given by the composition of a translation by $y \in \mathbb{R}^4$ with a homotety by $\lambda \in \mathbb{R}^+$. The pullback connection $t^*_{\lambda,y} A_0$ is still anti-self-dual; more explicitly:

$$A_{\lambda,y} = t^*_{\lambda,y} A_0 = \frac{\lambda^2}{\lambda^2 + |x - y|^2} \cdot \mathrm{Im}(q d\overline{q})$$

$$\text{and} \quad F_{A_{\lambda,y}} = \left(\frac{\lambda^2}{\lambda^2+|x-y|^2}\right)^2 \cdot \mathrm{Im}(dq \wedge d\overline{q}) \quad .$$

Note that the action density function $|F_A|^2$ has again a bell-shaped profile centered at y and decaying like r^{-4}; the parameter λ measures the concentration (or size) of instanton A.

Instantons of topological charge k can be obtained by "superimposing" k basic instantons, via the so-called *'t Hooft ansatz*. Consider the function $\rho : \mathbb{R}^4 \to \mathbb{R}$ given by:

$$\rho(x) = 1 + \sum_{j=1}^{k} \frac{\lambda_j^2}{(x-y_j)^2} \quad ,$$

where $\lambda_j \in \mathbb{R}$ and $y_j \in \mathbb{R}^4$. Then the connection $A = A_\mu dx_\mu$ with coefficients

$$A_\mu = i \sum_{\nu=1}^{4} \overline{\sigma}_{\mu\nu} \frac{\partial}{\partial x_\nu} \ln(\rho(x)) \tag{2.11}$$

are anti-self dual; here, $\overline{\sigma}_{\mu\nu}$ are the matrices given by $(\mu, \nu = 1, 2, 3)$:

$$\overline{\sigma}_{\mu\nu} = \frac{1}{4i}[\sigma_\mu, \sigma_\nu] \quad \overline{\sigma}_{\mu 4} = \frac{1}{2}\sigma_\mu$$

where σ_μ are the Pauli matrices.

The connection (2.11) correspond to k instantons centered at points y_i with size λ_i. The basic instanton (2.9) is exactly (modulo gauge transformation) what one obtains from (2.11) for the case $k = 1$.

$SU(2)$-instantons are also the building blocks for instantons with general structure group [1]. Let G be a compact semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Let $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ be any injective Lie algebra homomorphism. Then it is easy to see that:

$$\rho(A_0) = \frac{1}{1+|x|^2} \cdot \rho(\mathrm{Im}(q d\overline{q})) \tag{2.12}$$

is indeed a G-instanton on $\mathbb{R}^4$. while this guarantees the existence of G-instantons on $\mathbb{R}^4$, note that this instanton might be reducible (e.g. ρ can simply be the obvious inclusion of $\mathfrak{su}(2)$ into $\mathfrak{su}(n)$ for any n) and that its charge depends on the choice of representation ρ.

Furthermore, it is not clear whether every G-instanton can be obtained in this way, as the inclusion of a $SU(2)$-instanton through some representation $\rho : \mathfrak{su}(2) \to \mathfrak{g}$.

The ADHM construction All $SU(r)$-instantons on $\mathbb{R}^4$ can be obtained through a remarkable construction due to Atiyah, Drinfeld, Hitchin and Manin. It starts by considering hermitian vector spaces V and W of dimension c and r, respectively, and the following data:

$$B_1, B_2 \in \mathrm{End}(V) \quad , \quad i \in \mathrm{Hom}(W,V) \quad , \quad j \in \mathrm{Hom}(V,W) \quad ,$$

so-called ADHM data. Assume moreover that (B_1, B_2, i, j) satisfy the *ADHM equations*:

$$[B_1, B_2] + ij \quad = \quad 0 \tag{2.13}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j \quad = \quad 0 \tag{2.14}$$

Now consider the following maps

$$\alpha : V \times \mathbb{R}^4 \to (V \oplus V \oplus W) \times \mathbb{R}^4$$

$$\beta : (V \oplus V \oplus W) \times \mathbb{R}^4 \to V \times \mathbb{R}^4$$

given as follows:

$$\alpha(z_1, z_2) = \begin{pmatrix} B_1 + z_1 \\ B_2 + z_2 \\ j \end{pmatrix} \tag{2.15}$$

$$\beta(z_1, z_2) = \begin{pmatrix} -B_2 - z_2\mathbf{1} & B_1 + z_1\mathbf{1} & i \end{pmatrix} \tag{2.16}$$

where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ are complex coordinates on $\mathbb{R}^4$. The maps (2.15) and (2.16) should be understood as a family of linear maps parametrized by points in $\mathbb{R}^4$.

A straightforward calculation shows that the ADHM equations imply that $\beta\alpha = 0$ for every $(z_1, z_2) \in \mathbb{R}^4$. Therefore the quotient $E = \ker\beta/\mathrm{im}\alpha = \ker\beta \cap \ker\alpha^\dagger$ forms a complex vector bundle over $\mathbb{R}^4$ or rank r whenever (B_1, B_2, i, j) is such that α is injective and β is surjective for every $(z_1, z_2) \in \mathbb{R}^4$.

To define a connection on E, note that E can be regarded as a sub-bundle of the trivial bundle $(V \oplus V \oplus W) \times \mathbb{R}^4$. So let $\iota : E \to$

$(V\oplus V\oplus W)\times\mathbb{R}^4$ be the inclusion, and let $P:(V\oplus V\oplus W)\times\mathbb{R}^4\to E$ be the orthogonal projection onto E. We can then define a connection ∇ on E through the *projection formula*

$$\nabla s = P\underline{d}\iota(s)$$

where $\underline{d}$ denotes the trivial connection on the trivial bundle $(V\oplus V\oplus W)\times\mathbb{R}^4$.

To see that this connection is anti-self-dual, note that projection P can be written as follows

$$P = \mathbf{1} - \mathcal{D}^{\dagger}\Xi^{-1}\mathcal{D}$$

where

$$\mathcal{D} : (V\oplus V\oplus W)\times\mathbb{R}^4 \to (V\oplus V)\times\mathbb{R}^4$$
$$\mathcal{D} = \begin{pmatrix} \beta \\ \alpha^{\dagger} \end{pmatrix}$$

and $\Xi = \mathcal{D}\mathcal{D}^{\dagger}$. Note that $\mathcal{D}$ is surjective, so that Ξ is indeed invertible. Moreover, it also follows from (2.14) that $\beta\beta^{\dagger} = \alpha^{\dagger}\alpha$, so that $\Xi^{-1} = (\beta\beta^{\dagger})^{-1}\mathbf{1}$.

The curvature F_{∇} is given by:

$$\begin{aligned} F_{\nabla} &= P\left(\underline{d}(\mathbf{1}-\mathcal{D}^{\dagger}_{\mathrm{I}}\Xi^{-1}\mathcal{D})\underline{d}\right) = P\left(\underline{d}\mathcal{D}^{\dagger}\Xi^{-1}(\underline{d}\mathcal{D})\right) = \\ &= P\left((\underline{d}\mathcal{D}^{\dagger}\Xi^{-1})(\underline{d}\mathcal{D}) + \mathcal{D}^{\dagger}\underline{d}(\Xi^{-1}(\underline{d}\mathcal{D})\right) = \\ &= P\left((\underline{d}\mathcal{D}^{\dagger})\Xi^{-1}(\underline{d}\mathcal{D})\right) \end{aligned}$$

for $P\left(\mathcal{D}^{\dagger}\underline{d}(\Xi^{-1}(\underline{d}\mathcal{D})\right) = 0$ on $E = \ker\mathcal{D}$. Since Ξ^{-1} is diagonal, we conclude that F_{∇} is proportional to $d\mathcal{D}^{\dagger}\wedge d\mathcal{D}$, as a 2-form.

It is then a straightforward calculation to show that each entry of $d\mathcal{D}^{\dagger}\wedge d\mathcal{D}$ belongs to $\Omega^{2,-}$.

The extraordinary accomplishment of Atiyah, Drinfeld, Hitchin and Manin was to show that every instanton can be obtained in this way; see e.g. [3].

Chapter 3

Topology of smooth 4-manifolds

The magic of gauge theory resides on the fact that the geometry and topology of the moduli spaces $\mathcal{M}_X(r,k)$ capture a lot of the geometry and topology of the original manifold X.

Since $\mathcal{M}_X(r,k)$ as a whole is a topological invariant of X, one can use the classical topological invariants of $\mathcal{M}_X(r,k)$ (intersection form, Euler characteristic, etc) to define interesting nontrivial invariants of X. The paramount example of that are Donaldson's theorems on the existence and non-existence of smooth 4-manifolds with certain intersection forms: one assumes the existence of one of the manifold in question and finds that its instanton moduli space has impossible topological properties. It also leads to the existence of manifolds which are homeomorphic but not diffeomorphic to the usual Euclidean 4-space.

3.1 Donaldson's theorems

Let X be a closed (i.e. compact with no boundary) oriented 4-manifold, and let $V = H^2(X,\mathbb{Z})$. The cup product defines the following symmetric bilinear form on the $\mathbb{Z}$-module V:

$$\langle \alpha, \beta \rangle = (\alpha \cup \beta)[X] \quad ,$$

where $[X] \in H_4(X, \mathbb{Z})$ denotes the fundamental class. Moreover, this so-called *intersection form* $\langle \cdot, \cdot \rangle$ is *unimodular*, i.e. its determinant is 1.

The relevance of unimodular, symmetric bilinear forms is provided by a classical 1949 result of Whitehead: simply-connected, closed oriented 4-manifolds are classified, up to orientation-preserving homotopy equivalence, by their intersection forms (see [13], which also contains a classification of symmetric bilinear forms on $\mathbb{Z}$-modules).

This result immediately poses an existence question: which unimodular, symmetric bilinear forms on a $\mathbb{Z}$-module can arise as the intersection form of a simply-connected, closed oriented 4-manifolds? More interestingly, this question can be asked both in the topological and in the differential categories; it turns out, rather surprisingly, that the answers are quite different.

In the topological category, the answer was provided by Freedman in 1982: every unimodular symmetric bilinear form is the intersection form of a simply-connected, closed oriented 4-manifolds [4].

In contrast, it has been known since 1952 that there are some forms which cannot be realized as the intersection form of a *smooth* simply-connected, closed oriented 4-manifold. Indeed, a theorem due to Rokhlin asserts that if X is smooth simply-connected, closed oriented 4-manifold, whose intersection form is even, i.e. it satisfies:

$$\langle \alpha, \alpha \rangle = 0 \bmod 2 \quad \forall \alpha \in V$$

then its signature (the number of positive eigenvalues minus the number of negative eigenvalues) is congruent to $0 \bmod 16$. This rules out for instance the positive definite bilinear form induced by the following matrix:

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

which has signature 8.

Little progress was made beyond Rokhlin's theorem, until the following results due to Donaldson [3]. Recall that a bilinear for is said to be negative definite if

$$\langle \alpha, \alpha \rangle < 0 \quad \forall \alpha \neq 0 \in V$$

Theorem 5. Donaldson [3]. *The only negative definite forms realized as the intersection forms of smooth, simply-connected, closed oriented 4-manifolds are the standard diagonalizable forms $n(-1)$.*

Now let H denote the matrix:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ,$$

which is the intersection form for $S^2 \times S^2$.

Theorem 6. Donaldson [3]. *Assume that the form $n(-E_8) \oplus mH$ is realized by a smooth, simply-connected, closed oriented 4-manifold. If $n > 0$, then $m \geq 3$.*

In other words, the forms $2n(-E_8)$ are not realized, something that is not ruled out by Rokhlin's theorem.

The overall strategy for the proof of both non-existence theorems is the same. Supposing that a manifold X of the type in question does exist, one carefully chooses a vector bundle E over X and studies the moduli space of instantons $\mathcal{M}_X(E)$ on the bundle E. From the known topological features of $\mathcal{M}_X(E)$ and its embedding into $\mathcal{B}(\mathcal{E})$, the space of gauge equivalence classes of connections on E, one derives a contradiction.

For instance, we sketch an argument due to Fintushel and Stern which shows that there is no smooth, simply-connected, closed oriented 4-manifold X whose intersection form is $-E_8 \oplus -E_8$ [3, Theorem 8.1.1].

Suppose such a manifold exists, and let $e \in H^2(X, \mathbb{Z})$ be such that $e^2 = -2$. Let L be a complex line bundle over X with $c_1(L) = e$, and let $E = L \oplus \underline{\mathbb{R}}$, which is an $SO(3)$-bundle. Fixing a generic Riemannian metric on X, the general theory outlined in the previous chapter tells us that the moduli space $\mathcal{M}_X(E)$ of irreducible instantons on $E \to X$ is a compact 1-dimensional manifold with boundary. However, there can be one and only one reducible $U(1)$ connection on E; since reducible connection correspond to boundary points, $\partial\mathcal{M}_X(E)$ would consist of a single point, which is impossible.

3.2 Exotic $\mathbb{R}^4$'s

Finally, the results of Freedman and Donaldson may be combined to obtain the following remarkable result:

Theorem 7. *([5]) The exists infinitely many differentiable manifolds which are homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$.*

This phenomenon is really particular to dimension 4: there are no "exotic" $\mathbb{R}^n$'s for $n \neq 4$!

Let us sketch the simplest construction of an exotic $\mathbb{R}^4$, following Gompf [7]; in the process, we will also show the existence of an exotic $S^3 \times \mathbb{R}$. Before that, we must collect a few topological facts.

Recall that the *K3 surface* is the 4-dimensional smooth submanifold of $\mathbb{C}P^3$ defined by the equation $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$ ($[z_1 : z_2 : z_3 : z_4]$ are homogeneous coordinates in $\mathbb{CP}^3$); its intersection form is $2(-E_8) \oplus 3H$. Notice also that $3H$ is the intersection form of the compact manifold with boundary $X = \#_3(S^2 \times S^2) \setminus (\text{open } 4 - \text{ball})$.

The following statement is a particular case of Freedman's topological surgery.

Theorem 8. Freedman [4]. *There are embeddings $\imath : X \to K3$ and $\jmath : X \to \#_3(S^2 \times S^2)$ and diffeomorphic neighborhoods U of $\imath(X)$ and V of $\jmath(X)$ such that the following diagram commutes:*

$$\begin{tikzcd} & U \subset K3 \arrow[dd, "\text{diffeo}"] \\ X \arrow[ur, "\imath"] \arrow[dr, "\jmath"'] & \\ & V \subset \#_3(S^2 \times S^2) \end{tikzcd}$$

In particular, $U \setminus \imath(X)$ is diffeomorphic to $V \setminus \jmath(X)$; since the boundary of X is a 3-sphere, we can assume that $U \setminus \imath(X)$ is homeomorphic to $S^3 \times \mathbb{R}$.

Notice that $K3 \setminus \imath(X)$ is an open manifold collared by $U \setminus \imath(X)$, which is homeomorphic to $S^3 \times \mathbb{R}$. Thus we may topologically glue 4-ball B^4 on $K3 \setminus \imath(X)$ to obtain a closed 4-manifold with intersection form $-E_8 \oplus -E_8$. Remark that, by Theorem 6, this cannot be done smoothly, i.e. $(K3 \setminus \imath(X)) \cup B^4$ is not a smooth manifold.

Hence $U \setminus \imath(X)$ cannot be diffeomorphic to $S^3 \times \mathbb{R}$, i.e. it is an exotic $S^3 \times \mathbb{R}$. This is because it cannot contain any smooth S^3 separating its two ends; if it did, we could trim off the end of $K \setminus \imath(X)$ by cutting along such a 3-sphere, and then cap the new boundary smoothly with a 4-ball, obtaining an impossible smooth 4-manifold.

The last ingredient is the following topological characterization of the standard $\mathbb{R}^4$.

Theorem 9. Freedman [4]. *Any open 4-manifold X with $\pi_1(X) = H_2(X) = 0$ and whose end is topologically collared by $S^3 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.*

We are finally ready to exhibit our exotic $\mathbb{R}^4$. Take the embedding $\jmath : X \to \#_3(S^2 \times S^2)$ from Theorem 8 and let $F = \#_3(S^2 \times S^2) \setminus \jmath(X)$. It is not difficult to check that such F satisfies the hypothesis of Theorem 9, so that F is homeomorphic to $\mathbb{R}^4$.

Let us now argue that F is not diffeomorphic to the standard $\mathbb{R}^4$. Note that F is collared by $V \setminus \jmath(X)$, which is diffeomorphic to $U \setminus \imath(X)$, the exotic $S^3 \times \mathbb{R}$ constructed above which contains no smooth S^3 separating its ends. This means that there can be no smooth 3-sphere near the end of F.

Therefore, there exists a compact subset $C = F \setminus (V \setminus \jmath(X))$ in F which is not enclosed by any smooth 3-sphere. Such behavior is clearly impossible in the standard $\mathbb{R}^4$, hence F cannot be diffeomorphic to it.

In [7], Gompf modifies Theorem 8 to obtain, via a similar argument, two other examples of exotic $\mathbb{R}^4$'s embedded in $\mathbb{CP}^2$, rather than $\#_3(S^2 \times S^2)$.

Chapter 4

The connection with algebraic geometry

The second main result to be presented in this lectures is the Hitchin-Kobayashi correspondence. It says that the analytical problem of finding solutions to the anti-self-duality equation on a complex manifold is equivalent to the algebraic problem of finding holomorphic vector bundles satisfying a purely algebraic problem.

4.1 Stable holomorphic bundles

Let X be a compact Kähler surface, i.e. a complex manifold of complex dimension 2 equipped with a hermitian metric g such that the associated (1,1)-form:

$$\kappa = \frac{i}{2} \sum g_{ab} dz_a \wedge d\overline{z_b}$$

is closed; κ is called a *Kähler form*. More invariantly, we have for tangent vectors $v, w \in T_pX$:

$$\kappa(v, w) = g(v, Iw)$$

where I is the complex structure $I : TX \to TX$, $I^2 = -\mathbf{1}$.

Locally, we may take the hermitian metric g to be diagonalized; on the model space $\mathbb{C}^2$ with complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, we have that:

$$\begin{aligned} \kappa &= \frac{i}{2} dz_1 \wedge d\overline{z_1} + i dz_2 \wedge d\overline{z_2} = \\ &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \end{aligned} \tag{4.1}$$

A complex vector bundle $E \to X$ is said to be holomorphic if its transition functions $g_{\alpha\beta}$ are holomorphic. Alternatively, a holomorphic bundle $\mathcal{E} \to X$ is a pair $(E, \overline{\partial}_E)$ consisting of a complex vector bundle $E \to X$ and an operator

$$\overline{\partial}_E : \Gamma(E \otimes \Omega_X^{0,q}) \to \Gamma(E \otimes \Omega_X^{0,q+1})$$

satisfying the following conditions:

- $\overline{\partial}_E(f\sigma) = (\overline{\partial} f)\sigma + f(\overline{\partial}_E \sigma)$, where $f \in C^\infty(X)$ and $\sigma \in \Gamma(E \otimes \Omega_X^{0,q})$;
- $\overline{\partial}_E \sigma = 0$ on an open subset $U \subset X$ if and only if $\sigma|_U$ is holomorphic.

Let us now recall the notion of *stability* for holomorphic vector bundles over Kähler surfaces. Let $\mathcal{E}$ be a holomorphic vector bundle on X. The *degree* of $\mathcal{E}$ with respect to the Kähler form κ is defined by

$$\deg_\kappa(\mathcal{E}) = \int_X \kappa \wedge c_1(\mathcal{E}) \quad .$$

The *slope* of $\mathcal{E}$ is the rank-normalized degree, i.e.:

$$\mu_\kappa(\mathcal{E}) = \frac{\deg_\kappa(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} \quad .$$

The bundle $\mathcal{E}$ is said to be *stable* with respect to κ if every holomorphic sub-bundle $\mathcal{F} \hookrightarrow \mathcal{E}$ satisfies:

$$\mu_\kappa(\mathcal{F}) < \mu_\kappa(\mathcal{E}) \quad .$$

Moreover, the bundle $\mathcal{E}$ is said to be *semistable* with respect to κ if every holomorphic sub-bundle $\mathcal{F} \hookrightarrow \mathcal{E}$ satisfies:

$$\mu_\kappa(\mathcal{F}) \leq \mu_\kappa(\mathcal{E}) \quad .$$

In this context, the meaning of the word sub-bundle must properly understood, because in slightly more general than the usual meaning in differential geometry. We say that $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$ if there is a bundle map $\psi : \mathcal{F} \to \mathcal{E}$ such that $\psi(x)$ is injective for all but from finitely many points $x \in X$. In algebraic geometric terms, $\mathcal{F}$ is a sub-bundle of $\mathcal{E}$ if the sheaf of sections of $\mathcal{F}$ is a subsheaf of the sheaf of section of $\mathcal{E}$ such that the quotient sheaf $\mathcal{E}/\mathcal{F}$ is torsion-free.

Stable bundles on algebraic surfaces are of great interest within algebraic geometry. As we will see below, they sit at the crossroads of algebraic and differential geometry.

Let us know examine the link between stable bundles and anti-self-dual connections, known as the Hitchin-Kobayashi correspondence.

4.2 The Hitchin-Kobayashi correspondence

Let X be a compact Kähler surface as above, and let g denote the hermitian metric, while κ denotes the associated Kähler form. From (4.1) we see that κ is self-dual. Moreover:

$$dz_1 \wedge dz_2 = (dx_1 \wedge dx_3 - dx_2 \wedge dx_4) + i(dx_2 \wedge dx_3 + dx_1 \wedge dx_4)$$

Thus the real and imaginary parts of $dz_1 \wedge dz_2$ (and of $d\overline{z_1} \wedge d\overline{z_2}$) are also self-dual. Since the decomposition

$$\Omega^2_X = \Omega^{2,+}_X \oplus \Omega^{2,-}_X$$

is an orthogonal one, we conclude that:

$$\Omega^{2,+}_X = \Omega^{2,0}_X \oplus \Omega^0_X \cdot \kappa \oplus \Omega^{0,2}_X \quad ,$$

$$\text{and} \quad \Omega^{2,-}_X = (\Omega^0_X \cdot \kappa)^\perp$$

Here, $(\Omega^0_X \cdot \kappa)^\perp \subset \Omega^{1,1}_X$ consists of the $(1,1)$-forms orthogonal to the Kähler form κ. In other words, a 2-form ω is anti-self-dual if and only if ω is of type $(1,1)$ and $\langle \omega, \kappa \rangle = \omega \wedge \kappa = 0$.

Let E be a complex vector bundle over X. We begin by noticing that any connection ∇_A on E whose curvature F_A is of type $(1,1)$ induces a holomorphic structure $\overline{\partial}_A$ on E. Indeed, the covariant derivative

$$\nabla_A : \Gamma(E) \otimes \Omega^d_X \to \Gamma(E) \otimes \Omega^{d+1}_X$$

splits as a sum of operators

$$\overline{\partial}_A : \Gamma(E) \otimes \Omega_X^{p,q} \to \Gamma(E) \otimes \Omega_X^{p+1,q}$$

and

$$\overline{\partial}_A : \Gamma(E) \otimes \Omega_X^{p,q} \to \Gamma(E) \otimes \Omega_X^{p,q+1} \quad .$$

With respect to the decomposition:

$$\Omega_X^2 = \Omega_X^{2,0} \oplus \Omega_X^{1,1} \oplus \Omega_X^{0,2}$$

the curvature operator $F_A = \nabla_A \nabla_A$ is then written as:

$$F_A = (\partial_A)^2 + (\partial_A \overline{\partial}_A + \overline{\partial}_A \partial_A) + (\overline{\partial}_A)^2 \quad .$$

Thus if F_A is of type $(1,1)$, then $(\overline{\partial}_A)^2 = 0$ as desired.

This means, in particular, that anti-self-dual connections induce holomorphic structures because they are of type (1,1). Even more, the holomorphic structure induced by *anti-self-dual* connections are very special.

Theorem 10. *(*Hitchin-Kobayashi correspondence*) Let E be a hermitian vector bundle over a Kähler surface X. There is a 1-1 correspondence between the following objects:*

- *$SU(n)$-gauge equivalence classes of irreducible anti-self-dual connections on E;*
- *$SL(n,\mathbb{C})$-gauge equivalence classes of stable holomorphic structures on E.*

One side of this correspondence is relatively easy to demonstrate. Let ∇_A be an irreducible anti-self-dual connection on the hermitian vector bundle $E \to X$, and consider the associated holomorphic structure $\overline{\partial}_A$ on E; denote by $\mathcal{E}$ the holomorphic bundle $(E, \overline{\partial}_A)$. Since $c_1(E) = 0$ (E is hermitian), we have that $\mu(\mathcal{E}) = 0$. Thus the goal is to show that every holomorphic sub-bundle $\mathcal{F} \hookrightarrow \mathcal{E}$ must have $\deg_\kappa(\mathcal{F}) < 0$.

First, if $\mathrm{rk}(\mathcal{F}) = p$ note that $c_1(\mathcal{F}) = c_1(\bigwedge^p \mathcal{F})$, hence $\deg_\kappa(\mathcal{F}) = \deg_\kappa(\bigwedge^p \mathcal{F})$. The map $\mathcal{F} \hookrightarrow \mathcal{E}$ induces a map $\bigwedge^p \mathcal{F} \to \bigwedge^p \mathcal{E}$, which may be regarded as a section of the bundles $\bigwedge^p \mathcal{E} \otimes \bigwedge^p \mathcal{F}^*$.

Proposition 11. *([3, page 212]) Let $\mathcal{L} \to X$ be a holomorphic line bundle over a Kähler surface X; let κ denote its Kähler form and ν its volume. Then there is a compatible connection ∇_B on $\mathcal{L}$ so that:*

$$F_B \wedge \kappa = \frac{-2\pi}{\nu} \deg_\kappa(\mathcal{L}) \cdot \kappa^2 \quad .$$

As a holomorphic line bundle, we can find a connection ∇_B on $\bigwedge^p \mathcal{F}^*$ whose curvature is

$$F_B \wedge \kappa = \frac{2\pi}{\nu} \deg_\kappa(\mathcal{F}) \cdot \kappa^2$$

since $c_1(\bigwedge^p \mathcal{F}^*) = -c_1(\mathcal{F})$.

The anti-self-dual connection ∇_A on E naturally induces a connection $\nabla_A^{(p)}$ on $\bigwedge^p \mathcal{E}$, which is also anti-self-dual, hence $F_{A^{(p)}} \wedge \kappa = 0$. Tensoring $\nabla_A^{(p)}$ with ∇_B, we get a connection

$$\nabla_{\tilde{A}} = \nabla_A^{(p)} \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_B$$

on the tensor bundle $\bigwedge^p \mathcal{E} \otimes \bigwedge^p \mathcal{F}^*$; it follows that

$$F_{\tilde{A}} \wedge \kappa = F_{A^{(p)}} \wedge \kappa + F_B \wedge \kappa = \frac{2\pi}{\nu} \deg_\kappa(\mathcal{F}) \cdot \kappa^2 \quad . \tag{4.2}$$

Proposition 12. *Let $(V, \overline{\partial}_B) \to X$ be an holomorphic vector bundle over a compact Kähler surface. If $F_B \wedge \kappa > 0$, then V has no non-trivial holomorphic sections. If $F_B \wedge \kappa \geq 0$, then any holomorphic sections is covariantly constant (i.e. $\nabla_B \sigma = 0$).*

Together with (4.2), we conclude that if $\deg_\kappa(\mathcal{F}) > 0$ then $\bigwedge^p \mathcal{E} \otimes \bigwedge^p \mathcal{F}^*$ has no sections, which contradicts the fact that $\mathcal{F}$ is a subbundle of $\mathcal{E}$. If $\deg_\kappa(\mathcal{F}) = 0$, then $\mathcal{E}$ splits as a sum $\mathcal{F} \oplus \mathcal{F}'$, which contradicts the assumption that the original instanton connection A is irreducible. Thus the only possibility left is $\deg_\kappa(\mathcal{F}) > 0$, thus $\mathcal{E}$ is stable. That completes the proof of the easy part of the correspondence.

The other side of the correspondence is much harder, and it amounts to an existence result: given a stable holomorphic bundle $\mathcal{E} = (E, \overline{\partial}_E)$, one can find an unique compatible connection ∇_A on $\mathcal{E}$ (i.e. $\overline{\partial}_A = \overline{\partial}_E$) which is anti-self-dual. The proof of such statement involves hard analysis [3, 10].

Chapter 5

Hitchin's equations and integrable systems

Now let $X = \mathbb{R}^4$ provided with the flat Euclidean metric. A connection on a hermitian vector bundle over $\mathbb{R}^4$ of rank r can be regarded as 1-form

$$A = \sum_{k=1}^{4} A_k(x_1, \cdots, x_4) dx^k \quad , \quad A_k : \mathbb{R}^4 \to \mathfrak{u}(r)$$

Assuming that the connection components A_k are invariant under translation in one direction, say x_4, we can think of

$$\underline{A} = \sum_{k=1}^{3} A_k(x_1, x_2, x_3) dx^k$$

as a connection on a hermitian vector bundle over $\mathbb{R}^3$, with the fourth component $\phi = A_4$ being regarded as a bundle endomorphism $\phi : E \to E$, called a Higgs field. In this way, the anti-self-duality equation (2.3) reduces to the so-called **Bogomolny (or monopole) equation**:

$$F_{\underline{A}} = * d\phi \tag{5.1}$$

where $*$ is the Euclidean Hodge star in dimension 3.

Now assume that the connection components A_k are invariant under translation in two directions, say x_3 and x_4. Consider

$$\underline{A} = \sum_{k=1}^{2} A_k(x_1, x_2) dx^k$$

as a connection on a hermitian vector bundle over $\mathbb{R}^2$, with the third and fourth components combined into a complex bundle endomorphism:

$$\Phi = (A_3 + i \cdot A_4)(dx_1 - i \cdot dx_2)$$

taking values on 1-forms. The anti-self-duality equation (2.3) is then reduced to the so-called **Hitchin's equations**:

$$\begin{cases} F_{\underline{A}} = [\Phi, \Phi^*] \\ \overline{\partial}_{\underline{A}} \Phi = 0 \end{cases} \tag{5.2}$$

which were introduced by Hitchin in [8]. Conformal invariance of the anti-self-duality equation means that Hitchin's equations are well-defined over any Riemann surface.

Finally, assume that the connection components A_k are invariant under translation in three directions, say x_2, x_3 and x_4. After gauging away the first component A_1, the anti-self-duality equations (2.3) reduce to the so-called Nahm's equations:

$$\frac{dT_k}{dx_1} + \frac{1}{2} \sum_{j,l} \epsilon_{kjl} [T_j, T_l] = 0 \ , \quad j, k, l = \{2, 3, 4\} \tag{5.3}$$

where each T_k is regarded as a map $\mathbb{R} \to \mathfrak{u}(r)$.

We will now focus on Hitchin's equations, which have an extremely interesting relation with algebraic geometry and the theory of integrable systems. Those interested in monopoles and Nahm's equations are referred to [15] and the references therein. An interesting application of Nahm's equations is the existence of hyperkähler metrics on coadjoint orbits of complex semi-simple Lie groups, first established by Kronheimer [11].

It is also worth mentioning the beautiful book by Mason & Woodhouse [12], where other interesting reductions of the anti-self-duality equations are discussed, providing a deep relation between instantons and the general theory of integrable systems.

5.1 Hitchin's equations

Hitchin's equations (5.2) where first introduced in [8, 9], and have sparked a lot of interest from the algebraic-geometric point of view because of its relation with the theory of integrable systems. Our goal now is to expose such relation.

A completely integrable Hamiltonian system is a symplectic manifold (X, ω) of dimension $2n$ provided with n functionally independent, Poisson-commuting functions $h_1, \cdots, h_n$, that is:

- $dh_1 \wedge \cdots \wedge dh_n \neq 0$
- $\{h_k, h_l\} = X_{h_k}(h_l) = -X_{h_l}(h_k) = 0.$

where the Hamiltonian vector field X_f associated with a function $f : X \to \mathbb{C}$ is characterized by

$$\omega(X_f, Y) = Y(f) \quad , \quad \forall Y \in \Gamma(TX) \quad .$$

We can put the functions $h_1, \cdots, h_n$ together into a map $H : X \to \mathbb{C}^n$; the fibers $H^{-1}(v)$ are, for generic v, n-dimensional subvarieties of X admitting n linearly independent vector fields $X_{h_1}, \cdots, X_{h_n}$.

Furthermore, X is said to be *algebraically* completely integrable (ACI, for short) if X is an algebraic variety and the generic fiber of the map H is an open subset of an abelian variety. That will force the vector fields $X_{h_1}, \cdots, X_{h_n}$ to be linear along the fibers of H.

The surprising fact, noticed by Hitchin in [9], is that the moduli space of solutions of Hitchin's equations modulo gauge transformations has exactly the structure of an algebraically complete integrable system, so-called *Hitchin system*.

Indeed, let $V \to S$ be a rank r vector bundle over a Riemann surface S of genus $g \geq 2$. Let A be a connection on V, and $\Phi \in \Gamma(\mathrm{End}(V) \otimes K_S)$ be a Higgs field as above; here, $K_S = \Lambda_S^{1,0}$ denotes the canonical bundle of S, consisting of holomorphic 1-forms. In the same way as we did for instantons on 4-dimensional manifolds, we may consider the moduli space $\mathcal{H}(V)$ of solutions of the Hitchin's equations as the quotient of the subset of the configuration space

$$\mathcal{C}(V) = \mathcal{A}(V) \times \Gamma(\mathrm{End}(V) \otimes K_S)$$

consisting of solutions of (5.2) by the action of the gauge group $\mathcal{G}(V)$, given by:

$$g \cdot (A, \Phi) = (g^{-1}Ag + g^{-1}dg, g^{-1}\Phi g) \quad .$$

Analytical considerations similar to the ones in Chapter 2 will also provide $\mathcal{H}(V)$ with the structure of a Riemannian manifold, at least away from reducible solutions of (5.2). It turns out that, in the case $r = 2$, $\mathcal{H}(V)$ is a complex manifold of dimension $6(g-1)$.

Moreover, $\mathcal{H}(V)$ admits a complex symplectic form, constructed as follows. The space tangent to $(A, \Phi) \in \mathcal{C}(V)$ is just

$$\Omega^1_S \otimes \mathrm{End}(V) = \Omega^{0,1}_S \otimes \mathrm{End}(V) \oplus \Omega^{1,0}_S \otimes \mathrm{End}(V)$$

Then

$$\omega((a_1, b_1), (a_2, b_2)) = \int_S \mathrm{Tr}(a_1 \wedge b_2 - a_2 \wedge b_1)$$

defines a complex symplectic form on $\mathcal{C}(V)$, which descends to (the smooth part of) the quotient space $\mathcal{H}(V)$. The real and imaginary parts of ω define two real symplectic forms on $\mathcal{H}(V)$.

Any connection A on a vector bundle V over a Riemann surface induces a holomorphic structure $\overline{\partial}_A$ on V, since the curvature F_A is always a $(1,1)$-form. The second equation of (5.2) says exactly that Φ is holomorphic with respect to $\overline{\partial}_A$.

For each point $x \in X$, $\Phi(x)$ can be thought as a $r \times r$ matrix and we can consider its eigenvalues $\lambda_1(x), \cdots, \lambda_r(x)$. As x sweeps out the whole surface, we end up with a r-fold branched covering Σ of S living inside the total space of the canonical bundle K_S; Σ is a complex submanifold of K_S, and it is called the *spectral curve* associated with (V, A, Φ). The branch points of $\Sigma \to S$ correspond to the points $x \in S$ where the eigenvalues of $\Phi(x)$ come together.

We can go one step further, and define a "line bundle" $\mathcal{L} \to \Sigma$ by assigning to each point $s \in \Sigma$ the corresponding eigenspace of s as an eigenvalue of $\Phi(x)$. Remark that $\mathcal{L}$ is not an actual line bundle, because not all eigenvalues of $\Phi(x)$ are of multiplicity one; however, if Σ is smooth (and generically it is), then $\mathcal{L}$ is a well-defined line bundle. In algebraic geometric terms, the best way to describe $\mathcal{L}$, which is called the spectral data associated with (V, A, Φ), is as a torsion sheaf on K_S supported on Σ.

It turns out the moduli space $\mathcal{H}(V)$ of solutions of Hitchin's equations is diffeomorphic to the space of suitable spectral data (stable torsion sheaves of pure dimension 1 on K_S). In other words, given a spectral curve $\Sigma \subset K_S$ which is an r-fold covering of S and a "line bundle" on it we can reconstruct a rank r vector bundle $V \to S$ and a solution (A, Φ) of (5.2).

Considering $\mathcal{S}$ as the space of all possible (smooth) spectral curves within K_S, this diffeomorphism yields a map $\sigma : \mathcal{H}(V) \to \mathcal{S}$ which takes each (A, Φ) to the associated spectral curve Σ. The fiber $\sigma^{-1}(\Sigma)$ is the set of all (suitable) line bundles over Σ, i.e. is an open subset of the Jacobian of Σ. The miracle here is that the dimension of $\mathcal{S}$ coincides with the genus Σ, and it is given by $3(g-1)$ in the case $r = 2$, exactly half of the dimension of $\mathcal{H}(V)$; of course, the dimension of the fibers $\sigma^{-1}(\Sigma)$ also coincides with the genus of Σ.

In other words, $\sigma : \mathcal{H}(V) \to \mathcal{S}$ is a fibration in middle dimension. The picture is complete once one verifies that the $3(g-1)$ functionally independent holomorphic functions $\sigma_1, \cdots, \sigma_{3(g-1)}$ defining the map σ are Poisson-commuting. Once the moduli space is properly understood as a symplectic reduction, this follows from the general theory.

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