Tangents and Secants of Algebraic Varieties
Notes of a Course

Publicações Matemáticas

Tangents and Secants of Algebraic Varieties
Notes of a Course

Francesco Russo
UFPE

impa

24º Colóquio Brasileiro de Matemática

Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ

Impresso no Brasil / Printed in Brazil

Capa: Noni Geiger / Sérgio R. Vaz

24º Colóquio Brasileiro de Matemática

- Autovalores de Polinômios Matriciais: Sensibilidade, Computação e Aplicações - Fermín Viloche
- Existence and Stability of Solitary Wave Solutions to Nonlinear Dispersive Evolution Equations - Jaime Angulo Pava
- Geometria Diferencial das Curvas Planas - Walcy Santos, Hilário Alencar
- Geometry, Dynamics and Topology of Foliated Manifolds - Bruno Scárdua e Carlos Morales
- Integrabilidade de Folheações Holomorfas - Jorge Vitório Pereira
- Introduction to the Ergodic Theory of Chaotic Billiards - Nikolai Chernov e Roberto Markarian
- Some Recent Developments in the Theory of Minimal Surfaces in 3-Manifolds - Harold Rosenberg
- Spin Dynamics at Zero Temperature - Luiz Renato Fontes
- Statistical Analysis of Non-Uniformly Expanding Dynamical Systems - José Alves
- Symmetry Studies - An Introduction - Marlos Viana
- **Tangents and Secants of Algebraic Varieties** - Francesco Russo
- Tópicos em Algoritmos sobre Sequências - Alair Pereira do Lago e Imre Simon

Distribuição:
IMPA
Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ
e-mail: ddic@impa.br
http://www.impa.br

ISBN: 85-244-0213-X

a Cledvane e a Giulia,

ai miei genitori Annamaria e Sante e a mio fratello Alberto

Preface

The aim of these notes is to furnish an introduction to some classical and recent results and techniques in projective algebraic geometry. We treat the geometrical properties of varieties embedded in projective space, their secant and tangent lines, the behaviour of tangent linear spaces, the algebro-geometric and topological obstructions to their embedding into smaller projective spaces, the classification in the extremal cases.

We discuss some geometrical interpretations of various problems in terms of projective geometry such as the Waring problem and the canonical expression, subhomaloidal linear systems, the vanishing of the hessian determinant of a polynomial and its relations with dual varieties, the rationality over arbitrary fields of some determinantal (or cubic) hypersurfaces.

These are classical themes in algebraic geometry and the renewed interest at the beginning of the '80 of the last century came from some conjectures posed by Hartshorne, [**H2**], from an important connectedness theorem of Fulton and Hansen, [**FH**], and from its new and deep applications to the geometry of algebraic varieties, as shown by Fulton, Hansen, Deligne, Lazarsfeld and Zak, [**FH**], [**FL**], [**D2**], [**Z2**]. We shall try to illustrate these themes and results during the course and with more details through these notes.

There exists no introductory text on secant, tangent, dual varieties, Terracini Lemma, etc, and moreover, quite surprisingly, these notions are not well known today. Thus we were forced to recall their constructions at the beginning of the text and to prove their first properties. A more advanced reference on some topics presented here is [**Z2**], which influenced the presentation of many topics; these notes could be thought also as a natural preparation to parts of the above referred book.

Finally I apologize for the absence, only in the notes, of any figure as it should be the case in a text on geometry. It depends on my well known incompatibility with a normal use of this modern technology. I felt enough satisfied producing a document with an (automatic) index.

Ringraziamenti

Queste note sono basate su una versione preliminare scritta, fortunatamente a mano, durante il periodo passato all' Universitá di Roma Tor Vergata nell' ultimo trimestre 2002, insegnando un corso sull' argomento. Ringrazio Ciro (Ciliberto) per avermi invitato, per la splendida ospitalitá e per le innumerevoli e piacevoli discussioni, matematiche e non (e inoltre per quelle precedenti e successive); tutti i partecipanti per l' impegno e lo stimolo fornito; l' I.N.D.A.M. per aver finanziato il soggiorno e il viaggio.

Il CNPq e il PRONEX-Algebra Comutativa e Geometria Algebrica hanno finanziato negli ultimi cinque anni vari miei progetti di ricerca su questi argomenti, sia come borsista, sia con fondi diretti e esprimo qui la mia gratitudine per la fiducia concessa, spero almeno parzialmente ricambiata.

Sono riconoscente a Eduardo Esteves e agli organizzatori per avere avuto l' iniziativa di invitarmi ad offrire questo corso al 24° Coloquio Brasileiro de Matemática. Senza la loro richiesta non avrei mai intrapreso questo progetto.

Un ringraziamento incommensurabile a Cledvane per l' amore, la pazienza e l' incoraggiamento costanti e senza limiti, rafforzati ulteriormente in questo periodo in cui, con la scusa di scrivere queste note, mi sono frequentemente sottratto ad aiutarla nel difficile compito di convincere la nostra piccola Giulia che in certi orari sia meglio dormire. La nascita di Giulia, fonte di immensa allegria e continua felicitá, mi permette di scusarmi elegantemente con gli eventuali lettori per non avere avuto il tempo necessario per sottoporre il testo a una seria revisione, o almeno a una attenta rilettura.

Quanto scritto in queste note é frutto di quanto appreso da vari amici, colleghi e professori e in molte occasioni dai "classici", antichi e moderni. Innanzitutto dal mio amico e maestro Lucian (Bădescu), che per primo mi ha sapientemente incamminato verso questi temi sin dai tempi in cui ero studente attraverso gli illuminanti corsi [**B1**] e [**B2**], a cui in anteprima assistí in compagnia degli amici del Seminarul de Geometrie del giovedí . Quanto mi ha insegnato, non solo matematicamente, é tutt' oggi importantissimo per me e va ben oltre quanto queste parole possano esprimere.

Un ringraziamento particolare a Aron (Simis) e Fyodor (Zak). Il primo per avermi mostrato come l' algebra possa, talvolta, aiutare a comprendere meglio la geometria e per le innumerovoli discussioni sui piu' svariati temi. Il secondo per il suo grande spirito critico, per la sua visione dell' universo geometrico, che, sebbene qualche volta non condivisa, mi ha sempre influenzato; e anche per avermi spiegato e suggerito vari problemi molto interessanti.

Negli ultimi anni gli amici Alberto (Alzati), Ciro (Ciliberto) e Massimiliano (Mella) hanno avuto un ruolo fondamentale sia per l' amicizia sempre mostrata, sia per quanto mi hanno insegnato nelle nostre frequenti comunicazioni, non solo matematiche, e nei, purtroppo, piu' rari incontri. Queste frasi non potranno mai esprimere quanto sento. Senza il loro aiuto non avrei compiuto nessun passo, caso mai ne avessi fatto uno, nel tentativo di comprendere meglio una piccolissima parte dell' affascinante mondo proiettivo.

Recife, 30 maggio 2003 Francesco Russo

Contents

Introduction

After the period in which new and solid foundations to the principles of algebraic geometry were rebuilt especially by Zariski, Grothendieck and their schools, at the beginning of the '70 a new trend began. There was a renewed interest in solving concrete problems and in finding applications of the new methods and ideas. One can consult the beautiful book of Robin Hartshorne *Ample subvarieties of algebraic varieties*, [**H1**], to have a picture of that situation. In [**H1**] many outstanding questions, such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle (solved by Mori in [**Mo1**] and which cleared the path to the foundation of Mori theory, [**Mo2**]) were discussed and/or stated and a lot of other problems solved. In related fields we only mention Deligne proof of the Weil conjectures or, later, Faltings proof of the Mordell conjecture, which used the new machinery.

Let us quote a part of Zak' s introduction to his fascinating book [**Z2**]: "Among recent achievements in the field of multidimensional projective geometry we mention results of Hironaka, Matsumura, Ogus, and Hartshorne on formal neighborhoods and local cohomology, theorems of Barth, Goresky, and MacPherson on the topology of projective varieties, classification of Fano varieties given by Iskovskih, Mori, and others, and various versions of Schubert's enumerative geometry. One of the most important results of the last decade is the connectedness theorem of Fulton and Hansen (cf. [**FH**], [**FL**])".

The interplay between topology and algebraic geometry returned to flourish. Lefschetz theorem and Barth-Larsen theorem, see subsection 2.1.1, also suggested that smooth varieties, whose codimension is small with respect to their dimension, should have very strong restrictions both topological, both geometrical. For example a codimension 2 smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected. If $N \geq 6$, there are no known examples of codimension 2 smooth varieties with the exception of the trivial ones, the complete intersection of two hypersurfaces, i.e. the transversal intersection of two hypersurfaces, smooth along the subvariety. In fact, at least for the moment, one is able to construct only these kinds of varieties whose codimension is sufficiently small with respect to dimension.

Based on these empirical observations and, according to Fulton and Lazarsfeld, "on the basis of few examples", Hartshorne was led to formulate two conjectures in 1974, [**H2**]. The first one is the following.

"*Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \frac{3}{2}\dim(X)$, *i.e. if* $\operatorname{codim}(X) < \frac{1}{2}\dim(X)$, *then X is a complete intersection.*"

Let us quote Hartshorne: *While I am not convicted of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind* ([**H2**]). The conjecture is sharp as the example of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ shows.

It is not here the place to remark how many important results originated and still today arise from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety and k-normality and so on. The list of these achievements is too long that we preferred to avoid citations, being confidènt that everyone has met sometimes a problem or a result related to it. It is quite embarrassing that the powerful methods of modern algebraic geometry did not yet produced a solution (or a counterexample).

The second problem posed by Hartshorne, also suggested by the fact that complete intersections are linearly normal and by some examples in low dimension, is the following. We recall that a nonsingular variety $X \subset \mathbb{P}^N$ is called *linearly normal* if $h^0(X, \mathcal{O}_X(1)) = N+1$, i.e., there is no $X' \subset \mathbb{P}^{N'}$, $N' > N$, such that X' is not contained in a hyperplane and can be isomorphically projected onto X.

"*Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \frac{3}{2}\dim(X)+1$, *i.e. if* $\operatorname{codim}(X) < \frac{1}{2}\dim(X)+1$, *then X is linearly normal.*"

Let us quote once again Hartshorne point of view on this second problem: *Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3}{2}n+1$, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case $N = \frac{3n}{2}$. My feeling is that this conjecture should be easier to establish than the original one* ([**H2**]). Once again the bound is sharp taking into account the example of the projected Veronese surface in $\mathbb{P}^4$.

The conjecture on linear normality was proved by Zak at the beginning of the '80's and till now it is the major evidence for the possible truth of the complete intersection conjecture. Moreover, Zak classified all the extremal cases showing that there are only 4 varieties analogous to the Veronese surface in $\mathbb{P}^4$, see chapter 3. These varieties were dubbed *Severi varieties* in honor of Francesco Severi, who first established the case $n = 2$ in [**Sev1**].

Many theorems in classical projective geometry deal with "general" objects, as the Bertini theorem on hyperplane sections, see theorem 1.5.2 here. A more refined version of this theorem says that if $f : X \to \mathbb{P}^N$ is morphism, with X proper and such that $\dim(f(X)) \geq 2$, and if $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ is a general hyperplane, then $f^{-1}(H)$ is irreducible. The "Enriques-Zariski principle" says that "limits of connected varieties remain connected" and it is illustrated in the previous example because for an arbitrary $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$, one proves that $f^{-1}(H)$ is connected.

This last result is particularly interesting because, as shown by Deligne and Jouanolou, a small generalization of it proved by Grothendieck, [**Gr**] XIII 2.3, yields a simplified proof of a beautiful and interesting connectedness theorem of Fulton and Hansen in [**FH**], whose applications are deep and appear in different areas of algebraic geometry and topology as we survey in chapter 2. One of the most important is Zak's *theorem on tangencies*. In the simplest situation this theorem is formulated as follows. Let $X \subset \mathbb{P}^N$ be an irreducible n-dimensional projective variety over an algebraically closed field K that is not contained in any hyperplane, and let L be an m-dimensional linear subspace of $\mathbb{P}^N$ that is tangent to X along an r-dimensional subvariety $Y \subset X$ (this means that all (embedded) tangent spaces to X at the points of Y are contained in L, so that, in particular, $m \geq n$). Then $r \leq m - n$, see chapter 2.

In particular the classical theorem of Bertini can be improved and new statements appear; for example, for a nonsingular variety of dimension n, $X \subset \mathbb{P}^N$, each hyperplane section is reduced for $N < 2n$ and is normal for $N < 2n - 1$. Other applications furnished by Zak lead to the solution of many classical problems such as the finiteness of the Gauss map for smooth varieties or the fact that the dimension of the dual variety $X^* \subset \mathbb{P}^{N*}$ is not less than the dimension of $X \subset \mathbb{P}^N$.

The problems and results we exposed above and which are contained in these notes are examples of the themes treated in projective geometry. This means that we fix a variety, its embedding and we are studying the properties of this variety or of its projections onto smaller dimensional spaces. Thus only the different incarnations of the same variety embedded by a fixed very ample line bundle are studied, by considering various sublinear system of the complete one realizing it in projective spaces of different dimension. The existence of isomorphic projections onto smaller projective spaces translates into strong properties of the linearly normal embedded variety.

Let us quote excerpts from Hilbert presentation of projective geometry in **[HCV]**:

"..... we shall learn about geometrical facts that can be formulated and proved without any measurement or comparison of distances or of angles. It might be imagined that no significant properties of a figure could be found if we do without measurement of distances and angles and that only vague statements could be made. And indeed research was confined to the metrical side of geometry far a long time, and questions of the kind we shall discuss in this chapter arose only later, when the phenomena underlying perspective painting were being studied scientifically. Thus, if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure. In this way, the process of projecting led to a new theory, which was called projective geometry because of its origins. Since the 19th century, projective geometry has occupied a central position in geometric research. With the introduction of homogeneous coordinates, it became possible to reduce the theorems of projective geometry to algebraic equations in much the same way that Cartesian coordinates allow this to be done for the theorems of metric geometry. But projective analytic geometry is distinguished by the fact that it is far more symmetrical and general than metric analytic geometry, and when one wishes, conversely, to interpret higher algebraic relations geometrically, one often transforms the relations into homogeneous form and interprets the variables as homogeneous coordinates, because the metric interpretation in Cartesian coordinates would be too unwieldy."

Varieties which could be projected isomorphically to projective spaces of lower dimension such that their codimension became *small*, are very special. First of all the projected manifold is not a complete intersection, being not linearly normal, so that the principles cited above say that near the bound there should be very few examples, satisfying strong restrictions and, at least experimentally, they are very few and could be classified; for examples most of them are homogeneous. To study projections one naturally deals with secant and tangent lines to the variety and with the varieties described by these lines in the ambient space.

We recall that a nonsingular variety $X \subset \mathbb{P}^N$ can be isomorphically projected to $\mathbb{P}^{N-1}$ if and only if $SX \neq \mathbb{P}^N$, where SX is the secant variety of X, i.e., the closure of the union of chords joining pairs of distinct points of X. Thus, the minimal number m such that X can be isomorphically projected to $\mathbb{P}^m$ is equal to the dimension of the variety SX. The relationship between embedded tangent spaces to X and SX is given by *Terracini lemma*, see chapter 1.

If $z \in SX$, $z \in < x, y >$, where $x, y \in X$ and $< x, y >$ is the chord joining x with y, then the (embedded) tangent space $T_z SX$ contains the (embedded) tangent spaces $T_x X$ and $T_y X$. Moreover, if the ground field has characteristic zero and z is a general point of SX, then

$$T_z SX =< T_x X, T_y X > .$$

From this it follows that if X can be isomorphically projected to $\mathbb{P}^m$, then for each pair of points $x, y \in X$ there exists an m-dimensional linear subspace of $\mathbb{P}^N$ which is tangent to X at the points x and y (if the characteristic of the ground field is equal to zero, then the converse is also true).

Along with the secant variety SX one can consider higher secant varieties $S^k X$, $k \geq 1$, where $S^k X$ is the closure of the union of k-dimensional linear subspaces spanned by generic collections of $k+1$ points of X. Zak established a connection between geometric characteristics of the varieties $S^k X$ for various k. In particular, for an arbitrary nonsingular variety $X \subset \mathbb{P}^N$ such that $< X >= \mathbb{P}^N$ he proved that

$$S^{[\frac{n}{\delta}]} X = \mathbb{P}^N,$$

where

$$\delta = \delta(X) = 2n + 1 - \dim(SX)$$

and $[\frac{n}{\delta}]$ is the largest integer not exceeding $\frac{n}{\delta}$, see chapter 4. This yields a bound for the maximal (for given n and $r \leq 2n$) number N for which there exist a variety $X \subset \mathbb{P}^N$ that can be isomorphically projected to $\mathbb{P}^r$. This bound is sharp; the varieties for which it is attained are called *Scorza varieties*, in honor of Gaetano Scorza (Severi varieties are special cases of Scorza varieties for $\delta = \frac{n}{2}$). Zak obtained a complete classification of Scorza varieties, viz., there exist three series of such varieties and one special sixteen-dimensional variety, see chapter 4. In other words, for a smooth variety $X \subset \mathbb{P}^r$ such that $\mathrm{codim}(X) \leq \dim(X)$ there exists a sharp bound for $h^0((X, \mathcal{O}_X(1)))$ in terms of $\dim(X)$ and r and Zak classified all varieties for which this bound is attained.

Let us describe briefly the contents of the these notes. In the first chapter we recall the definitions of tangent cone, tangent space, tangent star to a variety at a fixed point, define the secant variety SX, the higher secant varieties $S^k X$, the tangent variety TX and the variety of tangent stars $T^* X$ of a variety $X \subset \mathbb{P}^N$. We consider its *join*, $S(X, Y)$, with another variety $Y \subset \mathbb{P}^N$ and prove Terracini lemma relating the dimension of SX, or more generally of $S^k X$, respectively $S(X, Y)$, with the intersection of general tangent spaces to the variety. We furnish the first consequence for linear tangency at $k+1$ general points, by defining the entry loci and by studying its first properties. We present, as a significant applications of the notions introduced, a short, elementary, self contained and "new" proof, using a suggestion of Gaetano Scorza contained in the footnote at page 197 of [S1], of a classical theorem of del Pezzo-Bertini-Severi characterizing the Veronese surface of $\mathbb{P}^5$ as the unique surface in $\mathbb{P}^N$, $N \geq 5$, not a cone, verifying one of the following equivalent conditions: two general tangent spaces intersect (del Pezzo), respectively, it contains a two dimensional family of smooth conics (Bertini), respectively, $\dim(SX) = 4$ (Severi). This is the finalization and simplification of the work began during the supervision of the master thesis [VA]. We also generalize it to a classification theorem of Edwards-Scorza, to the effect that n-dimensional varieties $X \subset \mathbb{P}^{n+3}$ such that $\dim(SX) = n+2$ are either cones over a curve or cones over the Veronese surface. We end the chapter recalling the definition of dual variety, its first properties, the definitions of Gauss maps and the relations with reflexivity.

In the second chapter we survey, following Fulton [Fu], the connectedness principle of Enriques-Severi and generalize it by reporting on the circle of ideas which led to the connectedness theorem of Fulton-Hansen and on some theorems related to it. Then we furnish a proof of Fulton-Hansen theorem according to Deligne and we describe its consequences for the geometry of embedded varieties proved by Zak (theorem of tangencies, finiteness of the Gauss map, dimension of the dual variety, hyperplane sections of low codimensional varieties, etc). Other applications to algebraic geometry are contained in [**FH**] and in [**FL**], see also [**B1**] and [**B2**].

In the third chapter we enounce Hartshorne conjectures, present Zak proof of the conjecture on linear normality, define Severi varieties and describe the 4 examples in dimension 2, 4, 8 and 16. We propose simple proofs of the classification of Severi varieties in dimension 2, 4 (and 8) and sketch a proof of the astonishing bound $n \leq 16$ for the dimension of a Severi variety (and of $n = 2, 4, 8, 16$), using results from [**Ru6**]. In particular, if correct, this approach reveals that the classification of Severi varieties becomes a consequence of the general theory of quadric varieties, a suitable generalization of quadric hypersurfaces.

In the fourth chapter we relate the dimensions of the higher secant varieties S^kX, $k \geq 2$, to the dimension of SX and prove Zak's additivity theorem for the higher secant defects. We define Scorza varieties as the extremal cases for the above referred maximal embedding and describe their classification. From the classification of Severi varieties it follows that only the cases $\delta = 1, 2, 4$ and 8 can exist. We propose simple proofs of the classification for $\delta = 1$ and 2 to illustrate the result, while for the remaining difficult cases we refer to [**Z2**].

In the last chapter we describe several applications and generalizations of the results contained in the previous chapters. Firstly we treat the case in which the varieties S^kX have the expected dimension and satisfy the property that through a general point of S^kX there passes a unique $\mathbb{P}^k$ generated by $k+1$ points on X. These varieties were named by Severi, [**Sev1**], *varieties with one apparent* $k+1$*-secant* $\mathbb{P}^{k-1}$ because they acquire such a secant space by a general projection, see section 5.1. Such varieties are the first interesting case of not k-defective varieties and enjoy special properties. For example they are rational by a result of [**CMR**], which proves an implication of a conjecture of Bronowski discussed in section 5.1. Moreover, we illustrate an interesting interplay between projective geometry and the expression of general degree d homogeneous polynomials as sums of d^{th}-powers of linear forms, the *Waring problem*, and on the number of these expressions, *canonical form*. Section 5.2 shows how it can be useful in many geometrical problems to study rational maps on projective space whose general fiber is a linear space, i.e. given by a *subhomaloidal linear system*. The third section treat an interesting algebraic problem formulated by Hesse [**He1**] to the effect that the determinant of the hessian matrix of a homogeneous polynomial vanishes identically if and only if, modulo a linear change of coordinates, the polynomial depends on less variables. Geometrically this means that the associated projective hypersurface is a *cone* and this interpretation immediately clears the path to counterexamples to the not obvious implication, the first ones being proposed by Gordan and Nöether [**GN**]. We analyze this problem geometrically, relate it to dual varieties and to some interesting questions on the divisibility of the hessian matrix of a homogeneous polynomial by the polynomial itself and we also look at their geometrical interpretations as suggested by the work of Beniamino Segre [**Se2**]. As always it appears the problem of describing the extremal cases and some open questions on the subject end the section. In the final section of chapter 5 an interesting relation between varieties with one apparent double point, or suitable generalizations, and the rationality of some cubic (or determinantal) hypersurfaces is discussed as an introduction to some important and difficult rationality problems for (cubic) hypersurfaces.

The notes end with excerpts from the obituaries of Gaetano Scorza and Alessandro Terracini, whose contributions in the field of algebraic geometry were deep, hoping their work will become more familiar. It is a kind of personal admiration, especially for the work of Gaetano Scorza, which probably has not yet been appreciated in the way it deserves as one can verify by reading the summary of his contributions given by Berzolari, or better directly through the reading of his *Opere Scelte*, [**S6**].

Scorza and Terracini together with classical algebraic Geometers, antique and modern, taught and teach to us also to experiment the "live rapport" with "the objects one studies" and showed us the "concrete intuition", described by Hilbert in his preface to the book "Geometry and the Imagination" [**HCV**]:

"*In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations. As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only far the research worker, but also far anyone who wishes to study and appreciate the results of research in geometry. In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual, intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof, without necessarily entering into the details connected with the strict definitions of concepts and with the actual calculations.*

In this manner, geometry being as many-faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains."

CHAPTER 1

Tangent cones, tangent spaces, tangent stars; secant, tangent and tangent star variety to an algebraic variety

1.1. Tangent cones to an algebraic variety and associated varieties

Let X be an algebraic variety, or more generally a scheme of finite type, over a fixed algebraically closed field K. Let $x \in X$ be a closed point. We briefly recall the definitions of *tangent cone to* X *at* x and of *tangent space to* X *at* x. For more details one can consult [Mu] or [Sh].

1.1.1. DEFINITION. **(Tangent cone at a point).** Let $U \subset X$ be an open neighbourhood of x, let $i : U \to \mathbb{A}^N$ be a closed immersion and let U be defined by the ideal $I \subset K[X_1, \ldots, X_N]$. There is no loss of generality in supposing $i(x) = (0, \ldots, 0) \in \mathbb{A}^N$. Given $f \in K[X_1, \ldots, X_N]$ with $f(0, \ldots, 0) = 0$, we can define the "leading form" of f, f^*, as the lower degree homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. Let

$$I^* = \{\text{the ideal generated by the "leading form"} f^*, \text{ for all } f \in I\}.$$

Then

$$C_xX := \operatorname{Spec}(K[X_1, \ldots, X_N]/I^*),$$

is called *the affine tangent cone to* X *at* x.

It could seem that it depends on the choice of $U \subset X$ and on the choice of $i : U \to \mathbb{A}^N$. It is not the case because if $(\mathcal{O}_x, m_x)$ is the local ring of regular functions of X at x, then it is immediate to see that

$$(k[X_1, \ldots, X_N]/I^*) \simeq \operatorname{gr}(\mathcal{O}_x) := \bigoplus_{n \geq 0} \frac{m_x^n}{m_x^{n+1}}.$$

This fact simply says that we can calculate C_xX by choosing an arbitrary set of generators of I and moreover that the definition is "local". It should be noticed that C_xX is a scheme, which can be neither irreducible nor reduced as the examples of plane cubic curves with a node and with a cusp show. We now get a geometrical interpretation of this cone and see some of its properties.

Since C_xX is "locally" defined by homogeneous forms, it can be naturally projectivized and thought as a subscheme of $\mathbb{P}^{N-1} = \mathbb{P}(\mathbb{A}^N)$. If we consider the blow-up of $x \in U \subset \mathbb{A}^N$, $\pi : Bl_xU \to U$, then Bl_xU is naturally a subscheme of $U \times \mathbb{P}^{N-1} \subset \mathbb{A}^N \times \mathbb{P}^{N-1}$ and the exceptional divisor $E := \pi^{-1}(x)$ is naturally a subscheme of $x \times \mathbb{P}^{N-1}$. With these identifications one shows that $E \simeq \mathbb{P}(C_xX) \subset \mathbb{P}^{N-1}$ as schemes, see [Mu], pg. 225. In particular, if X is equidimensional at x, then C_xX is an equidimensional scheme of dimension $\dim(X)$. Moreover, we deduce the following geometrical definition:

$$\mathcal{C}_xX = \overline{\bigcup_{y\in U} \lim_{y\to x} < y, x >}.$$

The cone $\mathcal{C}_xX$ can also be described geometrically in this way, see [Sh]. Let notations be as in the affine setting above and set

$$m = \min\{\text{mult}_x(l \cap V(f)),\ l \text{ line through } x\ ,\ f \in I\}.$$

Then $\mathcal{C}_xX$ is swept out locally by the lines l through x such that $\text{mult}_x(l \cap V(f)) > m$.

If $X \subset \mathbb{P}^N$ is quasi-projective, we define *the projective tangent cone to X at x*, indicated by C_xX, as the closure of $\mathcal{C}_xX \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. The same geometrical definition holds, remembering of the scheme structure,

$$C_xX = \overline{\bigcup_{y\in U} \lim_{y\to x} < y, x >} \subset \mathbb{P}^N.$$

We now recall the definition of tangent space to X at $x \in X$.

1.1.2. Definition. (**Tangent space at a point; Tangent variety to a variety**). Let notations be as in the previous definition. Given $f \in K[X_1, \dots, X_N]$ with $f(0, \dots, 0) = 0$, we can define the "linear term" of f, f^{lin}, as the degree one homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. In other words, $f^{lin} = \sum_{i=1}^{N} \frac{\partial f}{\partial X_i}(0)X_i$. Let

$$I^{lin} = \{\text{the ideal generated by the "linear terms"} f^{lin},\ \text{for all } f \in I\}.$$

Then

$$\mathcal{T}_xX := \text{Spec}(K[X_1, \dots, X_N]/I^{lin}),$$

is called *the affine tangent space to X at x.*

Geometrically it is the locus of *tangent lines to X at x*, where a line through x is tangent to X at x if it is tangent to the hypersurfaces $V(f) = 0$, $f \in I$, i.e. if the multiplicity of intersection of the line with $V(f)$ at $(0, \dots, 0)$ is greater than one. In particular this locus is a linear subspace of $\mathbb{A}^N$.

Since $I^{lin} \subseteq I^*$, we deduce the inclusion as schemes

$$\mathcal{C}_xX \subseteq \mathcal{T}_xX;$$

and that $\mathcal{T}_xX$ is the smallest *linear* subscheme of $\mathbb{A}^N$ containing $\mathcal{C}_xX$ as a subscheme (and not only as a set!). In particular for every $x \in X$ it holds $\dim(\mathcal{T}_xX) \geq \dim(X)$.

We recall that a point $x \in X$ is non-singular if and only $\mathcal{C}_xX = \mathcal{T}_xX$. Since $\mathcal{T}_xX$ is reduced and irreducible and since $\mathcal{C}_xX$ is of dimension $\dim(X)$, we have that $x \in X$ is non-singular if and only if $\dim(\mathcal{T}_xX) = \dim(X)$.

Once again there is an intrinsic definition of $\mathcal{T}_xX$

$$(K[X_1, \dots, X_N]/I^{lin}) \simeq S(m_x/m_x^2),$$

where $S(m_x/m_x^2)$ is the symmetric algebra of the K-vector space m_x/m_x^2.

If $X \subset \mathbb{P}^N$ is a quasi-projective variety, we define *the projective tangent space to X at x*, indicated by T_xX, as the closure of $\mathcal{T}_xX \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. Then T_xX is a linear projective space naturally attached to X and clearly $C_xX \subseteq T_xX$ as schemes. We also set, for a (quasi)-projective variety $X \subset \mathbb{P}^N$,

$$TX = \bigcup_{x \in X} T_xX,$$

the variety of tangents, or *the tangent variety of* X.

At a non-singular point $x \in X \subset \mathbb{P}^N$, the equality $C_xX = T_xX$ says that every tangent line to X at x is "limit" of a secant line $< x, y >$ with $y \in X$ approaching x. For singular points this is not the case as one sees in the simplest examples of singular points of an hypersurface.

An interesting question is to investigate what are the limits of secant lines $< y_1, y_2 >$, $y_i \in X$, $y_1 \neq y_2$, when y_i, $i = 1, 2$, approaches a fixed $x \in X$. As we will immediately see for a non-singular point $x \in X$, every tangent line to X at x arises in this ways but for singular points this is not the case. These limits generate a cone, the tangent star cone to X at x, which contains but does not usually coincide with C_xX (or $\mathcal{C}_xX$). From now on we restrict ourselves to the projective setting since we will not treat local questions related to tangent star cones but the situation can be "localized". Firstly we introduce the notion of secant variety to a variety $X \subset \mathbb{P}^N$.

1.1.3. Definition. **(Secant varieties to a variety).** For simplicity let us suppose that $X \subset \mathbb{P}^N$ is a closed irreducible subvariety.

Let

$$S_X^0 := \{((x_1, x_2), z) \ : \ z \in < x_1, x_2 >\} \subset (X \times X \backslash \Delta_X) \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S_X^0) = 2\dim(X) + 1$. Recall that, by definition, it is a $\mathbb{P}^1$-bundle over $X \times X \backslash \Delta_X$, which is irreducible. Let S_X be its closure in $X \times X \times \mathbb{P}^N$. Then S_X is an irreducible projective variety of dimension $2\dim(X) + 1$, called *the abstract secant variety to* X. Let us consider the projections of S_X onto the factors $X \times X$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times X & & \mathbb{P}^N. \end{tikzcd}$$

The secant variety to X, SX, is the scheme-theoretic image of S_X in $\mathbb{P}^N$, i.e.

$$SX = p_2(S_X) = \overline{\bigcup_{x_1 \neq x_2,\, x_i \in X} < x_1, x_2 >} \subseteq \mathbb{P}^N,$$

which is an irreducible algebraic variety of dimension $s(X) \leq 2\dim(X) + 1$, the variety swept out by the secant lines to X. If equality (does not).holds, then X is said to be (*defective*) *non-defective.*

Let now $k \geq 1$ be a fixed integer. We can generalize the construction to the case of $(k+1)$-secant $\mathbb{P}^k$, i.e. to the variety swept out by the linear spaces generated by $k + 1$ independent points on X.

Define

$$(S^k_X)^0 \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N$$

as the locally closed irreducible set

$$(S^k_X)^0 := \{((x_0, \ldots, x_k), z) \ : \ \dim(< x_0, \ldots, x_k >) = k \ , \ z \in < x_0, \ldots, x_k >\}.$$

Let S^k_X, *the abstract k-secant variety of* X, be

$$\overline{(S^k_X)^0} \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N.$$

The closed set S^k_X is irreducible and of dimension $(k+1)\dim(X) + k$. Consider the projections of S^k_X onto the factors $\underbrace{X \times \ldots \times X}_{k+1}$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S^k_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times \ldots \times X & & \mathbb{P}^N. \end{tikzcd}$$

The k-secant variety to X, $S^k X$, is the scheme-theoretic image of S^k_X in $\mathbb{P}^N$, i.e.

$$S^k X = p_2(S^k_X) = \overline{\bigcup_{x_i \in X \, , \dim(<x_0,\ldots,x_k>)=k} < x_0, \ldots, x_k >} \subseteq \mathbb{P}^N.$$

It is an irreducible algebraic variety of dimension $s_k(X) \leq (k+1)\dim(X) + k$. If equality (does not) holds, then X is said to be (*k-defective*)*not k-defective*.

We are now in position to define the last cone attached to a point $x \in X$. This notion was introduced by Johnson in [**Jo**] and further studied extensively by Zak. Algebraic properties of tangent star cones and of the algebras related to them are investigated in [**SUV**].

1.1.4. Definition. (**Tangent star at a point; Variety of tangent stars,** [**Jo**]). Let $X \subset \mathbb{P}^N$ be an irreducible projective variety.

The abstract variety of tangent stars to X, T^*_X, is defined by the following cartesian diagram

$$\begin{tikzcd} T^*_X \arrow[r, hook] \arrow[d, "p"'] & S_X \arrow[d, "p_2"] \\ \Delta_X \arrow[r, hook] & X \times X \end{tikzcd}$$

The tangent star to X *at* x, $T^*_x X$, is defined by

$$T^*_x X := p_2(p^{-1}((x,x))) \subseteq \mathbb{P}^N.$$

It is a scheme which can be described geometrically as follows:

$$T^*_x X = \overline{\bigcup_{(x_1,x_2) \in X \times X \setminus \Delta_X} \lim_{x_i \to x} < x_1, x_2 >} \subset \mathbb{P}^N.$$

The variety of tangent stars to X is by definition

$$T^* X = p_2(T^*_X) \subseteq \mathbb{P}^N,$$

so that by construction

$$T^*X \subseteq SX;$$

moreover letting only one point varying we deduce

$$C_xX \subseteq T^*_xX.$$

It is also clear that the limit of a secant line is a tangent line, i.e. that

$$T^*_xX \subseteq T_xX.$$

By what we have defined and studied we deduce that for a point $x \in X \subset \mathbb{P}^N$, there is the following relation between the cones we attached at X:

$$C_xX \subseteq T^*_xX \subseteq T_xX.$$

Moreover a point $x \in X$ is non-singular if and only if $C_xX = T^*_xX = T_xX$. We immediately show in the following class of examples that at singular points strict inequalities can hold, i.e. at singular points there could exist tangent lines which are not limit of secant lines.

1.1.5. EXAMPLE. **(Singular points for which $C_xX \subsetneq T^*_xX \subsetneq T_xX$).** Let $Y \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ be an irreducible, non-degenerate variety in $\mathbb{P}^N$. Consider a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ and let $X := S(p, Y)$ be *the cone over* Y *of vertex* p, i.e.

$$S(p,Y) = \bigcup_{y \in Y} < p, y > .$$

Then X is an irreducible, non-degenerate variety in $\mathbb{P}^{N+1}$. In fact, modulo a projective transformation, the variety X is defined by the same equations of Y, now thought as homogeneous polynomials with one variable more; in particular $\dim(X) = \dim(Y) + 1$.

The line $< p, y >$ is contained in X for every $y \in Y$, so that $X \subset T_xX$ and therefore $\mathbb{P}^N =< Y > \subset T_pX$. Since $p \in T_pX$, we get

$$T_pX = \mathbb{P}^{N+1}. \tag{1.1.1}$$

It follows from the definition of tangent cone to a variety that

$$C_pS(p,Y) = S(p,Y).$$

We also have that

$$S(p, SY) = SX. \tag{1.1.2}$$

Indeed, by projecting from p onto $\mathbb{P}^N$, it is clear that a general secant line to X projects onto a secant line to Y, proving $SX \subseteq S(p, SY)$. On the contrary if we get a general point $q \in S(p, SY)$, by definition it projects onto a general point $q' \in SY$, which belongs to a secant line $< p'_1, p'_2 >$, $p'_i \in Y$. The plane $< p, p'_1, p'_2 >$ contains the point q, while the lines $< p, p'_i >$, $i = 1, 2$, are contained in X by definition of cone; hence through q there pass infinitely many secant lines to X, yielding $S(p, SY) \subseteq SX$. The claim is proved.

The above argument proves the following general fact:

$$T^*_pS(p,Y) = S(p, SY).$$

Indeed by definition $T^*_pX \subseteq SX = S(p, SY)$ as schemes. On the other hand, by fixing two general points $p_1, p_2 \in X$, $p_1 \neq p_2$, $p_i \neq p$, the plane $< p, p_1, p_2 >$ is contained in T^*_pX as it is easily seen by varying the velocity of approaching p of two points $q_i \in < p, p_i >$. By the generality of the points p_i we get the inclusion $SX \subseteq T_p^* X$ as schemes and the proof of the claim.

As an immediate application one constructs example of irreducible singular varieties X with a point $p \in \mathrm{Sing}(X)$ for which

$$C_p X \subsetneq T_p^* X \subsetneq T_p X.$$

One can take as $Y \subset \mathbb{P}^4 \subset \mathbb{P}^5$ an irreducible, smooth, non-degenerate curve in $\mathbb{P}^4$ and consider the cone X over Y of vertex $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. Then $C_p X = S(p, Y) = X$, $T_p^* X = S(p, SY) = SX$ is an hypersurface in $\mathbb{P}^5$, because SY is an hypersurface in $\mathbb{P}^4$ (see 1.2.2 if you do not agree), while $T_p X = \mathbb{P}^5$. Every variety Y such that $SY \subsetneq \mathbb{P}^N$ (see the following exercise or take $N > 2\dim(Y) + 1$) will produce analogous examples.

1.1.6. Exercise. Let K be a(n algebraically closed) field. Recall that the linear combination of two (symmetric) matrixes of rank 1 has rank at most 2 and that every (symmetric) matrix of rank 2 can be written as the linear combination of two (symmetric) matrixes of rank 1.

Deduce the following geometrical consequences for the secant varieties of the varieties described below.

(1) Let $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the 2-Veronese surface in $\mathbb{P}^5$. Identify $\mathbb{P}^5$ with

$$\mathbb{P}(\{A \in M(3; K) \,:\, A = A^t\}),$$

and show that $X = \{[A] \,:\, \mathrm{rk}(A) = 1\}$. Prove that $SX = TX = V(det(A)) \subset \mathbb{P}^5$ is the cubic hypersurface given by the cubic polynomial $det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1} X \cap T_{x_2} X \neq \emptyset$ ((if you have a lot of energy and not enough patience to wait for the next section, prove that if the points are general, then the intersection consists of a point). Prove that $\mathrm{Sing}(SX) = X$.

(2) Let $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. Identify $\mathbb{P}^8$ with

$$\mathbb{P}(\{A \in M(3; K)\}),$$

and show that $X = \{[A] \,:\, \mathrm{rk}(A) = 1\}$. Prove that $SX = TX = V(det(A)) \subset \mathbb{P}^8$ is the cubic hypersurface given by the cubic polynomial $det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1} X$ and $T_{x_2} X$ intersect at least along a line (prove that if the points are general, then the intersection consists of a line). Take H be a general hyperplane in $\mathbb{P}^8$ and let $Y := X \cap H$. Then Y is a smooth, irreducible, non-degenerate 3-fold $Y \subset \mathbb{P}^7$ such that $SY \subseteq SX \cap H$ so that $\dim(SY) \leq 6$ (in fact one can prove that $SY = SX \cap H$ and hence that $\dim(SY) = 6$). Prove that given $y_1, y_2 \in Y$, then $T_{y_1} Y \cap T_{y_2} Y \neq \emptyset$ (consists of a point if the points are general). Prove that $\mathrm{Sing}(SX) = X$.

Let $p \in \mathbb{P}^9 \setminus \mathbb{P}^8$, let $Z = S(p, X) \subset \mathbb{P}^9$ and let $X' = X \cap W$, with $W \subset \mathbb{P}^9$ a general hypersurface, not an hyperplane, not passing through p. Then X' is a smooth, irreducible, non-degenerate 4-fold such that $SX' = SZ = S(p, SX)$. Conclude that $\dim(SX') = 8$ and use the fact that Z is a cone over X to deduce that two general tangent spaces to X' intersect.

(3) Generalize the previous exercise and find the relation between $SX \subset \mathbb{P}^N$ and $SX' \subset \mathbb{P}^{N+1}$ for $X' \subset \mathbb{P}^{N+1}$ a general intersection of $Z = S(p, X) \subset \mathbb{P}^{N+1}$ with a general hypersurface $W \subset \mathbb{P}^{N+1}$, not passing through $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$.

1.2. Join of varieties

We generalize to arbitrary irreducible varieties $X, Y \subset \mathbb{P}^N$ the notion of "cone" or of "join" of linear spaces.

Let us remember that if $L_i \simeq \mathbb{P}^{N_i} \subseteq \mathbb{P}^N$, $i = 1, 2$, is a linear subspace, then

$$< L_1, L_2 >:= \bigcup_{x_i \in L_i \ , \ x_1 \neq x_2} < x_1, x_2 >,$$

is a linear space called *the join of* L_1 *and* L_2. It is the smallest linear subspace of $\mathbb{P}^N$ containing L_1 and L_2. By Grassmann formula we have

$$\dim(< L_1, L_2 >) = \dim(L_1) + \dim(L_2) - \dim(L_1 \cap L_2), \tag{1.2.1}$$

where as always $\dim(\emptyset) = -1$. This shows that the dimension of the join depends on the intersection of the two linear spaces.

On the other hand, if $X \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ is an irreducible subvariety and if $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ is an arbitrary point, if we define as before

$$S(p, X) = \bigcup_{x \in X} < p, x >,$$

the cone of vertex p *over* X, then for every $z \in < p, x >$, $z \neq x$, $z \neq p$, we have by construction

$$T_z S(p, X) = < p, T_x X > = < T_p p, T_x X >, \tag{1.2.2}$$

i.e. the well known fact that the tangent space is constant along the ruling of a cone.

As we shall see in the next section, once we have defined the join of two varieties as the union of lines "joining" points of them, then we can "linearize" the problem looking at the tangent spaces and calculate the dimension of the "join" by looking at the affine cones over the varieties, exactly as in the proof of the formula 1.2.1. The dimension of the join of two varieties will depend on the intersection of a general tangent space of the first one with a general tangent space of the other one, a result known as Terracini Lemma, [T1]. Moreover a kind of property similar to the second tautological inequality in 1.2.2 will hold generically, at least in characteristic zero, see theorem 1.3.1.

1.2.1. Definition. **(Join of varieties; relative secant, tangent star and tangent varieties).** Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties.

Let

$$S^0_{X,Y} := \{((x, y, z), x \neq y : z \in < x_1, x_2 >\} \subset X \times Y \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S^0_{X,Y}) = \dim(X) + \dim(Y) + 1$. Let $S_{X,Y}$ be its closure in $X \times Y \times \mathbb{P}^N$. Then $S_{X,Y}$ is an irreducible projective variety of dimension $\dim(X)+\dim(Y)+1$, called *the abstract join of X and Y*. Let us consider the projections of $S_{X,Y}$ onto the factors $X \times Y$ and $\mathbb{P}^N$,

$$
\begin{tikzcd}
 & S_{X,Y} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \times Y & & \mathbb{P}^N.
\end{tikzcd}
\tag{1.2.3}
$$

The join of X and Y, $S(X,Y)$, is the scheme-theoretic image of $S_{X,Y}$ in $\mathbb{P}^N$, i.e.

$$S(X,Y) = p_2(S_{X,Y}) = \overline{\bigcup_{x \neq y,\, x\in X,\, y \in Y} < x, y >} \subseteq \mathbb{P}^N;$$

it is an irreducible algebraic variety of dimension $s(X,Y) \leq \dim(X)+\dim(Y)+1$, swept out by lines joining points of X with points of Y.

With these notations $S(X,X) = SX$ and $S(X, S^{k-1}X) = S^kX = S(S^lX, S^hX)$, if $h \geq 0$, $l \geq 0$, $h+l = k-1$. Moreover, for arbitrary irreducible varieties X, Y and Z, we have $S(X, S(Y,Z)) = S(S(X,Y), Z)$.

When $Y \subseteq X \subset \mathbb{P}^N$ is an irreducible closed subvariety, the variety $S(Y,X)$ is usually *the relative secant variety of X with respect to Y*. Analogously, $T(Y,X) = \bigcup_{y\in Y} T_yX$. In this case by taking $\Delta_Y \subset Y \times X$ and by looking at 1.2.3, we can define $T^*_{Y,X} := p_1^{-1}(\Delta_Y) \subseteq S_{Y,X}$ to be *the abstract relative tangent star variety* and finally

$$T^*(Y,X) := p_2(T^*_{Y,X}) \subseteq S(X,Y) \tag{1.2.4}$$

to be *the relative tangent star variety.* If

$$T^*_y(Y,X) = p_2(p_1^{-1}(y \times y)) = \overline{\bigcup_{(y_1,x_1)\in Y\times X\setminus \Delta_Y} \lim_{\substack{y_1 \to y \\ x_1 \to y}} < y_1, x_1 >} \subset \mathbb{P}^N,$$

then $T^*(Y,X) = \bigcup_{y\in Y} T^*_y(Y,X)$. With this terminology, $T^*_y(y,X) = C_yX$ and $T^*_y(X,X) = T^*_yX$ for every $y \in X$. In particular $C_yX = T^*_y(y,X) \subseteq T^*_y(X,X) = T^*_yX$.

We furnish some immediate applications of the definition of join to properties of S^kX and to characterizations of linear spaces.

1.2.2. PROPOSITION. ([**P2**]) *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties. The following holds:*

(1) *for every $x \in X$,*
$$Y \subseteq S(x,Y) \subseteq S(x, < Y >) \subseteq T_xS(X,Y);$$
(2) *if $S^kX = S^{k+1}X$ for some $k \geq 0$, then $S^kX = \mathbb{P}^{s_k(X)} \subseteq \mathbb{P}^N$;*
(3) *if $\dim(S^{k+1}X) = \dim(S^kX)+1$ for some $k \geq 0$, then $S^{k+1}X = \mathbb{P}^{s_{k+1}(X)}$ so that S^kX is an hypersurface in $\mathbb{P}^{s_{k+1}(X)}$;*
(4) *if $S^{k+1}X$, $k \geq 0$, is not a linear space, then $S^kX \subseteq \mathrm{Sing}(S^{k+1}X)$.*

PROOF. By definition of join we get the inclusion $S(x,Y) \subseteq S(X,Y)$ and hence $T_xS(x,Y) \subseteq T_xS(X,Y)$. Moreover for every $y \in Y$, $y \neq x$, the line $< x,y >$ is contained in $S(x,Y)$ and passes through x so that it is contained in $T_xS(x,Y)$ and part 1) easily follows.

Let $z \in S^kX$ be a smooth point of S^kX. From part 1) with $Y = S^kX$ we get $S^kX \subseteq T_zS(X, S^kX) = T_zS^{k+1}X = T_zS^kX = \mathbb{P}^{s_k(X)}$, which implies $S^kX = \mathbb{P}^{s_k(X)}$ since both are irreducible varieties of dimension $s_k(X)$.

To prove part 3), take $z \in S^{k+1}X \setminus S^kX$ be a general point and remark that by part 2) S^kX is not linear so that $\dim(< S^kX >) \geq s_k(X) + 1$. Then $\mathbb{P}^{s_k(X)+1} = T_zS^{k+1}X \supseteq < z, < S^kX >> \supseteq < S^kX > \supseteq \mathbb{P}^{s_k(X)+1}$, i.e. $p^{s_{k+1}(X)} = \mathbb{P}^{s_k(X)+1} = T_zS^{k+1}X =< S^kX >)$ for $z \in S^{k+1}X$ general. Then $< S^kX >= \mathbb{P}^{s_{k+1}(X)} \supseteq S^{k+1}X$ and hence equality holds.

Recall that in any case $S^{k+1}X \subseteq < X >$. Take $z \in S^kX$ and observe that by part 1) $T_zS^{k+1}X \supseteq < X > \supsetneq S^{k+1}X$ so that $\dim(T_zS^{k+1}X) > \dim(S^{k+1}X)$ and z is a singular point. $\square$

To a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we can associate an ascending filtration of irreducible projective varieties, whose inclusion are strict by 1.2.2, and an integer $k_0 = k_0(X) \geq 1$:

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N, \tag{1.2.5}$$

where k_0 is the least integer such that $S^kX = \mathbb{P}^N$.

The above immediate consequences of the definitions give also the following result, which was classically very well known, see for example [**P1**] footnote pg. 635, but considered as an open problem by Atiyah, [**At**] pg. 424. From the following corollary, an argument of Atiyah yields a proof of C. Segre and Nagata theorem about the minimal section of a geometrically ruled surface, see [**Ln**].

1.2.3. COROLLARY. ([**P1**]) *Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $s_k(C) = \min\{2k + 1, N\}$.*

PROOF. For $k = 0$ it is true and we argue by induction. Suppose $S^kC \subsetneq \mathbb{P}^N$. By proposition 1.2.2 $s_k(C) \geq s_{k-1}(C)+2$ and the description $S^k(C) = S(C, S^{k-1}C)$ yields $s_k(C) \leq s_{k-1}(C)+2$ so that $s_k(C) = 2(k-1)+1+2 = 2k+1$ as claimed. $\square$

We define and study linear projections with the terminology just introduced and generalize in a suitable way the dimension formula 1.2.1, in characteristic zero, i.e. to the case of arbitrary cones over the variety X. In the next section we deal with the general case.

1.2.4. DEFINITION. (**Linear projections and "linear" cones**) Let $L = \mathbb{P}^l \subset \mathbb{P}^N$ be a fixed linear space, $l \geq 0$, and let $M = \mathbb{P}^{N-l-1}$ be a linear space skew to L, i.e. $L \cap M = \emptyset$ and $< L, M >= \mathbb{P}^N$. Let $X \subsetneq \mathbb{P}^N$ be a closed irreducible variety not contained in L and let

$$\pi_L : X \dashrightarrow \mathbb{P}^{N-l-1} = M,$$

be the rational map defined on $X \setminus (L \cap X)$ by

$$\pi_L(x) =< L, x > \cap M.$$

The map is well defined by Grassmann formula, 1.2.1. Let $X' = \overline{\pi_L(X)} \subset \mathbb{P}^{N-l-1}$ be the closure of the image of X by π_L. The whole process can be described with the terminology of joins. Indeed we have

$$X' = S(L, X) \cap M,$$

i.e. X' is the intersection of M with the *cone over X of vertex L* and moreover $S(L,X) = S(L,X')$. The projective differential of π_L is the projection of the tangent spaces from L, i.e. if $x \in X \setminus (L \cap X)$, then $d_{\pi_L}(T_x X) =< L, T_x X > \cap M \subseteq T_{\pi_L(x)} X'$ as it is easily seen eventually passing to (local) coordinates.

Suppose $L \cap X = \emptyset$, then we claim that $\pi_L : X \to X'$ is a finite morphism, which implies $\dim(X) = \dim(X')$. Being a morphism between projective varieties, it is sufficient to show that it has finite fibers. By definition for $x' \in X'$,

$$\pi_L^{-1}(x') =< L, x' > \cap X \subset < L, x' >= \mathbb{P}^{l+1}.$$

If there exists an irreducible curve $C \subset < L, x' > \cap X \subset < L, x' >$, then $\emptyset \neq L \cap C \subseteq L \cap X$, contrary to our assumption.

In particular for an arbitrary L, the dimension of X' does not depend on the choice of the position of M, except for the requirement $L \cap M = \emptyset$.

The relation $S(L,X) = S(L,X')$ allows us to calculate the dimension of the irreducible variety $S(L,X)$ for an arbitrary L. Exactly as in 1.2.2 for $z \in S(L,X) \setminus L$,

$$z \in < L, x >=< L, \pi_L(z) >=< L, x' >,$$

with $x \in X$ and $\pi_L(z) = \pi_L(X) = x' \in X'$. Since $S(L,X')$ is, modulo a projective transformation, the variety defined by the same homogeneous polynomials of X' now though as polynomial in $N+1$ variables, we have

$$T_z S(L,X) =< L, T_{\pi_L(z)} X' > \supseteq < L, T_x X > . \tag{1.2.6}$$

Taking $z \in S(L,X)$ general and recalling that $L \cap M = \emptyset$ we deduce:

$$\dim(S(L,X)) = \dim(< L, T_{\pi_L(z)} X' >) = \dim(X') + l + 1. \tag{1.2.7}$$

Suppose till the end of the subsection $\mathrm{char}(K) = 0$. By generic smoothness, the differential map is surjective so that $T_{\pi_L(x)} X' = \pi_L(T_x X)$ for $x \in X$ general. In this case $\pi_L(x) = x' \in X'$ will be general on X' and finally

$$\dim(X') = \dim(T_{x'} X') = \dim(\pi_L(T_x X)) = \dim(X) - \dim(L \cap T_x X) - 1,$$

which combined with 1.2.7 gives the following generalization of 1.2.1:

$$\dim(S(L,X)) = \dim(L) + \dim(X) - \dim(L \cap T_x X), \tag{1.2.8}$$

$$x \in X \text{ general point.}$$

Moreover, we get the following refinement of 1.2.6

$$T_z S(L,X) =< L, T_x X >, \tag{1.2.9}$$

$$x \in X\ ,\ z \in < L, x > \text{ general points.}$$

We have generalized the notion of cone over a variety lying in a skew space with respect to the vertex by taking $S(L,X)$ and shown that by projecting the variety X from the vertex L, we can find the description of it as an "usual" cone, $S(L,X')$.

Now we investigate under which condition a variety is a "cone", i.e. there exists a "vertex" $L \simeq \mathbb{P}^l \subseteq X$ such that $X = S(L,X) = S(L,X')$, if X' is the section with a general $\mathbb{P}^{N-l-1}$ skew with the "vertex" L. Clearly the "vertex" is not uniquely determined if we not require some maximality condition. Let us begin with the definitions.

1.2.5. DEFINITION. **(Cone; vertex of a variety)** Let $X \subset \mathbb{P}^N$ be a closed (irreducible) subvariety. The variety is a *cone* if there exists $x \in X$ such that $S(x, X) = X$. Geometrically this means that given $y \in X$, $y \neq x$, the line $< x, y >$ is contained in X. In particular $x \in \bigcap_{y \in X} T_y X$.

This motivates the definition of vertex of a variety. Given $X \subset \mathbb{P}^N$ an irreducible closed subvariety, *the vertex of* X, $\mathrm{Vert}(X)$, is the set

$$\mathrm{Vert}(X) = \{ x \in X \ : \ S(x, X) = X \}.$$

In particular a variety X is a cone if and only if $\mathrm{Vert}(X) \neq \emptyset$; by definition $S(X, Y) = X$ if and only if $Y \subseteq \mathrm{Vert}(X)$.

We list some obvious consequences and leave to the interested reader the pleasure of showing that the hypothesis on the characteristic of the base field are necessary.

1.2.6. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a closed irreducible variety of dimension* $\dim(X) = n$. *The following holds:*

(1)
$$\mathrm{Vert}(X) = \mathbb{P}^l \subseteq \bigcap_{x \in X} T_x X,$$
$l \geq -1$;

(2) *if* $\mathrm{codim}(\mathrm{Vert}(X), X) \leq 1$, *then* $\mathrm{Vert}(X) = X = \mathbb{P}^n \subset \mathbb{P}^N$;

(3) *if* $\dim(S(X, Y)) = \dim(X) + 1$, *then* $Y \subseteq \mathrm{Vert}(S(X, Y))$;

(4) *if* $\mathrm{char}(K) = 0$,
$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l \subseteq X,$$
$l \geq -1$;

(5) *suppose* $\mathrm{char}(K) = 0$ *and* $\emptyset \neq \mathrm{Vert}(X) \subsetneq X$, *then* $X = S(\mathrm{Vert}(X), X')$ *is a cone, where X' is the projection of X from* $\mathrm{Vert}(X)$ *onto a* $\mathbb{P}^{N-l-1}$ *skew to* $\mathrm{Vert}(X)$ $(\dim(X') = n - l - 1)$.

PROOF. To prove 1) it is sufficient to show that, given two points $x_1, x_2 \in \mathrm{Vert}(X)$, the line $< x_1, x_2 >$ is contained in $\mathrm{Vert}(X)$, forcing $\mathrm{Vert}(X)$ irreducible and linear by proposition 1.2.2 part 2). Taken $y \in < x_1, x_2 > \setminus \{x_1, x_2\}$ and $x \in X \setminus \mathrm{Vert}(X)$, it is sufficient to prove that $< y, x > \subset X$. By definition the lines $< x_i, x >$ are contained in X and by varying the point $q \in < x_2, x > \subset X$ and by joining it with x_1 we see that the line $< x_1, q >$ is contained in X for every such q, i.e. the plane $\Pi_x = < x_1, x_2, x >$ is contained in X. Since y and x belong to Π_x, the claim follows.

If $\mathrm{Vert}(X) = X$, then $X = \mathbb{P}^n$ by part 1). If there exists $W = \mathbb{P}^{n-1} \subseteq \mathrm{Vert}(X) = \mathbb{P}^l \subseteq X$, i.e. if $l \geq n-1$, we can take $x \in X \setminus W$. Therefore $S(x, W) = \mathbb{P}^n$ and $S(W, x) \subseteq X$ forces $X = \mathbb{P}^n$.

To prove 3) take $y \in Y \setminus \mathrm{Vert}(X)$ and observe that for dimension reasons $S(y, X) = S(Y, X)$ and $S(y, S(X, Y)) = S(y, S(y, X)) = S(y, X) = S(Y, X)$ gives the desired conclusion.

Set $L = \bigcap_{x \in X} T_x X$ and assume $\mathrm{char}(K)$=0. By 1.2.8 $\dim(S(L, X)) = \dim(X)$, yielding $X = S(L, X)$ and $L \subseteq \mathrm{Vert}(X)$, which proves part 4). Part 5) follows in a straightforward way. $\square$

Later we will use the following result.

1.2.7. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension $n = \dim(X)$. Assume* $\mathrm{char}(K)=0$, $N \geq n+3$ *and* $\dim(SX) = n+2$. *If through the general point $x \in X$ there passes a line l_x contained in X, then X is a cone.*

PROOF. Let $x \in X$ be a general point. Then $x \notin \mathrm{Vert}(X)$ and $x \notin \mathrm{Vert}(SX)$ since X is non-degenerate, so that $X \subsetneq S(l_x, X) \subseteq SX$. If $\dim(S(l_x,X)) = n+2$, then $S(l_x, X) = SX$. Since $S(l_x, SX) = S(l_x, S(l_x, X)) = S(l_x, X) = SX$, we would deduce $x \in l_x \subseteq \mathrm{Vert}(SX)$. In conclusion l_x is not contained in $\mathrm{Vert}(SX)$ and $\dim(S(l_x,X)) = n+1$. Then the general tangent space to X, T_yX, will cut l_x in a point $p_{x,y} := l_x \cap T_yX$. If this point varies with y, then two general tangent spaces $T_{y_1}X$ and $T_{y_2}X$ would contain l_x so that $< l_x, < T_{y_1}X, T_{y_2}X >> = < T_{y_1}X, T_{y_2}X >$ would force $S(l_x, SX) = SX$, i.e. $l_x \subseteq \mathrm{Vert}(SX)$. So the point remain fixed, i.e. $p \in \cap_{y \in X} T_yX = \mathrm{Vert}(X)$ and X is a cone by proposition 1.2.6. □

We end this section by putting in relation the projections of a variety and the dimension of its secant or tangent varieties.

If $L = \mathbb{P}^l \subset \mathbb{P}^N$ is a linear space and if $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-l-1}$ is the projection onto a skew complementary linear space, then π_L restricts to a finite morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$, as soon as $L \cap X = \emptyset$. In the idea that studying varieties whose codimension is small with respect to the dimension is easier (from some points of view but not from others!), we can ask when this finite morphism is one-to-one, or a closed embedding. Let us examine this conditions in the following proposition.

1.2.8. PROPOSITION. *Let notations as above. Then:*

(1) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is one-to-one if and only if $L \cap SX = \emptyset$;*
(2) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is unramified if and only if $L \cap TX = \emptyset$;*
(3) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is a closed embedding if and only if $L \cap SX = L \cap TX = \emptyset$.*

PROOF. The morphism $\pi_L : X \to X' \subseteq \mathbb{P}^{N-l-1}$ is one-to-one if and only there exists no secant line to X cutting the center of projection: $< L, x > = < L, y >$ if and only if $< x, y > \cap L \neq \emptyset$. It is ramified at a point $x \in X$ if and only if $T_xX \cap L = \emptyset$ by looking at the projective differential of π_L. A morphism is a closed embedding if and only if it is one-to-one and unramified. □

We must state the following well known result, which only takes into account that for smooth varieties the equality $TX = T^*X$ furnishes $TX \subseteq SX$.

1.2.9. COROLLARY. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible closed subvariety. If $N > \dim(SX)$, then X can be isomorphically projected into $\mathbb{P}^{N-1}$. In particular if $N > 2\dim(X) + 1$, then X can isomorphically projected into $\mathbb{P}^{N-1}$.*

One could ask what is the meaning of $L \cap T^*X = \emptyset$. This means that π_L (or $d(\pi_L)$) restricted to T^*_xX is finite for every $x \in X$. This is the notion of J-unramified morphism, where J stands for Johnson [**Jo**], and it can be expressed in terms of affine tangent stars, see [**Z2**]. We take the above condition as the definition of J-unramified projection. In particular, if $L \cap SX = \emptyset$, then π_L is one-to-one and J-unramified and it is said to be *a J-embedding.* If the projection

$\pi_L : X \to X' \subset \mathbb{P}^{N-l-1}$, then $\mathrm{Sing}(\pi_L(X)) = \pi_L(\mathrm{Sing}(X))$ so that X' does not acquire singularities from the projection.

It is clearly weaker than the usual notion of embedding and it is well behaved to study the projections of singular varieties. For example take $C \subset \mathbb{P}^4 \subset \mathbb{P}^5$ a smooth non-degenerate curve in $\mathbb{P}^4$ and let $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. If $X = S(p, C)$ is the cone over C, then $T_pX = \mathbb{P}^5$, 1.1.1, and X cannot be projected isomorphically in $\mathbb{P}^4$. Since $SX = S(p, SC)$, 1.1.2, is an hypersurface in $\mathbb{P}^5$, there exists a point $q \in \mathbb{P}^5 \setminus X$ such that $\pi_q : X \to X'$ is a J-embedding and $X' = S(\pi_q(p), C)$ is a cone over C of vertex $\pi_q(p) = p'$. In this example the morphism π_q is one-to-one and unramified outside the vertex of the cones and maps the tangent star at p, $T^*_pX = S(p, SC)$, m-to-one onto $\mathbb{P}^4$, where $m = \deg(S(p, SC)) = \deg(SC) = \binom{d-1}{2} - g$, $d = \deg(C)$, g the genus of C.

The conditions $L \cap S(Y, X) = \emptyset$, respectively $L \cap T^*(Y, X) = \emptyset$ or $L \cap T(Y, X) = \emptyset$, with $Y \subseteq X$, mean that π_L is one-to-one in a neighbourhood of Y, respectively is J-unramified in a neighbourhood of Y or unramified in a neighbourhood of Y.

1.3. Terracini Lemma and its first applications

As we have seen the definition of secant variety is the "join" of X with itself and it is not clear how to calculate the dimension of SX, see exercise 1.1.6, or more generally the dimension of $S(X, Y)$. In fact, the circle of ideas, which allowed Alessandro Terracini to solve the problem of calculating the dimension of SX, or more generally of S^kX, originated exactly from the study of examples like the ones considered in 1.1.6 and from the pioneering work of Gaetano Scorza, [**S1**] and [**S4**]. Let Terracini explain us this process, by quoting the beginning of [**T1**]:

É noto, [**dP**], *che la sola V_2, non cono, di S_r, i cui S_2 tangenti si incontrano a due a due, é, se $r \geq 5$, la superficie di VERONESE; e che questa superficie,* [**Sev1**], *é pure caratterizzata dall' essere, in un tale S_r, la sola, non cono, le cui corde riempono una V_4. Recentemente lo SCORZA,* [**S3**] *pg. 265, disse di aver ragione di credere, sebbene non gli fosse venuto fatto di darne una dimostrazione, che le V_3 di S_7, o di uno spazio piú ampio, le cui corde non riempiono una V_7 << rientrino >> tra le V_3 a spazi tangenti mutuamente secantisi. Ora si puó dimostrare, piu' precisamente, che queste categorie di V_3 coincidono, anzi, piu' in generale, che: Se una V_k di S_r ($r > 2k$) gode di una delle due proprietá, che le corde riempiano una varietá di dimensione $2k-i$ ($i \geq 0$), o che due qualsiansi S_k tangenti si seghino in uno S_i, gode pure dell' altra. Questo teorema, a sua volta, non é se non un caso particolare di un teorema piú generale che ora dimostreremo, teorema che pone in relazione l' eventuale abbassamento di dimensione della varietá degli S_h $(h+1)$-seganti di una V_k immersa in uno spazio di dimensione $r \geq (h+1)k+h$, coll' esistenza di $h+1$ qualsiansi suoi S_k tangenti in uno spazio minore dell' ordinario.*

To calculate the dimension of $S(X, Y)$ in a simple way and to determinate the relation between $T_zS(X, Y)$, T_xX and T_yY, where $z \in \langle x, y \rangle$, $z \neq x$, $z \neq y$, $x \neq y$, we recall the definition of affine cone over a projective variety $X \subset \mathbb{P}^N$.

Let $\pi : \mathbb{A}^{N+1} \setminus \mathbf{0} \to \mathbb{P}^N$ be the canonical projection. If $X \subset \mathbb{P}^N$ is a closed subvariety, we indicate by $C_\mathbf{0}(X)$ *the affine cone over* X, i.e. $C_\mathbf{0}(X) = \pi^{-1}(X) \cup \mathbf{0}$ is the affine variety cut out by the homogeneous polynomials in $N+1$ variables defining X. If $\mathbf{x} \neq \mathbf{0}$ is a point such that $\pi(\mathbf{x}) = x \in X$, then

$$\pi(\mathcal{T}_{\mathbf{x}} C_0(X)) = T_x X.$$

Moreover, if $L_i = \pi(U_i)$, $i = 1, 2$, U_i vector subspace of $\mathbb{A}^{N+1}$, then by definition $< L_1, L_2 >= \pi(U_1 + U_2)$, where $+ : \mathbb{A}^{N+1} \times \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is the vector space operation. Therefore, thought as a morphism of algebraic varieties, the differential of the sum coincides with the operation, i.e.

$$d_{(\mathbf{x},\mathbf{y})} : \mathcal{T}_{(\mathbf{x},\mathbf{y})}(\mathbb{A}^{N+1} \times \mathbb{A}^{N+1}) = \mathcal{T}_{\mathbf{x}}\mathbb{A}^{N+1} \times \mathcal{T}_{\mathbf{y}}\mathbb{A}^{N+1} \to \mathcal{T}_{\mathbf{x}+\mathbf{y}}\mathbb{A}^{N+1}$$

is the sum of the corresponding vectors.

With the above notations we have

$$\overline{C_0(X) + C_0(Y)} = C_0(S(X,Y)). \tag{1.3.1}$$

We are now in position to prove the so called Terracini Lemma. The original proof of Terracini relies on the study of the differential of the second projection morphism $p_2 : S_{X,Y} \to S(X,Y)$. Here we follow Ådlandsvik, [**Åd**], to avoid the "difficulty", if any, of writing the tangent space at a point $(x, y, z) \in S^0_{X,Y}$. When writing $z \in < x, y >$, we always suppose $x \neq y$.

1.3.1. THEOREM. (**Terracini Lemma**) *Let $X, Y \subset \mathbb{P}^N$ be irreducible subvarieties. Then:*

(1) *for every $x \in X$, for every $y \in Y$, $x \neq y$, and for every $z \in < x, y >$,*
$$< T_x X, T_y Y > \subseteq T_z S(X, Y);$$
(2) *if $\operatorname{char}(K) = 0$, there exists an open subset U of $S(X, Y)$ such that*
$$< T_x X, T_y Y >= T_z S(X, Y)$$
for every $z \in U$, $x \in X$, $y \in Y$, $z \in < x, y >$. In particular
$$\dim(S(X, Y)) = \dim(X) + \dim(Y) - \dim(T_x X \cap T_y Y)$$
for $x \in X$ and $y \in Y$ general points.

PROOF. The first part follows from equation 1.3.1 and from the interpretation of the differential of the affine sum. The second part from generic smoothness applied to the affine cones over X, Y and $S(X, Y)$. □

Since we have quoted the original form given by Terracini, let us state it as an obvious corollary.

1.3.2. COROLLARY. ([T1]) *Let $X \subset \mathbb{P}^N$ be an irreducible subvariety of $\mathbb{P}^N$. Then:*

(1) *for every $x_0, \ldots, x_k \in X$ and for every $z \in < x_0, \ldots, x_k >$,*
$$< T_{x_0} X, \ldots, T_{x_k} X > \subseteq T_z S^k X;$$
(2) *if $\operatorname{char}(K) = 0$, there exists an open subset U of $S^k X$ such that*
$$< T_{x_0} X, \ldots, T_{x_k} X >= T_z S^k X$$
for every $z \in U$, $x_i \in X$, $i = 0, \ldots, k$, $z \in < x_0, \ldots, x_k >$. In particular
$$\dim(SX) = 2 \dim(X) - \dim(T_x X \cap T_y X)$$
for $x, y \in X$ general points.

The first application we give is the so called *Trisecant Lemma.* Let us recall that a line $l \subset \mathbb{P}^N$ is said to be a *trisecant line to* $X \subset \mathbb{P}^N$ if $\text{length}(l \cap X) \geq 3$.

1.3.3. PROPOSITION. **(Trisecant Lemma)** *Let $X \subset \mathbb{P}^N$ be a non-degenerate, irreducible closed subvariety. Suppose* $\text{char}(K)=0$ *and* $\text{codim}(X) > k$. *Then a general* $(k+1)$*-secant* $\mathbb{P}^k$, $< x_0, \dots, x_k >= L = \mathbb{P}^k$, *is not* $(k+2)$*-secant, i.e.* $L \cap X = \{x_0, \dots, x_k\}$ *as schemes. In particular, if* $\text{codim}(X) > 1$, *the projection of X from a general point on it,* $\pi_x : X \dashrightarrow X' \subset \mathbb{P}^{N-1}$, *is a birational map.*

PROOF. We claim that it is sufficient to prove the result for $k = 1$. Indeed X is not linear so that by taking a general $x \in X$ and projecting X from this point we get a non-degenerate, irreducible subvariety $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$ with $\text{codim}(X') = \text{codim}(X) - 1 > k - 1$. If the general $L =< x_0, \dots, x_k >$ as above were $k+2$-secant, by taking $x = x_k$, the linear space $< \pi_x(x_0), \dots, \pi_x(x_{k-1}) >= \mathbb{P}^{k-1} = L'$ would be a general k-secant $\mathbb{P}^{k-1}$, which results to be $(k+1) = ((k-1)+2)$-secant. So we can assume $k = 1$ and we set $n = \dim(X)$.

Take $x \in X \setminus \text{Vert}(X)$. Then a general secant line through x, $l =< x, y >$, is not tangent to X neither at x nor at y. If l is a trisecant line then necessarily it exists $u \in (l \cap X) \setminus \{x, y\}$. Consider the projection of X from x. Since $x \notin \text{Vert}(X)$, if $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$, then $\dim(X') = \dim(X)$ and $\pi_x(y) = \pi_x(u) = x'$ is a general smooth point of X'. By generic smoothness

$$< x, T_{x'}X' >=< x, T_y X >=< x, T_u X >$$

so that $T_y X$ and $T_u X$ are hyperplanes in $< x, T_{x'}X' >= \mathbb{P}^{n+1}$ so that

$$\dim(T_y X \cap T_u X) = n - 1.$$

Taking $z \in< x, y >=< y, u >$ general, we have a point in the set U specified in corollary 1.3.2 yielding $\dim(SX) = \dim(T_z SX) = \dim(< T_y X, T_u X >) = n + 1$. This implies $\text{codim}(X) = 1$ by proposition 1.2.2 part 3). The last part follows from the fact that a generically one-to one morphism is birational if $\text{char}(K)=0$, being generically étale. □

As a second application we reinterpret Terracini Lemma as tangency of tangent space to higher secant varieties at a general point along the locus described on X by the secant spaces passing through the point. To this aim we first define tangency along a subvariety and then the entry loci described above, studying their dimension.

1.3.4. DEFINITION. **(Tangencies along a subvariety)** Let $Y \subset X$ be a closed (irreducible) subvariety of X and let $L = \mathbb{P}^l \subset \mathbb{P}^N$, $l \geq \dim(X)$, be a linear subspace.

The linear space L is said to be *tangent to X along Y* if for every $y \in Y$

$$T_y X \subseteq L,$$

i.e. if and only if $T(Y, X) \subseteq L$.

The linear space L is said to be *J-tangent to X along Y* if for every $y \in Y$

$$T_y^* X \subseteq L,$$

i.e. if and only if $T^*(Y, X) \subseteq L$.

Clearly if L is tangent to X along Y, it is also J-tangent to X along Y.

In the case $L = \mathbb{P}^{N-1}$, the scheme-theoretic intersection $L \cap X = D$ is a divisor, i.e. a subscheme of pure dimension $\dim(X) - 1$. By definition, for every $y \in D$, we have $T_yD = T_yX \cap L$ so that, if X is a smooth variety, $L = \mathbb{P}^{N-1}$ is tangent to X exactly along $\mathrm{Sing}(D) = \{y \in D : \dim(T_yD) > \dim(D)\}$.

We define the important notions of entry loci and k-secant defect and we study their first properties.

1.3.5. DEFINITION. **(Entry loci and k-secant defect δ_k)** Let $X \subset \mathbb{P}^N$ be a closed irreducible non-degenerate subvariety. Let us recall the diagram defining the higher secant varieties S^kX as join of X with $S^{k-1}X$:

$$\begin{tikzcd} & S^k_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times S^{k-1}X & & \mathbb{P}^N. \end{tikzcd}$$

Let us define $\phi : X \times S^{k-1}X \to X$ to be the projection onto the first factor of this product.

Then, for $z \in S^kX$, *the k-entry locus of X with respect to z* is the scheme theoretic image

$$\Sigma^k_z = \Sigma^k_z(X) := \phi(p_2(p_1^{-1}(z))). \tag{1.3.2}$$

Geometrically, the support of Σ^k_z is the locus described on X by the $(k+1)$-secant $\mathbb{P}^k$ of X passing through $z \in S^kX$. If $z \in S^kX$ is general, then through z there passes an ordinary $(k+1)$-secant $\mathbb{P}^k$, i.e. given by $k+1$ distinct points on X and we can describe the support of Σ^k_z in this way

$$(\Sigma^k_z)_{\mathrm{red}} = \overline{\{x \in X : \exists x_1, \ldots, x_k \text{ distinct and } z \in < x, x_1, \ldots, x_k >\}}.$$

Moreover, by the theorem of the dimension of the fibers for general $z \in S^kX$, the support of Σ^k_z is equidimensional and every irreducible component contains ordinary $\mathbb{P}^k$'s since necessarily $\mathrm{codim}(X) > k$, see proposition 1.3.3. If char(K)=0 and for general $z \in S^kX$ the scheme $p_1^{-1}(z)$ is smooth so that Σ^k_z is reduced.

To recover the scheme structure of Σ^k_z geometrically, one could define Π_z as the locus of $(k+1)$-secant $\mathbb{P}^k$'s through z and define $\Sigma^k_z = \Pi_z \cap X$ as schemes. For example if through $z \in SX$ there passes a unique tangent line l to X, then in this way we get $\Pi_z = l$ and $\Sigma_z = l \cap X$ the point of tangency with the double structure.

Let us study the dimension of Σ^k_z for $z \in S^kX$ general. Before let us remark that if $x \in \Sigma^k_z$ is a general point in an irreducible component, $z \in S^kX$ general, then, as sets,

$$\phi^{-1}(x) = \dim(\{y \in S^{k-1}X : z \in < x, y >\} = < z, x > \cap S^{k-1}X \neq \emptyset$$

and $\dim(\phi^{-1}(x)) = 0$ because $z \in S^kX \setminus S^{k-1}X$ by the generality of z.

Then we define *the k-secant defect of* X, $1 \leq k \leq k_0(X)$, $\delta_k(X)$, as the integer

$$\delta_k(X) = \dim(\Sigma^k_z) = \dim(p_2(p_1^{-1}(z))) = s_{k-1}(X) + \dim(X) + 1 - s_k(X), \tag{1.3.3}$$

where $z \in S^kX$ is a general point.

For $k = 1$, we usually put $\Sigma_z = \Sigma^1_z$, $z \in SX$, and $\delta(X) = \delta_1(X) = 2\dim(X) + 1 - \dim(SX)$; for $k = 0$, $\delta_0(X) = 0$.

Let us reinterpret Terracini Lemma with these new definitions.

1.3.6. COROLLARY. **(Tangency along the entry loci)** *Let X be an irreducible non-degenerate closed subvariety. Let $k < k_0(X)$, i.e. $S^kX \subsetneq \mathbb{P}^N$, and let $z \in S^kX$ be a general point. Then:*

(1) *the linear space T_zS^kX is tangent to X along $(\Sigma_z^k)_{\rm red} \setminus \mathrm{Sing}(X)$;*
(2) *$\delta_k(X) < \dim(X)$;*
(3) *$\delta_{k_0}(X) = \dim(X)$ if and only if $s_{k_0-1}(X) = N-1$, i. e. if and only if $S^{k_0-1}X$ is an hypersurface;*
(4)
$$s_k(X) = (k+1)(n+1) - 1 - \sum_{i=1}^{k} \delta_i(X) = \sum_{i=0}^{k} (\dim(X) - \delta_i(X) - 1);$$
(5) (cfr. 1.2.3) *if X is a curve, $s_k(X) = 2k+1$.*

PROOF. Part 1) is clearly a restatement of part 1) of corollary 1.3.2 when we take into account the geometrical properties of Σ_z^k, $z \in S^kX$ general, described in the definition of entry loci and the fact that the locus of tangency of a linear space is closed in $X \setminus \mathrm{Sing}(X)$, see also definition 1.5.8. Recall that if char(K)=0, the scheme Σ_z^k is reduced.

If $\dim(\Sigma_z^k) = \delta_k(X) = \dim(X)$, then a general tangent space to S^kX would be tangent along X and X would be degenerated.

With regard to 3), we remark that $\delta_{k_0}(X) = s_{k_0-1}(X) + \dim(X) + 1 - N$ so that $\dim(X) - \delta_{k_0}(X) = N - 1 - s_{k_0-1}(X)$.

Part 4) is an easy computation by induction, while part 5) follows from part 4) since for a curve $\delta_k(X) < \dim(X)$ yields $\delta_k(X) = 0$. □

1.3.7. REMARK. The statement of part 1) cannot be improved. Take for example a cone $X \subset \mathbb{P}^5$ of vertex a point $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$ over a smooth non-degenerate projective curve $C \subset \mathbb{P}^4$. If $z \in S(p, SC) = SX$ is general and if $z \in < x, y >$, $x, y \in X$, it is not difficult to see that $\Sigma_z(X) =< p, x > \cup < p, y >$. The hyperplane T_zSX is tangent to X at x and y by Terracini Lemma, so that it is tangent to X along the rulings $< p, x >$ and $< p, y >$ minus the point p. Since $T_pX = \mathbb{P}^5$, the hyperplane T_zSX is not tangent to X at p (neither J-tangent to X at p).

A phenomenon studied classically firstly by Scorza, [**S1**], [**S2**], [**S4**], and then by Terracini, [**T2**] is the case in which imposing tangency of a hyperplane at $k+1$ general points, $k \geq 0$, of a variety $X \subset \mathbb{P}^N$ forces tangency along a positive dimensional variety, even if $\delta_k(X) = 0$. Indeed, Terracini Lemma says that if $\delta_k(X) > 0$, $k < k_0(X)$, than a hyperplane tangent at $k+1$ points, becomes tangent along the corresponding entry locus. The interesting and exceptional behaviour occurrs for varieties with $\delta_k(X) = 0$. The first examples are the tangent developable to a non-degenerate curve or cones of arbitrary dimension. Indeed they are 0-defective as every variety but by imposing tangency at a general point, we get tangency along the ruling passing through the point.

Varieties for which a hyperplane tangent at $k+1$, $k \geq 0$, general points is tangent along a positive dimensional subvariety are called *k-weakly defective varieties*, according to Chiantini and Ciliberto, [**CC**]. In [**CC**] many interesting properties of these varieties are investigated and a refined Terracini Lemma is proved, also putting in modern terms the classification of k-weakly defective irreducible surfaces obtained classically by Scorza, [**S2**], and Terracini, [**T2**]. Let us remark that, as shown in [CC], there exist smooth varieties of dimension greater than one which are k-weakly defective but not k-secant defective for every $k \geq 1$.

As another application, we study the dimension of the projection of a variety from linear subspaces generated by general tangent spaces. Terracini Lemma says that we are projecting from a general tangent space to the related higher secant variety. As we have seen when the center of projection L cuts the variety it is difficult to control the dimension of the image of X under projection because we do not know a priori how a general tangent space intersects L. In the case of $L = T_z S^{k-1}X$ this information is encoded in the dimension of S^kX and of the defect $\delta_k(X)$ as we immediately see. In chapter 5 we shall see how the degree of the projections from T_zS^kX is related to the number of $(k+2)$-secant $\mathbb{P}^{k+1}$ passing through a general point of $S^{k+1}X$, a problem dubbed as *Bronowski's conjecture*, [B1], and partially solved in [CMR]. Projections from tangent spaces, or more generally from T_zS^kX, were a classical tool of investigation, [Ca], [E1], [S1], [S4], [B1], [B2], and were recently used to study classical and modern problems, [CC], [CMR], [CR2].

1.3.8. PROPOSITION. **(Projections from tangent spaces)** Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate closed subvariety. Let $n = \dim(X)$ and suppose $\mathrm{char}(K)=0$ and $N \geq s_k$, $k \geq 1$, where $s_k = s_k(X)$. Set $\delta_k = \delta_k(X)$. Let $x_1, \ldots, x_k \in X$ be k general points and let $L =< T_{x_1}, \ldots, T_{x_k} >$ and $\pi_k = \pi_L$: $X \to X' \subset \mathbb{P}^{N-s_{k-1}(X)-1}$. Then $\dim(L) = s_{k-1}(X) = s_{k-1}$ and, if $X'_k = \pi_k(X) \subset \mathbb{P}^{N-s_{k-1}-1}$, then

(1) $\dim(X'_k) = s_k - s_{k-1} - 1 = n - \delta_k$;
(2) suppose $N \geq (k+1)n+k$ and and $s_{k-1} = kn+k-1$, i.e. if $\delta_{k-1} = 0$. Then $s_k = (k+1)n + k$ (or equivalently $\delta_k = 0$) if and only if $\dim(X'_k) = n$; if and only if $\pi_k : X \dashrightarrow X'_k \subset \mathbb{P}^{N-kn+k}$ is dominant. In particular if $N = (k+1)n + k$ and if $s_{k-1} = kn + k - 1$, then $S^kX = \mathbb{P}^{(k+1)n+k}$ if and only if $\pi_k : X \dashrightarrow \mathbb{P}^n$ is dominant.

PROOF. If $z \in< x_1, \ldots, x_k >$ is a general point, then z is a general point of $S^{k-1}X$ and by Terracini lemma $s_{k-1} = \dim(T_xS^{k-1}X) = \dim(< T_{x_1}, \ldots, T_{x_k} >)$. By equation 1.2.7 we get $\dim(X'_k) = \dim(S(T_zS^{k-1}X, X)) - s_{k-1} - 1 = s_k - s_{k-1} - 1 = n - \delta_k$. The other claims are only reformulations of part 1). □

A complete description of $S^lX'_k$ in terms of higher secant varieties of X is possible and the dimensions $s_l(X'_k)$ are easily expressible as functions of the $s_m(X)$, i.e. $\delta_k(X'_k)$ is controlled by $\delta_m(X)$ and viceversa. These remarks and the possibility of constructing explicit rational maps reveals the importance of projections from tangent spaces.

We study via Terracini Lemma the tangent space to the entry locus of SX at a general point of it. As a minimal generalization we can define the projections onto the i-factor $\phi_i : X_1 \times X_2 \to X_i$ and for $z \in S(X_1, X_2)$, define $\Sigma_z(X_i) = \phi_i(p_2(p_1^{-1}(z)))$, where the morphism p_i's are the map used for the definition of the join. We remark that $\dim(\Sigma_z(X_1)) = \dim(\Sigma_z(X_2)) = \dim(X_1) + \dim(X_2) + 1 - \dim(S(X_1, X_2))$. With these notations we get the following result.

1.3.9. PROPOSITION. *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties and assume* $\mathrm{char}(K)=0$. *Suppose $S(X,Y) \supsetneq X$ and $S(X,Y) \supsetneq Y$ to avoid trivialities.*

If $z \in S(X,Y)$ is a general point, if $x \in \Sigma_z(X)$ is a general point and if $< z, x > \cap Y = y \in \Sigma_z(Y)$, then y is a smooth point of $\Sigma_z(Y)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(Y) >= T_xX \cap < x, T_yY >,$$

$$T_y\Sigma_z(Y) = T_yY \cap < y, T_x\Sigma_z(X) >= T_yY \cap < y, T_xX >$$

and

$$T_xX \cap T_yY = T_x\Sigma_z(X) \cap T_y\Sigma_z(Y).$$

In particular for $z \in SX$ general point, X not linear, and for $x \in \Sigma_z(X)$ general point, we have that, if $< x, z > \cap X = y \in \Sigma_z(X)$, then y is a smooth point of $\Sigma_z(X)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(X) >= T_xX \cap < x, T_yY >$$

and

$$T_xX \cap T_yX = T_x\Sigma_z(X) \cap T_y\Sigma_z(X).$$

PROOF. Let us remark that by assumption and by the generality of z and of x, we can suppose that $y \notin T_xX$ and that $x \notin T_yY$.

Take $S(z, \Sigma_z(X)) = S(z, \Sigma_z(Y))$. Then $\dim(S(z, \Sigma_z(X))) = \dim(\Sigma_z(X)) + 1$. If $u \in < z, x >=< z, y >$ is a general point, then $T_uS(z, \Sigma_z(X)) =< z, T_x\Sigma_z(X) >= \mathbb{P}^{\dim(S(z,\Sigma_z(X)))}$ because $z \notin T_xX$. In particular u is a smooth point of $S(z, \Sigma_z(X))$. By Terracini Lemma, we get $T_uS(z, \Sigma_z(X)) \supseteq < z, T_y\Sigma_z(Y) >$, which together with $z \notin T_yY$ yields $\dim(T_y\Sigma_z(Y)) = \dim(\Sigma_z(Y))$ so that $y \in \Sigma_z(Y)$ is a smooth point. Moreover,

$$T_x\Sigma_z(X) \subseteq T_uS(z, \Sigma_z(Y)) =< z, T_y\Sigma_z(Y) >=< x, T_y\Sigma_z(Y) >\subseteq< x, T_yY > .$$

Since $T_x\Sigma_z(X) \subseteq T_xX$, to conclude it is enough to observe that

$$\begin{aligned}\dim(T_xX \cap < x, T_yY >) &= \dim(X) + \dim(Y) + 1 - \dim(< T_xX, T_yX >)\\ &= \dim(\Sigma_z(X)) = \dim(T_x\Sigma_z(X)).\end{aligned}$$

The other claims follows from symmetry between x and y or are straightforward. □

1.4. Characterizations of the Veronese surface in $\mathbb{P}^5$ according to del Pezzo, Bertini and Severi and classification of algebraic varieties in $\mathbb{P}^N$, $N \geq \dim(X) + 3$ with $\dim(SX) = \dim(X) + 2$

In this section, as a beautiful application of the definitions and tools introduced in this chapter, we prove various characterizations of the Veronese surface in $\mathbb{P}^5$ among irreducible non-degenerate surfaces in $\mathbb{P}^N$, not cones, $N \geq 5$, having special geometrical properties. We also classify varieties in $\mathbb{P}^N$, $N \geq \dim(X) + 3$ with $\dim(SX) = \dim(X) + 2$, a result due to Edwards for $\dim(X) \geq 3$, **[Ew]**, and outlined and essentially solved by Scorza in **[S1]**, as we shall see below. These results serve also as a motivation for the further generalizations of this classical material in the next chapters.

The proof we propose here is the most "elementary" we are aware of since it not based on any result involving dual varieties, contact loci, flatness and so on. We essentially use the previous results and the elementary fact that for an irreducible curve, not a line, supposing char(K)=0, a general tangent line to the curve at a point is tangent to it only at that point. This is an easy property which is immediately reduced to the analogous statement for plane curves by a linear projection. For plane curves it simply says that the dual curve of a plane curve has only a finite number of singular points. We followed a suggestion of Gaetano Scorza in [S1], footnote at page 197: *"Non mi sembra inutile far notare come partendo da un' osservazione analoga a quella del testo si possa arrivare alla dimostrazione del teorema del prof. Del Pezzo [n.d.A.: e del Prof. Edwards] in modo abbastanza rapido e semplice".*

1.4.1. THEOREM. **(Characterizations of the Veronese surface)** *Let $X \subset \mathbb{P}^N$, $N \geq 5$, be a non-degenerate irreducible surface, not a cone. Then $N = 5$ and X is projectively equivalent to the Veronese surface in $\mathbb{P}^5$, $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, if and only if one of the following equivalent conditions holds:*

(1) *if $x, y \in X$ are general points, then $T_xX \cap T_yX \neq \emptyset$* ([**dP**]);
(2) $\dim(SX) = 4$ ([**Sev1**]);
(3) *X contains a two dimensional family of irreducible conics* ([**Be**], pg. 392).

First of all, by Terracini Lemma if 1) holds, then $\dim(SX) \leq 4$ but since X is non-degenerate part 3) of proposition 1.2.2 implies $\dim(SX) = 4$. By Terracini Lemma 2) implies that $T_xX \cap T_yX$ consists of a point. Also condition 3) implies 1) (or 2)). Indeed, there exists at least a conic $C_{x,y}$ passing through the general points x and y so that $p_{x,y} := T_xC_{x,y} \cap T_yC_{x,y} \subseteq T_xX \cap T_yX$ and in fact equality holds. So it will sufficient to show that if $T_xX \cap T_yX = p_{x,y}$ is a point, then X is projectively equivalent to the Veronese surface, which is Del Pezzo's theorem, [**dP**] and [**Be**], pg. 394. During the proof of the preliminaries lemma the apparently more general fact that X contains a two dimensional linear system of Cartier divisors of selfintersection 1, which are conics in the fixed embedding, is seen to be a consequence of condition 1). This is essentially also Bertini's proof that 3) characterizes the Veronese surface, see [**Be**], pg. 392. So all the equivalences and the necessary tools will be established.

Let us recall that if $X \subset \mathbb{P}^N$ is an irreducible projective non-degenerate variety of dimension $n = \dim(X)$, then $\dim(SX) \geq n+1$ and that equality implies $N = n+1$, see proposition 1.2.2. Hence if $\operatorname{codim}(X) > 1$, $\dim(SX) \geq n+2$. Suppose $\dim(SX) = n+2$. If $N = n+2$, then $SX = \mathbb{P}^N$ and there is no particular restriction on X and clearly there exist infinitely many examples. If $N > n+2$ the complete classification of varieties with $\dim(SX) = n+2$ is contained in the following theorem of Scorza-Edwards.

1.4.2. THEOREM. (Scorza, [**S1**], Edwards,[**Ew**]) *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension $n = \dim(X) \geq 3$. Assume $N \geq n+3$ and that $\dim(SX) = n+2$. Then either X is a cone over a curve or $N = n+3$ and X is a cone over the Veronese surface in $\mathbb{P}^5$. On the contrary such varieties enjoy those geometrical properties.*

Equivalently, $X \subset \mathbb{P}^N$, $N \geq n+3$, is a cone over a curve or a cone over the Veronese surface in $\mathbb{P}^5$ if and only if it contains an irreducible two dimensional family of divisors which are quadric hypersurfaces in the fixed embedding. The general member of this family is a reducible quadric if and only if X is a cone over a curve.

Once again, it is clear that if through two general point there passes a quadric hypersurface of dimension $n-1$, then for a general point $z \in SX$,

$$2n + 1 - \dim(SX) = \dim(\Sigma_z(X)) \geq n - 1,$$

yields $\dim(SX) \leq n+2$ and hence equality by the non-degenerateness of $X \subset \mathbb{P}^N$ and by the hypothesis $N \geq n+3$. The other implication will follow once again by the next lemma of Scorza.

1.4.3. LEMMA. **(Scorza Lemma, [S1], footnote pg. 197)** *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension $n = \dim(X)$. Suppose $N \geq n+3$ and $\dim(SX) = n+2$. Then:*

(1) *the closure of the general fiber of the tangential projection of X from a general point $x \in X$ onto the irreducible curve $C_x \subset \mathbb{P}^{N-n-1}$, $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^{N-n-1}$ is either a $\mathbb{P}^{n-1}$ or an irreducible quadric hypersurface of dimension $n-1$. The first case occurs if and only if X is a cone over a curve. Moreover, X contains a two dimensional family of quadric hypersurfaces, whose general member is the union of two $\mathbb{P}^{n-1}$ if and only if X is a cone over a curve.*
(2) *If X is not a cone over a curve, then C_x is an irreducible conic, so that $N = n+3$ and the general fiber of $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ is an irreducible quadric hypersurface of dimension $n-1$.*

PROOF. Let $x, y \in X$ general points. The image of the tangential projection $\pi_x : X \dashrightarrow X'_x = C_x \subset \mathbb{P}^{N-n-1}$, or of $\pi_y : X \dashrightarrow X'_y = C_y \subset \mathbb{P}^{N-n-1}$, is an irreducible non-degenerate curve by proposition 1.3.8. Moreover π_x is defined at y, respectively π_y is defined at x, since being general points they do not belong to $\mathrm{Vert}(X)$. Let F_y denote the closure of the irreducible component of the fiber of π_x passing through y, respectively F_x denote the closure of irreducible component of the fiber of π_y passing through x. By generic smoothness they are reduced varieties of dimension $n-1$ since they are generically smooth irreducible varieties of dimension $n-1$. Moreover, by definition of π_x, respectively π_y, we have $F_y \subset < T_xX, y > \cap X$, respectively $F_x \subset < T_yX, x > \cap X$.

If $F_x \subseteq T_xX \cap X$, then by the generality of x and y, $F_y \subseteq T_yX \cap X$ so that $F_x \cup F_y \subset \Pi_{x,y} := < T_xX, y > \cap < T_yX, x > = \mathbb{P}^n$.

Suppose that F_x is not contained in $T_xX \cap X$ so that also F_y is not contained in $T_yX \cap X$. Let $\mathcal{C}_x = S(T_xX, C_x)$ be the cone over C_x of vertex T_xX and let $\mathcal{C}_y = S(T_yX, C_y)$ be the cone over C_y of vertex T_yX. By the generality of x, respectively y, the point $\pi_x(y) \in C_x$, respectively $\pi_y(x) \in C_y$, is a general point on C_x, respectively on C_y, so that the tangent space

$$< T_xX, T_{\pi_x(y)}C_x > = < T_xX, T_yX > = < T_{\pi_y(x)}C_y, T_yX >$$

is tangent to $\mathcal{C}_x$, respectively $\mathcal{C}_y$, exactly along $< T_xX, \pi_x(y) > \setminus T_xX = < T_xX, y > \setminus T_xX$, respectively $< T_yX, \pi_y(x) > \setminus T_yX = < T_yX, x > \setminus T_yX$. Recall that a general tangent line to a curve is tangent to it only at one point. Since $X \subseteq \mathcal{C}_x \cap \mathcal{C}_y$, the locus $Y \subset X$ of smooth points of $X \setminus ((T_xX \cap X) \cup (T_yX \cap X)$ at which the linear space $< T_xX, T_yX >$ is tangent is contained in the linear space $\Pi_{x,y} = < T_xX, y > \cap < T_yX, x > = \mathbb{P}^n$. In our hypothesis, F_x, respectively F_y, has an open dense set in common with Y, yielding that $F_x \cup F_y$ is contained in $\Pi_{x,y}$.

More generally, using the same argument, we get that the closure of the fibers $\pi_x^{-1}(\pi_x(y))$ and of $\pi_y^{-1}(\pi_y(x))$ are both contained in $\Pi_{x,y}$. In conclusion, in any case $F_x \cup F_y \subseteq \overline{\pi_y^{-1}(\pi_y(x))} \cup \overline{\pi_x^{-1}(\pi_x(y))} \subseteq \Pi_{x,y} = \mathbb{P}^n$.

Since the secant line $< x, y >$ is general, it is not a trisecant line. The line $< x, y >$ is contained in $\Pi_{x,y}$ so that

$$2 \geq \deg(\overline{\pi_x^{-1}(\pi_x(y))} \cup \overline{\pi_y^{-1}(\pi_y(x))}) \geq \deg(F_x \cup F_y) \geq 2,$$

where the last inequality holds since it cannot clearly be $F_x = F_y = \mathbb{P}^{n-1}$, the line $< x, y >$ being a proper secant line. In conclusion, either $\overline{\pi_y^{-1}(\pi_y(x))} = F_x = F_y = \overline{\pi_x^{-1}(\pi_x(y))}$ is an irreducible quadric hypersurface of dimension $n-1$ passing through x and y or $F_x = \overline{\pi_y^{-1}(\pi_y(x))}$ and $F_y = \overline{\pi_x^{-1}(\pi_x(y))}$ are $\mathbb{P}^{n-1}$'s intersecting in the linear space $L_{x,y} = T_xX \cap T_yX = \mathbb{P}^{n-2}$. In the last case the linear space $L = L_{x,y}$ does not vary by moving y in X because otherwise the linear spaces F_y would describe a $\mathbb{P}^n$ contained in X. Then $\mathbb{P}^{n-2} = L \subseteq \mathrm{Vert}(X) = \mathbb{P}^l$, $l \leq n-2$, forces $\mathrm{Vert}(X) = \mathbb{P}^{n-2}$ so that X is a cone over a curve by proposition 1.2.6. On the contrary if X is a cone over a curve, clearly the $\mathbb{P}^{n-1}$'s passing through two general points $x, y \in X$ are contracted by π_y, respectively π_x, so that $F_x \cup F_y$ is a reduced quadric hypersurface.

To prove part 2) it suffices to remark that if X is not a cone over a curve, then by the previous analysis two general points on X are connected by an irreducible quadric hypersurface, which dominates C_x, so that, since $X \subset \mathbb{P}^N$ is non-degenerate, $C_x \subset \mathbb{P}^{N-n-1}$ is an irreducible non-degenerate conic, yielding $N - n - 1 = 2$. □

As a corollary of Scorza Lemma we get the information about the entry locus of a variety $X \subset \mathbb{P}^N$, $N \geq n+3$ with $\dim(SX) = n+2$, the original key observation of Severi for $n = 2$ in his proof of the characterization of the Veronese surface, see [**Sev1**].

1.4.4. Corollary. ([**Sev1**]) *Let $X \subset \mathbb{P}^N$, $N \geq n+3$, be an irreducible non-degenerate projective variety of dimension n such that* $\dim(SX) = n+2$. *Let $z \in SX$ be a general point and let notations as in lemma 1.4.3. Then $\Sigma_z(X) = F_x \cup F_y$ is a quadric hypersurface which is reducible if and only if X is a cone over a curve.*

Proof. Let notations as in the above lemma. Then if $z \in < x, y >$ is general, by corollary 1.3.6 $T_zSX =< T_xX, T_yX >$ is tangent to X along $\Sigma_z(X) \setminus \mathrm{Sing}(X)$. By Scorza Lemma $\Sigma_z(X) \setminus \mathrm{Sing}(X) \subset \Pi_{x,y} = \mathbb{P}^n$ so that $\Sigma_z(X)$ is a hypersurface of degree at least 2 in $\Pi_{x,y}$ since $z \notin X$ (it it were a linear space, then $SX = X$ and X would be linear). Then $\Sigma_z(X)$ is a quadric hypersurface by the trisecant lemma and the conclusion follows by arguing as in the previous lemma. □

We restrict ourselves for a moment to the case of surfaces and prove that, if X is not a cone, two general fibers of $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ are linearly equivalent Cartier divisors intersecting transversally at x; and more precisely that every fiber of π_x is a smooth conic so that the closure of two arbitrary fibers are linear equivalent Cartier divisors, which are smooth conics in the fixed embedding.

1.4.5. Lemma. (Bertini, [**Be**]) *Let $X \subset \mathbb{P}^5$ be a non-degenerate irreducible projective surface, not a cone, such that* $\dim(SX) = 4$. *Then $(T_xX \cap X)_{\mathrm{red}} = x$, the closure of every fiber of $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$ is a smooth conic and two fibers of π_x are linearly equivalent Cartier divisors on X intersecting transversally at x.*

PROOF. Suppose that for a general point $x \in X$, there exists $p_x \in T_xX \cap X$, $p_x \neq x$. Fix a general x and take a general point $y \in X$. By lemma 1.4.3, if $\pi_x : X \dashrightarrow C_x$ is the tangential projection, then C_x is a smooth conic. Take the line $< y, p_y >$. Thus π_x is defined at y, since $\mathrm{Vert}(X) = \emptyset$ and since x, y are general points. It cannot be $\pi_x(y) \neq \pi_x(p_y)$, because otherwise the line $< y, p_y >$ would not cut T_xX so that it would project onto $T_{\pi_x(y)}C_x$ and this line would cut C_x at least in 3 points counted with multiplicity, contrary to the fact that C_x is a conic. If $\pi_x(y) = \pi_x(p_y)$, then the line $< y, p_y >$ cuts T_xX necessarily at $p_{x,y} = T_xX \cap T_yX$ and the line $< y, p_y >=< y, p_{x,y} >= T_yF_y$ would cut the smooth conic F_y in at least 3 points counted with multiplicity, which is impossible.

Therefore two fibers of π_x can intersect only at x and they are linearly equivalent divisors by definition. The closure of each fiber is then a Cartier divisor which is a conic on X passing necessarily through $x = (T_xX \cap X)_{\mathrm{red}}$. Since $(T_xX \cap X)_{\mathrm{red}} = x$, there is no line through X and the closure of every fiber is a smooth conic.

If two general fibers meet along a fixed tangent direction $l \subset T_xX$ at x, then the tangent spaces at two general points of these fibers, let us say $y \in X$, respectively $z \in X$, will cut the fixed line in different points since X is not a cone (otherwise $p_{x,y} = p_{x,z} \in \mathrm{Vert}(X)$ and X would be a cone). Then $S(l, SX) = SX$ by Terracini lemma since $\dim(< l, < T_zX, T_yX >>) = \dim(< T_zX, T_yX >) = \dim(SX)$. This forces $x \in l \subseteq \mathrm{Vert}(SX)$, which by the generality of $x \in X$, yields $X \subseteq \mathrm{Vert}(SX) = \mathbb{P}^l$, $l \leq 2$ (recall that SX is not linear and has dimension 4), i.e. $X = \mathbb{P}^2$. □

We can easily prove theorem 1.4.1.

PROOF. (1[st] proof of theorem 1.4.1). By Scorza Lemma and by lemma 1.4.5 the fibers of the tangential projections at x and at y, $x, y \in X$ general points, are linearly equivalent Cartier divisors of selfintersection 1. Moreover, since there exists a conic through x and y which is a fiber of both projections, we constructed a base point free two dimensional linear system of Cartier divisors on X of autointerction 1. The associated morphism $\phi : X \to \mathbb{P}^2$ is birational.

Let $\psi : \mathbb{P}^2 \dashrightarrow X \subset \mathbb{P}^5$ be the composition of $\phi^{-1} : \mathbb{P}^2 \dashrightarrow X$ with the inclusion $i : X \hookrightarrow \mathbb{P}^5$. Since lines in $\mathbb{P}^2$ are mapped into the two dimensional linear system of divisors constructed before, which are conics in the fixed embedding, the map ψ is given by a linear system of conics of dimension 5, i.e. by the complete linear system of conics, so that $\psi : \mathbb{P}^2 \to X$ is an isomorphism and X is projectively equivalent to $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. □

PROOF. (2[nd] proof of theorem 1.4.1). Fix a general point $x \in X$ and consider the tangential projection $\pi_x : X \dashrightarrow C_x \subset \mathbb{P}^2$. This rational map resolves to a morphism $\widetilde{\pi_x} : Bl_xX \to C_x \simeq \mathbb{P}^1$ such that every fiber is isomorphic to $\mathbb{P}^1$, i.e. it is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and in particular Bl_xX and hence X are smooth surfaces. Since Bl_xX contains the (-1)-curve E as a section of $\widetilde{\pi_x}$, then $Bl_xX \to C_x$ is isomorphic as a $\mathbb{P}^1$-bundle to $\pi : \mathbb{F}_1 \to \mathbb{P}^1$. By contracting E we get $X \simeq \mathbb{P}^2$. Since it contains conics in the fixed embedding and $N = 5$, it is necessarily the complete 2-Veronese embedding of $\mathbb{P}^2$. □

The reason for which we included the second proof, apparently more complicated, is for the analogy with the argument used by Mori to prove Hartshorne's conjecture that $\mathbb{P}^N$ is the only smooth projective variety of dimension N having ample tangent bundle, see [Mo1]. There one shows that for a general point $x \in X$ $\mathbb{P}(T_xX) \simeq \mathbb{P}^{N-1}$ and then by blowing-up x, it proves that $Bl_xX \to \mathbb{P}^{N-1}$ is a $\mathbb{P}^1$-bundle, see *loc. cit.*

Now we can prove theorem 1.4.2.

PROOF. (of theorem 1.4.2) Suppose X is not a cone over a curve. By lemma 1.4.3 we get $N = n+3$ and that through x there passes a line. In fact $T_xF_x \cap F_x \subset X$ is a quadric in $T_xF_x = \mathbb{P}^{n-1}$ and since $n-1 \geq 2$, through the point x there passes at least a line $l_x \subset T_xX \cap F_x \subset X$. Then X is a cone by lemma 1.2.7 and since it is not a cone over a curve, its linear section with a general $\mathbb{P}^5 \subset \mathbb{P}^{n+3}$ is an irreducible non-degenerate surface $Y \subset \mathbb{P}^5$, which is not a cone, and such $\dim(SY) \leq 4$. By theorem 1.1.6 $Y \subset \mathbb{P}^5$ is a Veronese surface and the conclusion follows. □

It is worth of note also the following geometrical characterization of the Veronese surface given by Ran: it is the unique smooth surface in $\mathbb{P}^5$ which is not contained in an irreducible 3-fold non-singular along the surface, see [Ra].

1.5. Dual varieties and contact loci of general tangent linear spaces

Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension n; let $\mathrm{Sm}(X) := X \setminus \mathrm{Sing}(X)$ be the locus of non-singular points of X. By definition $\mathrm{Sm}(X) = \{x \in X \,:\, \dim(T_xX) = n\}$.

If we take an hyperplane section of X, $Y = X \cap H$, where $H = \mathbb{P}^{N-1}$ is an arbitrary hyperplane, then for every $y \in Y$ we get

$$T_yY = T_yX \cap H. \tag{1.5.1}$$

Since Y is a pure dimensional scheme of dimension $n-1$, we see that $\mathrm{Sing}(Y) \setminus (\mathrm{Sing}(X) \cap H) = \{y \in Y \setminus \mathrm{Sing}(X) \cap Y \,:\, T_yX \subseteq H\}$, which is an open subset in the locus of points of X at which H is tangent. In particular to show that an hyperplane section has non-singular points, we have to exhibit an hyperplane H which is not tangent at all the points in which it intersects X. It naturally arises the need of patching together all the "bad" hyperplanes and eventually show that there always exists an hyperplane section of X, non-singular at least outside $\mathrm{Sing}(X)$. Since hyperplane can be naturally thought as points in the dual projective space $(\mathbb{P}^{N*}$, we can define a subvariety of $\mathbb{P}^{N*}$ parametrizing hyperplane sections which are singular also outside $\mathrm{Sing}(X)$. This locus is the so-called *dual variety*.

1.5.1. DEFINITION. **(Dual variety)** Let $X \subset \mathbb{P}^N$ be as above and let

$$\mathcal{P}_X := \overline{\{((x,H) \,:\, x \in \mathrm{Sm}(X),\, T_xX \subseteq H\}} \subset X \times \mathbb{P}^{N*},$$

the so called *conormal variety of* X.

Let us consider the projections of $\mathcal{P}_X$ onto the factors X and $\mathbb{P}^{N*}$,

$$\begin{tikzcd} & \mathcal{P}_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathbb{P}^{N*}. \end{tikzcd}$$

The dual variety to X, X^*, is the scheme-theoretic image of $\mathcal{P}_X$ in $\mathbb{P}^{N*}$, i.e. the algebraic variety

$$X^* := p_2(\mathcal{P}_X) \subseteq \mathbb{P}^{N*}.$$

The set $\mathcal{P}_X$ is easily seen to be a closed subset. For $x \in \mathrm{Sm}(X)$, we have $p_1^{-1}(x) \simeq (T_xX)^* = \mathbb{P}^{N-n-1} \subset \mathbb{P}^{N*}$. Then the set $\mathcal{P}_X$ is irreducible since $p_1^{-1}(\mathrm{Sm}(X)) \to \mathrm{Sm}(X)$ is a $\mathbb{P}^{N-n-1}$-bundle and clearly $\dim(\mathcal{P}_X) = N-1$. Then $\dim(X^*) \leq N-1$ and the *dual defect of* X, $\mathrm{def}(X)$, is defined as

$$\mathrm{def}(X) = N - 1 - \dim(X^*) \geq 0.$$

A variety is said to be *reflexive* if the natural isomorphism between $\mathbb{P}^N$ and $\mathbb{P}^{N**}$ induces an isomorphism between X and $X^{**} = (X^*)^*$.

Let us take $H \in X^*$. By definition

$$C_H := C_H(X) = p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \, : \, T_xX \subset H\}}$$

is exactly the closure of non-singular points of X where H is tangent to X, it is not empty so that $H \cap X$ is singular outside $\mathrm{Sing}(X)$. On the contrary if $H \notin X^*$, the hyperplane section $H \cap X$ can be singular only along $\mathrm{Sing}(X)$. This is the classical "Bertini theorem".

In particular we proved the following result.

1.5.2. THEOREM. *Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension $n =$. Then for every $H \in (\mathbb{P}^N)^* \setminus X^*$ the divisor $H \cap X$ is non-singular outside* $\mathrm{Sing}(X)$.

In particular if X has at most a finite number of singular points $p_1, \ldots, p_m$, then for every $H \notin X^* \cup (p_1)^* \cup \ldots \cup (p_m)^*$, the hyperplane section $H \cap X$ is a non-singular subscheme of pure codimension 1.

Later we shall see that if $n \geq 2$, then every hyperplane section is connected. For non-singular varieties, the hyperplane sections with hyperplanes $H \notin X^*$, being connected and non-singular are also irreducible so that are irreducible non-singular algebraic varieties.

To justify the name of conormal variety for $\mathcal{P}_X$ and to get some practice with the definitions, one could solve the following exercise. It is also a training for the language of locally free sheaves and their projectivizations.

1.5.3. EXERCISE. Prove the following facts.

(1) Let $X \subsetneq \mathbb{P}^M \subsetneq \mathbb{P}^N$ be a degenerate variety. Prove that $X^* \subset \mathbb{P}^{N*}$ is a cone of vertex $\mathbb{P}^{M*} = \mathbb{P}^{N-M-1} \subset \mathbb{P}^{N*}$ over the dual variety of X in $\mathbb{P}^M$. Suppose $X = S(L, X')$ is a cone of vertex $L = \mathbb{P}^l$, $l \geq 0$, over a variety $X' \subset M = \mathbb{P}^{N-l-1}$, $M \cap L = \emptyset$. Then $X^* \subset (\mathbb{P}^l)^* = \mathbb{P}^{N-l-1} \subset (\mathbb{P}^N)^*$ is degenerated. Is there any relation between X^* and the dual of X' in M?

Suppose $X \subset \mathbb{P}^N$ is a cone. Prove that $X^* \subset \mathbb{P}^{N*}$ is degenerated. Conclude that $X \subset \mathbb{P}^N$ is degenerated if and only if $X^* \subset \mathbb{P}^{N*}$ is a cone; and, dually, that $X \subset \mathbb{P}^N$ is a cone if and only if $X^* \subset \mathbb{P}^{N*}$ is degenerated.

(2) Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $p_2 : \mathcal{P}_C \to C^* \subset (\mathbb{P}^N)^*$ is a finite morphism so that $\mathrm{def}(C) = 0$.

(3) Let $X \subset \mathbb{P}^N$ be a non-singular variety, then $\mathcal{P}_X \simeq \mathbb{P}(\mathcal{N}^*_{X/\mathbb{P}^N}(1))$ (Grothendieck's notation), where $\mathcal{N}^*_{X/\mathbb{P}^N}(1)$ is the the twist of the conormal bundle of X in $\mathbb{P}^N$ by $\mathcal{O}_{\mathbb{P}^N}(1)$. Show that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is given by a sub-linear system of $|\mathcal{O}_{\mathcal{N}^*_{X/\mathbb{P}^N}(1)}(1)|$. (Hint: restrict Euler sequence to X and use the standard conormal sequence; interpret these sequences in terms

of the associated projective bundles and of the incidence correspondence defining $\mathcal{P}_X$).

(4) Let $X \subset \mathbb{P}^N$ be a smooth complete intersection. Deduce by the previous exercise that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism so that $\dim(X^*) = N-1$, i.e. $\mathrm{def}(X) = 0$ (Hint: show that $\mathcal{N}^*_{X/\mathbb{P}^N}(1)$ is a sum of very ample line bundles; deduce that $\mathcal{O}_{\mathcal{N}^*_{X/\mathbb{P}^N}(1)}(1)$ is very ample and finally that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism).

(5) Suppose char(K)=0 and let $C \subset \mathbb{P}^2$ be an irreducible curve, not a line. Show that C^* is an irreducible curve of degree at least 2. Take a tangent line at a point $x \in C$. Show that if T_xC is tangent at another point $y \in C$, $y \neq x$, then the point $(T_xC)^* \in C^*$ is a singular point of C^*. Deduce that if char(K)=0, then a general tangent line is tangent to C only at one point. Deduce that the same is true for an irreducible curve $C \subset \mathbb{P}^N$, $N \geq 3$.

(6) Let $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \geq 1$, be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Identify $\mathbb{P}^{2n+1}$ with the projectivization of the vector space of $2 \times n+1$ matrices and show that, due to the fact that there are only two orbits for the action of $GL(2)$ on $\mathbb{P}^N$ and on $(\mathbb{P}^N)^*$, $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ so that $\mathrm{def}((\mathbb{P}^1 \times \mathbb{P}^n)) = n-1$. Interpret this result geometrically and reverse the construction for $n = 2$ to show directly that $X = X^*$.

(7) Use the same argument as above to show that if $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, or if $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then $X^* \simeq SX$ and $SX^* \simeq X$.

As we have seen the dual varieties encode informations about the tangency of hyperplanes. Terracini Lemma says that linear spaces containing tangent spaces to higher secant varieties are tangent along $(\Sigma^k_z)_{\mathrm{red}} \setminus \mathrm{Sing}(X)$, see corollary 1.3.6. Hence if the maximal dimension of the fibers of $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is an upper bound for $\delta_k(X)$ as soon as $S^kX \subsetneq \mathbb{P}^N$, as we shall immediately see. More refined versions with the higher Gauss maps γ_m, see below, can be formulated but in those cases the condition expressed by the numbers $\varepsilon_m(X)$, which can be defined as below, is harder to control.

1.5.4. THEOREM. **(Dual variety and higher secant varieties)** *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Let $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ be as above and let $\varepsilon(X) = \max\{\dim(p_2^{-1}(H)), H \in X^*\}$. If $S^kX \subsetneq \mathbb{P}^N$, then $\delta_k(X) \leq \varepsilon(X)$. In particular if $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism, then* $\dim(S^kX) = \min\{(k+1)n + k, N\}$.

PROOF. Let $z \in S^kX$ be a general point. There exists $x \in \Sigma^k_z(X) \cap \mathrm{Sm}(X)$ and moreover T_zS^kX is contained in a hyperplane H. Then $p_1(p_2^{-1}(H)) \supseteq \mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H)$ (and more precisely $\mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H))$ contains the irreducible component of $\Sigma^k_z(X) \setminus (\mathrm{Sing}(X) \cap \Sigma^k_z(X))$ passing through x by corollary 1.3.6. Then $p_1(p_2^{-1}(H))$ has dimension at least $\delta_k(X) = \dim(\Sigma^k_z(X))$ and the conclusion follows. □

1.5.5. COROLLARY. *Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate curve. Then* $\dim(S^kC) = \min\{2k+1, N\}$ *(cfr. corollaries 1.2.3 and 1.3.6).*

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate complete intersection. Then $\dim(S^kX) = \min\{(k+1)n + k, N\}$.

PROOF. By exercise 1.5.3, we know that in both cases $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism. □

More generally one would study the locus of points at which a general hyperplane is tangent, the so called *contact locus*. For reflexive varieties it is a linear space of dimension $\mathrm{def}(X)$. This is an interpretation of the isomorphism $X \simeq (X^*)^*$. One should be careful in the interpretation of the result: it does not mean that the hyperplane remains tangent along the whole "contact locus", see remark 1.3.7 and adapt it to the more general situation of a ruling of a cone. This is true only for non-singular varieties. In particulat reflexive varieties of positive dual defect contain positive dimensional families of linear spaces.

1.5.6. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a reflexive variety. Then for $H \in \mathrm{Sm}(X^*)$,*

$$p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \, : \, T_xX \subset H\}} = (T_H X^*)^* = \mathbb{P}^{\mathrm{def}(X)}.$$

The following result will not be proved here but the reader can consult [Ha], pg. 208 for an elementary and direct proof. It is considered as a classical theorem, know at lest to C. Segre.

1.5.7. THEOREM. **(Reflexivity Theorem)** *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Suppose* $\mathrm{char}(K)=0$. *Then X is reflexive.*

Another natural and similar problem is to know if a general tangent space to a variety X is tangent at more than one point. During the discussion we will always suppose $\mathrm{char}(K)=0$ to avoid artificial problems, since the natural ones are enough interesting.

We have seen in exercise 1.5.3 that for irreducible curves a general tangent space is tangent only at one point. On the other hand if X is a cone over a curve, we know that a a general tangent space is tangent exactly along the ruling passing through the point. The unique common feature of irreducible algebraic varieties from this point of view seems to be the linearity of the locus of points at which a general linear space is tangent.

1.5.8. DEFINITION. **(Gauss maps)** Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and let $m \geq n$. Let

$$\mathcal{P}_X^m := \overline{\{((x, L) \; : \; x \in \mathrm{Sm}(X), \; T_xX \subseteq L\}} \subset X \times \mathbb{G}(m, N).$$

Let us consider the projections of $\mathcal{P}_X^m$ onto the factors X and $(\mathbb{P}^N)^*$,

$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "\gamma_m"] & \\ X & & \mathbb{G}(m, N). \end{tikzcd}$$

The variety of m-dimensional tangent subspaces to X, X_m^*, is the scheme-theoretic image of $\mathcal{P}_X^m$ in $\mathbb{G}(m, N)$, i.e. the algebraic variety

$$X_m^* := \gamma_m(\mathcal{P}_X^m) \subset \mathbb{G}(m, N).$$

For $m = N-1$, we recover the dual variety and its definition, while for $m = n$, we get the usual Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{G}(n, N)$ which associates to a point $x \in \mathrm{Sm}(X)$ its tangent space T_xX. For such $x \in \mathrm{Sm}(X)$ $\mathcal{G}_X(x) := \gamma_n(x) = T_xX$.

If $X \subset \mathbb{P}^N$ is an hypersurface, then $n = N-1$ and clearly the Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{P}^{N*}$ associates to a smooth point p of X its tangent hyperplane, so that in coordinates is given by

$$\mathcal{G}_X(p) = (\frac{\partial f}{\partial X_0}(p) : \ldots : \frac{\partial f}{\partial X_N}(p)).$$

The following theorem is once again a consequence of reflexivity and it is a generalization of proposition 1.5.6 and of the properties of cones. One can consult **[Z2]**, pg. 21, for a proof.

1.5.9. THEOREM. **(Linearity of general contact loci)** *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety. Assume* $\mathrm{char}(K)=0$. *The general fiber of the morphism $\gamma_m : \mathcal{P}_X^m \to X_m^*$ is a linear space of dimension* $\dim(\mathcal{P}_X^m) - \dim(X_m^*)$. *In particular the closure of a general fiber of $\mathcal{G}_X : X \dashrightarrow X_n^* \subset \mathbb{G}(n, N)$ is a linear space of dimension $n - \dim(\mathcal{G}_X(X))$ so that a general linear tangent space is tangent along an open subset of a linear space of dimension $n - \dim(\mathcal{G}_X(X))$.*

To conclude the section and the chapter, we prove via Terracini Lemma a relation between X^* and $(S^kX)^*$, $k < k_0(X)$, assuming $\mathrm{char}(K)=0$.

1.5.10. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Assume* $\mathrm{char}(K)=0$ *and $SX \subsetneq \mathbb{P}^N$. Then $(SX)^* \subseteq \mathrm{Sing}(X^*) \subsetneq X^*$, i.e. a general bitangent hyperplane represents a singular point of X^*. More generally for a given $k \geq 2$ such that $k < k_0(X)$, we have $(S^kX)^* \subseteq \mathrm{Sing}((S^{k-1}X)^*) \subsetneq (S^{k-1}X)^*$, i.e. a general $(k+1)$-tangent hyperplane represents a singular point of $(S^{k-1}X)^*$.*

PROOF. Take $H \in (SX)^*$ general point. Then $H \supseteq T_zSX$, with $z \in SX$ general point. By corollary 1.3.6, H is tangent to X along $\Sigma_z(X) \backslash (\Sigma_z(X) \cap \mathrm{Sing}(X))$ so that $H \in X^*$. Since X is non-degenerate, then $z \notin X$ implies that the contact locus of H is not linear, yielding $H \in \mathrm{Sing}(X^*)$ by proposition 1.5.6.

Take more generally $H \in (S^kX)^*$ general and write $S^kX = S(X, S^{k-1}X)$. Then $H \subseteq T_zS^kX$, with $z \in S^kX$ general point. Then there exists $y \in \mathrm{Sm}(S^{k-1}X)$ with $y \in \Sigma_z^k(X)$ and such that $z \in < x, y >$, $x \in X$, $x \neq y$. By Terracini Lemma $T_zS^kX \supseteq T_yS^{k-1}X$ so that $H \in (S^{k-1}X)^*$. Since $x \in X$, $x \in \mathrm{Sing}(H \cap S^{k-1}X)$, so that $p_2^{-1}(H) \subseteq S^{k-1}X$ is not linear since once again $z \in S^kX \setminus S^{k-1}X$ by the non-linearity of S^kX. □

Recall that to a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we associated an ascending filtration of irreducible projective varieties, see equation 1.2.5,

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N.$$

The above proposition says that at least over a filed of characteristic zero, there exists also a strictly descending dual filtration:

$$X^* \supsetneq \mathrm{Sing}(X^*) \supseteq (SX)^* \supsetneq \ldots \supseteq (S^{k_0-2}X)^* \supsetneq \mathrm{Sing}((S^{k_0-2}X)^*) \supseteq (S^{k_0-1}X)^*.$$

CHAPTER 2

Fulton-Hansen connectedness theorem and some applications to projective geometry

2.1. Connectedness principle of Enriques-Zariski-Grothendieck-Fulton-Hansen and some classical theorems in algebraic geometry

In the first chapter we introduced the main definitions of classical projective geometry and furnished rigorous proofs of many classical results. Many theorems in classical projective geometry deal with "general" objects. For example the classical Bertini theorem on hyperplane sections, see theorem 1.5.2. A more refined version of this theorem says that if $f : X \to \mathbb{P}^N$ is morphism, with X proper and such that $\dim(f(X)) \geq 2$, and if $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ is a general hyperplane, then $f^{-1}(H)$ is irreducible, see **[Ju]** theorem 6.10 for a modern reference. The "Enriques-Zariski principle" says that "limits of connected varieties remain connected" and it is for example illustrated in the previous example because for an arbitrary $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$, $f^{-1}(H)$ is connected as we shall prove below.

This result is particularly interesting because, as shown by Deligne and Jouanolou, a small generalization of it proved by Grothendieck, **[Gr]** XIII 2.3, yields a simplified proof of a beautiful and interesting connectedness theorem of Fulton and Hansen in **[FH]**, whose applications are deep and appear in different areas of algebraic geometry and topology. Moreover, Deligne's proof generalizes to deeper statements involving higher homotopy groups when studying complex varieties, see **[D1]**, **[D2]**, **[Fu]**, **[FL]**.

To illustrate this circle of ideas and the "connectedness principle", we describe how the theorem of Fulton-Hansen includes some classical theorems in algebraic geometry and generalizes them. In our treatment we strictly follow the surveys **[Fu]** and **[FL]**. Another interesting source, where the ideas of Grothendieck behind this theorem and their generalizations to d-connectedness and to weighted projective spaces are explained in great detail, are the notes of a course of Bădescu, **[B1]**, and his book **[B2]**.

Now we recall four classical theorem with emphasis on the connectedness results in the idea of looking for a common thread. When dealing with homotopy groups π_i, we are assuming $K = \mathbb{C}$ and referring to the classical topology.

2.1.1. **Four classical theorems.** Let us list the following more or less known theorems.

(1) (Bézout) Let X and Y be closed subvarieties of $\mathbb{P}^N$. If $\dim(X)+\dim(Y) \geq N$, then $X \cap Y \neq \emptyset$. If $\dim(X) + \dim(Y) > N$, then $X \cap Y$ is connected and more precisely $(\dim(X) + \dim(Y) - N)$-connected.

(2) (Bertini) Let $f : X \to \mathbb{P}^N$ be a morphism, with X proper variety, and let $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ be a linear space. If $l \leq \dim(f(X))$, then $f^{-1}(L) \neq \emptyset$. If $l < \dim(f((X))$, then $f^{-1}(L)$ is connected.

(3) (Lefschetz) If $X \subset \mathbb{P}^N$ is a closed irreducible subvariety of dimension n and if $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ is a linear space containing $\mathrm{Sing}(X)$, then

$$\pi_i(X, X \cap L) = 0 \quad \text{for } i \leq n - l.$$

Equivalently the morphism

$$\pi_i(X \cap L) \to \pi_i(X)$$

is an isomorphisms if $i \leq n - l$ and surjective if $i = n - l$.

(4) (Barth-Larsen) If $X \subset \mathbb{P}^N$ is a closed irreducible non-singular subvariety of dimension n, then

$$\pi_i(\mathbb{P}^N, X) = 0 \quad \text{for } i \leq 2n - N + 1.$$

(Recall that $\pi_i(\mathbb{P}^N) = \mathbb{Z}$ for $i = 0, 2$ and $\pi_i(\mathbb{P}^N) = 0$ for $i = 1, 3, 4, \ldots, 2N$).

As we said at the beginning usually the names of the classical theorem refers to properties of general linear sections, for which a better property can be expected, as in the case of Bertini theorem for example, or as in the case of Bézout theorem (when the intersection is transversal one usually computes $\#(X \cap Y)$). In the classical Lefschetz theorem the variety was non-singular and L was general.

Let us remark that the two parts of theorem 1) can be reformulated by mean of homotopy groups. The first part is equivalent to

$$\pi_0(X \cap Y) \to \pi_0(X \times Y)$$

is surjective, the second one to the fact that the above morphism is an isomorphism. Similarly theorem 2) can be reformulated as

$$\pi_0(f^{-1}(L)) \to \pi_0(X)$$

is an isomorphism.

A common look at the above theorems comes from the following observation of Hansen, [**FL**],[**FH**]. All the above theorems are statement about the not emptiness, respectively connectedness, of the inverse image of $\Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ under a proper morphism $f : W \to \mathbb{P}^N \times \mathbb{P}^N$ such that $\dim(f(W)) \geq N$, respectively $\dim(f(W)) > N$.

Suppose this is true and take $W = X \times Y$ for theorem 1) or $W = X \times L$ in theorem 2) and 3) at least to deduce the connectedness parts. Theorem 4) can be deduced by taking $W = X \times X$, see [**FL**] and [**Fu**].

These results can be explained from other points of view as consequences of the ampleness of the normal bundle of a smooth subvarieties, or of complete intersections in $\mathbb{P}^N$. On the other hand the same positivity holds for $\Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ since $N_{\Delta_{\mathbb{P}^N}/\mathbb{P}^N \times \mathbb{P}^N} \simeq T_{\mathbb{P}^N}$ and the tangent bundle to $\mathbb{P}^N$, $T_{\mathbb{P}^N}$, is ample by Euler sequence.

The above discussion and further generalizations by Faltings, Goldstein and Hansen revealed a connectedness principle, which we now state and later justify why one should expect its validity.

2.1.2. **Connectedness Principle,** [Fu], pg. 18. Let P be a smooth projective variety.

Given a "suitable positive" embedding $Y \hookrightarrow P$ of codimension l and a proper morphism $f : W \to P$, $n = \dim(W)$,

$$\begin{tikzcd} f^{-1}(Y) \arrow[r, hook] \arrow[d] & W \arrow[d, "f"] \\ Y \arrow[r, hook] & P, \end{tikzcd}$$

we should have

$$\pi_i(W, f^{-1}(Y)) \xrightarrow{\cong} \pi_i(P, Y) \quad \text{for } i \leq n - l - \text{"defect"}.$$

This defect should be measured by (a) lack of positivity of Y in P; (b) singularities of W; (c) dimensions of the fibers of f. Usually $\pi_i(P, Y) = 0$ for small i, so the conclusion is that, as regards connectivity, $f^{-1}(Y)$ must look like W. If the defect is zero we deduce that

$$f^{-1}(Y) \neq \emptyset \quad \text{if } n \geq l,$$

$$f^{-1}(Y) \quad \text{is connected and } \pi_1(f^{-1}(Y)) \to \pi_1(W) \quad \text{is surjective if } n > l.$$

The most basic case is with $P = \mathbb{P}^N$ and $Y = \mathbb{P}^{N-l}$ a linear subspace. In this case the principle furnishes the theorems of Bertini and Lefschetz by taking $W = X$. As we explained before the case which allows one to include all the classical theorems is $P = \mathbb{P}^N \times \mathbb{P}^N$, and $Y = \Delta_{\mathbb{P}^N}$ diagonally embedded in P. Indeed $W = X \times Y$ gives Bézout theorem, while theorems 2) and 3) are recovered by setting $W = X \times L$. Theorem 4) can be obtained with $W = X \times X$.

When $\mathbb{P}^N$ is replaced by other homogeneous spaces, one could measure the defect of positivity of its tangent bundle and one expects the principle to hold with this defect, see [Fa], [Go], [BS].

Why should one expect this connectedness principle to be valid? In some cases one can define a Morse function which measures distance from Y. Positivity should imply that all the Morse indices of this function are at least $n - l - 1$ (perhaps minus a defect). Then one constructs W from $f^{-1}(Y)$ by adding only cells of dimension at least $n - l - 1$, which yields the required vanishing of relative homotopy groups, see [Fu] for a proof giving theorems 3) and 4) above.

Before ending this long introduction to the connectedness theorem we recall for completeness the following statements for later reference. They particular forms or consequences of results of Barth and Barth and Larsen. Chronologically part 2) has been stated before than the Barth-Larsen theorem involving higher homotopy groups and recalled above.

2.1.3. THEOREM. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible projective variety and let $H \subset X$ be a hyperplane section.*

(1) *If $n \geq \frac{N+1}{2}$, then $\pi_1(X) = 1$* (Barth-Larsen).

(2) *If $n \geq \frac{N+i}{2}$, then the restriction map*

$$H^i(\mathbb{P}^N, \mathbb{Z}) \to H^i(X, \mathbb{Z})$$

is an isomorphism (Barth).

(3) *If* $n \geq \frac{N+2}{2}$, *then*

$$\mathrm{Pic}(X) \simeq \mathbb{Z} < H > \text{(Barth)}.$$

We come back to the algebraic setting and to the proof of the theorem of Fulton-Hansen and hence of the non-emptiness and connectedness parts ($i = 0$) of theorems 1), 2), 3) (and 4)). The theorem appears as a consequence of the connectedness of preimages of linear spaces under proper morphisms, a result due to Grothendieck and which follows from the "classical" Bertini theorem we quoted at the beginning. We start with the connectedness theorem and later prove some interesting results having their own interest and leading to its proof. In **[B1]**, Lucian Bădescu extends the connectedness theorem to weighted projective spaces using the original ideas of Grothendieck, so that many geometrical consequences of the result are valid also for this class of homogeneous varieties.

2.1.4. THEOREM. (**Fulton-Hansen Connectedness Theorem, [FH]**) *Let* X *be an irreducible variety, proper over an algebraically closed filed* K. *Let* $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ *be a morphism and let* $\Delta = \Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ *be the diagonal.*

(1) *If* $\dim(f(X)) \geq N$, *then* $f^{-1}(\Delta) \neq \emptyset$.
(2) *If* $\dim(f(X)) > N$, *then* $f^{-1}(\Delta)$ *is connected.*

We begin by recalling the following "classical" Bertini theorem in a more general form. For a proof we refer to **[Ju]**, theorem 6.10, where the hypothesis $K = \overline{K}$ is relaxed.

2.1.5. THEOREM. (**Bertini Theorem**, see **[Ju]**) *Let* X *be an irreducible variety and let* $f : X \to \mathbb{P}^N$ *be a morphism. For a fixed integer* $l \geq 1$, *let* $\mathbb{G}(N-l, N)$ *be the Grassmann variety of linear subspaces of* $\mathbb{P}^N$ *of codimension* l. *Then*

(1) *if* $l \leq \dim(\overline{f(X)})$, *then there is a non-empty open subset* $U \subseteq \mathbb{G}(N-l, N)$ *such that for every* $L \in U$,
$$f^{-1}(L) \neq \emptyset;$$
(2) *if* $l < dim(\overline{f(X)})$, *then there is a non-empty open subset* $U \subseteq \mathbb{G}(N-l, N)$ *such that for every* $L \in U$,
$$f^{-1}(L) \quad \textit{is irreducible.}$$

We now show that the Enriques-Zariski principle is valid in this setting by proving the next result, which is the key point towards theorem 2.1.4. We pass from general linear sections to arbitrary ones and for simplicity we suppose $K = \overline{K}$ as always.

2.1.6. THEOREM. (**[Gr]**, **[FH]**, **[Ju]**, theorem 7.1) *Let* X *be an irreducible variety and let* $f : X \to \mathbb{P}^N$ *be a morphism. Let* $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ *be an arbitrary linear space of codimension* l.

(1) *If* $l \leq \dim(\overline{f(X)})$ *and if* X *is proper over* K, *then*
$$f^{-1}(L) \neq \emptyset.$$
(2) *If* $l < dim(\overline{f(X)})$ *and if* X *is proper over* K, *then*
$$f^{-1}(L) \quad \textit{is connected.}$$

More generally for an arbitrary irreducible variety X, if $f : X \to \mathbb{P}^N$ is proper over some open subset $V \subseteq \mathbb{P}^N$, and if $L \subseteq V$, then, when the hypothesis on the dimensions are satisfied, the same conclusions hold for $f^{-1}(L)$.

PROOF. (According to [**Ju**]). We prove the second part of the theorem from which the statements in 1) and 2) follow.

Let $W \subseteq \mathbb{G}(N-l, N)$ be the open subset consisting of linear spaces contained in V and let

$$Z = \{(x, L') \in X \times W \,:\, f(x) \in L'\} \subset \{(x, L') \in X \times \mathbb{G}(N-l, N) \,:\, f(x) \in L'\} = \mathcal{I}.$$

The scheme Z is irreducible since it is an open subset of the Grassmann bundle $p_1 : \mathcal{I} \to X$. Since f is proper over V, the second projection $p_2 : Z \to W$ is a proper morphism. Consider its Stein factorization:

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "p_2"] & \\ W' \arrow[r, "r"'] & W; \end{tikzcd}$$

the morphism q is proper with connected fibers and surjective, while r is finite. By theorem 2.1.5 r is dominant and hence surjective if $l \leq \dim(\overline{f(X)})$, respectively generically one-to-one and surjective if $l < \dim(\overline{f(X)})$. In the first case $p_2 : Z \to W$ is surjective so that $f^{-1}(L) \neq \emptyset$ for every $L \in W$. In the second case, since W is smooth, it follows that r is one to one everywhere so that $f^{-1}(L) = q^{-1}(r^{-1}(L))$ is connected for every $L \in W$. □

2.1.7. REMARK. The original proof of Grothendieck used an analogous local theorem proved via local cohomology. His method has been used and extended by Hartshorne, Ogus, Speiser and Faltings. Faltings proved with similar techniques a connectedness theorem for other homogeneous spaces, see [**Fa**], at least in characteristic zero. A different proof of a special case of the above theorem was also given by Barth in 1969.

Now we are in position to prove the connectedness theorem.

PROOF. (of theorem 2.1.4, according to Deligne, [**D1**]). The idea is to pass from the diagonal embedding $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$ to a linear embedding $L = \mathbb{P}^N \subset \mathbb{P}^{2N+1}$, a well known classical trick.

In $\mathbb{P}^{2N+1}$ separate the $2N+2$ coordinates into $[X_0 : \ldots : X_N]$ and $[Y_0 : \ldots : Y_N]$ and think these two sets as coordinates on each factor of $\mathbb{P}^N \times \mathbb{P}^N$. The two N dimensional linear subspaces $H_1 : X_0 = \ldots = X_N = 0$ and $H_2 : Y_0 = \ldots = Y_N = 0$ of $\mathbb{P}^{2N+1}$ are disjoint. If $V = \mathbb{P}^{2N+1} \setminus (H_1 \cup H_2)$ since there is a unique secant line to $H_1 \cup H_2$ passing through each $p \in V$, there is a morphism

$$\phi : V \to H_1 \times H_2 = \mathbb{P}^N \times \mathbb{P}^N,$$

which to p associates the points $(p_1, p_2) = (< H_2, p > \cap H_1, < H_1, p > \cap H_2)$. In coordinates, $\phi([X_0 : \ldots : X_N : Y_0 : \ldots : Y_N] = ([X_0 : \ldots : X_N], [Y_0 : \ldots : Y_N])$. Then $\phi^{-1}(\phi(p)) =< p_1, p_2 > \setminus \{p_1, p_2\} \simeq \mathbb{A}^1_K \setminus 0$. Let $L = \mathbb{P}^N \subset V$ be the linear subspace of $\mathbb{P}^{2N+1}$ defined by $X_i = Y_i$, $i = 0, \ldots, N$. Then

$$\phi_{|L} : L \xrightarrow{\simeq} \Delta$$

is an isomorphism. Given $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ we construct the following Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "\phi"] & \mathbb{P}^N \times \mathbb{P}^N, \end{tikzcd}$$

where

$$X' = V \times_{\mathbb{P}^N \times \mathbb{P}^N} X.$$

Clearly ϕ' induces an isomorphism between $f'^{-1}(L)$ and $f^{-1}(\Delta)$. To prove the theorem it is sufficient to verify the corresponding assertion for $f'^{-1}(L)$. To this aim we apply theorem 2.1.6. Let us verify the hypothesis.

Since $\phi'^{-1}(x) \simeq \phi^{-1}(f(x)) = \mathbb{A}^1_K \setminus 0$ for every $x \in X$, the scheme X' is irreducible and of dimension $\dim(X) + 1$. The morphism f is proper, so that also $f' : X' \to V$ is proper and moreover $\dim(f(X')) = \dim(f(X)) + 1$. If $\dim(f(X)) \geq N$, then $\dim(f(X')) \geq N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. If $\dim(f(X')) > N$, then $\dim(f(X')) > N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. □

2.2. Zak's applications to Projective Geometry

In this section we come back to projective geometry and apply Fulton-Hansen theorem to prove some interesting and non-classical results in projective geometry. Most of the ideas and the results are due to Fyodor L. Zak, see [**Z2**], [**FL**], [**LV**], and they will be significant improvements of the classical material presented in the first chapter. Other applications to new results in algebraic geometry can be found in [**FH**], [**FL**], [**Fu**].

We begin with the following key result, which refines a result of Johnson, [**Jo**].

2.2.1. THEOREM. ([**FH**], [**Z2**]) *Let $Y \subseteq X \subset \mathbb{P}^N$ be a closed subvariety of dimension $r = \dim(Y) \leq \dim(X) = n$, with X irreducible and projective. Then either*

(1) $\dim(T^*(Y,X)) = r + n$ *and* $\dim(S(Y,X)) = r + n + 1$, *or*
(2) $T^*(Y,X) = S(Y,X)$.

PROOF. We can suppose Y irreducible and then apply the same argument to each irreducible component of Y. We know that $T^*(Y,X) \subseteq S(Y,X)$ and that $\dim(T^*(Y,X)) \leq r + n$ by construction. Suppose that $\dim(T^*(Y,X)) = r + n$. Since $S(Y,X)$ is irreducible and $\dim(S(Y,X)) \leq r + n + 1$, the conclusion holds.

Suppose now $\dim(T^*(Y,X)) = t < r + n$. We prove that $\dim(S(Y,X)) \leq t$ so that $T^*(Y,X) = S(Y,X)$ follows from the irreducibility of $S(Y,X)$. There exists $L = \mathbb{P}^{N-t-1}$ such that $L \cap T^*(Y,X) = \emptyset$. The projection $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^t$ restricts to a finite morphism on X and on Y, since $L \cap X = \emptyset$, see definition 1.2.4. Then $(\pi_L \times \pi_L)(X \times Y) \subset \mathbb{P}^t \times \mathbb{P}^t$ has dimension $r + n > t$ by hypothesis. By theorem 2.1.4, the closed set

$$\widetilde{\Delta} = (\pi_L \times \pi_L)^{-1}(\Delta_{\mathbb{P}^t}) \subset Y \times X$$

is connected and contains the closed set $\Delta_Y \subset Y \times X$ so that Δ_Y is closed in $\widetilde{\Delta}$.

We claim that

$$\Delta_Y = \widetilde{\Delta}.$$

This yields $L \cap S(Y,X) = \emptyset$ and hence $\dim(S(Y,X)) \leq N - 1 - \dim(L) = t$.

Suppose $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$. We find $y' \in Y$ such that $\emptyset \neq T^*_{y'}(Y,X) \cap L \subseteq T^*(Y,X) \cap L$ contrary to the assumption. If $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$, the connectedness of $\widetilde{\Delta}$ implies the existence of $(y',y') \in \overline{\widetilde{\Delta} \setminus \Delta_Y} \cap \Delta_Y$. Let notations be as in definition 1.2.1. i. e. $p_2(p_1^{-1}(y,x)) =< x,y >$ if $x \neq y$ and $p_2(p_1^{-1}(y,x)) = T^*_y(Y,X)$ if $x = y \in Y$. Since for every $(y,x) \in \widetilde{\Delta} \setminus \Delta_Y$ we have $< y,x > \cap L \neq \emptyset$ by definition of π_L ($\pi_L(y) = \pi_L(x)$, $y \neq x$, if and only if $< y,x > \cap L \neq \emptyset$), the same holds for (y',y') so that $p_2(p_1^{-1}(y,x)) \cap L \neq \emptyset$ forces $p_2(p_1^{-1}(y',y')) \cap L \neq \emptyset$. □

2.2.2. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Then either*

(1) $\dim(T^*X) = 2n$ *and* $\dim(SX) = 2n+1$, *or*
(2) $T^*X = SX$.

The following theorem well illustrates the passage from general to arbitrary linear spaces, as regards to tangency.

2.2.3. THEOREM. **(Zak's Theorem on Tangencies)** *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety of dimension n. Let $L = \mathbb{P}^m \subset \mathbb{P}^N$ be a linear subspace, $n \leq m \leq N-1$, which is J-tangent along the closed set $Y \subseteq X$. Then* $\dim(Y) \leq m - n$.

PROOF. Without loss of generality we can suppose that Y is irreducible and then apply the conclusion to each irreducible component. By hypothesis and by definition we get $T^*(Y,X) \subseteq L$. Since $X \subsetneq S(Y,X)$ and since X is non-degenerate, $S(Y,X)$ is not contained in L so that $T^*(Y,X) \neq S(Y,X)$. By theorem 2.2.1 we have $\dim(Y) + n = \dim(T^*(Y,X)) \leq \dim(L) = m$. □

We now come back to the problem of tangency and to contact loci of smooth varieties to furnish two beautiful applications of the theorem on Tangencies. We begin with the finiteness of the Gauss map of a smooth variety.

2.2.4. COROLLARY. **(Gauss map is finite for smooth varieties,** Zak) *Let $X \subsetneq \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension n. Then the Gauss map $\mathcal{G}_X : X \to \mathbb{G}(n,N)$ is finite. If moreover* char(K)=0, *the $\mathcal{G}_X$ is birational onto the image, i.e. X is a normalization of $\mathcal{G}_X(X)$.*

PROOF. As always it is sufficient to prove that $\mathcal{G}_X$ has finite fibers. For every $x \in X$, $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is the locus of points at which the tangent space T_xX is tangent. By theorem 2.2.3 it has dimension less or equal than $\dim(T_xX) - n = 0$.

If char(K)=0, then every fiber $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is linear by theorem 1.5.9 and of dimension zero by the first part, so that it reduces to a point. □

The next result reveals a special feature of non-singular varieties, since the result is clearly false for cones, see exercise 1.5.3.

2.2.5. COROLLARY. (Zak) *Let $X \subset \mathbb{P}^N$ be a smooth projective non-degenerate variety. Let $X^* \subset \mathbb{P}^{N*}$ be its dual variety. Then* $\dim(X^*) \geq \dim(X)$. *In particular, if also X^* is smooth, then* $\dim(X^*) = \dim(X)$.

PROOF. By the theorem of the dimension of the fiber, letting notations as in definition 1.5.1, $\dim(X^*) = N - 1 - \dim(p_2^{-1}(H))$, $H \in X^*$ general point. By theorem 2.2.3, $\dim(p_2^{-1}(H)) \leq N - 1 - \dim(X)$ and the conclusion follows. □

2.2.6. REMARK. In exercise 1.5.3, we saw that $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ for every $n \geq 1$. In [Ei], L. Ein shows that if $N \geq 2/3 \dim(X)$, if X is smooth, if char(K)=0 and if $\dim(X) = \dim(X^*)$, then $X \subset \mathbb{P}^N$ is either a hypersurface, or $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ Segre embedded, or $\mathbb{G}(1,4) \subset \mathbb{P}^9$ Plücker embedded, or the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$. In the last three cases $X \simeq X^*$.

We apply the theorem on Tangencies to deduce some strong properties of the hyperplane sections of varieties of small codimension. By the theorem of Bertini proved in the previous section we know that arbitrary hyperplane sections of varieties of dimension at least 2 are connected. When the codimension of the variety is small with respect to the dimension, some further restrictions for the scheme structure hold.

If $X \subset \mathbb{P}^N$ is a non-singular irreducible nondegenerate variety, we recall that for every $H \in X^*$

$$\mathrm{Sing}(H \cap X) = \{x \in X \,:\, T_x X \subset H\},$$

i.e. it is the locus of points at which H is tangent. By theorem 2.2.3 we get

$$\dim(\mathrm{Sing}(X \cap X) \leq N - 1 - \dim(X),$$

i.e.

$$\mathrm{codim}(\mathrm{Sing}(X \cap H), X \cap H) \geq 2\dim(X) - N.$$

Recall that $H \cap X$ is a Cohen-Macaulay scheme of dimension $\dim(X) - 1$ and that such a scheme is reduced as soon as it is generically reduced ($R_0 + S_1 \Leftrightarrow R_1$).

If $N \leq 2\dim(X) - 1$, then $H \cap X$ is a reduced scheme being non-singular in codimension zero and in particular generically reduced. The condition forces $\dim(X) \geq 2$, so that it is also connected by Bertini theorem.

If $N \leq 2\dim(X) - 2$, which forces $\dim(X) \geq 3$, then $H \cap X$ is also non-singular in codimension 1, so that it is normal being Cohen-Macaulay. Since it is connected and integral, it is also irreducible. The case of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ shows that this last result cannot be improved, since an hyperplane containing a $\mathbb{P}^2$ of the ruling yields a reducible, reduced, hyperplane section. Clearly in the same way, if $N \leq 2\dim(X) - k - 1$, $k \geq 0$, then $X \cap H$ is connected, Cohen-Macaulay and non-singular in codimension k. We summarize these result in the following corollary to the theorem on Tangencies.

2.2.7. COROLLARY. (Zak) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. Then*

(1) *if $N \leq 2n - 1$, then every hyperplane section is connected and reduced;*
(2) *if $N \leq 2n - 2$, then every hyperplane section is irreducible and normal;*
(3) *let $k \geq 2$. If $N \leq 2n - k - 1$, then every hyperplane section is irreducible, normal and non-singular in codimension k.*

CHAPTER 3

Hartshorne's conjectures and Severi varieties

3.1. Hartshorne's conjectures and Zak's theorem on linear normality

After the period in which new and solid foundations to the principles of algebraic geometry were rebuilt especially by Zariski, Grothendieck and their schools, at the beginning of the '70 a new trend began. There was a renewed interest in solving concrete problems and in finding applications of the new methods and ideas. One can consult the beautiful book of Robin Hartshorne, **[H1]**, to have a picture of that situation. In **[H1]** many outstanding questions, such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle (solved by Mori in **[Mo1]** and which cleared the path to the foundation of Mori theory, **[Mo2]**) were discusses, or stated and a lot of other problems solved. In related fields we only mention Deligne proof of the Weil conjectures or later Faltings proof of the Mordell conjecture, which used the new machinery.

The interplay between topology and algebraic geometry returned to flourish. Lefschetz theorem and Barth-Larsen theorem, see subsection 2.1.1 and theorem 2.1.3, also suggested that smooth varieties, whose codimension is small with respect to their dimension, should have very strong restrictions both topological, both geometrical. To have a feeling we remark that a codimension 2 smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected for example. If $N \geq 6$, there are no known examples of codimension 2 smooth varieties with the exception of the trivial ones, the complete intersection of two hypersurfaces, i.e. the transversal intersection of two hypersurfaces, smooth along the subvariety. In fact, at least for the moment, one is able to construct only these kinds of varieties whose codimension is sufficiently small with respect to dimension. Let us recall the following definition and some notable properties of complete intersections analogous to varieties whose codimension is small with respect to dimension.

3.1.1. DEFINITION. **(Complete intersection)** A variety $X \subset \mathbb{P}^N$ of dimension n is a *complete intersection* if there exist $N-n$ homogeneous polynomials $f_i \in K[X_0, \ldots, X_N]$ of degree $d_i \geq 1$, generating the homogeneous ideal $I(X) \subset K[X_0, \ldots, X_N]$, i.e. $I(X) = < f_1, \ldots, f_{N-n} >$.

Let us recall that since $f_1, \ldots, f_{N-n}$ form a regular sequence in $K[X_0, \ldots, X_N]$, the homogeneous coordinate ring $S(X) = K[X_0, \ldots, X_N]/I(X)$ has depth $n+1$, i.e. $X \subset \mathbb{P}^N$ is an arithmetically Cohen-Macaulay variety. Thus a complete intersection $X \subset \mathbb{P}^N$ is projectively normal, i.e. the restriction morphisms

$$H^0(\mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathcal{O}_X(m))$$

are surjective for every $m \geq 0$, so that X is connected, and $H^i(\mathcal{O}_X(m)) = 0$ for every i such that $0 < i < n$ and for every $m \in \mathbb{Z}$. Moreover, by Grothendieck theorem on complete intersections, $\mathrm{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}_X(1) >$, as soon as $n \geq 3$, see [**H1**]. By Lefschetz theorem complete intersections defined over $K = \mathbb{C}$ are simply connected, as soon as $n \geq 2$ and have the same cohomology $H^i(X, \mathbb{Z})$ of the projective spaces containing them for $i < n$.

Based on some empirical observations, inspired by the theorem of Barth and Larsen and, according to Fulton and Lazarsfeld, "on the basis of few examples", Hartshorne was led to formulate the following conjectures.

3.1.2. CONJECTURE. (**1^{st} Conjecture of Hartshorne, or Complete Intersection Conjecture,** [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \dfrac{3}{2}\dim(X)$, *i.e. if* $\mathrm{codim}(X) < \dfrac{1}{2}\dim(X)$, *then X is a complete intersection.*

Let us quote Hartshorne: *While I am not convicted of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind* ([**H2**]).

Hartshorne immediately remarks that the conjecture is sharp, due to the examples of the Grassmann variety of lines in $\mathbb{P}^4$, $\mathbb{G}(1,4) \subset \mathbb{P}^9$, Plücker embedded, and of the spinorial variety of dimension 10, $S^{10} \subset \mathbb{P}^{15}$; moreover, the examples of cones over curves in $\mathbb{P}^3$, not complete intersection, reveals the necessity of the non-singularity assumption. Varieties for which $N = \frac{3}{2}\dim(X)$ and which are not complete intersection are usually called *Hartshorne varieties.* No other example of Hartshorne variety is known till today. It is not a case that these varieties are homogeneous since a technique for constructing varieties of not too high codimension is exactly via algebraic groups, see for example [**Z2**], chapter 3, or the appendix to [**LV**].

One of the main difficulties of the problem is a good translation in geometrical terms of the algebraic condition of being a complete intersection and in general of dealing with the equations defining a variety.

It is not here the place to remark how many important results originated and still today arise from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety and k-normality and so on. The list of these achievements is too long that we preferred to avoid citations, being confident that everyone has met sometimes a problem or a result related to it.

Let us recall the following definition.

3.1.3. DEFINITION. (**Linear normality**) A non-degenerate irreducible variety $X \subset \mathbb{P}^N$ is said to be *linearly normal* if the linear of hyperplane sections is complete, i.e. if the injective, due to non-degenerateness, restriction morphism

$$H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{r} H^0(\mathcal{O}_X(1))$$

is surjective and hence an isomorphism.

If a variety $X \subset \mathbb{P}^N$ is not linearly normal, then the complete linear system $|\mathcal{O}_X(1)|$ is of dimension greater than N and embeds X as a variety $X' \subset \mathbb{P}^M$, $M > N$. Moreover, there exists a linear space $L = \mathbb{P}^{M-N-1}$ such that $L \cap X' = \emptyset$ and such that $\pi_L : X' \to X \subset \mathbb{P}^N$ is an isomorphism. Indeed, if $V = r(H^0(\mathcal{O}_{\mathbb{P}^N}(1))) \subsetneq H^0(\mathcal{O}_X(1))$ and if $U \subset H^0(\mathcal{O}_{\mathbb{P}^N}(1))$ is a complementary subspace of V in $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$, the one can take $\mathbb{P}^M = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^N}(1)))$, $L = \mathbb{P}(U)$ and the claim follows from the fact that $\pi_L : X' \simeq X \to X \subset \mathbb{P}^N = \mathbb{P}(V)$ is given by the very ample linear system $|V|$. On the contrary, if X is an isomorphic linear projection of a variety $X' \subset \mathbb{P}^M$, $M > N$, then X is not linearly normal.

In the same survey paper Hartshorne posed another conjecture, based on the fact that complete intersections are linearly normal and on some examples in low dimension.

3.1.4. Conjecture. (**2**[nd] **Conjecture of Hartshorne, or Linear Normality Conjecture,** [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

If $N < \dfrac{3}{2}\dim(X)+1$, i.e. if $\operatorname{codim}(X) < \dfrac{1}{2}\dim(X)+1$, then X is linearly normal.

Recalling proposition 1.2.8 and the above discussion, we can equivalently reformulate it by means of secant varieties putting "$N = N + 1$".

$$\text{If } N < \frac{3}{2}\dim(X) + 2, \text{ then } SX = \mathbb{P}^N.$$

Let us quote once again Hartshorne point of view on this second problem: *Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3n}{2} + 1$, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case $N = \frac{3n}{2}$. My feeling is that this conjecture should be easier to establish than the original one* ([**H2**]). Once again the bound is sharp taking into account the example of the projected Veronese surface in $\mathbb{P}^4$.

The conjecture on linear normality was proved by Zak at the beginning of the '80's and till now it is the major evidence for the possible truth of the complete intersection conjecture. As we shall see conjecture 3.1.4 is now an immediate consequence of Terracini Lemma and of theorem 2.2.1. Later we will furnish another proof of this theorem, cfr. theorem 4.1.4.

3.1.5. Theorem. (**Zak Theorem on Linear Normality**) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. If $N < \frac{3}{2}n + 2$, then $SX = \mathbb{P}^N$. Or equivalently if $SX \subsetneq \mathbb{P}^N$, then $\dim(SX) \geq \frac{3}{2}n + 1$ and hence $N \geq \frac{3}{2}n + 2$.*

Proof. Suppose that $SX \subsetneq \mathbb{P}^N$, then there exists a hyperplane H containing the general tangent space to SX, let us say T_zSX. Then by corollary 1.3.6, the hyperplane H is tangent to X along $\Sigma_z(X)$, which by the generality of z has pure dimension $\delta(X) = 2n + 1 - \dim(SX)$. Since $T(\Sigma_z(X), X) \subseteq H$, the non-degenerate variety $S(\Sigma_z(X), X) \supseteq X$ is not contained in H, yielding $T(\Sigma_z(X), X) \neq S(\Sigma_z(X), X)$. By theorem 2.2.1 we get

$$2n + 1 - \dim(SX) + n + 1 = \dim(S(\Sigma_z(X), X)) \leq \dim(SX),$$

i.e.

$$3n + 2 \leq 2\dim(SX)$$

implying

$$N - 1 \geq \dim(SX) \geq \frac{3}{2}n + 1.$$

□

3.2. Severi varieties

Theorem 3.1.5 opens the problem of investigating examples for which the result is sharp, i.e. to try to classify smooth varieties of dimension n, $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ such that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$, or equivalently smooth not linearly normal varieties of dimension n, $\widetilde{X} \subset \mathbb{P}^{\frac{3}{2}n+1}$. Clearly n is even so that the first case to be considered is $n = 2$ and so one would like to classify smooth surfaces in $\mathbb{P}^5$ such that $SX \subsetneq \mathbb{P}^5$. The answer is thus contained in the classical and well known theorem of Severi, **[Sev1]**, which is theorem 1.4.1 here, saying that X is projectively equivalent to the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. This justifies the name given by Zak to such varieties.

3.2.1. Definition. **(Severi variety)** A smooth irreducible non-degenerate variety of dimension n, $X \subset \mathbb{P}^{\frac{3}{2}n+2}$, is said to be a *Severi variety* if $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$.

By theorem 3.1.5, it follows that $SX \subset \mathbb{P}^{\frac{3}{2}n+2}$ is necessarily an hypersurface, i.e. $\dim(SX) = \frac{3}{2}n + 1$.

In exercise 1.1.6 we showed that the Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is an example of Severi variety of dimension 4. Indeed $N = 8 = \frac{3}{2} \cdot 4 + 2$ and SX is a cubic hyersurface, see *loc. cit.*. By the classical work of Scorza, last page of **[S1]**, it turns out that $\mathbb{P}^2 \times \mathbb{P}^2$ is the only Severi variety of dimension 4. We shall furnish a short, geometrical and elementary proof of this fact below, see theorem 3.2.6.

The realization of the Grassmann variety of lines in $\mathbb{P}^5$ Plücker embedded, $X = \mathbb{G}(1,5) \subset \mathbb{P}^{14}$, as the variety given by the pfaffians of the general antisymmetric 6×6 matrix, yields that $\mathbb{G}(1,5)$ is a Severi variety of dimension 8 such that its secant variety is a degree 3 hypersurface, see for example **[Ha]** pg. 112 and 145, for the last assertion.

A less trivial examples is a variety studied by Elie Cartan and also by Room. It is a homogeneous complex variety of dimension 16, $X \subset \mathbb{P}^{26}$, associated to the representation of E_6 and for this reason called *E_6-variety*, or *Cartan variety* by Zak. It has been shown by Lazarsfeld and Zak that its secant variety is a degree 3 hypersurface, see for example **[LV]** and **[Z2]**, chapter 3.

There is a unitary way to look at these 4 examples, by realizing them as "Veronese surfaces over the composition algebras over K", $K = \overline{K}$ and char(K)=0, **[Z2]** chapter 3. Let $\mathcal{U}_0 = K$, $\mathcal{U}_1 = K[t]/(t^2+1)$, $\mathcal{U}_2$ = quaternion algebra over K, $\mathcal{U}_3$ = Cayley algebra over K. For $K = \mathbb{C}$, we get $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions numbers $\mathbb{O}$. Let $\mathcal{I}_i$, $i = 0, \ldots, 3$, denote the Jordan algebra of Hermitian (3×3)-matrices over $\mathcal{U}_i$, $i = 0, \ldots, 3$. A matrix $A \in \mathcal{I}_i$ is called *Hermitian* if $\overline{A}^t = A$, where the bar denotes the involution in $\mathcal{U}_i$. Let

$$X_i = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \mathrm{rk}(A) = 1\} \subset \mathbb{P}(\mathcal{I}_i).$$

Then

$$N_i = \dim(\mathbb{P}(\mathcal{I})) = 3 \cdot 2^i + 2, \;\; n_i = \dim(X_i) = 2^{i+1} = 2\dim_K(\mathcal{U}_i),$$

and

$$SX = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \mathrm{rk}(A) \le 2\} = V(\det(A)) \subset \mathbb{P}(\mathcal{I}_i)$$

is a degree 3 hypersurface. By definition $X_i \subset \mathbb{P}(\mathcal{I}_i)$ is a Severi variety of dimension 2^{i+1}, which is seen to be one of the above examples.

A theorem of Jacobson states that over a fixed algebraically closed field K there are only four Jordan algebras, the algebras $\mathcal{U}_i$'s, and hence these are the only examples which can be constructed in this way.

The highly non-trivial and very beautiful result, which is essentially equivalent to Jacobson classification theorem, is the following classification theorem of Severi varieties proved by Zak.

3.2.2. THEOREM. **(Zak classification of Severi varieties, [Z1], [Z2], [LV], [La], [Ru6])** *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n, defined over an algebraically closed field K of characteristic 0. Then X is projectively equivalent to one of the following:*

(1) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
(2) *the Segre 4-fold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
(3) *the Grassmann variety $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$;*
(4) *the E_6-variety $X \subset \mathbb{P}^{26}$.*

A complete proof of this theorem for $n > 8$ is beyond the scope of these notes and of the lectures and it can be found in the above cited references. We prefer to sketch the basic ideas leading to the restriction $n = 2, 4, 8, 16$ for the dimension using some results from the theory of quadric varieties, [Ru6], and to study the analogies with the theorem we proved for $n = 2$, classifying Severi varieties in dimension 2,4 (and 8) and explaining why there exists only one case more. A proof of the classification can be found in above cited references. From now on we will suppose char(K)=0, or equivalently $K = \mathbb{C}$.

The following result is of fundamental importance and yields many interesting geometric restrictions for Severi varieties. For the first part of the proof we follow [Z2], IV, 2.1, while we provide a different way of getting the smoothness of $\Sigma_z(X)$.

3.2.3. PROPOSITION. **(Entry locus of a Severi variety, [Z1], [Z2])** *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n, defined over an algebraically closed field K of characteristic 0. Let $z \in SX$ be a general point. Then $\Sigma_z(X) = (C_{T_zSX}(SX) \cap X)_{red} = C_{T_zSX}(X)$ is a smooth quadric hypersurface of dimension $\frac{n}{2}$ in the contact locus of T_zSX, $C_{T_zSX}(SX) = \overline{\{u \in \mathrm{Sm}(SX) \,:\, T_uSX = T_zSX\}} = \mathbb{P}^{\frac{n}{2}+1}$. Moreover, a Severi variety contains a n-dimensional family of smooth quadric hypersurfaces such that through two general points of it there passes a unique member of the family and such that for a general point $x \in X$ two general quadric surfaces passing through it intersect transversally at x.*

PROOF. Let $H = T_zSX = \mathbb{P}^{\frac{3n}{2}+1}$, $z \in SX$ general point. Then since char(K)=0, we have $C_H(SX) = \mathbb{P}^{\mathrm{def}(SX)}$, see proposition 1.5.6. By definition

$$C_H(X) = \{x \in X \,:\, T_xX \subset H\} = \mathrm{Sing}(X \cap H).$$

Then $C_H(X)$ is a closed subvariety of X and $T(C_H(X), X) \subseteq H$. Since X is non-degenerate, $S(C_H(X), X)$ is non-degenerate so that $T(C_H(X), X) \neq S(C_H(X), X)$ yields, due to theorem 2.2.1,

$$\dim(C_H(X)) + n + 1 = \dim(S(C_H(X), X) \leq \dim(SX) = \frac{3}{2}n + 1,$$

i.e.

$$\dim(C_H(X)) \leq \frac{n}{2}. \tag{3.2.1}$$

On the other hand, by corollary 1.3.6, for $u \in C_H(SX)$ general, we have $T(\Sigma_u(X), X) \subseteq H$ and hence

$$\Sigma_u(X) \subseteq C_H(X) \tag{3.2.2}$$

for $u \in C_H(SX)$ general. Then

$$\dim(\Sigma_z(X)) = \dim(\Sigma_u(X)) = 2n + 1 - \dim(SX) = 2n + 1 - (\frac{3}{2} + 1) = \frac{n}{2}$$

so that by equation 3.2.2 $\Sigma_u(X)$ is a component of $C_H(X)$. Therefore for general $u \in C_H(SX) = \mathbb{P}^{\mathrm{def}(SX)}$,

$$\Sigma_u(X) = \Sigma_z(X)$$

implies that though a general point of $C_H(SX)$ there passes a secant line to $\Sigma_z(X)$, i.e.

$$S\Sigma_z(X) = C_H(SX) = \mathbb{P}^{\mathrm{def}(SX)} \tag{3.2.3}$$

and $\Sigma_u(\Sigma_z(X)) = \Sigma_z(X)$.

From these two facts we get,

$$\dim(C_H(SX)) = \dim(S\Sigma_z(X)) = 2\dim(\Sigma_z(X)) + 1 - \dim(\Sigma_u(\Sigma_z(X))) = \frac{n}{2} + 1,$$

as desired. Since $\Sigma_z(X)$ is an hypersurface in $C_H(SX)$, each point of $X \cap C_H(SX)$ is contained in $\Sigma_z(X)$, which together with 3.2.2 gives the equality of sets $\Sigma_z(X) = (X \cap C_H(X))_{\mathrm{red}}$. From 3.2.3 we deduce $\deg(\Sigma_z(X)) \geq 2$ and by the trisecant lemma and by the generality of z, a general secant line to X passing through z cuts X transversally only in two points and it is contained in $C_H(SX)$, so that $\deg(\Sigma_z(X)) = 2$ and $\Sigma_z(X)$ is a quadric surface.

Next we prove that $C_H(X) = \Sigma_z(X)$. Take a general point $x \in X$. Let $\pi_x : X \dashrightarrow \pi(X) = Y_x \subset \mathbb{P}^{\frac{n}{2}+1}$ be the projection from T_xX, and note that $\dim(Y_x) = \frac{n}{2}$ by proposition 1.3.8.

By arguing as above, $\dim(S(\Sigma_z(X), X)) = \frac{3}{2}n + 1 = \dim(SX)$ so that by Terracini lemma, we have $T_xX \cap T_w\Sigma_z(X) = \emptyset$ for $w \in \Sigma_z(X)$ general. This implies $\pi_x(\Sigma_z(X)) = Y_x$ and also that Y_x is an irreducible quadric hypersurface being a non-degenerate projection of a variety of degree 2. This immediately furnishes $C_H(SX) \cap T_xX = < \Sigma_z(X) > \cap T_xX = \emptyset$, $x \in X$ general, and that π_x restricts to an isomorphism on $\Sigma_z(X)$, which is now evident being an irreducible quadric hypersurface. Take $y \in X$ a general point and let $\pi_x(y) = y' \in Y_x$ be a general point. If Y_x were singular, it would be a quadric cone, so that

$$T_{y'}Y_x = \pi_x(T_yX) = < T_xX, T_yX > \cap \mathbb{P}^{\frac{n}{2}+1}$$

would be tangent at least along a line $l_{y'}$ passing through y'. Then the hyperplane $H = < T_xX, T_yX > = T_uSX$, $u \in SX$ general, would be tangent along $\pi_x^{-1}(l_{y'})$. Since $\dim(\pi_x^{-1}(l_{y'})) \geq \frac{n}{2} + 1$ (recall that $\dim(X) - \dim(Y_x) = \frac{n}{2}$), this would contradict 3.2.1. So we have shown that Y_x and $\Sigma_z(X)$ are smooth quadrics. Moreover, $C_H(X) \setminus T_xX \cap X$ has pure dimension $\frac{n}{2}$ being equal to $\pi_x^{-1}(y')$. By using the same argument as in Scorza Lemma, lemma 1.4.3, we immediately deduce that $C_H(X) \subseteq X \cap (< T_xX, y > \cap < T_yX, y >) = X \cap C_H(SX) = \Sigma_u(X)$.

By taking a general point $p \in X$, a general point $u \in < x, p >$ and a general point $w \in < y, p >$, the previous analysis shows that $\Sigma_u(X)$ and $\Sigma_w(X)$ intersect transversally at p. Indeed $\Sigma_u(X)$ is a fiber of π_x, while $\Sigma_w(X)$ is a section of π_x. □

Thus on an arbitrary Severi variety the picture is completely analogous to 2-dimensional case: there are $\infty^{\frac{n}{2}}$ smooth quadrics through a general point and two general ones intersect transversally at that point. For $n \geq 4$ the difficulty arises from the fact that these subvarieties are not divisors so that the geometric information encoded cannot be translated immediately into a classification result.

We list some interesting consequences of this proposition, which begin to put stronger and stronger restriction to Severi varieties.

3.2.4. COROLLARY. **(Rationality and linear normality of Severi varieties)** *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n and let notations as in proposition 3.2.3. Then X is a linearly normal rational variety, a birational explicit isomorphism being $\pi_x \times \pi_y : X \dashrightarrow Q_x \times Q_y$. Moreover, if $z \in SX$ is a general point, then $p_2 : S_{\Sigma_z(X),X} \to S(\Sigma_z(X), X) = SX$ is birational.*

PROOF. In the final part of the proof of proposition 3.2.3, we proved the birationality of $\pi_x \times \pi_y : X \dashrightarrow Q_x \times Q_y$, since we showed that a general fiber of pi_x and a general fiber of pi_y only intersect transversally at a point.

Let us prove that X is linearly normal. Suppose there exists a non degenerate variety $X' \subset \mathbb{P}^{\frac{3n}{2}+3}$ which projects isomorphically onto $X \subset \mathbb{P}^{\frac{3n}{2}+2}$. Then $\dim(SX') = \dim(SX) = \frac{3n}{2} + 1$, so that if $\pi_{x'} : X' \dashrightarrow Y'_{x'} \subset \mathbb{P}^{\frac{n}{2}+2}$ is the projection from the tangent space at a general point $x' \in X'$, by reasoning as in the proof of proposition 3.2.3 we would deduce that $Y'_{x'}$ is an isomorphic linear projection of a quadric hypersurface of dimension $\frac{n}{2}$ so that it would be degenerated. This contradiction proves the claim.

The last fact easily follows from the equality $S(\Sigma_z(X), X) = SX$ and the fact that if $u \in SX$ is general, then $\Sigma_z(X) \cap \Sigma_u(X)$ consists of a point. □

To put in the right perspective where the technical difficulties begin and what is the new and difficult part in the classification of Severi varieties, we derive as a consequence of proposition 3.2.3, a refined Scorza Lemma, a proof of the classification of 2-dimensional, respectively 4-dimensional Severi varieties, which was first given for irreducible surfaces in [**Sev1**], see theorem 1.4.1, respectively for 4-folds in [**S1**]. Here a "simplified" proof in the 2-dimensional case is possible due to the smoothness assumption. A non-elementary proof in the 4-dimensional case, based on the classification of del Pezzo varieties and on rather intricate arguments, was furnished in [**FR**], theorem 4, as one of the principal results of that paper. Here we simplify, due to the non-singularity assumption, the original argument of Scorza, who in fact classified irreducible 4-fold in $\mathbb{P}^N$, $N \geq 8$, such that $\dim(SX) = 7$, see [**S1**], pg. 204.

3.2.5. THEOREM. **(Severi classification of 2-dimensional Severi varieties, [Sev1])***Let $X \subset \mathbb{P}^5$ be a Severi surface. Then X is projectively equivalent to $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$.*

PROOF. Let $x_1, y_1 \in X$ general points and let C_{x_1,y_1} be the entry locus through x_1 and y_1. By proposition 3.2.3, $C_{x_1,y_1} \subset X$ is a smooth conic. Moreover, if $y_2 \in X$

is general, the conic C_{x_1,y_2} is linearly equivalent to C_{x_1,y_1}, being two general fibers of the tangential projection, $\pi_{x_1} : X \dashrightarrow C_{x_1} \subset \mathbb{P}^2$, of X from $T_{x_1}X$ onto the smooth conic C_{x_1}. If we fix y_1 and let x_1 vary, by projecting from $T_{y_1}X$, we obtain another pencil of conics linearly equivalent to C_{x_1,y_1}, so that C_{x_1,y_1} varies in a linear system of dimension at least 2 and $(C_{x_1,y_1})^2 = 1$ because two general entry loci intersect transversally only at one point by proposition 3.2.3. This two dimensional linear system defines a birational map

$$\phi : X \dashrightarrow \mathbb{P}^2,$$

which sends a general entry locus onto a line.

Let $i : X \hookrightarrow \mathbb{P}^5$ be the inclusion and define

$$\psi = i \circ \phi^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5.$$

The rational map ψ sends lines onto conics in X and hence onto conics of $\mathbb{P}^5$, so that it is given by a sublinear system of $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of dimension 5, i.e. by the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. This implies that ψ is an isomorphism and that X is projectively equivalent to $\nu_2(\mathbb{P}^2)$. □

The analysis in the 4 dimensional case is a little more difficult because the entry loci have codimension 2 so that they are not divisors on X.

3.2.6. THEOREM. **(Scorza classification of 4-dimensional Severi varieties, [S1], pg. 204)** *Let $X \subset \mathbb{P}^8$ be a Severi variety of dimension 4. Then X is projectively equivalent to $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ Segre embedded.*

PROOF. Let $\pi_x : X \dashrightarrow Q_x \subset \mathbb{P}^3$ be the projection from the tangent space at a general point $x \in X$. Let $l_1, l_2 \subset Q_x$ be two general lines belonging to different rulings of Q_x, let $D_i = \pi_x^*(l_i)$ be the corresponding divisors on X, let $p = l_1 \cap l_2$ and let $y \in \pi_x^{-1}(p)$ a general point. Let $Q_{x,y}$ be the entry locus through x and y, i.e. $Q_{x,y} = \Sigma_z(X)$ for $z \in <x, y>$ general point. Then

$$D_1 \cap D_2 = \overline{\pi_x^{-1}(l_1 \cap l_2)} = \overline{\pi_x^{-1}(p)} = Q_{x,y}. \tag{3.2.4}$$

Moreover,

$$D_1 + D_2 + E_x \sim \pi_x^*(\mathcal{O}_{Q_x}(1)) \sim H, \tag{3.2.5}$$

with H an hyperplane section of X and $E_x \geq 0$ the eventual fixed component of the linear system giving π_x, i.e. of the linear system of hyperplanes containing T_xX.

As we recalled above, two general points $y_1, y_2 \in X$ are joined by a smooth quadric surface Q_{y_1,y_2}. Let $l_i \subset Q_{y_1,y_2}$ be a line. Recall that π_x restricted to Q_{y_1,y_2} is an isomorphism. Let E_x be as in 3.2.5. Then $1 = H \cdot l = 1 + E_x \cdot l$ yields $E_x \cdot l = 0$. This means that if $E_x > 0$, then E_x cannot vary with $x \in X$ general point, i.e. $E_x = E \subset T_xX$, $x \in X$ general and X would be a cone. This contradiction furnishes

$$D_1 + D_2 \sim H,$$

and also, letting $H_p = < T_xX, l_1, l_2 > = \mathbb{P}^7 \subset \mathbb{P}^8$,

$$H_p \cap X = D_1 + D_2.$$

We now show that $D_i = \pi_y^*(l_i')$, with $\pi_y : X \dashrightarrow Q_y \subset \mathbb{P}^3$ the projection from T_yX and $l_1', l_2' \subset Q_y$ the lines of the two different rulings of Q_y passing through $\pi_y(x) = p' \in Q_y$. It is sufficient to prove that $T_yX \subseteq H_p$, because then

$\pi_y(D_1) + \pi_y(D_2) = \pi_y(H_p)$ will be a reducible hyperplane section of Q_y. By 3.2.4 $T_yX = T_y(D_1 + D_2) = H_p \cap T_yX$ and the claim follows.

Therefore there is a linear system of dimension 2 of divisors linearly equivalent to D_i, $i = 1, 2$. Each linear system defines a dominant rational map

$$\phi_i : X \dashrightarrow \mathbb{P}^2.$$

Indeed, by definition of π_z, a general entry locus $\Sigma_w(X) = Q$ of X is mapped isomorphically onto Q_z so that $\mathcal{O}_Q(D_1) = \mathcal{O}_Q(1,0)$. and $\mathcal{O}_Q(D_2) = \mathcal{O}_Q(0,1)$, modulo a renumbering. Thus $\phi_i(Q) = l_1 \subset \mathbb{P}^2$ is a line and two general points of $\phi_i(X)$ are joined by a line, so that $\phi_i(X) = \mathbb{P}^2$.

The two dominant rational maps $\phi_i : X \dashrightarrow \mathbb{P}^2$ yield a dominant rational map

$$\phi = \phi_1 \times \phi_2 : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

Let $\sigma_{2,2} : \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ be Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and let

$$\psi = \sigma_{2,2} \circ \phi : X \dashrightarrow \mathbb{P}^8.$$

From

$$\psi^*(\mathcal{O}_{\mathbb{P}^8}(1)) = \phi^*(p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) = \mathcal{O}_X(H),$$

we deduce that ψ is given a sublinear system of $|H|$ of dimension 8, i.e. by the complete linear system of hyperplane section by the linear normality of X, see proposition 3.2.4. This is a reformulation of the desired conclusion. □

By definition the dimension n of a Severi variety $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is an even number. We arrived at the central point of the whole analysis, bounding the dimension of a Severi variety. We sketch the proof contained in [**Ru6**], which, if correct, shows that this basic result is a corollary of the theory of quadric varieties.

3.2.7. THEOREM. **(Dimension of a Severi variety, [Z2], theorem 3.10, pg. 84), [Ru6]** *Let $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ be a Severi variety. Then $n = 2$, 4, 8 or 16. In particular, $\delta(X) = 1$, 2, 4 or 8.*

PROOF. In [**Ru6**] it is shown that a Severi variety $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is a quadric variety of type $\delta = \frac{n}{2}$. We can also suppose $n \geq 6$, i.e. $\delta \geq 3$.

For a smooth quadric variety of type $\delta \geq 3$, $X \subset \mathbb{P}^N$, one defines

$$r_X = \sup\{r \in \mathbb{N} : \delta \geq 2r + 1\}.$$

By the main result of [**Ru6**], 2^{r_X} divides $n - \delta = \frac{n}{2} = \delta$ so that 2^{r_X+1} divides n. Hence $\delta = \frac{n}{2}$ is even and, by definition of r_X, $\frac{n}{2} = 2r_X + 2$. Thus, for some integer $m \geq 1$,

$$m2^{r_X+1} = n = 4(r_X + 1).$$

Therefore either $r_X = 1$, i.e. $n = 8$, or $r_X = 3$, i.e. $n = 16$. □

The classification theorem 3.2.2 is now easy to deduce. Indeed, for $n = 8$ the variety $X \subset \mathbb{P}^{14}$ is a Mukai variety, being a Fano variety of index 6 with $b_2(X) = 1$. Indeed, by Barth theorem 2.1.3, $\mathrm{Pic}(X) = \mathbb{Z} < H >$ and $X \subset \mathbb{P}^{14}$ contains moving lines, so that it is Fano. In [**Ru6**], it is shown that for a quadric variety of type $\delta \geq 3$ and for a line $l \subset X$, $-K_X \cdot l = \frac{n+\delta}{2}$, so that $i(X) = 6$. It is classically well known that $X \simeq \mathbb{G}(1,5) \subset \mathbb{P}^{14}$ Plücker embedded. A uniform approach connecting the original ideas of Zak and a careful study of lines on X in dimension 4, 8 and 16 and leading to a quick classification of Severi varieties in dimension 4, 8 and 16 is described in [**Ru6**]. This approach does not depend on any previous classification result, with the exceptions of the rational representations on $\mathbb{P}^n$ of the varieties appearing, as done by Zak.

We would like to comment briefly the main differences between our approach, if correct, to the classification of Severi varieties and the ones present in the literature. Zak's approach, see also [**LV**], is based on some preliminary analysis of the geometry of SX and of the fact that all the entry loci are smooth quadric of dimension $\frac{n}{2}$. The central point in Zak's classification is a careful study of the linear spaces on the quadrics of the family $\widetilde{Q}_x$, i.e. of the entry loci of the variety to obtain $n \leq 16$ and $n \equiv 0 \pmod{4}$. This is very geometric but full of details and verifications. The first part of Zak's analysis was used by Chaput, [**Ch**], to prove a priori that X is homogeneous and then one deduces the classification from the known description of homogeneous varieties. There is a different proof of the classification by Landsberg, [**La**], via local differential geometry and second fundamental form. Landsberg derives some restrictions on the linear system of quadrics describing the second fundamental form, deduces the bound of the dimension from the classification of Clifford modules and then reconstruct the variety via moving frames.

In [**Ru6**] we use the fact that Severi varieties are defined by $\frac{3}{2}n + 3$ quadric hypersurfaces, yielding a special quadro-quadric transformations, both results being proved as preliminaries by Zak, and finally we concentrate on conics and lines contained in a Severi variety. Thus from this point of view the classification of Severi varieties is completely parallel to the classification of *quadro-quadric* special Cremona transformations, see [**ESB**] and section 5.2. Let us recall that Ein and Shepherd-Barron used Zak classification to conclude there exist only 4 quadro-quadric special Cremona transformations associated to the Severi varities. The classification of Severi varieties, or better of the possible dimension of such a variety, becomes a particular case of the study of conics and lines on varieties defined by quadratic equations, generalizing the case of the quadric hypersurface in $\mathbb{P}^N$, [**Ru6**]. The reconstruction of the Severi varieties in dimension 4, 8 and 16 in [**Ru6**] is analogous to Zak's one but follows the opposite direction, showing a priori, in the possible dimensions, the description of the cones $T_xX \cap X$.

CHAPTER 4

Extremal varieties and Scorza varieties

4.1. Additivity of higher secant defects and maximal embeddings

In this section we study the behaviour of the higher secant defects $\delta_k = \delta_k(X)$, $k \geq 1$, of an irreducible smooth non-degenerate variety of dimension n, $X \subset \mathbb{P}^N$.

Let us recall that, for $z \in S^kX$ general point,

$$\delta_k = \dim(\Sigma_z^k(X)) = \dim(S^{k-1}X) + n + 1 - \dim(S^kX) = s_{k-1} + n + 1 - s_k.$$

So higher the defect, smaller the dimension of S^kX. As we shall see below, if X is secant defective, i.e. if $\delta_1 = \delta > 0$, then its k-secant defect has to be at least $k\delta$, so that a secant defective variety has a minimum k-secant defect determined a priori. Of special interest will be secant defective varieties for which each δ_k will attain the minimal value $k\delta$. What is not at all clear at this point is the fact that these varieties can be completely classified in every dimension, at least in characteristic 0, and that they are suitable generalizations of Severi varieties.

Let us start with a general property of varieties defined over a field of characteristic 0.

4.1.1. Proposition. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety. Suppose* char(K)*=0. Let $k \geq 1$ be such that $S^kX \subsetneq \mathbb{P}^N$, let $x, y \in X$ and $u \in S^{k-1}X$ be general points. Then*

$$T_xX \cap T_yX \cap T_uS^{k-1}X = \emptyset.$$

Proof. Let $z \in S^kX$ be a general point and let $S^kX = S(X, S^{k-1}X)$. Then by corollary 1.3.6 the linear space $T_zS^k \neq \mathbb{P}^N$ is tangent along $\Sigma_z^k(X)$ so that it contains $T(\Sigma_z^k(X), X)$. Since X is non-degenerate, $S(\Sigma_z^k(X), X)$ is not contained in T_zS^kX. By theorem 2.2.1 we get $\dim(S(\Sigma_z^k(X), X) = \dim(\Sigma_z^k(X)) + \dim(X) + 1$ so that, for $x, y \in X$ general points,

$$T_xX \cap T_y\Sigma_z^k(X) = \emptyset. \tag{4.1.1}$$

By proposition 1.3.9, if $u \in S^{k-1}X$ is general, then

$$T_y\Sigma_z^k(X) = T_yX \cap < y, T_uS^{k-1}X > . \tag{4.1.2}$$

By combining equations 4.1.1 and 4.1.2, we finally obtain

$$T_xX \cap T_yX \cap T_uS^{k-1}X \subseteq T_xX \cap T_yX \cap < y, T_uS^{k-1}X > = T_xX \cap T_y\Sigma_z^k(X) = \emptyset.$$

□

By combining Terracini Lemma with the above proposition, we immediately obtain a proof in characteristic zero, the case we treat in the whole chapter, of the next theorem, which is true for arbitrary fields. For a proof valid in arbitrary characteristic one can consult [**Z2**], pg. 109.

4.1.2. THEOREM. **(Additivity of higher secant defects, Zak)** *Let $X \subset \mathbb{P}^N$ be an irreducible smooth non-degenerate projective variety. Let $k \in \mathbb{N}$, $1 \leq k \leq k_0$. Then*

$$\delta_k \geq \delta_{k-1} + \delta \geq k\delta.$$

PROOF. Fix k, $2 \leq k \leq k_0$, the result being trivial for $k = 1$. By definition $S^{k-1}X \subsetneq \mathbb{P}^N$, so that if $x, y \in X$ and $u \in S^{k-2}X$ are general points, by proposition 4.1.1, we get $T_xX \cap T_yX \cap T_uS^{k-2}X = \emptyset$.

Let

$$L_1 = T_xX \cap T_uS^{k-2}X \text{ and } L_2 = T_xX \cap T_yX.$$

By Terracini Lemma, $\dim(L_1) = \delta_{k-1} - 1$ and $\dim(L_2) = \delta - 1$ since $x, y \in X$ and $u \in S^{k-2}X$ are general points. Let $S^kX = S(X, S^{k-1}X)$ and set

$$L = T_xX \cap < T_yX, T_uS^{k-2}X > .$$

Once again by Terracini Lemma,

$$\dim(L) = \delta_k - 1.$$

Since $L_i \subseteq L$ and $L_1 \cap L_2 = T_xX \cap T_yX \cap T_uS^{k-2}X = \emptyset$, then

$$\delta_k - 1 = \dim(L) \geq \dim(< L_1, L_2 >) = \delta_{k-1} - 1 + \delta - 1 - (-1) = \delta_{k-1} + \delta - 1,$$

yielding the conclusions. □

We deduce some interesting corollaries of the above result. For a real number $r \in \mathbb{R}$, $[r]$ denotes the largest integer not exceeding r.

4.1.3. COROLLARY. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety of dimension n. Suppose $\delta > 0$. Then $k_0 \leq [\frac{n}{\delta}]$, i.e.*

$$S^{[\frac{n}{\delta}]}X = \mathbb{P}^N.$$

PROOF. Recall that $\delta_{k_0} \leq n$ by its definition so that

$$n \geq \delta_{k_0} \geq k_0\delta,$$

i.e.

$$\frac{n}{\delta} \geq k_0.$$

□

The second application is a different proof of Hartshorne conjecture on linear normality, cfr. theorem 3.1.5.

4.1.4. COROLLARY. **(Zak Theorem on Linear Normality)** *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. If $N < \frac{3}{2}n + 2$, then $SX = \mathbb{P}^N$. Or equivalently if $SX \subsetneq \mathbb{P}^N$, then $\dim(SX) \geq \frac{3}{2}n + 1$ and hence $N \geq \frac{3}{2}n + 2$.*

PROOF. Let us prove the last part. If $\delta > \frac{n}{2}$, then $1 = [\frac{n}{\delta}] \geq k_0 \geq 1$ yields $SX = \mathbb{P}^N$. So $\delta \leq \frac{n}{2}$ as soon as $SX \subsetneq \mathbb{P}^N$. This yields

$$N \geq \dim(SX) + 1 = 2n + 2 - \delta \geq \frac{3n}{2} + 2.$$

□

We begin a systematic study of smooth secant defective varieties in characteristic zero and try to determine the restrictions in terms of the embedding. We saw that if $\delta = 0$ and $N \geq 2n+1$, there always exist smooth non-degenerate varieties $X \subset \mathbb{P}^N$ of dimension n and with $\delta = 0$. For example one takes as X a smooth complete intersection of $N-n$ general hypersurfaces. Moreover, by corollary 1.5.5, for such varieties, if $k < k_0(X)$, then $s_k(X) = (k+1)n+k$, $\delta_k(X) = 0$ and $s_{k_0} = N$, so that N and k_0 are not determined or at least bounded by a function of n and δ and both can grow arbitrarily.

On the other hand, corollary 4.1.3 and theorem 4.1.2 say that for non-degenerate varieties of fixed dimension n and with fixed $\delta > 0$, k_0 and N are bounded from above by a function depending on n and δ. Indeed, $k_0 \leq [\frac{n}{\delta}]$, so that, by corollary 1.3.6 part 4 and by theorem 4.1.2,

$$N = s_{k_0} = (k_0+1)(n+1) - 1 - \sum_{i=1}^{k_0} \delta_i \leq (k_0+1)(n+1) - 1 - \delta \sum_{i=1}^{k_0} i, \tag{4.1.3}$$

is bounded by a function depending only on n, δ and k_0.

So a secant defective smooth non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n can be isomorphically projected in $\mathbb{P}^M$, $M \leq 2n$, but due to the secant deficiency it cannot be the isomorphic projection of a variety living in a projective space of arbitrary large dimension. The result of theorem 4.1.2 and the definition of S^kX and of k_0 say that linearly normal secant defective varieties with higher $N+1 = h^0(\mathcal{O}_X(1))$ are those for which δ_k is the minimum possible, i.e. varieties such that $\delta_k = k\delta$.

On the base of the previous discussion let us introduce some definitions and collect the above argument in a more systematic statement. We can think linear projection as a partial order in the set of the embeddings of a variety X in projective space. Of particular interest will be maximal and minimal elements with respect to this partial order.

4.1.5. DEFINITION. (**Functions $M(n,\delta)$, $m(n,\delta)$ and $f(n,\delta,k)$**) All varieties $X \subset \mathbb{P}^N$ are supposed to be smooth, non-degenerate and projective.

Let us define, it it exists (otherwise we put it equal to ∞), for $n \geq 1$ and for $\delta \geq 0$,

$$M(n,\delta) := \max\{N : \exists X \subset \mathbb{P}^N : \dim(X) = n,\, \delta(X) = \delta\}.$$

In the same way we define

$$m(n,\delta) := \min\{N : \exists X \subset \mathbb{P}^N : \dim(X) = n,\, \delta(X) = \delta\}.$$

Inspired by equation 4.1.3 we define for $k \geq 0$, for $n \geq 1$ and for $\delta \geq 0$:

$$f(k,n,\delta) := (k+1)(n+1) - \frac{k(k+1)\delta}{2} - 1.$$

We saw that $M(n,0) = \infty$. Clearly $m(n,\delta) = 2n+1-\delta$. Indeed general complete intersection of dimension n in $\mathbb{P}^{2n+1-\delta}$ are smooth non-degenerate varieties with $SX = \mathbb{P}^{2n+1-\delta}$ so that $\delta(X) = \delta$ and $m(n,\delta) \leq 2n+1-\delta$. On the other hand every variety $X \subset \mathbb{P}^N$ with $\delta(X) = \delta$ and of dimension n has $2n+1-\delta = \dim(SX) \leq N$, yielding $m(n,\delta) \geq 2n+1-\delta$.

The equation 4.1.3 can be read as, if $\delta > 0$, then $N \leq f(k_0,n,\delta)$.

Let us reinterpret corollary 4.1.3 in terms of these functions and study their first properties.

4.1.6. PROPOSITION. *Let the pairs (n,δ) in the statement be such that the functions $M(n,\delta)$ and $m(n,\delta)$ are defined. Then*

(1) *if $\delta > \frac{n}{2}$, then $M(n,\delta) = m(n,\delta) = 2n+1-\delta$;*
(2) $M(n,\delta-1) \geq M(n,\delta)+1$;
(3) $M(n-1,\delta-1) \geq M(n,\delta)-1$.

PROOF. By corollary 4.1.3, $\delta > \frac{n}{2}$ gives $k_0 = 1$ and hence $SX = \mathbb{P}^N$ so that $N = \dim(SX) = 2n+1-\delta = m(n,\delta)$ is determined by n and δ, yielding part 1).

Suppose given $X \subset \mathbb{P}^N$, $\dim(X) = n$ and $\delta(X) = \delta \geq 1$. Let $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$, set $Y = S(p,X)$ and take $X' = Y \cap H \subset \mathbb{P}^{N+1}$, $H \subset \mathbb{P}^{M(n,N+1}$ a general hypersurface of degree $d > 1$. The variety X' is smooth, non-degenerate, irreducible and of dimension n with $\delta(X') = \delta(X)-1$ and $SX' = S(p,SX)$. Indeed, $SX' \subseteq S(p,SX)$ so that it will be sufficient to prove the first part of the claim. Let $\pi_p : X' \to X$ be the projection from p onto $\mathbb{P}^N$. By Terracini lemma, if $p'1, p'2 \in X'$ are general points, then

$$\begin{aligned}
\delta(X')+1 &= \dim(<p, T_{p_1'}X> \cap <p, T_{p_2'}X>) \\
&= \dim(<p, T_{p_1'}X> \cap <p, T_{p_2'}X> \cap \mathbb{P}^N) + 1 \\
&= \dim((<p, T_{p_1'}X> \cap \mathbb{P}^N) \cap (<p, T_{p_2'}X> \cap \mathbb{P}^N)) + 1 \\
&= \dim(T_{\pi_p(p_1')}X \cap T_{\pi_p(p_2')}X) + 1 = \delta.
\end{aligned}$$

Suppose given $X \subset \mathbb{P}^N$, $\dim(X) = n$ and $\delta(X) = \delta \geq 1$. Let $X' = X \cap H \subset H = \mathbb{P}^{N-1}$ be a general hyperplane section. By Terracini Lemma and by the generality of H, if one takes $p_1, p_2 \in X' = X \cap H$ general, then $\delta(X') - 1 = \dim(T_{p_1}X' \cap T_{p_2}X') = \dim(T_{p_1}X \cap T_{p_2}X \cap H) = \delta - 2$ so that $\delta(X') = \delta - 1$ and $\dim(SX') = 2(n-1)+1-\delta(X') = 2n-\delta = \dim(SX)-1$. Since $SX' \subseteq SX \cap H$ we also deduce $SX' = SX \cap H$. $\square$

4.1.7. DEFINITION. **(Extremal variety)** A smooth irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n is said to be *an extremal variety* if $\delta(X) = \delta > 0$ and if $N = M(n,\delta)$.

In other words an extremal variety is a smooth secant defective variety, which is a maximal element in the partial order defined by isomorphic projection.

We are now in position to refine equation 4.1.3 in the sharpest form.

4.1.8. THEOREM. **(Maximal embedding of secant defective varieties, Zak, [Z2])** *Suppose $\delta > 0$. Then*

$$M(n,\delta) \leq f([\frac{n}{\delta}], n, \delta).$$

In particular a smooth non-degenerate irreducible projective variety $X \subset \mathbb{P}^N$ with $N = f([\frac{n}{\delta}], n, \delta)$ is linearly normal.

PROOF. By equation 4.1.3 we know that for a given variety $X \subset \mathbb{P}^N$ of dimension n and with $\delta(X) = \delta$, we have $N \leq f(k_0, n, \delta)$. On the other hand by corollary 4.1.3 we know that $k_0 \leq [\frac{n}{\delta}]$. Fixing n and δ, $y = f(k, n, \delta)$ is a parabola in the plane (k, y), whose vertex has coordinates $(\frac{2n-\delta+2}{2\delta}, \frac{(2n+\delta+2)^2}{8\delta})$; in particular it is an increasing function on the interval $0 \leq k \leq \frac{2n-\delta+2}{2\delta}$. So if $k_0 = [\frac{n}{\delta}]$, there is nothing to prove. If $k_0 < [\frac{n}{\delta}]$, then

$$k_0 \leq [\frac{n}{\delta}] - 1 \leq \frac{n}{\delta} - 1 < \frac{2n-\delta+2}{2\delta},$$

so that

$$N \leq f(k_0, n, \delta) \leq f(\frac{n}{\delta} - 1) = f(\frac{n}{\delta}, n, \delta) - 1 < f([\frac{n}{\delta}], n, \delta),$$

where the last inequality follows from the fact that $f(m, n, \delta) \in \mathbb{N}$ for every $m \in \mathbb{N}$. This finishes the proof. □

Theorem 4.1.8 says that secant defective varieties are allowed to live in a projective space of bounded dimension, the bound being expressed by the value $f([\frac{n}{\delta}], n, \delta)$.

Let us reinterpret some results in the light of the new definitions and of the first properties of the functions $M(n, \delta)$ and $f(k, n, \delta)$, following Zak [**Z2**].

4.1.9. REMARK. (Case $\delta > \frac{n}{2}$) If $\delta > \frac{n}{2}$, then $[\frac{n}{\delta}] = 1$, $SX = \mathbb{P}^N$ so that

$$m(n, \delta) = M(n, \delta) = 2n + 1 - \delta = f(1, n, \delta).$$

4.1.10. REMARK. (Case $\delta = \frac{n}{2}$) In this case $n \equiv 0 \pmod{2}$ and by theorem 4.1.8

$$N \leq f(2, n, \frac{n}{2}) = \frac{3n}{2} + 2.$$

There are two possibilities:

(1) $SX = \mathbb{P}^N$, $m(n, \frac{n}{2}) = \frac{3n}{2} + 1 = \dim(SX) = N$;
(2) $SX \subsetneq \mathbb{P}^N$, $N = \dim(SX) + 1 = M(n, \frac{n}{2}) = f(2, n, \frac{n}{2}) = \frac{3n}{2} + 2$.

Varieties in case 2) are clearly Severi varieties so the remark furnishes a new proof that Severi varieties are linearly normal, see corollary 3.2.4.

In the next remark we connect these results with the classical work of Gaetano Scorza.

4.1.11. REMARK. Suppose

$$n \equiv 1 \pmod{2} \text{ and } SX \subsetneq \mathbb{P}^N.$$

By remark 4.1.9

$$\delta < \frac{n-1}{2} \text{ and } s = \dim(SX) = 2n + 1 - \delta \geq \frac{3n+3}{2}.$$

We now discuss the extremal case $\delta = \frac{n-1}{2}$ in the above hypothesis.
Suppose $n = 3$ so that $\delta = 1$. By theorem 4.1.8

$$N \leq f(3, 3, 1) = 9 = s + 2.$$

By the main classification theorem of [**S1**], see also theorem 4.2.7, there is only one such 3-fold, $X = \nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$.

If $n > 3$, then

$$N \leq f(\frac{2n}{n-1}, n, \frac{n-1}{2}) = \frac{3n+7}{2} = s+2.$$

Therefore, for $n \equiv 1 \pmod{2}$, $\delta = \frac{n-1}{2}$ there are only the following cases:

(1) $SX = \mathbb{P}^N$, $N = s = m(n, \frac{n-1}{2}) = \frac{3n+3}{2}$;
(2) $N = s+1 = \frac{3n+5}{2}$;
(3) $N = s+2 = \frac{3n+7}{2} (= M(n, \frac{n-1}{2})$ if $n > 3)$;
(4) $n = 3$, $N = s+3 = M(3,1) = 9$.

All these cases really occur. Examples of case 2) are hyperplane sections of the Severi varieties. For an example as in case 3) one can take $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$ Segre embedded, while we saw above an example as in case 4).

4.2. Scorza varieties

In the previous section we defined extremal varieties and discussed various cases. In particular we saw that if $\delta > \frac{n}{2}$, then $SX = \mathbb{P}^N$ and $X \subset \mathbb{P}^N$ is an extremal variety such that $M(n,\delta) = f([\frac{n}{\delta}], n, \delta) = f(1, n, \delta) = 2n+1-\delta = N = m(n,\delta)$.

By definition of the function $f(k,n,\delta)$ and due to theorem 4.1.8, an extremal variety $X \subset \mathbb{P}^{M(n,\delta)}$, $\delta > 0$, satisfies $M(n,\delta) = f([\frac{n}{\delta}], n, \delta)$ if and only if $k_0 = [\frac{n}{\delta}]$ and $\delta_k = k\delta$ for $0 \leq k \leq k_0$.

The case of extremal varieties with $k_0 = 1 = [\frac{n}{\delta}]$, i.e. $M(n,\delta) = m(n,\delta)$ does not present particular restrictions and there are infinite examples. On the base of the examples in dimension 3 and 4, classically studied by Scorza in [S1] and [S4], Zak introduces the following definition.

4.2.1. Definition. **(Scorza variety, [Z2], pg. 121)** Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension n. Then X is said to be *a Scorza variety* if:

(i) $N > m(n,\delta)$, where $\delta = \delta(X) = 2n+1-\dim(SX)$;
(ii) $N = M(n,\delta) < \infty$, i.e. $\delta > 0$ and X is an extremal variety;
(iii) $M(n,\delta) = f([\frac{n}{\delta}], n, \delta)$, where $f(k,n,\delta) = (k+1)(n+1) - \frac{k(k+1)}{2}\delta - 1$.

From now on we will suppose char(K)=0. Under these hypothesis, a smooth non-degenerate irreducible projective variety $X \subset \mathbb{P}^N$ of dimension n is a Scorza variety if and only if $\delta \leq \frac{n}{2}$, $k_0 = [\frac{n}{\delta}]$ and $\delta_k = k\delta$ for $0 \leq k \leq [\frac{n}{\delta}]$.

As we saw in remark 4.1.10, Severi varieties are instances of Scorza varieties with $\delta = \frac{n}{2}$. So the class of Scorza varieties includes the four Severi varieties. The extraordinary and remarkable classification result due to Zak, which we will try to illustrate in this section, states that there are only few other examples. These examples form infinite series, whose first members are the three classical Severi varieties of dimensions 2,4 and 8. The classification result is the following.

4.2.2. Theorem. **(Classification of Scorza varieties, [Z2] chapter V)** *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension n. Then X is projectively equivalent to one of the following:*

(1) $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ $(\delta = 1)$;
(2) $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$, $a+b=n$, $|a-b| \leq 1$ $(\delta = 2)$;
(3) $\mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{8}}$, $n \equiv 0 \pmod{2}$ $(\delta = 4)$;

(4) *the E_6-variety $X \subset \mathbb{P}^{26}$ of dimension 16 ($\delta = 8$).*

There is uniform description of Scorza varieties with $n \equiv 0$ (mod. δ), "*the most interesting case*", according to Zak, [Z2] pg. 152. These varieties have a "determinantal" description as locus of rank 1 matrices in the projective space of suitable Jordan algebras of Hermitian matrices of order $\frac{n}{\delta}+1$ over composition algebras, generalizing the one furnished for Severi varieties, see [Z2] pg. 153 and [Ch2]; hence they are realized as suitable quadratic Veronese embedding of "generalized" projective spaces. From this point of view the classification of these Scorza varieties is completely parallel to the classification of the above algebras obtained algebraically by Albert, see for example [BK] or [Ja].

As in the case of Severi varieties we will furnish the ideas behind this classification result, proving some particular cases. Once again this will exactly reveal the points where the technical difficulties really arise.

The first important and significant result towards classification is the following.

4.2.3. THEOREM. **(Entry loci of Scorza varieties, [Z2]**, pg. 122) *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension n, with $\delta = \delta(X)$ and $N = f([\frac{n}{\delta}], n, \delta)$. Let $z \in S^kX$, $2 \le k \le k_0 - 1 = [\frac{n}{\delta}] - 1$. Then*

$$\Sigma_z^k(X) \subset C_{T_zS^kX}(S^kX) = \mathbb{P}^{f(k,k\delta,\delta)} = S^k\Sigma_z^k(X)$$

is a Scorza variety such that

$$\dim(\Sigma_z^k(X)) = k\delta,\ k_0(\Sigma_z^k(X)) = k,\ \delta_i(\Sigma_z^k(X)) = i\delta,\ 0 \le i \le k.$$

If $\frac{n}{\delta} > k_0 \ge 2$, then $\Sigma_z^{k_0}(X)$ is a Scorza variety of dimension $k_0\delta < n$ with $\delta(\Sigma_z^{k_0}(X)) = \delta$ and $< \Sigma_z^{k_0}(X) > = S^{k_0}\Sigma_z^{k_0}(X) = C_{T_zS^{k_0}X}(S^{k_0}X) = \mathbb{P}^{f(k_0,k_0\delta,\delta)}$.

For $z \in SX$ general point, $\Sigma_z(X) \subset \mathbb{P}^{\delta+1}$ is a non-singular quadric hypersurface of dimension δ.

The following corollary is a fundamental step for the classification of Scorza varieties since it drastically reduces the cases to be considered.

4.2.4. COROLLARY. **(Singular defect of Scorza varieties, [Z2]**, pg. 125) *Let $X \subset \mathbb{P}^{f([\frac{n}{\delta}],n,\delta)}$ be a Scorza variety. Then for $z \in S^2X$ general point, $\Sigma_z^2(X)$ is a Severi variety so that $\delta = 1$, 2, 4, or 8.*

In order to classify Scorza varieties we have to consider only the 4 cases: $\delta = 1$, 2, 4 or 8. We present a simplified approach to the classification of Scorza varieties with $\delta = 1$ and $\delta = 2$, based on projection from tangent linear spaces. These are the natural generalizations, in the smooth case, of Severi classification of 2-dimensional Severi varieties, i.e. of the well known characterizations of the Veronese surface, theorem 1.4.1 or theorem 3.2.5, and of Scorza classification of 4-dimensional Severi varieties, theorem 3.2.6. See [Z2], theorem 2.1 for different proofs, which directly or indirectly inspired ours.

We need two lemmas, the first reveals a strong property of Scorza varieties while the second one connects the entry loci and suitable projections from tangent linear spaces. A posteriori all Scorza varieties will be rational.

4.2.5. LEMMA. **(Regularity of Scorza varieties)** *Let $X \subset \mathbb{P}^N$ be a Scorza variety. Then $h^1(\mathcal{O}_X) = h^0(\Omega^1_X) = 0$.*

PROOF. By theorem 4.2.3 two general points on X are connected by a smooth conic, so that Scorza varieties are rationally connected. These conics varies in an irreducible family, whose general member is a smooth conic, let us say C, and such that fixing a general point $x \in X$, the members of the family passing through x cover a dense subset. In particular $T_{X|C} \simeq \sum_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_i > 0$ for every $i = 1, \ldots, n$, see for example [**K1**], II, 3.10. Therefore $\Omega^1_{X|C}$ is a sum of line bundles of negative degree, yielding $h^0(\Omega^1_{X|C}) = 0$. The conics C covers a dense subset of X, hence every section of Ω^1_X vanishes. Since char(K)=0, $h^1(\mathcal{O}_X) = h^0(\Omega^1_X) = 0$. □

4.2.6. LEMMA. **(Tangential projections of Scorza varieties)** *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension n and secant defect δ. Suppose $n \equiv 0$ (mod. δ). Let $z \in S^{k_0-2}X$ be a general point, let $\pi_z : X \dashrightarrow \pi_z(X) = Q_z \subset \mathbb{P}^{\delta+1}$ and let $u \in <z, x>$ be a general point of $S^{k_0-1}X$. Then*

1. *$k_0\delta = n$ and $S^{k_0-1}X$ is an hypersurface;*
2. *Q_z is a smooth quadric hypersurface of dimension δ and the closure of a general fiber of π_z is a Scorza variety of dimension $(k_0 - 1)\delta = n - \delta$;*
3. *every irreducible component of the contact locus $C_{T_uS^{k_0-1}X}(X)$ has dimension less than or equal to $n - \delta$. There is only one irreducible smooth component of dimension $n - \delta$, the entry locus $\Sigma^{k_0-1}_u(X)$. Furthermore, the other irreducible components of the contact locus $C_{T_uS^{k_0-1}X}(X)$, if any, are contained in $T_zS^{k_0-2}X \cap X$;*

PROOF. By definition of Scorza variety we get $k_0 = [\frac{n}{\delta}] = \frac{n}{\delta}$. Since $\delta_{k_0} = k_0\delta = n$, it follows from proposition 1.3.6 part 3) that $S^{k_0-1}X$ is a hypersurface, i.e. $s_{k_0-1} = N - 1$, and the first part is proved.

Let $z \in S^{k_0-2}X$ be a general point and let

$$\pi_z : X \dashrightarrow \mathbb{P}^{N-s_{k_0-2}-1} = \mathbb{P}^{n+1-(k_0-1)\delta} = \mathbb{P}^{\delta+1}$$

be the projection from the linear space $T_zS^{k_0-2}X$, see proposition 1.3.8. By proposition 1.3.8, $\pi_z(X) = Q_z \subset \mathbb{P}^{\delta+1}$ is an irreducible non-degenerate hypersurface. By corollary 4.2.3 through two general points $x_1, x_2 \in X$ there pass a smooth quadric hypersurface of dimension δ, let us say $\Sigma_w(X)$, which, as we shall immediately see, is isomorphic to Q_z via π_z. In particular the hypersurface Q_z is a smooth quadric hypersurface.

Indeed, if $T_zS^{k_0-2}X \cap \Sigma_w(X) \neq \emptyset$, then by the generality of $x_1, x_2 \in X$, $\pi_z(\Sigma_w(X))$ would be a positive dimensional linear space passing through two general points $\pi_z(x_i) \in Q_z$, so that Q_z would be linear. For the same reason we get $T_zS^{k_0-2}X \cap <\Sigma_w(X)> = \emptyset$, because otherwise $T_zS^{k_0-2}X \cap <\Sigma_w(X)> = p$, $p \notin \Sigma_w(X)$ by the previous analysis, and $\pi_z(\Sigma_w(X)) = \pi_p(\Sigma_w(X)) = \mathbb{P}^\delta \subseteq Q_z$, would force once again $Q_z = \mathbb{P}^\delta$, contrary to its non-degenerateness.

Let us show that a general entry locus $\Sigma_w(X)$ cuts the closure of every irreducible component F_j of F, the closure of a general fiber of π_z, only at a point, which is necessarily outside $T_zS^{k_0-2}X \cap X$, by the above analysis. This implies that the fiber of π_z is irreducible since we know that π_z restricts to an isomorphism on $\Sigma_w(X)$. If $x \in F_j$ is a general point and if $u \in <z, x>$ is a general point of $S^{k_0-1}X$ as above, then $T_uS^{k_0-1}X = <T_zS^{k_0-2}X, T_xX>$ is tangent along F_j, being tangent at all the points of the fiber by generic smoothness. Then $T(F_j, X) \subset T_u S^{k_0-1}X$ and since X is non-degenerate we get, for every irreducible component F_j of F, $S(F_j, X) \neq T(F_j, X)$. Hence by theorem 2.2.1

$$\dim(S(F_j, X)) = n - \delta + n + 1 = \dim(SX), \tag{4.2.1}$$

so that

$$S(F_j, X) = SX.$$

From 4.2.1 it follows $F_j \cap \Sigma_w(X) \neq \emptyset$ for every j, i.e. $\Sigma_w(X)$ intersects all the irreducible components of F, as claimed. Since π_z induces an isomorphic linear projection on the linear space $< \Sigma_w(X) >= \mathbb{P}^{\delta+1}_w$, F is irreducible and $(F \cdot \Sigma_w(X)) = 1$.

Since Q_z is a smooth quadric hypersurface, a tangent hyperplane is tangent to it only at one point, so that, with notations as above, the tangent hyperplane $T_u S^{k_0-1}X$ is tangent to X along F, which is irreducible, and eventually along a subvariety of $T_z S^{k_0-2}X \cap X$. By Terracini lemma, corollary 1.3.6, we know that $\Sigma_u^{k_0-1}(X) \subseteq F$, being not contained in $T_z S^{k_0-2}X \cap X$ by the generality of u. Then $\Sigma_u^{k_0-1}(X) = F$, since both have dimension $n - \delta = \delta_{k_0-1}$. By theorem 4.2.3, $F = \Sigma_z^{k_0-1}(X)$ is then a Scorza variety and in particular it is smooth. Let Z_j be an irreducible component of the contact locus $C_{T_u S^{k_0-1}X}(X)$ different from F, if any, and hence necessarily contained in $T_z S^{k_0-2}X \cap X$. Since

$$T(Z_j, X) \subseteq T_u S^{k_0-1}X,$$

by definition, we deduce $S(Z_j, X) \neq T(Z_j, X)$, so that

$$\dim(Z_j) + n + 1 = \dim(S(Z_j, X)) \leq \dim(SX) = 2n + 1 - \delta,$$

i.e. $\dim(Z_j) \leq n - \delta$. If $\dim(Z_j) = n - \delta$, then $S(Z_j, X) = SX$, which implies $\emptyset \neq \Sigma_w(X) \cap Z_j \subseteq \Sigma_w(X) \cap T_z S^{k_0-2}X$, $w \in SX$ general. We have previously shown that $\Sigma_w(X) \cap T_z S^{k_0-2}X = \emptyset$. Therefore this contradiction proves that $\dim(Z_j) < n - \delta$ for every j. □

With the previous lemmas we are in position to classify Scorza varieties with $\delta = 1, 2$. The proofs will be completely analogous to the proofs of the classification of 2-dimensional and 4-dimensional Severi varieties, see theorems 3.2.5 and 3.2.6. These proofs differs from the original ones by Zak.

4.2.7. THEOREM. **(Classification of Scorza varieties with $\delta = 1$)** *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate irreducible projective variety of dimension n such that* $\dim(SX) \leq 2n$. *Then $N \leq \frac{n(n+3)}{2}$ and equality holds if and only if X is projectively equivalent to $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.*

In particular $M(n, 1) = \frac{n(n+3)}{2}$ and $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^N$ is the only Scorza variety of dimension n with $\delta = 1$.

PROOF. By hypothesis $\delta(X) = \delta \geq 1$, so that $n \geq \frac{n}{\delta} \geq [\frac{n}{\delta}] \geq k_0$, and $N \leq f(k_0, n, \delta) \leq \frac{n(n+3)}{2}$. Suppose $N = \frac{n(n+3)}{2}$, then $\delta = 1$, $k_0 = n$ and X is a Scorza variety. By lemma 4.2.6, $S^{n-1}X$ is a hypersurface.

By theorem 3.2.5 we can suppose $n \geq 3$ and argue by induction on n. Let $u \in S^{n-1}X$, then $\Sigma_u^{n-1}(X) := D$ is a smooth divisor on X, which is a Scorza variety of dimension $n - 1$ and with $\delta = 1$ by theorem 4.2.3, so that $D \simeq \mathbb{P}^{n-1}$ by the induction hypothesis.

Consider the complete linear system $|D|$. We claim that $\dim|D| = m \geq n$. Indeed, if $x_0, \ldots, x_{n-2}$ are general points, if $z \in< x_0, \ldots, x_{n-2} >$ is general and if $u \in< z, x_{n-1} >$ is general, with x_{n-1} general, then

$$D = \Sigma_u^{n-1}(X) = \overline{\pi_z^{-1}(\pi_z(x_{n-1})}$$

by lemma 4.2.6, where $\pi_z : X \dashrightarrow C_z \subset \mathbb{P}^2$ is the projection from $T_z S^{n-2}X$. The smooth divisor D varies in a pencil of linearly equivalent divisors parametrized by the conic $\pi_z(X) = C_z$. By fixing $n-1$ points between $x_0, \ldots, x_{n-1}$, and projecting by the corresponding tangent space to $S^{n-1}X$ we construct n different pencils of linear equivalent divisors, each one containing D, so that $m \geq n$.

Let

$$\phi = \phi_{|D|} : X \dashrightarrow \phi(X) = X' \subseteq \mathbb{P}^m.$$

We claim that $m = n$ and that $\phi : X \dashrightarrow \mathbb{P}^n$ is dominant.

Since a conic $\Sigma_w(X) \subset X$, $w \in SX$ general, is mapped by ϕ onto a line ($(D \cdot \Sigma_w(X)) = 1$ by lemma 4.2.6), two general points on X' are joined by a line so that X' is a linear space. Thus $X' = \mathbb{P}^m$ and ϕ is dominant. On the other hand, $m = \dim(\phi(X)) \leq n$, yields together with the above analysis $m = n$. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0, \tag{4.2.2}$$

together with $D \simeq \mathbb{P}^{n-1}$, $h^0(\mathcal{O}_X(D)) = n+1$ and $h^1(\mathcal{O}_X) = 0$, recall lemma 4.2.5, yields $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, so that $(D)^n = 1$. The usual Castelnuovo's argument assures that $\mathcal{O}_X(D)$ is generated by global sections. Thus

$$\phi : X \to \mathbb{P}^n$$

is a birational morphism mapping conics in X onto lines in $\mathbb{P}^n$.

Let $i : X \hookrightarrow \mathbb{P}^{\frac{n(n+3)}{2}}$ be the inclusion and let

$$\psi = i \circ \phi^{-1} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\frac{n(n+3)}{2}}.$$

Since lines in $\mathbb{P}^n$ are sent into conics in $\mathbb{P}^{\frac{n(n+3)}{2}}$, the map ψ is given by a sublinear system of $|\mathcal{O}_{\mathbb{P}^n}(2)|$ of dimension $\frac{n(n+3)}{2}$, i.e. by the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(2)|$. The map ψ is an isomorphism, which is clearly a reformulation of the fact that X is projectively equivalent to $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$. □

We now proceed with the classification of Scorza varities with $\delta = 2$. Our proof is different from the one proposed by Zak in **[Z2]**, pg. 130. In particular we do not need to appeal to Barth's theorem as in **[Z2]**, pg. 132, and also we can avoid Zak's geometric but rather intricate analysis of the family of lines on X. The argument is exactly the same used in the classification of 4-dimensional Severi varieties, theorem 3.2.6.

4.2.8. THEOREM. **(Classification of Scorza varieties with $\delta = 2$)** *Let $X \subset \mathbb{P}^N$ be a variety of dimension $n \geq 4$ and such that* $\dim(SX) < 2n$. *Then*

(1) *if $n = 2m$, then $N \leq m(m+2) = (m+1)^2 - 1$;*
(2) *if $n = 2m+1$, then $N \leq (m+1)(m+2) - 1$.*

Moreover, the inequalities turn into equalities if and only if X is projectively equivalent to $\mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)}$ or to $\mathbb{P}^m \times \mathbb{P}^{m+1} \subset \mathbb{P}^{m^2+3m+1}$ Segre embedded.

In particular, if m is as above, $M(n,2) = \frac{n(n+4)-n-2m}{4}$ and the above ones are the only Scorza varieties with $\delta = 2$.

PROOF. We have $\delta \geq 2$, so that if $n = 2m$,

$$N \leq f(m, 2m, 2) = m(m+2) = (m+1)^2 - 1,$$

while if $n = 2m+1$, then

$$N \leq f(m, 2m+1, 2) = (m+1)(m+2) - 1$$

and the first part follows.

Suppose equality holds in the above inequalities. Then X is a Scorza variety of dimension $n = 2m$, respectively $n = 2m+1$, with $\delta = 2$.

Let us treat first the case $n = 2m = \delta m$, $k_0 = m$. By lemma 4.2.6, the image Q_z of the projection from $z \in S^{m-2}X$ general, $\pi_z : X \dashrightarrow Q_z \subset \mathbb{P}^3$, is a smooth quadric surface. Let $l_1, l_2 \subset Q_z$ be two general lines belonging to different rulings of Q_z and let $D_i = \pi_z^*(l_i)$ be the corresponding divisors on X. By lemma 4.2.6,

$$D_1 \cap D_2 = \overline{\pi_z^{-1}(l_1 \cap l_2)} = \overline{\pi_z^{-1}(p)} = F, \tag{4.2.3}$$

$p = l_1 \cap l_2$ general point on Q_z, with F a Scorza variety of dimension $n - \delta = 2m - 2 = 2(m-1)$. Moreover, we have $D_1 + D_2 + E_z \sim \pi_z^*(\mathcal{O}_{Q_z}(1)) \sim H_u$, with E_z the fixed component, if any, of the linear system of hyperplane containing $T_z S^{m-2}X$ and with H_u an hyperplane section of X containing $T_z S^{m-2}X$ and a general point $x \in F$ (i.e. if $u \in < z, x_{m-1} >$ is general, then $H_u = T_u S^{m-1}X \cap X$). The divisor H_u is connected since $n > 1$ so that if $E_z \neq \emptyset$, then there exists i such that $E_z \cap D_i \subset T_z S^{m-2}X \cap X$ has an irreducible component of dimension $n-2$, let us say G. By definition $G \subset \mathrm{Sing}(H_u)$. Since by lemma 4.2.6 F is the unique component of dimension $n - \delta = n-2$ of $\mathrm{Sing}(H_u)$ and since F is not contained in $T_z S^{m-2}X \cap X$, we obtain a contradiction. Therefore

$$D_1 + D_2 \sim \pi_z^*(\mathcal{O}_{Q_z}(1)) \sim H_u. \tag{4.2.4}$$

The point $z \in S^{m-2}X$ can be thought as a general point of the linear span of $m-1$ general points, $x_0, \ldots, x_{m-2}$ and $u \in S^{m-1}X$ as a general point of the linear space $< x_0, \ldots, x_{m-2}, x_{m-1} >$. We prove that each divisor D_i's, $i = 1,2$, varies in m different pencils of linearly equivalent divisors, yielding a linear system of dimension m of divisors linearly equivalent to D_i, $i = 1,2$. By 4.2.4, there exists an hyperplane

$$H_p = < T_z S^{m-2}X, l_1, l_2 >$$

such that

$$H_p \cap X = D_1 + D_2.$$

From this and from 4.2.3, we get $T_{x_i}X = T_{x_i}(D_1 + D_2) = T_{x_i}X \cap H_p$, i.e. $T_{x_i}X \subset H_p$ for each $i = 0, \ldots, m-1$. If $z_j \in < x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m-1} >$ is general, then it is a general point in $S^{m-2}X$. Thus $\pi_{z_j}(D_1) + \pi_{z_j}(D_2) = \pi_{z_j}(H_p)$ is a reducible hyperplane section of the quadric $\pi_{z_j}(X) = Q_{z_j} \subset \mathbb{P}^3$ and the claim follows.

The two linear system of linearly equivalent divisors we just constructed furnish two dominant rational maps

$$\phi_i : X \dashrightarrow \mathbb{P}^m.$$

Indeed, by definition of π_z, a general entry locus $\Sigma_w(X) = Q$ of X is mapped isomorphically onto Q_z so that $\mathcal{O}_Q(D_1) = \mathcal{O}_Q(1,0)$. and $\mathcal{O}_Q(D_2) = \mathcal{O}_Q(0,1)$, modulo a renumbering. Thus $\phi_i(Q) = l_1 \subset \mathbb{P}^m$ is a line and two general points of $\phi_i(X)$ are joined by a line, so that $\phi_i(X) = \mathbb{P}^m$.

The two dominant rational maps $\phi_i : X \dashrightarrow \mathbb{P}^m$ yield a dominant rational map

$$\phi = \phi_1 \times \phi_2 : X \dashrightarrow \mathbb{P}^m \times \mathbb{P}^m.$$

Let $\sigma_{m,m} : \mathbb{P}^m \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{m^2+2m}$ be the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^m$ and let

$$\psi = \sigma_{m,m} \circ \phi : X \dashrightarrow \mathbb{P}^{m^2+2m}.$$

From

$$\psi^*(\mathcal{O}_{\mathbb{P}^{m^2+2m}}(1)) = \phi^*(p_1^*(\mathcal{O}_{\mathbb{P}^m}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^m}(1))) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) = \mathcal{O}_X(H),$$

and from the fact that $\psi(X)$ is non-degenerate, we deduce that ψ is given a sublinear system of $|H|$ of dimension m^2+2m, i.e. by the complete linear system of hyperplane section, since X is linearly normal by proposition 3.2.4. This is a reformulation of the desired conclusion for $n = 2m$.

Let us treat the case $n = 2m+1$. By theorem 4.2.3 for $u \in S^m X = \mathbb{P}^{m^2+3m+1}$ general, the entry locus $D := \Sigma_u^m(X)$ is a Scorza variety of dimension $2m$ and with $\delta = 2$, i.e. $D = \mathbb{P}^m \times \mathbb{P}^m \subset < \Sigma_u^m(X) > = \mathbb{P}_u^{m^2+2m} \subset \mathbb{P}^N$ by the previous analysis. In particular D is a smooth divisor on X. Moreover, D varies in a positive dimensional families of divisors, whose general member is a Scorza variety of the form $\Sigma_{u'}^m(X)$, $u' \in \mathbb{P}^{m^2+3m+1}$ general. In particular $h^0(\mathcal{O}_D(D)) \geq m+1$, since through $m+1$ general points of X there passes a Scorza variety of the form $\Sigma_{u'}^m(X)$. From the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0, \tag{4.2.5}$$

together with $h^1(\mathcal{O}_X) = 0$, recall lemma 4.2.5, we deduce $h^0(\mathcal{O}_X(D)) = \alpha \geq m+2$.

Take a general hyperplane H through D and let

$$H \cap X = D + D'.$$

Then $D' > 0$ so that if $\Sigma_w(X) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a general entry locus, then $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \simeq \mathcal{O}_{\Sigma_w(X)}(H)$, yields $\mathcal{O}_{\Sigma_w(X)}(D) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)$ (or $\simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)$) , i.e. the general entry locus is contracted by the rational map associated to $|D|$ onto a line. The argument we repeated many times furnishes $\alpha = m+1$ and also that the rational map

$$\phi = \phi_{|D|} : X \to \mathbb{P}^{m+1},$$

is dominant. Without effort one immediately proves that it is in fact a morphism by the usual Castelnuovo's argument, observing that necessarily $\mathcal{O}_D(D) \simeq \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^m}(1,0)$ (or $\simeq \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^m}(0,1)$).

The linear system $|H - D|$ has dimension $\dim((< D >)^*) = (m^2+3m+1) - 1 - m^2 - 2m = m$ by the linear normality of X. This linear system defines a rational map

$$\psi = \psi_{|H-D|} : X \dashrightarrow \mathbb{P}^m,$$

which is dominant since $\mathcal{O}_{\Sigma_w(X)}(H - D) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)$ (or $\simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0)$).

As usual let $\sigma_{m,m+1} : \mathbb{P}^m \times \mathbb{P}^{m+1} \hookrightarrow \mathbb{P}^{m^2+3m+1}$ be the Segre embedding and let

$$\varphi = \sigma_{m,m+1} \circ (\psi \times \phi) : X \dashrightarrow \mathbb{P}^{m^2+3m+1}.$$

Since

$$\varphi^*(\mathcal{O}_{\mathbb{P}^{m^2+3m+1}}(1)) = \mathcal{O}_X(H - D) \otimes \mathcal{O}_X(D) = \mathcal{O}_X(H)$$

and since $\varphi(X)$ is non-degenerate, the rational map φ is given by a sublinear system of $|H|$ of dimension m^2+3m+1, i.e. by the complete linear system $|H|$ by the linear normality of X. Then X is projectively equivalent to $\mathbb{P}^m \times \mathbb{P}^{m+1}$ Segre embedded, as desired. $\square$

To conclude the classification of Scorza varieties one has to consider the cases $\delta = 4$ and $\delta = 8$. The proof is rather intricate and we refer to the final chapter of **[Z2]** for a complete treatment, hoping that our discussion served as a good motivation. Another approach to some cases of the classification of Scorza varieties ($n \equiv 0$ (mod. δ)), via the theory of homogeneous variety and Jordan algebras, is contained in **[Ch2]**.

CHAPTER 5

Miscellanea

5.1. Varieties with one apparent $(k+1)$-secant $\mathbb{P}^{k-1}$ and Bronowski conjecture; Waring' s problem and the canonical expression

Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate projective variety, which is not a linear space. For fixed $k \geq 1$, there exists a natural map

$$\phi_k : \underbrace{X \times \ldots \times X}_{k+1} \dashrightarrow \mathbb{G}(k, N),$$

defined by

$$\phi_k((x_0, \ldots, x_k)) = < x_0, \ldots, x_k > .$$

Let $\phi_k(X \times \ldots \times X) = (X)_k \subset \mathbb{G}(k, N)$. Then $(X)_k$ parametrizes the $(k+1)$-secant $\mathbb{P}^k$'s to $X \subset \mathbb{P}^N$. We now furnish a different description of $S^k X$ via $(X)_k$ and the incidence correspondence on $\mathbb{G}(k, N) \times \mathbb{P}^N$. Let

$$I = \{(\Pi, z) \in \mathbb{G}(k, N) \times \mathbb{P}^N \ : \ z \in \Pi\} \subset \mathbb{G}(k, N) \times \mathbb{P}^N.$$

Consider the projections of I onto the factors $\mathbb{G}(k, N)$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{G}(k, N) & & \mathbb{P}^N. \end{tikzcd}$$

Let $S_k X = p_1^{-1}((X)_k)$. Then $S_k(X)$ is an irreducible variety which is a $\mathbb{P}^k$ fibration over $(X)_k$ and there is a natural generically finite dominant rational map $\phi_k : S_X^k \dashrightarrow S_k X$ of degree $k!$. Moreover

$$p_2(S_k X) = S^k X.$$

The degree of $p_2 : S_k(X) \to S^k X$ has the following geometrical interpretation, when positive, *it is the number of $(k+1)$-secant $\mathbb{P}^k$ passing through the general point of $S^k X$.*

By convention, if $p_2 : S_k X \to S^k X$ is not generically finite, we put $\deg(p_2) = 0$. In particular $\deg(p_2) > 0$ if and only if $s_k(X) = (k+1)n + k$.

Let $X \subset \mathbb{P}^N$ be as above. By projecting $X \subset \mathbb{P}^N$ from a general linear space $L = \mathbb{P}^{N-(k+1)n-k}$ onto $\mathbb{P}^{(k+1)n+k-1}$, if $X' = \pi_L(X) \subset \mathbb{P}^{(k+1)n+k-1}$, the variety X' acquires exactly $\deg(p_2) \cdot \deg(S^k X)$ *new* $(k+1)$-secant $\mathbb{P}^{k-1}$, which are the projection of the $(k+1)$-secant $\mathbb{P}^k$ passing through the center of projection. For $k = 1$, a general projection of $X \subset \mathbb{P}^N$ in $\mathbb{P}^{2n}$ acquires a finite number of double points, which did not exist on X. This case was considered classically by Severi, **[Sev1]**, who dubbed the number $\deg(p_2) \cdot \deg(SX)$ as *the number of apparent double points of $X \subset \mathbb{P}^N$*, $N \geq 2n+1$. The word apparent is clearly justified by the fact that, a priori, $X \subset \mathbb{P}^N$ cannot have any $(k+1)$-secant $\mathbb{P}^{k-1}$ and these appear only on the projected variety. For $k=1$ we can consider for example the number of apparent double points of a smooth variety $X \subset \mathbb{P}^N$.

We are now in position to introduce the following definition.

5.1.1. DEFINITION. **(Number of apparent $(k+1)$-secant $\mathbb{P}^{k-1}$'s)** Let $X \subset \mathbb{P}^N$, $N \geq (k+1)n+k$, be an irreducible, non-degenerate, projective variety. Suppose $1 \leq k \leq k_0(X)$. We define *the number of apparent $(k+1)$-secant $\mathbb{P}^{k-1}$ to X*, $\nu_k(X)$, as

$$\nu_k(X) := \deg(p_2) \cdot \deg(S^k X),$$

where $p_2 : S_k X \to S^k X$ is as above.

By the previous discussion it is clear that $\nu_k(X)$ is the number of $(k+1)$-secant $\mathbb{P}^{k-1}$, which a general projection of X into $\mathbb{P}^{(k+1)n+k-1}$ acquires. For $k=1$ we let $\nu(X) = \nu_1(X)$.

The theory of varieties with $\nu_k(X) = 0$ coincides with the theory of k-defective varieties. As it was shown by Severi himself in [**Sev1**], by Edge, [**Ed**], by Bronowski, [**B1**], [**B2**], and more recently in [**Ru1**], [**AR2**], [**CMR**], [**CR2**] also the case of varieties with $\nu_k(X) = 1$ deserves special interest. These varieties share remarkable geometrical properties, as we shall briefly see below.

Firstly let us remark that the condition $\nu_k(X) = 1$ deals with a property of general projections, which, via Terracini lemma, can be related to projections from tangent spaces.

Indeed let us recall that in proposition 1.3.8 we proved that for a variety $X \subset \mathbb{P}^N$ with $\delta_{k-1}(X) = 0$, the condition $\delta_k(X) = 0$ is equivalent to the fact that the general tangential projection $\pi_k = \pi_L : X \dashrightarrow X' \subset \mathbb{P}^{N-kn+k}$, $L = \langle T_{x_1}, \ldots, T_{x_k} \rangle$, is dominant. In particular if $\nu_k(X) > 0$, then $\delta_k(X) = 0$, $\dim(X') = n - \delta_k(X) = n$ and $\pi_k = \pi_L : X \dashrightarrow X' \subset \mathbb{P}^{N-kn+k}$ is dominant and hence generically finite.

A very interesting and somehow intricate relation between the degree of π_k and $\nu_k(X)$ was proposed by Jaroslaw Bronowski in [**B1**], theorem 4 at page 82, at least for $\nu_k(X) = 1$.

Let us remark that $\nu_k(X) = 1$ implies $\deg(S^k X) = 1$, so that $N = (k+1)n+k$, and also $\deg(p_2) = 1$. Therefore $p_2 : S_k X \to S^k X = \mathbb{P}^{(k+1)n+k}$ is a birational morphism in this case.

5.1.2. QUESTION. **(Bronowski claim, [B1])** *Let $X \subset \mathbb{P}^{(k+1)n+k}$ be an irreducible, non-degenerate projective variety. Then $\nu_k(X) = 1$ if and only if $\pi_k : X \dashrightarrow \mathbb{P}^n$ is birational.*

We called the above claim *Bronowski claim* since the proof proposed by Bronowski is obscure, as far as I know, to all modern algebraic geometers who read it.

From a theoretical point of view the most interesting implication seems to be $\nu_k(X) = 1 \Rightarrow \deg(\pi_k) = 1$, having the strong consequence that varieties with one apparent $(k+1)$-secant $\mathbb{P}^{k-1}$ are rational. On the other hand, in order to construct explicit examples of varieties with $\nu_k(X) = 1$, the other implication would be very useful since otherwise the condition $\nu_k(X) = 1$ (and sometimes also $s_k(X) = (k+1)n+k$) is quite hard to verify, especially for $k > 1$.

The following result is essentially established in [**CMR**] via the method of degenerations of projections, even if there only the case $k=1$ was treated. The formulation for arbitrary $k \geq 1$ is stated in [**CR2**], although it was known to the three authors of [**CMR**]. It is useless to remark the deep link between an extrinsic property of X, $\nu_k(X) = 1$, and its rationality.

5.1.3. THEOREM. ([**CMR**], [**CR2**]) *Let $X \subset \mathbb{P}^N$, $N \geq (k+1)n + k$, be a smooth irreducible non-degenerate projective variety. Then $\nu_k(X) \geq \deg(\pi_k)$. In particular, if $\nu_k(X) = 1$, then $\pi_k : X \dashrightarrow \mathbb{P}^n$ is a birational isomorphism so that X is a rational variety.*

This explains why, in principle, it could be possible, at least for small n and small k, to classify all smooth varieties with $\nu_k(X) = 1$. The first and highly non-trivial case appears for $k = 1$, i.e. for varieties with one apparent double point. For varieties with one apparent double point there some additional geometrical properties, especially their linear normality, which help a lot. The case of 3-folds with one apparent double point is already rather complicated, see [**CMR**]. The known results are the classification of smooth surfaces with one apparent double point (the classification of smooth curves with $\nu_k = 1$ being trivial for every $k \geq 1$), see [**Sev1**], [**Ru1**] and also [**CMR**]. These are only rational normal scrolls of degree four and the del Pezzo surface of degree 5. For $n = 3$ there are 5 types of 3-folds with one apparent double point: the rational normal scrolls of degree 5, two hyperquadric fibrations which are divisors of type $(0, 2)$, respectively $(1, 2)$, on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and a scroll over a surface of degree 8. What is really remarkable is that, due to the strong restrictions on their geometry, very few examples of varieties with $\nu(X) = 1$ are known, also in higher dimension. There are three series we now construct for every $n \geq 2$, following a beautiful geometric idea of Edge, also rediscovered by F. L. Zak.

Outside these series only other 6 examples of smooth varieties with one apparent double point are know: one in dimension 3, referred above, one of dimension 4, the linear section of the spinor variety $S^{10} \subset \mathbb{P}^{15}$, and the four *Lagrangian Grassmannians* $\mathbb{G}_K^{lag}(2,5)$ over the four composition algebras $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, embedded via their Plücker embedding, of dimension 6, respectively 9, 15, 27.

It is an open problem if the varieties described above are the only varieties with one apparent double point of dimension $n \geq 4$.

We now sketch Edge's argument from [**Ed**] to the effect that smooth irreducible divisors of type $(0, 2)$, $(1,2)$ and $(2, 1)$ on the Segre varieties $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \geq 2$, have one apparent double point. This generalizes the trivial fact that the only smooth curves, not necessarily irreducible, on a smooth quadric in $\mathbb{P}^3$ having one apparent double point are of the above types.

5.1.4. PROPOSITION. ([**Ed**]) *Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible projective variety contained as a divisor of type (a, b) in $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \geq 2$. Then X has one apparent double point if and only if $(a, b) \in \{(2, 1), (0, 2), (1, 2)\}$.*

PROOF. As we know for, $p \notin Y := \mathbb{P}^1 \times \mathbb{P}^n$, the entry locus $\Sigma_p(Y)$ has the form $\mathbb{P}^1 \times \mathbb{P}^1_p$ for some $\mathbb{P}^1_p \subset \mathbb{P}^n$ and spans a linear $\mathbb{P}^3_p$. So if X is a divisor of type (a, b) of Y, the secant lines of X passing through p are exactly the secant lines of $X \cap \mathbb{P}^3_p$ passing through p. For a general $p \in \mathbb{P}^{2n+1}$, $X \cap \mathbb{P}^3_p$ is a reduced, not necessarily irreducible curve and it is a divisor of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1_p$. Hence X has one apparent double point if and only if $(a, b) \in \{(1, 2),\ (2, 1),\ (2, 0),\ (0, 2)\}$. If $(a, b) = (2, 0)$, then $X = \mathbb{P}^n \amalg \mathbb{P}^n$ is reducible. $\square$

The divisors of type $(2,1)$ are the rational normal scrolls of minimal degree in $\mathbb{P}^{2n+1}$. The divisors of type $(0,2)$ are isomorphic to $\mathbb{P}^1 \times Q^{n-1}$, where $Q^{n-1} \subset \mathbb{P}^n$ is a quadric hypersurface of maximal rank, so that they admit a structure of *twisted cubic over a split cubic Jordan algebra*, see [Mk]; divisors of type $(1,2)$ are hyperquadric fibrations of special kind. The above varieties are usual called *Edge varieties.* Edge varieties have degree $d = n+2$, respectively $2n$, $2n+1$ and in **[AR2]** are characterized as the only varieties with one apparent double point of dimension n and degree $d \leq 2n+1$ for every $n \geq 2$. Moreover in **[AR2]** it is shown that for $2n+2 \leq d \leq 2n+4$ there are only 3 varieties with one apparent double point: for $n=3$ and $d=8$ it is the scroll over a surface we cited above; for $n=4$ and $d=12$ the linear section of $S^{10} \subset \mathbb{P}^{15}$ and for $n=6$ and $d=16$ the variety $\mathbb{G}^{lag}_{\mathbb{R}}(2,5) \subset \mathbb{P}^{13}$.

In **[CR2]**, a classification of linearly normal surfaces with $\nu_k(X) = 1$ is proposed. Moreover, some examples and series of varieties with $\nu_k(X) = 1$ of arbitrary dimensions are constructed. For example, it is shown that some smooth variety of minimal degree $X \subset \mathbb{P}^{(k+1)n+k}$ have $\nu_k(X) = 1$, see the next section. Moreover, it is also considered the case of suitable hyperquadric fibrations similar to the other Edge varieties. An interesting case we discuss below is the surface given by the 5$^{\text{th}}$ Veronese embedding of the plane, $v_5(\mathbb{P}^2) \subset \mathbb{P}^{20}$ (and also by its tangential projections from $1,2$ or 3 points).

Instead of entering into the details of the above classifications, contained in the quoted papers, I prefer to continue the discussion about the property $\nu_k(X) = 1$ by relating it to another problem, known in the literature as *Waring problem.* This furnishes also a geometrical interpretation of $k_0(X)$ and of $\nu_{k_0}(X)$ for the Veronese embedding of $\mathbb{P}^n$, $v_d(\mathbb{P}^n) \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))) = \mathbb{P}^{N(d)}$, $N(d) = \binom{n+d}{n} - 1$.

Let us recall the following elementary fact. The variety $v_d(\mathbb{P}^n) \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$ is the locus of the classes of homogeneous polynomial $[f] \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$, which are d^{th}-powers of linear forms in the variables $x_0, \ldots, x_n$. Thus $[f] \in v_d(\mathbb{P}^n)$ if and only if $[f] = [l^d]$ with $l \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. Let us set $X = v_d(\mathbb{P}^n) \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$, $d \geq 2$.

The interpretation of $S^kX \subset \mathbb{P}^{N(d)}$ is the following: $[f] \in S^kX$ if and only if $[f] = [l_0^d] + \ldots + [l_k^d]$ with $l_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. The interpretation of $k_0(d,n) := k_0(v_d(\mathbb{P}^n))$ in this case is the following: $k_0(d,n))+1$ is *the minimal number of d^{th}-powers of linear forms necessary to write a general $[f] \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ as a linear combinations of them.*

5.1.5. Problem. **(Waring problem, [P3])** *Compute $k_0(v_d(\mathbb{P}^n))$ as a function of d and n.*

The problem was stated by Palatini in **[P3]**, studied in **[P4]** and **[B1]** and finally settled in **[AH]**, see theorem 5.1.7 below.

Let us define $w_0(n,d) = k_0(d,n)+1$. From $S^{k_0(d,n)}v_d(\mathbb{P}^n) = \mathbb{P}^{N(d)}$ we get

$$w_0(n,d)(n+1)-1 = (k_0(d,n)+1)n+k_0(d,n) \geq s_{k_0}(v_d(\mathbb{P}^n)) = N(d) = \binom{n+d}{n} - 1,$$

so that

$$w_0(d,n) \geq \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil. \tag{5.1.1}$$

For $n = 1$, since a curve is not k-defective for every $k \geq 1$ by proposition 1.2.3, or exercise 1.5.3, the answer is simple.

5.1.6. PROPOSITION. *If $n = 1$, then $w_0(d,1) = \lceil \frac{d+1}{2} \rceil$ for every $d \geq 1$. Thus $w_0(d,1) = \frac{d}{2}$ if d is even and $w_0(d,1) = \frac{d-1}{2}$ if d is odd.*

The final step was the characterization, proved by Alexander and Hirschowitz in [**AH**], of the known examples as the unique exceptions to equality in 5.1.1. The result is the following.

5.1.7. THEOREM. ([**AH**]) *Suppose $n \geq 2$. Then*

$$w_0(d,n) = \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil,$$

unless (d,n) is one of the following
- *$(2,n)$, where $w_0(2,n) = n+1 > \lceil \frac{n+2}{2} \rceil$;*
- *$(3,4)$, where $w_0(3,4) = 8 > \lceil \frac{1}{5}\binom{7}{3} \rceil = 7$;*
- *$(4,2)$, where $w_0(4,2) = 6 > \lceil \frac{1}{3}\binom{6}{2} \rceil = 5$;*
- *$(4,3)$, where $w_0(4,3) = 10 > \lceil \frac{1}{4}\binom{7}{3} \rceil = 9$;*
- *$(4,4)$, where $w_0(4,4) = 15 > \lceil \frac{1}{5}\binom{8}{4} \rceil = 14$.*

Once the Waring problem has been solved, one may ask if there are finitely many ways of expressing a general homogeneous form of degree d in $n+1$ variables, $f(x_0,\ldots,x_n)$, as the sum of $w_0(d,n) = \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$ d^{th}-powers of linear forms in the $n+1$ variables. A necessary condition is that $\frac{1}{n+1}\binom{n+d}{n}$ is an integer. Suppose $\frac{1}{n+1}\binom{n+d}{n}$ is an integer, equal to $w_0(d,n)$, i.e. $\frac{1}{n+1}\binom{n+d}{n}$ is an integer such that (d,n) is different from $(2,n)$, $(3,4)$, $(4,2)$ or $(4,4)$, see theorem 5.1.7. Under these hypothesis $S^{w_0(d,n)-1} = \mathbb{P}^{N(d)}$ so that the number of expression of a general form is exactly $\deg(p_2)$, where $p_2 : S_{w_0(d,n)-1}X \to S^{w_0(d,n)-1}X = \mathbb{P}^{N(d)}$. Thus $\deg(p_2)$ equals $\nu_{w_0(d,n)-1}(v_d(\mathbb{P}^n)) := \nu(d,n)$. If $\nu(d,n) = 1$, then we say that a general homogeneous form of degree d in $n+1$ variables has *a canonical expression.*

Let us now see geometrically that that for the examples listed in theorem 5.1.7 equality in 5.1.1 does not hold.

The first example we already met is the Scorza variety $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$, $n \geq 2$. Indeed in this case we saw that $k_0(2,n) = n$, so that $w_0(2,n) = n+1$ is strictly greater than $\lceil \frac{1}{n+1}\binom{n+2}{2} \rceil = \lceil \frac{n+2}{2} \rceil$, as soon as $n \geq 2$.

An interesting example, which was considered a remarkable result at the end of the 19^{th} century, was the case $(d,n) = (4,2)$ studied by Clebsh, [**Cl1**].

5.1.8. EXAMPLE. (Clebsch, [**Cl1**]) Let $d = 4$ and $n = 2$ and consider the surface $X = v_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$. Then $s(X) = 5$, $s_2(X) = 8$, $s_3(X) = 11$, $s_4(X) = 13$. In particular $k_0(4,2) = 5$, so that $w_0(4,2) = 6 > \lceil \frac{1}{3}\binom{6}{2} \rceil = 5$.

To prove the above assertion we use proposition 1.3.8. Let $1 \leq k \leq 4$ be fixed and let $\pi_k : X \dashrightarrow \mathbb{P}^{13-s_{k-1}(X)}$ be the general tangential projection from

$L =< T_{x_1}X, \ldots, T_{x_k}X >$. The composition $\phi_k = \pi_k \circ v_4 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{13-s_{k-1}(X)}$ is given by the linear system $|4l - 2x_1 - \ldots - 2x_k|$ of quartic plane curves having double points at $x_1, \ldots, x_k$. Moreover, $s_k(X) = \dim(\phi_k(\mathbb{P}^2)) + s_{k-1}(X) + 1$.

To prove geometrically the above assertion about $s_i(X)$, it is enough to remark that the linear system of plane quartics with 0,1,2 or 3 general double points has positive degree, i.e. if $C \in |4l - 2x_1 - \ldots - 2x_k|$, $0 \leq k \leq 3$, then $C^2 > 0$. Hence for $0 \leq k \leq 3$, $\dim(\phi_k(\mathbb{P}^2)) = 2$ and $s_k(X) = 3k + 2$.

The linear system of quartics with 4 general double points is composed with the pencil of the conics through the 4 points, so that it has degree 0. Thus $\dim(\phi_4(\mathbb{P}^2)) = 1$ and $s_4(X) = s_3(X) + 2 = 13$, so that $S^4X \subset \mathbb{P}^{14}$ is an hypersurface and clearly $S^5X = \mathbb{P}^{14}$.

By looking at $v_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ as the quadratic reembedding of $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, we can obtain $S^4v_4(\mathbb{P}^2)$ as the intersection of $S^4v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$ with the $\mathbb{P}^{14}$ spanned by $v_4(\mathbb{P}^2) \subset \mathbb{P}^{20}$. Since $S^4v_2(\mathbb{P}^5) \subset \mathbb{P}^{20}$ is an hypersurface of degree 6, whose equation is the determinant of the general symmetric 6×6 matrix of linear forms, we deduce $\deg(S^4v_4(\mathbb{P}^2)) = 6$ and eventually the equation of $S^4v_4(\mathbb{P}^2)$. This equation expresses the condition on the coefficients of a quartic form in 3 variables to be, or not, the sum of five 4^{th}-powers of a linear form in the three variables.

The other exceptions to equality in 5.1.1 were known classically to Reye, [**Re2**], Sylvester, [**Sy2**], Richmond, see [**Ri**], Palatini, [**P4**], and Dixon, [**Di**]. A geometrical proof along the lines of the above argument can be furnished also in these cases by using tangential projections, i.e. Terracini Lemma. We briefly discuss them in the next example, following Bronowski, see [**B1**] pg. 73.

5.1.9. Example. (**Cases** $(d,n) = (3,4)$, $(4,3)$, $(4,4)$)) Suppose $d = 3$ and $n = 3$ or 4. Consider the linear system $|D|$ of cubic hypersurfaces in $\mathbb{P}^n$ having $\frac{1}{n+1}\binom{n+d}{n} - 1 := k - 1$ assigned general double points. The $\frac{1}{2}(k-1)(k-2)$ lines joining pairs of base points form a base curve c of $|D|$ of order $\frac{1}{2}(k-1)(k-2)$, having a point of multiplicity $k-2$ at each of the $k-1$ assigned base points. Let $\widetilde{D} = D - c$, which can be thought as the strict transform of D on the blow-up of $\mathbb{P}^n$ along c, $\widetilde{\mathbb{P}^n}$. Then $(\widetilde{D})^{n-1} = d$ is a curve of degree $3^{n-1} - \frac{1}{2}(k-1)(k-2)$, which has a point of multiplicity $2^{n-1} - (k-2)$ at each of the base points. Moreover D and d meet at $(D \cdot d) = 3^n - \frac{3}{2}(k-1)(k-2)$, of which $2^n(k-1) - 2(k-1)(k-2)$ are absorbed at the base points. Thus $\widetilde{D}^n = 3^n - 2^n(k-1) + \frac{1}{2}(k-1)(k-2) = 3^n - 2^{n-1}\frac{1}{3}n(n+5) + \frac{1}{72}n(n-1)(n+5)(n+6)$. For $n = 4$, $(\widetilde{D})^4 = 0$, while for $n = 3$, $(\widetilde{D})^3 = 1$, i.e. the well known fact that cubics surface with 4 general double points define a Cremona transformation in $\mathbb{P}^3$. In conclusion $\nu(3,3) = 1$, $\nu(3,4) = 0$ and one can show with analogous arguments that $\nu(3,n) > 0$ for $n \neq 4$.

Suppose $d = 4$ and $n = 3$. Geometrically this case and the case $(4,4)$ are completely analogue to $(4,2)$. Indeed $\lceil \frac{1}{4}\binom{7}{3} \rceil = 9$. Take $9 - 1 = 8$ general points $p_1, \ldots, p_9 \in \mathbb{P}^3$ and let $|D| = |4H - 2p_1 - \ldots - 2p_8|$. Since the linear system of quadric surfaces in $\mathbb{P}^3$, $|Q|$, has dimension 9 and self intersection $Q^3 = 8$, through 8 general points of p^3, there passes a pencil of quadric surfaces, let us say spanned by $Q_1, Q_2 \in |Q - p_1 - \ldots - p_8|$ such that $Q_1 + Q_2 \sim D$. Then it immediately follows $D^3 = 0$, so that by proposition 1.3.8, $\nu_9(4,3) = 0$. It is easy to verify that $k_0(4,3) = 9$.

In the same way for $d = 4$ and $n = 4$, given $\lceil \frac{1}{5}\binom{8}{4} \rceil - 1 = 14 - 1 = 13$ general points in $\mathbb{P}^4$, the linear system of quadric hypersurfaces of $\mathbb{P}^4$ through 13 general points is a pencil. This yields $\nu_{13}(4,4) = 0$. It is easy to deduce that $k_0(4,4) = 14$.

Classically it was known to Sylvester, [**Sy1**], Reye, [**Re1**], Hilbert, [**Hi**], Richmond, [**Ri**] and Palatini, [**P3**], that there are cases for which a canonical expression exist. Till today there is no uniform way of proving that the examples listed below are the unique cases for which there exist a canonical expression. We have furnished in 5.1.9 the geometrical reason, via Bronowski claim 5.1.2, of $\nu(3,3) = 1$. A modern algebro-geometric proof can be found also in [**SB1**]. In the next section we will prove geometrically that $\nu(5,2) = 1$, a result first stated by Hilbert in [**Hi**], and also of the result $\nu(2m+1,1) = 1$, which is more classical and more difficult to attribute to someone. The basic question with regard to canonical expression is the following.

5.1.10. QUESTION. (**Canonical expression**) *Is it true that $\nu(d,n) = 1$ if and only if (d,n) is one of the following*

- $(2m+1,1)$, $m \geq 1$, *where* $w_0(2m+1,1) = m+1$;
- $(3,3)$, *where* $w_0(3,3) = 5$;
- $(5,2)$, *where* $w_0(5,2) = 7$?

In [**B1**] Bronowski proposes an affirmative solution to the question. Most of the claims he made are uncertain and its proofs are quite dubious. A modern approach seems to be necessary also for this result of Bronowski. The cases $n \leq 2$ are treated geometrically in [**CR2**], verifying that in this range these are the only examples. To our knowledge the questions remains open for $n \geq 3$. We remark that it is easy to construct a lot of couples (d,n) for which $\frac{1}{n+1}\binom{n+d}{n}$ is an integer.

Any case we would like to stress that the open implication in 5.1.2 would offer a smooth, simple and uniform proof that for the values listed above $\nu(d,n) = 1$. Indeed in each case, letting notations as in example 5.1.8, the map $\phi_{k_0(d,n)} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is easily seen to be birational. This is a further motivation for trying to prove the open implication in Bronowski claim 5.1.2. We would like to remark that the enumerative methods developed recently did not cover the cases above, neither for $n = 2$, see [**ES**] and [**Le**], even if they calculate $\nu(d,2)$ for other values of d.

5.2. (Special) Subhomaloidal systems

In this section we illustrate an interesting interplay between birational and projective geometry by studying (special) subhomaloidal linear systems on projective space. The notion of homaloidal linear system is classical while the definition of subhomaloidal linear system was introduced by Semple and Tyrrell in [**ST1**], see also [**ST2**], [**ST3**], [**CK1**], [**CK2**], [**ESB**], [**HKS**], [**AR2**], [**AR3**].

5.2.1. DEFINITION. (**(Special) Subhomaloidal linear system**) Consider a birational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$, not an isomorphism, i.e. a Cremona transformation. If the map ϕ is given by the homogeneous forms $F_0, \ldots, F_N$ of degree $d \geq 2$, then the associated linear system is said a *homaloidal linear system.* If $F_0, \ldots, F_N$ define a smooth, irreducible variety $X = V(F_0, \ldots, F_N) \subset \mathbb{P}^N$, then ϕ is said a *special Cremona transformation* and $X \subset \mathbb{P}^N$ its *center*, i.e. its locus of indetermination. In this case the associated linear system is called a *special homaloidal system.* The

Cremona transformation $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is said to be *of type* (d_1, d_2), if ϕ is given by forms of degree d_1 and ϕ^{-1} by forms of degree d_2.

More generally, a linear system of hypersurfaces of $\mathbb{P}^N$ is said to be *subhomaloidal* if the (closure of a) general fiber of the associated rational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^s$, $s \leq N$, is a linear projective space and homaloidal if ϕ is birational onto the image. It is said to be *special* if the base locus scheme of the linear system is smooth and irreducible; it is said to be *completely subhomaloidal* if (the closure of) every fiber is a linear projective space. These definitions can also be extended to linear systems of hypersurfaces such that the associated map $\phi : \mathbb{P}^N \dashrightarrow Z \subseteq \mathbb{P}^M$, $Z = \overline{Im(\phi)}$, is birational onto the image, respectively has a linear projective space as general fiber, where $\dim(Z) \leq N$.

Subhomaloidal systems always carry a remarkable geometric information and can be used to prove some interesting geometrical properties of a given variety $X \subset \mathbb{P}^N$, when the forms defining the linear systems are somehow related to X. These constructions have been applied to varieties with one apparent double point or more generally to varieties with one apparent $(k+1)$-secant $\mathbb{P}^{k-1}$, to varieties with one apparent s-ple point and in particular to canonical expression, 5.1.10. One can consult [**AR2**], [**AR3**], [**AR1**] and [**CR2**] for more examples and applications of this kind. In particular in [**AR2**] the following *principle* was observed: *a way of verifying the property that through a general point of the space there passes a unique $(k+1)$-secant $\mathbb{P}^k$ (or more generally a s-secant line, etc, etc) to a (smooth) variety $X \subset \mathbb{P}^N$, is to construct a suitable subhomaloidal system from the equations of $S^l X$ for some $l \geq 0$, whose general fibers are the linear spaces considered.*

Very interesting applications of some subhomaloidal systems to the rationality of some moduli spaces were furnished by Shepherd-Barron in [**SB1**] and [**SB2**].

An interesting problem, whose study was began in [**CK1**] and [**CK2**] and continued in [**ESB**], is the classification of special Cremona transformations. With regard to this problem we quote the beginning of [**ST3**]: *Any Cremona transformation defined by a homaloidal system of primals with a single irreducible non-singular base variety is a rare enough phenomenon to merit special study.* Regardless the classifications of special Cremona transformations the following is known. If $X \subset \mathbb{P}^N$ is the center, the classification is known for $\dim(X) \leq 2$, see [**CK1**] (whose result is incomplete as observed by the author and Massimiliano Mella; a quick proof of the complete classification is contained in [**Ru4**]), for $\operatorname{codim}(X) = 2$ and for *quadro-quadric*, i.e. of type $(2,2)$, the classification is given in [**ESB**]. One immediately realizes that there exist very few examples, which carry an interesting and peculiar geometry together with a lot of deep properties such as many linear syzygies for the defining equations, projective normality or in most cases arithmetically Cohen-Macaulayness, etc, etc, see for example [**Ru4**]. As noted in [**ESB**] varieties defining special Cremona transformations are not complete intersections and being of "small codimension" in most case are very particular. For this reason one could hope, at least in principle, to classify all special Cremona transformations. The classification could be somehow complicated due to the degenerations from the general case into specialized ones. The specialized special Cremona transformations can be used to construct interesting birational morphism between smooth varieties, which are extremal elementary contractions in the sense of Mori theory, clearly not inverse of a blow-up, see [**AR3**]. In this way there appears an interesting bridge between classical results such as [To], [Fn],[ST2], [ST3], [ST1] and modern higher dimensional geometry as shown in [**AR3**].

To illustrate the relation between the themes discussed in the previous sections and special subhomaloidal system we quote the following very beautiful result.

5.2.2. PROPOSITION. ([**ESB**], [**Z2**]) *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible projective, non-degenerate variety. The following conditions are equivalent:*

i) *$N = \frac{3n}{2} + 2$ and $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ defines a quadro-quadric special Cremona transformation.*
ii) *$N = \frac{3n}{2} + 2$ and $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is a Severi variety.*

Using Zak's classification of Severi varieties Ein and Shepherd-Barron then conclude that there are only for examples of quadro-quadric Cremona transformations, given by equations defining the four Severi varieties.

The fact that the four Severi varieties define quadro-quadric Cremona transformations can be verified in this way. Recall that the Severi varieties have the following uniform description. Let $\mathbb{A}_{\mathbb{R}}$ denote $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, i.e., the four real division algebras of real dimension, respectively, 1,2,4, 8. Let $\mathbb{A} = \mathbb{A}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and let $\mathcal{H}_{\mathbb{R}}$ denote the $\mathbb{A}_{\mathbb{R}}$-hermitian forms on $\mathbb{A}^3_{\mathbb{R}}$, i.e., the 3×3 $\mathbb{A}_{\mathbb{R}}$-hermitian matrices. If $x \in \mathcal{H}_{\mathbb{R}}$, then we may write

$$x = \begin{pmatrix} \alpha_1 & \overline{\beta}_1 & \overline{\beta}_2 \\ \beta_1 & \alpha_2 & \overline{\beta}_3 \\ \beta_2 & \beta_3 & \alpha_3 \end{pmatrix}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{A}_{\mathbb{R}}$. Let $\mathcal{H} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and let $X \subset \mathbb{P}(\mathcal{H})$ the locus of rank one elements.

The four Severi varieties are exactly $X \subset \mathbb{P}(\mathcal{H})$ for $\mathbb{A}_{\mathbb{R}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and SX is the locus of rank 2 matrices; clearly X is defined by $\dim(\mathbb{P}(\mathcal{H}))+1$ quadrics. These quadrics define a rational map, $T\colon \mathbb{P}(\mathcal{H}) \dashrightarrow \mathbb{P}(\mathcal{H})$. One sees that T is a birational, involutory map that is an isomorphism on $\mathbb{P}(\mathcal{H}) \setminus SX$. Indeed, by writing down the equations defining X, one verifies that T is the composition of the map sending a matrix in $\mathcal{H}$ to the matrix of its cofactors and of an involutory projectivity of $\mathbb{P}(\mathcal{H})$. The cases of the Veronese surface and of the Segre 4-fold are classical (see for example [**SR**]).

In our recent paper [**Ru6**], we showed that, via the theory of quadric varieties, this point of view can be reverted and we essentially classified the centers of quadro-quadric Cremona transformations, obtaining a new, simple and direct geometric proof of the classification of Severi varieties. The details, if correct, will be published elsewhere.

It is worth remarking that Cremona transformation of the above kind appear also in connection with most of the remaining Scorza varieties, by considering the transformation which associates to a $(n+1) \times (n+1)$ (symmetric, antisymmetric with $n = 2m$) matrix, the matrix of its cofactors. In the first case we take $X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n(n+2)}$, $n \geq 2$, in the second case $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ and finally for the antisymmetric case $\mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{2}}$, $n \equiv 0 \pmod{2}$. The Cremona transformation is an isomorphism outside the hypersurface $S^{k_0-1}X$ and the map is given by the partial derivatives of an equation defining it.

We return to the principle we quoted at the beginning of the section to furnish two explicit applications, whose proofs are for the first time, as far as we know, geometric. Particular cases for $n = 1$ of the next examples were proved at the end of the 19^{th} century by Sylvester, Clebsch, Reye and Richmond, see references in the previous section.

5.2.3. EXAMPLE. **(Rational normal scrolls in $\mathbb{P}^{(k+1)n+k}$ with $a_1 \geq k$ have $\nu_k = 1$, [CR2])** Let $0 \leq a_1 \leq a_1 \leq \dots \leq a_n$ be integers and set $N = a_1 + \dots + a_n + n - 1$. Recall that a *rational normal scroll* $S(a_1, \dots, a_n)$ in $\mathbb{P}^N$ is the image of the projective bundle $\mathbb{P}(a_1, \dots, a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ via the linear system $|\mathcal{O}_{\mathbb{P}}(1)|$. The dimension of $S(a_1, \dots, a_n)$ is n, its degree is $a_1 + \dots + a_n = N - n + 1$ and $S(a_1, \dots, a_n)$ is smooth if and only if $a_1 > 0$. Otherwise, if $0 = a_1 = \dots = a_i < a_{i+1}$, it is the cone over $S(a_{i+1}, \dots, a_n)$ with vertex a $\mathbb{P}^{i-1}$.

Rational normal scrolls, the (cones over) Veronese surface in $\mathbb{P}^5$, and quadrics, can be characterized as those non-degenerate irreducible varieties in projective space having minimal degree $d = \operatorname{codim}(X) + 1$, see for example [**EH**].

Let $X = S(a_1, \dots, a_n) \subset \mathbb{P}^N$ be as above. Then $\dim(S^k X) = (k+1)n + k$ if and only if $a_1 \geq k$ and more precisely we have that

$$\dim(S^k X) = \min\{N, N + k + 1 - \sum_{1 \leq j \leq n;\, k \leq a_j} (a_j - k)\},$$

by applying Terracini Lemma and induction to projections from general tangent spaces or by writing equations of $S^k X$, see [**Ro**] or [**CR2**]. From the above description it also follows that, whenever $S^k X \subsetneq \mathbb{P}^N$, $\operatorname{Sing}(S^k X) = S^{k-1} X$, $k \geq 1$.

Let $X = S(a_1, \dots, a_n) \subset \mathbb{P}^N$ as above with $a_1 \geq k$ and with $N = (k+1)n + k$. We have $S^k X = \mathbb{P}^N$ and we prove that $\nu_k(X) = 1$. Let $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ and let F be a fiber of the structural morphism $\pi : X \to \mathbb{P}^1$. Then $|H - kF|$ is generated by global sections and $h^0(H - kF) = \sum_{i=1}^n (a_i + 1 - k) = k(n+1) + 1 - n(k-1) = k + n + 1$. Let

$$\phi_1 = \phi_{|H-kF|} : X \to \mathbb{P}^{k+n}$$

and let

$$\phi_2 = \phi_{|kF|} : X \to \mathbb{P}^k.$$

Clearly $\phi_1(X) = S(a_1 - k, \dots, a_n - k)$. Let $\phi = (\phi_1, \phi_2)$. We get the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{P}^k \times \mathbb{P}^{k+n} \\ \downarrow & & \downarrow \\ \mathbb{P}^N & \hookrightarrow & \mathbb{P}^{(k+1)(n+1)-1} := \mathbb{P}_{n,n+k}. \end{array}$$

Now let $\psi : \mathbb{P}_{n,n+k} \dashrightarrow \mathbb{G}(k, n+k)$ be the map which associates to a $(k+1) \times (n+k+1)$ matrix its equivalence class under the action of $GL(k+1)$. It immediately follows that the closure of the fibers of ψ are linear spaces of dimension $k^2 + 2k$. Let us remark that ψ is given by forms of degree $k+1$ vanishing with order at least k along $\mathbb{P}^k \times \mathbb{P}^{n+k} \subset \mathbb{P}_{k,n+k}$ and it is not defined along $S^{k-1}(\mathbb{P}^k \times \mathbb{P}^{n+k})$. Moreover, a linear fiber through a general point $p \in \mathbb{P}_{k,n+k}$, $\mathbb{P}_p^{k^2+k}$, can be interpreted as the linear span of a $\mathbb{P}^k \times \mathbb{P}_p^k \subset \mathbb{P}^k \times \mathbb{P}^{n+k}$, which is the closure of the locus of points of $\mathbb{P}^k \times \mathbb{P}^{n+k}$ described by the $(k+1)$-secant $\mathbb{P}^k$ to $\mathbb{P}^k \times \mathbb{P}^{n+k}$ passing through p.

Let us indicate by $\widetilde{\psi} : \mathbb{P}^N \dashrightarrow \mathbb{G}(k, n+k)$ the restriction of ψ to $\mathbb{P}^N \subset \mathbb{P}_{k,n+k}$. The morphism $\widetilde{\psi}$ is defined on $\mathbb{P}^N$. Indeed, by the above description it is sufficient to show that a general point $p \in \mathbb{P}^N$ has *rank* $k+1$, thought as a point of $\mathbb{P}_{k,n+k}$.

We have to go back through the commutative diagram above. The point p belongs to a $k+1$-secant $\mathbb{P}^k$, let us say $< p_0, \ldots, p_k >$ with p_i's general on X. Then also their images through ϕ_1 and ϕ_2 will generate a $\mathbb{P}^k$ in the corresponding space. Modulo a projective change of coordinates in $\mathbb{P}^k$, respectively $\mathbb{P}^{n+k}$, we can suppose $\phi_i(p_j) = (0 : \ldots : 0 : \underbrace{1}_{j} : 0 : \ldots : 0)$. The claim easily follows. Moreover, $\widetilde{\psi}$ is dominant since for a general fiber F of ψ we have $\dim(F \cap \mathbb{P}^N) \geq k^2 + k + N - (k+1)(n+k+1) + 1 = k$. By the theorem of the dimension of the fibers, the general fiber of $\widetilde{\psi}$ has dimension $k = (k+1)n + k - (k+1)n$ and its closure, being the intersection of two linear spaces, is a $\mathbb{P}^k$, which is $(k+1)$-secant to X. Since $\widetilde{\psi}$ is defined by forms of degree $k+1$ vanishing with order at least k along X, a $(k+1)$-secant $\mathbb{P}^k$ passing through a general point of $\mathbb{P}^N$ is contracted by $\widetilde{\psi}$ so that it coincides with the fiber passing through the general point. Then through a general point of $\mathbb{P}^N$ there passes a unique $(k+1)$-secant $\mathbb{P}^k$, i.e. $\nu_k(X) = 1$.

We conclude by proving that a general quintic form in three variables has a canonical form. This result was known to Hilbert, [Hi], but we never found a "modern", or geometric proof of it.

5.2.4. EXAMPLE. **(Canonical expression for the quintic form in three variables, i.e. $\nu_6(v_5(\mathbb{P}^2)) = 1$)** We prove that the 5-Veronese embedding of $\mathbb{P}^2$, $X = \nu_5(\mathbb{P}^2) \subset \mathbb{P}^{20}$ is a smooth surface with $\nu_6(X) = 1$.

We slightly modify and adapt to our need a construction due to N. Shepherd-Barron, [SB1]. Let $F \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(1,1)|$ and let p_1 and p_2 indicate the projections (or their restrictions to F) of $\mathbb{P}^2 \times (\mathbb{P}^2)^*$. Let us recall that $F = \{(x, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^* : x \in l\}$. Let $\phi = \phi_{|\mathcal{O}_F(1,2)|} : F \hookrightarrow \mathbb{P}^{14}$. Since every fiber of $p_2 : F \to (\mathbb{P}^2)^*$ is embedded as a line in $\mathbb{P}^{14}$, we get an isomorphism of $(\mathbb{P}^2)^*$ with a subvariety of $\mathbb{G}(1,14)$. Let $X \subset \mathbb{G}(1,14) \subset \mathbb{P}^{104}$ be the image of $(\mathbb{P}^2)^* \subset \mathbb{G}(1,14)$ under the Plücker embedding of $\mathbb{G}(1,14)$. We claim that X is the 5-Veronese embedding of $\mathbb{P}^2$.

To prove this let us introduce the following Schubert cycles in $\mathbb{G} = \mathbb{G}(1,N)$. $A = \{l \in \mathbb{G} : l$ lies in a given hyperplane $\}$, $B = \{l \in \mathbb{G} : l$ meets a given linear space of codimension $3\}$, $C = \{l \in \mathbb{G} : l$ meets a given linear space of codimension $2\}$. Then C is a hyperplane section of $\mathbb{G}$ in its Plücker embedding and $C^2 = A + B$. Note that $\deg(X) = X \cdot C^2 = X \cdot A + X \cdot B$. The embedding of X is given by a complete linear system, because it is G-equivariant (see [**SB2**]), so that it is enough to prove that $\deg(X) = 25$. By definition in our example, $X \cdot A = \deg(F) = (p_1^*\mathcal{O}(1) + p_2^*\mathcal{O}(2))^3 = 18$. Let $H \subset \mathbb{P}^{14}$ be a general hyperplane and let $S \in |F \cap H|$. Then $X \cdot A$ is equal to the number of fibers of p_2 that lie in H, i.e. the number of exceptional curves contracted by $p_2 : S \to (\mathbb{P}^2)^*$. Then $X \cdot A = 9 - K_S^2 = 7$, since $K_S = \mathcal{O}_S(-1,0)$ and $K_S^2 = (p_1^*\mathcal{O}(-1))^2 \cdot (p_1^*\mathcal{O}(1) + p_2^*\mathcal{O}(2)) = 2$. Finally $\deg(X) = 18 + 7 = 25$ and the conclusion follows. By the above discussion the linear span of $X \subset \mathbb{P}^{104}$ is $< X >= \mathbb{P}^{20}$.

Let us recall that given a vector space W of odd dimension $2k+1$, there is a natural rational map $\psi : \mathbb{P}(\Lambda^2 W^*) \dashrightarrow \mathbb{P}(W^*)$, associating to a skew 2-form its kernel; recall that a skew 2-form has even rank. Then the general fiber of ψ is a linear space and if $\dim(W) = 2k+1$, then the map is given by forms of degree k vanishing with order al least $k-1$ along $\mathbb{G}(1,2k) \subset \mathbb{P}(\Lambda^2 W^*)$.

For $W = H^0(\mathcal{O}_F(1,2))$ we get a rational map $\psi : \mathbb{P}^{104} \dashrightarrow \mathbb{P}^{14}$ for which the closure of a general fiber F is a $\mathbb{P}^{90}$. In **[SB2]**, lemma 12, it is shown that the locus of indetermination of ψ does not contain $S^6X =< X >$, the last equality being well known. The general fiber F will cut $< X >= \mathbb{P}^{20}$ in a linear space of dimension at least 90+20-104=6, so that the restriction of ψ to $< X >$, $\psi : \mathbb{P}^{20} \dashrightarrow \mathbb{P}^{14}$ is dominant and the closure of a general fiber is then a linear space of dimension 6 by the above analysis and by the theorem of the dimension of the fibers. Then a 7-secant $\mathbb{P}^6$ passing through a general point of $\mathbb{P}^{20}$ is contracted by ψ so that it coincides with the general fiber of ψ restricted to $\mathbb{P}^{20}$, i.e. $\nu_6(X) = 1$.

5.3. Hessian of a polynomial and dual variety

In this section we discuss a beautiful problem originated by a question posed by Hesse, **[He1]** and **[He2]**, treated by Gordan and Nöether, **[GN]**, and more recently by Beniamino Segre, **[Se2]** and **[Se4]**, Franchetta, **[Fr1]** and **[Fr2]**, Permutti, **[Pt1]**, **[Pt2]** and **[Pt3]**, Ciliberto, **[Ci]** and Zak, **[Z3]**. We illustrate through it some interesting connections between algebra and algebraic and differential geometry. Also restricting to algebraic geometry some subtle relations between varieties without dual defect, *complete divisibility* of the equation of their dual hypersurface, their Gauss maps and Scorza varieties *accidentally* appear.

I was introduced to this beautiful subject through discussions with Ciro Ciliberto and Aron Simis.

First let us recall some definitions, remarking that the nowadays called *hessian matrix* of a function was introduced by Hesse exactly to treat and analyze the problem we describe below.

Let $f(x_0,\ldots,x_n) \in K[x_0,\ldots,x_N]$, char($K$)=0, $N \geq 1$, be a homogeneous polynomial of degree $d \geq 1$ in the $N+1$ variables $x_0,\ldots,x_N$, without multiple factors but not necessarily irreducible. We define the *Hessian of* f, indicated by $h(f)$, to be the determinant of the $(N+1)\times(N+1)$ hessian matrix of f, $H(f) := [\frac{\partial^2}{\partial x_i \partial x_j} f]$.

Clearly $h(f)$ is a homogeneous polynomial of degree $(N+1)(d-2)$, if it does not vanish identically. In the last case we say that $f(x_0,\ldots,x_N)$, or the corresponding hypersurface $X = V(f) \subset \mathbb{P}^N$, has *vanishing*, or *indeterminate hessian.*

If, modulo a linear change of variables, f does not depend on all the variables, clearly $h(f) \equiv 0$. Geometrically, if $X = V(f) \subset \mathbb{P}^N$ is (projectively equivalent to) a cone, then $h(f) \equiv 0$. The condition $h(f) \equiv 0$ is invariant under linear change of coordinates.

The question considered by Hesse in **[He1]** and **[He2]** was if the converse holds. Let us state it more precisely.

5.3.1. QUESTION. (**Hesse problem, [He1]** and **[He2]**) *Is it true that if $h(f) \equiv 0$, then there exists a linear change of variables such that the new polynomial depends on $r \leq N$ variables? Equivalently, is it true that if $h(f) \equiv 0$, then $X = V(F) \subset \mathbb{P}^N$ is a cone? Or algebraically, is it true that $h(f) \equiv 0$ implies that $\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_N}$ are linearly dependent?*

Let us analyze the geometrical reasons and results in low dimension and degree, which lead Hesse to consider the above problem. The cases $N \leq 2$ are easy to study and the answer to the question is positive, as we shall see below. For $N = 2$,

$X = V(F) \subset \mathbb{P}^2$ is a plane curve. It is easy to see that $h(f) \equiv 0$ if and only if $X \subset \mathbb{P}^2$ is the union of d lines passing through a fixed point, see also proposition 5.3.3. In particular, if $N = 2$ and $h(f) \equiv 0$, $X \subset \mathbb{P}^2$ is a cone. We immediately observe the difference between the condition $h(f) \equiv 0$ and $h(f) = 0$ (mod. f). Also for the case $N = 2$, the two conditions control different geometrical phenomena. In the second case one can only say that the plane curve $X = V(f) \subset \mathbb{P}^2$ is the union of lines, not necessarily concurrent. The easiest examples being $f(x_0, x_1, x_2) = x_0x_1x_2$. This seems to be overlooked by modern differential geometers, see the introduction of [**FW**].

For $N = 3$, the condition $h(f) \equiv 0$ characterizes once again cones. We briefly explain the geometrical reason by introducing some more definitions and notations, useful for a better understanding of the problem and also to have an intuition why it is false for $N \geq 4$.

To $f(x_0, \ldots, x_n) \in K[x_0, \ldots, x_N]$ as above it is naturally associated the *(first) polar map of f* (or of $X = V(f) \subset \mathbb{P}^N$):

$$\phi_f := (\frac{\partial f}{\partial x_0} : \ldots : \frac{\partial f}{\partial x_N}) : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N*}.$$

It is not defined along $\mathrm{Sing}(X) \subset X = V(f) \subset \mathbb{P}^N$. Let $Z = \phi_f(\mathbb{P}^N) \subseteq \mathbb{P}^{N*}$ be the closure of the image of ϕ_f. The restriction of ϕ_f to $X \setminus \mathrm{Sing}(X)$ is the Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{P}^{N*}$ of the hypersurface $X = V(f) \subset \mathbb{P}^N$. If $h(f) = det(H(f)) \equiv 0$ is identically zero, then $r = \dim(Z) = rk(H(f)) - 1 \leq N - 1$ (the hessian matrix of f is the projective differential of ϕ_f).

The following remarks yield a geometrical interpretation of the condition $h(f) \equiv 0$ and their proof is straightforward using the map ϕ_f. Part iii) was known to [**GN**], even if not stated in this form, and comes from the fact that the (projective) Jacobian matrix of ϕ_f is symmetric, being $H(f)$. One can also see [**Z3**], 4.9.

5.3.2. PROPOSITION. *Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface and let $\phi_f : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N*}$ be the associated polar map. Then:*

- i) *$Z \subset \mathbb{P}^{N*}$ is non-degenerate if and only if $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}$ are linearly independent if and only if X is not a cone;*
- iii) *$h(f) \equiv 0$ if and only if $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}$ are algebraically dependent, i.e. if and only if there exists $g(x_0, \ldots, x_N) \in K[x_0, \ldots, x_N]$ such that*

$$g(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}) \equiv 0;$$

- iv) *if $h(f) \equiv 0$, then $Z^* \subset \mathrm{Sing}(X)$.*

Proposition 5.3.2 says that Hesse question 5.3.1 can be translated algebraically in the following way: *is it true that if $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}$ are algebraically dependent, then they are linearly dependent?*

A deeper geometrical interpretation, which clears the path to counterexamples is the following:*is it true that $h(f) \equiv 0$ if and only if $Z \subset \mathbb{P}^{N*}$ is degenerate?*

Put in this way it should be clear that there are no reasons for the validity of the claim for arbitrary N, even if Hesse published two papers, [**He1**] and [**He2**] with a "proof" of it. Indeed a whole class of counterexamples were constructed by Gordan and Nöether, who also put the problem in a larger context and showed its relations to algebraic solution of some partial differential equations. Before proceeding to construct in a geometric way a large class of counterexamples for $N \geq 4$, let us furnish a cheap and easy proof that for $N = 1, 2$ the question has a positive answer, without using the notion of flex of a curve. Later we shall see that also for $N = 3$, the answer to Hesse question is positive.

5.3.3. PROPOSITION. *Let $X = V(f) \subset \mathbb{P}^N$, $N = 1, 2$, be a reduced hypersurface of degree d such that $h(f) \equiv 0$. Then $X \subset \mathbb{P}^N$ is a cone. More precisely for $N = 1$ we get that $h(f) \equiv 0$ if and only if $d = 1$ and for $N = 2$ we have $h(f) \equiv 0$ if and only if $X = V(f) \subset \mathbb{P}^2$ is the union of d concurrent lines.*

PROOF. Since $h(f) \equiv 0$, $r = \dim(Z) < N$. If $N = 1$, then $Z \subset \mathbb{P}^1$ is a point, so that the partial derivatives of $f(x_0, x_1)$ are constant and $d = 1$. Thus $X = V(f) \subset \mathbb{P}^1$ is a point, i.e. a 0-dimensional cone.

Suppose $N = 2$. Then $\dim(Z) \leq 1$. If Z is a point, then $d = 1$ and viceversa. Suppose $\dim(Z) = 1$. From $Z^* \subset \mathrm{Sing}(X)$ we deduce that Z^* is a point, so that $Z = \phi_f(\mathbb{P}^2)$ is a line and hence degenerate. Thus $X = V(f) \subset \mathbb{P}^2$ consists of a finite number of lines varying in the pencil $Z \subset \mathbb{P}^{2*}$, so that it is a cone. $\square$

As we recalled above, a whole class of counterexamples to Hesse question for $N \geq 4$ were constructed by Gordan and Nöether in **[GN]**, see also **[Pt1]**. We construct geometrically the easiest counterexamples following the ideas developed by Umberto Perazzo in **[Pe]**. Surely if $d = 2$, i.e. for quadric hypersurfaces in any dimensions, $h(f) \equiv 0$ if and only if $X = V(f) \subset \mathbb{P}^N$ is a cone of vertex a $\mathbb{P}^{N+1-\mathrm{rk}(H(f))}$. Thus the first case for which Hesse claim fails could appear for $N = 4$ and $d = 3$. Perazzo's first remark was the following, see pg. 343 of **[Pe]**: *un' esempio semplicissimo.*

5.3.4. PROPOSITION. (Perazzo, **[Pe]**) *Let $X = V(f) \subset \mathbb{P}^N$, $N \geq 4$, be a reduced cubic hypersurface. Suppose $\mathrm{Sing}(X) = \mathbb{P}^m$ with $m > \frac{N-1}{2}$. Then $h(f) \equiv 0$.*

PROOF. Let $p \in \mathbb{P}^N$ be a general point and let $\sum_{i,j=0}^N x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p) = 0$ be the polar hyperquadric Q_p of p with respect to $X = V(f) \subset \mathbb{P}^N$. We prove that for general p, the quadric Q_p is singular, which clearly implies $h(f) \equiv 0$. For $q \in \mathrm{Sing}(X)$, the polar hyperplane H_q of q with respect to X has equation $\sum_{i=0}^N x_i \frac{\partial f}{\partial x_i}(q) = 0$, so that it is all $\mathbb{P}^N$. Since a general $p \in \mathbb{P}^N$ belongs to H_q, by the "reciprocity law" of polarity, $q \in Q_p$ for general p, i.e. for general $p \in \mathbb{P}^N$ the polar quadric Q_p contains $\mathrm{Sing}(X) = \mathbb{P}^m$. Since a linear space contained in a smooth quadric hypersurface in $\mathbb{P}^N$, $N \geq 2$, has dimension less or equal to $\frac{N-1}{2}$, we obtain the desired conclusion. $\square$

Thus cubic hypersurfaces of $\mathbb{P}^N$, $N \geq 4$, with $\mathrm{Sing}(X) = \mathbb{P}^{N-2}$ have vanishing hessian and clearly are not cones. Even if it seems a large class of examples, there is essentially only one such cubic hypersurface in $\mathbb{P}^4$. In the second section of his paper, **[Pe]** pg. 343–351, Perazzo studies in details the class of cubic hypersurfaces in $\mathbb{P}^N$ having as singular locus a linear space, giving some "canonical" forms for $N \leq 6$ and in particular producing explicit examples.

5.3.5. PROPOSITION. (Perazzo, **[Pe]**, **[Ru5]**) *Let $X = V(f) \subset \mathbb{P}^N$, $N \geq 4$ be an irreducible cubic hypersurface such that $\mathrm{Sing}(X) = \mathbb{P}^{N-2}$. Then $N = 4$, X has vanishing hessian, $X^* \subset \mathbb{P}^4$ is a smooth rational normal scroll of degree 3 and*

$X = V(f) \subset \mathbb{P}^4$ is the projection of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ from a point not on it. The image of the corresponding polar map is a rank 3 quadric hypersurface. Furthermore, modulo projective transformations, the equation of X is of the form $x_3^2 x_0 + 2x_3x_4x_1 + x_4^2 x_2 + \Phi(x_3, x_4) = 0$. In conclusion the dual of a smooth rational normal scroll of degree 3, modulo projective transformations, is the only irreducible cubic hypersurface in $\mathbb{P}^N$, $N \geq 4$, not a cone and having $\mathrm{Sing}(X) = \mathbb{P}^{N-2}$.

PROOF. (sketch) First of all $X = S(1,2)^* \subset \mathbb{P}^4$ is a cubic hypersurface, not a cone, with $\mathrm{Sing}(X) = \mathbb{P}^2$, the partial derivatives of an equation of X giving a double structure on $\mathbb{P}^2$. Indeed, since $S(1,2) \subset \mathbb{P}^4$ is a hyperplane section of the Segre 3-fold $Y = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, the variety $S(1,2)^*$ is the projection of Y from the point $h \in \mathbb{P}^5$ corresponding to the hyperplane H cutting $S(1,2)$ on Y, i.e. $S(1,2)^* = \pi_h(Y) \subset \mathbb{P}^4$. The entry locus of Y with respect to h is isomorphic to a quadric hypersurface $\mathbb{P}^1 \times \mathbb{P}^1_h$, so that $\mathrm{Sing}(S(1,2)^*) = \pi_h(\mathbb{P}^1 \times \mathbb{P}^1_h) = \mathbb{P}^2$. Since $\deg(Y) = 3$, we get that $\deg(S(1,2)^*) = 3$, so that $S(1,2)^*$ is a cubic hypersurface, not a cone (also because $S(1,2) \subset \mathbb{P}^4$ is non-degenerate), with vanishing hessian (we remark the fact that we proved this without writing any explicit equation of $S(1,2)^*$!). From the geometrical description it also follows that the scheme structure on $\mathrm{Sing}(S(1,2)^*)$ is "double".

Let $\{\mathbb{P}^{N-1}\}_{\lambda \in \mathbb{P}^1}$ be the pencil of hyperplanes through $\mathrm{Sing}(X) = \mathbb{P}^{N-2}$. The intersection $\mathbb{P}^{N-1}_\lambda \cap X$ is a cubic hypersurface in $\mathbb{P}^{N-1}_\lambda$ having a $\mathbb{P}^{N-2}$ as singular locus, i.e. $\mathbb{P}^{N-1}_\lambda \cap X = 2\,\mathrm{Sing}(X) \cup \mathbb{P}^{N-2}_\lambda$. Therefore X is swept out by a pencil $\mathbb{P}^{N-2}_\lambda$ such that through every point $p \in X \setminus \mathrm{Sing}(X)$ there passes a unique $\mathbb{P}^{N-2}_\lambda$ of the family. Since it is not a cone and it has degree 3, it easily follows that X is a rational normal scroll which is the projection of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ from a point not on it, yielding $N = 4$. By duality it immediately follows that $X^* \subset \mathbb{P}^4$ is an irreducible hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2$, not a cone, so that X^* is projectively equivalent to $S(1,2) \subset \mathbb{P}^4$.

If $N = 4$, then $h(F) \equiv 0$ by Perazzo' s remark 5.3.4 and it is easy to see that $rk(H(F)) = 4$. Therefore the polar map associated to such an $X \subset \mathbb{P}^4$ has a 3 dimensional image $Z = \phi(\mathbb{P}^4)$, which we claim being a cone over a smooth conic, i.e. a rank 3 quadric hypersurface in $(\mathbb{P}^4)^*$. For $p \in \mathbb{P}^4$ a general point, the polar quadric Q_p has rank 4 and contains $Sing(X) = \mathbb{P}^2$ so that $\mathrm{Sing}(Q_p)$ is a point in $\mathbb{P}^2 = \mathrm{Sing}(X)$. On the other hand $\mathrm{Sing}(Q_p)$ describes a conic as p varies in $\mathbb{P}^4$ as it easily follows (it is the intersection of a pencil of hyperplanes). If $z \in Z$ is a general point, then $\mathrm{Sing}(Q_p) = (T_{\phi(p)}Z)^* \subseteq Z^* \cap \mathbb{P}^2$, which gives that $Z^* \subsetneq \mathbb{P}^2$, i.e. Z^* is a plane curve and in fact the above conic so that Z is a quadric cone over a smooth conic. For the equations one can consult **[Pe]** or **[Ru5]**. □

An explicit counterexample to Hesse question is produce by taking $\Phi(x_3, x_4) = 0$ in the above proposition, i.e. by considering $f(x_0, \ldots, x_4) = x_0x_3^2 + 2x_1x_3x_4 + x_2x_4^2$.

Let us state the following consequence, which is the classification of *codegree* 3 irreducible varieties, not cones, such that either $\mathrm{codim}(X^*) = 2$ or X^* is a non-normal hypersurface.

5.3.6. COROLLARY. (**[Ru5]**) *Let $X \subset \mathbb{P}^N$, $N \geq 3$, be an irreducible non-degenerate variety such that X^* is a non-normal hypersurface of degree 3. Then $X = S(1,2)$ or one of its projections from an external point, i.e. a cubic surface with a double line. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety, not a cone, such* *that X^* is a degree 3 variety of codimension 2. Then either $X \simeq X^* \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or $X = S(1,2)^*$.*

PROOF. Let us recall the following easy fact: a cubic hypersurface $Y \subset \mathbb{P}^N$ is non-normal if and only if $\mathrm{Sing}(Y) = \mathbb{P}^{N-2}$. Then the first part follows from proposition 5.3.5 since X^* is not a cone. If X^* is not a hypersurface, by projecting from a general linear space we get a non-normal cubic hypersurface Y^* where Y is the *dual linear section* of X. The conclusion follows by applying the same argument as in proposition 5.3.5. Further details and applications are contained in [**Ru5**]. □

Many other applications of the geometrical ideas of Perazzo can be furnished. One interesting characterization concerns cubic hypersurfaces with $\mathrm{Sing}(X) = \mathbb{P}^2$ and with vanishing hessian. They are classified as those for which the image of the polar map is a rank 3 quadric hypersurface in $\mathbb{P}^{N*}$ (or equivalently those for which $Z^* \subset \mathrm{Sing}(X) = \mathbb{P}^2$ is a smooth conic), see [**Pe**] and [**Ru5**]. We state only the following consequence of this result.

5.3.7. PROPOSITION. (Perazzo, [**Pe**] pg. 351 (and 352)) *Let $X = V(f) \subset \mathbb{P}^4$ be an irreducible cubic hypersurface, which is not a cone and such that $h(f) \equiv 0$. Then $\mathrm{Sing}(X) = \mathbb{P}^2$, Z^* is a conic, Z is a quadric hypersurface of rank 3 and $X = S(1,2)^*$. In particular, for an irreducible cubic hypersurfaces in $\mathbb{P}^4$, not a cone, we have that $h(f) \equiv 0$ if and only $X = S(1,2)^*$.*

Even if Hesse problem has a negative answer in general, it opened an interesting area of research dealing with hypersurfaces, not cones, for which $h(f) \equiv 0$. This class of varieties appears naturally in various areas, such as differential geometry and approximation theory, see for examples [**Se3**], [**FW**] and [**PW**].

To explain the relations with curvature and differential geometry let us recall a formula, also quite unknown to differential geometers according to [**FW**], explaining the relations between the hessian of a (holomorphic or C^∞) function and the gaussian curvature.

Let $z_1, \dots, z_N$ be coordinates on $\mathbb{C}^N$ (or $\mathbb{R}^N$). If $\mathbb{C}^N \subset \mathbb{P}^N$ is the complement of $x_0 = 0$, then $z_j = \frac{x_j}{x_0}$ for $j = 1, \dots, N$. Let $f(z_1, \dots, z_N)$ be a holomorphic function on $\mathbb{C}^N$ (for $f(x_0, \dots, x_N)$ homogeneous polynomial, we clearly take $f(1 : \frac{x_1}{x_0} : \dots : \frac{x_N}{x_0})$). Let $f_j = \frac{\partial f}{\partial x_j}$ and let $f_{j,l} = \frac{\partial^2 f}{\partial x_j \partial x_l}$. Let

$$\widetilde{H}(f) = \begin{pmatrix} 0 & f_1 & \cdots & f_N \\ f_1 & f_{1,1} & \cdots & f_{1,N} \\ \vdots & \vdots & & \vdots \\ f_N & f_{N,1} & \cdots & f_{N,N} \end{pmatrix}.$$

If $f(x_0, \dots, x_N)$ is a homogeneous polynomial of degree d, then using Euler formula one proves that

$$x_0^2 h(f) = (d-1)^2 \begin{vmatrix} \frac{d}{d-1} f & f_1 & \cdots & f_N \\ f_1 & f_{1,1} & \cdots & f_{1,N} \\ \vdots & \vdots & & \vdots \\ f_N & f_{N,1} & \cdots & f_{N,N} \end{vmatrix},$$

so that, for the a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_N]$, on $X = V(f) \cap \mathbb{C}^N$, we get

$$h(f) = (d-1)^2 \left|\widetilde{H}(f(1, z_1, \ldots, z_N)\right|,$$

or in other words that, for a homogeneous polynomial $f(x_0, \ldots, x_N)$, the vanishing of the hessian modulo f, $h(f) = 0$ (mod. f), is equivalent to $\left|\widetilde{H}(f(1, z_1, \ldots, z_N)\right| = 0$.

The differential geometrical meaning of the last condition is that at every point smooth point of the analytic (or differential) hypersurface $V(f) \subset \mathbb{C}^N$ (or $\mathbb{R}^N$), the *gaussian curvature* of $V(f)$ is zero. To avoid technical definitions, we limit ourselves to real manifolds, where the following result, which can be suitable extended to complex manifolds, holds.

5.3.8. PROPOSITION. (Segre formula, see [**Se3**], pg. 13 or [**FW**]) *Let $f(z_1, \ldots, z_N)$ be a real C^∞ function. Let $V(f) \subset \mathbb{R}^N$ be the associated hypersurface. For a non-singular point $p \in V(f)$, taking the induced Riemannian metric,*

$$K(p) = \frac{(-1)^{N+1}}{(f_1^2 + \ldots + f_N^2)^2} \left|\widetilde{H}(f)\right|,$$

where $K(p)$ is the gaussian curvature.

At least for $N = 3$ it is well known that under some regularity assumptions, real hypersurfaces with $K(p) = 0$ for every smooth point $p \in V(f)$ are *developable*, so that are either cones (or cilinders) or the tangent developable of a curve, see for example [**DC**]. In algebraic geometry the relation between the vanishing of the hessian and developable manifolds was investigated in [**Se2**] and in [**Fr1**], [**Fr2**], as we shall briefly describe below.

We return to the algebro-geometrical investigation of the condition of vanishing hessian and, motivated by the above results, of the condition of $h(f) = 0$ (mod. f), or more generally of the meaning of $\mathrm{rk}(H(f))$ (mod. f) for a homogeneous polynomial in $N+1$ variables over a field of characteristic zero. It remains the problem of trying to characterize some classes of hypersurfaces with vanishing hessian as in [**Fr1**], [**Fr2**] or more recently in [**Z3**]. We survey the work of Beniamino Segre in [**Se2**] and [**Se4**] on the condition of divisibility of $h(f)$ by the polynomial f.

To this aim let us recall that the restriction of the polar map ϕ_f to the hypersurface $X = V(f) \subset \mathbb{P}^N$ is the Gauss map of X, $\mathcal{G}_X : X \dashrightarrow \mathbb{P}^{N*}$, so that $\mathcal{G}_X(X) = X^*$. The restriction of $H(f)$ to X represents the (projective) differential of $\mathcal{G}_X$ so that one could expect some relation between $\mathrm{rk}(H(f))$ (mod. f) and the dimension of X^*. This was established by B. Segre in [**Se2**], see also [**Z3**] for another proof. For simplicity we assume $X = V(f) \subset \mathbb{P}^N$ irreducible, even if the result holds also for reduced hypersurface, suitable formulated. We indicate $\mathrm{rk}(H(f))$ (mod. f) by $\mathrm{rk}_f(H(f))$.

5.3.9. THEOREM. ([**Se2**]) *Let $X = V(f) \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface. Then*

i) $\mathrm{rk}_f(H(f)) = \dim(X^*) + 2 = \mathrm{rk}(d\mathcal{G}_X) + 2$.

ii) $f^{\mathrm{def}(X)}$ *divides* $h(f)$.

In particular if $h(f) \not\equiv 0$, *then*

$$d \geq 2\frac{N+1}{\dim(X^*)+2} \qquad \text{(Segre inequality)}.$$

For the proof we refer to [Se2], or [Z3]. Only a word about the deduction of part ii) and of the Segre inequality from part i). Let us recall that $h(f)$, if not identically zero, is a homogeneous polynomial of degree $(N+1)(d-2)$, while $\dim(X^*) = N-1-\operatorname{def}(X)$. Since $H(f)$ is a square matrix of order $N+1$, from $\operatorname{rk}_f(H(f)) = \dim(X^*)+2$, one deduces that $h(f) \in (f^{N+1-\operatorname{rk}_f H(f)}) = (f^{\operatorname{def}(X)})$. Comparing the degrees of $h(f)$ and $f^{\operatorname{def}(X)}$, when $h(f) \not\equiv 0$, one gets

$$d(N-1-\dim(X^*)) \leq (N+1)(d-2),$$

which is equivalent to Segre inequality.

Returning to developable hypersurfaces, by Segre theorem, we deduce that if $\operatorname{rk}_f H(F) < N+1$, then $\operatorname{def}(X) > 0$ and X is ruled by linear spaces $\mathbb{P}^{\operatorname{def}(X)}$ which, outside $\operatorname{Sing}(X)$, coincide with the fibers of the Gauss map and along which the tangent hyperplane is constant (recall that fibers of the Gauss map are linear by theorem 1.5.9). Naturally a *complex foliation* appears.

This discussion furnishes also an explanation why Hesse question has a positive answer for $N = 3$. Indeed if $N = 3$ and $h(f) \equiv 0$, then $\mathcal{G}_X(X)$ is a curve, unless $d = 1$, and $X = V(f) \subset \mathbb{P}^3$ is a developable surface, i.e. it is either a cone or the tangent developable of a curve by a well known theorem of C. Segre, see also [GH]. One verifies by a direct calculation that for the tangent developable of a curve $h(f) \not\equiv 0$, even if $h(f) = 0 \pmod{f}$, so that the conclusion holds. One again the conclusion $h(f) \equiv 0$ and $h(f) = 0 \pmod{f}$ control two different geometrical situations and exprime two different types of developable conditions.

It is worth remarking that by reverting the construction yielding the Segre inequality, one could deduce a bound for the degree of the dual variety of a variety. Let us suppose that $X \subset \mathbb{P}^N$ is an irreducible variety. Let $X^* \subset \mathbb{P}^{N*}$ be its dual variety. Define the *codegree of* X, $d^* = \deg(X^*)$. By reducing to the case in which X^* is an hypersurfaces with projections on $\mathbb{P}^{N*}$ and section on X, which do not change neither d^*, we deduce, under the hypothesis that X^* has not vanishing hessian, the following inequality for the codegree of X,

(5.3.1)
$$d^* \geq 2\frac{N+1}{n+2}.$$

Moreover, in [Z3], Zak proves that if, after the reduction to the case X^* hypersurface, we get a hypersurface with vanishing hessian, then $d^* > 2\frac{N+1}{n+2}$, unless $X = V(f) \subset \mathbb{P}^N$ is a quadric cone. More precisely we have the following result.

5.3.10. PROPOSITION. (Zak, [Z3]) *Let* $X \subset \mathbb{P}^N$ *be a non-degenerate irreducible variety. Then the following conditions are equivalent:*

i)
$$d^* = 2\frac{N+1}{n+2};$$

ii) *Either* $N = n+1$, $X = V(f) \subset \mathbb{P}^N$, $h(f) \equiv 0$ *and* X *is a quadratic cone of vertex* $\mathbb{P}^{\operatorname{def}(X)-1}$ *or* $\operatorname{def}(X) = 0$ *and for* $X^* = V(f) \subset \mathbb{P}^{N*}$, $h(f) = f^{N-n-1}$, *where* f *is a suitably choosen equation for* X^*.

The above result says that inequality 5.3.1 turns into an equality if and only if $\operatorname{def}(X) = 0$ and X^* has non-vanishing hessian, unless X is a quadric cone. Moreover in this case we have *complete divisibility* of the hessian of X^*. As one could imagine, this condition is quite strong and very rarely satisfied. In order to see what it does mean geometrically let us relate it to nothing less than Hartshorne conjecture and Severi varieties. If $d^* = 2$, then X is a quadric hypersurface. Suppose $d^* = 3$, the first non-trivial case.

By proposition 5.3.10,

$$3 = d^* = 2\frac{N+1}{n+2}$$

if and only if

$$N = \frac{3n}{2} + 2.$$

Moreover if $d^* = 2\frac{N+1}{n+2}$, the variety $X \subset \mathbb{P}^N$ is non-degenerate and linearly normal. So that by definition a smooth variety $X \subset \mathbb{P}^N$ with $d^* = 3$ and for which $d^* = 2\frac{N+1}{n+2}$ is a Severi variety. In particular there are only 4 examples.

In **[Se2]** and **[Se4]**, B. Segre proves that the Scorza varieties $v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$, $\mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)}$ and $\mathbb{G}(1, 2m+1) \subset \mathbb{P}^{m(2m+3)}$ are varieties of codegree $n+1$, respectively $m+1$ for which the hessian is completely divisible, i.e. such that $d^* = 2\frac{N+1}{n+2}$. A geometrical and unitary way of looking at these examples, proving the above claim, is to observe that in all examples $S^{k_0-1}X$ is a hypersurface defined as a suitable locus of matrices of rank less than the maximal allowed. On the space there is a natural Cremona transformation sending a symmetric (general, respectively antisymmetric) matrix to its inverse. This Cremona transformation is the polar map of $S^{k_0-1}X$ and it is an isomorphism outside $S^{k_0-1}X$, giving the assertion.

On the base of the strong restrictions a variety $X \subset \mathbb{P}^N$ has to verify in order to satisfy the condition $d^* = 2\frac{N+1}{n+2}$, one naturally poses the following question. This arose from a discussion on the subject with Fyodor Zak.

5.3.11. QUESTION. **(Complete divisibility)** *Let $X \subset \mathbb{P}^N$ be a smooth variety such that $d^* = 2\frac{N+1}{n+2}$. Is it true that $X \subset \mathbb{P}^N$ is one of the following:*

i) *a smooth quadric hypersurface* $(d^* = 2)$;
ii) $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ $(d^* = n+1)$;
iii) $\mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)}$, $m \geq 2, (d^* = m+1)$;
iv) $\mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{2}}$, $n \equiv 0 \pmod{2}$, $(d^* = \frac{n}{2}+1)$;
v) *the E_6-variety $X \subset \mathbb{P}^{26}$ of dimension 16* $(d^* = 3)$*?*

Related to the above question it naturally appears the following. Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface. The polar map

$$\phi_f : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N*}$$

is dominant if and only if $h(f) \not\equiv 0$. Then one can ask under which conditions the polar map is birational, i.e. it defines a so called *polar*, or *integrable Cremona* *transformation.* The polynomials defining polar Cremona transformations are usually dubbed as *homaloidal polynomials.* The next question is related to the above discussion and quite natural.

5.3.12. QUESTION. **(Homaloidal polynomials)** *Is the degree of a square free polynomial defining a polar Cremona transformation bounded by $N+1$? If it is irreducible, is it true that $d \leq N+1$? What are the irreducible (or more generally the square free) polynomials of degree $N+1$ defining polar Cremona transformations?*

Very little is known about polar Cremona transformations. There is a classification for $N = 2$ due to Dolgachev, see **[Do]**, who also prove that polynomials of degree $N+1$, product of $N+1$ general linear factors, define polar Cremona transformations for $N = 2, 3$.

Another fascinating question, related to varieties with one apparent double points, is the analysis of polynomials $f \in K[x_0, \ldots, x_N]$ such that $h(f) = \beta f^\alpha$ with $\alpha > \operatorname{def}(X)$, $X = V(f) \subset \mathbb{P}^N$, $\beta \in K^*$, see **[Se2]** pg. 172 and **[Ci]**. The first instance of this phenomenon was known to Cayley, see **[Cy]**, who observed that the tangent developable to the twisted cubic $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is a degree 4 surface $X = V(f) \subset \mathbb{P}^3$, with $\operatorname{def}(X) = 1$ and such that $h(f) = \beta f^2$. This can be generalized to the class of *twisted cubics over the cubic Jordan algebras.* We described them also as Lagrangian Grassmannian over the four composition algebras in the previous section and also as the Edge varieties $\mathbb{P}^1 \times \mathbb{Q}^{n-1}$, $n \geq 2$.

All these varieties have the property that their tangent variety $TX = V(f) \subset \mathbb{P}^{2n+1}$ is a quartic hypersurface isomorphic to X^* and such that $h(f) = \beta f^{n+1}$. In this case $n+1 = 2n+1-1-n+1 = \operatorname{def}(TX)+1$. So the condition $h(f) = \beta f^{\operatorname{def}(X)+1}$ is shared by duals varieties of some smooth varieties with one apparent double point, which are also homogeneous varieties. This opens the question of understanding why the completely divisibility or the higher divisibility control phenomena related to secant and tangent lines and forces isomorphism between SX and X^*, respectively TX and X^*. It is interesting to note, as always, the presence of some remarkable polar Cremona transformation, defined by the equations of TX. It is an involutive Cremona transformation associated to all varieties with one apparent double point $X \subset \mathbb{P}^{2n+1}$, which to a geneal $p \in \mathbb{P}^{2n+1}$ associates the harmonic conjugate to it with respect to the two points of intersection with X of the unique secant line to X passing through p. For varieties with one apparent double point, which are not homogeneous, in general one could not expect complete divisibility for the equation of X^* neither the isomorphism between TX and X^*. On the base of this and of the results proved for $n = 1$ by Franchetta, **[Fr1]**, and for $n = 2$ by Ciliberto, **[Ci]**, motivated by the question below, supposing as always $\beta \in K^*$, one could ask:

5.3.13. QUESTION. **(Higher divisibility, [Se2] pg. 172)** *What are the smooth varieties $X \subset \mathbb{P}^N$ for which the dual is a hypersurface $X^* = V(f) \subset \mathbb{P}^{N*}$ such that $h(f) = \beta f^{N-n}$?*

Is it true that $\deg(f) = 4$ *and $h(f) = \beta f^{N-n}$ if and only if $X \subset \mathbb{P}^N$ is a twisted cubic over a cubic Jordan algebra?*

Given a (smooth) non-degenerate variety $X \subset \mathbb{P}^N$ with $\operatorname{def}(X) = 0$, *what is the maximum $\alpha > N-1-n$ such that, letting $X^* = V(f) \subset \mathbb{P}^N$, $h(f) = \beta f^\alpha$?*

Many other interesting questions and relations between polynomials, dual varieties and varieties with remarkable projective geometry can be formulated. We hope to have convinced the reader of the ubiquity of interesting connections between varieties with special or extremal geometric properties, algebra, topology and some subtle questions in arithmetic.

5.4. Some (determinantal) hypersurfaces and their rationality

In this section we study geometrically the rationality over arbitrary fields K of some hypersurface in $\mathbb{P}^N_K$, whose equation can be written as the determinant of a square matrix of order $N+1$ of linear forms on $\mathbb{P}^N$. In particular we prove that a smooth cubic surfaces in $\mathbb{P}^3$ has such an equation if and only if it is rational over K, a result of B. Segre, [Se1]. Clearly the point here is that we do not suppose a priori that such a surface is the blow-up of $\mathbb{P}^2$ at six distinct points defined over $\overline{K}$. On the contrary, following a suggestion of Semple and Roth, we deduce from the above mentioned result that an arbitrary smooth cubic surface is the blow-up of $\mathbb{P}^2$ at six point defined over $\overline{K}$ because, over $\overline{K}$, it has a determinantal equation of the above type. The result is suitably generalized to higher N.

Let us recall the following construction of [**AR3**]. Let $A(\mathbf{x})$ be a $(N+1)\times N$ matrix of linear forms on $\mathbb{P}^M$ with coordinates $\mathbf{x} = (x_0 : \ldots : x_M)$, $M \geq N$. Suppose the $N+1$ minors of maximal order N of $A(\mathbf{x})$, $F_0, \ldots, F_N$ define a smooth geometrically irreducible codimension 2 subvariety $X \subset \mathbb{P}^M$. By definition, the variety $X \subset \mathbb{P}^M$ is scheme theoretically defined by the forms F_i's and, for example by using the Hilbert-Burch theorem (or the Eagon-Northcott complex), the ideal sheaf $\mathcal{I}_X$ has the minimal resolution

$$0 \to \mathcal{O}(-N-1)^N \stackrel{A(\mathbf{x})}{\to} \mathcal{O}(-N)^{N+1} \to \mathcal{I}_X \to 0.$$

In particular $X \subset \mathbb{P}^M$ is an arithmetically Cohen-Macaulay variety. The homogeneous forms $F_0, \ldots, F_N$ define a rational map $\phi : \mathbb{P}^M \dashrightarrow \mathbb{P}^N$.

Let $\mathbf{y} = (y_0 : \ldots : y_N)$ be coordinates on the target $\mathbb{P}^N$. The closure of the graph of ϕ, Γ_ϕ, in $\mathbb{P}^M \times \mathbb{P}^N$ is $Bl_X(\mathbb{P}^M) = \widetilde{\mathbb{P}^M}$, which coincides outside the exceptional divisor E with the scheme of equation

$$\sum a_{i,j}(\mathbf{x})y_j = A(\mathbf{x})^t \cdot \mathbf{y}^t = \mathbf{0}.$$

In fact for $\mathbf{x} \in \mathbb{P}^M \setminus X$ by elementary linear algebra we get:

$$A(\mathbf{x})^t \cdot \mathbf{y}^t = \mathbf{0} \Leftrightarrow \mathbf{y} = (F_0(\mathbf{x}) : \ldots : F_N(\mathbf{x})) \Leftrightarrow (\mathbf{x}, \mathbf{y}) \in \Gamma_\phi.$$

There exists a $N \times (M+1)$ matrix of linear forms $B(\mathbf{y})$ such that

$$A(\mathbf{x})^t \cdot \mathbf{y}^t = B(\mathbf{y}) \cdot \mathbf{x}^t.$$

Let $\widetilde{\phi} : \widetilde{\mathbb{P}^M} \to \mathbb{P}^N$ be the resolution of the rational map ϕ. Let $\mathbf{y} = \phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{P}^M \setminus X$. The above discussion yields

$$\phi^{-1}(\mathbf{y}) = \{ \ \mathbf{x} \in \mathbb{P}^M \setminus X \ : \ B(\mathbf{y}) \cdot \mathbf{x}^t = \mathbf{0} \ \},$$

whose closure in $\mathbb{P}^M$ is clearly a linear space of dimension $M - \mathrm{rk}(B(\mathbf{y}))$, let us say $\mathbb{P}^{M-\mathrm{rk}(B(\mathbf{y}))}_{\mathbf{y}}$. In particular we deduce that the fibers of ϕ are all irreducible and that the rank of $B(\mathbf{y})$ at the general point of $\mathrm{Im}(\phi)$ is equal to $\dim(\mathrm{Im}(\phi))$ by the theorem of the dimension of the fibers.

It is also clear that, when $\mathbb{P}_{\mathbf{y}}^{M-\mathrm{rk}(B(\mathbf{y}))}$ is positive dimensional, it cuts X along a determinantal hypersurface of degree N. We leave the details to the reader. The geometrical interpretation is that if through a point $p \in \mathbb{P}^M \setminus X$ there passes a N-secant line to X, since this line is contracted by ϕ, the locus of N-secant lines to $X \subset \mathbb{P}^M$ through p is the projective space $\mathbb{P}_{\phi(p)}^{M-\mathrm{rk}(B(\phi(p))}$ cutting X along a hypersurface whose equation is determinantal in this projective space. Moreover this geometric description implies that, by looking at $\widetilde{\phi}$, we can say that the closure of a general fiber of $\widetilde{\phi}$ is a linear space.

The above discussion about the locus of N-secant lines through a general point of $\mathbb{P}^M$ is a particular version of proposition 2.9.1 of [**Ro**]. This result of Room can be considered as the first case of the general situation described also in [**HKS**], [**Ve**] and [**RS**]. In fact Room's result is true for more general situations and determinantal varieties, see *loc. cit.*

Suppose $M = N+1$ (and to have smooth $X \subset \mathbb{P}^{N+1}$, $2 \leq N \leq 4$), the description of the fibers of ϕ yields $\mathrm{Sec}_N X = \mathbb{P}^{N+1}$ and the fact that the general fiber F of the associated rational map $\phi : \mathbb{P}^{N+1} \dashrightarrow \mathbb{P}^N$ is a N-secant line. Take now an arbitrary hypersurface $Y = V(G) \subset \mathbb{P}^{N+1}$ of degree $N+1$ through $X \subset \mathbb{P}^{N+1}$. The restriction of ϕ to $Y \subset \mathbb{P}^N$ is a birational isomorphism $\phi_{|Y} : Y \dashrightarrow \mathbb{P}^n$ because a general fiber of ϕ is a N-secant line to X, not contained in Y, so that this general fiber residually cuts Y in a point outside X. Moreover the equation of an Y as above can be written as the determinant of a $(N+1) \times (N+1)$ matrix of linear forms on $\mathbb{P}^{N+1}$. Indeed, since $G(\mathbf{x}) = \sum_{i=0}^{N} l_i(\mathbf{x}) F_i(\mathbf{x})$, the graph of $\phi_{|Y}$ inside $\mathbb{P}^{N+1} \times \mathbb{P}^N$ coincides outside $E \cap Y$ with the scheme given by the $N+1$ bihomogeneous $(1,1)$ forms

$$A(\mathbf{x})^t \cdot \mathbf{y}^t = 0,$$

$$\sum_{i=0}^{N} l_i(\mathbf{x}) y_i = 0,$$

which can be written as

$$\widetilde{A}(\mathbf{x})^t \cdot \mathbf{y}^t = 0, \tag{5.4.1}$$

where $\widetilde{A}(\mathbf{x})$ is the desired $(N+1) \times (N+1)$ matrix of linear forms on $\mathbb{P}^{N+1}$. The conclusion easily follows since $\det(\widetilde{A}(\mathbf{x}))$ is a homogeneous polynomial of degree $N+1$ vanishing at the general point of Y. We remark that the same construction applies also for a geometrically reduced, not necessarily geometrically irreducible, determinantal variety $X \subset \mathbb{P}^{N+1}$, as soon as $\mathrm{Sec}_N X = \mathbb{P}^{N+1}$. By reverting the construction we have that a hypersurface of degree M in $\mathbb{P}^M$ containing a codimension 2 geometrically reduced variety of the above type has an equation given by the determinant of a $M \times M$ matrix of linear forms on $\mathbb{P}^M$.

We collect the above discussion in the following proposition.

5.4.1. Proposition. ([**AR3**]) *Let $A(\mathbf{x})$ be a $N+1 \times N$ matrix of linear forms on $\mathbb{P}_K^{N+1}$, K an arbitrary field, with coordinates $\mathbf{x} = (x_0 : \ldots : x_{N+1})$. Suppose the $N+1$ minors of maximal order N of $A(\mathbf{x})$, $F_0, \ldots, F_N$ define a (smooth) geometrically (irreducible) reduced, geometrically connected, codimension 2 subvariety $X \subset \mathbb{P}^M$ such that* $\mathrm{Sec}_N X = \mathbb{P}^M$.

Then any hypersurface $Y \subset \mathbb{P}^{N+1}_K$ of degree $N+1$ through X has an equation given by the determinant of a $(N+1) \times (N+1)$ matrix of linear forms on $\mathbb{P}^{N+1}_k$ and the restriction of ϕ to Y is birational.

5.4.2. COROLLARY. **(Smooth cubic surfaces over algebraically closed fields are determinantal)** *Let $X \subset \mathbb{P}^3_K$ be a pure one-dimensional, non-degenerate, geometrically reduced and geometrically connected scheme of degree 3. Then X has one apparent double point and a cubic surface defined over K through X has an equation given by the determinant of a 3×3 matrix of linear forms defined over K. In particular a smooth irreducible cubic surface defined over an algebraically closed field K contains three lines cutting in two points, or an irreducible conic and a line cutting at a point, so that it has an equation given by the determinant of a 3×3 matrix of linear forms defined over K.*

PROOF. Such an X is either irreducible so that it is a twisted cubic, or it has 2 or 3 irreducible components. In the last case the irreducible components are lines, which can intersect only at 2 points (over $K = \overline{K}$), because otherwise X would be planar. In the remaining case it consists of a line and an irreducible conic, intersecting only at a point, because otherwise the curve X would be planar. It is now immediate to see that the above curves are defined by the minors of a 2×3 matrix of linear forms on $\mathbb{P}^3_K$, or deduce it from the fact that are suitable linear sections of the Segre 3-fold, $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Moreover, clearly $SX = \mathbb{P}^3$.

If $K = \overline{K}$, then it is a well known elementary geometric fact that a smooth cubic surface $S \subset \mathbb{P}^3_K$ contains two skew lines l_1, l_2. By cutting S with a plane passing through a line l_1 we get a reduced curve of degree 3, $l_1 \cup C$. Then $C \cup l_2$ is a curve in S as the above X so that the result follows. □

In conclusion a smooth cubic surface over an algebraically closed field K has always an equation given by the determinant of a 3×3 matrix of linear forms on $\mathbb{P}^3_K$, a classical result known to Grassmann and Clebsch, see [Ga], [Cl2], [SR], pg. 123 and also [Bea], 6.4. Here the proof is completely elementary and, for example, we did not use the stronger fact that a smooth cubic surface over an algebraically closed field is the blow-up of $\mathbb{P}^2$ in 6 points. In fact we shall immediately see that this fact is a consequence of the above analysis, as suggested in [SR] pg. 123. The extension to arbitrary fields was considered in [Se1] and also in [Bea], 6.5. Surely for smooth cubic surfaces the point is to show geometrically the existence of the curve without knowing a priori the plane representation of the cubic surface. The morphism to $\mathbb{P}^2$, i.e. the plane representation for the ones which are not cones, is obtained by solving the linear system 5.4.1. We give a direct easy proof of it.

Let $N = 2$ and $M = 5$ and let notations be as above. Let $\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ be the associated completely subhomaloidal linear system, by abusing language, of quadrics through $Y = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Modulo projective transformations we can suppose that $A(\mathbf{x})$ is the following 3×2 matrix of linear forms on $\mathbb{P}^5$, whose minors give rise to Y:

$$\begin{pmatrix} x_0 & x_3 \\ x_1 & x_4 \\ x_2 & x_5 \end{pmatrix}.$$

The 2×6 matrix $B(\mathrm{y})$ is

$$\begin{pmatrix} y_0 & y_1 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_0 & y_1 & y_2 \end{pmatrix},$$

so that every fiber of $\widetilde{\phi} : \widetilde{\mathbb{P}^5} \to \mathbb{P}^2$ is a $\mathbb{P}^3$ cutting Y along a quadric surface.

Suppose from now on that $S \subset \mathbb{P}^3$ is a smooth cubic surface, defined over k and containing a curve $X \subset \mathbb{P}^3$ as above. Let us restrict $\widetilde{\phi} : \widetilde{\mathbb{P}^3} \to \mathbb{P}^2$ to $\widetilde{S} \simeq S$. This yields a morphism $\sigma : S \to \mathbb{P}^2$. By the above description of the fibers, we get that σ is birational since a general secant line to X is not contained in S and it cuts S in a point outside X. Moreover, the positive dimensional fibers of σ are the secant lines to X contained in S. Hence they are finite in number since σ is dominant, disjoint and are (-1)-curves by adjunction formula. Therefore σ is the inverse of the blow-up of $\mathbb{P}^2$ at a finite number of distinct points $p_1, \ldots, p_m$, defined as scheme over k. The inverse map $\sigma^{-1} : \mathbb{P}^2 \dashrightarrow S$ is clearly given by cubic forms passing simply through $p_1, \ldots, p_m$, so that $m = 6$. No three are collinear and that they do not lie on a smooth conic because S is smooth.

In conclusion we have given an elementary proof of the following theorem of B. Segre, see also [**Bea**], 6.5.

5.4.3. THEOREM. ([Se1]) *Let $S \subset \mathbb{P}^3_K$ be a smooth, geometrically irreducible cubic surface defined over an arbitrary field K. The following conditions are equivalent:*

i) *$S \subset \mathbb{P}^3_K$ has an equation given by the determinant of a 3×3 matrix of linear forms on $\mathbb{P}^3_K$;*
ii) *S contains a degree 3 geometrically reduced and geometrically connected non-degenerate curve defined over K;*
iii) *S is the blow-up of $\mathbb{P}^2$ at six distinct points defined over K;*
iv) *S contains a k-point and a set of six disjoint lines defined over k.*

Continuing in this direction one could discuss the rationality and the representation on $\mathbb{P}^4$ of quartic determinantal hypersurfaces through a Bordiga surface (or one of its degenerations) and so on. This is treated by different methods by Room, [**Ro**] chapters XV and XVI. We prefer to describe briefly an interesting relation between varieties with one apparent double points and rational smooth cubic hypersurfaces, generalizing corollary 5.4.2.

Let us begin with a general discussion on the rationality of smooth cubic hypersurfaces. Let $X^3_m \subset \mathbb{P}^{m+1}$ be a smooth cubic hypersurface. If $m = 1$, then the smooth cubic curve $X^3_1 \subset \mathbb{P}^2$ is not rational and neither unirational. For $m \geq 2$, Max Nöether proved that a smooth cubic hypersurface $X^3_m \subset \mathbb{P}^{m+1}$ is unirational over the field of definition of a line contained in it. Recently this was generalized by Kollár, see [**K2**], who showed that a cubic hypersurface X is unirational over a field K if and only if $X(K) \neq \emptyset$.

If K is algebraically closed, then it is well known, a short and simple proof being furnished above, that a smooth cubic surface over K is rational. As in the above proof one can argue that $X^3_2 \subset \mathbb{P}^3$ contains two skew lines L_1 and L_2. Then one defines directly a rational map $\psi : L_1 \times L_2 \dashrightarrow X^3_2$ by associating to the point (p, q) the third point of intersection between the line $< p, q >$ and X^3_2. Since through every point of $\mathbb{P}^3 \setminus (L_1 \cup L_2)$ there passes only one line intersecting L_1 and L_2, the map ψ is birational and hence X_2^3 is rational. Over an arbitrary field K, if X_2^3 contains two lines L_1 and L_2 defined over K, then ψ would be defined over K and X_2^3 would be rational over K.

Shepherd-Barron in [**SB3**] gave a geometrical proof that a smooth quintic Del Pezzo surface X_2^5 defined over an arbitrary field K is K-rational (a result known to Enriques, see [**E2**], and also proved by Swinnerton-Dyer in [**SD**]), by using the fact that such a surface projects birationally onto a cubic surface with two skew lines defined over K. His method is based on the study of the subhomaloidal systems of quadrics through X_2^5 and closely related to some results reported here. Let us briefly recall his proof.

It is easy to see that $X_2^5 \subset \mathbb{P}^5$ is the scheme-theoretic intersection of 5 quadrics giving, after blowing-up X_2^5, a morphism $\phi : Bl_{X_2^5}(\mathbb{P}^5) \to \mathbb{P}^4$ whose general fiber is a secant line to X_2^5. Since ϕ is defined over K, a general secant line $l \subset \mathbb{P}^5$ cuts X_2^5 in two points p_1 and p_2, "defined over K". The projection of X_2^5 from l gives a birational morphism of $X_2^5 \setminus \{p_1, p_2\}$ onto the image which extends to an isomorphism between the blow-up of X_2^5 in p_1 and p_2 and a cubic surface $X_2^3 \subset \mathbb{P}^3$. Since p_1 and p_2 are "defined over K", the resulting skew lines on X_2^3 are "defined over K". Thus X_2^3 is K-rational by the above construction and X_2^5 will be K-rational.

A fundamental theorem of Clemens and Griffiths, [**CG**], shows that, over the complex field, a smooth cubic hypersurface in $\mathbb{P}^4$ is not rational. On the other hand, in every space of odd dimension, let us say $\mathbb{P}^{2n+1}$, there exist examples of rational smooth cubic hypersurfaces. Indeeed there exist smooth irreducible cubic hypersurface containing a pair of skew $\mathbb{P}^n$, which, by generalizing the construction given above for $n = 1$, are rational. One immediately realizes that such hypersurfaces are special and special when n increases. A pair of skew $\mathbb{P}^n$ is an example of reducible smooth variety with one apparent double point and the same construction can be generalized to smooth cubics hypersurfaces of $\mathbb{P}^{2n+1}$ containing a smooth irreducible variety with one apparent double point, see [**Ru1**]. Let us recall the details and make some historical remarks.

Let $X \subset \mathbb{P}^{2n+1}$ be a (smooth) cubic hypersurface containing a (smooth) variety W of dimension n. We can define a rational map

$$\psi : W^{(2)} \dashrightarrow X,$$

which to the distinct points $p, q \in W$ associates the third point of intersection of the line $< p, q >$ with X.

Taking a general point $x \in X \setminus W$. The point x is in the image of ψ when it lies on a secant line to W. The number of preimages of x by ψ is equal to the number of secant lines passing through x. Thus ψ is finite and dominant if the generic projection of W from $x \in X$ has a finite (non zero) number of double points.

In particular if W has only one apparent double point, then through the general point of X there passes only a secant line to W, since the fundamental locus of the birational map $p_2^{-1} : \mathbb{P}^{2n+1} = SW \dashrightarrow S_1W$, see 5.1 for the notation, has codimension at least 2. Moreover $W^{(2)}$ is rational since from $p_1 \circ p_2^{-1}$ we get a dominant rational map $\mathbb{P}^{2n+1} \dashrightarrow W^{(2)}$ with fiber a general secant line, so that a restriction to a general $\mathbb{P}^{2n} \subset \mathbb{P}^{2n+1}$ yields the desired birational isomorphism. Therefor if $\nu(W) = 1$, X is a rational smooth cubic hypersurface.

When the variety $W \subset \mathbb{P}^{2n+1}$ is defined by quadratic equations defining a special subhomaloidal system on $\mathbb{P}^{2n+1}$, whose general fiber is a secant line to W, the above map is simply the restriction to $X \subset \mathbb{P}^{2n+1}$ of the map $\phi_{|H^0(\mathcal{I}_W(2))|}$: $\mathbb{P}^{2n+1} \dashrightarrow Z \subseteq \mathbb{P}(H^0(\mathcal{I}_W(2)))$, inducing a birational isomorphism between X and Z. It is clear that in this case Z is a rational variety birational to $W^{(2)}$.

We proved the following classical and rather well known result, generalizing corollary 5.4.2.

5.4.4. PROPOSITION. *Let $X \subset \mathbb{P}^{2n+1}$ be a smooth irreducible cubic hypersurface defined over a filed K, containing a (smooth irreducible) variety W defined over K and having one apparent double point. Then X is rational over K.*

5.4.5. REMARK. The above idea for proving rationality of some cubic hypersurfaces in $\mathbb{P}^{2n+1}$ is classical, see for example [**Rt**], pg. 101.

Morin erroneously claimed that a general smooth cubic hypersurface in $\mathbb{P}^5$ contains a rational normal scroll of degree 4 and hence it is rational (see [**Mr**]). Fano then proved that if a smooth cubic hypersurface $X_4 \subset \mathbb{P}^5$ contains a rational normal scroll of degree 4, then it contains also a Del Pezzo surface of degree 5 (and viceversa) and that the family of such surfaces contained in X_4 has dimension 2, respectively 5, contrary to the "expected" dimension 1, respectively 4.

Thus the locus of smooth cubic hypersurfaces in $\mathbb{P}^5$ containing smooth surfaces with one apparent double point has codimension one in the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^5$ (see [**Fn**], [**Rt**], pg. 101, [**Tr2**] and [**Bea**], propositon 9.2 a) and b)). Let us recall that the locus of smooth cubic hypersurfaces in $\mathbb{P}^5$ containing a pair of skew planes has codimension two in the corresponding moduli space.

Fano also constructs explicit birational maps from a smooth cubic hypersurface $X \subset \mathbb{P}^5$ as above to $\mathbb{P}^4$, considering (the restriction to X of) the linear systems of quadrics through the surfaces S (see [**Fn**], [**Tr1**] and [**AR3**]).

Tregub describes a different codimension two locus in the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^5$, which parametrizes cubic hypersurfaces containing a Veronese surface and a plane intersecting in three points and shows that each of these cubic hypersurfaces is rational with a slight modification of the above construction (see [**Tr2**]).

Let Π be the divisor of smooth cubic hypersurfaces of $\mathbb{P}^5$ parametrizing smooth cubic hypersurfaces containing a plane. Recently Hasset constructed a countably infinite collection of divisors in Π, each of which parametrizes rational cubic hypersurfaces (see [**Hs**]).

Other example of smooth cubic hypersurfaces in $\mathbb{P}^7$ and the rational representations of the cases presented above is contained in the recent paper [**AR3**].

Gaetano Scorza

In this section we transcribe some excerpts from the obituary of Gaetano Scorza, written by Luigi Berzolari and published at the beginning of the first volume of the "Opere Scelte" di Gaetano Scorza, [S6]. As it will be transparent from the words of Berzolari, the work of Gaetano Scorza is very deep, vast and dealt with various themes including (abstract) algebra and geometry. His contributions has not been yet put in the right light and only recently his pioneering work on secant defective variety was appreciated by modern projective geometers. At the end of his life he wrote an interesting book on group theory, which was published posthumously, [S5].

"Nel pomeriggio del 6 agosto di quest'anno, dopo lunga malattia sopportata con cristiana rassegnazione, spegnevasi in Roma all' etá di 62 anni, il senatore prof. BERNARDINO GAETANO SCORZA, ordinario di Geometria analitica, con elementi di proiettiva e Geometria descrittiva con disegno, in quell' Universitá, il quale era Socio Corrispondente del nostro Istituto [n.d.c(nota del copiatore) Reale Lombardo di Scienze e Lettere] sin dal giugno 1922.

Gravi sventure l' avevano colpito in questi ultimi anni. Il 3 dicembre 1934 eragli improvvisamente mancata la diletta consorte ANGELA DRAGONI, giá Sua compagna di Studi all' Ateneo pisano; il 26 giugno 1937 un morbo crudele, ribelle ad ogni tentativo della scienza, aveva troncato la fiorente esistenza del figlio DINO, appena ventisettenne, da poco salito alla cattedra di Diritto Commerciale nell'Universitá di Bari, al quale il vivido ingegno sembrava assicurare il piú brillante avvenire.

Nello strazio del Suo cuore, trasse conforto dall'affetto e dalle cure amorose dei figli superstiti, e, com'ebbe a scrivermi in altra recente dolorosa circostanza, dalla fede, profondamente sentita, in una superiore esistenza ultraterrena.

Forte e aitante nella persona, nel perfetto equilibrio di ogni facoltá, nulla avrebbe potuto far presagire una Sua prossima fine. Ma le dure prove subite avevano forse indebolita la Sua fibra e resa meno vigorosa la Sua reazione alla violenza del male.

Se l'immatura scomparsa di GAETANO SCORZA getta nel lutto la nostra scienza, dove nei vari campi dell' Algebra e della Geometria Egli aveva impresso orme durature, non meno pungente é il cordoglio che grava sull'animo di quanti ebbero la fortuna di avvicinarlo, e ne ricordano con intensa commozione l'austera dignitá della vita, il fervido patriottismo, la saldezza e il calore nelle amicizie, la signorile semplicitá dei modi, l'ampiezza della cultura scientifica e letteraria, il finissimo intuito da cui era guidato verso ogni manifestazione del bello e del vero.

Nato a Morano Calabro (in provincia di Cosenza) il 29 settembre 1876, frequentó le scuole medie in parte al Collegio Nazareno di Roma, in parte alle Scuole pie degli Scolopi di Firenze. Nel 1898 conseguí con lode la laurea in Matematica all'Universitá di Pisa, e lá si trattenne durante il 1898-99 come assistente di Geometria analitica e proiettiva. Nell'anno successivo, per uno scambio di posti di assistente fatto da Lui e da ALBERTO TANTURRI in accordo con i loro Maestri rispettivi EUGENIO BERTINI e CORRADO SEGRE, fu assistente di Geometria proiettiva e descrittiva all' Universitá di Torino, ció che Gli diede agio di avvicinare il SEGRE e di vederne in atto le alte doti d'insegnante. Alla fine del 1900 fece ritorno a Pisa, e in quella R. Scuola Normale Superiore ottenne l'abilitazione all'insegnamento, coprendo, sino al 1902, l'ufficio di assistente.

Passato alle Scuole medie, insegnó successivamente negli Istituti tecnici di Terni, Bari e Palermo; nel 1907 prese la libera docenza in Geometria proiettiva e descrittiva, e nel 1912-13, in seguito a concorso, fu nominato straordinario di queste discipline all' Universitá di Cagliari. Si trasferí l'anno appresso alla stessa cattedra dell'Universitá di Parma; dal 1916-17 al 1920-21 fu a Catania con l'insegnamento di Geometria analitica e proiettiva, poi sino al 1934-35 a Napoli con quello di Geometria analitica, dopo di che fu chiamato all' Universitá di Roma.

Nelle varie sedi ebbe altresí incarichi di materie cosí del primo come del secondo biennio: Analisi algebrica, Analisi infinitesimale, Geometria superiore, Matematiche complementari. Meritano, tra gli altri, di essere ricordati i corsi svolti a Roma sulla teoria dei corpi numerici e delle algebre, sulla teoria dei numeri e su quella dei gruppi e delle equazioni algebriche.

Ampi riconoscimenti dei Suoi meriti eminenti di uomo e di scienziato ebbe nella nomina a membro del Consiglio superiore della pubblica istruzione (1923-1932), a Presidente del Comitato Matematico nel Consiglio nazionale delle ricerche (1928-1931), a Vice-presidente della Commissione internazionale per l'insegnamento matematico, a membro della Commissione internazionale per la cooperazione intellettuale della Societá delle Nazioni, della Commissione scientifica dell'Unione matematica italiana, e della Commissione che preparó la prima edizione del testo unico per le scuole elementari, a Socio di numerose nostre Istituzioni scientifiche.

Conseguí nel 1909 il premio ministeriale della R. Accademia dei Lincei per le Scienze matematiche, e nel 1922 Gli fu conferita la medaglia d'oro della Societá italiana delle Scienze, detta dei XL. Nel giugno di quest'anno [n.d.c. 1939] era stato nominato Senatore del Regno.

Della larghezza della Sua cultura si ha un chiaro riflesso nella varietá degli argomenti che formano oggetto degli scritti da Lui dati alle stampe.

Di tali scritti, tre, del 1902 e del 1903, concernono questioni di Economia politica, e un altro é il testo della conferenza sul principio di causalitá e sulle applicazioni della Matematica alle scienze sociali, tenuta al Congresso della "Mathesis del 1921.

Parecchi sono necrologie di nostri matematici, o recensioni di opere spettanti alla matematica, alla filosofia, alla letteratura, o conferenze sulla matematica in rapporto allo sviluppo storico di alcune sue teorie, alla filosofia, all'arte, alla didattica.

Altri ancora, originati dal Suo insegnamento nelle Scuole medie, si riferiscono a particolari questioni relative agli elementi dell'algebra e della geometria, e con i trattati di cui si dirá tra poco, fanno fede della scrupolosa serietá con la quale Egli sempre consideró i vari aspetti del proprio ufficio di maestro.

Le pubblicazioni rimanenti, in numero di circa 70, appartengono tutte alle parti piú elevate della Matematica pura e Gli assicurano un posto assai onorevole nella storia della Matematica italiana dell'ultimo quarantennio.

Chi si accinga allo studio di codesti lavori rimane subito colpito - oltre che dall'importanza dei risultati, conseguiti il piú delle volte come frutto di generali vedute unificatrici, alle quali fa singolare riscontro l'esauriente finitezza di ogni particolare - dalla succinta eleganza dei procedimenti cosí sintetici come algoritmici, rappresentati con tanto cristallina limpidezza di forma da costituire una vera opera d'arte. E invero alla vigoria del pensiero si accompagnava in Lui un temperamento schiettamente estetico, del quale si hanno manifestazioni assai significative pur nei trattati destinati all'insegnamento medio e all'insegnamento superiore.

Queste doti si ravvisano giá nella Dissertazione di laurea (1898), in cui vien ricostruita in forma semplice, e con aggiunte notevoli, la teoria delle figure polari delle curve piane algebriche e si approfondisce il caso delle curve del quart'ordine.

Collegata con essa, una breve Nota dell' anno successivo fa conoscere una bella proprietá relativa al "covariante S" di una quartica piana, mentre altri contributi alla teoria delle quartiche trovansi in un gruppo di lavori (1900, 1901 e 1907), nei quali l'Autore si propone di estendere la teoria delle corrispondenze algebriche biunivoche tra i punti di una curva ellittica alle corrispondenze algebriche di indici p, p esistenti sopra una curva di genere p a moduli generali.

Problemi di tutt'altra natura sono affrontati in pubblicazioni di poco posteriori.

In due Note del 1908 e del 1909 si ricerca come possa estendersi alle varietá di tre o di quattro dimensioni di uno spazio S_r (con $r \geq 7$ e rispettivamente con $r \geq 9$) il teorema del DEL PEZZO, secondo il quale la superlcie di VERONESE é la sola superficie di S_r (con $r > 4$), che non sia un cono e i cui piani tangenti s'incontrino a due a due. Attraverso ingegnose discussioni, l'estensione é ottenuta in modo esauriente per le varietá di tre dimensioni, con qualche restrizione per quelle di quattro dimensioni.

Nello stesso torno di tempo lo SCORZA assegnó tutti i tipi di superficie di un S_r (con $r > 5$), per le quali avviene che l'S_5 ad esse tangente in due punti generici risulti tangente in altri ∞^1 punti; e in aggiunta a risultati precedenti del CASTELNUOVO e dell'ENRIQUES sulle superficie e sulle varietá tridimensionali di cui le curve sezioni siano ellittiche, determinó in modo completo tutte le varietá di dimensione arbitraria dotate di tale proprietá. Fin dal 1890 il CASTELNUOVO aveva assegnato diverse classi di superficie non rigate, a sezioni piane (o iperpiane) di genere 3, e la loro rappresentazione sul piano semplice o doppio. Dieci anni dopo, il CASTELNUOVO e l'ENRIQUES, in una Memoria comune, stabilirono il teorema fondamentale, che una superficie di ordine superiore a 4, a sezioni di genere 3, é razionale, o rigata, o birazionalmente equivalente a una rigata di genere 1 oppure 2. Ponendo a base questo teorema, lo SCORZA, in due elaborate Memorie (1909 e 1910), condusse a compimento la determinazione delle superficie di quel tipo, e ne studió la rappresentazione sul cono cubico ellittico per mezzo di trasformazioni birazionali dello spazio.

Ma l'argomento che lo SCORZA ha coltivato con maggior predilezione, e al quale, con dovizia di mezzi geometrici, algebrici e aritmetici, ha fatto compiere progressi essenziali, é quello delle funzioni abeliane, particolarmente delle funzioni abeliane singolari.

Un classico teorema, enunciato dal RIEMANN (1860) e dal WEIERSTRASS (1880), e dimostrato per la prima volta da R. POINCARÉ e E. PICARD, assegna condizioni necessarie e sufficienti perché una tabella di numeri, di $2p$ orizzontali e p verticali, possa pensarsi come la tabella di $2p$ sistemi di periodi primitivi per una funzione abeliana a p variabili indipendenti. Di esso lo SCORZA (1913) diede una nuova dimostrazione estremamente semplice e perspicua, generalizzando un teorema che G. BAGNERA e M. DE FRANCHIS avevano stabilito per il caso $p = 2$.

Della teoria delle funzioni abeliane singolari erano scarsi i risultati generali giá noti, e soltanto il caso iperellittico a due variabili era stato oggetto di profonde ricerche da parte di G. HUMBERT, e, dal punto di vista geometrico, studiato in modo esauriente nelle Memorie premiate di F. ENRIQUES e F. SEVERI e di G. BAGNERA e M. DE FRANCHIS. Nelle ricerche di HUMBERT aveva ufficio essenziale la considerazione di un certo invariante, dotato della propriet fondamentale che a seconda del segno attribuitogli per definizione, esso é un numero (intero) essenzialmente positivo o essenzialmente negativo.

Ma i procedimenti prevalentemente aritmetici seguiti dal geometra francese lasciavano ben poca speranza che si potessero estendere alle funzioni abeliane singolari con un numero qualunque di variabili indipendenti. Nel cercare di raggiungere codesta estensione, lo SCORZA (1914) riprese dapprima lo studio del caso iperellittico, e partendo dal teorema di esistenza per le funzioni iperellittiche nella forma semplice data da G. BAGNERA e M. DE FRANCHIS nella Memoria premiata, mostró come per mezzo di un'opportuna rappresentazione geometrica il teorema di HUMBERT si riducesse ad una propriet elementare delle quadriche a punti ellittici.

Guidato da un'analoga veduta, due anni appresso pervenne nel modo piú luminoso alla desiderata estensione, mediante una rappresentazione geometrica, con lá quale tutti i problemi di esistenza delle funzioni abeliane singolari sono ridotti a problemi d'indole piú elementare, relativi all'esistenza di sistemi lineari, formanti gruppo, di omografie razionali di un iperspazio, aventi come unito un determinato spazio in esso contenuto.

Fondamento della ricerca fu un' interpretazione geometrica del teorema d' esistenza delle funzioni abeliane a un numero qualunque di variabili, e il risultato fu conseguito attraverso ingegnose considerazioni di Geometria proiettiva iperspaziale e una delicata discussione intorno alle proprietá topologiche di una certa ipersuperficie, che é una particolare varietá di SEGRE.

La considerazione delle operazioni degeneri Lo condusse ad approfondire, in una serie di Note del 1915 e 1916, lo studio degli integrali abeliani riducibili appartenenti ad una varietá algebrica, e, tra l' altro, a dare dimostrazioni mirabilmente semplici di classici teoremi del PICARD e del POINCARÉ.

Il ricorso ad una rappresentazione iperspaziale per lo studio degli integrali abeliani riducibili era giá stato attuato l'anno precedente dal SEVERI. Essa gli aveva permesso di precisare il concetto di sistema regolare di integrali riducibili appartenenti ad una varietá algebrica, e quelli del sistema congiungente e del sistema intersezione di due tali sistemi come sistemi ancora regolari, e lo aveva condotto a dare una dimostrazione assai semplice ed elegante di uno dei teoremi del POINCARÉ, ed anzi un' ampia generalizzazione del medesimo.

Ad ulteriori risultati, per mezzo di convenienti rappresentazioni geometriche (che possono essere di vario tipo), pervennero simultaneamente il ROSATI e lo

SCORZA: il primo in relazione con la teoria delle corrispondenze algebriche tra i punti di una curva algebrica, e prendendo a fondamento una rappresentazione iperspaziale delle note equazioni di HURWITZ. Lo SCORZA, ponendosi da un punto di vista piú generale, in una magistrale Memoria del 1916 mise in luce il fondo aritmetico comune alla teoria degli integrali riducibili e alle teorie affini (trasformazione delle funzioni abeliane, funzioni abeliane a moltiplicazione complessa, corrispondenze algebricbe tra curve algebriche) e mostró come tutte s' inquadrino in una medesima teoria, che chiamó delle matrici di RIEMANN (matrici che possono pensarsi come tabelle dei periodi per un corpo di funzioni abeliane).

Di esse lo SCORZA fece uno studio sistematico profondo, introducendo, accanto al genere, i due caratteri che disse "indice di singolaritá" e "indice di moltiplicabilitá", e assegnando i limiti entro i quali possono variare e le relazioni da cui sono legati.

Stabilita la distinzione di quelle matrici in pure e impure, e la conseguente definizione di asse di una matrice impura, investigó a fondo la configurazione di tali assi, e mostró come il problema fondamentale della teoria - che consiste nella determinazione di tutte le possibili matrici di RIEMANN - si riconduca a quello della determinazione delle matrici riemanniane pure. Introdotto poi il concetto di pseudoasse di un' arbitraria matrice riemanniana, ne pose in rilievo l' importanza nello studio della matrice stessa.

Nella seconda parte della Memoria fece l'applicazione dei risultati della prima ad uno studio esauriente del gruppo delle trasformazioni birazionali di una superficie iperellittica in sé, assegnando per ciascun tipo di tali superficie le propriet fondamentali del relativo gruppo di trasformazioni.

Poiché un sistema lineare di omografie, formante un gruppo, puó interpretarsi come un' algebra di numeri complessi a piú unitá, in un' ampia Memoria del 1921 lo SCORZA presentó una succinta esposizione sistematica di questo argomento, e ne fece notevolissime applicazioni alle funzioni abeliane, esaurendo altresí l'esame di talune questioni, che nei lavori precedenti non era riuscito a risolvere. Appunto dalla teoria delle algebre fu condotto all' introduzione di un nuovo carattere fondamentale di una matrice riemanniana, che chiamó *rango* della matrice e che nelle ricerche accennate compie un ufficio essenziale.

La teoria generale delle algebre, largamente studiata all'estero, ha tuttora in Italia scarsi cultori, e a partire dal 1921 fu l'oggetto pressoché esclusivo delle ricerche dello SCORZA. Risale appunto a quell'anno la pubblicazione di un Suo poderoso trattato, nel quale la teoria é presentata, con la consueta elegante perspicuitá e compiutezza, in un' esposizione dove sono raccolti in armonica unitá tutti i risultati piú essenziali conseguiti per vie disparate dai ricercatori precedenti, e numerosi nuovi ne sono aggiunti, dovuti all'Autore medesimo.

Tra gli ultimi meritano particolare rilievo quelli che si riferiscono alle algebre, reali o complesse, legate ai gruppi di ordine finito, e furono anche oggetto di Suoi lavori di poco posteriori.

Altri scritti contengono ricerche varie sia sulla teoria generale delle algebre, sia su classi notevoli di esse (particolarmente sulle cosiddette "algebre pseudonulle"), inoltre la completa classificazione delle algebre del terzo e del quarto ordine, qualunque sia il corpo numerico nel quale esse s'intendano definite. Né é da omettere il ricordo della bella osservazione (1926), secondo la quale - per mezzo di proprietá dei corpi numerici a cui é dedicata la prima parte del trattato - la formola di M.

CIPOLLA esprimente la risoluzione apiristica delle congruenze binomie puó dedursi, in un corpo numerico finito arbitrario, dalla classica formola d'interpolazione del LAGRANGE. In un'elegante Memoria del 1936 viene studiata, per un'algebra "regolare", reale o complessa, una rappresentazione geometrica, che é in intima connessione con una notevole classe di variet di SEGRE, e, tra altro, conduce ad una proprietá di queste varietá, che é del tutto analoga al classico teorema del LIE sulla superficie di STEINER, esteso da vari Autori alla superficie di VERONESE, e dallo stesso SCORZA (1935) a tutte le varietá di VERONESE.

Questo rapido cenno intorno all'attivitá scientifica dello SCORZA sarebbe troppo incompleto se non facesse menzione di un altro argomento, a cui da lungo tempo Egli aveva consacrato studi perseveranti. Alludo alla teoria dei gruppi, d'ordine finito o no, sulla quale, com'é avvertito al principio di una Nota del 1927, intendeva redigere un trattato, che avrebbe dovuto contenere anche il frutto di Sue ricerche personali. Di queste, qualcuna ha formato oggetto di brevi Note a partire dal 1926, particolarmente di quella testé citata e di altre due successive, che recano utili complementi a ricerche del CIPOLLA sulla struttura dei gruppi di ordine finito.

Desta tristezza il ricordo del rammarico, da Lui piú volte espressomi, perché altre cure spesso Lo distogliessero dal compimento del Suo lavoro. É da augurare che dalle carte da Lui lasciate si trovi modo di estrarre e render di pubblica ragione almeno una parte di quell'opera. Sarebbe questo il piú degno omaggio alla memoria dell'Uomo insigne, che cosí nella vita privata come nella vita pubblica é stato esempio costante di integra probitá, e con gli scritti, ispirati ad un'altissima concezione del lavoro scientifico, ha onorato il nome italiano."

Luigi Berzolari

Alessandro Terracini

In this section we transcribe some excerpts from the obituary of Alessandro Terracini, written by Eugenio Togliatti and published on the Bollettino dell' Unione Matematica Italiana, [**Tg**], in order to let known more informations about the life and work of Alessandro Terracini. In particular, his Lemma, for which he is mostly remembered today, was written, when 22 years old, as a part of his undergraduate thesis at the end of the 4^{th} year of study at the University of Torino.

" Il 2 aprile 1968, dopo lunga e penosa malattia, mancava a Torino ALESSANDRO TERRACINI, professore emerito di geometria all' Universitá di Torino. Era nato il quella cittá il 19 ottobre 1889.

La sua vita fu piuttosto movimentata. Laureato a Torino il 5 luglio 1911, fu assunto subito come assistente alla cattedra di geometria proiettiva dell' Universitá di Torino, tenuta allora da GINO FANO. Durante la guerra 1915–18 fu ufficiale del genio in varie sedi, tra cui il fronte a Gorizia.

....Nel 1919, finita la guerra, si trasferí all' Universitá di Modena chiamatovi da ERMENEGILDO DANIELE, come assistente e incaricato di analisi algebrica; ritornando a Torino nell' autunno del 1923 come assistente e come incaricato all' Universitá e al Politecnico. Nel febbraio del 1925, in seguito a concorso, fu nominato professore di geometria analitica a Catania, cattedra che egli preferí a quella di geometria superiore dell' Unoversitá di Cagliari, per la quale aveva pure vinto il concorso. Dopo pochi mesi, alla fine dello stesso anno 1925, fu trasferito all' Universitá di Torino, come titolare della cattedra di geometria (analitica con elementi di proiettiva e geometria descrittiva con disegno), che tenne, insieme con svariati carichi di insegnamento, fino al collocamento a riposo (1° novembre 1964); salvo una interruzione di quasi 10 anni, a causa delle persecuzioni razziali, dal 1938–39, quando fu allontanato brutalmente dall' insegnamento, fino al febbraio 1948. In questi anni fu professore all' Univeristá di Tucuman [n.d.t. Argentina].

....Nel febbraio-marzo 1968 avviene la pubblicazione dei "Selecta", [**T3**], ove sono ristampate 63 delle sua circa 176 pubblicazioni scientifiche. La scelta é stata fatta da lui stesso. I due volumi dovrebbero dovuto costituire un omaggio di amici e di colleghi alla sua eccezionale attivitá di Maestro e di Scienziato in occasione del compimento del suo 80° anno di etá; una sorte dolorosa ha voluto diversamente.

ALESSANDRO TERRACINI appartiene alla scuola matematica torinese, a quella scuola ben nota, che tanto ha operato in tutti i rami della nostra scienza, e che vantava quando egli é entrato studente all' universitá, i nomi di CORRADO SEGRE, GINO FANO, ENRICO D' OVIDIO, GIUSEPPE PEANO, CARLO SOMIGLIANA, GUIDO FUBINI. Il rispetto riverenziale del Nostro per sí illustri maestri traspare nel volume dei "Ricordi", [**T4**], nei capitoli a ciascuno di essi singolarmente dedicati e che ne tratteggiano magistralmente le figure.

Scorrendo l' elenco delle sue pubblicazioni, che si estendono senza interruzioni per un sessantennio, dal 1908 al 1967, si notano innanzittutto la grande quantitá e varietá delle questioni alle quali egli ha rivolto la sua attenzione; prevalgono tuttavia in modo decisivo le ricerche di geometria differenziale, sia metrica, sia, e soprattutto, proiettiva. Intorno al 1908 era la geometria differenziale proiettiva un campo di ricerche geometriche di formazione recente, prima e molto significativa realizzazione, nel campo della geoemtria differenziale, del programma di Erlangen.
......

.....É ben noto che, a partire da questi inizi, la geometria differenziale proiettiva ebbe, nei primi decenni di questo secolo, uno sviluppo assai rapido, al quale matematici italiani hanno portato un contributo sostanziale con caratteristiche proprie e svariate; basti ricordare, oltre i nomi giá citati di CORRADO SEGRE e di GUIDO FUBINI, quelli di TERRACINI appunto e di ENRICO BOMPIANI. .....

.... Gi nel 1926-27 compariva il trattato di FUBINI e CECH su questa materia con appendici di ŢIŢEICA, di BOMPIANI e di TERRACINI, seguito ben presto dalla bella edizione francese che consente un rapido e facile orientamento nel nuovo campo di studi.

L'argomento assegnato da C. SEGRE a TERRACINI per la sua tesi di laurea rientra in quest'ordine d'idee, e ha dato luogo ad alcuni lavori suoi, comparsi tra il 1911 ed il 1912, e coi quali egli assume subito una posizione di ricercatore di prima linea; essi riguardano alcuni tipi particolari di variet´ iperspaziali (che hanno gli spazi tangenti mutuamente secanti, o che rap preslntano molte equazioni di LAPLACE, o che sono luoghi di spazi con carattere di sviluppabili). Da questi primi lavori appaiono giá le tendenze caratteristiche del suo modo di lavorare in questo campo, che si ritroveranno poi sempre in seguito. .....

.... i metodi di TERRACINI sono piú vicini a quelli di CORRADO SEGRE, di cui era allievo, anche se egli, nei "Ricordi", dichiara di sentirsi allievo pure di FUBINI, del quale non aveva mai seguito i corsi. Ed infatti, pur servendosi sia di forme differenziali che di sistemi di equazioni alle derivate parziali, egli dá sempre molto peso alla visione geometrica dei problemi, alla quale si collega in armonica collaborazione lo strumento analitico con un ufficio che per lui non é solo di controllo, ma che ha anche parte costruttiva nelle ricerche. "*Un metodo* ", egli scrive, "*al quale ci si attenga fedelmente, é un'ottima guida, é, direi, come un buon monarca illuminato; ma assoluto, che alle volte rischia di diventare un tiranno; ed allora puó essere necessario cercare di evadere*". In armonia con ció, lo vediamo sempre fondere il procedimento sintetico con quello analitico, contribuendo potentemente a quella geometrizzazione della geometria differenziale proiettiva che illumina i processi prevalentemente analitici sopra ricordati (che spesso erano stati la via delle scoperte), senza assumere, all'opposto, atteggiamenti troppo esclusivamente sintetici. ....

......Infine, non si possono passare sotto silenzio alcuni studi di carattere storico......

... Son lavori che rivelano grande acume critico e scrupolositá eccezionale, nella valutazione dei documenti e delle testimonianze; vi traspare quell'amore per la storia della scienza intesa come "studio del divenire delle conquiste dello spirito, necessario per l'intelligenza del pensiero scientifico". Del resto, il senso storico affiora assai spesso nella produzione scientifica di TERRACINI. Gli stessi caratteri presentano anche i vari necrologi e le commemorazioni di: C, SEGRE, G. FUBINI, F. ENRIQUES, G. CASTELNUOVO, G. FANO, G. LORIA, U. AMALDI,

B. LEVI; nonché le numerose recensioni. Si notano talora in queste ultime, insieme coi meritati elogi per gli autori, anche critiche severe, se pur sempre obiettive, che egli, nei "Ricordi", qualifica bonariamente di " crudeltá proprie della gioventú".

.....

.....La sua opera d'insegnante é ben nota; i suoi allievi ben ricordano la limpidezza e l'efficacia delle sue lezioni, sempre diligentemente preparate su appunti manoscritti, che egli rileggeva poco prima di ciascuna lezione. Alcuni dei suoi allievi hanno raggiunto la cattedra universitaria; molti altri presentarono, da lui guidati, tesi di laurea od altri lavori che furono pubblicati. L'apprezzamento del suo insegnamento multiforme e delle sue doti didattiche appare anche nelle parole entusiaste di FELIX HERRERA nelle onoranze tributategli a Tucumán nel 1962. Le sue lezioni di geometria analitica e proiettiva restano nel ben noto trattato pubblicato in collaborazione con GINO FANO, in prima edizione nel 1929 ed in varie edizioni successive. Nel 1939 egli scrisse pure, particolare poco noto, un trattato di algebra per le scuole secondarie, recante il nome di TRICOMI, anziché il suo, a causa delle persecuzioni razziali.

Fuori del campo strettamente scientifico, la figura di ALESSANDRO TERRACINI, ben nota a tutti noi che lo abbiamo conosciuto, appare viva e direi quasi parlante nel volume dei "Ricordi" piú volte citato, coi suoi profondi affetti familiari, con la sua consuetudine di vedere la matematica anche nelle cose piú comuni della vita quotidiana, con la sua vastissima cultura che si estendeva anche al campo letterario ed artistico, con le sue sane e varie abitudini sportive.

Dalle ultime pagine di quel volume apprendiamo poi, cosa che noi, colleghi a lui pi vicini, ben sapevamo, che nei suoi ultimi anni, di fronte al mutamento profondo che il mondo universitario andava subendo, egli si rendeva conto assai bene che le necessitá di una universitá divenuta scuola di massa non potevano piú essere quelle dei tempi passati, e che era inutile "*ostinarsi a voler tenere in vita ció che morto*"; frase che segue alla dichiarazione, di sapore malinconico, che "*é forse bene che l'antica generazione di docenti vada scomparendo*". E forse si pu sentire qui un poco di senso nostalgico proveniente dalla sua scarsa "*simpatia per la cosidetta matematica moderna, o almeno per l'uso e l'abuso dell'aggettivo moderno applicato alla matematica, come se si trattasse di qualcosa di effettivamente nuovo*". Effettivamente, l'opera scientifica di ALESSANDRO TERRACINI appartiene al passato, ma ad un passato glorioso e recente, che é soltanto in apparenza cosí lontano nel tempo come sembrerebbe; non solo, ma un'opera eclettica e ricchissima come la sua, ove una potente intuizione, diciamo pure geometrica anche se questo vocabolo ha ormai solo un significato convenzionale, domina e guida con mano sicura gli strumenti di ricerca piú appropriati, rester pur sempre un esempio ed un modello ben degno."

Eugenio G. Togliatti

Bibliography

[Åd] B. Ådlandsvik, *Joins and higher secant varieties*, Math. Scand. 61 (1987), 213–222.

[AH] J. Alexander, A. Hirschowitz, *Polynomial interpolation in several variables*, J. Alg. Geometry 4 (1995), 201–222.

[AR1] A. Alzati, F. Russo, *On the k-normality of projected algebraic varieties*, Bull. Braz. Math. Soc. 33 (2002), 27–48.

[AR2] A. Alzati, F. Russo, *Special subhomaloidal systems of quadrics and varieties with one apparent double point*, Math. Proc. Camb. Phil. Soc. 134 (2003), 65-82.

[AR3] A. Alzati, F. Russo, *Some elementary extremal contractions between smooth varieties arising from projective geometry*, submitted.

[At] M. F. Atiyah, *Complex fiber bundles and ruled surfaces*, Proc. London Math. Soc. 5 (1955), 407–434.

[AW1] M. Andreatta, J. Wisniewski, *On contractions of smooth varieties*, J. Alg. Geometry 7 (1998), 253–312.

[AW2] M. Andreatta, J. Wisniewski, *Contractions of smooth varieties II. Computations and applications.*, Boll. Unione Mat. Ital. B (8) 1 (1998), 343–360.

[B1] L. Bădescu, Special chapters of projective geometry, Rend. Sem. Matem. Fisico Milano, 69 (1999-2000), 239–326.

[B2] L. Bădescu, Formal Geometry and Projective Geometry, preliminary version of a book.

[BS] L. Bădescu, M. Schenider, *A criterion for extending meromorphic functions*, Math. Ann. 305 (1996), 393–402.

[Bea] A. Beauville, *Determinantal hypersurfaces*, Mich. Math. J. 48 (2000), 39–64.

[Be] E. Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*, seconda edizione, Principato, 1923.

[BK] H. Braun, M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin and New York, 1966.

[B1] J. Bronowski, *The sum of powers as canonical expressions*, Proc. Cam. Phil. Soc. 29 (1933), 69–82.

[B2] J. Bronowski, *Surfaces whose prime sections are hyperelliptic*, J. London Mat. Soc. 8 (1933), 308–312.

[Ca] G. Castelnuovo, *Massima dimensione dei sistemi lineari di curve piane di dato genere*, Annali di Mat. 18 (1890).

[CJ] M. L. Catalano-Johnson, *The homogeneous ideal of higher secant varieties*, J. Pure Appl. Algebra 158 (2001), 123–129.

[Cy] A. Cayley, *On certain developable surfaces*, Quart. J. Pure Appl. Math. 6 (1894), 108–126.

[Ch] P. E. Chaput, *Severi Varieties*, Math. Z. 240 (2002), 451–459, or math.AG/0102042.

[Ch2] P. E. Chaput, *Scorza varieties and Jordan algebras*, math.AG/0208207

[CC] L. Chiantini, C. Ciliberto, *Weakly defective varieties*, Trans. A.M.S. 354 (2001), 151–178.

[Ci] C. Ciliberto, *Ipersuperficie algebriche a punti parabolici e relative hessiane*, Rend. Acc. Naz. Scienze 98 (1979-80), 25–42.

[CMR] C. Ciliberto, M. Mella, F. Russo, *Varieties with one apparent double point*, J. Alg. Geometry, to appear.

[CR1] C. Ciliberto, F. Russo, *On surfaces with two apparent double points*, Advances Geom., to appear.

[CR2] C. Ciliberto, F. Russo, *Castelnuovo, Enriques, Reid, Bronowski; k-secant degree, etc, etc.*, preprint Roma/Bucureşti/Recife, 2002-2003, 60 pages.

[Cl1] A. Clebsch, *Über Kurven vierter dritter Ordnung*, J. Reine Angew. Math. 59 (1861), 125–145.

[Cl2] A. Clebsch, *Die Geometrie auf den Flächen dritter Ordnung*, J. Reine Angew. Math. 65 (1866), 359–380.

[CG] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95 (1972), 281–356.

[CKM] H. Clemens, J. Kollár, S. Mori, *Higher dimensional complex Geometry*, Astérisque 166, 1988.

[CK1] B. Crauder, S. Katz, *Cremona transfromations with smooth irreducible fundamental locus*, Amer. J. Math. 111 (1989), 289–309.

[CK2] B. Crauder, S. Katz, *Cremona transformations and Hartshorne's conjecture*, Amer. J. Math. 113 (1991), 289–309.

[D1] P. Deligne, *Letters to W. Fulton*, july/november 1979.

[D2] P. Deligne, Le groupe fondamental du complément d' une courbe plane n' ayant que des points doubles ordinaries est abélien, Sém. Bourbaki, n. 543, 1979/80, L.N.M. 842, Springer -Verlag, 1981, 1–10.

[dP] P. del Pezzo *Sulle superficie dell' n-esimo ordine immerse nello spazio a n dimensioni*, Rend. Circ. Mat. Palermo 12 (1887), 241–271.

[Di] A. Dixon, *The canonical form of the ternary sextic and the quaternary quartic*, Proc. London Math. Soc. 4 (1906), 160–168.

[Do] I. Dolgachev, *Polar Cremona transformations*, Mich. Math. J., 48 (2000), 191–202.

[DC] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, New Jersey, 1976.

[Ed] W. L. Edge, *The number of apparent double points of certain loci*, Proc. Cambridge Philos. Soc. 28 (1932),285–299.

[Ew] B. A. Edwards, *Algebraic loci of dimension $r-3$ in space of r dimensions, of which a chord cannot be drawn through an arbitrary point of space*, J. London Math. Soc. 2 (1927), 155–158.

[Ei] L. Ein, *Varieties with small dual variety I*, Invent. Math. 86 (1986), 63–74.

[ESB] L. Ein, N. Shepherd-Barron, *Some special Cremona transformations*, Amer. J. Math. 11 (1989), 783–800.

[EH] D. Eisenbud, J. Harris, *On varieties of minimal degree*, Alg. Geometry Bowdin 1985, Proc. Symp. in Pure Math. 46 (1987), 3–13.

[ES] G. Ellingsrud, A. Strømme, *Bott's formula and Enumerative Geometry*, J. Amer. Math. Soc. 9 (1996), 175–193.

[E1] F. Enriques, *Sulla massima dimensione dei sistemi lineari di dato genere appartenenti a una superficie algebrica*, Atti Reale Acc. Scienze Torino 29 (1894), 275–296.

[E2] F. Enriques, *Sulle irrazionalità da cui può farsi dipendere la risoluzione di una equazione algebrica $f(x,y,z)=0$ con funzioni razionali di due parametri*, Math. Ann. 49 (1897), 1–23.

[Fa] G. Faltings, *Formale Geometrie und homogene Räume*, Inv. Math. 64 (1981), 123–165.

[Fn] G. Fano, *Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine*, Comm. Math. Helv. 15 (1942), 71–80.

[FW] G. Fischer, H. Wu, *Developable complex analytic submanifolds*, Inter. Jour. Math. 6 (1995), 229–272.

[Fr1] A. Franchetta, *Forme algebriche sviluppabili e relative hessiane*, Atti Acc. Lincei 10 (1951), 1–4.

[Fr2] A. Franchetta, *Sulle forme algebriche di S_4 aventi lhessiana indeterminata*, Rend. Mat. 13 (1954), 1–6.

[FR] T. Fujita, J. Roberts, *Varieties with small secant varieties: the extremal case*, Amer. Jour. Math. 103 (1981), 953–976.

[Fu] W. Fulton, *On the Topology of Algebraic Varieties*, Proc. Symp. Pure Math. 46 (1987), 15–46.

[FH] W. Fulton, J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. 110 (1979), 159–166.

[FL] W. Fulton, R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, L.N.M. 862, Springer-Verlag, 1981, 26–92.

[Go] N. Goldstein, *Ampleness and connectedness in complex G/P*, Trans. Amer. Math. Soc. 274 (1982), 361–373.

[GN] P. Gordan, M. Nöther, *Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet*, Math. Ann. 10 (1876), 547–568.

[Ga] H. Grassmann, *Die stereometrischen Gleichungen dritten grades, und die dadurch erzeugten Oberflächen*, J. Reine Angew. Math. 49 (1855), 47–65.

[GH] P. Griffiths, J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. Ecole Norm. Sup. 12 (1979), 355-432.

[Gr] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théormes de Lefschetz locaux et globaux*, Séminaire de Géomé trie Algébrique (1962), North Holland, 1968.

[Ha] J. Harris, *Algebraic Geometry, A First Course*, G. T. M. 133, Springer Verlag, 1992.

[H1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, L.N.M. 156, Springer-Verlag, 1970.

[H2] R. Hartshorne, *Varieties of small codimension in projective space*, Bull. A. M. S. 80 (1974), 1017-1032.

[Hs] B. Hasset, *Some rational cubic fourfolds*, J. Alg. Geom. 8 (1999), 103–114.

[He1] O. Hesse, *Über die Bedingung, unter welche eine homogene ganze Function von n unabhängigen Variabeln durch Lineäre Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurück-führen lässt, die eine Variable weniger enthält*, J. reine angew. Math. 42 (1851), 117–124.

[He2] O. Hesse, *Zur Theorie der ganzen homogenen Functionen*, J. reine angew. Math. 56 (1859), 263–269.

[Hi] D. Hilbert, *Letter adresseé á M. Hermite*, Gesam. Abh. vol. II, 148–153.

[HCV] D. Hilbert, S. Cohn-Vossen, *Geometry and the Imagination*, Chelsea Pub. Co., New York, 1952 (translated from German *Anschauliche Geometrie*, 1932).

[HKS] K. Hulek, S. Katz, F.O. Schreyer, *Cremona transformations and syzygies*, Math. Z. 209 (1992), 419–443.

[Ja] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., 39, Amer. Math. Soc., Providence, RI, 1968.

[Jo] K. Johnson, *Immersion and embedding of projective varieties*, Acta Math. 140 (1978), 49–74.

[Ju] J. P. Jouanolou, *Théorémes de Bertini et applications*, Prog. Math. 42, Birkauser, 1983.

[K1] J. Kollár, *Rational Curves on Algebraic Varieties*, Erg. der Math. 32, Springer Verlag, Germany, 1996.

[K2] J. Kollár, *Unirationality of cubic hypersurfaces*, e-print alg-geom/0005146.

[KM] J. Kollár, S. Mori, *Birational Geometry of Algebraic Varieties*, Camb. Tracts in Math. 134, Camb. University Press, Cambridge, U. K., 1998.

[La] J. M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. Jour. 85 (1996), 605–634.

[Ln] H. Lange, *Higher secant varieties of curves and the theorem of Nagata on ruled surfaces*, Manuscripta Math. 47 (1984), 263–269.

[LV] R. Lazarsfeld, A. van de Ven, *Topics in the geometry of projective space, Recent work by F. L. Zak*, DMV Seminar 4, Birkhäuser, (1984).

[Le] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Inv. Math. 136 (1999), 157–207.

[Mo1] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. Math. 110 (1979), 593–606.

[Mo2] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. 116 (1982), 133–176.

[Mr] U. Morin, *Sulla razionalità dell' ipersuperficie cubica dello spazio lineare S_5*, Rend. Sem. Mat. Univ. Padova 11 (1940), 108–112.

[Mk] S. Mukai, *Simple Lie algebra and Legendre variety*, preprint (1998), http://www.math.nagoya-u.ac.jp/~mukai

[Mu] D. Mumford, *The Red Book of Varieties and Schemes*, L. N. M. 1358, Srpinger-Verlag, 1988.

[P1] F. Palatini, *Sulle superficie algebriche i cui S_h $(h+1)$-seganti non riempiono lo spazio ambiente*, Atti Accad. Torino 41 (1906), 634–640.

[P2] F. Palatini, *Sulle varietá algebriche per le quali sono minori dell' ordinario, senza riempire lo spazio ambiente, una o alcune delle varietá formate da spazi seganti*, Atti Accad. Torino 44 (1909), 362–375.

[P3] F. Palatini, *Sulla rappresentazione delle forme ed in particolare della cubica quinaria con la somma di potenze di forme lineari*, Atti Accad. Torino 38 (1902), 43–50.

[P4] F. Palatini, *Sulla rappresentazione delle forme ternarie mediante la somma di potenze di forme lineari*, Rend. Accad. Lincei 5 (1903), 378–385.

[Pe] U. Perazzo, *Sulle varietá cubiche la cui hessiana svanisce identicamente*, Atti R. Acc. Lincei (1900), 337–354.

[Pt1] R. Permutti, *Su certe forme a hessiana indeterminata*, Ricerche di Mat. 6 (1957), 3–10.

[Pt2] R. Permutti, *Sul teorema di Hesse per forme sopra un campo a caratteristica*, Le Matematiche 13 (1963), 115–128.

[Pt3] R. Permutti, *Su certe classi di forme a hessiana indeterminata*, Ricerche di Mat. 13 (1964), 97–105.

[PW] A. Pinkus, B. Wajnryb, *A problem of approximation using multivariate polynomials*, Russian Math. Surveys 50 (1995), 319–340.

[Ra] Z. Ran, *Absence of the Veronese from smooth threefolds in* $\mathbb{P}^5$, Contem. Math. 116, 1991, 125–128.

[Re1] T. Reye, *Geometrischer Beweis des Sylvesterschen Satzes:" Jede quaternäre cubische Forme ist darstellbar als Summe von fünf Cuben linearer Formen*, J. Reine Angew. Math. 78 (1874), 114–122.

[Re2] T. Reye, *Darstellung quaternärer biquadratischer Formen als Summen von zehn Biquadraten*, J. Reine Angew. Math. 78 (1874), 123–129.

[Ri] H. W. Richmond, *On canonical forms*, Quart. J. Math. 33 (1902), 331-340.

[Ro] T. G. Room, *The Geometry of Determinantal Loci*, Cambridge Univerity Press, Cambridge, U. K., 1938.

[Rt] L. Roth, *Algebraic threefolds with special regard to problems of rationality*, Springer-Verlag, 1955.

[Ru1] F. Russo, *On a theorem of Severi*, Math. Ann. 316 (2000), 1–17.

[Ru2] F. Russo, *The antibirational involutions of the plane and the classification of real del Pezzo surfaces*, in "Algebraic Geometry", Proceedings Conference in Memory of Paolo Francia, De Gruyeter, 2002, 289–312.

[Ru3] F. Russo, *A characterization of nef and good divisors by asymptotic multiplier ideals*, 9 pages, 2000, unpublished.

[Ru4] F. Russo, *Pensieri Cremoniani*, 15 pages, 2001, unpublished.

[Ru5] F. Russo, *Some remarks on the classical work of Umberto Perazzo*, 10 pages, 2002, unpublished.

[Ru6] F. Russo, *Quadric varieties, tangential projections and second fundamental form*, in preparation.

[RS] F. Russo, A. Simis, *On birational maps and Jacobian matrices*, Comp. Math. 126 (2001), 335–358.

[S1] G. Scorza, *Sulla determinazione delle varietá a tre dimensioni di* S_r $(r \geq 7)$ *i cui* S_3 *tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo 25 (1908), 193–204.

[S2] G. Scorza, *Un problema sui sistemi lineari di curve appartenenti a una superficie algebrica*, Rend. Reale Ist. Lombardo Scienze e Lettere 41 (1908), 913–920.

[S3] G. Scorza, *Le varietá a curve sezioni ellittiche*, Ann. Mat. Pura Appl. 15 (1908), 217–273.

[S4] G. Scorza, *Sulle varietá a quattro dimensioni di* S_r $(r \geq 9)$ *i cui* S_4 *tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo 27 (1909), 148–178.

[S5] G. Scorza, *Gruppi Astratti*, Ed. Cremonese, Roma, 1942.

[S6] G. Scorza, Opere Scelte, I–IV, Ed. Cremonese, 1960.

[Se1] B. Segre, *On the rational solutions of homogeneous cubic equations in four variables*, Math. Notae 11 (1951), 1–68.

[Se2] B. Segre, *Bertini forms and hessian matrices*, J. London Math. Soc. 26 (1951), 164–176.

[Se3] B. Segre, *Some Properties of Differentiable Varieties and Transformations*, Erg. Math., Springer-Verlag, 1957.

[Se4] B. Segre, *Sull' hessiano di taluni polinomi (determinanti, pfaffiani, discriminanti, risultanti, hessiani) I, II*, Atti Acc. Lincei 37 (1964), 109–117 e 215–221.

[SR] J. G. Semple, L. Roth, *Introduction to Algebraic Geometry*, Oxford University Press, 1949 and 1986.

[ST1] J. G. Semple, J. A. Tyrrell, *Specialization of Cremona transformations*, Mathematika 15 (1968), 171–177.

[ST2] J. G. Semple, J. A. Tyrrell, *The Cremona transformation of S_6 by quadrics through a normal elliptic septimic scroll* $^1R^7$, Mathematika 16 (1969), 88–97.

[ST3] J. G. Semple, J. A. Tyrrell, *The $T_{2,4}$ of S_6 defined by a rational surface* $^3F^8$, Proc. Lond. Math. Soc. 20 (1970), 205–221.

[Sev1] F. Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni e ai suoi punti tripli apparenti*, Rend. Circ. Mat. Palermo 15 (1901), 33–51.

[Sev2] F. Severi, *Sulle intersezioni delle varietá algebriche e sopra i loro caratteri e singolaritá proiettive*, Mem. Accad. Sci. Torino 52 (1902), 61–118.

[Sh] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, 1974.

[SB1] N. I. Shepherd-Barron, *The rationality of Certain Spaces Associated to Trigonal Curves*, Proc. Symp. in Pure Math. 46 (1987), 165–171.

[SB2] N. I. Shepherd-Barron, *The rationality of some moduli spaces of plane curves*, Comp. Math. 67 (1988), 51–88.

[SB3] N. I. Shepherd-Barron, *The rationality of quintic Del Pezzo surfaces- a short proof*, Bull. London Math. Soc. 24 (1992), 249–250.

[SUV] A. Simis, B. Ulrich, W. V. Vasconcelos, *Tangent star cones*, J. Reine angew. Math. 483 (1997), 23–59.

[SD] H. P. F. Swinnerton-Dyer, *Rational points on Del Pezzo surfaces of degree 5*, Algebraic Geometry, Oslo 1970, Wolthers-Nordhoff, 1972, 287–290.

[Sy1] J. J. Sylvester, *Sur une extension d' un théorème de Clebsch relatif aux courbes de quartième degré*, Comptes Rendus Acad. Sciences 102 (1886), 203–216.

[Sy2] J. J. Sylvester, *An essay on canonical forms, supplemented by a sketch of a memoir on elimination, transformation and canonical form*, Collected Works, Vol. I, Cambridge Univ. Press, 1904, 203–216.

[T1] A. Terracini, *Sulle V_k per cui la varieta' degli S_h $(h+1)$-secanti ha dimensione minore dell' ordinario*, Rend. Circ. Mat. Palermo 31 (1911), 392–396.

[T2] A. Terracini, *Su due problemi concernenti la determinazione di alcune classi di superficie, considerate da G. Scorza e F. Palatini.*, Atti Soc. Natur. e Matem. Modena 6 (1921-22), 3–16.

[T3] A. Terracini, *Selecta*, I–II, Ed. Cremonese, Roma, 1968.

[T4] A. Terracini, *Ricordi di un matematico. Un sessantennio di vita universitaria*, Ed. Cremonese, Roma, 1968.

[To] J. A. Todd, *The quarto-quartic transformation of four-dimensional space associated with certain projectively generated loci*, Proc. Camb. Phil. Soc. 26 (1929), 323–333.

[Tg] E. G. Togliatti, *Alessandro Terracini*, Boll. Unione Mat. Ital. 2 (1969), 145–152.

[Tr1] S. L. Tregub, *Three constructions of rationality of cubic fourfold*, Moscow Univ. Math. Bull. 39 (3) (1984), 8–16.

[Tr2] S. L. Tregub, *Two remarks on four dimensional cubics*, Russian Math. Surveys, 48 (2) (1993), 206–208.

[VA] K. Vasconcelos de Araujo, *A superfí cie de Veronese*, Tese de Mestrado, UFPE, 2002.

[Ve] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction*, Comp. Math. 125 (2001), 263–282 .

[Z1] F. L. Zak, *Severi varieties*, Math. USSR Sbornik 54 (1986), 113–127.

[Z2] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monografhs, vol. 127, Amer. Math. Soc. 1993.

[Z3] F. L. Zak, *Determinants of projective varieties and their degrees*, preprint, 2002.

Index