Symmetry Studies
An Introduction

Publicações Matemáticas

Symmetry Studies
An Introduction

Marlos Viana
University of Illinois at Chicago

Este texto foi aumentado, revisto, e publicado pela Cambridge University Press em 2008 com o título Symmetry Studies - An introduction to the analysis of structured data in applications.

impa

24º Colóquio Brasileiro de Matemática

Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ

Impresso no Brasil / Printed in Brazil

Capa: Noni Geiger / Sérgio R. Vaz

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Distribuição:
IMPA
Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ
e-mail: ddic@impa.br
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ISBN: 85-244-0205-9

Preface

This monography introduces the theory and methods of symmetry studies, including the notions of structured data and data reduction by symmetrically equivalent components. The broad objective is to apply these notions to explore more fully the interplay between research questions and scientific explanation. The motivation for this comes from a variety of disciplines, including physics, physiology, chemistry and molecular biology, where notions of symmetry continue to play a significant role in exploring natural phenomena, and from the goal of applying these principles to the analysis of experimental data. With the language and methods of symmetry studies, newer questions and potential answers may be identified.

This introductory text is intended for students and collaborating scientists in areas where mathematical, statistical and probability applications and arguments are routinely required. The prerequisites are at the level of upper undergraduate training. For example, the algebraic aspects of the method should be accessible to students who have had an introductory-level course in algebra, whereas the statistical and probabilistic aspects require the basic notions of probability models and distributions of quadratic forms, e.g., Fisher-Cochran Theorem.

The two introductory chapters of this text are where the basic language of structured data and symmetries studies is introduced and illustrated with a number of examples. Chapter 3 introduces the foundations of the algebraic component of these studies, while the remaining three chapters are dedicated to specific symmetry studies. Selected chapters include a briefly annotated list of suggested readings. Large matrices, tables and graphics were occasionally moved to the end of the corresponding chapter, to eliminate unnecessary page-breaks. An italicized word in the text indicates a technical term that is introduced at that time to prompt the reader to review its definition. A short table of symbols is included in the Appendix.

These notes reflect, with my gratitude, the dedicated comments, conversations, suggestions and enthusiasm of many colleagues including, among others, S. Andersson, A. de Bucchianico, S. Evans, I. Helland, K. Mallesh, P. McCullagh, I. Olkin, M. Perlman, D. Richards, S. Smith and A. Takemura.

I greatly appreciate the opportunity of teaching the material covered in the present text during the 24th Brazilian Mathematics Colloquium in my hometown, Rio de Janeiro. To all my colleagues at IMPA, I express my appreciation for their assistance and support.

Marlos A. G. Viana
The University of Illinois

Chicago, April 23, 2003.

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CHAPTER 1

Introduction

This chapter is an overview of symmetry studies for structured data, including an introduction to the main concepts and tools developed in the sequence. Structured data (and probability models for these data) arise from the simple observation that in many experimental conditions there is an outstanding structural connection between the measured object and the corresponding measurements, which then suggests natural ways of summarizing and analyzing these data. Equally important, for prospective investigations, is the fact that these structural relationships provide a richer domain within which newer research questions might be properly formulated. Consequently, a higher level of scientific explanation might be obtained.

Data analysts tend to represent a potential set of observations as $x_1, x_2, \ldots, x_n$. In the analysis of structured data, we would write, instead, $x(1), x(2), \ldots, x(n)$, bringing forth the important fact that the set

$$V = \{1, 2, \ldots, n\},$$

which provides the labels for the possible data units, is only one among the many other sets of labels, and that, consequently, different choices of V may correspond to different functional relationships among observable measurements and the structure of the space indexing these potential measurements. The labels are no longer static but have the capability of interacting with (the interpretation of) the events. The consequence is a broader framework within which data can be queried and interpreted.

In the next sections we will introduce an example of structured data, followed by the basic algebraic, probabilistic and observational aspects of potential symmetry studies suggested by the structure of interest.

EXAMPLE 1.1 (A simple structure from molecular biology). A *biological sequence* is a finite string of symbols from a finite alphabet ($\mathcal{A}$) of residues, such as the linear string

ctccttgggatattgatgatctgtagtgctacagaaaaattgtgggtcacagtctattat,

in which the symbols are letters in the alphabet $\mathcal{A} = \{A, G, T, C\}$. Here the symbols represent adenine (A), guanine (G), thymine (T) and cytosine (C) molecules in DNA (deoxyribonucleic acid) sequences. The adjacency of two symbols in the linear string means that the two molecules are chemically binded to each other. There are many more common alphabets, representing

- the nucleotides adenine (A), guanine (G), cytosine (C), uracil (U) in RNA (ribonucleic acid) sequences: $\mathcal{A} = \{A, G, T, U\}$;
- the classs $u = \{A, G\}$ of purine and $y = \{C, T\}$ of pyrimidine residues: $\mathcal{A} = \{u, y\}$,

or the larger class of amino acids: Alanine (A), Arginine (R), Asparagine (N), Aspartic (D), Cysteine (C), Glutamic (E), Glutamine (Q), Glycine (G), Histidine (H), Isoleucine (I), Leucine (L), Lysine (K), Methionine (M), Phenylalanine (F), Proline (P), Serine (S), Threonine (T), Tryptophan (W), Tyrosine (Y), Valine (V), in protein sequences, in which case

$$\mathcal{A} = \{A, R, N, D, C, E, Q, G, H, I, L, K, M, F, P, S, T, W, Y, V\}.$$

The length of *global* or complete sequences, in base pairs, ranges from 10^3 (single-stranded virus) to 10^9 (mammals). Well-defined *codes*, such as the standard code,

```
amino acids = FFLLSSSSYY**CC*WLLLLPPPPHHQQRRRRIIIMTTTTNNKKSSRRVVVVAAAADDEEGGGG
      Base1 = TTTTTTTTTTTTTTTTCCCCCCCCCCCCCCCCAAAAAAAAAAAAAAAAGGGGGGGGGGGGGGGG
      Base2 = TTTTCCCCAAAAGGGGTTTTCCCCAAAAGGGGTTTTCCCCAAAAGGGGTTTTCCCCAAAAGGGG
      Base3 = TCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAGTCAG
```

then translate DNA triplets into specific amino acids. The set, V, of DNA triplets constitutes an example of a simple structure in which the points in the structure are analytically and experimentally important. In many ways, common to molecular biologists, these triplets may be seen as labels or indices for experimental and analytical studies. We may say that the structure

$$V = \{ttt, ttc, tta, \ldots, gga, ggg\}$$

defined by these 64 simple sequences (s) in length of three written with a four-letter alphabet $\mathcal{A} = \{A, G, C, T\}$, is a structure indexing potential molecular constructs or measurements, x(s), such as the triplet's molecular weight. In that sense, then,

$$x(ttt), x(ttc), x(tta), \ldots, x(gga), x(ggg),$$

are data indexed by the structure V, or, shortly, a *structured data.* The structure may be amalgamated, for example, by rewriting each word with the shorter alphabet $\mathcal{A} = \{u, y\}$ of purine-pyrimidine residues. The new structure

$$V = \{yyy, yyy, yyu, \ldots, uuu\}$$

of triplets of purine-pyrimidines has $2^4 = 16$ points or labels, and

$$x(yyy), x(yyy), x(yyu), \ldots, x(uuu),$$

are the corresponding structured data. Here is another simple structured data: the structure is the set

$$V = \{A, G, C, T\} \times \{A, G, C, T\}$$

of ordered DNA nucleotides. It has $4 \times 4 = 16$ points in it. Given two DNA local sequences

I = ttttcgctatggaacctgggatctttagtttgaaatgggagagcccattccgcctggaaaaaattagataaggtaag,

J = tttccgtcatggaacctggaatagttgctcaaaagtgggagcaaccgcttaggtttgaaaaaattagataagggcgg,

we measure, in each point (i, j) of V, the frequency x(i, j) with which i in I *aligns* with j in J along the two sequences. Here are the resulting structured data:

$$\left[\begin{array}{c|cccc} i\backslash j & A & C & G & T \\ \hline A & 17 & 2 & 4 & 0 \\ C & 3 & 5 & 1 & 3 \\ G & 3 & 1 & 15 & 1 \\ T & 1 & 4 & 2 & 15 \end{array}\right].$$

This simple structured entails a core concept in molecular biology, that of biological *homology.* It refers to shared characteristics among species that have been inherited (the earlier Darwin's view), or derived (post Darwinian or phylogenetic[1]) from a common ancestor. More recently evolved species are expected to have more, and more similar, homologies (e.g., the form and function of DNA and protein sequences). It implies the notion of constructs which are similar but not identical. □

1.1. Algebraic aspects.

First we observe that any biological sequence ℓ base pairs long is representable by a function or mapping

$$\mathrm{s} : \mathrm{L} \to \mathcal{A},$$

where $\mathrm{L} = \{1, 2, \ldots, \ell\}$ is the set for the ordered positions in which the residues in the alphabet $\mathcal{A}$ are located. Typical alphabets, as illustrated above, are $\mathcal{A} = \{\mathrm{A}, \mathrm{G}, \mathrm{T}, \mathrm{C}\}$ in DNA sequences, $\mathcal{A} = \{\mathrm{A}, \mathrm{G}, \mathrm{T}, \mathrm{U}\}$ in RNA sequences, or simply a two-letter alphabet $\mathcal{A} = \{\mathrm{u}, \mathrm{y}\}$ of purine (u=A or u=G) and pyrimidine (y=C or y=T) residues. The set V, of all mappings $\mathrm{s} : \{1, 2, 3\} \to \{\mathrm{A}, \mathrm{G}, \mathrm{T}, \mathrm{C}\}$, defined by the entries of the matrix

$$(1.1)\qquad \begin{bmatrix} aaa & ggg & ccc & ttt & aag & aac & aat & gga \\ ggc & ggt & cca & ccg & cct & tta & ttg & ttc \\ aga & aca & ata & gag & gcg & gtg & cac & cgc \\ ctc & tat & tgt & tct & gaa & caa & taa & agg \\ cgg & tgg & acc & gcc & tcc & att & gtt & ctt \\ agc & gac & cga & acg & gca & cag & agt & atg \\ tga & gat & gta & tag & act & atc & tca & cat \\ cta & tac & gct & gtc & tcg & cgt & ctg & tgc \end{bmatrix}$$

is an example of a *structure.* Indicating by $|\mathcal{A}|$ the number of elements in the set $\mathcal{A}$, we note that there are $|\mathcal{A}|^\ell$ sequences in V and that each element in V is referred to as a $|\mathcal{A}|$-sequence in length of ℓ. For example, every two-sequence in length of four, with $\mathcal{A} = \{\mathrm{u}, \mathrm{y}\}$, is a mapping

$$\mathrm{s} : \{1, 2, 3, 4\} \to \{\mathrm{u}, \mathrm{y}\}.$$

The 16 points in the space V of all two-sequences in length of four may be represented by
(1.2)

$$\mathrm{V} = \left[\begin{array}{c|cccccccccccccccc} \mathrm{s} & 1 & 16 & 15 & 14 & 12 & 8 & 13 & 11 & 7 & 10 & 6 & 4 & 9 & 5 & 3 & 2 \\ \hline \mathrm{s}(1) & y & u & y & u & u & u & y & y & y & u & u & u & y & y & y & u \\ \mathrm{s}(2) & y & u & u & y & u & u & y & u & u & y & y & u & y & y & u & y \\ \mathrm{s}(3) & y & u & u & u & y & u & u & y & u & y & u & y & y & u & y & y \\ \mathrm{s}(4) & y & u & u & u & u & y & u & u & y & u & y & y & u & y & y & y \end{array}\right].$$

[1]Phylum refers to a tribe or race of organisms, related by descent from a common ancestral form; a series of animals or plants genetically related. See also *Homology: A Concept in Crisis* by J. W. and P. Nelson, in Critical Perspective Origins and Design 18:2: http://www.arn.org/docs/odesign/od182/hobi182.htm

The numbers in the first row are labels for each mapping.

Permutation symmetries. A permutation τ over a finite set L is an injective (one-to-one) mapping $\tau : \mathrm{L} \to \mathrm{L}$. We indicate the set of all permutations on a finite set L with ℓ elements by S_ℓ.

EXAMPLE 1.2. The set S_3 of all permutations of 3 symbols includes the identity (1) transformation

$$1 = \begin{bmatrix} 1 \to & 1 \\ 2 \to & 2 \\ 3 \to & 3 \end{bmatrix},$$

three transpositions,

$$(12) = \begin{bmatrix} 1 \to & 2 \\ 2 \to & 1 \\ 3 \to & 3 \end{bmatrix}, \quad (13) = \begin{bmatrix} 1 \to & 3 \\ 2 \to & 2 \\ 3 \to & 1 \end{bmatrix}, \quad (23) = \begin{bmatrix} 1 \to & 1 \\ 2 \to & 3 \\ 3 \to & 2 \end{bmatrix},$$

and two cyclic permutations,

$$(123) = \begin{bmatrix} 1 \to & 2 \\ 2 \to & 3 \\ 3 \to & 1 \end{bmatrix}, \quad (132) = \begin{bmatrix} 1 \to & 3 \\ 2 \to & 1 \\ 3 \to & 2 \end{bmatrix}.$$

In summary,

$$S_3 = \{1, (12), (13), (23), (123), (132)\}.$$

□

We say that a permutation τ *fixes* the element $j \in L$ if $\tau(j) = j$. The *identity* transformation, indicated by 1, fixes all elements in L. Given two permutations τ and σ, the *composite* function $\tau\sigma$ takes the element $j \in L$ to the element $\tau(\sigma(j))$ in L, and is also a permutation. For every permutation τ in L, the equation $\tau(j') = j$ has a unique solution j' for each $j \in L$. The resulting function $j \mapsto j'$ is also a permutation, called the *inverse* permutation, and is indicated by τ^{-1}. It holds that $\tau\tau^{-1} = \tau^{-1}\tau = 1$. Adding the fact that composition of functions is an associative operation, that is $(\tau\sigma)\eta = \tau(\sigma\eta)$, we observe that the set S_ℓ together with the operation of function composition, defines a *permutation group* of order ℓ. We refer to permutations, permutation symmetries and symmetries without distinction. This equivalence will become clear in the sequence. The subset C_3 of all cyclic permutations of 3 symbols is indicated by $C_3 = \{1, (123), (132)\}$. Permutations and permutation groups are considered later on in Chapter 3.

Composing sequences and symmetries. Given a sequence s in length of ℓ and a symmetry τ in S_ℓ then the composite $s\tau^{-1}$ is also a sequence in length of ℓ in V (using the inverse permutation will be justified later). Say that $\mathcal{A} = \{A, C, G, T\}$, so that $\ell = 4$,

$$\tau = \begin{bmatrix} 1 \to & 2 \\ 2 \to & 3 \\ 3 \to & 4 \\ 4 \to & 1 \end{bmatrix} \quad \text{and } s = \begin{bmatrix} 1 \to & A \\ 2 \to & A \\ 3 \to & G \\ 4 \to & C \end{bmatrix}. \quad \text{Then,} \quad s\tau^{-1} = \begin{bmatrix} 1 \to & C \\ 2 \to & A \\ 3 \to & A \\ 4 \to & G \end{bmatrix}.$$

Given a mapping $s : L \mapsto C$, where the set L has ℓ elements and the set C has c elements, and permuations $\tau \in S_\ell$ and $\sigma \in S_c$, the composition $s\tau^{-1}$ is called a *composition on the left*, whereas σs is a *composition on the right.*. These compositions are studied in detail later on in Chapter 3. From the fact that the composition $s\tau^{-1}$ of a mapping $s \in V$ and a permutation $\tau \in S_\ell$ leads to another mapping in V suggests the notion of *permutation orbits* of the mapping s generated by the symmetries of interest. This is the set of all mappings sharing the symmetries of interest. Equivalently, two mappings are in the same orbit when one is obtained from the other by composing it with a permutation from the symmetries of interest. For notation, we write

$$\mathcal{O} = \{s\tau^{-1}; \tau \in S_\ell\}$$

to indicate the permutation orbit of a mapping s resulting from composing it with S_ℓ on the left. Similarly, $\mathcal{O} = \{\sigma s; \sigma \in S_c\}$ is a permutation orbit. When the symmetries of interest are the cyclic permutations, we obtain the corresponding *cyclic orbits.* For example, starting with the sequence CGG in length of three and composing on the left with all three cyclic permutations in C_3 we obtain the orbit

$$\mathcal{O}_{\mathrm{CGG}} = \{\mathrm{CGG}, \mathrm{GCG}, \mathrm{GGC}\}.$$

Similarly, starting with the sequence uuyuuy in length of six and composing on the left with all six cyclic permutations in C_6 we obtain the orbit

$$\mathcal{O}_{\mathrm{uuyuuy}} = \{\mathrm{uuyuuy}, \mathrm{yuuyuu}, \mathrm{uyuuyu}\}.$$

EXAMPLE 1.3 (Permutation orbits for two-sequences in length of four). The mapping space V of all two-sequences in length of four ($|L| = 4, |\mathcal{A}| = 2$) has has $2^4 = 16$ points, each representing one sequence, as shown in Matrix (1.2). Consider the left composition ($s\tau^{-1}$) of sequences in V with the symmetries in S_4. The group S_4 is detailed later on in Chapter 3. It has 6 transpositions, 3 cycles or order 2, 8 cycles of order 3 and 6 cycles of order 4. These permutations are indicated in the first column of Matrix (1.21), shown at the end of this chapter. The sequences $s \in V$ are represented by their labels on the first row. These are the same labels shown in Matrix (1.2). The resulting compositions $s\tau^{-1}$ are shown in the adjacent 16 columns. For example, if

$$\tau = (1234) = \begin{bmatrix} 1 \rightarrow & 2 \\ 2 \rightarrow & 3 \\ 3 \rightarrow & 4 \\ 4 \rightarrow & 1 \end{bmatrix}, \quad \text{and } s = \begin{bmatrix} 1 \rightarrow & u \\ 2 \rightarrow & u \\ 3 \rightarrow & y \\ 4 \rightarrow & u \end{bmatrix}, \quad \text{then } s\tau^{-1} = \begin{bmatrix} 1 \rightarrow & u \\ 2 \rightarrow & u \\ 3 \rightarrow & u \\ 4 \rightarrow & y \end{bmatrix},$$

so that the composition with τ^{-1} takes the sequence uuyu (label 12) into the sequence uuuy (label 8). In particular, these two sequences are in the same orbit. The resulting orbits (indicating the sequences by their labels) may be expressed as

$$\begin{aligned} \mathcal{O}_0 &= \{1\}, \\ \mathcal{O}_1 &= \{9, 5, 3, 2\}, \\ \mathcal{O}_2 &= \{13, 11, 7, 10, 6, 4\}, \\ \mathcal{O}_3 &= \{15, 14, 12, 8\}, \\ \mathcal{O}_4 &= \{16\}, \end{aligned}$$

so that

$$V = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \tag{1.3}$$

forms a disjoint partition of V. Also note that the orbit $\mathcal{O}_k$ may be characterized by the number of purines (symbols 'u') in the sequences, that is,

$$\mathcal{O}_k = \{s \in V; |s^{-1}(u)| = k\}, \quad k = 0, \dots, 4.$$

Note that

$$|\mathcal{O}_k| = \binom{\ell}{k}.$$

□

The effect of composing V with S_ℓ on the left is that of removing the order of the positions- equivalently, any two sequences are then equivalent, similar or indistinguishable, when they differ only by reordering the location of the letters or residues. As a result, we obtain another space, called the *quotient space*, in which the elements are the resulting 5 permutation orbits $\mathcal{O}_0, \mathcal{O}_1, \dots \mathcal{O}_4$. These orbits are characterized by the number of, say, purines. That is, orbit $\mathcal{O}_i$ is composed of those sequences with exactly i purines in it.

1.2. Probabilistic aspects

In this section we will introduce some probabilistic aspects of the analysis of structured data. In particular, we will discuss the interplay among symmetry relations, the structure and probability models. Initially, we consider the case in which the uncertainty that is described by the model is derived from the points in the structure V.

Let P indicate a probability model in the space V of two-sequences in length of ℓ. We say that P has the symmetry of the group G if P is constant over each one of the orbits of V. For example, if

$$P(s) = P(s\tau^{-1}) \tag{1.4}$$

for all sequences s in V and permutations τ in G. Because s is now a random variable, the purine-pyrimidine levels

$$(\text{number of purines, number of pyrimidines}) = (i, \ell - i)$$

are also random variables, and consequently, the probability laws

$$\mathcal{L}_i = (\frac{i}{\ell}, \frac{\ell - i}{\ell}), \quad i = 0, 1, \dots, \ell, \tag{1.5}$$

associated with the orbits described in (1.3) are also random. Here are the possible probability laws for purine-pyrimidine levels from two-sequences in length of four:

$$\mathcal{L}_0 = (0,1),\ \mathcal{L}_1 = (\frac{1}{4}, \frac{3}{4}),\ \mathcal{L}_2 = (\frac{2}{4}, \frac{2}{4}),\ \mathcal{L}_3 = (\frac{3}{4}, \frac{1}{4}),\ \mathcal{L}_4 = (1,0).$$

The likelihood of each law is therefore determined by the probability of seeing a sequence which is associated with the law- because all sequences in the orbit $\mathcal{O}_i$ lead to the law $\mathcal{L}_i$ and conversely, we see that $\mathcal{L}_i$ occurs with probability $P(\mathcal{O}_i)$; shortly,

$$\text{Probability of law } \mathcal{L}_i = P(\mathcal{O}_i).$$

Clearly, if the law P is such that all sequences are equally likely (P is said to be uniform), then condition (1.4) is satisfied and

$$\text{(1.6)} \qquad \text{Probability of law } \mathcal{L}_i = \mathrm{P}(\mathcal{O}_i) = \frac{|\mathcal{O}_i|}{|\mathrm{V}|} = \frac{\binom{\ell}{i}}{|\mathrm{V}|}.$$

We have, for two-sequences in length of four,

$$\mathrm{P}(\mathcal{O}_0) = \frac{1}{16},\ \mathrm{P}(\mathcal{O}_1) = \frac{4}{16},\ \mathrm{P}(\mathcal{O}_2) = \frac{6}{16},\ \mathrm{P}(\mathcal{O}_3) = \frac{4}{16},\ \mathrm{P}(\mathcal{O}_4) = \frac{1}{16},$$

so that the most likely distribution of purine-pyrimidine levels, under uniformly distributed sequences, is

$$\mathcal{L}_2 = (\frac{2}{4}, \frac{2}{4}).$$

EXAMPLE 1.4 (Four-sequences in length of three). Let $\mathcal{A} = \{A, C, G, T\}$. The space V of all four-sequences in length of three has $|V| = 4^3 = 64$ sequences. The random variables generated by similarity of the positions are the frequencies of

$$(\text{adenines, cytosines, guanines, thymines}) = (f_a, f_c, f_g, f_t),$$

with $f_a + f_c + f_g + f_t = 3$. Consequently, the corresponding probability laws

$$\mathcal{L}_\lambda = (\frac{f_a}{3}, \frac{f_c}{3}, \frac{f_g}{3}, \frac{f_t}{3})$$

are also random. The index λ in $\mathcal{L}_\lambda$ indicates the corresponding orbit, in analogy with expression (1.5). We obtain these indices as the possible *integer partitions* of 3 in length of 4, so that there are three *types* of orbits, and corresponding laws:

$$\lambda = 3000 \to \mathcal{L}_{3000} = (1, 0, 0, 0),$$
$$\lambda = 2100 \to \mathcal{L}_{2100} = (\frac{2}{3}, \frac{1}{3}, 0, 0),$$
$$\lambda = 1110 \to \mathcal{L}_{1110} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0).$$

Similarly to expression (1.6) we now obtain

$$\text{(1.7)} \qquad \text{Probability of law } \mathcal{L}_{3000} = \mathrm{P}(\mathcal{O}_{3000}) = \frac{\binom{3}{3,0,0,0}}{|\mathrm{V}|} = \frac{3!}{3!0!0!0!}\frac{1}{64} = \frac{1}{64},$$

$$\text{(1.8)} \qquad \text{Probability of law } \mathcal{L}_{2100} = \mathrm{P}(\mathcal{O}_{2100}) = \frac{\binom{3}{2,1,0,0}}{|\mathrm{V}|} = \frac{3!}{2!1!0!0!}\frac{1}{64} = \frac{3}{64},$$

$$\text{(1.9)} \qquad \text{Probability of law } \mathcal{L}_{1110} = \mathrm{P}(\mathcal{O}_{1110}) = \frac{\binom{3}{1,1,1,0}}{|\mathrm{V}|} = \frac{3!}{1!1!1!0!}\frac{1}{64} = \frac{6}{64},$$

so that, under the assumption that all 4-sequences in length of 3 are equally likely (uniform probability), the most probable distribution by levels of nucleotides comes from the *class* of distribution given by $\mathcal{L}_{1110}$, each of which has the highest probability, 6/64. Simple combinatorics (discussed later on in Chapter 3) show that there are

$$\text{(1.10)} \qquad \frac{4!}{3!1!} = 4$$

such most probable probability laws describing the nucleotide levels, namely,

$$\text{(1.11)} \qquad (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0),\ (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}),\ (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}),\ (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

□

EXAMPLE 1.5 (Maxwell-Boltzmann Equilibrium Distribution). The following is a quote from von Mises (1957, p.200) which describes the context of Boltzmann's arguments. Only the notation was partially adapted to conform with the present one. The *velocity space* is as usually described in the physics literature, e.g., Ruhla (1989, p.79).

> The assumption of the classical theory is that equal probabilities are assigned to equal volumes in this velocity space. We will call each element of the volume in the velocity space a possible 'position' or 'place' of the molecule. If we now consider a collective whole elements are distributions of certain number ℓ of molecules over c positions in the velocity space, it follows that all possible c^ℓ distributions have the same probability. For example, imagine two molecules A and B, and three different positions a,b,c. The number of different distributions is 9, since each of the three positions of A, namely Aa,Ab,Ac can be combined with each of B. According to the classical theory, all these distributions have the same probability, 1/9. A new theory, first suggested by the Indian physicist Bose[2], and developed by Einstein, chooses another assumption regarding the equal probabilities. Instead of considering single molecules and assuming that each molecule can occupy all positions in the velocity space with equal probability, the new theory starts with the concept of 'repartition'. This is given by the *number* of molecules at each place of the velocity space, without paying attention to the individual molecules. From this point of view, only six 'partitions' are possible for two molecules on three places, namely, both molecules may be together at a, at b, or at c, or they may be separated, one at a and one at b, one at a and one at c, or one at b and one at c. According to the Bose-Einstein theory, each of these six cases has the same probability, 1/6. In the classical theory, each of these three possibilities would have the probability of 1/9, each of the other three, however, 2/9, because, in assuming individual molecules, each of the last three possibilities can be realized in two different ways: A can be in a, and B in b, or vice versa, B can be in a, and A in b.
>
> The Italian physicist Fermi[3] advanced still another hypothesis. He postulated that only such distributions are possible- and possess equal probabilities- in which all molecules occupy different places. In our example of two molecules and three positions, there would only be three possibilities, each having the probability 1/3; i.e., one molecule in a and one in b; one in a and one in c; one in b and one in c.

[2]Satyendranath Bose, Born: 1 Jan 1894 in Calcutta, India Died: 4 Feb 1974 in Calcutta, India

[3]Enrico Fermi was born in Rome on 29th September, 1901. The Nobel Prize for Physics was awarded to Fermi for his work on the artificial radioactivity produced by neutrons, and for nuclear reactions brought about by slow neutrons. He died in Chicago on 29th November, 1954.

> In testing these and other hypotheses it is assumed, according to Boltzmann's entropy theorem, that the probability of the state of a gas is a measure of its entropy, and the object of the investigation is to find which theory best approximates the actually observed dependence of entropy on temperature and mass.

The arguments in von Mises' narrative can be summarized, using the orbit method, as follows: Let $L = \{A, B\}$, $C = \{a, b, c\}$ and V the set of all mappings $s : L \to C$, that is,

$$V = \left[\begin{array}{c|ccccccccc} s & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline s(A) & a & b & c & a & b & a & c & b & c \\ s(B) & a & b & c & b & a & c & a & c & b \end{array}\right].$$

Under the Maxwell-Boltzmann (MB) model, it is assumed that all points or configurations in the space V are equally likely, or *uniformly distributed*, that is:

$$P(s) = \frac{1}{|V|} = \frac{1}{9}, \quad \text{for all } s \in V.$$

Under the Fermi-Dirac (FD) model, it is assumed that all points in the quotient space V/S_2 of V by the action $s\tau^{-1}$ of shuffling the molecules' labels (in $L = \{A, B\}$) are uniformly distributed. Thus, in the FD model the uniform probability is defined on the *orbits* of V under the label symmetry. The following matrix summarizes the action $s\tau^{-1}$ on V:

$$\left[\begin{array}{c|ccccccccc} \sigma\backslash s & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ (12) & 1 & 2 & 3 & 5 & 4 & 7 & 6 & 9 & 8 \end{array}\right],$$

so that the six orbits in the quotient space V/S_2 are

$$\mathcal{O}_{11} = \{1\},\ \mathcal{O}_{12} = \{2\},\ \mathcal{O}_{13} = \{3\},\ \mathcal{O}_{21} = \{4,5\},\ \mathcal{O}_{22} = \{6,7\},\ \mathcal{O}_{23} = \{8,9\},$$

each one of these having probability of 1/6. A probability law in V such as

$$P(s) = \begin{cases} 1/6 & \text{when } s \in \{\mathcal{O}_{11}, \mathcal{O}_{12}, \mathcal{O}_{13}\}, \\ 1/12 & \text{when } s \in \{\mathcal{O}_{21}, \mathcal{O}_{22}, \mathcal{O}_{23}\}, \end{cases}$$

would be consistent with the assumptions of the FD model.

The Bose-Einstein (BE) model assumes that only the injective mappings

$$V_I \equiv \left[\begin{array}{c|cccccc} s & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline s(A) & a & b & a & c & b & c \\ s(B) & b & a & c & a & c & b \end{array}\right] \subset V$$

are admissible representations of the physical system, and that a uniform probability law is assigned to the resulting orbits in the quotient space of V_I by the action $s\tau^{-1}$ of shuffling the molecules' labels. Therefore, starting with

$$\left[\begin{array}{c|cccccc} \sigma\backslash s & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 4 & 5 & 6 & 7 & 8 & 9 \\ (12) & 5 & 4 & 7 & 6 & 9 & 8 \end{array}\right],$$

we obtain the three orbits

$$\mathcal{O}_1 = \{4,5\},\ \mathcal{O}_2 = \{6,7\},\ \mathcal{O}_3 = \{8,9\}$$

in the quotient space V_I/S_2. To each of these, a probability of 1/3 is assigned. In the present example, a probability law in V given by

$$P(s) = \begin{cases} 1/6 & \text{when } s \in \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}, \\ 0 & \text{otherwise}, \end{cases}$$

would be consistent with the assumptions of the FD model. Thus, in summary:

Model	Domain of the Uniform Law
Maxwell-Boltzmann	V
Fermi-Dirac	V/G
Bose-Einstein	V_I/G

Here is an analogy within the molecular biology context we started with. This is similar to Example 1.4. Consider the space of four-sequences in length of six. In the present context, we have six numbered molecules indexed by the set $L = \{1, 2, 3, 4, 5, 6\}$ and four energy levels, indicated by the set $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$. The energy configurations are mappings

$$s : L \to \mathcal{E},$$

so that there is a total of $|\mathcal{E}|^{|L|} = 4^6 = 4096$ accessible microstates. We pass from microstates to measurable macrostates by dividing the space by similarities that result among the molecules when their numbers are erased. This is in analogy to erasing the position of the nucleotides in a biological sequence. Algebraically, this is obtained by letting the (group S_6 of) permutations act on (by shuffling) the molecule labels in the set L. The composition rule is $s\tau^{-1}$. The resulting classes $\mathcal{O}_\lambda$ of orbits are then the energy macrostates realized by the system. Here are the resulting classes, their volume $|\mathcal{O}_\lambda|$, usually indicated by Ω_λ in the thermodynamics context, and their number Q_λ of quantal states:

λ	Ω_λ	Q_λ	$\Omega_\lambda \times Q_\lambda$
6000	1	4	4
5100	6	12	72
4200	15	12	180
4110	30	12	360
3300	20	6	120
3210	60	24	1440
3111	120	4	480
2220	90	4	360
2211	180	6	1080
total	522	84	4096

Similar calculations are outlined in Example 3.16 of Chapter 3. There are $Q = 6$ quantal states associated with the most probable ($\Omega = 180$) orbit type, Ω_{2211}.

In Boltzmann model all particles are considered to be distinguishable, so that a uniform probability can be assigned to each one of them. However, the passage from the accessible microstates to macrostates is equivalent to obtaining a partition of the ensemble V of accessible microstates into orbits of symmetry realized by the symmetric group acting on V according to the composition rule $s\tau^{-1}$. It is an important observation that the mean energy level

$$\overline{\mathcal{E}} = \frac{1}{\ell} \sum_{i} \mathcal{E}_i f_i, \tag{1.12}$$

where $f_i = |s^{-1}(\mathcal{E}_i)|$ indicates the number of molecules at the energy level $\mathcal{E}_i$, of any configuration in V, is an invariant under the composition rule $s\tau^{-1}$ and, therefore, depends only on the orbit (macrostate) realized by the configuration. Boltzmann reasoned that the molecule-energy configurations in V evolved from least probable configurations to most probable configurations, so that the quest for describing the equilibrium energy distribution in the ensemble requires the determination of the most likely configurations in V. This, in turn, requires the determination of the macrostate (orbit) with the largest volume Ω, conditioned on the fact that mean energy of the isolated ensemble must remain constant. Given a configuration s with f_1 particles at the energy level $\mathcal{E}_1$, f_2 particles at the level $\mathcal{E}_2$, f_3 particles at the level $\mathcal{E}_3$, etc, its orbit $\mathcal{O}_s$ has volume

$$|\mathcal{O}_s| = \frac{\ell!}{f_1!f_2!f_3!\ldots}. \tag{1.13}$$

We have then a well-defined mathematical problem: find the macrostate identified by $f_1, f_2, \ldots$ which maximizes (1.13) for a given mean energy level $\overline{\mathcal{E}}$. The solution is the Maxwell-Boltzmann canonical distribution,

$$P(\mathcal{E}_i) = \frac{e^{-\beta\mathcal{E}_i}}{\sum_j e^{-\beta\mathcal{E}_j}}. \tag{1.14}$$

The canonical distribution (its classical derivation is outlined in the appendix to this chapter) is the most likely energy distribution of the ensemble. Similar calculations can be obtained for the models of Fermi-Dirac and Bose-Einstein.

We conclude this example noting that the constrained minimization of $\sum f_i \ln f_i$ is equivalent to the constrained maximization of

$$H = -\sum_{i} \frac{f_i}{\ell} \ln(\frac{f_i}{\ell})$$

which is the *entropy* of the probability law associated with the orbit of $f_1, f_2, f_3, \ldots$. The entropy, usually indicated by S in thermodynamics, is a physical characteristic (e.g., temperature, mass) of the gas and at the same time, a measure of uniformity in its thermodynamical probability law. The canonical distribution corresponds to an ensemble configured to its maximum entropy. Boltzmann's statistical expression

$$S = k \ln \Omega$$

for the equilibrium entropy relates the equilibrium or limit number of accessible microstates, Ω, and k, the (known now as) Boltzmann constant 1.3807×10^{-23} K J/molecule. A volume of gas, left to itself, will almost always be found in the state of the most probable distribution. □

EXAMPLE 1.6 (Maxwell-Boltzmann Law for Velocities in a Perfect Gas). In this example we outline the classical derivation of Maxwell-Boltzmann Law. In the context of the orbit method, Maxwell's assumptions e.g., Ruhla (1989, Ch.4) led to the searching of a probability law, indicated here by F, for the random velocity vector (v) satisfying the following conditions: First, the component-velocities are statistically independent and identically distributed, so that the law F should have the form

$$F(v) = f(v_x)f(v_y)f(v_z),$$

where f indicated the common probability law for the component-velocities. The *isotropic condition* states that F should be invariant under all central rotations, indicated here by U, in the three-dimensional Euclidian space $\mathbb{R}^3$. Denoting by $S(3, \mathbb{R})$ the collection of all such rotations, we write the isotropic condition as,

$$F(Uv) = F(v), \text{ for all } U \in S(3, \mathbb{R}). \tag{1.15}$$

Note the analogy between the isotropic condition and the invariance condition described by Expression (1.4). These two conditions lead to the probability law which has the form

$$F(v) = A^3 e^{-\mu ||v||^2}, \tag{1.16}$$

where $v = ||v|| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the *speed* in the velocity vector v. The constants are determined from additional physical considerations. The orbits $\mathcal{O}_v$ in the quotient space are determined by the velocity vectors v in $\mathbb{R}^3$ with common speed v. □

The following table summarizes the symmetry relations introduced in Examples 1.3, 1.4, 1.5 and 1.6. In each case, a uniform probability law is assigned to the points within the corresponding orbits in the quotient space.

Model	Symmetries	Orbits
DNA sequences	all cyclic permutations, C_ℓ	V/C_ℓ
Maxwell-Boltzmann	1 (the identity)	V
Fermi-Dirac	all permutations, S_ℓ	V/S_ℓ
Bose-Einstein	all permutations, S_ℓ	V_I/S_ℓ
Velocity in a perfect gas	all rotations, $O(3, \mathbb{R})$	$\mathbb{R}^3/O(3, \mathbb{R})$

1.3. Observational aspects

We observe, measure or annotate something, indicated in the sequence by x(s), in each point s on the structure V. These measurements are mappings $x : V \to \mathcal{V} \subset \mathbb{R}^v$, or points in the (real or complex) vector space (indicated in the sequence by $\mathcal{F}$) of functions with domain V and image $\mathcal{V}$. Ttypically, $\mathcal{V}$ is a linear subspace of $\mathbb{R}^v$.

EXAMPLE 1.7. In each four-sequence in length of three, s, measure the number $x(s) = |s^{-1}(a)|$ of adenine residues in the sequence s. That is,

$$\begin{bmatrix} aaa & aac & aag & aat & caa & cac & cag & cat \\ aca & acc & acg & act & cca & ccc & ccg & cct \\ aga & agc & agg & agt & cga & cgc & cgg & cgt \\ ata & atc & atg & att & cta & ctc & ctg & ctt \\ gaa & gac & gag & gat & taa & tac & tag & tat \\ gca & gcc & gcg & gct & tca & tcc & tcg & tct \\ gga & ggc & ggg & ggt & tga & tgc & tgg & tgt \\ gta & gtc & gtg & gtt & tta & ttc & ttg & ttt \end{bmatrix} \xrightarrow{x} \begin{bmatrix} 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

□

EXAMPLE 1.8. If m_a, m_g, m_c, m_t indicate the molecular weight of the corresponding residue in the alphabet $\mathcal{A} = \{A, G, C, T\}$, then

$$x(s) = m_a|s^{-1}(a)| + m_c|s^{-1}(c)| + m_g|s^{-1}(g)| + m_t|s^{-1}(t)|$$

is the molecular weight of a four-sequence in lenth of four with $|s^{-1}(a)|$ adenines, $|s^{-1}(c)|$ cytosines, $|s^{-1}(g)|$ guanines and $|s^{-1}(t)|$ thymines. Consequently, relative to the probability law

$$\mathcal{L}_\lambda = (\frac{f_a}{3}, \frac{f_c}{3}, \frac{f_g}{3}, \frac{f_t}{3}),$$

the mean molecular weight

$$\bar{m}_\lambda = m_a\frac{f_a}{3} + m_c\frac{f_c}{3} + m_g\frac{f_g}{3} + m_t\frac{f_t}{3}$$

of s may be obtained. The molecular weights[4] are:

- $m_a = 135.128$ g/mol for adenine (amino-6-purine $C_5H_5N_5$);
- $m_g = 150.12$ g/mol for guanine (amino-2-hydroxy-6-purine $C_5H_4N_5O$);
- $m_c = 111.1$ g/mol for cytosine (2-hydroxy, 4 Amino-pyrimidine $C_4H_5N_3O$);
- $m_t = 126.1$ g/mol for thymine (2,4-dihydroxy-5-methyl-pyrimidine $C_5H_6N_2O_2$),

so that

$$\bar{m} = 132.116 \text{g/mol}, \quad \bar{m} = 137.116 \text{g/mol}, \quad \bar{m} = 127.109 \text{g/mol}, \quad \bar{m} = 129.106 \text{g/mol},$$

corresponding to the laws in (1.11). Intuitively, the molecular weight should not depend on the ordering of the sequence. In fact, $x(s\tau^{-1}) = x(s)$ for all $s \in V$, $\tau \in S_3$. □

EXAMPLE 1.9. Consider again the structure V of four-sequences in length of three, in which $\mathcal{A} = \{A, G, C, T\}$. Given a reference nucleotide sequence

ggtctctctggttagaccagatttgagcctgggagctctctggctaac

tagggaacccactgcttaagcccaataaagcttgccttgagtgcttcaagta

(in length of 100) from the same alphabet, measure the frequency

$$x(s) = \sum_{k=1}^{98} I_s(f^k),$$

[4]Refer to http://hamers.chem.wisc.edu/chapman/Other/chem.html for an on-line molecular weight calculator.

with which each sequence $s \in V$ occurs in the reference sequence. Here,

$$I_s(f^k) = \begin{cases} 1 & \text{if } s = f^k \\ 0 & \text{if } s \neq f^k \end{cases}$$

is the indicator function for a match between the sequence s and the sequence in length of three f^k starting at position k in the reference string. The following matrices summarize the structure and the resulting structured data:

$$\begin{bmatrix} aaa & aac & aag & aat & caa & cac & cag & cat \\ aca & acc & acg & act & cca & ccc & ccg & cct \\ aga & agc & agg & agt & cga & cgc & cgg & cgt \\ ata & atc & atg & att & cta & ctc & ctg & ctt \\ gaa & gac & gag & gat & taa & tac & tag & tat \\ gca & gcc & gcg & gct & tca & tcc & tcg & tct \\ gga & ggc & ggg & ggt & tga & tgc & tgg & tgt \\ gta & gtc & gtg & gtt & tta & ttc & ttg & ttt \end{bmatrix} \xrightarrow{x} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 3 & 2 & 0 & 2 \\ 2 & 4 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 4 & 4 & 4 \\ 1 & 1 & 3 & 1 & 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 5 & 1 & 0 & 0 & 5 \\ 2 & 1 & 2 & 2 & 2 & 3 & 3 & 0 \\ 1 & 1 & 1 & 1 & 2 & 1 & 3 & 1 \end{bmatrix}.$$

□

1.4. Connecting structures and data

We now have introduced an example of a structure V (e.g., all two-sequences in length of four), an example of a measurement $x : V \rightarrow \mathbb{R}$, such as the molecular weight, and shown how symmetries and rules for symmetry compositions (e.g., $s\tau^{-1}$) in V lead to a disjoint partition

$$V = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$$

of the initial structure into orbits of symmetrically similar objects. It is then of natural interest *representing* this same factorization into the vector space $\mathbb{R}^v$, where the observations, annotations or data vector $(x(s))_{s \in V}$ reside, and study the data accordingly. This is the role of the basic methods for linear representation of finite groups, to be discussed later on in Chapter 3. The following example will illustrate one of the consequences of linearly representing similarity relations into the data space.

EXAMPLE 1.10 (Averaging). The averaging of v real numbers is an implicit statement that, under certain conditions, the labels indexing these numbers have become irrelevant and, therefore, are all *similar* ($\sim$) to each other. A self-similarity relationship, on the other hand, dictates that each label is similar only to itself. These simplest forms of label relationship determine the important connection between the set of labels, V, and the space, $\mathcal{V}$, where the data $\{x(s); s \in V\}$ are represented. In the present case, the label similarities generate two matrices (say $v = 3$ and $\mathcal{V} = \mathbb{R}^3$, the usual Euclidean real vector space)

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

in correspondence to the similarity definitions indicated above, namely

$$J : i \sim j \iff J_{ij} = 1, \quad I : i \sim j \iff I_{ij} = 1.$$

Writing $\mathcal{P}_1 = \frac{1}{3}\mathrm{J}$ and $\mathcal{P}_2 = \mathrm{I} - \frac{1}{3}\mathrm{J}$, we note that

$$\mathrm{I} = \mathcal{P}_1 + \mathcal{P}_2, \tag{1.17}$$

with $\mathcal{P}_1^2 = \mathcal{P}_1$, $\mathcal{P}_2^2 = \mathcal{P}_2$ and $\mathcal{P}_1\mathcal{P}_2 = 0$. If we denote $e' = (1,1,1)$ and let $\bar{x}$ indicate the average of the components of x, then a consequence of these properties is that any vector $x \in \mathcal{V}$ can be expressed as the sum $m + r$ of a vector m in the one-dimensional space $\mathcal{V}_1$ generated by $\mathcal{P}_1 x = \bar{x}e$, and a vector r in the $(v-1)$-dimensional orthogonal complement $\mathcal{V}_2$ of $\mathcal{V}_1$ in $\mathcal{V}$. The decomposition

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2, \tag{1.18}$$

or $\mathrm{I} = \mathcal{P}_1 + \mathcal{P}_2$, in turn, leads to the decomposition of the corresponding square distances

$$x'x = x'\mathcal{P}_1 x + x'\mathcal{P}_2 x, \tag{1.19}$$

or

$$\sum_{s \in V} x(s)^2 = v\bar{x}^2 + \sum_{s \in V}(x(s) - \bar{x})^2,$$

which is characteristic of the Fisher-Cochran argument determining the probability distribution of quadratic forms and resulting statistical inference. □

1.5. Summary

In this chapter we introduced an example of a structure V on which the points $s \in V$ have the interpretation of indices or labels for measurements x(s). Some measurements or assessments are naturally subject to statistical variability, like the alignment frequencies described in Example 1.1. In fact, if the reference string is a string sampled from a global, larger, sequence then the resulting frequency x(s) is also random. Other measurements are intrinsically constant, such as the molecular weight, e.g., x(tta) = 387.328 g/mol. of the triplet tta. However, should one consider these triplets as random points s in the structure V, then x(s) becomes a random variable due to the randomness in the label space V.

The structure is then subject to a decomposition or factorization, the result of classifying as equivalent those points in the structure that are related according to a given definition of symmetry. These classes of equivalent points are the orbits of V and may lead to the determination of the *invariants* of the natural phenomena. The speed of the perfect gas and the number of purines in the biological sequences are examples of invariants determined from studying the structural orbits. Probability models that respect the factorization are said to be invariant under the factorization. These models assign a common probability to symmetrically equivalent points. In the next chapter we give continuation to the examples introduced above, providing additional detail to the connections among structure, data, symmetries and the factorization of the structured data.

1.6. Appendix

1.6.1. The canonical distribution. Using Stirling's approximation $\ln t! \equiv t\ln t - t$, we have,

$$\ln|\mathcal{O}_s| = \ln \ell! - \sum \ln f_i = \ell\ln\ell - \ell - \sum(f_i \ln f_i - f_i) = \ell\ln\ell - \sum f_i \ln f_i.$$

Equivalently, then, we seek to minimize $\sum f_i \ln f_i$ subject to (1.12). These two conditions lead to

$$(\ell + \sum_i \ln f_i) df_i = 0, \quad \sum_i \mathcal{E}_i df_i = 0.$$

A sufficient condition for the existence of a solution (using Lagrange multipliers argument) is that there are constants α and β satisfying $\sum_i(\mathcal{E}_i + \alpha + \beta \ln f_i) df_i = 0$, in which case the solutions take the form $f_i = \alpha e^{-\beta\mathcal{E}_i}$. The condition $\sum_i f_i = \ell$ implies $\alpha = \ell / \sum_i e^{-\beta\mathcal{E}_i}$, so that

$$f_i = \ell \frac{e^{-\beta\mathcal{E}_i}}{\sum_j e^{-\beta\mathcal{E}_j}}. \tag{1.20}$$

The value of β follows from the condition $\frac{1}{\ell}\sum_i f_i\mathcal{E}_i = \overline{\mathcal{E}}$. That is, β is a solution of

$$\frac{\sum_i e^{-\beta\mathcal{E}_i}\mathcal{E}_i}{\sum_j e^{-\beta\mathcal{E}_j}} = \overline{\mathcal{E}}.$$

From (1.20) we then obtain Maxwell-Boltzmann canonical distribution shown in equation (1.14).

1.6.2. Position-symmetry orbits in the structure V of two-sequences in length of four.

(1.21)

V =

$S_4\backslash s$	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
1	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
(34)	1	16	15	14	8	12	13	7	11	6	10	4	5	9	3	2
(23)	1	16	15	12	14	8	11	13	7	10	4	6	9	3	5	2
(24)	1	16	15	8	12	14	7	11	13	4	6	10	3	5	9	2
(12)	1	16	14	15	12	8	13	10	6	11	7	4	9	5	2	3
(13)	1	16	12	14	15	8	10	11	4	13	6	7	9	2	3	5
(14)	1	16	8	14	12	15	6	4	7	10	13	11	2	5	3	9
(234)	1	16	15	12	8	14	11	7	13	4	10	6	3	9	5	2
(243)	1	16	15	8	14	12	7	13	11	6	4	10	5	3	9	2
(123)	1	16	14	12	15	8	10	13	6	11	4	7	9	2	5	3
(124)	1	16	14	8	12	15	6	10	13	4	7	11	2	5	9	3
(132)	1	16	12	15	14	8	11	10	4	13	7	6	9	3	2	5
(134)	1	16	12	14	8	15	10	4	11	6	13	7	2	9	3	5
(142)	1	16	8	15	12	14	7	4	6	11	13	10	3	5	2	9
(143)	1	16	8	14	15	12	6	7	4	13	10	11	5	2	3	9
(12)(34)	1	16	14	15	8	12	13	6	10	7	11	4	5	9	2	3
(13)(24)	1	16	12	8	15	14	4	11	10	7	6	13	3	2	9	5
(14)(23)	1	16	8	12	14	15	4	6	7	10	11	13	2	3	5	9
(1234)	1	16	14	12	8	15	10	6	13	4	11	7	2	9	5	3
(1243)	1	16	14	8	15	12	6	13	10	7	4	11	5	2	9	3
(1324)	1	16	12	8	14	15	4	10	11	6	7	13	2	3	9	5
(1342)	1	16	12	15	8	14	11	4	10	7	13	6	3	9	2	5
(1432)	1	16	8	15	14	12	7	6	4	13	11	10	5	3	2	9
(1423)	1	16	8	12	15	14	4	7	6	11	10	13	3	2	5	9

Further reading

(1) The structure of short nucleotide sequences introduced this chapter appears, implicitly, in the work of Doi (1991) on evolutionary molecular biology. Related examples will be discussed in later chapters, with references to the works of Evans and Speed (1993) on phylogenetic trees and Dudoit and Speed (1999) on linkage analysis, among others;

(2) The argument of decomposing the data space according to similarity relations implied by the space of labels has a long history. It appears in the early work of James (1957) and Hannan (1965) under the name of relationship algebra, where the analyses of variance for classical experimental designs are obtained in analogy to the simple example described above. An early application of group theory is found in James (1954). The monographs of Diaconis (1988) and Eaton (1989) are now classic references in

the field of group representations and group invariance applications in probability and statistics;

(3) The notions of symmetry and symmetry in science are discussed in detail in the works of Joe Rosen (1975, 1995), including an accessible introduction to the mathematics of symmetry and the formulation of the *Symmetry Principle.* The classical introductory work of Hermann Weyl (1952) includes the notions of bilateral, translatory, rotational, ornamental and crystal symmetry;

(4) The interplay between the many aspects of knowledge and explanation in science is discussed by many contemporary authors, including Wilson (1998), also Howson and Urbach (1989), Gower (1997).

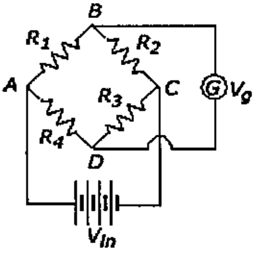
Exercises

1.1. Moments of the canonical distribution. Show that the mean $(\overline{\mathcal{E}})$ and variance (var $(\mathcal{E})$) of the canonical distribution can be expressed in terms of the *partition function* $Z = \sum_j e^{-\beta \mathcal{E}_j}$ as

$$\overline{\mathcal{E}} = -\frac{\partial \ln Z}{\partial \beta}, \quad \text{var } (\mathcal{E}) = \frac{\partial^2 \ln Z}{\partial \beta^2}.$$

1.2. The diagram of a basic Wheatstone bridge[5] circuit, shown in Figure 1.1, contains four resistances $\{r_1, r_2, r_3, r_4\}$, a constant voltage input V_{in}, and a voltage

FIGURE 1.1. Wheatstone bridge circuit



V_g, related by

$$V_g = \frac{r_1 r_3 - r_2 r_4}{(r_1 + r_2)(r_3 + r_4)} V_{in}.$$

Given a fixed set of resistors for which $V_g \neq 0$, consider the set

$$K = \{1, (12)(34), (13)(24), (14)(23)\}$$

of permutations of the index set $\{1, 2, 3, 4\}$ and define, for $\tau \in K$, the function

$$x(\tau) = \frac{r_{\tau 1} r_{\tau 3} - r_{\tau 2} r_{\tau 4}}{(r_{\tau 1} + r_{\tau 2})(r_{\tau 3} + r_{\tau 4})}.$$

Assume that $V_{in} = 1$ so that $x(\tau)$ is then the voltage measurement V_g when the resistors in the bridge are permuted according to $\tau \in K$.

[5] e.g., wttp://www.efunda.com/designstandards/sensors/methods/.

(1) Show that K, together with the operation of composition of functions, is a group, and conclude that x is an example of a scalar function defined on a group;

(2) Show that x can be written as $x(\tau) = \chi(\tau)x(1)$, where $\chi(\tau) \in \{1, -1\}$ and satisfies

$$\chi(\tau\sigma) = \chi(\tau)\chi(\sigma)$$

for all τ, σ in K;

(3) Indicate by V the set of all Wheatstone bridge circuits b_τ generated when the resistors are shuffled according to the permutations $\tau \in K$ and define two circuits b_σ and b_τ in V as *equivalent* when $x(\tau) = x(\sigma)$. Calculate the set V/K of orbits in V and then, conclude that the electric current *direction* through the measuring instrument is an invariant of the circuit;

(4) Open the wire between the node B and the instrument, and use the voltage potential V_g at that point to feed a second bridge inserted therein, that is V_g is V_{in} for the new bridge. Formulate the algebraic representation of the new combined, or cascaded, circuit and determine its invariants.

CHAPTER 2

Examples of structured data

2.1. Introduction

In the previous chapter we have introduced the following basic components of a symmetry study:

(1) a structure, or set of labels, V, such as the set C^L of all mappings

$$s : L = \{1, \ldots \ell\} \to C = \{1, \ldots, c\};$$

(2) a notion of *symmetry*, generally a group G of permutations acting on or composing with the elements of V;
(3) a similarity rule for pairs of points s, f in V, such as the letter-symmetry ($f = \sigma s$) or the position-symmetry ($f = s\tau^{-1}$) *compositions*;
(4) the resulting *orbits* $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of similar points in V;
(5) a scalar-valued *measurement* x defined in each point $s \in V$;
(6) *probability laws* for
   (a) labels $s \in V$;
   (b) data x(s) on a given label s;
   (c) data x;
   (d) symmetries $\sigma, \tau, \ldots$,
   necessary to describe the uncertainty in compositions such as $x(s\tau^{-1})$ or $x(\sigma s)$;
(7) a v-dimensional *linear representation* of the similarity relations and resulting orbits of the label space V into a data space such as $\mathbb{R}^v$;
(8) a set of *projections* $\mathcal{P}_1, \mathcal{P}_2, \ldots$ decomposing a linear subspace $\mathcal{V}$ of $\mathbb{R}^v$ where the data vector $x = (x(s))_{s \in V}$ obtains, and subsequent analyses and interpretation of these structured data.

In the following sections we will further detail these concepts with additional examples and comments.

2.2. Similarities and their linear representations

In Mendelian genetics, a character or trait that is produced by two alleles $\{A, a\}$ may be represented by the simple structure of all mappings $s : \{1, 2\} \to \{A, a\}$. The fact that, genetically, the relative position of the alleles is irrelevant, says that this space is subject to location-symmetry (S_2 acting on the left), with resulting orbits

$$\mathcal{O}_0 = \{aa\}, \ \ \mathcal{O}_1 = \{Aa, aA\} \ \ \mathcal{O}_2 = \{AA\}.$$

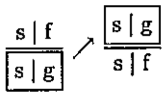
In terms of their phenotype, if $\mathcal{O}_0$ is the recessive trait, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are the dominant traits. Genotypically, $\mathcal{O}_1$ is the heterozygous pair, whereas the other two orbits contain the homozygous pairs. These traits, in turn, produce, as the result of the cell meiosis process, a new space of allele combinations consisting of all mappings $s : \{1,2,3,4\} \to \{A, a\}$ with exactly two alleles of each type. This space is a substructure of the space V of two-sequences in length of four, under the position-symmetries of S_4, introduced earlier on in Chapter 1, Example 1.3. More specifically, V decomposes as

$$V = V_{40} \cup V_{31} \cup V_{22},$$

with $V_{40} = \mathcal{O}_0 \cup \mathcal{O}_4$, $V_{31} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $V_{22} = \mathcal{O}_2$. The substructure of interest is exactly V_{22}. It corresponds to all mappings s with $|s^{-1}(a)| = |s^{-1}(A)| = 2$, namely,

$$V_{22} = \{AAaa, AaAa, AaaA, aAAa, aAaA, aaAA\}$$

indicated, respectively, by the labels $\{13, 11, 7, 10, 6, 4\}$ in Matrices (1.2) and (1.21) of Chapter 1. Fisher (1947) uses an argument of similarity relations to discover the number, k, of classes of equivalent alleles, when two alleles are considered similar if, after a crossover,



the two alleles f and g differ only by a location-symmetry, that is, $f = g\tau^{-1}$ for some $\tau \in S_4$, e.g., Hannan (1965). In what follows, we will outline the solution to Fisher's problem. Take for example the permutation $\tau = (34)$, which permutes the position of the third and forth alleles with each other. The resulting similarities are then

$$AAaa \sim AAaa, \quad aAAa \sim aAaA, \quad AaaA \sim AaAa, \quad aaAA \sim aaAA.$$

As a consequence, now we can associate to τ a permutation matrix $\rho(\tau)$ that operates in the associated space $\mathcal{V} = \mathbb{R}^6$, defined by the similarity relation between any two labels i, j in V, namely,

$$\rho_{ij} = 1 \iff i \sim j,$$

and otherwise $\rho_{ij} = 0$. In the present case, with $\tau = (34)$, the resulting similar pairs of labels are (13, 13), (10, 6), (7, 11) and (4, 4), so that we obtain the matrix representation

$$\rho(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

relative to the ordered set of labels $\{13, 11, 7, 10, 6, 4\}$. The matrix ρ is such that each row and column has exactly one entry equal to 1 and all other entries equal to 0, and is, therefore, a *permutation matrix*. We note that alleles 13 (AAaa) and 4 (aaAA) are self-similar. The collection of these permutation matrices, when τ varies over all 4! permutations in S_4, constitutes a *linear representation* of S_4 and establishes the connection between the set of labels, V_{22}, and the space of potential observations, $\mathcal{V}$, through the position-symmetries defined on V_{22}. Linear representations are studied later on in Chapter 3.

Fisher explored the fact that the trace $\chi(\rho)$ of each permutation matrix ρ describes exactly the number of pairs of alleles that are self-similar under the given permutation (or the number of points fixed by the permutation), and verified that the number k of classes (or orbits) of equivalent quads is then given by the average number

$$k = \frac{1}{4!}\sum_{\tau}\chi(\rho(\tau))$$

of fixed points $\chi(\rho(\tau))$. A systematic reading of Matrix (1.21) shows that the identity permutation fixes all 6 points in V_{22}, each one of the 6 transposition and each one of the 3 permutations conjugate to (12)(23) fix 2 points, whereas all the remaining have no fixed points. The total number of fixed points is then $6 + 9 \times 2$ and hence, the number of orbits is

$$\frac{1}{4!}(6 + 9 \times 2) = 1,$$

thus saying that all quads are equivalent. This is a direct and typical application of Burnside Lemma, which is introduced later on in Chapter 3 (see also Exercise 3.12). Clearly, this simply says that for any two given alleles, there is one symmetry in S_4 which makes them similar. The answer is certainly more interesting when m loci are simultaneously considered, with 6^m labels in the new (product) set $V_{22} \times V_{22} \times \ldots V_{22}$ of m copies of the original space V_{22}. The labels are all the (ordered) sets of four chromosomes obtained from two chromosomes allowing for meiosis for all forms of crossing over (keeping the loci in the same order shown the chromosome). The action, or composition rule, φ in this product space is now defined as

$$(s\tau^{-1}, g\tau^{-1}, \ldots, h\tau^{-1}), \quad \tau \in S_4, \ \ s, g, \ldots, h \in V_{22}.$$

This action on the product of the mapping spaces leads to a *tensor* representation of S_4 in $\mathbb{R}^{6m}$, in which the trace calculations are simply the product of the component traces. Thus, there are exactly

$$k = \frac{1}{4!}[6^m + 9.2^m]$$

orbits or classes of inequivalent types.

2.3. Data partitioning and amalgamation

The structured defined by the product set

$$V = \{\text{low, high}\} \times \{\text{slow, fast}\}$$

is characteristic of a two-factor factorial experiment of, say, a chemical compound tested under these conditions of temperature and reaction time. Similarity relations among the labels in V are discussed in detail later on in Chapter 5. These relations and their linear representations in the data space $\mathcal{V} = \mathbb{R}^4$, lead to four natural projections in this space, namely, $\mathcal{P}_1 = \frac{1}{4}ee'$, where $e' = (1, 1, 1, 1)$,

$$\mathcal{P}_2 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad \mathcal{P}_3 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

and

$$\mathcal{P}_4 = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Note, characteristically, that

$$\mathrm{I} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4, \quad \mathcal{P}_i\mathcal{P}_j = 0, i \neq j, \quad \mathcal{P}_i^2 = \mathcal{P}_i, \quad \mathcal{P}_i' = \mathcal{P}_i, \ i, j = 1, 2, 3, 4.$$

In the (balanced) case in which the same number of measurements is obtained in each point of the structure, the usual main effects and first-order interactions for temperature and reaction time are then determined by calculating the corresponding values of $\mathcal{P}x$, where $x \in \mathcal{V}$ is the chemical compound measurements observed in the $\mathcal{V}$ space. Similarly to the decomposition shown in (1.19), the resulting analysis of variance associated with the partition $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_4$ of the (vector) space of observables, $\mathcal{V}$, is

$$x'x = x'\mathcal{P}_1x + x'\mathcal{P}_2x + x'\mathcal{P}_3x + x'\mathcal{P}_4x.$$

In classical Mendelian genetics breeding experiments, the same structure and similarity argument leads to the usual *partitioning of deviations* concept, as known to geneticists. For example, the set of labels

$$V = \{\text{pale, not pale}\} \times \{\text{ruby, not ruby}\}$$

may refer to ear and eye trait characteristics of four kinds of mice. The data in the $\mathcal{V}$ space may be the observed frequencies of breeding types when a given number of mice are studied, e.g., Green (1981, p.36). Here, if we denote by $x' = (136, 3, 4, 133)$

ears	eyes	frequency
not pale	not ruby	136
pale	not ruby	3
not pale	ruby	4
pale	ruby	133

the vector of observed frequencies, the interpretation that is given to each of the projections $\mathcal{P}_i x$ depends on whether or not there is a well-defined notion of *addition* among the components of x; that is, whether or not the space $\mathcal{V}$ carries with it a definition of vector space consistent with the interpretation of the data at hand. As discussed later, special considerations will be required when the data reside on a lower-dimension subspace of the space $\mathcal{V}$. This is the case when there are natural restrictions, such as

$$n = x(1) + \ldots + x(v),$$

among the components of the data indexed by the structure V. The broad interpretation of each projection matrix is that of informing which components of the data vector should be *compared* or *amalgamated.* This is an important concept, which ties the notions of structured data and their symmetry relations with the basic operation of amalgamation of data defined at lower-dimension subspaces. Statistical inferences under such restrictions is termed *simplicial* or *compositional* inference, and have a long tradition in fields such as chemistry and geology, e.g., Aitchison

(2001) and comments in Chapter 4. For example, the interpretation given to the one-dimensional projection

$$\mathcal{P}_2 \mathrm{x} = \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 136 \\ 3 \\ 4 \\ 133 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \end{bmatrix}$$

is that of comparing the joint frequency ($140 = 136 + 4$) of mice without pale ears with the frequency ($136 = 3 + 133$) of mice with pale ears. Similarly, $\mathcal{P}_3\mathrm{x}$ leads to comparing the joint frequency (139) of mice without ruby eyes with the frequency (137) of mice with ruby eyes, whereas $\mathcal{P}_4\mathrm{x}$ leads to comparing the joint frequency (269) of mice in which the ear-eye traits are *positively* associated with the frequency (7) of mice in which the traits are *negatively* associated. The geneticist, based on the usual χ^2 analysis, interprets these data as indicating that the ear types and the eye types apparently did not combine at random.

2.4. Bilateral symmetries

Symmetry-asymmetry questions are very frequently encountered in biomedical areas, and most commonly when bilateral biological systems are investigated. Examples include symmetry perception in normal vision, visual signaling by asymmetry, modeling of auditory spatial receptive fields, molecular basis of left-right asymmetry, patterns of plenum temporale symmetry/asymmetry and many more. Normal vision requires that objects remain identifiable when rotated, scaled or translated in a space-time domain. Symmetry also plays a (visible!) role in beauty and evolution, as recently argued by Enquist and Arak (1994). The following examples will illustrate how these questions translate into similarity relations adequate to suggest the factorization of the data structures associated to them.

EXAMPLE 2.1 (Lateral and contralateral comparisons). Consider an experiment in which two measurements, such as the visual acuity, indicated here by the symbols ($\triangleleft\ \triangleright$) and the intra ocular pressure, denoted by ($<\ >$), can be observed in the left $\{\triangleleft,\ <\}$ eye and in right $\{\triangleright,\ >\}$ eye. Indicate by A the set {acuity, pressure} of these ocular assessments, and by E the set {left eye, right eye} indicating the eye in which the assessments in A are obtained. Relative comparisons between two localized assessments can be represented by the structure

$$\mathrm{V} = \mathrm{C}^{\mathrm{L}} = \left[\begin{array}{cccccccccccccccccc} \mathrm{s} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \mathrm{s}(1) = & \triangleleft & \triangleright & < & > & \triangleleft & \triangleleft & \triangleleft & \triangleright & \triangleright & \triangleright & < & < & < & > & > & > \\ \mathrm{s}(2) = & \triangleleft & \triangleright & < & > & \triangleright & < & > & \triangleleft & < & > & \triangleleft & \triangleright & > & \triangleleft & \triangleright & < \end{array} \right]$$

of all mappings $\mathrm{s} : \mathrm{L} \to \mathrm{C}$, where $\mathrm{L} = \{1, 2\}$ and $\mathrm{C} = \mathrm{A} \times \mathrm{E}$. The structure V has $\mathrm{v} = \mathrm{c}^{\ell} = 16$ mappings, labeled $1, \ldots, 16$. A commonly used relative comparison x(s) of interest is the covariance, or any other measure of association, between the (random) measurements represented by s(1) and s(2). For example, if s is the mapping number 9 in the table above, then x(s) stands for the covariance between the acuity in the right eye and the pressure in the left eye. Equivalently, in this case, we may write the structure V in the form of the 4×4 matrix

$$
\mathrm{V} = \left[\begin{array}{ccccc}
 & \triangleright & \triangleleft & > & < \\
\triangleright & . & . & . & . \\
\triangleleft & . & . & . & . \\
> & . & . & . & . \\
< & . & . & . & .
\end{array} \right],
$$

and interpret the entries as the labels in $\mathrm{A} \times \mathrm{E}$. The symmetries of interest are those permuting the location of the assessment or those permuting the type of assessment. Therefore, these symmetries consist of a simple transposition (σ) and the identity transformation (1). Suppose first that these symmetries act in each of the two sets with the result that when transposition of eye location is to be followed with a transposition in the type of visual assessment. The corresponding composition rule, or action, φ_1 of $\mathrm{S}_2 = \{1, (12)\}$ on V may be represented as

$$
\varphi_1(\sigma, \mathrm{s}) = (\sigma \mathrm{s}_\mathrm{a}, \sigma \mathrm{s}_\mathrm{e}), \quad \sigma \in \mathrm{S}_2,
$$

where $(\mathrm{s}_\mathrm{a}, \mathrm{s}_\mathrm{e})$ are the components of the mapping s. In turn, the action φ_1 classifies the points in V by similarity or equivalence: two points s and f in V are equivalent when related by

$$
\mathrm{f} = \varphi_1(\sigma, \mathrm{s}),
$$

for some σ in S_2. The resulting classes of equivalence, as will be shown in Chapter 3, can be linearly represented by assigning to each $\sigma \in \mathrm{S}_2$ a linear transformation $\rho(\sigma)$, and forming certain averages of these representations. The result is a family of projection matrices, $\mathcal{P}_1, \ldots, \mathcal{P}_\mathrm{r}$, which will leave invariant exactly those subspaces of $\mathcal{V} = \mathbb{R}^\mathrm{v}$ associated with the equivalence classes in V. For example, the trivial projection

$$
\mathcal{P}_1 = \frac{1}{2}[\rho(1) + \rho(\sigma)] = \frac{1}{2}
\left[\begin{array}{cccccccccccccccc}
1 & 1 & & & & & & & & \cdot & & & & & & \\
1 & 1 & & & & & & & & \cdot & & & & & & \\
 & & 1 & 1 & & & & & & & & & & & & \\
 & & 1 & 1 & & & & & & & & & & & & \\
 & & & & 1 & 0 & 0 & 1 & 0 & 0 & & & & & & \\
 & & & & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & \\
 & & & & 0 & 0 & 1 & 0 & 1 & 0 & & & & & & \\
 & & & & 1 & 0 & 0 & 1 & 0 & 0 & & & & & & \\
 & & & & 0 & 0 & 1 & 0 & 1 & 0 & & & & & & \\
 & & & & 0 & 1 & 0 & 0 & 0 & 1 & & & & & & \\
 & & & & & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 1 & 0 & 0 & 1 \\
 & & & & & & & & & & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 1 & 0 & 0 & 1
\end{array} \right]
$$

is, in the present case, a direct sum of four component projections. There are 8 resulting classes of equivalence, or orbits

$$
\mathcal{O}_1 = \{1, 2\}, \quad \mathcal{O}_2 = \{3, 4\},
$$

$$
\mathcal{O}_{31} = \{5, 8\}, \quad \mathcal{O}_{32} = \{6, 10\}, \quad \mathcal{O}_{33} = \{7, 9\},
$$

$$
\mathcal{O}_{41} = \{11, 15\}, \quad \mathcal{O}_{42} = \{12, 14\}, \quad \mathcal{O}_{43} = \{13, 16\},
$$

in V, indicated by the mappings' labels. These orbits may be summarized in the matrix below, in which repeated letters indicate equivalent points (sharing the same orbit):

$$\mathrm{S}_2 \to \begin{bmatrix} & \triangleright & \triangleleft & > & < \\ \triangleright & \mathrm{a} & \mathrm{u} & \mathrm{x} & \mathrm{z} \\ \triangleleft & \mathrm{u} & \mathrm{a} & \mathrm{z} & \mathrm{x} \\ > & \mathrm{y} & \mathrm{w} & \mathrm{b} & \mathrm{v} \\ < & \mathrm{w} & \mathrm{y} & \mathrm{v} & \mathrm{b} \end{bmatrix}. \tag{2.1}$$

Additional symmetry may be considered by allowing separate transformations acting on each of the component sets A and E. In this case, it is possible to fix one component and apply a transposition on the other component. The resulting set of transformations is the *product* group

$$\mathrm{S}_2 \times \mathrm{S}_2 = \{(1,1),(1,\mathrm{t}),(\mathrm{t},1),(\mathrm{t},\mathrm{t})\},$$

with componentwise multiplication. The resulting action, φ_2, on V is now given by

$$\varphi_2((\sigma,\tau),\mathrm{s}) = (\sigma \mathrm{s_a}, \tau \mathrm{s_e}), \quad (\sigma,\tau) \in \mathrm{S}_2 \times \mathrm{S}_2.$$

Similarly, two points s and f in V are equivalent when they are related by $\mathrm{f} = \varphi_2((\sigma,\tau),\mathrm{s})$, for some (σ,τ) in $\mathrm{S}_2 \times \mathrm{S}_2$. The trivial projection is now

$$\mathcal{P}_1 = \frac{1}{4}\begin{bmatrix}
1&1&1&1&&&&&&&&&&&& \\
1&1&1&1&&&&&&&&&&&& \\
1&1&1&1&&&&&&&&&&&& \\
1&1&1&1&&&&&&&&&&&& \\
&&&&1&0&0&1&0&0&0&0&1&0&0&1 \\
&&&&0&1&0&0&0&1&1&0&0&0&1&0 \\
&&&&0&0&1&0&1&0&0&1&0&1&0&0 \\
&&&&1&0&0&1&0&0&0&0&1&0&0&1 \\
&&&&0&0&1&0&1&0&0&1&0&1&0&0 \\
&&&&0&1&0&0&0&1&1&0&0&0&1&0 \\
&&&&0&1&0&0&0&1&1&0&0&0&1&0 \\
&&&&0&0&1&0&1&0&0&1&0&1&0&0 \\
&&&&1&0&0&1&0&0&0&0&1&0&0&1 \\
&&&&0&0&1&0&1&0&0&1&0&1&0&0 \\
&&&&0&1&0&0&0&1&1&0&0&0&1&0 \\
&&&&1&0&0&1&0&0&0&0&1&0&0&1
\end{bmatrix},$$

which defines the equivalence relations among the data points in $\mathcal{V}$ indexed by V. There are 4 resulting orbits,

$$\mathcal{O}_1 = \{1,2,3,4\}, \quad \mathcal{O}_2 = \{5,8,13,16\}, \quad \mathcal{O}_1 = \{6,10,11,15\}, \quad \mathcal{O}_1 = \{7.9.12.14\},$$

illustrated in the diagram that follows;

$$\mathrm{S}_2 \times \mathrm{S}_2 \to \begin{bmatrix} & \triangleright & \triangleleft & > & < \\ \triangleright & \mathrm{a} & \mathrm{u} & \mathrm{x} & \mathrm{z} \\ \triangleleft & \mathrm{u} & \mathrm{a} & \mathrm{z} & \mathrm{x} \\ > & \mathrm{x} & \mathrm{z} & \mathrm{a} & \mathrm{u} \\ < & \mathrm{z} & \mathrm{x} & \mathrm{u} & \mathrm{a} \end{bmatrix}. \tag{2.2}$$

We conclude this example by pointing that the reductions or factorizations indicated in Matrices (2.1) and (2.2) are the only ones that result from symmetry actions on the components of $A \times E$. In particular, we learn that neither one of these reductions lead to comparisons of data points indexed by the x and z components shown in Matrices (2.1) and (2.2). That is, data indexed by $(\triangleright, >)$ and $(\triangleright, <)$ are not related by similarity relations. The same is true for $(\triangleleft, <)$ and $(\triangleleft, >)$. In the case, as illustrated in the present example, that x(s) defines the covariance between s(1) and s(2), the interpretation is that the acuity-pressure covariance block cannot be reduced beyond the form

$$\begin{bmatrix} x & z \\ z & x \end{bmatrix}, \quad x \neq z.$$

Consequently, we also learn that a parametric hypothesis such as $x = z$ does not follow from component-symmetry considerations alone. □

EXAMPLE 2.2 (Symmetry and homogeneity). Now think of the components of $A \times E$ in Example 2.1 as labels for a common bilateral observation (e.g., a measure of localized retinal thickness) taken at two different points in time. Denote the bilateral measurements taken before treatment by $(\triangleleft, \triangleright)$ and the post treatment measurements by $(<, >)$. A larger set of symmetries may then be considered, relative to the rules φ_1 and φ_2 discussed above. In the extreme case, we may now apply the full set of permutations on the 4 components of $C = \{\triangleleft, \triangleright, <, >\}$, with the rule

$$\varphi_3(\sigma, s) = \sigma s, \quad \sigma \in S_4, \quad s \in V.$$

The space V has $v = 2^4$ labels and the resulting 2 orbits or classes of equivalent mappings are indicated in Matrix (2.3). All *variances* or *closed loops* are pooled into one orbit and all *covariances* or *open loops* are combined into the other orbit:

$$(2.3) \qquad S_4 \rightarrow \begin{bmatrix} & \triangleright & \triangleleft & > & < \\ \triangleright & a & u & u & u \\ \triangleleft & u & a & u & u \\ > & u & u & a & u \\ < & u & u & u & a \end{bmatrix}.$$

This is a nearly homogeneous (single orbit) decomposition of the initial structure V. Here are other symmetries of interest, resulting from simple transpositions and the identity transformation. They decompose the space V into 8 orbits indicated in the corresponding matrices by the entries with common letters:

$$(2.4) \qquad \{1, (12)(34)\} \rightarrow \begin{bmatrix} & \triangleright & \triangleleft & > & < \\ \triangleright & a & u & x & z \\ \triangleleft & u & a & z & x \\ > & y & w & b & v \\ < & w & y & v & b \end{bmatrix},$$

$$(2.5) \qquad \{1, (13)(24)\} \rightarrow \begin{bmatrix} & \triangleright & \triangleleft & > & < \\ \triangleright & a & v & w & y \\ \triangleleft & u & b & z & x \\ > & w & y & a & v \\ < & z & x & u & b \end{bmatrix},$$

$$\{1,(14)(23)\} \to \begin{bmatrix} & \rhd & \lhd & > & < \\ \rhd & a & v & y & w \\ \lhd & u & b & z & x \\ > & x & z & b & u \\ < & w & y & v & a \end{bmatrix}. \tag{2.6}$$

Each set of symmetries leads to potentially distinct interpretations of the underlying structured data. □

2.5. Cyclic symmetries

Consider the set of permutations

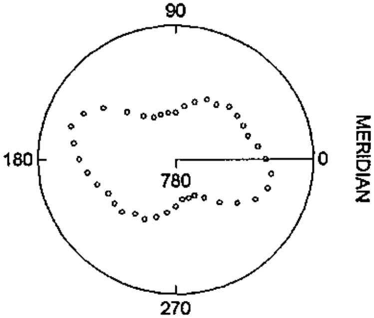
$$C_4 = \{1,(1234),(13)(24),(1432)\}$$

acting on the mapping space V of Example 2.1 by cyclically permuting the entries in the set C according to the rule $\varphi(\sigma, s) = \sigma s$. The resulting action leads to 4 orbits, indicated by Matrix (2.7),

$$C_4 \to \begin{bmatrix} & \rhd & \lhd & > & < \\ \rhd & a & v & x & u \\ \lhd & u & a & v & x \\ > & x & u & a & v \\ < & v & x & u & a \end{bmatrix}. \tag{2.7}$$

The interpretation of these orbits are suggested by applications in which the points of C are thought of labels for sensors located $\pi/4$ radians apart along a circle. In this case, the space V are labels for paired comparisons between points s(1) and s(2) located in two concentric rings, with, say, s(1) in the outer ring and s(2) in the inner ring. These comparisons arise, for example, in the analysis of data from human corneal curvature, and are discussed later on in Chapter 6. Figure 2.1 illustrates a sample of corneal curvatures (in units proportional to the curvature in mm.) obtained at points $\pi/20$ radians apart along a imaginary ring centered at the cornea's apex. The structure in the present example represents two of these rings,

FIGURE 2.1. Corneal curvature data with 9-degree meridian separation.



each one with $\ell = 4$ points. The resulting orbits or classes of equivalence,

$$\mathcal{O}_0 \rightarrow \begin{bmatrix} s(1) = & \triangleleft & \triangleright & < & > \\ s(2) = & \triangleleft & \triangleright & < & > \end{bmatrix}, \quad \mathcal{O}_1 \rightarrow \begin{bmatrix} s(1) = & \triangleleft & \triangleright & < & > \\ s(2) = & > & \triangleleft & \triangleright & < \end{bmatrix},$$

$$\mathcal{O}_2 \rightarrow \begin{bmatrix} s(1) = & \triangleleft & \triangleright & < & > \\ s(2) = & < & > & \triangleleft & \triangleright \end{bmatrix}, \quad \mathcal{O}_3 \rightarrow \begin{bmatrix} s(1) = & \triangleleft & \triangleright & < & > \\ s(2) = & \triangleright & < & > & \triangleleft \end{bmatrix},$$

describe potential comparisons

$$x(s) = (x(s(1)), x(s(2)))$$

when the two rings are off-phase by $\pi i/4$ radians, $i = 0, 1, 2, 3$. A data summary of interest is the within-orbit sums

$$\sum_{s \in \mathcal{O}_i} x(s(1))x(s(2)) = \sum_{j=1}^{4} x(j)x(i-j),$$

$i = 0, 1, 2, 3$. This is an example of a function defined on the group $\mathbb{Z}_3$ of integers with addition modulo 3. Its Fourier transform at the principal frequency is simply

$$\hat{x} = \sum_{i} \sum_{s \in \mathcal{O}_i} x(s(1))x(s(2)) = \sum_{j} (x \star x)(j) = 4\bar{x}^2, \tag{2.8}$$

where $x \star x$ indicates the two-fold convolution (in $\mathbb{Z}$ with addition modulo 3) of x.

EXAMPLE 2.3 (Cyclic symmetries). The mapping structure from the previous example is helpful for describing the following data, discussed by Wit and McCullagh (2001). The mapping structure $V = C^L$ is defined with

TABLE 2.1. Frequency of mining disasters between 1851-1962.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	
Autumn	7	10	5	5	6	7	1	41
Winter	5	9	10	10	11	7	0	52
Spring	3	7	10	12	13	9	2	56
Summer	4	8	8	9	5	6	2	42
	19	34	33	36	35	29	5	

$$L = \{1, 2\}, \quad C = \{\text{seasons of the year }\} \times \{\text{days of the week}\},$$

so that there are $v = c^\ell = 784$ labels or indices for potential paired annotations. The symmetries are now naturally imposed by sets of cyclic transformations applied to seasons of the year and to days of the week, indicated, respectively, by C_4 and C_7. The (product) action of these symmetries is given by

$$\varphi(\sigma, s) = (\sigma_1 s_1, \sigma_2 s_2), \quad \sigma = (\sigma_1, \sigma_2) \in C_4 \times C_7,$$

where (s_1, s_2) are the (season, day)-components of a mapping $s \in V$. A possible annotation of interest is $x(s) = |x(s(1)) - x(s(2))|$, where $x(s(j))$ is the frequency at entry $(s_1(j), s_2(j))$ given by Table 2.1.In contrast, the same data may be decomposed by considering the similarities defined in the product structure

$$V = \{\text{seasons of the year }\} \times \{\text{days of the week}\},$$

with 28 labels for potential comparisons. □

These examples illustrate the extent with which simple label structures such as the mapping space C^L or the product space $C \times L$ can be modified to accommodate different conditions and interpretations of similarities, including the set of symmetries (G), the rule (φ) with which these symmetries are applied to the structure V, and the annotations $x(s) \in \mathcal{V}$ of interest. In the following examples, we briefly describe other interpretations for these structures, related to the techniques to be developed in the sequence.

2.6. Linkage analysis

The segregation products or inheritance vectors for a sibship of size 2 can be represented by the set V of all mappings

$$s : C = \{1,2\} \to L_p \times L_m \equiv \{1,2\} \times \{3,4\},$$

where each mapping represents a possible configuration of chromosome pairs for the two sibships. The paternal chromosomes are labeled by $L_p = \{1,2\}$ and the maternal chromosomes by $L_m = \{3,4\}$. Indicate the components of each mapping s by

$$s(j) = (p(j), m(j)) \in L_p \times L_m, \quad j = 1, 2.$$

The *identity by descent* (IBD) of the inheritance vector s is given by

$$x(s) = \delta_{p(1)}(p(2)) + \delta_{m(1)}(m(2)),$$

where $\delta_S(\cdot)$ denotes the indicator function for the symbol S. Table 2.2 shows a number of inheritance vectors and their corresponding IBD values evaluated as x(s). With the additional notation introduced in Table 2.2 we write the space

TABLE 2.2

s(1)	s(2)	s	x(s)
a=(1,3)	a=(1,3)	1313 -aa	2
c=(2,3)	a=(1,3)	2313 -ca	1
c=(2,3)	b=(1,4)	2314 -cb	0
c=(2,3)	d=(2,4)	2324 -cd	1

of inheritance vectors as the space of all mappings $s : \{1,2\} \to \{a,b,c,d\}$. The space V has 16 labels or indices for possible IBD evaluations. Next, we consider the symmetries of interest. These are defined by the permutations

$$\sigma_1 = (ac)(bd), \quad \text{paternal chromosome symmetry,}$$

$$\sigma_2 = (ab)(cd), \quad \text{maternal chromosome symmetry,}$$

$$\sigma_3 = (bc), \quad \text{parental origin symmetry,}$$

which together generate the *dihedral group* D_4, e.g., Example 3.14, Chapter 3. These symmetries are realized as a three-dimensional rigid motion of the regular rectangle with vertices the chromosome pairs in the set $\{a,b,c,d\}$. The motions are defined as rotations of the rectangle through 90 deg angles about a perpendicular axis through its center, and as 180 deg rotations about an axis of symmetry on its plane. There are 8 elements in this group. These symmetries act on the structure V according to the rule $(\sigma, s) \to \sigma s$. The resulting orbits are shown in Matrix

(2.10). In addition, the sibship symmetry is generated by the single transposition $\tau = (12)$. It generates the symmetric group S_2 which then acts on the left by $(\tau, s) \to s\tau^{-1}$. Altogether, the symmetries acting on the label space define a group action or composition rule $\varphi : (S_2, D_4) \times V \to V$ given by

$$\varphi : ((\tau, \sigma), s) \mapsto \sigma s \tau^{-1}. \tag{2.9}$$

It then follows, as shown in Dudoit and Speed (1999), that the resulting classes of similar IBD-valued inheritance vectors will coincide with the orbits obtained from studying the group action given by Expression (2.9).

(2.10)

$D_4 \backslash s$	*aa*	*bb*	*cc*	*dd*	*ac*	*ca*	*ab*	*ba*	*cd*	*dc*	*bd*	*db*	*ad*	*da*	*bc*	*cb*
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$(abdc)$	2	4	1	3	8	7	11	12	5	6	10	9	15	16	14	13
$(ad)(bc)$	4	3	2	1	12	11	10	9	8	7	6	5	14	13	16	15
$(acdb)$	3	1	4	2	9	10	6	5	12	11	7	8	16	15	13	14
$(ac)(bd)$	3	4	1	2	6	5	9	10	7	8	12	11	16	15	14	13
$(ab)(cd)$	2	1	4	3	11	12	8	7	10	9	5	6	15	16	13	14
(ad)	4	3	2	1	10	9	12	11	6	5	8	7	14	13	15	16
(cd)	1	3	2	4	7	8	5	6	11	12	9	10	13	14	16	15
IBD	2	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0
$\lvert\mathcal{O}\rvert$	4	4	4	4	8	8	8	8	8	8	8	8	8	8	8	8
$\mathcal{O}$	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3

,

The trivial projection, corresponding to these orbits, is given by

$$\mathcal{P}_1 = \frac{1}{16}\left[\begin{array}{cccccccccccccccc}
2&2&2&2&&&&&&&&&&&& \\
2&2&2&2&&&&&&&&&&&& \\
2&2&2&2&&&&&&&&&&&& \\
2&2&2&2&&&&&&&&&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&1&1&1&1&1&1&1&1&&&& \\
&&&&&&&&&&&&2&2&2&2 \\
&&&&&&&&&&&&2&2&2&2 \\
&&&&&&&&&&&&2&2&2&2 \\
&&&&&&&&&&&&2&2&2&2
\end{array}\right].$$

2.7. Binary mappings

Consider the space V of all *Boolean mappings* $s : L_1 \times L_2 \to \{0, 1\}$ together with a *topography* data, or gray scale,

$$h : L_1 \times L_2 \to \{0, 1, \ldots, 256\},$$

defined in the chart $L = L_1 \times L_2$. A binary mapping is simply a subset s of $L_1 \times L_2$. Each binary mapping $s \in V$ selects or marks (with a 1, say,) a subset of locations in the chart L where a survey (or average gray level)

$$x(s) = \sum_{j \in L} s(j)h(j)$$

of heights h(j) is obtained. Surveys can be defined over the *contours*

$$\Gamma_0 = \{j \in L : s(j) = 0\}, \quad \Gamma_1 = \{j \in L : s(j) = 1\}.$$

The case

$$L = \{j\frac{\pi}{2} : \ j = 0,1,2,3\} \times \{1,2,3\},$$

for example, represents a 12-point discrete chart based on 3 concentric rings and 4 semimeridians (90-degree separation). The space V of binary mappings has 2^{12} distinct labels for potential surveys $x(s) \in \mathcal{V}$. Natural reductions of the data space $\mathcal{V}$ follow from considerations of semimeridian-symmetry in the V space, in which two mappings s and f are considered equivalent if

$$f(j_1, j_2) = s(j_1, \tau(j_2)), \quad \tau \in S_3,$$

or from ring-symmetry, in which the equivalence is given by

$$f(j_1, j_2) = s(\alpha(j_1), j_2), \quad \alpha \in C_4.$$

Also note all data indexed by a matrix structure can be framed within the present construction. In fact, any $c \times c$ matrix A with entries $a_{ii'}$ can be interpreted as a realization of a scalar function x defined on the structure $V = L \times C$, with $L = \{1, 2\}$ and $C = \{1, 2, \ldots, c\}$, in which $x(s) = a_{s(1),s(2)}$ is defined.

2.8. Exchangeability

Here is a well-known setting: Consider an urn with 5 distinct marbles numbered $1, 2, 3, 4, 5$, each one of color, say, yellow (y) or green (g). The possible urn structures may be considered as non-observable events whereas the color or the number of a marble drawn from the urn are observable events. The urn compositions are represented by all mappings (s) in $V = C^L$, with $L = \{1,2,3,4,5\}$ and $C = \{y, g\}$. Here it is natural to classify the possible structures by the number of, say, yellow marbles. That is, making the marbles distinguishable by color only. To accomplish this, we let the permutations in S_5 act on the label space V according to the rule

$$\varphi(\tau, s) = s\tau^{-1}$$

and count two configurations s and f as equivalent when there is a permutation τ connecting (via $f = s\tau^{-1}$) the two mappings. The resulting classes of equivalent mappings are exactly the sets of urn compositions with $0, 1, \ldots, 5$ yellow marbles, and the *exchangeable probability laws* in V are convex combinations of those laws assigning equal or uniform probability to equivalent members. All binary sequences are representable in this space, and finite forms of exchangeability can be defined, with the resulting finite-type De Finetti theorems.

EXAMPLE 2.4 (Partial exchangeability). Consider again the urn with 5 distinct marbles with the added information that marbles 1, 2 and 3 are larger in volume than marbles 4 and 5. The size-related or partial exchangeability erases the number-labels within each one of the two groups separately. The equivalent mappings representative of partially exchangeable sequences arise from the product of the space $V_1 = L_1^C$ and $V_2 = L_2^C$, where $L_1 = \{1, 2, 3\}$ and $L_2 = \{4, 5\}$. Two mappings (s, f) and (u, v) in $V_1 \times V_2$ are equivalent when there are permutations τ in S_3 and σ in S_2 such that

$$(s, f) = (u\tau^{-1}, v\sigma^{-1}).$$

The resulting classes of similar mappings are exactly the sets of configurations with k_1 yellow smaller marbles and k_2 yellow larger marbles, $k_1 = 0, 1, 2, 3$, $k_2 = 0, 1, 2$. If we indicate by $|\mathcal{O}_i|$ the size of the equivalence class (under the corresponding partial symmetry) for marbles of the same size, then the partially exchangeable probability laws are convex combinations of laws assigning equal or uniform probability

$$\frac{1}{|\mathcal{O}_1||\mathcal{O}_2|}$$

to equivalent members. □

EXAMPLE 2.5 (Bilateral exchangeability). Indicate the left eye by OS and the right eye by OD, and let $L = \{OS, OD\}$. Also, let $C = \{1, 0\}$, where 1 stands for the condition *the eye is examined with lense refraction*, and 0 stands for the condition *the eye is examined without lense refraction.* At each point s in the mapping space $V = C^L$ we annotate a numerical expression of the resulting visual acuity $x(s) \in \mathcal{V}$, or some frequency data related to the acuity response from a group of subjects. The hypothesis that the visual acuity response of the visual system due to lense refracting is indifferent to left-right indexing is described by making the permutations in S_2 act on V by $(\tau, s) \mapsto s\tau^{-1}$. Similarly to the previous examples, this action simplifies or factors the original space V (with 4 labels) into 3 equivalency classes, or orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, namely:

(1) $\mathcal{O}_1$ bilateral refraction, with 1 label,
(2) $\mathcal{O}_2$ monocular refraction, with 2 labels,
(3) $\mathcal{O}_3$ without refraction, with 1 label.

These orbits in the V space, in turn, will define the corresponding summaries and analysis in the data space $\mathcal{V}$. This example is applicable to any bilateral biological system as well. □

2.9. Homogeneous spaces

An important subset of the mapping space $V = C^L$ is the class S_ℓ of permutation mappings (bijective mappings, $|C| = |L|$). As indicated earlier, the mappings in S_ℓ, together with the operation of function composition, form a finite group. An example of a scalar function defined in, or indexed by, S_ℓ is voting preferences data where ℓ candidates are (completely) ranked according to the voters' preferences. The frequencies, $x(s)$, of each possible ranking $s \in S_\ell$ among the voters are the available data, as illustrated in Table 2.3, from Diaconis (1989). In this case, we will find that the symmetries imposed by S_ℓ on itself result in a single-orbit (homogeneous or transitive) space. The theory needed to decompose these elementary (regular) actions is the basis for all other cases illustrated in the previous examples. The basic results are detailed in Chapter 3.

EXAMPLE 2.6 (Frequency analysis). This illustration is a discrete version of the light polarization experiment,

$$\text{light particle} \rightarrow \text{polarizer sheet} \xrightarrow{\phi} \text{analyzer sheet} \xrightarrow{\cos^2 \phi} \text{detector},$$

TABLE 2.3. Three-candidate (228 voters) partial election data by rankings, corresponding permutations, s, and observed frequencies, x(s).

Ranking	s	x(s)
321	(13)	29
312	(132)	67
231	(123)	37
213	(12)	24
132	(23)	43
123	1	28

or Malus' law, e.g., Ruhla (1989, Section 7.3), in which a single photon leaves from the polarizer sheet and emerges from the analyzer sheet with a transmission probability $\cos^2\phi$, and fails to emerge with probability $\sin^2\phi$, where ϕ is the angle between the two sheets. Let $L = \{1,2\}$ and $C = \{i\frac{\pi}{2} : \quad i = 0,1,2,3\}$. The elements (s) in the structure V describe (a finite number of) the relative angles, $\phi = s(1) - s(2)$, between the polarizer and the analyzer, whereas the annotation

$$x(s) = \cos^2[s(1) - s(2)] \in \mathcal{V}$$

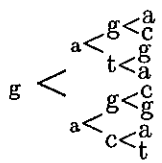
describes the (transmission) probability with which the particle reaches the detector. Of interest here is the relation between probability laws in V and probability laws in the space $\mathcal{V}$ of plausible states (described by corresponding probabilities). There are several tools to work with. Because the space C, together with the operation (equivalent to that) of adding integers modulo 3, is now a finite group, the structure V becomes endowed with a corresponding operation

$$(s+f)(j) = s(j) + f(j), \quad s, f \in V,$$

of sum of functions, and is also a finite group. Therefore, as suggested by equation (2.8), convolutions, deconvolutions and frequency analysis can be defined for the interpretation of data in the $\mathcal{V}$ space indexed by the mapping structure V. □

2.10. Trees

The diagram below illustrates a realization of a *binary tree* in which the *nodes* are indexed by the symbols in the alphabet (of nucleotide residues) $\mathcal{A} = \{A, G, C, T\}$.



The number of *generations* shown in the diagram in 3. The set $\mathcal{T}$ of all m-generation binary trees can be obtained by defining $L_j = \{1, \ldots, 2^j\}$, $V_j = \mathcal{A}^{L_j}$ and

$$\mathcal{T} = V_0 \times V_1 \times \ldots, \times V_m.$$

Any point $s = (s_0, s_1, \ldots, s_m) \in \mathcal{T}$ is a realization of a m-generation binary tree. In the tree illustrated in the above diagram, we have

$$s_0 = G \in V_0, \quad s_1 = AA \in V_1, \quad s_2 = GTGC \in V_2, \quad s_3 = ACGACGAT \in V_3.$$

Any *branch* in this tree has the form

$$s_0(1) \to s_1(i_1) \to s_2(i_2) \to s_3(i_3), \qquad i_j \in L_j.$$

A base $c \geq 2$ tree is defined similarly, starting with $L_j = \{1, \ldots, c^j\}$. The structure $\mathcal{T}$ has

$$|\mathcal{T}| = 4^{(c^{m+1}-1)/(c-1)}$$

points and may be factored or simplified, for example, by composing a permutation $\sigma \in S_4$ with each tree $s \in \mathcal{T}$ according to the rule

$$\varphi(\sigma, s) = (\sigma s_0, \sigma s_1, \ldots, \sigma s_m) \in \mathcal{T}.$$

EXAMPLE 2.7 (Phylogenetic trees). The problem of determining phylogenetic relations among certain species using nucleotide sequence data is central to studies in molecular evolution. In the work of Evans and Speed (1993), each tree is a candidate for the true phylogenetic tree describing the m-generation evolution of the observed present-day species corresponding to nucleotide sequence s_m. *Transition probabilities* along any branch can be obtained from the addition table

$$\left[\begin{array}{c|cccc} + & A & G & C & T \\ \hline A & A & G & C & T \\ G & G & A & T & C \\ C & C & T & A & G \\ T & T & C & G & A \end{array}\right], \tag{2.11}$$

defining a commutative group (called the Klein four-group) in the alphabet or base set $\mathcal{A} = \{A, G, C, T\}$. In fact, let x indicate a probability law in $\mathcal{A}$ and define,

$$P_t(u) = x(u - t), \quad t, u \in \mathcal{A}.$$

Then

$$\sum_{u \in \mathcal{A}} P_t(u) = \sum_{u \in \mathcal{A}} x(t - u) = \sum_{u \in \mathcal{A}} x(t + u) = \sum_{u \in \mathcal{A}} x(u) = 1,$$

the next to last equality being justified by observing that the orbit $\mathcal{O}_t = \{t + u; u \in \mathcal{A}\}$ of t coincides with $\mathcal{A}$ for all $t \in \mathcal{A}$. Consequently, P_t is a transition probability for each $t \in \mathcal{A}$, and the matrix X with entries $X_{tu} = x(t-u)$ determines a transition probability matrix for each fixed probability law x in $\mathcal{A}$. This leads to certain classes of transition matrices, such as the Kimura three-parameter model

$$X = \begin{bmatrix} & A & G & C & T \\ \hline A & 1-\sum_{u\in\mathcal{A}} x(u) & x(G) & x(C) & x(T) \\ G & x(A) & 1-\sum_{u\in\mathcal{A}} x(u) & x(T) & x(C) \\ C & x(C) & x(T) & 1-\sum_{u\in\mathcal{A}} x(u) & x(G) \\ T & x(T) & x(C) & x(G) & 1-\sum_{u\in\mathcal{A}} x(u) \end{bmatrix}.$$

The factorization of branching probabilities can be obtained from the action of the Klein group on $\mathcal{T}$. For example, the branch GATA from the tree illustrated in the above diagram has probability

$$\begin{aligned} P(GATA|x) &= x(A-G)x(T-A)x(A-T) = x(A+G)x(T+A)x(A+T) \\ &= x(G)x(T)x(T) = x(G)x^2(T), \end{aligned}$$

and is mapped into the branch (GG)(GA)(GT)(GA) = AGCG under the composition with $\sigma = G$. In turn,

$$\begin{aligned} P(AGCG|x) &= x(G-A)x(C-G)x(G-C) = x(G+A)x(C+G)x(G+C) \\ &= x(G)x(T)x(T) = x(G)x^2(T), \end{aligned}$$

so that these two branches are similar and the probability law P preserves the invariance. The study of these and other invariants is present in the work of Evans and Speed (1993). □

EXAMPLE 2.8 (Numerical trees). The Euclidean Algorithm for determining the greatest common divisor of two integers y_1, y_2 is based on repeated iterations of

$$F(y_1, y_2) = (\min\{y_1, y_2\}, \max\{y_1, y_2\} - \min\{y_1, y_2\}).$$

Starting at $y' = (y_1, y_2)$, after ℓ steps the calculation is at the point

$$F^{\ell}(y) = Ms(\ell)\dots Ms(2)Ms(1)y,$$

for some mapping or branch $s : \{1, 2, \dots, \ell\} \mapsto \{t, 1\}$, where

$$t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

At each step, s(j) permutes components of y into increasing order and M subtracts the first (smallest) component from the second (largest) component. The calculation stops when the algorithm reaches (d, 0), in which case d is the GCD of y_1, y_2. Selecting y according to a probability law turns the number ℓ of steps necessary to reaching the GCD is a random variable and s(j) into a random permutation matrix, $j = 1, \dots, \ell$. For example, if $y' = (24, 67)$, then, after $\ell = 11$ steps determined by the mapping or branch

$$s = (s(1), \dots, s(11)) = (1, 1, t, t, 1, 1, t, t, 1, 1, 1),$$

we obtain

$$F^{11}(y) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = M^4 g M^1 g M^3 g M^1 g M^2 y,$$

so that 1 is the GCD of (24, 64). The exponents [2, 1, 3, 1, 4] in M generate the simple continued fraction expansion

$$\frac{67}{24} = [2,1,3,1,4] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{4}}}}$$

of y_1/y_2 and can be obtained by the lengths of rises (runs of 1's) and falls (runs of t's) in s. Conversely, after ℓ steps starting at a given point (d, 0), there are 2^ℓ branches

$$\mathrm{Bt}^{s(1)}\mathrm{Bt}^{s(2)}\ldots\mathrm{Bt}^{s(\ell)}, \quad \mathrm{B} = \mathrm{M}^{-1},$$

connecting $y_0 = (d, 0)$ to 2^ℓ points y(s), and such that the GCD of y(s) is d. One event of interest is, for example, the random distance $x(s) = ||y(s) - y_0||$, where y_0 is selected according to a probability law. Then, the set of 2^ℓ branches can be factored by equivalence relations

$$\mathrm{Bg}^{s(1)}\mathrm{Bg}^{s(2)}\ldots\mathrm{Bg}^{s(\ell)} \sim \mathrm{Bg}^{s\sigma(1)}\mathrm{Bg}^{s\sigma(2)}\ldots\mathrm{Bg}^{s\sigma(\ell)}, \quad \sigma \in S_\ell$$

imposed by symmetry transformations $\sigma \in S_\ell$, as outlined in the previous examples.
□

2.11. Ordered structures

Let C be a completely ordered set with c objects, $\{t, u, v, \ldots\}$. The set V of all mappings from L (with ℓ labels) describes all the possible patterns of ordered observations for finite sets such as C. More specifically, observed $s \in V$, there is one permutation $\tau \in S_\ell$ ordering the components of s, that is,

$$s\tau(1) \leq s\tau(2), \leq \ldots \leq s\tau(\ell).$$

These are called *ranking permutations* and arise in the linear representation of order statistics and induced order statistics. In particular, the space of all mappings from L to L describes all possible patterns of repeated entries on any vector with ℓ objects. Also of interest here is the study of the relation between the probability law for s in V and the resulting probability law for the ordering permutations τ in S_ℓ.

2.12. Superimposed processes

Let $j \in \{1, 2, \ldots, \ell\}$ be the label for ℓ distinct cell types and $i \in \{0, 1, \ldots, c\}$ indicate the frequency with which these cell types are present in a cell configuration s in V. For example, in any cell configuration $s \in V$:

(1) s(j) is the number of j-type cells in s;
(2) $\sum_j s(j)$ is the total number of cells in s;
(3) $|s^{-1}(i)|$ is the number of distinct cell labels present exactly i times.

The only available data, $x = x(s)$, are the *cell variety* $x_i = |s^{-1}(i)|$, $i = 1, \ldots, c$. The objective is making inferences about the superimposed random process $\sum_{j=1}^{\ell} s(j)$, which describes the total number of cells, given the data x_i and a probability model in V.

2.13. Summary

We have introduced the basic components for the analysis of data indexed by a discrete structure. In the examples above, we considered structures such as the mapping space $V = C^L$, which includes the three basic operations in data reduction, namely,

(1) Selection, when $\ell < c$ and V is restricted to injective mappings;
(2) Amalgamation, when $\ell > c$ and V is restricted to surjective mappings;
(3) Relabeling, when $\ell = c$ and C^L is restricted to permutation mappings. This includes all problems of data indexed by (the permutation) groups.

Other discrete structures of interest are $V = L \times C$, and $V = L \cup C$, for example, where L and C are finite sets. The points $s \in V$ are labels making possible the identification of potential events, where some annotation or realization

$$x : s \in V \to x(s) \in \mathcal{V}$$

is obtained. Typically, $\mathcal{V}$ is an affine subspace of a real or complex vector space. Experimental questions dictate the symmetries of interest- typically those defined by a finite group (G) of transformations. These symmetries when applied to the labels in V, according to a definition rule φ, simplify or factor these labels into disjoint similarity classes

$$V = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_h,$$

which, in turn, can be represented as linear transformations in the data vector space $\mathcal{V}$. These linear transformations are permutation matrices, $\rho(\sigma)$, defined by the similarity relations, or, equivalently, defined by the changing of the canonical basis $\{e_s; s \in V\}$ of $\mathcal{V} = \mathbb{R}^v$ into the basis $\{e_{\varphi(\sigma,s)}; s \in V\}$, for each $\sigma \in G$. The resulting factorization

$$\mathcal{V} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_h$$

in the data vector space is the consequence of defining a set of orthogonal projections ($\mathcal{P}$), which are linear combinations of these permutation matrices $\rho(\sigma)$, for $\sigma \in G$, and real or complex scalar coefficients. In particular, if there are h of these linear combinations (projections), then

$$I = \mathcal{P}_1 + \mathcal{P}_2 + \ldots + \mathcal{P}_h,$$

so that the classic analysis of variance,

$$x'x = x'\mathcal{P}_1 x + x'\mathcal{P}_2 x + \ldots + x'\mathcal{P}_h x,$$

obtains.

The probabilistic setting includes a probability law, P, for the observable random component $x(s) \in \mathcal{V}$, which is then indexed by s, φ and G. The labels $s \in V$ may also be random, and a probability law, w, in V describes the uncertainty in s. Thus, the probability model for observables x taking values in $\mathcal{V}$ is

$$\mathcal{P} = \{P(\cdot \mid s) : s \in V, \varphi, G\}.$$

Within the Bayesian framework, observables, x, and non-observables indices (or parameters), s, are treated with equal relevance. The goal is deriving inferences for x and s, given observed data and a factorization of $\mathcal{V}$ determined by the symmetries in G and the composition rule φ.

Further reading

(1) The reader my consult Snedecor and Cochran (1989) for the basic notions of classical statistical inference, including the analysis of variance;
(2) The statistical aspects of quadratic forms (needed for the second-order analysis associated with the canonical projections) are developed, for example, in Rao (1973);
(3) The notion of points as labels identifying potential events appears in modern-day physics, in contrast to Newton's views in which points are essentially indistinguishable. A comment in that direction is found in Cartier (2001);
(4) The classic work of Fisher (1947) on the theory of linkage in polysomic inheritance;
(5) The characterization of cyclic symmetries in the study of purine and pyrimidine contents of local nucleotide sequences for evolution of human immunodeficiency virus type 1 is present in the work of Doi (1991);
(6) The works of Dudoit and Speed (1999) on linkage analysis and of Evans and Speed (1993), Billera, Holmes and Vogtmann (2001) on phylogenetic trees;
(7) The literature on the many aspects of symmetry in science and methodology is overly extensive. It ranges from studies considering the role of symmetry in beauty and evolution, e.g., Enquist and Arak (1994), anatomic symmetry between fellow eyes, e.g., Pauleikhoff, Wormald, Wright, Wessing and Bird (1992), parallel visual processes in symmetry perception in normal vision, e.g., Wagemans (1999), symmetry discrimination, e.g., Szlyk, Seiple and Xie (1995), Szlyk, Rock and Fisher (1995), visual signalling by asymmetry, e.g., Swaddle (1999), Tyler (1996), to the works of Graf and Schachman (1996) on random circular permutations of genes and polypeptide chains and of Hellige (1993) on hemispherical asymmetry. See also
hppp://mathforum.org/geometry/rugs/resources/biblio/ss.html
for a bibliography on symmetry and pattern;
(8) Symmetry and the covariance structure of symmetrically dependent observations, e.g., Lee and Viana (1999), Viana and Olkin (1998), Viana and Olkin (2000), Lee (1998);
(9) Invariant measures in groups and their uses in statistics e.g., Wijsman (1990);
(10) Invariance in factorial models, McCullagh (2000), quotient spaces and statistical models, McCullagh (1999);
(11) Algebraic statistics, e.g., Pistone, Riccomagno and Wyinn (2000).

Exercises

2.1. Following the notion from Section 2.10, consider the space $\mathcal{T}$ of two-generation binary trees with nodes indexed by the alphabet $\{u, v\}$. Here is one element $s \in \mathcal{T}$;

$$u <^{y}_{u} .$$

(1) Determine the number $|\mathcal{T}|$ of elements in $\mathcal{T}$;

(2) Determine the orbits of $\mathcal{T}$ under the composition rules

(a) $\varphi_1 : \quad (s_0, s_1) \to (\sigma s_0, \sigma s_1), \quad \sigma \in S_2$;

(b) $\varphi_2 : \quad (s_0, s_1) \to (\sigma s_0, \eta s_1), \quad \sigma, \eta \in S_2$;

(c) $\varphi_3 : \quad (s_0, s_1) \to (s_0, s_1 \tau^{-1}), \quad \tau \in S_2$.

CHAPTER 3

Algebraic aspects

3.1. Introduction

In this chapter we will review those concepts of linear representations of finite groups which are of interest to the analysis of structured data. In earlier chapters we have discussed a number of examples introducing a structure V, e.g., the mapping space, a group G of symmetries, and a rule φ for composing the symmetries (τ) with the elements (s) in V, such as position symmetries $s\tau^{-1}$. In addition, at each point s of V we measure something, obtaining the data $\{x(s); s \in V\}$. For example, Matrix 3.1 shows a frequency data x(s) observed at each one of the 16 two-sequences in length of four.

(3.1)

s(1)	y	u	y	u	u	u	y	y	y	u	u	u	y	y	y	u
s(2)	y	u	u	y	u	u	y	u	u	y	y	u	y	y	u	y
s(3)	y	u	u	u	y	u	u	y	u	y	u	y	y	u	y	y
s(4)	y	u	u	u	u	y	u	u	y	u	y	y	u	y	y	y
x(s) →	5	52	18	12	15	17	16	6	10	11	9	12	5	1	4	5

Fix any $s \in V$, say s = yyuu, and consider the sequences obtained by composing yyuu with each permutation (τ) in S_4 according to the rule $s\tau^{-1}$ (position symmetry). The resulting sequences are shown in the column corresponding to s = yyuu in Matrix (3.43) in Section 3.15, Tables and Graphic Displays, at the end of this chapter. The following matrices show the resulting frequency data $x(s\tau^{-1})$, as a function of the permutations $\tau \in S_4$:

τ	$x(s\tau^{-1})$	τ	$x(s\tau^{-1})$	τ	$x(s\tau^{-1})$
1	16	(243)	10	(13)(24)	12
(34)	16	(123)	11	(14)(32)	12
(23)	6	(124)	9	(1234)	11
(24)	10	(132)	6	(1243)	9
(12)	16	(134)	11	(1324)	12
(13)	11	(142)	10	(13)(42)	16
(14)	9	(143)	9	(14)(32)	10
(234)	6	(12)(34)	16	(1423)	12

Consequently, we may associate to each $s \in G$ a scalar-valued function $\tau \mapsto x(s\tau^{-1})$ defined in the group G. The study of the vector space $\mathcal{G}$ of scalar (vector, matrix or operator)-valued functions defined on a group G is, therefore, of natural interest to the analysis of structured data and their symmetries. In the following sections we will study a number of these functions, leading to the theorem describing the projections in the vector space of the measurements x(s) obtained at the structure of interest, V. In the sequence, unless indicated otherwise, all groups are finite. We write $|G|$ or g to indicate the number of elements in the group G.

3.2. Groups and homomorphisms

Given two finite sets C and L, we indicate by C^L the set of all mappings s defined on L with values in C. We indicate by S_L the set of all bijective mappings defined on the set L. These mappings are the *permutations* of L. In particular, when $L = \{1, 2, \ldots, \ell\}$, we write S_ℓ to indicate these permutations.

DEFINITION 3.1. A group is a nonempty set G equipped with an associative operation, $* : G \times G \to G$, and an element $1 \in G$, satisfying:

(1) $1 * \tau = \tau * 1$, for all $\tau \in G$;
(2) for every $\tau \in G$, there is an element $\tau^{-1} \in G$ such that $\tau * \tau^{-1} = \tau^{-1} * \tau = 1$.

A *commutative* group is one in which the operation is commutative (the term *abelian* is also common in the literature). A subset of G which is a group under $*$ is called a *subgroup* of G.

EXAMPLE 3.1. The set S_ℓ, together with the operation of mapping composition, defines the group of permutations on the integers $\{1, \ldots, \ell\}$. Similarly, S_L together with the operation of composition of functions is a finite group. □

EXAMPLE 3.2. The group of permutations S_3 is a group of *order* $3! = 6$ (the number of elements in the group). We write (12) to indicate the permutation $\begin{bmatrix} 1 & \to & 2 \\ 2 & \to & 1 \\ 3 & \to & 3 \end{bmatrix}$, and similarly

$$(13) \equiv \begin{bmatrix} 1 & \to & 3 \\ 2 & \to & 3 \\ 3 & \to & 1 \end{bmatrix}, \quad (23) \equiv \begin{bmatrix} 1 & \to & 1 \\ 2 & \to & 3 \\ 3 & \to & 2 \end{bmatrix},$$

$$(123) \equiv \begin{bmatrix} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 1 \end{bmatrix}, \quad (132) \equiv \begin{bmatrix} 1 & \to & 3 \\ 2 & \to & 1 \\ 3 & \to & 2 \end{bmatrix}.$$

Usually, 1 denotes the identity permutation. The permutation $\tau * \sigma$ is obtained by first applying τ followed by σ, e.g., $(13) * (23) = (123)$,

$$(123) = \begin{bmatrix} 1 & \to & 3 \\ 2 & \to & 2 \\ 3 & \to & 1 \end{bmatrix} \text{ followed by } \begin{bmatrix} 1 & \to & 1 \\ 2 & \to & 3 \\ 3 & \to & 2 \end{bmatrix}.$$

The multiplication table for S_3 is shown in Matrix (3.2), where the permutations are indicated by $1 = a, (12) = b, (13) = c, (23) = d, (123) = e, (132) = f$.

$$\left[\begin{array}{c|cccccc} * & a & b & c & d & e & f \\ \hline a & a & b & c & d & e & f \\ b & b & a & e & f & c & d \\ c & c & f & a & e & d & b \\ d & d & e & f & a & b & c \\ e & e & d & b & c & f & a \\ f & f & c & d & b & a & e \end{array}\right]. \tag{3.2}$$

The *cyclic* group G is characterized by the fact that all its elements have the form c^k, for some element $c \in G$. Clearly, then, clyclic groups are commutative. The cyclic group $C_3 = \{1, (123), (132)\}$ is a commutative subgroup of S_3, of order 3. □

EXAMPLE 3.3 (Matrix groups). The set $GL(\mathbb{R}^n)$ of $n \times n$ nonsingular (i.e., nonzero determinant) matrices with real entries is a group under the operation of matrix multiplication. The order of this group is not finite. The set M_n of $n \times n$ permutation matrices[1] is a finite subgroup of $GL_n(\mathbb{R})$, of order n!. □

EXAMPLE 3.4 (Groups given by a presentation). The following groups are defined by their presentation relations (see Exercise 3.2):

(1) The dihedral groups, $D_{2n} =< a, b : a^n = 1, b^2 = 1, (ab)^2 = 1 >$ of order 2n;
(2) The quaternion group $Q_8 =< a, b; a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} >$ of order 8;
(3) The quaternion groups $Q_{4m} =< a, b; a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} >$ of order 4m, often called *generalized quaternions*.

□

EXAMPLE 3.5 (Direct product of groups). Given two groups $(G, \cdot)$ and $(H, *)$, the set product $G \times H = \{(\tau, \sigma); \tau \in G, \sigma \in H\}$ together with the operation

$$(\tau, \sigma) \times (\tau_1, \sigma_1) = (\tau \cdot \tau_1, \sigma * \sigma_1),$$

is a group, called the *product* group of $(G, \cdot)$ and $(H, *)$. □

DEFINITION 3.2. Given two groups G, H, a *homomorphism* from G to H is a function $\rho : G \to H$ preserving the group structure, that is,

$$\rho(\tau \cdot \sigma) = \rho(\tau) * \rho(\sigma), \quad \text{for all } \tau, \sigma \in G.$$

An *isomorphism* is an invertible homomorphism. When G = H, the isomorphism ρ is called an *automorphism* in H. The *kernel* of a homomorphism ρ is the set of those elements in G mapped into the identity element of H, that is, $\ker \rho = \{\tau \in G; \rho(\tau) = 1\}$.

Note that when $\ker \rho = \{1\}$ then the homomorphism ρ is an isomorphism onto its image in H. In fact, $\rho(\tau) = \rho(\sigma)$ implies

$$\rho(\tau\sigma^{-1}) = \rho(\tau)\rho(\sigma^{-1}) = \rho(\sigma)\rho(\sigma^{-1}) = \rho(1) = 1,$$

[1] A permutation matrix has exactly one entry equal to one in each row and column, and all other entries equal to zero

so that $\tau\sigma^{-1} \in \{1\}$, or, $\sigma = \tau$.

EXAMPLE 3.6. The permutation group S_ℓ is isomorphic to the matrix group M_ℓ: to each permutation τ in S_ℓ we associate the permutation matrix r_τ in M_ℓ defined by the changing of basis

$$\{e_1, e_2, \ldots, e_\ell\} \to \{e_{\tau 1}, e_{\tau 2}, \ldots, e_{\tau \ell}\}$$

in $\mathbb{R}^\ell$. We then have $r_{\tau\sigma} = r_\tau r_\sigma$, for all $\tau, \sigma \in S_\ell$. In S_3, for instance, $(123) = (13) * (23)$ and

$$r_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = r_{(23)} r_{(13)}.$$

□

EXAMPLE 3.7. Let $\mathcal{V}$ indicate a vector space over the field $\mathbb{C}$ of the complex numbers, and $GL(\mathcal{V})$ the group of automorphisms of $\mathcal{V}$. The elements of $GL(\mathcal{V})$ are the *linear mappings*, α, with inverse denoted by α^{-1}. When $\mathcal{V}$ has a finite basis $\{e_1, e_2, \ldots, e_v\}$, each linear mapping α is represented by a matrix A, with coefficients a_{ij} defined by

$$\alpha(e_j) = \sum_i a_{ij} e_i.$$

□

EXAMPLE 3.8. Recall that a scalar product on a vector space $\mathcal{V}$ over the complex field $\mathbb{C}$ is a function $(\ ,\ ) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that, for all $x, y, z \in \mathcal{V}$ and $\lambda \in \mathbb{C}$,

(1) $(x + y, z) = (x, z) + (y, z)$,
(2) $(\lambda x, y) = \lambda(x, y)$,
(3) $(x, y) = \overline{(y, x)}$,
(4) $(x, x) > 0$ if $x \neq 0$.

For each $y \in \mathcal{V}$, define $y^* : \mathcal{V} \to \mathcal{V}$ by $y^*(x) = (x, y)$. Then clearly y^* is a homomorphism of $\mathcal{V}$ and the mapping $y \mapsto y^*$ is an isomorphism from $\mathcal{V}$ into $GL(\mathcal{V})$. In fact,

$$(x+y)^*(z) = (z, x+y) = \overline{(x+y, z)} = \overline{(x, z)} + \overline{(y, z)} = (z, x) + (z, y) = x^*(z) + y^*(z),$$

and because $[y^*(y) = (y, y) > 0$ if $y \neq 0]$ implies that $[\ker (y \mapsto y^*) = \{0\}]$, the isomorphism obtains. □

EXAMPLE 3.9. Fix any member τ of a group G and define the mapping $\tau^* : G \to G$ by $\tau^*(\sigma) = \tau\sigma\tau^{-1}$. Then, for every $\tau \in G$, the mapping τ^* is a isomorphism in G, and the mapping $\tau \mapsto \tau^*$ is a homomorphism of G, taking values in the set Aut (G) of automorphisms in G. The mapping τ^* is usually called the *conjugation by* τ. □

DEFINITION 3.3 (Semidirect product of groups). Given two groups G and H, let η^* indicate an isomorphism of G indexed by an element $\eta \in H$. For (τ, σ) and (τ_1, σ_1) in $G \times H$, define the operation

$$(\tau, \sigma) \times_\eta (\tau_1, \sigma_1) = (\tau\eta^*(\tau_1), \sigma\sigma_1).$$

Then, $G \times H$, together with $\times_\eta$, is a group, called the *semidirect product* of G and H under η.

3.3. Group actions

DEFINITION 3.4. Given a set V and a group G, a *group action* is a function $\varphi : \mathrm{G} \times \mathrm{V} \to \mathrm{V}$ satisfying

(1) $\varphi(1, \mathrm{s}) = \mathrm{s}$, for all s in V,
(2) $\varphi(\sigma, \varphi(\tau, \mathrm{s})) = \varphi(\sigma\tau, \mathrm{s})$, for all $\mathrm{s} \in \mathrm{V}$, τ, σ in G.

The *orbit* $\mathcal{O}_\mathrm{s}$ of an element $\mathrm{s} \in \mathrm{V}$ generated by G under the action φ is the set

$$\mathcal{O}_\mathrm{s} = \{\varphi(\tau, \mathrm{s}); \tau \in \mathrm{G}\}. \tag{3.3}$$

We also define the set

$$\text{fix } (\tau) = \{\mathrm{s} \in \mathrm{V}; \varphi(\tau, \mathrm{s}) = \mathrm{s}\} \tag{3.4}$$

of elements in V that remain fixed by τ under the action φ, and the set

$$\mathrm{G}_\mathrm{s} = \{\tau \in \mathrm{G}; \varphi(\tau, \mathrm{s}) = \mathrm{s}\}, \tag{3.5}$$

of elements $\tau \in \mathrm{G}$ fixing the point $\mathrm{s} \in \mathrm{V}$. This set is the *stabilizer* of s by G under φ. It is then easy to check that

$$|\mathrm{G}| = |\mathcal{O}_\mathrm{s}||\mathrm{G}_\mathrm{s}|. \tag{3.6}$$

Moreover, note that G_s is a subgroup of G:

(1) $1 \in \mathrm{G}_\mathrm{s}$;
(2) $\tau, \sigma \in \mathrm{G}_\mathrm{s}$ implies $\varphi(\tau\sigma, \mathrm{s}) = \varphi(\tau, \varphi(\sigma, \mathrm{s})) = \varphi(\tau, \mathrm{s}) = \mathrm{s}$;
(3) $\tau \in \mathrm{G}_\mathrm{s}$ implies $\mathrm{s} = \varphi(1, \mathrm{s}) = \varphi(\tau^{-1}\tau, \mathrm{s}) = \varphi(\tau^{-1}, \varphi(\tau, \mathrm{s})) = \varphi(\tau^{-1}, \mathrm{s})$, that is, $\tau^{-1} \in \mathrm{G}_\mathrm{s}$.

G_s is also called the *isotropy* group of s in G under φ.

When the orbit $\mathcal{O}_\mathrm{s}$ of an element $\mathrm{s} \in \mathrm{V}$ generated by G under the action φ coincides with V we say that the action φ is *transitive*, or that G acts transitively on V.

EXAMPLE 3.10. Let V indicate the set $\mathrm{V} = \{\mathrm{uu}, \mathrm{yy}, \mathrm{uy}, \mathrm{yu}\}$ of two-sequences in length of two, equivalently, the set of all mappings s from $\{1, 2\}$ into $\{1, 2, \}$. Let the group G be $\mathrm{S}_2 = \{1, (12)\}$. The reader may verify that $\varphi_1(\tau, \mathrm{s}) = \mathrm{s}\tau^{-1}$ and $\varphi_2(\sigma, \mathrm{s}) = \sigma\mathrm{s}$ are actions of S_2 on V. Action φ_1 classifies the sequences by symmetries in the position of the residues, whereas φ_2 classifies the sequences by symmetries in the labels or names of the residues. The evaluations of these actions are summarized in the following matrices:

$$\varphi_1 : \left[\begin{array}{c|cccc} \tau\backslash\mathrm{s} & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ \hline 1 & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ (12) & \mathrm{uu} & \mathrm{yy} & \mathrm{yu} & \mathrm{uy} \end{array}\right], \quad \varphi_2 : \left[\begin{array}{c|cccc} \tau\backslash\mathrm{s} & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ \hline 1 & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ (12) & \mathrm{yy} & \mathrm{uu} & \mathrm{yu} & \mathrm{uy} \end{array}\right].$$

The action on the left, $\mathrm{s}\tau^{-1}$, generates 3 orbits $\{\mathrm{uu}\}$, $\{\mathrm{yy}\}$ and $\{\mathrm{uy}, \mathrm{yu}\}$, whereas the action on the right, $\sigma\mathrm{s}$, generates 2 orbits, $\{\mathrm{uu}, \mathrm{yy}\}$ and $\{\mathrm{uy}, \mathrm{yu}\}$. Now let $\mathrm{G} = \mathrm{S}_2 \times \mathrm{S}_2$, the product group, and $\varphi_3((\tau, \sigma), \mathrm{s}) = \sigma\mathrm{s}\tau^{-1}$. We obtain,

$$\varphi_3 : \left[\begin{array}{c|cccc} (\tau, \sigma)\backslash\mathrm{s} & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ \hline (1,1) & \mathrm{uu} & \mathrm{yy} & \mathrm{uy} & \mathrm{yu} \\ (1,(12)) & \mathrm{uu} & \mathrm{yy} & \mathrm{yu} & \mathrm{uy} \\ ((12),1) & \mathrm{yy} & \mathrm{uu} & \mathrm{yu} & \mathrm{uy} \\ ((12),(12)) & \mathrm{yy} & \mathrm{uu} & \mathrm{uy} & \mathrm{yu} \end{array}\right],$$

with two resulting orbits $\{uu, yy\}$ and $\{uy, yu\}$ (see Exercise 3.10). □

EXAMPLE 3.11. Let $\mathcal{F}(V)$ indicate the vector space of scalar-valued functions, x, defined in the structure V. Defining $\varphi : G \times \mathcal{F}(V) \to \mathcal{F}(V)$ by $\varphi(\sigma, x)(s) = x(\sigma s)$, we obtain an action of G on $\mathcal{F}(V)$. To see this, $\varphi(1, x) = x$ and

$$\varphi(\sigma, \varphi(\tau, x))(s) = \varphi(\tau, x)(\sigma s) = x(\tau(\sigma s)) = x((\tau\sigma)s) = \varphi(\tau\sigma, x)(s),$$

for all $s \in V$. Similarly, $\gamma : G \times \mathcal{F}(V) \to \mathcal{F}(V)$ given by $\gamma(\tau, x)(s) = x(s\tau^{-1})$, is a (*contravariant*) action of G on $\mathcal{F}(V)$. In fact, $\gamma(1, x) = x$ and

$$\gamma(\sigma, \gamma(\tau, x))(s) = \gamma(\tau, x)(s\sigma^{-1}) = x(s\sigma^{-1}\tau^{-1}) = x(s(\tau\sigma)^{-1}) = \gamma(\tau\sigma, s)(x),$$

for all $s \in V$. □

3.4. Counting orbits

The following classical result says that the number of orbits in V generated by the action φ of G equals the average number of fixed points of φ.

LEMMA 3.1 (Burnside[2]). If a finite group G acts on V according to φ, then

$$\text{Number of orbits in } V = \frac{1}{|G|} \sum_{\tau \in G} |\text{fix } (\tau)|.$$

PROOF. Let $A = \{(\tau, s) \in G \times V; \varphi(\tau, s) = s\}$. First calculate the number $|A|$ of elements in A as

$$|A| = \sum_{\tau \in G} |\text{fix } (\tau)|. \tag{3.7}$$

Secondly, writing $\eta =$ number of orbits in V, evaluate this same number as

$$|A| = \sum_{s \in V} |G_s| = \sum_{i=1}^{\eta} |\mathcal{O}_i||G_s| = \sum_{i=1}^{\eta} |\mathcal{O}_i| \frac{|G|}{|\mathcal{O}_i|} = \eta \times |G|. \tag{3.8}$$

From equalities 3.7 and 3.8 the result then follows. □

Matrix (3.43) in Section 3.15 shows the left action of S_4 on the space V of all two-sequences in length of four. Also shown are the volumes of $\text{fix}(\sigma)$ and of G_s.

EXAMPLE 3.12. From Matrix (3.43), with $G = S_4$, it follows that

$$\text{Number of orbits of } V = \frac{1}{|G|} \sum_{G} |\text{ fix}(\sigma) | = \frac{120}{24} = 5,$$

namely (indicating the mappings by their labels),

$$\mathcal{O}_0 = \{1\},$$
$$\mathcal{O}_1 = \{9, 5, 3, 2\},$$
$$\mathcal{O}_2 = \{13, 11, 7, 10, 6, 4\},$$
$$\mathcal{O}_3 = \{15, 14, 12, 8\},$$
$$\mathcal{O}_4 = \{16\}.$$

[2]William Burnside, Born: 2 July 1852 in London, England. Died: 21 Aug 1927 in West Wickham, London, England. Among his applied mathematics teachers at Cambridge were Stokes, Adams and Maxwell. The Lemma was actually proved by Frobenius in 1887

In addition, because $|\mathcal{O}_i| = |G|/|G_{s_i}|$, we have

$$
\begin{aligned}
|\mathcal{O}_0| &= 24/24 = 1,\\
|\mathcal{O}_1| &= 24/6 = 4,\\
|\mathcal{O}_2| &= 24/4 = 6,\\
|\mathcal{O}_3| &= 24/6 = 4,\\
|\mathcal{O}_4| &= 24/24 = 1.
\end{aligned}
$$

□

EXAMPLE 3.13 (Order-four orbits of cyclic subgroups for two-sequences in length of four). These orbits are also called *cyclic orbits.* Following Example 3.12, consider the left action of $C_4 = \{1, (1234), (13)(24), (1432)\}$ on the mapping space V. The resulting action, shown in Matrix (3.9),

(3.9)

$C_4\backslash s$	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
1	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
(13)(24)	1	16	12	8	15	14	4	11	10	7	6	13	3	2	9	5
(1234)	1	16	14	12	8	15	10	6	13	4	11	7	2	9	5	3
(1432)	1	16	8	15	14	12	7	6	4	13	11	10	5	3	2	9

lead to the orbits (indicating the mappings by their labels)

$$
\begin{aligned}
&\mathcal{O}_0 = \{1\},\\
&\mathcal{O}_1 = \{9, 5, 3, 2\},\\
&\mathcal{O}_{21} = \{13, 7, 10, 4\}, \quad \mathcal{O}_{22} = \{11, 6\},\\
&\mathcal{O}_3 = \{15, 14, 12, 8\},\\
&\mathcal{O}_4 = \{16\}.
\end{aligned}
$$

We note that C_4 splits the original orbit $\mathcal{O}_2$ under S_4 into two new orbits, $\mathcal{O}_{21}$ and $\mathcal{O}_{22}$, so that $\mathcal{O}_{21} \cup \mathcal{O}_{22} = \mathcal{O}_2$. There are many more cyclic symmetries and orbit configurations. □

EXAMPLE 3.14. *Dihedral orbits* for two-sequences in length of four. Following Example 3.12, consider the left action of the group D_4 on the mapping space V. Recall that D_4 may also be realized as the group of rotational and axial symmetries of the regular rectangle. The resulting actions are shown in Matrix (3.10):

(3.10)

$D_4\backslash s$	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
1	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
(24)	1	16	15	8	12	14	7	11	13	4	6	10	3	5	9	2
(13)	1	16	12	14	15	8	10	11	4	13	6	7	9	2	3	5
(12)(34)	1	16	14	15	8	12	13	6	10	7	11	4	5	9	2	3
(13)(24)	1	16	12	8	15	14	4	11	10	7	6	13	3	2	9	5
(14)(23)	1	16	8	12	14	15	4	6	7	10	11	13	2	3	5	9
(1234)	1	16	14	12	8	15	10	6	13	4	11	7	2	9	5	3
(1432)	1	16	8	15	14	12	7	6	4	13	11	10	5	3	2	9

which shows that D_4 and C_4 generate the same set of orbits. □

EXAMPLE 3.15. Orbits generated by cyclic permutations of order 2, for two-sequences in length of four. Following Example 3.12, consider the left action of

$G = \{1, (13)(24)\}$ on the mapping space V. The resulting action, shown in Matrix (3.11),

$$\text{(3.11)} \quad \left[\begin{array}{c|cccccccccccccccc} G\backslash s & 1 & 16 & 15 & 14 & 12 & 8 & 13 & 11 & 7 & 10 & 6 & 4 & 9 & 5 & 3 & 2 \\ \hline 1 & 1 & 16 & 15 & 14 & 12 & 8 & 13 & 11 & 7 & 10 & 6 & 4 & 9 & 5 & 3 & 2 \\ (13)(24) & 1 & 16 & 12 & 8 & 15 & 14 & 4 & 11 & 10 & 7 & 6 & 13 & 3 & 2 & 9 & 5 \end{array}\right],$$

and corresponding orbits,

$$\begin{aligned} \mathcal{O}_0 &= \{1\}, \\ \mathcal{O}_{11} &= \{9,3\}, \quad \mathcal{O}_{12} = \{5,2\}, \\ \mathcal{O}_{211} &= \{13,4\}, \quad \mathcal{O}_{212} = \{7,10\}, \quad \mathcal{O}_{221} = \{11\}, \quad \mathcal{O}_{222} = \{6\}, \\ \mathcal{O}_{31} &= \{14,8\}, \quad \mathcal{O}_{32} = \{15,12\}, \\ \mathcal{O}_4 &= \{16\}, \end{aligned}$$

show that G further splits the original order-4 cyclic orbits into additional, smaller orbits. □

EXAMPLE 3.16 (Maxwell-Boltzmann and Bose-Einstein counts). Following Example 1.5 of Chapter 1, define two mappings s and f in the mapping space $V = C^L$ as equivalent whenever $s\tau^{-1} = f$ for some permutation $\tau \in S_\ell$. That is, s and f differ only by a permutation of the position of the $(c = |C|)$ symbols in C. The orbits $O(s) = \{s\tau^{-1}; \tau \in S_\ell\}$ decompose the space V into the sum

$$V = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_m, \tag{3.12}$$

where each of these components is some $\mathcal{O}(s)$, m is the number of *elementary frames* $\lambda = (n_1, \ldots, n_c)$ or non-negative integer solutions $n_1, \ldots, n_c$ of $\ell = n_1 + \ldots + n_c$ subject to $\ell \geq n_1 \geq \ldots \geq n_c \geq 0$, and $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are disjoint unions of orbits whose members share a particular frame. The orbits in the *quotient space*

$$V/S_\ell = \{\mathcal{O}_1, \ldots, \mathcal{O}_m\}$$

defined by (3.12) are *left* permutation orbits. Permutation orbits obtained from the action σs, $\sigma \in S_c$, are defined similarly. To illustrate further the orbit decomposition, consider the case in which the mapping space $V = C^L$ represents the possible compositions of an urn with four marbles with labels in the set $L = \{1, 2, 3, 4\}$ and colored with colors in the set $C = \{\text{red } (\circ), \text{blue } (\bullet), \text{green } (\diamond)\}$. Two urn compositions are defined as equivalent when they differ only by relabeling of the marbles. That is, S_4 acts on the left of V according to $s\tau^{-1}$. Start with the elementary frames: There are $m = 4$ of those (see Exercise 3.20), namely,

$$\lambda_1 = (4,0,0), \ \ \lambda_2 = (3,1,0), \ \ \lambda_3 = (2,2,0), \ \ \lambda_4 = (2,1,1).$$

Define the *components* of frame $\lambda = (n_1, n_2, \ldots, n_c)$ to be the *distinct* integers a_i in the frame λ, and $k = k(\lambda)$ to be the number of such components. Indicate by $(m_1, \ldots, m_k)$ the multiplicities with which the k distinct components of each frame occur. With that notation in mind, each frame can be written as

$$\lambda = (a_1^{m_1}, \ldots, a_k^{m_k}).$$

We then write $\lambda_1 = 40^2$, $\lambda_2 = 310$, $\lambda_3 = 2^2 0$, $\lambda_4 = 21^2$. In correspondence with (3.12), we obtain the decomposition

$$|V| = c^\ell = \sum_{t=1}^{m} |\mathcal{O}_t| = \sum_{\lambda} \frac{\ell!}{(a_1!)^{m_1}(a_2!)^{m_2}\ldots(a_k!)^{m_k}} \, \frac{c!}{m_1!m_2!\ldots m_k!}$$

of the Maxwell-Boltzmann count c^ℓ into the volumes of the orbits in V/S_ℓ and their multiplicities. In the above decomposition, λ varies over the m different frames, that is, $m_1a_1 + \ldots + m_ka_k = \ell$ and $m_1 + \ldots + m_k = c$. Moreover,

$$\sum_\lambda \frac{c!}{m_1!m_2!\ldots m_k!} = \binom{c+\ell-1}{\ell}$$

decomposes the Bose-Einstein count $\binom{c+\ell-1}{\ell}$ into the sum of the number

$$Q_\lambda = \frac{c!}{m_1!m_2!\ldots m_k!}$$

of quantal states associated to frame λ. Writing

$$\Omega_\lambda = \frac{\ell!}{(a_1!)^{m_1}(a_2!)^{m_2}\ldots(a_k!)^{m_k}}, \tag{3.13}$$

and $v(\lambda) = \Omega_\lambda Q_\lambda$ we have $c^\ell = \sum_\lambda v(\lambda)$. Direct computation leads to

$$|\mathcal{O}_1| = 3, \quad |\mathcal{O}_2| = 24, \quad |\mathcal{O}_3| = 18, \quad |\mathcal{O}_4| = 36.$$

The reader may verify the following correspondence among frames, orbits and urn compositions:

$$\begin{aligned}
\lambda &= 40^2, & \mathcal{O}_1, &\quad \{\square \ \circ \ \circ \ \circ\}\\
\lambda &= 310, & \mathcal{O}_2, &\quad \{\circ \ \circ \ \circ \ \bullet\}\\
\lambda &= 2^20, & \mathcal{O}_3, &\quad \{\circ \ \circ \ \bullet \ \bullet\}\\
\lambda &= 21^2, & \mathcal{O}_4, &\quad \{\circ \ \circ \ \bullet \ \diamond\}.
\end{aligned}$$

Moreover, the volume of the orbit associated with frame $\lambda = (n_1, \ldots, n_c)$ further decomposes in terms of its subframes of size $c - 1$. In the present example,

$$\begin{aligned}
|\mathcal{O}_1| &= v(4,0,0) = \tbinom{4}{4}v(0,0) + \tbinom{4}{0}v(4,0) = 1 + 2,\\
|\mathcal{O}_2| &= v(3,1,0) = \tbinom{4}{3}v(1,0) + \tbinom{4}{1}v(3,0) + \tbinom{4}{0}v(3,1) = 8 + 8 + 8,\\
|\mathcal{O}_3| &= v(2,2,0) = \tbinom{4}{2}v(2,0) + \tbinom{4}{0}v(2,2) = 12 + 6,\\
|\mathcal{O}_4| &= v(2,1,1) = \tbinom{4}{2}v(1,1) + \tbinom{4}{1}v(2,1) = 12 + 24.
\end{aligned}$$

The resulting partition of V according to *color-only* attribute is defined by the number $|A_{ij}|$ of configurations in which all j marbles have color i. In the present example, $j = 0, 1, 2, 3, 4$, $i \in \{\text{red}, \text{blue}, \text{green}\}$, and

$$\begin{aligned}
|A_{i0}| &= \tbinom{4}{0}[v(4,0) + v(3,1) + v(2,2)] = 16,\\
|A_{i1}| &= \tbinom{4}{1}[v(3,0) + v(2,1)] = 32,\\
|A_{i2}| &= \tbinom{4}{2}[v(2,0) + v(1,1)] = 24,\\
|A_{i3}| &= \tbinom{4}{3}v(1,0) = 8,\\
|A_{i4}| &= \tbinom{4}{4}v(0,0) = 1.
\end{aligned}$$

If the urn compositions are equally likely, or, equivalently, if the points in V are uniformly distributed, the resulting probabilities

$$w_i(j) = P[\text{all j marbles have the same color i}]$$

in V/S_4 are

$$(w_i(0), w_i(1), w_i(2), w_i(3), w_i(4)) = \frac{1}{81}(16, 32, 24, 8, 1).$$

3.5. Linear representations

Given $\tau \in G$ and an action $\varphi : G \times V \to V$, note that the evaluations $\tau^*(s) = \varphi(\tau, s)$, $s \in V$, define a permutation in V. In fact, $\varphi(\tau, s) = \varphi(\tau, f)$ implies

$$s = \varphi(\tau^{-1}, \varphi(\tau, s)) = \varphi(\tau^{-1}, \varphi(\tau, f)) = f.$$

Moreover,

$$\tau^*\sigma^*(s) = \varphi(\tau, \varphi(\sigma, s)) = \varphi(\tau\sigma, s) = (\tau\sigma)^*(s),$$

so that the mapping $\tau \in G \mapsto \tau^* \in S_V$ is a group homomorphism. We thus have

PROPOSITION 3.1. If G acts on V according to φ, then the evaluations $\tau^*(s) = \varphi(\tau, s)$, $s \in V$, are permutations in V, for all $\tau \in G$. Conversely, given a homomorphism $\tau \mapsto \tau^*$, the mapping $\tau^*(s) = \varphi(\tau, s)$ defines a group action on V.

The argument justifying Proposition 3.1, when applied with V = G leads to Cayley's Theorem:

THEOREM 3.1 (Cayley, 1878). Every group G is isomorphic to a subgroup of S_G. If G is finite with ℓ elements, then G is isomorphic to a subgroup of S_ℓ.

Proposition 3.1 shows that the mapping $\tau \mapsto \tau^*$ defined in G with values in S_V is a homomorphism from G into S_V. Correspondingly, let $\{e_s; s \in V\}$ indicate a basis for the vector space $\mathbb{R}^V$, indexed by the elements of V, $\{e_{\tau^*(s)}; s \in V\}$ the new basis determined by τ^*, and

$$\rho(\tau) : \quad \{e_s; s \in V\} \to \{e_{\tau^*(s)}; s \in V\},$$

the nonsingular matrix representing the changing of basis. Then, ρ is a group homomorphism from G into $GL(\mathbb{R}^V)$.

DEFINITION 3.5. A linear representation of a group G in a vector space $\mathcal{V}$ is a group homomorphism from G into $GL(\mathcal{V})$.

Note that every linear representation maps the identity of G into the identity matrix (or operator) of $GL(\mathcal{V})$, that is, $\rho(1) = I$. Also, it maps the inverse τ^{-1} of τ into the inverse $\rho(\tau)^{-1}$ of the linear operator $\rho(\tau)$, that is $\rho(\tau^{-1}) = \rho(\tau)^{-1}$. Usually, the dimension of ρ indicates the dimension of the corresponding vector space. Also note that if ρ is a representation of G, then β, defined by $\beta(\tau) = B^{-1}\rho(\tau)B$ is also a representation of G, for every non-singular matrix B of dimension equal to the dimension of ρ. Any two such representations, obtained one from another by a changing of basis, are called *equivalent* or *isomorphic* representations. We write $\rho \simeq \beta$ to indicate that ρ and β are equivalent[3].

EXAMPLE 3.17 (One-dimensional representations). The *principal* or trivial representation: $\rho(\tau) = 1$, for all $\tau \in G$. The antisymmetric or *signature* representation of S_ℓ:

$$\rho(\tau) = \begin{cases} 1 & \text{if the permutation } \tau \text{ is even;} \\ -1 & \text{if the permutation } \tau \text{ is odd.} \end{cases}$$

□

[3]Most of the theory of linear representation of finite groups have the equivalent result formulated for infinite groups, in which we would look at ρ as linear operators. In the present discussion, we often write or think of ρ as the notation indicating the representation in its matrix form. At times, however, the broader interpretation of ρ as a linear operator also applies. We may write ρ_τ or $\rho(\tau)$ without distinction. For example, $\rho(\tau)_{12}$ indicates the entry $(1, 2)$ of the representation ρ evaluated at τ.

EXAMPLE 3.18 (The regular and the permutation representations). The *regular* representation is defined by the action $\varphi(\tau, \sigma) = \tau\sigma$ of G on itself. The matrix representation is that changing the basis $\{e_\sigma; \sigma \in G\}$ into $\{e_{\tau\sigma}; \sigma \in G\}$. The dimension of the representation is $|G|$. More generally, if G acts of a set V with v elements, the representation defined by G acting on the basis $\{e_s; s \in V\}$ of $\mathbb{R}^v$ according to $\varphi(\tau, e_s) = e_{\varphi(\tau,s)}$ is referred to as the *permutation* representation of G acting on V. The dimension of the representation is v. When $G = S_\ell$ acts on $V = \{1, 2, \ldots, \ell\}$ according to $\varphi(\tau, i) = \tau i$, the matrix representation is that changing the basis $\{e_i; i = 1, 2, \ldots, \ell\}$ into $\{e_{\tau i}; i = 1, 2, \ldots, \ell\}$. Its dimension is ℓ. To illustrate, consider S_3 and the element $\tau = (132)$. From the multiplication matrix of S_3 shown in Matrix (3.2), the basis indexed by $\{1, (12), (13), (23), (123), (132)\}$ is changed, under the regular action, to the basis indexed by $\{(132), (13), (23), (12), 1, (123)\}$. The regular representation then maps (132) to the matrix

$$\rho_{(132)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Similarly, the basis indexed by $\{1, 2, 3\}$ is changed, under the permutation action, to the basis indexed by $\{1, 3, 2\}$, so that the permutation representation maps (132) to the matrix

$$\rho_{(132)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The permutation representation of S_2 acting on the space of two-sequences in length of two according to

$$\varphi_1 : \left[\begin{array}{c|cccc} \tau\backslash s & uu & yy & uy & yu \\ \hline 1 & uu & yy & uy & yu \\ t = (12) & uu & yy & yu & uy \end{array}\right]$$

as discussed in Example 3.10, is given by

$$\rho_1(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The representation has dimension 4. Similarly, the permutation representation of S_2 acting according to

$$\varphi_2 : \left[\begin{array}{c|cccc} \tau\backslash s & uu & yy & uy & yu \\ \hline 1 & uu & yy & uy & yu \\ (12) & yy & uu & yu & uy \end{array}\right],$$

leads to the representation

$$\rho_2(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho_2(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Its dimension is also 4. □

EXAMPLE 3.19. The reader may verify that

$$\beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \beta_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \beta_{13} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$\beta_{23} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \beta_{123} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \beta_{132} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

is a two-dimensional representation of S_3.

EXAMPLE 3.20 (The tensor representation). Let ρ indicate the regular representation of $G = S_2$. Write $S_2 = \{1, t = (12)\}$, so that $\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\rho(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The permutation representation of G acting on $G \times G$ according to $\varphi(\tau, (\sigma_1, \sigma_2)) = (\tau\sigma_1, \tau\sigma_2)$ is given by

$$1 \mapsto I_4 = \rho(1) \otimes \rho(1), \quad t \mapsto \begin{bmatrix} 0 & \rho(t) \\ \rho(t) & 0 \end{bmatrix} = \rho(t) \otimes \rho(t),$$

where $\otimes$ indicates the Kronecker product of two matrices. Equivalently, for each $\tau \in S_2$, these matrices represent the changing from a basis indexed by $G \times G$ to the basis indexed by $\{(\tau\sigma_1, \tau\sigma_2); (\sigma_1, \sigma_2) \in G \times G\}$. This new representation is called the *tensor* representation of ρ with itself, and is indicated by $\rho \otimes \rho$. Its dimension is $(\dim \rho)^2 = 4$. Similarly, $\varphi(\tau, (\sigma_1, \sigma_2)) = (\varphi_1(\tau, \sigma_1), \varphi_2(\tau, \sigma_2))$ defines the tensor representation of two representations ρ_1 and ρ_2 of G, given, respectively, by actions φ_1 and φ_2 (the same construction applies to the tensor representation of three or more representations). □

EXAMPLE 3.21 (The action of G on cosets of a subgroup H). Given a subgroup H of G, consider the set $G/H = \{\sigma H; \sigma \in G\}$ of (left) cosets of H in G. Then, it is easy to verify that $\varphi : G \times G/H \to G/H$ defined by

$$\varphi(\tau, \sigma H) = \sigma(\tau^{-1}H) = \sigma\tau^{-1}H$$

is an action of G on G/H. □

Consequently, from Proposition 3.1, we obtain a representation ρ of G into $GL(R^n)$, where $n = [G : H]$ is the number of (left) cosets of H in G. In particular, note that $\rho(\tau) = I_n$ for all $\tau \in H$, so that the restriction of ρ to H is the trivial representation of S_n. Moreover, if $\tau \in \ker \rho$, we must have

$$\sigma_i H = \sigma_i \tau^{-1} H, \quad i = 1, \ldots, n,$$

or $H = \tau^{-1}H$, which implies $\eta = \tau^{-1}\gamma$ for some $\eta, \gamma \in H$. Consequently $\tau = \eta\gamma^{-1} \in H$. This proves

THEOREM 3.2. If $H \subset G$ and $[G : H] = n$, then there is a representation ρ of G into $GL(\mathbb{R}^n)$, with $\ker \rho \subseteq H$.

EXAMPLE 3.22. Using the notation of the multiplication table (3.2) for S_3, write $1 = a, (12) = b, (13) = c, (23) = d, (123) = e, (132) = f$. In Theorem 3.2, let $H = \{a, b\}$ and $G = S_3$, so that the cosets of H in G are $aH = \{a, b\}$, $cH = \{c, f\}$ and $dH = \{d, e\}$. The index $[G : H]$ is 3. The action $(\tau, \sigma H) = \sigma\tau^{-1}H$ of S_3 on S_3/S_2, summarized in the following matrix,

$$\left[\begin{array}{c|ccc} S_3 & aH & cH & dH \\ \hline a & aH & cH & dH \\ b & aH & cH & dH \\ c & dH & aH & cH \\ d & cH & dH & aH \\ e & dH & aH & cH \\ f & cH & dH & aH \end{array}\right],$$

leads to the respresentation ρ of S_3 into $GL(\mathbb{R}^3)$ determined by

$$\rho_1 = \rho_{12} = I_3, \quad \rho_{13} = \rho_{123} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho_{23} = \rho_{132} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

in which $\ker \rho = H \subseteq H$. The representation ρ is equivalent to the permutation representation of C_3. □

EXAMPLE 3.23 (Action by conjugation). Indicate by $\mathcal{G}$ the set of all complex-valued functions defined on a given group G. It follows that $\mathcal{G}$ is a vector space over the same field, $\mathbb{C}$. Define, for $(\sigma, x) \in G \times \mathcal{G}$,

$$\varphi(\sigma, x) = \sigma^*(x) \in \mathcal{G},$$

where the mapping $\sigma^*(x)$ takes $\tau \in G$ into $x(\sigma\tau\sigma^{-1}) \in \mathbb{C}$. That is, $\sigma^*(x)(\tau) = x(\sigma\tau\sigma^{-1})$. The reader may verify that

(1) φ is a group action;
(2) $\sigma^* : \mathcal{G} \to \mathcal{G}$ is a linear mapping in $\mathcal{G}$, with inverse $\sigma^{-1*} \in \mathcal{G}$, for all $\sigma \in G$. That is, σ^* is an element of the group $GL(\mathcal{G})$ of invertible linear mappings;
(3) $\phi : \sigma \in G \mapsto \sigma^* \in GL(\mathcal{G})$ is a group homomorphism.

The result is a representation ϕ of G in $GL(\mathcal{G})$. □

EXAMPLE 3.24 (Category representations[4]). Consider the vector space $\mathcal{L}(V)$ of formal linear combinations $\sum_{s \in V} x(s)s$ of points in a finite set V, with, say, real-valued coefficients $x(s)$. If S_V acts on V according to φ, then

$$\varphi'(\tau, \sum x(s)s) = \sum x(s)\varphi(\tau, s)$$

is an action of S_V on $\mathcal{L}(V)$. Note that all actions φ' have the same dimension, namely, $|V| = v$. This is due to the fact that the permutations in S_V cannot *almagamate* the (vectors indexed by) points in V. An example of amalgamation

[4]P. McCullagh, personal communication.

was described earlier on in Chapter 1, in which the initial alphabet of residues $\{A, G, C, T\}$ is amalgamated into $\mathcal{A} = \{u, y\}$ by the surjective mapping

$$f: \; A \to u, \;\; G \to u, \;\; C \to y, \;\; T \to y.$$

Similarly, S_V cannot *select* and *relabel* a proper subset of V. A natural extension to sets of permutations which includes all amalgamations is the category $\mathcal{S}$ of all finite sets and all surjections between those sets. Similarly, the category $\mathcal{I}$ of all all finite sets and all injective mappings between these sets includes all operations of *selection* or *sampling*. For instance, the mapping $\phi : \{1,2\} \to \{a,b,c\}$ defined by $\phi(1) = c, \phi(2) = b$ selects the proper subset $\{c, b\}$ and relabels it as $\{1,2\}$. In what follows, we will outline the construction of a category representation of $\mathcal{S}$.

A category representation of $\mathcal{S}$ is a special functor F from $\mathcal{S}$ into the category $\mathcal{K}$ of linear transformations on vector spaces. Recall that the objects of $\mathcal{S}$ are finite sets $(V, V', V'', \ldots)$, the morphisms are the surjective mappings $V \xrightarrow{\phi} V' \xrightarrow{\phi'} V''$ and the composition is mapping composition. The objects of $\mathcal{K}$ are vector spaces, the morphisms are linear mappings, and the composition is mapping composition. We now describe the object component F_o and the arrow component F_a of the functor F:

(1) F_o maps objects $V \in \mathcal{S}$ into the set $\mathcal{L}(V)$ of all formal linear combinations $\sum_{s \in V} x(s)s$ of elements $s \in V$ with coefficients scalar fuctions x defined on V. Because $\mathcal{L}(V)$ is a vector space, we have $F_o(V) \in \mathcal{K}$;

(2) F_a maps the morphism $[V \xrightarrow{\phi} V'] \in \mathcal{S}$ into

$$[\mathbb{R}^V \xleftarrow{\phi^* = F_a(\phi)} \mathbb{R}^{V'}] \in \mathcal{K}, \tag{3.14}$$

where, by definition, $\phi^*(f) = f \circ \phi$ is the pullback mapping. To verify (3.14), first note that $\mathbb{R}^{V'}$ and $\mathbb{R}^V$ are vector spaces. Moreover, φ^* is linear. To see this, for all $f, g \in \mathbb{R}^{V'}$, $s \in V$, and scalars λ,

$$\begin{aligned}\phi^*(f+g)(s) &= (f+g)(\phi(s)) = f(\phi(s)) + g(\phi(s)) = \phi^*(f)(s) + \phi^*(g)(s) \\ &= (\phi^*(f) + \phi^*(g))(s),\end{aligned}$$

that is $\phi^*(f+g) = \phi^*(f) + \phi^*(g)$, and

$$\phi^*(\lambda f)(s) = (\lambda f)\phi(s) = \lambda f(\phi(s)) = \lambda \phi^*(f)(s),$$

that is $\phi^*(\lambda f) = \lambda \phi^*(f)$. Therefore, (3.14) obtains;

(3) If $[V \xrightarrow{\phi} V' \xrightarrow{\phi'} V''] \in \mathcal{S}$ then

$$\begin{array}{ccccc} \mathbb{R}^V & \xleftarrow{\phi^*} & \mathbb{R}^{V'} & \xleftarrow{\phi'^*} & \mathbb{R}^{V''} \\ (g \circ \phi') \circ \phi & & g \circ \phi' & & g \end{array}$$

satisfies $F_a(\phi' \circ \phi) = F_a(\phi)F_a(\phi')$. To see this,

$$\begin{aligned}F_a(\phi' \circ \phi)(g) &= g \circ (\phi' \circ \phi) = (g \circ \phi') \circ \phi = \phi^*(g \circ \phi') \\ &= F_a(\phi)(g \circ \phi') = F_a(\phi)\phi'^*(g) = F_a(\phi)F_a(\phi')(g),\end{aligned}$$

so that $F_a(\phi' \circ \phi) = F_a(\phi)F_a(\phi')$.

Conditions (1)-(3), in addition to the fact that $F(1_V) = 1_{F(V)}$, show that F is a contravariant functor from $\mathcal{S}$ into $\mathcal{K}$. It is a category representation. □

3.6. Unitarily equivalent representations

In this section we review the notion of unitary representations and outline the argument showing that every representation of a finite group is equivalent to an unitary representation. Consequently, one may assume that a representation is unitary.

Let ρ be a representation of G (with $|G|$ elements) on a finite dimensional vector space $\mathcal{V}$ over the complex field, in which a scalar product (|) is defined e.g., Example 3.8. It is a simple verification that

$$(x, y) = \frac{1}{|G|} \sum_{\tau \in G} (\rho_\tau x | \rho_\tau y) \tag{3.15}$$

is then an *invariant* scalar product in $\mathcal{V}$. That is, (,) is a scalar product and $(\rho_\tau x, \rho_\tau y) = (x, y)$ for all $\tau \in G$ and all $x, y \in \mathcal{V}$. Consequently, $(\rho_\tau x, y) = (x, \rho_{\tau^{-1}} y)$, for all $\tau \in G$ and all $x, y \in \mathcal{V}$. Since the *adjoint* representation, ρ^*, of ρ is the (unique) representation defined by the equation

$$(\rho_\tau x, y) = (x, \rho_\tau^* y), \quad \text{for all } \tau \in G, \quad \text{for all } x, y \in \mathcal{V}. \tag{3.16}$$

we conclude that

$$\rho_\tau^* = \rho_{\tau^{-1}}, \tag{3.17}$$

that is, the representation ρ is *unitary.* Moreover,

PROPOSITION 3.2. The matrix form $[\rho_\tau]$ of ρ_τ is unitary with respect to the scalar product (x, y) just defined, for all $\tau \in G$.

PROOF. Let $\mathcal{B} = \{y_1, \ldots, y_\ell\}$ be an orthonormal basis for $\mathcal{V}$, relative to the invariant scalar product. Recall that the coordinate column vector $[y]$ of y is given by

$$[y]' = ((y, y_1), (y, y_2), \ldots, (y, y_\ell)). \tag{3.18}$$

First we show that if $[\rho_\tau]_\mathcal{B} = R_\tau$ then $[\rho_\tau^*]_\mathcal{B} = R_\tau^*$, where R_τ^* indicates the conjuage-transpose, or *Hermitian* transpose of matrix R. In fact, if

$$\rho_\tau^* y_j = \sum_{k=1}^{\ell} d_{kj} y_k,$$

then, from equations (3.16) and (3.18), we have

$$d_{ji} = (\rho_\tau^* y_i, y_j) = \overline{(y_j, \rho_\tau^* y_i)} = \overline{(\rho_\tau y_j, y_i)} = \overline{(\sum_{k=1}^{\ell} \rho_{kj}(\tau) y_k, y_i)} = \overline{\rho}_{ij}, \quad \text{for all } x, y \in \mathcal{V},$$

that is, $[\rho_\tau^*]_\mathcal{B} = \overline{R}_\tau' = R_\tau^*$. As a consequence, from (3.17)

$$R_\tau^* = [\rho_\tau^*]_\mathcal{B} = [\rho_\tau^{-1}]_\mathcal{B} = [\rho_\tau]_\mathcal{B}^{-1} = R_\tau^{-1},$$

that is, $R_\tau^* R_\tau = I$, for all $\tau \in G$, concluding the proof. □

EXAMPLE 3.25. We will construct the representation unitarily equivalent to the two-dimensional representation of S_3,

$$\beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \beta_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \beta_{13} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$\beta_{23} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \beta_{123} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \beta_{132} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

illustrated earlier on in Example 3.19. The invariant scalar product derived from the Euclidean scalar product $(\,|\,)$ in $\mathbb{R}^2$ is

$$(\mathrm{x},\mathrm{y}) = \sum_\tau (\beta_\tau \mathrm{x}|\beta_\tau \mathrm{y}) = \sum_\tau \mathrm{x}'\beta_\tau'\beta_\tau \mathrm{y} = \mathrm{x}' \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \mathrm{y} \equiv \mathrm{x}'\mathrm{F}\mathrm{y}.$$

Next, starting with the canonical basis $\mathrm{v}_1 = (1,0)$, $\mathrm{v}_2 = (0,1)$ for $\mathbb{R}^2$, use Gram-Schmidt to construct a basis $\{\mathrm{w}_1, \mathrm{w}_2\}$ that is orthonormal relative to the invariant scalar product:

(1) $||\mathrm{v}_1||^2 = \mathrm{v}_1'\mathrm{F}\mathrm{v}_1 = 8$. Let $\mathrm{w}_1 = \mathrm{v}_1/||\mathrm{v}_1|| = (\sqrt{2}/4, 0)$;

(2) w_2 is the normalized version of $\mathrm{v}_2 - \mathrm{w}_1'\mathrm{F}\mathrm{w}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, which has norm $\sqrt{6}$. That is $\mathrm{w}_2 = (-\sqrt{6}/12, \sqrt{6}/6)$.

The resulting new (unitarily equivalent) representation is then $\gamma_\tau = \mathrm{H}^{-1}\beta_\tau\mathrm{H}$, where

$$\mathrm{H} = \begin{bmatrix} 1/4\,\sqrt{2} & -1/12\,\sqrt{6} \\ 0 & 1/6\,\sqrt{6} \end{bmatrix}.$$

We obtain $\gamma_1 = \mathrm{I}_2$,

$$\gamma_{12} = \begin{bmatrix} 1/2 & 1/4\,\sqrt{2}\sqrt{6} \\ 1/4\,\sqrt{2}\sqrt{6} & -1/2 \end{bmatrix}, \quad \gamma_{13} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\gamma_{23} = \begin{bmatrix} 1/2 & -1/4\,\sqrt{2}\sqrt{6} \\ -1/4\,\sqrt{2}\sqrt{6} & -1/2 \end{bmatrix}, \quad \gamma_{123} = \begin{bmatrix} -1/2 & 1/4\,\sqrt{2}\sqrt{6} \\ -1/4\,\sqrt{2}\sqrt{6} & -1/2 \end{bmatrix},$$

and

$$\gamma_{132} = \begin{bmatrix} -1/2 & -1/4\,\sqrt{2}\sqrt{6} \\ 1/4\,\sqrt{2}\sqrt{6} & -1/2 \end{bmatrix}.$$

In each case we have $\gamma_\tau\gamma_\tau' = \mathrm{I}_2$. □

3.7. Stable subspaces

DEFINITION 3.6. Let ρ be a representation of G on $GL(\mathcal{V})$ and $\mathcal{W}$ be a linear subspace of $\mathcal{V}$ with the property that if $\mathrm{x} \in \mathcal{W}$ then $\rho(\tau)\mathrm{x} \in \mathcal{W}$ for all $\tau \in$ G. In this case, $\mathcal{W}$ is called a *stable* subspace of $\mathcal{V}$.

Note that $\{0\}$ and $\mathcal{V}$ are stable subspaces of $\mathcal{V}$.

EXAMPLE 3.26. Let $G = S_2 = \{1, (12) = t\}$ and ρ its regular representation. Starting with the basis $\{e_{1,1}, e_{1,t}, e_{t,1}, e_{t,t}\}$ for the tensor representation $\rho\otimes\rho$, form the new basis with components

$$\mathrm{v}_1 = 2e_{1,1}, \quad \mathrm{v}_2 = 2e_{t,t}, \quad \mathrm{v}_3 = e_{1,t} + e_{t,1}, \quad \mathrm{v}_4 = e_{1,t} - e_{t,1}.$$

The regular representation, β, of S_2 acting on this new basis is given by

$$\beta(1) = \begin{bmatrix} \boxed{I_3} & 0 \\ 0 & \boxed{1} \end{bmatrix}, \quad \beta(t) = \begin{bmatrix} \boxed{F} & 0 \\ 0 & \boxed{-1} \end{bmatrix},$$

where $F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We observe that the representation β *decomposes* as the sum of two components β_1 and β_2, given by

$$\beta_1(1) = I_3, \ \beta_1(t) = F, \quad \beta_2(1) = 1, \ \beta_2(t) = -1.$$

This means that the corresponding subspaces

$$\mathcal{V}_1 =< v_1, v_2, v_3 >, \quad \mathcal{V}_2 =< v_4 >$$

are stable subspaces of $\mathbb{R}^4$ under β. The direct-sum decomposition $\mathbb{R}^4 = \mathcal{V}_1 \oplus \mathcal{V}_2$ justifies the notation $\beta = \beta_1 \oplus \beta_2$. We say that β is *decomposable* and that β_1 and β_2 are its *components*. In addition, because, β and $\rho \otimes \rho$ are equivalent, we write

$$\rho \otimes \rho \simeq \beta = \beta_1 \oplus \beta_2.$$

The subspaces (and corresponding representations) $\mathcal{V}_1$ and $\mathcal{V}_2$ are called, respectively, the *symmetric square* (Sym^2) and *alternating square* (Alt^2) subspaces or representations[5]. The study of group representations is concerned with describing all inequivalent, indecomposable representations of a group G. □

EXAMPLE 3.27 (The Sym^2 and Alt^2 representations). Let ρ indicate a representation of G acting on V (with v elements). The basis for the tensor representation $\rho \otimes \rho$ is indexed by the entries of the matrix $V \times V = \{(s, f); s, f \in V\}$. Let D indicate the main diagonal of $V \times V$ and U its upper triangular part. The permutation representation of G acting on the indices of the basis

$$\{e_{(s,f)} + e_{(f,s)}; (s, f) \in D \cup U\}$$

is the Sym^2 (symmetric square) representation of G. Its dimension is $v(v+1)/2$. The permutation representation of G acting on the indices of the basis

$$\{e_{(s,f)} - e_{(f,s)}; (s, f) \in U\}$$

is the Alt^2 (alternating square) representation of G. Its dimension is $v(v-1)/2$. Moreover,

$$\rho \otimes \rho \simeq Sym^2 \oplus Alt^2.$$

Here are the derivations when ρ is the permutation representation of the cyclic group C_4: We have $\rho(1) = I_4$,

$$\rho((1234)) = r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \rho((13)(24)) = r^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and

$$\rho(1432) = r^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

[5]The respresentation Alt^2 is also called *exterior square* representation.

Correspondly, $\rho \otimes \rho(1) = I_{16}$, $\rho \otimes \rho(1234) = r \otimes r$, $\rho \otimes \rho((13)(24)) = r^2 \otimes r^2$ and $\rho \otimes \rho(1432) = r^3 \otimes r^3$. The basis for $\rho \otimes \rho$ may be indexed by $V \times V = \{(0,0),(0,1),\ldots(3,2),(3,3)\}$, from which the two bases for the Sym-square and Alt-square representations can be obtained. The indices for these bases are, respectively,

$$D \cup U = \{(0,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\},$$

$$U = \{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}.$$

The matrix generating the two new basis $\{e_{(s,f)} + e_{(f,s)}; (s,f) \in D \cup U\}$ and $\{e_{(s,f)} - e_{(f,s)}; (s,f) \in U\}$ is

$$B = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0
\end{bmatrix},$$

from which we obtain,

$$= B(\rho \otimes \rho)(\tau)B^{-1} \simeq (\rho \otimes \rho)(\tau) = \begin{bmatrix} \boxed{\text{Alt}^2(\tau)} & 0 \\ 0 & \boxed{\text{Sym}^2(\tau)} \end{bmatrix},$$

or, $\rho \otimes \rho \simeq \text{Alt}^2 \oplus \text{Sym}^2$. Denoting $\tau = (1234)$, the components of the decomposition are given by

$$\text{Sym}^2(\tau) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\mathrm{Sym}^2(\tau^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathrm{Sym}^2(\tau^3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathrm{Alt}^2(\tau) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{Alt}^2(\tau^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathrm{Alt}^2(\tau^3) = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$\mathrm{Sym}^2(1) = \mathrm{I}_{10}$ and $\mathrm{Alt}^2(1) = \mathrm{I}_6$. □

Note that when $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ and $\mathcal{V}_1$ is a stable subspace of V, of dimension v_1, under ρ, it is necessary and sufficient that the pattern of $\rho(\tau)$ takes the matrix form

$$\rho(\tau) = \begin{bmatrix} \mathrm{R}_1(\tau) & 0 \\ \mathrm{M}(\tau) & \mathrm{R}_2(\tau) \end{bmatrix},$$

for matrices $\mathrm{R}_1(\tau)$, $\mathrm{R}_2(\tau)$ and $\mathrm{M}(\tau)$ of dimensions $\mathrm{v}_1 \times \mathrm{v}_1$, $\mathrm{v}_2 \times \mathrm{v}_2$ and $\mathrm{v}_2 \times \mathrm{v}_1$, respectively. In this case, $\tau \mapsto \mathrm{R}_1(\tau)$ and $\tau \mapsto \mathrm{R}_2(\tau)$ are representations of G, of dimensions v_1 and v_2, respectively. In this case, we say that ρ is a *reducible* representation. An *irreducible* representation, equivalently, implies that the only proper stable linear subspace of $\mathcal{V}$ is the $\{0\}$ subspace, that is,

$$\rho(\tau)\mathrm{v} \in \mathcal{W} \text{ for all } \tau \in \mathrm{G}, \text{ for some } \mathcal{W} \subset \mathcal{V} \Longrightarrow \mathcal{W} = \{0\}.$$

EXAMPLE 3.28. The regular representation of S_2 is given by $\rho(1) = I_2$ and $\rho(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where $t = (12)$. Now take $B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, so that relative to the new basis $\{e_t - e_1, e_t\}$, the equivalent representation $\beta(\tau) = B^{-1}\rho(\tau)B$ is given by

$$\beta(1) = B^{-1}\rho(1)B = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}, \quad \beta(t) = B^{-1}\rho(t)B = \begin{bmatrix} \boxed{-1} & 0 \\ -1 & \boxed{1} \end{bmatrix}.$$

In this case, the representation ρ reduces isomorphically to two components ρ_1 and ρ_2 determined by

$$\rho_1(1) = 1, \quad \rho_1(t) = 1 \quad \text{and} \quad \rho_2(1) = 1, \quad \rho_1(t) = -1.$$

Note that these are one-dimensional representations and hence irreducible components. The component ρ_2 is the signature representation (Sgn), often also called *sign* representation, whereas ρ_1 is the principal (1) representation. In summary,

$$\begin{bmatrix} & 1 & t \\ \hline 1 & \boxed{1} & \boxed{1} \\ \text{Sgn} & \boxed{1} & \boxed{-1} \end{bmatrix}. \tag{3.19}$$

Note that $\boxed{1}, \boxed{-1}$ are 1×1 matrices of action in each of the representations 1 and Sgn. □

EXAMPLE 3.29. Let $V = \{uu, yy, uy, yu\}$, e.g., Example 3.10, where S_2 acts according to $s\tau^{-1}$, $s \in V$, $\tau \in S_2$. Let ρ indicate the corresponding representation. We observed that ρ is decomposable as $\rho_1 \oplus \rho_2$, where each component has dimension 2: $\rho_1 = 1_2$ and ρ_2 is the regular representation of S_2, which (e.g., Example 3.28) further reduces into the (irreducible) one-dimensional components 1_1 and Sgn. That is,

$$\rho = 1_2 \oplus 1_1 \oplus \text{Sgn}.$$

Correspondingly, note that

$$\mathcal{P}_1 = \frac{1}{2}(\rho(1) + \rho(t)) = \begin{bmatrix} I_2 & 0 \\ 0 & \frac{1}{2}ee' \end{bmatrix},$$

and

$$\mathcal{P}_2 = \frac{1}{2}(\rho(1) - \rho(t)) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix},$$

satisfying the following properties: $\mathcal{P}_1 + \mathcal{P}_2 = I_4$, $\mathcal{P}_1\mathcal{P}_2 = 0$, $\mathcal{P}_1^2 = \mathcal{P}_1$ and $\mathcal{P}_2^2 = \mathcal{P}_2$. These *canonical projections* associated with the representation of S_2 acting on V were obtained as linear combinations of $\rho(1), \rho(t)$ in which the coefficients were taken from Matrix 3.19 of irreducible components of S_2. These projections, as indicated earlier in the chapter, and in the examples of Chapters 1 and 2, are the primary applications of the theory being reviewed in this chapter. □

EXAMPLE 3.30. (A two-dimensional irreducible representation of S_3) We will construct a two-dimensional irreducible representation of S_3. Let ρ indicate the permutation representation of S_3, e.g., Example 3.18. The representation has dimension 3 and is given by

$$\rho_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\rho_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \rho_{123} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \rho_{132} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Start with the subspace W_1 generated by the sum $e = e_1 + e_2 + e_3$ of the vectors in the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. That is, W_1 is generated by $e' = (1, 1, 1)$. Clearly, W_1 is a stable subspace of ρ, that is,

$$\rho_\tau y \in W_1, \quad \text{for all } y \in W_1, \quad \text{for all } \tau \in S_3.$$

Let $W_0 = \{y \in \mathbb{R}^3; e'y = 0\}$ be the orthogonal complement of W_1 in $\mathbb{R}^3$ and $\mathcal{P} = \frac{1}{3}ee'$ the projection on W_1 along W_0, that is, $\mathbb{R}^3 = W_0 \oplus W_1$ and $\mathcal{P}_y = 0$ for all $y \in W_0$. Similarly, let

$$Q = I_3 - \mathcal{P} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{3.20}$$

indicate the projection on W_0 along W_1. The important point in this construction is observing that the matrix Q commutes with ρ_τ, for all $\tau \in S_3$, that is,

$$\rho_\tau Q \rho_{\tau^{-1}} = Q, \quad \text{for all } \tau \in S_3.$$

As a consequence, if $y \in W_0$ then $y \in Qz$ for some $z \in \mathbb{R}^3$, and $\rho_\tau y = \rho_\tau Qz = Q\rho_\tau z \in W_0$, for all $\tau \in S_3$. That is, W_0 is a stable complement of W_1 in $\mathbb{R}^3$, with $\dim W_0 = 3 - 1 = 2$. To construct a 2-dimensional representation (β) in W_0, note, from the corresponding projection in (3.20), that a basis $\{v_1, v_2\}$ for Im Q is $v_1 = 2e_1 - e_2 - e_3$, $v_2 = -e_1 + 2e_2 - e_3$. The resulting representation of $\tau = (12)$, for example, is obtained from the fact that

$$\tau v_1 = 2e_{\tau 1} - e_{\tau 2} - e_{\tau 3} = 2e_2 - e_1 - e_3 = v_2,$$

$$\tau v_2 = -e_{\tau 1} + 2e_{\tau 2} - e_{\tau 3} = -e_2 + 2e_1 - e_3 = v_1,$$

that is, $\beta_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Similar calculations (noting, from 3.20, that $-e_1 - e_2 + 2e_3 = -v_1 - v_2$) leads to the linear representation (shown along with their corresponding traces)

$$\beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathrm{tr}\ \beta_1 = 2,$$

$$\beta_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathrm{tr}\ \beta_{12} = 0,$$

$$\beta_{13} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \quad \mathrm{tr}\ \beta_{13} = 0,$$

$$\beta_{23} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad \mathrm{tr}\ \beta_{23} = 0,$$

$$\beta_{123} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \mathrm{tr}\ \beta_{123} = -1,$$

$$\beta_{132} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathrm{tr}\ \beta_{132} = -1.$$

The two-dimensional representation β is irreducible. In fact, if there were a proper one-dimensional stable subspace W, with generator y, then it would verify $\beta_{12}\mathrm{y} = \lambda\mathrm{y}$ for some scalar λ, which implies $\mathrm{y}_2 = \lambda\mathrm{y}_1$, $\mathrm{y}_1 = \lambda\mathrm{y}_2$. The non-zero eigenvalue solutions to $\mathrm{y}_2 = \lambda^2\mathrm{y}_2$ are $\lambda = \pm 1$, that is, $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_1)$ or $\mathrm{y} = (\mathrm{y}_1, -\mathrm{y}_1)$. Sice the subspace W must also be stable under β_{13} then we would have

$$\beta_{13}\mathrm{y} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{y}_1 \\ \mathrm{y}_1 \end{bmatrix} = \begin{bmatrix} -2\mathrm{y}_1 \\ \mathrm{y}_1 \end{bmatrix} \in \mathrm{W}$$
$$\iff \mathrm{y}_1 = 0, \text{ using } \mathrm{y} = (\mathrm{y}_1, \mathrm{y}_1) \text{or } \mathrm{y} = (\mathrm{y}_1, -\mathrm{y}_1) \implies \mathrm{W} = \{0\}.$$

Because $\{0\}$ is the only proper stable subspace, β is irreducible.

In analogy to Table 3.19, we may summarize these results according to the trace (indicated by χ), of the corresponding (classes of) representations as follows:

(3.21)

χ	1	(12)	(123)
χ_1	1	1	1
χ_β	2	0	−1
χ_{Sgn}	1	−1	1

Note that we added the trace of the one-dimensional Sgn representation. Later on we will see that this table completely describes the representations of S_3. □

EXAMPLE 3.31. To appreciate the role of the field of scalars in Example 3.30, restrict the search for a one-dimensional stable subspace to the cyclic subgroup $C_3 = \{1, (123), (132)\}$ of S_3. In this case, we have the two-dimensional representation

$$\gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma_{123} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \gamma_{132} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

The equations $\gamma_\tau\mathrm{y} = \lambda\mathrm{y}$ for $\tau \in C_3$ lead to the characteristic equations $(1-\lambda)^2 = 0$ and $\lambda^2 + \lambda + 1 = 0$. When the field of scalars is the complex field $\mathbb{C}$ we find two one-dimensional irreducible representations, corresponding to the roots $\omega = \frac{2\pi i}{6}$ and ω^2. If the field of scalars is the reals then γ is irreducible. Here is the summary for three irreducible representations of C_3,

$$\begin{bmatrix} \chi & 1 & (123) & (132) \\ \hline \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & \omega & \omega^2 \\ \chi_2 & 1 & \omega^2 & \omega \end{bmatrix}. \tag{3.22}$$

□

THEOREM 3.3. Let $\rho : G \to GL(\mathcal{V})$ be a linear representation of G in $\mathcal{V}$ and let W_1 be a vector subspace of $\mathcal{V}$ stable under G. Then there is a complement W_0 of W_1 in $\mathcal{V}$ which is also stable under G.

PROOF. Let $\mathcal{P}_1$ be a projection on W_1 along some vector space complement of W_1 in $\mathcal{V}$. Form the average

$$\overline{\mathcal{P}}_1 = \frac{1}{|G|} \sum_{\tau \in G} \rho(\tau)\mathcal{P}_1\rho(\tau^{-1})$$

of projections on W_1 along that vector space complement. Then, Im $\overline{\mathcal{P}}_1 = \{\overline{\mathcal{P}}_1 z; z \in \mathcal{V}\} = W_1$. To see this, first note that for $z \in \mathcal{V}$ we have $\mathcal{P}_1\rho_{\tau^{-1}}z \in W_1$, and because W_1 is a stable subspace, $\rho_\tau[\mathcal{P}_1\rho_{\tau^{-1}}z] \in W_1$, so that $\overline{\mathcal{P}}_1 z \in W_1$, that is, Im $\overline{\mathcal{P}}_1 \subseteq W_1$. Secondly, if $z \in W_1$, which is stable, we have $\rho_{\tau^{-1}}z \in W_1$ for all $\tau \in G$, so that $\mathcal{P}_1\rho_{\tau^{-1}}z = \rho_{\tau^{-1}}z$. This implies

$$\overline{\mathcal{P}}_1 z = \frac{1}{|G|} \sum_{\tau \in G} \rho_\tau \mathcal{P}_1 \rho_{\tau^{-1}} z = \frac{1}{|G|} \sum_{\tau \in G} \rho_\tau \rho_{\tau^{-1}} z = z,$$

that is, if $z \in W_1$ then $z = \overline{\mathcal{P}}_1 z \in$ Im $\overline{\mathcal{P}}_1$, and hence $W_1 \subseteq$ Im $\overline{\mathcal{P}}_1$. Therefore, $W_1 =$ Im $\overline{\mathcal{P}}_1$. Let then $W_0 = \ker \overline{\mathcal{P}}_1 = \{z \in \mathcal{V}; \overline{\mathcal{P}}_1 z = 0\}$, so that $\mathcal{V} = W_1 \oplus W_0$. To conclude the proof, we must show that W_0 is G-stable: In fact, for all $\tau \in G$,

$$\begin{aligned} \rho_\tau \overline{\mathcal{P}}_1 \rho_{\tau^{-1}} &= \frac{1}{|G|} \sum_{\sigma \in G} \rho_\tau \rho_\sigma \mathcal{P}_1 \rho_{\sigma^{-1}} \rho_{\tau^{-1}} = \frac{1}{|G|} \sum_{\sigma \in G} \rho_{\tau\sigma} \mathcal{P}_1 \rho_{(\tau\sigma)^{-1}} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \rho_\sigma \mathcal{P}_1 \rho_{\sigma^{-1}} = \overline{\mathcal{P}}_1, \end{aligned}$$

so that $y \in W_0 = \ker \overline{\mathcal{P}}_1$ implies $\overline{\mathcal{P}}_1 y = 0$ and hence $\overline{\mathcal{P}}_1 \rho_\tau y = \rho_\tau \overline{\mathcal{P}}_1 y = 0$, thus showing that $\rho_\tau y \in W_0$, for all $\tau \in G$. Consequently, W_0 is a stable subspace of $\mathcal{V}$ under G. □

THEOREM 3.4. Every representation is a direct sum of irreducible representations.

PROOF. Let $\mathcal{V}$ be (the vector space associated to) a linear representation of G. The argument is by induction on the dimension of $\mathcal{V}$. Suppose dim $\mathcal{V} \geq 1$. If $\mathcal{V}$ is irreducible, the proof is complete. Otherwise, from Theorem 3.3, $\mathcal{V} = V' \oplus V''$ with dim $V' <$ dim V and dim $V'' <$ dim V. By the induction hypothesis, V' and V'' are direct sum of irreducible representations, and then so is $\mathcal{V}$. □

3.8. Characters

Given a representation ρ, the complex-valued function

$$\chi_\rho : \tau \to \text{tr } \rho_\tau$$

is called the *character* of the representation. It plays an important role in the characterization of the representation. Since $\rho(1) = I_\ell$ and $\ell = \dim \rho$, we note that $\chi_\rho(1) = \dim \rho$.

If λ is an eigenvalue of ρ, then, relative to the invariant scalar product 3.15, we have

$$(y, y) = (\rho_\tau y, \rho_\tau y) = (\lambda y, \lambda y) = \lambda\overline{\lambda}(y, y),$$

so that $\lambda\overline{\lambda} = 1$. Let $\lambda_1, \ldots, \lambda_m$ indicate the eigenvalues of ρ_τ (over $\mathbb{C}$). Then

$$\chi_\rho(\tau^{-1}) = \text{tr } \rho_{\tau^{-1}} = \text{tr } \rho_\tau^{-1} = \sum \lambda_i^{-1} = \sum \overline{\lambda}_i = \text{tr } \overline{\rho}_\tau = \overline{\chi}_\rho(\tau). \tag{3.23}$$

Also note, since trace is invariant under similarity, that

$$\chi_\rho(\tau\sigma\tau^{-1}) = \chi_\rho(\sigma), \quad \text{for all } \tau, \sigma \in G. \tag{3.24}$$

DEFINITION 3.7 (Class functions). A scalar-valued function h defined on G and satisfying $h(\tau\sigma\tau^{-1}) = h(\sigma)$, for all $\sigma, \tau \in G$ is called a *class function.*

Class functions are constant within each conjugacy class of G. We will study these functions with detail in Section 3.11.

PROPOSITION 3.3. Let $\rho^i : G \to GL(\mathcal{V}_i)$ be a linear representation of G, with corresponding character χ_i, $i = 1, 2$. Then

$$\chi_{\rho^1\oplus\rho^2} = \chi_1 + \chi_2, \quad \chi_{\rho^1\otimes\rho^2} = \chi_1 \times \chi_2.$$

PROOF. We have

$$\chi_{\rho^1\oplus\rho^2} = \text{tr } (\rho^1 \oplus \rho^2) = \text{tr } \begin{bmatrix} \rho^1 & 0 \\ 0 & \rho^2 \end{bmatrix} = \text{tr } \rho^1 + \text{tr } \rho^2 = \chi_1 + \chi_2,$$

whereas, noting that the diagonal of $\chi_{\rho^1\otimes\rho^2}$ is

$$([\rho^1]_{11}\text{diag } \rho^2, [\rho^1]_{22}\text{diag } \rho^2, \ldots, [\rho^1]_{n_1n_1}\text{diag } \rho^2),$$

we obtain

$$\chi_{\rho^1\otimes\rho^2} = \text{tr } (\rho^1 \otimes \rho^2) = \sum_i [\rho^1]_{ii} \times \sum_j [\rho^2]_{jj} = \text{tr } \rho^1 \times \text{tr } \rho^2 = \chi_1 \times \chi_2.$$

□

EXAMPLE 3.32. In Example 3.27 we considered the Sym^2 and Alt^2 representations, where we showed that

$$\rho \otimes \rho \simeq \text{Sym}^2 \oplus \text{Alt}^2.$$

From the decomposition for the tensor representation of C_4 discussed in that example, we obtain the following characters:

C_4	$\chi_\rho(\tau)$	$\chi_{\text{Sym}^2}(\tau)$	$\chi_{\text{Alt}^2}(\tau)$
1	4	10	6
(1234)	0	0	0
(13)(24)	0	2	−2
(1432)	0	0	0

Note that, for all $\tau \in C_4$,

$$\chi^2_\rho(\tau) = \chi_{\rho\otimes\rho}(\tau) = \chi_{\mathrm{Sym}^2}(\tau) + \chi_{\mathrm{Alt}^2}(\tau),$$

and

$$\chi_{\mathrm{Sym}^2}(\tau) = \frac{1}{2}(\chi^2_\rho(\tau) + \chi_\rho(\tau^2)), \quad \chi_{\mathrm{Alt}^2}(\tau) = \frac{1}{2}(\chi^2_\rho(\tau) - \chi_\rho(\tau^2)).$$

It can be shown that these two equalities hold in general for any linear representation ρ of G. □

3.9. Schur's lemma and applications

LEMMA 3.2 (Schur). Let $\rho_i : G \to \mathcal{V}_i$ be irreducible representations of G, $i = 1, 2$, and let $f : \mathcal{V}_1 \to \mathcal{V}_2$ be a non-zero linear mapping satisfying $f \cdot \rho_1(\tau) = \rho_2(\tau) \cdot f$ for all $\tau \in G$. Then,

(1) ρ_1 and ρ_2 are isomorphic;
(2) If $\mathcal{V}_1 = \mathcal{V}_2$ and $\rho_1 = \rho_2$ then f is a scalar multiple of the identity mapping.

PROOF. Let $W_1 = \ker f = \{x; f(x) = 0\}$. If $x \in W_1$ then $f(x) = 0$ and $f\rho_2(\tau)x = \rho_1(\tau)f(x) = 0$, which implies $\rho_1(\tau)x \in W_1$, for all $\tau \in G$. That is, W_1 is a stable subspace. Since ρ is irreducible, we must have $W_1 = \{0\}$ or $W_1 = \mathcal{V}_1$. If $W_1 = \mathcal{V}_1$ then $f = 0$, contrary to the hypothesis, hence $W_1 = \{0\}$. Similarly, we obtain Image f is stable and equal to $\mathcal{V}_2$. Hence, f is an isomorphism, and the two representations are equivalent or isomorphic. For the second part, let λ be an eigenvalue of f (the field is $\mathbb{C}$, so there is at least one) and define $f' = f - \lambda$, understanding that $\lambda \equiv \lambda I$. If $f(x) = \lambda x$ then $(f - \lambda)x = 0$, so that $\ker (f - \lambda) \neq \{0\}$, and equivalently, $f - \lambda$ is not an isomorphism. Moreover,

$$(f - \lambda)\rho(\tau) = f\rho(\tau) - \lambda\rho(\tau) = \rho(\tau)f - \rho(\tau)\lambda = \rho(\tau)(f - \lambda), \quad \text{for all } \tau \in G.$$

From the first part of the Lemma, it follows that $f - \lambda = 0$, or $f = \lambda I$. □

In the study of the linear representations of a finite group G, it is often necessary to consider the vector space $\mathcal{G}$ of all scalar functions defined on G. An important element in $\mathcal{G}$ is the character $\chi_\rho(\tau) = \operatorname{tr} \rho_\tau$ of a representation ρ, introduced earlier on in Section 3.8. In general, note that each entry ρ_{ij} of a linear representation ρ defines a scalar function $\tau \to \rho_{ij}(\tau)$.

EXAMPLE 3.33. Let $G = S_3$. In Example 3.30 we identified an irreducible representation, β,

$$\beta(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \beta(12) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \beta(13) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\beta(23) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \beta(123) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \beta(132) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

of dimension 2, and two irreducible non-equivalent representations of dimension one: the trivial representation, $1(\tau) = 1$ for all $\tau \in G$ and the signature representation

$$\mathrm{Sgn}(1) = 1, \quad \mathrm{Sgn}(12) = \mathrm{Sgn}(13) = \mathrm{Sgn}(23) = -1, \quad \mathrm{Sgn}(123) = \mathrm{Sgn}(132) = 1.$$

All together, these representations illustrate $24 + 1 + 1 = 26$ scalar functions defined on G, or 26 points in the vector space $\mathcal{G}$. These functions have a number of characteristic properties. For example, note that

$$\sum_{\tau\in G} 1(\tau)\text{hSgn}(\tau^{-1}) = 0, \quad \text{for all scalar h},$$

$$\sum_{\tau\in G} 1(\tau)\text{H}\beta(\tau^{-1}) = 0, \quad \sum_{\tau\in G}\text{Sgn}(\tau)\text{H}\beta(\tau^{-1}) = 0, \quad \text{for all H} : \mathbb{R}^2 \to \mathbb{R}.$$

In fact, we have,

PROPOSITION 3.4. For every non-equivalent irreducible representations ρ_1, ρ_2 and every linear mapping $\text{H} : \mathcal{V}_1 \to \mathcal{V}_2$, it holds that $\sum_{\tau\in G}\rho_1(\tau)\text{H}\rho_2(\tau^{-1}) = 0$.

PROOF. Note that $\text{H}_0 = \sum_{\tau\in G}\rho_1(\tau)\text{H}\rho_2(\tau^{-1})$ is a linear mapping from $\mathcal{V}_1$ into $\mathcal{V}_2$ which intertwines with $\rho_1(\tau)$ and $\rho_2(\tau)$ for all $\tau \in G$, that is, $\rho_1(\tau)\text{H}_0 = \text{H}_0\rho_2(\tau)$ for all $\tau \in G$. From Schur's Lemma (the representations are non-equivalent irreducible) it follows that $\text{H}_0 = 0$. □

□

Now take any linear mapping $\text{H} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and consider the two-dimensional irreducible representation β of S_3 reviewed above. Direct evaluation shows that

$$\frac{1}{6}\sum_{\tau\in S_3}\beta_\tau\text{H}\beta_{\tau^{-1}} = \frac{a+d}{2}I_2 = \frac{\text{tr H}}{2}I_2.$$

In fact, we have,

PROPOSITION 3.5. Let ρ be an irreducible representation of G into $GL(\mathcal{V})$ with $\dim \rho = n$. Then, for any linear mapping H in $\mathcal{V}$,

$$\frac{1}{|G|}\sum_{\tau\in G}\rho_\tau\text{H}\rho_{\tau^{-1}} = \frac{\text{tr H}}{n}I_n.$$

PROOF. Schur's Lemma implies that $\text{H}_0 = \frac{1}{|G|}\sum_{\tau\in G}\rho_\tau\text{H}\rho_{\tau^{-1}} = \lambda I_n$ for some scalar λ. Taking the trace on both sides (and using its invariance under similarity) the result $\lambda = \text{tr H}/n$ obtains. □

EXAMPLE 3.34 (A n − 1-dimensional irreducible representation). Consider the permutation representation, ρ, of S_3. We know that $\rho \simeq 1 \oplus \beta$, where β is the 2-dimensional irreducible obtained in Example 3.30. That is,

$$\rho_\tau = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{\beta_\tau} \end{bmatrix}.$$

Then, for any linear transformation matrix $\text{H} = \begin{bmatrix} h_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ in $\mathbb{R}^3$ (in block form of dimensions 1 and 2), the reader may verify that

$$\frac{1}{3!}\sum_{\tau\in S_3}\rho_\tau\text{H}\rho_{\tau^{-1}} = \begin{bmatrix} \boxed{h_{11}} & 0 \\ 0 & \boxed{\frac{\text{tr } H_{22}}{2}I_2} \end{bmatrix}.$$

Example 3.30 suggests that we consider the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$ transforming from the canonical basis of $\mathbb{R}^n$ into the new basis $\mathcal{B} = \{e_1 + e_2 + e_2, 2e_1 - e_2 - e_3, -e_1 + 2e_2 - e_3\}$. Then, it follows that

$$\frac{1}{3!}\sum_{\tau \in S_3}(P\rho_\tau P^{-1})H(P\rho_{\tau^{-1}}P^{-1}) = \begin{bmatrix} \boxed{h_{11}} & 0 \\ 0 & \boxed{\frac{\text{tr } H_{22}}{2} I_2} \end{bmatrix}.$$

The above construction applies to S_n in general, thus leading to the existence of a $n-1$ irreducible representation on the subspace complement to the trivial one-dimensional representation. □

Proposition 3.6. If ρ is the permutation representation of S_n, then, for every real or complex $n \times n$ matrix H,

$$\frac{1}{n!}\sum_{\tau \in S_n}\rho_\tau H \rho_{\tau^{-1}} = a_0 ee' + a_1 I_n,$$

where the coefficients a_0 and a_1 are scalars defined by the relations $n(a_0+a_1) = \text{tr } H$ and $n(n-1)a_0 = e'He - \text{tr } H$, in which $e'He$ is the sum of the entries in H.

Proof. Let $M = \frac{1}{n!}\sum_{\tau \in S_n}\rho_\tau H \rho_{\tau^{-1}}$ and let $J = PHP^{-1}$ where P is the $n \times n$ matrix

$$P = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ n-1 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & n-1 & -1 \end{bmatrix}.$$

Consequently, applying Proposition 3.34 to the $(1, n-1)$- irreducible decomposition of $P\rho P^{-1}$, we have

$$PMP^{-1} = \frac{1}{n!}\sum_{\tau \in S_n}(P\rho_\tau P^{-1})J(P\rho_{\tau^{-1}}P^{-1}) = \begin{bmatrix} J_{11} & 0 \\ 0 & \frac{\text{tr } J_{22}}{n-1} I_{n-1} \end{bmatrix},$$

from which we obtain

$$M = P^{-1}\begin{bmatrix} J_{11} & 0 \\ 0 & \frac{\text{tr } J_{22}}{n-1} I_{n-1} \end{bmatrix} P.$$

Direct evaluation, using the definition of the matrix P, shows that M is the matrix with entries

$$M_{ij} = \begin{cases} \text{tr } H/n & \text{if } i = j; \\ (e'He - \text{tr } H)/(n-1) & \text{if } i \neq j \end{cases},$$

which is the proposed result. □

Consider again the irreducible representations 1,Sgn and β of S_3, discussed earlier on in Example 3.33. Let $H = (h_{11}, h_{12})$ be any linear mapping from $\mathbb{R}^2$ into $\mathbb{R}$. From Schur's Lemma we know that

$$\sum_{\tau \in G} \text{Sgn }(\tau) H \beta(\tau^{-1}) = 0.$$

That is, the linear forms

$$\sum_{\tau\in G} \mathrm{Sgn}\,(\tau)[h_{11}\beta_{11}(\tau^{-1}) + h_{12}\beta_{21}(\tau^{-1})], \quad \sum_{\tau\in G} \mathrm{Sgn}\,(\tau)[h_{11}\beta_{12}(\tau^{-1}) + h_{12}\beta_{22}(\tau^{-1})]$$

in h_{11} and h_{12} vanish for all values of h_{11} and h_{12}. Therefore, the corresponding coefficients must be zero, that is,

$$\sum_{\tau\in G} \mathrm{Sgn}\,(\tau)\beta_{11}(\tau^{-1}) = 0, \quad \sum_{\tau\in G} \mathrm{Sgn}\,(\tau)\beta_{21}(\tau^{-1}) = 0, \tag{3.25}$$

$$\sum_{\tau\in G} \mathrm{Sgn}\,(\tau)\beta_{21}(\tau^{-1}) = 0, \quad \sum_{\tau\in G} \mathrm{Sgn}\,(\tau)\beta_{22}(\tau^{-1}) = 0. \tag{3.26}$$

The reader may verify relations (3.25) and (3.26) from Matrix (3.27).

(3.27)

τ	1	$\mathrm{Sgn}(\tau^{-1})$	$\beta_{11}(\tau^{-1})$	$\beta_{21}(\tau^{-1})$	$\beta_{12}(\tau^{-1})$	$\beta_{22}(\tau^{-1})$
1	1	1	1	0	0	1
(12)	1	−1	0	1	1	0
(13)	1	−1	−1	0	−1	1
(23)	1	−1	1	−1	0	−1
(123)	1	1	0	−1	1	−1
(132)	1	1	−1	1	−1	0

This is the argument that proves

COROLLARY 3.1. For any two non-equivalent irreducible representations ρ, β of G, the relation

$$\sum_{\tau\in G} \rho_{ij}(\tau)\beta_{k\ell}(\tau^{-1}) = 0$$

holds for all i, j, k, ℓ indexing the entries of these representations.

Consider again the irreducible two-dimensional representation, β, of S_3 discussed in Example 3.33. From Proposition 3.4, we know that

$$\frac{1}{|G|}\sum_{\tau\in G} \beta_\tau \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \beta_{\tau^{-1}} = \frac{\mathrm{tr}\, H}{2} I_2,$$

implying that, for all scalars $h_{11}, h_{12}, h_{21}, h_{22}$, we must have

$$\frac{1}{|G|}\sum_{\tau\in G}\sum_{j,k=1}^{2} \beta_{ij}(\tau)h_{jk}\beta_{ki}(\tau^{-1}) = \frac{1}{2}h_{11} + \frac{1}{2}h_{22}, \quad i = 1, 2.$$

or, equivalently,

$$\frac{1}{|G|}[\sum_{\tau\in G} \beta_{i1}(\tau)\beta_{1i}(\tau^{-1})]h_{11} + \frac{1}{|G|}[\sum_{\tau\in G} \beta_{i1}(\tau)\beta_{2i}(\tau^{-1})]h_{12} +$$
$$\frac{1}{|G|}[\sum_{\tau\in G} \beta_{i2}(\tau)\beta_{1i}(\tau^{-1})]h_{21} + \frac{1}{|G|}[\sum_{\tau\in G} \beta_{i2}(\tau)\beta_{2i}(\tau^{-1})]h_{22} = \frac{1}{2}h_{11} + \frac{1}{2}h_{22}, \quad i = 1, 2,$$

for all scalars $h_{11}, h_{12}, h_{21}, h_{22}$. Consequently, equating the coefficients of the linear forms, the equality $\sum_{\tau\in G} \beta_{ij}(\tau)\beta_{k\ell}(\tau^{-1}) = \frac{1}{2}$ when $i = \ell, j = k$ (and 0 otherwise) must obtain. This is the argument proving the following result:

PROPOSITION 3.7. For any n-dimensional irreducible representation, ρ, of G we have

$$\sum_{\tau\in G} \rho_{ij}(\tau)\rho_{k\ell}(\tau^{-1}) = \begin{cases} \frac{1}{n} & \text{if } i=\ell, j=k; \\ 0 & \text{otherwise.} \end{cases}$$

Matrix (3.27) provides the numerical values for applying Proposition 3.7 to the irreducible representations of S_3.

3.10. Orthogonality relations for characters

Given the complex-valued functions f and g defined on a group G, let

$$(f \mid g) = \frac{1}{|G|}\sum_{\tau\in G} f(\tau)\overline{g(\tau)}. \tag{3.28}$$

We note that $(\cdot \mid \cdot)$ is linear in the first argument, *semilinear* (or *conjugate linear*) in the second argument, and $(f \mid f) > 0$ if $f \neq 0$, and hence is a scalar product (e.g., Example 3.8) in the vector space $\mathcal{G}$ of complex-valued functions defined in G. In particular, if χ_1 and χ_2 are characters of a representation of G, then $\chi_1, \chi_2 \in \mathcal{G}$ and, from Expression (3.23),

$$(\chi_1 \mid \chi_2) = \frac{1}{|G|}\sum_{\tau\in G}\chi_1(\tau)\overline{\chi_2(\tau)} = \frac{1}{|G|}\sum_{\tau\in G}\chi_1(\tau)\chi_2(\tau^{-1}).$$

Recall from Section 3.6, that we may assume that the representation ρ is unitary. Consequently, Proposition 3.7 can be expressed as
(3.29)

$$\frac{1}{|G|}\sum_{\tau\in G}\rho_{ij}(\tau)\rho_{k\ell}(\tau^{-1}) = \frac{1}{|G|}\sum_{\tau\in G}\rho_{ij}(\tau)\overline{\rho_{k\ell}(\tau)} = (\rho_{ij} \mid \rho_{k\ell}) = \begin{cases} \frac{1}{n} & \text{if } i=\ell,\ j=k \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, Corollary 3.1 becomes

$$(\rho_{ij} \mid \beta_{k\ell}) = 0, \quad \text{for all } i,j,k,\ell, \tag{3.30}$$

where ρ and β are two non-equivalent irreducible representations of G.

THEOREM 3.5. (a) If χ is the character of an irreducible representation then $(\chi \mid \chi) = 1$; (b) If χ_1 and χ_2 are the characters of two non-equivalent irreducible representations of a group G, then $(\chi_1 \mid \chi_2) = 0$.

PROOF. From expression (3.29), we have

$$(\chi \mid \chi) = \frac{1}{|G|}\sum_{\tau\in G}(\sum_{i=1}^{n}\rho_{ii}(\tau) \mid \sum_{j=1}^{n}\rho_{jj}(\tau)) = \sum_{i=1}^{n}(\rho_{ii} \mid \rho_{ii}) = \sum_{i=1}^{n}\frac{1}{n} = 1,$$

whereas, from Expression (3.30), similarly, we obtain $(\chi_1 \mid \chi_2) = 0$, concluding the proof. □

We refer to the character of an irreducible representation as an *irreducible character*.

EXAMPLE 3.35. The following matrix shows three irreducible characters for S_3, corresponding to the component irreducible representations 1, Sgn, and β of the permutation representation ρ, discussed earlier on in Example 3.33:

$$\left[\begin{array}{c|cccc} \tau & \chi_\rho & \chi_1 & \chi_{\rm Sgn} & \chi_\beta \\ \hline 1 & 3 & 1 & 1 & 2 \\ (12) & 1 & 1 & -1 & 0 \\ (13) & 1 & 1 & -1 & 0 \\ (23) & 1 & 1 & -1 & 0 \\ (123) & 0 & 1 & 1 & -1 \\ (132) & 0 & 1 & 1 & -1 \end{array}\right]; \tag{3.31}$$

It also shows the character χ_ρ of the permutation representation ρ. □

The reader may verify, from Matrix 3.31, that

$$(\chi_1 \mid \chi_\rho) = (\chi_\beta \mid \chi_\rho) = (\chi_{\rm Sgn} \mid \chi_\rho) = 1.$$

In fact, $(\chi_\theta \mid \chi_\rho)$ is the number of irreducible representations isomorphic to θ in the decomposition of ρ. We have, then,

PROPOSITION 3.8. If ρ is a linear representation of G with character χ, and ρ decomposes as a sum $\rho_1 + \ldots + \rho_r$ of irreducible representations with characters $\chi_1, \ldots, \chi_r$, then $(\chi_i \mid \chi)$ is the number of representations in the decomposition that are isomorphic to ρ_i.

EXAMPLE 3.36. Let S_2 act on $V = \{uu, yy, uy, yu\}$ according to $s\tau^{-1}$ (location symmetry). The representation, denoting $S_2 = \{1, t = (12)\}$, is

$$\rho_1 = I_4, \quad \rho_t = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \end{bmatrix},$$

and, accordingly, we write, $\rho = 1 \oplus 1 \oplus \gamma$. Let $u = v_{uy} + v_{yu}$, $v = v_{uy} - v_{yu}$. Then $1v = v$, $tv = -v$, so that $\gamma = 1 \oplus \text{Sgn}$. In total,

$$\rho = 1 \oplus 1 \oplus 1 \oplus \text{Sgn} .$$

Here is the character table

$$\left[\begin{array}{c|ccc} \tau & \chi_\rho & \chi_1 & \chi_{\rm Sgn} \\ \hline 1 & 4 & 1 & 1 \\ t & 2 & 1 & -1 \end{array}\right],$$

from which we obtain

$$(1 \mid \rho) = \frac{1}{2}(\chi_1(1)\chi_\rho(1) + \chi_1(t)\chi_\rho(t)) = \frac{1}{2}(4+2) = 3,$$

which is the multiplicity of the trivial representation in the decomposition of ρ. Similarly, the signature representation appears with multiplicity

$$(\text{Sgn} \mid \rho) = \frac{1}{2}(1 \times 4 + (-1) \times 2) = 1.$$

Note that the multiplicity of a given irreducible component does not depend on the underlying decomposition. Moreover, two representations with the same character are isomorphic, because they contain each irreducible component with exactly the same multiplicity. These arguments reflect the importance of characters in the study of linear representations. □

We may then restrict our attention to the set $\chi_1, \ldots, \chi_h$ of *distinct* irreducible characters of G, and write,

$$\mathcal{V} = m_1\mathcal{V}_1 \oplus \ldots \oplus m_h\mathcal{V}_h,$$

or, equivalently, $\rho = m_1\rho_1 \oplus \ldots \oplus m_h\rho_h$. In this case, we have

$$\chi_\rho = m_1\chi_1 + \ldots + m_h\chi_h. \tag{3.32}$$

The multiplicities m_i are given by the integers $(\chi_\rho \mid \chi_i) \geq 0$, $i = 1, \ldots h$. In the previous example

$$\chi_\rho = 3\chi_1 + \chi_{\text{Sgn}}.$$

Consequently, the orthogonality relations among the irreducible components imply that $(\chi_\rho \mid \chi_\rho) = \sum_{i=1}^{h} m_i^2$. The following result is a useful characterization of the irreducible representations.

THEOREM 3.6. $(\chi_\rho \mid \chi_\rho) = 1$ if and only if ρ is irreducible.

PROOF. We have $(\chi_\rho \mid \chi_\rho) = \sum_{i=1}^{h} m_i^2 = 1$ if and only if exactly one of the m_i's is equal to 1 and all the others are equal to 0, in which case ρ is isomorphic to that irreducible component. □

EXAMPLE 3.37. Consider the irreducible representations 1, β and Sgn of S_3, along with the tensor $\beta \otimes \beta$ representation. Matrix 3.33 shows the corresponding characters:

(3.33)

τ	β	$\beta \otimes \beta$	1	Sgn
1	2	4	1	1
(12)	0	0	1	−1
(13)	0	0	1	−1
(23)	0	0	1	−1
(123)	−1	1	1	1
(132)	−1	1	1	1

The reader may verify that

$$(\chi_\beta \mid \chi_\beta) = (\chi_1 \mid \chi_1) = (\chi_{\text{Sgn}} \mid \chi_{\text{Sgn}}) = 1;$$

As for the tensor representation, $(\chi_{\beta\otimes\beta} \mid \chi_{\beta\otimes\beta}) = 18/6 = 3$, so it must be reducible. On the other hand,

$$(\chi_{\beta\otimes\beta} \mid \chi_\beta) = (\chi_{\beta\otimes\beta} \mid \chi_1) = (\chi_{\beta\otimes\beta} \mid \chi_{\text{Sgn}}) = 1,$$

so that these representations appear in the decomposition of the tensor representation with single multiplicity. In fact, $\beta \otimes \beta = 1 \oplus \beta \oplus \text{Sgn}$, with the corresponding character decomposition. □

Of particular interest in the study of group representations is the *regular* representation, introduced earlier on in Example 3.18. It is defined by the action $\varphi(\tau,\sigma)=\tau\sigma$ in $G\times G$. Its dimension is $|G|$. Since, for all $\sigma\in G$, $\varphi(\tau,\sigma)=\varphi(\eta,\sigma)$ if and only if $\tau=\sigma$, and $\varphi(\tau,1)=\tau$ for all $\tau\in G$, it follows that its character is given by

$$\chi_{\text{reg}}(\tau)=\begin{cases}0 & \text{if } \tau\neq 1;\\ |G| & \text{if } \tau=1.\end{cases}$$

Consequently, for any irreducible representation ρ of G with character χ_ρ, we have

$$(\chi_{\text{reg}},\chi_\rho)=\frac{1}{|G|}\sum_{\tau\in G}\chi_{\text{reg}}(\tau)\chi_\rho(\tau^{-1})=\chi_\rho(1)=\dim\rho, \tag{3.34}$$

that is, *every irreducible representation is contained in the regular representation with multiplicity equal to its dimension.*

PROPOSITION 3.9. The dimensions $n_1,\ldots,n_h$ of the h distinct irreducible representations of G, satisfy the relation

$$|G|=\sum_{i=1}^{h}n_i^2.$$

PROOF. From relations (3.32) and (3.34), we have $\chi_{\text{reg}}(\tau)=\sum_{i=1}^{h}n_i\chi_i(\tau)$, for all $\tau\in G$. Taking $\tau=1$, the proposed equality obtains. □

Note that for $\tau\neq 1$, the defining property of χ_{reg} implies that

$$\sum_{i=1}^{h}n_i\chi_i(\tau)=0. \tag{3.35}$$

EXAMPLE 3.38. Let $G=S_3$. The irreducible non-equivalent representations 1, β and Sgn are contained in the regular representation with multiplicities 1, 2, 1, respectively. Because $|G|=6=1^2+2^2+1^2$, these must be all the distinct irreducible non-equivalent representations of S_3. □

3.11. Class functions

A scalar-valued function x defined on G and satisfying

$$x(\sigma\tau\sigma^{-1})=x(\tau),\quad \text{for all } \tau,\sigma\in G,$$

is called a *class function.* Indicate by $\mathcal{C}$ the set of class functions on G. Note that $\mathcal{C}$ is a linear subspace of the vector space $\mathcal{G}$ of scalar-valued functions defined on G. All characters belong to $\mathcal{C}$. From Example 3.23 we observe that $\mathcal{C}$ is a stable subspace of $\mathcal{G}$ under the representation $\sigma\xrightarrow{\phi}\sigma^*$, that is,

$$x\in\mathcal{C}\Longrightarrow\phi(\sigma)x=x,\quad \text{for all } \sigma\in G.$$

More precisely, $\mathcal{C}$ is the subspace of $\mathcal{G}$ of functions invariant under this conjugation action. For each class function, x, and any representation ρ, define the linear mapping

$$\widehat{x}(\rho)=\sum_{\tau\in G}x(\tau)\rho(\tau).$$

Note that $\widehat{x}(\rho)$ commutes with $\rho(\tau)$ for all $\tau \in G$. In fact,

$$\begin{aligned}\rho_\tau \widehat{x}(\rho)\rho_{\tau^{-1}} &= \rho_\tau \sum_\sigma x(\sigma)\rho_\sigma\rho_{\tau^{-1}} = \sum_\sigma x(\sigma)\rho_\tau\rho_\sigma\rho_{\tau^{-1}} = \sum_\sigma x(\sigma)\rho_{\tau\sigma\tau^{-1}} \\ &= \sum_\sigma x(\tau\sigma\tau^{-1})\rho_{\tau\sigma\tau^{-1}} = \sum_\sigma x(\sigma)\rho_\tau = \widehat{x}(\rho).\end{aligned}$$

Therefore, if ρ is an irreducible representation, it follows from Schur's Lemma that $\widehat{x}(\rho) = \lambda I$. To evaluate λ we take the trace in each side of the above equality, to obtain

$$\begin{aligned}\mathrm{tr}\ \widehat{x}(\rho) &= \sum_{\tau\in G} x(\tau)\mathrm{tr}\ \rho(\tau) = \sum_{\tau\in G} x(\tau)\chi_\rho(\tau) = \sum_{\tau\in G} x(\tau)\overline{\chi_\rho(\tau^{-1})} \\ &= |G|(x,\overline{\chi_\rho}) = \mathrm{tr}\ \lambda I_n = n\lambda,\end{aligned}$$

so that $\lambda = |G|(x,\overline{\chi_\rho})/n$. This proves

PROPOSITION 3.10. If ρ is an n-dimensional irreducible representation of G, then

$$\widehat{x}(\rho) = \frac{|G|}{n}(x,\overline{\chi_\rho})I_n.$$

THEOREM 3.7. The distinct irreducible characters form an orthonormal basis for $\mathcal{C}$.

PROOF. From Theorem 3.5 we know that the set of distinct irreducible characters form an orthonormal set of functions in $\mathcal{C}$. We need to show that this set generates $\mathcal{C}$. Suppose that $x \in \mathcal{C}$ and that x is orthogonal to $\overline{\chi}_1, \ldots, \overline{\chi}_h$. Therefore, for any irreducible n-dimensional representation ρ of G, we have

$$\widehat{x}(\rho) = \frac{|G|}{n}(x \mid \overline{\chi}_\rho)I_n = 0.$$

Because every representation decomposes as a sum of irreducible components, it follows that $\widehat{x}(\rho) = 0$ for every representation ρ. In particular, $\widehat{x}(\rho_{reg}) = 0$, in which case

$$0 = \widehat{x}(\rho_{reg})e_1 = \sum_{\tau\in G} x(\tau)\rho_{reg}(\tau)e_1 = \sum_{\tau\in G} x(\tau)e_\tau,$$

which implies $x(\tau) = 0$ for all $\tau \in G$. That is, $x = 0$. □

Note that the dimension of the subspace $\mathcal{C}$ of class functions is determined both by the number of distinct irreducible representations of G and by the number of orbits, or *conjugacy classes*, of G under the action $\sigma\tau\sigma^{-1}$, in which the class functions can be arbitrarily defined. Consequently, the number of distinct irreducible representations coincide with the number of conjugacy classes of G.

EXAMPLE 3.39. If G is a commutative group, then G has $|G|$ conjugacy classes and hence $|G|$ distinct irreducible representations. Moreover, because

$$|G| = \sum_{j=1}^{g} \dim^2 \rho_j,$$

we conclude that these representations are all one-dimensional. In particular, if G is cyclic, they are given by $\rho_j(\tau^k) = e^{2\pi jki/|G|}$. □

PROPOSITION 3.11. If $\chi_1, \dots, \chi_h$ are the distinct irreducible characters of group G, then

$$\sum_i \overline{\chi}_i(\eta)\chi_i(\tau) = \begin{cases} \frac{|G|}{|\mathcal{O}_\tau|} & \text{if } \eta = \tau; \\ 0 & \text{if } \eta \notin \mathcal{O}_\tau, \end{cases}$$

where $|\mathcal{O}_\tau|$ is the number of elements in the conjugacy class $\mathcal{O}_\tau = \{\sigma\tau\sigma^{-1}, \ \sigma \in G\}$ of $\tau \in G$.

PROOF. Define

$$x_\tau(\eta) = \begin{cases} 1 & \text{if } \eta \in \mathcal{O}_\tau; \\ 0 & \text{if } \eta \notin \mathcal{O}_\tau. \end{cases}$$

Then x_τ is a class function and, consequently, can be expressed as a linear combination $\sum_i c_i\chi_i$ of the distinct irreducible characters $\chi_1, \dots, \chi_h$ of G. The reader may verify that, in this case, $c_i = (x_\tau \mid \chi_i) = |\mathcal{O}_\tau| \, \overline{\chi}_i(\tau)/|G|$, so that

$$x_\tau(\eta) = \sum_i \frac{|\mathcal{O}_\tau|}{|G|}\overline{\chi}_i(\tau)\chi_i(\eta) = \begin{cases} 1 & \text{if } \eta = \tau \\ 0 & \text{if } \eta \notin \mathcal{O}_\tau, \end{cases}$$

from which the result follows. □

EXAMPLE 3.40. Matrix (3.36) shows the irreducible characters $\chi_1, \chi_{Sgn}, \chi_\beta$ of S_3, along with the characters $\chi_\rho, \chi_{\beta\otimes\beta}, \chi_{reg}$ of the permutation, tensor $\beta \otimes \beta$ and regular representations, respectively.

(3.36)

τ	χ_ρ	χ_1	χ_{Sgn}	χ_β	$\chi_{\beta\otimes\beta}$	χ_{reg}
1	3	1	1	2	4	6
(12)	1	1	−1	0	0	1
(13)	1	1	−1	0	0	1
(23)	1	1	−1	0	0	1
(123)	1	1	−1	0	0	1
(132)	0	1	1	−1	1	0

S_3 has three conjugate orbits (and hence three distinct irreducible representations),

$$\mathcal{O}_1 = \{1\}, \quad \mathcal{O}_t = \{(12), (13), (23)\}, \quad \mathcal{O}_c = \{(123), (132)\}.$$

We obtain

$$\overline{\chi}_1(\tau)\chi_1(\tau)+\overline{\chi}_{Sgn}(\tau)\chi_{Sgn}(\tau)+\overline{\chi}_\beta(\tau)\chi_\beta(\tau) = \begin{cases} 4+1+1 = 6 = |G|/|\mathcal{O}_1|, & \text{if } \tau \in \mathcal{O}_1; \\ 0+1+1 = 2 = |G|/|\mathcal{O}_t|, & \text{if } \tau \in \mathcal{O}_t; \\ 1+1+1 = 3 = |G|/|\mathcal{O}_c|, & \text{if } \tau \in \mathcal{O}_c, \end{cases}$$

whereas

$$\overline{\chi}_1(\tau)\chi_1(\eta) + \overline{\chi}_{Sgn}(\tau)\chi_{Sgn}(\eta) + \overline{\chi}_\beta(\tau)\chi_\beta(\eta) =$$

$$\begin{cases} 2 \times 0 + 1 \times (-1) + 1 \times 1 = 0, & \text{if } \tau = 1, \ \eta = (12); \\ 2 \times (-1) + 1 \times 1 + 1 \times 1 = 0, & \text{if } \tau = 1, \ \eta = (123); \\ 0 \times (-1) + (-1) \times 1 + 1 \times 1 = 0, & \text{if } \tau = (12), \ \eta = (123). \end{cases}$$

To decompose, say, the character of $\beta \otimes \beta$, we write

$$\chi_{\beta\otimes\beta} = c_1\chi_1 + c_{Sgn}\chi_{Sgn} + c_\beta\chi_\beta,$$

in which the coefficients are determined by

$$c_1 = (\chi_{\beta\otimes\beta} \mid \chi_1) = 6/6 = 1, \quad c_{\mathrm{Sgn}} = (\chi_{\beta\otimes\beta} \mid \chi_{\mathrm{Sgn}}) = 6/6 = 1,$$

and

$$c_\beta = (\chi_{\beta\otimes\beta} \mid \chi_\beta) = 6/6 = 1.$$

In fact, $\chi_{\beta\otimes\beta} = \chi_1 + \chi_{\mathrm{Sgn}} + \chi_\beta$. □

3.12. The canonical projections

THEOREM 3.8 (Canonical Decomposition). Let ρ be a linear representation of G into GL($\mathcal{V}$), $\rho_1, \ldots, \rho_h$ the distinct irreducible representations of G, with corresponding characters $\chi_1, \ldots, \chi_h$ and dimensions $n_1, \ldots, n_h$. Then,

$$\mathcal{P}_i = \frac{n_i}{|G|} \sum_{\tau \in G} \overline{\chi}_i(\tau)\rho(\tau),$$

is a projection of $\mathcal{V}$ onto a subspace $\mathcal{V}_i$, sum of m_i isomorphic copies of the irreducible subspaces associated with ρ_i, $i = 1, \ldots, h$. Moreover, $\mathcal{P}_i\mathcal{P}_j = 0$, for $i \neq j$, $\mathcal{P}_i^2 = \mathcal{P}_i$ and $\sum_i \mathcal{P}_i = I_v$, where $v = \dim \mathcal{V} = \sum_{i=1}^h m_i n_i$.

PROOF. Write $\rho = \sum_{i=1}^h m_j\rho_j$, where $\rho_1, \ldots, \rho_h$ are the distinct irreducible representations of G. Therefore, from Proposition 3.10,

$$\begin{aligned}
\mathcal{P}_i &= \frac{n_i}{|G|} \sum_{\tau \in G} \overline{\chi}_i(\tau)[\sum_{j=1}^h m_j\rho_j(\tau)] = \frac{n_i}{|G|} \sum_{j=1}^h m_j[\sum_{\tau \in G} \overline{\chi}_i(\tau)\rho_j(\tau)] \\
&= \frac{n_i}{|G|} \sum_{j=1}^h \frac{|G|}{n_i}(\overline{\chi}_i \mid \overline{\chi}_j)I_v = \frac{n_i}{|G|} m_i \frac{|G|}{n_i} I_v = m_i I_v,
\end{aligned}$$

so that $\mathcal{P}_i$ acts as the identity operator in (each of the m_i copies of) $\mathcal{V}_i$, and as the null operator elsewhere. That is, $\mathcal{P}_i$ is a projection of $\mathcal{V}$ into $\mathcal{V}_i$. Similar argument shows that $\mathcal{P}_i\mathcal{P}_j = 0$ for $j \neq i$. Moreover,

$$\sum_{i=1}^h \mathcal{P}_i = \sum_{i=1}^h \frac{n_i}{|G|} \sum_{\tau \in G} \overline{\chi}_i(\tau)\rho(\tau) = \frac{1}{|G|} \sum_{\tau \in G} [\sum_{i=1}^h n_i\overline{\chi}_i(\tau)]\rho(\tau).$$

From Proposition 3.11, we know that $n_i = \chi_i(1)$, so that

$$\sum_{i=1}^h n_i\overline{\chi}_i(\tau) = \sum_{i=1}^h \overline{\chi}_i(\tau)\chi_i(1) = \begin{cases} 0, & \text{if } \tau \neq 1; \\ \frac{|G|}{|\mathcal{O}_1|} = 1, & \text{if } \tau = 1. \end{cases}$$

Consequently,

$$\sum_{i=1}^h \mathcal{P}_i = \frac{1}{|G|}|G|\rho(1) = I_v,$$

concluding the proof. □

EXAMPLE 3.41. Following Example 3.10, let V indicate the set V = {uu, yy, uy, yu} of two-sequences in length of two, equivalently, the set of all mappings s from

$L = \{1,2\}$ into L. Let $G = S_2$. Consider the action $\varphi(\tau, s) = s\tau^{-1}$ which classifies the sequences by symmetries in the position of the residues:

$$\varphi : \begin{bmatrix} \tau\backslash s & \text{uu} & \text{yy} & \text{uy} & \text{yu} \\ \hline 1 & \text{uu} & \text{yy} & \text{uy} & \text{yu} \\ t = (12) & \text{uu} & \text{yy} & \text{yu} & \text{uy} \end{bmatrix}.$$

The regular representation of S_2 defined by the left action is then

$$\rho(1) = I_4, \quad \rho(t) = \begin{bmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \end{bmatrix}.$$

The irreducible characters are the characters χ_1 and χ_2 of the trivial and the signature representations, respectively:

$$\begin{bmatrix} \tau & \chi_1 & \chi_2 \\ \hline 1 & 1 & 1 \\ t & 1 & -1 \end{bmatrix}.$$

These representations have dimension equal to 1, so that $n_1 = n_2 = 1$. Also, $|G| = 2$. Therefore

$$\mathcal{P}_1 = \frac{1}{2}[\overline{\chi}_1(1)\rho(1) + \overline{\chi}_1(t)\rho(t)] = \frac{1}{2}[\rho(1) + \rho(t)] = \begin{bmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}} \end{bmatrix},$$

$$\mathcal{P}_2 = \frac{1}{2}[\overline{\chi}_2(1)\rho(1) + \overline{\chi}_2(t)\rho(t)] = \frac{1}{2}[\rho(1) - \rho(t)] = \begin{bmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}} \end{bmatrix}.$$

We have $I_4 = \mathcal{P}_1 + \mathcal{P}_2$, $\mathcal{P}_1\mathcal{P}_2 = 0$, $\mathcal{P}_1^2 = \mathcal{P}_1$, $\mathcal{P}_2^2 = \mathcal{P}_2$. Note that, correspondingly, $\mathcal{V} \doteq \mathbb{R}^4$ decomposes into the sum

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3,$$

of stable subspaces, of dimensions $1, 1, 2$, respectively. The irreducible decomposition, as shown earlier in Example 3.30, corresponds to the sum

$$\rho = (\chi_\rho \mid \chi_1)\rho_1 + (\chi_\rho \mid \chi_{\text{Sgn}})\rho_{\text{Sgn}} = 3\rho_1 + \rho_{\text{Sgn}}.$$

□

EXAMPLE 3.42 (The left action $s\tau^{-1}$ of S_3 on the set V of four-sequences in length of three). There are $h = 3$ projections $\mathcal{P}_k$ in $\mathbb{R}^{64}$, each one of the form

$$\mathcal{P}_k = \begin{bmatrix} P_k^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_k^{\lambda_m} \end{bmatrix}, \quad k = 1, \ldots, h.$$

In the present case, m = 3 (the number of integer partitions of c in length of ℓ), and, up to isomorphisms,

$$P_k^{\lambda_1} = \begin{bmatrix} Q_1^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_1}^{\lambda_1} \end{bmatrix}, \quad P_k^{\lambda_2} = \begin{bmatrix} Q_1^{\lambda_2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_2}^{\lambda_2} \end{bmatrix},$$

and

$$P_k^{\lambda_3} = \begin{bmatrix} Q_1^{\lambda_3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_3}^{\lambda_3} \end{bmatrix}.$$

In addition, each projection:

(1) $Q_i^{\lambda_1}$ acts on a subspace of dimension $\ell_1 = 1$, $i = 1, \ldots, c_1 = 4$;
(2) $Q_i^{\lambda_2}$ acts on a subspace of dimension $\ell_2 = 3$, $i = 1, \ldots, c_2 = 12$;
(3) $Q_i^{\lambda_3}$ acts on a subspace of dimension $\ell_3 = 4$, $i = 1, \ldots, c_3 = 6$,

so that $\dim \mathbb{R}^V = 64 = 4 \times 1 + 12 \times 3 + 6 \times 4$. More specifically, the first projection $\mathcal{P}_1$ is determined by (indicating by J_n the $n \times n$ matrix with all entries equal to 1)

$$Q_1^{\lambda_1} = \ldots = Q_4^{\lambda_1} = 1,$$
$$Q_1^{\lambda_2} = \ldots = Q_{12}^{\lambda_2} = \frac{1}{3} J_3,$$
$$Q_1^{\lambda_3} = \ldots = Q_6^{\lambda_3} = \frac{1}{6} J_6;$$

The second projection $\mathcal{P}_2$ is determined by

$$Q_1^{\lambda_1} = \ldots = Q_4^{\lambda_1} = 0,$$
$$Q_1^{\lambda_2} = \ldots = Q_{12}^{\lambda_2} = 0,$$
$$Q_1^{\lambda_3} = \ldots = Q_6^{\lambda_3} = \frac{1}{6} \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix},$$

and the third projection, $\mathcal{P}_3$, is determined by

$$Q_1^{\lambda_1} = \ldots = Q_4^{\lambda_1} = 0,$$

$$Q_1^{\lambda_2} = \ldots = Q_{12}^{\lambda_2} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$$Q_1^{\lambda_3} = \ldots = Q_6^{\lambda_3} = \frac{1}{3}\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

It holds that $\sum_k \mathcal{P}_k = I$, $\mathcal{P}_k\mathcal{P}_p = 0$, $p \neq k$ and $\mathcal{P}_k^2 = \mathcal{P}_k$, $k = 1, \ldots, 3$. □

EXAMPLE 3.43 (The action σs of S_4 on the set of four-sequences in length of three). There are $h = 5$ projections $\mathcal{P}_k$ in $\mathbb{R}^{64}$, corresponding to the 5 irreducible characters

$\chi\backslash\tau$	1	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	−1	0	−1
χ_3	2	0	2	−1	0
χ_4	3	−1	−1	0	1
χ_5	1	−1	1	1	−1

of S_4. Each projection has the form

$$\mathcal{P}_k = \begin{bmatrix} P_k^{\lambda_1} & 0 & 0 \\ 0 & P_k^{\lambda_2} & 0 \\ 0 & 0 & P_k^{\lambda_3} \end{bmatrix}, \quad k = 1, \ldots, 5,$$

with

$$P_k^{\lambda_1} = \begin{bmatrix} Q_1^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_1}^{\lambda_1} \end{bmatrix}, \quad P_k^{\lambda_2} = \begin{bmatrix} Q_1^{\lambda_2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_2}^{\lambda_2} \end{bmatrix},$$

and

$$P_k^{\lambda_3} = \begin{bmatrix} Q_1^{\lambda_3} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{c_3}^{\lambda_3} \end{bmatrix}.$$

In addition:

(1) $Q_i^{\lambda_1}$ acts on a subspace of dimension $\ell_1 = 4$, $i = 1, \ldots, c_1 = 1$;
(2) $Q_i^{\lambda_2}$ acts on a subspace of dimension $\ell_2 = 3$, $i = 1, \ldots, c_2 = 12$;
(3) $Q_i^{\lambda_3}$ acts on a subspace of dimension $\ell_3 = 24$, $i = 1, \ldots, c_3 = 1$,

so that $\dim \mathbb{R}^{\mathsf{V}} = 64 = 1 \times 4 + 3 \times 12 + 1 \times 24$. More specifically, the projection $\mathcal{P}_1$ is given by

$$Q_1^{\lambda_1} = \frac{1}{4} J_4,$$
$$Q_1^{\lambda_2} = \ldots = Q_3^{\lambda_2} = \frac{1}{12} J_{12},$$
$$Q_1^{\lambda_3} = \frac{1}{24} J_{24}.$$

The projection $\mathcal{P}_2$ is determined by

$$Q_1^{\lambda_1} = 0,$$
$$Q_1^{\lambda_2} = \ldots = Q_3^{\lambda_2} = 0,$$

and

$$Q\lambda_{3_1} = \frac{1}{24}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix}
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1
\end{bmatrix}.$$

The projection $\mathcal{P}_3$ is given by

$Q_1^{\lambda_1} = 0,$

$$Q_1^{\lambda_2} = \ldots = Q_3^{\lambda_2} = \frac{1}{12}\begin{bmatrix}
2 & -1 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 \\
-1 & -1 & 2 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\
-1 & -1 & 2 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2 \\
-1 & -1 & 2 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & 2 & -1 \\
2 & -1 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & 2
\end{bmatrix},$$

and $Q_1^{\lambda_2} = \frac{1}{12}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 2 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 2 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 2 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

[illegible]tained from

$$
\begin{bmatrix}
1 & -1 \\
1 & -1 \\
3 & -1 \\
-1 & 3
\end{bmatrix},
$$

$$
[illegible] = \frac{1}{8}
\begin{bmatrix}
4 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & -2 & -1 & 1 & -2 \\
1 & 4 & 1 & -1 & 1 & -2 & 0 & -1 & -1 & -1 & -2 & 1 \\
1 & 1 & 4 & -1 & -2 & 1 & -1 & -2 & 1 & 0 & -1 & -1 \\
0 & -1 & -1 & 4 & 1 & 1 & 1 & -1 & -2 & 1 & -1 & -2 \\
-1 & 1 & -2 & 1 & 4 & 1 & -1 & 0 & -1 & -2 & -1 & 1 \\
-1 & -2 & 1 & 1 & 1 & 4 & -2 & -1 & 1 & -1 & 0 & -1 \\
-1 & 0 & -1 & 1 & -1 & -2 & 4 & 1 & 1 & 1 & -2 & -1 \\
1 & -1 & -2 & -1 & 0 & -1 & 1 & 4 & 1 & -2 & 1 & -1 \\
-2 & -1 & 1 & -2 & -1 & 1 & 1 & 1 & 4 & -1 & -1 & 0 \\
-1 & -1 & 0 & 1 & -2 & -1 & 1 & -2 & -1 & 4 & 1 & 1 \\
1 & -2 & -1 & -1 & -1 & 0 & -2 & 1 & -1 & 1 & 4 & 1 \\
-2 & 1 & -1 & -2 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & 4
\end{bmatrix},
$$

$[illegible] = \frac{1}{8} \times \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

$$
A =
\begin{bmatrix}
3 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 \\
1 & 3 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & 3 & 0 & 1 & 1 & 0 & -1 & 1 & -1 & 0 & -1 \\
1 & 0 & 0 & 3 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & 0 \\
0 & 1 & 1 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & -1 & 1 \\
1 & -1 & 0 & 0 & -1 & -1 & 3 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & -1 & 0 & 1 & 3 & 0 & 0 & 1 & 1 \\
0 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & 3 & 0 & 1 & 1 \\
-1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 3 & 1 & 1 \\
-1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 1 & 1 & 3 & 0 \\
-1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 3
\end{bmatrix},
$$

$$
B =
\begin{bmatrix}
0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\
-1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & -1 & 0 \\
-1 & -1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & -1 \\
0 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 \\
1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & -1 & 0 \\
0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \\
0 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 \\
-1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 0 & -1 \\
-1 & -1 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & -1 & 0 & 0
\end{bmatrix},
$$

$$C = \begin{bmatrix} 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & -1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 3 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 1 & 0 & -1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 & 0 & -1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 3 & -1 & 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & 0 & -1 & -1 & 3 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & 3 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 3 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & 1 & 3 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 3 \end{bmatrix}.$$

The final projection, $\mathcal{P}_5$, is determined from

$Q_1^{\lambda_1} = 0,$

$$Q_1^{\lambda_2} = \ldots = Q_3^{\lambda_2} = \frac{1}{8} \begin{bmatrix} 2 & -1 & -1 & -2 & 1 & 1 & 1 & -1 & 0 & 1 & -1 & 0 \\ -1 & 2 & -1 & 1 & -1 & 0 & -2 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 2 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 2 & -1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 2 & -1 & 1 & -2 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & -1 & -1 & 2 & 0 & 1 & -1 & 1 & -2 & 1 \\ 1 & -2 & 1 & -1 & 1 & 0 & 2 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 & -2 & 1 & -1 & 2 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & -1 & -1 & 2 & 1 & 1 & -2 \\ 1 & 1 & -2 & -1 & 0 & 1 & -1 & 0 & 1 & 2 & -1 & -1 \\ -1 & 0 & 1 & 1 & 1 & -2 & 0 & -1 & 1 & -1 & 2 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 1 & 1 & -2 & -1 & -1 & 2 \end{bmatrix},$$

and $Q_1^{\lambda_3} = \frac{1}{8} \times \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

$$A = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 3 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 3 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 3 & 1 & 0 & -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 3 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & -1 & 3 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 3 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & -1 & 3 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & 3 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 3 & 0 & 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 3 & 1 & 0 & 0 & -1 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 3 & -1 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & -1 & -1 & 3 & 0 \\ 1 & 0 & 0 & -1 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}.$$

It holds that $\sum_k \mathcal{P}_k = I$, $\mathcal{P}_k \mathcal{P}_p = 0$, $p \neq k$ and $\mathcal{P}_k^2 = \mathcal{P}_k$, $k = 1, \ldots, h$. □

3.13. Projections in the data space

Recall that $\mathcal{F}$ is the vector space of scalar-valued functions defined in the structure of interest, V, upon which the finite group G acts according to φ. For each $\tau \in G$, define the operator $\tau^* : \mathcal{F} \to \mathcal{F}$ which takes $x \in \mathcal{F}$ into $\tau^*(x) : V \to V$ given by

$$\tau^*(x)(s) = x(\varphi(\tau, s)).$$

The operator τ^* is linear. In fact,

$$\begin{aligned}\tau^*(x+y)(s) &= (x+y)(\varphi(\tau,s)) = x(\varphi(\tau,s)) + y(\varphi(\tau,s)) = \tau^*(x)(s) + \tau^*(y)(s)\\ &= (\tau^*(x) + \tau^*(y))(s),\end{aligned}$$

for all $s \in V$, so that $\tau^*(x+y) = \tau^*(x) + \tau^*(y)$. Moreover, for any scalar λ in the field of $\mathcal{F}$,

$$\tau^*(\lambda x)(s) = (\lambda x)(\varphi(\tau, s)) = \lambda \tau^*(x)(s),$$

for all $s \in V$. That is, $\tau^*(\lambda x) = \lambda \tau^*(x)$.

The mapping $\tau \to \tau^*$ is a homomorphism in G into Aut $(\mathcal{F})$. In fact,

$$(\tau^*(x)\sigma^*(x))(s) = x(\varphi(\tau, \varphi(\sigma, s))) = x(\varphi(\tau\sigma, s)) = (\tau\sigma)^*(x)(s),$$

for all $x \in \mathcal{F}$, $s \in V$, $\sigma, \tau \in G$, that is, $\tau^*\sigma^* = (\tau\sigma)^*$. In addition, $1^*(x)(s) = x(\varphi(1, s)) = x(s)$, so that $1^*(x) = x$. In particular, τ^* is invertible, with inverse τ^{-1^*}, that is, $\tau^* \in GL(\mathcal{F})$.

In $\mathcal{F}$, define the scalar products

$$(x, y)_s = \frac{1}{|G|} \sum_{\tau \in G} x(\varphi(\sigma, s))\overline{y(\varphi(\sigma, s))}, \quad s \in V.$$

Then, τ^* is unitary with respect to these scalar products. In fact,

$$\begin{aligned}(\tau^*(x), \tau^*(y))_s &= \frac{1}{|G|} \sum_{\sigma \in G} \tau^*(x)(\varphi(\sigma, s))\overline{\tau^*(y)(\varphi(\sigma, s))}\\ &= \frac{1}{|G|} \sum_{\sigma \in G} x(\varphi(\tau, \varphi(\sigma, s))\overline{y(\varphi(\tau, \varphi(\sigma, s))}\\ &= \frac{1}{|G|} \sum_{\sigma \in G} x(\varphi(\tau\sigma, s))\overline{y(\varphi(\tau\sigma, s))}\\ &= \frac{1}{|G|} \sum_{\sigma \in G} x(\varphi(\sigma, s))\overline{y(\varphi(\sigma, s))}\\ &= (x, y)_s.\end{aligned}$$

Applying Theorem 3.8 to the representation τ^*, we have,

PROPOSITION 3.12. The mapping $\tau \to \tau^*$ is a unitary linear representation of G in $GL(\mathcal{F})$, and, for each irreducible character, χ, of G,

$$\mathcal{P}_\chi = \frac{\chi(1)}{|G|} \sum_{\tau \in G} \chi(\tau)\tau^*$$

is a projection in $\mathcal{F}$.

Direct verification shows that the principal projection $\mathcal{P}_1(x)$ evaluates at $s \in V$ as

$$\mathcal{P}_1(x)(s) = \frac{1}{|G|}\sum_{\tau\in G} x(\varphi(\tau, s)) = \frac{1}{|G|}|G_s|\sum_{f\in\mathcal{O}_s} x(f) = \frac{1}{|G|}\frac{|G|}{|\mathcal{O}_s|}\sum_{f\in\mathcal{O}_s} x(f)$$
$$= \frac{1}{|\mathcal{O}_s|}\sum_{f\in\mathcal{O}_s} x(f),$$

the average of x in the orbit $\mathcal{O}_s$ of s under action φ. We also note that Proposition 3.12 applies to the vector space $\mathcal{G}$ of scalar-valued functions defined in G, with $\varphi : G \times G \to G$, and scalar product

$$(x, y) = \frac{1}{|G|}\sum_{\tau\in G} x(\tau)\overline{y(\tau)} = (x, y)_1.$$

EXAMPLE 3.44. Consider the space V of two-sequences in length of two and let φ be the action of S_2 on the left. We observe the data $x(s) \in \{a, b, c, d\} \subset \mathbb{R}$, as indicated in the matrix 3.37.

(3.37)

s	uu	yy	uy	yu
x (s)	a	b	c	d
1	uu	yy	uy	yu
(12)	uu	yy	yu	uy

Write s′ to indicate $s\tau^{-1}$ when $\tau = (12)$. The two canonical projections are evaluated as

$$\mathcal{P}_1(x)(s) = \frac{1}{2}\{x(s) + x(s')\} \quad \mathcal{P}_2 = \frac{1}{2}\{x(s) - x(s')\},$$

so that

$$\mathcal{P}_1(x)(uu) = a, \quad \mathcal{P}_1(x)(yy) = b, \quad \mathcal{P}_1(x)(uy) = \mathcal{P}_1(x)(yu) = \frac{c+d}{2}.$$

Similarly

$$\mathcal{P}_2(x)(uu) = \mathcal{P}_2(x)(yy) = 0, \quad \mathcal{P}_2(x)(uy) = \frac{c-d}{2}, \quad \mathcal{P}_2(x)(yu) = \frac{d-c}{2}.$$

We verify that

(1) $\mathcal{P}_1(x) + \mathcal{P}_2(x)$ is the identity operator in $\mathcal{F}$;

(2) $\mathcal{P}_2(x)\mathcal{P}_2(x)(s)$ is evaluated by iterating $\mathcal{P}_2$ on the image data

(3.38) $$\{0, 0, \frac{c-d}{2}, \frac{d-c}{2}\},$$

thus obtaining $\mathcal{P}_2^2(x)(uu) = \mathcal{P}_2^2(x)(yy) = 0$,

$$\mathcal{P}_2^2(x)(uy) = \frac{(c-d)/2 - (d-c)/2}{2} = \frac{c-d}{2},$$

$$\mathcal{P}_2^2(x)(yu) = \frac{(d-c)/2 - (c-d)/2}{2} = \frac{d-c}{2}.$$

That is, $\mathcal{P}_2^2(x) = \mathcal{P}_2$. Similarly, $\mathcal{P}_1^2(x) = \mathcal{P}_1(x)$;

(3) $\mathcal{P}_1(x)\mathcal{P}_2(x)$ is evaluated by applying $\mathcal{P}_1$ to the image data (3.38), obtaining

$$\mathcal{P}_1\mathcal{P}_2(x)(uu) = \mathcal{P}_1\mathcal{P}_2(x)(yy) = 0,$$

and

$$\begin{aligned}\mathcal{P}_1\mathcal{P}_2(x)(uy) &= \frac{(c-d)/2+(d-c)/2}{2} = \mathcal{P}_1\mathcal{P}_2(x)(yu)\\ &= \frac{(d-c)/2+(c-d)/2}{2} = 0.\end{aligned}$$

That is, $\mathcal{P}_1\mathcal{P}_2(x) = 0$.

□

EXAMPLE 3.45. Consider the election data where three candidates are (completely) ranked according to the voters' preferences. The frequencies, $x(\sigma)$, of each possible ranking $\sigma \in S_3$ among the voters are the available data, as illustrated in Table 3.1 (from Diaconis (1989)). For example, 67 respondents preferred candidate 3 ranked first, candidate 1 ranked second and candidate 2 ranked last. The action

TABLE 3.1. Three-candidate (228 voters) Partial election data by rankings, corresponding permutation σ and observed frequencies, $x(\sigma)$.

Ranking (1st, 2nd, 3rd)		σ	label	$x(\sigma)$
	3,2,1	(13)	3	29
	3,1,2	(132)	6	67
	2,3,1	(123)	5	37
	2,1,3	(12)	2	24
	1,3,2	(23)	4	43
reference choice	1,2,3	1	1	28

is simply $\varphi(\sigma,\tau) = \sigma\tau$ so that the representation in $GL(\mathcal{F})$ evaluates according to $\tau^*(x)(\sigma) = x(\sigma\tau)$ and the corresponding canonical projections

$$\mathcal{P}_\chi = \frac{\chi(1)}{|G|}\sum_{\tau\in G}\chi(\tau)\tau^*$$

evaluate as

$$\mathcal{P}_\chi(x)(\sigma) = \frac{\chi(1)}{|G|}\sum_{\tau\in G}\chi(\tau)x(\sigma\tau).$$

There are three canonical projections, associated with the irreducible characters

$$\left[\begin{array}{c|ccc} \chi & 1 & (12) & (123) \\ \hline 1:\chi_1 & 1 & 1 & 1 \\ 2:\chi_\beta & 2 & 0 & -1 \\ 3:\chi_{Sgn} & 1 & -1 & 1 \end{array}\right]$$

of S_3. Matrices (3.39) show the multiplication table of S_3 and the corresponding data $x(\sigma\tau)$,

$$(3.39) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \\ 3 & 6 & 1 & 5 & 4 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 4 & 2 & 3 & 6 & 1 \\ 6 & 3 & 4 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{x} \begin{bmatrix} 28 & 24 & 29 & 43 & 37 & 67 \\ 24 & 28 & 37 & 67 & 29 & 43 \\ 29 & 67 & 28 & 37 & 43 & 24 \\ 43 & 37 & 67 & 28 & 24 & 29 \\ 37 & 43 & 24 & 29 & 67 & 28 \\ 67 & 29 & 43 & 24 & 28 & 37 \end{bmatrix},$$

from which the canonical projections

label	σ	$x(\sigma)$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
1	1	28	38.0	−16.0	6.0
2	(12)	24	38.0	−8.0	−6.0
3	(13)	29	38.0	−3.0	−6.0
4	(23)	43	38.0	11.0	−6.0
5	(123)	37	38.0	−7.0	6.0
6	(132)	67	38.0	23.0	6.0

are obtained. The interpretation of these projections follows from the irreducible characters (as class functions) and their values over the conjugacy classes of S_3. Equivalently, the reader may keep in mind the direct-sum decomposition

$$\mathbb{R}^6 = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$$

associated with these projections. For example, reading from the above matrix, $28 = 38 - 16 + 6$ with $38 \in \mathcal{V}_1, -16 \in \mathcal{V}_2, 6 \in \mathcal{V}_3$. More generally, Matrix (3.40)

(3.40)

$6\,\mathcal{P}_1$	$3\,\mathcal{P}_2$	$6\,\mathcal{P}_3$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(1)-x(5)-x(6)$	$x(1)-x(2)-x(3)-x(4)+x(5)+x(6)$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(2)-x(3)-x(4)$	$x(2)-x(1)-x(5)-x(6)+x(3)+x(4)$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(3)-x(4)-x(2)$	$x(2)-x(1)-x(5)-x(6)+x(3)+x(4)$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(4)-x(2)-x(3)$	$x(2)-x(1)-x(5)-x(6)+x(3)+x(4)$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(5)-x(6)-x(1)$	$x(1)-x(2)-x(3)-x(4)+x(5)+x(6)$
$x(1)+x(2)+x(3)+x(4)+x(5)+x(6)$	$2x(6)-x(1)-x(5)$	$x(1)-x(2)-x(3)-x(4)+x(5)+x(6)$

shows that (the subspace image of) $\mathcal{P}_1$, of dimension one, describes the overall mean frequency

$$\frac{1}{6}[x(1)+x(2)+x(3)+x(4)+x(5)+x(6)],$$

$\mathcal{P}_2$, of dimension two, describes all *within-class* frequency comparisons, e.g.,

$$\frac{1}{3}[2x(1)-x(5)-x(6)], \quad \frac{1}{3}[2x(2)-x(3)-x(4)]$$

whereas $\mathcal{P}_3$, of dimension one, describes the *between-class* frequency comparisons, e.g.,

$$\frac{1}{6}[x(1)-x(2)-x(3)-x(4)+x(5)+x(6)].$$

In particular, the within-class comparisons

$$\frac{1}{3}[2x(6)-x(5)-x(1)] = 23, \quad \frac{1}{3}[2x(4)-x(2)-x(3)] = 11$$

reflect assessments or decisions between preferences which change the leading candidate while (cyclically to the right or to the left) preserving the relative ranking order of the three candidates, e.g., (using the ranking notation of Table 3.1),

$$\frac{1}{3}[x(3,1,2) - x(2,3,1) + x(3,1,2) - x(1,2,3)] = 23,$$

or decisions between preferences which change the leading candidate while partially preserving the relative order of the three candidates, e.g.,

$$\frac{1}{3}[x(1,3,2) - x(2,1,3) + x(1,3,2) - x(3,2,1)] = 11.$$

whereas the between-class frequency comparisons reflect decisions between runner-ups, e.g.,

$$\frac{1}{6}[x(1,3,2) - x(1,2,3) + x(2,1,3) - x(2,3,1) + x(3,2,1) - x(3,1,2)] = 6.$$

The numerical evaluation of $\frac{1}{3}[x(3,1,2) - x(2,3,1) + x(3,1,2) - x(1,2,3)] = \frac{30+39}{3} = 23$ shows that voters relatively fail to prefer those rankings in which candidate 3 does not appear in the leading position. Clearly, this conclusion depends on the reference ranking, namely $(1,2,3)$. However, the analysis also shows that the relatively higher preference for rankings $3,1,2$ ($x = 67$) and $1,3,2$ ($x = 42$) is largely due to their component in the subspace image of $\mathcal{P}_2$, thus saying that there is a significant tendency to cyclically moving the rankings. Because these are stable subspaces, the conclusion is independent of the reference ranking and constitutes an invariant aspect of the voting results. This, of course, is not evident directly from a simple inspection of the original frequencies. The reader may also note that the component in $\mathcal{V}_3$ is of the same magnitude for all observed frequencies, and therefore may serve as a background, or residual value. These arguments are characteristic of a *first-order* analysis. Additional *second-order* inferences for these data, based on decompositions of the form $x'x = x'Ix = x'\mathcal{P}_1x + x'\mathcal{P}_2x + x'\mathcal{P}_3x$, are discussed later on in Chapters 4 and 5. A computer source ©MAPLE code for the canonical projections of the regular representation of S_3 is available at the address

`http://tigger.uic.edu/~viana/s3_regular_projections.html.`

□

EXAMPLE 3.46. Consider again the APA election data described in Diaconis (1989). Now we look at rankings of candidates 1, 2, 3 and 4. Table 3.2 shows the number of votes for each one of the 24 different rankings. For example, 54 voters (out of 851) ranked candidate 1 as third choice, candidate 2 as second, and candidate 3 as first choice. Matrix (3.44), Section 3.15, shows the multiplication table for S_4. The character table of S_4 is shown in Matrix (3.41). Matrix (3.42) summarizes the resulting canonical projections, in analogy to the analysis of the three-candidate data. The results are also illustrated in Figure 3.1. The display on the top shows the projection of each frequency (sequentially in the order shown in Matrix (3.42)) into the subspaces image of $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{P}_5$, indexed in the horizontal axis by $2,3,4,5$ respectively. The displays in the bottom aggregate these components by subspace or projection (indicated in the vertical axis by $P_2, \ldots P_4$). Here are some general comments:

(1) Note that the canonical projections provide a basis for *explanation* of data that goes beyond the originally observed frequencies- e.g., the observed

TABLE 3.2. Four-candidate partial APA election data by rankings, corresponding permutation σ, conjugacy class (expressed in partition notation) and number of voters $x(\sigma)$.

Ranking	σ	class	$x(\sigma)$	Ranking	σ	class	$x(\sigma)$
4321	(14)(23)	2^2	29	2431	(124)	3, 1	54
4312	(1423)	4	67	2413	(1243)	4	44
4231	(14)	$2, 1^2$	37	2341	(1234)	4	26
4213	(143)	3, 1	24	2314	(123)	3, 1	24
4132	(142)	3, 1	43	2143	(12)(34)	2^2	35
4123	(1432)	4	28	2134	(12)	$2, 1^2$	50
3421	(1324)	4	57	1432	(24)	$2, 1^2$	50
3412	(13)(24)	2^2	49	1423	(243)	3, 1	46
3241	(134)	3, 1	22	1342	(234)	3, 1	25
3214	(13)	$2, 1^2$	22	1324	(23)	$2, 1^2$	19
3142	(13)(24)	$2, 1^2$	34	1243	(34)	$2, 1^2$	11
3124	(132)	3, 1	26	1234	1	1^4	29

frequencies of permutations (24), (12) and (13)(24) are essentially equal (50, 50, 49) and yet their components into the five subspaces are *clearly* distinct. Characteristically, these differences may suggest a much broader range of *explanation*;

(2) The display on the top part of Figure 3.1 is useful for identifying responses that have similar projection patterns (and then, necessarily, similar observed frequencies), e.g., displays number (counting sequentially from the top) 14, 17 and 20 have *similar* projections and similar frequencies (43, 49, 44) respectively;

(3) The aggregate display on the bottom part of Figure 3.1 is useful for comparing among the subspaces (image[6] of the corresponding projections indicated on the vertical axes). The reader may want to say that the election is being decided in the subspace $\mathcal{V}_2$ image of $\mathcal{P}_2$! Very little contribution comes from $\mathcal{V}_3$, which, uniformly, assigns nearly vanishing components to frequencies of permutations in the conjugacy class of (123);

(4) To understand the role of $\mathcal{P}_2$ we have to suggest an interpretation of the conjugacy classes in the context of ranked data. Note that each permutation in the conjugacy class $C_{(12)}$ of (12) represents the *up-ranking* of exactly one candidate, the *down-ranking* of exactly one candidate, and the *keeping* of exactly two rankings. In general, we have,

C	up-ranking	down-ranking	keeping	count
1	0	0	4	1
(12)	1	1	2	6
(123)	1	2	1	4
	2	1	1	4
(12)(34)	2	2	0	3
(1234)	3	1	0	1
	1	3	0	1
	2	2	0	4

,

[6] That is, $\mathcal{V} = \text{Image } \mathcal{P}$.

showing that, in average (m), permutations in the conjugacy classes of (1234) and (12)(34) up-rank or down-rank m = 2 candidates, permutations in the class of (123) up rank or Dow rank m = 1.5 candidates, whereas the transpositions up rank or Dow rank m = 1 candidate. Inspection of the character table of S_4 reveals that $\mathcal{P}_2$ is about comparing single transpositions (m = 1) with $\{(12)(34),(1234)\}$ (m = 2); $\mathcal{P}_3$ compares (12)(34) with (123); $\mathcal{P}_4$ compares (1234) with $\{(12),(12)(34)\}$, whereas $\mathcal{P}_5$ compares $\{(12),(1234)\}$ with $\{(12)(34),(123)\}$. Consequently, the relatively larger components in $\mathcal{V}_2$ suggest a stronger, more radical, disliking of the reference ranking. However, and most importantly, because these subspaces are invariant subspaces, these interpretations do not depend on the adopted $(1,2,3,4)$ reference ranking and constitute an invariant analysis. Clearly, other conclusions are specific to the reference ranking, e.g., permutation (1423) is the top choice and permutation (34) the least preferred.

A computer source ©MAPLE code for the canonical projections of the regular representation of S_4 is available at the address

`http://tigger.uic.edu/~viana/s4_regular_projections.html.`

(3.41)

$\chi\backslash\tau$	1	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	−1	0	−1
χ_3	2	0	2	−1	0
χ_4	3	−1	−1	0	1
χ_5	1	−1	1	1	−1

(3.42)

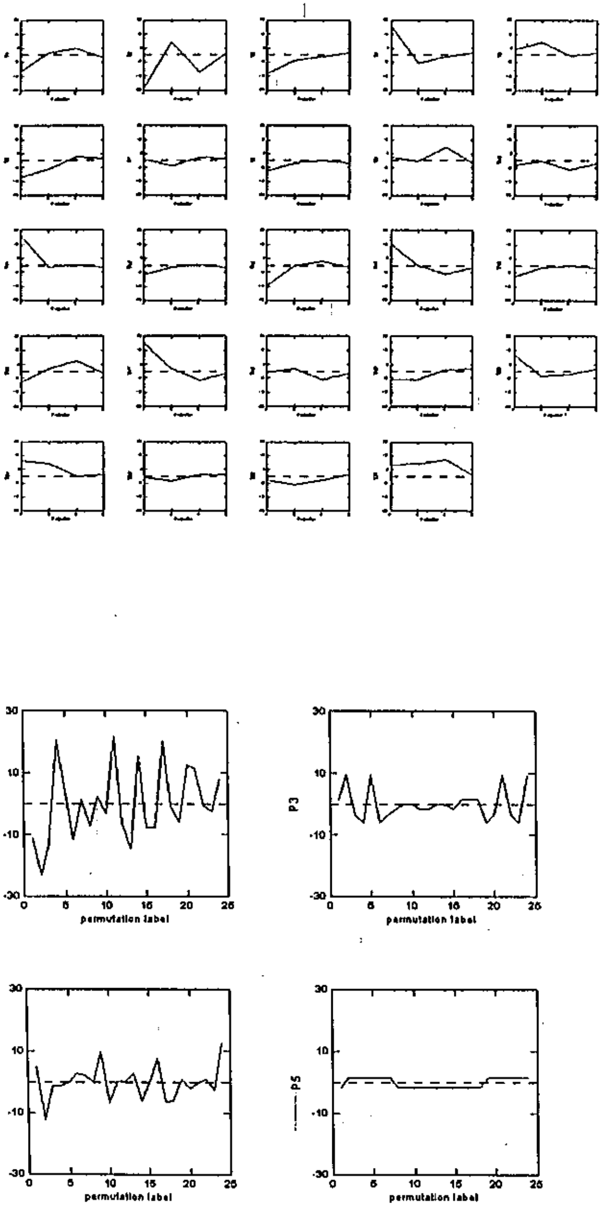
label	σ	$x(\sigma)$	$\mathcal{P}_1x$	$\mathcal{P}_2x$	$\mathcal{P}_3x$	$\mathcal{P}_4x$	$\mathcal{P}_5x$
1	1	29	35.38	−11.38	1.667	4.875	−1.542
2	(34)	11	35.38	−23.12	9.333	−12.12	1.542
3	(23)	19	35.38	−13.25	−3.417	−1.250	1.542
4	(24)	50	35.38	20.25	−5.917	−1.250	1.542
5	(12)	50	35.38	3.875	9.333	−0.1250	1.542
6	(13)	22	35.38	−11.75	−5.917	2.750	1.542
7	(14)	37	35.38	1.250	−3.417	2.250	1.542
8	(234)	25	35.38	−7.250	−1.583	0.0	−1.542
9	(243)	46	35.38	2.500	−0.08333	9.750	−1.542
10	(123)	24	35.38	−3.250	−0.08333	−6.500	−1.542
11	(124)	54	35.38	21.50	−1.583	0.2500	−1.542
12	(132)	26	35.38	−6.500	−1.583	0.2500	−1.542
13	(134)	22	35.38	−14.50	−0.08333	2.750	−1.542
14	(142)	43	35.38	15.25	−0.08333	−6.0	−1.542
15	(143)	24	35.38	−7.750	−1.583	−0.5000	−1.542
16	(12)(34)	35	35.38	−7.875	1.667	7.375	−1.542
17	(13)(24)	49	35.38	19.88	1.667	−6.375	−1.542
18	(14)(23)	29	35.38	−0.6250	1.667	−5.875	−1.542
19	(1234)	26	35.38	−6.0	−5.917	1.0	1.542
20	(1243)	44	35.38	12.50	−3.417	−2.0	1.542
21	(1324)	57	35.38	11.12	9.333	−0.3750	1.542
22	(1342)	34	35.38	−0.5000	−3.417	1.0	1.542
23	(1432)	26	35.38	−2.500	−5.917	−2.500	1.542
24	(1423)	67	35.38	8.125	9.333	12.62	1.542

A similar analysis for the frequencies of nucleotides sequences in length of four with all bases distinct is shown in Examples 4.6 and 4.7 in Chapter 4. □

3.14. Summary

In this chapter we reviewed the introductory concepts of linear representation of finite groups, with the objective of connecting the symmetries imposed on a structure V, usually defined by a finite group G acting on V according to a rule φ, with the structured data $\{x(s); s \in V\}$, where x is a scalar valued function defined on V. In particular, we are also interested in the case in which $V = G$, so that the data are indexed by the group G. If the group G acts on V, with v elements, according to the rule φ, we showed that the linear operator $\rho(\tau)$ defining the changing of basis from $\{e_s; s \in V\} \subset \mathbb{R}^v$ to $\{e_{\varphi(\tau,s)}; s \in V\}$ is an homomorphism from G to $GL(\mathbb{R}^v)$, that is, ρ is a linear representation of G. When $V = \{1, \ldots, \ell\}$, $G = S_\ell$ and $\varphi(\tau, i) = \tau i$ we obtained the permutation representation; when $V = G$ and $\varphi(\tau, \sigma) = \tau\sigma$ we obtained the regular representation of G. We defined the vector space $\mathcal{G}$ of scalar-valued functions defined on G and showed that the set $\{\chi_1, \cdots \chi_h\}$ of distinct irreducible characters of G is an orthonormal basis for $\mathcal{G}$. The connection among the structure V, the symmetries in G and the data points x in the vector space $\mathcal{F}$ of scalar-valued functions defined on V was then obtained

FIGURE 3.1. Canonical projections of each observed frequency $x(\sigma)$, $\sigma \in S_4$, into the four non-trivial subspaces $\mathcal{V}_2, \ldots, \mathcal{V}_5$ (top) and corresponding joint spectra (bottom) .



by showing that the operators

$$\mathcal{P}_\chi = \frac{\chi(1)}{|G|} \sum_{\tau \in G} \chi(\rho)\rho(\tau)$$

are projections in $\mathbb{R}^V$. The resulting first-order analysis $\mathcal{P}_{\chi_1}x, \ldots, \mathcal{P}_{\chi_h}x$ and second-order analysis $x'x = \sum_\chi x'\mathcal{P}_\chi x$ of the structured data x, as described in the following chapters, may then be obtained.

3.15. Tables and graphic displays

3.15.1. The left action $s\tau^{-1}$ of S_4 on two-sequences of length four.

(3.43)

s(1)	y	u	y	u	u	u	y	y	y	u	u	u	y	y	y	u	
s(2)	y	u	u	y	u	u	y	u	u	y	y	u	y	y	u	y	
s(3)	y	u	u	u	y	u	u	y	u	y	u	y	y	u	y	y	
s(4)	y	u	u	u	u	y	u	u	y	u	y	y	u	y	y	y	
label →	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2	|fix(σ)|
$\mu = 1111$																	
1	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2	16
$\mu = 2110$																	
(34)	1	16	15	14	8	12	13	7	11	6	10	4	5	9	3	2	8
(23)	1	16	15	12	14	8	11	13	7	10	4	6	9	3	5	2	8
(24)	1	16	15	8	12	14	7	11	13	4	6	10	3	5	9	2	8
(12)	1	16	14	15	12	8	13	10	6	11	7	4	9	5	2	3	8
(13)	1	16	12	14	15	8	10	11	4	13	6	7	9	2	3	5	8
(14)	1	16	8	14	12	15	6	4	7	10	13	11	2	5	3	9	8
$\mu = 3100$																	
(234)	1	16	15	12	8	14	11	7	13	4	10	6	3	9	5	2	4
(243)	1	16	15	8	14	12	7	13	11	6	4	10	5	3	9	2	4
(123)	1	16	14	12	15	8	10	13	6	11	4	7	9	2	5	3	4
(124)	1	16	14	8	12	15	6	10	13	4	7	11	2	5	9	3	4
(132)	1	16	12	15	14	8	11	10	4	13	7	6	9	3	2	5	4
(134)	1	16	12	14	8	15	10	4	11	6	13	7	2	9	3	5	4
(142)	1	16	8	15	12	14	7	4	6	11	13	10	3	5	2	9	4
(143)	1	16	8	14	15	12	6	7	4	13	10	11	5	2	3	9	4
$\mu = 2200$																	
(12)(34)	1	16	14	15	8	12	13	6	10	7	11	4	5	9	2	3	4
(13)(24)	1	16	12	8	15	14	4	11	10	7	6	13	3	2	9	5	4
(14)(23)	1	16	8	12	14	15	4	6	7	10	11	13	2	3	5	9	4
$\mu = 4000$																	
(1234)	1	16	14	12	8	15	10	6	13	4	11	7	2	9	5	3	2
(1243)	1	16	14	8	15	12	6	13	10	7	4	11	5	2	9	3	2
(1324)	1	16	12	8	14	15	4	10	11	6	7	13	2	3	9	5	2
(1342)	1	16	12	15	8	14	11	4	10	7	13	6	3	9	2	5	2
(1432)	1	16	8	15	14	12	7	6	4	13	11	10	5	3	2	9	2
(1423)	1	16	8	12	15	14	4	7	6	11	10	13	3	2	5	9	2
$\|G_\mu\|$	24	24	6	6	6	6	4	4	4	4	4	4	6	6	6	6	

3.15.2. The multiplication matrix for S_4. The correspondence between the letters and the permutations is indicated by the first two columns of the multiplication table.

(3.44)

1	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	x	z
(34)	b	a	h	i	p	m	o	c	d	s	t	v	f	x	g	e	u	z	j	k	q	l	n	r
(23)	c	i	a	h	j	l	r	d	b	e	s	f	u	z	x	t	v	g	k	p	m	q	o	n
(24)	d	h	l	a	k	q	n	b	c	t	e	u	v	g	z	s	f	x	p	j	i	m	r	o
(12)	e	p	l	n	a	j	k	v	x	f	g	c	s	d	t	b	z	u	m	o	r	h	i	q
(13)	f	o	j	q	l	a	m	z	t	c	u	e	g	v	b	x	d	s	r	i	k	n	p	h
(14)	g	m	r	k	n	o	a	s	u	z	d	x	b	e	f	v	t	c	h	q	i	p	l	j
(234)	h	d	b	c	s	v	z	i	a	p	j	m	q	r	n	k	l	o	t	e	f	u	g	x
(243)	i	c	d	b	t	u	x	a	h	k	p	q	l	o	r	j	m	n	e	s	v	f	z	g
(123)	j	t	f	z	c	e	s	q	o	l	r	a	k	h	p	i	n	m	u	x	g	d	b	v
(124)	k	s	u	g	d	t	e	m	r	q	n	i	p	a	j	h	o	l	v	z	x	b	c	f
(132)	l	x	e	v	f	c	u	n	p	a	m	j	r	q	i	o	h	k	g	b	s	z	t	d
(134)	m	g	s	u	v	b	f	r	k	h	q	p	o	l	a	n	i	j	z	d	t	x	e	c
(142)	n	v	x	e	g	z	d	p	l	o	a	r	h	k	q	m	j	i	b	f	c	s	u	t
(143)	o	f	z	t	x	g	b	j	q	r	i	n	a	p	m	l	k	h	c	u	d	e	v	s
(12)(34)	p	e	v	x	b	s	t	l	n	m	o	h	j	i	k	a	r	q	f	g	z	c	d	u
(13)(24)	q	z	t	f	u	d	v	o	j	i	l	k	n	m	h	r	a	p	x	c	e	g	s	b
(14)(32)	r	u	g	s	z	x	c	k	m	n	h	o	l	j	i	q	p	a	d	v	b	t	f	e
(1234)	s	k	m	r	h	p	j	u	g	v	z	b	t	c	e	d	x	f	q	n	o	l	a	i
(1243)	t	j	q	o	l	k	p	f	z	u	x	d	e	b	s	c	g	v	i	r	n	a	h	m
(1324)	u	r	k	m	q	i	l	g	s	d	v	t	x	f	c	z	b	e	n	h	p	o	j	a
(1342)	v	n	p	l	m	h	q	x	e	b	f	s	z	u	d	g	c	t	o	a	j	r	k	i
(1432)	x	l	n	p	o	r	i	e	v	g	b	z	c	t	u	f	s	d	a	m	h	j	q	k
(1423)	z	q	o	j	r	n	h	t	f	x	c	g	d	s	v	u	e	b	i	l	a	k	m	p

Further reading

(1) The presentation of the material in this chapter closely follows the program of Serre (1977). All the basic results of functions on groups can be seen in the classic text Naimark (1982). The basic facts about projections and vector spaces are found in Halmos (1987)'s classic text. See also Rotman (1995) on general facts about the theory of groups, and Simon (1996) for a more contemporary text on representations of finite and compact groups. The reader will enjoy reading the historial account, by Lam (1998), of representations of finite groups in the past century;

(2) On permutation groups, Cameron (1999) or Dixon and Mortimer (1996);

(3) On combinatorics, Cameron (1994) or Stanton and White (1986);

(4) Matrix groups, e.g., Curtis (1984);

(5) Random matrices, functions, and permutations, Diaconis and Shahshahani (1986), Tuljapurkar (1986), Kolchin (1971), Diaconis and Freedman (1999), Arratia and Simon (1992), Lalley (1996) ;

(6) Permutation groups e.g., Dixon and Mortimer (1996);

Exercises

3.1. Show that, for any member τ of group G, the mapping $\tau^* : G \to G$ by $\tau^*(\sigma) = \tau\sigma\tau^{-1}$ is an isomorphism in G, and the mapping $\tau \mapsto \tau^*$ is a homomorphism of G, taking values in the set Aut (G) of isomorphisms in G.

3.2. Use the presentation relations shown in Example 3.4 to construct the multiplication table of groups D_3, Q_2 and Q_3.

3.3. Following Definition 3.3, show that $G \times H$, together with $\times_\eta$, is a group in which;

(1) The identity is $(1_G, 1_H)$;
(2) The inverse $(\tau,\sigma)^{-1}$ of (τ,σ) is given by $(\eta^{-1*}(\tau^{-1}), \sigma^{-1})$.

3.4. Following Exercise 3.3, show that $G \times \{1_H\}$ is a normal subgroup of $G \times_\eta H$ (recall that N is a normal subgroup of G whenever $\tau N = N\tau$ for all $\tau \in G$).

3.5. Following Example 3.24, show that $\varphi'(\tau, \sum x(s)s) = \sum x(s)\varphi(\tau, s)$ is an action of G on $\mathcal{L}(V)$.

3.6. Indicate by G the set of all non-singular $n \times n$ real doubly-stochastic matrices, that is,

$$G = \{A \in GL(n, \mathbb{R}); Ae = e, e'A = e'\}.$$

Given $A \in G$ and a real vector $a = (a_1, a_2)$ define the $n \times n$ matrix $[a, A] = a_1 ee' + a_2 A$, where e indicates the n-component vector of ones. The equality

$$[a, A][b, B] = (na_1b_1 + a_1b_2 + a_2b_1)ee' + a_2b_2AB,$$

suggests the operation $* : (a, b) \in \mathbb{R}^2 \times \mathbb{R}^2 \to a * b = (na_1b_1 + a_1b_2 + a_2b_1,\ a_2b_2) \in \mathbb{R}^2$, so that $[a, A][b, B] = [a * b, AB]$. Show that $(\mathbb{R}^2, +, *)$ is an algebra. That is, for all $a, b, c \in \mathbb{R}^2$ and all numbers γ in the scalar field ($\mathbb{R}$ or $\mathbb{C}$) of the vector space $(\mathbb{R}^2, +)$, we have $a * b \in (\mathbb{R}^2, +)$, $a * (b + c) = a * b + a * c$, $(a + b) * c = a * c + b * c$, and $\gamma(a * b) = a * (\gamma b) = (\gamma a) * b$.

3.7. Show that $W = \{(a_1, a_2) \in \mathbb{R}^2;\ a_2 \neq 0,\ na_1 + a_2 \neq 0\}$, together with $*$ of Exercise 3.6, is a commutative group in which the unit is $1_* = (0, 1) \in W$ and, for $a \in W$,

$$a_*^{-1} = \left(\frac{-a_1}{a_2(na_1 + a_2)}, \frac{1}{a_2}\right) \in W$$

and $a * a_*^{-1} = a_*^{-1} * a = 1_*$.

3.8. Show that $WG = \{[a, A]; a \in W, A \in G\}$ is a subgroup of $GL_n(\mathbb{R})$. *hint:* First note that G is a subgroup of $GL_n(\mathbb{R})$ and define the direct product group $W \times G$ of W and G, with the multiplication given by

$$(a, A), (b, B) \;\to\; (a * b, AB).$$

Then show that this operation is an homomorphism between $W \times G$ and $GL(n, \mathbb{R})$ which takes $(1_*, 1_G) = ((0, 1), I_n)$ into I_n. Therefore WG is an isomorphic image of the product group and hence a subgroup of $GL(n, \mathbb{R})$.

3.9. With the notation of Exercise 3.8, show that

(1) when $G = \{\ I_n\}$, WG is the subgroup of all equicorrelated covariance matrices;
(2) when $W = \{1_*\}$ and $G = S_n$, WG generates the subgroup S_n of $n \times n$ permutation matrices;
(3) when $W = \{1_*\}$ and

$$G = \{w_0 I_n + w_1 g + w_2 g^2 + \ldots + w_{n-1} g^{n-1};\ \sum_{i=0}^{n-1} w_i = 1,\ w_i \in \mathbb{R}\},$$

where g is an element of order n in S_n, WG generates the subgroup of stochastic circulants with first row $w' = (w_0, \ldots, w_{n-1})$. For example, take $n = 4$ and let F be a stochastic circulant with first row w'. Then $F' = w_0 I + w_1 g^3 + w_2 g^2 + w_3 g \in G$ and

$$FF' = \alpha_0 I + \alpha_1 g + \alpha_2 g^2 + \alpha_1 g^3 = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}$$

is a symmetric stochastic circulant with first row determined by $\alpha_i = w' g^i w$.

3.10. Based on Example 3.10 calculate $|\text{fix } (\tau)|$, $|G_s|$, and $|\mathcal{O}_s|$ and verify that

$$\text{number of orbits in } V = \frac{1}{|G|} \sum_{\tau \in G} |\text{fix } (\tau)|.$$

3.11. From Matrix 3.43, calculate the isotropy group for the set of sequences {yyuu, yuyu}.

3.12. Refer to Example 3.10 and the Mendelian genetics example introduced in Section 2.2 of Chapter 2. We consider the action σs of S_2 on the right (shuffling the names of the residues) and the orbit $\mathcal{O} = \{uy, yu\}$. Write $t = (12)$. Show that fix $(1) = 2$ and that fix $(t) = 0$, so that the number of orbits, given by the average number of fixed points, is (fix $(1) +$ fix $(t)/2 = 1$. Now consider the product space $\mathcal{O}^m$ of all mappings, f, from $\{1, \ldots, m\}$ into $\mathcal{O}$. Each point in $\mathcal{O}^m$ is a string of length m with entries from $\mathcal{O}$. Apply the action σs componentwise:

$$(\sigma, (f(1), \ldots, f(m)) \to (\sigma f(1), \ldots, \sigma f(m)).$$

Show that the number k of orbits in $\mathcal{O}^m$ is $k = 2^m - 1$. Consequently, the downstring *distance* (m) where a crossover occured may be estimated from the number of observed distinct genotypes (k) by $\hat{m} = 1 + \log_2 k$.

3.13. Let ρ be a representation of G (with g elements) on a finite dimensional vector space $\mathcal{V}$, in which a scalar product $(\,|\,)$ is defined e.g., Example 3.8. Show that

$$(x, y) = \frac{1}{g} \sum_{\tau \in G} (\rho_\tau x | \rho_t auy)$$

is a scalar product in $\mathcal{V}$ and that it satisfies $(\rho_\tau x, \rho_\tau y) = (x, y)$ for all $\tau \in G$ and all $x, y \in \mathcal{V}$.

3.14. Apply Proposition 3.12 to evaluate the projections $\mathcal{P}(x)$ of x under the action of S_4 (location symmetry), for each one of the five frequency data shown in Matrix 3.45. The location number indicates the sequence location of the starting residue of the generating string.

(3.45)

s	1	16	15	14	12	8	13	11	7	10	6	4	9	5	3	2
s(1)	y	u	y	u	u	u	y	y	y	u	u	u	y	y	y	u
s(2)	y	u	u	y	u	u	y	u	u	y	y	u	y	y	u	y
s(3)	y	u	u	u	y	u	u	y	u	y	u	y	y	u	y	y
s(4)	y	u	u	u	u	y	u	u	y	u	y	y	u	y	y	y
location	x(s)															
200	5	52	18	12	15	17	16	6	10	11	9	12	5	1	4	5
1000	7	30	20	20	21	20	10	6	10	8	7	8	8	7	8	8
3000	1	43	20	15	17	19	12	8	8	11	10	10	6	5	7	6
5000	6	24	18	19	21	18	11	9	12	12	12	9	6	6	9	6
6000	4	18	14	15	14	14	11	20	12	8	19	12	11	8	7	11

3.15. In Expression 3.20 show that $Q = B(B'B)^{-1}B'$, with

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Generalize to arbitrary dimensions.

3.16. In Example 3.30, show that the matrices describing the two-dimensional representation of S_3 can be obtained as $\rho(\beta) = H'\rho(\beta)H(H'H)^{-1}$, where

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}.$$

3.17. For an arbitrary representation ρ, show that $\rho(\sigma)$ commutes with

$$\sum_{\tau \in G} \rho(\tau) M \rho(\tau^{-1})$$

for every $\sigma \in G$ matrix M of dimension equal to the dimension of ρ.

3.18. Show that $\chi_\rho(\sigma\tau\sigma^{-1}) = \chi_\rho(\tau)$.

3.19. Matrix (3.46)

(3.46)

$\chi \backslash \tau$	1	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	−1	0	−1
χ_3	2	0	2	−1	0
χ_4	3	−1	−1	0	1
χ_5	1	−1	1	1	−1

is the character matrix for S_4. Show that the canonical projections for the permutation representation of S_4 are given by:

$$\mathcal{P}_1 = \frac{1}{24}\sum_{\tau} \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

of dimension 1, and

$$\mathcal{P}_2 = \frac{3}{24}\sum_{\tau}\chi_2(\tau^{-1})\rho(\tau) = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix},$$

of dimension 3, with the remaining 3 projections being equal to zero. That is, the only non-zero projections for the permutation representation of S_4 are given by $\mathcal{P}_1 = \frac{1}{4}J_4$ and $\mathcal{P}_2 = I - \mathcal{P}_1$. Is is true that the only non-zero canonical projections for the permutation representation of S_n are given by $\mathcal{P}_1 = \frac{1}{n}J_n$ and $\mathcal{P}_2 = I - \mathcal{P}_1$?

3.20. Show that m in (3.12) is determined by the number of conjugacy classes in S_ℓ, when $c \geq \ell$. For $c < \ell$, m is the number of unordered decompositions of a positive integer ℓ as a sum of c non-negative integers, and is given by $m = \sum_{t=1}^{c} p(l,t)$, where, recursively, $p(n,k) = \sum_{t=1}^{k} p(n-1,t)$. Tables are available, e.g., Takács (1984).

3.21. Show that the Klein group defined by Table 2.11 in Chapter 1 is isomorphic to the subgroup

$$\{1, (14)(32), (12)(34), (13)(24)\}$$

of the permutation group S_4. Equivalently, identify $A = (1,1), G = (1,t), C = (t,1), T = (t,t)$, where $C_2 = \{1,t\}$, with $t = (12)$ and show that the Klein group can be realized as the product group $C_2 \times C_2$. Show that its character table is given by

χ	A	G	C	T
$\chi_{1\otimes 1}$	1	1	1	1
$\chi_{1\otimes t}$	1	−1	1	−1
$\chi_{t\otimes 1}$	1	1	−1	−1
$\chi_{t\otimes t}$	1	−1	−1	1

where the entries are obtained as

$$\chi_{\tau\otimes\sigma}(\tau',\sigma') = \chi_\tau(\tau')\chi_\sigma(\sigma'), \quad \tau,\tau',\sigma,\sigma' \in C_2,$$

and χ_1, χ_t are the characters of C_2. Conclude that the canonical projections associated with the permutation representation of the Klein group is given by the matrices $\{uu, ut, tu, tt\}$ of Example 5.6 of Chapter 5.

CHAPTER 4

Applications: short nucleotide sequences

4.1. Introduction

In this chapter we apply the mapping structure to the study of symmetries in short nucleotide sequences. A number of descriptive summaries of these sequences are based on measurements defined on certain permutation orbits of interest, as illustrated in the following Examples 4.1 and 4.2.

EXAMPLE 4.1 (Frequency diversity). The work of Doi (1991) on the evolutionary strategy of the HIV-1 virus defines the *frequency diversity* in each cyclic orbit $\mathcal{O}_f = \{f\tau^{-1}; \tau \in C_\ell\}$, as the ratio

$$\max_{s \in \mathcal{O}_f} n(s) / \min_{s \in \mathcal{O}_f} n(s)$$

between the largest and the smallest of the observed frequencies n(s), as s varies within the orbit $\mathcal{O}_f$ of f. Here s and f are short DNA sequences or mappings defined in $L = \{1, 2, \ldots, \ell\}$ with values in the alphabet of residues $\mathcal{A} = \{A, G, C, T\}$. These sequences are, therefore, points in the structure $V = \mathcal{A}^L$, where the cyclic group C_ℓ acts on the left according to $(\tau, f) = f\tau^{-1}$ by cyclically moving the *positions* of the residues on the sequence. The frequencies n(s) are calculated within a given fixed region of interest, such as conservative or hyper variable regions, which may lead to different interpretations of the virus' evolutionary strategies. Figure 4.1 shows the observed diversity for the cyclic orbits

$$\mathcal{O}_{acg} = \{acg, cga, gac\}, \quad \mathcal{O}_{aac} = \{aac, caa, aca\}, \quad \mathcal{O}_{atg} = \{atg, gat, tga\}, \quad \mathcal{O}_{cgt} = \{cgt, tcg, gtc\}$$

in the space V of four-sequences in length of three, along the BRU isolate K02013 described in Section 4.6. The frequencies n(s) are evaluated at each of 45 intervals in length of 200 residues. For convenience of display we show the diversity range $\max_{\mathcal{O}_s} n(s) - \min_{\mathcal{O}_s} n(s)$ instead. The global sequence is approximately 9000 bp-long. We remark that the sequence diversity can be extended to different symmetries and actions, which may then lead to different biological interpretations. This is, in fact, the major reason for formulating a given problem with a new language- the possibility of proposing new questions.

EXAMPLE 4.2 (Baseline variation). In Section 1.2 of Chapter 1 we observed that if P is a probability law in V such that $P(s) = P(s\tau^{-1})$ for all permutation $\tau \in S_\ell$, then P is constant in each one of the *orbits* of V determined by S_ℓ. For two-sequences from the alphabet $\mathcal{A} = \{u, y\}$, these orbits are defined by collecting together the sequences with the same number of, say, purines (u). There are, in this case, $\ell + 1$ orbits, as the number of purines ranges from 0 to ℓ. The uniformity of probability within each orbit may serve as a baseline or reference variation e.g., Durbin, Eddy, Krogh and Mitchison (1998). Figures 4.2, 4.3, 4.5 and 4.6 illustrate the relative frequency ratios and diversity within each one of the two cyclic orbits

$$\text{single-purine: } u_1 = \{uyy, yuy, yyu\}, \quad \text{single-pyrimidine: } u_2 = \{yuu, uyu, uuy\}.$$

The amount with which these ratios deviate from 1 and with which diversities deviate from 0 imply the eventual inadequacy of the independent letters model (which, in particular, satisfies the invariance condition $P(s) = P(s\tau^{-1})$. In addition, these graphs also reveal the striking effect of the component yuy (alternating purine-pyrimidine) relative to the other two components yyu and uyy (non-alternating purine-pyrimidine) components in the orbit. The similarity between figures 4.2 and 4.5 suggests that this pattern is invariant under the transposition of the two letters (S_2 acting on the right). Figures 4.2 and 4.5 also suggest an association between the two non-alternating single-letter frequencies. Figure 4.4 (top) shows the relative frequency ratio f_{uuy}/f_{yyu}, the single-pyrimidine to single-purine or non-alternating single-letter ratio, which is about 1.5 across the genome. However, the non-alternating to alternating single-letter ratio,

$$\frac{f_{uuy}}{f_{uyu}} / \frac{f_{yyu}}{f_{yuy}},$$

shown at the bottom part of Figure 4.4, portrays a much stable distribution of relative frequencies along the BRU isolate K02013. These results are also observed along the isolate M26727, as Figures 4.5, 4.6 and 4.7 show.

4.2. The structure of four-sequences in length of three

Matrix 4.1 shows elements in the structure V of four-sequences in length of three. For computational purposes it is convenient to label the mappings according to the base-c representation

$$I(s) = \sum_{j=1}^{\ell} (s(j) - 1)c^{j-1}.$$

For example, mapping 33 is $s(1) = 1, s(2) = 2, s(3) = 1$. In the present context, c is the number $|\mathcal{A}| = 4$ of letters in the alphabet of interest, and $\ell = 3$, the length of each sequence. There are 64 points in the mapping space V.

(4.1)

V =

s	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48
s(1)	1	2	3	4	1	1	2	1	1	3	1	1	4	2	2	1
s(2)	1	2	3	4	1	2	1	1	3	1	1	4	1	2	1	2
s(3)	1	2	3	4	2	1	1	3	1	1	4	1	1	1	2	2
s	5	9	13	18	26	30	35	39	47	52	56	60	2	3	4	21
s(1)	2	2	3	2	2	4	3	3	1	3	3	2	3	3	4	4
s(2)	2	3	2	2	4	2	3	1	3	3	2	3	3	4	3	4
s(3)	3	2	2	4	2	2	1	3	3	2	3	3	4	3	3	1
s	23	24	41	42	44	61	62	63	37	34	7	25	10	19	53	29
s(1)	4	1	4	4	2	4	4	3	1	2	3	1	2	3	1	1
s(2)	1	4	4	2	4	4	3	4	2	1	2	3	3	1	2	4
s(3)	4	4	2	4	4	3	4	4	3	3	1	2	1	2	4	2
s	8	50	14	20	57	45	12	51	15	36	58	46	28	55	31	40
s(1)	4	2	2	4	1	1	4	3	3	4	2	2	4	3	3	4
s(2)	2	1	4	1	3	4	3	1	4	1	3	4	3	2	4	2
s(3)	1	4	1	2	4	3	1	4	1	3	4	3	2	4	2	3

FIGURE 4.1. Diversity, expressed as the range $\max_{\mathcal{O}_s} n(s) - \min_{\mathcal{O}_s} n(s)$, of selected orbits $\mathcal{O}_{acg}$, $\mathcal{O}_{aac}$, $\mathcal{O}_{atg}$ and $\mathcal{O}_{cgt}$, indicated in the vertical axis, along the BRU isolate K02013, evaluated at each of 45 intervals in length of 200 residues, indicated in the horizontal axis.

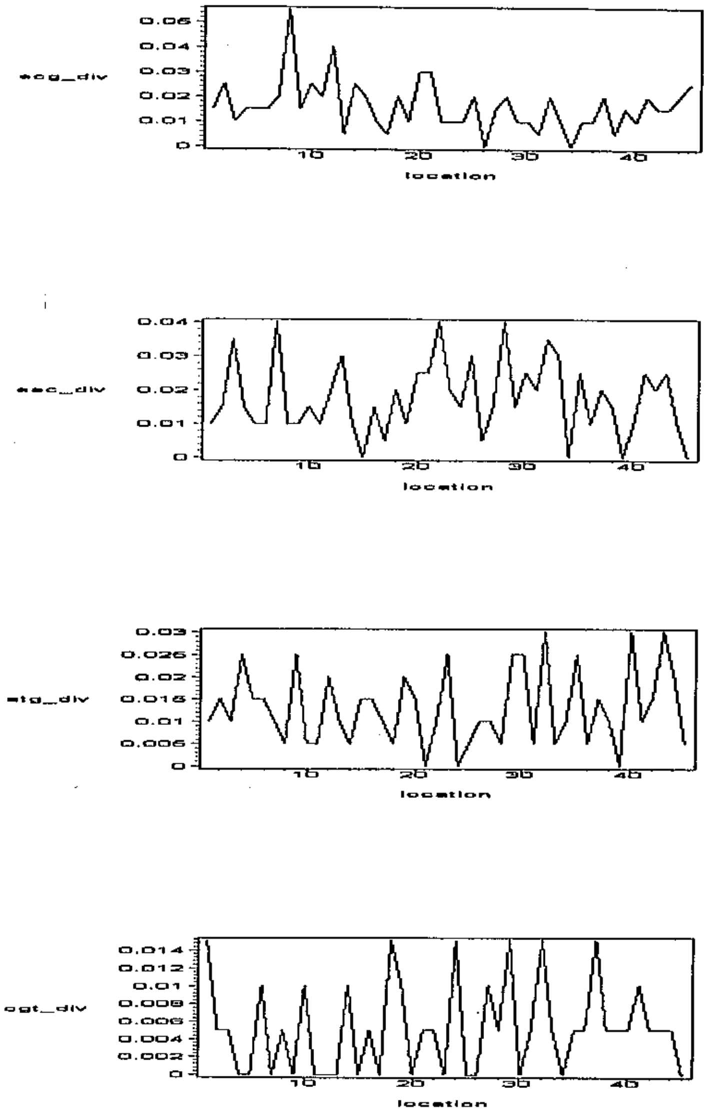


FIGURE 4.2. Relative frequency ratios f_{yyu}/f_{uyy} (the relatively more stable curve) and f_{yyu}/f_{yuy} in the single-purine orbit u_1 (top) and corresponding orbit diversity (bottom), along the BRU isolate K02013, evaluated at each of 45 intervals in length of 200 residues.

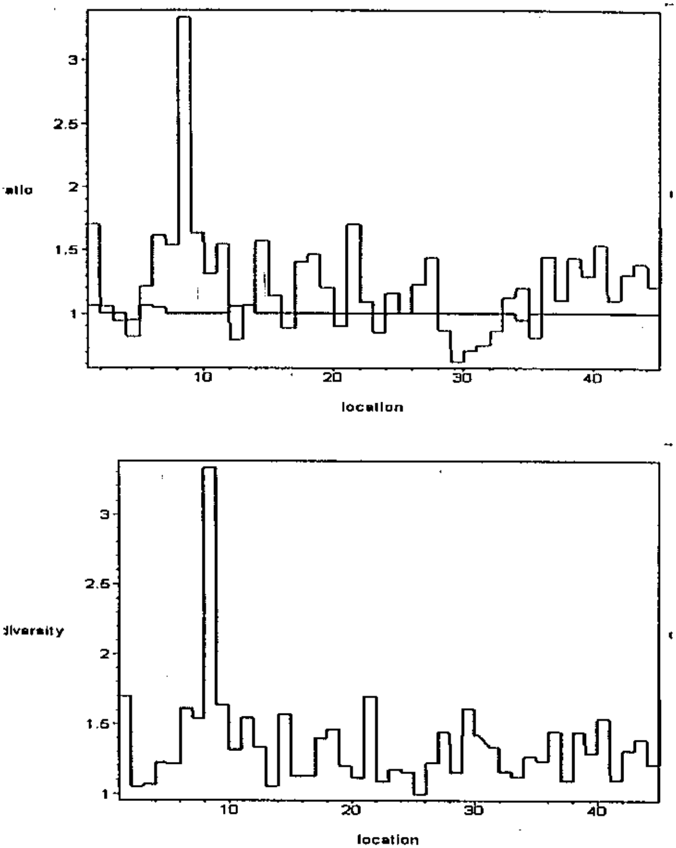


FIGURE 4.3. Relative frequency ratios f_{uuy}/f_{yuu} (the relatively more stable curve) and f_{uuy}/f_{uyu} in the single-pyrimidine orbit u_2 (top) and corresponding orbit diversity (bottom), along the BRU isolate K02013.

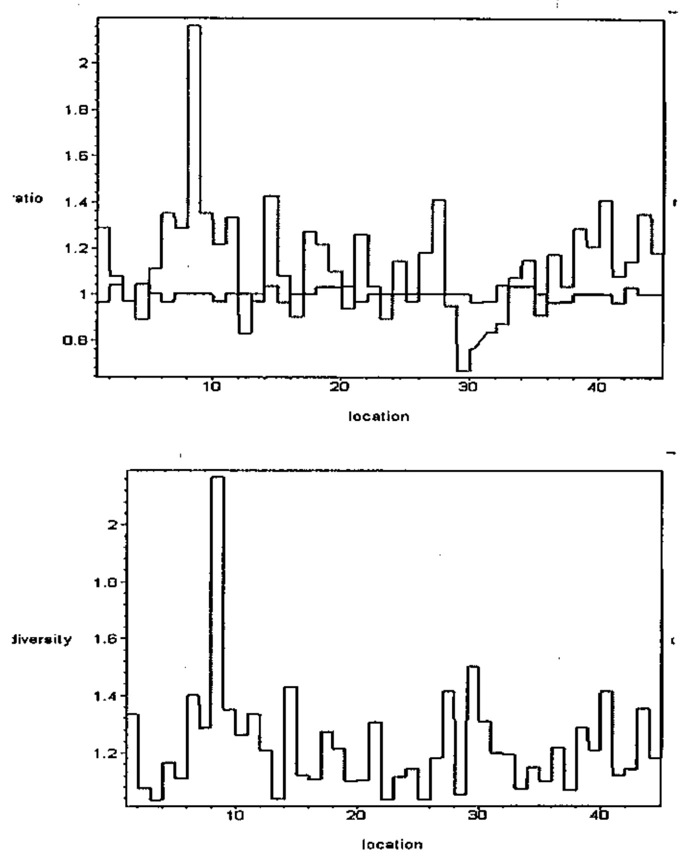
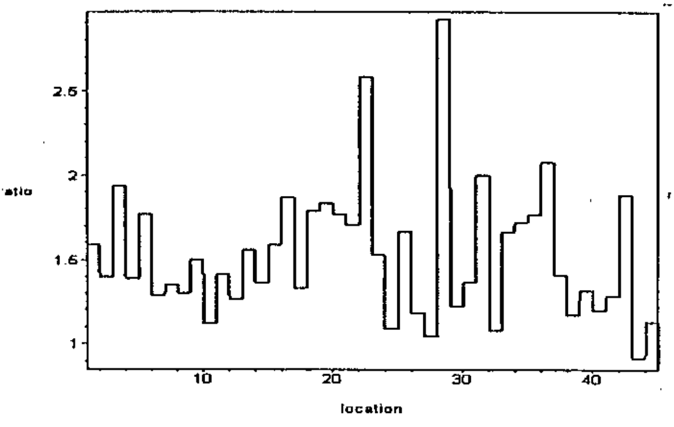


FIGURE 4.4. Relative frequency ratio f_{uuy}/f_{yyu} (top) and relative frequency ratio $(f_{uuy}/f_{uyu})/(f_{yyu}/f_{yuy})$ (bottom), along the BRU isolate K02013, evaluated at each of 45 intervals in length of 200 residues.



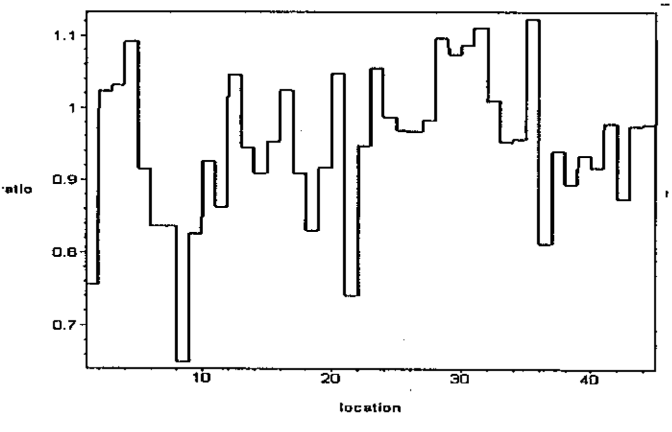
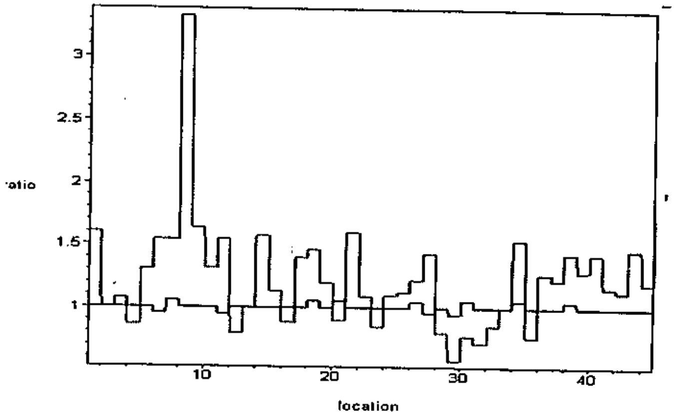


FIGURE 4.5. Relative frequency ratios f_{yyu}/f_{uyy} (green, steady curve) and f_{yyu}/f_{yuy} (red, variable curve) in the single-purine orbit u_I (top) and corresponding orbit diversity (bottom), along the isolate M26727, evaluated at each of 45 intervals in length of 200 residues.



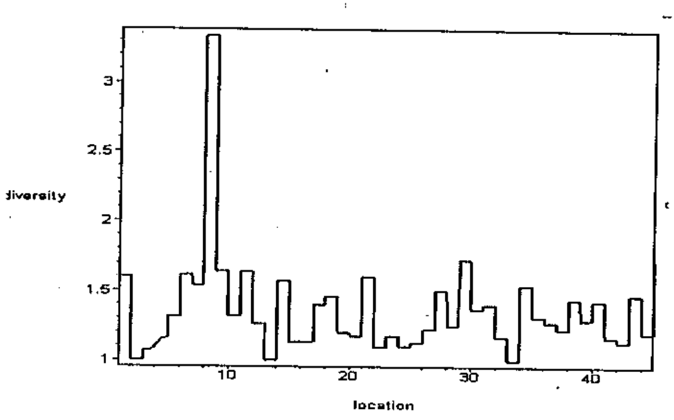


FIGURE 4.6. Relative frequency ratios f_{uuy}/f_{yuu} (green, steady curve) and f_{uuy}/f_{uyu} (red, variable curve) in the single-pyrimidine orbit u_2 (top) and corresponding orbit diversity (bottom), along the isolate M26727, evaluated at each of 45 intervals in length of 200 residues.

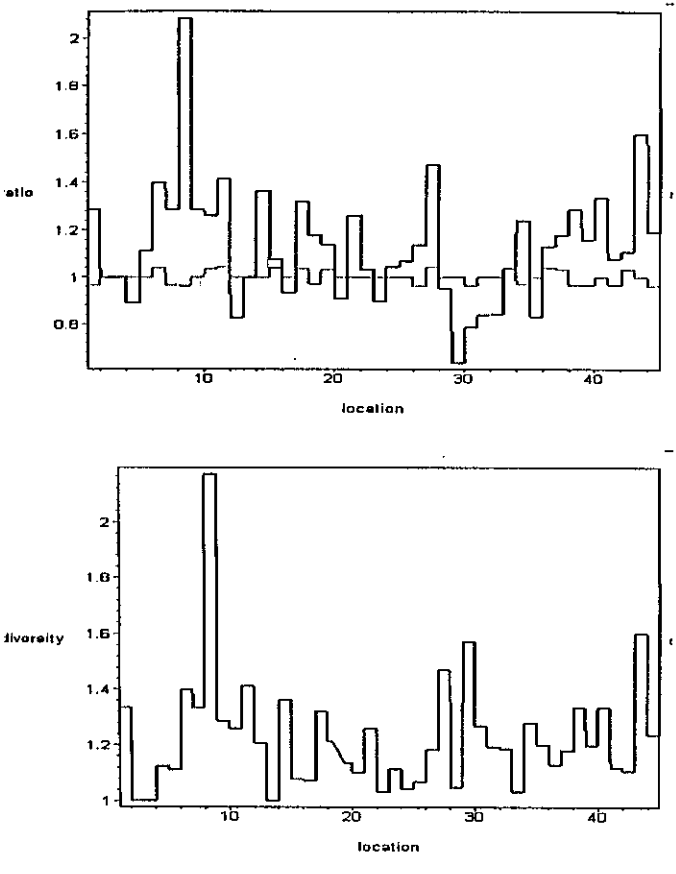
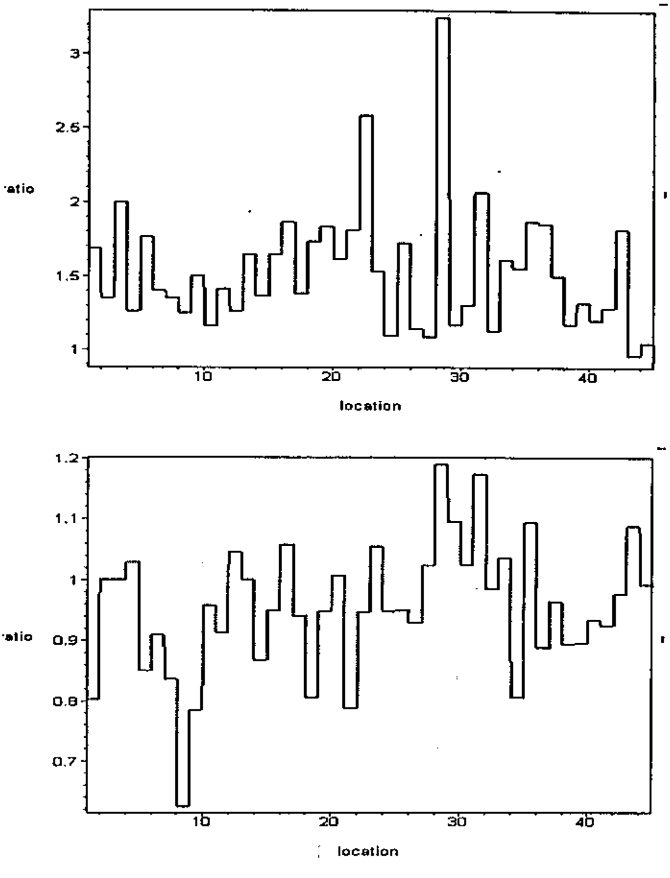


FIGURE 4.7. Relative frequency ratio f_{uuy}/f_{yyu} (top) and relative frequency ratio $(f_{uuy}/f_{uyu})/(f_{yyu}/f_{yuy})$ (bottom), along the isolate M26727, evaluated at each of 45 intervals in length of 200 residues.



4.3. Letter-symmetry on four-sequences in length of three

Matrices (4.9),(4.10),(4.11), and (4.12) in Section 4.6 summarize the letter-symmetry factorization of the space of four-sequences in length of three when S_4 acts on that space according to σs. The tables show the resulting orbits, fixed points and orbit stabilizers. In each matrix, : $\text{fix}^*(\tau)$ is the partial number of fixed points in that matrix. Following Example 3.43 from Chapter 3, there are $r = 5$ projections $\mathcal{P}_k$ in $\mathbb{R}^{64}$, each one of the form

$$\mathcal{P}_k = \begin{bmatrix} \boxed{Q_1^{k,\lambda_1}} & & & & \\ & \boxed{Q_1^{k,\lambda_2}} & & & \\ & & \boxed{Q_2^{k,\lambda_2}} & & \\ & & & \boxed{Q_3^{k,\lambda_2}} & \\ & & & & \boxed{Q_1^{k,\lambda_3}} \end{bmatrix}, \quad k = 1, \ldots, 5,$$

indexed by the partitions $\lambda_1 = 3000$, $\lambda_2 = 2100$ and $\lambda_3 = 1110$. The matrices $Q_c^{k,\lambda}$ and their multiplicities are defined in Example 3.43 of Chapter 3. Table 4.1 summarizes the dimensions of the resulting stable subspaces of $\mathcal{V} = \mathbb{R}^{64}$.

TABLE 4.1. Stable subspaces and corresponding dimensions under the action σs of S_4.

$\mathcal{P}$	$Q_1^{\lambda_1}$	$Q_1^{\lambda_2}$	$Q_2^{\lambda_2}$	$Q_3^{\lambda_2}$	$Q_1^{\lambda_3}$	total
$\mathcal{P}_1$	1	1	1	1	1	5
$\mathcal{P}_2$	0	0	0	0	1	1
$\mathcal{P}_3$	0	2	2	2	4	10
$\mathcal{P}_4$	3	6	6	6	9	30
$\mathcal{P}_5$	0	3	3	3	9	18
total	4	12	12	12	24	64

4.4. Position-symmetry on four-sequences in length of three

Matrices 4.16, 4.17, 4.18 and 4.19, Exercise 4.3, show the mapping space V and the resulting action $s\tau^{-1}$ of S_3 on V. There are 3 projections $\mathcal{P}_k$ in $\mathbb{R}^{64}$, each one of the form

$$\mathcal{P}_k = \begin{bmatrix} \boxed{\begin{bmatrix} Q_1^{k,\lambda_1} & & \\ & \ddots & \\ & & Q_4^{k,\lambda_1} \end{bmatrix}} & & \\ & \boxed{\begin{bmatrix} Q_1^{k,\lambda_2} & & \\ & \ddots & \\ & & Q_{12}^{k,\lambda_2} \end{bmatrix}} & \\ & & \boxed{\begin{bmatrix} Q_1^{k,\lambda_3} & & \\ & \ddots & \\ & & Q_6^{k,\lambda_3} \end{bmatrix}} \end{bmatrix},$$

$k = 1, 2, 3$, indexed by the partitions $\lambda_1 = 3000$, $\lambda_2 = 2100$ and $\lambda_3 = 1110$, where the matrices $Q_c^{k,\lambda}$ are defined in Example 3.42 of Chapter 3.

4.5. Multinomial large-sample analysis

The results about large-sample distributions that are used in this section are described in detail, for example, in Bishop, Fienberg and Holland (1975, p.469). Let $n = \sum_s x(s)$ and consider the law $\mathcal{L}(x)$ of x to be multinomial $\mathcal{M}(n, p)$, where p indicates the vector with components the probabilities assigned to the v components of s. Let also

$$u_n = \frac{\sqrt{v}}{\sqrt{n}}(x - np), \tag{4.2}$$

so that $\mathcal{L}(u_n) \to \mathcal{L}(u)$, where the law of u is multivariate normal, $\mathcal{N}(0, \Omega)$, with covariance matrix $\Omega = v(D_p - pp')$. The notation D_p indicates a diagonal matrix formed with the components of p. Consequently, for any given projection $\mathcal{P}$ we have $\mathcal{L}(u_n'\mathcal{P}u_n) \to \mathcal{L}(u'\mathcal{P}u)$, which is the law of $\sum f_m z_m^2$, where $z_1^2, \ldots, z_m^2$ are random variables independent and identically distributed as Chi-square with one degree of freedom, χ_1^2, and the coefficients f_m are the eigenvalues of $\mathcal{P}^{1/2}\Omega\mathcal{P}'^{1/2}$. Because $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P} = \mathcal{P}'$, we consider the eigenspace of $v\mathcal{P}(D_p - pp')\mathcal{P}$. We the have

PROPOSITION 4.1. Under the hypothesis of uniform probabilities, that is $p = e/v$, we have

(1) $\mathcal{L}(u_n'\mathcal{P}u_n) \to \chi^2_{\text{tr}\,\mathcal{P}}$, for all $\mathcal{P} \perp \frac{1}{v}ee'$,
(2) $u_n'\mathcal{P}u_n = 0$ for $\mathcal{P} = \frac{1}{v}ee'$,
(3) $\mathcal{L}(u_n'u_n) \to \chi^2_{v-1}$.

PROOF. When $p = e/v$ and $\mathcal{P} \perp \frac{1}{v}ee'$, we have $v\mathcal{P}(D_p - pp')\mathcal{P} = v\mathcal{P}(\frac{1}{v}I - \frac{1}{v^2}ee')\mathcal{P} = \mathcal{P}^2 - \frac{1}{v}\mathcal{P}e(\mathcal{P}e)' = \mathcal{P}^2 = \mathcal{P}$. Moreover, because $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}' = \mathcal{P}$ the eigenvalues of $\mathcal{P} \in \{0, 1\}$ and hence $\mathcal{L}(\sum f_m z_m^2)$ is the law of the sum of tr $\mathcal{P}$ independent χ_1^2 laws, that is, $\chi^2_{\text{tr}\,\mathcal{P}}$. When $\mathcal{P} = \frac{1}{v}ee'$, direct calculation shows that $u_n'\mathcal{P}u_n = 0$, and consequently, $\mathcal{L}(u_n'u_n) \to \chi_{v-1}$. □

EXAMPLE 4.3 (Position-symmetry decomposition). This is the result of the left action $s\tau^{-1}$ of S_3 on the space V of all four-sequences in length of three. Matrix

(4.13), Section 4.6, shows the space V of four-sequences in length of three, the observed frequency data x, and corresponding projections $\mathcal{P}$x, in each of the partitions λ of V, based on the 2586 bp sequence described in Subsection 4.6.1 of the same section. Consider, to illustrate, the frequency data $x' = (61, 83, 76)$, corresponding to the orbit, or stable subspace $\mathcal{W}$ indexed by, {aag, aga, gaa} in V. The resulting decomposition of $\mathcal{W}$ is obtained from

$$I_3 = \mathcal{P}_1 + \mathcal{P}_3 = \frac{1}{3}J_3 + (I - \frac{1}{3}J_3).$$

From Example 3.42 of Chapter 3 we note that there are 12 of these subspaces under the partition $\lambda = 2100$, namely $I_3 = Q_i^{1,\lambda_2} + Q_i^{3,\lambda_2}$, $i = 1, \ldots 12$. From Proposition 4.1, and the fact that $\mathcal{P}_2 = 0$, the law of u′u is χ_2^2, and this is the only component available. The decomposition is then simply

$$\left[\begin{array}{cccc} u'u & u'\mathcal{P}_1u & u'\mathcal{P}_2u & u'\mathcal{P}_3u \\ \hline 3.4454 & 0.0 & 0.0 & 3.4454 \end{array}\right].$$

Matrix (4.3) summarizes all 12 decompositions within frame $\lambda = 2100$:

(4.3)

$\mathcal{O}$	$u'u$	$u'\mathcal{P}_1u$	$u'\mathcal{P}_2u$	$u'\mathcal{P}_3u$
aag	3.4454	0.0	0.0	3.4454
aac	8.2222	0.0	0.0	8.2222
aat	7.9234	0.0	0.0	7.9234
gga	4.0221	0.0	0.0	4.0221
ggc	15.167	0.0	0.0	15.167
ggt	16.074	0.0	0.0	16.074
cca	0.82051	0.0	0.0	0.82051
ccg	8.9714	0.0	0.0	8.9714
cct	0.59375	0.0	0.0	0.59375
tta	4.8212	0.0	0.0	4.8212
ttg	9.8000	0.0	0.0	9.8000
ttc	1.4054	0.0	0.0	1.4054
df	2	0	0	2

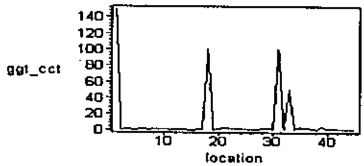
Figure 4.8 shown the relative frequency ratios of GGT and GGC to CCT along the along the BRU isolate K02013, evaluated at each of 45 intervals in length of 200 residues, indicated in the horizontal axis. These ratios were identified by the corresponding u′u ratios in the analysis of variance for frame $\lambda = 2100$. The location of the peaks within the genome remain invariant within the orbit of GGC or GGT. The reader may also refer each sum of square (u′u) in Matrix (4.3) against the critical point $\chi^2_{2,0.95} = 5.99$ of the χ^2 distribution with two degrees of freedom[1] in identifying statistically *salient* orbits.

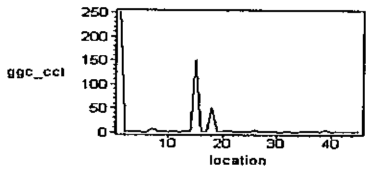
Similar analysis can be derive from the partition $\lambda = 3000$. Take for example the frequency data $x' = (30, 8, 20, 45, 48, 63)$, corresponding to the stable subspace indexed by the points in the orbit

$$\{gac, cga, acg, agc, gca, cag\}$$

[1]Other critical points are $\chi^2_{1,0.95} = 3.84$, $\chi^2_{3,0.95} = 7.81$, $\chi^2_{4,0.95} = 9.488$ and $\chi^2_{5,0.95} = 11.07$, for example.

FIGURE 4.8. Relative frequency ratios of GGT and GGC to CCT, identified by the corresponding u'u ratios in the analysis of variance for frame $\lambda = 2100$.





in V. This is a stable subspace of dimension 6, and the decomposition is

$$I_6 = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = \frac{1}{6}J_6 + \frac{1}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes J_3 + [I_2 \otimes 3(I_3 - \frac{1}{3}J_3)].$$

From Example 3.42, we note that there are 4 such decompositions, namely $I_6 = Q_i^{1,\lambda_3} + Q_i^{2,\lambda_3} + Q_i^{3,\lambda_3}$, $i = 1, \ldots 4$. The resulting analyses of variance are summarized in Matrix (4.4).

(4.4)

$\mathcal{O}$	u'u	$u'\mathcal{P}_1u$	$u'\mathcal{P}_2u$	$u'\mathcal{P}_3u$
cag	71.353	0.0	57.176	14.176
atg	7.7944	0.0	3.7944	4.0
atc	3.0318	0.0	0.42857	2.6032
gtc	65.970	0.0	63.164	2.8060
df	5	0	1	4

Matrix (4.5) shows the F-ratios

$$F_3 = \frac{u'u/5}{u'\mathcal{P}_3u/4}, \quad F_2 = \frac{u'u/5}{u'\mathcal{P}_2u/1}$$

of u′u relative to the residuals from projections $\mathcal{P}_2$ and $\mathcal{P}_3$.

$$(4.5)\qquad \left[\begin{array}{cc} F_3 & F_2 \\ \hline 4.028 & 0.249 \\ 1.558 & 0.411 \\ 0.932 & 1.442 \\ 18.848 & 0.208 \end{array}\right].$$

For reference, we observe that $F_{0.95,5,4} = 6.26$. □

EXAMPLE 4.4 (Letter-symmetry decomposition). This is the result of the action σs of S_4 on the space V of all four-sequences in length of three. Matrices (4.14) and (4.15) in Section 4.6 show the space V of four-sequences in length of three, the observed frequency data x, and corresponding projections $\mathcal{P}x$, in each of the λ-partitions of V. We will detail the analyses for partitions $\lambda_2 = 2100$ and $\lambda_3 = 3000$. Consider, to illustrate, the stable subspace of $\mathbb{R}^{64}$ indexed by the orbit

$$\mathcal{O} = \{aag, aac, aat, gga, ggc, ggt, cca, ccg, cct, tta, ttg, ttc\}$$

with motif $* * +$ in λ_2. The corresponding observed frequency data are

$$x' = [\; 61 \;\; 46 \;\; 98 \;\; 73 \;\; 23 \;\; 31 \;\; 43 \;\; 7 \;\; 24 \;\; 43 \;\; 55 \;\; 20 \;].$$

Note that $|\mathcal{O}| = 12$. The decomposition in $\mathbb{R}^{12}$, from Example 3.43, is

$$I_{12} = Q_i^{1,\lambda_2} + Q_i^{2,\lambda_2} + Q_i^{3,\lambda_2} + Q_i^{4,\lambda_2} + Q_i^{5,\lambda_2}, \quad i = 1, 2, 3.$$

From Proposition 4.1, the decomposition of each one of the three stable subspaces is shown in Matrix 4.6.

$$(4.6)\qquad \left[\begin{array}{cc|ccccc} \mathcal{O} & u'u & u'\mathcal{P}_1u & u'\mathcal{P}_2u & u'\mathcal{P}_3u & u'\mathcal{P}_4u & u'\mathcal{P}_5u \\ \hline **+ & 163.21 & 0.0 & 0.0 & 0.14122 & 127.22 & 35.845 \\ *+* & 168.57 & 0.0 & 0.0 & 15.719 & 151.94 & 0.90577 \\ +** & 138.11 & 0.0 & 0.0 & 4.5954 & 109.21 & 24.298 \\ \hline df & 11 & 0 & 0 & 2 & 6 & 3 \end{array}\right].$$

We note that all quadratic sums are very stable relative to the six-dimensional residual sum $u'\mathcal{P}_4u$. We conclude with the analysis for the single orbit within frame $\lambda_3 = 3000$ and $|\mathcal{O}| = 24$. The decomposition in $\mathbb{R}^{24}$ is $I_{24} = Q^{1,\lambda_3} + Q^{2,\lambda_3} + Q^{3,\lambda_3} + Q^{4,\lambda_3} + Q^{5,\lambda_3}$. The observed frequency data

$$x' = (45, 30, 8, 10, 48, 63, 51, 53, 35, 39, 56, 53, 28, 29, 34, 36, 26, 36, 37, 13, 4, 4, 40, 36),$$

leads to the analysis of variance

$$\left[\begin{array}{cc|ccccc} \mathcal{O} & u'u & u'\mathcal{P}_1u & u'\mathcal{P}_2u & u'\mathcal{P}_3u & u'\mathcal{P}_4u & u'\mathcal{P}_5u \\ \hline abc & 187.81 & 0.0 & 0.83047 & 6.2703 & 71.528 & 109.18 \\ \hline df & 23 & 0 & 1 & 4 & 9 & 9 \end{array}\right].$$

□

EXAMPLE 4.5 (An independent-sample analysis). Indicate by $\hat{x}(s)$ the observed relative frequency of sequence s in a randomly selected string of length 200 from the BRU isolate described in Subsection 4.6.1. More precisely, we sampled from the string shown in Subsection 4.6.1, which is 2586 bp-long, as follows: first selecting a number between 1 and 800, extracting the string of length 200 starting at that number, and determining $\hat{x}(s)$ based on this 200 bp-long string. The sampling is repeated for each four-sequence in length of three, s, in V. We assume that the observed frequencies are approximately independent binomial random variables. Let $x(s) = \arcsin\sqrt{\hat{x}(s)}$, so that the joint probability law of x is approximately $\mathcal{N}_v(p, \nu I)$, where p indicates the vector with components the probabilities assigned to the v components of s. First we describe the analysis of the position-symmetry study. The observed data are shown in Section 4.6.5. The overall analysis of variance,

k	$x'\mathcal{P}_k x$	tr $\mathcal{P}_k$	$x'\mathcal{P}_k x/\text{tr } \mathcal{P}_k$
3000	0.94433	20	0.047216
2100	0.027141	4	0.00678
1110	0.057365	40	0.00143
total	1.02884	64	

can be decomposed further,

λ	k	$x'P_k x$	tr P_k	$x'P_k x/\text{tr } P_k$
3000	1	0.11879	4	0.02969
3000	2	0.0	0.0	–
3000	3	0.0	0.0	–
	total	0.11879	4	0.02969
2100	1	0.58872	12	0.04906
2100	2	0.0	0	0.0
2100	3	0.020662	24	0.00086
	total	0.60939	36	0.01692
1110	1	0.23680	4	0.05920
1110	2	0.02714	4	0.00678
1110	3	0.03670	16	0.00229
	total	0.30065	24	0.01252

according to the motives, or frames, indexed by λ. The data from the letter-symmetry study are shown in Section 4.6.6. The overall analysis of variance is

k	$x'\mathcal{P}_k x$	tr $\mathcal{P}_k$	$x'\mathcal{P}_k x/\text{tr } \mathcal{P}_k$
1	0.91303	5	0.18260
2	0.00068	1	0.00068
3	0.02412	10	0.00241
4	0.07848	30	0.00261
5	0.03930	18	0.00218
total	1.05564	64	

The motif decomposition leads to

λ	k	$x'P_kx$	tr P_k	$x'P_kx$/tr P_k
3000	1	0.11424	1	0.11424
3000	4	0.00454	3	0.00151
	total	0.11879	4	0.02969
2100	1	0.51375	3	0.17125
2100	3	0.01217	6	0.00202
2100	4	0.05969	18	0.00331
2100	5	0.00646	9	0.00071
	total	0.59208	36	0.01644
1110	1	0.285033	1	0.28503
1110	2	0.000687	1	0.00068
1110	3	0.011953	4	0.00298
1110	4	0.014245	9	0.00158
1110	5	0.032844	9	0.00364
	total	0.34476	24	0.01436

□

EXAMPLE 4.6 (Spectral analysis for the 2586 bp-long DNA sequence). Matrix 4.7 shows the spectral analysis for the 2586 bp-long DNA sequence shown in Section 4.6. The spectral analysis is charaterized by transitive actions, or single-orbit actions. In this example we consider the substructure V_I of all four-sequences in length of four in which the nuclueotides are all distinct, and let S_4 act according to σs, $\sigma \in S_4$, $s \in V_I$. Since V_I is a realization of S_4, this is similar to Example 3.46 from Chapter 3. The reference sequence is s=*agct*. □

EXAMPLE 4.7 (Spectral analysis for the entire 9000 bp-long DNA sequence). Matrix 4.8 shows the spectral analysis for the entire 9000 bp-long DNA sequence for the immunodeficiency virus type 1, isolate BRU e.g., Section 4.6. The decomposition is also illustrated in Figure 4.9. □

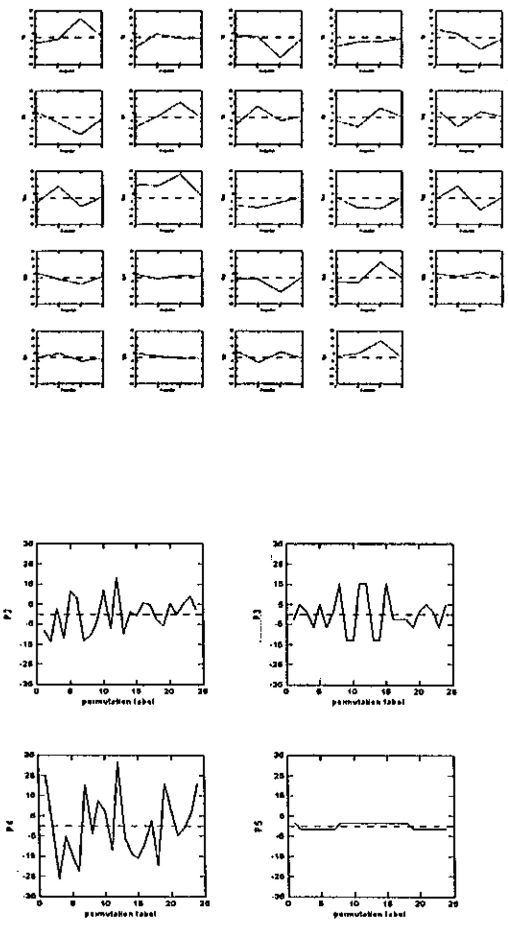
(4.7)

σs	σ	$x(\sigma s)$	$\mathcal{P}_1 x$	$\mathcal{P}_2 x$	$\mathcal{P}_3 x$	$\mathcal{P}_4 x$	$\mathcal{P}_5 x$
agct	1	12	7.458	−2.375	−1.500	8.375	0.04167
agtc	(34)	5	7.458	−0.1250	−1.167	−1.125	−0.04167
acgt	(23)	2	7.458	1.0	2.083	−8.500	−0.04167
atcg	(24)	2	7.458	−2.750	−0.9167	−1.750	−0.04167
gact	(12)	1	7.458	0.3750	−1.167	−5.625	−0.04167
cgat	(13)	3	7.458	0.7500	−0.9167	−4.250	−0.04167
tgca	(14)	10	7.458	−4.0	2.083	4.500	−0.04167
actg	(234)	8	7.458	−1.625	3.500	−1.375	0.04167
atgc	(243)	12	7.458	2.125	−2.0	4.375	0.04167
gcat	(123)	8	7.458	1.375	−2.0	1.125	0.04167
gtca	(124)	6	7.458	−1.875	3.500	−3.125	0.04167
caga	(132)	23	7.458	3.125	3.500	8.875	0.04167
cgta	(134)	1	7.458	−1.375	−2.0	−3.125	0.04167
tacg	(142)	1	7.458	−2.125	−2.0	−2.375	0.04167
tgac	(143)	7	7.458	0.3750	3.500	−4.375	0.04167
gatc	(12)(34)	7	7.458	2.625	−1.500	−1.625	0.04167
ctag	(13)(24)	4	7.458	0.3750	−1.500	−2.375	0.04167
tcga	(14)(23)	1	7.458	−0.6250	−1.500	−4.375	0.04167
gcta	(1234)	12	7.458	−0.7500	−0.9167	6.250	−0.04167
gtac	(1243)	15	7.458	2.500	2.083	3.0	−0.04167
ctga	(1324)	7	7.458	0.8750	−1.167	−0.1250	−0.04167
catg	(1342)	11	7.458	0.5000	2.083	1.0	−0.04167
tagc	(1432)	9	7.458	2.750	−0.9167	−0.2500	−0.04167
tcag	(1423)	12	7.458	−1.125	−1.167	6.875	−0.04167

(4.8)

σs	σ	$x(\sigma s)$	$\mathcal{P}_1 x$	$\mathcal{P}_2 x$	$\mathcal{P}_3 x$	$\mathcal{P}_4 x$	$\mathcal{P}_5 x$
agct	1	44	27.833	−8.0	−2.5000	25.250	1.4167
agtc	(34)	16	27.833	−13.625	5.0833	−1.8750	−1.4167
acgt	(23)	4	27.833	2.7500	1.3333	−26.500	−1.4167
atcg	(24)	3	27.833	−11.625	−6.4167	−5.3750	−1.4167
gact	(12)	28	27.833	11.375	5.0833	−14.875	−1.4167
cgat	(13)	6	27.833	8.3750	−6.4167	−22.375	−1.4167
tgca	(14)	35	27.833	−13.250	1.3333	20.500	−1.4167
actg	(234)	30	27.833	−10.625	15.250	−3.8750	1.4167
atgc	(243)	25	27.833	−3.8750	−12.750	12.375	1.4167
gcat	(123)	36	27.833	12.125	−12.750	7.3750	1.4167
gtca	(124)	26	27.833	−6.8750	15.250	−11.625	1.4167
caga	(132)	94	27.833	18.375	15.250	31.125	1.4167
cgta	(134)	1	27.833	−9.6250	−12.750	−5.8750	1.4167
tacg	(142)	4	27.833	1.3750	−12.750	−13.875	1.4167
tgac	(143)	28	27.833	−0.87500	15.250	−15.625	1.4167
gatc	(12)(34)	24	27.833	5.7500	−2.5000	−8.5000	1.4167
ctag	(13)(24)	34	27.833	4.7500	−2.5000	2.5000	1.4167
tcga	(14)(23)	5	27.833	−2.5000	−2.5000	19.250	1.4167
gcta	(1234)	35	27.833	−5.8750	−6.4167	20.875	−1.4167
gtac	(1243)	40	27.833	5.5000	1.3333	6.7500	−1.4167
ctga	(1324)	27	27.833	0.12500	5.0833	−4.6250	−1.4167
catg	(1342)	32	27.833	5.0	1.3333	−0.75000	−1.4167
tagc	(1432)	36	27.833	9.1250	−6.4167	6.8750	−1.4167
tcag	(1423)	55	27.833	2.1250	5.0833	21.375	−1.4167

FIGURE 4.9. Canonical projections of each observed DNA frequency $x(\sigma s)$, $\sigma \in S_4$, into the four non-trivial subspaces $\mathcal{V}_2, \ldots, \mathcal{V}_5$ (top) and corresponding joint spectra (bottom). The displays correspond to the frequencies shown, in the same order, in Matrix 4.8. The data are based on the 9000 bp-long sequence.



4.6. Tables and graphic displays

4.6.1. The Human Immunodeficiency Virus Type I. Here is a fragment of the entire 9229 bp (base-pair) long nucleotide sequence. To locate the sequence in the NCBI[2] data base, use the accession number K02013.

```
LOCUS       HIVBRUCG      2586 bp ss-RNA     linear   VRL 02-AUG-1993
DEFINITION  Human immunodeficiency virus type 1, isolate BRU
complete genome  (LAV-1).
ACCESSION   K02013 REGION: 5803..8388

        1 atgagagtga aggagaaata tcagcacttg tggagatggg ggtggaaatg gggcaccatg
       61 ctccttggga tattgatgat ctgtagtgct acagaaaaat tgtgggtcac agtctattat
      121 ggggtacctg tgtggaagga agcaaccacc actctatttt gtgcatcaga tgctaaagca
      181 tatgatacag aggtacataa tgtttgggcc acacatgcct gtgtacccac agaccccaac
      241 ccacaagaag tagtattggt aaatgtgaca gaaaatttta acatgtggaa aaatgacatg
      301 gtagaacaga tgcatgagga tataatcagt ttatgggatc aaagcctaaa gccatgtgta
      361 aaattaaccc cactctgtgt tagtttaaag tgcactgatt tggggaatgc tactaatacc
      421 aatagtagta ataccaatag tagtagcggg gaaatgatga tggagaaagg agagataaaa
      481 aactgctctt tcaatatcag cacaagcata agaggtaagg tgcagaaaga atatgcattt
      541 ttttataaac ttgatataat accaatagat aatgatacta ccagctatac gttgacaagt
      601 tgtaacacct cagtcattac acaggcctgt ccaaaggtat cctttgagcc aattcccata
      661 cattattgtg ccccggctgg ttttgcgatt ctaaaatgta ataataagac gttcaatgga
      721 acaggaccat gtacaaatgt cagcacagta caatgtacac atggaattag gccagtagta
      781 tcaactcaac tgctgttgaa tggcagtcta gcagaagaag aggtagtaat tagatctgcc
      841 aatttcacag acaatgctaa aaccataata gtacagctga accaatctgt agaaattaat
      901 tgtacaagac ccaacaacaa tacaagaaaa agtatccgta tccagagggg accagggaga
      961 gcatttgtta caataggaaa aataggaaat atgagacaag cacattgtaa cattagtaga
     1021 gcaaaatgga atgccacttt aaaacagata gctagcaaat taagagaaca atttggaaat
     1081 aataaaacaa taatctttaa gcaatcctca ggaggggacc cagaaattgt aacgcacagt
     1141 tttaattgtg gaggggaatt tttctactgt aattcaacac aactgtttaa tagtacttgg
     1201 tttaatagta cttggagtac tgaagggtca aataacactg aaggaagtga cacaatcaca
     1261 ctcccatgca gaataaaaca atttataaac atgtggcagg aagtaggaaa agcaatgtat
     1321 gcccctccca tcagcggaca aattagatgt tcatcaaata ttacagggct gctattaaca
     1381 agagatggtg gtaataacaa caatgggtcc gagatcttca gacctggagg aggagatatg
     1441 agggacaatt ggagaagtga attatataaa tataaagtag taaaaattga accattagga
     1501 gtagcaccca ccaaggcaaa gagaagagtg gtgcagagag aaaaaagagc agtgggaata
     1561 ggagctttgt tccttgggtt cttgggagca gcaggaagca ctatgggcgc acggtcaatg
     1621 acgctgacgg tacaggccag acaattattg tctggtatag tgcagcagca gaacaatttg
     1681 ctgagggcta ttgaggcgca acagcatctg ttgcaactca cagtctgggg catcaagcag
     1741 ctccaggcaa gaatcctggc tgtggaaaga tacctaaagg atcaacagct cctggggatt
     1801 tggggttgct ctggaaaact catttgcacc actgctgtgc cttggaatgc tagttggagt
     1861 aataaatctc tggaacagat ttggaataac atgacctgga tggagtggga cagagaaatt
     1921 aacaattaca caagcttaat acattcctta attgaagaat cgcaaaacca gcaagaaaag
     1981 aatgaacaag aattattgga attagataaa tgggcaagtt tgtggaattg gtttaacata
     2041 acaaattggc tgtggtatat aaaaatattc ataatgatag taggaggctt ggtaggttta
     2101 agaatagttt ttgctgtact ttctatagtg aatagagtta ggcagggata ttcaccatta
     2161 tcgtttcaga cccacctccc aaccccgagg ggacccgaca ggcccgaagg aatagaagaa
     2221 gaaggtggag agagagacag agacagatcc attcgattag tgaacggatc cttagcactt
     2281 atctgggacg atctgcggag cctgtgcctc ttcagctacc accgcttgag agacttactc
     2341 ttgattgtaa cgaggattgt ggaacttctg ggacgcaggg ggtgggaagc cctcaaatat
     2401 tggtggaatc tcctacagta ttggagtcag gaactaaaga atagtgctgt tagcttgctc
     2461 aatgccacag ccatagcagt agctgagggg acagataggg ttatagaagt agtacaagga
     2521 gcttgtagag ctattcgcca catacctaga agaataagac agggcttgga aaggattttg
     2581 ctataa
```

[2] National Center for Biotechnology Information, http://www.ncbi.nlm.nih.gov/entrez/query.fcgi?db=Nucleotide

4.6.2. Letter-symmetry and orbits for four-sequences in length of three. Matrices (4.9), (4.10), (4.11) and (4.12) show the resulting action of S_4 from the right, fixed points and orbit stabilizers. In each matrix, $\text{fix}^* = \text{fix}^*(\tau)$ denotes the partial number of points fixed by τ in that matrix.

(4.9)

$\tau\backslash s$	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48	fix*
1	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48	16
(34)	1	22	64	43	17	49	33	6	54	38	16	32	48	11	27	59	4
(23)	1	43	22	64	33	17	49	11	27	59	6	38	54	16	48	32	4
(24)	1	64	43	22	49	33	17	16	48	32	11	59	27	6	54	38	4
(12)	22	1	43	64	6	38	54	17	33	49	27	11	59	32	16	48	4
(13)	43	22	1	64	27	11	59	38	6	54	33	17	49	48	32	16	4
(14)	64	22	43	1	32	48	16	54	38	6	59	27	11	49	17	33	4
(234)	1	43	64	22	33	49	17	11	59	27	16	48	32	6	38	54	1
(243)	1	64	22	43	49	17	33	16	32	48	6	54	38	11	59	27	1
(123)	22	43	1	64	38	6	54	27	11	59	17	33	49	32	48	16	1
(124)	22	64	43	1	54	38	6	32	48	16	27	59	11	17	49	33	1
(132)	43	1	22	64	11	27	59	33	17	49	38	6	54	48	16	32	1
(134)	43	22	64	1	27	59	11	38	54	6	48	32	16	33	17	49	1
(142)	64	1	43	22	16	48	32	49	33	17	59	11	27	54	6	38	1
(143)	64	22	1	43	32	16	48	54	6	38	49	17	33	59	27	11	1
(12)(34)	22	1	64	43	6	54	38	17	49	33	32	16	48	27	11	59	0
(13)(24)	43	64	1	22	59	11	27	48	16	32	33	49	17	38	54	6	0
(14)(23)	64	43	22	1	48	32	16	59	27	11	54	38	6	49	33	17	0
(1234)	22	43	64	1	38	54	6	27	59	11	32	48	16	17	33	49	0
(1243)	22	64	1	43	54	6	38	32	16	48	17	49	33	27	59	11	0
(1324)	43	64	22	1	59	27	11	48	32	16	38	54	6	33	49	17	0
(1342)	43	1	64	22	11	59	27	33	49	17	48	16	32	38	6	54	0
(1432)	64	1	22	43	16	32	48	49	17	33	54	6	38	59	11	27	0
(1423)	64	43	1	22	48	16	32	59	11	27	49	33	17	54	38	6	0
$\lvert G_s \rvert$	6	6	6	6	2	2	2	2	2	2	2	2	2	2	2	2	

(4.10)

$\tau\backslash s$	5	9	13	18	26	30	35	39	47	52	56	60	2	3	4	21	fix*
1	5	9	13	18	26	30	35	39	47	52	56	60	2	3	4	21	16
(34)	5	13	9	18	30	26	52	56	60	35	39	47	2	4	3	21	4
(23)	9	5	13	35	39	47	18	26	30	52	60	56	3	2	4	41	3
(24)	13	9	5	52	60	56	35	47	39	18	30	26	4	3	2	61	3
(12)	18	26	30	5	9	13	39	35	47	56	52	60	21	23	24	2	2
(13)	39	35	47	26	18	30	9	5	13	60	56	52	42	41	44	23	2
(14)	56	60	52	30	26	18	47	39	35	13	5	9	62	63	61	24	2
(234)	9	13	5	35	47	39	52	60	56	18	26	30	3	4	2	41	0
(243)	13	5	9	52	56	60	18	30	26	35	47	39	4	2	3	61	0
(123)	26	18	30	39	35	47	5	9	13	56	60	52	23	21	24	42	0
(124)	30	26	18	56	60	52	39	47	35	5	13	9	24	23	21	62	0
(143)	35	39	47	9	5	13	26	18	30	60	52	56	41	42	44	3	0
(134)	39	47	35	26	30	18	60	56	52	9	5	13	42	44	41	23	0
(142)	52	60	56	13	9	5	47	35	39	30	18	26	61	63	62	4	0
(143)	56	52	60	30	18	26	13	5	9	47	39	35	62	61	63	24	0
(12)(34)	18	30	26	5	13	9	56	52	60	39	35	47	21	24	23	2	0
(13)(24)	47	35	39	60	52	56	9	13	5	26	30	18	44	41	42	63	0
(14)(23)	60	56	52	47	39	35	30	26	18	13	9	5	63	62	61	44	0
(1234)	26	30	18	39	47	35	56	60	52	5	9	13	23	24	21	42	0
(1243)	30	18	26	56	52	60	5	13	9	39	47	35	24	21	23	62	0
(1324)	47	39	35	60	56	52	26	30	18	9	13	5	44	42	41	63	0
(1342)	35	47	39	9	13	5	60	52	56	26	18	30	41	44	42	3	0
(1432)	52	56	60	13	5	9	30	18	26	47	35	39	61	62	63	4	0
(1423)	60	52	56	47	35	39	13	9	5	30	26	18	63	61	62	44	0
$\lvert G_s \rvert$	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	

(4.11)

$\tau\backslash s$	23	24	41	42	44	61	62	63	37	34	7	25	10	19	53	29	fix*
1	23	24	41	42	44	61	62	63	37	34	7	25	10	19	53	29	16
(34)	24	23	61	62	63	41	42	44	53	50	8	29	14	20	37	25	0
(23)	42	44	21	23	24	61	63	62	25	19	10	37	7	34	57	45	1
(24)	63	62	41	44	42	21	24	23	45	36	15	57	12	51	29	53	1
(12)	3	4	42	41	44	62	61	63	34	37	19	10	25	7	50	14	2
(13)	21	24	3	2	4	63	62	61	7	10	37	19	34	25	55	31	2
(14)	23	21	44	42	41	4	2	3	40	46	55	28	58	31	8	20	2
(234)	44	42	61	63	62	21	23	24	57	51	12	45	15	36	25	37	0
(243)	62	63	21	24	23	41	44	42	29	20	14	53	8	50	45	57	0
(123)	41	44	2	3	4	62	63	61	10	7	25	34	19	37	58	46	0
(124)	63	61	42	44	41	2	4	3	46	40	31	58	28	55	14	50	0
(143)	2	4	23	21	24	63	61	62	19	25	34	7	37	10	51	15	0
(134)	24	21	63	62	61	3	2	4	55	58	40	31	46	28	7	19	0
(142)	3	2	44	41	42	24	21	23	36	45	51	12	57	15	20	8	0
(143)	21	23	4	2	3	44	42	41	8	14	53	20	50	29	40	28	0
(12)(34)	4	3	62	61	63	42	41	44	50	53	20	14	29	8	34	10	0
(13)(24)	61	62	3	4	2	23	24	21	15	12	45	51	36	57	31	55	0
(14)(23)	42	41	24	23	21	4	3	2	28	31	58	40	55	46	12	36	0
(1234)	44	41	62	63	61	2	3	4	58	55	28	46	31	40	10	34	0
(1243)	61	63	2	4	3	42	44	41	14	8	29	50	20	53	46	58	0
(1324)	62	61	23	24	21	3	4	2	31	28	46	55	40	58	15	51	0
(1342)	4	2	63	61	62	23	21	24	51	57	36	15	45	12	19	7	0
(1432)	2	3	24	21	23	44	41	42	20	29	50	8	53	14	36	12	0
(1423)	41	42	4	3	2	24	23	21	12	15	57	36	51	45	28	40	0
$\lvert G_s \rvert$	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	

(4.12)

$\tau \backslash s$	8	50	14	20	57	45	12	51	15	36	58	46	28	55	31	40	fix*	fix
1	8	50	14	20	57	45	12	51	15	36	58	46	28	55	31	40	16	64
(34)	7	34	10	19	45	57	15	36	12	51	46	58	31	40	28	55	0	8
(23)	12	51	15	36	53	29	8	50	14	20	55	31	40	58	46	28	0	8
(24)	14	20	8	50	25	37	10	19	7	34	28	40	58	31	55	46	0	8
(12)	20	53	29	8	58	46	28	55	31	40	57	45	12	51	15	36	0	8
(13)	40	58	46	28	51	15	36	57	45	12	50	14	20	53	29	8	0	8
(14)	53	14	50	29	12	36	57	15	51	45	10	34	25	7	19	37	0	8
(234)	10	19	7	34	29	53	14	20	8	50	31	55	46	28	40	58	0	1
(243)	15	36	12	51	37	25	7	34	10	19	40	28	55	46	58	31	0	1
(123)	28	55	31	40	50	14	20	53	29	8	51	15	36	57	45	12	0	1
(124)	29	8	20	53	10	34	25	7	19	37	12	36	57	15	51	45	0	1
(143)	36	57	45	12	55	31	40	58	46	28	53	29	8	50	14	20	0	1
(134)	37	10	34	25	15	51	45	12	36	57	14	50	29	8	20	53	0	1
(142)	50	29	53	14	28	40	58	31	55	46	25	37	10	19	7	34	0	1
(143)	55	46	58	31	36	12	51	45	57	15	34	10	19	37	25	7	0	1
(12)(34)	19	37	25	7	46	58	31	40	28	55	45	57	15	36	12	51	0	0
(13)(24)	46	28	40	58	19	7	34	25	37	10	20	8	50	29	53	14	0	0
(14)(23)	57	15	51	45	8	20	53	14	50	29	7	19	37	10	34	25	0	0
(1234)	25	7	19	37	14	50	29	8	20	53	15	51	45	12	36	57	0	0
(1243)	31	40	28	55	34	10	19	37	25	7	36	12	51	45	57	15	0	0
(1324)	45	12	36	57	7	19	37	10	34	25	8	20	53	14	50	29	0	0
(1342)	34	25	37	10	31	55	46	28	40	58	29	53	14	20	8	50	0	0
(1432)	51	45	57	15	40	28	55	46	58	31	37	25	7	34	10	19	0	0
(1423)	58	31	55	46	20	8	50	29	53	14	19	7	34	25	37	10	0	0
$\lvert G_s \rvert$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		120

4.6.3. Position-symmetry and orbits for four-sequences in length of three. Matrix (4.13) shows the space V of four-sequences in length of three, the observed data x, and corresponding projections $\mathcal{P}$x, in each of the partitions of V.

			$\lambda = 2100$				
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x
a	a	g	17	61	$\frac{220}{3}$	0	$-\frac{37}{3}$
a	g	a	5	83	$\frac{220}{3}$	0	$\frac{29}{3}$
g	a	a	2	76	$\frac{220}{3}$	0	8/3
a	a	c	33	46	63	0	−17
a	c	a	9	78	63	0	15
c	a	a	3	65	63	0	2
a	a	t	49	98	$\frac{235}{3}$	0	$\frac{59}{3}$
a	t	a	13	73	$\frac{235}{3}$	0	−16/3
t	a	a	4	64	$\frac{235}{3}$	0	$-\frac{43}{3}$
g	g	a	6	73	$\frac{181}{3}$	0	$\frac{38}{3}$
g	a	g	18	55	$\frac{181}{3}$	0	−16/3
a	g	g	21	53	$\frac{181}{3}$	0	$-\frac{22}{3}$
g	g	c	38	23	12	0	11
g	c	g	26	6	12	0	−6
c	g	g	23	7	12	0	−5
g	g	t	54	31	$\frac{136}{3}$	0	$-\frac{43}{3}$
g	t	g	30	38	$\frac{136}{3}$	0	$-\frac{22}{3}$
t	g	g	24	67	$\frac{136}{3}$	0	$\frac{65}{3}$
c	c	a	11	43	39	0	4
c	a	c	35	39	39	0	0
a	c	c	41	35	39	0	−4
c	c	g	27	7	$\frac{35}{3}$	0	−14/3
c	g	c	39	8	$\frac{35}{3}$	0	−11/3
g	c	c	42	20	$\frac{35}{3}$	0	$\frac{25}{3}$
c	c	t	59	24	$\frac{64}{3}$	0	8/3
c	t	c	47	21	$\frac{64}{3}$	0	−1/3
t	c	c	44	19	$\frac{64}{3}$	0	−7/3
t	t	a	16	43	$\frac{151}{3}$	0	$-\frac{22}{3}$
t	a	t	52	45	$\frac{151}{3}$	0	−16/3
a	t	t	61	63	$\frac{151}{3}$	0	$\frac{38}{3}$
t	t	g	32	55	$\frac{130}{3}$	0	$\frac{35}{3}$
t	g	t	56	48	$\frac{130}{3}$	0	14/3
g	t	t	62	27	$\frac{130}{3}$	0	$-\frac{49}{3}$
t	t	c	48	20	$\frac{74}{3}$	0	−14/3
t	c	t	60	26	$\frac{74}{3}$	0	4/3
c	t	t	63	28	$\frac{74}{3}$	0	10/3

(4.13)

			$\lambda = 3000$				
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x
a	a	a	1	83	83	0	0
g	g	g	22	53	53	0	0
c	c	c	43	22	22	0	0
t	t	t	64	44	44	0	0
			$\lambda = 1110$				
g	a	c	34	30	34	−18	14
c	g	a	7	8	34	−18	−8
a	c	g	25	10	34	−18	−6
a	g	c	37	45	34	18	−7
g	c	a	10	48	34	18	−4
c	a	g	19	63	34	18	11
a	t	g	29	53	$\frac{287}{6}$	−11/2	$\frac{32}{3}$
t	g	a	8	35	$\frac{287}{6}$	−11/2	$-\frac{22}{3}$
g	a	t	50	39	$\frac{287}{6}$	−11/2	−10/3
a	g	t	53	51	$\frac{287}{6}$	11/2	−7/3
g	t	a	14	56	$\frac{287}{6}$	11/2	8/3
t	a	g	20	53	$\frac{287}{6}$	11/2	−1/3
a	t	c	45	29	$\frac{63}{2}$	3/2	−4
t	c	a	12	34	$\frac{63}{2}$	3/2	1
c	a	t	51	36	$\frac{63}{2}$	3/2	3
a	c	t	57	28	$\frac{63}{2}$	−3/2	−2
c	t	a	15	26	$\frac{63}{2}$	−3/2	−4
t	a	c	36	36	$\frac{63}{2}$	−3/2	6
g	t	c	46	13	$\frac{67}{3}$	$-\frac{46}{3}$	6
t	c	g	28	4	$\frac{67}{3}$	$-\frac{46}{3}$	−3
c	g	t	55	4	$\frac{67}{3}$	$-\frac{46}{3}$	−3
g	c	t	58	37	$\frac{67}{3}$	$\frac{46}{3}$	−2/3
c	t	g	31	40	$\frac{67}{3}$	$\frac{46}{3}$	7/3
t	g	c	40	36	$\frac{67}{3}$	$\frac{46}{3}$	−5/3

4.6.4. Letter-symmetry, orbits and multinomial data for four-sequences in length of three. Matrices (4.14) and (4.15) show the space V of four-sequences in length of three, the observed data x, and corresponding projections $\mathcal{P}$x, in each of the partitions of V, obtained with the action of S_4 on V from the right.

(4.14)

s(1)	s(2)	s(3)	$\lambda = 2100$ label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
a	a	g	17	61	$\frac{131}{3}$	0	5/6	$\frac{111}{4}$	$-\frac{45}{4}$
a	a	c	33	46	$\frac{131}{3}$	0	1/12	$\frac{67}{8}$	$-\frac{49}{8}$
a	a	t	49	98	$\frac{131}{3}$	0	$-\frac{11}{12}$	$\frac{303}{8}$	$\frac{139}{8}$
g	g	a	6	73	$\frac{131}{3}$	0	5/6	$\frac{69}{4}$	$\frac{45}{4}$
g	g	c	38	23	$\frac{131}{3}$	0	$-\frac{11}{12}$	$-\frac{203}{8}$	$\frac{45}{8}$
g	g	t	54	31	$\frac{131}{3}$	0	1/12	$\frac{33}{8}$	$-\frac{135}{8}$
c	c	a	11	43	$\frac{131}{3}$	0	1/12	$-\frac{55}{8}$	$\frac{49}{8}$
c	c	g	27	7	$\frac{131}{3}$	0	$-\frac{11}{12}$	$-\frac{241}{8}$	$-\frac{45}{8}$
c	c	t	59	24	$\frac{131}{3}$	0	5/6	-20	$-1/2$
t	t	a	16	43	$\frac{131}{3}$	0	$-\frac{11}{12}$	$\frac{141}{8}$	$-\frac{139}{8}$
t	t	g	32	55	$\frac{131}{3}$	0	1/12	$-\frac{45}{8}$	$\frac{135}{8}$
t	t	c	48	20	$\frac{131}{3}$	0	5/6	-25	1/2
a	g	a	5	83	$\frac{130}{3}$	0	$\frac{35}{12}$	$\frac{317}{8}$	$-\frac{23}{8}$
a	c	a	9	78	$\frac{130}{3}$	0	$\frac{89}{12}$	$\frac{199}{8}$	$\frac{19}{8}$
a	t	a	13	73	$\frac{130}{3}$	0	$-\frac{31}{3}$	$\frac{79}{2}$	1/2
g	a	g	18	55	$\frac{130}{3}$	0	$\frac{35}{12}$	$\frac{47}{8}$	$\frac{23}{8}$
g	c	g	26	6	$\frac{130}{3}$	0	$-\frac{31}{3}$	$-\frac{103}{4}$	$-5/4$
g	t	g	30	38	$\frac{130}{3}$	0	$\frac{89}{12}$	$-\frac{89}{8}$	$-\frac{13}{8}$
c	a	c	35	39	$\frac{130}{3}$	0	$\frac{89}{12}$	$-\frac{75}{8}$	$-\frac{19}{8}$
c	g	c	39	8	$\frac{130}{3}$	0	$-\frac{31}{3}$	$-\frac{105}{4}$	5/4
c	t	c	47	21	$\frac{130}{3}$	0	$\frac{35}{12}$	$-\frac{211}{8}$	$\frac{9}{8}$
t	a	t	52	45	$\frac{130}{3}$	0	$-\frac{31}{3}$	$\frac{25}{2}$	$-1/2$
t	g	t	56	48	$\frac{130}{3}$	0	$\frac{89}{12}$	$-\frac{35}{8}$	$\frac{13}{8}$
t	c	t	60	26	$\frac{130}{3}$	0	$\frac{35}{12}$	$-\frac{153}{8}$	$-\frac{9}{8}$
g	a	a	2	76	$\frac{131}{3}$	0	1/3	$\frac{107}{4}$	$\frac{21}{4}$
c	a	a	3	65	$\frac{131}{3}$	0	$\frac{29}{6}$	23/2	5
t	a	a	4	64	$\frac{131}{3}$	0	$-\frac{31}{6}$	$\frac{143}{4}$	$-\frac{41}{4}$
a	g	g	21	53	$\frac{131}{3}$	0	1/3	$\frac{57}{4}$	$-\frac{21}{4}$
c	g	g	23	7	$\frac{131}{3}$	0	$-\frac{31}{6}$	$-\frac{85}{4}$	$-\frac{41}{4}$
t	g	g	24	67	$\frac{131}{3}$	0	$\frac{29}{6}$	3	$\frac{31}{2}$
a	c	c	41	35	$\frac{131}{3}$	0	$\frac{29}{6}$	$-17/2$	-5
g	c	c	42	20	$\frac{131}{3}$	0	$-\frac{31}{6}$	$-\frac{115}{4}$	$\frac{41}{4}$
t	c	c	44	19	$\frac{131}{3}$	0	1/3	$-\frac{79}{4}$	$-\frac{21}{4}$
a	t	t	61	63	$\frac{131}{3}$	0	$-\frac{31}{6}$	$\frac{57}{4}$	$\frac{41}{4}$
g	t	t	62	27	$\frac{131}{3}$	0	$\frac{29}{6}$	-6	$-\frac{31}{2}$
c	t	t	63	28	$\frac{131}{3}$	0	1/3	$-\frac{85}{4}$	$\frac{21}{4}$

(4.15)

			$\lambda = 3000$						
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
a	a	a	1	83	$\frac{101}{2}$	0	0	$\frac{65}{2}$	0
g	g	g	22	53	$\frac{101}{2}$	0	0	5/2	0
c	c	c	43	22	$\frac{101}{2}$	0	0	$-\frac{57}{2}$	0
t	t	t	64	44	$\frac{101}{2}$	0	0	−13/2	0
			$\lambda = 1110$						
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
a	g	c	37	45	$\frac{407}{12}$	$-\frac{13}{12}$	−13/3	$-\frac{13}{4}$	$\frac{79}{4}$
g	a	c	34	30	$\frac{407}{12}$	$\frac{13}{12}$	$\frac{15}{4}$	$\frac{59}{8}$	$-\frac{129}{8}$
c	g	a	7	8	$\frac{407}{12}$	$\frac{13}{12}$	$-\frac{13}{4}$	$-\frac{47}{8}$	$-\frac{143}{8}$
a	c	g	25	10	$\frac{407}{12}$	$\frac{13}{12}$	−1/2	−5/4	$-\frac{93}{4}$
g	c	a	10	48	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{29}{12}$	$-\frac{19}{8}$	$\frac{121}{8}$
c	a	g	19	63	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{23}{12}$	$\frac{47}{8}$	$\frac{179}{8}$
a	g	t	53	51	$\frac{407}{12}$	$\frac{13}{12}$	$\frac{15}{4}$	$\frac{63}{8}$	$\frac{35}{8}$
a	t	g	29	53	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{29}{12}$	$\frac{129}{8}$	$\frac{13}{8}$
t	g	a	8	35	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{23}{12}$	$\frac{57}{8}$	$-\frac{55}{8}$
g	a	t	50	39	$\frac{407}{12}$	$-\frac{13}{12}$	−13/3	$\frac{37}{2}$	−8
g	t	a	14	56	$\frac{407}{12}$	$\frac{13}{12}$	−1/2	15	13/2
t	a	g	20	53	$\frac{407}{12}$	$\frac{13}{12}$	$-\frac{13}{4}$	$\frac{151}{8}$	$\frac{19}{8}$
a	c	t	57	28	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{23}{12}$	$-\frac{53}{8}$	−1/8
a	t	c	45	29	$\frac{407}{12}$	$\frac{13}{12}$	$-\frac{13}{4}$	−3/8	$-\frac{19}{8}$
t	c	a	12	34	$\frac{407}{12}$	$\frac{13}{12}$	$\frac{15}{4}$	$-\frac{59}{8}$	$\frac{21}{8}$
c	a	t	51	36	$\frac{407}{12}$	$\frac{13}{12}$	−1/2	1/2	1
c	t	a	15	26	$\frac{407}{12}$	$-\frac{13}{12}$	−13/3	−3	1/2
t	a	c	36	36	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{29}{12}$	$\frac{19}{8}$	$-\frac{13}{8}$
g	c	t	58	37	$\frac{407}{12}$	$\frac{13}{12}$	$-\frac{13}{4}$	$-\frac{101}{8}$	$\frac{143}{8}$
g	t	c	46	13	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{23}{12}$	$-\frac{51}{8}$	$-\frac{123}{8}$
t	c	g	28	4	$\frac{407}{12}$	$-\frac{13}{12}$	−13/3	$-\frac{49}{4}$	$-\frac{49}{4}$
c	g	t	55	4	$\frac{407}{12}$	$-\frac{13}{12}$	$\frac{29}{12}$	$-\frac{129}{8}$	$-\frac{121}{8}$
c	t	g	31	40	$\frac{407}{12}$	$\frac{13}{12}$	$\frac{15}{4}$	$-\frac{63}{8}$	$\frac{73}{8}$
t	g	c	40	36	$\frac{407}{12}$	$\frac{13}{12}$	−1/2	$-\frac{57}{4}$	$\frac{63}{4}$

4.6.5. Position-symmetry analysis for the independent-sample data.

			$\lambda = 2100$				
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x
a	a	g	17	9	$\frac{25}{3}$	0	2/3
a	g	a	5	8	$\frac{25}{3}$	0	−1/3
g	a	a	2	8	$\frac{25}{3}$	0	−1/3
a	a	c	33	8	8	0	0
a	c	a	9	6	8	0	−2
c	a	a	3	10	8	0	2
a	a	t	49	13	$\frac{40}{3}$	0	−1/3
a	t	a	13	16	$\frac{40}{3}$	0	8/3
t	a	a	4	11	$\frac{40}{3}$	0	−7/3
g	g	a	6	8	6	0	2
g	a	g	18	5	6	0	−1
a	g	g	21	5	6	0	−1
g	g	c	38	5	7/3	0	8/3
g	c	g	26	0	7/3	0	−7/3
c	g	g	23	2	7/3	0	−1/3
g	g	t	54	1	11/3	0	−8/3
g	t	g	30	6	11/3	0	7/3
t	g	g	24	4	11/3	0	1/3
c	c	a	11	2	3	0	−1
c	a	c	35	3	3	0	0
a	c	c	41	4	3	0	1
c	c	g	27	1	2/3	0	1/3
c	g	c	39	0	2/3	0	−2/3
g	c	c	42	1	2/3	0	1/3
c	c	t	59	2	10/3	0	−4/3
c	t	c	47	6	10/3	0	8/3
t	c	c	44	2	10/3	0	−4/3
t	t	a	16	5	$\frac{20}{3}$	0	−5/3
t	a	t	52	7	$\frac{20}{3}$	0	1/3
a	t	t	61	8	$\frac{20}{3}$	0	4/3
t	t	g	32	6	4	0	2
t	g	t	56	4	4	0	0
g	t	t	62	2	4	0	−2
t	t	c	48	1	8/3	0	−5/3
t	c	t	60	4	8/3	0	4/3
c	t	t	63	3	8/3	0	1/3

			$\lambda = 3000$				
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x
a	a	a	1	9	9	0	0
g	g	g	22	3	3	0	0
c	c	c	43	6	6	0	0
t	t	t	64	3	3	0	0
			$\lambda = 1110$				
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x
g	a	c	34	3	11/3	−8/3	2
c	g	a	7	0	11/3	−8/3	−1
a	c	g	25	0	11/3	−8/3	−1
a	g	c	37	2	11/3	8/3	−13/3
g	c	a	10	8	11/3	8/3	5/3
c	a	g	19	9	11/3	8/3	8/3
a	t	g	29	14	$\frac{20}{3}$	5/3	$\frac{17}{3}$
t	g	a	8	4	$\frac{20}{3}$	5/3	−13/3
g	a	t	50	7	$\frac{20}{3}$	5/3	−4/3
a	g	t	53	2	$\frac{20}{3}$	−5/3	−3
g	t	a	14	6	$\frac{20}{3}$	−5/3	1
t	a	g	20	7	$\frac{20}{3}$	−5/3	2
a	t	c	45	5	14/3	1/3	0
t	c	a	12	7	14/3	1/3	2
c	a	t	51	3	14/3	1/3	−2
a	c	t	57	6	14/3	−1/3	5/3
c	t	a	15	2	14/3	−1/3	−7/3
t	a	c	36	5	14/3	−1/3	2/3
g	t	c	46	3	5/2	−3/2	2
t	c	g	28	0	5/2	−3/2	−1
c	g	t	55	0	5/2	−3/2	−1
g	c	t	58	4	5/2	3/2	0
c	t	g	31	4	5/2	3/2	0
t	g	c	40	4	5/2	3/2	0

4.6.6. Letter-symmetry analysis for the independent-sample data.

			$\lambda = 2100$						
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
a	a	g	17	9	$\frac{73}{12}$	0	$-4/3$	$7/2$	$3/4$
a	a	c	33	8	$\frac{73}{12}$	0	$-\frac{7}{12}$	$\frac{39}{8}$	$-\frac{19}{8}$
a	a	t	49	17	$\frac{73}{12}$	0	$\frac{23}{12}$	$\frac{59}{8}$	$\frac{13}{8}$
g	g	a	6	5	$\frac{73}{12}$	0	$-4/3$	1	$-3/4$
g	g	c	38	8	$\frac{73}{12}$	0	$\frac{23}{12}$	$-\frac{19}{8}$	$\frac{19}{8}$
g	g	t	54	4	$\frac{73}{12}$	0	$-\frac{7}{12}$	$1/8$	$-\frac{13}{8}$
c	c	a	11	7	$\frac{73}{12}$	0	$-\frac{7}{12}$	$-\frac{7}{8}$	$\frac{19}{8}$
c	c	g	27	0	$\frac{73}{12}$	0	$\frac{23}{12}$	$-\frac{45}{8}$	$-\frac{19}{8}$
c	c	t	59	3	$\frac{73}{12}$	0	$-4/3$	$-7/4$	0
t	t	a	16	7	$\frac{73}{12}$	0	$\frac{23}{12}$	$5/8$	$-\frac{13}{8}$
t	t	g	32	3	$\frac{73}{12}$	0	$-\frac{7}{12}$	$-\frac{33}{8}$	$\frac{13}{8}$
t	t	c	48	2	$\frac{73}{12}$	0	$-4/3$	$-11/4$	0
a	g	a	5	15	$\frac{61}{12}$	0	$\frac{35}{12}$	$\frac{51}{8}$	$5/8$
a	c	a	9	9	$\frac{61}{12}$	0	$-1/3$	$\frac{15}{4}$	$1/2$
a	t	a	13	4	$\frac{61}{12}$	0	$-\frac{31}{12}$	$\frac{21}{8}$	$-\frac{9}{8}$
g	a	g	18	7	$\frac{61}{12}$	0	$\frac{35}{12}$	$-3/8$	$-5/8$
g	c	g	26	1	$\frac{61}{12}$	0	$-\frac{31}{12}$	$-\frac{19}{8}$	$\frac{7}{8}$
g	t	g	30	1	$\frac{61}{12}$	0	$-1/3$	$-7/2$	$-1/4$
c	a	c	35	3	$\frac{61}{12}$	0	$-1/3$	$-5/4$	$-1/2$
c	g	c	39	1	$\frac{61}{12}$	0	$-\frac{31}{12}$	$-5/8$	$-\frac{7}{8}$
c	t	c	47	5	$\frac{61}{12}$	0	$\frac{35}{12}$	$-\frac{35}{8}$	$\frac{11}{8}$
t	a	t	52	4	$\frac{61}{12}$	0	$-\frac{31}{12}$	$3/8$	$\frac{9}{8}$
t	g	t	56	6	$\frac{61}{12}$	0	$-1/3$	1	$1/4$
t	c	t	60	5	$\frac{61}{12}$	0	$\frac{35}{12}$	$-\frac{13}{8}$	$-\frac{11}{8}$
g	a	a	2	10	$\frac{35}{6}$	0	$-5/6$	$9/2$	$1/2$
c	a	a	3	8	$\frac{35}{6}$	0	$2/3$	3	$-3/2$
t	a	a	4	15	$\frac{35}{6}$	0	$1/6$	8	1
a	g	g	21	4	$\frac{35}{6}$	0	$-5/6$	$-1/2$	$-1/2$
c	g	g	23	1	$\frac{35}{6}$	0	$1/6$	$-9/2$	$-1/2$
t	g	g	24	8	$\frac{35}{6}$	0	$2/3$	$1/2$	1
a	c	c	41	7	$\frac{35}{6}$	0	$2/3$	-1	$3/2$
g	c	c	42	3	$\frac{35}{6}$	0	$1/6$	$-7/2$	$1/2$
t	c	c	44	3	$\frac{35}{6}$	0	$-5/6$	0	-2
a	t	t	61	5	$\frac{35}{6}$	0	$1/6$	0	-1
g	t	t	62	3	$\frac{35}{6}$	0	$2/3$	$-5/2$	-1
c	t	t	63	3	$\frac{35}{6}$	0	$-5/6$	-4	2

s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
			$\lambda = 3000$						
a	a	a	1	9	$\frac{21}{4}$	0	0	$\frac{15}{4}$	0
g	g	g	22	3	$\frac{21}{4}$	0	0	$-9/4$	0
c	c	c	43	6	$\frac{21}{4}$	0	0	$3/4$	0
t	t	t	64	3	$\frac{21}{4}$	0	0	$-9/4$	0
			$\lambda = 1110$						
s(1)	s(2)	s(3)	label	x	$\mathcal{P}_1$x	$\mathcal{P}_2$x	$\mathcal{P}_3$x	$\mathcal{P}_4$x	$\mathcal{P}_5$x
a	g	c	37	4	$\frac{95}{24}$	$3/8$	$-\frac{13}{12}$	$-3/2$	$9/4$
g	a	c	34	1	$\frac{95}{24}$	$-3/8$	$-\frac{7}{12}$	$5/4$	$-\frac{13}{4}$
c	g	a	7	0	$\frac{95}{24}$	$-3/8$	$\frac{5}{12}$	$-3/2$	$-5/2$
a	c	g	25	0	$\frac{95}{24}$	$-3/8$	$1/6$	$-5/8$	$-\frac{25}{8}$
g	c	a	10	8	$\frac{95}{24}$	$3/8$	$-1/12$	0	$\frac{15}{4}$
c	a	g	19	9	$\frac{95}{24}$	$3/8$	$7/6$	$5/8$	$\frac{23}{8}$
a	g	t	53	4	$\frac{95}{24}$	$-3/8$	$-\frac{7}{12}$	$1/2$	$1/2$
a	t	g	29	3	$\frac{95}{24}$	$3/8$	$-1/12$	$1/2$	$-7/4$
t	g	a	8	4	$\frac{95}{24}$	$3/8$	$7/6$	$-1/8$	$-\frac{11}{8}$
g	a	t	50	7	$\frac{95}{24}$	$3/8$	$-\frac{13}{12}$	$\frac{13}{4}$	$1/2$
g	t	a	14	6	$\frac{95}{24}$	$-3/8$	$1/6$	$\frac{9}{8}$	$\frac{9}{8}$
t	a	g	20	7	$\frac{95}{24}$	$-3/8$	$\frac{5}{12}$	2	1
a	c	t	57	7	$\frac{95}{24}$	$3/8$	$7/6$	$5/8$	$\frac{7}{8}$
a	t	c	45	5	$\frac{95}{24}$	$-3/8$	$\frac{5}{12}$	$-1/4$	$5/4$
t	c	a	12	3	$\frac{95}{24}$	$-3/8$	$-\frac{7}{12}$	0	0
c	a	t	51	5	$\frac{95}{24}$	$-3/8$	$1/6$	$\frac{15}{8}$	$-5/8$
c	t	a	15	2	$\frac{95}{24}$	$3/8$	$-\frac{13}{12}$	$-1/4$	-1
t	a	c	36	5	$\frac{95}{24}$	$3/8$	$-1/12$	$5/4$	$-1/2$
g	c	t	58	4	$\frac{95}{24}$	$-3/8$	$\frac{5}{12}$	$-1/4$	$1/4$
g	t	c	46	2	$\frac{95}{24}$	$3/8$	$7/6$	$-\frac{9}{8}$	$-\frac{19}{8}$
t	c	g	28	0	$\frac{95}{24}$	$3/8$	$-\frac{13}{12}$	$-3/2$	$-7/4$
c	g	t	55	1	$\frac{95}{24}$	$3/8$	$-1/12$	$-7/4$	$-3/2$
c	t	g	31	4	$\frac{95}{24}$	$-3/8$	$-\frac{7}{12}$	$-7/4$	$11/4$
t	g	c	40	4	$\frac{95}{24}$	$-3/8$	$1/6$	$-\frac{19}{8}$	$\frac{21}{8}$

Further reading

(1) The book by Durbin et al. (1998) includes an introduction to biologic sequences and sequencing, including many probabilistics aspects;

(2) The work of Doi (1991) on local nuclotide sequences, of Graf and Schachman (1996) on random permutation of genes and expressed polypetide chains, of Finkel (1992) on HIV-1 ancestry primordial expansions. The notion of covariability in amino acid chains is developed in Bickel, Cosman, Olshen, Spector, Rodrigo and Mullins (1996);

(3) The long-range correlations in DNA sequences is discussed in e.g., Voss (1992), Peng, Buldyrev, Goldberger, Havlin, Sciotino, Simons and Stanley (1992). See also Herzel, W.Ebeling and Schmitt (1994) and Salamon and Konopka (1992) for the notion of entropy in biosequences.

Exercises

4.1. For the actions defined in this chapter, show that $\sum_k \mathcal{P}_k = I$, $\mathcal{P}_k\mathcal{P}_p = 0$, $p \neq k$ and $\mathcal{P}_k^2 = \mathcal{P}_k$, $k = 1, \ldots, r$.

4.2. For the actions defined in this chapter, show that $\sum_k P_k^\lambda = I$, $P_k^\lambda P_p^\lambda = 0$, $p \neq k$ and $(P_k^\lambda)^2 = P_k^\lambda$, $k = 1, \ldots, r$, for each partition λ.

4.3. The following matrices show the action from the left on the set of all four-sequences in length of four.

(4.16)

σ\s	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48
1	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48
(12)	1	22	43	64	17	33	49	6	38	54	11	27	59	16	32	48
(13)	1	22	43	64	2	3	4	21	23	24	41	42	44	61	62	63
(23)	1	22	43	64	5	9	13	18	26	30	35	39	47	52	56	60
(123)	1	22	43	64	2	3	4	21	23	24	41	42	44	61	62	63
(132)	1	22	43	64	5	9	13	18	26	30	35	39	47	52	56	60

(4.17)

σ\s	5	9	13	18	26	30	35	39	47	52	56	60	2	3	4	21
1	5	9	13	18	26	30	35	39	47	52	56	60	2	3	4	21
(12)	2	3	4	21	23	24	41	42	44	61	62	63	5	9	13	18
(13)	5	9	13	18	26	30	35	39	47	52	56	60	17	33	49	6
(23)	17	33	49	6	38	54	11	27	59	16	32	48	2	3	4	21
(123)	17	33	49	6	38	54	11	27	59	16	32	48	5	9	13	18
(132)	2	3	4	21	23	24	41	42	44	61	62	63	17	33	49	6

(4.18)

σ\s	23	24	41	42	44	61	62	63	37	34	7	25	10	19	53	29
1	23	24	41	42	44	61	62	63	37	34	7	25	10	19	53	29
(12)	26	30	35	39	47	52	56	60	34	37	10	19	7	25	50	20
(13)	38	54	11	27	59	16	32	48	7	19	37	10	25	34	8	14
(23)	23	24	41	42	44	61	62	63	25	10	19	37	34	7	29	53
(123)	26	30	35	39	47	52	56	60	19	7	25	34	37	10	20	50
(132)	38	54	11	27	59	16	32	48	10	25	34	7	19	37	14	8

(4.19)

σ\s	8	50	14	20	57	45	12	51	15	36	58	46	28	55	31	40
1	8	50	14	20	57	45	12	51	15	36	58	46	28	55	31	40
(12)	14	53	8	29	51	36	15	57	12	45	55	40	31	58	28	46
(13)	53	20	29	50	12	15	57	36	45	51	28	31	58	40	46	55
(23)	20	14	50	8	45	57	36	15	51	12	46	58	40	31	55	28
(123)	29	8	53	14	36	51	45	12	57	15	40	55	46	28	58	31
(132)	50	29	20	53	15	12	51	45	36	57	31	28	55	46	40	58

Determine the fixed points and the orbit stabilizers.

4.4. Consider the following data set described in Cox and Snell (1989, p.6), from an original study reported by Lombard and Doering (1947). In this study, *the*

TABLE 4.2. A 2×2^4 structure. Study of cancer knowledge.

s	No. successes	No. trials	s	No. successes	No. trials
1	84	477	d	2	12
a	75	231	ad	7	13
b	13	63	bd	4	7
ab	35	94	abd	8	12
c	67	150	cd	3	11
ac	201	378	acd	27	45
bc	16	32	bcd	1	23
abc	102	169	abcd	23	31

response concerned individual's knowledge of cancer, as measured in a test, a 'good' score being a succsee and a 'bad' score a failure. There were four factors expected to account for the variation in the probability of success, the individuals being classified into 2^4 cells depending on the presence of exposure to (a) newspapers; (b) radio; (c) solid reading; (d) lectures.

(1) Identify the mapping structure in these data (recall the analysis of two-sequences in length of four discussed earlier on in Section mmm);
(2) Formulate the symmetries of interest;
(3) Formulate an inference basis for analysis.

CHAPTER 5

Applications: data with set product structure

5.1. Introduction

In this chapter we consider data that are undexed by a set product structure V. The symmetries of interest are those defined by the direct product group $G \times H$ acting on V. Typically, G and H are subgroups of permutation groups. In particular, to decompose the structured data when ρ^g and ρ^h are linear representations of G and H, respectively, we will construct and apply the canonical projections (5.1)

$$\mathcal{P}_{mn} = \frac{d_m d_n}{|G||H|} \sum_{\sigma,\tau} \chi^g_m(\sigma^{-1})\chi^h_n(\tau^{-1})(\rho^g \otimes \rho^h)(\sigma,\tau), \quad n = 1,\ldots,N, \quad m = 1,\ldots,M.$$

associated with the tensor representation $\rho^g \otimes \rho^h$ of ρ^g and ρ^h. From Theorem 3.8, Chapter 3, M and N indicate the number of irreducible representations of G and H, with corresponding characters χ^g_m and χ^h_n, and dimensions d_m and d_n.

5.2. Permutation symmetry studies

EXAMPLE 5.1. Consider the simple set product structure $V = L_1 \times L_2$ with $L_1 = \{1,2\}$ and $L_2 = \{1,2,3\}$, which is the index set for a 2×3 data table such as

$$y = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \in \mathcal{V}.$$

We will obtain a representation of V into a six-dimensional vector space $\mathcal{V}$ realized by the direct product group $S_2 \times S_3$ acting on V according to the action $(\tau i, \sigma j)$, where $(\tau,\sigma) \in S_2 \times S_3$ and $(i,j) \in V$. The observables are written as $y' = (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathcal{V}$. Following Chapter 3, we start with the character tables

$\chi_2\backslash\tau$	1	t
χ_{21}	1	1
χ_{22}	1	−1

$\chi_3\backslash\tau$	1	t	r
χ_{31}	1	1	1
χ_{32}	2	0	−1
χ_{33}	1	−1	1

of S_2 and S_3, in which 1 indicates the appropriate identity, t the corresponding (conjugacy class of) transpositions and r the (class of) order 3 cyclic permutations. Also, indicate by ρ^2 and ρ^3 the permutation representations of S_2 and S_3, respectively. It then follows from Theorem 3.8 that the canonical projections for S_2 are given by:

$$\mathcal{P}_{21} = \frac{1}{2}\sum_{\tau} \chi_{11}(\tau^{-1})\rho^2(\tau) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_{22} = \frac{1}{2}\sum_{\tau}\chi_{12}(\tau^{-1})\rho^2(\tau) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Similarly, we obtain, for S_3,

$$\mathcal{P}_{31} = \frac{1}{3}\sum_{\tau}\chi_{31}(\tau^{-1})\rho^3(\tau) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_{32} = \frac{1}{3}\sum_{\tau}\chi_{31}(\tau^{-1})\rho^3(\tau) = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$$\mathcal{P}_{33} = \sum_{\tau}\chi_{33}(\tau^{-1})\rho^3(\tau) = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{P}_i\mathcal{P}_j = 0$, for $i \neq j$, and that $\sum \mathcal{P}_i = I$, for each one of the two sets $\{\mathcal{P}_{21}, \mathcal{P}_{22}\}$ and $\{\mathcal{P}_{31}, \mathcal{P}_{32}, \mathcal{P}_{33}\}$ of projections. To obtain the proposed canonical projections we form the tensor product of these marginal canonical representations, thus obtaining four projections, namely;

$$\mathcal{P}_1 = \mathcal{P}_{21} \otimes \mathcal{P}_{31} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_2 = \mathcal{P}_{21} \otimes \mathcal{P}_{32} = \frac{1}{6}\begin{bmatrix} 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \\ 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 \end{bmatrix},$$

$$\mathcal{P}_3 = \mathcal{P}_{22} \otimes \mathcal{P}_{31} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_4 = \mathcal{P}_{22} \otimes \mathcal{P}_{32} = \frac{1}{6} \begin{bmatrix} 2 & -1 & -1 & -2 & 1 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 \\ -1 & -1 & 2 & 1 & 1 & -2 \\ -2 & 1 & 1 & 2 & -1 & -1 \\ 1 & -2 & 1 & -1 & 2 & -1 \\ 1 & 1 & -2 & -1 & -1 & 2 \end{bmatrix}.$$

These projections are pairwise orthogonal and verify the decomposition

$$\mathrm{I}_6 = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4. \tag{5.2}$$

Table 5.1 shows the dimensions ($\mathrm{d} = \mathrm{tr}\ \mathcal{P}$) of the corresponding subspaces and indices for the respective bases. These indices carry the first-order interpretation of the data summarized in the subspaces generated by these bases. We conclude

TABLE 5.1. Canonical subspaces of $\rho^2 \otimes \rho^3$, respective dimensions ($\mathrm{d} = \mathrm{tr}\ \mathcal{P}$) and corresponding bases.

$\mathcal{P}$	d	basis	interpretation
$\mathcal{P}_1$	1	$u_1 + u_2 + u_3 + v_1 + v_2 + v_3$	baseline average
$\mathcal{P}_2$	2	$2u_1 - u_2 - u_3 + 2v_1 - v_2 - v_3,\ -u_1 + 2u_2 - u_3 - v_1 + 2v_2 - v_3$	column effect
$\mathcal{P}_3$	1	$u_1 + u_2 + u_3 - v_1 - v_2 - v_3$	row effect
$\mathcal{P}_4$	2	$2u_1 - u_2 - u_3 - 2v_1 + v_2 + v_3,\ -u_1 + 2u_2 - u_3 + v_1 - 2v_2 + v_3$	remainder ϵ

this example with a selected number of decompositions realized by the set $(\mathcal{P})$ of projections described above. In each case we present a particular data along with the corresponding decomposition of the sum of squares. The reader should find the correspondence between these data and the resulting relative magnitude of row and column effects. The column indicated by $\mathcal{Q}$ describes the resulting decomposition under the equivalent set of projections associated with $\rho^3 \otimes \rho^2$ (see Exercise 5.1).

(1)

$$\begin{bmatrix} 3.1 & 4.5 & 6.7 \\ 4.1 & 2.3 & 7.9 \end{bmatrix} :$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	136.3	136.3
c	19.30	0.0
r	0.0	19.30
ϵ	3.62	3.62
total	159.3	159.3

;

(2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} :$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	73.50	73.50
c	4.0	13.50
r	13.50	4.0
ϵ	0.0	0.0
total	91.0	91.0

;

(3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 3 \end{bmatrix}:$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	66.67	66.67
c	0.333	10.67
r	10.67	0.333
ϵ	6.330	6.330
total	84.0	84.0

;

(4)

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 6 \end{bmatrix}:$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	60.167	60.167
c	2.333	13.500
r	13.500	2.333
ϵ	3.0	3.0
total	79.0	79.0

;

(5)

$$\begin{bmatrix} 1 & 2 & 5 \\ 4 & 5 & 8 \end{bmatrix}:$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	104.2	104.2
c	17.33	13.50
r	13.50	17.33
ϵ	0.01	0.01
total	135.0	135.0

;

(6)

$$\begin{bmatrix} 1 & 2 & 5 \\ 1.1 & 2.1 & 5.1 \end{bmatrix}:$$

V	ss($\mathcal{P}$)	ss($\mathcal{Q}$)
1	44.28	44.28
c	17.33	0.015
r	0.015	17.33
ϵ	0.001	0.001
total	61.63	61.63

.

□

EXAMPLE 5.2. Consider the set product structure $V = L_1 \times L_2 \times L_3$ with $L_1 = \{1, 2\}$, $L_2 = \{1, 2, 3\}$ and $L_3 = \{1, \ldots, n\}$. This is similar to Example 5.2, now with n observations in each one of the cells of the 2×3 data table. The data space $\mathcal{V}$ has dimension $\ell_1 \times \ell_2 \times n$. If we assume that all observations within each cell of $L_1 \times L_2$ are experimentally *equivalent*, we may decide (or propose to test) that such equivalence relation is best represented by shuffling these within-cell data according to the action of the symmetries in S_n. The important point is observing that this is one of many choices here. The context may suggest that these data are *only cyclically* equivalent, in which case we would shuffle the data using the cyclic symmetries of C_n. This is characteristic of a symmetry study. We assume, to continue, that S_n is suitable and choose the canonical one-dimensional and n − 1-dimensional projections

$$\mathcal{A} = \frac{1}{n}ee', \quad \mathcal{A}^{\perp} = I - \mathcal{A}$$

of S_n and tensor them with the initial decomposition of $L_1 \times L_2$, expressed in (5.2). The result is a representation defined on the 6n-dimensional vector space $\mathcal{V}$ realized by the direct product group $S_2 \times S_3 \times S_n$ acting on V according to $(\tau i, \sigma j, \eta k)$, where (τ, σ, η) are in the direct product group and (i, j, k) in V. The observables are written as $y' = (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathcal{V}$, with the understanding that each entry is a vector in $\mathbb{R}^n$. The resulting canonical (pairwise orthogonal) decomposition is

$$I = \mathcal{P}_1 \otimes \mathcal{A} + \ldots \mathcal{P}_4 \otimes \mathcal{A} + \mathcal{P}_1 \otimes \mathcal{A}^{\perp} + \ldots \mathcal{P}_4 \otimes \mathcal{A}^{\perp}.$$

Moreover (see Exercise 5.5) the sum of squares decomposition is given by

$$y'y = z'\mathcal{P}_1 z + \ldots z'\mathcal{P}_4 z + \sum_{i=1}^{3}(u_i'\mathcal{A}^{\perp}u_i + v_i'\mathcal{A}^{\perp}v_i),$$

where $z' = (\sqrt{u_1'\mathcal{A}u_1}, \sqrt{u_2'\mathcal{A}u_2}, \sqrt{u_3'\mathcal{A}u_3}, \sqrt{v_1'\mathcal{A}v_1}, \sqrt{v_2'\mathcal{A}v_2}, \sqrt{v_3'\mathcal{A}v_3})$. Here is one numerical example, with the resulting decomposition followed by the standard analysis of variance:

$$y = \left[\begin{array}{c|c|c} 3.1, 3.5, 3.2 & 4.5, 4, 4.7 & 6.7, 4.5, 6.8 \\ \hline 4.1, 4, 4.4 & 2.3, 1.9, 1.5 & 7.9, 7.7, 8 \end{array}\right] :$$

V	ss	dim V
1	380.870	1
c	49.958	2
r	0.035	1
cr	15.779	2
e_1	0.086	2
e_2	0.260	2
e_3	3.380	2
e_4	0.086	2
e_5	0.319	2
e_6	0.045	2
total	450.840	18

.

Source	Sum-of-Squares	df	Mean-Square	F-ratio
ROW	0.036	1	0.036	0.102
COL	49.963	2	24.982	71.718
ROW*COL	15.781	2	7.891	22.652
Error	4.180	12	0.348	

□

Example 5.3 (A 2 × 2 Latin Square structure). Consider the following simple Latin Square assignment of two treatments {a, b} to the 2 × 2 product space V, with a resulting data point y in the space $\mathcal{V}$:

$$V = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad y = \begin{bmatrix} 10 & 14 \\ 7 & 18 \end{bmatrix} \in \mathcal{V}.$$

The observables are written as $y' = (a_1, b_1, b_2, a_2) \in \mathcal{V}$, of dimension $v = 4$. The V space is subject to row-symmetry, column-symmetry and entry-symmetry conditions. First we consider with the row-column full permutation symmetry under the action of the direct product group $S_2 \times S_2$. Following Chapter 3, we start with the character table of S_2, given by

$$\left[\begin{array}{c|cc} \chi\backslash\tau & 1 & (12) \\ \hline \chi_1 & 1 & 1 \\ \chi_2 & 1 & -1 \end{array}\right].$$

From the canonical projections

$$\mathcal{P}_1 = \frac{1}{2}\sum_\tau \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{P}_2 = \frac{1}{2}\sum_\tau \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

of S_2 we obtain the canonical projections

$$\mathcal{P}_{11} = \mathcal{P}_1 \otimes \mathcal{P}_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{P}_{12} = \mathcal{P}_1 \otimes \mathcal{P}_2 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

and

$$\mathcal{P}_{21} = \mathcal{P}_2 \otimes \mathcal{P}_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad \mathcal{P}_{22} = \mathcal{P}_2 \otimes \mathcal{P}_2 = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

of $S_2 \times S_2$ into $\mathcal{V}$. These are one-dimensional pairwise orthogonal projections, with respective bases indexed by

$$\left[\begin{array}{c|c} \mathcal{P}_{11} & a_1 + b_1 + b_2 + a_2 \\ \mathcal{P}_{12} & a_1 - b_1 + b_2 - a_2 \\ \mathcal{P}_{21} & a_1 + b_1 - b_2 - a_2 \\ \mathcal{P}_{22} & a_1 - b_1 - b_2 + a_2 \end{array}\right].$$

The reader may describe the interpretation of the corresponding subspaces. The resulting decomposition of the sum of squares for $y = \begin{bmatrix} 10 & 14 \\ 7 & 18 \end{bmatrix}$ is

trait	$\mathcal{P}$	$y'\mathcal{P}y$	dim = tr $\mathcal{P}$
1	$\mathcal{P}_{11}$	600.25	1
row	$\mathcal{P}_{21}$	.25	1
column	$\mathcal{P}_{12}$	56.25	1
treatment + ϵ	$\mathcal{P}_{22}$	12.25	1
total	I	669	4

Note that all resulting subspaces are one-dimensional and hence irreducible. □

EXAMPLE 5.4 (A 4 × 4 Latin Square structure). Consider the following Latin Square assignment of four treatments $\{a, b, c, d\}$ to the 4×4 product space V, and a resulting data point $y \in \mathcal{V}$:

$$V = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}, \quad y = \begin{bmatrix} 10 & 14 & 7 & 8 \\ 7 & 18 & 11 & 8 \\ 5 & 10 & 11 & 9 \\ 10 & 10 & 12 & 14 \end{bmatrix} \in \mathcal{V}.$$

The observables are written as $y' = (a_1, b_1, c_1, d_1, \ldots, b_4, c_4, d_4, a_4) \in \mathcal{V}$, of dimension $v = 16$. The structure V is subject to row-symmetry, column-symmetry and entry-symmetry. To derive the projections associated with the row-column symmetries under the action of $S_4 \times S_4$ we start with the character table of S_4, given by

$\chi\backslash\tau$	1	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	−1	0	−1
χ_3	2	0	2	−1	0
χ_4	3	−1	−1	0	1
χ_5	1	−1	1	1	−1

From Theorem 3.8, the canonical projections associated with the permutation representation, ρ, of S_4 are given by:

$$\mathcal{P}_1 = \frac{1}{24}\sum_{\tau} \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

of dimension 1, and

$$\mathcal{P}_2 = \frac{3}{24}\sum_{\tau} \chi_2(\tau^{-1})\rho(\tau) = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix},$$

of dimension 3, with the remaining 3 projections being equal to zero[1]. The resulting canonical projections determined by the corresponding tensor representation of $S_4 \times S_4$ are then

$$\mathcal{P}_{11} = \mathcal{P}_1 \otimes \mathcal{P}_1 = \frac{1}{16} J_{16},$$

$\mathcal{P}_{12} = \mathcal{P}_1 \otimes \mathcal{P}_2$, or

$$\mathcal{P}_{12} = \frac{1}{16} \left[\begin{array}{rrrr|rrrr|rrrr|rrrr}
3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 \\
\hline
3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 \\
\hline
3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 \\
\hline
3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3
\end{array} \right],$$

$\mathcal{P}_{21} = \mathcal{P}_2 \otimes \mathcal{P}_1$, or

$$\mathcal{P}_{21} = \frac{1}{16} \left[\begin{array}{rrrr|rrrr|rrrr|rrrr}
3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\hline
-1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 & -1 & -1 & -1 & -1 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 & 3
\end{array} \right],$$

[1]This is a consequence of the existence of a $(n-1)$-dimensional irreducible representation for S_n, e.g., Example 3.34, Chapter 3

$\mathcal{P}_{22} = \mathcal{P}_2 \otimes \mathcal{P}_2$, or

$$\mathcal{P}_{22} = \frac{1}{16}\left[\begin{array}{rrrr|rrrr|rrrr|rrrr}
9 & -3 & -3 & -3 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
-3 & 9 & -3 & -3 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 \\
-3 & -3 & 9 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 \\
-3 & -3 & -3 & 9 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 \\
\hline
-3 & 1 & 1 & 1 & 9 & -3 & -3 & -3 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & -3 & 9 & -3 & -3 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 \\
1 & 1 & -3 & 1 & -3 & -3 & 9 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 \\
1 & 1 & 1 & -3 & -3 & -3 & -3 & 9 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 \\
\hline
-3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 9 & -3 & -3 & -3 & -3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 9 & -3 & -3 & 1 & -3 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & -3 & 9 & -3 & 1 & 1 & -3 & 1 \\
1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & 9 & 1 & 1 & 1 & -3 \\
\hline
-3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 9 & -3 & -3 & -3 \\
1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 9 & -3 & -3 \\
1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & -3 & 9 & -3 \\
1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & 9
\end{array}\right].$$

From $\mathrm{I} = \mathcal{P}_{11} + \mathcal{P}_{12} + \mathcal{P}_{21} + \mathcal{P}_{22}$, we obtain the orthogonal decomposition for the total sum of squares

trait	$\mathcal{P}$	$y'\mathcal{P}y$	dim = tr $\mathcal{P}$
1	$\mathcal{P}_{11}$	1681.0	1
row	$\mathcal{P}_{21}$	18.50	3
column	$\mathcal{P}_{12}$	51.50	3
treatment + e	$\mathcal{P}_{22}$	83.0	9
total	I	1834	16

Next, we have to decompose the 9-dimensional space, image of $\mathcal{V}$ under $\mathcal{P}_{22}$. This can be obtained by letting S_4 act on the four labels of each treatment and utilizing the $(n-1)$-dimensional irreducible decomposition studied in Chapter 3, Example 3.34, and Example 5.2 in this chapter. More specifically, we define

$$\mathcal{P}_t = \begin{bmatrix} \mathcal{P}_{t:a} & 0 & 0 & 0 \\ 0 & \mathcal{P}_{t:b} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{t:c} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{t:d} \end{bmatrix}, \quad \mathcal{P}_e = \begin{bmatrix} \mathcal{P}_{e:a} & 0 & 0 & 0 \\ 0 & \mathcal{P}_{e:b} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{e:c} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{e:d} \end{bmatrix},$$

where

$$\mathcal{P}_{t:i} = \frac{1}{24}\sum_{\tau} \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{24}\begin{bmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix} = \frac{1}{4}J_4, \quad i = a, b, c, d,$$

and

$$\mathcal{P}_{e:i} = \frac{3}{24}\sum_{\tau}\chi_2(\tau^{-1})\rho(\tau) = \frac{1}{8}\begin{bmatrix} 6 & -2 & -2 & -2 \\ -2 & 6 & -2 & -2 \\ -2 & -2 & 6 & -2 \\ -2 & -2 & -2 & 6 \end{bmatrix} = \mathrm{I}_4 - \frac{1}{4}\mathrm{J}_4, \quad \mathrm{i} = \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}.$$

The resulting decomposition

$$\mathrm{y}'\mathrm{g}'\mathcal{P}_{22}'\mathcal{P}_{\mathrm{t}}\mathcal{P}_{22}\mathrm{gy} + \mathrm{y}'\mathrm{g}'\mathcal{P}_{22}'\mathcal{P}_{\mathrm{e}}\mathcal{P}_{22}\mathrm{gy}$$

of $\mathrm{y}'\mathcal{P}_{22}\mathrm{y}$, where g is the (changing of basis) transformation defined by

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \\ \mathrm{d} & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{c} & \mathrm{d} & \mathrm{a} & \mathrm{b} \\ \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{a} \end{bmatrix} \rightarrow \begin{bmatrix} \mathrm{a} & \mathrm{a} & \mathrm{a} & \mathrm{a} \\ \mathrm{b} & \mathrm{b} & \mathrm{b} & \mathrm{b} \\ \mathrm{c} & \mathrm{c} & \mathrm{c} & \mathrm{c} \\ \mathrm{d} & \mathrm{d} & \mathrm{d} & \mathrm{d} \end{bmatrix},$$

is described in the following matrix:

trait	$y'\mathcal{P}y$	dim
1	1681.000	1
row	18.500	3
column	51.500	3
t : a	36.000	*
t : b	2.250	*
t : c	30.250	*
t : d	4.000	*
e : a	2.375	*
e : b	1.125	*
e : c	3.875	*
e : d	3.125	*
total	1834	16

.

The standard analysis of the same data set is described in the following table.

Analysis of Variance

Source	m-of-Squares	df	Mean-Square	F-ratio
ROW	18.500	3	6.167	3.524
COL	51.500	3	17.167	9.810
TREAT	72.500	3	24.167	13.810
Error	10.500	6	1.750	

□

EXAMPLE 5.5 (A 3 × 3 Latin Structure). This example is similar to Examples 5.4 and 5.3. We outline the results for the following label structure and a data point $y \in \mathcal{V}$:

$$V = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}, \quad y = \begin{bmatrix} 10 & 14 & 7 \\ 7 & 18 & 11 \\ 5 & 10 & 11 \end{bmatrix} \in \mathcal{V}.$$

From the character table of S_3,

$$\left[\begin{array}{c|ccc} \chi\backslash\tau & 1 & (12) & (123) \\ \hline \chi_1 & 1 & 1 & 1 \\ \chi_2 & 2 & 0 & -1 \\ \chi_3 & 1 & -1 & 1 \end{array}\right],$$

we obtain the canonical projections for the permutation representation, ρ, of S_3 as:

$$\mathcal{P}_1 = \frac{1}{6}\sum_\tau \chi_1(\tau^{-1})\rho(\tau) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_2 = \frac{2}{6}\sum_\tau \chi_2(\tau^{-1})\rho(\tau) = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The remaining projection is equal to zero. The resulting canonical projections for $\rho \otimes \rho$ are:

$$\mathcal{P}_{11} = \mathcal{P}_1 \otimes \mathcal{P}_1 = \frac{1}{9}J_9,$$

$$\mathcal{P}_{12} = \mathcal{P}_1 \otimes \mathcal{P}_2 = \frac{1}{9}\left[\begin{array}{ccc|ccc|ccc} 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 \\ \hline 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 \\ \hline 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 \end{array}\right],$$

$$\mathcal{P}_{21} = \mathcal{P}_2 \otimes \mathcal{P}_1 = \frac{1}{9}\left[\begin{array}{ccc|ccc|ccc} 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \end{array}\right],$$

$$\mathcal{P}_{22} = \mathcal{P}_2 \otimes \mathcal{P}_2 = \frac{1}{9}\left[\begin{array}{ccc|ccc|ccc} 4 & -2 & -2 & -2 & 1 & 1 & -2 & 1 & 1 \\ -2 & 4 & -2 & 1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -2 & 4 & 1 & 1 & -2 & 1 & 1 & -2 \\ \hline -2 & 1 & 1 & 4 & -2 & -2 & -2 & 1 & 1 \\ 1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1 \\ 1 & 1 & -2 & -2 & -2 & 4 & 1 & 1 & -2 \\ \hline -2 & 1 & 1 & -2 & 1 & 1 & 4 & -2 & -2 \\ 1 & -2 & 1 & 1 & -2 & 1 & -2 & 4 & -2 \\ 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 & 4 \end{array}\right].$$

The corresponding bases are shown in Table 5.2: From $\mathrm{I} = \mathcal{P}_{11} + \mathcal{P}_{12} + \mathcal{P}_{21} + \mathcal{P}_{22}$

TABLE 5.2. Canonical bases for $\rho \otimes \rho$ and respective dimensions ($\mathrm{d} = \mathrm{tr}\ \mathcal{P}$), where ρ is the permutation representation of S_3 .

$\mathcal{P}$	d	basis	trait
$\mathcal{P}_{11}$	1	$a_1 + b_1 + c_1 + c_2 + a_2 + b_2 + b_3 + c_3 + a_3$	1
$\mathcal{P}_{12}$	2	$2a_1 - b_1 - c_1 + 2c_2 - a_2 - b_2 + 2b_3 - c_3 - a_3,$	
		$-a_1 + 2b_1 - c_1 - c_2 + 2a_2 - b_2 - b_3 + 2c_3 - a_3$	c
$\mathcal{P}_{21}$	2	$2a_1 + 2b_1 + 2c_1 - c_2 - a_2 - b_2 - b_3 - c_3 - a_3,$	
		$-a_1 - b_1 - c_1 + 2c_2 + 2a_2 + 2b_2 - b_3 - c_3 - a_3$	r

we obtain the orthogonal decomposition for the total sum of squares

trait	$\mathcal{P}$	$y'\mathcal{P}y$	dim = tr $\mathcal{P}$
1	$\mathcal{P}_{11}$	961	1
row	$\mathcal{P}_{21}$	16.666	2
column	$\mathcal{P}_{12}$	68.666	2
treatment + e	$\mathcal{P}_{22}$	38.666	4
total	I	1085	9

A decomposition of the 4-dimensional space, image of $\mathcal{V}$ under $\mathcal{P}_{22}$ is

$$\mathcal{P}_t = \begin{bmatrix} \mathcal{P}_{t:a} & 0 & 0 \\ 0 & \mathcal{P}_{t:b} & 0 \\ 0 & 0 & \mathcal{P}_{t:c} \end{bmatrix}, \quad \mathcal{P}_e = \begin{bmatrix} \mathcal{P}_{e:a} & 0 & 0 \\ 0 & \mathcal{P}_{e:b} & 0 \\ 0 & 0 & \mathcal{P}_{e:c} \end{bmatrix},$$

where

$$\mathcal{P}_{t:i} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{P}_{e:i} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad i = a, b, c, d.$$

The additional decomposition

$$y'g'\mathcal{P}_{22}'\mathcal{P}_t\mathcal{P}_{22}gy + y'g'\mathcal{P}_{22}'\mathcal{P}_e\mathcal{P}_{22}gy$$

of $y'\mathcal{P}_{22}y$, where g is the transformation corresponding to that defined in Example 5.4, leads to the complete decomposition,

trait	$y'\mathcal{P}y$	dim
1	961.000	1
row	16.667	2
column	68.667	2
t : a	21.333	*
t : b	0.333	*
t : c	16.666	*
e : a	0.222	*
e : b	0.222	*
e : c	0.222	*
total	1085	9

,

which corresponds to the standard analysis of variance,

Source	Sum-of-Squares	df	Mean-Square	F-ratio
row	16.667	2	8.333	25.000
column	68.667	2	34.333	103.000
treat	38.000	2	19.000	57.000
Error	0.667	2	0.333	

for these data. □

EXAMPLE 5.6 (Decomposing the standard 2^p factorial data). For the purpose of this example, let $G = S_2$ and $\tau = (12)$. Starting with $p = 1$, there are two natural projections, namely,

$$u = \frac{1}{2}(\rho(1) + \rho(\tau)) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad t = \frac{1}{2}(\rho(1) - \rho(\tau)) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The projections for the 2^2 factorial experiment are given by

$$\mathrm{uu} = \mathrm{u} \otimes \mathrm{u} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathrm{ut} = \mathrm{u} \otimes \mathrm{t} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathrm{tu} = \mathrm{t} \otimes \mathrm{u} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad \mathrm{tt} = \mathrm{t} \otimes \mathrm{t} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

These projections act on points $\mathrm{v}' = (00, 10, 01, 11)$ in V, in which the low (0) and high (1) levels of each one of the two factors are represented. The data are indexed by these labels. For the 2^3 experiment we have

$$\mathrm{v}' = (000, 100, 010, 110, 001, 101, 011, 111)$$

and

$$\mathrm{uuu} = \mathrm{u} \otimes \mathrm{u} \otimes \mathrm{u} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathrm{tuu} = \mathrm{t} \otimes \mathrm{u} \otimes \mathrm{u} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathrm{uut} = \mathrm{u} \otimes \mathrm{u} \otimes \mathrm{t} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix},$$

$$\mathrm{tut} = \mathrm{t} \otimes \mathrm{u} \otimes \mathrm{t} = \frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathrm{utu} = \mathrm{u} \otimes \mathrm{t} \otimes \mathrm{u} = \frac{1}{8}\begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix},$$

$$\mathrm{ttu} = \mathrm{t} \otimes \mathrm{t} \otimes \mathrm{u} = \frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathrm{utt} = \mathrm{u} \otimes \mathrm{t} \otimes \mathrm{t} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix},$$

$$\text{ttt} = \text{t} \otimes \text{t} \otimes \text{t} = \frac{1}{8} \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

We conclude this example with a numerical illustration in which two observations are obtained at each of the 8 labels of a 2^3 factorial space. We have

$$\text{V} = \{0,1\}^3 \times \{1,2\} = \begin{bmatrix} \text{a} & \text{b} & \text{c} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \times \{1,2\}, \text{ and } \text{y} = \begin{bmatrix} 15 & 779 \\ 999 & 990 \\ 499 & 212 \\ 286 & 611 \\ 438 & 239 \\ 926 & 787 \\ 871 & 303 \\ 891 & 663 \end{bmatrix} \in \mathcal{V}.$$

Because the label space V has the form $\{\text{Low}, \text{High}\}^3 \times \{1 \ldots \text{n}\}$ indexing the n observations in each configuration of the factorial sub-space, we add the 1-dimensional and the (n-1)-dimensional projections

$$\text{A} = \frac{1}{\text{n}}\text{ee}', \quad \text{A}^{\perp} = \text{I} - \text{A},$$

obtaining the canonical (pairwise orthogonal) decomposition of $\mathcal{V}$

$$\text{I} = \text{uuu} \otimes \text{A} + \ldots \text{ttt} \otimes \text{A} + \text{uuu} \otimes \text{A}^{\perp} + \ldots \text{ttt} \otimes \text{A}^{\perp}.$$

The numerical components of the decomposition are shown in the following matrix:

trait	$\mathcal{P}$	$y'(\mathcal{P} \otimes A)y$	$\text{tr}\, \mathcal{P} \otimes A$
1	*uuu*	5651317.56	1
c	*uut*	488950.56	1
b	*utu*	43785.56	1
bc	*utt*	173264.06	1
a	*tuu*	33033.06	1
ac	*tut*	76.56	1
ab	*ttu*	143073.06	1
abc	*ttt*	7788.06	1
		$y'(\mathcal{P} \otimes A^{\perp})y$	$\text{tr}\, \mathcal{P} \otimes A^{\perp}$
e	*uuu*	7267.56	1
e	*uut*	3570.06	1
e	*utu*	86289.06	1
e	*utt*	173264.06	1
e	*tuu*	232083.06	1
e	*tut*	19670.06	1
e	*ttu*	4192.56	1
e	*ttt*	76314.06	1
total	I	7143939.00	16

All subspaces are one-dimensional so that the decomposition is irreducible and the analysis is complete. Here is the condensed standard analysis:

```
                        Analysis of Variance

Source    Sum-of-Squares  df  Mean-Square    F-ratio
a              33033.063   1    33033.063      0.439
b              43785.563   1    43785.563      0.581
c             488950.563   1   488950.563      6.491
a*b           143073.062   1   143073.062      1.899
a*c               76.563   1       76.563      0.001
b*c           173264.062   1   173264.062      2.300
a*b*c           7788.063   1     7788.063      0.103
Error         602650.500   8    75331.313
```

□

EXAMPLE 5.7 (Fractional factorial experiments). The projections defining a half fraction (2^{3-1}) of the 2^3 experiment described in Example 5.6 are obtained as

$$\mathcal{P}_1 = (\mathrm{u}+\mathrm{t}) \otimes \mathrm{u} \otimes \mathrm{u} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_2 = (\mathrm{u}+\mathrm{t}) \otimes \mathrm{u} \otimes \mathrm{t} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathcal{P}_3 = (\mathrm{u}+\mathrm{t}) \otimes \mathrm{t} \otimes \mathrm{t} = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix},$$

$$\mathcal{P}_4 = (\mathrm{u}+\mathrm{t}) \otimes \mathrm{t} \otimes \mathrm{u} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}.$$

The resulting experimental assessments, obtained by symbolically multiplying the projection matrices by

$$\mathrm{v}' = (000, 100, 010, 110, 001, 101, 011, 111)$$

can be indicated by

$$\begin{aligned} \mathrm{L}_1 &= 000 + 100 + 010 + 110, \\ \mathrm{L}_2 &= 000 - 100 + 010 - 110, \\ \mathrm{L}_3 &= 000 - 100 - 010 + 110, \\ \mathrm{L}_4 &= 000 + 100 - 010 - 110, \end{aligned}$$

thus showing that only the half fraction of the original 8 labels in V are needed in the fractional experiment. □

EXAMPLE 5.8. In general, the fractional experiments for the 2^4 factorial experiment are obtained as the solutions to the equations

$$\mathrm{P}_1 + \mathrm{P}_2 = \mathrm{I}, \quad 2^{-3} \text{ fraction}, \tag{5.3}$$

$$\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_4 = \mathrm{I}, \quad 2^{-2} \text{ fraction}, \tag{5.4}$$

$$\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_4 + \mathrm{P}_5 + \mathrm{P}_6 + \mathrm{P}_7 + \mathrm{P}_8 = \mathrm{I}, \quad 2^{-1} \text{ fraction}, \tag{5.5}$$

where the unknown are pairwise disjoint projection matrices in the original 2^4-dimensional space. Equation (5.3) has three sets of non-isomorphic solutions, each one corresponding to a 2^{4-3} fractional experiment, namely:

$$\begin{aligned} &\mathcal{P}_{3,11} = \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \oplus 1 \otimes 1, \quad \mathcal{P}_{3,12} = \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes 1 \otimes 1, \\ &\mathcal{P}_{3,21} = \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}) \otimes 1, \quad \mathcal{P}_{3,22} = \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}) \otimes 1, \\ &\mathcal{P}_{3,31} = \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}), \quad \mathcal{P}_{3,32} = \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}). \end{aligned}$$

Equation (5.4) leads to two sets of non-isomorphic 2^{4-2} fractional experiments, given by

$$\begin{aligned} &\mathcal{P}_{2,11} = \mathrm{u} \otimes \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}), \quad \mathcal{P}_{2,12} = \mathrm{u} \otimes \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}), \\ &\mathcal{P}_{2,13} = \mathrm{t} \otimes \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}), \quad \mathcal{P}_{2,14} = \mathrm{t} \otimes \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes (\mathrm{u}+\mathrm{t}), \end{aligned}$$

$$\begin{aligned} &\mathcal{P}_{2,21} = \mathrm{u} \otimes \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes 1, \quad \mathcal{P}_{2,22} = \mathrm{u} \otimes \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes 1, \\ &\mathcal{P}_{2,23} = \mathrm{t} \otimes \mathrm{u} \otimes (\mathrm{u}+\mathrm{t}) \otimes 1, \quad \mathcal{P}_{2,24} = \mathrm{t} \otimes \mathrm{t} \otimes (\mathrm{u}+\mathrm{t}) \otimes 1. \end{aligned}$$

Equation (5.5) has one set of solutions, defining the 2^{4-1} fractional experiment, given by

$$\mathcal{P}_{1,1} = u \otimes u \otimes u \otimes (u+t), \quad \mathcal{P}_{1,2} = t \otimes u \otimes u \otimes (u+t),$$
$$\mathcal{P}_{1,3} = u \otimes u \otimes t \otimes (u+t), \quad \mathcal{P}_{1,4} = t \otimes u \otimes t \otimes (u+t),$$
$$\mathcal{P}_{1,5} = u \otimes t \otimes u \otimes (u+t), \quad \mathcal{P}_{1,6} = t \otimes t \otimes u \otimes (u+t),$$
$$\mathcal{P}_{1,7} = u \otimes t \otimes t \otimes (u+t), \quad \mathcal{P}_{1,8} = t \otimes t \otimes t \otimes (u+t).$$

□

5.3. Cyclic symmetry studies

In this section we consider the set product space $V = C \times L$ subject to the action (??) when G and H are the cyclic subgroups C_c and C_ℓ, respectively. There are $c\ell$ one-dimensional irreducible representations of $C_c \times C_\ell$ with projection matrices given by

$$(5.6) \qquad \mathcal{P}_{mn} = \frac{1}{c\ell} \sum_{i,j} \omega_c^{mi} \omega_\ell^{nj} (\rho_c^i \otimes \rho_\ell^j), \quad n = 1, \ldots, c, \quad m = 1, \ldots, \ell,$$

where $\omega_f = e^{2\pi i/f}$ and ρ_f is the permutation representation of the generating cyclic permutation (12...f). As commented earlier on in Chapter 3, it is important to distinguish the role of the field of scalars defining the vector space $\mathcal{V}$, where these projections operate on. Let us consider the case of C_3 first: The resulting canonical projections are given by

$$\mathcal{P}_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{P}_2 = \frac{1}{3}\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}, \quad \mathcal{P}_3 = \frac{1}{3}\begin{bmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix},$$

where $\omega = \omega_3 = e^{2\pi i/3}$. In general, these matrices are in $GL(\mathbb{C}^3)$, so that the resulting linear operations upon the vectors in $\mathcal{V}$ then require that $\mathcal{V}$ is regarded as a vector space over the field $\mathbb{C}$ of the complex numbers. When the field of interest is the real field (e.g., in applying Fisher-Cochran's argument), then the irreducible decomposition, now in $GL(\mathbb{R}^3)$, is $I = \mathcal{Q}_1 + \mathcal{Q}_2$, with $\mathcal{Q}_1 = \mathcal{P}_1$ of dim = 1 and

$$\mathcal{Q}_2 = \mathcal{P}_2 + \mathcal{P}_3 = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

of dim = 2. Note that $\mathcal{P}^2 = \mathcal{P}_3'$, so that

$$y'\mathcal{P}_2 y = (y'\mathcal{P}_2 y)' = y'\mathcal{P}_3 y,$$

leading to the total decomposition

$$y'y = y'\mathcal{Q}_1 y + y'\mathcal{Q}_2 y = y'\mathcal{P}_1 y + 2y'\mathcal{P}_2 y.$$

Similarly, with C_4, we would obtain the canonical decomposition $I = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$, with

$$\mathcal{Q}_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{Q}_2 = \frac{1}{4}\begin{bmatrix} 1 & \omega^2 & 1 & \omega^2 \\ \omega^2 & 1 & \omega^2 & 1 \\ 1 & \omega^2 & 1 & \omega^2 \\ \omega^2 & 1 & \omega^2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

both with dim = 1, and

$$\mathcal{Q}_3 = \mathcal{P}_3 + \mathcal{P}_4 = \frac{1}{4}\left(\begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega^3 & 1 & \omega & \omega^2 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega & \omega^2 & \omega^3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & \omega^3 & \omega^2 & \omega \\ \omega & 1 & \omega^3 & \omega^2 \\ \omega^2 & \omega & 1 & \omega^3 \\ \omega^3 & \omega^2 & \omega & 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix},$$

with dim = 2. Here (see also Exercise 5.10), $\mathcal{P}_3 = \mathcal{P}_4'$, and the decomposition of the sum of squares is

$$\mathrm{y}'\mathrm{y} = \mathrm{y}'\mathcal{Q}_1\mathrm{y} + \mathrm{y}'\mathcal{Q}_2\mathrm{y} + \mathrm{y}'\mathcal{Q}_3\mathrm{y} = \mathrm{y}'\mathcal{Q}_1\mathrm{y} + \mathrm{y}'\mathcal{Q}_2\mathrm{y} + 2\mathrm{y}'\mathcal{P}_3\mathrm{y}.$$

In the following example we consider the product action of $C_4 \times C_7$ to examine a data set subject to weekly and seasonal cycles.

EXAMPLE 5.9. The data shown in Table 5.3 are discussed in Wit and McCullagh (2001), where the authors propose the family of models

TABLE 5.3. Frequency of mining disasters between 1851-1962.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	*total*
Autumn	7	10	5	5	6	7	1	41
Winter	5	9	10	10	11	7	0	52
Spring	3	7	10	12	13	9	2	56
Summer	4	8	8	9	5	6	2	42
total	19	34	33	36	35	29	5	191

$$\pi_{ij} = \alpha_i\beta_j(1 + \lambda\cos\theta_i\cos\phi_j), \tag{5.7}$$

where θ, ϕ are ordered angles on $[0, 2\pi)$, α_i, β_j, and $\lambda \geq 0$ and $\sum_i \alpha_i = \sum_j \beta_j = 1$, to assess particular deviations from independence. The model is closed under cyclic relabeling, and, if two neighboring levels (in a cyclic sense) are merged, say i and i + 1 into level c, then the result is of the same functional form as equation (5.7), namely,

$$\begin{aligned} \pi_{cj} &= \pi_{ij} + \pi_{i+1,j} \\ &= \beta_j(\alpha_i + \alpha_{i+1})(1 + \lambda\cos\varphi_j\frac{\alpha_i\cos\theta_i + \alpha_{i+1}\cos\theta_{i+1}}{\alpha_i + \alpha_{i+1}}) \\ &= \beta_j\alpha_c(1 + \lambda\cos\phi_j\cos\theta_c), \end{aligned}$$

where $\alpha_c = \alpha_i + \alpha_{i+1}$. The angle θ_c is between θ_i and θ_{i+1} because the cosine is a continuous function. The mining disaster data can then be fitted to the two-way model (5.7) by means of maximum likelihood.

To decompose these data, we apply (5.6) with $\ell = 4$ and $c = 7$. There are 28 one-dimensional projections defined in $\mathcal{V}$ regarded as a complex vectors space, decomposing the total sum of squares, 1607. The results are summarized in the following matrix in which the (m,n) entry corresponds to the projection $\mathcal{P}_{mn}$.

$$
[\mathcal{P}_{mn}] = \left[\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 13.378 & 3.325 & 2.419 & 0.539 & 2.218 & 16.762 & 11.607 \\
2 & 2.784 & 1.165 & 0.014 & 0.014 & 1.165 & 2.784 & 0.321 \\
3 & 16.762 & 2.218 & 0.539 & 2.419 & 3.325 & 13.378 & 11.607 \\
4 & 68.450 & 23.760 & 5.510 & 5.510 & 23.760 & 68.450 & 1302.900
\end{array}\right].
$$

The following table shows the resulting 15 components of the irreducible (in $\mathbb{R}$) decomposition of the original (x'x) sums of squares and the transformed (u'u) sum of squares based on the multinomial vector $u' = \sqrt{v}(x - np)/\sqrt{n}$, as defined in Section 4.5, expression (4.2), and Proposition 3.8.

	m,n	m',n'	$x'(\mathcal{P}_{mn}+\mathcal{P}_{m'n'})x$	dim	$u'(\mathcal{P}_{mn}+\mathcal{P}_{m'n'})u$
weekly-quarterly	1,1	3,6	26.756	2	3.922
	1,2	3,5	6.650	2	0.975
	1,3	3,4	4.838	2	0.709
	1,4	3,3	1.078	2	0.157
	1,5	3,2	4.436	2	0.650
	1,6	3,1	33.524	2	4.914
	2,1	2,6	5.568	2	0.816
	2,2	2,5	2.330	2	0.341
	2,3	2,4	4.838	2	0.004
weekly	4,1	4,6	136.900	2	20.064
	4,2	4,5	47.520	2	6.963
	4,3	4,4	11.020	2	1.612
quarterly	1,7	3,7	23.214	2	3.403
	2,7		0.321	1	0.047
	4,7		1302.900	1	(0)
total			$x'x = 1607.000$	28 (27)	$u'u = 44.584$

Recall from Proposition 3.8 that the $\mathcal{L}(u'(\mathcal{P}_{mn} + \mathcal{P}_{m'n'})u)$ is approximately χ^2_2 relative to the *DC component*, so that the decompositions identify:

(1) the *DC component* $m = 4, n = 7$: note that under the multinomial transformation, as expected, this component is zero;
(2) the weekly cycle $n = 1$, with $u'(\mathcal{P}_{mn} + \mathcal{P}_{m'n'})u = 20.064$ and the weekly cycle $n = 2$, with $u'(\mathcal{P}_{mn} + \mathcal{P}_{m'n'})u = 6.963$, corresponding to angular phases $\theta = 2\pi/7$ and $\theta = 4\pi/7$, respectively;
(3) a suppressed, not significant, quarterly cycle m=1, with $u'(\mathcal{P}_{mn}+\mathcal{P}_{m'n'})u = 3.403$;
(4) a significant quarterly-weekly cycle $m = 1, n = 6$ (equivalently $m = 1, n = 1$), with $u'(\mathcal{P}_{mn} + \mathcal{P}_{m'n'})u = 4.9146$ and angular phase $\theta = 2\pi/28$.

□

Exercises

5.1. Show that the set of projections $\mathcal{Q}_1 = \mathcal{P}_{31} \otimes \mathcal{P}_{21}$, $\mathcal{Q}_2 = \mathcal{P}_{31} \otimes \mathcal{P}_{22}$, $\mathcal{Q}_3 = \mathcal{P}_{32} \otimes \mathcal{P}_{21}$, $\mathcal{Q}_4 = \mathcal{P}_{32} \otimes \mathcal{P}_{22}$ is isomorphic to the set $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ in Example 5.1, with the effect of interchanging rows with columns.

5.2. The set of projections $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ in Example 5.1 is a basis for the center C of the group algebra $\mathbb{C}(\rho^2 \otimes \rho^3)$. Describe the resulting 2^4 elements generated by this basis.

5.3. From equation (5.1) show that $\mathcal{P}_{mn} = \mathcal{P}_m \otimes \mathcal{P}_n$, where $\mathcal{P}_m$ and $\mathcal{P}_n$ are the associated projections in C and L, respectively.

5.4. Given a canonical decomposition $I_m = \mathcal{P}_1 + \mathcal{P}_2$ and the standard averaging decomposition $I_n = \mathcal{A} + \mathcal{A}^\perp$, show that

$$I_{mn} = \mathcal{P}_1 \otimes \mathcal{A} + \mathcal{P}_1 \otimes \mathcal{A}^\perp + \mathcal{P}_2 \otimes \mathcal{A} + \mathcal{P}_2 \otimes \mathcal{A}^\perp$$

is also a canonical decomposition. Give the dimensions of the component subspaces.

5.5. With the notation of Exercise 5.4, show that

(1) $\sum_i y'(\mathcal{P}_i \otimes \mathcal{A})y = \sum_i z'\mathcal{P}_i z$, where the j − th component of z_j of z is given by $\sqrt{y_j'\mathcal{A}y_j}$, for $j = 1, \ldots, n$;

(2) $\sum_i y'(\mathcal{P}_i \otimes \mathcal{A}^\perp)y = \sum_j y_i'\mathcal{A}^\perp y_j$.

5.6. Show that the 9-dimensional projection $\mathcal{P}_{22}$ of Example 5.4 decomposes into 9 one-dimensional (irreducible) projections, given by the pairwise tensor products of

$$\mathcal{P}_{221} = \frac{1}{8}\begin{bmatrix} 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 \\ 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 \end{bmatrix}, \quad \mathcal{P}_{222} = \frac{1}{8}\begin{bmatrix} 2 & 2 & -2 & -2 \\ 2 & 2 & -2 & -2 \\ -2 & -2 & 2 & 2 \\ -2 & -2 & 2 & 2 \end{bmatrix},$$

and

$$\mathcal{P}_{223} = \frac{1}{8}\begin{bmatrix} 2 & -2 & -2 & 2 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 2 & -2 & -2 & 2 \end{bmatrix}.$$

5.7. Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ of Example 5.3, define the bases $\beta_1 = \{e_1 + e_2\}$ and $\beta_2 = \{e_1 + e_2, -e_1 + e_2\}$, where $e_1' = (1, 0)$ and $e_2' = (0, 1)$. Let S_2 act on β_1 and β_2 regularly and construct the two one-dimensional irreducible representations of S_2. Recover its character table.

5.8. Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ of Example 5.5, define the bases $\beta_1 = \{e_1 + e_2 + e_3\}$, $\beta_2 = \{2e_1 - e_2 - e_3, -e_1 + 2e_2 - e_3\}$. Let S_3 act on β_1 and β_2 regularly and construct a one-dimensional and a two-dimensional irreducible representations of S_3. Recover its character table (hint: use the orthogonality relations for characters or add the one-dimensional sign (τ) representation).

5.9. Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ of Example 5.4, define the bases $\beta_1 = \{e_1+e_2+e_3+e_4\}$ and $\beta_2 = \{3e_1 - e_2 - e_3 - e_4, -e_1 + 3e_2 - e_3 - e_4, -e_1 + -e_2 - 3e_3 - e_4\}$. Let S_3 act on β_1 and β_2 regularly and construct a three-dimensional irreducible representations of

S_4. Add the two one-dimensional irreducibles. Find a two-dimensional irreducible. Recover its character table by adding the two one-dimensional irreducibles (the other three-dimensional is the tensor product of the sign (τ) with the first three-dimensional irreducible.

5.10. Show that the number of distinct irreducible representations of C_n into $GL_n(\mathbb{R})$ is $(n+2)/2$ when n is even, with two representations of dim $= 1$ and $(n-2)/2$ of dim $= 2$. When n is odd, there are $(n+1)/2$ irreducibles, with one representation of dim $= 1$ and $(n-1)/2$ of dim $= 2$.

5.11. With the notation and definitions of Example 5.6, with $p = 1$, show that $I = u + t$, $t^2 = t$, $u^2 = u$, $tu = ut = 0$.

5.12. Based on the results of Problem 5.11, show that

$$\{u \otimes u \otimes u,\ t \otimes u \otimes u,\ u \otimes u \otimes t,\ t \otimes u \otimes t,\ u \otimes t \otimes u,\ t \otimes t \otimes u,\ u \otimes t \otimes t,\ t \otimes t \otimes t\}$$

is a set of pairwise orthogonal projections (that is, for any two distinct matrices a and b in the set, we have $a^2 = a, b^2 = b, ab = ba = 0$).

5.13. Based on the results of Problem 5.11, show that

$$\{(u+t) \otimes u \otimes u,\ (u+t) \otimes u \otimes t,\ (u+t) \otimes t \otimes t,\ (u+t) \otimes t \otimes u\}$$

is a set of pairwise orthogonal projections.

5.14. For $k = 4$ (see Example 5.8), show that the set of solutions

$$\mathcal{P}_{p,1}, \mathcal{P}_{p,2}, \ldots, \mathcal{P}_{p,2^{k-p}}$$

to the (generating) equation $P_1 + \ldots + P_{2^{k-p}} = I$ constitutes a system of orthogonal projections, realized as tensors of rank 2^p, $p = 1, 2, 3$.

5.15. Show that the 2^p factorial design is defined by the p-fold tensor product of the permutation representation in S_2.

5.16. Consider the data, described in Cox and Snell (1989, p.87), from an experiment comparing a standard detergent (M) and a new product (X). The three factors are water softness, temperature and previous experience with product M. For each of the factor combinations, a number n of subjects indicate whether X is preferred to M. The following table shows the number x of positive responses and the number of subjects considered in each factor combination:

TABLE 5.4. Number of preferences for brand X and number of subjects.

Previous user of M		no	no	yes	yes
Temperature		low	high	low	high
Water softness: Hard	x	68	42	37	27
Hard	n	110	72	89	67
Medium	x	66	33	47	23
Medium	n	116	56	102	70
Soft	x	63	29	57	19
Soft	n	116	56	106	48

(1) Identify the set product structure underlying the data;
(2) Identify the symmetries of interest;
(3) Obtain the projections and corresponding sum of squares decompositions.

CHAPTER 6

Applications: geometric optics

6.1. Introduction

In this chapter we will discuss a symmetry study of a structure common in ophthalmic geometric optics, and certain invariance properties of probability laws defined on data indexed by these structures, that are consistent with the optic properties of interest. The random measurements of interest are discrete-valued curvature levels measured on the anterior surface of the human cornea. These measurements, indicated in the chapter by

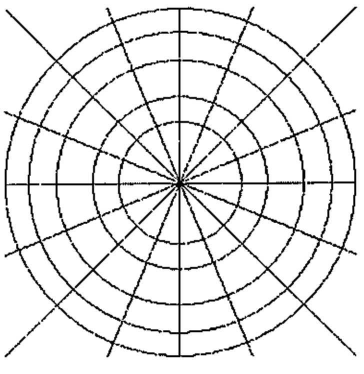
$$y = (y(1), y(2), \ldots, y(\ell)),$$

take values on a totally ordered, finite set, C. The data y are structured according to the space $V = C^L$ of all mappings from $L = \{1, \ldots, \ell\}$ to C. The symmetry of interest are location symmetries, resulting from the action $(\tau, y) \to y\tau^{-1}$ of S_ℓ on V. The case in which $C = \mathbb{R}$ is briefly discussed later on in the chapter. First we define the experimental conditions within which the structure obtains, the data associated with the structure, the symmetries of interest, and the restriction that these symmetries impose on probability models for these data.

6.2. Keratometry

Keratometry is the measurement of corneal curvature of a small area using a sample of four reflected points of light along an annulus 3 to 4 mm. in diameter, centered about the line of sight. For normal cornea this commonly approximates the apex of the cornea. The fundamental principle of computerized keratometry is similar in that the relative separation of reflected points of light along concentric rings are used to calculate the curvature of the measured surface. Using a pattern of concentric light-reflecting rings and sampling at specific circularly equidistant intervals, a numerical model for the curvature of the measured surface may be obtained. Sampling takes place at equally-spaced ring-semimeridian intersections, similar to those shown in Figure 6.1. In ophthalmic geometric optics it is of interest the determination of the angular variation between the extreme corneal curvature values along a given ring. The difference between the steep (maximum) and the flat (minimum) curvatures, as well as the angular variation between these extremes, are related to the amount of regular astigmatism present in the optics of the eye. The different curvatures of these various refractive surfaces diffuse light rays and interfere with a sharp formation of the image on the retina. The chart of concentric rings shown in Figure 6.1 represents the locations in the anterior surface of the cornea where these random curvature measurements, indicated by y in this chapter,

FIGURE 6.1. The concentric label structure V.



are obtained in a computerized corneal topography setting, e.g., Viana, Olkin and McMahon (1993), Klyce and Wilson (1989).

6.3. Astigmatic and stigmatic probability laws

The simplest geometrical representation of the corneal curvature surface, indexed by this structure, corresponds to a spherical-cylindrical surface with the location of steep and flat curvatures oriented with a 90 deg angular separation (Euler's theorem of classical defferential geometry). Typically, these curvatures vary according to approximating functions with argument

$$\alpha \cos^2 \frac{2\pi k}{\ell} + \beta \sin^2 \frac{2\pi k}{\ell}, \quad k = 0, \ldots, \ell - 1, \tag{6.1}$$

defined along ℓ equally-spaced points on a given ring, where the coefficients are determined by the geometry of the optical surface. This right-angle angular separation between the extreme curvatures is a characteristic of an optically astigmatic ($\alpha \neq \beta$) surface. As a consequence, we have:

DEFINITION 6.1 (Astigmatic property). The probability law $\mathcal{L}(y)$ of y satisfies the astigmatic property when the mean angular variation between two order statistics is 90 deg if and only if these are the extreme (flat and steep) order statistics.

In contrast, a spherical ($\alpha = \beta$) surface would lead to a constant mean curvature and, in particular, the mean angular variation between any two ordered curvatures should be a constant. This suggests:

DEFINITION 6.2 (Stigmatic property). The probability law $\mathcal{L}(y)$ of y satisfies the stigmatic property when the mean angular variation between any two order statistics is functionally independent of these order statistics.

Implicit in the above definitions is the understanding that, for finitely many (ℓ) points along any given ring in V, the mean angular variation is only approximately 90 deg. The limit mean angular variation, as $\ell \to \infty$, should reach the right-angle value. Because the components of y are in a totally ordered set, it is possible to order these components according to the specified total order relation ($\leq$) in C. We indicate by $\widetilde{y}$ the ordered version of y and write

$$\widetilde{\tau} = \{y \in V; y\tau^{-1} = \widetilde{y}\}$$

to indicate the set of all mappings (or vectors) y in V which are ordered by the permutation τ^{-1}, that is,

$$y(\tau^{-1}1) \leq y(\tau^{-1}2) \leq \ldots \leq y(\tau^{-1}\ell).$$

In particular, $\widetilde{1} = \{y \in V; y = \widetilde{y}\}$ is the set of all ordered mappings and, clearly, $\widetilde{\tau} = \widetilde{1}\tau^{-1}$, for all τ in S_ℓ.

Because of the random nature of y, the permutations involved in the ordering of the components of y are also random. We refer to these *random permutations* as ranking permutations. Note that when C is a finite set then $\widetilde{\tau}$ is obviously measurable with respect to $\mathcal{L}(y)$, for all $\tau \in S_\ell$.

DEFINITION 6.3. If for any two distinct permutations τ and σ we have $P(\widetilde{\tau} \cap \widetilde{\sigma}) = 0$ with respect to $P \equiv \mathcal{L}(y)$, we say that

$$V = \bigcup_{\tau \in S_\ell} \widetilde{\tau}$$

is a stochastically disjoint partition of V.

Clearly, if $y \in \widetilde{\tau} \cap \widetilde{\sigma}$ then both σ and τ are ranking permutations for y. The following proposition establishes a basic connection between the symmetries in the probability law $\mathcal{L}(y)$ of $y \in V$ (describing the observable curvatures) and the probability law $\mathcal{L}(\tau)$ of the ranking permutations $\tau \in S_\ell$.

PROPOSITION 6.1. If $\mathcal{L}(y) = \mathcal{L}(y\tau^{-1})$ for all $\tau \in S_\ell$, and $\bigcup_\tau \widetilde{\tau}$ is a stochastically disjoint partition of V, then $\mathcal{L}(\tau)$ is the uniform (Haar) probability law in S_ℓ.

PROOF. Under the stated assumptions, $P = \mathcal{L}(y)$ induces the law $\pi = \mathcal{L}(\tau)$ in S_ℓ given by $\pi(\tau) = P(\widetilde{\tau})$, so that

$$1 = \sum_{\tau \in S_\ell} P(\widetilde{\tau}) = \sum_{\tau \in S_\ell} (\widetilde{1}\tau) = \ell! P(\widetilde{1}),$$

and consequently

$$\pi(\tau) = P(\widetilde{\tau}) = P(\widetilde{1}) = \frac{1}{\ell!}$$

for all $\tau \in S_\ell$, that is, $\mathcal{L}(\tau)$ is uniform in S_ℓ. □

EXAMPLE 6.1. Consider the simplest case in which $C = \{a, b\}$, such as with binary-colored topography mappings, and $L = \{1, 2\}$, e.g., Section 2.7 of Chapter 2. Suppose also that $a \leq b$. The mapping space V has 4 points, namely,

$$V = \{aa,\ bb,\ ab,\ ba\},$$

and decomposes, in the natural way, according to the frames $\lambda = 20$ and $\lambda = 11$ (the two integer partitions of $\ell = 2$), that is

$$V = V_{20} \oplus V_{11},$$

and each of these components decomposes into isomorphic orbits,

$$V_{20} = \mathcal{O}_{11} \oplus \mathcal{O}_{12} = \{aa\} \oplus \{bb\}, \quad V_{11} = \mathcal{O}_{21} = \{ab,\ ba\}.$$

For each $\tau \in S_2 = \{1, (12)\}$, we apply the definition of

$$\widetilde{\tau} = \{y \in V;\ y(\tau^{-1}1) \leq y(\tau^{-1}2)\},$$

to obtain

$$\widetilde{1} = \{aa,\ bb,\ ab\}, \quad \widetilde{(12)} = \{aa, bb, ba\} = \widetilde{1}(12).$$

Next we introduce the probability laws $P \equiv \mathcal{L}(y)$ satisfying the permutation-symmetry $\mathcal{L}(y\tau^{-1}) = \mathcal{L}(y)$ for all τ in S_ℓ. As briefly outlined in Chapter 1, these laws have the form of convex linear combinations

$$P = f_{11}w_{11} + f_{12}w_{12} + f_{21}w_{21},$$

where

$$w_i = \begin{cases} \frac{1}{|\mathcal{O}_i|}, & y \in \mathcal{O}_i, \\ 0, & y \notin \mathcal{O}_i, \end{cases}$$

and $\sum_i f_i = 1$, $f_i \geq 0$, for $i \in \{11, 12, 21\}$. It then follows that

$$P(\widetilde{1}) = f_{11} + f_{12} + \frac{1}{2}f_{21} = P(\widetilde{(12)}),$$

and, because $f_{11} + f_{12} + f_{21} = 1$, the condition

$$P(\widetilde{1}) + P(\widetilde{(12)}) = 1$$

is equivalent to $f_{21} = 1$, or $f_{11} = f_{12} = 0$. In this case, P induces a well-defined probability law π in S_2, given by

$$\pi(\tau) = P(\widetilde{\tau}),$$

which is also invariant, and hence uniform. Here we see that when $\mathcal{L}(y)$ is S_2-invariant then the law $\mathcal{L}(\tau)$ of the ranking permutations τ in S_2 is uniform if and only if

$$V = \widetilde{1} \cup \widetilde{(12)}$$

is a stochastic partition. □

EXAMPLE 6.2. Consider the case $c = \ell = 3$, so that C has three *levels of gray*, say. For simplicity, write these levels symbolically as $\{1, 2, 3\}$ and suppose that the total order of C leads to $1 \leq 2 \leq 3$. We observe a map $y = (y(1), y(2), y(3))$. The space V decomposes according to the frames $\lambda \in \{300, 210, 111\}$, the three integer partitions of $\ell = 3$, as

$$V = V_{300} \oplus V_{210} \oplus V_{111},$$

whereas

$$V_{300} = \mathcal{O}_{11} \oplus \mathcal{O}_{12} \oplus \mathcal{O}_{13},$$

with each orbit carrying one single mapping,

$$V_{210} = \mathcal{O}_{21} \oplus \ldots \oplus \mathcal{O}_{26},$$

with three mappings in each orbit, and

$$V_{111} = \mathcal{O}_{31},$$

with six mappings in the orbit. Matrix 6.2 summarizes the orbits and classes $\tilde{\tilde{\tau}}$ generated by position-symmetries $(\tau, \mathrm{y}) \to \mathrm{y}\tau^{-1}$. In each column, the boxed labels indicate the mappings composing the corresponding class $\tilde{\tau}$.

(6.2)

$\mathcal{O}$	s(1)	s(2)	s(3)	label	τ : 1	(12)	(13)	(23)	(123)	(132)
11	1	1	1	1	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$
12	2	2	2	14	$\boxed{14}$	$\boxed{14}$	$\boxed{14}$	$\boxed{14}$	$\boxed{14}$	$\boxed{14}$
13	3	3	3	27	$\boxed{27}$	$\boxed{27}$	$\boxed{27}$	$\boxed{27}$	$\boxed{27}$	$\boxed{27}$
21	1	1	2	10	$\boxed{10}$	$\boxed{10}$	2	4	2	4
21	1	2	1	4	4	2	4	$\boxed{10}$	$\boxed{10}$	2
21	2	1	1	2	2	4	$\boxed{10}$	2	4	$\boxed{10}$
22	2	2	1	5	5	5	$\boxed{13}$	11	$\boxed{13}$	11
22	2	1	2	11	11	$\boxed{13}$	11	5	5	$\boxed{13}$
22	1	2	2	13	$\boxed{13}$	11	5	$\boxed{13}$	11	5
23	1	1	3	19	$\boxed{19}$	$\boxed{19}$	3	7	3	7
23	1	3	1	7	7	3	7	$\boxed{19}$	$\boxed{19}$	3
23	3	1	1	3	3	7	$\boxed{19}$	3	7	$\boxed{19}$
24	3	3	1	9	9	9	$\boxed{25}$	21	$\boxed{25}$	21
24	3	1	3	21	21	$\boxed{25}$	21	9	9	$\boxed{25}$
24	1	3	3	25	$\boxed{25}$	21	9	$\boxed{25}$	21	9
25	2	2	3	23	$\boxed{23}$	$\boxed{23}$	15	17	15	17
25	2	3	2	17	17	15	17	$\boxed{23}$	$\boxed{23}$	15
25	3	2	2	15	15	17	$\boxed{23}$	15	17	$\boxed{23}$
26	3	3	2	18	18	18	$\boxed{26}$	24	$\boxed{26}$	24
26	3	2	3	24	24	$\boxed{26}$	24	18	18	$\boxed{26}$
26	2	3	3	26	$\boxed{26}$	24	18	$\boxed{26}$	24	18
31	1	2	3	22	$\boxed{22}$	20	6	16	12	8
31	1	3	2	16	16	12	8	$\boxed{22}$	20	6
31	2	1	3	20	20	$\boxed{22}$	12	8	6	16
31	3	1	2	12	12	16	20	6	8	$\boxed{22}$
31	2	3	1	8	8	6	16	20	$\boxed{22}$	12
31	3	2	1	6	6	8	$\boxed{22}$	12	16	20

Note that within each one of the 10 orbits there is exactly one element from the set $\tilde{\mathrm{I}}$ of ordered elements of V, a fact which characterizes $\tilde{\mathrm{I}}$, and $\tilde{\tilde{\tau}}$ in general, as *cross sections* in V. More precisely, a subset $\Gamma \subset \mathrm{V}$ is a cross section if, for each $\mathrm{y} \in \mathrm{V}$, $\Gamma \cap \mathcal{O}(\mathrm{y})$ consists of exactly one point (see Eaton (1989, p.58) on conditions under which there is a stochastic representation of the form $\mathcal{L}(\mathrm{y}) = \mathcal{L}(\mathrm{x}\tau)$ for the law of y, where x is a random variable defined in $\tilde{\mathrm{I}}$ and independent of τ uniformly distributed in S_ℓ).

The invariant laws in V are convex combinations $P = \sum_i f_i w_i$, where

$$w_i = \begin{cases} \frac{1}{|\mathcal{O}_i|}, & y \in \mathcal{O}_i, \\ 0, & y \notin \mathcal{O}_i, \end{cases}$$

and $\sum_i f_i = 1$, $f_i \geq 0$, for $i = 1, \ldots, 10$. More precisely,

$$P = \sum_{i=1}^{3} f_{1i} w_{1i} + \sum_{j=1}^{6} f_{2j} w_{2j} + f_{31} w_{31},$$

where $w_{1i} = 1$ inside each orbit in $\widetilde{\lambda}_{300}$ and $w_{1i} = 0$ elsewhere; $w_{2j} = 1/3$ inside each orbit in $\widetilde{\lambda}_{210}$ and zero elsewhere; $w_{31} = 1/6$ inside the single orbit in $\widetilde{\lambda}_{111}$ and zero elsewhere, and

$$\sum_{i=1}^{3} f_{1i} + \sum_{j=1}^{6} f_{2j} + f_{31} = 1.$$

As a consequence, we obtain

$$P(\widetilde{\tau}) = \sum_{i=1}^{3} f_{1i} + \frac{1}{3} \sum_{j=1}^{6} f_{2j} + \frac{1}{6} f_{31}, \quad \text{for all } \tau \in S_3. \tag{6.3}$$

Similarly to the previous example, the condition $\sum_{\tau \in S_3} P(\widetilde{\tau}) = 1$ is obtained (and hence the law of the ranking permutations is uniform in S_3) if and only if

$$V = \bigcup_{\tau \in S_3} \widetilde{\tau}$$

is a stochastic partition. □

Expression (6.3) reflects the fact that the space V decomposes as the sum of three orbits of size 1, corresponding to frame $3^1 0^2$, six orbits of size 3 corresponding to frame $2^1 1^1 0^1$ and one orbit of size 6 corresponding to frame 1^3, so that $|V| = 27 = 3 \times 1 + 6 \times 3 + 1 \times 6$. Combinatorial results discussed earlier on in Chapter 3 show that, in general, there are n_λ orbits of size m_λ corresponding to frame λ, with $|V| = c^\ell = \sum_\lambda m_\lambda n_\lambda$, where

$$m_\lambda = \frac{\ell!}{(a_1!)^{m_1} (a_2!)^{m_2} \ldots (a_k!)^{m_k}}, \quad n_\lambda = \frac{c!}{m_1! m_2! \ldots m_k!},$$

so that (6.3) extends to

$$P(\widetilde{\tau}) = \sum_\lambda \frac{1}{m_\lambda} \sum_{j=1}^{n_\lambda} f_{j\lambda},$$

where λ varies over the (m) different frames $\lambda = a_1^{m_1} \ldots a_k^{m_k}$, with $m_1 a_1 + \ldots + m_k a_k = \ell$ and $m_1 + \ldots + m_k = c$. This leads to

PROPOSITION 6.2. Let $y \in C^L$ for a finite totally ordered set C and $\mathcal{L}(y)$ be S_ℓ-invariant. Then the law $\mathcal{L}(\tau)$ of the ranking permutations τ in S_ℓ is uniform if and only if

$$V = \bigcup_{\tau \in S_\ell} \widetilde{\tau}$$

is a stochastic partition.

6.4. A constructive principle for astigmatic probability laws

Propositions 6.1 and 6.2 suggest that the assumption of permutation invariance (as described by the symmetries of the entire group S_ℓ) will, in general, lead to stigmatic laws. This was apparent earlier on when discussing expression (6.5). In this section, we outline the construction of astigmatic laws for a finite-valued random variable. The construction extends to the case $c = \infty$ by defining two points y and x in C^L as similar whenever they share the same frame of repeated symbols, that is, $\lambda(y) = \lambda(x)$.

We will consider a simple case in which y has $\ell = 8$ components and $C = \{a, b, c\}$, with the ordered relation $a \leq b \leq c$. The only assumption here is that C is a totally ordered (finite) set. This is the case when dealing with topographic maps colored by shades of gray, say. The case of numerically-valued data is discussed at the end of this section. The label space V has $3^8 = 6561$ points and, under the full set (S_8) of permutation symmetries, decomposes according to $|V| = \sum_\lambda m_\lambda n_\lambda$, with the corresponding components given by Table 6.1. We want to define a smaller

TABLE 6.1. Decomposition of $V = C^L$, with $c = 3$and $\ell = 8$, under the full set of permutation symmetries (S_8). There are n_λ orbits of size m_λ corresponding to frame λ.

λ	m_λ	n_λ	$m_\lambda n_\lambda$
800	1	3	3
710	8	6	48
620	28	6	168
611	56	3	168
530	56	6	336
521	168	6	1008
440	70	3	210
431	280	6	1680
422	420	3	1260
332	560	3	1680
total			6561

subset $G \subseteq S_8$ of permutation symmetries which is consistent with the astigmatic property. The natural candidate, consistently with (6.1), is one which leaves invariant the components of y when these components vary according to the sequence (6.4)

$$y(1) = a, \;\; y(2) = b, \;\; y(3) = c, \;\; y(4) = b, \;\; y(5) = a, \;\; y(6) = b, \;\; y(7) = c, \;\; y(8) = b,$$

along one of the rings as illustrated in Figure 6.1, starting at one of the horizontal semimeridians, say. In this case, the flat (a) and the steep (c) curvatures are displaced by right angles with the intermediate curvature value (b) assigned to the other 4 semimeridians. The set of symmetries associated with (6.4) defines a commutative group, indicated here by $G \subseteq S_8$, with generators the permutations

$h = (2468)$, $t = (15)$ and $v = (37)$. The order of G is 16, with elements

$$G = \{1, t, v, h, h^2, h^3, tv, th, th^2, th^3, vh, vh^2, vh^3, thv, th^2v, th^3v\},$$

isomorphic to the product group $C_4 \times C_2 \times C_2$, where C_n indicates the cyclic group of order n. Table 6.2 shows the proposed sets of generators for a selected number of partitions. As an example of an invariant probability law for y consider the

TABLE 6.2. Symmetry generators for selected number of partitions (ℓ).

partitions	|G|	generators
3	2	(12)
4	4	(13),(24)
5	4	(14),(23)
6	8	(1245),(36)
7	8	(16),(25),(34)
8	16	(2468),(15),(37)
9	16	(18),(27),(45),(36)

following points $y \in V$,

$$\begin{aligned} y_{11} &= (a, b, c, b, a, b, c, b), \\ y_{21} &= (a, c, c, c, a, c, c, c), \\ y_{22} &= (a, a, c, a, a, a, c, a), \\ y_{31} &= (a, a, a, a, a, a, a, a), \\ y_{32} &= (b, b, b, b, b, b, b, b), \\ y_{33} &= (c, c, c, c, c, c, c, c), \end{aligned}$$

each of which is fixed by all permutations in G, so that these points define single-element orbits in V. Next, assign G-invariant probabilities to $P = \mathcal{L}(y)$ according to convex combinations $P = \sum_i f_i w_i$, where

$$w_i = \begin{cases} \frac{1}{|\mathcal{O}_i|}, & y \in \mathcal{O}_i, \\ 0, & y \notin \mathcal{O}_i, \end{cases}$$

and $\sum_i f_i = 1$, $f_i \geq 0$, for $i \in \{11, 21, 22, 31, 32, 33\}$. Note that each orbit $\mathcal{O}_i$ consists of exactly one element. For example, take

$$f_i = \begin{cases} \frac{16}{20}, & i \in \{11\}, \\ \frac{3}{40}, & i \in \{21, 22\}, \\ \frac{1}{60}, & i \in \{31, 32, 33\}. \end{cases}$$

Direct computation then shows that the mean natural angular variation between flat and steep curvatures, under $\mathcal{L}(y)$, is $0.93\pi/2$, with a standard deviation of $0.178\pi/2$. In contrast, the mean natural angular variation between flat and "next to flat" curvatures is $0.5375\pi/2$, thus reflecting the approximately astigmatic property of $\mathcal{L}(y)$. In the next section we discuss the construction of astigmatic laws for random curvatures with values in $\mathbb{R}$.

6.5. Real-valued astigmatic laws

Let $C = \mathbb{R}$, so that C^L is a (real) vector space and the usual operations with vectors are defined. In particular, the vector of means $\mu = E(y) \in V$ of y and the covariance matrix Σ of y, based on $\mathcal{L}(y)$ are well-defined. We retain the same set G of symmetries defined in the previous section. Of interest now is the invariant parametric structure to be imposed into μ and Σ, consistent with the astigmatic property. These symmetry conditions are obtained, in general, by linearly representing the symmetries of interest, that is, through the permutation representation

$$\tau \rightarrow \rho(\tau)$$

of G into $GL(8, \mathbb{R})$ and calculating the first canonical projection

$$\mathcal{P}_1 = \frac{1}{16}\sum_{\tau \in G} \rho(\tau) = \frac{1}{16}\begin{bmatrix} 8 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 8 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix},$$

according to Theorem 3.8 of Chapter 3. The equation $\mathcal{P}_1\mu = \mu$ then implies that μ should have the form

$$\mu = (\mu_a, \mu_b, \mu_c, \mu_b, \mu_a, \mu_b, \mu_c, \mu_b)$$

with the additional astigmatic condition

$$\mu_a < \mu_b < \mu_c$$

imposed. Similarly, to determine the pattern of the parametric structure of the covariance matrix Σ, we make use of the fact that Σ has the symmetry of $G \subseteq S_\ell$ if and only if Σ commutes will all the permutation matrices representing G. Equivalently, Σ should satisfy the condition

$$\Sigma = \sum_{\tau \in G} \rho(\tau)\Sigma\rho(\tau)'.$$

The solution is the class of all covariance matrices Σ patterned according to

$$\Sigma = \begin{bmatrix} A & B & C & B & D & B & C & B \\ B & E & F & G & B & J & F & G \\ C & F & H & F & C & F & I & F \\ B & G & F & E & B & G & F & J \\ D & B & C & B & A & B & C & B \\ B & J & F & G & B & E & F & G \\ C & F & I & F & C & F & H & F \\ B & G & F & J & B & G & F & E \end{bmatrix},$$

described by at most 10 distinct covariance parameters $\{A, B, \ldots, J\}$.

6.6. The uniform mean angular variation

We observe the data $y = (y(1), \ldots, y(\ell))$ according to a probability law $\mathcal{L}(y)$ and assume that $\bigcup_{\tau \in S_\ell} \widetilde{\tau}$ is a stochastically disjoint partition of V. Let τ be the ranking permutation in S_ℓ ordering the components of y, that is, $y(\tau 1) \leq y(\tau 2) \leq \ldots \leq y(\tau \ell)$, so that $\tau 1$ is the location of the flat (minimum) curvature and $\tau \ell$ the location of the steep (maximum) curvature. The angular displacement between the extreme curvatures may be defined as follows: fix a *north pole* $m \in \{1, 2, \ldots, \ell\}$ and a walking direction $\sigma = (12 \ldots \ell)$ in S_ℓ for counting the steps along V. If, in the direction determined by σ, it takes n steps to walk from the flat curvature to the north pole and N steps to walk from the steep curvature to the north pole, we express this by writing $\sigma^n \tau 1 = m$ and $\sigma^N \tau \ell = m$. Consequently $n - N = \tau 1 - \tau \ell$, in the direction given by σ and $n - M = \tau \ell - \tau 1$, in the direction defined by σ^{-1}. This justifies the definition of the angular variation $\alpha(\tau)$ between the extreme values of y as

$$\alpha_{\ell 1}(\tau) = |\tau \ell - \tau 1| \frac{2}{\ell} \pi.$$

Proposition 6.1 suggests that we consider the first moments of the angular variation under uniformly distributed ranking permutations. More precisely:

PROPOSITION 6.3. Under the uniform probability law for $\tau \in S_\ell$, we have

$$E(\alpha_{\ell 1}) = \frac{2}{3} \frac{\ell + 1}{\ell} \pi \xrightarrow{\ell \to \infty} \frac{2}{3} \pi, \quad \mathrm{Var}(\alpha_{\ell 1}) = \frac{2}{9} \frac{(\ell+1)(\ell-2)}{\ell^2} \pi^2 \xrightarrow{\ell \to \infty} \frac{2}{3} \pi^2.$$

PROOF. Let $\phi(\tau) = |\tau \ell - \tau 1|$, so that $\phi^{-1}(1) \cup \ldots \cup \phi^{-1}(\ell - 1)$ forms a disjoint partition of S_ℓ and, under the uniform law in S_ℓ,

$$E(\alpha_{\ell 1}(\tau)) = \sum_{k=1}^{\ell-1} \frac{2k\pi}{\ell} |\phi^{-1}(k)| \frac{1}{\ell!},$$

where $|\phi^{-1}(k)|$ is the number of permutations $\tau \in S_\ell$ such that $\phi(\tau) = k$. From the fact that $|\phi^{-1}(k)| = 2(\ell - k)[(\ell - 2)!]$, direct computation then shows that

$$E(\alpha_{\ell 1}(\tau)) = \frac{2}{3} \frac{\ell + 1}{\ell} \pi, \quad E(\alpha_{\ell 1}^2(\tau)) = \frac{2}{3} \frac{\ell + 1}{\ell} \pi^2,$$

from which the proposed results follow. □

Similarly, we define the *natural* angular variation between the extreme values of y as

$$a_{\ell 1}(\tau) = \min \{\alpha_{\ell 1}(\tau), \ell - \alpha_{\ell 1}(\tau)\},$$

so that $0 \leq a(\tau) \leq \pi$. Let $f(\tau) = \min \{|\tau \ell - \tau 1|, \ell - |\tau \ell - \tau 1|\}$. It then follows that for even values of $\ell \geq 3$, the number $|f^{-1}(k)|$ of permutations in S_ℓ satisfying $f(\tau) = k$ is given by

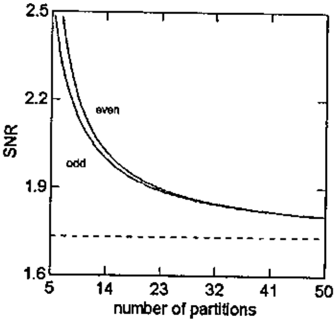
$$|f^{-1}(k)| = \begin{cases} 2\ell[(\ell-2)!], & k = 1, \ldots, \frac{\ell}{2} - 1, \\ \ell[(\ell-2)!], & k = \frac{\ell}{2}, \end{cases}$$

whereas, for odd values of $\ell \geq 3$,

$$|f^{-1}(k)| = 2\ell[(\ell - 2)!], \quad k = 1, \ldots, \frac{\ell - 1}{2}.$$

With proof similar to that of Proposition 6.3, we obtain the first moments for the natural angular variation:

FIGURE 6.2. Signal-to-noise ratio (SNR) for the uniform natural angular variation, as a function of the number ℓ of partitions.



PROPOSITION 6.4. Under the uniform probability law for $\tau \in S_\ell$, for odd $\ell \geq 3$, we have

$$E(a_{\ell 1}) = \frac{1}{2}\frac{\ell+1}{\ell}\pi \xrightarrow{\ell\to\infty} \frac{1}{2}\pi, \quad \mathrm{Var}(a_{\ell 1}) = \frac{1}{12}\frac{(\ell+1)(\ell-3)}{\ell^2}\pi^2 \xrightarrow{\ell\to\infty} \frac{1}{12}\pi^2,$$

and, for even $\ell \geq 3$, we have

$$E(a_{\ell 1}) = \frac{1}{2}\frac{\ell}{\ell-1}\pi \xrightarrow{\ell\to\infty} \frac{1}{2}\pi, \quad \mathrm{Var}(a_{\ell 1}) = \frac{1}{12}\frac{\ell^3-4\ell^2+8\ell^2-8}{\ell(\ell-1)^2}\pi^2 \xrightarrow{\ell\to\infty} \frac{1}{12}\pi^2.$$

Figure 6.2 shows the signal-to-noise ratio (inverse of the coefficient of variation) for the uniform natural angular variation, as a function of the number ℓ of partitions in the ring V. For a large number ℓ of partitions, the CV is of the order of $\sqrt{3}/3$.

The key observation to Propositions 6.3 and 6.4 is that the proofs do not depend on which two order statistics are involved in the definition of $\phi(\tau)$ or of $f(\tau)$. In fact, the same results would be obtained for the mean angular variation between the smallest and next-to-smallest values of y, say. More precisely, for all $\sigma \in S_\ell$ fixed,

$$\lim_{\ell\to\infty}(E[\phi(\tau)] - E[\phi(\tau\sigma)]) = \lim_{\ell\to\infty}(E[f(\tau)] - E[f(\tau\sigma)]) = 0.$$

As a consequence, we learn that as long as the probability law $\mathcal{L}(\tau)$ for the ranking permutations τ in S_ℓ is uniform, the resulting probability law for the angular variation preserves the stigmatic property of $\mathcal{L}(y)$. In summary,

PROPOSITION 6.5. If the probability law $\mathcal{L}(\tau)$ of τ is uniform in S_ℓ, then the probability law $\mathcal{L}(y)$ of y satisfies the stigmatic property.

From Proposition 6.1 we then obtain

PROPOSITION 6.6. If $\mathcal{L}(y) = \mathcal{L}(y\tau^{-1})$ for all $\tau \in S_\ell$, and $\bigcup_\tau \tilde{\tau}$ is a stochastically disjoint partition of V then $\mathcal{L}(y)$ satisfies the stigmatic property.

For example, if $\mathcal{L}(y)$ is multivariate normal with mean (μ) and covariance matrix (Σ) which are invariant under the symmetry of S_ℓ, then the law of y is consistent with the stigmatic property. More precisely, if

$$\frac{1}{\ell!}\sum_{\tau\in S_\ell}\rho(\tau)\mu=\mu,\quad\text{and}\quad\frac{1}{\ell!}\sum_{\tau\in S_\ell}\rho(\tau)\Sigma\rho(\tau)'=\Sigma,\tag{6.5}$$

where $\rho(\tau)$ is the linear (permutation) representation of S_ℓ, then $\mathcal{L}(y)$ satisfies the stigmatic property. As a consequence, the construction of μ and Σ that is consistent with the astigmatic property will depend on defining a smaller subset $G\subseteq S_\ell$ of symmetries. This construction was described earlier on in Section 6.4.

Further reading

(1) The basic statistical aspects of corneal curvature data are discussed in e.g., Viana et al. (1993);
(2) The early work of Votaw (1948) and Wilks (1946) on special patterns of symmetry on covariance structures, also Olkin and Viana (1995);
(3) Group symmetry covariance models, e.g., Perlman (1987). See also the work of Gao and Marden (2001) where the argument of testing for certain patterns of symmetry by averaging over a class of permutation matrices is applied. In particular, the reader may consult Diaconis (1990) for specific aspects of group invariance applied to the characterization of certain patterned matrices;
(4) The algebraic formulation of visual perception e.g., Hoffman (1966).

Exercises

6.1. Show that the factor $|\tau\ell-\tau 1|$ present in the angular variation can be expressed as $d'\rho(\tau)r$, where $d'=(-1,0,\ldots,0,1)$, $r'=(1,2,\ldots,\ell)$ and ρ is the linear (permutation) representation of S_ℓ, and that, consequently,

$$|\tau\ell-\tau 1|^2=d'\rho(\tau)(rr')\rho(\tau)'d.$$

6.2. Following Problem 6.1, show that the derivation of the (uniform) mean squared angular variation can be obtained from the fact, (e.g., Chapter 3 or Viana (2001)), that

$$\frac{1}{\ell!}\sum_{\tau\in S_\ell}\rho(\tau)H\rho(\tau)'=v_0\frac{ee'}{n}+v_1(I-\frac{ee'}{n}),$$

where $E'=(1,\ldots,1)$ with ℓ components, $v_0=e'He/n$ is the sum of the components of H, and $v_0+(n-1)v_1=\text{tr } H$, for any given (real or complex) matrix H of dimension ℓ (in this case $H=rr'$).

6.3. Average linear ranks. The derivation of the mean angular variation between flat and steep curvatures of Section 6.4 makes use of the usual averaging of ranks. The notion of linear representation of order statistics is useful to describe the averaging process and linearly represent the derivation of the corresponding ranks. Here is the outline: Note from Matrix 6.2 that the map $y=(1,1,2)$ belongs to both $\widetilde{1}$ and $\widetilde{(12)}$. To indicate this, let $\widetilde{y}=\{\tau;y\in\widetilde{\tau}\}$. Show that the mean linear rank R(y) of y, under a uniform law in S_ℓ, is given by $R(y)=\frac{1}{|\widetilde{y}|}\sum_{\widetilde{y}}\rho(\tau)'r$, where

$r' = (1, 2, \ldots, \ell)$ and ρ is the permutation representation of S_ℓ. For $y = (1,1,2)$, show that $\tilde{y} = \{1, (12)\}$ and consequently

$$\frac{1}{2}[\rho(1)' + \rho((12))'] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix}.$$

For $y = (1,2,1)$, show that $\tilde{y} = \{(23), (132)\}$, and

$$\frac{1}{2}[\rho((23))' + \rho((132))'] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \\ 1.5 \end{bmatrix}.$$

6.4. Evaluate the orbits for the letter-symmetry action $(\sigma, s) \to \sigma s$ of S_3 on the space V of three-sequences in length of three, described by the matrix below and Example 6.2:

s(1)	s(2)	s(3)	label	σ : 1	(12)	(13)	(23)	(123)	(132)
1	1	1	1	1	14	27	1	14	27
2	2	2	14	14	1	14	27	27	1
3	3	3	27	27	27	1	14	1	14
1	1	2	10	10	5	18	19	23	9
1	2	1	4	4	11	24	7	17	21
2	1	1	2	2	13	26	3	15	25
2	2	1	5	5	10	23	9	18	19
2	1	2	11	11	4	17	21	24	7
1	2	2	13	13	2	15	25	26	3
1	1	3	19	19	23	9	10	5	18
1	3	1	7	7	17	21	4	11	24
3	1	1	3	3	15	25	2	13	26
3	3	1	9	9	18	19	5	10	23
3	1	3	21	21	24	7	11	4	17
1	3	3	25	25	26	3	13	2	15
2	2	3	23	23	19	5	18	9	10
2	3	2	17	17	7	11	24	21	4
3	2	2	15	15	3	13	26	25	2
3	3	2	18	18	9	10	23	19	5
3	2	3	24	24	21	4	17	7	11
2	3	3	26	26	25	2	15	3	13
1	2	3	22	22	20	6	16	8	12
1	3	2	16	16	8	12	22	20	6
2	1	3	20	20	22	8	12	6	16
3	1	2	12	12	6	16	20	22	8
2	3	1	8	8	16	20	6	12	22
3	2	1	6	6	12	22	8	16	20

List of symbols

- V a structured, a finite set of labels or indices
- $\tau, \sigma, \ldots$ symmetry transformations
- G a group of symmetries
- S_ℓ the group of permutations on $\{1, 2, \ldots, \ell\}$
- S_V the group of permutations on V
- C_ℓ the cyclic group of order ℓ
- M_ℓ the group of $\ell \times \ell$ permutation matrices
- $\varphi_1, \varphi_2, \ldots$ group actions
- $\mathcal{O}_1, \mathcal{O}_2, \ldots$ orbits
- ρ_τ or $\rho(\tau)$ the representation ρ evaluated at $\tau \in G$
- $|G|$ the number of elements in G
- C^L the set of mappings from defined in L with values in C
- $\mathcal{F}$ the vector space of scalar-valued functions (measurements) x
- $\mathcal{V}$ a linear subspace of $\mathbb{R}^V$
- $\mathcal{P}$ projections
- $\mathcal{L}(\mathcal{V})$ linear mappings defined on the vector space $\mathcal{V}$
- $GL(\mathcal{V})$ invertible linear mappings defined on $\mathcal{V}$
- $\mathcal{G}$ scalar-valued functions defined on G
- $\mathcal{C}$ class functions
- $\mathcal{B}$ basis for $\mathcal{V}$
- ee' or J the matrix with all entries equal to 1- when needed, the dimension of the matrix is indicated, e.g., J_3

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SYMMETRY STUDIES- AN INTRODUCTION

*

***** CORRIGENDA *****

MARLOS VIANA

(1) p. 1, Example 1.1: the alphabet should be $\{A, G, C, U\}$;
(2) p. 2: replace *melecular constructs* by **molecular constructs**;
(3) p. 9: The first citation *Under the Fermi-Dirac (FD) ...* should read **Under the Bose-Einstein (BE) ...**... Thus, in the **BE** model... would be consistent with the assumptions of the **BE** model. The final citation *The Bose-Einstein ...* should read **The Fermi-Dirac (FD)**;
(4) p. 10: Rows 2 and 3, column 1, of the summary table should be interchanged;
(5) p. 12: Rows 3 and 4, column 1, of the summary table should be interchanged;
(6) p. 12: Last line *Ttypically* should read **Typically**;
(7) p. 12: the notation should be SO(3) instead of $O(3, \mathbb{R})$ or $S(3, \mathbb{R})$;
(8) p. 13: Example 1.7 should read *four-sequences in length of three...*;
(9) p. 22: line 6 should read $V_{31} = \mathcal{O}_1 \cup \mathcal{O}_3$;
(10) p. 25, line 3: $\mathcal{P}_2 x = \ldots = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$;
(11) p. 29: The interpretation of these orbits *is* ...
(12) p. 32, Matrix 2.10 row (ad) should read: $(ad)|4\ 2\ 3\ 1|\ldots$ and row (cd) should read: $(cb)|1\ 3\ 2\ 4|\ldots$;
(13) p. 35: replace *sequence* by **sequel**;
(14) p. 36, line 1: The number ... in the diagram *is* 3;
(15) p. 37, Example 2.8: Selecting y according ... to reaching the GCD *into* a random variable ...;
(16) p. 38, line 1: so that 1 is the GCD of (24, *67*);
(17) p. 44: Bottom of page- Replace ... first applying τ followed by σ, e.g., $(13)*(23) = (123)$ by ... first applying σ followed by τ, e.g., $(23)*(13) = (123)$;
(18) p. 74, Expression (3.34): replace $(\chi_{\text{reg}}, \chi_\rho)$ by $(\chi_{\text{reg}} \mid \chi_\rho)$;
(19) p. 76: replace *as shown earlier in Example 3.30* by **as shown earlier in Example 3.36**;
(20) p. 77: should read $v = \dim \mathcal{V} = \sum_{i=1}^{h} n_i$;
(21) p. 79: ...so that $\dim \mathbb{R}^v = 64 = 4 \times 1 + 12 \times 3 + 4 \times 6$;
(22) p. 82, line 1: replace Q^{λ_2} by Q^{λ_3};
(23) p. 86, line 4, should read: which takes $x \in \mathcal{F}$ into $\tau^*(x) \in \mathcal{F}$ given by ...;
(24) p. 98, Problem 3.12: replace (fix (1) + fix (2)/2 = 1 by [fix (1) + fix (2)]/2 = 1;
(25) p. 98, Problem 3.13: replace $(\rho_\tau x \mid \rho_t auy)$ by $(\rho_\tau x \mid \rho_\tau y)$;
(26) p. 111, just preceeding Prop. 4.1 should read: We *then* have:;
(27) p. 112 following Matrix 4.3 should read: Figure 4.8 *shows ... CCT along the BRU isolate*;

Date: October 10, 2004.

(28) p. 112, last paragraph should read: Similar analysis can be **derived for** the partition $\lambda = \mathbf{1110}$;
(29) p. 131, Further Reading (2): replace *nuclotide* by **nucleotide**;
(30) p. 132, Problem 4.3: replace *in length of four* by **in length of three**;
(31) p. 135: replace *In this chapter we consider data that are undexed...* by **In this chapter we consider data that are indexed...**;
(32) p. 154: first line of section 5.3 should read: $V = C \times L$ subject to the action of the product group $G \times H$ when G and H are ...;
(33) p. 157, Problem 5.5: replace *component of* z_j of z by **component** z_j of z;
(34) p. 157, Problems 5.7 and 5.9: replace *regularly* by **naturally**;
(35) p. 157, Problem 5.9: replace *representations* by **representation**;
(36) p. 161: replace *totally ordered, finite set,* C, by **totally ordered, finite, set** C;
(37) p. 172, Problem 6.2: replace $E' = (1, \ldots, 1)$ by $e' = (1, \ldots, 1)$;
(38) p. 175: replace *V a structured* by **V a structure**;
(39) p. 175: replace C^L *the set of mappings from defined in ...* by C^L **the set of mappings defined in ...**;