Statistical Analysis of Non-Uniformly Expanding Dynamical Systems

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Introduction

In general terms, Dynamical Systems theory has two major goals: to describe the typical behavior of trajectories, specially as time goes to infinity; to understand how this behavior changes when the system is modified, and to what extent it is stable under small perturbations. Even in cases of very simple transformations the orbits may have a rather complicated behavior. Moreover, systems may display sensitivity on the initial conditions, i.e. a small variation on the initial point gives rise to a completely different behavior of its orbit. Among many others, these are obstacles that we find when we try to predict the long-term behavior of a system and its stability.

Related to the first goal we have mentioned are the *Sinai-Ruelle-Bowen (SRB)* or *physical measures* which characterize asymptotically, in time average, a large set of orbits of the phase space. These SRB measures may be understood as equilibrium states for a probabilistic description of the system. Significant information on the dynamical properties of a system is given by the *correlation decay* of an SRB measure, which in particular tells the velocity at which the equilibrium is reached. Connected to this is the *statistical stability* of a system, which means continuous variation of the SRB measures under small modifications of the law that governs the system; this naturally points in the direction of the second goal, as does the *stochastic stability* of a system. Briefly, this may be understood as the characterization of the stability of the statistical properties of the system when small random errors are incorporated in measurements along the way.

Systems displaying uniformly expanding/contracting behavior on Riemannian manifolds have been exhaustively studied in the last three decades, and several results on their statistical properties have been obtained, starting with Sinai, Ruelle and Bowen; see [**Si72, Si68, Ru, BR, Bo70, Bo75**] and also [**Ki86, Ki88, Yo86**]. The study of systems exhibiting expansion only in asymptotic terms has been done in the pioneer work of Jakobson [**Ja**], where the existence of SRB measures for many quadratic transformations of the interval is established; see also [**BC85, BY92**]. In addition, decay of correlations and stochastic stability for this kind of systems have been obtained in several significant works by Baladi, Benedicks, Carleson, Viana and Young [**BC85, BaY, BaV, BY92**]. Moreover, related to this is the remarkable work of Benedicks and Carleson [**BC91**] for Hénon two dimensional maps exhibiting strange attractors; see also [**MV, Vi1, BY93, BY00, BeV01, BeV02**]. Motivated by the results for multidimensional non-uniformly expanding systems in [**Vi2, Al00**], general conclusions on the existence of SRB measures for systems exhibiting non-uniformly expanding behavior are drawn in [**ABV**]; see [**Car, BoV**] as well. Subsequent works gave rise to several results on the statistical properties of those SRB measures, for instance [**AV, AA1, Cas, ALP3, Ol, Va**].

The aim of this text is to present recent developments on the study of the statistical behavior of non-uniformly expanding transformations in Riemannian compact manifolds in finite dimension, thus unifying some of the results mentioned above and presenting a global theory for this sort of transformations. Our analysis will be focused on SRB measures, decay of correlations, statistical and stochastic stability. Readers should be acquainted with concepts from Measure Theory, Integration and Ergodic Theory such as measure spaces, L^p spaces, absolute continuity, weak* convergence, invariant measures and ergodicity. For the sake of completeness we include a preliminary chapter where we present all these concepts, and in appendices we guide an overview on functions of bounded variation in higher dimensions, and on the statistical properties of Markov towers.

In Chapter 1 we introduce the notion of non-uniformly expanding map, in the sense of [**ABV**]. This comprises the case of maps with (non-degenerate) critical sets whose orbits have slow recurrence to the critical set. In Section 1.2 we study a few examples of non-uniformly expanding maps defined in higher dimensional manifolds: the class of local diffeomorphisms introduced in [**ABV**], and the class of transformations (with critical) sets introduced in [**Vi2**]. Following [**AAS**], in Section 1.3 we show that if non-uniform expansion condition is verified on a set of total probability, then the transformation must be uniformly expanding.

A powerful tool for the study of ergodic properties of non-uniformly expanding maps has been introduced in [**Al00**] through the notion of hyperbolic times. This concept, that we define in Chapter 2, has been put into an abstract setting in [**ABV**], and plays a key role in a large part of the theory we explain here. Besides, we believe that some properties of hyperbolic times have also a mathematical interest on their own. In the first sections of Chapter 2 we present the two main features of hyperbolic times: namely, the local control of distortion in a uniform way (not depending on the point nor on the iterate), and their existence with positive frequency for non-uniformly expanding maps. In Section 2.3 we prove that if the first hyperbolic time map is integrable with respect to the Lebesgue measure, then there exists an absolutely continuous invariant measure. Finally, in Section 2.4 we disclose some results from [**AA2**] connecting the positive frequency of hyperbolic times to the integrability of the first hyperbolic time map. In particular, we study an example which evinces that the integrability of the first hyperbolic time map is not a consequence of an almost everywhere positive frequency of hyperbolic times. On the opposite direction we point out that if the integrability is "sufficiently strong", then there is some "large portion" of the phase space whose points have positive frequency of hyperbolic times.

Sinai-Ruelle-Bowen (SRB) measures are defined in Chapter 3. In Section 3.1 we give results from **[ABV]** indicating that the phase space of a non-uniformly expanding system is covered (up to a zero Lebesgue measure subset of points) by the basins of finitely many SRB measures. Using the notion of functions of bounded variation in general dimension, in Section 3.2 we prove the existence of absolutely continuous invariant measures for piecewise expanding maps in higher dimensions **[GB89, Al00]**; and moreover there is a finite number of ergodic measures for maps of that type. The results of Section 3.2 are used in Section 3.3 to prove (via return maps) the existence of absolutely continuous invariant measures for certain classes of maps exhibiting non-uniform expanding behavior. These results yield, in Section 3.4, to the framework developed in **[AV]** supplying sufficient conditions for the statistical stability of some classes of non-uniformly expanding transformations.

In Chapter 4 we show that some induced Markov structures exist for non-uniformly expanding maps **[ALP3]**. This can be thought of as a partial generalization, to the framework of non-uniformly expanding maps, of the main classical statement that uniformly hyperbolic systems may be endowed with a finite Markov partition; see **[Bo70]** and also **[AW67, AW70, Si68]**. The significance of such Markov structures goes well beyond the consequences for the statistical properties of the map. A first consequence of the existence of these Markov structures is established in Section 4.2, where sufficient conditions are given for the statistical stability of non-uniformly hyperbolic maps. One of the purposes of Chapter 4 is to give rates for the correlation decay of non-uniformly expanding transformations. Therefore we relate the time generic points need to attain some uniformly expanding behavior with the tail of the Markovian structure (that part of the set that has not been partitioned yet). A key role in this context is played by the results of Young **[Yo99]** for Markovian towers.

In the last chapter we study random perturbations of a non-uniformly expanding transformation. We start by introducing, in the first two sections, the concept of random perturbation of a map and the notion of stationary measure. In Section 5.1 we define physical measures and stochastic stability, and verify that non-uniformly expanding transformations possess only a finite number of physical measures; see **[Ar00, AA1]**. We also show that the number of such physical measures is bounded by the number of SRB measures of the unperturbed system, at least for small noise level. In the last section we present the results from **[AA1]** giving both necessary conditions and sufficient conditions for the stochastic stability of non-uniformly expanding transformations. As a corollary we settle the stochastic stability of the non-uniformly expanding maps from Section 1.2.

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Porto, June 5, 2003

Para ser grande, sê inteiro: nada
Teu exagera ou exclui.
Sê todo em cada coisa. Põe quanto és
No mínimo que fazes.
Assim em cada lago a lua toda
Brilha, porque alta vive.

Ricardo Reis

Contents

Preliminaries

In this chapter we present the basic concepts and results from Measure Theory, Integration and Ergodic Theory that are relevant for a good understanding of this text. All theorems we present here are standards and proofs may easily be found in several books in these subjects. We just mention **[Mu]**, **[Br68]**, **[Ma87]** and **[Wa]**.

1. Measure spaces

Let X be set and $\mathcal{A}$ be a collection of subsets of X. We say that $\mathcal{A}$ is a *σ-algebra* if the following conditions hold:

(1) $X \in \mathcal{A}$;
(2) if $A \in \mathcal{A}$ then $X \backslash A \in \mathcal{A}$;
(3) if $A_1, A_2, A_3 \cdots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

We will refer to a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a σ-algebra on X as a *measurable space.* Let $\mu \colon \mathcal{A} \to [0, +\infty]$ be a function defined on a σ-algebra $\mathcal{A}$ of X.We say that μ is a *measure* if the following conditions hold:

(1) $\mu(\emptyset) = 0$.
(2) If $A_1, A_2, \cdots \in \mathcal{A}$ are pairwise disjoint, then $\mu\big(\cup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{n} \mu(A_i)$.

We say $(X, \mathcal{A}, \mu)$ is a *measure space* if $(X, \mathcal{A})$ is a measurable space and μ is a measure on $\mathcal{A}$. If $\mu(X) = 1$ then we say that μ is a *probability measure* and $(X, \mathcal{A}, \mu)$ is a *probability space.*

EXAMPLE 0.1 (Counting measure). Let X be a set. We define a function $\#$ in the σ-algebra $\mathcal{P}(X)$ (the collection of all subsets of X) taking $\#(A)$ as the number of elements of A ($+\infty$ if A infinite) for each $A \subset X$. $\#$ defines a measure on $\mathcal{P}(X)$ that will be called the *counting measure* on X.

EXAMPLE 0.2 (Dirac measure). Let X be a set and fix a point $x \in X$. Given $A \subset X$ we define δ_x in $\mathcal{P}(X)$ as $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ otherwise. δ_x defines a probability measure on X that will be called the *Dirac measure* supported on x.

Let $(X, \mathcal{A}, \mu)$ be measure space. We say that $A \subset X$ has *null measure* if there is $B \in \mathcal{A}$ such that $A \subset B$ and $\mu(B) = 0$. We say that some property on the elements of X holds *almost everywhere* (*a.e.* for short), if the set of points for which that property does not hold has null measure.

Now we assume that X is a metric space. We define $\mathcal{B}(X)$, the *Borel σ-algebra* on X, as the σ-algebra generated by the open sets of X, i.e. the smallest (in terms of inclusion) σ-algebra that contains the open sets of X. This smallest σ-algebra always exists, since $\mathcal{P}(X)$ is a σ-algebra containing the open subsets of X and the intersection of σ-algebras with this property still has this property. A measure defined on the Borel σ-algebra of a metric space is said to be a *Borel measure*. The *support* of a Borel measure μ, which is denoted by $\operatorname{supp}(\mu)$, is defined as

$$\operatorname{supp}(\mu) = \{x \in X : \mu(U) > 0 \text{ for each neighborhood } U \text{ of } x\}.$$

A Borel probability measure μ on a compact metric space X is said to be *regular* if for all $A \in \mathcal{B}(X)$ and $\epsilon > 0$ there are a closed set $F_\epsilon \subset A$ and an open set $U_\epsilon \supset A$ such that $\mu(U_\epsilon \setminus F_\epsilon) < \epsilon$.

THEOREM 0.3. *Every Borel probability measure of a compact metric space is regular.*

One interesting problem in Measure Theory is to decide when a certain function defined over a class of subsets of a given set can be extended to a measure defined on the σ-algebra generated by those sets.

THEOREM 0.4. *Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^d$. There is a unique measure m defined on $\mathcal{B}$ such that for intervals $I_1, \ldots, I_d \subset \mathbb{R}$ one has*

$$m\left(\prod_{i=1}^{n} I_i\right) = |I_1| \times \cdots \times |I_n|,$$

where each $|I_i|$ denotes the length of I_i.

The measure m given by the previous theorem is said to be the *Lebesgue measure* on $\mathbb{R}^d$. Using a volume form and the exponential map we introduce the Lebesgue measure on Riemannian manifolds in a similar way.

2. Integration

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $\varphi\colon X \to \mathbb{R}$ is *measurable function*, if $\varphi^{-1}(B) \in \mathcal{A}$ for every Borel set $B \subset \mathbb{R}$. A function $\varphi\colon X \to \mathbb{C}$ is said to be measurable if both its real part and its imaginary part are measurable functions. We say that $\varphi\colon X \to \mathbb{C}$ is a *simple function* if there are $A_1, \ldots, A_n \in \mathcal{A}$ and $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$\varphi = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i},$$

where $\mathbf{1}_A$ denotes the characteristic function of $A \subset M$. A simple function φ is said to be an *integrable function* if $\sum_{i=1}^{n} a_i \mu(A_i) < \infty$ (we assume that $0 \cdot \infty = 0$).

In such case we define the *integral* of φ with respect to μ as

$$\int_X \varphi d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

This value does not depend on the way we write φ as a combination of characteristic functions. We say that a measurable $\varphi\colon X \to \mathbb{C}$ is an *integrable function* if there is a sequence of simple functions $\varphi_n\colon X \to \mathbb{C}$ such that

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \text{for almost every } x \in X,$$

and

$$\lim_{n\to\infty} \int_X |\varphi_n - \varphi_m| d\mu = 0.$$

In this case we define the *integral* of φ with respect to μ as

$$\int_X \varphi d\mu = \lim_{n\to\infty} \int_X \varphi_n d\mu.$$

One can prove that this limit exists and is independent of the sequence we take. Moreover, a function φ is integrable if and only if $|\varphi|$ is integrable. Given $A \in \mathcal{A}$ we say that φ is integrable on A if $\varphi 1_A$ is integrable. In such case we write

$$\int_A \varphi d\mu = \int_X \varphi 1_A d\mu.$$

We drop X when the integral is over the whole space.

THEOREM 0.5 (Dominated Convergence). *Let $(\varphi_n)_n$ be a sequence of measurable functions such that $|\varphi_n| \le \psi$, where ψ is integrable. If $\varphi = \lim_{n\to\infty} \varphi_n$ almost everywhere, then φ is integrable and*

$$\int \varphi d\mu = \lim_{n\to\infty} \int \varphi_n d\mu.$$

Let $(X, \mathcal{A}, \mu)$ be a measure space. Given $p \ge 1$ we define $L^p(\mu)$ as the set of those $\varphi\colon X \to \mathbb{C}$ such that $|\varphi|^p$ is integrable, identifying two function that coincide almost everywhere. Then

$$\|\varphi\|_p \equiv \left(\int_X |\varphi|^p \right)^{1/p}$$

defines a norm in $L^p(\mu)$. The integral is well-defined, since maps that coincide almost everywhere have the same integral. We define $L^\infty(\mu)$ as the set of those measurable functions φ for which there is Then

$$\varphi \longmapsto \|\varphi\|_\infty \equiv \inf \big\{ C \ge 0 \colon |f(x)| \le C \text{ almost everywhere} \big\}$$

also defines a norm on $L^\infty(\mu)$. $L^p(\mu)$ endowed with the norm $\|\ \ \|_p$ is a Banach space for $1 \le p \le \infty$.

THEOREM 0.6 (Hölder Inequality). *Let $f \in L^p(\mu)$ for some $p > 1$, and let $g \in L^q(\mu)$ for $q > 1$ with $p^{-1} + q^{-1} = 1$. Then $fg \in L^1(\mu)$ and*

$$\int |fg| d\mu \leq \|f\|_p \|g\|_q.$$

Let μ and ν be finite measures defined on a same σ-algebra $\mathcal{A}$. We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$. The measures μ and ν are said to be *equivalent* if both $\mu \ll \nu$ and $\nu \ll \mu$.

THEOREM 0.7 (Radon-Nykodim). *The measure ν is absolutely continuous with respect to μ if and only if there is $\varphi \colon X \to \mathbb{R}$ nonnegative and integrable with respect to μ such that*

$$\nu(A) = \int_A \varphi d\mu \quad \textit{for each } A \in \mathcal{A}.$$

Moreover, any two functions with this property coincide μ almost everywhere.

The function given by the previous theorem is called the *Radon-Nykodim derivative* of ν with respect to μ and denoted by $d\nu/d\mu$.

Let ν and ν be measures defined on a σ-algebra $\mathcal{A}$ of X. We say that μ and ν are *singular measures*, and write $\mu \perp \nu$, if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0 = \nu(X \setminus A)$.

THEOREM 0.8 (Lebesgue Decomposition). *Let μ and ν be finite measures defined on a σ-algebra $\mathcal{A}$. Then there are (finite) measures μ_a and μ_s with $\mu_a \ll \nu$ and $\mu_s \perp \nu$ such that $\mu = \mu_a + \mu_s$.*

3. Invariant measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $f \colon X \to X$ is a *measurable transformation* if $f^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. The measure μ is said to be *invariant* by f (or f *preserves* μ) if $\mu(f^{-1}(A))$ for all $A \in \mathcal{A}$. We may associate to a measurable transformation f and a measure μ a new measure that we denote by $f_*\mu$ and call the *push-forward* of the measure μ by f, and is defined as $f_*\mu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{A}$. Note that μ is invariant by f if and only if $f_*\mu = \mu$.

Let X be a compact metric space. We denote by $\mathbb{P}(X)$ the space of probability measures defined on the Borel σ-algebra of X. Note that $\mathbb{P}(X)$ is a convex space. We introduce the weak* topology on $\mathbb{P}(X)$ in the following way: a sequence $(\mu_n)_n$ in $\mathbb{P}(X)$ converges to $\mu \in \mathbb{P}(X)$ if and only if

$$\int \varphi d\mu_n \to \int \varphi d\mu, \quad \text{for each continuous } \varphi \colon X \to \mathbb{R}.$$

Since we are taking X a compact metric space, then $C(X)$ is separable, and so we may find a sequence $(\psi_n)_n$ dense in $C(X)$. The function

$$\mathrm{d}_{\mathbb{P}}(\mu,\nu)=\sum_{k=1}^{\infty}\frac{1}{2^n}\left|\int\psi_n\,d\mu-\int\psi_n\,d\nu\right| \tag{0.1}$$

defines a metric on $\mathbb{P}(X)$ which gives the weak* topology.

We may associate to a measurable transformation $f\colon X\to X$ an operator $f_*\colon\mathbb{P}(X)\to\mathbb{P}(X)$, assigning to each $\mu\in\mathbb{P}(X)$ the push-forward $f_*\mu$ of μ by f. If f is a continuous transformation of a compact metric space X, then taking some measure $\mu\in\mathbb{P}(X)$, a Dirac measure for instance, we define a sequence of measures in $\mathbb{P}(X)$,

$$\mu_n=\sum_{j=0}^{n-1}f_*^j\mu.$$

If f is continuous then it happens that f_* is also continuous, and a weak* accumulation point of the above sequence is a fixed point for f_*.

THEOREM 0.9. *Let X be a compact metric space. If $f\colon X\to X$ is a continuous transformation, then f has some invariant Borel probability measure.*

Let μ be a measure invariant by $f\colon X\to X$. We say that μ is an *ergodic measure* if the phase space cannot be decomposed into invariant regions that are relevant in terms of the measure μ, i.e. if $A\in\mathcal{A}$ satisfies $f^{-1}(A)=A$, then $\mu(A)\mu(X\setminus A)=0$.

THEOREM 0.10 (Birkhoff). *Let $f\colon X\to X$ preserve a probability measure μ. Given any $\varphi\in L^1(\mu)$ there exists $\varphi^*\in L^1(\mu)$ with $\varphi^*\circ f=\varphi^*$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi\circ f^n(x)=\varphi^*(x)$$

for μ almost every $x\in X$. Moreover, if f is ergodic, then $\varphi^*=\int\varphi d\mu$ almost everywhere.

The following result shows that any probability measure which is invariant by a continuous transformation of a compact metric space can be decomposed into ergodic components.

THEOREM 0.11 (Ergodic Decomposition). *Suppose that X is a compact metric space and $f\colon X\to X$ is a continuous transformation that preserves a probability measure μ. Then there is a family of ergodic probability measures $(\mu_x)_{x\in X}$ defined for μ almost every $x\in X$ such that for each $\varphi\in L^1(\mu)$*

$$\int\varphi\,d\mu=\int\left[\int\varphi(y)\,d\mu_x(y)\right]\,d\mu(x).$$

In particular, continuous transformations of compact metric spaces always have ergodic probability measures.

CHAPTER 1

Non-uniformly expanding maps

Let $f\colon M \to M$ be a smooth map of a compact connected, and let m denote the Lebesgue measure on M. The map f is said to be *uniformly expanding* if there is some $\sigma > 1$ such for some choice of a metric in M one has

$$\|Df(x)v\| \geq \sigma\|v\|, \quad \text{for all } x \in M \text{ and all } v \in T_xM. \tag{1.1}$$

The ergodic properties of uniformly expanding maps are quite well understood, with several results making a good description of the statistics of their orbits. Our aim here is to try to obtain similar results for maps displaying some weaker forms of expansion. This weakness may be carried out in two directions: on the one hand, restricting the set of points for which expansion holds; on the other hand, assuming that expansion holds only in time average over orbits.

1. Definition and examples

Let $f\colon M \to M$ be a continuous map which is a local diffeomorphism in the whole manifold except at a set $\mathcal{C} \subset M$ of critical points. This set $\mathcal{C}$ may be taken as some set of points where the derivative of f fails to be invertible or simply does not exist.

DEFINITION 1.1. We say that $\mathcal{C} \subset M$ is a *non-degenerate critical set* if the following conditions hold. The first one says that f *behaves like a power of the distance* to $\mathcal{C}$: there are constants $B > 1$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{C}$

$$(\mathrm{s}_1) \qquad \frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq B\operatorname{dist}(x,\mathcal{C})^{-\beta} \text{ for all } v \in T_xM.$$

Moreover, the functions $\log|\det Df|$ and $\log\|Df^{-1}\|$ are *locally Lipschitz* at points $x \in M \setminus \mathcal{C}$, with Lipschitz constant depending on $\operatorname{dist}(x,\mathcal{C})$: for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ we have

$$(\mathrm{s}_2) \qquad \big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\,\big| \leq \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y);$$

$$(\mathrm{s}_3) \qquad \big|\log|\det Df(x)| - \log|\det Df(y)|\,\big| \leq \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y).$$

Given $\delta > 0$ and $x \in M \setminus \mathcal{C}$ we define the *δ-truncated distance* from x to $\mathcal{C}$

$$\operatorname{dist}_{\delta}(x,\mathcal{C}) = \begin{cases} 1, & \text{if } \operatorname{dist}(x,\mathcal{C}) \geq \delta; \\ \operatorname{dist}(x,\mathcal{C}), & \text{otherwise.} \end{cases}$$

Note that this is not really a distance function: $\operatorname{dist}(x,y) + \operatorname{dist}_\delta(y,C)$ may be smaller than $\operatorname{dist}_\delta(x,C)$.

DEFINITION 1.2. Let $f\colon M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set C. We say that f is *non-uniformly expanding on a set* $H \subset M$ if the following conditions hold:

(1) there is $\lambda > 0$ such that for each $x \in H$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda; \tag{1.2}$$

(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), C) < \epsilon. \tag{1.3}$$

We will refer to the second condition above by saying that the orbits of points in H have *slow recurrence* to C. The case $C = \emptyset$ may also be considered, and in such case the definition reduces to the first condition. A map is said to be *non-uniformly expanding* if it is non-uniformly expanding on a set of full Lebesgue measure.

REMARK 1.3. It is worthwhile to stress that condition (1.3) is not needed in all its strength. The only place where we will be using (1.3) is in Proposition 2.12. As we shall see it is enough to have condition (1.3) for some sufficiently small $\epsilon > 0$ and conveniently chosen $\delta > 0$; see Proposition 2.12 and Remark 2.13.

Notice that in the one-dimensional case the condition (1.2) is equivalent to the existence of one positive *Lyapunov exponent* at x:

$$\liminf_{n\to\infty} |(f^n)'(x)| = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^n(x))| > \lambda > 0.$$

In dimension greater than one condition (1.2) is not equivalent to say that f has $\dim(M)$ positive Lyapunov exponents at $x \in M$, as Example 1.4 below illustrates. The formulation in the higher dimensional case is motivated by the fact that we want to make an assumption about the average expansion in every direction. Indeed for a linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, the condition $\|A\| > 1$ only provides information about the existence of some expanded direction, whereas the condition $\|A^{-1}\| < 1$ says that every direction is expanded by A.

EXAMPLE 1.4. Consider a period 2 orbit $\{p, q\}$ for a local diffeomorphism f on a surface which, for a given choice of local basis at p and q, satisfies

$$Df(p) = \begin{pmatrix} 1/2 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad Df(q) = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Then it is clear that both Lyapunov exponents at p or q are $\log(3/2)/2 > 0$ and the limit in (1.2) with $x = p$ or q equals $\log 2 > 0$.

The following result gives an useful criterium for proving the non-uniform expansion of a map as we shall illustrate after it.

PROPOSITION 1.5. *Let $f : M \to M$ be a C^2 local diffeomorphism outside a critical set $\mathcal{C}$. Suppose that f has an invariant measure $\mu \ll m$ such that:*

(1) $\int \log \|Df(x)^{-1}\| \, d\mu < 0$;

(2) $\log \operatorname{dist}(x, \mathcal{C})$ *is μ-integrable.*

Then there exists a set $H \subset M$ with positive Lebesgue measure where f is non-uniformly expanding.

PROOF. The assumption on the integral of $\log \|Df(x)^{-1}\|$ with respect to μ together with Birkhoff's ergodic theorem ensure that there is some $c > 0$ for which the set

$$E = \left\{ x \in M : \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c \right\} \tag{1.4}$$

satisfies $\mu(E) > 0$, and so $m(E) > 0$. It remains to show that a positive Lebesgue measure subset of points in E have slow recurrence to $\mathcal{C}$. We start by fixing $\alpha > 0$ such that

$$\prod_{n \geq 1} (1 - e^{-\alpha n}) \geq 1 - m(E).$$

The integrability of $\log \operatorname{dist}(x, \mathcal{C})$ with respect to the measure μ and the definition of the δ-truncated distance $\operatorname{dist}_\delta$ ensure that for each $k \in \mathbb{N}$ we may find $\delta_k > 0$ for which

$$\int_M - \log \operatorname{dist}_{\delta_k}(x, \mathcal{C}) \, d\mu \leq \frac{1}{k 2^{k+1}}.$$

We define for each $k \in \mathbb{N}$

$$\varphi_k(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \operatorname{dist}_{\delta_k}(f^j(x), \mathcal{C}).$$

This φ_k is well-defined μ almost everywhere in M by Birkhoff's ergodic theorem. Moreover

$$\int_M \varphi_k \, d\mu = \int_M - \log \operatorname{dist}_{\delta_k}(x, \mathcal{C}) \, d\mu \leq \frac{1}{k 2^{k+1}}.$$

Let

$$R_k = \left\{ x \in M : \varphi_k(x) > \frac{1}{k} \right\}.$$

Since $\varphi_k \geq 0$ we have

$$\frac{\mu(R_k)}{k} \leq \int_{R_k} \varphi_k \, d\mu \leq \int_M \varphi_k \, d\mu \leq \frac{1}{k 2^{k+1}},$$

which implies that $\mu(R_k) \le 2^{-(k+1)}$. By the absolute continuity of μ with respect to m we may find $k_1 \in \mathbb{N}$ such that $m(M \setminus R_{k_1}) \ge 1 - e^{-\alpha}$. This is the first step in the following construction by induction on n. Assuming that we have chosen $k_1 < k_2 < \cdots < k_n$ satisfying

$$m(M \setminus (R_{k_1} \cup \cdots \cup R_{k_j})) \ge (1 - e^{-\alpha j}) m(M \setminus (R_{k_1} \cap \cdots \cap R_{k_{j-1}}))$$

for all $j = 2, \ldots, n$, then we may find a big enough $k_{n+1} > k_n$ such that

$$m(M \setminus (R_{k_1} \cap \cdots \cap R_{k_n} \cap R_{k_{n+1}})) \ge (1 - e^{-\alpha(n+1)}) m(M \setminus (R_{k_1} \cap \cdots \cap R_{k_n})).$$

Now we take

$$S = M \setminus (R_1 \cup R_2 \cup \cdots).$$

Let us show that points in S have slow recurrence to $\mathcal{C}$. Given $\varepsilon > 0$ we choose $k_0 \in \mathbb{N}$ for which $\varepsilon > 1/k_0$. If $x \in S$, then in particular $x \notin R_{k_0}$, and this implies

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_{\delta_{k_0}}(f^j(x), \mathcal{C}) = \varphi_{k_0}(x) \le \frac{1}{k_0} \le \varepsilon.$$

Since

$$m(S) \ge \prod_{n \ge 1} (1 - e^{-\alpha n}) \ge 1 - m(A),$$

then $H = A \cap S$ has positive Lebesgue measure. □

REMARK 1.6. Note that we have used the assumption on the integral of $\log \|Df(x)^{-1}\|$ just to guarantee that the set E in (1.4) has positive Lebesgue measure. Thus, that assumption can be omitted if we know in advance that there is some set $H \subset M$ with $m(H) > 0$ such that (1.2) holds for all $x \in H$.

Quadratic maps have served as a model that inspired many of the statistical results about non-uniformly expanding maps. Although it might seem unnatural from that point of view to use the knowledge of those statistical properties for quadratic maps to prove that these are non-uniformly expanding, we include it here just to illustrate that the model fits the general theory.

EXAMPLE 1.7 (Quadratic maps). Consider $f_a \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = a - x^2$, with $a \in \mathbb{R}$. The most interesting part on study of the dynamics of f_a occurs for $-1/4 < a \le 2$, with f_a restricted to the interval $[f^2(0), f(0)]$. It is well known that there is a positive Lebesgue measure subset of parameters $-1/4 < a \le 2$ for which f_a has an ergodic absolutely continuous invariant measure μ, which is also equivalent to m on $[f^2(0), f(0)]$, see e.g. **[Ja, BC85, BY92]**. Moreover, $d\mu_a/dm$ belongs to some $L^p(m)$ with $p > 1$, and f_a has a positive Lyapunov exponent μ almost everywhere. This gives the first part of Definition 1.2. For showing the slow recurrence to the critical set $\mathcal{C} = \{0\}$, we observe that since $\log|x|$ belongs to all $L^q(m)$ with $1 \le q < \infty$, we have by Hölder inequality that $\log|x|$ is μ_a integrable. Thus, by Proposition 1.5 we have that f_a is non-uniformly expanding on a set with positive μ (thus m) measure. Ergodicity gives non-uniform expansion Lebesgue almost everywhere.

The example we present below has some special features that we will explore later on. By now we just prove that there is a positive Lebesgue measure subset of points where the map has non-uniformly expanding behavior. Actually, in our later study we will see that the non-uniform expansion holds Lebesgue almost everywhere.

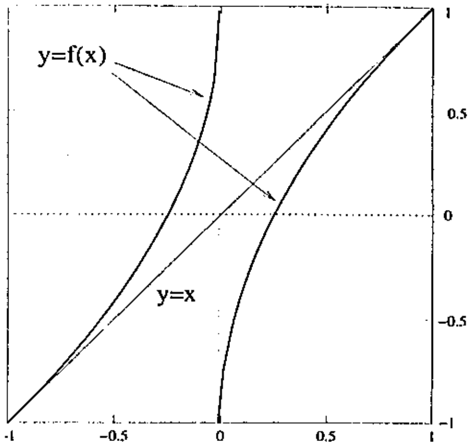

FIGURE 1.1. The map of Example 1.8.

EXAMPLE 1.8. Let I denote the interval $[-1,1]$, and consider the map from I into itself whose graph is drawn in Figure 1.1, given by

$$x \mapsto \begin{cases} 2\sqrt{x}-1 & \text{if } x \geq 0, \\ 1-2\sqrt{|x|} & \text{otherwise.} \end{cases}$$

This map induces a continuous local homeomorphism $f: S^1 \to S^1$ through the identification $S^1 = I/\sim$, where $-1 \sim 1$, differentiable everywhere except at the point 0. One can easily verify that $C = \{0, \pm 1\}$ is a non-degenerate critical set for f, in the sense of Definition 1.1.

Let us show that f preserves Lebesgue measure. It is enough to check that f preserves the Lebesgue measure of the intervals, since these generate the Borel σ-algebra. Observe that f has two inverse branches $g_1 : (-1,1) \to (0,1)$ and $g_2 : (-1,1) \to (-1,0)$, given by

$$g_1(x) = \left(\frac{1+x}{2}\right)^2 \quad \text{and} \quad g_2(x) = -\left(\frac{1-x}{2}\right)^2.$$

Thus the pre-image of an interval (a,b) is made by two intervals whose lengths are

$$\left(\frac{1+b}{2}\right)^2 - \left(\frac{1+a}{2}\right)^2 \quad \text{and} \quad -\left(\frac{1-b}{2}\right)^2 + \left(\frac{1-a}{2}\right)^2.$$

We immediately verify that the sum of these two lengths is precisely $b-a$, and so we conclude that f preserves Lebesgue measure.

Observe now that $\log|(f'(x))^{-1}| = \log\sqrt{|x|}$ is integrable with respect to the Lebesgue measure on S^1, and

$$\int_{S^1} \log|(f'(x))^{-1}|\,dm = \int_{-1}^{1} \frac{1}{2}\log|x| \left(\frac{1}{2}dx\right) = \frac{1}{2}\int_0^1 \log x\,dx = -\frac{1}{2}$$

(we are taking measure m normalized on S^1). We also have that

$$\log \operatorname{dist}(x, \mathcal{C}) = \begin{cases} \log|x|, & \text{if } |x| < 1/2, \\ \log(1-|x|), & \text{otherwise,} \end{cases}$$

is integrable with respect to the Lebesgue measure on S^1. Hence, we may apply Proposition 1.5 and thus conclude that f is non-uniformly expanding on some set with positive Lebesgue measure.

2. Some open classes

In this section we present two open classes of non-uniformly expanding maps in higher dimensional spaces. The first class was introduced in [**ABV**], and its maps display non-uniform expansion Lebesgue almost everywhere but are not uniformly expanding. The maps of the second class, introduced in [**Vi2**], have a non-empty critical sets (points with non-invertible derivative) and display non-uniform expansion at Lebesgue almost all points.

2.1. Local diffeomorphisms. Here we present robust (C^1 open) classes of local diffeomorphisms (with no critical sets) that are non-uniformly expanding. Such classes of maps and can be obtained, e.g. through deformation of a uniformly expanding map by isotopy inside some small region. In general, these maps are not uniformly expanding: deformation can be made in such way that the new map has periodic saddles.

Let M be a compact manifold supporting some uniformly expanding map f_0. M could be the d-dimensional torus T^d, for instance. Let $V \subset M$ be some small compact domain, so that the restriction of f_0 to V is injective. Let f be any map in a sufficiently small C^1-neighborhood $\mathcal{N}$ of f_0 so that:

(1) f is *volume expanding everywhere*: there exists $\sigma_1 > 1$ such that

$$|\det Df(x)| > \sigma_1 \quad \text{for every } x \in M;$$

(2) f is *expanding outside* V: there exists $\sigma_0 > 1$ such that

$$\|Df(x)^{-1}\| < \sigma_0 \quad \text{for every } x \in M \setminus V;$$

(3) f is *not too contracting on* V: there is some small $\delta > 0$ such that

$$\|Df(x)^{-1}\| < 1 + \delta \quad \text{for every } x \in V.$$

We are going to show that every map f in such a C^1-neighborhood $\mathcal{N}$ of f_0 is non-uniformly expanding.

LEMMA 1.9. *Let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of M into domains such that f is injective on B_j, for $1 \le j \le p+1$. There exists $\theta > 0$ such that the orbit of Lebesgue almost every point $x \in M$ spends a fraction θ of the time in $B_1 \cup \cdots \cup B_p$, that is,*

$$\#\{0 \le j < n : f^j(x) \in B_1 \cup \cdots \cup B_p\} \ge \theta n$$

for every large n.

PROOF. Let n be fixed. Given a sequence $\underline{i} = (i_0, i_1, \ldots, i_{n-1})$ in $\{1, \ldots, p+1\}$, we denote

$$[\underline{i}] = B_{i_0} \cap f^{-1}(B_{i_1}) \cap \cdots \cap f^{-n+1}(B_{i_{n-1}}).$$

Moreover, we define $g(\underline{i})$ to be the number of values of $0 \le j \le n-1$ for which $i_j \le p$. We begin by noting that, given any $\theta > 0$, the total number of sequences $\underline{i}$ for which $g(\underline{i}) < \theta n$ is bounded by

$$\sum_{k<\theta n} \binom{n}{k} p^k \le \sum_{k \le \theta n} \binom{n}{k} p^{\theta n}$$

A standard application of Stirling's formula (see e.g. [**BoV**, Section 6.3]) gives that the last expression is bounded by $e^{\gamma n} p^{\theta n}$, where γ depends only on θ and goes to zero when θ goes to zero. On the other hand, since we are assuming that f is volume expanding everywhere and not too contracting on B_{p+1}, we have $m([\underline{i}]) \le m(M)\,\sigma_1^{-(1-\theta)n}$. Then the measure of the union I_n of all the sets $[\underline{i}]$ with $g(\underline{i}) < \theta n$ is less than $m(M)\sigma_1^{-(1-\theta)n} e^{\gamma n} p^{\theta n}$ Since $\sigma_1 > 1$, we may fix θ small so that $e^{\gamma} p^{\theta} < \sigma_1^{1-\theta}$. This means that the Lebesgue measure of I_n goes to zero exponentially fast as $n \to \infty$. Thus, by the lemma of Borel-Cantelli, Lebesgue almost every point $x \in M$ belongs in only finitely many sets I_n. Clearly, any such point x satisfies the conclusion of the lemma. □

Let $\theta > 0$ be the constant given by Lemma 1.9, and fix $\delta > 0$ small enough so that $\sigma_0^{\theta}(1+\delta) \le e^{-\lambda}$ for some $\lambda > 0$. Let x be any point satisfying the conclusion of the lemma. Then

$$\prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma_0^{\theta n}(1+\delta)^{(1-\theta)n} \le e^{-\lambda n}$$

for every large enough n. This implies that x satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -\lambda\,.$$

and since the conclusion of Lemma 1.9 holds Lebesgue almost everywhere we have that f is a non-uniformly expanding map.

REMARK 1.10. We have also proved that, for any such f, the Lebesgue measure of the set

$$\{x \in M : \|Df^j(x)^{-1}\| > e^{-\lambda j} \text{ for some } j \geq n\}$$

goes to zero exponentially fast when $n \to \infty$.

2.2. Viana maps. In what follows we study an open classes of non-uniformly expanding maps with critical sets in higher dimensions. We skip the most technical points and refer to **[Vi2]** for details. We study the two-dimensional case, and then say how the arguments can be adapted to higher dimensions.

2.2.1. *Two-dimensional case.* Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map

$$\begin{array}{rccc} \hat{f}: & S^1 \times \mathbb{R} & \longrightarrow & S^1 \times \mathbb{R} \\ & (s,x) & \longmapsto & (\hat{g}(s), \hat{q}(s,x)) \end{array}$$

where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = ds$ (mod $\mathbb{Z}$) for some $d \geq 16$, and $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps f close to $\hat{f}$ restricted to $S^1 \times I$. Taking into account the expression of $\hat{f}$ it is not difficult to check that $\hat{f}$ (and any map f close to $\hat{f}$ in the C^2 topology) behaves like a power of the distance close to the critical set.

The results in **[Vi2]** show that if the map f is sufficiently close to $\hat{f}$ in the C^3 topology then f has two positive Lyapunov exponents almost everywhere: there is a constant $\lambda > 0$ for which

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|Df^n(s,x)v\| \geq \lambda$$

for Lebesgue almost every $(s,x) \in S^1 \times I$ and every non-zero $v \in T_{(s,x)}(S^1 \times I)$. This does not necessarily imply that f is non-uniformly expanding. However, a slightly deeper use of Viana's arguments enables us to prove the non-uniform expansion of f. For the sake of clearness, we start by assuming that f has the special form

$$f(s,x) = (g(s), q(s,x)), \quad \text{with} \quad \partial_x q(s,x) = 0 \quad \text{if and only if} \quad x = 0, \tag{1.5}$$

and describe how the conclusions in **[Vi2]** are obtained for each C^2 map f satisfying

$$\|f - \hat{f}\|_{C^2} \leq \alpha \quad \text{on} \quad S^1 \times I. \tag{1.6}$$

Then we explain how these conclusions extend to the general case, using the existence of a central invariant foliation, and we show how the results in **[Vi2]** give the non-uniform expansion and slow approximation of orbits to the critical set for each map f as in (1.6).

The estimates on the derivative rely on a statistical analysis of the returns of orbits to the neighborhood $S^1\times(-\sqrt{\alpha},\sqrt{\alpha})$ of the critical set $\mathcal{C}=\{(s,x):x=0\}$. We set

$$J(0)=I\setminus(-\sqrt{\alpha},\sqrt{\alpha})\quad\text{and}\quad J(r)=\{x\in I:|x|<e^{-r}\}\quad\text{for } r\geq 0.$$

From here on we only consider points $(s,x)\in S^1\times I$ whose orbit does not hit the critical set $\mathcal{C}$. This constitutes no restriction in our results, since the set of those points has full Lebesgue measure. For each integer $j\geq 0$ we define $(s_j,x_j)=f^j(s,x)$ and

$$r_j(s,x)=\min\{r\geq 0:x_j\in J(r)\}.$$

Consider, for some small constant $0<\eta<1/4$,

$$G=\left\{0\leq j<n\colon r_j(s,x)\geq\left(\frac{1}{2}-2\eta\right)\log\frac{1}{\alpha}\right\}.$$

Fix some integer $n\geq 1$ sufficiently large (only depending on $\alpha>0$). The results in [**Vi2**] show that if we take

$$B_2(n)=\{(s,x)\colon\ \text{there is } 1\leq j<n \text{ with } x_j\in J([\sqrt{n}])\},$$

where $[\sqrt{n}]$ is the integer part of $\sqrt{n}$, then we have

$$m(B_2(n))\leq\text{const}\,e^{-\sqrt{n}/4}, \tag{1.7}$$

and for small $\lambda>0$ (only depending on the quadratic map Q)

$$\log\prod_{j=0}^{n-1}|\partial_x q(s_j,x_j)|\geq 2\lambda n-\sum_{j\in G}r_j(s,x)\quad\text{for}\quad(s,x)\notin B_2(n), \tag{1.8}$$

see [**Vi2**, pp. 75 & 76]. Moreover, defining for $\varepsilon>0$

$$B_1(n)=\left\{(s,x)\notin B_2(n):\sum_{j\in G}r_j(s,x)\geq\varepsilon n\right\},$$

then for small $\varepsilon>0$ there is a constant $\xi>0$ for which

$$m\big(B_1(n)\big)\leq e^{-\xi n}, \tag{1.9}$$

see [**Vi2**, p. 77]. Taking into account the definitions of $J(r)$ and r_j, this shows that if we take $\delta=(1/2-2\eta)\log(1/\alpha)$, then

$$\sum_{j=0}^{n-1}-\log\operatorname{dist}_\delta(f^j(s,x),\mathcal{C})\leq\varepsilon n\quad\text{for}\quad(s,x)\notin B_1(n)\cup B_2(n). \tag{1.10}$$

On the other hand, we have for $(s,x)\in S^1\times I$

$$Df(s,x)^{-1}=\frac{1}{\partial_x q(s,x)\partial_s g(s)}\begin{pmatrix}\partial_x q(s,x) & 0\\ -\partial_s q(s,x) & \partial_s g(s)\end{pmatrix}. \tag{1.11}$$

Since all the norms are equivalent in finite dimensional Banach spaces, it is no restriction for our purposes to take the norm of $Df(s,x)^{-1}$ as the maximum of the absolute values of its entries. From (1.5) and (1.6) we deduce that for small $\alpha > 0$

$$|\partial_s g| \geq d - \alpha, \quad |\partial_s q| \leq \alpha|b'| + \alpha \leq 8\alpha \quad \text{and} \quad |\partial_x q| \leq |2x| + \alpha \leq 4,$$

which together with (1.11) gives

$$\left\|Df(s,x)^{-1}\right\| = |\partial_x q(s,x)|^{-1},$$

as long as $\alpha > 0$ is taken sufficiently small. This implies

$$\sum_{j=0}^{n-1} \log \|Df(s_j,x_j))^{-1}\| = -\sum_{j=0}^{n-1} \log |\partial_x q(s_j,x_j)| \tag{1.12}$$

for every $(s,x) \in S^1 \times I$. Taking $\varepsilon < \lambda$, then we have

$$\sum_{j=0}^{n-1} \log |\partial_x q(s_j,x_j)| = \log \prod_{j=0}^{n-1} |\partial_x q(s_j,x_j)| \geq \lambda n \tag{1.13}$$

for every $(s,x) \notin B_1(n) \cup B_2(n)$ (recall (1.8) and the definition of $B_1(n)$). We conclude from (1.12) and (1.13) that

$$\sum_{j=0}^{n-1} \log \|Df(s_j,x_j))^{-1}\| \leq -\lambda n \quad \text{for} \quad (s,x) \notin B_1(n) \cup B_2(n). \tag{1.14}$$

If we take $E_n = B_1(n) \cup B_2(n)$ and $\Gamma_n = \cup_{k \geq n} E_k$, then in view of the estimates (1.7) and (1.9) on the Lebesgue measure of the sets $B_1(n)$ and $B_2(n)$, there is $\gamma > 0$ for which

$$m(\Gamma_n) \leq \text{const}\, e^{-\gamma\sqrt{n}}, \quad \text{for all } n \geq 1. \tag{1.15}$$

Moreover, it follows from (1.10) and (1.14) that for each $(s,x) \notin \Gamma_n$

$$\frac{1}{k}\sum_{j=0}^{k-1} -\log \operatorname{dist}_\delta(f^j(s,x),\mathcal{C}) \leq \varepsilon \quad \text{for all } k \geq n,$$

and

$$\frac{1}{k}\sum_{j=0}^{k-1} \log \|Df(s_j,x_j))^{-1}\| \leq -\lambda \quad \text{for all } k \geq n.$$

This shows that f is non-uniformly expanding.

REMARK 1.11. Let us observe, for future reference, that the constants δ, λ and γ only depend on the quadratic map Q and $\alpha > 0$. In particular, the decay estimate (1.15) on the Lebesgue measure of Γ_n only depends on the quadratic map Q and $\alpha > 0$.

Now we describe how in **[Vi2]** the same conclusions are obtained without assuming (1.5). Since $\hat{f}$ is strongly expanding in the horizontal direction, it follows from the methods of **[HPS]** that any map f sufficiently close to $\hat{f}$ admits a unique invariant central foliation $\mathcal{F}^c$ of $S^1 \times I$ by smooth curves uniformly close to vertical segments; see [**Vi2**, Section 2.5]. Actually, $\mathcal{F}^c$ is obtained as the set of integral curves of a vector field $(\xi^c, 1)$ in $S^1 \times I$ with ξ^c uniformly close to zero. The previous analysis can then be carried out in terms of the expansion of f along this central foliation $\mathcal{F}^c$. More precisely, $|\partial_x q(s,x)|$ is replaced by

$$|\partial_c q(s,x)| \equiv |Df(s,x)v_c(s,x)|,$$

where $v_c(s,x)$ is a unit vector tangent to the foliation at the point (s,x). The previous observations imply that v_c is uniformly close to $(0,1)$ if f is close to $\hat{f}$. Moreover, cf. [**Vi2**, Section 2.5], it is no restriction to suppose $|\partial_c q(s,0)| \equiv 0$, so that $\partial_c q(s,x) \approx |x|$, as in the unperturbed case. Indeed, if we define the *critical set* of f by

$$\mathcal{C} = \{(s,x) \in S^1 \times I : \partial_c q(s,x) = 0\}.$$

by an easy implicit function argument it is shown in [**Vi2**, Section 2.5] that $\mathcal{C}$ is the graph of some C^2 map $\eta : S^1 \to I$ arbitrarily C^2-close to zero if α is small. This means that up to a change of coordinates C^2-close to the identity we may suppose that $\eta \equiv 0$ and, hence, write for $\alpha > 0$ small

$$\partial_c q(s,x) = x\psi(s,x) \quad \text{with } |\psi + 2| \text{ close to zero.}$$

This provides an analog to the second part of assumption (1.5). At this point, the arguments apply with $\partial_x q(s,x)$ replaced by $\partial_c q(s,x)$, to show that orbits have slow approximation to the critical set $\mathcal{C}$ and $\prod_{i=0}^{n-1} |\partial_c q(s_i,x_i)|$ grows exponentially fast for Lebesgue almost every $(s,x) \in S^1 \times I$. A matrix formula for $Df^n(s,x)^{-1}$ similar to that in (1.11) can be obtained if we replace the vector $(0,1)$ in the canonical basis of the space tangent to $S^1 \times I$ at (s,x) by $v_c(s,x)$, and consider the matrix of $Df^n(s,x)^{-1}$ with respect to the new basis.

2.2.2. *Higher dimensions.* Here we explain how the previous construction can be adapted to higher dimensions. Consider $\hat{f}\colon T^m \times \mathbb{R} \to T^m \times \mathbb{R}$ given by

$$\hat{f}(\theta,x) = (\hat{g}(\theta), \hat{h}(\theta,x)),$$

where $\hat{g}$ is an expanding map on the m-torus T^m and $\hat{h}(\theta,x) = a_0 + (\alpha b(\theta) - x^2$. As before, and $a_0 \in (1,2)$ is such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. For simplicity, we take $\hat{g}$ to be linear and to have a unique largest eigenvalue λ_u. Then we suppose the function b to vary in a Morse fashion along the corresponding eigendirection V_u. In this setting an admissible curve is a curve of the form $\{(\Theta(t), X(t))\} \subset T^m \times \mathbb{R}$ with $\Theta(t) = \theta_0 + tV_u$ and $|X'|$, $|X''|$ small. Then, up to assuming λ_u sufficiently large (depending only on the Morse function b), the same arguments as before prove that for small enough α the map $\hat{f}_\alpha$ has $m+1$ positive Lyapunov exponents at

$\hat{f}_a(\Theta(\theta), X(t))$, for almost every t. Moreover, the same remains true for all small perturbations f of f_a, as long as every eigenvalue of $\hat{g}$ is larger than 4, in order to assure that the invariant foliation $\{\theta = \text{const}\}$ is normally expanding.

3. Uniform vs. non-uniform expansion

From [Ma85] one knows that C^2 local diffeomorphisms in one-dimensional manifolds are necessarily uniformly expanding. The situation is completely different in higher dimensions. As we have seen in Subsection 2.1 there are examples of maps in dimension greater than one which are non-uniformly expanding and not uniformly expanding.

Uniformly expanding maps are many times defined as differentiable maps for which the hypothesis of the proposition below holds. Since we are working in finite dimensional compact manifolds, then all norms are equivalent, and so that other possible definition is equivalent to the one we have given in (1.1). A metric for which the constant $C > 0$ below may be taken equal to 1 is said to be an *adapted metric* for f.

PROPOSITION 1.12. *Let $f\colon M \to M$ be a differentiable map for which there are $C > 0$ and $0 < \lambda < 1$ such that*

$$\|Df^n(x)v\| \geq C\lambda^n \|v\|, \tag{1.16}$$

for all $n \geq 1$, $x \in M$ and $v \in T_xM$. Then f is uniformly expanding.

PROOF. Assume that there are $C > 0$ and $0 < \lambda < 1$ for which (1.16) holds. Taking $1 < \sigma < \lambda$, we define the norm $|\cdot|$ on the tangent space of $x \in M$ by

$$|v| = \sum_{n=0}^{\infty} \sup_{y \in f^{-n}(x)} \sigma^n \left\|Df^n(y)^{-1}v\right\|, \quad \text{for all } v \in T_xM.$$

This is well defined by (1.21) and by the choice of σ. We have for each $v \in T_xM$

$$\begin{aligned} |Df(x)v| &= \sum_{n=0}^{\infty} \sup_{y \in f^{-n}(f(x))} \sigma^n \left\|Df^n(y)^{-1}Df(x)v\right\| \\ &\geq \sum_{n=1}^{\infty} \sup_{y \in f^{-n}(f(x))} \sigma^n \left\|Df^n(y)^{-1}Df(x)v\right\| \\ &\geq \sum_{n=1}^{\infty} \sup_{y \in f^{-(n-1)}(x)} \sigma^n \left\|Df^{n-1}(y)^{-1}v\right\| \\ &= \sigma|v| \end{aligned}$$

This concludes the proof of the proposition. □

Our aim now is to investigate whether (apparently) weaker forms of expansion might still imply uniform expansion or not. This will be the case if we replace the set of full Lebesgue measure by a set of total probability in the definition of non-uniformly expanding map. Recall that a Borel set in a topological space is said to have *total probability* if it has probability one for every f-invariant probability measure. Let us start with a couple of auxiliary lemmas.

LEMMA 1.13. *Let X be a compact metric space and $f\colon X \to X$ a continuous map. If $\varphi\colon X \to \mathbb{R}$ is continuous and*

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) < 0 \tag{1.17}$$

holds in a subset of points $x \in X$ with total probability, then it holds for all $x \in X$.

PROOF. Arguing by contradiction, let us suppose that there is some $x \in X$ such that

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \geq 0.$$

Then, for every $k \in \mathbb{N}$, there is some integer n_k for which

$$\frac{1}{n_k}\sum_{j=0}^{n_k-1} \varphi(f^j(x)) > -\frac{1}{k}.$$

It is no restriction to assume that $n_1 < n_2 < \cdots$ and we do it. We define the sequence of probability measures

$$\mu_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1} \delta_{f^j(x)}, \quad k \geq 1,$$

where each $\delta_{f^j(x)}$ is the Dirac measure on $f^j(x)$. Let μ be a weak* accumulation point of this sequence when $k \to +\infty$. Taking a subsequence, if necessary, we assume that μ_{n_k} converges to μ. Standard arguments show that μ is f invariant. Since the function φ is continuous we have

$$\int \varphi\, d\mu = \lim_{k\to+\infty} \frac{1}{n_k}\sum_{j=0}^{n_k-1} \varphi(f^j(x)) \geq 0$$

by definition of μ and the way we have chosen the sequence $(n_k)_k$. However, since we are assuming that (1.17) holds in a set of total probability measure, we have that μ almost every y is such that

$$\bar\varphi(y) := \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j y) < 0.$$

On the other hand,

$$\int \bar\varphi\, d\mu = \int \varphi\, d\mu \geq 0$$

by Birkhoff's ergodic theorem. This gives a contradiction, thus proving the lemma. □

LEMMA 1.14. *Let X be a compact metric space and $f\colon X \to X$ a continuous map. If $\varphi\colon X \to \mathbb{R}$ is a continuous function and for all $x \in X$*

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) < 0,$$

then there are $C, c > 0$ and $p \in \mathbb{N}$ such that for all $x \in X$ and $m \geq 1$

$$\sum_{j=0}^{mp-1}\varphi(f^j(x)) \leq -cm + C.$$

PROOF. According to the hypothesis of the lemma, for each $x \in X$ there are $q(x) \in \mathbb{N}$ and $c(x) > 0$ such that

$$\frac{1}{q(x)}\sum_{j=0}^{q(x)-1}\varphi(f^j(x)) < -2c(x).$$

Thus, by continuity, for each $x \in X$ there is a neighborhood V_x of x such that for every $y \in V_x$ one has

$$\frac{1}{q(x)}\sum_{j=0}^{q(x)-1}\varphi(f^j(y)) < -c(x).$$

Since X is compact, there is a finite cover $V_{x_1}, \ldots, V_{x_s}$ of X by neighborhoods of this type. Let

$$p = \max\{q(x_1), \ldots, q(x_s)\} \quad \text{and} \quad c = \min\{c(x_1), \ldots, c(x_s)\}. \tag{1.18}$$

We define for $x \in X$

$$q_1(x) = \min\{q(x_i)\colon x \in V_{x_i},\ i = 1, \ldots, s\}$$

and a sequence of maps $q_k : M \to \{1, \ldots, p\}$, $k \geq 0$, in the following way:

$$q_0(x) = 0, \quad q_{k+1}(x) = q_k(x) + q_1(f^{q_k(x)}x). \tag{1.19}$$

Observe that there is no conflict in the definition of q_1. Now it will be useful to take

$$\alpha = \max_{x\in M}\varphi(x).$$

Let us fix $x \in X$. Given $m \geq 1$ we define

$$h = \max\{k \geq 1 : q_k(x) \leq mp\}.$$

We must have $mp - q_h(x) \le p$. It follows from (1.18) and (1.19) that for $k \ge 0$

$$\begin{aligned}\sum_{j=q_k(x)}^{q_{k+1}(x)-1} \varphi(f^j(x)) &= \sum_{j=0}^{q_1\left(f^{q_k(x)}(x)\right)-1} \varphi(f^{q_k(x)+j}(x)) \\ &\le -cq_1(f^{q_k(x)}(x)) \\ &= -c\big(q_{k+1}(x) - q_k(x)\big)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j=0}^{mp-1} \varphi(f^j(x)) &= \sum_{j=q_0(x)}^{q_1(x)-1} \varphi(f^j(x)) + \cdots + \sum_{j=q_{h-1}(x)}^{q_h(x)-1} \varphi(f^j(x)) + \sum_{j=q_h(x)}^{mp-1} \varphi(f^j(x)) \\ &\le -c\sum_{j=1}^{h}[q_j(x) - q_{j-1}(x)] + \alpha(mp - q_h(x)) \\ &\le -cq_h(x) + \alpha p \\ &\le -cm + \alpha p\end{aligned}$$

(observe that $q_h(x) \ge mp/p = m$). Thus we just have to take $C = \alpha p$. □

The next result gives in particular that if a local diffeomorphism f is non-uniformly expanding on a set with total probability, then f is uniformly expanding; observe that condition (1.20) in Theorem (1.15) is weaker than condition (1.2).

THEOREM 1.15. *Let $f : M \to M$ be a C^1 local diffeomorphism. If for all $x \in M$ in a set with total probability*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^{-1}_{f^i(x)}\| < 0, \tag{1.20}$$

then f is uniformly expanding.

PROOF. The fact that f is a local diffeomorphism implies that the map

$$\lambda(x) = \log \|Df(x)^{-1}\|$$

is a continuous function from M to $\mathbb{R}$. Fix $x \in M$ and $0 \ne v \in T_xM$. Observe that

$$\|Df^{mp}(x)^{-1}\| \le \exp\left(\sum_{j=0}^{mp-1} \lambda(f^jx)\right).$$

Let $C_0 > 0$ be as in Lemma 1.14. Taking $K_0 = e^{C_0}$ and $\rho = e^{-c/2}$, we have for all $m \ge 1$

$$\|v\| = \|Df^{mp}(x)^{-1}Df^{mp}(x)v\| \le K\rho^m\|Df^{mp}(x)v\|,$$

which is equivalent to

$$\|Df^{mp}(x)v\| \geq \frac{1}{K}(\rho^{-1})^m\|v\|.$$

On the other hand, we have for all $x \in M$, $v \in T_xM$ and $n \geq 1$

$$\|Df^n(x)v\| \geq e^{-\alpha n}\|v\|.$$

Let an integer $n \geq 1$ be given. There are $m \geq 1$ and $r \in \{0, \dots, p-1\}$ such that $n = mp + r$. Hence

$$\begin{aligned}
\|Df^n(x)v\| &= \|Df^r(f^{mp}(x))Df^{mp}(x)v\| \\
&\geq e^{-r\alpha}\|Df^{mp}(x)v\| \\
&\geq \frac{e^{-p\alpha}}{K}(\rho^{-1})^m\|v\| \\
&= \frac{e^{-p\alpha}\rho^{r/p}}{K}(\rho^{-1/\overline{N}})^{mp+r}\|v\|
\end{aligned}$$

Thus we have proved that for all $n \geq 1$, $x \in M$ and $v \in T_xM$

$$\|Df^n(x)v\| \geq C\lambda^n\|v\|, \tag{1.21}$$

where $C = e^{-p\alpha}\rho/K > 0$ and $\lambda = \rho^{-1/p} > 1$. □

CHAPTER 2

Hyperbolic times

A powerful tool for the study of the ergodic properties of Viana maps has been introduced in **[Al00]** through the notion of *hyperbolic times.* This notion has been extended to a very general class of maps in **[ABV]**. Roughly speaking, hyperbolic times are iterates of a given point at which some uniform (not depending on the point nor on the iterate) backward contraction holds, thus implying uniformly bounded distortion on some small neighborhood of that point. This makes hyperbolic times play a key role in the study of the statistical properties of many classes of dynamical systems. We fix once and for all $B > 1$ and $\beta > 0$ as in Definition 1.1, and a constant $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$.

DEFINITION 2.1. Given $\sigma < 1$ and $\delta > 0$, we say that n is a (σ, δ)-*hyperbolic time* for a point $x \in M$ if for all $1 \leq k \leq n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), C) \geq \sigma^{bk}. \tag{2.1}$$

In the case $C = \emptyset$ the definition of (σ, δ)-hyperbolic time reduces to the first condition in (2.1) and we simply call it a σ-*hyperbolic time.*

As we shall see below, hyperbolic times appear with positive frequency at Lebesgue almost all points for non-uniformly expanding maps. In such a case we are able to define at Lebesgue almost every point a first hyperbolic time map. Our goal in this chapter is to present the most important features of hyperbolic times and to connect the positive frequency of hyperbolic times to the integrability of the first hyperbolic time map positive frequency of hyperbolic times and to the existence of absolutely continuous invariant measures.

1. Bounded distortion

Hyperbolic times of a given point correspond to iterates where the map locally behaves as if it were an expanding map, namely with uniform expansion and uniform bounded distortion. This is stated precisely in Proposition 2.3. We start with a preliminary technical result.

LEMMA 2.2. *Given $\delta > 0$ fix $\delta_1 > 0$ so that $4\delta_1 < \delta$ and $4B\delta_1 < \delta^\beta |\log \sigma|$. If n is a (σ, δ)-hyperbolic time for x, then*

$$\|Df(y)^{-1}\| \leq \sigma^{-1/2} \|Df(f^{n-j}(x))^{-1}\|$$

for any $1 \le j < n$ and any point y in the ball of radius $2\delta_1\sigma^{j/2}$ around $f^{n-j}(x)$.

PROOF. Since n is a (σ,δ)-hyperbolic time for x we have for any $1 \le j < n$

$$\operatorname{dist}_\delta(f^{n-j}(x), C) \ge \sigma^j.$$

According to the definition of the truncated distance, this means that

$$\operatorname{dist}(f^{n-j}(x), C) = \operatorname{dist}_\delta(f^{n-j}(x), C) \ge \sigma^{bj} \quad \text{or else} \quad \operatorname{dist}(f^{n-j}(x), C) \ge \delta.$$

In either case, we have $\operatorname{dist}(y, f^{n-j}(x)) < \operatorname{dist}(f^{n-j}(x), C)/2$ for any $1 \le j < n$, because we chose $b < 1/2$ and $\delta_1 < \delta/4 < 1/4$. Therefore, we may use (s_2) to conclude that

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(f^{n-j}(x))^{-1}\|} \le B \frac{\operatorname{dist}(y, f^{n-j}(x))}{\operatorname{dist}(f^{n-j}(x)), C)^\beta} \le B \frac{2\delta_1\sigma^{j/2}}{\min\{\sigma^{b\beta j}, \delta^\beta\}}.$$

Since δ and σ are smaller than 1, and we took $b\beta < 1/2$, the term on the right hand side is bounded by $2B\delta_1\delta^{-\beta}$. Moreover, our second condition on δ_1 means that this last expression is smaller than $\log \sigma^{-1/2}$. □

The previous lemma will be very useful in proving one the main features of hyperbolic times, namely the existence of inverse branches with uniform backward contraction, as stated in the next result.

PROPOSITION 2.3. *Given $0 < \sigma < 1$ and $\delta > 0$, there exists $\delta_1 > 0$ such that if n is a (σ,δ)-hyperbolic time for x, then there exists a neighborhood V_n of x such that:*

(1) *f^n maps V_n diffeomorphically onto the ball of radius δ_1 around $f^n(x)$;*
(2) *for all $1 \le k < n$ and $y, z \in V_n$,*

$$\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z)).$$

PROOF. Let δ_1 be given by Lemma 2.2. We shall prove, by induction on $j \ge 1$, that there exists a well defined branch of f^{-j} on the ball of radius δ_1 around $f^n(x)$, mapping $f^n(x)$ to $f^{n-j}(x)$. In addition, this branch is a $\sigma^{j/2}$-contraction.

Starting the induction argument, we note that for $j = 1$ the Lemma 2.2 gives

$$\|Df(y)^{-1}\| \le \sigma^{-1/2} \|Df(f^{n-1}(x))^{-1}\| \le \sigma^{1/2},$$

since n is a hyperbolic time for x. This means that f is a $\sigma^{-1/2}$-dilation in the ball of radius $2\delta_1\sigma^{1/2}$ around $f^{n-1}(x)$. As a consequence, there exists some neighborhood $V(n-1)$ of $f^{n-1}(x)$ contained in that ball of radius $2\delta_1\sigma^{1/2}$, that is mapped diffeomorphically onto the ball of radius δ_1 around $f^n(x)$.

Now, given any $j > 1$, let us suppose that we have constructed a neighborhood $V(n-j+1)$ of $f^{n-j+1}(x)$ such that the restriction of f^{j-1} to $V(n-j+1)$ is a diffeomorphism onto the ball of radius δ_1 around $f^n(x)$, with

$$\|Df(f^i(z))^{-1}\| \le \sigma^{-1/2} \|Df(f^{n-j+i+1}(x))^{-1}\| \tag{2.2}$$

for all z in $V(n-j+1)$ and $0 \le i \le j-1$. Then, by Lemma 2.2 and the hypothesis that n is a hyperbolic time for x,

$$\|Df^j(y)^{-1}\| \le \prod_{i=0}^{j-1} \|Df(f^i(y))^{-1}\| \le \prod_{i=0}^{j-1} \sigma^{-1/2} \|Df(f^{n-j+i}(x))^{-1}\| \le \sigma^{j/2}$$

for any point y in the ball of radius $2\delta_1\sigma^{j/2}$ whose image $z = f(y)$ is in $V(n-j+1)$.

Now we can construct an inverse branch of f^j on the ball of radius δ_1 around $f^n(x)$, by lifting geodesics in the following way. Given a geodesic γ connecting $f^n(x)$ to a point in the boundary of the ball, there is a well defined lift of the restriction of γ to a small neighborhood of $f^n(x)$, into a curve starting at $f^{n-j}(x)$. Moreover, as far as this curve does not leave the ball of radius $2\delta_1\sigma^{j/2}$, the derivative on it is a $\sigma^{-j/2}$-dilation. This means that the length of the lifted curve is less than $\delta_1\sigma^{j/2}$, and so the curve is actually contained in a smaller ball. This proves that the lift is well defined on the whole geodesic γ. Thus, we have a well defined branch of f^{-j} on the ball of radius δ_1 around $f^n(x)$ as we claimed. We call $V(n-j)$ the image of that inverse branch. By construction, $V(n-j)$ is contained in the $2\delta_1\sigma^{j/2}$-ball around $f^{n-j}(x)$ and its image under f coincides with $V(n-j+1)$. So, in view of Lemma 2.2, we also recovered the induction assumption (2.2) for points in $V(n-j)$ and times $0 \le i \le j$.

In this way, we construct neighborhoods $V(n-j)$ of $f^{n-j}(x)$ as above, for all $1 \le j \le n$. The lemma follows taking $V_n = V(0)$. □

We shall often refer to the sets V_n as *hyperbolic pre-balls* and to their images $f^n(V_n)$ as *hyperbolic balls*. Notice that the latter are indeed balls of radius $\delta_1 > 0$.

REMARK 2.4. It follows from the proof of the previous proposition that for every x belonging to a hyperbolic pre-ball V_n associated to a (σ, δ)-hyperbolic time n we have $\|Df^n(x)^{-1}\| \le \sigma^{n/2}$.

One of the most important properties of hyperbolic times is the uniformly bounded distortion on hyperbolic pre-balls given by the next result.

COROLLARY 2.5 (Bounded Distortion). *There exists $C_0 > 0$ such that for every hyperbolic pre-ball V_n and every $y, z \in V_n$*

$$\log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le C_0 \operatorname{dist}(f^n(y), f^n(z)).$$

PROOF. Let $x \in M$ be the point having n as a (σ, δ)-hyperbolic time with associated hyperbolic pre-ball V_n. By Proposition 2.3 we have for each $y, z \in V_n$ and each $0 \le k < n$

$$\operatorname{dist}(f^k(y), f^k(z)) \le \delta_1 \sigma^{(n-k)/2}.$$

On the other hand, since n is a hyperbolic time for x

$$\begin{aligned}
\operatorname{dist}(f^k(y),\mathcal{C}) &\geq \operatorname{dist}(f^k(x),\mathcal{C}) - \operatorname{dist}(f^k(x),f^k(y)) \\
&\geq \sigma^{b(n-k)} - \delta_1\sigma^{(n-k)/2} \\
&\geq \frac{1}{2}\sigma^{b(n-k)} \qquad\qquad (2.3)\\
&\geq 2\delta_1\sigma^{(n-k)/2},
\end{aligned}$$

as long as $\delta_1 < 1/4$; recall that $b < 1/2$. Thus we have

$$\operatorname{dist}(f^k(y),f^k(z)) \leq \frac{1}{2}\operatorname{dist}(f^k(y),\mathcal{C}),$$

and so we may use (s_3) to obtain

$$\log\frac{|\det Df(f^k(y))|}{|\det Df(f^k(z))|} \leq \frac{B}{\operatorname{dist}(f^k(y),\mathcal{C})^\beta}\operatorname{dist}(f^k(y),f^k(z)).$$

Hence, by (2.3) and Proposition 2.3

$$\begin{aligned}
\log\frac{|\det Df^n(y)|}{|\det Df^n(z)|} &= \sum_{k=0}^{n-1}\log\frac{|\det Df(f^k(y))|}{|\det Df(f^k(z))|} \\
&\leq \sum_{k=0}^{n-1} 2^\beta B\frac{\sigma^{(n-k)/2}}{\sigma^{b\beta(n-k)}}\operatorname{dist}(f^n(y),f^n(z)).
\end{aligned}$$

It suffices to take $C_0 \geq \sum_{k=1}^\infty 2^\beta B\sigma^{(1/2-b\beta)k}$; recall that $b\beta < 1/2$. □

Many times along this text it will be useful to have the following weaker form of the previous corollary.

COROLLARY 2.6. *There exists $C_1 > 0$ such that for every hyperbolic pre-ball V_n and every $y,z \in V_n$*

$$\frac{1}{C_1} \leq \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1.$$

PROOF. Take $C_1 = \exp(C_0 D)$, where D is the diameter of M. □

COROLLARY 2.7. *There is a constant $C_2 > 0$ (only depending on δ_1 and C_1) such that for every $n \geq 0$*

$$\frac{d}{dm}f_*^n\big(m \mid H_n\big) \leq C_2,$$

where H_n is the set of points that have $n \in \mathbb{N}$ as a (σ,δ)-hyperbolic time.

PROOF. Take $\delta_1 > 0$ given by Proposition 2.3. It suffices to show that there is some uniform constant $C > 0$ such that if $A \subset M$ is a Borel set with diameter smaller than $\delta_1/2$, then

$$m\big(f^{-n}(A) \cap H_n\big) \leq Cm(A).$$

Let A be a Borel set in M with diameter smaller than $\delta_1/2$ and B an open ball of radius $\delta_1/2$ containing A. Taking the connected components of $f^{-n}(B)$ we may write

$$f^{-n}(B) = \bigcup_{k\geq 1} B_k,$$

where $(B_k)_{k\geq 1}$ is a (possibly finite) family of two-by-two disjoint open sets in M. Considering only those B_k that intersect H_n, we choose, for each $k \geq 1$, a point $x_k \in H_n \cap B_k$. For each $k \geq 1$ let V_{x_k} be the neighborhood of x_k given by Proposition 2.3. Since B is contained in $B(f^n(x_k), \delta_1)$, the ball of radius δ_1 around $f^n(x_k)$, and f^n is a diffeomorphism from V_{x_k} onto $B(f^n(x_k), \delta_1)$, we must have $B_k \subset V_{n_k}$ (recall that by the choice of B_k we have $f^n(B_k) \subset B$). As a consequence of this and Proposition 2.3 we have that $f^n \mid B_k \colon B_k \to B$ is a diffeomorphism with uniformly bounded distortion for all $n \geq 1$ and $k \geq 1$:

$$\frac{1}{C_1} \leq \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1 \quad \text{for all } y, z \in B_k.$$

This finally gives

$$\begin{aligned} m(f^{-n}(A) \cap H_n) &\leq \sum_{k\geq 1} m(f^{-n}(A \cap B) \cap B_k) \\ &\leq \sum_{k\geq 1} C_1 \frac{m(A \cap B)}{m(B)} m(B_k) \\ &\leq C_2 m(A), \end{aligned}$$

for some constant $C_2 > 0$ only depending on $C_1 > 0$ and on the volume of the ball B of radius $\delta_1/2$. □

2. Positive frequency

In the first half of this section we will derive some consequences of the existence of many points in the phase space with positive frequency of hyperbolic times. In the second half we show that non-uniformly expanding maps have positive frequency of hyperbolic times at Lebesgue almost every point.

DEFINITION 2.8. We say that the *frequency of (σ, δ)-hyperbolic times* for $x \in M$ is positive, if there is some $\theta > 0$ such that for large $n \in \mathbb{N}$ there are $\ell \geq \theta n$ and integers $1 \leq n_1 < n_2 \cdots < n_\ell \leq n$ which are (σ, δ)-hyperbolic times for x.

We will extract several interesting consequences from the lemma that we present next.

LEMMA 2.9. *Let $A \subset M$ be a set with positive Lebesgue measure whose points have positive frequency of (σ, δ)-hyperbolic times. Then there are $\theta > 0$ and*

$n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{m(A\cap H_j)}{m(A)} \geq \theta,$$

where H_j is the set of points that have j as a (σ,δ)-hyperbolic time.

PROOF. Since we are assuming that points in A have positive frequency of (σ,δ)-hyperbolic times, then there are $\theta > 0$, a set $B \subset A$ with $m(B) \geq m(A)/2$, and $n_0 \in \mathbb{N}$ such that for every $x \in B$ and $n \geq n_0$ there are (σ,δ)-hyperbolic times $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ for x with $\ell \geq 2\theta n$. Take now $n \geq n_0$ and let ξ_n be the measure in $\{1,\ldots,n\}$ defined by $\xi_n(J) = \#J/n$, for each subset J. Then, using Fubini's Theorem

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} m(B\cap H_j) &= \int\left(\int_B \mathbf{1}(x,i)\,dm(x)\right)d\xi_n(i)\\ &= \int_B\left(\int \mathbf{1}(x,i)\,d\xi_n(i)\right)dm(x),\end{aligned}$$

where $\mathbf{1}(x,i) = 1$ if $x \in H_i$, and $\mathbf{1}(x,i) = 0$ otherwise. Since for every $x \in B$ and $n \geq n_0$ there are $0 < n_1 < n_2 < \cdots < n_\ell \leq n$ with $\ell \geq 2\theta n$ such that $x \in H_{n_i}$ for $1 \leq i \leq \ell$, then the integral with respect to $d\xi_n$ is larger than $2\theta > 0$. So, the last expression in the formula above is bounded from below by $2\theta m(B) = \theta m(A)$. $\square$

Before we state our next result, let us recall that a set $H \subset M$ is said to be *positively invariant* by $f\colon M \to M$ if $f(H) \subset H$.

THEOREM 2.10. *Let $f\colon M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C} \subset M$. If there is $H \subset M$ with $m(H) > 0$ whose points have positive frequency of (σ,δ)-hyperbolic times, then f has some absolutely continuous invariant probability measure μ. Moreover, if H is a positively invariant closed set, then μ has support contained in $\cap_{j\geq 1} f^j(H)$.*

PROOF. We let $(\mu_n)_n$ be the sequence of the averages of the positive iterates of the Lebesgue measure restricted to H,

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m \mid H).$$

Letting H_j be the set of points in H that have $j \geq 1$ as a (σ,δ)-hyperbolic time, we define

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m \mid H_j).$$

By Lemma 2.9 we have some $\theta > 0$ and $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\nu_n(M) \geq \frac{1}{n}\sum_{i=0}^{n-1} m(H_i) \geq \frac{1}{n}\sum_{i=0}^{n-1} m(H \cap H_i) \geq \theta m(H). \tag{2.4}$$

Moreover, by Corollary 2.7, every $f_*^j(m|H_j)$ is absolutely continuous with respect to Lebesgue measure, with density uniformly bounded from above, and so the same is true for every ν_n.

Since we are working with a continuous map in the compact space M, we know that sequences of probability measures in M have weak* accumulation points. Take $n_k \to \infty$ such that both μ_{n_k} and ν_{n_k} converge in the weak* sense to measures μ and ν, respectively. Then μ is an invariant probability measure, $\mu = \nu + \eta$ for some measure η, ν is absolutely continuous with respect to Lebesgue measure, and $\nu(H) > 0$ by (2.4). Now, if $\eta = \eta_{ac} + \eta_s$ denotes the Lebesgue decomposition of η (as the sum of an absolutely continuous and a singular measure, with respect to Lebesgue measure), then $\mu_{ac} = \nu + \eta_{ac}$ gives the absolutely continuous component in the corresponding decomposition of μ. By uniqueness of the Lebesgue decomposition, and the fact that the push-forward under f preserves the class of absolutely continuous measures, we may conclude that μ_{ac} is an invariant measure. Clearly, $\mu_{ac}(H) \geq \nu(H) > 0$. Normalizing μ_{ac} we obtain an absolutely continuous f-invariant probability measure.

For the second statement of the theorem we just have to observe that since H is a positively invariant closed set and the measures μ_n are supported in the positive images of H by f, then the support of μ must be contained in the maximal invariant set contained in H, which is precisely $\bigcap_{j\geq 1} f^j(H)$. □

Our aim now is to show that hyperbolic times appear with positive frequency for non-uniformly expanding maps. The following lemma, due to Pliss [Pl], plays a crucial role in the main result that we are going to prove in that direction.

LEMMA 2.11. *Given $0 < c_1 < c_2 < A$ let $\theta = (c_2 - c_1)/(A - c_1)$. Given real numbers $a_1, \ldots, a_N$ satisfying $a_j \leq A$ for every $1 \leq j \leq N$ and*

$$\sum_{j=1}^{N} a_j \geq c_2 N,$$

there are $l > \theta N$ and $1 < n_1 < \cdots < n_l \leq N$ so that

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n)$$

for every $0 \leq n < n_i$ and $i = 1, \ldots, l$.

PROOF. Define for each $1 \leq n \leq N$,

$$S_n = \sum_{j=1}^{n}(a_j - c_1), \quad \text{and also} \quad S_0 = 0.$$

Then define $1 < n_1 < \cdots < n_l \le N$ to be the maximal sequence such that $S_{n_i} \ge S_n$ for every $0 \le n < n_i$ and $i = 1, \ldots, l$. Note that l cannot be zero, since $S_N > S_0$. Moreover, the definition means that

$$\sum_{j=n+1}^{n_i} a_j \ge c_1(n_i - n), \quad \text{for } 0 \le n < n_i \text{ and } i = 1, \ldots, l.$$

So, we only have to check that $l > \theta_0 N$. Observe that, by definition,

$$S_{n_i - 1} < S_{n_{i-1}} \quad \text{and so} \quad S_{n_i} < S_{n_{i-1}} + (A - c_1)$$

for every $1 < i \le l$. Moreover,

$$S_{n_1} \le (A - c_1) \quad \text{and} \quad S_{n_l} \ge S_N \ge N(c_2 - c_1).$$

This gives,

$$N(c_2 - c_1) \le S_{n_l} = \sum_{i=2}^{l} \left(S_{n_i} - S_{n_{i-1}} \right) + S_{n_1} < l(A - c_1),$$

which completes the proof. □

PROPOSITION 2.12. *Assume that $f\colon M \to M$ is non-uniformly expanding on $H \subset M$. Then there are $0 < \sigma < 1$, $\delta > 0$ and $\theta > 0$ (depending only on λ and on the map f) such that the frequency of (σ, δ)-hyperbolic times for points in H is greater than θ.*

PROOF. The strategy is to use Lemma 2.11 twice, first for the sequence given by $a_j = -\log \|Df(f^{j-1}(x))^{-1}\|$ (up to a cut off that makes it bounded from above), and then with $a_j = \log \operatorname{dist}_\delta(f^{j-1}(x), \mathcal{C})$ for a convenient $\delta > 0$. We prove that there exist many times n_i for which the conclusion of Lemma 2.11 holds, simultaneously, for both sequences. Then we check that any such n_i is a (σ, δ)-hyperbolic time for x.

Assuming that (1.2) holds for $x \in H$, then for large $N \in \mathbb{N}$ we have

$$\sum_{j=0}^{N-1} -\log \|Df(f^j(x))^{-1}\| \ge \lambda N.$$

Take $\beta > 0$ given by Definition 1.1, and fix any $\rho > \beta$. Then (s_2) implies that

$$\big| \log \|Df(x)^{-1}\| \big| \le \rho \,|\log \operatorname{dist}(x, \mathcal{C})| \tag{2.5}$$

for every x in a neighborhood V of $\mathcal{C}$. Fix $\varepsilon_1 > 0$ so that $\rho\varepsilon_1 \le \lambda/2$, and let $r_1 > 0$ be so that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_1}(f^j(x), \mathcal{C}) \ge -\varepsilon_1 N. \tag{2.6}$$

The assumption of slow recurrence to the critical set ensures that this is possible. Fix any $K_1 \ge \rho |\log r_1|$ large enough so that it is also an upper bound for $-\log\|Df^{-1}\|$ on the complement of V. Then let J be the subset of times $1 \le j \le N$ such that $-\log\|Df(f^{j-1}(x))^{-1}\| > K_1$, and define

$$a_j = \begin{cases} -\log\|Df(f^{j-1}(x))^{-1}\| & \text{if } j \notin J \\ 0 & \text{if } j \in J. \end{cases}$$

By construction, $a_j \le K_1$ for $1 \le j \le N$. Note that if $j \in J$ then $f^{j-1}(x) \in V$. Moreover, for each $j \in J$

$$\rho\,|\log r_1| \le K_1 < -\log\|Df(f^{j-1}(x))^{-1}\| < \rho\,|\log \operatorname{dist}(f^{j-1}(x),\mathcal{C})|,$$

which shows that $\operatorname{dist}(f^{j-1}(x),\mathcal{C}) < r_1$ for every $j \in J$. In particular,

$$\operatorname{dist}_{r_1}(f^{j-1}(x),\mathcal{C}) = \operatorname{dist}(f^{j-1}(x),\mathcal{C}) < r_1, \quad \text{for all } j \in J.$$

Therefore, by (2.5) and (2.6),

$$\sum_{j\in J} -\log\|Df(f^{j-1}(x))^{-1}\| \le \rho \sum_{j\in J} |\log \operatorname{dist}(f^{j-1}(x),\mathcal{C})| \le \rho\,\varepsilon_1 N.$$

We have chosen $\varepsilon_1 > 0$ in such a way that the last term is less than $\lambda N/2$. As a consequence,

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} -\log\|Df(f^{j-1}(x))^{-1}\| - \sum_{j\in J} -\log\|Df(f^{j-1}(x))^{-1}\| \ge \frac{\lambda}{2}N\,.$$

Thus, we have checked that we may apply Lemma 2.11 to the numbers $a_1,\dots,a_N$, with $c_1 = \lambda/4$, $c_2 = \lambda/2$, and $A = K_1$. The lemma provides $\theta_1 > 0$ and $l_1 \ge \theta_1 N$ times $1 \le p_1 < \cdots < p_{l_1} \le N$ such that

$$\sum_{j=n+1}^{p_i} -\log\|Df(f^{j-1}(x))^{-1}\| \ge \sum_{j=n+1}^{p_i} a_j \ge \frac{\lambda}{4}(p_i - n) \tag{2.7}$$

for every $0 \le n < p_i$ and $1 \le i \le l_1$.

Now fix $\varepsilon_2 > 0$ small enough so that $\varepsilon_2 < \theta_1 b\lambda/4$, and let $r_2 > 0$ be such that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_2}(f^j(x),\mathcal{C}) \ge -\varepsilon_2 N\,. \tag{2.8}$$

Let $c_1 = -b\lambda/4$, $c_2 = -\varepsilon_2$, $A = 0$, and

$$\theta_2 = \frac{c_2 - c_1}{A - c_1} = 1 - \frac{4\varepsilon_2}{b\,\lambda}\,.$$

Applying Lemma 2.11 to $a_j = \log \operatorname{dist}_{r_2}(f^{j-1}(x),\mathcal{C})$, with $1 \le j \le N$, we conclude that there are $l_2 \ge \theta_2 N$ times $1 \le q_1 < \cdots < q_{l_2} \le N$ such that

$$\sum_{j=n}^{q_i-1} \log \operatorname{dist}_{r_2}(f^j(x),\mathcal{C}) \ge -\frac{b\lambda}{4}\,(q_i - n) \tag{2.9}$$

for every $0 \le n < q_i$ and $1 \le i \le l_2$.

Finally, our condition on ε_2 means that $\theta_1+\theta_2>1$. Let $\theta=\theta_1+\theta_2-1$. Then there exist $l=(l_1+l_2-N)\geq\theta N$ times $1\leq n_1<\cdots<n_l\leq N$ at which (2.7) and (2.9) occur simultaneously:

$$\sum_{j=n}^{n_i-1}-\log\|Df(f^j(x))^{-1}\|\geq\frac{\lambda}{4}(n_i-n)$$

and

$$\sum_{j=n}^{n_i-1}\log\operatorname{dist}_{r_2}(f^j(x),\mathcal{C})\geq-\frac{b\lambda}{4}(n_i-n),$$

for every $0\leq n<n_i$ and $1\leq i\leq l$. Letting $\sigma=e^{-\lambda/4}$ we easily obtain from the inequalities above

$$\prod_{j=n_i-k}^{n_i-1}\|Df(f^j(x))^{-1}\|\leq\sigma^k\quad\text{and}\quad\operatorname{dist}_{r_2}(f^{n_i-k}(x),\mathcal{C})\geq\sigma^{bk}$$

for every $1\leq i\leq l$ and $1\leq k\leq n_i$. In other words, all those n_i are (σ,δ)-hyperbolic times for x, with $\delta=r_2$. □

REMARK 2.13. From the proof of Proposition 2.12 one easily sees that condition (1.3) in the definition of non-uniformly expanding map is not needed in all its strength for the proof work. Actually, the only places where we have used (1.3) are (2.6) and (2.8). Hence, it is enough that (1.3) holds for $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$ and $\delta=\max\{r_1,r_2\}$.

REMARK 2.14. Observe that the proof of Proposition 2.12 gives more precisely that if for some $x\in M$ and $N\in\mathbb{N}$

$$\sum_{j=0}^{N-1}-\log\|Df(f^j(x))^{-1}\|\geq\lambda N\quad\text{and}\quad\sum_{j=0}^{N-1}\log\operatorname{dist}_\delta(f^j(x),\mathcal{C})\geq-\varepsilon N$$

(where ε and δ are chosen according to in Remark 2.13), then there exist integers $0<n_1<\cdots<n_l\leq N$ with $l\geq\theta N$ such that n_i is a (σ,δ)-hyperbolic time for x for each $1\leq i\leq l$.

3. First hyperbolic time map

The existence of (σ,δ)-hyperbolic times for Lebesgue almost all points in M allows us to introduce a map $h\colon M\to\mathbb{Z}^+$ defined Lebesgue almost everywhere and assigning to each $x\in M$ its first (σ,δ)-hyperbolic time. Related to integrability properties of this first hyperbolic time map are some statistical properties of several classes of dynamical systems, such as stochastic stability and correlation decay. The same conclusion of Theorem 2.10 can be obtained under the assumption of integrability of the first hyperbolic time map.

Consider $(\mu_n)_n$ the sequence of averages of forward iterates of Lebesgue measure on M

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m.$$

Since we are dealing with a continuous map of a compact manifold, we know that the sequence $(\mu_n)_n$ has accumulation points, which belong to the space of probability measures invariant by f. Now the idea is to show that such accumulation points are absolutely continuous with respect to the Lebesgue measure.

Defining, for each $n \geq 1$,

$$H_n^* = \{x \in M\colon\ n \text{ is the \textit{first} } (\sigma,\delta)\text{-hyperbolic time for } x\},$$

we immediately have

$$\int_M h\,dm = \sum_{k=1}^{\infty} k\,m(H_k^*). \tag{2.10}$$

It will be useful to define, for each $n, k \geq 1$,

$$H_n = \{x \in M\colon\ n \text{ is a } (\sigma,\delta)\text{-hyperbolic time for } x\},$$

and

$$R_{n,k} = \left\{\, x \in H_n :\ f^n(x) \in H_k^* \,\right\}.$$

Observe that $R_{n,k}$ is precisely the set of points $x \in M$ for which n is a (σ,δ)-hyperbolic time and $n+k$ is the next (σ,δ)-hyperbolic time for x after n. Defining the measures

$$\nu_n = f_*^n(m \mid H_n) \tag{2.11}$$

and

$$\eta_n = \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} f_*^{n+j}(m \mid R_{n,k}), \tag{2.12}$$

we may write

$$\mu_n \leq \frac{1}{n}\sum_{j=0}^{n-1}(\nu_j + \eta_j).$$

It follows from Corollary 2.7 that

$$\frac{d\nu_n}{dm} \leq C_2 \tag{2.13}$$

for every $n \geq 0$, with C_2 not depending on n. Our goal now is to control the densities of the measures η_n.

PROPOSITION 2.15. *Given $\varepsilon > 0$, there is $C_3(\varepsilon) > 0$ such that for every $n \geq 1$ we may bound η_n by the sum of two non-negative measures, $\eta_n \leq \omega + \rho$, with*

$$\frac{d\omega}{dm} \leq C_3(\varepsilon) \quad \textit{and} \quad \rho(M) < \varepsilon.$$

PROOF. Let A be some Borel set in M. For each $n \geq 0$ we have

$$\begin{aligned}\eta_n(A) &= \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} m\big(f^{-n-j}(A) \cap R_{n,k}\big)\\ &\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} m\big(f^{-n}\big(f^{-j}(A) \cap H_k^*\big) \cap H_n\big)\\ &\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} C_2 m\big(f^{-j}(A) \cap H_k^*\big).\end{aligned}$$

(in this last inequality we have used the bound (2.13) above). Let now $\varepsilon > 0$ be some fixed small number. By the integrability of h and since (2.10) holds, we may choose some integer $\ell = \ell(\varepsilon)$ for which

$$\sum_{j=\ell}^{\infty} k\, m(H_k^*) < \frac{\varepsilon}{C_2}.$$

We take

$$\omega = C_2 \sum_{k=2}^{\ell-1}\sum_{j=1}^{k-1} f_*^j(m \mid H_k^*)$$

and

$$\rho = C_2 \sum_{k=\ell}^{\infty}\sum_{j=1}^{k-1} f_*^j(m \mid H_k^*). \tag{2.14}$$

This last measure satisfies

$$\rho(M) = C_2 \sum_{k=\ell}^{\infty}\sum_{j=1}^{k-1} m(H_k^*) \leq C_2 \sum_{k=\ell}^{\infty} k\, m(H_k^*) < \varepsilon.$$

On the other hand, we have

$$\omega \leq C_2 \sum_{k=2}^{\ell-1}\sum_{j=1}^{k-1} f_*^j m,$$

and this last measure has density bounded by some constant since we are taking a finite number of push-forwards of Lebesgue measure by the non-degeneracy conditions of f. □

It follows from this last proposition and (2.13) that weak* accumulation points of $(\mu_n)_n$ cannot have singular part, thus being absolutely continuous with respect to the Lebesgue measure. Since such weak* accumulation points are invariant with respect to f, we have proved the following result:

THEOREM 2.16. *Let $f: M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C} \subset M$. If the first (σ, δ)-hyperbolic time map $h: M \to \mathbb{Z}^+$ is Lebesgue integrable, then f has some absolutely continuous invariant probability measure.*

4. Integrability vs. positive frequency

Here we relate the integrability of the first hyperbolic time map with the existence of a positive frequency of hyperbolic times. This is far from being completely understood, but we have already some results depicting a good part of the situation. A first result in this direction may be obtained for local diffeomorphisms.

THEOREM 2.17. *Let $f: M \to M$ be a C^2 local diffeomorphism. If for some $0 < \sigma < 1$ the first σ-hyperbolic time map is Lebesgue integrable, then there is $\bar{\sigma} > 0$ such that Lebesgue almost every $x \in M$ has positive frequency of $\bar{\sigma}$-hyperbolic times.*

PROOF. Observe that by definition of σ-hyperbolic time, if n is a σ-hyperbolic time for x and k is a σ-hyperbolic time for $f^n(x)$, then $n + k$ is a σ-hyperbolic time for x. Moreover, since h is well defined Lebesgue almost everywhere and f preserves sets of Lebesgue measure zero, then Lebesgue almost all points must have infinitely many hyperbolic times. Thus we have

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le \log \sigma < 0 \tag{2.15}$$

for Lebesgue almost every $x \in M$. Let μ be the absolutely continuous f-invariant measure given by Theorem 2.16. We have that (2.15) also holds for μ almost every $x \in M$. Since we are assuming f a local diffeomorphism we have that $\log \|Df(x)^{-1}\|$ is continuous and hence integrable with respect to μ. Birkhoff's ergodic theorem then ensures that the limit in (2.15) exists μ almost everywhere, and so

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le \log \sigma < 0 \tag{2.16}$$

for μ almost every $x \in M$. Since μ is absolutely continuous with respect to m we have that condition (2.16) holds for a set of points in M with positive Lebesgue measure. Actually condition (2.16) holds for Lebesgue almost every $x \in M$. Indeed, let H be the set of points for which (2.16) holds. Since H is positively invariant by f and $h|_H$ is integrable with respect to $m|H$, then if $m(M \setminus H) > 0$, by the previous argument we would prove the existence of some $A \subset M \setminus H$ with $m(A) > 0$ such that (2.16) holds for every $x \in A$. This would naturally give a contradiction.

Now since (2.16) holds for Lebesgue almost every $x \in M$, applying Proposition 2.12 we obtain the desired conclusion. $\square$

We are interested in relating the integrability of the first hyperbolic time map to the existence of positive frequency of hyperbolic times for maps with critical sets as well. We assume until the end of this section that $f : M \to M$ is a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C} \subset M$.

PROPOSITION 2.18. *If $\mathcal{C}$ is a compact submanifold of M with* $\dim(\mathcal{C}) < \dim(M)$, *then* $\log \operatorname{dist}(x, \mathcal{C})$ *belongs to* $L^p(m)$ *for every* $1 \le p < \infty$.

PROOF. We may assume without loss of generality that $\mathcal{C}$ is connected. Let $\dim(\mathcal{C}) = k < n = \dim(M)$. We may cover $\mathcal{C}$ with finitely many images of charts $\psi_i(U_i)$ $(i = 1, \dots, p)$ such that $U_i \subset \mathbb{R}^n$ is a bounded open set and $\psi_i^{-1}(\mathcal{C}) \subset U_i \cap (\mathbb{R}^k \times 0^{n-k})$. Denoting by λ the usual n-dimensional volume on $\mathbb{R}^n$ and by d the standard Euclidean distance on $\mathbb{R}^n$, then there are constants $C, K > 0$ such that for all $i = 1, \dots, p$

$$\frac{1}{C} \le \frac{d(\psi_i^{-1})_* m}{d\lambda} \le C,$$

and for all $w, z \in U_i$

$$\frac{1}{K} d(w, z) \le \operatorname{dist}(\psi_i(w), \psi_i(z)) \le K d(w, z).$$

Hence, for showing that $\log \operatorname{dist}(x, \mathcal{C})$ is integrable with respect to m, it is enough to show that $\log d(x, U \cap (\mathbb{R}^k \times 0^{n-k}))$ is integrable with respect to λ for any open and bounded neighborhood U of the origin in $\mathbb{R}^n$. We may assume without loss of generality that U is sufficiently small in order to $U \subset B_k \times B_{n-k}$, where B_k and B_{n-k} are the unit balls around the origin in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively. For $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ we have

$$d(z, \mathbb{R}^k \times 0^{n-k}) = (z_{k+1}^2 + \dots + z_n^2)^{1/2}.$$

Hence, we have for $1 \le p < \infty$

$$\int_U |\log d(z, \mathbb{R}^k \times 0^{n-k})|^p d\lambda \le$$

$$\le \frac{1}{2^p} \int_{B_k} \left(\int_{B_{n-k}} |\log(z_{k+1}^2 + \dots + z_n^2)|^p dz_{k+1} \cdots dz_n \right) dz_1 \cdots dz_k.$$

Now it is enough to show that the inner integral in the last expression is finite. Actually, denoting by S_ρ^{n-k-1} the $(n-k-1)$-sphere with radius ρ around the origin in $\mathbb{R}^{n-k}$, dA its area element and a the total area of S_1^{n-k-1}, we have

$$\begin{aligned} \int_{B_{n-k}} |\log(z_{k+1}^2 + \dots + z_n^2)|^p dz_{k+1} \cdots dz_n &= \int_0^1 \left(\int_{S_\rho^{n-k-1}} |2 \log \rho|^p \, dA \right) d\rho \\ &= a \int_0^1 \rho^{n-k-1} |\log \rho|^p \, d\rho. \end{aligned}$$

Since this last integral is finite, we finish the proof of the result. □

Assume now that h is integrable with respect to the Lebesgue measure. By Theorem 2.16 there exists an absolutely continuous invariant probability measure μ for f.

COROLLARY 2.19. *If the density $d\mu/dm$ belongs to $L^q(m)$ for some $q > 1$, then* $\log \operatorname{dist}(x, \mathcal{C})$ *is μ-integrable.*

PROOF. This is an immediate application of Hölder inequality. Actually, since

$$\int \log \operatorname{dist}(x, \mathcal{C}) d\mu = \int \log \operatorname{dist}(x, \mathcal{C}) \frac{d\mu}{dm}\, dm,$$

and we have $d\mu/dm$ belonging to $L^q(m)$ for some $q > 1$, and $\log \operatorname{dist}(x, \mathcal{C})$ belonging $L^p(m)$ for every p, then taking p equal to the conjugate of q, that is $p^{-1} + q^{-1} = 1$, then Hölder inequality gives that the integral above is finite. □

Our aim now is to get the same conclusion of the previous corollary under an integrability hypothesis on the first hyperbolic time map. Observe that the absolutely continuous f-invariant measure μ may be obtained as a weak* accumulation point of the sequence $(\mu_n)_n$ of averages of push-forwards of Lebesgue measure. As shown in Section 3, we may write

$$\mu_n \leq \frac{1}{n} \sum_{j=0}^{n-1} (\nu_j + \eta_j),$$

where ν_j and η_j are given by (2.11) and (2.12).

LEMMA 2.20. *If the first (σ, δ)-hyperbolic time map $h : M \to \mathbb{Z}^+$ belongs to $L^p(m)$ for some $p > 4$, then* $\log \operatorname{dist}(x, \mathcal{C})$ *is μ-integrable.*

PROOF. We take any $\varepsilon > 0$ and use Corollary 2.7 to ensure the existence of two non-negative measures ω and ρ bounding η_n, where ω has density bounded by some constant and ρ has total mass bounded by ε. Recall that ρ was defined in (2.14) by

$$\rho = C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} f_*^j (m \mid H_k^*),$$

where ℓ is some large integer.

Let us compute now the weight ρ gives to some special family of neighborhoods of $\mathcal{C}$. For $i \geq 1$ let $d_i = \sigma^{bi}$ where $0 < \sigma < 1$ comes from the definition of (σ, δ)-hyperbolic time. Define for $i \geq 1$

$$B_i = \{x \in M : \operatorname{dist}(x, \mathcal{C}) \leq d_i\}.$$

If n is a (σ,δ)-hyperbolic time for $x \in M$, then $f^j(x) \in M \setminus B_i$ for all $j \in \{n-i,\dots,n-1\}$. This implies that

$$\begin{aligned}
\rho(B_i) &= C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} m(H_k^* \cap f^{-j}(B_i)) \\
&= C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-i} m(H_k^* \cap f^{-j}(B_i)) \\
&\le C_2 \sum_{k=\max\{\ell,i\}}^{\infty} \sum_{j=1}^{k-i} m(H_k^* \cap f^{-j}(B_i)) \\
&\le C_2 \sum_{k=i}^{\infty} k\, m(H_k^*),
\end{aligned}$$

for all $i \ge 1$. Now by Corollary 2.7 and Proposition 2.15 we know that

$$\mu_n \le \frac{1}{n}\sum_{j=0}^{n-1} \nu_j + \omega + \rho \le \nu + \rho$$

where ν is a measure with uniformly bounded density. Hence any weak* accumulation point μ of the sequence $(\mu_n)_n$ is bounded by $\nu+\rho$. Since we are assuming that $\mathcal{C}$ is a submanifold of M, then $\log \operatorname{dist}(x,\mathcal{C})$ is integrable with respect to ν by Proposition 2.18. On the other hand,

$$\int_M -\log \operatorname{dist}_\delta(x,\mathcal{C})\, d\rho \le \sum_{i=1}^{\infty} -\rho(B_i)\log d_{i+1} \le -b\log\sigma \sum_{i=1}^{\infty}(i+1)\sum_{k=i}^{\infty} k\, m(H_k^*).$$

We have $h \in L^p(m)$ by assumption, which is equivalent to

$$\sum_{k\ge 1} k^p m(H_k^*) < \infty.$$

This implies that there is some constant $C > 0$ such that $m(H_k^*) \le Ck^{-p}$ for all $k \ge 1$. Thus we have for $i \ge 2$

$$\sum_{k=i}^{\infty} k\, m(H_k^*) \le \sum_{k=i}^{\infty} \frac{C}{k^{p-1}} \le \int_{i-1}^{\infty} \frac{C}{x^{p-1}}\, dx = \frac{C}{(p-2)(i-1)^{p-2}},$$

and so

$$\sum_{i=2}^{\infty}(i+1)\sum_{k=i}^{\infty} k\, m(H_k^*) \le \frac{C}{p-2}\sum_{i=2}^{\infty} \frac{i+1}{(i-1)^{p-2}}.$$

This last quantity is finite whenever $p > 4$. Hence $\log \operatorname{dist}(x,\mathcal{C})$ is integrable with respect to μ for all $p > 4$. $\square$

REMARK 2.21. As a consequence of the definition of non-degenerate critical set, if $\log \operatorname{dist}(x, \mathcal{C})$ is μ-integrable, then $\log \|Df(x)^{-1}\|$ is also μ-integrable. Actually, by condition (s_1) we have for some $\zeta > \beta$

$$\big|\log \|Df(x)^{-1}\|\big| \le \zeta \big|\log \operatorname{dist}(x, \mathcal{C})\big|$$

for all x in a small open neighborhood V of $\mathcal{C}$. Since $\log \|Df(x)^{-1}\|$ is bounded on the compact set $M \setminus V$, this function is necessarily integrable with respect to μ on M as long as $\log \operatorname{dist}(x, \mathcal{C})$ is μ-integrable.

Now we are able to prove a result for maps with critical sets similar to the one we have in Theorem 2.17, under an integrability assumption on the first hyperbolic time map.

THEOREM 2.22. *Let $f\colon M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C} \subset M$. If for some $0 < \sigma < 1$ and $\delta > 0$ the first (σ, δ)-hyperbolic time map $h : M \to \mathbb{Z}^+$ belongs to $L^p(m)$ with $p > 4$, then there are $\bar\sigma > 0$ and $\theta > 0$ such that the frequency of $(\bar\sigma, \delta)$-hyperbolic times is bigger than θ for Lebesgue almost every $x \in M$.*

PROOF. Assuming that the first (σ, δ)-hyperbolic time map h belongs to $L^p(m)$ for some $p > 4$, it follows from Corollary 2.19, Lemma 2.20 and Remark 2.21 that both

(a) $\log \|Df(x)^{-1}\|$ is integrable with respect to μ;
(b) $\log \operatorname{dist}(x, \mathcal{C})$ is integrable with respect to μ.

Observe that by definition of (σ, δ)-hyperbolic time, if n is a (σ, δ)-hyperbolic time for x and if k is a (σ, δ)-hyperbolic time for $f^n(x)$, then $n+k$ is a (σ, δ)-hyperbolic time for x. Moreover, since h is well defined Lebesgue almost everywhere and f preserves sets of Lebesgue measure zero, then Lebesgue almost all points must have infinitely many hyperbolic times. Thus we have

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le \log \sigma < 0 \tag{2.17}$$

for Lebesgue almost every $x \in M$, and hence for μ almost every $x \in M$. The μ-integrability of $\log \|Df(x)^{-1}\|$ and Birkhoff's ergodic theorem then ensure that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le \log \sigma < 0 \tag{2.18}$$

for μ almost every $x \in M$.

Then by Proposition 1.5 (see also Remark 1.6) there is $H \subset M$ with $m(H) > 0$ where f is non-uniformly expanding. Actually, we will to prove that f is non-uniformly expanding Lebesgue almost every $x \in M$. Let H be the set of points where f is non-uniformly expanding, and take $B = M \setminus H$. Observe that B is invariant by f and also $h \in L^p(m|B)$ with $p > 4$. If $m(B) > 0$, then by the previous argument there would be some $A \subset B$ with $m(A) > 0$ and f non-uniformly expanding on A. This would naturally give a contradiction. Thus we have proved that f is non-uniformly expanding Lebesgue almost everywhere in M. Now we apply Proposition 2.12 and obtain the result. $\square$

We do not know if the strong integrability condition in the hypotheses of the previous theorem is really necessary, or it is just a lack of the method we have used to prove it. It remains an interesting open question to find the smallest value of $p \geq 1$ for which the conclusion of Theorem 2.22 holds.

REMARK 2.23. Observe that the hypothesis of h belonging to $L^p(m)$ for some $p > 4$ can be replaced by $d\mu/dm \in L^q(m)$ for some $q > 1$. In fact, the integrability of h has only been used to prove that $\log \operatorname{dist}(x, \mathcal{C})$ is integrable with respect to μ, which implies that $\log \|Df(x)^{-1}\|$ is also integrable with respect to μ by Remark 2.21. As stated in Corollary 2.19 this is a consequence of $d\mu/dm \in L^q(m)$ for some $q > 1$.

In the opposite direction, one could ask whether the positive frequency of hyperbolic times is enough for assuring integrability of the first hyperbolic time map or not. As we shall see below, the map we have introduced in Example 1.8 shows that the answer to this question is negative. It remains an interesting open problem to know if there is a map with those properties and no critical set, i.e. does positive frequency of hyperbolic times imply integrability of the first hyperbolic time map for transformations with no critical sets?

EXAMPLE 2.24. Let us consider again the map f from S^1 into itself that we have introduced in Example 2.24. Recall that f has been defined as que quotient of the map of the interval into itself

$$x \mapsto \begin{cases} 2\sqrt{x} - 1 & \text{if } x \geq 0, \\ 1 - 2\sqrt{|x|} & \text{otherwise.} \end{cases}$$

We have already seen that f preserves Lebesgue measure and that there is a set $H \subset S^1$ with positive Lebesgue measure where f is non-uniformly expanding. Here we will show that the uniform expansion holds Lebesgue almost everywhere (thus deducing that Lebesgue almost every point has positive frequency of hyperbolic times by Proposition 2.12), and that no first hyperbolic time map is integrable with respect to Lebesgue measure. It will be useful to have shown that f is topologically mixing.

Topological mixing. We will show that given any open interval $J \subset S^1$ there is some $N \in \mathbb{N}$ such that $f^N(J) = S^1$. As we have observed before, f has two inverse branches $g_1 : (-1, 1) \to (0, 1)$ and $g_2 : (-1, 1) \to (-1, 0)$, given by

$$g_1(x) = \left(\frac{1+x}{2}\right)^2 \quad \text{and} \quad g_2(x) = -\left(\frac{1-x}{2}\right)^2.$$

Let $X = \{g_1^n(0), g_2^n(0) : n \geq 0\}$ be a set of points in the pre-orbit of $1 \in S^1$ and let $\emptyset \neq J \subseteq S^1$ be an open interval. We note that if $X \cap J \neq \emptyset$, then there is $n \geq 1$ such that $1 \in f^n(J)$, thus the interval $f^n(J)$ would contain a neighborhood of 1 in S^1 . We easily see that this implies $f^{n+k}(J) = S^1$ for some $k \in \mathbb{N}$. Hence to prove topological mixing for f it is enough to show that given any open interval $J \subset S^1$ there is $j \in \mathbb{N}$ such that $f^j(J) \cap X \neq \emptyset$.

Let us take an interval $J \subset S^1$ such that $J \cap X = \emptyset$. Hence $0 \notin J$. Assume for definiteness that $J \subset (-1, 0)$. Thus there is $n \in \mathbb{N}$ such that $f^n|_J$ is a diffeomorphism and $f^n(J) \subset (0,1)$. It is clear that there is $\sigma > 1$ independent of n such that $m(f^{n+1}(J)) \geq \sigma m(J)$. Now let $J_1 = f^{n+1}(J) \subseteq (0,1)$. If $J_1 \cap X \neq \emptyset$, then we are done. Otherwise, by the symmetry of f, we repeat the argument obtaining an iterate $J_2 \subset (-1,0)$ of J with $m(J_2) \geq \sigma^2 m(J)$. Since $(\sigma^k)_{k\geq 1}$ is unbounded, after a finite number of iterates the image of J will eventually hit X.

Positive frequency of hyperbolic times almost everywhere. As we have shown before, there is a positive Lebesgue measure set $H \subset S^1$ where f is non-uniformly expanding. Then by Let us now prove that Lebesgue measure is ergodic with respect to f. Then by Theorem 2.10 there exists some absolutely continuous f-invariant probability measure μ. Using an ergodic component if necessary (Theorem 0.11) we may assume that μ is ergodic. Moreover, taking

$$G = \left\{ x \in S^1 : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \xrightarrow{w*} \mu, \quad \text{when } n \to \infty \right\},$$

we have that there is an interval $J \subset G$, up to a null Lebesgue measure subset; see Lemma 3.3. Due to the topological mixing property and the regularity of f (preserves null Lebesgue measure sets) this implies that G equals S^1 up to a Lebesgue measure zero subset. Hence, by Birkhoff's ergodic theorem $m = \mu$. Now, by the ergodicity of m, the limits in (1.2) and (1.3) are constant m almost everywhere. Since $m(H) > 0$, then f must be non-uniformly expanding Lebesgue almost everywhere.

Non-integrability of the first hyperbolic time map. We now show that the map $h : S^1 \to \mathbb{N}$ assigning to each $x \in S^1$ the first (σ, δ)-hyperbolic time of x cannot be Lebesgue integrable in S^1 for any $0 < \sigma < 1$ and $\delta > 0$. We observe that if n is a (σ, δ)-hyperbolic time for x, then in particular it must be

$$|f'(f^{n-1}(x))| \geq \sigma^{-1} > 1.$$

Hence the first (σ, δ)-hyperbolic time for a given $x \in S^1 \setminus C$ is at least the number of iterates needed for x to enter into a neighborhood of 0. Considering the inverse branch g_1 of f and iterate a point $x_1 \in (0, 1/2)$ under g_1, we obtain a sequence $(x_n)_{n\geq 1}$ in $(0,1)$ satisfying

$$x_{n+1} = \frac{(1 + x_n)^2}{4}, \quad n \geq 1. \tag{2.19}$$

According to the observation above, we must have

$$\int_{S^1} h\,dm \geq \sum_{n\geq 1} n(x_{n+1}-x_n).$$

In order to see that h is not integrable with respect to m, it suffices to show that

$$\sum_{n\geq 1} n(x_{n+1}-x_n) = +\infty. \tag{2.20}$$

We first prove (by induction) that

$$0 \leq x_n \leq 1 - \frac{1}{2n} \quad \text{for every } n \geq 1. \tag{2.21}$$

This obviously holds for $n=1$ since we have chosen $x_1 \in (0, 1/2)$. Assuming that (2.21) holds for $n \geq 1$ we then have

$$\begin{aligned} 0 \leq x_{n+1} &= \frac{(1+x_n)^2}{4} \\ &\leq \frac{(2-1/(2n))^2}{4} \\ &= 1 - \frac{1}{2n} + \frac{1}{16n^2} \\ &= 1 - \frac{1}{2n+2}\left(\frac{n+1}{n} - \frac{n+1}{8n^2}\right). \end{aligned}$$

It is enough to observe that

$$\frac{n+1}{n} - \frac{n+1}{8n^2} = \frac{8n^2+7n-1}{8n^2} > 1, \quad \text{for all } n \geq 1.$$

Using the recurrence relation (2.19), a simple calculation now shows that

$$x_{n+1} - x_n = \frac{(1-x_n)^2}{4},$$

which together with (2.21) leads to

$$x_{n+1} - x_n \geq \frac{1}{16n^2}.$$

This is enough for concluding (2.20), which implies that h in non-integrable with respect to Lebesgue measure.

CHAPTER 3

SRB measures

One effective way for studying the statistical properties of dynamical systems is by determining the time (in average) typical orbits spend in different regions of the phase space. According to the ergodic theorem of Birkhoff, such times are well defined for almost all points, with respect to any invariant probability measure. However, the notion of typical orbit is usually meant in the sense of Lebesgue measure, which is not always captured by invariant measures. Indeed, it is a fundamental problem to understand under which conditions the behavior of typical points is well defined from this statistical point of view. This problem can be precisely formulated by means of the following notion, introduced by Sinai, Ruelle, and Bowen.

DEFINITION 3.1. Let μ be a probability measure invariant by f. We say that μ is an *Sinai-Ruelle-Bowen (SRB) measure* if for a positive Lebesgue measure set os points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \tag{3.1}$$

This is equivalent to say that the averages of Dirac measures along the orbit of x converge to μ in the weak* sense. We define $B(\mu)$, the *basin* of μ, as the set of those points $x \in M$ for which (3.1) holds.

From Birkhoff's ergodic theorem one easily deduces that if μ is an ergodic probability measure which is absolutely continuous with respect to the Lebesgue measure, then μ is an SRB measure. Actually, Birkhoff's ergodic theorem guarantees that if μ is ergodic then $B(\mu)$ has full μ measure. Since μ is absolutely continuous with respect to the Lebesgue measure, then the basin of μ cannot have zero Lebesgue measure.

1. Ergodicity and finiteness

Here we study the existence of SRB measures for maps with non-uniform expansion on subsets of positive Lebesgue measure. Throughout this section we assume that f is a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{C} \subset M$. The main result of this section is the following one.

THEOREM 3.2. *Assume that* $f\colon M \to M$ *is a non-uniformly expanding map. Then there are ergodic absolutely continuous probability measures* $\mu_1, \dots, \mu_p$ *whose* *basins cover a full Lebesgue measure subset of M. Moreover, if μ is an invariant probability measure, then μ is a convex linear combination of those SRB measures: there $\alpha_1 \geq 0, \dots, \alpha_p \geq 0$ with $\alpha_1 + \cdots + \alpha_p = 1$ such that $\alpha_1\mu_1 + \cdots + \alpha_p\mu_p = \mu$.*

The existence of absolutely continuous invariant measures is a consequence of Proposition 2.12 together with Theorem 2.10. The next lemma will allow us to show that M is covered by the basins of finitely many ergodic absolutely continuous invariant measures.

LEMMA 3.3. *Let $G \subset M$ with positive Lebesgue measure be such that f is non-uniformly expanding on G. Then there exists some disk Δ with radius $\delta_1/4$ such that $m(\Delta \setminus G) = 0$.*

PROOF. It suffices to prove that there exist disks of radius $\delta_1/4$ where the relative measure of G is arbitrarily close to 1. Let $\epsilon > 0$ be some small number, G_c be a compact subset of G, and G_o be a neighborhood of G_c such that $G_o \setminus G_c$ has Lebesgue measure less than $\epsilon m(G)$. Then by Proposition 2.12 and Lemma 2.9 there are $\theta > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\frac{1}{n}\sum_{j=0}^{n-1} m(G_j) \geq \frac{1}{n}\sum_{j=0}^{n-1} m(G \cap G_j) \geq \theta m(G),$$

where G_j is the subset of points in G for which j is a (σ, δ)-hyperbolic time. Hence, there exist arbitrarily large values of $j \geq 1$ such that $m(G_j) \geq \theta m(G)$, and so

$$m(G_c \cap G_j) \geq \frac{\theta}{2} m(G). \tag{3.2}$$

Assume that j is large enough so that for any point x in $G_c \cap G_j$, the neighborhood $V_j(x)$ is contained in G_o. Here $V_j(x)$ is the neighborhood of x constructed in Proposition 2.3: it is mapped diffeomorphically onto the ball of radius δ_1 around $f^j(x)$ by f^j. Let $W_x \subset V_j(x)$ be the pre-image of the ball of radius $\delta_1/4$ under this diffeomorphism. Let $x_1, \dots, x_N$ be points in $G_c \cap G_j$ such that $W_{x_1}, \dots, W_{x_N}$ cover the compact set $G_c \cap G_j$. Up to reordering, we may suppose that $W_{x_1}, \dots, W_{x_n}$, some $n \leq N$, is a maximal sub-family whose elements are two-by-two disjoint. Notice that the $V_j(x_1), \dots, V_j(x_n)$ cover $G_c \cap G_j$, since their union contains every W_{x_i}, $1 \leq i \leq N$. Indeed, every W_{x_i} must intersect some W_{x_k} with $k \leq n$. Then its image under f^j intersects the ball of radius $\delta_1/4$ around $f^j(x_k)$ and so it is contained in the corresponding ball of radius δ_1. This means, precisely, that W_{x_i} is contained in $V_j(x_k)$.

By the bounded distortion property given by Corollary 2.6, $m(W_x)$ is larger than the product of $m(V_j(x))$ by some uniform constant $\tau > 0$ (independent of x or j). So, the Lebesgue measure of $W_{x_1} \cup \cdots \cup W_{x_n}$ is larger than $\tau m(G_c \cap G_j)$. If $\xi > 0$ is such that $m(W_{x_i} \setminus (G_c \cap G_j)) \geq \xi m(W_{x_i})$ for each $1 \leq i \leq n$, then by

(3.2)

$$m\big(W_{x_1}\cup\cdots\cup W_{x_n})\setminus(G_c\cap G_j)\big)\geq\xi\tau m(G_c\cap G_j)\geq\xi\tau\frac{\theta}{2}m(G).$$

On the other hand, since each W_{x_i} is contained in G_o and $G_c\cap G_j\subset G$, this measure must be smaller than $\epsilon m(G)$. This means that by reducing ϵ (which we may, by increasing j), we can force ξ to be arbitrarily small. In other words, we may find j and W_{x_i} such that the relative Lebesgue measure of $W_{x_i}\cap G_c\cap G_j$ in W_{x_i} is arbitrarily close to 1. Then, by bounded distortion, the relative Lebesgue measure of $G\supset f^j(G_c\cap G_j)$ in the ball of radius $\delta_1/4$ around $f^j(x_i)$ is also arbitrarily close to 1. So the proof of the lemma is complete. □

PROOF OF THEOREM 3.2. Let μ_0 be any absolutely continuous invariant probability measure. If μ_0 is not ergodic, then we may decompose M into two disjoint invariant sets H_1 and H_2 both with positive μ_0-measure. In particular, both H_1 and H_2 have positive Lebesgue measure. Let μ_1 and μ_2 be the normalized restrictions of μ_0 to H_1 and H_2, respectively. Clearly, they are also absolutely continuous invariant measures. If they are not ergodic, we continue decomposing them, in the same way as we did for μ_0. On the other hand, by Lemma 3.3, each one of the invariant sets we find in this decomposition has full Lebesgue measure in some disk with fixed radius. Since these disks must be disjoint, and the ambient manifold is compact, there can only be finitely many of them. So, the decomposition must stop after a finite number of steps, giving that μ_0 can be written $\mu_0=\sum_{i=1}^{s}\mu_0(H_i)\mu_i$ where $H_1,\ldots,H_s$ is a partition of M into invariant sets with positive measure and each $\mu_i=(\mu_0|H_i)/\mu_0(H_i)$ is an ergodic probability measure. □

COROLLARY 3.4. *Assume that $f\colon M\to M$ is non-uniformly expanding map. If f is transitive, then M is covered (Lebesgue mod 0) by the basin of a unique SRB measure, which is ergodic and absolutely continuous.*

PROOF. Assume by contradiction that there two distinct SRB measures μ_1 and μ_2 as in Theorem 3.2. Since $B(\mu_1)$ and $B(\mu_2)$ are positively invariant sets, then by Lemma 3.3 there are disks Δ_1 and Δ_2 such that $m(\Delta_i\setminus B(\mu_i))=0$ for $i=1,2$. The transitivity of f and the invariance of $B(\mu_1)$ and $B(\mu_2)$ imply that $m(B(\mu_1)\cap B(\mu_2))>0$. Since distinct SRB measures have disjoint basins we have a contradiction. □

2. Piecewise expanding maps

One possible way for proving the existence of invariant measures for certain dynamical systems may be by choosing conveniently some region in the phase space and studying an induced return map to that region. This method can also be very efficient in proving the existence of absolutely continuous invariant measures. In this section we are particulary interested in the study of the return maps themselves. Later on we will make several applications of the results of this section.

Let Δ be a (topological) disk in $\mathbb{R}^d$, for some $d \geq 1$, and consider a map $F : \Delta \to \Delta$. Using local charts we can easily derive the same conclusions of this section for maps defined on disks of any d-dimensional manifold.

DEFINITION 3.5. We say that $F : \Delta \to \Delta$ is a C^2 *piecewise expanding map*, if there is a countable partition $\mathcal{P}$ of a full Lebesgue measure subset of Δ, such that F is a C^2 bijection from the interior of each $U \in \mathcal{P}$ onto its image, admitting a C^2 extension to the closure of U, and the following conditions hold:

(1) *Expansion:* there is $0 < \kappa < 1$ such that for x in the interior of the elements of $\mathcal{P}$
$$\|DF(x)^{-1}\| < \kappa.$$
(2) *Bounded distortion:* there is some constant $K > 0$ such that for every $U \in \mathcal{P}$ and $x, y \in U$
$$\log\left|\frac{\det DF(x)}{\det DF(y)}\right| \leq K \operatorname{dist}(F(x), F(y)).$$
(3) *Long branches:* the elements of $\mathcal{P}$ have piecewise C^2 boundaries with finite $(d-1)$-dimensional volume, and there are constants $\rho > 0$ and $\beta > 0$ with $\kappa(1-\kappa)^{-1} < \beta \leq 1$ such that for each $U \in \mathcal{P}$:
   (i) the boundary of $F(U)$ has a tubular neighborhood of size ρ inside $F(U)$;
   (ii) the C^2 components of the boundary of $F(U)$ meet at angles greater than $\arcsin(\beta) > 0$.

The main goal of this section is the theorem below, which assures the existence and finiteness of absolutely continuous invariant probability measures for piecewise expanding maps. Despite of its own interest it will also be very useful in forthcoming sections.

THEOREM 3.6. *Let $F : \Delta \to \Delta$ be a C^2 piecewise expanding map. There is a finite number of absolutely continuous F-invariant probability measures such that any absolutely continuous F-invariant probability measure can be written as a convex linear combination of those measures.*

The proof of Theorem 3.6 uses the notion of variation for functions in multidimensional spaces. Let F be as in the statement of Theorem 3.6 and let $\{U_i\}_{i=1}^{\infty}$ be its domains of smoothness. For each $i \geq 1$ we let F_i be the C^2 bijection from the interior of U_i onto its image. We introduce the transfer operator $\mathcal{L} : L^1(\mathbb{R}^d) \longrightarrow L^1(\mathbb{R}^d)$ associated to F, defined by
$$\mathcal{L}\varphi = \sum_{i=1}^{\infty} \frac{\varphi \circ F_i^{-1}}{|J \circ F_i^{-1}|} \mathbf{1}_{F(U_i)}.$$

By change of variables

$$\int (\mathcal{L}\varphi)\psi dm = \int \varphi(\psi \circ F) dm, \tag{3.3}$$

whenever these integrals make sense. In particular, each fixed point of $\mathcal{L}$ is the density of an absolutely continuous F-invariant finite measure. We will use the fact that $\mathcal{L}$ never expands L^1 norms:

$$\int |\mathcal{L}\varphi| dm \le \int \mathcal{L}|\varphi| dm = \int |\varphi| dm, \tag{3.4}$$

The next lemma provides a useful estimate on the distortion of F restricted to its domains of smoothness. Actually, the estimate given by the lemma could be taken as the definition of bounded distortion as well, since one can easily prove that it is equivalent to the one given in Definition 3.5.

LEMMA 3.7. *There is a constant $K_1 > 0$ such that for every $i \ge 1$*

$$\frac{\left\| D\left(J \circ F_i^{-1}\right)\right\|}{\left|J \circ F_i^{-1}\right|} < K_1,$$

where $J = \det DF$ is the Jacobian of F.

PROOF. First we observe that

$$\frac{\left\| D\left(J \circ F_i^{-1}\right)\right\|}{\left|(J \circ F_i^{-1})\right|} = \left\| D\left(\log\left|J \circ F_i^{-1}\right|\right)\right\|.$$

Thus we just have to prove that the functions $\log\left|J \circ F_i^{-1}\right|$, with $i \ge 1$, have uniformly bounded derivatives. Take any point x in the interior of Δ and v a vector of the canonical basis of $\mathbb{R}^d$. By the bounded distortion condition in Definition 3.5 we have for small $t \in \mathbb{R}$

$$\begin{aligned}\log\left|J \circ F_i^{-1}\right|(x+tv) - \log\left|J \circ F_i^{-1}\right|(x) &\le K \operatorname{dist}(F(F_i^{-1}(x+tv)), F(F_i^{-1}(x)))\\ &= Kt.\end{aligned}$$

This implies uniformly bounded derivatives of $\log\left|J \circ F_i^{-1}\right|$, $i \ge 1$. □

Next we prove a *Lasota-Yorke type inequality* for functions in $BV(\mathbb{R}^d)$, which plays a crucial role in the proof of the existence of fixed points for $\mathcal{L}$.

PROPOSITION 3.8. *There are constants $0 < \lambda < 1$ and $K_2 > 0$ such that for every $\varphi \in BV(\mathbb{R}^d)$*

$$\operatorname{var}(\mathcal{L}\varphi) \le \lambda \operatorname{var}(\varphi) + K_2 \int |\varphi| dm.$$

PROOF. We start by proving the result in the case that φ is a C^1 function. We have

$$\mathcal{L}\varphi = \sum_{i=1}^{\infty} \phi_i 1_{F(U_i)}, \quad \text{where} \quad \phi_i = \frac{\varphi \circ F_i^{-1}}{J \circ F_i^{-1}}.$$

Hence, using the subaditivity of variation and Lemma A.6 we deduce

$$\begin{aligned}\operatorname{var}(\mathcal{L}\varphi) &\leq \sum_{i=1}^{\infty}\operatorname{var}(\phi_i 1_{F(U_i)})\\ &= \sum_{i=1}^{\infty}\left(\int_{F(U_i)}\|D\phi_i\|dm + \int_{\partial F(U_i)}|\phi_i|dm_0\right),\end{aligned}$$

where m_0 denotes the induced $(d-1)$-dimensional Lebesgue measure on the boundaries of the elements of the partition. Let us now estimate each one of the terms involved in this last sum. For the first one we have

$$\begin{aligned}&\int_{F(U_i)}\|D\phi_i\|dm \leq\\ &\leq \int_{F(U_i)}\frac{\|D(\varphi\circ F_i^{-1})\|}{|J\circ F_i^{-1}|}dm + \int_{F(U_i)}\left\|(\varphi\circ F_i^{-1})\cdot D\left(\frac{1}{J\circ F_i^{-1}}\right)\right\|dm\\ &\leq \kappa\int_{F(U_i)}\frac{\|D\varphi(F_i^{-1})\|}{|J\circ F_i^{-1}|}dm + \int_{F(U_i)}K_1\left|\frac{\varphi\circ F_i^{-1}}{J\circ F_i^{-1}}\right|dm,\end{aligned}$$

where $K_1 > 0$ is the constant given by Lemma 3.7. By a change of variables induced by F in these last two integrals we obtain

$$\int_{F(U_i)}\|D\phi_i\|dm \leq \kappa\int_{U_i}\|D\varphi\|dm + K_1\int_{U_i}|\varphi|dm.$$

For the second term in the sum above, we have by Lemma A.7

$$\begin{aligned}\int_{\partial F(U_i)}|\phi_i|dm_0 &\leq \frac{1}{\beta}\left(\frac{1}{\rho}\int_{F(U_i)}|\phi_i|dm + \int_{F(U_i)}\|D\phi_i\|dm\right)\\ &\leq \frac{1}{\beta\rho}\int_{U_i}|\varphi|dm + \frac{1}{\beta}\int_{F(U_i)}\|D\phi_i\|dm\\ &\leq \left(\frac{1}{\beta\rho}+\frac{K_1}{\beta}\right)\int_{U_i}|\varphi|dm + \frac{\kappa}{\beta}\int_{U_i}\|D\varphi\|dm.\end{aligned}$$

Altogether, this yields

$$\begin{aligned}\operatorname{var}(\mathcal{L}\varphi) &\leq \sum_{i=1}^{\infty}\left[\left(\kappa+\frac{\kappa}{\beta}\right)\int_{U_i}\|D\varphi\|dm + \left(K_1+\frac{1}{\beta\rho}+\frac{K_1}{\beta}\right)\int_{U_i}|\varphi|dm\right]\\ &\leq \kappa\left(1+\frac{1}{\beta}\right)\operatorname{var}(\varphi) + \left(K_1+\frac{1}{\beta\rho}+\frac{K_1}{\beta}\right)\int|\varphi|dm,\end{aligned}$$

from which we deduce the result for the special case $\varphi \in C^1(\mathbb{R}^d)$, simply by taking $\lambda = \kappa(1+1/\beta)$ and $K_2 = K_1 + 1/(\beta\rho) + K_1/\beta$. This proves this case, because the long branches condition (e_5) implies that $\kappa(1+1/\beta) < 1$.

For the general, case we observe that by Proposition A.3, given $\varphi \in BV(\mathbb{R}^d)$ we may choose a sequence $(\varphi_k)_k$ of functions in $C^1(\mathbb{R}^d)$ such that

$$\lim_{k\to\infty} \|\varphi_k - \varphi\|_1 = 0 \quad \text{and} \quad \lim_{k\to\infty} \operatorname{var}(\varphi_k) = \operatorname{var}(\varphi).$$

As a consequence of what we have seen for the case $\varphi \in C^1(\mathbb{R}^d)$, we have in particular that $\mathcal{L}(C^1(\mathbb{R}^d)) \subset BV(\mathbb{R}^d)$. By (3.4), the sequence $(\mathcal{L}\varphi_k)_k$ also converges in $L^1(\mathbb{R}^d)$ to $\mathcal{L}\varphi$, and so we may apply Proposition A.2 to get

$$\begin{aligned}
\operatorname{var}(\mathcal{L}\varphi) &\leq \liminf_{k\to+\infty} \operatorname{var}(\mathcal{L}\varphi_k) \\
&\leq \liminf_{k\to+\infty} \left(\kappa(1+1/\beta)\operatorname{var}(\varphi_k) + K_0\|\varphi_k\|_1\right) \\
&= \kappa(1+1/\beta)\operatorname{var}(\varphi) + K_0\|\varphi\|_1.
\end{aligned}$$

This proves the general case, again by the long branches condition. □

REMARK 3.9. The proof of the last lemma gives $\lambda = \kappa(1+1/\beta)$ and $K_2 = K_1 + 1/(\beta\rho) + K_1\beta$. The knowledge of these expressions for λ and K_2 will be useful in the future.

Consider for each $k \geq 1$ the function

$$\varphi_k = \frac{1}{k}\sum_{j=0}^{k-1} \mathcal{L}^j 1_\Delta.$$

Using (3.3) and the fact that $\varphi_k \geq 0$ we have

$$\int |\varphi_k|\,dm = 1, \quad \text{for every } k \geq 1.$$

By Proposition 3.8 we have $\operatorname{var}(\varphi_k) \leq K_3$ for $k \geq 1$, where $K_3 = \operatorname{var}(1_\Delta) + K_2\sum_{k=0}^{\infty}\lambda^k + 1$. It follows from Proposition A.4 that $(\varphi_k)_k$ has a subsequence converging in the L^1-norm to some ρ with $\operatorname{var}(\rho) \leq K_3$. Hence, $\mu_F = \rho m$ is an absolutely continuous F-invariant probability measure. This proves the first part of Theorem 3.6.

LEMMA 3.10. *Given any $\varphi \in L^1(\mathbb{R}^d)$, the sequence $1/n\sum_{j=0}^{n-1}\mathcal{L}^j\varphi$ has some accumulation point in $L^1(\mathbb{R}^d)$. Moreover, such accumulation point is a function with variation bounded by $4K_2\|\varphi\|_1$.*

PROOF. Let $\varphi \in L^1(\mathbb{R}^d)$ and take a sequence $(\varphi_n)_n$ in $BV(\mathbb{R}^d)$ converging to φ in the L^1-norm. It is no restriction to assume that $\|\varphi_n\|_1 \leq 2\|\varphi\|_1$ for every $n \geq 1$ and we do it. For each $n \geq 1$ we have

$$\operatorname{var}(\mathcal{L}^j\varphi_n) \leq \lambda^j \operatorname{var}(\varphi_n) + K_2\|\varphi_n\|_1 \leq 3K_2\|\varphi\|_1$$

for large j. So, taking k large enough we have

$$\operatorname{var}\left(\frac{1}{k}\sum_{j=0}^{k-1}\mathcal{L}^j\varphi_n\right) \leq 4K_2\|\varphi\|_1.$$

Moreover

$$\Big\|\frac{1}{k}\sum_{j=0}^{k-1}\mathcal{L}^j\varphi_n\Big\|_1 \le \frac{1}{k}\sum_{j=0}^{k-1}\|\mathcal{L}^j\varphi_n\|_1 \le 2\|\varphi\|_1$$

for every $j \ge 1$. It follows from Proposition A.4 that there is some $\hat{\varphi}_n \in BV(\mathbb{R}^d)$ and a sequence $(k_i)_i$ for which

$$\lim_{i\to\infty}\Big\|\frac{1}{k_i}\sum_{j=0}^{k_i-1}\mathcal{L}^j\varphi_n - \hat{\varphi}_n\Big\|_1 = 0$$

and, moreover, $\mathrm{var}(\hat{\varphi}_n) \le 4K_2\|\varphi\|_1$. Now we apply the same argument to the sequence $(\hat{\varphi}_n)_n$ in order to obtain a subsequence $(n_l)_l$ such that $(\hat{\varphi}_{n_l})_l$ converges in the L^1-norm to some $\hat{\varphi}$ with $\mathrm{var}(\hat{\varphi}) \le 4K_2\|\varphi\|_1$. Since

$$\Big\|\frac{1}{k}\sum_{j=0}^{k-1}\mathcal{L}^j\varphi_{n_l} - \hat{\varphi}\Big\|_1 \le \Big\|\frac{1}{k}\sum_{j=0}^{k-1}\mathcal{L}^j\varphi_{n_l} - \hat{\varphi}_{n_l}\Big\|_1 + \|\hat{\varphi}_{n_l} - \hat{\varphi}\|_1,$$

there is some sequence $(k_l)_l$ for which

$$\lim_{l\to\infty}\Big\|\frac{1}{k_l}\sum_{j=0}^{k_l-1}\mathcal{L}^j\varphi_{n_l} - \hat{\varphi}\Big\|_1 = 0.$$

On the other hand,

$$\Big\|\frac{1}{k_l}\sum_{j=0}^{k_l-1}\left(\mathcal{L}^j\varphi_{n_l} - \mathcal{L}^j\varphi\right)\Big\|_1 \le \frac{1}{k_l}\sum_{j=0}^{k_l-1}\|\varphi_{n_l} - \varphi\|_1 = \|\varphi_{n_l} - \varphi\|_1$$

and this last term goes to 0 as $l \to \infty$. Finally, this implies that

$$\lim_{l\to\infty}\Big\|\frac{1}{k_l}\sum_{j=0}^{k_l-1}\mathcal{L}^j\varphi - \hat{\varphi}\Big\|_1 = 0,$$

thus proving that $\hat{\varphi}$ is an accumulation point for the sequence $1/n\sum_{j=0}^{n-1}\mathcal{L}^j\varphi$. $\square$

Observe that any accumulation point $\hat{\varphi}$ of a sequence as in the lemma is a fixed point for the transfer operator.

COROLLARY 3.11. *Let $A \subset \Delta$ be an F-invariant set with positive Lebesgue measure. There is an absolutely continuous F-invariant probability measure $\mu_A = \varphi_A m$ for which $\mu_A(A) = 1$. Moreover, φ_A may be taken with* $\mathrm{var}(\varphi_A) \le 4K_2$.

PROOF. Let $A \subset \Delta$ be an F-invariant set with positive Lebesgue measure. Considering in the previous lemma $\varphi = \mathbf{1}_A \in L^1(\mathbb{R}^d)$, we find $\varphi_A \in BV(\mathbb{R}^d)$ and a sequence $(k_l)_l$ for which $\mathrm{var}(\varphi_A) \le 4K_2\|1_A\|_1 \le 4K_2$ and

$$\lim_{l\to\infty}\Big\|\frac{1}{k_l}\sum_{j=0}^{k_l-1}\mathcal{L}^j 1_A - \varphi_A\Big\|_1 = 0.$$

In particular $\|\varphi_A\|_1 = m(A) > 0$. Then $\mu_A = (\varphi_A/m(A))m$ is a probability, and it is F-invariant because φ_A is a fixed point of $\mathcal{L}$. Since

$$\begin{aligned} m(A)\mu_A(\Delta \setminus A) &= \lim_{l\to\infty} \frac{1}{k_l}\sum_{j=0}^{k_l-1}\int_{\Delta\setminus A} \mathcal{L}^j 1_A \, dm \\ &= \lim_{l\to\infty} \frac{1}{k_l}\sum_{j=0}^{k_l-1}\int_{\Delta} (1_{\Delta\setminus A}\circ F^j) 1_A \, dm = 0 \end{aligned}$$

we have that μ_A gives full weight to A, thus concluding the proof of the result. □

COROLLARY 3.12. *There is a constant $\alpha > 0$ such that if $A \subset \Delta$ is an F-invariant set with positive Lebesgue measure, then $m(A) \geq \alpha$.*

PROOF. Let $A \subset \Delta$ be a F-invariant set with positive Lebesgue measure and $\mu_A = \varphi_A m$ a measure as in Corollary 3.11. Since $\varphi_A \in BV(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ (recall Proposition A.5) and μ_A gives full weight to A, it follows from Minkowski's inequality that

$$1 = \int_A \varphi_A dm \leq \|\varphi_A\|_p \cdot \|1_A\|_q \leq K_1 4K_2 \, m(A)^{1/d}.$$

We take $\alpha = (K_1 4K_2)^{-d}$. □

It immediately follows that Δ can be decomposed into finitely many minimal F-invariant sets $A_1, \ldots, A_p$ with positive Lebesgue measure. By minimality, for each $i = 1, \ldots p$, the absolutely continuous F-invariant measure μ_{A_i} giving full weight to A_i is ergodic. Moreover, any absolutely continuous F-invariant probability measure μ can be written as $\mu = \sum_{i=1}^{p} \mu(A_i)\, \mu_{A_i}$. This completes the proof of Theorem 3.6.

3. Return maps

Here we will apply the results of the previous section to the setting of return maps. Let $f\colon M \to M$ be a map from some d-dimensional Riemannian manifold into itself, such that the push forward of Lebesgue measure $f_* m$ is absolutely continuous with respect to m. Let $F : \Delta \to \Delta$ be a *return map* for f in some topological disk $\Delta \subset M$. This means that there exists a countable partition $\mathcal{P}$ of a full Lebesgue measure subset of Δ, and there exists a *return time* function $R\colon \mathcal{P} \to \mathbb{Z}^+$ such that

$$F|_U = f^{R(U)}|_U \quad \text{for each} \quad U \in \mathcal{P}.$$

We will assume throughout this section that F is a C^2 piecewise expanding map with bounded distortion and long branches. Thus, by Theorem 3.6 it has some invariant probability measure μ_F which is absolutely continuous with respect to the Lebesgue measure on Δ (henceforth denoted by m and assumed to be normalized). Moreover, from Lemma 3.10 and Proposition A.5 one easily deduces that the density $d\mu/dm$ belongs to $L^p(\Delta)$ for $p = d/d-1$. Observe that $p^{-1} + d^{-1} = 1$. Now we define

$$\mu_f^* = \sum_{j=0}^{\infty} f_*^j \left(\mu_F | \{R > j\}\right). \tag{3.5}$$

PROPOSITION 3.13. *If $R \in L^d(\Delta)$, then μ_f^* is an absolutely continuous f-invariant probability measure with support contained in $\cup_{j\geq 0} f^j(\Delta)$.*

PROOF. We first show that μ_f^* is f-invariant. Let A be an arbitrary Borel subset of M. We have

$$\begin{aligned}
\mu_f^*(f^{-1}(A)) &= \sum_{j=0}^{\infty} \mu_F\left(f^{-j}(f^{-1}(A)) \cap \{R > j\}\right) \\
&= \sum_{j=0}^{\infty} \mu_F\left(f^{-(j+1)}(A) \cap (\{R = j+1\} \cup \{R > j+1\})\right) \\
&= \sum_{j=0}^{\infty} \mu_F\left(f^{-(j+1)}(A) \cap \{R = j+1\}\right) + \\
&\qquad + \sum_{j=0}^{\infty} \mu_F\left(f^{-(j+1)}(A) \cap \{R > j+1\}\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_F\left(f^{-(j+1)}(A) \cap \{R = j+1\}\right) &= \mu_F\left(\cup_{j\geq 1}\left(f^{-j}(A) \cap \{R = j\}\right)\right) \\
&= \mu_F(F^{-1}(A)) = \mu(A),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu\left(f^{-(j+1)}(A) \cap \{R > j+1\}\right) &= \mu_f^*(A) - \mu_F(A \cap \{R > 0\}) \\
&= \mu_f^*(A) - \mu_F(A),
\end{aligned}$$

which altogether give

$$\mu^*(f^{-1}(A)) = \mu_f^*(A).$$

Thus the measure μ_f^* is f-invariant.

The absolute continuity of μ_f^* is a consequence of the absolute continuity of μ_F. Note that since we are assuming f_*m absolutely continuous with respect to m, if A is a Borel set in M with $m(A) = 0$, then $m(f^{-j}(A)) = 0$ for every $j \geq 0$. Then, by the absolute continuity of μ_F, we have $\mu_F(f^{-j}(A)) = 0$ for every $j \geq 0$, and so $\mu_f^*(A) = 0$ by the expression of μ_f^*.

Finally we prove that μ_f^* is finite. Observe that

$$\mu_f^*(M) = \sum_{j=0}^{\infty} \mu_F\{R > j\} = \int R\mu_F = \int R\frac{d\mu_F}{dm}dm.$$

Since the density $d\mu_F/dm$ belongs $L^p(\Delta)$ with $p = d/(d-1)$, and we are assuming that $R \in L^d(\Delta)$, then this last integral is finite by Hölder inequality. □

PROPOSITION 3.14. *There are ergodic absolutely continuous f-invariant probability measures $\mu_1^*, \dots, \mu_r^*$ with supports contained in $\cup_{j\geq 0} f^j(\Delta)$, and there are $\alpha_1, \dots, \alpha_r \geq 0$ with $\alpha_1 + \cdots + \alpha_r = 1$ such that $\mu_f^* = \alpha_1\mu_1^* + \cdots + \alpha_r\mu_r^*$.*

PROOF. Normalizing μ_f^* if necessary we assume that μ_f^* is a probability measure. If μ_f^* is ergodic, it is enough to take $r = 1$, $\alpha_1 = 1$, and $\mu_1^* = \mu_f^*$. Otherwise, there exists some f-invariant set A such that $0 < \mu_f^*(A) < 1$. Let us observe that $A \cap \Delta$ is necessarily F-invariant:

$$F^{-1}(A\cap\Delta) = \{x \in \Delta \colon F(x) \in A\} = \bigcup_{j\geq 1} \left(f^{-j}(A) \cap \{R_f = j\}\right) = A \cap \Delta,$$

Because of the assumption $\mu_f^*(A) > 0$ and the definition of μ_f^*, there exists $j \geq 0$ such that $\mu_F(f^{-j}(A) \cap \{R = j\}) > 0$. Then $\mu_F(A) = \mu_F(f^{-j}(A))$ is also positive. Since μ_F is supported in Δ, this is the same as saying that $\mu_F(A\cap\Delta) > 0$. Then, by absolute continuity, $m(A \cap \Delta) > 0$. So, by Corollary 3.12, we have

$$m(A \cap \Delta) \geq \alpha(d). \tag{3.6}$$

Now, either A is minimal, in the sense that there is no f-invariant set $B \subset A$ with $\mu_f^*(A) > \mu_f^*(B) > 0$, or else we apply the same arguments as before, with B and $A \setminus B$ in the place of A. Of course, all this can be said about the complement $\cup_{j\geq 0} f^j(\Delta) \setminus A$ as well. The important point is that at all stages we have an uniform lower bound as in (3.6). Thus this subdivision must stop after a finite number of steps. That is, we find a decomposition of $\cup_{j\geq 0} f^j(\Delta)$ into a finite number of f-invariant sets $A_1, \dots, A_r$ with positive μ_f-measure, such that $m(A_i \cap \Delta) \geq \alpha(d)$ for $1 \leq i \leq r$ and, most important, each A_i is minimal in the above sense. Define $\alpha_i = \mu_f^*(A_i)$ and μ_i^* to be the restriction of μ_f^* to A_i, divided by α_i. Clearly, each μ_i is absolutely continuous and f-invariant (because A_i is f-invariant). Moreover, μ_i^* is ergodic, because A_i was taken minimal. □

COROLLARY 3.15. *If f has a unique SRB measure μ_f in $\cup_{j\geq 0} f^j(\Delta)$, then $\mu_f = \mu_f^*$.*

PROOF. Each of the measures μ_i^* in Proposition 3.14 is an SRB measure. Therefore, the assumption implies that $r = 1$ and $\mu_f^* = \mu_1^* = \mu_f$. □

4. Statistical stability

Let $\mathcal{F}$ be a family of C^k maps, for some $k \geq 2$, from a d-dimensional manifold M into itself, and endow $\mathcal{F}$ with the C^k topology. We assume that each $f \in \mathcal{F}$ admits a unique absolutely continuous f-invariant probability measure μ_f in some forward invariant (by every $f \in \mathcal{F}$) region U containing Δ.

DEFINITION 3.16. We say that $f_0 \in \mathcal{F}$ is *statistically stable*, if the map

$$\mathcal{F} \ni f \longmapsto \frac{d\mu_f}{dm}$$

is continuous in f_0 with respect to the L^1-norm in $L^1(m)$.

The goal of this section is to give sufficient conditions for the statistical stability of maps in such families.

Suppose that we may associate to each $f \in \mathcal{F}$ a C^2 piecewise expanding map $F_f \colon \Delta \to \Delta$ with bounded distortion and long branches as in Definition 3.5. For $f \in \mathcal{F}$, let $\mathcal{P}_f$ denote the partition into domains of smoothness of F_f and $R_f \colon \mathcal{P}_f \to \mathbb{Z}^+$ be the corresponding return time function. We assume that these maps are in the conditions of Proposition 3.13. In particular, we have $R \in L^d(\Delta)$, which then implies that $\mu_f^* = \sum_{j=0}^{\infty} f_*^j(\mu_F|\{R_f > j\})$ is an absolutely continuous f-invariant finite measure, where μ_F is an absolutely continuous F-invariant probability measure given by Theorem 3.6. Moreover, from Lemma 3.10 and Proposition A.5 one has that the density $d\mu_F/dm$ belongs to $L^p(\Delta)$ for $p = d/(d-1)$.

We consider elements f_0 of $\mathcal{F}$ satisfying the following uniformity conditions:

(u_1) given $\epsilon > 0$ there is $\delta > 0$ such that for any $f \in \mathcal{F}$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \|R_f - R_{f_0}\|_d < \epsilon.$$

(u_2) the constants κ, K, β, ρ as in Definition 3.5 may be chosen uniformly for f in a C^k neighborhood of f_0.

We also assume that the maps in a neighborhood of f_0 satisfy the following non-degeneracy condition: given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$m(E) \leq \delta \quad \Rightarrow \quad m(f^{-1}(E)) \leq \epsilon \tag{3.7}$$

for any measurable subset E of U and any f in $\mathcal{U}$. This can often be enforced by requiring some jet of order $l \leq k$ of f_0 to be everywhere non-degenerate.

THEOREM 3.17. *Let $\mathcal{U}$ be as above, and suppose that every $f \in \mathcal{U}$ admits a unique SRB measure μ_f in U. Then*

(1) *μ_f is absolutely continuous with respect to the Lebesgue measure m;*
(2) *if $f_0 \in \mathcal{U}$ satisfies (u_1) and (u_2) then f_0 is statistically stable.*

In the remaining of this section we will obtain several results that altogether will give the proof of Theorem 3.17. Take $\mathcal{F}$ a uniform family of C^k maps such that each $f \in \mathcal{F}$ admits a unique SRB measure μ_f. Denote by F_0 be the return map of f_0, and by R_0 the corresponding return time.

At this point we also introduce the transfer operator $\mathcal{L}_f$ associated to $f \in \mathcal{F}$, defined for each $\varphi \in L^1(\mathbb{R}^d)$ as

$$\mathcal{L}_f\varphi(y) = \sum_{x \in f^{-1}(y)} \frac{\varphi(x)}{|\det Df(x)|}. \tag{3.8}$$

The function $\mathcal{L}_f\varphi(y)$ fails to be defined only when y is a critical value of φ. We have

$$\int (\mathcal{L}_f\varphi)\psi \, dm = \int \varphi(\psi \circ f) \, dm$$

for all $\varphi, \psi \in L^1(\mathbb{R}^d)$ such that integrals make sense. For the sake of notational simplicity we will denote by $\mathcal{L}$ the transfer operator $\mathcal{L}_f$ associated to $f \in \mathcal{F}$. Similarly, we will simply denote by $\mathcal{L}_0$ the transfer operator $\mathcal{L}_{f_0}$ associated to $f_0 \in \mathcal{F}$.

LEMMA 3.18. *Let $f_0 \in \mathcal{F}$. Given $\epsilon > 0$ there is $\delta > 0$ such that for any $f \in \mathcal{F}$ with $\|f - f_0\|_{C^1} < \delta$ we have*

$$\int |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm \leq \epsilon \left(\operatorname{var}(\varphi) + \|\varphi\|_1\right),$$

for every $\varphi \in BV(\mathbb{R}^d)$ with support contained in Δ.

PROOF. Our assumptions, namely the existence of a piecewise expanding return map, imply that the critical set of f_0 (the set of points where f_0 fails to be a local diffeomorphism) intersects Δ in a zero Lebesgue measure set. Given any $\epsilon_1 > 0$, define $\mathcal{C}(\epsilon_1)$ as the ϵ_1-neighborhood of this intersection. Clearly, $m(f(\mathcal{C}(\epsilon_1))) \leq \text{const}\, m(\mathcal{C}(\epsilon_1))$ for some constant that may be taken uniform in a C^1 neighborhood of f_0. So, using (3.7) we may fix ϵ_1 small enough so that

$$m\big(f^{-1}(\hat{f}(\mathcal{C}(\epsilon_1)))\big) \leq \frac{1}{2}\left(\frac{\epsilon}{8K_1}\right)^d, \tag{3.9}$$

for every f, $\hat{f}$ in some neighborhood of f_0, where K_1 is the constant in Proposition A.5. We decompose $\Delta \setminus \mathcal{C}(\epsilon_1)$ into a finite collection $\mathcal{D}(f_0)$ of domains of injectivity of f_0. Observe that if f is close enough to f_0, in the C^1 sense, then $\mathcal{C}(\epsilon_1)$ also contains the critical set of f. Hence, we may define a corresponding collection $\mathcal{D}(f)$ of domains of injectivity for f in $\Delta \setminus \mathcal{C}(\epsilon_1)$, and there is a natural bijection associating to each $D_0 \in \mathcal{D}(f_0)$ a unique $D \in \mathcal{D}(f)$ such that the Lebesgue measure of $D \,\triangle\, D_0$ is small, where $D \,\triangle\, D_0$ denotes the symmetric difference of the two sets D and D_0. Observe that $\mathcal{L}_f$ is supported in

$$f(\Delta) = f(\mathcal{C}(\epsilon_1)) \cup \bigcup_{D \in \mathcal{D}(f)} f(D),$$

and analogously for $\mathcal{L}_0$. So,

$$\begin{aligned}\int |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm &\leq \int_{f_0(\mathcal{C}(\epsilon_1))\cup f(\mathcal{C}(\epsilon_1))} \big(|\mathcal{L}\varphi| + |\mathcal{L}_0\varphi|\big) dm &(3.10)\\ &+ \sum_{D_0\in\mathcal{D}(f_0)} \int_{f_0(D_0)\cap f(D)} |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm &(3.11)\\ &+ \sum_{D_0\in\mathcal{D}(f_0)} \int_{f_0(D_0)\Delta f(D)} \big(|\mathcal{L}\varphi| + |\mathcal{L}\varphi_0|\big) dm, &(3.12)\end{aligned}$$

where D always denotes the element of $\mathcal{D}(f)$ associated to each $D_0 \in \mathcal{D}(f_0)$. Let us now estimate the expressions on the right hand side of this inequality. We start with (3.10). For notational simplicity, we write $E = f_0(\mathcal{C}(\epsilon_1)) \cup f(\mathcal{C}(\epsilon_1))$. Then

$$\int_E |\mathcal{L}\varphi| dm \leq \int \mathbf{1}_E\big(\mathcal{L}|\varphi|\big) dm = \int (\mathbf{1}_E \circ f)|\varphi| dm.$$

It follows from Minkowski's inequality, Proposition A.5, and (3.9) that

$$\int (\mathbf{1}_E \circ f)|\varphi| dm \leq m\big(f^{-1}(E)\big)^{1/d} \|\varphi\|_p \leq \frac{\epsilon}{8K_1} K_1 \operatorname{var}(\varphi) = \frac{\epsilon}{8} \operatorname{var}(\varphi).$$

The case $f = f_0$ gives a similar bound for the second term in (3.10). So,

$$\int_{f_0(\mathcal{C}(\epsilon_1))\cup f(\mathcal{C}(\epsilon_1))} \big(|\mathcal{L}\varphi| + |\mathcal{L}_0\varphi|\big) dm \leq \frac{\epsilon}{4} \operatorname{var}(\varphi). \tag{3.13}$$

Making the change of variables $y = f_0(x)$ in (3.11), we may rewrite it as

$$\int_{\widehat{D}_0} \left| \frac{\varphi}{|\det Df|} \circ (f^{-1} \circ f_0) - \frac{\varphi}{|\det Df_0|} \right| \cdot |\det Df_0| dm,$$

where $\widehat{D}_0 = f_0^{-1}\big(f_0(D_0) \cap f(D)\big) = D_0 \cap \big(f_0^{-1} \circ f\big)(D)$. For notational simplicity, we introduce $g = f^{-1} \circ f_0$. The previous expression is bounded by

$$\int_{\widehat{D}_0} \left(|\varphi \circ g - \varphi| \cdot \frac{|\det Df_0|}{|\det Df| \circ g} + |\varphi| \cdot \left| \frac{|\det Df_0|}{|\det Df| \circ g} - 1 \right| \right) dm.$$

Choosing $\delta > 0$ sufficiently small, the assumption $\|f - f_0\|_{C^1} < \delta$ implies

$$\left| \frac{|\det Df_0|}{|\det Df| \circ g} - 1 \right| \leq \epsilon, \quad \text{and so} \quad \frac{|\det Df_0|}{|\det Df| \circ g} \leq 2$$

on $\Delta \setminus \mathcal{C}(\epsilon_1)$ (which contains $\widehat{D}_0$). Hence, using Lemma A.8,

$$\begin{aligned}\int_{f_0(D_0)\cap f(D)} |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm &\leq 2\int_{\widehat{D}_0} |\varphi \circ g - \varphi| dm + \epsilon \int |\varphi| dm\\ &\leq 2K_4 \|g - \mathrm{id}\,\|_0^d \operatorname{var}(\varphi) + \epsilon \int |\varphi| dm.\end{aligned}$$

Reducing $\delta > 0$, we can make $\|g - \mathrm{id}\,\|_0^d$ arbitrarily small, so that

$$\int_{f_0(D_0)\cap f(D)} |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm \leq \frac{\epsilon}{4}\,\mathrm{var}(\varphi) + \epsilon\|\varphi\|_1. \tag{3.14}$$

We estimate the terms in (3.12) in much the same way as we did for (3.10). For each D_0 let E be $f_0(D_0) \,\Delta\, f(D)$. The properties of the transfer operator, followed by Minkowski's inequality, yield

$$\begin{aligned}\int_E |\mathcal{L}\varphi| dm \;&\leq\; \int \mathbf{1}_E\big(\mathcal{L}|\varphi|\big) dm \\ &=\; \int \mathbf{1}_{f^{-1}(E)}|\varphi| dm \leq m\big(f^{-1}(E)\big)^{1/q}\|\varphi\|_p.\end{aligned}$$

Fix $\epsilon_2 > 0$ such that $\#\mathcal{D}(\varphi_0)4\epsilon_2 < \epsilon$. Taking δ sufficiently small, we may ensure that the Lebesgue measure of all the sets

$$f^{-1}(E) = f^{-1}(f_0(D_0) \,\Delta\, f(D))$$

is small enough so that, using also Proposition A.5, the right hand side is less than $\epsilon_2\,\mathrm{var}(f)$. In this way we get

$$\int_{f_0(D_0)\Delta f(D)} \big(|\mathcal{L}_0\varphi| + |\mathcal{L}\varphi|\big) dm \leq 2\epsilon_2\,\mathrm{var}(\varphi) \tag{3.15}$$

(the second term on the left is estimated in the same way as the first one). Putting (3.13), (3.14), (3.15) together, we obtain

$$\int |\mathcal{L}\varphi - \mathcal{L}_0\varphi| dm \leq \Big(\frac{\epsilon}{2} + \#\mathcal{D}(f_0)2\epsilon_2\Big)\,\mathrm{var}(\varphi) + \epsilon\|\varphi\|_1$$

and this is smaller than $\epsilon(\mathrm{var}(\varphi) + \|\varphi\|_1)$. □

LEMMA 3.19. *Given $\epsilon > 0$, there are $N \geq 1$ and $\delta = \delta(\epsilon, N) > 0$ for which*

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \Big\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_f > j\}}\Big\|_d < \epsilon.$$

PROOF. For the sake of notational simplicity we denote R_f by R and R_{f_0} by R_0. Let $\epsilon > 0$ be given, and take $N \geq 1$ in such a way that

$$\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_0 > j\}}\|_d < \epsilon/3.$$

This is possible because we are implicitly assuming that $R_0 \in L^d(m)$. Then we have

$$\Big\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R>j\}}\Big\|_d = \Big\|R - R_0 + R_0 - \sum_{j=0}^{N-1} \mathbf{1}_{\{R_0>j\}} + \sum_{j=0}^{N-1} \mathbf{1}_{\{R_0>j\}} - \sum_{j=0}^{N-1} \mathbf{1}_{\{R>j\}}\Big\|_d$$
$$\leq \|R - R_0\|_d + \Big\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_0>j\}}\Big\|_d + \sum_{j=0}^{N-1} \big\|\mathbf{1}_{\{R_0>j\}} - \mathbf{1}_{\{R>j\}}\big\|_d,$$

and so, if we take $\delta = \delta(N,\epsilon) > 0$ sufficiently small then, under assumption (u_2), the first and third terms in the sum above can be made smaller than $\epsilon/3$. $\square$

Fix $f_0 \in \mathcal{F}$ and let $(f_n)_n$ be a sequence of maps in $\mathcal{F}$ converging to f_0 in the C^k topology. Let μ_0 be an absolutely continuous F_0-invariant probability measure and μ_0^* be the φ_0-invariant measure obtained from it as in Proposition 3.13. We represent by ρ_0 the density of μ_0. Moreover, we denote by F_n, R_n, μ_n, μ_n^*, ρ_n the corresponding objects for each f_n, and we denote by $\mathcal{L}_0$ the operator $\mathcal{L}_{f_0}$ associated to $f_0 \in \mathcal{F}$ for every $n \geq 0$.

Our goal now is to prove that the density of μ_n^* converges in the L^1-norm (with respect to the Lebesgue measure) to μ_0^* as n goes to infinity. We remark that, as a consequence of our construction,

$$\operatorname{var}(\rho_n) \leq K_3 \quad \text{and} \quad \int \rho_n dm \leq 1$$

for every $n \geq 1$ (recall Proposition 3.8). Thus, by Proposition A.4, the sequence of densities $(\rho_n)_n$ is relatively compact with respect to the L^1 norm: any subsequence contains another subsequence which is L^1 convergent. This means that we only have to prove that $(\mu_{n_i}^*)_i$ converges to μ_0^* for every subsequence $(n_i)_i$ such that $(\rho_{n_i})_i$ converges in the L^1-norm to some function ρ_∞. The previous remark also gives $\operatorname{var}(\rho_\infty) \leq K_3$. We consider $\mu_\infty = \rho_\infty m$ and define

$$\mu_\infty^* = \sum_{j=0}^{\infty} f_*^j \left(\mu_\infty | \{R_0 > j\}\right).$$

We want to show that the densities of μ_n^* with respect to the Lebesgue measure converge in the L^1-norm to the density of μ_∞^* and, moreover, the measure μ_∞^* coincides with μ_0^*.

LEMMA 3.20. *$\dfrac{d\mu_{n_i}^*}{dm}$ converges to $\dfrac{d\mu_\infty^*}{dm}$ in the norm of $L^1(m)$.*

PROOF. We are going to prove that given $\epsilon > 0$ there is $\delta > 0$ for which

$$\left\|\frac{d\mu_{n_i}^*}{dm} - \frac{d\mu_\infty^*}{dm}\right\|_1 < \epsilon \quad \text{whenever} \quad \|f_{n_i} - f_0\|_{C^1} < \delta.$$

We have

$$\mu_\infty^* = \sum_{j=0}^{\infty} (f_0^j)_* \left(\mu_\infty \mid \{R_0 > j\}\right) \quad \text{and} \quad \mu_{n_i}^* = \sum_{j=0}^{\infty} (f_{n_i}^j)_* \left(\mu_{n_i} \mid \{R_{n_i} > j\}\right). \tag{3.16}$$

By Lemma 3.19 there is an integer $N \geq 1$ and $\delta = \delta(\epsilon, N) > 0$ for which

$$\|f - f_0\|_{C^1} < \delta \quad \Rightarrow \quad \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_f > j\}} \Big\|_q < \frac{\epsilon}{4K_1K_3}. \tag{3.17}$$

In what follows we take $i \geq 1$ to be sufficiently large so that $\|f_{n_i} - f_0\| < \delta$. We split each one of the sums in (3.16) as

$$\mu_\infty^* = \sum_{j=0}^{N} \nu_{\infty,j} + \eta_{\infty,N} \quad \text{and} \quad \mu_{n_i}^* = \sum_{j=0}^{N} \nu_{n_i,j} + \eta_{n_i,N}, \tag{3.18}$$

where

$$\nu_{\infty,j} = (f_0)_*^j \left(\mu_\infty | \{R_0 > j\}\right), \quad \eta_{\infty,N} = \sum_{j=N+1}^{\infty} (f_0)_*^j \left(\mu_\infty | \{R_0 > j\}\right),$$

and $\nu_{n_i,j}$ and $\eta_{n_i,N}$ are defined similarly, with φ_{n_i}, μ_{n_i}, R_{n_i} in the place of f, μ_∞, R_0, respectively. We have

$$\eta_{\infty,N}(M) = \sum_{j=N}^{\infty} \mu_\infty\{R_0 > j\} = \sum_{j=N}^{\infty} \int \rho_\infty \mathbf{1}_{\{R_0 > j\}} dm \leq \|\rho_\infty\|_p \cdot \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_0 > j\}} \Big\|_q,$$

and, analogously,

$$\eta_{n_i,N}(M) \leq \|\rho_{n_i}\|_p \cdot \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_{n_i} > j\}} \Big\|_q$$

which together with Proposition A.5 and (3.17) yield

$$\left\| \frac{d\eta_{n_i,N}}{dm} - \frac{d\eta_{\infty,N}}{dm} \right\|_1 \leq \eta_{n_i,N}(M) + \eta_{\infty,N}(M) < \epsilon/2. \tag{3.19}$$

On the other hand, for $j = 1, \ldots, N$

$$\left\| \frac{d\nu_{n_i,j}}{dm} - \frac{d\nu_{\infty,j}}{dm} \right\|_1 = \left\| \mathcal{L}_{f_{n_i}^j}(\rho_{n_i} \mathbf{1}_{\{R_{n_i} > j\}}) - \mathcal{L}_{f_0^j}(\rho_\infty \mathbf{1}_{\{R_0 > j\}}) \right\|_1. \tag{3.20}$$

Denote

$$A = \|\mathcal{L}_{f_{n_i}^j}(\rho_{n_i} \mathbf{1}_{\{R_{n_i} > j\}}) - \mathcal{L}_{f_{n_i}^j}(\rho_\infty \mathbf{1}_{\{R_0 > j\}})\|_1$$

and

$$B = \|\mathcal{L}_{f_{n_i}^j}(\rho_\infty \mathbf{1}_{\{R_0 > j\}}) - \mathcal{L}_{f_0^j}(\rho_\infty \mathbf{1}_{\{R_0 > j\}})\|_1.$$

Here we also use the transfer operators for the iterated maps $f_{n_i}^j$ and f_0^j defined in the same way as for f in (3.8). Then

$$\begin{aligned} A &\leq \|\rho_{n_i}\mathbf{1}_{\{R_{n_i}>j\}} - \rho_\infty\mathbf{1}_{\{R_0>j\}}\|_1 \\ &\leq \|\rho_{n_i}\mathbf{1}_{\{R_{n_i}>j\}} - \rho_\infty\mathbf{1}_{\{R_{n_i}>j\}}\|_1 + \|\rho_\infty\mathbf{1}_{\{R_{n_i}>j\}} - \rho_\infty\mathbf{1}_{\{R_0>j\}}\|_1 \\ &\leq \|\rho_{n_i} - \rho_\infty\|_1 + \|\rho_\infty(\mathbf{1}_{\{R_{n_i}>j\}} - \mathbf{1}_{\{R_0>j\}})\|_1 \end{aligned}$$

and the last term is bounded by $\|\rho_\infty\|_p\|\mathbf{1}_{\{R_{n_i}>j\}} - \mathbf{1}_{\{R_0>j\}}\|_q$. Taking into account ($u_1$), we get $A \leq \epsilon/(4N)$ if i is sufficiently large. Using Proposition 3.18 we also get $B \leq \epsilon/(4N)$, for large i. It follows that (3.20) is less than $A + B \leq \epsilon/(2N)$ for each $1 \leq j \leq N$. Thus the sum over all these j's is less than $\epsilon/2$. Together with (3.19), this completes the proof of the proposition. □

LEMMA 3.21. *μ_∞^* is an f_0-invariant measure.*

PROOF. It follows from Lemma 3.20 that $(\mu_{n_i}^*)_i$ converges to μ_∞^* in the weak* topology. Hence, given any $\varphi\colon M \to \mathbb{R}$ continuous we have

$$\int \varphi d\mu_{n_i}^* \to \int \varphi d\mu_\infty^* \quad \text{when} \quad i \to \infty.$$

On the other hand, since $\mu_{n_i}^*$ is f_{n_i}-invariant we have

$$\int \varphi d\mu_{n_i}^* = \int (\varphi \circ f_{n_i}) d\mu_{n_i}^* \quad \text{for every } i.$$

So, it suffices to prove that

$$\int (\varphi \circ f_{n_i}) d\mu_{n_i}^* \to \int (\varphi \circ f_0) d\mu_\infty^* \quad \text{when} \quad i \to \infty. \tag{3.21}$$

We have

$$\begin{aligned} &\Big|\int (\varphi \circ f_{n_i}) d\mu_{n_i}^* - \int (\varphi \circ f_0) d\mu_\infty^*\Big| \leq \\ &\qquad \Big|\int (\varphi \circ f_{n_i}) d\mu_{n_i}^* - \int (\varphi \circ f_0) d\mu_{n_i}^*\Big| + \Big|\int (\varphi \circ f_0) d\mu_{n_i}^* - \int (\varphi \circ f_0) d\mu_\infty^*\Big|. \end{aligned}$$

Since $\varphi \circ f_{n_i} - \varphi \circ f_0$ is uniformly close to zero when i is large, the first term in the sum above is close to zero for i sufficiently large. On the other hand, since $(\mu_{n_i}^*)_i$ converges to μ_∞^* in the weak* topology we also have that the second term in the sum above is close to zero if i is large. □

It follows from this last result and the uniqueness of the absolutely continuous φ-invariant measure that $\mu_\infty^* = \mu_0^*$. So, Lemma 3.20 really states that the measures $\mu_{n_i}^*$ have densities converging in the L^1-norm to the density of μ_0^*. This completes the proof of Theorem 3.17.

CHAPTER 4

Markov structures

One successful way for studying the statistical properties of uniformly expanding transformations is by codification of the system via Markov partitions. This strategy gives very good results in the setting of uniformly expanding transformations. For non-uniformly expanding maps the existence of the classical Markov partitions is still unknown. A partial answer to this problem has been given in [**ALP3**], where it was shown that some Markov structures exist for non-uniformly expanding maps. The existence of these structures will play a key role in proving some statistical stability results and in obtaining rates for the correlation decay of the system, as we shall see below.

DEFINITION 4.1. Let f be some map from a manifold M into itself. We say that f induces a *Markov structure* on a disk $\Delta_0 \subset M$ if there is a countable partition $\mathcal{P}$ (mod 0) of Δ_0, and a return time function $R : \Delta_0 \to \mathbb{N}$ constant on elements of $\mathcal{P}$ such that the following properties hold:

(1) *Piecewise expansion*: the induced map $F : \Delta_0 \to \Delta_0$ given by $F(x) = f^{R(x)}(x)$ (which is defined almost everywhere) is a piecewise expanding map; cf. Definition 3.5.
(2) *Markov:* the map F is a C^2 diffeomorphism (and in particular a bijection) from each $U \in \mathcal{P}$ onto Δ_0.

The induced $F\colon \Delta_0 \to \Delta_0$ is said to be a *Markov map.*

Observe that, by definition, an induced Markov structure gives rise to a piecewise expanding map where the geometric long branches condition (see Definition 3.5) is replaced by the stronger Markov condition.

A problem that much interests us in this subject, because of the implications it has, is to understand how the Lebesgue measure of the set of points $x \in \Delta_0$ for which $R(x) > n$ decays as n tends to infinity. The Lebesgue measure of these sets will be related to the Lebesgue measure of some tail sets for non-uniformly expanding maps that we will introduce latter on. If $f\colon M \to M$ is non-uniformly expanding map, then condition (1.2) implies that the *expansion time* function

$$\mathcal{E}(x) = \min\left\{ N \geq 1\colon \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \leq -\lambda, \text{ for all } n \geq N \right\} \qquad (4.1)$$

is defined and finite almost everywhere in M. We think of this as the waiting time before the exponential derivative growth kicks in. Then, according to Remark 2.13, we fix $\varepsilon > 0$ and $\delta > 0$ in (1.3) in such a way that the proof of Proposition 2.12 works. The *recurrence time* function

$$\mathcal{R}(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leq \varepsilon, \quad \text{for all } n \geq N \right\} \tag{4.2}$$

is also defined and finite almost everywhere in M. Again this is an asymptotic statement and we have no a-priori knowledge about how fast this limit is approached or with what degree of uniformity for different points x. We introduce the *tail set (at time n)*

$$\Gamma_n = \{x : \mathcal{E}(x) > n \text{ or } \mathcal{R}(x) > n\}. \tag{4.3}$$

This is the set of points which at time n have not yet achieved either the uniform exponential growth of derivative or the uniform subexponential recurrence given by conditions (1.2) and (1.3). If the critical set is empty, we simply ignore the recurrence time function and consider only the expansion time function in the definition of Γ_n.

1. Inducing Markov structures

Our aim here is to show that Markov structures exist for non-uniformly expanding maps, and control the decay of the return map in terms of the decay of the tail set. If such a Markov structure exists, then necessarily there will be some recurrence to the domain of such Markov structure. This remark should make natural the transitivity assumption on the hypotheses of the main result in this section.

THEOREM 4.2. *Let $f : M \to M$ be a transitive C^2 non-uniformly expanding map. Then f induces some Markov structure on a disk contained in M. Moreover, if there exist $C, \gamma > 0$ such that $m(\Gamma_n) \leq Cn^{-\gamma}$ for all $n \in \mathbb{N}$, then there exists $\bar{C} > 0$ such that $m\{R > n\} \leq \bar{C}n^{-\gamma}$ for all $n \in \mathbb{N}$.*

As we shall see in Remark 4.14, if the map f is uniformly expanding then $m\{R > n\}$ decays exponentially fast with n.

We do not need transitivity in all its strength. Before we tell what is the weaker form of transitivity that is enough for our purposes, let us recall that given $\delta > 0$, a subset A of M is said to be *δ-dense* if any point in M is at a distance smaller than δ from A. For the proof of the theorem above it is enough that there is some point $p \in M$ whose pre-orbit does not hit the critical set of f and is δ-dense for some sufficiently small $\delta > 0$ (depending on the radius of hyperbolic balls for f). As the lemma below shows, in our setting of non-uniformly expanding maps this is a consequence of the usual transitivity of f.

LEMMA 4.3. *Let $f\colon M \to M$ be a transitive non-uniformly expanding map. Given $\delta > 0$ there is $p \in M$ and $N_0 \in \mathbb{N}$ such that $\cup_{j=0}^{N_0} f^{-j}\{p\}$ is δ-dense in M and disjoint from the critical set $\mathcal{C}$.*

PROOF. Observe that the assumptions on f imply that the images and preimages of sets with zero Lebesgue measure still have zero Lebesgue measure. Hence, the set $\mathcal{B} = \cup_{n\geq 0} f^{-n}\left(\cup_{m\geq 0} f^m(\mathcal{C})\right)$ has Lebesgue measure equal to zero. On the other hand, since f is transitive, we have by Corollary 3.4 that there is a unique SRB measure for μ, which is ergodic and absolutely continuous with respect to Lebesgue measure, and whose support is the whole manifold M. This implies that μ almost every point in M has a dense orbit. Since μ is absolutely continuous with respect to Lebesgue, then there is a positive Lebesgue measure subset of points in M with dense orbit. Thus there must be some point $q \in M \setminus \mathcal{B}$ with dense orbit. Take $N_0 \in \mathbb{N}$ for which $q, f(q), \dots, f^{N_0}(q)$ is δ-dense. The point $p = f^{N_0}(q)$ satisfies the conclusions of the lemma. $\square$

The proof of the Theorem 4.2 requires several technical constructions which we will explain along this section. Assuming that f is non-uniformly expanding, then by Proposition 2.12 there are σ, δ and θ such that Lebesgue almost every $x \in M$ has frequency of (σ, δ)-hyperbolic times greater than θ. We fix once and for all $p \in M$ and $N_0 \in \mathbb{N}$ for which

$$\cup_{j=0}^{N_0} f^{-j}\{p\} \quad \text{is } \delta_1/3\text{-dense in } M \text{ and disjoint from } \mathcal{C}, \tag{4.4}$$

where $\delta_1 > 0$ is the radius of hyperbolic balls given by Proposition 2.3. Take constants $\varepsilon > 0$ and $\delta_0 > 0$ so that

$$\sqrt{\delta}_0 \ll \delta_1/2 \quad \text{and} \quad 0 < \varepsilon \ll \delta_0.$$

We start the proof of Theorem 4.2 with a couple of auxiliary lemmas.

LEMMA 4.4. *There are constants $K_0, D_0 > 0$ depending only on f, σ, δ_1 and the point p, such that for any ball $B \subset M$ of radius δ_1 there are an open set $V \subset B$ and an integer $0 \leq m \leq N_0$ for which:*

(1) *f^m maps V diffeomorphically onto $B(p, 2\sqrt{\delta}_0)$;*
(2) *for each $x, y \in V$*

$$\log\left|\frac{\det Df^m(x)}{\det Df^m(y)}\right| \leq D_0 \operatorname{dist}(f^m(x), f^m(y))$$

Moreover, for each $0 \leq j \leq N_0$ the j-preimages of $B(p, 2\sqrt{\delta}_0)$ are all disjoint from $\mathcal{C}$, and for x belonging to any such j-preimage we have

$$\frac{1}{K_0} \leq \|Df^j(x)\| \leq K_0.$$

PROOF. Since $\cup_{j=0}^{N_0} f^{-j}\{p\}$ is $\delta_1/3$ dense in M and disjoint from $\mathcal{C}$, choosing $\delta_0 > 0$ sufficiently small we have that each connected component of the preimages of $B(p, 2\sqrt{\delta_0})$ up to time N_0 are bounded away from the critical set C and are contained in a ball of radius $\delta_1/3$.

This immediately implies that any ball $B \subset M$ of radius δ_1 contains a preimage V of $B(p, 2\sqrt{\delta_0})$ which is mapped diffeomorphically onto $B(p, 2\sqrt{\delta_0})$ in at most N_0 iterates. Moreover, since the number of iterations and the distance to the critical region are uniformly bounded, the volume distortion is uniformly bounded.

Observe that δ_0 and N_0 have been chosen in such a way that all the connected components of the preimages of $B(p, 2\sqrt{\delta_0})$ up to time N_0 are uniformly bounded away from the critical set C, and so there is some constant $K_0 > 1$ such that

$$\frac{1}{K_0} \leq \|Df^m(x)\| \leq K_0.$$

for all $1 \leq m \leq N_0$ and x belonging to an m-preimage of $B(p, 2\sqrt{\delta_0})$. □

Next we prove a useful and non-obvious consequence of the existence of hyperbolic times, namely that if we fix some $\varepsilon > 0$ then there exist some N_ε depending only on ε such that any ball of radius ε has some subset which grows to a fixed size with bounded distortion within N_ε iterates.

LEMMA 4.5. *There exists $N_\varepsilon > 0$ such that any ball $B \subset M$ of radius ε contains a hyperbolic pre-ball $V_n \subset B$ with $n \leq N_\varepsilon$.*

PROOF. Take any $\varepsilon > 0$ and a ball $B(z, \varepsilon)$. By Proposition 2.3 we may choose $n_\varepsilon \in \mathbb{N}$ large enough so that any hyperbolic pre-ball V_n associated to a hyperbolic time $n \geq n_\varepsilon$ will have diameter not exceeding $\varepsilon/2$. Now notice that by Proposition 2.12 Lebesgue almost every point has an infinite number of hyperbolic times and therefore

$$m\left(M \setminus \bigcup_{j=n_\varepsilon}^{n} H_j\right) \to 0 \quad \text{as } n \to \infty.$$

Hence, it is possible to choose $N_\varepsilon \in \mathbb{N}$ such that

$$m\left(M \setminus \bigcup_{j=n_\varepsilon}^{N_\varepsilon} H_j\right) < m(B(z, \varepsilon/2)).$$

This ensures that there is a point $\hat{x} \in B(z, \varepsilon/2)$ with a hyperbolic time $n \leq N_\varepsilon$ and associated hyperbolic pre-ball $V_n(x)$ contained in $B(z, \varepsilon)$. □

REMARK 4.6. Observe that if n is a hyperbolic time for f, then it is also a hyperbolic time for every map in a C^1 neighborhood of f. Hence, for given $\varepsilon > 0$ the integer N_ε may be taken uniform in a whole C^1 neighborhood of f, and only depending on ε, σ and δ_1.

1.1. The partitioning algorithm. Here we describe the construction of the partition (mod 0) of $\Delta_0 = B(p, \delta_0)$. The basic intuition is that we wait for some iterate $f^k(\Delta_0)$ to cover Δ_0 completely, and then define the subset $U \subset \Delta_0$ such that $f^k : U \to \Delta_0$ is a diffeomorphism, as an element of the partition with return time k. We then continue to iterate the complement $\Delta_0 \setminus U$ until this complement covers again Δ_0 and repeat the same procedure to define more elements of the final partition with higher return times. Using the fact that small regions eventually become large due to the expansivity condition, it follows that this process can be continued and that Lebesgue almost every point eventually belongs to some element of the partition. Moreover, the return time function depends on the time that it takes small regions to become large on average and this turns out to depend precisely on the measure of the tail set.

Now we introduce neighborhoods of p

$$\Delta_0^0 = B(p,\delta_0), \quad \Delta_0^1 = B(p,2\delta_0), \quad \Delta_0^2 = B(p,\sqrt{\delta_0}) \quad \text{and} \quad \Delta_0^3 = B(p,2\sqrt{\delta_0}).$$

For $0 < \sigma < 1$ given by Proposition 2.12, let

$$I_k = \left\{x \in \Delta_0^1 \,:\, \delta_0(1+\sigma^{k/2}) < \operatorname{dist}(x,p) < \delta_0(1+\sigma^{(k-1)/2})\right\}, \quad k \geq 1,$$

be a partition (mod 0) into countably many rings of $\Delta_0^1 \setminus \Delta_0$.

The construction of the partition of Δ_0 is inductive and we give precisely the general step of the induction below. For the sake of a better visualization of the process, and to motivate the definitions, we start with the first step.

First step of the induction. Take R_0 some large integer to be determined below; we ignore any dynamics occurring up to time R_0. Let $k \geq R_0+1$ be the first time that $\Delta_0 \cap H_k \neq \emptyset$. For $j < k$ we define formally the objects $\Delta_j, A_j, A_j^\epsilon$ whose meaning will become clear in the next paragraph, by $A_j = A_j^\epsilon = \Delta_j = \Delta_0$. Let $(U_{k,j}^3)_j$ be the connected components of $f^{-k}(\Delta_0^3) \cap A_{k-1}^\epsilon$ contained in hyperbolic pre-balls V_{k-m} with $k - N_0 \leq m \leq k$ which are mapped diffeomorphically onto Δ_0^3 by f^k. Now let

$$U_{k,j}^i = U_{k,j}^3 \cap f^{-k}\Delta_0^i, \quad i = 0,1,2$$

and set $R(x) = k$ for $x \in U_{k,j}^0$. Now take

$$\Delta_k = \Delta_{k-1} \setminus \{R = k\}.$$

We define also a function $t_k : \Delta_k \to \mathbb{N}$ by

$$t_k(x) = \begin{cases} s & \text{if } x \in U_{k,j}^1 \text{ and } f^k(x) \in I_s \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}$$

Finally let

$$A_k = \{x \in \Delta_k : t_k(x) = 0\}, \quad B_k = \{x \in \Delta_k : t_k(x) > 0\}$$

and

$$A_k^\epsilon = \{x \in \Delta_k : \operatorname{dist}(f^{k+1}(x), f^{k+1}(A_k)) < \epsilon\}.$$

General step of the induction. The general inductive step of the construction now follows by repeating the arguments above with minor modifications. More precisely we assume that sets Δ_i, A_i, A_i^ε B_i, $\{R=i\}$ and functions $t_i:\Delta_i\to\mathbb{N}$ are defined for all $i\le n-1$. For $i\le R_0$ we just let $A_i=A_i^\varepsilon=\Delta_i=\Delta_0$, $B_i=\{R=i\}=\emptyset$ and $t_i\equiv 0$. Now let $(U_{n,j}^3)_j$ be the connected components of $f^{-n}(\Delta_0)\cap A_{n-1}^\varepsilon$ contained in hyperbolic pre-balls V_m, with $n-N_0\le m\le n$, which are mapped onto Δ_0^3 by f^n. Take

$$U_{n,j}^i=U_{n,j}^3\cap f^{-n}\Delta_0^i,\quad i=0,1,2,$$

and set $R(x)=n$ for $x\in U_{n,j}^0$. Take also

$$\Delta_n=\Delta_{n-1}\setminus\{R=n\}.$$

The definition of the function $t_n:\Delta_n\to\mathbb{N}$ is slightly different in the general case:

$$t_n(x)=\begin{cases} s & \text{if } x\in U_{n,j}^1\setminus U_{n,j}^0 \text{ and } f^n(x)\in I_s \text{ for some } j,\\ 0 & \text{if } x\in A_{n-1}\setminus \cup_j U_{n,j}^1,\\ t_{n-1}(x)-1 & \text{if } x\in B_{n-1}\setminus\cup_j U_{n,j}^1.\end{cases}$$

Finally let

$$A_n=\{x\in\Delta_n: t_n(x)=0\},\quad B_n=\{x\in\Delta_n:t_n(x)>0\}$$

and

$$A_n^\varepsilon=\{x\in\Delta_n:\operatorname{dist}(f^{n+1}(x),f^{n+1}(A_n))<\varepsilon\}.$$

At this point we have completely described the inductive construction of the sets A_n, A_n^ε, B_n and $\{R=n\}$.

The construction detailed before provides an algorithm for the definition of a family of topological balls contained in Δ_0 and satisfying the Markov property as required. Next we show that this algorithm does indeed produce a partition mod 0 of Δ_0 as required.

REMARK 4.7. Associated to each component U_{n-k}^0 of $\{R=n-k\}$, for some $k>0$, we have a collar $U_{n-k}^1\setminus U_{n-k}^0$ around it; knowing that the new components of $\{R=n\}$ do not intersect "too much" $U_{n-k}^1\setminus U_{n-k}^0$ is important for preventing overlaps on sets of the partition.

In order to see that the sets we construct at each step do not intersect the previously constructed sets, it is enough to show that if $\varepsilon>0$ is sufficiently small, then $U_n^1\cap\{t_{n-1}>1\}=\emptyset$ for each component U_n^1. Indeed, take some $k>0$ and let U_{n-k}^0 be a component of $\{R=n-k\}$ such that its collar Q_k (the part of U_{n-k}^1 that is mapped by f^{n-k} onto I_k) intersects U_n^1. Recall that Q_k is precisely the collar around U_{n-k}^0 on which t_{n-1} takes the value 1. Letting q_1 and q_2 be any two points in distinct components (inner and outer) of the boundary of Q_k, we have by Proposition 2.3 and Lemma 4.4

$$\operatorname{dist}(f^{n-k}(q_1),f^{n-k}(q_2))\le K_0\sigma^{(k-N_0)/2}\operatorname{dist}(f^n(q_1),f^n(q_2)).\qquad(4.5)$$

We also have

$$\begin{aligned} \operatorname{dist}(f^{n-k}(q_1), f^{n-k}(q_2)) &\geq \delta_0(1+\sigma^{(k-1)/2}) - \delta_0(1+\sigma^{k/2}) \\ &= \delta_0\sigma^{k/2}(\sigma^{-1/2}-1), \end{aligned}$$

which combined with (4.5) gives

$$\operatorname{dist}(f^n(q_1), f^n(q_2)) \geq K_0^{-1}\sigma^{N_0/2}\delta_0(\sigma^{-1/2}-1).$$

On the other hand, since $U_n^1 \subset A_{n-1}^\varepsilon$ by construction of U_n^1, taking

$$\varepsilon < K_0^{-1}\sigma^{N_0/2}\delta_0(\sigma^{-1/2}-1) \tag{4.6}$$

we have $U_n^1 \cap \{t_{n-1} > 1\} = \emptyset$.

1.2. Expansion. Recall that by construction, the return time R for an element U of the partition $\mathcal{P}$ of Δ_0 is formed by a certain number n of iterations given by the hyperbolic time of a hyperbolic pre-ball $V_n \supset U$, and a certain number $m \leq N_0$ of additional iterates which is the time it takes to go from $f^n(V_n)$ which could be anywhere in M, to $f^{n+m}(V_n)$ which covers Δ_0 completely. It follows from Remark 2.4 and Lemma 4.4 that

$$\|Df^{n+m}(x)^{-1}\| \leq \|Df^m(f^n(x))^{-1}\| \cdot \|Df^n(x)^{-1}\| < K_0\sigma^{n/2} \leq K_0\sigma^{(R_0-N_0)/2}.$$

By taking R_0 sufficiently large we can make this last expression smaller than 1.

1.3. Bounded distortion. For the bounded distortion estimate required in Definition 3.5 we need to show that there exists a constant $K > 0$ such that for any x, y belonging to an element $U \in \mathcal{P}$ with return time R, we have

$$\log\left|\frac{\det Df^R(x)}{\det Df^R(y)}\right| \leq K \operatorname{dist}(f^R(x), f^R(y)).$$

Recall that by construction, the return time R for an element U of the partition $\mathcal{P}$ of Δ_0 is formed by a certain number n of iterations given by the hyperbolic time of a hyperbolic pre-ball $V_n \supset U$, and a certain number $m = R - n \leq N_0$ of additional iterates which is the time it takes to go from $f^n(V_n)$ (which could be anywhere in the manifold M) to Δ_0 and cover it completely. By the chain rule

$$\log\left|\frac{\det Df^R(x)}{\det Df^R(y)}\right| = \log\left|\frac{\det Df^{R-n}(f^n(x))}{\det Df^{R-n}(f^n(y))}\right| + \log\left|\frac{\det Df^n(x)}{\det Df^n(y)}\right|.$$

For the first term in this last sum we observe that by Lemma 4.4 we have

$$\log\left|\frac{\det Df^{R-n}(f^n(x))}{\det Df^{R-n}(f^n(y))}\right| \leq D_0 \operatorname{dist}(f^R(x), f^R(y)).$$

For the second term in the sum above, we may apply Corollary 2.5 and obtain

$$\log\left|\frac{\det Df^n(x)}{\det Df^n(y)}\right| \leq C_0 \operatorname{dist}(f^n(x), f^n(y)).$$

Also by Lemma 4.4 we may write

$$\operatorname{dist}(f^n(x), f^n(y)) \le K_0 \operatorname{dist}(f^R(x), f^R(y)).$$

Thus we just have to take $K = D_0 + C_0 K_0$.

1.4. Metric estimates. Here we prove that the construction defined in Subsection 1.1 does indeed produce a partition of Δ_0 as in the Theorem 4.2, modulo a zero Lebesgue measure set. We split our argument into two parts.

1.4.1. *Estimates obtained from the construction.* In this first part we obtain some estimates relating the Lebesgue measure of the sets A_n, B_n and $\{R > n\}$ with the help of specific information extracted from the inductive construction we performed in Subsection 1.1.

LEMMA 4.8. *There exists a constant $a_0 > 0$ (not depending on δ_0) such that $m(B_{n-1} \cap A_n) \ge a_0 m(B_{n-1})$ for every $n \ge 1$.*

PROOF. It is enough to see that this holds for each connected component of B_{n-1} at a time. Let C be a component of B_{n-1} and Q be its outer ring corresponding to $t_{n-1} = 1$. Observe that by Remark 4.7 we have $Q = C \cap A_n$. Moreover, there must be some $k < n$ and a component U_k^0 of $\{R = k\}$ such that f^k maps C diffeomorphically onto $\cup_{i=k}^{\infty} I_i$ and Q onto I_k, both with distortion bounded by C_1 and $e^{D_0 L}$, where L is the diameter of M; cf. Corollary 2.6 and Lemma 4.4. Thus, it is sufficient to compare the Lebesgue measures of $\cup_{i=k}^{\infty} I_i$ and I_k. We have

$$\frac{m(I_k)}{m(\cup_{i=k}^{\infty} I_i)} \approx \frac{[\delta_0(1+\sigma^{(k-1)/2})]^d - [\delta_0(1+\sigma^{k/2})]^d}{[\delta_0(1+\sigma^{(k-1)/2})]^d - \delta_0^d} \approx 1 - \sigma^{1/2}.$$

Clearly this proportion does not depend on δ_0. □

LEMMA 4.9. *There exist $b_0, c_0 > 0$ with $b_0 + c_0 < 1$ such that for every $n \ge 1$*

(1) *$m(A_{n-1} \cap B_n) \le b_0 m(A_{n-1})$;*
(2) *$m(A_{n-1} \cap \{R = n\}) \le c_0 m(A_{n-1})$.*

Moreover $b_0 \to 0$ and $c_0 \to 0$ as $\delta_0 \to 0$.

PROOF. It is enough to prove these estimates for each neighborhood of a component U_n^0 of $\{R = n\}$. Observe that by construction we have $U_n^3 \subset A_{n-1}^{\varepsilon}$, which means that $U_n^2 \subset A_{n-1}$, because $\varepsilon < \delta_0 < \sqrt{\delta_0}$. Using the distortion bounds of f^n on U_n^3 given by Corollary 2.6 and Lemma 4.4 we obtain

$$\frac{m(U_n^1 \setminus U_n^0)}{m(U_n^2 \setminus U_n^1)} \approx \frac{m(\Delta_0^1 \setminus \Delta_0^0)}{m(\Delta_0^2 \setminus \Delta_0^1)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

which gives the first estimate. Moreover,

$$\frac{m(U_n^0)}{m(U_n^2 \setminus U_n^1)} \approx \frac{m(\Delta_0^0)}{m(\Delta_0^2 \setminus \Delta_0^1)} \approx \frac{\delta_0^d}{\delta_0^{d/2}} \ll 1,$$

and this gives the second one. □

The next result asserts that a fixed proportion of $A_{n-1} \cap H_n$ gives rise to new elements of the partition within a finite number of steps (not depending on n).

PROPOSITION 4.10. *There exist $c_1 > 0$ and a positive integer $N = N(\varepsilon)$ such that $m\left(\cup_{i=0}^{N}\{R = n+i\}\right) \geq c_1 m(A_{n-1} \cap H_n)$ for every $n \geq 1$.*

PROOF. Take $r = 5\delta_0 K_0^{N_0}$, where N_0 and K_0 are given by Lemma 4.4. Let $\{z_j\}$ be a maximal set in $f^n(A_{n-1} \cap H_n)$ with the property that $B(z_j, r)$ are pairwise disjoint. By maximality we have

$$\cup_j B(z_j, 2r) \supset f^n(A_{n-1} \cap H_n).$$

Let x_j be a point in H_n such that $f^n(x_j) = z_j$ and consider the hyperbolic pre-ball $V_n(x_j)$ associated to x_j. Observe that f^n sends $V_n(x_j)$ diffeomorphically onto a ball of radius δ_1 around z_j as in Proposition 2.12. In what follows, given $B \subset B(z_j, \delta_1)$, we will simply denote $(f^n|V_n(x_j))^{-1}(B)$ by $f^{-n}(B)$.

CLAIM. *The set $f^{-n}(B(z_j, r))$ contains some component of $\{R = n + k_j\}$ with $0 \leq k_j \leq N_\varepsilon + N_0$ for each j.*

We start by showing that

$$t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) > 0 \quad \text{for some } 0 \leq k_j \leq N_\varepsilon + N_0. \tag{4.7}$$

Assume by contradiction that $t_{n+k_j}|f^{-n}(B(z_j,\varepsilon)) = 0$ for all $0 \leq k_j \leq N_\varepsilon + N_0$. This implies that $f^{-n}(B(z_j, \varepsilon)) \subset A^{\varepsilon}_{n+k_j}$ for all $0 \leq k_j \leq N_\varepsilon + N_0$. Using Lemma 4.5 we may find a hyperbolic pre-ball $V_m \subset B(z_j, \varepsilon)$ with $m \leq N_\varepsilon$. Now, since $f^m(V_m)$ is a ball B of radius δ_1 it follows from Lemma 4.4 that there is some $V \subset B$ and $m' \leq N_0$ with $f^{m'}(V) = \Delta_0$. Thus, taking $k_j = m + m'$ we have that $0 \leq k_j \leq N_\varepsilon + N_0$ and $f^{-n}(V_m)$ is an element of $\{R = n+k_j\}$ inside $f^{-n}(B(z_j, \varepsilon))$. This contradicts the fact that $t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) = 0$ for all $0 \leq k_j \leq N_\varepsilon + N_0$, and so (4.7) holds. Let k_j be the smallest integer $0 \leq k_j \leq N_\varepsilon + N_0$ for which $t_{n+k_j}|f^{-n}(B(z_j, \varepsilon)) > 0$. Since

$$f^{-n}(B(z_j, \varepsilon)) \subset A^{\varepsilon}_{n-1} \subset \{t_{n-1} \leq 1\},$$

there must be some element $U^0_{n+k_j}(j)$ of $\{R = n + k_j\}$ for which

$$f^{-n}(B(z_j, \varepsilon)) \cap U^1_{n+k_j}(j) \neq \emptyset.$$

Recall that by definition f^{n+k_j} sends $U^1_{n+k_j}(j)$ diffeomorphically onto Δ^1_0, the ball of radius $(1+s)\delta_0$ around p. From time n to $n+k_j$ we may have some final "bad" period of length at most N_0 where the derivative of f may contract, however being bounded from below by $1/K_0$ in each step. Thus, the diameter of $f^n(U^1_{n+k_j}(j))$ is at most $4\delta_0 K_0^{N_0}$. Since $B(z_j, \varepsilon)$ intersects $f^n(U^1_{n+k_j}(j))$ and $\varepsilon < \delta_0 < \delta_0 K_0^{N_0}$, we have by definition of r

$$f^{-n}(B(z_j, r)) \supset U^0_{n+k_j}(j).$$

Thus we have shown that $f^{-n}(B(z_j,r))$ contains some component of $\{R=n+k_j\}$ with $0\le k_j\le N_\varepsilon+N_0$, and so we have proved the claim.

Since n is a hyperbolic time for x_j, we have by the distortion control given by Corollary 2.6

$$\frac{m(f^{-n}(B(z_j,2r)))}{m(f^{-n}(B(z_j,r)))}\le C_1\frac{m(B(z_j,2r))}{m(B(z_j,r))} \tag{4.8}$$

and

$$\frac{m(f^{-n}(B(z_j,r)))}{m(U^0_{n+k_j}(j))}\le C_0\frac{m(B(z_j,r))}{m(f^n(U^0_{n+k_j}(j)))}. \tag{4.9}$$

Here we are implicitly assuming that

$$r=r(\delta_0)<\delta_1/2. \tag{4.10}$$

This can be done by taking δ_0 small enough. Note that estimates on N_0 and K_0 improve when we diminish δ_0.

From time n to time $n+k_j$ we have at most $k_j=m_1+m_2$ iterates with $m_1\le N_\varepsilon$, $m_2\le N_0$ and $f^n(U^0_{n+k_j}(j)))$ containing some point $w_j\in H_{m_1}$. By the definition of (σ,δ)-hyperbolic time we have $\operatorname{dist}_\delta(f^i(x),\mathcal{C})\ge\sigma^{bN_\varepsilon}$ for every $0\le i\le m_1$, which implies that there is some constant $D=D(\varepsilon)>0$ such that $|\det(Df^i(x))|\le D$ for $0\le i\le m_1$ and $x\in f^n(U^0_{n+k_j}(j))$. On the other hand, since the first N_0 preimages of Δ_0 are uniformly bounded away from $\mathcal{C}$ we also have some $D'>0$ such that $|\det(Df^i(x))|\le D'$ for every $0\le i\le m_2$ and x belonging to an i preimage of Δ_0. Hence,

$$m(f^n(U^0_{n+k_j}(j)))\ge\frac{1}{DD'}m(\Delta_0),$$

which combined with (4.9) gives

$$m(f^{-n}(B(z_j,r)))\le Cm(U^0_{n+k_j}(j)),$$

with C only depending on C_1, D, D', δ_0 and the dimension of M. We also deduce from (4.8) that

$$m(f^{-n}(B(z_j,2r)))\le C'm(f^{-n}(B(z_j,r)))$$

with C' only depending on C_1 and the dimension of M. Finally let us compare the Lebesgue measure of the sets $\cup_{i=0}^N\{R=n+i\}$ and $A_{n-1}\cap H_n$. We have

$$m\big(A_{n-1}\cap H_n\big)\le\sum_j m(f^{-n}(B(z_j,2r)))\le C'\sum_j m(f^{-n}(B(z_j,r))).$$

On the other hand, by the disjointness of the balls $B(z_j,r)$ we have

$$\sum_j m(f^{-n}(B(z_j,r)))\le C\sum_j m(U^0_{n+k_j}(j))\le Cm\left(\cup_{i=0}^N\{R=n+i\}\right).$$

It is enough to take $c_1=(CC')^{-1}$. $\square$

REMARK 4.11. It follows from the choice of the constants D and D' (hence C and C') that c_1 only depends on the constants σ, b, N_ε, C_1 and N_0.

1.4.2. *General estimates.* For the time being we have taken a disk Δ_0 of radius $\delta_0 > 0$ around a point $p \in M$ with certain properties, and defined inductively the subsets A_n, B_n, $\{R = n\}$ and Δ_n which are related in the following way:

$$\Delta_n = \Delta_0 \setminus \{R \leq n\} = A_n \dot{\cup} B_n.$$

Since we are dealing with a non-uniformly expanding map, for each $n \in \mathbb{N}$ we also have defined the set H_n of points that have n as a (σ, δ)-hyperbolic time, and the tail set Γ_n as in (4.3). From the definition of Γ_n, Remark 2.14 and Lemma 2.9 we deduce that:

(m_1) there is $\theta > 0$ such that for every $n \geq 1$ and every $A \subset M \setminus \Gamma_n$ with $m(A) > 0$
$$\frac{1}{n} \sum_{j=1}^{n} \frac{m(A \cap H_j)}{m(A)} \geq \theta.$$

Moreover, we have proved in Lemma 4.8, Lemma 4.9 and Proposition 4.10 that the following metric relations also hold:

(m_2) there is $a_0 > 0$ (bounded away from 0 with δ_0) such that for all $n \geq 1$
$$m(B_{n-1} \cap A_n) \geq a_0 m(B_{n-1});$$

(m_3) there are $b_0, c_0 > 0$ with $b_0 + c_0 < 1$ and $b_0, c_0 \to 0$ as $\delta_0 \to 0$, such that for all $n \geq 1$
$$\frac{m(A_{n-1} \cap B_n)}{m(A_{n-1})} \leq b_0 \quad \text{and} \quad \frac{m(A_{n-1} \cap \{R = n\})}{m(A_{n-1})} \leq c_0;$$

(m_4) there is $c_1 > 0$ and an integer $N \geq 0$ such that for all $n \geq 1$
$$m\left(\cup_{i=0}^{N} \{R = n + i\}\right) \geq c_1 m(A_{n-1} \cap H_n).$$

In the inductive process of construction of the sets A_n, B_n, $\{R = n\}$ and Δ_n we have fixed some large integer R_0, being this the first step at which the construction began. Recall that $A_n = \Delta_n = \Delta_0$ and $B_n = \{R = n\} = \emptyset$ for $n \leq R_0$. For technical reasons we will assume that

$$R_0 > \max\{2(N+1), 12/\theta\}. \tag{4.11}$$

Note that since N and θ do not depend on R_0 this is always possible.

This is the abstract setting under which we will be completing the proof of Theorem 4.2. From now on we will only make use of the metric relations (m_1)-(m_4) and will not be concerned with any other properties about these sets. This may be useful for future applications.

LEMMA 4.12. *There is $a_1 > 0$, with $a_1 \to 0$ as $\delta_0 \to 0$, such that for all $n \geq 1$*
$$m(B_n) \leq a_1 m(A_n).$$

PROOF. We have by (m_3)

$$m(A_n \cap A_{n-1}) \geq \eta m(A_{n-1}), \tag{4.12}$$

where $\eta = 1 - b_0 - c_0$. Then we take

$$\widehat{a} = \frac{b_0 + c_0}{a_0} \quad \text{and} \quad a_1 = \frac{(1+a_0)b_0 + c_0}{a_0 \eta}. \tag{4.13}$$

The fact that $a_1 \to 0$ when $\delta_0 \to 0$ is a consequence of $b_0, c_0 \to 0$ when $\delta_0 \to 0$, and a_0 being bounded away from 0. Observe that $0 < \eta < 1$ and $\widehat{a} < a_1$.

Now the proof follows by induction. The result is obviously true for n up to R_0. Assuming that it holds for $n - 1 \geq R_0$ we will show that it also holds for n, by considering separately the cases $m(B_{n-1}) > \widehat{a} m(A_{n-1})$ and $m(B_{n-1}) \leq \widehat{a} m(A_{n-1})$.

Assume first that we have $m(B_{n-1}) > \widehat{a} m(A_{n-1})$. We may write $m(B_{n-1}) = m(B_{n-1} \cap A_n) + m(B_{n-1} \cap B_n)$, which by (m_2) gives

$$m(B_{n-1} \cap B_n) \leq (1 - a_0) m(B_{n-1}). \tag{4.14}$$

Since we have $m(B_n) = m(B_n \cap B_{n-1}) + m(B_n \cap A_{n-1})$, it follows from (4.14) and (m_3) that $m(B_n) \leq (1 - a_0) m(B_{n-1}) + b_0 m(A_{n-1})$. According to the case we are considering this leads to

$$m(B_n) \leq (1 - a_0) m(B_{n-1}) + b_0 \widehat{a}\, m(B_{n-1}) \leq m(B_{n-1}). \tag{4.15}$$

On the other hand, we have $m(A_n) = m(A_n \cap A_{n-1}) + m(A_n \cap B_{n-1})$, which together with (m_2) and (4.12) gives $m(A_n) \geq \eta m(A_{n-1}) + a_0 m(B_{n-1})$. Again by the case we are considering we have

$$m(A_n) \geq \eta m(A_{n-1}) + a_0 \widehat{a}\, m(A_{n-1}) \geq m(A_{n-1}). \tag{4.16}$$

Inequalities (4.15), (4.16) together with the inductive hypothesis yield the result in this first case.

Assume now that $m(B_{n-1}) \leq \widehat{a} m(A_{n-1})$. Since $m(B_n) = m(B_n \cap B_{n-1}) + m(B_n \cap A_{n-1})$, it follows from (m_3) that $m(B_n) \leq m(B_{n-1}) + b_0 m(A_{n-1})$. On the other hand, (4.12) implies $m(A_n) \geq \eta m(A_{n-1})$. Hence

$$\frac{m(B_n)}{m(A_n)} \leq \frac{m(B_{n-1}) + b_0 m(A_{n-1})}{\eta m(A_{n-1})} \leq \frac{\widehat{a} + b_0}{\eta} = a_1,$$

which proves the result also in this case. $\square$

COROLLARY 4.13. *There exists $c_2 > 0$ such that for every $n \geq 1$*

$$m(\Delta_n) \leq c_2 m(\Delta_{n+1}).$$

PROOF. Using (m_3) we obtain

$$m(\Delta_{n+1}) \geq m(A_{n+1}) \geq (1 - b_0 - c_0) m(A_n).$$

On the other hand, by Lemma 4.12,

$$m(\Delta_n) = m(A_n) + m(B_n) \leq (1 + a_1^{-1}) m(A_n).$$

It is enough to take $c_2 = (1 + a_1^{-1})/(1 - b_0 - c_0)$. □

At this point we are able to specify the choice of δ_0. First of all, let us recall that the number θ in (m_1) does not depend on δ_0. Assume that

$$m(\Gamma_n) \leq Cn^{-\gamma}, \quad \text{for some } C, \gamma > 0,$$

and pick $\alpha > 0$ such that

$$\alpha < \left(\frac{\theta}{12}\right)^{\gamma+1}. \tag{4.17}$$

Then we choose $\delta_0 > 0$ small enough so that

$$a_1 < 2\alpha. \tag{4.18}$$

This is possible because $a_1 \to 0$ as $\delta_0 \to 0$ by Lemma 4.12.

Since $m(\Delta_n) = m(A_n) + m(B_n)$, we easily deduce from (m_4) and Lemma 4.12 that if we take

$$b_1 = \frac{c_1}{1 + a_1}, \tag{4.19}$$

then

$$m\left(\cup_{i=0}^N \{R = n + i\}\right) \geq b_1 \frac{m(A_{n-1} \cap H_n)}{m(A_{n-1})} m(\Delta_{n-1}).$$

This immediately implies that

$$m\left(\Delta_{n+N}\right) \leq \left(1 - b_1 \frac{m(A_{n-1} \cap H_n)}{m(A_{n-1})}\right) m(\Delta_{n-1}). \tag{4.20}$$

At this point we have some recurrence relation for the Lebesgue measure of the sets Δ_n. Now take any large n and let $k_0 \geq 1$ be the smallest integer for which $n - 1 - k_0(N+1) \leq R_0$. Assumption (4.11) on R_0 and N implies that $n-(k_0+1)(N+1) \geq 1$. Consider the partition of $\{n-(k_0+1)(N+1), \dots, n-1\}$ into the sets

$$\begin{aligned}
J_N &= \{n-1, n-1-(N+1), \dots, n-1-k_0(N+1)\}, \\
&\vdots \\
J_1 &= \{n-N, n-N-(N+1), \dots, n-N-k_0(N+1)\}, \\
J_0 &= \{n-(N+1), n-2(N+1), \dots, n-(k_0+1)(N+1)\}.
\end{aligned}$$

Applying (4.20) repeatedly we arrive at the following set of $N+1$ inequations:

$$\begin{aligned}
m\left(\Delta_{n+N}\right) &\leq \prod_{j \in J_N} \left(1 - b_1 \frac{m(A_j \cap H_{j+1})}{m(A_j)}\right) m(\Delta_0), \\
&\vdots \\
m\left(\Delta_n\right) &\leq \prod_{j \in J_0} \left(1 - b_1 \frac{m(A_j \cap H_{j+1})}{m(A_j)}\right) m(\Delta_0).
\end{aligned}$$

Multiplying the terms in the inequations above and ignoring the factors from $n-(k_0+1)(N+1)$ to R_0-1 on the right hand side (observe that those factors are smaller than 1), we obtain

$$\prod_{j=0}^{N} m\left(\Delta_{n+j}\right) \leq \prod_{j=R_0}^{n-1}\left(1-b_1\frac{m(A_{j-1}\cap H_j)}{m(A_{j-1})}\right) m(\Delta_0)^{N+1}.$$

Since $(\Delta_n)_n$ forms a decreasing sequence of sets we finally have

$$m\left(\Delta_{n+N}\right) \leq \exp\left(-\frac{b_1}{N+1}\sum_{j=R_0}^{n}\frac{m(A_{j-1}\cap H_j)}{m(A_{j-1})}\right) m(\Delta_0). \tag{4.21}$$

We will complete the proof of Theorem 4.2 by considering several different cases, according to the behavior of the proportions $m(A_{j-1}\cap H_j)/m(A_{j-1})$.

REMARK 4.14. A straightforward calculation shows that if the average

$$\frac{1}{n}\sum_{j=1}^{n}\frac{m(A_{j-1}\cap H_j)}{m(A_{j-1})}$$

is bounded away from 0 for large n, then $m\{R>n\}=m(\Delta_n)$ decays exponentially fast to 0. This happens, for instance, when f is uniformly expanding.

Now we are in conditions to give the last result of this section, from which we will conclude the proof of Theorem 4.2. It will be useful to introduce

$$E_n=\left\{j\leq n\colon \frac{m(A_{j-1}\cap H_j)}{m(A_{j-1})}<\alpha\right\},\quad \text{for each } n\geq 1,$$

and

$$F=\left\{n\in\mathbb{N}\colon \frac{\#E_n}{n}>1-\frac{\theta}{12}\right\}.$$

An issue we have to address is the link between the statistics of hyperbolic times, the spatial distribution of points having hyperbolic time at some given time, and the geometrical structure of sets arising from the construction of the partition described before. We are able to implement a partially successful strategy in this respect: in the polynomial case we establish an essentially optimal link between the rate of decay of the tail set and the rate of the return time map. The nature of the argument does not immediately extend to the exponential case.

PROPOSITION 4.15. *Take any $n\in F$ with $n\geq R_0$. If $m(A_n)\geq 2m(\Gamma_n)$, then there is some $0<k=k(n)<n$ for which $m(A_n)<(k/n)^\gamma m(A_k)$.*

PROOF. Assuming that $m(A_n)\geq 2m(\Gamma_n)$, we easily deduce that

$$\frac{m(A_n\setminus\Gamma_n)}{m(A_n)}=1-\frac{m(A_n\cap\Gamma_n)}{m(A_n)}\geq\frac{1}{2}.$$

Thus we have for $1 \le j \le n$

$$\frac{m(A_n \cap H_j)}{m(A_n)} \ge \frac{m(A_n \setminus \Gamma_n)}{m(A_n)} \cdot \frac{m((A_n \setminus \Gamma_n) \cap H_j)}{m(A_n \setminus \Gamma_n)} \ge \frac{1}{2} \cdot \frac{m((A_n \setminus \Gamma_n) \cap H_j)}{m(A_n \setminus \Gamma_n)},$$

which together with the conclusion of (m_1) for the set $A = A_n \setminus \Gamma_n$ gives

$$\frac{1}{n}\sum_{j=1}^{n} \frac{m(A_n \cap H_j)}{m(A_n)} \ge \frac{\theta}{2}. \tag{4.22}$$

Define

$$G_n = \left\{ j \in E_n \colon \frac{m(A_{j-1})}{m(A_n)} > \frac{\theta}{12\alpha} \right\}.$$

Since $n \in F$, we have

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} \frac{m(A_n \cap H_j)}{m(A_n)} &\le \frac{\theta}{12} + \frac{1}{n}\sum_{j \in E_n} \frac{m(A_n \cap H_j)}{m(A_n)} \\ &\le \frac{\theta}{12} + \frac{1}{n}\sum_{j \in E_n \setminus G_n} \frac{m(A_n \cap H_j)}{m(A_n)} + \frac{\#G_n}{n}. \end{aligned}$$

Now, for $j \in E_n \setminus G_n$ we have

$$\begin{aligned} \frac{m(A_n \cap H_j)}{m(A_n)} &= \frac{m(A_n \cap H_j)}{m(A_{j-1})} \cdot \frac{m(A_{j-1})}{m(A_n)} \\ &\le \left[\frac{m(A_n \cap A_{j-1} \cap H_j)}{m(A_{j-1})} + \frac{m((A_n \setminus A_{j-1}) \cap H_j)}{m(A_{j-1})} \right] \frac{m(A_{j-1})}{m(A_n)} \\ &\le \left[\frac{m(A_{j-1} \cap H_j)}{m(A_{j-1})} + a_1 \right] \frac{\theta}{12\alpha}. \end{aligned}$$

For this last inequality we used the fact that $(A_n \setminus A_{j-1}) \subset B_{j-1}$ and $j \notin G_n$. Hence

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} \frac{m(A_n \cap H_j)}{m(A_n)} &\le \frac{\theta}{12} + \frac{1}{n}\sum_{j \in E_n \setminus G_n} \frac{m(A_{j-1} \cap H_j)}{m(A_{j-1})} \frac{\theta}{12\alpha} + a_1 \frac{\theta}{12\alpha} + \frac{\#G_n}{n} \\ &< \frac{\theta}{12} + \alpha \frac{\theta}{12\alpha} + a_1 \frac{\theta}{12\alpha} + \frac{\#G_n}{n}. \end{aligned}$$

By the choice of a_1 we have that the third term in the last sum above is smaller than $\theta/6$. So, using (4.22) we obtain

$$\frac{\#G_n}{n} > \frac{\theta}{6}. \tag{4.23}$$

Now, defining $k = \max(G_n) - 1$, we have $m(A_n) < 12\alpha\theta^{-1} m(A_k)$. It follows from (4.23) that $k + 1 > \theta n/6$, and so $k/n > \theta/12$, because $n \ge R_0 > 12/\theta$. Since we have chosen $\alpha < (\theta/12)^{\gamma+1}$, we finally have

$$\left(\frac{k}{n}\right)^{\gamma} > \frac{12}{\theta}\left(\frac{\theta}{12}\right)^{\gamma+1} > \frac{12\alpha}{\theta},$$

which completes the proof of the result. □

Now we are able to conclude the proof of Theorem 4.2. It easily follows from Lemma 4.12 that

$$m(\Delta_n) \leq (1 + a_1)m(A_n). \tag{4.24}$$

Hence, it is enough to derive the tail estimate of Theorem 4.2 for $m(A_n)$ in the place of $m\{R > n\} = m(\Delta_n)$.

Given any large integer n, we consider the following two cases:

(1) If $n \in \mathbb{N} \setminus F$, then by (4.21) and Corollary 4.13 we have

$$m(\Delta_n) \leq c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(n - R_0)\right) m(\Delta_0).$$

(2) If $n \in F$, then we distinguish the following two subcases:
   (a) If $m(A_n) < 2m(\Gamma_n)$, then nothing has to be done.
   (b) If $m(A_n) \geq 2m(\Gamma_n)$, then we apply Proposition 4.15 and get some $k_1 < n$ for which

$$m(A_n) < \left(\frac{k_1}{n}\right)^{\gamma} m(A_{k_1}).$$

The only case we are left to consider is 2(b). In such case, either k_1 is in situation 1 or 2(a), or by Proposition 4.15 we can find $k_2 < k_1$ for which

$$m(A_{k_1}) < \left(\frac{k_2}{k_1}\right)^{\gamma} m(A_{k_2}).$$

Arguing inductively we are able to show that there is a sequence of integers $0 < k_s < \cdots < k_1 < n$ for which one of the following situations eventually holds:

(A) $$m(A_n) < \left(\frac{k_s}{n}\right)^{\gamma} c_2^N \exp\left(-\frac{b_1\theta\alpha}{12(N+1)}(k_s - R_0)\right) m(\Delta_0).$$

(B) $$m(A_n) < \left(\frac{k_s}{n}\right)^{\gamma} m(\Gamma_{k_s}).$$

(C) $$m(A_n) < \left(\frac{R_0}{n}\right)^{\gamma} m(\Delta_0).$$

In all these three situations we arrive at the desired conclusion of Theorem 4.2. Situation (C) corresponds to falling in case 2(b) successively until $k_s \leq R_0$. For the first part of the theorem, observe that for a non-uniformly expanding map, $m(\Gamma_n)$ always goes to zero when n goes to infinity, even if we do not have any a priori knowledge of the rate at which it decays.

REMARK 4.16. As one can easily see from case (B) above, the constant $\bar{C} > 0$ in Theorem 4.2 will depend on the constant $C > 0$. Moreover, from (4.24) and all the three possible cases one deduces that $\bar{C}$ also depends on some previous constants, namely α, a_1, b_1, c_1, θ, N and R_0. It is possible to check that all these constants ultimately depend on the constants B, β, b and λ associated to the non-uniformly expanding map f. Naturally they also depend on the first and second derivatives of f. For future reference we explicit the dependence of the various constants in the table below:

Constant	Dependence	Reference
σ, δ, θ	λ	Proposition 2.12
δ_1	B, β, σ, δ	Lemma 2.2
α	θ	(4.17)
N_0	δ_1	(4.4)
D_0, K_0	σ, δ_1	Lemma 4.4
C_0	B, β, b, σ	Corollary 2.5
C_1	C_0	Corollary 2.6
δ_0	δ_1, α	Lemma 4.4, (4.10), (4.17)
a_0	σ, C_1, D_0	Lemma 4.8
b_0, c_0	C_1, D_0, δ_0	Lemma 4.9
a_1	a_0, b_0, c_0, α	(4.13), (4.18)
c_1	σ, b, N_ε, C_0, N_0	Remark 4.11
b_1	a_1, c_1	(4.19)
c_2	a_1, b_0, c_0	Corollary 4.13
ε	K_0, N_0, δ_0, σ	(4.6)
N_ε	ε, σ, δ_1	Remark 4.6
N	N_0, N_ε	Proposition 4.10
R_0	K_0, σ, N_0, N, θ	Subsection 1.2, (4.11)

For a better understanding of the dependencies above we used the principle that no constant depends on a constant from a line below. Consequently we have all constants depending on B, β, b and λ.

2. Uniformness

In this section we show that if the rate of expansion decays with some uniformity in a family $\mathcal{F}$ of C^k, $k \geq 2$, non-uniformly expanding maps of a d-dimensional manifold M, then each $f \in \mathcal{F}$ is statistically stable.

DEFINITION 4.17. We say that a family $\mathcal{F}$ as above is a *uniform family* if the constants ε, δ and λ (cf. Definition 1.2 and Remark 2.13) can be chosen uniformly for all $f \in \mathcal{F}$.

Observe that an induced Markov structure for f gives, by definition, a piecewise expanding map defined in some ball Δ_0 of the ambient manifold; cf. Definition 3.5. Moreover, the ball may be taken the same for every map belonging to a sufficiently small C^2 neighborhood of f in a uniform family. Indeed, the ball Δ_0 is centered at a point $p \in M$ which has been chosen in (4.4) in such a way that for some $N_0 \in \mathbb{N}$ the set $\cup_{j=0}^{N_0} f^{-j}\{p\}$ is $\delta_1/3$-dense in M and disjoint from the critical set, where $\delta_1 > 0$ is the radius of hyperbolic balls given by Proposition 2.3. Since δ_1 may be taken the same for every map f in a uniform family and the radius δ_0 of the ball Δ_0 may be taken uniform in a neighborhood of f (see Remark 4.16), then the point p and N_0, and hence the ball Δ_0, may be taken the same for every map belonging to a sufficiently small C^2 neighborhood of f. By an implicit function argument one can prove that the critical set varies continuously with the map in the C^2 topology.

The construction of the Markov structure as described in Section 4.1 can be performed in such a way that the following uniformity condition holds:

(u_0) given any integer $N \geq 1$ and any $\epsilon > 0$, there is $\delta = \delta(\epsilon, N) > 0$ such that for $j = 1, \ldots, N$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad m\big(\{R_f = j\} \,\Delta\, \{R_{f_0} = j\}\big) < \epsilon, \tag{4.25}$$

where Δ represents the symmetric difference of two sets.

This is just by continuity of the inductive construction for maps in a C^k neighborhood of the original map. In fact, the construction of the partition on which the map R_f takes constant values is based on a finite number of iterations of f. By continuity, we can perform the construction of the partition in such a way that for some fixed integer N the Lebesgue measure of $\{R_f = j\}$ varies continuously with the map f for $j \leq N$. Moreover, the Lebesgue measures of the auxiliary sets A_j and B_j also vary continuously with the map f for $j \leq N$. Hence, the construction can be carried out with R_f depending continuously on f as stated in (u_0).

The following lemma gives a useful criterium for checking uniformity condition (u_1) presented in Section 3.4. Let us recall that we have proved in Lemma 3.19 that condition (u_1) implies the hypothesis (4.26) in the lemma below. Here we prove that the converse holds under assuming (u_0).

LEMMA 4.18. *Assume that (u_0) holds for f_0, and that given any $\epsilon > 0$ there are $N \geq 1$ and $\delta > 0$ for which*

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_f > j\}} \Big\|_d < \epsilon. \tag{4.26}$$

Then uniformity condition (u_1) holds for f_0.

PROOF. For the sake of notational simplicity we will write R instead of R_f and R_0 instead of R_{f_0}. Let $\epsilon > 0$ be given, and take $N \geq 1$ in such a way that

$$\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_0>j\}}\|_d < \epsilon/3.$$

This is possible because we are (implicitly) assuming that $R_0 \in L^d(m)$. Since

$$R_0 = \sum_{j=0}^{\infty} \mathbf{1}_{\{R_0>j\}},$$

then we have

$$\begin{aligned}\Big\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R>j\}}\Big\|_d &= \Big\|R - R_0 + R_0 - \sum_{j=0}^{N-1} \mathbf{1}_{\{R_0>j\}} + \sum_{j=0}^{N-1} \mathbf{1}_{\{R_0>j\}} - \sum_{j=0}^{N-1} \mathbf{1}_{\{R>j\}}\Big\|_d \\ &\leq \|R - R_0\|_d + \Big\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_0>j\}}\Big\|_d + \sum_{j=0}^{N-1} \Big\|\mathbf{1}_{\{R_0>j\}} - \mathbf{1}_{\{R>j\}}\Big\|_d.\end{aligned}$$

So, if we take $\delta = \delta(N, \epsilon) > 0$ sufficiently small, then under assumption (u_0) and by (4.25), the first and third terms in the sum above can be made smaller than $\epsilon/3$. This gives the desired conclusion. □

Let $\mathcal{F}$ be a uniform family of non-uniformly expanding maps. Given $f \in \mathcal{F}$ we let the expansion time function $\mathcal{E}^f$ and the recurrence time function $\mathcal{R}^f$ be defined as in (4.1) and (4.2) respectively. The tail set Γ_n^f is also defined for $f \in \mathcal{F}$ as in (4.3) for $n \geq 1$.

THEOREM 4.19. *Let $\mathcal{F}$ be a uniform family of C^k non-uniformly maps, with $k \geq 2$, for which there are $C > 0$ and $\gamma > d$ such that*

$$m(\Gamma_n^f) \leq Cn^{-\gamma}, \quad \text{for all } n \geq 1 \text{ and } f \in \mathcal{F}.$$

Then uniformity conditions (u_1) and (u_2) hold for each $f \in \mathcal{F}$.

PROOF. Take any $f_0 \in \mathcal{F}$. If we assume that there are $C > 0$ and $\gamma > d$ such that $m(\Gamma_n^f) \leq Cn^{-\gamma}$ for all $n \geq 1$ and all $f \in \mathcal{F}$, then by Theorem 4.2 there is a constant $\bar{C} > 0$ such that $m(\{R_f > j\}) \leq \bar{C}n^{-\gamma}$ for all $n \geq 1$ and all $f \in \mathcal{F}$, as long as f is taken in a sufficiently small C^k neighborhood of f_0 in $\mathcal{F}$, say $f \in \mathcal{F}$ with $\|f - f_0\|_{C^k} < \delta$. Actually, as we have observed in Remark 4.16 the constant $\bar{C}$ may be taken uniformly in a neighborhood of the map f_0. Thus, given $f \in \mathcal{F}$ with $\|f - f_0\|_{C^k} < \delta$ and an integer $N \geq 1$, we have

$$\|\sum_{j=N}^{\infty} \mathbf{1}_{\{R_f>j\}}\|_d \leq \sum_{j=N}^{\infty} m\{R_f > j\}^{1/d} \leq \sum_{j=N}^{\infty} \bar{C}^{1/d} n^{-\gamma/d}.$$

Since we are assuming $\gamma > d$, this last sum can be made arbitrarily small by taking N sufficiently large. Applying Lemma 4.18 we obtain uniformity condition (u_1).

For proving that (u_2) holds, we just have to show that the constants κ and K in Definition 3.5 may be chosen uniformly for f in a C^k neighborhood of f_0 in the uniform family $\mathcal{F}$. This is because in our setting the images of the elements of the partition are all equal to the ball Δ_0, which then implies that the long branches condition is trivially verified.

The constant K is given in Subsection 4.1.3. As it has been shown there, it only depends on C_0, D_0 and K_0. From Remark 4.16 we see that these constants may be chosen uniformly in $\mathcal{F}$. On the other hand, the constant κ is given in Subsection 1.2 σ, N_0, K_0 and R_0 which again may be chosen uniformly in $\mathcal{F}$. □

As an immediate consequence of Theorem 4.19 and Theorem 3.17 we obtain:

COROLLARY 4.20. *If $\mathcal{F}$ is a uniform family of C^k ($k \geq 2$) non-uniformly maps where non-degeneracy condition* (3.7) *holds, then each $f \in \mathcal{F}$ is statistically stable.*

EXAMPLE 4.21. Let $\mathcal{V}$ be the class of Viana maps introduced in 1.2.2, which has been described as a small neighborhood of a map $\hat{f}$ from the cylinder into itself. It follows from the construction that $\mathcal{V}$ is a uniform family. Moreover, (3.7) holds in this context, as long as the neighborhood $\mathcal{V}$ is chosen sufficiently small. Indeed, denoting $J_f = \det Df$, we have

$$J\hat{f} = D\hat{g}\,\frac{\partial \hat{q}}{\partial x} \quad \text{and} \quad \frac{\partial J\hat{f}}{\partial x} = D\hat{g}\,\frac{\partial^2 \hat{q}}{\partial x^2}\,.$$

Our assumptions give that the last expression is bounded away from zero. So, choosing $\mathcal{V}$ small enough, then there exists $c_1 > 0$ such that $|\partial J_f/\partial x| \geq c_1$ for any $f \in \mathcal{V}$. Consequently, $m(f^{-1}(E)) \leq \operatorname{const} m(E)^{1/2}$ for any $f \in \mathcal{V}$ and any measurable set E. It follows from Corollary 4.20 that Viana maps are statistically stable.

3. Rates of mixing

An invariant measure μ for a map f is said to be *mixing* if for all measurable sets $A, B \subset M$ we have $\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$. Defining the *correlation function*

$$C_n(\varphi, \psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

it is sometimes possible to obtain specific rates of decay which depend only on the map f (up to a multiplicative constant which is allowed to depend on φ, ψ) as long as the *observables* φ, ψ belong to some appropriate functional space. Notice that choosing these observables to be characteristic functions this gives exactly the definition of mixing.

The precise dynamical features which cause mixing, and in particular the dynamical features which cause different rates of decay of the correlation function, are still far from understood. Exponential mixing for uniformly expanding and uniformly hyperbolic systems has been known since the work of Sinai, Ruelle and Bowen [**Si68, Bo70, Bo75, BR**] and may not seem surprising in view of the fact that all quantities involved are exponential. However the subtlety of the question is becoming more apparent in the light of recent examples which satisfy asymptotic exponential expansion estimates but only subexponential decay of correlations.

We are also interested in conditions for the validity of the Central Limit Theorem, which states that the probability of a given deviation of the average values of an observable along an orbit from the asymptotic average is essentially given by a Normal Distribution: given a Hölder continuous function φ which is not a coboundary ($\varphi \neq \psi \circ f - \psi$ for any $\psi \in L^2(\mu)$) there exists $\sigma > 0$ such that for every interval $J \subset \mathbb{R}$,

$$\mu\left\{x \in X : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\varphi(f^j(x)) - \int \varphi d\mu\right) \in J\right\} \to \frac{1}{\sigma\sqrt{2\pi}}\int_J e^{-t^2/2\sigma^2}dt.$$

Here we study of the correlation decay and validity of the Central Limit Theorem for non-uniformly expanding maps in terms of the tail set. We will use the results in [**Yo99**] for Markov towers.

THEOREM 4.22. *Let $f : M \to M$ is a transitive C^2 non-uniformly expanding map, and suppose that there exists $\gamma > 1$ such that*

$$m(\Gamma_n) \leq \mathcal{O}(n^{-\gamma}).$$

Then f has a unique ergodic absolutely continuous invariant probability measure μ. Some finite power of f is mixing with respect to μ, and the correlation function for $\varphi \in L^\infty(m)$ and Hölder continuous ψ satisfies

$$C_n(\varphi, \psi) \leq \mathcal{O}(n^{-\gamma+1}).$$

Moreover, if $\gamma > 2$ then the Central Limit Theorem holds.

PROOF. The existence of a unique ergodic absolutely continuous probability measure μ is a consequence of Corollary 3.4. We are left to prove the correlation decay and the Central Limit Theorem for (f, μ).

From Theorem 4.2 we know that there is some disk $\Delta_0 \subset M$, a countable partition $\mathcal{P}$ into sub-disks of a full Lebesgue measure subset of Δ_0, and a return time function $R : \Delta_0 \to \mathbb{N}$ constant on elements of $\mathcal{P}$, for which

$$m\{R > n\} \leq \mathcal{O}(n^{-\gamma}), \tag{4.27}$$

such that the following conditions on the induced map $F : \Delta_0 \to \Delta_0$ given by $F(x) = f^{R(x)}(x)$ hold:

(1) there is $0 < \beta < 1$ such that for x in the interior of the elements of $\mathcal{P}$

$$\|DF(x)^{-1}\| < \beta.$$

(2) there is some constant $K>0$ such that for every $U\in\mathcal{P}$ and $x,y\in U$
$$\log\left|\frac{\det DF(x)}{\det DF(y)}\right|\le K\,\mathrm{dist}(F(x),F(y)).$$
(3) the map F is a C^2 diffeomorphism (and in particular a bijection) from each $U\in\mathcal{P}$ onto Δ_0.

It immediately follows from the uniform expansivity property (1) above that for any $x,y\in\Delta_0$ which have the same combinatorics, i.e. which remain in the same elements of the partition $\mathcal{P}$ for some number $s(x,y)$ of iterates of the induced map F, we have
$$\mathrm{dist}(x,y)\le\beta^{s(x,y)}.$$
Together with properties (1)-(3) above this implies that F is in the conditions of Appendix B. Since we have no a priori knowledge of $\gcd\{R_i\}$, we may need to take some finite power of f in order to assure that $\gcd\{R_i\}=1$.

Then we introduce the Markov tower,
$$\Delta=\big\{(x,n)\in\Delta_0\times\mathbb{N}:\ 0\le n<R(x)\big\},$$
and the tower map $T\colon\Delta\to\Delta$ given by
$$T(x,n)=\begin{cases}(x,n+1), & \text{if } n+1<R(x);\\ (F(x),0), & \text{if } n+1=R(x).\end{cases}$$
We have by construction
$$T^{R(x)}(x,0)=(F(x),0)=(f^{R(x)}(x),0).$$
We take m_0 the Lebesgue measure on Δ_0, and take $\bar m$ the push-forward of m_0 to the higher levels of the tower by the action of T as described in Appendix B. (We use $\bar m$ instead of m in order to distinguish it from the Lebesgue measure on M). Theorem B.1 gives that T has an invariant probability measure ν which is equivalent to $\bar m$. Letting
$$\mathcal{H}_\beta=\left\{\varphi:\Delta\to\mathbb{R}\mid\ \exists C>0\text{ such that }|\varphi(x)-\varphi(y)|\le C\beta^{s(x,y)}\ \forall x,y\in\Delta\right\},$$
we have by Theorem B.2 that the correlation function for $\varphi\in\mathcal{H}_\beta$ and $\psi\in L^\infty(\bar m)$ in the underlying space (Δ,ν) satisfies
$$C_n^T(\varphi,\psi)\le\mathcal{O}(n^{-\gamma+1}).\tag{4.28}$$
(We use the superscript T in order to distinguish this from that associated to f). Moreover, if $\gamma>2$ then the Central Limit Theorem holds for T.

Now we define a projection
$$\begin{array}{rccl}\pi & : & \Delta & \longrightarrow & \bigcup_{n\ge0}f^n(\Delta_0).\\ & & (x,n) & \longmapsto & f^n(x)\end{array}$$
Observe that this map π satisfies
$$f\circ\pi=\pi\circ T.$$

Letting $\mu^* = \pi_* \nu$, we easily verify that μ^* is an absolutely continuous f invariant probability measure. Thus μ^* must coincide with the unique ergodic absolutely continuous f invariant probability measure μ.

Let now $\varphi \in L^\infty(m)$ and $\varphi\colon M \to \mathbb{R}$ be Hölder continuous with Hölder exponent $\eta > 0$. Defining $\bar\psi = \psi \circ \pi$ and $\bar\varphi = \varphi \circ \pi$, then $\bar\psi \in L^\infty(\bar m)$ and $\bar\varphi \in \mathcal{H}_\beta$ where $\beta = \max \|(Df^R)^{-1}\|^\eta$. Moreover,

$$\int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu = \int (\bar\varphi \circ T^n)\bar\psi d\nu - \int \bar\varphi d\nu \int \bar\psi d\nu,$$

which together with (4.28) gives the desired estimate for the correlation decay. A similar observation leads to the Central Limit Theorem by application of Theorem B.3. □

As we have observed in Remark 4.14, the Lebesgue measure of $\{R > n\}$ decays exponentially fast when f is uniformly expanding. Since the proof of Theorem 4.22 also works in that case, this method also gives the well-known exponential decay of correlations for uniformly expanding maps.

CHAPTER 5

Random perturbations

The goal of this chapter is the study of the statistical behavior of a system evolving under random perturbations of a fixed transformation, and to understand how the statistical behavior of this random system relates to the statistical behavior of the original system when we consider small perturbations.

The stability of a dynamical system f under random perturbations can heuristically be introduced in the following way. Assume that instead of time averages of Dirac measures supported on the iterates of $x_0 \in M$, we consider time averages of Dirac measures δ_{x_j}, where at each iteration we take x_{j+1} close to $f(x_j)$ with a controlled error. One is interested in studying the existence of limit measures for these time averages and their relation to the analogous ones for unperturbed orbits, that is, the stochastic stability of the initial system. Stability results have been established in **[Ki86, Ki88]** for uniformly expanding maps, and in **[BY92, BaV]** for certain quadratic maps of the interval. Another important contribution is the announced work of Benedicks and Viana for Hénon-like strange attractors. Here we follow the approach of **[AA1]** in the study of the stochastic stability of non-uniformly expanding dynamical systems, and give sufficient conditions and necessary conditions for their stochastic stability.

We will use the approach of **[Vi3, Ar00]** for putting in precise mathematical terms the heuristic description of random perturbation introduced above. We take a continuous map

$$\begin{array}{rccc} F: & T & \longrightarrow & C^2(M,M) \\ & t & \longmapsto & f_t \end{array}$$

from a metric space T into the space of C^2 maps from M to M, with $f = f_{t^*}$ for some fixed $t^* \in T$. Given $\underline{t} = (t_1, t_2, t_3, \ldots)$ in the infinite product space $T^{\mathbb{N}}$ we define $f^0_{\underline{t}} = \mathrm{id}_M$, and for $n \geq 1$

$$f^n_{\underline{t}} = f_{t_n} \circ \cdots \circ f_{t_1}.$$

Given $x \in M$ we call the sequence $\left(f^n_{\underline{t}}(x)\right)_{n\geq 1}$ a *random orbit* of x. We will restrict the class of perturbations we are going to consider for maps with critical sets: we take all the maps f_t with the same critical set $\mathcal{C}$ by imposing that

$$Df_t(x) = Df(x), \quad \text{for every } x \in M \setminus \mathcal{C} \text{ and } t \in T. \tag{5.1}$$

Perturbations of this type may be implemented, for instance, in Lie groups; see Example 5.3 below.

We also take a family $(\theta_\epsilon)_{\epsilon>0}$ of probability measures on T such that their supports form a nested family of connected compact sets, and

$$\operatorname{supp}(\theta_\epsilon) \to \{t^*\}, \quad \text{when} \quad \epsilon \to 0.$$

Given an integer $n \geq 1$ and $x \in M$, we define the map

$$\tau_x^n \colon T^{\mathbb{N}} \to M, \quad \tau_x^n(\underline{t}) = f_{\underline{t}}^n(x).$$

The family $(\tau_x)_{x\in M}$ gives the possible transitions from the point x to points in M through f_t with $t \in T$. This enables us to introduce the probability measure $(\tau_x^n)_*\theta_\epsilon^{\mathbb{N}}$, which is the push-forward of the probability measure $\theta_\epsilon^{\mathbb{N}}$ from $T^{\mathbb{N}}$ to M via τ_x^n. The value of $(\tau_x)_*\theta_\epsilon^{\mathbb{N}}$ at Borel set $A \subset M$ gives the probability of the point x fall into the set A when iterated by the maps f_t with $t \in \operatorname{supp}(\theta_\epsilon)$.

DEFINITION 5.1. We will refer to a pair $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ as above as a *random perturbation* of f. We say that $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ is a *non-degenerate random perturbation*, if for small $\epsilon > 0$ we may take $\xi = \xi(\epsilon) > 0$ for which the following holds for all $x \in M$:

(1) $\{f_t(x) \colon t \in \operatorname{supp}(\theta_\epsilon)\}$ contains the ball of radius ξ around $f(x)$;
(2) $(\tau_x)_*\theta_\epsilon^{\mathbb{N}}$ is absolutely continuous with respect to m.

The first condition in the definition above means that perturbed iterates cover a full neighborhood of the unperturbed ones for all sufficiently small noise levels, while the second one means that sets of perturbation vectors with positive $\theta_\epsilon^{\mathbb{N}}$ measure send any point $x \in M$ onto subsets of M with positive Lebesgue measure. The example below shows that we can always construct a non-degenerate random perturbation of any smooth map $f : M \to M$ on a compact manifold.

EXAMPLE 5.2. Let $f : M \to M$ be a continuous map of a d-dimensional compact Riemannian manifold. We take an open cover of M given by the images of finitely many local charts $\psi_i : B_3(0) \to M$, for $i = 1, \ldots, k$, with the property that

$$\bigcup_{i=1}^{k} \psi_i(B_1(0)) \supset M, \tag{5.2}$$

where $B_r(0)$ denotes the ball of radius $r > 0$ around 0 in $\mathbb{R}^d$. For each $1 \leq i \leq k$ we take orthonormal C^1 vector fields $X_i^1, \ldots, X_i^d$ in $\psi_i(B_3(0))$, and extend them respectively to C^1 vector fields $Y_i^1, \ldots, Y_i^d$ defined in the whole M, in such a way that for each $1 \leq i \leq k$ and $1 \leq j \leq d$ both:

(1) Y_i^j is null outside $\psi_i(B_2(0))$;
(2) Y_i^j coincides with X_i^j in $\psi_i(B_1(0))$.

Since (5.2) holds, then at each $x \in M$ there is some $1 \leq i \leq k$ such that $Y_i^d(x), \ldots, Y_i^d(x)$ is an orthonormal basis for T_xM. Let $\Gamma : TM \times \mathbb{R} \to M$ be the geodesic flow associated to the Riemannian metric. Then we define a continuous map

$$F : \left(\mathbb{R}^d\right)^k \longrightarrow C^2(M, M)$$

as

$$F(U_1, \dots, U_k)(x) = \Gamma\left(\left(f(x), \sum_{i=1}^{k}\sum_{j=1}^{d} u_{ij}X_i^j(f(x))\right), 1\right)$$

where $U_i = (u_{i1}, \dots, u_{id})$ for $1 \le i \le k$. Now let the family of probability measures $(\theta_\epsilon)_\epsilon$ in $\mathbb{R}^{kd}$ be defined for each $\epsilon > 0$ as

$$\theta_\epsilon = \frac{1}{\lambda(B_\epsilon(0))}(\lambda | B_\epsilon(0)),$$

where λ is the Lebesgue measure on $\mathbb{R}^{kd}$. It is straightforward to check that $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ is a non-degenerate random perturbation of f.

EXAMPLE 5.3. Let $f : M \to M$ be a map from some Lie group M into itself, for instance $M = \mathbb{T}^d$ the d-dimensional torus, and let $(\theta_\epsilon)_{\epsilon>0}$ be a family of probability measures on M such that $\operatorname{supp}(\theta_\epsilon)$ contains an open neighborhood of the identity e and is contained in a ϵ-neighborhood of e. We define a function

$$F : M \longrightarrow C^2(M, M)$$

associating to each $t \in M$ the C^2 map f_t from M into itself, defined as

$$f_t(x) = t \cdot f(x),$$

where the operation is that of the group structure on M. Then $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ is a non-degenerate random perturbation of f for which (5.1) holds.

1. Stationary measures

Let a non-degenerate random perturbation $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ of a map $f: M \to M$ from a Riemannian manifold into itself be given. It will be useful for our purposes to introduce the skew-product map

$$\begin{array}{rccc} S: & T^{\mathbb{N}} \times M & \longrightarrow & T^{\mathbb{N}} \times M \\ & (\underline{t}, z) & \longmapsto & (\sigma(\underline{t}), f_{t_1}(z)) \end{array} \tag{5.3}$$

where σ is the left shift on the elements $\underline{t} = (t_1, t_2, \dots) \in T^{\mathbb{N}}$, defined as

$$\sigma(t_1, t_2, t_3 \dots) = (t_2, t_3, \dots).$$

The notion below replaces the usual invariance of a measure with respect to a transformation in the present context of random perturbations.

DEFINITION 5.4. Given $\epsilon > 0$, we say that a probability measure μ^ϵ on the Borel sets of M is a *stationary measure*, if

$$\iint \varphi(f_t(x))\, d\mu^\epsilon(x)\, d\theta_\epsilon(t) = \int \varphi\, d\mu^\epsilon \tag{5.4}$$

for every $\varphi: M \to \mathbb{R}$ integrable with respect to μ^ϵ.

We leave it as an exercise to reader to check that μ^ϵ is a stationary measure if and only if the measure $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ on $T^{\mathbb{N}} \times M$ is invariant by S.

LEMMA 5.5. *If $(\mu^\epsilon)_{\epsilon>0}$ is a family of stationary measures having μ_0 as a weak* accumulation point when $\epsilon \to 0$, then μ_0 is invariant by f.*

PROOF. By the weak* of $(\mu^\epsilon)_{\epsilon>0}$ to μ_0 when $\epsilon \to 0$ we have that for any continuous $\varphi\colon M \to \mathbb{R}$

$$\int \varphi d\mu^\epsilon \to \int \varphi d\mu_0 \quad \text{when} \quad \epsilon \to 0.$$

On the other hand, since μ^ϵ is stationary we have

$$\iint \varphi(f_t(x))\, d\theta_\epsilon(t)\, d\mu^\epsilon(x) = \int \varphi\, d\mu^\epsilon \quad \text{for every } \epsilon.$$

Therefore, it suffices to prove that

$$\iint \varphi(f_t(x))\, d\theta_\epsilon(t)\, d\mu^\epsilon(x) \to \int \varphi \circ f d\mu_0 \quad \text{when} \quad \epsilon \to 0.$$

We have

$$\left|\iint \varphi(f_t(x))\, d\theta_\epsilon(t)\, d\mu^\epsilon(x) - \int \varphi \circ f d\mu_0\right| \leq$$
$$\left|\iint \varphi(f_t(x))\, d\theta_\epsilon(t)\, d\mu^\epsilon(x) - \int \varphi \circ f d\mu^\epsilon\right| + \left|\int \varphi \circ f d\mu^\epsilon - \int \varphi \circ f d\mu_0\right|.$$

Since $\operatorname{supp}(\theta_\epsilon) \to \{t^*\}$ when $\epsilon \to 0$, we have that

$$\int \varphi(f_t(x))\, d\theta_\epsilon(t) - \varphi \circ f$$

converges uniformly to 0 when $\epsilon \to 0$. Then the first term in the sum above is close to zero for small $\epsilon > 0$. On the other hand, since μ^ϵ converges in the weak* topology to μ_0, we also have that the second term in the sum above is also close to zero for small $\epsilon > 0$. □

Let us now fix $x \in M$, and consider the sequence of measures

$$\mu_n^\epsilon = \frac{1}{n}\sum_{j=0}^{n-1}(\tau_x^j)_*\theta_\epsilon^{\mathbb{N}}. \tag{5.5}$$

Since this is a sequence of probability measures on the compact manifold M, then it has weak* accumulation points when $n \to \infty$. We will show that such accumulation points are always absolutely continuous with respect to the Lebesgue measure. Let us remark that the assumption on the density of $(\tau_x)_*\theta_\epsilon$ in the result below is quite general, holding for instance for the random perturbations of Examples 5.2 and 5.3.

LEMMA 5.6. *Every weak* accumulation point of $(\mu_n^\epsilon)_n$ is a stationary measure, which is absolutely continuous with respect to the Lebesgue measure.*

PROOF. Let μ^ϵ be a weak* accumulation point of $(\mu_n^\epsilon)_n$. Then, there is some sequence $n_k \to \infty$ such that we may write

$$\iint \varphi(f_t(y))\, d\mu^\epsilon(y)\, d\theta_\epsilon(t) = \int \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \varphi(f_t(f_{\underline{t}}^j(x)))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t})\, d\theta_\epsilon(t)$$

for each continuous $\varphi : M \to \mathbb{R}$. We have by dominated convergence

$$\begin{aligned}
&\int \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \varphi(f_t(f_{\underline{t}}^j(x)))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t})\, d\theta_\epsilon(t) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int\int \varphi(f_t(f_{\underline{t}}^j(x)))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t})\, d\theta_\epsilon(t) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \varphi(f_{\underline{t}}^{j+1}(x))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \varphi(f_{\underline{t}}^j(x))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}) + \lim_{k\to\infty} \frac{1}{n_k} \int \left[\varphi(f_{\underline{t}}^{n_k}(x)) - \varphi(x)\right]\, d\theta_\epsilon^{\mathbb{N}}(\underline{t})
\end{aligned}$$

Since φ is bounded, then the second limit in this last expression is equal to 0, while the first one gives precisely the integral of φ with respect to μ^ϵ. Thus we have shown that μ^ϵ is a stationary measure.

Let us now check the absolute continuity. We have for every measurable set $A \subset M$

$$\begin{aligned}
\int 1_A\, d\mu^\epsilon &= \iint 1_A(f_t(x))\, d\mu^\epsilon(x)\, d\theta_\epsilon(t) \\
&= \iint 1_A(f_t(x))\, d\theta_\epsilon(t)\, d\mu^\epsilon(x) \\
&= \int (\tau_x)_*\theta_\epsilon^{\mathbb{N}}(A)\, d\mu^\epsilon(x).
\end{aligned}$$

This shows that

$$\mu^\epsilon(A) = \int (\tau_x)_*\theta_\epsilon^{\mathbb{N}}(A)\, d\mu^\epsilon(x).$$

Since $(\tau_x)_*\theta_\epsilon^{\mathbb{N}}$ is absolutely continuous with respect to m (recall Definition 5.1), then we must have $\mu^\epsilon(A) = 0$ whenever $m(A) = 0$. □

The next result shows that stationary measures must be supported in topologically relevant regions of the phase space. A Borel set $A \subset M$ is said to be *invariant (by a random perturbation)* if $f_t(A) \subset A$ for every $t \in \operatorname{supp}(\theta_\epsilon)$, at least for small $\epsilon > 0$.

LEMMA 5.7. *If μ^ϵ is a stationary measure, then* $\operatorname{supp}(\mu^\epsilon)$ *is an invariant set with nonempty interior.*

PROOF. Let μ^ϵ be a stationary measure. Take any $z \in \mathrm{supp}(\mu^\epsilon)$ and let U be some small neighborhood of $f_s(z)$ in M, for $s \in \mathrm{supp}(\theta_\epsilon)$. We have

$$\mu^\epsilon(U) = \int \mathbf{1}_U d\mu^\epsilon = \iint \mathbf{1}_{f_t^{-1}(U)}(x) d\mu^\epsilon(x) d\theta_\epsilon(t) = \int \mu^\epsilon(f_t^{-1}(U)) d\theta_\epsilon(t).$$

Since $f_s^{-1}(U)$ is a neighborhood of z in M, then by continuity, $f_t^{-1}(U)$ must be a neighborhood of z. Then we have $\mu^\epsilon(f_t^{-1}(U)) > 0$ for all t in a small neighborhood of s, because $z \in \mathrm{supp}(\mu^\epsilon)$. Since $s \in \mathrm{supp}(\theta_\epsilon)$, the last integral in the expression above is positive, and so $\mu^\epsilon(U) > 0$. This shows that $\mathrm{supp}(\mu^\epsilon)$ is an invariant set. Since $f_t(\mathrm{supp}(\mu^\epsilon)) \subset \mathrm{supp}(\mu^\epsilon)$ for all $t \in \mathrm{supp}(\theta_\epsilon)$, by Definition 5.1 this implies that the interior of $\mathrm{supp}(\mu^\epsilon)$ is nonempty. $\square$

2. Physical measures

The notion that we present below plays in the present setting a role similar to that played by SRB measures in deterministic systems. SRB measures are frequently called physical measures in several places. Here we use this last term only in the context of random perturbations of a map, in order to distinguish them from the previously introduced SRB measures.

DEFINITION 5.8. Let μ^ϵ be a stationary measure for a random perturbation $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ of $f: M \to M$. We say that μ^ϵ is a *physical measure* if, for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f_{\underline{t}}^n(x)\big) = \int \varphi \, d\mu^\epsilon \tag{5.6}$$

for all continuous $\varphi: M \to \mathbb{R}$ and $\theta_\epsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. The set of points $x \in M$ for which (5.6) holds for all continuous $\varphi: M \to \mathbb{R}$ is denoted by $B(\mu^\epsilon)$ and called the *basin of* μ^ϵ.

The result below guarantees that every stationary measure can be decomposed into a combination of ergodic measures. We say that a stationary measure μ^ϵ is *ergodic* if the measure $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ is ergodic for the skew-product dynamical system $S: T^{\mathbb{N}} \times M \to T^{\mathbb{N}} \times M$ defined in (5.3).

PROPOSITION 5.9. *Let μ^ϵ be a stationary measure for a random perturbation $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ of $f: M \to M$. Then there is a family of ergodic probability measures $(\mu_x^\epsilon)_{x\in M}$ such that for every μ^ϵ-integrable $\varphi : M \to \mathbb{R}$*

$$\int \varphi \, d\mu^\epsilon = \int \int \varphi(y) \, d\mu_x^\epsilon(y) \, d\mu^\epsilon(x).$$

PROOF. Let μ^ϵ be a stationary measure as in the statement. Then $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ is invariant by S. By the ergodic decomposition of Theorem 0.11 we know that there is a family $\eta^\epsilon_{\underline{t},x}$ of S ergodic probability measures defined for $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ such that for any $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$-integrable $\psi : T^{\mathbb{N}} \times M \to \mathbb{R}$

$$\int \psi \, d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon) = \int \left(\int \psi \, d\eta^\epsilon_{\underline{t},x} \right) d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon)(\underline{t}, x). \tag{5.7}$$

We now show that $\eta^\epsilon_{\underline{t},x} = \theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon_x$ for a probability measure μ^ϵ_x which will be S ergodic by construction, for $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. Indeed, since $\theta^{\mathbb{N}}_\epsilon$ is a product measure on $T^{\mathbb{N}}$ we may write (5.7) as

$$\int \psi \, d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon) = \int \left[\int \left(\int \psi \, d\eta^\epsilon_{u\underline{t},x} \right) d\theta_\epsilon(u) \right] d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon)(\underline{t}, x),$$

where $u\underline{t}$ is the vector $(u, t_1, t_2, \dots)$ with $\underline{t} = (t_1, t_2, \dots)$. Since (5.7) holds for every $\psi \in L^1(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon)$ we must have

$$\eta^\epsilon_{\underline{t},x} = \int \eta^\epsilon_{u\underline{t},x} \, d\theta_\epsilon(u),$$

for $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. This argument may be reapplied any number of times, thus proving

$$\eta^\epsilon_{\underline{t},x} = \int \eta^\epsilon_{u_1 u_2 \dots u_k \underline{t},x} \, d\theta^k_\epsilon(u_1, u_2, \dots, u_k),$$

for $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ and every $k \geq 1$. Hence

$$\eta^\epsilon_{\underline{t},x} = \int \eta^\epsilon_{\underline{u},x} \, d\theta^{\mathbb{N}}_\epsilon(\underline{u}),$$

for $\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. This is equivalent to say that there is a family $(\eta^\epsilon_x)_{x \in M}$ of probability measures in $T^{\mathbb{N}} \times M$ such that

$$\eta^\epsilon_{\underline{t},x} = \eta^\epsilon_x, \quad \text{for } \theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon \text{ almost all } (\underline{t}, x) \in T^{\mathbb{N}} \times M.$$

Using this back in (5.7) we obtain

$$\begin{aligned} \int \psi \, d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon) &= \int \left(\int \psi \, d\eta^\epsilon_x \right) d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon)(\underline{t}, x) \\ &= \int \left(\int \psi \, d\eta^\epsilon_x \right) d\mu^\epsilon(x). \end{aligned} \tag{5.8}$$

Applying this to the characteristic function of an arbitrary Borel set $A \subset M$ we have

$$\mu^\epsilon(A) = \int (1_A \circ \tau) \, d(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon) = \int \left(\int (1_A \circ \tau) \, d\eta^\epsilon_x \right) d\mu^\epsilon(x),$$

where $\pi : T^{\mathbb{N}} \times M \to M$ is the projection on the second coordinate. Hence for any measurable $U \subset T^{\mathbb{N}}$ we also have

$$\left(\theta^{\mathbb{N}}_\epsilon \times \mu^\epsilon \right) (U \times A) = \theta^{\mathbb{N}}_\epsilon(U) \, \mu^\epsilon(A) = \int \theta^{\mathbb{N}}_\epsilon(U) \left(\int (1_A \circ \pi) \, d\eta^\epsilon_x \right) d\mu^\epsilon(x).$$

Defining the push-forward $\mu_x^\epsilon = \pi_* \eta_x^\epsilon$ we arrive at

$$\left(\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon\right)(U \times A) = \int \left(\theta_\epsilon^{\mathbb{N}} \times \mu_x^\epsilon\right)(U \times A)\, d\mu^\epsilon(x)$$

for all measurable sets $U \subset T^{\mathbb{N}}$ and $A \subset M$, thus showing that $\eta_{\underline{t},x} = \eta_x = \theta_\epsilon^{\mathbb{N}} \times \mu_x^\epsilon$ for $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. The statement of the proposition follows from this by taking $\psi = \varphi \circ \pi$ in (5.8). □

LEMMA 5.10. *If μ^ϵ is an ergodic absolutely continuous stationary measure for a non-degenerate random perturbation of $f\colon M \to M$, then μ^ϵ is a physical measure. Moreover, there is $\xi > 0$ (depending only on $\epsilon > 0$) such that the basin of any physical measure contains Lebesgue almost every point in some ball of radius ξ.*

PROOF. Letting $\pi\colon T^{\mathbb{N}} \times M \to M$ denote the projection onto M, by Birkhoff's Ergodic Theorem we have for every μ^ϵ-integrable $\varphi\colon M \to \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (\varphi \circ \pi)\left(S^j(\underline{t}, x)\right) = \int (\varphi \circ \pi)\, d(\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon)$$

for $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, where S is the skew-product map defined in (5.3). This is equivalent to say that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f_{\underline{t}}^j(x)\right) = \int \varphi\, d\mu^\epsilon$$

for $\theta_\epsilon^{\mathbb{N}} \times \mu^\epsilon$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. Hence μ^ϵ almost every $x \in M$ belongs to the basin of μ^ϵ. Since μ^ϵ is absolutely continuous with respect to the Lebesgue measure, then μ^ϵ is a physical measure. Thus we have proved the first part of the lemma.

Take now x an arbitrary point in the basin of a physical measure μ^ϵ. Since we are assuming that the random perturbation is non-degenerate, then there is $\xi = \xi(\epsilon) > 0$ such that $\{f_t(x)\colon t \in \operatorname{supp}(\theta_\epsilon)\}$ contains the ball $B(f(x), \xi)$ of radius ξ around $f(x)$. Also by non-degeneracy of the random perturbation we have that $(\tau_x)_* \theta_\epsilon^{\mathbb{N}}$ is equivalent to m on $\tau_x(\operatorname{supp}(\theta_\epsilon^{\mathbb{N}}))$. Then m almost every $y \in B(f(x), \xi)$ belongs in the basin of μ^ϵ, by the invariance of basin under random iterations. □

PROPOSITION 5.11. *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a non-degenerate random perturbation of $f\colon M \to M$. Given $\epsilon > 0$ there are $l = l(\epsilon) \in \mathbb{N}$ and physical measures $\mu_1^\epsilon, \dots, \mu_l^\epsilon$ such that for any absolutely continuous stationary measure μ^ϵ there are $\alpha_1, \dots, \alpha_l \geq 0$ with $\alpha_1 + \dots + \alpha_l = 1$ and $\mu^\epsilon = \alpha_1 \mu_1^\epsilon + \dots + \alpha_l \mu_l^\epsilon$.*

PROOF. Let μ^ϵ be an absolutely continuous stationary measure (it exists by Lemma 5.6), and let $(\mu_x^\epsilon)_x$ be the family of ergodic stationary measures given by Proposition 5.9. Since μ^ϵ is absolutely continuous with respect to m, then μ_x^ϵ is also absolutely continuous with respect to m for μ^ϵ almost every $x \in M$. From Lemma 5.10 we have that μ_x^ϵ is a physical measure for μ^ϵ almost every $x \in M$. In particular, the set of physical measures is non-empty. Since distinct physical measures have disjoint basins, we also have by Lemma 5.10 that for a noise level $\epsilon > 0$ the number of physical measures is finite (possibly depending on ϵ).

Let now $\mu_1^\epsilon, \ldots, \mu_l^\epsilon$ be the set of all physical measures of f for a given noise level $\epsilon > 0$, where $l = l(\epsilon) \geq 1$. Let μ^ϵ be an absolutely continuous stationary measure, and consider $(\mu_x^\epsilon)_{x \in M}$ the family of ergodic stationary measures of the decomposition given by Proposition 5.9. Since μ_x^ϵ is a physical measure for μ^ϵ almost every $\in M$, we must have μ_x^ϵ coinciding with some of the physical measures $\mu_1^\epsilon, \ldots, \mu_l^\epsilon$ for μ^ϵ almost every $x \in M$. Defining for each $1 \leq j \leq l$

$$P_j = \left\{x \in M : \mu_x^\epsilon = \mu_j^\epsilon\right\},$$

then the sets $P_1, \ldots, P_l$ partition M up to a null μ^ϵ measure subset of points. Given any μ^ϵ-integrable $\varphi : M \to \mathbb{R}$ we have by Proposition 5.9

$$\begin{aligned}
\int \varphi \, d\mu^\epsilon &= \int \left[\int \varphi(y) \, d\mu_x^\epsilon(y)\right] d\mu^\epsilon(x) \\
&= \sum_{j=1}^{l} \int_{P_j} \left[\int \varphi(y) \, d\mu_j^\epsilon(y)\right] d\mu^\epsilon(x) \\
&= \sum_{j=1}^{l} \mu^\epsilon(P_j) \int \varphi \, d\mu_j^\epsilon.
\end{aligned}$$

This gives the conclusion of the proposition with $\alpha_j = \mu^\epsilon(P_j)$ for $1 \leq j \leq l$. □

Recall that up until now we used no more than the continuity of the map. For the proof of the next theorem we need non-uniform expansion. It may be understood as the version of Theorem 3.2 in this context. Observe that, depending on the perturbation vector we choose, ergodic averages of a same point may approximate distinct physical measures. More properties of these physical measures are given in [**Ar00**]. In particular,

$$\operatorname{supp}(\mu_i^\epsilon) \subset B(\mu_i^\epsilon), \quad \text{for } 1 \leq i \leq l. \tag{5.9}$$

The method for proving this inclusion involves minimal invariant domains for random perturbations and it is out of the scope of the present work; see [**Ar00**, Proposition 7.3] for details.

THEOREM 5.12. *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a non-degenerate random perturbation of a C^2 non-uniformly expanding map $f : M \to M$. Then there is $l \geq 1$ such that for small enough $\epsilon > 0$ the physical measures $\mu_1^\epsilon, \ldots, \mu_l^\epsilon$ satisfy:*

(1) *for each* $x \in M$ *there exists a* $\theta_\epsilon^{\mathbb{N}}$ *mod* 0 *partition* $T_1(x), \dots, T_l(x)$ *of* $T^{\mathbb{N}}$ *such that*

$$\mu_i^\epsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \textit{for every } \underline{t} \in T_i(x);$$

(2) *if* $m|B(\mu_i^\epsilon)$ *denotes the normalized restriction of the Lebesgue measure to the basin of* μ_i^ϵ, *then*

$$\mu_i^\epsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int (f_{\underline{t}}^j)_* (m|B(\mu_i^\epsilon))\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}).$$

Both limits are taken in the weak sense.*

PROOF. Fix small $\epsilon > 0$ and let $\mu_1^\epsilon, \dots, \mu_l^\epsilon$ be the physical measures given by Proposition 5.11. Take any $x \in M$ and let μ^ϵ be a weak* accumulation point of the sequence $(\mu_n^\epsilon)_n$ defined in (5.5). We will prove that this is the only weak* accumulation point of (5.5) by showing that the values of the $\alpha_1, \dots, \alpha_l$ in the decomposition given by Proposition 5.11 depend only on x and not on the subsequence that converges to μ^ϵ. From (5.9) we deduce that the physical measures have disjoint supports, and so

$$\alpha_i = \mu^\epsilon(\mathrm{supp}(\mu_i^\epsilon)), \quad \text{for } 1 \le i \le l.$$

We define for $1 \le i \le l$

$$T_i(x) = \left\{ \underline{t} \in \mathrm{supp}(\theta_\epsilon^{\mathbb{N}}) \colon f_{\underline{t}}^j(x) \in \mathrm{supp}(\mu_i^\epsilon) \text{ for some } j \ge 1 \right\}.$$

We clearly have

$$T_i(x) = \cup_{j\ge 1} T_i^j(x), \quad \text{with} \quad T_i^j(x) \subset T_i^{j+1}(x) \quad \text{for each } j \ge 1, \tag{5.10}$$

where $T_i^j(x) = \left\{ \underline{t} \in \mathrm{supp}(\theta_\epsilon^{\mathbb{N}}) : f_{\underline{t}}^j(x) \in \mathrm{supp}(\mu_i^\epsilon) \right\}$, since the supports of physical measures are themselves invariant by Lemma 5.7. Now we fix some $1 \le i \le l$ and take any small $\eta > 0$. Since μ^ϵ is a regular probability measure, we may find an open set $U \supset \mathrm{supp}(\mu_i^\epsilon)$ and a closed set $K \subset \mathrm{supp}(\mu_i^\epsilon)$ such that $\mu^\epsilon(U \setminus K) < \eta$ and $\mu^\epsilon(\partial U) = \mu^\epsilon(\partial K) = 0$. In fact, there is an at most countable number of δ-neighborhoods of $\mathrm{supp}(\mu_i^\epsilon)$ whose boundaries have positive μ^ϵ measure, and likewise for the compacts coinciding with the complement of the δ-neighborhood of $M \setminus \mathrm{supp}(\mu_i^\epsilon)$. Then, taking $\alpha_i = \mu^\epsilon(\mathrm{supp}(\mu_i^\epsilon))$ we have

$$\begin{aligned} \alpha_i + \eta \ge \mu^\epsilon(U) &= \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\epsilon^{\mathbb{N}}\{\underline{t} \in T^{\mathbb{N}} : f_{\underline{t}}^j(x) \in U\} \\ &\ge \limsup_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\epsilon^{\mathbb{N}}\big(T_i^j(x)\big) \end{aligned}$$

for some sequence of integers $1 \le n_1 < n_2 < n_3 < \cdots$, and likewise for

$$\begin{aligned} \alpha_i - \eta \le \mu^\epsilon(K) &= \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\epsilon^{\mathbb{N}}\{\underline{t} \in T^{\mathbb{N}} : f_{\underline{t}}^j(x) \in K\} \\ &\le \liminf_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\epsilon^{\mathbb{N}}\big(T_i^j(x)\big). \end{aligned}$$

Since $\eta > 0$ was arbitrary, this shows

$$\alpha_i = \mu^\epsilon(\mathrm{supp}(\mu_i^\epsilon)) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\epsilon^{\mathbb{N}}\big(T_i^j(x)\big).$$

Moreover, from (5.10) we have that

$$\theta_\epsilon^{\mathbb{N}}\big(T_i(x)\big) = \lim_{j\to\infty} \theta_\epsilon^{\mathbb{N}}\big(T_i^j(x)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \theta_\epsilon^{\mathbb{N}}\big(T_i^j(x)\big) = \alpha_i,$$

which shows that each α_i depends only on the random orbits of x and not on the particular sequence $(n_k)_k$. Thus we see that the sequence of measures in (5.5) effectively converges in the weak* topology. Moreover, the sets $T_1(x), \dots, T_l(x)$ are pairwise disjoint, by definition, and their total $\theta_\epsilon^{\mathbb{N}}$ measure equals $\alpha_1 + \cdots + \alpha_l = 1$, thus forming a $\theta_\epsilon^{\mathbb{N}}$ modulo zero partition of $T^{\mathbb{N}}$. We observe that if $\underline{t} \in T_i(x)$, then $f_{\underline{t}}(x) \in \mathrm{supp}(\mu_i^\epsilon) \subset B(\mu_i^\epsilon)$, for each $i = 1, \dots, l$. This means that the $\theta_\epsilon^{\mathbb{N}}$ modulo zero partition of $T^{\mathbb{N}}$ satisfies the first item of the proposition.

Now fix $1 \le i \le l$. For all $x \in B(\mu_i^\epsilon)$ it holds that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_{\underline{t}}^j(x)) = \int \varphi \, d\mu_i^\epsilon$$

for $\theta_\epsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. Recall that $m(B(\mu_i^\epsilon)) > 0$ by the definition of physical measure. Using dominated convergence and integrating both sides of the above equality twice, first with respect to m and then with respect to $\theta_\epsilon^{\mathbb{N}}$, we arrive at the statement of the second item.

We are left to prove that $l = l(\epsilon)$ does not depend on ϵ for small enough $\epsilon > 0$. Fixing $1 \le i \le l$ we let x in the interior of $\mathrm{supp}(\mu_i^\epsilon)$ be such that the orbit $(f^j(x))_j$ has infinitely many hyperbolic times. Recall that $f \equiv f_{t^*}$ is non-uniformly expanding. Then there is a big enough hyperbolic time n so that $V_n \subset \mathrm{supp}(\mu_i^\epsilon)$, where V_n is the hyperbolic pre-ball containing x given by Proposition 2.3. Since $t^* \in \mathrm{supp}(\theta_\epsilon)$ and $\mathrm{supp}(\mu_i^\epsilon)$ is invariant under f_t for all $t \in \mathrm{supp}(\theta_\epsilon)$, we must have

$$\mathrm{supp}(\mu_i^\epsilon) \supset f_{\underline{t}^*}^n(V_n) = B\big(f_{t^*}^n(x), \delta_1\big), \tag{5.11}$$

where $\delta_1 > 0$ is the constant given by Proposition 2.3 and $B\big(f_{t^*}^n(x), \delta_1\big)$ is the ball of radius δ_1 around $f_{t^*}^n(x) = f^n(x)$. Then, we easily deduce that the number

$l = l(\epsilon)$ is bounded from above by some uniform constant since M is compact. On the other hand, since each invariant set must contain some physical measure, we see that for $0 < \epsilon' < \epsilon$ there must be some physical measure $\mu^{\epsilon'}$ with $\mathrm{supp}(\mu^{\epsilon'})$ contained in $\mathrm{supp}(\mu^{\epsilon})$. In fact, $\mathrm{supp}(\mu^{\epsilon})$ is invariant under f_t for every $t \in \mathrm{supp}(\theta_{\epsilon'}) \subset \mathrm{supp}(\theta_{\epsilon})$. Thus we conclude that there must be $\epsilon_0 > 0$ such that $l = l(\epsilon)$ is constant for $0 < \epsilon < \epsilon_0$. □

REMARK 5.13. Let us point out that from (5.11) one easily deduces that the Lesbesgue measure of the basin of each physical measure is uniformly bounded from below, since the support of such a measure is always contained in its basin.

REMARK 5.14. If the map $f : M \to M$ is topologically transitive, then every stationary measure must be supported on the whole of M, since the support is invariant and has nonempty interior. According to the discussion above, there must be only one such stationary measure, which must be physical.

We note that the number l of physical measures for small $\epsilon > 0$ and the number p of SRB measures for f are obtained by different existential arguments. It is natural to ask whether there is any relation between l and p.

PROPOSITION 5.15. *Let $f : M \to M$ is a C^2 non-uniformly expanding map. If $p \geq 1$ is the number of SRB measures of f and $l \geq 1$ is the number of physical measures of a non-degenerate random perturbation of f, then for $\epsilon > 0$ small enough we have $l \leq p$.*

PROOF. Observe that $\mathrm{supp}(\mu^{\epsilon})$ is forward invariant under $f = f_{t^*}$ and, moreover, f is non-uniformly expanding for Lebesgue almost every x in $\mathrm{supp}(\mu^{\epsilon})$, because it holds almost everywhere in M (by assumption) and $\mathrm{supp}(\mu^{\epsilon})$ has nonempty interior. Thus by Theorem 3.2 we assure the existence of at least one SRB measure μ with $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu^{\epsilon})$. Hence, we have seen that each support of a physical measure μ^{ϵ} must contain the support of at least one SRB measure for the unperturbed map f. Since the number of SRB measures is finite we have $l \leq p$, where p is the number of SRB measures. □

The reverse inequality does not hold in general, as the following example shows: it is possible for two distinct SRB measures to have intersecting supports and, in this circumstance, the random perturbations will mix their basins and there will be some physical measure whose support overlaps the supports of both SRB measures.

EXAMPLE 5.16. The first example is the map $f : [-3, 1] \to [-3, 1]$ whose graph is given in Figure 5.1

$$f(x) = \begin{cases} 1 - 2x^2 & \text{if } -1 \leq x \leq 1 \\ 2(x+2)^2 - 3 & \text{if } -3 \leq x \leq -1 \end{cases}.$$

The dynamics of f on $[-1, 1]$ and $[-3, -1]$ is conjugated to the tent map $T(x) = 1 - 2|x|$ from $[-1, 1]$ into itself. We may interpret f as a circle map through

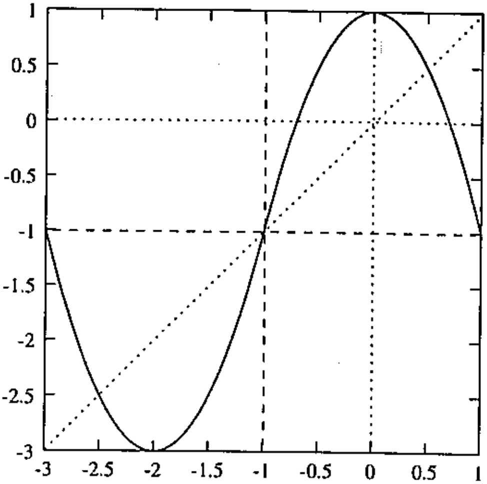

FIGURE 5.1. map for which $1 = l < p = 2$

the identification $S^1 = [-3, 1]/\{-3, 1\}$. This is a non-uniformly expanding map (with a critical set), and there are two ergodic absolutely continuous (thus SRB) invariant measures μ_1 and μ_2 whose supports are $[-3, -1]$ and $[-1, 1]$ respectively. Moreover, defining $F(t) = R_t \circ f$, where $R_t : S^1 \to S^1$ is the rotation of angle t and $\theta_\epsilon = (2\epsilon)^{-1}(m|[-\epsilon, \epsilon])$ for small $\epsilon > 0$, we have that $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ is non-degenerate random perturbation of f. Since $\operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2) = \{-1\}$ we have that for $\epsilon > 0$ small enough there must be a unique physical measure. Indeed, any weak* accumulation point of a family of physical measures must have the point -1 in its support.

It is possible that a random perturbation of a system have more than one physical measure for small noise level, as the next example shows.

EXAMPLE 5.17. The second example is defined on the interval $I = [-7, 2]$. We take the map $q_a(x) = a - x^2$ on $[-2, 2]$ for some parameter $a \in (1, 2)$ satisfying the good conditions of [BC85], and the "same" map on $[-7, -3]$, conveniently conjugated: $p_a(x) = (x + 5)^2 - 5 - a$. Then the two pieces of graph are glued together in such a way that we obtain a smooth map $f : I \to I$ sending I into its interior, as Figure 5.2 shows. The intervals $I_q = [q_a^2(0), q_a(0)]$ and $I_p = [p_a(-5), p_a^2(-5)]$ are forward invariant for f, and then we can find slightly larger intervals $I_1 \supset I_p$ and $I_2 \supset I_q$ that become trapping regions for the map f. So, taking $F(t) = f + t$, and θ_ϵ as in the previous example with $0 < \epsilon < \epsilon_0$ for some $\epsilon_0 > 0$ small enough, then $\{F, (\theta_\epsilon)_\epsilon\}$ is a random perturbation of f leaving the intervals I_1 and I_2 invariant by each $F(t)$. Moreover, Lebesgue almost every $x \in I$ eventually arrives at one of these intervals. Then by [BY93] the map f

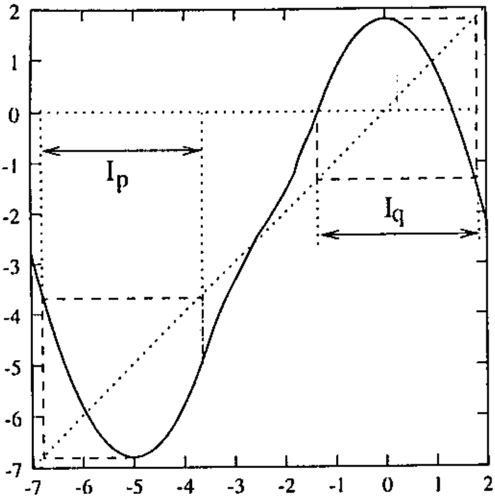

FIGURE 5.2. A map for which $l = p = 2$

has two SRB measures with supports contained in each trapping region, and f admits two distinct physical measures whose supports are contained in I_1 and I_2 respectively, for small enough noise level.

3. Non-uniform expansion on random orbits

In this section we introduce some definitions and results which have a similar in Chapter 2. Proofs of results will use essentiality the same ideas.

DEFINITION 5.18. Let $f\colon M \to M$ be a local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$, and let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ as a random perturbation of f. We say that f is *non-uniformly expanding on random orbits* if the following conditions hold, at least for small $\epsilon > 0$:

(1) there is $c > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| < -c. \tag{5.12}$$

(2) given any small $\gamma > 0$ there is $\delta > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) < \gamma. \tag{5.13}$$

From here on we assume that $f\colon M \to M$ is a local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. When $\mathcal{C}$ is equal to the empty set, then we naturally disregard the second condition in the definition above. For the case of $\mathcal{C} = \emptyset$ we have to be particularly careful in the proof of Proposition 5.20 due to the fact that we are not assuming condition (5.1) for maps with no critical sets. For the next definition we fix $B > 1$ and $\beta > 0$ as in Definition 1.1, and take a constant $b > 0$ such that $2b < \min\{1, \beta^{-1}\}$.

DEFINITION 5.19. Given $0 < \alpha < 1$ and $\delta > 0$, we say that $n \in \mathbb{Z}^+$ is a (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ if for every $1 \le k \le n$

$$\prod_{j=n-k}^{n-1} \|Df_{t_{j+1}}(f_{\underline{t}}^j(x))^{-1}\| \le \alpha^k \quad \text{and} \quad \operatorname{dist}_\delta(f_{\underline{t}}^{n-k}(x), \mathcal{C}) \ge \alpha^{bk}. \tag{5.14}$$

In the case of $\mathcal{C} = \emptyset$ the definition of (α, δ)-hyperbolic time reduces to the first condition in (5.14) and we simply call it a *α-hyperbolic time.*

The results we present below will be proved by mimicking the corresponding ones in the deterministic situation; cf. Sections 2.1 and 2.2.

PROPOSITION 5.20. *Let $\{F, (\theta_\epsilon)_{\epsilon>0}\}$ be a random perturbation of f and assume that f is non-uniformly expanding on random orbits. Then there are $0 < \alpha < 1$ and $\delta > 0$ such that $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ has some (α, δ)-hyperbolic time.*

PROOF. Assume first that the critical set $\mathcal{C}$ is nonempty. Thus we are taking random perturbations of f which satisfy (5.1). Let $(\underline{t}, x)$ be a point in $T^{\mathbb{N}} \times M$ satisfying (5.12). For large N we have

$$\sum_{j=0}^{N-1} -\log \left\|Df(f_{\underline{t}}^j(x))^{-1}\right\| \ge cN > 0.$$

Take $\beta > 0$ given by Definition 1.1, and fix any $\rho > \beta$. Then (s_2) implies that

$$\left|\log \left\|Df(x)^{-1}\right\|\right| \le \rho \left|\log \operatorname{dist}(x, \mathcal{C})\right| \tag{5.15}$$

for every x in a neighborhood V of $\mathcal{C}$. Now we take $\gamma_1 > 0$ so that $\rho\gamma_1 \le c/2$ and let $\delta_1 > 0$ be small enough to have

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{\delta_1}(f_{\underline{t}}^j(x), S) \ge -\gamma_1 N, \tag{5.16}$$

which is possible for large N after condition (5.13). Moreover, fixing $H \ge \rho|\log \delta_1|$ sufficiently large in order that it be also an upper bound for

$$\left\{-\log \|Df(x)^{-1}\| : \ x \in M \setminus V\right\},$$

then the set

$$E = \left\{1 \le j \le N : -\log \|Df(f_{\underline{t}}^{j-1}(x))^{-1}\| > H\right\}$$

is such that $f_{\underline{t}}^{j-1}(x) \in V$ for all $j \in E$, and

$$\rho \left|\log \operatorname{dist}(f_{\underline{t}}^{j-1}(x), \mathcal{C})\right| > -\log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| > H \geq \rho |\log \delta_1|.$$

This implies $\operatorname{dist}(f_{\underline{t}}^{j-1}(x), \mathcal{C}) < \delta_1$, and so

$$\operatorname{dist}_{\delta_1}(f_{\underline{t}}^{j-1}(x), \mathcal{C}) = \operatorname{dist}(f_{\underline{t}}^{j-1}(x), \mathcal{C}) < \delta_1, \quad \text{for all } j \in E.$$

Hence, defining

$$a_j = \begin{cases} -\log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| & \text{if } j \notin E \\ 0 & \text{if } j \in E \end{cases}$$

we have $a_j \leq H$ for $1 \leq j \leq N$, and from (5.15) and (5.16)

$$-\sum_{j \in E} \log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| \leq \rho \sum_{j \in E} \left|\log \operatorname{dist}(f_{\underline{t}}^{j-1}(x), \mathcal{C})\right| \leq \rho \gamma_1 N.$$

Since $\gamma_1 > 0$ has been chosen so that $\rho\gamma_1 \leq c/2$ we deduce

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} -\log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| - \sum_{j \in E} -\log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| \geq \frac{c}{2} N.$$

Thus we may apply Lemma 2.11 to the sequence $a_1, \ldots, a_N$, with $c_1 = c/4$, $c_2 = c/2$, and $A = H$ (we may also assume $H > c_2$ by increasing H if needed). Thus there are $\theta_1 > 0$ and $l_1 > \theta_1 N$ times $1 \leq q_1 < \ldots < q_{l_1} \leq N$ such that

$$\sum_{j=n+1}^{q_i} -\log \left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| \geq \sum_{j=n+1}^{q_i} a_j \geq \frac{c}{4}(q_i - n) \tag{5.17}$$

for every $0 \leq n < q_i$ and $1 \leq i \leq l_1$. We observe that (5.17) is just the first part of the requirements on (α, δ)-hyperbolic times for $(\underline{t}, x)$ if $\alpha = \exp(c/4)$.

Now we apply again Lemma 2.11, this time to the sequence

$$a_j = \log \operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j-1}(x), \mathcal{C}),$$

where $\delta_2 > 0$ has been chosen so that for $\gamma_2 > 0$ with $\gamma_2 < \theta_1 bc/4$ we have by (5.13)

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j}(x), \mathcal{C}) \geq -\gamma_2 N.$$

Defining $c_0 = bc/4$, $c_1 = -\gamma_2$, $A = 0$ and

$$\theta_2 = \frac{c_2 - c_1}{A - c_1} = 1 - \frac{4\gamma_2}{bc},$$

then Lemma 2.11 ensures that there are $l_2 \geq \theta_2 N$ times $1 \leq r_1 < \ldots < r_{l_2} \leq N$ satisfying

$$\sum_{j=n+1}^{r_i} \log \operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j+1}(x), \mathcal{C}) \geq \frac{bc}{2}(r_i - n) \tag{5.18}$$

for every $0 \le n < r_i$ and $1 \le i \le l_2$. Let us note that the condition on γ_2 assures $\theta_1 + \theta_2 > 1$. So if $\theta = \theta_1 + \theta_2 - 1$, then there must be $l = (l_1 + l_2 - N) \ge \theta N$ and $1 \le n_1 < \ldots < n_l \le N$ for which (5.17) and (5.18) both hold. Thus if we take $\delta = \delta_2$ and $\alpha = \exp(-c/4)$ then we have for $1 \le i \le l$ and $1 \le k \le n_i$

$$\prod_{j=n_i-k}^{n_i} \|Df(f_{\underline{t}}^j(x))^{-1}\| \le \alpha^k \quad \text{and} \quad \mathrm{dist}_{\delta_2}(f_{\underline{t}}^{n_i-k}(x), \mathcal{C}) \ge \alpha^{bk}. \tag{5.19}$$

In the case of a map f with critical set this means that these n_i are (α, δ)-hyperbolic times for $(\underline{t}, x)$, since we have assumed in (5.1) that in this case the perturbed maps satisfy

$$Df_t(x) = Df(x) \quad \text{for every } x \in M \setminus \mathcal{C} \text{ and } t \in T.$$

Let us now prove also the result in the setting of random perturbations of a local diffeomorphism without assuming (5.1). Actually, taking the perturbations f_t in a sufficiently small C^1-neighborhood of f, then

$$\|Df_t(y)^{-1}\| \le \frac{1}{\sqrt{\alpha}} \|Df(y)^{-1}\|$$

for every $y \in M$, which together with (5.19) gives

$$\prod_{j=n_i-k}^{n_i-1} \|Df_t(f_{\underline{t}}^j(x))^{-1}\| \le \prod_{j=n_i-k}^{n_i-1} \frac{1}{\sqrt{\alpha}} \|Df(f_{\underline{t}}^j(x))^{-1}\| \le \alpha^{k/2}.$$

This shows that n_i is a $\sqrt{\alpha}$-hyperbolic time for $(\underline{t}, x)$. □

REMARK 5.21. Similarly to the deterministic case, one easily sees that condition (5.13) is not needed in all its strength in the proof of Proposition 5.20. Actually, it is enough that (5.13) holds for some sufficiently small $\gamma > 0$ and some convenient $\delta > 0$.

Observe that the proof of Proposition 5.20 also gives a definite positive fraction of hyperbolic times for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. In the present context of random perturbations of a map we will not make use of the existence of such a positive frequency.

LEMMA 5.22. *Given $\delta > 0$ fix $\delta_1 > 0$ so that $4\delta_1 < \delta$ and $4B\delta_1 < \delta^\beta |\log \alpha|$. If n is a (α, δ)-hyperbolic time for $(\underline{t}, x)$, then*

$$\|Df(y)^{-1}\| \le \alpha^{-1/2} \|Df(f_{\underline{t}}^{n-j}(x))^{-1}\|$$

for any $1 \le j < n$ and any point y in the ball of radius $2\delta_1 \sigma^{j/2}$ around $f_{\underline{t}}^{n-j}(x)$.

PROOF. Since n is a (σ, δ)-hyperbolic time for $(\underline{t}, x)$ we have for any $1 \le j < n$

$$\mathrm{dist}_\delta(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \ge \alpha^j.$$

According to the definition of the truncated distance, this means that

$$\mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C}) = \mathrm{dist}_\delta(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \ge \alpha^{bj} \text{ or else } \mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \ge \delta.$$

In any case, we have for all y in the ball of radius $2\delta_1\alpha^{j/2}$ around $f_{\underline{t}}^{n-j}(x)$

$$\operatorname{dist}(y, f_{\underline{t}}^{n-j}(x)) \geq \operatorname{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C})/2,$$

because we have chosen $b < 1/2$ and $\delta_1 < \delta/4 < 1/4$. Therefore condition (s_2) implies

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(f_{\underline{t}}^{n-j}(x))^{-1}\|} \leq B\frac{\operatorname{dist}(f_{\underline{t}}^{n-j}(x), y)}{\operatorname{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C})^{\beta}} \leq B\frac{2\delta_1\alpha^{j/2}}{\min\{\alpha^{b\beta j}, \delta^{\beta}\}}.$$

Since δ and α are smaller than 1, and we took $b\beta < 1/2$, the term on the right hand side is bounded by $2B\delta_1\delta^{-\beta}$. Moreover, our second condition on δ_1 means that this last expression is smaller than $\log\alpha^{-1/2}$. $\square$

PROPOSITION 5.23. *There is $\delta_1 > 0$ such that if n is (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, then there is a neighborhood $V_n(\underline{t}, x)$ of x in M such that:*

(1) *$f_{\underline{t}}^n$ maps $V_n(\underline{t}, x)$ diffeomorphically onto $B_{\delta_1}(f_{\underline{t}}^n(x))$;*
(2) *for every $y, z \in V_n(\underline{t}, x)$ and $1 \leq k \leq n$*

$$\operatorname{dist}(f_{\underline{t}}^{n-k}(y), f_{\underline{t}}^{n-k}(z)) \leq \alpha^{k/2} \operatorname{dist}(f_{\underline{t}}^n(y), f_{\underline{t}}^n(z)).$$

PROOF. Let $\delta_1 > 0$ be given by Lemma 5.22. The proof will be by induction on $j \geq 1$. First we show that there is a well defined branch of f^{-j} on a ball of small enough radius around $f_{\underline{t}}^j(x)$. We observe that Lemma 5.22 gives for $j = 1$

$$\|Df(y)^{-1}\| \leq \alpha^{-1/2}\|Df(f_{\underline{t}}^{n-1}(x))^{-1}\| \leq \alpha^{1/2},$$

because n is a (α, δ)-hyperbolic time for $(\underline{t}, x)$. This means that f is a $\alpha^{-1/2}$-dilation in the ball of radius $2\delta_1\alpha^{1/2}$ around $f_{\underline{t}}^{n-1}(x)$. Consequently there is some neighborhood $V_1(\underline{t}, x)$ of $f_{\underline{t}}^{n-1}(x)$ inside the ball of radius $2\delta_1\alpha^{1/2}$ that is diffeomorphic to the ball of radius δ_1 around $f_{\underline{t}}^n(x)$ through f_{t_n}, when f is a map with critical set satisfying (5.1).

For $j \geq 1$ let us suppose that we have obtained a neighborhood $V_j(\underline{t}, x)$ of $f_{\underline{t}}^{n-j}(x)$ such that $f_{t_n} \circ \cdots \circ f_{t_{n-j+1}} \mid V_j(\underline{t}, x)$ is a diffeomorphism onto the ball of radius δ_1 around $f_{\underline{t}}^n(x)$ with

$$\|Df(f_{t_{n-j+i+1}} \circ \cdots \circ f_{t_{n-j+1}}(z))^{-1}\| \leq \alpha^{-1/2}\|Df(f_{\underline{t}}^{n-j+i+1}(x))^{-1}\| \tag{5.20}$$

for all $z \in V_j(\underline{t}, x)$ and $0 \leq i < j$. Then, by Lemma 5.22 and under the assumption that n is a (α, δ)-hyperbolic time for x,

$$\begin{aligned}
\left\|D\big(f_{t_n} \circ \cdots \circ f_{t_{n-j}}(y)\big)^{-1}\right\| &\leq \prod_{i=0}^{j} \left\|Df_{t_{n-j+i}}\big(f_{t_{n-j+i-1}} \circ \cdots \circ f_{t_{n-j}}(y)\big)^{-1}\right\| \\
&\leq \prod_{i=0}^{j} \alpha^{-1/2}\left\|Df_{t_{n-j+i}}\big(f_{\underline{t}}^{n-j+i-1}(x)\big)^{-1}\right\| \\
&\leq (\alpha^{-1/2})^{j+1} \cdot \alpha^{j+1} = \alpha^{(j+1)/2}
\end{aligned}$$

for every y on the ball of radius $2\delta_1\alpha^{(j+1)/2}$ around $f_{\underline{t}}^{n-j-1}(x)$ whose image $f_{t_{n-j}}(y)$ is in $V_j(\underline{t},x)$ (above we convention $f_{t_{n-j+i-1}}\circ\cdots\circ f_{t_{n-j}}(y)=y$ for $i=0$).

This shows that the derivative of $f_{t_n}\circ\cdots\circ f_{t_{n-j}}$ is a $\alpha^{-(j+1)/2}$-dilation on the intersection of $f_{t_{n-j}}^{-1}\big(V_j(\underline{t},x)\big)$ with the ball of radius $2\delta_1\alpha^{(j+1)/2}$ around $f_{\underline{t}}^{n-j-1}(x)$, and hence there is an inverse branch of $f_{t_n}\circ\cdots\circ f_{t_{n-j}}$ defined on the ball of radius δ_1 around $f_{\underline{t}}^{n}(x)$. Thus we may define $V_{j+1}(\underline{t},x)$ as the image of the ball of radius δ_1 around $f_{\underline{t}}^{n}(x)$ under this inverse branch, and recover the induction hypothesis for $j+1$. In this way we obtain neighborhoods $V_j(\underline{t},x)$ of $f_{\underline{t}}^{n-j}(x)$ as above for all $1\le j\le n$. □

COROLLARY 5.24. *There is a constant $C_1>0$ such that if n is a (α,δ)-hyperbolic time for $(\underline{t},x)\in T^{\mathbb{N}}\times M$ and $y,z\in V_n(\underline{t},x)$, then*

$$\frac{1}{C_1}\le\frac{|\det Df_{\underline{t}}^{n}(y)|}{|\det Df_{\underline{t}}^{n}(z)|}\le C_1.$$

PROOF. For $1\le k\le n$ the distance between $f_{\underline{t}}^{k}(x)$ and either $f_{\underline{t}}^{k}(y)$ or $f_{\underline{t}}^{k}(z)$ is smaller than $\alpha^{(n-k)/2}$ which is smaller than $\alpha^{b(n-k)}\le\operatorname{dist}(f_{\underline{t}}^{k}(x),\mathcal{C})$. So, by ($s_3$) we have

$$\begin{aligned}\log\frac{|\det Df_{\underline{t}}^{n}(y)|}{|\det Df_{\underline{t}}^{n}(z)|} &= \sum_{k=0}^{n-1}\log\frac{|\det Df_{t_{k+1}}(f_{\underline{t}}^{k}(y))|}{|\det Df_{t_{k+1}}(f_{\underline{t}}^{k}(z))|}\\ &\le \sum_{k=1}^{n-1}\log\frac{|\det Df(f_{\underline{t}}^{k}(y))|}{|\det Df(f_{\underline{t}}^{k}(z))|}\\ &\le \sum_{k=0}^{n-1}2B\frac{\alpha^{(n-k)/2}}{\alpha^{b\beta(n-k)}},\end{aligned}$$

and it is enough to take $C_1\le\exp\big(\sum_{i=1}^{\infty}2B\alpha^{(1/2-b\beta)i}\big)$, recalling that $b\beta<1/2$ and also (5.1). □

4. Stochastic stability

Here we give both necessary conditions and sufficient conditions for the physical measures of a random perturbation of a non-uniformly expanding map to accumulate on the SRB measures of that map, when the noise level goes to zero.

DEFINITION 5.25. A C^2 non-uniformly expanding map $f\colon M\to M$ is said to be *stochastically stable* if for any non-degenerate random perturbation $\{F,(\theta_\epsilon)_{\epsilon>0}\}$ of f the weak* accumulation points (when $\epsilon>0$ goes to zero) of the physical measures are convex linear combinations of the SRB measures of f.

A necessary condition for the stochastic stability of a non-uniformly expanding map with no critical set will be given below. Let us prove first an auxiliary lemma.

LEMMA 5.26. *Let $f\colon M \to M$ be a stochastically stable non-uniformly expanding map. Given $\varphi\colon M \to \mathbb{R}$ continuous and $\delta > 0$ there is $\epsilon_0 > 0$ such that for $0 \le \epsilon < \epsilon_0$ we have*

$$\left|\int \varphi d\mu^\epsilon - \int \varphi d\mu_\epsilon\right| < \delta,$$

where μ_ϵ is a convex linear combination of the SRB measures of f.

PROOF. We will use the following auxiliary result: *Let X be a compact metric space, $K \subset X$ a closed (compact) subset and $(x_t)_{t>0}$ a curve in X (not necessarily continuous) such that all its accumulation points (as $t \to 0^+$) lie in K. Then for every open neighborhood U of K there is $t_0 > 0$ such that $x_t \in U$ for every $0 \le t < t_0$.* Indeed, supposing not, there is a sequence $(t_n)_n$ with $t_n \to 0^+$ when $n \to \infty$ such that $x_{t_n} \notin U$. Since X is compact this means that $(x_t)_{t>0}$ has some accumulation point in $X \setminus U$, thus outside K, contrary to the assumption.

Now, the space $X = \mathbb{P}(M)$ of the probability measures on M with the weak* topology is a compact metric space, and the convex hull K of the finitely many SRB measures of f is closed. Hence, considering a curve of physical measures $(\mu^\epsilon)_{\epsilon>0}$ in $\mathbb{P}(M)$, we are in the context of the above result, since we are assuming f stochastically stable. A metric on $\mathbb{P}(M)$ giving the weak* topology is

$$\mathrm{d}_{\mathbb{P}}(\mu,\nu) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left|\int \varphi_n\, d\mu - \int \varphi_n\, d\nu\right|,$$

where $\mu,\nu \in \mathbb{P}(M)$ and $(\varphi_n)_n$ is a dense sequence in $C^0(M,\mathbb{R})$; see (0.1).

Take any continuous map $\varphi\colon M \to \mathbb{R}$ and any $\delta > 0$. By the density of $(\varphi_n)_n$ in $C^0(M,\mathbb{R})$, there must be some $k \in \mathbb{N}$ such that

$$\|\varphi - \varphi_k\|_0 < \frac{\delta}{3}.$$

By the aforementioned auxiliary result, there must be some $\epsilon_0 > 0$ with the following property: for every $0 \le \epsilon < \epsilon_0$ there exists $\mu_\epsilon \in \mathbb{P}(M)$ which is a convex linear combination of the finitely many SRB measures of f such that $\mathrm{d}_{\mathbb{P}}(\mu^\epsilon,\mu_\epsilon) < \delta(3\cdot 2^k)^{-1}$. This in particular implies that

$$\frac{1}{2^k}\left|\int \varphi_k\, d\mu^\epsilon - \int \varphi_k\, d\mu_\epsilon\right| < \frac{\delta}{3\cdot 2^k},$$

by the definition of the distance $\mathrm{d}_{\mathbb{P}}$, and so

$$\left|\int \varphi_k\, d\mu^\epsilon - \int \varphi_k\, d\mu_\epsilon\right| < \frac{\delta}{3}.$$

Hence we get

$$
\begin{aligned}
&\left|\int \varphi\, d\mu^\epsilon - \int \varphi\, d\mu_\epsilon\right| \le \\
&\le \quad \left|\int \varphi\, d\mu^\epsilon - \int \varphi_k\, d\mu^\epsilon\right| + \left|\int \varphi_k\, d\mu^\epsilon - \int \varphi_k\, d\mu_\epsilon\right| + \left|\int \varphi_k\, d\mu_\epsilon - \int \varphi\, d\mu_\epsilon\right| \\
&< \quad \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,
\end{aligned}
$$

which completes the proof of the lemma. □

The next result shows that non-uniform expansion on random orbits is a necessary condition for the stochastic stability of a non-uniformly expanding local diffeomorphisms.

THEOREM 5.27. *Let $f\colon M \to M$ be a non-uniformly expanding C^2 local diffeomorphism. If f is stochastically stable, then f is non-uniformly expanding on random orbits.*

PROOF. We know from Theorem 5.12 that there is $l \ge 1$ such that for small $\epsilon > 0$ there are physical measures $\mu_1^\epsilon, \dots, \mu_l^\epsilon$ with the following property: for each $x \in M$ there is a $\theta_\epsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \dots, T_l(x)$ of $T^{\mathbb{N}}$ such that

$$\mu_i^\epsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \text{for each} \quad \underline{t} \in T_i(x),$$

where the convergence is in the weak* sense. Since f is a local diffeomorphism, then $\log \|Df(x)^{-1}\|$ is a continuous map. Thus, we have for each $x \in M$ and $\theta_\epsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$ there is some physical measure μ_i^ϵ with $1 \le i \le l$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df\big(f_{\underline{t}}^j(x)\big)^{-1}\| = \int \log \|Df(x)^{-1}\| d\mu_i^\epsilon. \tag{5.21}$$

Now it suffices to show that there is $c > 0$ such that if $\mu^\epsilon = \mu_i^\epsilon$ for any $1 \le i \le l$, then

$$\int \log \|Df(x)^{-1}\| d\mu^\epsilon < -c \quad \text{for small } \epsilon > 0. \tag{5.22}$$

Let $\lambda > 0$ be the constant given by the non-uniform expansion of f; cf. Definition 1.2. Applying Lemma 5.26 to the continuous map $\varphi(x) = \log \|Df(x)^{-1}\|$ and $\delta = \lambda/2$, then we obtain $\epsilon_0 > 0$ such that for each $0 \le \epsilon < \epsilon_0$ there is μ_ϵ a convex linear combination of the SRB measures $\mu_1, \dots, \mu_p$ for which

$$\left|\int \log \|Df(x)^{-1}\| d\mu^\epsilon - \int \log \|Df(x)^{-1}\| d\mu_\epsilon\right| < \frac{\lambda}{2}. \tag{5.23}$$

Since μ_i is an SRB measure for $1 \le i \le p$, then we have for Lebesgue almost every $x \in B(\mu_i)$

$$\int \log \|Df(x)^{-1}\| d\mu_i = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -\lambda.$$

Since μ_ϵ is a convex linear combination of $\mu_1, \ldots, \mu_p$, then we also have

$$\int \log \|Df(x)^{-1}\| d\mu_\epsilon < \lambda,$$

which together with (5.23) gives

$$\int \log \|Df(x)^{-1}\| d\mu^\epsilon < -\frac{\lambda}{2}.$$

Thus we have proved (5.22) for $0 \le \epsilon < \epsilon_0$ with $c = \lambda/2$. □

The previous result legitimates the assumption of non-uniform expansion on random orbits, if we want to prove the stochastic stability of a non-uniformly expanding map. Under such assumption Proposition 5.20 allows us to introduce a map

$$h_\epsilon \colon T^{\mathbb{N}} \times M \to \mathbb{Z}^+,$$

by taking $h_\epsilon(\underline{t}, x)$ the first (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. Integrability properties of h_ϵ will play an important role in the proof of the stochastic stability of f.

Definition 5.28. Assume that h_ϵ is integrable with respect to the product measure $\theta_\epsilon^{\mathbb{N}} \times m$. This means that

$$\|h_\epsilon\|_1 = \sum_{k=0}^{\infty} k\, (\theta_\epsilon^{\mathbb{N}} \times m)\{(\underline{t}, x)\colon h_\epsilon(\underline{t}, x) = k\} < \infty.$$

We say that the family $(h_\epsilon)_{\epsilon>0}$ has *uniform L^1-tail* if the series above converges uniformly, as a series of functions on the variable ϵ.

Assume that f is non-uniformly expanding on random orbits and fix $\delta > 0$ and $\gamma > 0$ in such a way that the proof of Proposition 5.20 works; recall Remark 5.21. Choose for $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ a positive integer $N_\epsilon = N_\epsilon(\underline{t}, x)$ for which

$$\sum_{j=0}^{N_\epsilon - 1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| \le -cN_\epsilon \quad \text{and} \quad \sum_{j=0}^{N_\epsilon - 1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) \le \gamma N_\epsilon.$$

Take $N_\epsilon(\underline{t}, x)$ the smallest integer with this property. This allows us to introduce a map

$$N_\epsilon \colon T^{\mathbb{N}} \times M \to \mathbb{Z}^+ \tag{5.24}$$

such that $h_\epsilon \le N_\epsilon$ (recall the proof of Proposition 5.20). Thus, the integrability of the map h_ϵ is implied by the integrability of the map N_ϵ, which can in practice be easier to handle. The following lemma gives a useful criterium for obtaining the uniform L^1-tail for the family of first hyperbolic time maps. This is an immediate consequence of the considerations above and Weierstrass criterium for the uniform convergence of a series of functions.

LEMMA 5.29. *Assume that there is a sequence $(a_n)_n$ of nonnegative numbers such that for small $\epsilon > 0$*

$$m\{x \in M : N_\epsilon(\underline{t}, x) > n\} \le a_n \quad \text{and} \quad \sum_{k=1}^{\infty} n a_n < \infty$$

for $\theta_\epsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. Then $(h_\epsilon)_\epsilon$ has uniform L^1-tail.

Next we present some results heading in the direction of Theorem 5.32, that gives sufficient conditions for the stochastic stability of a non-uniformly expanding map. We will prove our results in the context of maps with critical sets; then the proofs will follow for local diffeomorphisms, since all we use from now on is the existence of hyperbolic times for random orbits given by Proposition 5.20 and the properties of those hyperbolic times.

Let $f: M \to M$ be a C^2 non-uniformly expanding map, and assume that f is non-uniformly expanding on random orbits and $(h_\epsilon)_\epsilon$ has uniform L^1-tail. Let μ^ϵ be a physical measure of level ϵ for some small $\epsilon > 0$ and define for each $n \ge 1$

$$\mu_n^\epsilon = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{m(B(\mu^\epsilon))} \int (f_{\underline{t}}^j)_* \big(m | B(\mu^\epsilon)\big)\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}).$$

We know from Theorem 5.12 that μ^ϵ is the weak* limit of the sequence of probability measures $(\mu_n^\epsilon)_n$. Define for each $\underline{t} \in T^{\mathbb{N}}$ and $n \ge 1$

$$H_n(\underline{t}) = \{x \in B(\mu^\epsilon) :\ n \text{ is a } (\alpha,\delta)\text{-hyperbolic time for } (\underline{t}, x)\ \},$$

and

$$H_n^*(\underline{t}) = \{x \in B(\mu^\epsilon) :\ n \text{ is the first } (\alpha,\delta)\text{-hyperbolic time for } (\underline{t}, x)\ \}.$$

$H_n^*(\underline{t})$ is precisely the set of those points $x \in B(\mu^\epsilon)$ for which $h_\epsilon(\underline{t}, x) = n$. For $n, k \ge 1$ we also define $R_{n,k}(\underline{t})$ as the set of those points $x \in M$ for which n is a (α, δ)-hyperbolic time and $n + k$ is the first (α, δ)-hyperbolic time after n, that is

$$R_{n,k}(\underline{t}) = \left\{x \in H_n(\underline{t}) :\ f_{\underline{t}}^n(x) \in H_k^*(\sigma^n \underline{t})\ \right\},$$

where $\sigma: T^{\mathbb{N}} \to T^{\mathbb{N}}$ is the shift map $\sigma(t_1, t_2, \dots) = (t_2, t_3, \dots)$. Considering the measures

$$\nu_n^\epsilon = \int (f_{\underline{t}}^n)_* \big(m | H_n(\underline{t})\big) d\theta_\epsilon^{\mathbb{N}}(\underline{t})$$

and

$$\eta_n^\epsilon = \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \int (f_{\underline{t}}^{n+j})_* \big(m | R_{n,k}(\underline{t})\big) d\theta_\epsilon^{\mathbb{N}}(\underline{t}),$$

we may write

$$\mu_n^\epsilon \leq \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{m(B(\mu^\epsilon))}(\nu_j^\epsilon + \eta_j^\epsilon).$$

PROPOSITION 5.30. *There is a constant $C_2 > 0$ such that*

$$\frac{d}{dm}(f_{\underline{t}}^n)_*(m \mid H_n(\underline{t})) \leq C_2$$

for every $n \geq 0$ and $\underline{t} \in T^{\mathbb{N}}$.

PROOF. Take $\delta_1 > 0$ given by Proposition 2.3. It is sufficient to prove that there is some uniform constant $C > 0$ such that if A is a Borel set in M with diameter smaller than $\delta_1/2$ then

$$m\big(f_{\underline{t}}^{-n}(A) \cap H_n(\underline{t})\big) \leq Cm(A).$$

Let A be a Borel set in M with diameter smaller than $\delta_1/2$ and B an open ball of radius $\delta_1/2$ containing A. We may write

$$f_{\underline{t}}^{-n}(B) = \bigcup_{k\geq 1} B_k,$$

where $(B_k)_{k\geq 1}$ is a (possibly finite) family of two-by-two disjoint open sets in M. Discarding those B_k that do not intersect $H_n(\underline{t})$, we choose for each $k \geq 1$ a point $x_k \in H_n(\underline{t}) \cap B_k$. For $k \geq 1$ let $V_n(\underline{t}, x_k)$ be the neighborhood of x_k in M given by Proposition 2.3. Since B is contained in $B\big(f_{\underline{t}}^n(x_k), \delta_1\big)$, the ball of radius δ_1 around $f_{\underline{t}}^n(x_k)$, and $f_{\underline{t}}^n$ is a diffeomorphism from $V_n(\underline{t}, x_k)$ onto $B\big(f_{\underline{t}}^n(x_k), \delta_1\big)$, we must have $B_k \subset V_n(\underline{t}, x_k)$ (recall that by our choice of B_k we have $f_{\underline{t}}^n(B_k) \subset B$). As a consequence of this and Corollary 2.6, we have for every k that the map $f_{\underline{t}}^n|_{B_k}\colon B_k \to B$ is a diffeomorphism with bounded distortion:

$$\frac{1}{C_1} \leq \frac{|\det Df_{\underline{t}}^n(y)|}{|\det Df_{\underline{t}}^n(z)|} \leq C_1$$

for all $y, z \in B_k$. This finally gives

$$\begin{aligned} m\big(f_{\underline{t}}^{-n}(A) \cap H_n(\underline{t})\big) &\leq \sum_k m\big(f_{\underline{t}}^{-n}(A \cap B) \cap B_k\big) \\ &\leq \sum_k C_1 \frac{m(A \cap B)}{m(B)} m(B_k) \\ &\leq C_2 m(A), \end{aligned}$$

where $C_2 > 0$ is a constant only depending on C_1, on the volume of the ball B of radius $\delta_1/2$, and on the volume of M. □

It follows immediately from Proposition 5.30 that

$$\frac{d\nu_n^\epsilon}{dm} \leq C_2 \quad \text{for every } n \geq 0 \text{ and small } \epsilon > 0. \tag{5.25}$$

Our goal now is to control the density of the measures η_n^ϵ, in order to have the absolute continuity of the weak* accumulation points of the measures μ^ϵ when $\epsilon > 0$ goes to zero.

PROPOSITION 5.31. *Given $\zeta > 0$, there is $C_3(\zeta) > 0$ such that for every $n \geq 0$ and $\epsilon > 0$ we may bound η_n^ϵ by the sum of two non-negative measures, $\eta_n^\epsilon \leq \omega^\epsilon + \rho^\epsilon$, with*

$$\frac{d\omega^\epsilon}{dm} \leq C_3(\zeta) \quad \textit{and} \quad \rho^\epsilon(M) < \zeta.$$

PROOF. Let A be some Borel set in M. We have for each $n \geq 0$

$$\begin{aligned}
\eta_n^\epsilon(A) &= \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int m\big(f_{\underline{t}}^{-n-j}(A)\cap R_{n,k}(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t})\\
&\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int m\big(f_{\underline{t}}^{-n}\big(f_{\sigma^n\underline{t}}^{-j}(A)\cap H_k^*(\sigma^n\underline{t})\big)\cap H_n(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t})\\
&\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}C_2\int m\big(f_{\underline{t}}^{-j}(A)\cap H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t}).
\end{aligned}$$

(in this last inequality we used Proposition 5.30 and the fact that $\theta_\epsilon^{\mathbb{N}}$ is σ-invariant). Let now $\zeta > 0$ be some fixed small number. Since we are assuming $(h_\epsilon)_\epsilon$ with uniform L^1-tail, then there is some integer $N = N(\zeta)$ for which

$$\sum_{j=N}^{\infty} k\int m\big(H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t}) < \frac{\zeta}{C_2}.$$

We take

$$\omega^\epsilon = C_2\sum_{k=2}^{N-1}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^j)_*\big(m|H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t})$$

and

$$\rho^\epsilon = C_2\sum_{k=N}^{\infty}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^j)_*\big(m|H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t}).$$

For this last measure we have

$$\rho^\epsilon(M) = C_2\sum_{k=N}^{\infty}\sum_{j=1}^{k-1}\int m\big(H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t}) \leq C_2\sum_{k=N}^{\infty} k\int m\big(H_k^*(\underline{t})\big)\,d\theta_\epsilon^{\mathbb{N}}(\underline{t}) < \zeta.$$

On the other hand, it follows from the definition of (α,δ)-hyperbolic times that there is some constant $a = a(N) > 0$ such that $\operatorname{dist}\big(H_k(\underline{t}),\mathcal{C}\big) \geq a$ for $1 \leq k \leq N$.

Defining $K \subset M$ as the set of those points in M whose distance to C is greater than a, we have

$$\omega' \le C_2 \sum_{k=2}^{N-1} \sum_{j=1}^{k-1} \int (f_{\underline{t}}^j)_*(m|K)\, d\theta_\epsilon^{\mathbb{N}}(\underline{t}),$$

and this last measure has density bounded by some uniform constant, as long as we take maps f_t in a sufficiently small neighborhood of f in the C^1 topology. □

It follows from Remark 5.13, Proposition 5.31 and (5.25) that the weak* accumulation points of μ^ϵ when $\epsilon \to 0$ cannot have singular part, thus being absolutely continuous with respect to the Lebesgue measure. Moreover, Lemma 5.5 implies that the weak* accumulation points of a family of stationary measures are f-invariant measures. Thus we have proved the following result.

THEOREM 5.32. *Let $f\colon M \to M$ be a C^2 non-uniformly expanding map. If f is non-uniformly expanding on random orbits and $(h_\epsilon)_\epsilon$ has uniform L^1-tail, then f is stochastically stable.*

Next we show that the local diffeomorphisms of Subsection 1.2.1 and Viana maps from Subsection 1.2.2 satisfy the hypotheses of Theorem 5.32, thus being stochastically stable.

4.1. Local diffeomorphisms. Let us show that the non-uniformly expanding local diffeomorphisms presented in Subsection 1.2.1 are stochastically stable. Recall that any such f has been obtained through deformation of a uniformly expanding map by isotopy inside some small region of compact Riemannian manifold M. Consider a continuous

$$\begin{array}{rccc} F: & T & \longrightarrow & C^2(M,M) \\ & t & \longmapsto & f_t \end{array}$$

where T is a metric space, and take a family $(\theta_\epsilon)_{\epsilon>0}$ of probability measures on T such that their supports $\mathrm{supp}(\theta_\epsilon)$ form a nested family of connected compact sets and

$$\mathrm{supp}(\theta_\epsilon) \to \{t^*\} \quad \text{when} \quad \epsilon \to 0,$$

where $t^* \in T$ is such that $f_{t^*} = f$. The construction of f has been made in such a way that we may assume that there is some small compact domain $V \subset M$ so that the restriction of f_t to V is injective for every $t \in T$ reducing T if necessary to a small neighborhood of t^*. We may assume moreover that for every $t \in T$:

(1) f_t is *volume expanding everywhere*: there exists $\sigma_1 > 1$ such that

$$|\det Df_t(x)| > \sigma_1 \quad \text{for every } x \in M;$$

(2) f is *expanding outside* V: there exists $\sigma_0 > 1$ such that

$$\|Df(x)^{-1}\| < \sigma_0 \quad \text{for every } x \in M \setminus V;$$

(3) f_t is *not too contracting on* V: there is some small $\delta > 0$ such that

$$\|Df_t(x)^{-1}\| < 1 + \delta \quad \text{for every } x \in V.$$

Now we are going to show that f is non-uniformly expanding on random orbits. We begin with an essentially combinatorial lemma.

LEMMA 5.33. *Let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of M into domains such that f is injective on B_j, for $1 \le j \le p+1$. Then there is $\zeta > 0$ such that for $\theta_\epsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ and large enough $n \ge 1$*

$$\#\left\{0 \le j < n\colon f_{\underline{t}}^j(x) \in B_1 \cup \ldots \cup B_p\right\} \ge \zeta n; \tag{5.26}$$

Moreover, there are $0 < \tau < 1$ and $N \in \mathbb{N}$ such that for each $\underline{t} \in T^{\mathbb{N}}$ the set $I_n(\underline{t})$ of points $x \in M$ for which (5.26) *does not hold satisfies $m(I_n(\underline{t})) \le \tau^n$ for $n \ge N$.*

PROOF. Let us fix $n \ge 1$ and $\underline{t} \in T^{\mathbb{N}}$. Given a sequence $\underline{i} = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, p+1\}^n$ we write

$$[\underline{i}] = B_{i_0} \cap (f_{\underline{t}}^1)^{-1}(B_{i_1}) \cap \cdots \cap (f_{\underline{t}}^{n-1})^{-1}(B_{i_{n-1}})$$

and define $g(\underline{i}) = \#\{0 \le j < n : i_j \le p\}$. We start by observing that for $\zeta > 0$ the number of sequences $\underline{i}$ such that $g(\underline{i}) < \zeta n$ is bounded by

$$\sum_{k < \zeta n} \binom{n}{k} p^k \le \sum_{k \le \zeta n} \binom{n}{k} p^{\zeta n}.$$

Using Stirling's formula (cf. [**BoV**, Section 6.3]) the expression on the right hand side is bounded by $e^{\gamma n} p^{\zeta n}$, where $\gamma > 0$ depends only on ζ and goes to zero when ζ goes to zero.

On the other hand, assumptions (a) and (b) on the maps f_t ensure that $m([\underline{i}]) \le \sigma^{-(1-\zeta)n}$ (recall that we are assuming $m(M) = 1$). Hence, given $\underline{t} \in T^{\mathbb{N}}$ the measure of the union $I_n(\underline{t})$ of all the sets $[\underline{i}]$ with $g(\underline{i}) < \zeta n$ is bounded by

$$\sigma^{-(1-\zeta)n} e^{\gamma n} p^{\zeta n}.$$

Since $\sigma > 1$ we may choose ζ so small that $e^\gamma p^\zeta < \sigma^{(1-\zeta)}$. Then $m(I_n(\underline{t})) \le \tau^n$, with $\tau = e^{\gamma+\zeta-1} p^\zeta < 1$, for n greater than some $N \in \mathbb{N}$. Note that τ and N do not depend on $\underline{t} \in T^{\mathbb{N}}$. This proves the second part of the lemma. Setting

$$I_n = \bigcup_{\underline{t} \in T^{\mathbb{N}}} \left(\{\underline{t}\} \times I_n(\underline{t})\right),$$

we have $(\theta_\epsilon^{\mathbb{N}} \times m)(I_n) \le \tau^n$ by Fubini's Theorem, for $n \ge N$. Since

$$\sum_{n \ge 1} (\theta^{\mathbb{N}} \times m)(I_n) < \infty,$$

then Borel-Cantelli's Lemma implies that

$$(\theta_\epsilon^{\mathbb{N}} \times m)\left(\cap_{n \ge 1} \cup_{k \ge n} I_k\right) = 0,$$

and this means that $\theta_\epsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ satisfies (5.26). $\square$

Let $\zeta > 0$ be the constant provided by Lemma 5.33. We fix $\eta > 0$ sufficiently small so that $\sigma_0^\zeta(1+\delta) \le e^{-c_0}$ holds for some $c_0 > 0$, and take $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ satisfying (5.26). The assumptions on f_t for $t \in T$ imply

$$\prod_{j=0}^{n-1} \|Df(f_{\underline{t}}^j(x))^{-1}\| \le \sigma_0^{\zeta n}(1+\delta)^{(1-\zeta)n} \le e^{-c_0 n}.$$

for large enough n. This implies that f is non-uniformly expanding on random orbits:

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| \le -c_0$$

for $\theta_\epsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. Moreover, defining N_ϵ as in (5.24), from the second part of Lemma 5.33 we have

$$m\{x \in M : N_\epsilon(\underline{t}, x) > n\} \le \tau^n,$$

for $n \ge N$ and $\theta_\epsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. From Lemma 5.29 we deduce that the family $(h_\epsilon)_\epsilon$ of first hyperbolic time maps has uniform L^1-tail. Thus we have proved the following result:

THEOREM 5.34. *There are open sets $\mathcal{U} \subset C^2(M, M)$ such that every $f \in \mathcal{U}$ is non-uniformly expanding on random orbits and the family $(h_\epsilon)_\epsilon$ of first hyperbolic time functions has uniform L^1-tail.*

In particular, every $f \in \mathcal{U}$ is stochastically stable, by Theorem 5.32.

4.2. Viana maps. Let f be a Viana map as described in Subsection 1.2.2. As we have seen before, it is no restriction to assume that $\mathcal{C} = \{(s,x) \in S^1 \times I : x = 0\}$ is the critical set of f and we do so. Fix $\{F, (\theta_\epsilon)_\epsilon\}$ a random perturbation of f for which (5.1) holds. Our goal now is to prove that any such f satisfies the hypotheses of Theorem 5.32 for $\epsilon > 0$ sufficiently small, and thus conclude that f is stochastically stable. So, we want to show that if $\epsilon > 0$ is small enough then

- f is non-uniformly expanding on random orbits;
- the family of hyperbolic time maps $(h_\epsilon)_\epsilon$ has uniform L^1-tail.

We observe that in the estimates we have obtained in Subsection 1.2.2 for $\log \operatorname{dist}_\delta(x_j, \mathcal{C})$ and $\log \|(Df(s_j, x_j))^{-1}\|$ over the orbit of a given point $(s,x) \in S^1 \times I$, we can easily replace iterates (s_j, x_j) by random iterates $(s_{\underline{t}}^j, x_{\underline{t}}^j) = f_{\underline{t}}^j(s,x)$. Actually, the method we used for obtaining estimate (1.8) rely on a delicate decomposition of the orbit of the point (s,x) from time 0 until time n into finite pieces according to its returns to the neighborhood $S^1 \times (-\sqrt{\alpha}, \sqrt{\alpha})$ of the critical set. The main tools are [**Vi2**, Lemma 2.4] and [**Vi2**, Lemma 2.5] whose proofs may easily be mimicked for random orbits. Indeed, the important fact in the proof of the referred lemmas is that orbits of points in the central direction stay close to orbits of the quadratic map Q for long periods, as long as $\alpha > 0$ is taken sufficiently small. Hence, such results can easily be obtained for random orbits as long as we take $\epsilon > 0$ with $\epsilon \ll \alpha$ and perturbation vectors $\underline{t} \in \operatorname{supp}(\theta_\epsilon)$.

Thus, the procedure of described in Subsection 1.2.2 applies to this situation, and we are able to prove that there exists some $c > 0$, and for $\gamma > 0$ there is $\delta > 0$ such that

$$\sum_{j=0}^{n-1} \log \|Df(s_{\underline{t}}^j, x_{\underline{t}}^j))^{-1}\| \le -cn \quad \text{and} \quad \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(x_{\underline{t}}^j, C) \le \gamma n$$

for $(s,x) \notin B_1(n) \cup B_2(n)$, where $B_1(n)$ and $B_2(n)$ are subsets of $S^1 \times I$ with

$$m\big(B_1(n)\big) \le e^{-\xi n} \quad \text{and} \quad m(B_2(n)) \le \operatorname{const} e^{-\sqrt{n}/4}$$

for some constant $\xi > 0$ only depending on γ. This gives the non-uniform expansion on random orbits. Moreover, defining N_ϵ as in (5.24), we have for $\underline{t} \in T^{\mathbb{N}}$

$$m\big\{(\underline{t}, x) \in T^{\mathbb{N}} \times M : N_\epsilon(\underline{t}, x) > n\big\} \le \operatorname{const} e^{-\sqrt{n}/4},$$

thus giving that the family of first hyperbolic time maps has uniform L^1-tail by Lemma 5.29.

For the sake of completeness let us explain how the Markov partitions $\mathcal{P}_n$ of S^1 can be defined in this case, in order to obtain the estimates on the Lebesgue measure of $B_1(n)$ and $B_2(n)$. We consider $M = S^1 \times I$ and define the skew-product map

$$\begin{array}{rccc} F: & T^{\mathbb{N}} \times M & \longrightarrow & T^{\mathbb{N}} \times M, \\ & (\underline{t}, z) & \longmapsto & \big(\sigma(\underline{t}), f_{t_1}(z)\big) \end{array}$$

where σ is the left shift map. Writing $f_t(z) = \big(g_t(z), q_t(z)\big)$ for $z = (s,x) \in S^1 \times I$, we have that $q_t(s,\cdot)$ is a unimodal map close to $\hat{q}$ for all $s \in S^1$ and $t \in \operatorname{supp}(\theta_\epsilon)$ with $\epsilon > 0$ small.

PROPOSITION 5.35. *Given $\underline{t} \in T^{\mathbb{N}}$ there is a C^1 foliation $\mathcal{F}_{\underline{t}}^c$ of M such that if $L_{\underline{t}}(z)$ is the leaf of $\mathcal{F}_{\underline{t}}^c$ through a point $z \in M$, then*

(1) *$L_{\underline{t}}(z)$ is a C^1 submanifold of M close to a vertical line in the C^1 topology;*
(2) *$f_{t_1}\big(L_{\underline{t}}(z)\big)$ is contained in $L_{\sigma\underline{t}}\big(f_{t_1}(z)\big)$.*

PROOF. This will be obtained as a consequence of the fact that the set of vertical lines constitutes a normally expanding invariant foliation for $\hat{f}$. Let $\mathcal{H}$ be the space of continuous maps $\xi : T^{\mathbb{N}} \times M \to [-1,1]$ endowed with the sup norm, and define the map $A : \mathcal{H} \to \mathcal{H}$ by

$$A\xi(\underline{t}, z) = \frac{\partial_x q_{t_1}(z)\xi(F(\underline{t},z)) - \partial_x g_{t_1}(z)}{-\partial_s q_{t_1}(z)\xi(F(\underline{t},z)) + \partial_s g_{t_1}(z)}, \quad \underline{t} = (t_1, t_2, \dots) \in T^{\mathbb{N}} \quad \text{and} \quad z \in M.$$

Note that A is well-defined, since

$$|A\xi(\underline{t},z)| \le \frac{(4+\alpha+\epsilon)+\alpha+\epsilon}{-(\text{const }\alpha+\epsilon)+(d-\alpha-\epsilon)} < 1$$

for small $\alpha > 0$ and $\epsilon > 0$. Moreover, A is a contraction on $\mathcal{H}$: given $\xi, \zeta \in \mathcal{H}$ and $(\underline{t}, z) \in T^{\mathbb{N}} \times M$ then

$$\begin{aligned}
&|A\xi(\underline{t},z) - A\zeta(\underline{t},z)| \\
&\quad\le \frac{|\det Df_{t_1}(z)|\cdot|\xi(\underline{t},z)-\zeta(\underline{t},z)|}{\big|\big(-\partial_s q_{t_1}(z)\xi(F(\underline{t},z))+\partial_s g_{t_1}(z)\big)\cdot\big(-\partial_s q_{t_1}(z)\zeta(F(\underline{t},z))+\partial_s g_{t_1}(z)\big)\big|} \\
&\quad\le \frac{\big((d+\alpha+\epsilon)(4+\alpha+\epsilon)+\alpha+\epsilon\big)\cdot\big|\xi(\underline{t},z)-\zeta(\underline{t},z)\big|}{(d-\text{const}\alpha-\epsilon)^2}.
\end{aligned}$$

This last quantity can be made smaller than $|\xi(\underline{t},z)-\eta(\underline{t},z)|/2$, as long as α and ϵ are chosen sufficiently small. This shows that A is a contraction on the Banach space $\mathcal{H}$, and so it has a unique fixed point $\xi^c \in \mathcal{H}$.

It is no restriction for our purposes if we think of T as being equal to $\text{supp}(\theta_\epsilon)$ for some small ϵ. Note that the map A depends continuously on F and for $\epsilon > 0$ small enough the fixed point of A is close to the zero constant map. This holds because we are choosing $\text{supp}(\theta_\epsilon)$ close to $\{t^*\}$, $f_{t^*} = f$ and f close to $\hat{f}$. Then, for $\epsilon > 0$ small enough, we have $\xi^c(\underline{t},\cdot)$ uniformly close to $\xi^c(\underline{t}^*,\cdot)$ and it is not hard to check that $\xi_0^c = \xi^c(\underline{t}^*,\cdot)$ is precisely the map whose integral leaves of the vector field $(\xi_0^c, 1)$ give the invariant foliation $\mathcal{F}^c$ associated to $f_{t^*} = f$. Since this foliation depends continuously on the dynamics and for $f = \hat{f}$ we have $\xi_0^c \equiv 0$ (see [Vi2, Section 2.5]), we finally deduce that $\xi^c(\underline{t},\cdot)$ is uniformly close to zero for small $\epsilon > 0$.

We have defined A in such a way that if we take $E^c(\underline{t},z) = \text{span}\{(\xi^c(\underline{t},z),1)\}$, then for every $\underline{t} \in T^{\mathbb{N}}$ and $z \in S^1 \times I$

$$Df_{t_1}(z)E^c(\underline{t},z) \subset E^c(F(\underline{t},z)). \tag{5.27}$$

Now, for fixed $\underline{t} \in T^{\mathbb{N}}$, we take $\mathcal{F}_{\underline{t}}^c$ to be the set of integral curves of the vector field $z \to (\xi^c(\underline{t},z),1)$ defined on $S^1 \times I$. Since the vector field is taken of class C^0, it does not follow immediately that through each point in $S^1 \times I$ passes only one integral curve. We will prove uniqueness of solutions by using the fact that the map f has a big expansion in the horizontal direction.

Assume, by contradiction, that there are two distinct integral curves $Y, Z \in \mathcal{F}_{\underline{t}}^c$ with a common point. So we may take three distinct nearby points $z_0, z_1, z_2 \in S^1 \times I$ such that $z_0 \in Y \cap Z$, $z_1 \in Y$, $z_2 \in Z$ and z_1, z_2 have the same x-coordinate. Let X be the horizontal curve joining z_1 to z_2. If we consider $X_n = \pi_2 \circ F^n(\underline{t}, X)$ for $n \ge 1$, where π_2 is the projection from $T^{\mathbb{N}} \times S^1 \times I$ onto $S^1 \times I$, we have that the curves X_n are nearly horizontal and grow in the horizontal direction (when n increases) by a factor close to d for small α and ϵ, see [Vi2, Section 2.1]. Hence, for large n, X_n wraps many times around the cylinder $S^1 \times I$. On the other hand, since $Y_n = \pi_2 \circ F^n(\underline{t}, Y)$ and $Z_n = \pi_2 \circ F^n(\underline{t}, Z)$ are always tangent to the vector field $z \to \big(\xi^c(\sigma^n \underline{t}, z), 1\big)$ on $S^1 \times I$, it follows that all the iterates of Y_n and Z_n have small amplitude in the s-direction. This gives a contradiction, since the closed curve made by Y, Z and X is homotopic to zero in $S^1 \times I$ and the closed curve made by Y_n, Z_n and X_n cannot be homotopic to zero for large n. Thus, for fixed $\underline{t} \in T^{\mathbb{N}}$ we have uniqueness of solutions of the vector field $z \to (\xi^c(\underline{t}, z), 1)$, and from (5.27) it follows that $\mathcal{F}^c_{\underline{t}}$ is an F-invariant foliation of M by nearly vertical leaves. □

Now, using the foliations given by the previous proposition we are able to define the Markov partitions of S^1 also in this setting. Given any smooth map $X : S^1 \to I$ whose graph is nearly horizontal, denote $\widehat{X}^n_{\underline{t}}(s) = f^n_{\underline{t}}\big(s, X(s)\big)$ for $n \geq 0$ and $s \in S^1$. Take some leaf $L^0_{\underline{t}}$ of the foliation $\mathcal{F}^c_{\underline{t}}$. Letting $L^n_{\underline{t}} = f^n_{\underline{t}}(L_{\underline{t}})$ for $n \geq 1$, we define the sequence of Markov partitions $(\mathcal{P}^n_{\underline{t}})_n$ of S^1 as

$$\mathcal{P}^n_{\underline{t}} = \Big\{[s', s'') \colon (s', s'') \text{ is a connected component of } (\widehat{X}^n_{\underline{t}})^{-1}\big((S^1 \times I) \setminus L^n_{\underline{t}}\big)\Big\}.$$

It is easy to check that $\mathcal{P}^{n+1}_{\underline{t}}$ refines $\mathcal{P}^n_{\underline{t}}$ for each $n \geq 1$ and, taking $\epsilon \ll \alpha$,

$$(d + \operatorname{const}\alpha)^{-n} \leq |\omega| \leq (d - \operatorname{const}\alpha)^{-n}$$

for each $\omega \in \mathcal{P}^n_{\underline{t}}$. This permits to obtain estimates (1.7) and (1.9) for the Lebesgue measure of the sets $B_1(n)$ and $B_2(n)$ exactly in the same way as in Subsection 1.2.2, also with the constants only depending on the quadratic map Q (cf. Remark 1.11). Thus we have proved the following result:

THEOREM 5.36. *Viana maps are non-uniformly expanding on random orbits and the family $(h_\epsilon)_\epsilon$ of first hyperbolic time functions has uniform L^1-tail.*

In particular, Viana maps are stochastically stable, by Theorem 5.32.

APPENDIX A

Functions of bounded variation

In this appendix we introduce the notion of variation for functions defined on higher dimensional spaces, and we present some results in this subject. The definition we present here corresponds to a generalization of the classical definition of bounded variation for functions defined in one-dimensional spaces. Recall that the usual definition in dimension one uses the fact that $\mathbb{R}$ is well-ordered and so it cannot be immediately generalized.

DEFINITION A.1. Let $\varphi \in L^1(\mathbb{R}^d)$ have compact support. We define the *variation* of φ as

$$\operatorname{var}(\varphi) = \sup\left\{\int_{\mathbb{R}^d} \varphi \operatorname{div}(\psi) dm_d : \ \psi \in C_0^1(\mathbb{R}^d, \mathbb{R}^d) \quad \text{and} \quad \|\psi\|_0 \le 1\right\},$$

where $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ is the set of C^1 maps with compact support from $\mathbb{R}^d$ to $\mathbb{R}^d$, $\| \ \ \|_0$ is the sup norm in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$, and $\operatorname{div}(\psi)$ denotes the divergence of ψ.

It is not hard to check that if $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ is a C^1 function, then

$$\operatorname{var}(\varphi) = \int_{\mathbb{R}^d} \|D\varphi\| \, dm_d$$

(see e.g. [**Gi**, Example 1.2]). We define the space of *bounded variation* functions on $\mathbb{R}^d$

$$BV(\mathbb{R}^d) = \left\{\varphi \in L^1(\mathbb{R}^d) \colon \ \operatorname{var}(\varphi) < +\infty\right\}.$$

Next we present some of the most important results on functions of bounded variation. For a complete and general exposition in this subject we recommend [**Gi**] or [**EG**]. Also [**GB89**] and [**AV**] contain some results on functions of bounded variation that are useful for us.

PROPOSITION A.2. *If $(\varphi_k)_k$ is a sequence of functions in $BV(\mathbb{R}^d)$ converging to $\varphi \in L^1(\mathbb{R}^d)$ in the L^1 norm, then* $\operatorname{var}(\varphi) \le \liminf_k \operatorname{var}(\varphi_k)$.

PROOF. See [**Gi**, Theorem 1.9]. □

The next proposition gives some kind of density of the C^∞ functions in the space of functions of bounded variation. It can be shown that $BV(\mathbb{R}^d)$ is a Banach space with the norm

$$\|\varphi\|_{BV} = \|\varphi\|_1 + \operatorname{var}(\varphi).$$

However the proposition does not say that C^∞ functions are dense in $BV(\mathbb{R}^d)$ with the norm above.

PROPOSITION A.3. *Given $\varphi \in BV(\mathbb{R}^d)$, there is a sequence $(\varphi_n)_n$ of C^∞ maps such that*

$$\lim_{n\to\infty}\int|\varphi-\varphi_n|dm=0 \quad \textit{and} \quad \lim_{n\to\infty}\int\|D\varphi_n\|dm=\operatorname{var}(\varphi).$$

PROOF. See [**Gi**, Theorem 1.17]. □

Another important result in this subject is the following compactness result.

PROPOSITION A.4. *If $(\varphi_k)_k$ is a sequence of functions in $BV(\mathbb{R}^d)$ such that there is a constant $K_0>0$ for which*

$$\operatorname{var}(\varphi_k)\le K_0 \quad \textit{and} \quad \int|\varphi_k|dm\le K_0 \quad \textit{for every } k,$$

then there is a subsequence of $(\varphi_k)_k$ converging in the L^1-norm to some φ_0 with $\operatorname{var}(\varphi_0)\le K_0$.

PROOF. See [**Gi**, Theorem 1.19]. □

In higher dimensions a function of bounded variation need not be bounded; see [**GB92**]. The result below gives in particular that $BV(\mathbb{R}^d)\subset L^p(\mathbb{R}^d)$ for $p=d/(d-1)$.

PROPOSITION A.5. *Let $\varphi\in BV(\mathbb{R}^d)$ and take $p=d/(d-1)$. Then*

$$\|\varphi\|_p\le K_1\operatorname{var}(\varphi),$$

where $K_1>0$ is a constant depending only on d.

PROOF. See [**Gi**, Theorem 1.28]. □

In the next two lemmas we use m_{d-1} for the Lebesgue measure induced on $(d-1)$-dimensional submanifolds of $\mathbb{R}^d$.

LEMMA A.6. *Let $S\subset\mathbb{R}^d$ be a closed domain with piecewise smooth $(d-1)$-dimensional boundary and take $\varphi\in L^1(\mathbb{R}^d)$. Assume that φ is equal to 0 in $\mathbb{R}^d\setminus S$, continuous in S and C^1 in* $\operatorname{int}(S)$. *Then*

$$\operatorname{var}(\varphi)=\int_{\operatorname{int}(S)}\|D\varphi\|dm_d+\int_{\partial S}|\varphi|dm_{d-1}.$$

PROOF. See [**Gi**, Remark 2.14]. □

LEMMA A.7. *Let $S\subset\mathbb{R}^d$ be a closed domain with piecewise smooth $(d-1)$-dimensional boundary of class C^2. Assume that ∂S has a tubular neighborhood of size $\rho>0$ inside S, and the C^2 components of the boundary of S meet at angles greater than* $\arcsin(\beta)>0$. *If $\varphi\colon S\to\mathbb{R}$ is C^1, then*

$$\int_{\partial S}\varphi dm_{d-1}\le\frac{1}{\beta}\left(\frac{1}{\rho}\int_S\varphi dm_d+\int_S\|D\varphi\|dm_d\right).$$

PROOF. See [**GB89**, Lemma 3]. □

We finally present the last result we need in this subject.

LEMMA A.8. *There is* $K_4 = K_4(d) > 0$ *such that, for any* $\varphi \in BV(\mathbb{R}^d)$ *and any* C^1 *embedding* $f\colon D \to \mathbb{R}^d$ *of a compact domain* $D \subset \mathbb{R}^d$,

$$\int_D |\varphi \circ f - \varphi| dm \le K_4 \| f - \mathrm{id} \|_0^d \operatorname{var}(\varphi).$$

PROOF. See [**AV**, Lemma 3.1]. □

APPENDIX B

Markov towers

In this appendix we present some results from **[Yo98]** which are used in these notes. We start with a set Δ_0, a measure m_0 defined in some σ-algebra $\mathcal{B}_0$ of Δ_0 such that $m_0(\Delta_0) < \infty$, and an integrable function $R : \Delta_0 \to \mathbb{N}$. Using R we introduce a partition $\{\Delta_{0,i}\}_{i\geq 1}$ of Δ_0 by

$$\Delta_{0,i} = R^{-1}(i) \quad \text{for each } i \geq 1.$$

We define $R_i = R \mid \Delta_{0,i}$ and assume that that $\gcd\{R_i\} = 1$. We also assume that there is a function $F : \Delta_0 \to \Delta_0$ satisfying some properties that we will explicit below. Before that, we introduce $s : \Delta_0 \times \Delta_0 \to \mathbb{N}_0$, called the *separation time* function, given by

$$s(x, y) = \min \left\{n \geq 0\colon F^n(x) \text{ and } F^n(y) \text{ lie in distinct } \Delta_{0,i}\right\}.$$

We assume that this function is well-defined m_0 almost everywhere on Δ_0, and the following conditions hold:

(1) *Markov*: $F|_{\Delta_{0,i}} : \Delta_{0,i} \to \Delta_0$ is a bijection.

(2) *Regularity*: F has a *Jacobian* JF with respect to m_0, i.e.

$$m_0(F(B)) = \int_B JF dm_0, \quad \text{for all } B \in \mathcal{B}_0,$$

which is positive m_0 almost everywhere.

(3) *Bounded distortion*: there are $C > 0$ and $0 < \beta < 1$ such that for all $i \geq 1$ and all $x, y \in \Delta_{0,i}$ we have

$$\left|\frac{JF(x)}{JF(y)} - 1\right| \leq C\beta^{s(F(x),F(y))}. \tag{B.1}$$

We introduce a set Δ, called a *Markov tower*,

$$\Delta = \big\{(x, n)\colon x \in \Delta_0 \text{ and } 0 \leq n < R(x)\big\},$$

and a map $T\colon \Delta \to \Delta$, called a *tower map*, given by

$$T(x,n) = \begin{cases} (x, n+1), & \text{if } n+1 < R(x); \\ (F(x), 0), & \text{if } n+1 = R(x). \end{cases}$$

We have by construction $T^{R(x)}(x, 0) = (F(x), 0)$. From here on we make no distinction between $\Delta_0 \times \{0\}$ and Δ_0. We call $\Delta_\ell = \Delta \cap \{n = \ell\}$ the ℓ^{th} level of the tower, and define

$$\Delta_{\ell,i} = \Delta_\ell \cap \{x \in \Delta_{0,i}\},$$

so that Δ_{R_i-1} is precisely the last level of the tower above $\Delta_{0,i}$. We introduce $\mathcal{B}$ the σ-algebra on Δ such that $\mathcal{B}|\Delta_{\ell,i}$ is formed by sets of the type $\{\ell\} \times B_0$ with $B_0 \in \mathcal{B}_0|\Delta_{0,i}$. Finally we use T to push the measure m_0 to the upper levels of the tower. Thus, we obtain a measure m on $\mathcal{B}$ such that $m|\Delta_{\ell+1,i} = T_*(m|\Delta_{\ell,i})$ for $\ell < R_i - 1$, and $m|\Delta_0 = m_0$.

It will be convenient to have the separation time function s defined in Δ. This may be done by taking $s(x,y) = s(x',y')$ if x and y lie in a same $\Delta_{l,i}$, where x', y' are the corresponding elements of $\Delta_{0,i}$, and $s(x,y) = 0$ otherwise.

Now we introduce some spaces of functions. Letting $\beta < 1$ be as in (B.1) we define

$$\mathcal{H}_\beta = \left\{\varphi : \Delta \to \mathbb{R} \mid \exists C > 0 \text{ such that } |\varphi(x) - \varphi(y)| \le C\beta^{s(x,y)}\ \forall x,y \in \Delta\right\}$$

$$\mathcal{H}_\beta^+ = \Big\{\varphi \in \mathcal{H}_\beta \mid \exists C > 0 \text{ such that on each } \Delta_{l,i}, \text{ either } \varphi \equiv 0, \text{ or}$$
$$\varphi > 0 \text{ and } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le C\beta^{s(x,y)}\ \forall x,y \in \Delta_{l,i}\Big\}$$

The following result gives the existence of an equilibrium probability measure for the tower map, and gives its basic properties.

THEOREM B.1. *If R is integrable with respect to m_0, then*
1. *T has an invariant probability measure ν which is equivalent to m;*
2. *$d\nu/dm$ belongs to $\mathcal{H}_\beta^+$ and is bounded from below by some $c > 0$;*
3. *(T,ν) is exact and, hence ergodic and mixing.*

PROOF. See [**Yo99**, Theorem 1]. □

We are interested in studying the correlation decay of the random variables $\{\varphi \circ T^n : n \ge 0\}$ where $\varphi : \Delta \to \mathbb{R}$ is an observable and the underlying space is (Δ, ν). Recall that the correlation function is defined as

$$C_n(\varphi,\psi) = \left|\int (\varphi \circ T^n)\psi d\nu - \int \varphi d\nu \int \psi d\nu\right|,$$

for observables φ, ψ and $n \ge 0$.

THEOREM B.2. *Assume that $\varphi \in L^\infty(m)$ and $\psi \in \mathcal{H}_\beta$. Then we have:*
1. *if $m\{R > n\} = \mathcal{O}(n^{-\gamma})$ for some $\gamma > 1$, then $C_n(\varphi,\psi) = \mathcal{O}(n^{-\gamma+1})$;*
2. *if $m\{R > n\} = \mathcal{O}(\theta^n)$ for some $0 < \theta < 1$, then there is $0 < \tilde\theta < 1$ such that $C_n(\varphi,\psi) = \mathcal{O}(\tilde\theta^n)$.*

PROOF. See [**Yo98**, Theorem 2 & Theorem 3]. □

It is also possible to obtain conditions for the validity of the Central Limit Theorem, which states that the probability of a given deviation of the average values of an observable along an orbit from the asymptotic average is essentially given by a Normal Distribution.

THEOREM B.3. *Assume that $m\{R > n\} = \mathcal{O}(n^{-\gamma})$ for some $\gamma > 2$, and take $\varphi \in C_\beta$ with $\int \varphi d\nu = 0$. Then there exists $\sigma > 0$ such that for every interval $I \subset \mathbb{R}$,*

$$\mu\left\{x \in X : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu\right) \in I\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_I e^{-t^2/2\sigma^2} dt$$

if and only if $\varphi \neq \psi \circ f - \psi$ for any ψ.

PROOF. See [Yo98, Theorem 4]. □

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