Spin Dynamics at Zero Temperature

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Preface

These are notes for a course at the 24th meeting of the Brazilian Mathematical Society (24º Colóquio Brasileiro de Matemática) on July 27–August 1/2003 at IMPA/CNPq, Rio de Janeiro.

They were prepared in April-June/2003, and are based on a number of papers involving the author. There is a didactic intention, on one side, as well as a desire to lead interested readers to the sources, on another side. This ambiguity may have led to some notation inconsistencies throughout the text, in which case there will at least be consistency with the originals.

The co-authors of the above mentioned papers, M. Isopi, C. M. Newman, K. Ravishankar, R. H. Schonmann, V. Sidoravicius, and D. L. Stein are thankfully acknowledged, and exempted from any responsibility for the style of presentation.

Contents

Chapter 1

Introduction

Spin dynamics at low temperature exhibit interesting physical features associated to initial disordered state and/or disordered environment, such as *aging* (ever longer metastable slowing down of the dynamics) and *chaotic time dependence* (long term oscilations of correlations as functions of the disorder). The former one takes place, for example, in the case where, after an observation of the state of the system is made at a given *age* of the system (time from the beginning of the experiment), the time till we get a second observation which is minimally decorrelated from that first observation scales with the age of the system at the first observation. The second feature manifests itself in the non weak convergence of the dynamics, as time increases, combined with the predictability at large times of the state of the system as a function of the disorder.

It is natural to begin the mathematical analysis of models for those dynamics, with the aim of verifying those features rigorously, and the mechanism for their occurrence, at the extreme case of zero temperature. In the second and third parts of these notes, we will consider one dimensional dynamics where one or more of those features can be established rigorously. In the case of aging, we will show that it follows from suitable scaling limits of the dynamics.

We will also treat, in the first part, the issue of *fixation* (strong convergence of the dynamics to a fixed configuration), which is a phenomenon purely of zero temperature, and in principle unrelated to disordered parameters, for dynamics in arbitrary dimension.

1.1 Models and questions

The point of departure is Glauber dynamics for the Ising model. We have spins ± 1 in $\mathbb{Z}^d$. Given a initial configuration $\sigma_0 \in \Sigma := \{-1,+1\}^{\mathbb{Z}^d}$, we evolve it according to a Markov process given by a generator of the form

$$(Lf)(\sigma) = \sum_{x\in\mathbb{Z}^d} \lambda(x)c(x,\sigma)(f(\sigma^x) - f(\sigma)), \tag{1.1}$$

acting on local functions f, where the rate function $c(\cdot,\cdot)$ is reversible with respect to (finite volume) Gibbs measures associated to the (formal) Hamiltonian

$$H(\sigma) = -\frac{1}{2} \sum_{\substack{x,y:\\ \|x-y\|=1}} \sigma(x)\sigma(y) \tag{1.2}$$

and inverse temperature β. See [31] for Ising models and Gibbs measures (in equilibrium). We will have a *change of time scale* function λ, which is positive, and will be taken either identically 1, as usual, or random. $\lambda(x)$ can be interpreted as the rate of spin flip attempts at site x. It will be denoted sometimes below as λ_x. See next section for a construction/more explicit description of the above dynamics. There are several possible choices for $c(\cdot,\cdot)$, and we focus on one of them, and sometimes consider another one. The former possibility, one that leads to the so called *heat-bath* dynamics, is

$$c(x,\sigma) = (1+\exp\{-\beta[H(\sigma^x) - H(\sigma)]\})^{-1} \tag{1.3}$$

and the latter possibility, leading to the *Metropolis* dynamics, is

$$c(x,\sigma) = \exp\{-\beta[H(\sigma^x) - H(\sigma)]^+\}, \tag{1.4}$$

where

$$\sigma^x(y) = \begin{cases} \sigma(y), & \text{if } y \neq x, \\ -\sigma(y), & \text{if } y = x, \end{cases} \tag{1.5}$$

is the *spin flip at site x* operator, and $a^+ = \max(a,0)$ for all $a \in \mathbb{R}$. Notice that, even though (1.2) does not make sense as an infinite sum, the difference

$$H(\sigma^x) - H(\sigma) = \sum_{y:\|x-y\|=1} \sigma(x)\sigma(y) \tag{1.6}$$

does make sense and is a local function of σ.

These are well known and much studied dynamics. See e.g. [40] (in this reference the models go under the name of *stochastic Ising models*).

In these notes, we will consider the zero temperature limit of these dynamics. These will be dynamics on Σ with the following rates.

$$c(x,\sigma) = \begin{cases} 1, & \text{if } \mathcal{N}(x,\sigma) > d; \\ \frac{1}{2}, & \text{if } \mathcal{N}(x,\sigma) = d; \\ 0, & \text{if } \mathcal{N}(x,\sigma) < d, \end{cases} \tag{1.7}$$

for heat-bath, and

$$c(x,\sigma) = \begin{cases} 1, & \text{if } \mathcal{N}(x,\sigma) \geq d; \\ 0, & \text{if } \mathcal{N}(x,\sigma) < d, \end{cases} \tag{1.8}$$

for Metropolis, where $\mathcal{N}(x,\sigma)$ is the number of nearest neighbors of x whose spin is opposite to the one in x, i.e.,

$$\mathcal{N}(x,\sigma) = \#\{y : \|x-y\| = 1 \text{ and } \sigma(y) \neq \sigma(x)\}. \tag{1.9}$$

The dynamics given by (1.8) is the *threshold voter model* (see [40, 41]) with threshold d. The one in (1.7) coincides in $d = 1$ with the simple symmetric *linear voter model*.

We will study some aspects of the asymptotic in time behavior of the above dynamics. Let σ_t be the configuration at time t of the process started from σ_0 chosen from a product Bernoulli measure with parameter p.

In Part I, Chapter 2, we will address the issue of *fixation*, that is, whether or not the dynamics *fixates*, by which we mean whether or not σ_t converges as $t \to \infty$, for almost every realization of the initial condition and dynamical random variables, to a (possibly random) limiting spin configuration. Here $\lambda \equiv 1$. We will refer to results about nonfixation (that occurs when the spin at some site flips infinitely often with positive probability) and, in Theorem 2.1, establish a fixation result in $d \geq 2$ and for p close enough to either 1 or 0.

In Parts II and III, we will analyse the one dimensional voter model (1.7) only. In Part II, Chapter 3, we will address the issue of *chaotic time dependence* (CTD), which is the non weak convergence of σ_t as $t \to \infty$. Here we take $\lambda(x), x \in \mathbb{Z}$, i.i.d., and prove in Theorem 3.4 that when $\lambda(0)^{-1}$ is in the basin of attraction of a stable law with index $\alpha < 1$, then the dynamics exhibits CTD. More precisely, we show, under this condition, that, for almost every realization of σ_0 and λ, $\mathbb{E}(\sigma_t(0)|\sigma_0, \lambda)$ does not converge as $t \to \infty$. This nonconvergence means that the dynamical correlations of the system are nontrivial functions of σ_0 and λ which oscilate indefinitely as $t \to \infty$. We also refer to results on conditions for the absence of CTD. The occurrence of CTD in the voter model is related to the behavior of the distribution of its *dual* random walk, in this case, with random rates, namely CTD in the voter model for a given realization of λ (and a typical realization of σ_0) is equivalent to *localization* of the distribution of the random walk, i.e., to the non convergence to 0 as $t \to \infty$ of the maximum in x of $\mathbb{P}(X_t = x|\lambda)$, where $X_t, t \geq 0$, is a simple symmetric random walk with rates λ. We will then get a CTD result for the voter model with random rates from a localization result for the random walk with random rates.

In Part III, we study the aging phenomenon. First, in Chapter 4, for the random walk with random rates (RWRR) of Chapter 3. Then, in Chapter 5, for the ordinary uniform rate voter model. *Aging* is characteristic of low temperature dynamics of systems starting from a high temperature configuration, and is signaled by certain scaling relations obeyed by correlations, or more generally response functions, of the system at two large, widely separated times (see (3.1) below). The usual approach in the literature to obtaining aging results for given systems is to analyse individual correlations, and prove scaling relations for those. Our approach is to establish scaling limits for the dynamics, and thence, aging results as corollaries.

In Chapter 4, Theorem 4.13, we establish the scaling limit of the RWRR, which turns out to be a singular diffusion, whose singularity explains both the aging *and* the localization phenomenons.

In Chapter 5, we consider the ordinary voter model, which can be described via coalescing random walks starting from every space-time point (more on this in the next section), and discuss briefly coalescing Brownian motions as scaling limits for that system, both in a restricted sense, where the usual invariance principle can be invoked, and in a broader sense, and compute limiting aging functions from that (see (5.23) and (5.30)).

Finally, in Chapter 6, we discuss in detail the continuum model for coalescing random walks starting from every space-time point, which we call the *Brownian web*. It consists roughly of coalescing Brownian motions starting from every space-time point. We characterize (Theorem 6.1) and construct the Brownian web in a suitable space, where weak convergence can be analysed, and then establish criteria for weak convergence to the Brownian web (Theorem 6.6), which are verified in the case of coalescing random walks

(Theorem 6.8).

The Brownian web has an interest of its own and as a basic object underlying the scaling limits of voter models, coalescing random walks, their variants, and other models (like stochastic flows). We close Chapter 6 with two sections. In Section 6.2, we discuss duality in the Brownian web, introduce the *dual* Brownian web, and classify points of $\mathbb{R}^2$ according to multiplicity, i.e., the number paths of the web starting out from a point, an issue which is related to duality. In the final Section 6.3, we consider a modification of the voter model, which is also a Glauber dynamics, now at positive temperature, and describe its scaling limit (where the temperature is also scaled) in terms of the joint Brownian web together with the dual Brownian web, plus Poisson marks, that account for the extra noise from the temperature. This allows for the explicit computation of limiting aging functions (like (6.34)), and also for the definition of a continuum spin dynamics (at the final Subsection 6.3.2), of which the continuum Ising model is an invariant measure in a particular case.

1.2 Glauber dynamics and the voter model

Consider the following stochastic process σ_t, $t \geq 0$, in the state space $\Sigma = \{-1,+1\}^{\mathbb{Z}^d}$ of *spin* configurations in $\mathbb{Z}^d$. At every site of $\mathbb{Z}^d$, there is a clock whose alarm sets off at intervals which form an independent and identically distributed family of exponential random variables. The families of exponential random variables of the different sites are all independent of each other. Every time the alarm sets off at an arbitrary site $x \in \mathbb{Z}^d$, the spin of x, $\sigma(x)$, will flip with probability $c(x,\sigma)$, where σ is the spin configuration immediately before the alarm and $c(x,\sigma)$ is given in (1.7).

This *spin dynamics* can be regarded as the zero temperature limit of the well known *(heat-bath) Glauber dynamics* (as discussed in the previous section).

Let λ_x be the rate of the exponential waiting time at site x. We will assume throughout (except at Chapter 4) that

$$0 < \lambda_x \leq 1 \quad \text{for all} \quad x \in \mathbb{Z}^d, \tag{1.10}$$

so that the process is well defined for all $t \geq 0$ (see e.g. [40], Section I.3) and obeys (1.1).

Voter model and coalescing random walks

Consider now the following alternative spin flip rule (within the same framework of alarm clocks). Each time the alarm sets off at site x, $\sigma(x)$ adopts the spin of a nearest neighboring site chosen uniformly at random, with all the choices for all the sites, and at all times, independent of each other. This is the d-dimensional (linear) *voter model* [40].

It is easy to convince oneself that in $d = 1$, the voter model and the zero temperature Glauber dynamics are equivalent, and so we can regard the voter model itself as a zero temperature dynamics (in all d, but in these notes we will only treat explicitly the 1-dimensional case).

The voter model has an important and very useful duality property with respect to coalescing random walks, which we describe next.

First of all, coalescing random walks are particles performing simple symmetric (continuous time) random walks in $\mathbb{Z}^d$, such that when two particles meet in the same site, they coalesce into a single particle, which keeps on performing a simple symmetric random walk from then on. Let $X_t(x)$ denote the position at time $t \geq 0$ of a particle that was initially at site x (we assume that the exponential waiting times at site x have rate λ_x, as before). Then, for every $t \geq 0$,

$$\{\sigma_t(x),\, x \in \mathbb{Z}^d\} = \{\sigma_0(X_t(x)),\, x \in \mathbb{Z}^d\} \tag{1.11}$$

in distribution, where $\sigma_{\cdot}(\cdot)$ is the voter model.

For a justification of (1.11), consider the following graphical representation of the voter model (see also [40, 18]).

In Figure 1.1, the portion up to time t between sites -2 and 3 of the 1D voter model is represented. Time goes up. In the time line of each site, outgoing arrows of length 1 are located at the i.i.d. exponential distances. Once one such location, say in the time line of site x, is chosen (according to an exponential random variable of rate λ_x for the distance to the previous such point in the time line), then, whether it is an arrow to the right or to the left, is equally likely.

The voter model can be constructed using this graph, which is known as a *Harris diagram*, in the following way. Given a initial assignment of spins $+1$ or -1 to the sites of $\mathbb{Z}$ at time 0, the spin updates at each site are done successively going upward its time line until an outgoing arrow is met, and, from that point till the next one in the time line, assigning to x the spin of the neighboring site where the first arrow ends, at that time. This

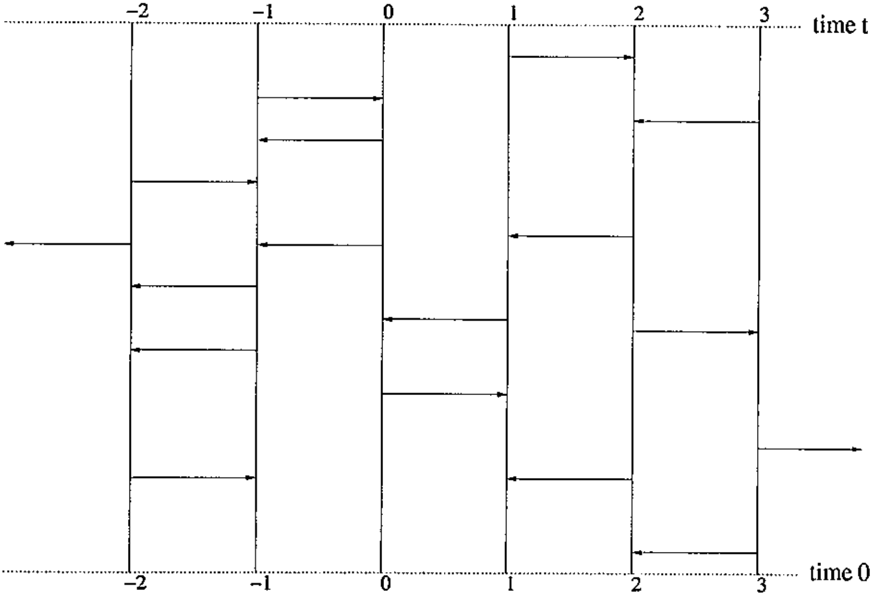

Figure 1.1: Harris diagram for the 1D voter model

procedure almost surely uniquely defines the evolution for all times, under assumption (1.10).

Now, the duality with coalescing random walks. In order to find out what is the spin of a given site x at a given time t in the voter model, using the Harris diagram, follow the time line backwards from that site at that time and follow the outgoing arrows to neighboring sites from each site of the way, down to time 0. The spin of site x at time t is the spin at time 0 of the site arrived at in this way. For example, in Figure 1.1, the spin of the origin at time t is the one of site -1 at time 0; the one of site 1 at time t is the one of site 1 at time 0. Notice that the path followed from (x, t) to time 0 is a random walk path. Furthermore, two paths, one from (x, t) and another from (y, t), are independent until they meet, when they coalesce. In Figure 1.1, the backward paths starting in -2, -1 and 0 at time t are all coalesced together by time 0; so are the ones starting in 1, 2 and 3 at time t, by time 0. We conclude that, for every fixed $t \geq 0$, the backward paths starting at $\{(x, t), x \in \mathbb{Z}\}$ are coalescing random walk paths, as described a few paragraphs above, and the duality relation (1.11) follows.

Remark 1.2.1 *If we want to study different time joint distributions or correlations of σ_t, then the duality relation is as follows (as is clear from the above discussion). Suppose $B \subset \mathbb{Z}^d$ and $T \subset \mathbb{R}^+$ are finite sets. Then*

$$\{\sigma_t(x),\, x \in B, t \in T\} = \{\sigma_0(X_t^{(t)}(x)),\, x \in B, t \in T\} \tag{1.12}$$

in distribution, where $\{X_r^{(t)}(x),\, x \in B, 0 \le r \le t, t \in T\}$ are backward in time coalescing random walks, with $X_r^{(t)}(x), 0 \le r \le t$, a backward in time random walk starting at time t on site x.

Say we want to look at the two-time two-point correlation function

$$C(x,y;s,t) := \mathbb{E}(\sigma_s(x)\sigma_t(y)). \tag{1.13}$$

Then, by (1.12),

$$\begin{aligned} C(x,y;s,t) &= \mathbb{P}(A(x,y;s,t)) + p\,\mathbb{P}(A^c(x,y;s,t)) \\ &= p + (1-p)\mathbb{P}(A(x,y;s,t)), \end{aligned} \tag{1.14}$$

where $A(x,y;s,t)$ is the event that the backward walks $X_\cdot^{(s)}(x)$ and $X_\cdot^{(t)}(y)$ coalesce before time 0. We will come back to C in Chapter III.

We finish this section with an application of (1.11) to the the asymptotic distribution of the voter model when started with a translation invariant product Bernoulli measure. Let σ_0 have such distribution, that is, for every finite $A \subset \mathbb{Z}^d$,

$$\mathbb{P}(\sigma_0(x) = 1 \text{ for all } x \in A) = p^{|A|}, \tag{1.15}$$

where $p \in [0,1]$ is a parameter.

Using (1.11), we get, for every $A \subset \mathbb{Z}^d$,

$$\mathbb{P}(\sigma_t(x) = 1 \text{ for all } x \in A) = \mathbb{P}(\sigma_0(X_t(x)) = 1 \text{ for all } x \in A). \tag{1.16}$$

Now, let $N_A(t)$ denote the number of particles into which the particles initially at A have coalesced by time t. It is clear that the latter probability equals

$$\sum_{k=1}^{|A|} p^k \mathbb{P}(N_A(t) = k) = \mathbb{E}(p^{N_A(t)}). \tag{1.17}$$

Since $N_A(t)$ is nonincreasing in t, we have that the latter expectation converges as $t \to \infty$, and we conclude that the distribution of σ_t converges as $t \to \infty$ (when integrated with respect to the initial condition[1]). Furthermore, in $d = 1$ and 2, $N_A(t) \to 1$ as $t \to \infty$ almost surely. This follows from the recurrence of the random walk in those dimensions. We conclude that, in $d = 1$ and 2

$$\sigma_t \Rightarrow p\delta_{+\mathbf{1}} + (1-p)\delta_{-\mathbf{1}} \tag{1.18}$$

as $t \to \infty$, where $\Rightarrow$ denotes weak convergence, $\delta_{+\mathbf{1}}$ is the point mass at $+\mathbf{1}$, the (invariant) configuration where all spins are $+1$, and $\delta_{-\mathbf{1}}$ is the point mass at $-\mathbf{1}$, the (invariant) configuration where all spins are -1.

The transience of the random walk in $d \geq 3$ implies that the limiting measure of σ_t in those dimensions is more complicated than a mixture of $\delta_{+\mathbf{1}}$ and $\delta_{-\mathbf{1}}$.

Remark 1.2.2 *In $d \geq 2$, Glauber dynamics and voter model behave in a substantially different way. For example, (1.18) does not hold for the former dynamics in $d \geq 2$, if p is close enough to 0 or 1 (see [30]; we will argue in detail the $d = 2$ case in Chapter 2).*

[1] In Chapter 3 we will see a situation where, not integrated with respect to the initial configuration, for almost every such configuration, σ_t does not converge weakly as $t \to \infty$.

Part I

Fixation

Chapter 2

Fixation in 2 and higher dimensions

We begin the discussion of our models and questions with the issue of fixation/nonfixation for the Glauber dynamics at zero temperature in $\mathbb{Z}^d$, started from a Bernoulli product measure with $\mathbb{P}_p(\sigma_0(x) = +1) = p$ for each $x \in \mathbb{Z}^d$, introduced in Section 1.2. In this chapter, we take $\lambda_x \equiv 1$, and we will make the p dependence on the overall probability measure $\mathbb{P} = \mathbb{P}_p$ explicit.

For each value of p, we will say that the system fixates if for each $x \in \mathbb{Z}^d$ the spin at x flips only finitely many times (locally the system becomes trapped).

As discussed in Section 1.2, in $d = 1$ our system is the (linear) voter model, which is known to almost surely not fixate for any $p \in (0, 1)$ (and any d).

As also pointed in that section, in $d \geq 2$, our model is not any longer the linear voter model (it is rather closely related to a *threshold voter model*, which is a subject of considerable interest; see [41] and references therein), and the long time behavior is more challenging. In $d = 2$, it is proved in [45] that if $p = 1/2$, the system does not fixate:

$$\mathbb{P}_{1/2}(\sigma_t(0) \text{ flips i.o.}) = 1. \tag{2.1}$$

Recently, stronger results were obtained in this case in [14]. It is not known if (2.1) is true in higher dimensions (see [46] and references therein). See also [15] for an analysis of the model in the hexagonal lattice, where fixation occurs at $p = 1/2$.

In contrast to (2.1), we will argue below (for $d = 2$) the following.

Theorem 2.1 (Fixation) *For $d \geq 2$ there exists $p_0 \equiv p_0(d) < 1$, such that for any $p > p_0$ and $\epsilon > 0$*

$$\mathbb{P}_p(\sigma_t(0) = -1) \;\leq\; \mathbb{P}_p(\sigma_s(0) = -1 \text{ for some } s \geq t) \;\leq\; e^{-t^{(1/d)-\epsilon}}, \qquad (2.2)$$

for t large enough. In particular, for any $p > p_0$

$$\mathbb{P}_p(\sigma_t(0) = +1 \text{ eventually}) \;=\; 1. \qquad (2.3)$$

Remark 2.2 *Theorem 2.1 shows that, for large p, fixation occurs as fast as a stretched exponential. In [30], it is shown that in $d = 2$ it occurs no faster than that, and the correct "stretch" of the exponential as a power $1/2$ is identified, i.e., we have that*

$$\lim_{t\to\infty} \frac{\log(-\log(\mathbb{P}_p(\sigma_t(0) = -1)))}{\log(t)} \;=\; \frac{1}{2}.$$

See [30] for a discussion of this issue and the related literature.

Comparing (2.1) with Theorem 2.1 raises the question of the values of p for which there is fixation in the state with all spins $+1$. We believe that this happens for every $p > 1/2$. Similarly, it seems reasonable to conjecture that at low positive temperatures the Glauber dynamics (heat-bath, Metropolis, or other kinds) started from homogeneous product measure with $\mathbb{P}_p(\sigma_0(0) = +1) = p > 1/2$, converges weakly to the $(+)$-phase (see Open Problem 7, p.225, of [40]). The results of [30] also suggest the conjecture that in $d = 2$ this convergence occurs as fast as, but not faster than, a stretched exponential.

The proof of Theorem 2.1 will be based on a multiscale analysis.

From this point on, we restrict attention to $d = 2$, which already contains the main arguments for the general case [30].

A basic ingredient in the proof of the above theorem is an estimate on the time needed for the dynamics to "eliminate droplets", by which we mean the following. Suppose that the process is started from the configuration in which the cube $R = \{1, 2, ..., L\}^2$ is occupied by spins -1 and all other spins are $+1$. It is clear that the spins outside R will never flip and that eventually the configuration with all spins $+1$ will be reached, with the system becoming trapped there. Let T be the random time when the system reaches this trapping configuration. We need to estimate how T grows with L. Heuristically, one expects the droplet to shrink with its boundary being well approximated, when L is large, by a surface moving by mean-curvature (see, e.g., [39], [32], [53], [17] and references therein). This leads to T being of the order of L^2. The following result contains a rigorous upper bound consistent with this picture.

Proposition 2.3 (Single-droplet erosion time estimate) *There exist* $C, \gamma \in (0, \infty)$ *such that for all large* L

$$\mathbb{P}(T > CL^2) \ \leq \ e^{-\gamma L}.$$

Remark 2.4 *In [30], Theorem 1.3, we derive upper and lower bounds for* T *in general* $d \geq 2$. *In* $d = 2$, *our lower bound matches the upper bound in the above proposition, but for a logarithmic denominator.*

This chapter is organized as follows. In the following section, we introduce some terminology, notation and basic tools, including the bootstrap percolation process. In Section 2.2, we argue Proposition 2.3. In Section 2.3, we prove Theorem 2.1, after explaining the heuristics behind the proof.

2.1 Definitions and tools

We begin with several standard definitions. As already mentioned we shall consider models on the lattice $\mathbb{Z}^2$. The cardinality of a set $\Lambda \subset \mathbb{Z}^2$ will be denoted by $|\Lambda|$. The distance between vertices of $\mathbb{Z}^2$ will be measured by the L^1 norm $||\cdot||$, and denoted $\text{dist}(x, y) = ||x - y||$. A (selfavoiding) path is a sequence of distinct vertices $x_1, x_2, \ldots, x_n$, such that $||x_i - x_{i+1}|| = 1$, and the vertices x_1 and x_n are called the start- and end-points of the path.

Graphical construction. It is convenient to make the following graphical construction of the process, which is a little more complicated than the one in Section 1.2. It provides a version of the process either on the infinite lattice $\mathbb{Z}^2$ or on any of its finite subsets with arbitrary boundary conditions and starting from any initial configuration, which are all naturally coupled (that is, all live on the same probability space), so as to have good ordering properties, which will be useful in what follows.

To each vertex $x \in \mathbb{Z}^2$ we associate two independent Poisson processes, each one of rate 1. We will denote the successive arrival times (after time

0) of these Poisson processes by $\{\tau^+_{x,n}\}_{n=1,2...}$ and $\{\tau^-_{x,n}\}_{n=1,2...}$, assuming that the Poisson processes associated to different vertices are also mutually independent. We say that at each space-time point of the form $(x, \tau^+_{x,n})$ there is an upward mark and at each point of the form $(x, \tau^-_{x,n})$ there is a downward mark. We associate to each arrival time $\tau^*_{x,n}$ (where $*$ stands for $+$ or $-$) a random variable $U^*_{x,n}$ with uniform distribution between 0 and 1. All these random variables are supposed independent among themselves and independent from the previously introduced Poisson processes. This finishes the construction of the probability space. The process then is defined as follows: we update the state of the process at each space-time location where there is a mark according to the following rules. If the mark that we are considering is at the point $(x, \tau^*_{x,n})$, and the configuration immediately before time $\tau^*_{x,n}$ restricted to x and its neighboring sites was identical to a configuration σ, then

i) if $\sigma(x) = -1$ (resp. $\sigma(x) = +1$), then the spin at x can flip only if the mark is an upward (resp. downward) one;

ii) if the mark is upward and $\sigma(x) = -1$, or if the mark is downward and $\sigma(x) = +1$, then the spin at x flips if and only if $U^*_{x,n} < c'(x, \sigma)$, where

$$c'(x,\sigma) = \begin{cases} 1, & \text{if at least 2 neighbors of } x \text{ have its opposite spin;} \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Remark 2.5 *With this $c'(\cdot,\cdot)$, our process is not exactly the zero temperature heat-bath Glauber dynamics for the Ising model described in Section 1.2 and assumed in the previous section; it corresponds instead to the zero temperature Metropolis dynamics for the Ising model (see paragraph of (1.7,1.8)). This difference*[1] *is nevertheless immaterial as regards Theorem 2.1 and its proof. See [30]. The choice of this dynamics is for convenience of exposition only.*

Remark 2.6 *The use of upward and downward marks, combined with the attractiveness of the flip probabilities*[2] *guarantees that if we compare the evolution started from two configurations which are comparable in the sense that wherever the former has a spin +1, so does the latter, then this property is*

[1]Notice that it ammounts to a difference in the flip probability in x only when there is a tie in σ of the numbers of spins '+' and '−' at the neighbors of x; in the latter case, it is 1; in the former, 1/2.

[2]The probability of a flip $-1 \to +1$ in a spin configuration σ is less or equal to the one in a spin configuration σ', if $\sigma(x) \le \sigma'(x)$ for all $x \in \mathbb{Z}^2$.

preserved by the evolution. We will make extensive use of such comparisons. For more on this fundamental technique and the related notion of stochastic order, we refer the reader to Sections II.2 and III.2 of [40].

We will refer to the Poisson marks and the associated uniformly distributed random variables, as the "graphical marks".

$\mathbb{P}_p$ is then the probability measure over the initial Bernoulli distribution with density p of spins $+1$ and over the graphical marks.

Bootstrap percolation. In the definition below we consider configurations in $\{u, s\}^{\mathbb{Z}^2}$, where u (for "unstable") and s (for "stable") are arbitrary symbols.

Definition 2.7 *The 2-dimensional ($u \to s$) bootstrap percolation process with threshold 2, defined in a finite or infinite volume $\Lambda \subseteq \mathbb{Z}^2$, starting from the initial configuration $\eta_0 \in \{u, s\}^\Lambda$ is a cellular automaton which evolves in discrete time $t = 0,\ 1,\ 2, \ldots$, and such that at each time unit $t \geq 1$ the current configuration is updated according to the following rules. For each $x \in \Lambda$,*

1. If $\eta_{t-1}(x) = s$, then $\eta_t(x) = s$.

2. If $\eta_{t-1}(x) = u$, and at time $t-1$ the vertex x has at least 2 nearest neighbors in Λ in state s, then $\eta_t(x) = s$; otherwise the spin at vertex x remains unchanged, i.e., $\eta_t(x) = u$.

If $|\Lambda| < \infty$, it is immediate from the Definition 2.7, that the procedure described above converges in a finite time to a fixed configuration. In this context, we say that we *bootstrap* the initial configuration η_0, the result of which is the final configuration of the bootstrap percolation dynamics.

Remark 2.8 *There is a close relation of this process with the spin flip system that we are considering. One can see the latter dynamics as follows. When the Poisson clock rings at site x, if $\sigma(x) = +1$, then the ($1+ \to -1$) bootstrap percolation rule is applied; if $\sigma(x) = -1$, then the ($-1 \to +1$) bootstrap percolation rule is applied instead. The spin system can thus be seen as a competition of a ($1+ \to -1$) bootstrap percolation process with a ($-1 \to +1$) one (ignoring that one is in continuous time, and the other in discrete time). We will come back to this in the heuristics background to the proof of Theorem 2.1 (see the beginning of Section 2.3). It goes however beyond the heuristics, being also of technical importance in the proof of Theorem 2.1.*

When Λ is a rectangle, the bootstrapped configuration can be described as follows (see [20], [3]). Say that a family of disjoint subsets of $\mathbb{Z}^2$ is *well separated* if there is no vertex in $\mathbb{Z}^2$ at distance less than or equal to 1 from two sets in the family. For any initial η_0, the bootstrapped configuration takes the value s on the smallest collection of well separated rectangles contained in Λ which contains all the sites x at which $\eta_0(x) = s$. We say that the rectangle $R \subset \Lambda$ is *internally spanned* by the configuration $\eta \in \{u, s\}^\Lambda$, if bootstrap percolation restricted to the volume R, started from $\eta_0 = \eta|_R$, ends up with all sites in R in state s.

The following lemma provides us with a crucial estimate.

Lemma 2.9 (Aizenman-Lebowitz, [3]) *Suppose that $n \in \{1, 2, ...\}$ is fixed, that Λ is a finite rectangle and $\eta \in \{u, s\}^\Lambda$ is a configuration with the property that when we perform $(u \to s)$ bootstrap percolation with threshold 2 in Λ started from η, we obtain a final configuration which includes a rectangle with one of its sides larger than n fully occupied by s. Then there must exist some rectangle $R \subset \Lambda$ with larger side in $\{\lfloor n/2 \rfloor - 1, \dots, n\}$ which is internally spanned by η.*

This lemma may be seen as providing the existence of a "bottleneck event": if a large rectangle full of s occurs in the bootstrapped configuration, then, for arbitrary n between 1 and the size of the larger side of this rectangle, a rectangle of "size of order n" has to be internally spanned. One can then optimize on the choice of n to obtain an upper bound on the probability of large rectangles occurring in the bootstrapped configuration. (The "bottleneck event" is the occurrence of an internally spanned rectangle of "size of the order of the optimal n", i.e., the n which minimizes the probability of this event.) And the point of looking at internally spanned rectangles is that one needs to consider the dynamics inside those rectangles only.

2.2 Proof of Proposition 2.3

We want to analyze the evolution of the process started from the configuration in which the cube $R = \{1, 2, ..., L\}^2$ is occupied by spins -1 and all other spins are $+1$. We will compare this evolution with the one in which the initial configuration has spins -1 in the quadrant $\{1, 2, ...\}^2$, and spins $+1$ at the other sites. We will refer to this comparison system as the quadrant evolution. Note that in the quadrant evolution, when the site (x, y), $x, y \geq 1$ has a spin $+1$, the sites (x', y') with $x' \leq x$ and $y' \leq y$ also have spins $+1$. Let T' be the first time that in the quadrant evolution the site (L, L) has a spin $+1$, and note that in this evolution at time T' the square R is fully occupied by spins $+1$. Therefore, by attractiveness, T' is stochastically larger than T (see Remark 2.6), i.e., for any $t \geq 0$,

$$\mathbb{P}(T > t) \ \leq \ \mathbb{P}(T' > t).$$

In order to estimate T', we use a well-known relation between the quadrant evolution and the exclusion process. In the relevant version of the exclusion process, particles are initially located at each negative site in $\mathbb{Z}$, with the positive sites and the origin being empty. Particles try to jump independently of everything else one step to the right at rate $1/2$ and one step to the left at rate $1/2$. The only interaction among the particles being that a jump is suppressed if a particle is attempting to jump to an occupied site. After accelerating the quadrant evolution by a factor of $1/2$, it can be mapped into the evolution of this exclusion process in the following way, as explained in [40], pp. 411, 412. Let $X_i(t)$ be the position at time t of the exclusion process particle initially at the site $-i$, $i = 1, 2, \ldots$. In the quadrant evolution, the site (x, y), $x, y \geq 1$ is occupied by a spin $+1$ at time t if and only if, in the exclusion process at time t the displacement $X_y(t) + y$ of the particle initially at $-y$ is at least x. In particular,

$$\mathbb{P}(T' > CL^2) \ \leq \ \mathbb{P}(X_L(2CL^2) < 0) \ = \ \mathbb{P}\left(\sum_{i \geq 1} \mathbf{1}\{X_i(2CL^2) \geq 0\} < L\right),$$

where $\mathbf{1}A$ is the indicator of the event A.

If it were not for the exclusion constraint, the quantities $X_i(\cdot)$ in the sum above would be independent simple symmetric random walks, and in that case the result would follow by standard arguments about sums of independent random variables and the single simple symmetric random walk distribution (which we leave for the reader to check). However, under the exclusion constraint, the estimate obtained for independent particles is an upper bound for the latter probability above. This is due to a negative association property of exclusion particles; see Proposition VIII.1.7 in [40]. So, we get the result. □

2.3 Proof of Theorem 2.1

The proof of this theorem will be rather complicated. To make it more readable, we first explain some ideas behind it.

At first sight, one could think that if p is close to 1, so that the spins -1 form initially only finite clusters dispersed in a sea of spins $+1$, the spins $+1$ would easily take over and eliminate the spins -1. The actual subtlety in the behavior of the system, even in this case, may be seen as resulting from the subtlety of the behavior of bootstrap percolation. As pointed out in Remark 2.8, one can roughly see the system as a competition between a $(-1 \rightarrow +1)$ threshold 2 bootstrap percolation process and a $(+1 \rightarrow -1)$ threshold 2 bootstrap percolation process. (We say "roughly" because both bootstrap percolation processes are running in continuous time.) In particular, if the spins -1 did not flip at all, and the spins $+1$ flipped with the rules of our dynamics, then the set of sites which would eventually be in state -1 would be precisely the set resulting from the $(+1 \rightarrow -1)$ threshold 2 bootstrap percolation dynamics applied to the initial configuration. And it is known (see [20][3]) that, contrary to what one might first guess, even when $p < 1$ is very close to 1, this dynamics leads to each site being eventually in state -1. This means that to show that our dynamics in reality leads to each site being eventually in state $+1$, one must show that the significantly larger initial density of spins $+1$ allows them, as they expand via their own bootstrap percolation mechanism, to prevent the -1 spins from expanding.

Motivated by the picture discussed above, it is natural to try to use results and techniques which allow one to control from above the speed with which -1 spins can expand via their bootstrap percolation mechanism. Results of this kind are available from [3], specifically, in the form of Lemma 2.9 (here) and its uses. Once control is gained on how fast -1 spins can expand, one can hope to use the estimate in Proposition 2.3 to show that before the -1's can expand, the spins $+1$ will typically eliminate them.

The need to consider a hierarchy of space and time scales in the analysis (multiscale analysis) arises then as follows. The approach mentioned in the previous paragraph can be used to show that there are space and time scales, l_1 and t_1, so that for regions of size l_1, the spins $+1$ will typically eradicate the spins -1 in a time t_1. But on the infinite lattice $\mathbb{Z}^2$, there will exist rare regions of size l_1 which at time t_1 still contain spins -1. Our approach

[3] and [52], for extensions

relies then on considering a coarser (rescaled) lattice, with rescaled sites corresponding to original blocks of size l_1. One starts then at time t_1 with a very low density of rescaled sites containing spins -1. The arguments from the previous paragraph can be applied at this rescaled level, and produce space and time scales, l_2 and t_2, so that for regions of size l_2 in the rescaled lattice, typically only rescaled sites with no spins -1 are left in a time t_2. The procedure can be iterated, producing rapidly growing sequences of space and time scales. To control the eradication of spins -1 in the k-th scale, one uses induction in k, taking as input the results of the analysis of the $(k-1)$-th scale.

One further idea is needed in the implementation of the scheme sketched above. In order to analyze the behavior of the system at each scale, one needs to know not only that the starting configuration in this step has a low density of rescaled sites which contain spins -1, but one needs also to know that the joint distribution of these rare "bad" rescaled sites is well behaved. To recursively obtain this sort of control, we will be considering a modified dynamics in which spin interaction is allowed (since it is naturally introduced by the dynamics in the original model) but is restricted in range. The comparison between the original dynamics and this modified one will be obtained from attractiveness and from estimates on the speed with which effects can propagate.

We turn now to the specification of the space and time scales that will be used in the analysis, and various related objects.

We will use the notation $q = 1 - p$. For $\epsilon > 0$, and small positive $\varkappa > 0$, to be chosen later, we set

$$q_0 = q, \qquad \ell_0 = 1, \qquad t_0 = 0,$$

and inductively define

$$q_k = e^{-\varkappa/q_{k-1}}, \qquad \ell_k = \left\lfloor \left(\frac{1}{q_{k-1}}\right)^{2+2\epsilon} \right\rfloor, \qquad t_k = \left(\frac{1}{q_{k-1}}\right)^{2+\epsilon}. \tag{2.5}$$

Remark 2.10 *Note that, setting $Q_k = 1/q_k$ for all $k \geq 0$, and $\lambda = e^{\varkappa} > 1$, we have $Q_k = \lambda^{Q_{k-1}}$ for all $k \geq 1$, and thus*

$$q_k = \lambda^{-\lambda^{\cdot^{\cdot^{\lambda^{1/q}}}}}, \qquad \ell_k = \left\lfloor \lambda^{(2+2\epsilon)\lambda^{\cdot^{\cdot^{\lambda^{1/q}}}}} \right\rfloor, \qquad t_k = \lambda^{(2+\epsilon)\lambda^{\cdot^{\cdot^{\lambda^{1/q}}}}}. \tag{2.6}$$

Set also

$$L_k = \ell_0 \cdot \ell_1 \cdot \ldots \cdot \ell_k, \qquad T_k = t_0 + t_1 + \ldots + t_k. \tag{2.7}$$

Now we define the hypercubes of scale k, $k = 0, 1, \ldots$ as

$$B_k^i = \{0, ..., L_k - 1\}^2 + L_k i, \quad i \in \mathbb{Z}^2.$$

Note that the hypercubes of scale 0 are of size 1, that for each k, $\{B_k^i,\ i \in \mathbb{Z}^2\}$ is a partition of $\mathbb{Z}^2$, and that for $k \geq 1$, each hypercube of scale k is the disjoint union of $(l_k)^2$ hypercubes of scale $k-1$.

For $k \geq 1$, with each B_k^i, $i \in \mathbb{Z}^2$, we also associate a larger hypercube, denoted by $\widetilde{B}_k^i$, which roughly speaking is concentric with B_k^i and has side-length $(1+\frac{2}{3})L_k$ (see Fig. 2.1). We will need $\widetilde{B}_k^i$ to be a union of hypercubes of scale $k-1$, and with this in mind we define

$$\widetilde{B}_k^i = (\cup_{j \in \bar{B}_k} B_{k-1}^j) + L_k i, \text{ with } \bar{B}_k = \left\{ -\left\lfloor \frac{1}{3} l_k \right\rfloor, \ldots, l_k + \left\lfloor \frac{1}{3} l_k \right\rfloor \right\}^2. \tag{2.8}$$

For simplicity we abbreviate $B_k = B_k^0$ and $\widetilde{B}_k = \widetilde{B}_k^0$.

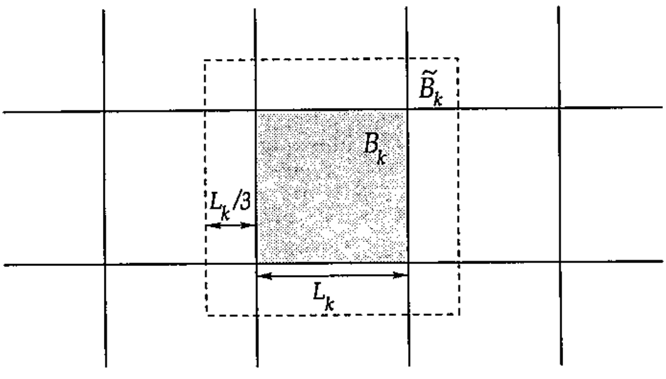

Figure 2.1: Solid lines separate blocks B_k^i, $i \in \mathbb{Z}^2$; dashed lines indicate the block $\widetilde{B}_k$, associated with the block B_k, which is shaded. The outer corridors between the boundaries of $\widetilde{B}_k$ and B_k are of width $L_k/3$.

Theorem 2.1 will be obtained through a comparison of the original system with a modified dynamics, further referred to as *block-dynamics*, coupled to the original dynamics in that it will be constructed on the same probability space, using the same graphical marks, and defined by Rules 1-3 below. The block-dynamics will be so constructed as to have the following properties.

(A) The block-dynamics favors -1 spins, in the sense that at any site and time where the original dynamics has a -1 spin, also the block-dynamics has a -1 spin.

(B) In the block-dynamics at the time T_k all hypercubes of the k^{th} scale will be 'monochromatic', i.e., they will be entirely filled in either by $+1$ or -1 spins. In the former case we will say that the block is in state $+1$, and in the latter case that it is in state -1.

(C) For each $k \geq 1$, the random field η_k that associates to each $i \in \mathbb{Z}^2$ a random variable $\eta_k(i)$ which takes the value $+1$ (resp. -1) if at time T_k the block B_k^i is in state $+1$ (resp. -1) is a 1-dependent random field. This means that for each n, if $i_1, ..., i_n \in \mathbb{Z}^2$ are such that $\text{dist}(x_i, x_j) > 1$ for $i \neq j$, then the random variables $\eta_k(i_1), ..., \eta_k(i_n)$ are independent.

To assure that (C) above is satisfied, in the block-dynamics, the evolution of the spins in each box B_k^i, $i \in \mathbb{Z}^2$, during the interval of time $[T_{k-1}, T_k]$, will depend only on the configuration at time T_{k-1} and the graphical marks inside the corresponding box $\widetilde{B}_k^i$.

To define the block-dynamics so that (A) above is also satisfied, we first introduce a notion of *influence time*, associated with the box $\widetilde{B}_k^i$. Let $(\sigma_{\widetilde{B}_k^i,\zeta;s}^{\xi,T_{k-1}})_{s \geq T_{k-1}}$ be the evolution in the box $\widetilde{B}_k^i$, with boundary condition ζ outside this box, started at time T_{k-1} from the configuration ξ inside the box and obtained using the graphical marks. Set, now,

$$\tau_k^i = \inf\Big\{ s \geq T_{k-1} : \sigma_{\widetilde{B}_k^i,+;s}^{\xi,T_{k-1}}(x) \neq \sigma_{\widetilde{B}_k^i,-;s}^{\xi,T_{k-1}}(x) \text{ for some } x \in B_k^i \text{ and some } \xi \in \{-1,+1\}^{\widetilde{B}_k^i} \Big\}. \tag{2.9}$$

In words, consider two spin configurations which at time T_{k-1} are the same inside $\widetilde{B}_k^i$ (the enlarged hypercube B_k^i), but outside $\widetilde{B}_k^i$ one is all $+$, and the other all $-$. Then τ_k^i is the first time after T_{k-1} when, evolving under the same graphical marks, these two configurations become different inside B_k^i. Note that by attractiveness, evolutions with other boundary conditions will be bounded from above and below by the two evolutions which appear in the definition of τ_k^i. Therefore we can think of τ_k^i as the first (random) time after

T_{k-1}, when spins in B_k^i can suffer any influence from the spins outside of $\widetilde{B}_k^i$ at time T_{k-1}.

In order to define the evolution in the block-dynamics of the spins inside each box B_k^i, $i \in \mathbb{Z}^2$, from T_{k-1} to T_k, we use now the following rules.

Rule 1. During the interval of time $[T_{k-1}, T_k)$ we observe the evolution inside the box $\widetilde{B}_k^i$ with $+1$ boundary conditions. We assign to the spins in the box B_k^i up to time $\min\{\tau_k^i, T_k\}$ the values that we see in that evolution.

Rule 2. If it is the case that $\tau_k^i < T_k$, then at the time τ_k^i all spins in B_k^i will be declared to be -1, and persist at this state without change up to time T_k.

Rule 3. If, following the two rules above, there is any spin in state -1 in B_k^i at times which are arbitrarily close to T_k, then at time T_k all the spins in B_k^i are declared to be in state -1. Otherwise, at time T_k all the spins in B_k^i are declared to be in state $+1$.

It is immediate from *Rules 1-3* that the block-dynamics satisfies properties (A), (B) and (C) above. (Note that for this we need the initial distribution to be 1-dependent, which is the case, since it is a product measure. The properties can then be verified by induction in k.)

Let $\widetilde{q}_k$, $k \geq 0$, denote the probability that at the time T_k the block B_k is in the state -1. Note that $\widetilde{q}_0 = q_0$. The following is our main technical estimate in order to prove Theorem 2.1.

Lemma 2.11 *If q is small enough, then $\widetilde{q}_m \leq q_m$ for all $m \geq 0$.*

The following estimates will be needed in the proof of Lemma 2.11 and Theorem 2.1. Let

$$\hat{q} = \sup\{x > 0 : \text{if } q \in (0, x), \text{ then } q_k \leq q,\ k = 0, 1, \ldots\}.$$

Note that if $q > 0$ is small enough, then $q_1 \leq q_0 = q$ and then, by induction, q_k is decreasing in k. Therefore $\hat{q} > 0$. Several times we will need to take $q \in (0, \hat{q})$, to assure that certain inequalities hold regardless of the value of k. In other words, we will need this assumption in order to assure uniformity over scales in the choice of constants in the multiscale scheme. So every time below when we mention that q is small enough, we in particular mean that $q \in (0, \hat{q})$.

One can show [30] that for arbitrary $\delta > 0$, if q is small enough, then

$$q_{k-1} q_{k-2} \cdots q_1 q_0 \ \geq\ (q_k)^\delta, \quad \text{for } k \geq 1. \tag{2.10}$$

From the definitions of L_k and l_k, it is clear that for small q,

$$L_k \geq l_k \geq \frac{1}{2}\left(\frac{1}{q_{k-1}}\right)^{2+2\epsilon}, \tag{2.11}$$

for $k \geq 1$. For a bound in the opposite direction, we use (2.10) to obtain the following. For arbitrary $\delta > 0$, if q is small enough, then for $k \geq 1$

$$L_k \leq \left(\frac{1}{q_{k-1}q_{k-2}\cdots q_1 q_0}\right)^{2+2\epsilon} \leq \left(\left(\frac{1}{q_k}\right)^{\frac{\delta}{2+2\epsilon}}\right)^{2+2\epsilon} = \left(\frac{1}{q_k}\right)^{\delta}. \tag{2.12}$$

2.3.1 Proof of Lemma 2.11

We use induction on m. The statement is obviously true for $m = 0$. Assume now that it is true for $m = k$, and we will show that $\widetilde{q}_{k+1} \leq q_{k+1}$.

Following Rule 1, we observe the evolution inside the box $\widetilde{B}_{k+1}$ with +1 boundary conditions, during the interval of time $[T_k, T_{k+1})$. Let F_{k+1} be the event that in this evolution −1 spins are present in the box B_{k+1} at times which are arbitrarily close to T_{k+1}. We will show that

$$\mathbb{P}_p(F_{k+1}) \leq \frac{q_{k+1}}{2}. \tag{2.13}$$

We will also show that

$$\mathbb{P}_p(\tau_{k+1} < T_{k+1}) \leq \frac{q_{k+1}}{2}, \tag{2.14}$$

where $\tau_{k+1} = \tau^0_{k+1}$. Combining (2.13) and (2.14) yields the desired inequality $\widetilde{q}_{k+1} \leq q_{k+1}$, since $\widetilde{q}_{k+1} \leq \mathbb{P}_p(F_{k+1}) + \mathbb{P}_p(\tau_{k+1} < T_{k+1})$.

The proof of (2.13) will be divided into two steps. In the first step, we will analyze the random configuration inside $\widetilde{B}_{k+1}$ at the time T_k, and use methods from the study of bootstrap percolation to show that all the −1 spins in this configuration are likely to be contained in a collection of well separated rectangles which are not too large. In the second step, we will analyze the evolution inside the box $\widetilde{B}_{k+1}$ with +1 boundary conditions, during the interval of time $[T_k, T_{k+1})$, conditioned on the configuration at time T_k having that property. In this step, we make use of Proposition 2.3.

The third step in the proof of Lemma 2.11 will be the proof of (2.14).

Step 1. Control of bootstrapping at time T_k. We will cover all the sites in $\widetilde{B}_{k+1}$ which have a spin -1 at time T_k with a collection R_1,R_2,...,R_N of well separated rectangles. (Recall that a family of disjoint subsets of $\mathbb{Z}^2$ is well separated if there is no vertex in $\mathbb{Z}^2$ at distance less than or equal to 1 from two sets in the family.) At time T_k all blocks B_k^i of scale k are monochromatic, i.e., entirely occupied by spins -1 or $+1$, and now they will play the role of "renormalized sites" of $\widetilde{B}_{k+1}$. For the sake of notation, we identify these "renormalized sites" with elements of $\bar{B}_{k+1}$ (see (2.8)). For $i \in \bar{B}_{k+1}$, let $\eta_k(i)$ be $+1$ (resp. -1) if the block B_k^i is in state $+1$ (resp. -1) at time T_k. We obtain first the collection of rectangles $\bar{R}_1$,$\bar{R}_2$,...,$\bar{R}_N$, by applying the $(+1 \to -1)$ threshold 2 bootstrap percolation rule to the random field η_k in $\bar{B}_{k+1}$. Recall that this means that $\bar{R}_1$, $\bar{R}_2$,...,$\bar{R}_N$ is the smallest collection of well separated rectangles in $\bar{B}_{k+1}$ which contain all the renormalized sites $i \in \bar{B}_{k+1}$ which have $\eta_k(i) = -1$. Let now $R_n = \cup_{j\in\bar{R}_n} B_k^j$, $n = 1, ..., N$, and note that $R_1, R_2, ..., R_N$ have the desired properties.

By translation invariance and from the induction hypothesis, for each i

$$\mathbb{P}_p(\eta_k(i) = -1) \ = \ \widetilde{q}_k \leq q_k. \tag{2.15}$$

For $b > 0$ to be chosen later (small enough), we define the following event.

$$E_{k+1} = \left\{ \bar{R}_1,\ \bar{R}_2,\ ...,\ \bar{R}_N \text{ have sides of length at most } \left\lfloor \frac{b}{q_k} \right\rfloor \right\}. \tag{2.16}$$

Our goal in this step is to show that if b and $\varkappa$ are chosen properly (independently of k) and q is small enough, then (2.15) implies

$$\mathbb{P}_p((E_{k+1})^c) \ \leq \ \frac{e^{-\varkappa/q_k}}{4} \ = \ \frac{q_{k+1}}{4}. \tag{2.17}$$

In order to show (2.17), we will use Lemma 2.9, with the choice

$$n \ = \ \left\lfloor \frac{b}{q_k} \right\rfloor. \tag{2.18}$$

We have

$$\begin{array}{l}\text{\# of rectangles inside } \bar{B}_{k+1} \text{ with the length} \\ \text{of the larger side being in } \{\lfloor n/2 \rfloor - 1, \dots, n\}\end{array} \ \leq \ \left(\frac{5}{3}\ell_{k+1}\right)^2 \left(\left\lfloor \frac{b}{q_k} \right\rfloor\right)^2, \tag{2.19}$$

where the first term on the right hand side of (2.19) is an upper bound on the number of choices of positions for the corner of the rectangle with all maximal coordinates, and the second term is an upper bound on the number of choices of sidelengths of the rectangle.

We claim that if $R \subset \bar{B}_k$ is a rectangle of size $n_1 \times n_2$, with $n_1 \leq n_2$, then for the bootstrap percolation process that we are considering,

$$\mathbb{P}_p(R \text{ is internally spanned}) \;\leq\; (2n_1 q_k)^{\lfloor n_2/3 \rfloor}. \tag{2.20}$$

To see this, let $R'_1, R'_2, ..., R'_{n_2}$ be the segments of size n_1 which partition R according to the value of the 2-nd coordinate, and ordered according to increasing values of this coordinate. If R is internally spanned, then for each $1 \leq j \leq n_2 - 1$, $R'_j \cup R'_{j+1}$ must contain some vertex i at which $\eta_k(i) = -1$ (otherwise each of these vertices will have at most 1 neighboring vertex $i' \in R$ with $\eta_k(i') = -1$ and therefore will remain in state $+1$ in the bootstrap percolation process in R that we are considering). We can now use the fact that the random field $(\eta_k(i))_{i\in\mathbb{Z}^2}$ is 1-dependent, to conclude that for $j = 1, 4, 7, 10, ..., 3(\lfloor n_2/3\rfloor - 1) + 1$, the corresponding events are independent. In combination with (2.15), this leads to (2.20), since for each j, $|R'_j \cup R'_{j+1}| = 2n_1$.

Maximizing the right hand side of (2.20) over choices of $n_1 \leq n_2$, with $n_2 \in \{\lfloor n/2 \rfloor - 1, \ldots, n\}$ and n given by (2.18), we obtain

$$\begin{aligned} &\mathbb{P}_p(R \text{ is internally spanned}) \\ &\leq\; (2(n_2)q_k)^{\lfloor n_2/3 \rfloor} \;\leq\; (2b)^{b/(7q_k)} = \exp\left\{\frac{b\,\log(2b)}{7q_k}\right\}, \end{aligned} \tag{2.21}$$

provided $2b < 1$ and q is small enough. Using Lemma 2.9 with the choice made above of n, (2.19) and (2.21), we have

$$\begin{aligned} \mathbb{P}_p((E_{k+1})^c) \;&\leq\; \left(\frac{5b\ell_{k+1}}{3q_k}\right)^2 \exp\left\{\frac{b\,\log(2b)}{7q_k}\right\} \\ &\leq\; \left(\frac{5b}{3(q_k)^{4+2\epsilon}}\right)^2 \exp\left\{\frac{b\,\log(2b)}{7q_k}\right\}. \end{aligned} \tag{2.22}$$

For small $b > 0$, $b\log(2b)$ is negative. So, choosing $b > 0$ which maximizes $-b\log(2b)$ and $\varkappa = -\frac{1}{8}b\log(2b) > 0$, yields (2.17).

Step 2. Erosion of (-1)-rectangles. We will show in this step that, for q small enough,

$$\mathbb{P}_p(F_{k+1}|E_{k+1}) \;\leq\; \frac{q_{k+1}}{4}. \tag{2.23}$$

In combination with (2.17), this implies (2.13),

$$\mathbb{P}_p(F_{k+1}) \leq \mathbb{P}_p(F_{k+1}|E_{k+1}) + \mathbb{P}_p((E_{k+1})^c).$$

By the Markov property, we need to estimate the probability that starting at time T_k from a configuration in $\widetilde{B}_{k+1}$ compatible with the event E_{k+1} and letting the system evolve with $+1$ boundary conditions, some spin -1 will be present at time T_{k+1}. By attractiveness, an upper bound can be obtained on this probability by starting the evolution inside $\widetilde{B}_{k+1}$ at time T_k with -1 spins at all sites of the rectangles $R_1, ..., R_N$ described in Step 1. Clearly, no spin -1 can appear in this evolution outside of the rectangles $R_1, ..., R_N$. Also, once a rectangle is "destroyed", meaning that it contains no -1 spins, no -1 spins will ever be created inside of it again, in the evolution that we are considering here.

If E_{k+1} occurs, then each rectangle R_n, $n = 1, ..., N$, is contained in a cube of sidelength bounded above by

$$\left\lfloor \frac{b}{q_k} \right\rfloor L_k \;\leq\; \left(\frac{1}{q_k}\right)^{1+\epsilon/4},$$

for small q, where we used (2.12). By attractiveness, the time needed to erode R_n is therefore stochastically bounded above by the time needed to erode a cube with sidelength equal to this upper bound. From Proposition 2.3 we conclude then that, for small q, for each $n = 1, ..., N$, the probability that at time $T_{k+1} = T_k + t_{k+1} = T_k + (1/q_k)^{(2+\epsilon)}$ there is any spin -1 inside R_n is bounded above by

$$\exp\left(-\gamma\left(\frac{1}{q_k}\right)^{1+\epsilon/4}\right),$$

where γ is a positive constant. But, clearly, for small q,

$$N \;\leq\; \left|\widetilde{B}_{k+1}\right| \;\leq\; \left(\frac{5}{3}L_{k+1}\right)^2 \;\leq\; \frac{1}{q_{k+1}},$$

by (2.12). From the bounds in the last two displays, we obtain

$$\mathbb{P}_p(F_{k+1}|E_{k+1}) \;\leq\; \frac{1}{q_{k+1}} \exp\left(-\gamma\left(\frac{1}{q_k}\right)^{1+\epsilon/4}\right) \leq \frac{q_{k+1}}{4},$$

for small q. This completes the proof of (2.23), and hence of (2.13).

Step 3. Control of the outer influence. In this step we will prove (2.14).

We begin with the notion of a *discrepancy process*. Consider two evolutions $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}$ and $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}$, $t \geq T_k$, starting at time T_k from the same configuration ξ in $\widetilde{B}_{k+1}$, with (+) and (−) boundary condition outside $\widetilde{B}_{k+1}$, respectively, and using the same graphical marks. We say that at time $t \geq T_k$ there is a *discrepancy* at a vertex $x \in \widetilde{B}_{k+1}$ if $\sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}(x) \neq \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}(x)$, for some ξ. Otherwise, the vertex x is called an *agreement* vertex.

According to this terminology and our choice of the conditions for the dynamics at T_k, all vertices in $\widetilde{B}_{k+1}$ at time T_k are agreement vertices and all vertices outside $\widetilde{B}_{k+1}$ have discrepancies. Moreover, the influence time τ_{k+1} is the time of occurrence of the first discrepancy in B_{k+1}.

We observe that the occurrence of a discrepancy at time s at vertex $z \in \widetilde{B}_{k+1}$ implies, that at time $s-$ there must be at least one neighboring vertex of z occupied by a discrepancy. This follows readily from the fact that the two evolutions that we are considering use the same graphical marks. For $x \in \widetilde{B}_{k+1}$, we define

$$T(x) \;=\; \inf\left\{ s \geq T_k :\; \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},+;t}(x) \neq \sigma^{\xi,T_k}_{\widetilde{B}_{k+1},-;t}(x) \text{ for some } \xi \right\}.$$

We recall that a (self-avoiding) path π on $\mathbb{Z}^2$ is a sequence of distinct vertices $\pi = \{x_1, x_2, \ldots, x_n\}$, such that $||x_i - x_{i+1}|| = 1$, $1 \leq i < n$. A *chronological path* in the epoch (T_k, T_{k+1}) is a path $\pi = \{x_1, x_2, \ldots, x_n\}$ such that

$$T_k \;<\; T(x_1) \;<\; T(x_2) \;<\cdots<\; T(x_n) \;<\; T_{k+1}.$$

In this case we say that the chronological path starts at x_1 and ends at x_n.

Let us introduce some notation. For any $\Lambda \subset \mathbb{Z}^2$, let

$$\partial\Lambda \;=\; \{x \in \Lambda :\; \text{there is a nearest neighbor of } x \text{ not in } \Lambda\}; \quad (2.24)$$

$$\bar{\partial}\Lambda \;=\; \{x \notin \Lambda :\; \text{there is a nearest neighbor of } x \text{ in } \Lambda\}. \quad (2.25)$$

Now we claim that if some vertex $x \in \partial B^i_{k+1}$ at time $t > T_k$ is occupied by a discrepancy, then there exists a chronological path in the epoch (T_k, t) which starts at some vertex of $\bar{\partial}\widetilde{B}^i_{k+1}$ and ends at $x \in \partial B^i_{k+1}$. Once this is realized, the proof of (2.14) is completed by estimating the probability of such an event.

To prove the above claim we will construct a chronological path ending at $x \in \partial B^i_{k+1}$ by moving backwards in time. Assume that $x = \tilde{x}_1 \in \partial B^i_{k+1}$

and $T(\tilde{x}_1) < t$. So at time $T(\tilde{x}_1)$ there must exist at least one neighboring vertex $\tilde{x}_2 \in \widetilde{B}^i_{k+1}$ of x_1 occupied by a discrepancy with $T(\tilde{x}_2) < T(\tilde{x}_1)$. At time $T(\tilde{x}_2)$ there must exist at least one neighboring vertex $\tilde{x}_3 \in \widetilde{B}^i_{k+1}$ of $\tilde{x}_2$, occupied by a discrepancy with $T(\tilde{x}_3) < T(\tilde{x}_2)$. Note that $\tilde{x}_3 \neq \tilde{x}_1$. It also follows that $T(\tilde{x}_3) < T(\tilde{x}_2)$. Now we iterate the procedure, thus producing a self-avoiding path moving backwards in time, such that $\tilde{x}_j \neq \tilde{x}_i$ for $j \neq i$ and $T(\tilde{x}_j) < T(\tilde{x}_{j-1}) < \cdots < T(\tilde{x}_2) < T(\tilde{x}_1) < t$.

This time-reversed path can be made to end in a site x_r in $\bar{\partial}\widetilde{B}^i_{k+1}$ after finitely many steps. For the finiteness of the number of steps, notice that during the epoch (T_k, t), with $t < T_{k+1}$, only finitely many Poisson marks occur in the interior of $\widetilde{B}^i_{k+1}$ $\mathbb{P}$-a.s. Due to the coupling we are using, a discrepancy at some vertex cannot be created without the presence of at least one other discrepancy at some neighboring vertex and at time T_k, the discrepancies are located only outside $\widetilde{B}^i_{k+1}$. So our path must be traced back to some $x_r \in \bar{\partial}\widetilde{B}^i_{k+1}$.

To get a forwards in time path, we invert the order and set $x_j = \tilde{x}_{r+1-j}$, $1 \leq j \leq r$, and obtain a chronological path starting at $x_1 \in \bar{\partial}\widetilde{B}^i_{k+1}$ and ending at $x_r \in \partial B^i_{k+1}$.

From the claim it follows that, if $\tau_{k+1} \leq T_{k+1}$, then there exists a chronological path connecting $\bar{\partial}\widetilde{B}_{k+1}$ to ∂B_{k+1}, thus covering a distance which, for small q, is at least

$$r_k = \left\lfloor \frac{1}{4} L_{k+1} \right\rfloor \geq \frac{1}{9}\left(\frac{1}{q_k}\right)^{2+2\epsilon},$$

where in the last inequality we used (2.11).

The number of possible starting points for this chronological path equals

$$\left|\bar{\partial}\widetilde{B}_{k+1}\right| \leq C L_{k+1} \leq \frac{C}{q_{k+1}},$$

for small q, where C is a constant, and we used (2.12) in the last inequality.

On the other hand, the probability that any given path of length r is a chronological path during the epoch (T_k, T_{k+1}) is clearly bounded above by $\mathbb{P}(Z \geq r-1)$, where Z is a Poisson random variable with mean $T_{k+1} - T_k = t_{k+1} = (q_k)^{-(2+\epsilon)}$. Using the standard large deviation estimate for Poisson random variables (see, e.g., (A.2) in [34], p.467) $\mathbb{P}\{Z \geq r-1\} \leq e^{-(\log(r/t_{k+1})-1)r}$, together with the upper bound 4^r on the number of self-avoiding paths of length r starting from a given vertex, we obtain the following estimate. If q is chosen small enough, then for appropriate constants C' and C'', for any $k \geq 1$, $\mathbb{P}_p(\tau_{k+1} \leq T_{k+1})$ is bounded above by

$$\sum_{r \geq r_k} \frac{C}{q_{k+1}} 4^r \, e^{-2r \log 4} \;\leq\; \frac{C'}{q_{k+1}} \exp\left(-C'' \left(\frac{1}{q_k}\right)^{2+2\epsilon}\right) \;\leq\; \frac{q_{k+1}}{2},$$

where we used the fact that r/t_{k+1} can be made arbitrarily large, uniformly in k and $r \geq r_k$, by taking q small enough.

This proves (2.14), and completes the proof of Lemma 2.11. □

2.3.2 Conclusion of proof of Theorem 2.1

Set $a = 1/(2+\epsilon)$ and note for later use that for any $\epsilon > 0$

$$a > \frac{1}{2} - \epsilon. \tag{2.26}$$

We will first prove that if q is small enough, then there are $C_1, C_2 \in (0, \infty)$ such that

$$\mathbb{P}_p(\sigma_t(0) = -1) \;\leq\; C_1 e^{-C_2 t^a}, \tag{2.27}$$

for $t > 0$.

We consider first times of the form $t = T_k$ for some $k \geq 1$. From (2.5) it follows that $q_k = e^{-\varkappa (t_k)^a}$. Comparing the original dynamics with the block-dynamics at time T_k (recall Property (A) of the block dynamics, and note that the origin is in the block B_k) and using Lemma 2.11, we get

$$\mathbb{P}_p(\sigma_t(0) = -1) \;=\; \mathbb{P}_p(\sigma_{T_k}(0) = -1) \;\leq\; \tilde{q}_k \;\leq\; q_k \;=\; e^{-\varkappa (t_k)^a}. \tag{2.28}$$

In order to replace t_k with T_k in the exponent, we observe that $T_1 = t_1$, and there exists $0 < c < 1$, such that, for small q, for all $k \geq 2$,

$$\frac{t_{k-1}}{t_k} = \left(\frac{q_{k-1}}{q_{k-2}}\right)^{2+\epsilon} = \left(\frac{e^{-\varkappa/(q_{k-2})}}{q_{k-2}}\right)^{2+\epsilon} \leq c, \tag{2.29}$$

and thus

$$T_k \;=\; t_1 + \cdots + t_k \;\leq\; t_k\big(1 + c + c^2 + \dots\big) \;=\; (1-c)^{-1} t_k. \tag{2.30}$$

So, for $t = T_k$, from (2.28) and (2.29) we get

$$\mathbb{P}_p(\sigma_t(0) = -1) \le e^{-\varkappa(t_k)^a} \le e^{-\varkappa((1-c)T_k)^a} = e^{-C(T_k)^a} = e^{-Ct^a}, \quad (2.31)$$

where we take $C = \varkappa(1-c)^a$.

In order to extend the result to all positive times $t \ge 0$ we will be comparing evolutions started from product measures with different values of q. For this purpose, we write $q_k(q)$, $t_k(q)$ and $T_k(q)$ for the corresponding values of q_k, t_k and T_k defined by (2.5) and (2.7) with $q_0 = q$. Summarizing the conclusion in (2.31), we know that there exists $\widetilde{q} \in (0, \hat{q})$ and $C > 0$, such that for all $q \in (0, \widetilde{q}]$, if for some $k \ge 1$, $t = T_k(q)$, then

$$\mathbb{P}_p(\sigma_t(0) = -1) \le e^{-Ct^a}. \quad (2.32)$$

We will write $\widetilde{q}_k = q_k(\widetilde{q})$, $\widetilde{t}_k = t_k(\widetilde{q})$ and $\widetilde{T}_k = T_k(\widetilde{q})$. Since $\widetilde{q} \in (0, \hat{q})$, $\widetilde{q}_k$ decreases with k, and therefore $\widetilde{t}_k$ increases with k.

Note that, for each fixed $k \ge 1$, if we imagine the parameter q decreasing continuously from $\widetilde{q}$ to $\widetilde{q}_1$, we will have the corresponding $T_k(q)$ increasing continuously from $T_k(\widetilde{q}) = \widetilde{T}_k$ to $T_k(\widetilde{q}_1) = t_1(\widetilde{q}_1) + ... + t_k(\widetilde{q}_1) = t_2(\widetilde{q}) + ... + t_{k+1}(\widetilde{q}) = T_{k+1}(\widetilde{q}) - t_1(\widetilde{q}) = \widetilde{T}_{k+1} - \widetilde{t}_1$. Thus, by continuity, any $t > 0$ which is not in $\cup_{k\ge1}[\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k)$, can be written as $t = T_{k(t)}(q(t))$, for some $k(t) \ge 1$ and some $q = q(t) \in (\widetilde{q}_1, \widetilde{q}]$. Set $p(t) = 1 - q(t)$. Then for any $q \in (0, \widetilde{q}_1)$ we have $p = 1 - q \ge 1 - \widetilde{q}_1 \ge 1 - q(t) = p(t)$. Therefore, by attractiveness and (2.32), we have

$$\mathbb{P}_p(\sigma_t(0) = -1) \le \mathbb{P}_{p(t)}(\sigma_t(0) = -1) \le e^{-Ct^a}. \quad (2.33)$$

This establishes the validity of (2.27) for $q < \widetilde{q}_1$ and $t > 0$ which is not in $\cup_{k\ge1}[\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k)$. To extend the result to t in this excluded set, observe that for each k and $t \in [\widetilde{T}_k - \widetilde{t}_1, \widetilde{T}_k]$, if $\sigma_t(0) = -1$ and the spin at the origin does not flip between times t and $\widetilde{T}_k$, then $\sigma_{\widetilde{T}_k}(0) = -1$. Using the Markov property, we obtain then

$$\mathbb{P}_p(\sigma_t(0) = -1) \le \frac{1}{e^{-\widetilde{t}_1}} \mathbb{P}_p(\sigma_{\widetilde{T}_k}(0) = -1) \le C' e^{-C''(\widetilde{T}_k)^a} \le C' e^{-C''t^a}, \quad (2.34)$$

where the term $e^{-\widetilde{t}_1}$ is a lower bound on the probability of the event that no flips occur at the origin from t to $\widetilde{T}_k$. This completes the proof of (2.27).

To derive (2.2) from (2.27), we first note that an argument similar to the one used to derive the first inequality in (2.34) (but now using the strong Markov property) gives, for $t \geq 1$,

$$\mathbb{P}_p(\sigma_s(0) = -1 \text{ for some } s \in [t-1,\, t)) \;\leq\; e\,\mathbb{P}_p(\sigma_t(0) = -1) \;\leq\; e\,C_1 e^{-C_2 t^a}.$$

Consequently

$$\begin{aligned}
&\mathbb{P}_p(\sigma_s(0) = -1 \text{ for some } s \geq t) \\
\leq\;& \sum_{n=0}^{\infty} \mathbb{P}_p(\sigma_s(0) = -1 \text{ for some } s \in [t+n, t+n+1)) \\
\leq\;& \sum_{n=0}^{\infty} e\,C_1 e^{-C_2(t+n+1)^a} \;\leq\; e\,C_1 \int_t^{\infty} e^{-C_2 s^a}\, ds. \qquad (2.35)
\end{aligned}$$

Observe now that for arbitrary $\delta > 0$,

$$\int_t^{\infty} e^{-C_2 s^a}\, ds \leq e^{-C_2 t^{a-\delta}},$$

for large t. To see this, note that $e^{-C_2 s^a} \leq C_2(a-\delta)s^{a-\delta-1}e^{-C_2 s^{a-\delta}}$, for large s, and integrate both sides from t to ∞.

From the last two displays, we obtain

$$\mathbb{P}_p(\sigma_s(0) = -1 \text{ for some } s \geq t) \leq e\,C_1\, e^{-C_2 t^{a-\delta}} \leq e^{-t^{a-2\delta}},$$

for large t.

This completes the proof of (2.2), since from (2.26) we have $a - 2\delta > (1/2) - \epsilon$, for small $\delta > 0$. □

Part II

Chaotic time dependence

Chapter 3

Voter model with random rates

In this chapter we come back to the voter model introduced in Section 1.2. We will be interested now in the absence of weak convergence as time diverges of the dynamics with fixed initial configuration chosen from a translation invariant product symmetric Bernoulli measure[1]

To observe such a phenomenon, which we call chaotic time dependence (for reasons that will become clear), we introduce random rates, as follows. We will have the rates $\{\lambda_i,\, i \in \mathbb{Z}^d\}$ be a family of independent and identically distributed random variables. We then have what we call the voter model with random rates (VMRR). For convenience, we will focus rather on the inverses of the rates $\tau := \{\tau_i,\, i \in \mathbb{Z}^d\}$, where $\tau_i = \lambda_i^{-1}$. τ is then a family of independent and identically distributed random variables with the property that

$$\mathbb{P}(\tau_0 \geq 1) = 1 \tag{3.1}$$

(coming from (1.10); in fact, we have that, for *every* realization of τ, $\tau_i \geq 1$ for every i).

The initial configuration, which is a product of symmetric Bernoullis, will be denoted ξ.

Definition 3.1 *For given realizations of the initial configuration of spins ξ and configuration of rates τ, we say that σ_t exhibits* chaotic time dependence *(CTD) if it does* not *converge weakly as $t \to \infty$ (i.e., if some finite dimensional distribution — or correlation function — of σ_t does not converge to a single limit as $t \to \infty$).*

[1] As argued at the end of Section 1.2, when integrated with respect to such measure, the voter model does converge weakly as time diverges.

Remark 3.2 *CTD may be seen as the nonequilibrium counterpart of the equilibrium phenomenon of* chaotic size dependence, *which is the non convergence of finite volume Gibbs measures as volume goes to infinity for disordered parameter/boundary conditions spin systems (like spin glasses, but also ferromagnetic systems with random boundary conditions). See [42, 43, 44, 21].*

Remark 3.3 *The results below raise the issue of whether CTD occurs in other more physical spin systems in $d \geq 2$ such as the Edwards-Anderson spin glass [19] and the homogeneous Ising ferromagnet with random initial conditions, under appropriate dynamics.*

We now state our result on the occurrence of CTD.

Theorem 3.4 *In $d = 1$, if the distribution of τ_0 is such that there exist constants $0 < c, c' < \infty$ and $0 < \alpha < 1$ for which*

$$ct^{-\alpha} \leq \mathbb{P}(\tau_0 > t) \leq c't^{-\alpha} \tag{3.2}$$

for all large t, then the VMRR exhibits chaotic time dependence for almost every ξ and τ.

Remark 3.5 *In [24], further results, all establishing weak convergence in various degrees of strength, are derived. It is shown, for example, that if τ_0 is a little more than integrable, then the VMRR converges weakly for almost every τ and ξ in every dimension. The same holds in $d \geq 3$, if a positive real moment of τ_0 exists. In $d = 2$, the situation is more delicate; under the same condition on τ as in Theorem 3.4, with $\alpha > 0$, weak convergence is proved, but not for almost every τ and ξ, rather in probability with respect to these families of random variables (see [24] for more details).*

The argument for Theorem 3.4 is based on the dual representation of the voter model in terms of coalescing random walks (1.11) seen in Section 1.2. In this context, we will call $\{X_t(i),\, t \geq 0,\, i \in \mathbb{Z}^d\}$ coalescing random walks with random rates (CRWRR).

Remark 3.6 *We note, for use in the proofs below, that for each i fixed, $X_t(i)$, $t \geq 0$, is an ordinary simple symmetric random walk in $\mathbb{Z}^d$ (with inhomogeneous rates), which we call in this context a random walk with random rates (RWRR). We will mostly denote $X_t(0)$ more compactly by X_t. We will also consider the discrete-time random walk embedded in X_t, denoted by $\tilde{X}_n$.*

In the proof of Theorem 3.4 in the next section, we use the representation (1.11) to show that the expected value of the spin at the origin at time t, given τ and σ_0, does not converge as $t \to \infty$. As noted in Remark 3.6, the object that comes into play is the random walk X_t. The quantity we will study, already using the representation (1.11), is thus

$$\mathbb{E}[\sigma_t(0)|\tau, \sigma_0 = \xi] = \mathbb{E}[\xi_{X_t}|\tau, \xi] = \sum_{i \in \mathbb{Z}^d} \xi_i \, \mathbb{P}(X_t = i|\tau), \tag{3.3}$$

where τ refers to the rate configuration and ξ is the initial spin configuration of the VMRR.

Remark 3.7 *The almost sure CTD of Theorem 3.4 follows from the following properties of the RWRR, to be argued below. For almost every τ, for all i fixed*

$$\lim_{t\to\infty} \mathbb{P}(X_t = i|\tau) = 0, \quad \textit{but} \quad \limsup_{t\to\infty} \sup_i \mathbb{P}(X_t = i|\tau) > 0. \tag{3.4}$$

The mechanism behind these facts, as we will show below, is the localization *or* concentration *at large times of $\mathbb{P}(X_t = \cdot|\tau)$ in sites where τ is large. The confinement of $\mathbb{Z}$ makes it predictable (with probability bounded away from 0) which large τ sites X_t is located in.*

Remark 3.8 *As we will see in next chapter, this localization phenomenon is related to an aging phenomenon exhibited by the RWRR.*

3.1 Proof of Theorem 3.4

In what follows we will express the time spent by X_t at a given site i between jumps by $\tau_i T$, where T is an exponential random variable with mean one. In this way, we have three independent families of random variables involved in X_t. The embedded walk $\tilde{X}$, τ and a sequence of i.i.d. exponentials of mean one, $T_1, T_2, \ldots$

We next state a result about the scaling of the time spent by X_t in the first n jumps, denoted S_n. We can write

$$S_n = \sum_{i=1}^{n} \tau_{\tilde{X}_i} T_i. \tag{3.5}$$

Let $\gamma = \frac{1}{2\alpha} + \frac{1}{2}$.

Lemma 3.9 *For $d = 1$ and $\alpha < 1$, suppose that the distribution of τ_0 satisfies the right hand side inequality of (3.2) for some finite constant c' and all large enough t. Then S_n/n^γ is tight; i.e.,*

$$\lim_{B\to\infty} \mathbb{P}\left(S_n > Bn^\gamma\right) = 0 \quad \textit{uniformly in } n. \tag{3.6}$$

The proof of Lemma 3.9 is given later in this section. We next state another lemma that will be used together with Lemma 3.9 to prove Theorem 3.4; its proof is given at the end of the section.

Lemma 3.10 *To prove Theorem 3.4, it is sufficient to show that*

$$\mathbb{P}\left(\limsup_{t\to\infty}\sup_i \mathbb{P}(X_t = i|\tau) > 0\right) = 1. \tag{3.7}$$

Proof of Theorem 3.4.

Let $I = I(n,\tau)$ be the leftmost integer i in $[-\sqrt{n},\sqrt{n}]$ where τ_i achieves its maximum. That is,

$$I = \min\{i : -\sqrt{n} \le i \le \sqrt{n} \text{ and } \tau_i = \max_{-\sqrt{n}\le j\le\sqrt{n}} \tau_j\}. \tag{3.8}$$

Let B and ϵ be positive numbers. We will first show that

$$\lim_{B\to\infty}\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\mathbb{P}\left(\frac{1}{Bn^\gamma}\int_0^{Bn^\gamma} 1\{X_s = I\}\,ds > \epsilon\right) = 1, \tag{3.9}$$

and from this conclude that (3.7) holds.

The above probability is bounded from below by

$$\mathbb{P}\left(\frac{1}{n^\gamma}\int_0^{S_n} 1\{X_s = I\}\,ds > B\epsilon\right) - \mathbb{P}\left(S_n > Bn^\gamma\right). \tag{3.10}$$

By Lemma 3.9, the last probability in (3.10) can be made arbitrarily small uniformly in n by choosing B large. Thus

$$\limsup_{B\to\infty}\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\mathbb{P}\left(S_n > Bn^\gamma\right) = 0. \tag{3.11}$$

The left hand side of the expression inside the first probability in (3.10) (which we will later denote by U) equals

$$\frac{1}{n^\gamma}\tau_I G(L_{n,I}) \tag{3.12}$$

in distribution, where $L_{n,k}$ denotes the local time of $\tilde{X}$ up to time n at site k, that is

$$L_{n,k} = \sum_{i=0}^{n} 1\{\tilde{X}_i = k\}, \tag{3.13}$$

and, given $L_{n,k} = m$, $G(m)$ is a gamma random variable with mean and variance m, which is independent of τ and of $L_{n,k}$. Now (3.12) is the product of

$$\frac{\tau_I}{(\sqrt{n})^{\frac{1}{\alpha}}} \quad \text{and} \quad \frac{G(L_{n,I})}{\sqrt{n}}, \tag{3.14}$$

which are asymptotically strictly positive in probability (this is easy to check for the former random variable from the left hand side inequality of (3.2); for the latter, it follows from Theorems 9.13 and 10.1 in [49] and from the right hand side inequality of (3.2)). Thus

$$\liminf_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{n^\gamma}\int_0^{S_n} 1\{X_s = I\}\, ds > B\epsilon\right) = 1 \quad . \tag{3.15}$$

for all B, and (3.9) follows from (3.15) and (3.11), via (3.10).

Next it follows from (3.9) by fairly standard arguments that

$$\lim_{B\to\infty} \liminf_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{Bn^\gamma}\int_0^{Bn^\gamma} \mathbb{P}(X_s = I|\tau)\, ds > \epsilon\right) = 1. \tag{3.16}$$

To see this, replace the ϵ in (3.9) by $\epsilon/(1-\sqrt{\delta})$ for some small δ and use the fact that

$$\mathbb{P}\left(U \geq \frac{\epsilon}{1-\sqrt{\delta}}\right) \geq 1-\delta \quad \text{implies} \quad \mathbb{P}\left(E[U|\tau] \geq \epsilon\right) \geq \sqrt{\delta}.$$

The estimate (3.16) now implies that

$$\lim_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{n}\int_0^{n} \sup_i \mathbb{P}(X_s = i|\tau)\, ds > \epsilon\right) = 1 \tag{3.17}$$

and thus that

$$\lim_{\epsilon\to 0} \lim_{t\to\infty} \mathbb{P}\left(\sup_{s>t} \sup_i \mathbb{P}(X_s = i|\tau) > \epsilon\right) = 1, \tag{3.18}$$

which implies the condition (3.7) of Lemma 3.10. □

It remains to prove Lemmas 3.9 and 3.10.

Proof of Lemma 3.9.

We will argue that

$$\lim_{\lambda\to 0+} \liminf_{n\to\infty} \mathbb{E}\left(\exp\left\{-\frac{\lambda}{n^\gamma} S_n\right\}\right) = 1. \tag{3.19}$$

This immediately implies the desired result.

We rewrite the expectation in (3.19) as

$$\mathbb{E}\left[\mathbb{E}\left(\exp\{-\lambda S_n/n^\gamma\}|\tau, \tilde{X}\right)\right]. \tag{3.20}$$

From (3.5) and Jensen's inequality, the expectation inside the brackets can be bounded below by

$$\exp\left\{-\frac{\lambda}{n^\gamma}\sum_{i=1}^{n} \tau_{\tilde{X}_i}\right\}. \tag{3.21}$$

The expectation of (3.21) can be expressed as

$$\mathbb{E}\left[\mathbb{E}\left(\exp\left\{-\frac{\lambda}{n^\gamma}\sum_k \tau_k L_{n,k}\right\}\middle| \tilde{X}\right)\right] \tag{3.22}$$

and the expectation inside brackets in (3.22) equals

$$\prod_k \left\{1 - \mathbb{E}\left(1 - \exp\{-\ell\tau_0\}|\tilde{X}\right)\right\}, \tag{3.23}$$

where $\ell = \ell(n,k) = \lambda L_{n,k}/n^\gamma$.

It follows from the right hand side inequality of (3.2) that there exists a finite constant c'' such that the last expectation can be bounded above by $c''\ell^\alpha$. Using the facts that given $\delta' > 0$, $1 - x_k \geq \exp\{-(1+\delta')x_k\}$ for small enough nonnegative x_k and that

$$\sup_k L_{n,k}/n^\gamma \leq \sum_k L_{n,k}/n^\gamma = n^{1-\gamma} \to 0 \tag{3.24}$$

as $n \to \infty$ (since $\gamma > 1$), we conclude that (3.22) is bounded below by

$$\mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{n^{\gamma\alpha}}\sum_k L_{n,k}^\alpha\right\}\right], \tag{3.25}$$

where c''' is a finite constant.

By the Law of the Iterated Logarithm for $\tilde{X}$ and the fact that $L_{n,k}$ can be approximated by the Brownian local time, denoted $\mathcal{L}_{n,k}$, within a margin of error of $n^{1/4}$ ([49], Theorem 10.1), (3.25) becomes

$$\mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{\sqrt{n}}\sum_k(\mathcal{L}_{n,k}/\sqrt{n})^\alpha\right\}\right] = \mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{\sqrt{n}}\sum_k \mathcal{L}_{1,k/\sqrt{n}}^\alpha\right\}\right] \tag{3.26}$$

plus an $o(1)$ error term. (The identity in (3.26) follows by a simple change of variables.) Now

$$\frac{1}{\sqrt{n}}\sum_k \mathcal{L}_{1,k/\sqrt{n}}^\alpha \to \int \mathcal{L}_{1,x}^\alpha dx < \infty \tag{3.27}$$

almost surely as $n \to \infty$, since $\mathcal{L}_{1,x}$ is continuous and with bounded support almost surely. (3.19) follows by dominated convergence. □

Remark 3.11 *Arguments like those in the proof of Lemma 3.9 show that under the assumption of the left hand side inequality of (3.2), we have*

$$\lim_{\lambda\to\infty}\limsup_{n\to\infty}\mathbb{E}(\exp\{-\lambda S_n/n^\gamma\}) = 0. \tag{3.28}$$

This and (3.19) together imply that if both sides of (3.2) are valid, then $\{S_n/n^\gamma, n \geq 1\}$ *is a tight sequence and every weak limit is supported on* $(0,\infty)$. *We further note that if* c *and* c' *can be taken arbitrarily close in (3.2) (that is, if* $t^\alpha\mathbb{P}(\tau_0 > t)$ *has a positive finite limit as* $t \to \infty$*), then* S_n/n^γ *converges weakly as* $n \to \infty$.

Proof of Lemma 3.10.

We will argue that (3.7) implies that for almost every τ and ξ, $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ does not converge as $t \to \infty$. We begin by showing, through a renewal theory argument, that (3.4) holds. It is sufficient to show that for all i and τ

$$\lim_{t\to\infty}\mathbb{P}(X_t = i|X_0 = i,\tau) = 0. \tag{3.29}$$

Indeed, denoting the first hitting time of i (starting from 0) by $\mathcal{H}_i$, we have

$$\mathbb{P}(X_t = i|\tau) = \int_0^t \mathbb{P}(X_{t-s} = i|X_0 = i, \tau)\mathbb{P}(\mathcal{H}_i \in ds) \tag{3.30}$$

and (3.4) follows from (3.29) through (3.30).

Now the probability in (3.29), which we now denote by $H(t)$, satisfies the following renewal equation:

$$H(t) = \mathbb{P}(\tau_i T > t|\tau) + \int_0^t H(t-s)\, dF(s), \tag{3.31}$$

where T is an exponential random variable with mean one and F is the distribution function of the sum of the independent random variables $\tau_i T$ and $\mathcal{H}_i'$ (conditional on τ), the latter being the return time to i of X_t (after leaving i).

We claim that $\mathbb{E}(\mathcal{H}_i'|\tau) = \infty$ for all τ. This can be argued by coupling (in the natural way) $\mathcal{H}_i' = \mathcal{H}_i'(\tau)$ and $\mathcal{H}_i'(\tilde{\tau})$, with $\tilde{\tau} \equiv 1$, so that $\mathcal{H}_i'(\tau) \geq \mathcal{H}_i'(\tilde{\tau})$. But $\mathcal{H}_i'$ is the return time in the homogeneous continuous time simple symmetric random walk on $\mathbb{Z}$ which is well known to have infinite mean. Going back to (3.31), since $\mathbb{P}(\tau_i T > t|\tau)$ is (directly) Riemann integrable, the Renewal Theorem ([22], Chapter XI) applies and (3.29) follows.

Returning now to $\mathbb{E}[\sigma_t(0)|\tau, \xi]$ and using the representation (1.11), we have (as previously noted in (3.3)) that

$$\mathbb{E}[\sigma_t(0)|\tau, \xi] = \sum_i \xi_i \mathbb{P}(X_t = i|\tau). \tag{3.32}$$

(3.4) now implies that, for every τ, the convergence of $\mathbb{E}[\sigma_t(0)|\tau, \xi]$ is in the tail sigma-field of the variables in ξ and thus, by the Kolmogorov 0-1 Law, is a trivial event. In other words, if

$$A_1 = \{\mathbb{E}[\sigma_t(0)|\tau, \xi] \text{ converges}\}, \tag{3.33}$$

then $\mathbb{P}(A_1|\tau) = 0$ or 1 for every τ.

Now let

$$A_2 = \{\mathbb{P}(A_1|\tau) = 1\}. \tag{3.34}$$

Notice that A_2 involves τ only. Our aim is to prove that, as a consequence of (3.7), $\mathbb{P}(A_2) = 0$. On A_2, $\mathbb{E}[\sigma_t(0)|\tau, \xi]$ converges to a constant for almost every ξ, again by triviality. The constant has to be 0, since $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ is uniformly bounded (between -1 and 1) and $\mathbb{E}[\sigma_t(0)1_{A_2}|\tau] = 1_{A_2}\mathbb{E}[\sigma_t(0)|\tau] = 0$ for every τ.

The uniform boundedness now implies that on A_2,

$$\mathbb{E}\left\{\mathbb{E}^2[\sigma_t(0)|\tau,\xi]\middle|\,\tau\right\} \to 0 \tag{3.35}$$

as $t \to \infty$. But the left hand side of (3.35) equals

$$\sum_i \mathbb{P}^2(X_t = i|\tau), \tag{3.36}$$

which is greater than or equal to $\sup_i \mathbb{P}^2(X_t = i|\tau)$. Thus (3.35) implies that on A_2, $\sup_i \mathbb{P}(X_t = i|\tau) \to 0$ as $t \to \infty$ and hence (3.7) implies that $\mathbb{P}(A_2) = 0$, as desired. □

Part III

Scaling limits and aging in 1 dimension

Introduction

Aging of dynamical systems is a memory effect observed (in laboratory) in disordered (inhomogeneous) physical systems [12]. The system is initiated at high temperature, and suddenly cooled down. It then evolves at low temperature from the frozen in (quenched) disordered initial condition. The system then *ages* for a time $t_w > 0$, and is later observed at time t_w+t, $t > 0$, when a quantity $R(t_w, t_w + t)$ is measured. $R(t_w, t_w + t)$ could be a response function at t_w+t to a change of parameters (like the temperature or magnetic field) at t_w, or a two-time correlation function, or a persistence function, etc. It will sometimes below be called an *aging function*. The signature of aging is the following scaling behavior of R.

$$\lim_{\substack{t_w,t\to\infty \\ t^\eta/t_w\to\theta}} R(t_w, t_w + t) = \mathcal{R}(\theta), \tag{3.1}$$

where $\eta > 0$ is a scale parameter, $\theta > 0$ and $\mathcal{R}$ is a nontrivial function. When $\eta = 1$, we have *normal* aging; if $\eta < 1$, then we have *sub*-aging; and *super*-aging, if $\eta > 1$. For example, normal aging associated to a two-time correlation function of a ± 1 ferromagnetic spin system (with $\lim_{\theta\to 0}\mathcal{R}(\theta) = 1$) corresponds roughly speaking to the condition that, to obtain a second observation minimally uncorrelated to a first observation made when the system has an old age t_w, it is necessary to wait a time proportional to t_w. In this context (and in general), aging refers to the ever longer delay to observe changes in the system.

Aging is a phenomenon which can only occur out of equilibrium, since in equilibrium $R(t_w, t_w + t)$ would be independent of t_w and thus the limit in (3.1) would not depend on θ. It was first noticed and studied in intrinsically disordered systems, like spin glasses, but it also occurs in homogeneous systems, like the ferromagnetic Ising model (in this case, the disorder is present only at the initial condition).

The phenomenology in the case of spin glasses is the following. In the energy landscape where the dynamics takes place, there are wells of varying depths. At low temperature, the landscape is very rough, and the process spends most of the time in deep wells. At time t_w, the process is at a well of depth $D(t_w)$. From then on, a minimally uncorrelated observation will be possible only when the process leaves that well, which takes time proportional to t_w.

In the case of the ferromagnetic Ising model at low temperature with a disordered initial condition (product of Bernoullis(1/2), say), the situation is similar, and can also be viewed as follows. The initial clusters of $+1$'s and -1's evolve, some getting larger, some shrinking and disappearing. At time t_w, the region around the origin is in a cluster of size $V(t_w)$. From then on, a minimally uncorrelated observation will be possible only when another cluster surrounds the origin, which takes time proportional to t_w.

Many studies and results on aging in the theoretical physics and mathematics literature are based in the models we consider in these notes, and concentrate in establishing the phenomenon for some functions of interest, like correlation or persistence (which is the probability of no spin flip in $[t_w, t_w + t]$). Our approach aims at identifying aging as a feature of a scaling relation *of the dynamics* (and not only of a specific function). We try to establish scaling limits for the whole dynamics.

The study of more realistic models like the Edwards-Anderson spin glass in dimension at least 3 or the Ising model in dimension at least 2 meets with substantial technical difficulty. For this reason, simplified models are considered in the mathematics literature, and also in the theoretical physics one.

The models we study in this chapter are all one dimensional and are related to the voter model, also in one dimension (which is equivalent to the heat bath Glauber dynamics at zero temperature, as seen in Chapter 1). The first one is the random walk with random rates (RWRR), in next chapter. In the case of heavy tailed inverse rates, this model behaves like in the phenomenological description of aging for spin glasses (as just discussed) — in fact, RWRR's on various graphs have been proposed as approximate models for the dynamics of mean field spin glasses (see discussion and references in next section). Sites with large inverse rates (on which the process takes longer to jump from) concentrate the distribution of the random walk, and this is at the root of aging.

The RWRR was introduced in the previous chapter and the different (but related) dynamical issue of chaotic time dependence (CTD) was studied. In next chapter, we obtain and characterize the scaling limit of the RWRR as a diffusion in a random environment, which is itself the scaling limit of the (inverse) rates, given by the Lévy process associated to that scaling limit. The increments of the Lévy process, as is well known, can be described by means of a (Poisson) point process in $\mathbb{R} \times \mathbb{R}^+$ or, alternatively, as a random measure which is almost surely locally finite and purely atomic (with dense support) in $\mathbb{R}$. The last feature is behind the aging results that follow (and also of the CTD exhibited by the model).

The ordinary Ising model (ferromagnetic, homogeneous) under the voter model dynamics can be described by coalescing random walks starting at arbitrary/all space-time locations (in $\mathbb{Z}\times\mathbb{R}$), as seen in Chapter 1, Section 1.2. The scaling limit of that dynamics is a continuum model, which can be described as coalescing Brownian motions starting from arbitrary/all space-time locations (in $\mathbb{R} \times \mathbb{R}$). In Chapter 5, we use this scaling limit (in limited forms) to obtain aging results for the original dynamics. Finally, in Chapter 6 we discuss the full continuum model, the *Brownian web*, the underlying space where it is realized, and weak convergence issues in that context, from which aging as well as other limits can be derived. We also explore features of the Brownian web on its own. Finally, we introduce a spin dynamics at *very low* temperature, and discuss its scaling limit, in connection with a variant of the Brownian web.

Chapter 4

Random walk with random rates

In this chapter, we treat the random walk with random rates of the previous chapter. From a scaling limit result we derive for it, we get aging results as well as a form of the localization result behind the chaotic time dependence (CTD) of the voter model with random rates of Chapter 3 (see Lemma 3.10).

4.1 Introduction

We come back to the random walk with random rates (RWRR) in one dimension with heavy tailed inverse rates. See Remark 3.6. We will sometimes below make the rates explicit and denote it by (X, τ). Besides the relationship with Glauber dynamics at zero temperature, via the equivalent voter model with random rates (VMRR) — see Subsection 1.2 — the RWRR is more loosely related also to spin-glass dynamics at low temperature. In an effort to understand the latter dynamics, in particular the aging phenomenon, *trap models* [13] have been proposed as approximate models for the dynamics of some mean field spin glasses, like the *random energy model.* Trap models are random walks with random rates whose distribution has heavy tails (at low temperature for the corresponding spin glass) in various graphs, depending e.g. on the dynamics. The RWRR we are considering here is a particular case, then, with the graph $\mathbb{Z}$ (with nearest neighbor bonds). See also [8, 9, 10, 11, 16].

The localization result of last chapter (Lemma 3.10) as well as aging results (some of which will be derived below) can be related to an appropriate scaling limit of (X, τ). When τ_0 has a finite mean, it can be shown (e.g., by the convergence results of [54], as discussed below) that (for a.e. τ) there is a central limit theorem for X_t, and more generally an invariance principle, i.e., that $\epsilon X_{t/\epsilon^2}$ converges to a Brownian motion as $\epsilon \to 0$. In the case of heavy tail for τ_0, in a sense to be precised, it turns out that the Brownian motion is replaced by a singular diffusion Z (in a random environment) — singular here meaning that the single time distributions of Z are discrete.

A convenient quantity, with which to express the relation between localization and the scaling limit, is the "amount of localization" at time t, as measured by

$$q_t = \mathbb{E} \sum_{i \in \mathbb{Z}} [\mathbb{P}(X_t = i|\tau)]^2, \tag{4.1}$$

where the expectation is with respect to τ. Theorem 4.2 below says that as $t \to \infty$, q_t converges to $q_\infty \in (0,1)$ (depending on $\alpha \in (0,1)$), which can itself be expressed by a formula (see (4.5) and (4.7) below) analogous to (4.1) with the singular diffusion Z replacing the random walk X. In terms of the VMRR, we can express

$$q_t = \mathbb{E}\{\mathbb{E}^2[\sigma_t(0)|\xi, \tau]\}. \tag{4.2}$$

In view of the close relationship of $\sum_{i\in\mathbb{Z}}[\mathbb{P}(X_t = i|\tau)]^2$ and $\sup_i \mathbb{P}(X_t = i|\tau)$, we can interpret q_∞ as an amount of chaotic time dependence/predictability of the VMRR. See Remark 3.7. It is thus a natural dynamical order parameter for CTD in that spin dynamics.

Our analysis of the scaling limit of (X, τ) will also yield results about aging of the RWRR. One interesting example of a response function of the RWRR for which the limit (3.1) follows from our results is $R(t_w+t, t) = q_t(t_w)$, where

$$q_t(t_w) = \mathbb{E} \sum_{i \in \mathbb{Z}} [\mathbb{P}(X_{t_w+t} = i|\tau, X_{t_w})]^2. \tag{4.3}$$

Of course, $q_t(0) = q_t$, corresponding to the amount of localization after time t, starting from a fresh ($t_w = 0$) system with $X_0 = 0$ that has not been aged. As with q_∞, the limit function $\mathcal{R}(\theta)$ will be given by a formula (see (4.8)) like (4.3), but with X replaced by the diffusion Z. It follows from (4.8) that $\mathcal{R}(\theta)$ tends to 1 as $\theta \to 0$ and to q_∞ as $\theta \to \infty$. Other examples of RWRR quantities that exhibit normal aging are the (unconditional) probabilities

$\mathbb{P}(X_{t_w+t} = X_{t_w})$, which we discuss below, and

$$\mathbb{P}(\max_{t_w \le t' \le t_w+t} \tau_{X_{t'}} > \max_{0 \le t' \le t_w} \tau_{X_{t'}}), \tag{4.4}$$

which measures the prospects for "novelty" in this aging system.

The scaling limit of (X, τ) will be given by a (singular) one-dimensional diffusion Z in a random environment ρ, denoted (Z, ρ). As a corollary, we get for example that

$$q_t \to \mathbb{E} \sum_{z \in \mathbb{R}} [\mathbb{P}(Z_s = z | \rho)]^2 > 0. \tag{4.5}$$

(Other examples, related to aging, will be discussed below.) Here $s > 0$ is arbitrary, and by the singularity of Z, we mean that (conditional on ρ) the distribution of Z_s is discrete, even though Z is a bona-fide diffusion with continuous sample paths. We shall see why the above expression for q_∞, which describes the amount of localization of (Z, ρ) at time s does not in fact depend on s (as long as $s \neq 0$), a fact that may at first seem surprising (since $Z_s \to 0$ as $s \to 0$, almost surely). Indeed this lack of dependence follows from the scaling/self-similarity properties of (Z, ρ) which imply that (conditioned on ρ) the distribution of $s^{\alpha/(\alpha+1)} Z_s$ is a *random* measure on $\mathbb{R}$ whose distribution (arising from its dependence on ρ) does not depend on $s > 0$. We now give a precise definition of this diffusion in a random environment, (Z, ρ).

Definition 4.1 (Diffusion with random speed measure, (Z, ρ)) *The random environment ρ, the spatial scaling limit of the original environment τ of rates on $\mathbb{Z}$, is a random discrete measure, $\sum_i W_i \delta_{Y_i}$, where the countable collection of (Y_i, W_i)'s yields an inhomogeneous Poisson point process on $\mathbb{R} \times (0; \infty)$ with density measure $dy\, \alpha w^{-1-\alpha} dw$. Conditional on ρ, Z_s is a diffusion process (with $Z_0 = 0$) that can be expressed as a time change of a standard one-dimensional Brownian motion $B(t)$ with speed measure ρ, as follows [33]. Letting $\ell(t, x)$ denote the local time at x of $B(t)$, define*

$$\phi_t^\rho := \int \ell(t, y)\, d\rho(y) \tag{4.6}$$

and the stopping time ψ_s^ρ as the first time t when $\phi_t^\rho = s$ (so that ψ^ρ is the inverse function of ϕ^ρ); then $Z_s = B(\psi_s^\rho)$.

Note that although ρ is discrete, the set of Y_i's is a.s. dense in $\mathbb{R}$ because the density measure is non-integrable at $w = 0$. For (a deterministic) $s > 0$, the distribution of Z_s is a discrete measure whose atoms are precisely those of ρ; this is essentially because the set of times when Z is anywhere else than these atoms has zero Lebesgue measure.

The next theorem gives the limit (4.5) as part of the convergence of the rescaled random walk (X, τ) to the diffusion (Z, ρ). A more complete result explaining the nature of this convergence is provided later in Theorem 4.13.

Theorem 4.2 *Assume that* $\mathbb{P}(\tau_0 > 0) = 1$ *and* $\mathbb{P}(\tau_0 > t) = L(t)/t^\alpha$, *where* L *is a nonvanishing slowly varying function at infinity and* $\alpha < 1$. *Then for* $\epsilon > 0$, *there exists* $c_\epsilon > 0$ *with* $c_\epsilon \to 0$ *as* $\epsilon \to 0$, *so that for any fixed* $s > 0$, *the distribution of* $Z_s^{(\epsilon)} = \epsilon X_{s/(c_\epsilon \epsilon)}$, *conditioned on* τ *and thus regarded as a random probability measure on* $\mathbb{R}$, *converges to the distribution of* Z_s, *conditioned on* ρ, *in such a way that*

$$q_{s/(c_\epsilon\epsilon)} = \mathbb{E}\sum_{i\in\mathbb{Z}}[P(Z_s^{(\epsilon)} = \epsilon i|\tau)]^2 \to \mathbb{E}\sum_{z\in\mathbb{R}}[P(Z_s = z|\rho)]^2. \tag{4.7}$$

Remark 4.3 *Notice that the first assumption on the distribution of* τ_0 *is more relaxed than the one made in Chapter 3 in (3.1). It is not difficult to see that the RWRR as well as the VMRR is almost surely well defined under the weaker assumption. The only thing that could go wrong would be if there would be explosions (that is, an infinite number of jumps in finite time of the/some random walk). But that would mean that the sum in (3.5) would converge as* $n \to \infty$, *and this almost surely does not happen. Theorem 3.4 of course holds under the weaker assumption.*

The second assumption on the distribution of τ_0 *is also different from the one in Chapter 3 on Theorem 3.4. This time, however, the assumptions are not comparable.*

We now return to a discussion of aging in the RWRR. Analogously to (4.5), we have $\mathcal{R}(\theta)$ of (4.3) given by

$$\lim_{t'\to\infty} q_{\theta t'}(t') = \mathbb{E}\sum_{z\in\mathbb{R}}[\mathbb{P}(Z_{s+\theta s} = z|\rho, Z_s)]^2. \tag{4.8}$$

The validity of this limit (as well as of the analogous one for (4.4) and others) also follows from the results and techniques below — see Remark 4.10. Here, the self-similarity properties of (Z, ρ) imply that the RHS of (4.8) depends only on θ and *not* on s (for $0 < s < \infty$), explaining the basic signature of normal aging — that the asymptotics of $q_t(t_w)$ depend only on the asymptotic ratio of t/t_w.

Another example of an RWRR localization quantity with normal aging behavior is

$$q'_t(t_w) = \mathbb{P}(X_{t_w+t} = X_{t_w}) = \mathbb{E}\mathbb{P}(X_{t_w+t} = X_{t_w} | \tau, X_{t_w}). \qquad (4.9)$$

In this case, the asymptotic aging function, $\mathcal{R}'(\theta)$, would have limits 1 and 0, respectively, as $\theta \to 0$ and ∞.

To see the lack of dependence of the RHS's of (4.5) and (4.8) on s, we may proceed as follows. For $\lambda > 0$, consider the rescaled Brownian motion and environment,

$$B^\lambda(t) = \lambda^{-1/2} B(\lambda t); \quad \rho^\lambda = \sum_i (\lambda^{-1/2})^{1/\alpha} W_i \delta_{\lambda^{-1/2} Y_i}. \qquad (4.10)$$

Since B^λ and ρ^λ are equidistributed with B and ρ, it follows that if we define a diffusion Z^λ as the time-changed B^λ using speed measure ρ^λ, then $(Z^\lambda, \rho^\lambda)$ is equidistributed with the original diffusion in a random environment (Z, ρ). On the other hand, on the original probability space on which B and ρ are defined, one has $Z_s^\lambda = \lambda^{-1/2} Z_{\lambda^{(\alpha+1)/(2\alpha)} s}$, so that the RHS's of (4.5) and (4.8) remain the same when s is replaced by $\lambda^{(\alpha+1)/(2\alpha)} s$, and thus cannot depend on s.

To best understand how (Z, ρ) arises as the scaling limit of (X, τ), one should take into account the fact that not only diffusions, but also random walks (or more accurately, birth-death processes) can be expressed as time-changed Brownian motions [54, 33]. In particular, if for any $\epsilon > 0$, we take as speed measure

$$\rho^{(\epsilon)} := \sum_{i \in \mathbb{Z}} c_\epsilon \tau_i \delta_{\epsilon i}, \qquad (4.11)$$

where the parameter $c_\epsilon > 0$ is yet to be determined, and then do the time-change on the rescaled Brownian motion B^{1/ϵ^2}, the resulting process is a rescaling of the original random walk X, namely $Z_s^{(\epsilon)} = \epsilon X_{s/(c_\epsilon \epsilon)}$. When the distribution F of the τ_i's has a finite mean, then by the Law of Large Numbers, taking $c_\epsilon = \epsilon$, $\rho^{(\epsilon)}$ converges to (the mean of F times) Lebesgue measure and $Z^{(\epsilon)}$ converges to a Brownian motion as $\epsilon \to 0$ [54, 35]. On the other hand, if $1 - F(u) = L(u)/u^\alpha$ with $\alpha < 1$ and $L(u)$ is slowly varying at infinity [22], then by choosing c_ϵ appropriately one has (from the classical theories of domains of attraction and extreme value statistics) convergence (in various senses, to be discussed) of $\rho^{(\epsilon)}$ to the random measure ρ.

The idea that also $(Z^{(\epsilon)}, \rho^{(\epsilon)})$ should converge to (Z, ρ) in some sense should by now be quite clear. And indeed the basic convergence results of [54] are enough to imply, for example, that a functional like

$$\mathbb{E}\{[P(a \leq Z_s^{(\epsilon)} \leq b \,|\, \rho^{(\epsilon)})]^2\} \tag{4.12}$$

(for *deterministic* a, b) converges to the corresponding quantity for (Z, ρ). But they are not sufficient to get the convergence of localization quantities like

$$q_{s/(c_\epsilon \epsilon)} = \mathbb{E} \sum_{z \in \mathbb{R}} [P(Z_s^{(\epsilon)} = z \,|\, \rho^{(\epsilon)})]^2.$$

The problem in our case is not primarily with the randomness of $\rho^{(\epsilon)}$ (i.e., of τ) and ρ, but occurs already when considering the nature of convergence of a process $Y^{(\epsilon)}(t)$ that is a Brownian motion time-changed with a *deterministic* speed measure $\mu^{(\epsilon)}$. The convergence results of [54] imply that if $\mu^{(\epsilon)} \to \mu$ vaguely, then (for example) one has weak convergence of the distribution $\bar{\mu}^{(\epsilon)}$ of $Y^{(\epsilon)}(t_0)$ to the corresponding $\bar{\mu}$. But we need stronger convergence.

This stronger convergence is the subject of next section, which contains the main technical result of the chapter, Theorem 4.9, in which weak convergence is combined with *point process convergence.* By point process convergence for (say) a discrete measure $\sum_i w_i^{(\epsilon)} \delta_{y_i^{(\epsilon)}}$ to $\sum_i w_i \delta_{y_i}$ (where we have expressed each sum so that the atoms are not repeated), we mean that the subset of $\mathbb{R} \times (0, \infty)$ consisting of all the $(y_i^{(\epsilon)}, w_i^{(\epsilon)})$'s converges to the set of all (y_i, w_i)'s — in the sense that every open disk (whose closure is a compact subset of $\mathbb{R} \times (0, \infty)$) containing exactly m of the (y_i, w_i)'s $(m = 0, 1, \dots)$ with none on its boundary, contains also exactly m of the $(y_i^{(\epsilon)}, w_i^{(\epsilon)})$'s for all small ϵ. Our technical result is that vague plus point process convergence for the speed measures $\mu^{(\epsilon)} \to \mu$ implies the same for the distributions at a fixed time t_0; i.e., $\bar{\mu}^{(\epsilon)} \to \bar{\mu}$.

Going from this result for a sequence of deterministic speed measures to our context of random speed measures requires a bit more work, which is presented in Sections 4.3 and 4.4 below. The way we handle that is to replace the random measures $\rho^{(\epsilon)}$ which only converge (in our two senses) in distribution, by a different (but also natural) coupling for the various ϵ's than that provided by the space of the original τ_i's, so that convergence becomes almost sure. This coupling is presented in Section 4.3.

In the next sections, we describe our results in more detail. Due to their technical character, we either omit or do a simpler case of some of the proofs, and refer to [25] for the complete arguments. See that reference also for a broader discussion on the issues of this section.

4.2 The continuity theorem

Let μ, $\mu^{(\epsilon)}$, $\epsilon > 0$, be non-identically-zero, locally finite measures on $\mathbb{R}$. Let Y_t, $Y_t^{(\epsilon)}$, $t \geq 0$, $Y_0^{(\epsilon)} = Y_0 = x$, be the Markov processes in one dimension obtained by time changing a standard Brownian motion through $\mu, \mu^{(\epsilon)}$; i.e., let $B = B(s)$, $s \geq 0$, be a standard Brownian motion (with $B(0) = 0$) and let

$$\phi_s^{(\epsilon)}(x) := \int \ell(s, y - x)\, d\mu^{(\epsilon)}(y), \tag{4.13}$$

where ℓ is the Brownian local time of B [54, 33] and $\mu^{(0)} = \mu$; let also

$$\psi^{(\epsilon)}(x) = \psi_t^{(\epsilon)}(x) := \inf\{s \geq 0 : \phi_s^{(\epsilon)}(x) = t\}, \tag{4.14}$$

and make

$$Y_t^{(\epsilon)} = B(\psi_t^{(\epsilon)}(x)) + x. \tag{4.15}$$

Let $\psi(\cdot) = \psi^{(0)}(\cdot)$ and $Y_\cdot = Y_\cdot^{(0)}$. Notice that, since $\ell(s, y)$ is nondecreasing in s for all y, $\phi_s^{(\epsilon)}(x)$ is nondecreasing in s, and so its (right-continuous) inverse $\psi_t^{(\epsilon)}(x)$ is well-defined. Processes described in this way are known in the literature as *quasidiffusions*, *gap diffusions* or *generalized diffusions* ([36, 37, 38] and references therein). They generalize the usual diffusions in that the *speed measures* μ can be zero in intervals, thus including birth and death and other processes.

We discuss now the types of convergence we will need for our results. Let $\mathcal{M}$ be the space of locally finite measures on $\mathbb{R}$ and $\mathcal{P}$ its subspace of probability measures.

Definition 4.4 (Vague convergence) *Given a family ν, $\nu^{(\epsilon)}$, $\epsilon > 0$, in $\mathcal{M}$, we say that $\nu^{(\epsilon)}$ converges* vaguely *to ν, and write $\nu^{(\epsilon)} \stackrel{v}{\to} \nu$, as $\epsilon \to 0$, if for all continuous real-valued functions f on $\mathbb{R}$ with bounded support*

$$\int f(y)\, d\nu^{(\epsilon)}(y) \to \int f(y)\, d\nu(y) \ \text{as } \epsilon \to 0.$$

Definition 4.5 (Point process convergence) *For the same family, we say that* $\nu^{(\epsilon)}$ *converges* in the point process sense *to* ν, *and write* $\nu^{(\epsilon)} \overset{pp}{\to} \nu$, *as* $\epsilon \to 0$, *provided the following is valid: if the atoms of* ν, $\nu^{(\epsilon)}$ *are, respectively, at the distinct locations* y_i, $y_{i'}^{(\epsilon)}$ *with weights* w_i, $w_{i'}^{(\epsilon)}$, *then the subsets* $V^{(\epsilon)} \equiv \cup_{i'}\{(y_{i'}^{(\epsilon)}, w_{i'}^{(\epsilon)})\}$ *of* $\mathbb{R} \times (0,\infty)$ *converge to* $V = \cup_i\{(y_i, w_i)\}$ *as* $\epsilon \to 0$ *in the sense that for any open* U *whose closure* $\bar{U}$ *is a compact subset of* $\mathbb{R}\times(0,\infty)$ *such that its boundary contains no points of* V, *the number of points* $|V^{(\epsilon)} \cap U|$ *in* $V^{(\epsilon)} \cap U$ *(necessarily finite since* U *is bounded and at a finite distance from* $\mathbb{R}\times\{0\}$*) equals* $|V \cap U|$ *for all* ϵ *small enough.*

These notions can be related to the following condition, where for $\nu \in \mathcal{P}$ we order the (y_i, w_i)'s (the locations and weights of the atoms of ν) so that $w_{i_1} \geq w_{i_2} \geq \ldots$, where w_{i_1} is the largest weight, w_{i_2} is the second largest, and so forth. For a measure not in $\mathcal{P}$, we use an arbitrary ordering of the atoms.

Condition 4.6 *For each* $l \geq 1$, *there exists* $j_l(\epsilon)$ *such that*

$$(y_{j_l(\epsilon)}, w_{j_l(\epsilon)}) \to (y_{i_l}, w_{i_l}) \quad \textit{as } \epsilon \to 0. \tag{4.16}$$

We now state a useful relationship among the above notions.

Proposition 4.7 *For any family* ν, $\nu^{(\epsilon)}$, $\epsilon > 0$, *in* $\mathcal{M}$, *the following two assertions hold. If* $\nu^{(\epsilon)} \overset{pp}{\to} \nu$ *as* $\epsilon \to 0$, *then Condition 1 holds. If Condition 1 holds and* $\nu^{(\epsilon)} \overset{v}{\to} \nu$ *as* $\epsilon \to 0$, *then* $\nu^{(\epsilon)} \overset{pp}{\to} \nu$ *as* $\epsilon \to 0$.

We leave it to the reader to find an example where Condition 1 holds but point process convergence does not. The following is a useful corollary of Proposition 4.7.

Proposition 4.8 *Let* ν, $\nu^{(\epsilon)}$, $\epsilon > 0$, *be any family in* $\mathcal{P}$. *If as* $\epsilon \to 0$ *both* $\nu^{(\epsilon)} \overset{pp}{\to} \nu$ *and* $\nu^{(\epsilon)} \overset{v}{\to} \nu$, *then as* $\epsilon \to 0$

$$\sum_{i'} [w_{i'}^{(\epsilon)}]^2 \to \sum_i [w_i]^2. \tag{4.17}$$

Proof.

By the first assertion of Proposition 4.7, Condition 4.6 holds. This in turn implies that

$$\liminf_{\epsilon\to 0} \sum_j [w_j^{(\epsilon)}]^2 \geq \sup_k \sum_{l=1}^{k} [w_{i_l}]^2 = \sum_i [w_i]^2. \tag{4.18}$$

This together with the distinctness of the (y_i, w_i)'s also implies that for any k the indices $j_1(\epsilon), \ldots, j_k(\epsilon)$ are distinct for small enough ϵ. Furthermore, it implies that if k and δ are such that $w_{i_k} > \delta > w_{i_{k+1}}$, then for ϵ small enough

$$\sup\{w_j^{(\epsilon)} : j \notin \{j_1(\epsilon), \ldots, j_k(\epsilon)\}\} < \delta. \tag{4.19}$$

To see this, note that otherwise along some subsequence $\epsilon = \epsilon_l \to 0$ there would be an index $j^*(\epsilon) \notin \{j_1(\epsilon), \ldots, j_k(\epsilon)\}$ with $\liminf y_{j^*(\epsilon)}^{(\epsilon)} \geq \delta$ and either (i) $y_{j^*(\epsilon)}^{(\epsilon)} \to y^* \in (-\infty, +\infty)$ or else (ii) $|y_{j^*(\epsilon)}^{(\epsilon)}| \to \infty$. Case (i) would contradict $\nu^{(\epsilon)} \xrightarrow{pp} \nu$, while case (ii) would imply that the family $\{\nu^{(\epsilon)}\}$ is not tight, which would contradict $\nu^{(\epsilon)} \xrightarrow{v} \nu$ since $\nu^{(\epsilon)}$ and ν are all probability measures. Using the above choice of k and δ, we thus have

$$\limsup_{\epsilon \to 0} \sum_j [w_j^{(\epsilon)}]^2 \leq \sum_{l=1}^k [w_{i_l}]^2 + \limsup_{\epsilon \to 0} \sum_j \delta w_j^{(\epsilon)} = \sum_{l=1}^k [w_{i_l}]^2 + \delta. \tag{4.20}$$

Letting $k \to \infty$ and $\delta \to 0$ completes the proof. □

We are ready to state the main result of this section.

Theorem 4.9 *Let $\mu^{(\epsilon)}, \mu, Y^{(\epsilon)}, Y$ be as above and fix any deterministic $t_0 > 0$ and $x \in \mathbb{R}$. Let $\bar{\mu}^{(\epsilon)}$ denote the distribution of $Y_{t_0}^{(\epsilon)}$ (with $Y_0^{(\epsilon)} = x$) and define $\bar{\mu}$ similarly for Y_{t_0}. Note that $\bar{\mu}^{(\epsilon)} = D_{t_0,x}(\mu^{(\epsilon)})$ and $\bar{\mu} = D_{t_0,x}(\mu)$, where $D_{t_0,x}$ is some deterministic function from the non-identically-zero measures in $\mathcal{M}$ to $\mathcal{P}$. Suppose*

$$\mu^{(\epsilon)} \xrightarrow{v} \mu \quad \textit{and} \quad \mu^{(\epsilon)} \xrightarrow{pp} \mu \quad \textit{as } \epsilon \to 0. \tag{4.21}$$

Then, as $\epsilon \to 0$,

$$\bar{\mu}^{(\epsilon)} \xrightarrow{v} \bar{\mu} \quad \textit{and} \quad \bar{\mu}^{(\epsilon)} \xrightarrow{pp} \bar{\mu}. \tag{4.22}$$

Remark 4.10 *To study limits involving two (or more) times (see, e.g., (4.3), (4.8), (4.9)), some straightforward extensions of Theorem 4.9 are useful. One of these is that (4.22) remains valid if $Y_0^{(\epsilon)} = x^{(\epsilon)}$ with $x^{(\epsilon)} \to x$. Another is that the single-time distribution $\bar{\mu}^{(\epsilon)}$ of $Y_{t_0}^{(\epsilon)}$ can be replaced by the multi-time distribution of $(Y_{t_1}^{(\epsilon)}, \ldots, Y_{t_m}^{(\epsilon)})$, with point process convergence for measures on $\mathbb{R}^m$ defined in the obvious way.*

The following is an immediate consequence of Theorem 4.9 and Proposition 4.8.

Corollary 4.11 *Under the same hypotheses, the weights of the atoms of $\bar{\mu}^{(\epsilon)}$ and $\bar{\mu}$ satisfy*

$$\sum_j [\bar{w}_j^{(\epsilon)}]^2 \to \sum_i [\bar{w}_i]^2 \quad \text{as } \epsilon \to 0. \tag{4.23}$$

Remark 4.12 *More explicitly, (4.23) takes the form*

$$\sum_{y \in \mathbb{R}} [\mathbb{P}(Y_{t_0}^{(\epsilon)} = y)]^2 \to \sum_{y \in \mathbb{R}} [\mathbb{P}(Y_{t_0} = y)]^2 \quad \text{as } \epsilon \to 0, \tag{4.24}$$

or, equivalently, if $Y_t^{(\epsilon)\prime}$ (resp. Y_t') is an independent copy of $Y_t^{(\epsilon)}$ (resp. Y_t), then

$$\mathbb{P}(Y_{t_0}^{(\epsilon)\prime} = Y_{t_0}^{(\epsilon)}) \to \mathbb{P}(Y_{t_0}' = Y_{t_0}) \quad \text{as } \epsilon \to 0. \tag{4.25}$$

4.3 A coupling for the scaled random rates

As discussed briefly at the end of Section 4.1, the rescaled random walk with random rates, $Z_\cdot^{(\epsilon)} = \epsilon X_{\cdot/(c_\epsilon \epsilon)}$ is a quasidiffusion whose (random) speed measure $\rho^{(\epsilon)}$, given by (4.11), only converges *in distribution* to the (random) speed measure ρ of the scaling limit diffusion Z. To take advantage of the results of Section 4.2, it is convenient to find random measures $\tilde{\rho}^{(\epsilon)}$ equidistributed (for each ϵ) with $\rho^{(\epsilon)}$ and such that $\tilde{\rho}^{(\epsilon)}$ converges *almost surely* as $\epsilon \to 0$ to ρ, in both the vague and point process senses of Section 4.2. In this section we will construct such $\tau^{(\epsilon)}$, equidistributed with τ for each $\epsilon > 0$, on the natural probability space where ρ is defined, in a *very special case* of the distribution of τ_0, namely when it is itself a positive stable distribution of index α, where the desired properties follow easily. For the general case, we refer to [25].

Consider the Lévy process (see, e.g., [48, 50, 51]) V_x, $x \in \mathbb{R}$, $V_0 = 0$, with stationary and independent increments given by

$$\mathbb{E}\left[e^{ir(V_{x+x_0} - V_{x_0})}\right] = \exp\left\{\alpha x \int_0^\infty (e^{irw} - 1)\, w^{-1-\alpha}\, dw\right\} \tag{4.26}$$

for any $r, x_0 \in \mathbb{R}$ and $x \geq 0$. It satisfies

$$\lim_{y \to \infty} y^\alpha \mathbb{P}(V_1 > y) = 1 \tag{4.27}$$

([22], Theorem XVII.5.3). Let ρ be the (random) Lebesgue-Stieltjes measure on the Borel sets of $\mathbb{R}$ associated to V, i.e.,

$$\rho((a,b]) = V_b - V_a,\ a,b \in \mathbb{R},\ a < b, \tag{4.28}$$

where we have chosen the process V to have sample paths that are right-continuous (with left-limits). Then

$$\frac{d\rho}{dx} = \frac{dV}{dx} = \sum_j w_j\, \delta(x - x_j), \tag{4.29}$$

where the (countable) sum is over the indices of an inhomogeneous Poisson point process $\{(x_j, w_j)\}$ on $\mathbb{R} \times (0,\infty)$ with density $dx\, \alpha w^{-1-\alpha}\, dw$.

For each $\epsilon > 0$, we want to define, in the fixed probability space on which V and ρ are defined, a sequence $\tau_i^{(\epsilon)}$, $i \in \mathbb{Z}$, of independent random variables such that

$$\tau_i^{(\epsilon)} \sim \tau_0 \quad \text{for every } i \in \mathbb{Z} \tag{4.30}$$

(where $\sim$ denotes equidistribution) and with the following property: for a given family of constants c_ϵ, $\epsilon > 0$, let

$$\tilde{\rho}^{(\epsilon)} := \sum_{i=-\infty}^{\infty} c_\epsilon \tau_i^{(\epsilon)} \delta_{\epsilon i}; \tag{4.31}$$

we demand that constants c_ϵ be chosen so that

$$\tilde{\rho}^{(\epsilon)} \xrightarrow{v} \rho \quad \text{and} \quad \tilde{\rho}^{(\epsilon)} \xrightarrow{pp} \rho \quad \text{as } \epsilon \to 0, \text{ almost surely.} \tag{4.32}$$

If τ_0 is equidistributed with the positive α-stable random variable V_1, then, according to (4.27), $\mathbb{P}(\tau_0 > t) = L(t)/t^\alpha$ with $L(t) \to 1$ as $t \to \infty$. Now, we may simply choose $c_\epsilon = \epsilon^{1/\alpha}$ and take $\tau^{(\epsilon)}$ to be the sequence of scaled increments of the Lévy process V:

$$\tau_i^{(\epsilon)} = \frac{1}{c_\epsilon}\left(V_{\epsilon(i+1)} - V_{\epsilon i}\right). \tag{4.33}$$

The validity of (4.30) and (4.32) are then elementary exercises.

4.4 Scaling limit for the RWRR

Let X_t, $t \geq 0$, $X_0 = 0$, be a continuous time random walk on $\mathbb{Z}$ with rates given by $\lambda_i = \tau_i^{-1}$, $i \in \mathbb{Z}$, where τ_i, $i \in \mathbb{Z}$, are i.i.d. random variables such that $\mathbb{P}(\tau_0 > 0) = 1$ and $\mathbb{P}(\tau_0 > t) = L(t)/t^\alpha$, where L is a nonvanishing slowly varying function at infinity and $\alpha < 1$.

We consider now the scaling limit of the random walk X_t. Let

$$Z_t^{(\epsilon)} = \epsilon X_{t/(c_\epsilon \epsilon)}, \quad t \geq 0. \tag{4.34}$$

To study the limit of $Z^{(\epsilon)}$, in the presence of the random rates, which themselves converge vaguely and in the point process sense, but only in distribution, we will need a weak notion of vague and point process convergence, as follows. Let $\tilde{C}_b$ be the class of bounded real functions f on the space $\mathcal{P}$ of probability measures on $\mathbb{R}$ that are *weakly continuous* in the sense that $f(\mu_n) \to f(\mu)$ as $n \to \infty$ for all μ, μ_n, $n \geq 1$, in $\mathcal{P}$ such that *both* $\mu_n \stackrel{v}{\to} \mu$ *and* $\mu_n \stackrel{pp}{\to} \mu$ as $n \to \infty$.

Let Z_t be the (random) quasidiffusion Y_t as in (4.13)-(4.14) above, but with speed measure μ taken to be the (random) discrete measure ρ of (4.28)–(4.29) associated with the Lévy process V. For $t_0 > 0$ fixed, let $\bar{\rho}$ and $\bar{\rho}^{(\epsilon)}$ be the (random) probability distributions of Z_{t_0} and $Z_{t_0}^{(\epsilon)}$, respectively; i.e., $\bar{\rho}$ is the conditional distribution of Z_{t_0} given ρ while $\bar{\rho}^{(\epsilon)}$ is the conditional distribution of $Z_{t_0}^{(\epsilon)}$ given τ. We can now state the following theorem, which is a consequence of (4.30), (4.32) and Theorem 4.9.

Theorem 4.13 *As* $\epsilon \to 0$,

$$\mathbb{E}(f(\bar{\rho}^{(\epsilon)})) \to \mathbb{E}(f(\bar{\rho})) \tag{4.35}$$

for all $f \in \tilde{C}_b$; *in particular,*

$$\begin{aligned} \mathbb{E}\sum_{x\in\mathbb{R}} \left[\bar{\rho}^{(\epsilon)}(\{x\})\right]^2 &= \mathbb{E}\sum_{x\in\mathbb{R}}[\mathbb{P}(Z_{t_0}^{(\epsilon)} = x|\tau)]^2 \to \\ &\to \mathbb{E}\sum_{x\in\mathbb{R}} [\bar{\rho}(\{x\})]^2 = \mathbb{E}\sum_{x\in\mathbb{R}}[\mathbb{P}(Z_{t_0} = x|\rho)]^2. \end{aligned} \tag{4.36}$$

Proof of Theorem 4.13

$Z^{(\epsilon)}$ is distributed as a standard Brownian motion time changed through the speed measure $\rho^{(\epsilon)}$ (see (4.11) and the beginning of Section 4.2). Let

ρ and $\tilde{\rho}^{(\epsilon)}$ be as in (4.28) and (4.31) and let $\tilde{Z}^{(\epsilon)}$ be a standard Brownian motion time changed through $\tilde{\rho}^{(\epsilon)}$. By (4.30),

$$(Z^{(\epsilon)}, \rho^{(\epsilon)}) \sim (\tilde{Z}^{(\epsilon)}, \tilde{\rho}^{(\epsilon)}). \tag{4.37}$$

To obtain (4.35), it is thus enough, by (4.37) and dominated convergence, to show that $\bar{\tilde{\rho}}^{(\epsilon)}$, the probability distribution of $\tilde{Z}^{(\epsilon)}_{t_0}$ (which is $D_{t_0,0}(\tilde{\rho}^{(\epsilon)})$ in the notation of Theorem 4.9 and is random because of its dependence on $\tilde{\rho}^{(\epsilon)}$ and hence on the Lévy process V), satisfies: $\bar{\tilde{\rho}}^{(\epsilon)} \xrightarrow{v} \bar{\rho}$ and $\bar{\tilde{\rho}}^{(\epsilon)} \xrightarrow{pp} \bar{\rho}$ almost surely. But that follows from (4.30), (4.32) and Theorem 4.9.

Then (4.36) follows from (4.35) with the function f on $\mathcal{P}$ defined by $f(\mu) = \sum_{x\in\mathbb{R}} [\mu(\{x\})]^2$, which belongs to $\tilde{\mathcal{C}}_b$ by Proposition 4.8. □

Chapter 5

Ordinary voter model

We now discuss the ordinary one dimensional voter model with homogeneous rates. In Section 1.2, where the (general) model was introduced, we saw that it is equivalent to the zero temperature heat-bath Glauber dynamics, and can be represented in terms of backwards in time coalescing random walks, with homogeneous rates in the present case. Some of the material here was taken from [28] and [29], to which we also refer for some literature about it, and more discussion.

We discuss now the forwards dynamics. As pointed out at the introduction of this part, the initial clusters of spins (+1's and −1's) evolve by growing, and shrinking and disappearing. Their boundaries perform *annihilating random walks*, i.e., they evolve as independent random walks up until two of them meet, and then both disappear.

We can, and will, consider the model with more than two kinds of spins (or *colors*). The (voter model) dynamics is defined in the same way: when the alarm goes off at site x, the color there is changed to the one of a nearest neighbor chosen uniformly at random. In particular, the dual representations in terms of coalescing random walks (1.11) and the more general (1.12) are valid. It will be convenient to look at the case where initially there is a different color at each site. In this case, the boundaries between same color clusters perform *coalescing* random walks, rather than annihilating ones. If we have initially more than 2 but, say, finitely many colors only, then the boundaries will perform coalescing *or* annihilating random walks, depending on wheter the colors of the clusters which are adjacent to the disappearing one are different or not, respectively. See Figure 5.1.

In any case, the dynamics starting with any initial configuration, with any

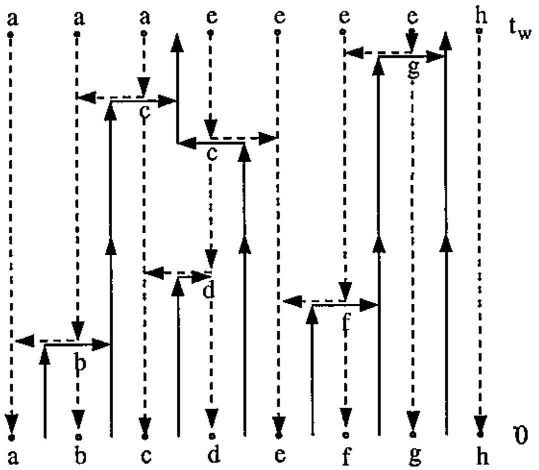

Figure 5.1: A representation of the dynamics, both forwards (solid lines) and backwards (dashed lines), in a lattice with eight sites, with one different color in each site initially.

number of colors, can be readily obtained from dynamics starting with the "one different color per site" configuration, by properly identifying the colors in the latter initial configuration. For example, in the diagram of Figure 5.1, if we want to run the dynamics with two colors, the initial color identification might be "abdgh with +1" and "cef with −1"; by replacing these values for all times one obtains the corresponding diagram for the two-color dynamics.

Below, we will consider only product initial configurations with q colors, that is,

$$\mathbb{P}(\sigma(i_1) = j_1, \ldots, \sigma(i_n) = j_n) = q^{-n} \tag{5.1}$$

for all $n = 1, 2, \ldots,$; $j_1, \ldots, j_n$ in $\{1, \ldots, q\}$; and distinct $i_1, \ldots, i_n$ in $\mathbb{Z}$, where $q \geq 2$ is an integral parameter. The "one different color per site" configuration will be considered the $q = \infty$ case.

One also refers to the voter model with q colors as the *Potts model with q states* (under zero temperature/voter model dynamics) [28].

As aging functions, we will initially consider the following quantities

$$G_q(x, y; s, t) := \mathbb{P}(\sigma_s(x) = \sigma_t(y)), \tag{5.2}$$

the probability that the spins at (x, s) and (y, t) are the same. This is related (for dinite q) to the two-site two-time correlation function, introduced in Section 1.2 (but here denoted)

$$C_q(x,y;s,t) := \mathbb{E}(\vec{\sigma}_s(x)\cdot\vec{\sigma}_t(y)), \tag{5.3}$$

where $\vec{\sigma}(\cdot)$ denotes the *tetrahedral representation* of q-state Potts spins, as unit vectors in $\mathbb{R}^q$ pointing only to the vertices of a $(q-1)$-dimensional *tetrahedron*. The dot product between to such vectors satisfies

$$\vec{\sigma}(x)\cdot\vec{\sigma}(y) = (q\delta_{\sigma(x),\sigma(y)} - 1)/(q-1), \tag{5.4}$$

where δ is the Kronecker delta. (C of Section 1.2 is denoted C_2 here.) In particular, we have, from (5.2), (5.3) and (5.4),

$$C_q(x,y;s,t) = (qG_q(x,y;s,t) - 1)/(q-1). \tag{5.5}$$

As pointed out above, we can relate quantities like G_q for all finite q to the $q=\infty$ case. For G_q, we get a particularly simple relationship. Indeed, by (1.12) and (5.2), for finite q

$$\begin{aligned} G_q(x,y;s,t) &= \mathbb{P}(\sigma_0(X_s^{(s)}(x)) = \sigma_0(X_t^{(t)}(y))) \\ &= \mathbb{P}(X_s^{(s)}(x) = X_t^{(t)}(y)) \\ &+ \mathbb{P}(\sigma_0(X_s^{(s)}(x)) = \sigma_0(X_t^{(t)}(y)), X_s^{(s)}(x) \neq X_t^{(t)}(y)) \\ &= \mathbb{P}(A_{x,y;s,t}) + \frac{1}{q}\,\mathbb{P}(A^c_{x,y;s,t}) \qquad (5.6) \\ &= \frac{1}{q}\,[(q-1)\,\mathbb{P}(A_{x,y;s,t}) + 1], \qquad (5.7) \end{aligned}$$

where $A_{x,y;s,t} := \{X_s^{(s)}(x) = X_t^{(t)}(y)\}$ is the event that the two backward random walks $X_\cdot^{(s)}(x)$ and $X_\cdot^{(t)}(y)$ have coalesced by time 0. Notice that (5.6) is valid when $q=\infty$, by making $1/q = 0$. We then have that

$$G_\infty(x,y;s,t) = \mathbb{P}(A_{x,y;s,t}) =: C_\infty(x,y;s,t), \tag{5.8}$$

and from (5.5) and (5.6),

$$C_q(x,y;s,t) = C_\infty(x,y;s,t) \tag{5.9}$$

for all $q \geq 2$. Notice that the right hand side of (5.9) is independent of q.

We also get from (5.7-5.9)

$$G_q(x,y;s,t) = \frac{1}{q}\left[(q-1)\,C_q(x,y;s,t)+1\right]. \tag{5.10}$$

The proper space-time scaling for this model, in particular in order to derive aging results is the usual diffusive one

$$s = t_w; \quad t = (1+\theta)t_w; \quad x = \lfloor v\sqrt{t_w}\rfloor; \quad y = \lfloor z\sqrt{t_w}\rfloor, \tag{5.11}$$

where $\theta \geq 0$, v, z are real numbers. By the space translation invariance of the model, it is sufficient to take $v = 0$.

Now, in order to obtain the asymptotics of $C_\infty(0, \lfloor z\sqrt{t_w}\rfloor; t_w, (1+\theta)t_w)$ as $t_w \to \infty$, it is enough, by (5.8), to understand the scaling limit of

$$(X_r^{(t_w)}(0), X_{r'}^{((1+\theta)t_w)}(\lfloor z\sqrt{t_w}\rfloor))_{0\leq r\leq t_w, 0\leq r'\leq(1+\theta)t_w} \tag{5.12}$$

as $t_w \to \infty$. By Donsker's invariance principle

$$t_w^{-1/2}\left(X_r^{(t_w)}(0), X_{r'}^{((1+\theta)t_w)}(\lfloor z\sqrt{t_w}\rfloor)\right) \Rightarrow (B_t, B'_{t'}) \tag{5.13}$$

as $t_w \to \infty$, where (B, B') are backwards in time coalescing Brownian motions[1] starting at space-time points $(0,1)$ and $(z, 1+\theta)$, respectively. Thus

$$C_\infty(0, \lfloor z\sqrt{t_w}\rfloor; t_w, (1+\theta)t_w) = \mathbb{P}(A_{0,\lfloor z\sqrt{t_w}\rfloor; t_w, (1+\theta)t_w}) \to \mathbb{P}(\tilde{A}_{z,\theta}) \tag{5.14}$$

as $t_w \to \infty$, where $\tilde{A}_{z,\theta}$ is the event that the two backward Brownian motions B and B' coalesce by time 0. See Figure 5.2.

5.1 An expression for $\mathbb{P}(\tilde{A}_{z,\theta})$

From the above, we can obtain an expression for $g(z,\theta) := \mathbb{P}(\tilde{A}_{z,\theta})$, and we do that now.

We will first condition on B'_θ, which has probability density function $f(x) = (1/\sqrt{2\pi\theta})\exp[-(x-z)^2/2\theta]$. Given $B'_\theta = x$, the probability of $\tilde{A}_{z,\theta}$ is the probability that two independent backward Brownian motions, one

[1] I.e., Brownian motions which are independent up until they meet, when they coalesce, and continue from then on as a single Brownian motion.

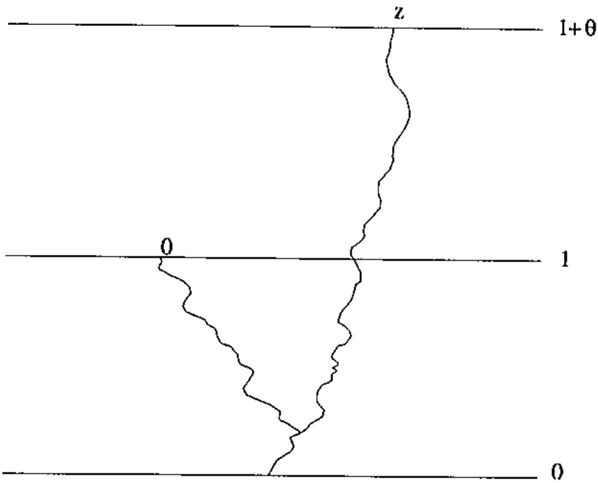

Figure 5.2: A realization of B and B' where $\tilde{A}_{z,\theta}$ occurs.

starting at $(0, 1)$ and the other at $(x, 1)$, meet before time 0. Let us denote by $\tilde{B}_0$ and $\tilde{B}_x$ these Brownian motions. Then, if $\tilde{A}_x$ is the event that $\tilde{B}_0$ and $\tilde{B}_x$ meet at some $s \in [0, 1]$, we have

$$g(z,\theta) = \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} dx\, e^{-(x-z)^2/2\theta} g(x)\,, \tag{5.15}$$

where $g(x) = \mathbb{P}(\tilde{A}_x)$.

Now, the event $\tilde{A}_x$ in (5.15) can be seen as that in which $\tilde{B}_0(s) - \tilde{B}_x(s) = 0$ for some $s \in [0, 1]$. Since the difference of two independent Brownian motions with diffusion coefficient 1 starting at x and y, respectively, is a Brownian motion with diffusion coefficient 2 starting at $x - y$, we can rewrite the probability $g(x)$ in (5.15) as

$$\begin{aligned} g(x) &= \mathbb{P}(B_x(s) = 0 \text{ for some } s \in [0,1]) \\ &= 1 - \mathbb{P}(B_x \text{ does not touch 0 during } [0,1]), \end{aligned} \tag{5.16}$$

where $B_x(s)$ is a Brownian motion with diffusion coefficient 2 starting at x. By the symmetry in x, and then the Reflection Principle [18], the latter probability can be expressed as

$$\mathbb{P}(B_{|x|}(1) > 0) - \mathbb{P}(B_{-|x|}(1) > 0).$$

Using translation invariance, this expression equals

$$\mathbb{P}(B_0(1) > -|x|) - \mathbb{P}(B_0(1) > |x|) = 2\mathbb{P}(0 < B'(1) < |x|/\sqrt{2}) = \phi(|x|/\sqrt{2}),$$

where $B'(\cdot)$ is a Brownian motion with diffusion coefficient 1 starting at the origin and

$$\phi(x) := \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt. \tag{5.17}$$

So,

$$\begin{aligned} g(z,\theta) &= \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} dx e^{-(x-z)^2/2\theta} \left[1 - \phi\left(|x|/\sqrt{2}\right)\right] \\ &= 1 - \frac{1}{\sqrt{2\pi\theta}} \int_0^{\infty} dx \left(e^{-(x-z)^2/2\theta} + e^{-(x+z)^2/2\theta}\right) \phi\left(x/\sqrt{2}\right). \end{aligned} \tag{5.18}$$

This can be further simplified as follows. Let

$$h(z) = \int_{-z}^{\infty} dx\, e^{-x^2/2\theta} \phi\left((x+z)/\sqrt{2}\right).$$

Then,

$$\sqrt{2\pi\theta}\,(1 - g(z,\theta)) = h(z) + h(-z). \tag{5.19}$$

Now,

$$h'(z) = \frac{1}{\sqrt{2}} \int_{-z}^{\infty} dx\, e^{-x^2/2\theta} \phi'\left((x+z)/\sqrt{2}\right) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} dx\, e^{-(x-z)^2/2\theta} e^{-x^2/4}. \tag{5.20}$$

Combining (5.19) and (5.20), we obtain after some rearrangement

$$\begin{aligned} g(z,\theta) &= \frac{1}{\pi\sqrt{2\theta}} \left[\int_{|z|}^{\infty} dt \int_0^{\infty} dx\; e^{-(x-t)^2/2\theta} e^{-x^2/4} \right. \\ &\quad \left. - \int_{-\infty}^{-|z|} dt \int_0^{\infty} dx\, e^{-(x-t)^2/2\theta} e^{-x^2/4}\right] \\ &= \frac{1}{\pi\sqrt{2\theta}} \int_{|z|}^{\infty} dt\; e^{-t^2/2\theta} \int_0^{\infty} dx\; e^{-\frac{x^2}{4\theta}(2+\theta)} \left[e^{xt/\theta} - e^{-xt/\theta}\right]. \end{aligned} \tag{5.21}$$

After some further algebra, we find

$$g(z,\theta) = \psi\left(|z|/(\sqrt{2+\theta}), \sqrt{2/\theta}\right), \tag{5.22}$$

where $\psi(a,b) = \sqrt{\frac{2}{\pi}}\int_a^\infty dt e^{-t^2/2}\phi(bt)$, $a, b > 0$. Because

$$\frac{\partial}{\partial b}\psi(a,b) = 2e^{-(1+b^2)a^2/2}/[\pi(1+b^2)],$$

we finally have

$$g(z,\theta) = \frac{2}{\pi}\int_0^{\sqrt{2/\theta}} dt\, \frac{e^{-z^2(1+t^2)/(2(2+\theta))}}{1+t^2}. \tag{5.23}$$

(5.22-5.23) reduce to more explicit formulas in particular cases:

$$g(0,\theta) = \tfrac{2}{\pi}\mathrm{arctg}\sqrt{2/\theta}, \tag{5.24}$$

$$g(z,0) = 1 - \phi(|z|/\sqrt{2}),\ g(z,2) = 1 - \phi^2(|z|/2). \tag{5.25}$$

5.2 A scaling limit for the full dynamics

We saw above how to obtain (quite explicit) aging results for a two-point two-time correlation function, namely C_q and the variant G_q, from the scaling limit of two coalescing random walks, in terms of two coalescing Brownian motions. It is straightforward to extend this analysis and results to n-point n-time correlation functions, which can, via (1.12), be related to n coalescing random walks, and the aging limits will be given in terms of the associated quantities for n coalescing Brownian motions. This will follow from the weak convergence of the properly rescaled random walks to the corresponding Brownian motions, which in turn can be seen as an application of Donsker's invariance principle for a single random walk.

If on the other hand we are interested in aging functions which depend on spins of a set of space-time locations which is not fixed or bounded in number as $t_w, t \to \infty$ (we will see an example right away), then we will not in general be able to get the aging result from Donsker's invariance principle alone. We will need something else.

For example, the *persistence function* $F_q(t_w, t_w+t)$, $0 < t_w, t$, is defined as the probability that there is no spin flip at the origin during the time interval

(t_w, t_w+t). Let $\hat{N} = \hat{N}(t_w, t_w+t)$ be the number of distinct backward random walks remaining at time zero from all those starting at $(0, s)$, $s \in (t_w, t_w+t)$. When $\hat{N} = k$ and $q < \infty$, the probability of no flips at 0 in this time interval is $(1/q)^{k-1}$ and so the persistence probability $F_q(t_w, t_w+t)$ is $\mathbb{E}[(1/q)^{\hat{N}-1}]$, and $F_\infty = \mathbb{P}(\hat{N} = 1)$. Now, $\hat{N}$ involves random walks starting from unboundedly many space-time points as $t_w, t \to \infty$.

In cases like these, the usual invariance principle for single or boundedly many random walks will not suffice. One thing we will have to consider is what kind of object could be the limit of unboundedly many random walks starting from a set of space time points which scales like t_w as $t_w \to \infty$ (like the $\{0\} \times (t_w, t_w + t)$, with $t = O(t_w)$, of the persistence function example). It is natural thus to aim for coalescing Brownian motions starting from *all* space time points in $\mathbb{R}^2$. This object in fact exists, can be constructed, and we will do that in the next chapter. In that context, we will call it the *Brownian web*. It consists of a network of Brownian paths starting from all space time points in $\mathbb{R}^2$ which are independent till they meet, and then they coalesce and continue as single Brownian paths until they meet again with another path, and coalesce again, and so on.

In the remainder of this section we will presume the existence of the Brownian web, assume that the proper scaling limit of the coalescing random walks starting from *all* space-time points of $\mathbb{Z} \times \mathbb{R}$ to the Brownian web holds (this will be discussed in the next chapter as well), and discuss some of the properties of the Brownian web, which will lead to an explicit computation of the persistence function in the case when $q = \infty$. In fact we will need to consider an extended continuum object, consisting of the Brownian web together with a dual object, which is itself a Brownian web, but a forward one (i.e., it is a network of coalescing Brownian paths running forwards in time; the Brownian web we referred to before arises from backward in time random walks, so it consists of backward in time Brownian paths).

As discussed in the beginning of the chapter, in the case of the voter dynamics with $q = \infty$, besides the backward coalescing random walks establishing color history/ancestry, we have forward coalescing random walks of the boundaries, which take place on the *dual lattice* $\mathbb{Z}^* := \mathbb{Z} + 1/2$. Notice that forward and backward random walk paths interact by reflection. That is, given a backward path, this path acts as a reflecting barrier for a forward path, which evolves as an independent random walk when it is at distance greater than $1/2$ away from that path. The continuum limit will be two versions of the Brownian web, one backwards, one forwards, which interact by reflection. We will refer to this *duality* of the two Brownian webs, and call the joint object the *double Brownian web.*

Returning now to the persistence function, the continuum version of $\hat{N}$ is $N = N(\theta)$, the number of distinct points at $\mathbb{R} \times \{0\}$ which are crossed by backward paths of the (backward) Brownian web starting at $\{0\} \times (1, 1+\theta)$.

In the scaling limit as $t_w, t \to \infty$ with $t/t_w \to \theta$, $\hat{N}$ should converge to N (more about this point in the next chapter). Thus, $F_q(t_w, t_w + t)$ should converge to

$$f_q(\theta) = \begin{cases} \mathbb{E}[(1/q)^{N-1}], & \text{if } q < \infty, \\ \mathbb{P}(N = 1), & \text{if } q = \infty. \end{cases} \tag{5.26}$$

We can obtain an explicit expression for $\mathbb{P}(N = 1)$, playing with the properties of the Brownian web and its dual, and we do that next, closing this chapter, and postponing a more careful discussion on the nature of Brownian web and the scaling limit of the coalescing random walks to the next one.

Let $(-X, Y)$ denote the (random) spatial interval with the same ($q = \infty$) spin value at time 1 as the origin. The event $A_{x,y}$ that $X > x$ and $Y > y$ occurs if and only if the backward Brownian paths starting at $(-x, 1)$ and $(y, 1)$ coalesce before time 0. See Figure 5.3.

Proceeding as in our analysis of (5.15), we see that

$$P(A_{x,y}) = 1 - \phi\left((x+y)/\sqrt{2}\right). \tag{5.27}$$

The probability density of (X, Y) is then

$$\mu(x, y) = \frac{1}{2\sqrt{\pi}}(x+y)e^{-(x+y)^2/4} \tag{5.28}$$

for $x, y > 0$. Now, given $(X, Y) = (x, y)$, $f_\infty(\theta)$ is the probability that the *forward* Brownian paths starting at $(-x, 1)$ and $(y, 1)$ do not touch the origin during $(1, 1+\theta)$, and thus

$$\mathbb{P}(N = 1) = \int_0^\infty \int_0^\infty dx\, dy\, \mu(x, y)\, \phi\left(\frac{x}{\sqrt{\theta}}\right) \phi\left(\frac{y}{\sqrt{\theta}}\right). \tag{5.29}$$

After further analysis of the same kind used to derive (5.22)-(5.23), one finds

$$f_\infty(\theta) = \frac{2}{\pi} \arcsin \frac{1}{1+\theta}. \tag{5.30}$$

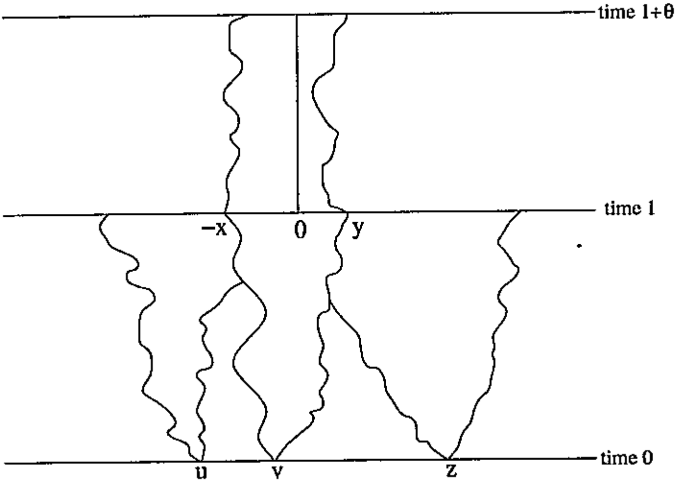

Figure 5.3: A (forward) path configuration where $A_{x,y}$ occurs. The paths represented before time 1 determine the color cluster of the origin, which has the color of v at time 0, and the ones adjacent to it. The paths starting between u and v at time 0 all coalesce before time 1, and so do all those starting between v and z. The backward paths starting on (-x,y) at time 1 all coalesce together by time 1. The backward paths starting on $\{0\} \times (1, 1+\theta)$ have to touch (-x,y) because they cannot cross the forward paths starting at -x and y.

Note that $\{N = 1\}$ as an event of the single backward Brownian web depends on infinitely many paths (all those starting at $\{0\} \times (1, 1+\theta)$), but in the above calculation, using the dual web, we needed to consider only four Brownian paths. By a similar argument, using the same duality which exists for the backward and forward families of coalescing random walks before the limit, we can show, using Donsker's invariance principle for single random walks only, that

$$F_\infty(t_w, t_w + t) = \mathbb{P}(\hat{N} = 1) \to \mathbb{P}(N = 1) = f_\infty(\theta) \qquad (5.31)$$

as $t_w, t \to \infty$ with $t/t_w \to \theta$.

Remark 5.1 *A similar approach to compute* $\mathbb{P}(\hat{N} = k)$, $k > 1$, *would require the distribution of boundaries at time* 1 *(after all the coalescing from the continuum infinity of colors at time* 0*). This is actually a simple point process [4], and it is known that the random walk version converges weakly to the Brownian one [4]. See also [5, 6]. So one can actually, based on the results in [4], get the convergence of* $\mathbb{P}(\hat{N} = k)$ *to* $\mathbb{P}(N = k)$ *for* $k > 1$, *and thence of* $F_q(t_w, t_w + t)$ *to* $f_q(\theta)$ *for finite* q, *by using additionally only Donsker's invariance principle for single random walks.*

Remark 5.2 *From the discussion in the previous remark, we conclude that a recourse to the full Brownian web, and weak convergence thereto, can be avoided, but one will need then to consider a family of coalescing Brownian paths starting from all points of* $\mathbb{R} \times \{0\}$ *and weak convergence to one of its functions, namely, the boundaries at time* 1. *Actually, the continuum process for the original dynamics, the voter model with* $q = \infty$, *which can be represented fully by the boundary dynamics consisting of coalescing random walks starting from every point in* $\mathbb{Z}^* \times \{0\}$, *is exactly the restricted web of coalescing Brownian paths starting from all points of* $\mathbb{R} \times \{0\}$ *(rather than the full one starting from all points of* $\mathbb{R}^2$*), so all we need to consider is that object and weak convergence thereto. At the end of the next chapter we will introduce a variant of the voter model dynamics, corresponding to a spin dynamics at* very low *temperature, for which the continuum limit involves necessarily the full double Brownian web.*

Chapter 6

The Brownian web

In this chapter we construct a network of coalescing one dimensional Brownian paths starting from every space-time point in $\mathbb{R}^2$. We call it the *Brownian web*. This object arises as (a candidate for) the scaling limit of coalescing one dimensional random walks starting from every space-time point in $\mathbb{Z} \times \mathbb{R}$, which appear in the dual/backward representation of the voter model. It is then related to aging issues for that dynamics. The limiting aging functions are potentially expected values of variables of the Brownian web.

A system of coalescing one dimensional Brownian paths starting from every space-time point in $\mathbb{R}^2$ has been introduced and studied in [6] in connection also with coalescing random walks. A variant of it, with reflection and absorption boundary conditions, was constructed and analysed in [55], where it was used as an auxiliary process, in another context. Neither of these works considered issues of characterization of or weak convergence (of rescaled coalescing random walks, for example) to the constructed system. Our construction can be seen as yet another variant of those constructions, this time with characterization and weak convergence results in mind. In particular, it sets the stage for the derivation of weak convergence criteria, which are applicable to coalescing random walks and other coalescing systems (more on this later).

Even though the final object of our construction is different in important ways from the previous constructions, all of them start out in the same way, by defining first a network of Brownian paths starting from a dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$. From then on, each construction takes a different way to obtain paths starting out from $\mathbb{R}^2 \setminus \mathcal{D}$. In [6, 55], different, but similar in spirit, kinds of semicontinuity conditions are imposed, and those lead to the final object. In particular, each point of $\mathbb{R}^2$ has exactly one path starting out from it in those constructions. In our construction, we regard the preliminary set of paths starting from $\mathcal{D}$ as a subset of a suitable metric space of paths (Π, d), with good topological properties. Our final object will be the closure of the preliminary set in (Π, d). As it turns out, the object thus obtained, the *Brownian web*, is almost surely a compact set of (Π, d). We can regard the Brownian web then as a random variable living in the space of compact subsets of (Π, d). This is the Hausdorff metric space $(\mathcal{H}, d_{\mathcal{H}})$ associated to (Π, d), from which it inherits good topological properties, and thus we have a set up for characterizing and deriving weak convergence to the Brownian web, which we proceed to do. We can then apply those convergence criteria to coalescing random walks and other processes. This approach is inspired in that of [1] to study scaling limits of two dimensional critical statistical physics systems in equilibrium. See also [2]. One other feature of our construction of the Brownian web, which distinguishes it from the previous ones, is that there will be *multiple (starting) points*, i.e., points where more than one path from the Brownian web start out from.

In the next section, we give more details of the construction, characterization of and weak convergence to the Brownian web. We discuss then briefly the application of those criteria to coalescing random walks. In the section following the next one, we discuss duality, introduce the *double Brownian web* (already discussed in the previous seciton), and give some results about types of multiple points (according to multiplicity). In the final section we discuss a spin dynamics which could be described as a dynamics at very low temperature, for which the proper scaling limit will involve a Brownian web *with marks*.

6.1 Characterization and convergence

We start with some topological background, specifying the probability space where we make the construction. Since we want to define a family of random paths starting from all points of the plane, we find convenient to take a compactified space. Let $(\bar{\mathbb{R}}^2, \rho)$ be the completion (or compactification) of $\mathbb{R}^2$ under the metric ρ given by

$$\rho((x_1, t_1), (x_2, t_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \quad (6.1)$$

$\bar{\mathbb{R}}^2$ may be thought of as the image of $[-\infty,\infty]\times[-\infty,\infty]$ under the mapping

$$(x,t) \rightsquigarrow (\Phi(x,t),\Psi(t)) \equiv \left(\frac{\tanh(x)}{1+|t|},\tanh(t)\right). \tag{6.2}$$

For $t_0 \in [-\infty,\infty]$, let $C[t_0]$ denote the set of functions f from $[t_0,\infty]$ to $[-\infty,\infty]$ such that $\Phi(f(t),t)$ is continuous. Then define

$$\Pi = \bigcup_{t_0\in[-\infty,\infty]} C[t_0]\times\{t_0\}, \tag{6.3}$$

where $(f,t_0)\in\Pi$ represents a path in $\bar{\mathbb{R}}^2$ starting at $(f(t_0),t_0)$. For(f,t_0) in Π, we denote by $\hat{f}$ the function that extends f to all $[-\infty,\infty]$ by setting it equal to $f(t_0)$ for $t<t_0$. Then we take

$$d((f_1,t_1),(f_2,t_2)) = (\sup_t |\Phi(\hat{f}_1(t),t) - \Phi(\hat{f}_2(t),t)|) \vee |\Psi(t_1)-\Psi(t_2)|. \tag{6.4}$$

(Π,d) is a complete separable metric space.

Let now $\mathcal{H}$ denote the set of compact subsets of (Π,d), with $d_{\mathcal{H}}$ the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1,K_2) = \sup_{g_1\in K_1}\inf_{g_2\in K_2} d(g_1,g_2) \vee \sup_{g_2\in K_2}\inf_{g_1\in K_1} d(g_1,g_2). \tag{6.5}$$

$(\mathcal{H},d_{\mathcal{H}})$ is also a complete separable metric space.

The probability space where the Brownian web will live is $(\mathcal{H},\mathcal{F}_{\mathcal{H}})$, where $\mathcal{F}_{\mathcal{H}}$ the Borel σ-field associated to $(\mathcal{H},d_{\mathcal{H}})$. We can now give an existence and characterization result for the Brownian web.

Theorem 6.1 *There is an $(\mathcal{H},\mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{\mathcal{W}}$ whose distribution is uniquely determined by the following three properties.*

(o) From any deterministic point (x,t) in $\mathbb{R}^2$, there is almost surely a unique path $W_{x,t}$ starting from (x,t).

(i) For any deterministic $n,(x_1,t_1),\ldots,(x_n,t_n)$, the joint distribution of $W_{x_1,t_1},\ldots,W_{x_n,t_n}$ is that of coalescing Brownian motions (with unit diffusion constant).

(ii) For any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\bar{\mathcal{W}}$ is the closure in $(\mathcal{H},d_{\mathcal{H}})$ of $\{W_{x,t}:(x,t)\in\mathcal{D}\}$.

Remark 6.2 *There are natural $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables satisfying (o) and (i) but not (ii). An instance of such a random variable can be shown to arise as the scaling limit of some stochastic flows, extending work of Piterbarg [47].*

An explicit construction of the Brownian web follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where an i.i.d. family of standard Brownian motions $(B_j)_{j\geq 1}$ is defined. Let $\mathcal{D} = \{(x_j, t_j),\, j \geq 1\}$ be a countable dense set in $\mathbb{R}^2$. Let

$$W_j(t) = x_j + B_j(t - t_j),\ t \geq t_j \tag{6.6}$$

be a Brownian path starting at position x_j at time t_j.

We now construct coalescing Brownian paths out of the family of paths $(W_j)_{j\geq 1}$ by specifying coalescing rules. When two paths meet for the first time, they coalesce into a single path, which is that of the Brownian motion with the lower label. We denote the coalescing Brownian paths by $\tilde{W}_j$, $j \geq 1$.

We now give a more careful inductive definition of the coalescing paths. First we set

$$\tilde{W}_1 = W_1. \tag{6.7}$$

For $j \geq 2$, let $\tilde{W}_j$ be the mapping from $[t_j, \infty)$ to $\mathbb{R}$ defined as follows. Let

$$\tau_j = \inf\{t \geq t_j : W_j(t) = \tilde{W}_i(t) \text{ for some } 1 \leq i < j\}, \tag{6.8}$$
$$I_j = \min\{1 \leq i < j : W_j(\tau_j) = \tilde{W}_i(\tau_j)\} \tag{6.9}$$

be the first time when W_j meets any of the paths $\tilde{W}_i$ with $i < j$, and the smallest label among such paths, respectively. Set

$$\begin{aligned} \tilde{W}_j(t) &= W_j(t), \text{ if } t_j \leq t \leq \tau_j \\ &= \tilde{W}_{I_j}(t), \text{ if } t > \tau_j. \end{aligned} \tag{6.10}$$

We then define the *Brownian web skeleton* $\mathcal{W}(\mathcal{D})$ with *starting set* $\mathcal{D}$ as

$$\tilde{\mathcal{W}}_k = \tilde{\mathcal{W}}_k(\mathcal{D}) = \{\tilde{W}_j : 1 \leq j \leq k\} \tag{6.11}$$
$$\tilde{\mathcal{W}} = \tilde{\mathcal{W}}(\mathcal{D}) = \bigcup_k \tilde{\mathcal{W}}_k. \tag{6.12}$$

Definition 6.3 *Let $\bar{\mathcal{W}}(\mathcal{D})$ be the closure in (Π, d) of $\tilde{\mathcal{W}}(\mathcal{D})$. $\bar{\mathcal{W}}(\mathcal{D})$ is the object we will call the* Brownian web.

Proposition 6.4 *$\bar{\mathcal{W}}(\mathcal{D})$ has the following properties.*

a. *It is almost surely a compact subset of (Π, d).*

b. *Almost surely, $\bar{\mathcal{W}}(\mathcal{D}) = \lim_{k\to\infty} \tilde{\mathcal{W}}_k(\mathcal{D})$, where the limit is taken in $\mathcal{H}$.*

c. *The distribution of $\bar{\mathcal{W}}(\mathcal{D})$ does not depend on $\mathcal{D}$ (nor on its ordering).*

d. *$\bar{\mathcal{W}}(\mathcal{D})$ satisfies properties (o) and (i) of Theorem 6.1; i.e., its finite dimensional distributions (whether from points in $\mathcal{D}$ or not) are those of coalescing Brownian motions. Furthermore, it also satisfies property (ii) of that theorem.*

e. *Almost surely, for every $\epsilon > 0$ and every $\theta = (f(s), t_0)$ in $\bar{W}(\mathcal{D})$, there exists a path $\theta_\epsilon = (g(s), t_0')$, in the skeleton $\tilde{\mathcal{W}}(\mathcal{D})$ such that $g(s) = f(s)$ for all $s \geq t_0 + \epsilon$.*

The latter property says that even though there are many more paths in $\bar{\mathcal{W}}(\mathcal{D})$ than in $\tilde{\mathcal{W}}(\mathcal{D})$ (a continuum infinity in the former vis-a-vis a countable infinity in the latter), almost surely every path in $\bar{\mathcal{W}}(\mathcal{D})$ coincides with paths in $\tilde{\mathcal{W}}(\mathcal{D})$ after every time strictly above its starting time.

We next state a convergence theorem. It is restricted to random variables with noncrossing paths (that is, random variables living in the subset of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ consisting of compact sets of paths where no two paths cross each other). More general results, as well as proofs, are given in Sections 6 and 7 of [27]. Our convergence criteria involve a *counting random variable*, which we define next.

Definition 6.5 *For $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$, let $\eta(t_0, t; a, b)$ be the number of* distinct *points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths in $\bar{\mathcal{W}}$ which also touch some point in $[a, b] \times \{t_0\}$. Let also $\hat{\eta}(t_0, t; a, b) = \eta(t_0, t; a, b) - 1$.*

We note that by duality arguments (see Section 6.2), it can be shown that for deterministic t_0, t, a, b, $\hat{\eta}$ is equidistributed with the number of distinct points in $[a, b] \times \{t_0 + t\}$ that are touched by paths in $\bar{\mathcal{W}}$ which also touch $\mathbb{R} \times \{t_0\}$.

Theorem 6.6 *Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ are $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables with noncrossing paths. If, in addition, the following three conditions are valid, then the distribution μ_n of $\mathcal{X}_n$ converges to the distribution $\mu_{\bar{\mathcal{W}}}$ of the standard Brownian web.*

(I_1) *For any deterministic* $y_1, \ldots, y_m \in \mathcal{D}$*, where* $\mathcal{D}$ *is an arbitrary deterministic countable dense subset of* $\mathbb{R}^2$*, there exist* $\theta_n^{y_1}, \ldots, \theta_n^{y_m} \in \mathcal{X}_n$ *such that* $\theta_n^{y_1}, \ldots, \theta_n^{y_m}$ *converge in distribution as* $n \to \infty$ *to coalescing Brownian motions (with unit diffusion constant) starting at* $y_1, \ldots, y_m$*.*

(B_1) $\forall t > 0, \ \limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mu_n(\hat{\eta}(t_0, t; a, a+\epsilon) \geq 1) \to 0$ *as* $\epsilon \to 0+$*;*

(B_2) $\forall t > 0, \ \epsilon^{-1}\limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mu_n(\hat{\eta}(t_0, t; a, a+\epsilon) \geq 2) \to 0$ *as* $\epsilon \to 0+$*.*

Remark 6.7 *Condition* I_1 *and the noncrossing property imply tightness of* $\mathcal{X}_1, \mathcal{X}_2, \ldots$ *By* I_1*, any subsequence limit contains a version of the Brownian web. Conditions* B_1 *and* B_2 *insure that it contains nothing else. See [27].*

We next apply Theorem 6.6 to coalescing random walks.

Convergence of coalescing random walks

We will consider both continuous and discrete time families of coalescing simple symmetric random walks in dimension 1, denoted X and $\tilde{X}$, respectively, and their diffusive rescalings $X^{(\delta)}$ and $\tilde{X}^{(\delta)}$, resp., where δ is an arbitrary positive real number. X as well as a backward version of it was defined in Chapter 1 and then used at other chapters, including this one. Let us recall/remake the definition of X, and then define $\tilde{X}$. Let us consider a full space Harris diagram in $\mathbb{Z} \times \mathbb{R}$, such that for each $x \in \mathbb{Z}$, we have a doubly infinite time line, and in that a Poisson point process with rate 1. The Poisson point processes of different sites are independent. At each Poisson mark, we flip a fair coin to decide whether to put a leftward or rightward outgoing arrow of unit length. For $(x,t) \in \mathbb{Z} \times \mathbb{R}$, let $X'_{x,t}(s), s \geq t$, be the continuous time random walk starting at (x,t) and following the arrows upwards. In order to have continuous trajectories, we make the jump from the Poisson mark at a site, say (y,r), not horizontally to, say, $(y+1,r)$ (in which case, we would have a discontinuity of the path at that point), but rather to $(y+1,r')$, where r' is the instant of the next Poisson mark of the Poisson process at $y+1$'s time line after r. We then have random walks $X_{x,t}$ defined for the Poisson mark times, and we define it at the other times, either by linear interpolation, in the case of times after the first Poisson mark time at the time line of x after t, or as $X_{x,t}(s)$ otherwise. $\{X_{x,t}(s), s \geq t : (x,t) \in \mathbb{Z} \times \mathbb{R}\}$ is then a family of continuous time, continuous path coalescing random walks.

Adding to these some boundary paths in order to have a compact subset of (Π, d), we get X. See Figure 6.1 below.

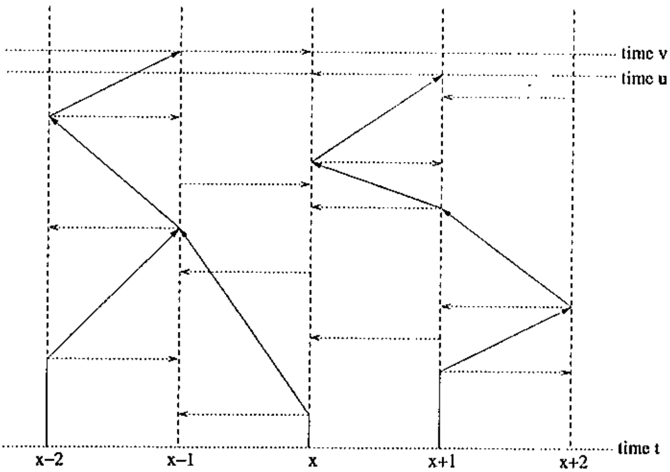

Figure 6.1: Coalescing random walks in continuous time. The horizontal coordinate is space and the vertical one is time. Time lines are dashed. We portray in full lines the paths starting at (x-2,t) and (x,t) (till time v), and at (x+1,t) (till time u).

Now for the discrete time family. Consider the lattice $\mathbb{Z}^2_e$ consisting of the sites $(i, j) \in \mathbb{Z}^2$ such that $i+j$ is even. For each site of $\mathbb{Z}^2_e$ flip an independent fair coin to decide wheteher to put a northeast or a northwest arrow of length $\sqrt{2}$. For $(x,t) \in \mathbb{Z}^2_e$, let $\tilde{X}_{x,t}(s), s = t, t+1, \dots$ be the random walk starting at (x, t) and following the arrows upwards, and define $\tilde{X}_{x,t}(s)$ for noninteger times by linear interpolation. $\{\tilde{X}_{x,t}(s), s \geq t : (x,t) \in \mathbb{Z} \times \mathbb{R}\}$ is then a family of continuous time continuous path coalescing random walks. Adding to these some boundary paths in order to have a compact subset of (Π, d), we get $\tilde{X}$. Figure 6.2 below represents a portion of $\tilde{X}$.

Let us now rescale X and $\tilde{X}$. We make the usual diffusive rescaling of time with $\delta > 0$ and space with $\sqrt{\delta}$, and let $\delta \to 0$.

For $(x,t) \in \sqrt{\delta}\,\mathbb{Z} \times \mathbb{R}$, let

$$X^{(\delta)}_{x,t}(s) = \sqrt{\delta}\, X_{\delta^{-1/2}x,\delta^{-1}t}(\delta^{-1}s), \tag{6.13}$$

$s \geq t$, and $X^{(\delta)} = \{X^{(\delta)}_{x,t}(s), s \geq t : (x,t) \in \sqrt{\delta}\,\mathbb{Z} \times \mathbb{R}\}$ plus boundary paths.

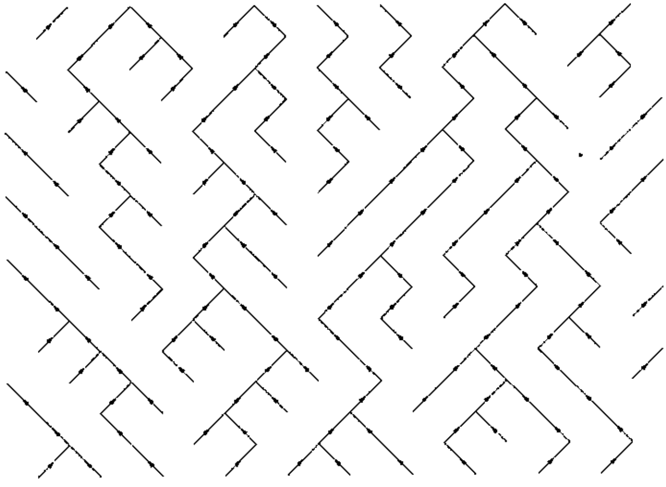

Figure 6.2: Coalescing random walks in discrete time; the horizontal coordinate is space and the vertical one is time.

And for $(x,t) \in \mathbb{Z}^2_{e,\delta} := \{(\sqrt{\delta}\,x', \delta t') : (x',t') \in \mathbb{Z}^2_e\}$, let

$$\tilde{X}^{(\delta)}_{x,t}(s) = \sqrt{\delta}\,\tilde{X}_{\delta^{-1/2}x,\delta^{-1}t}(\delta^{-1}s), \tag{6.14}$$

$s \geq t$, and $\tilde{X}^{(\delta)} = \{\tilde{X}^{(\delta)}_{x,t}(s), s \geq t : (x,t) \in \mathbb{Z}^2_{e,\delta}\}$ plus boundary paths.

We now state our convergence theorem for the rescaled coalescing random walks $X^{(\delta)}$ and $\tilde{X}^{(\delta)}$.

Theorem 6.8 *Each of the collections of rescaled coalescing random walk paths, $X^{(\delta)}$ (in continuous time) and $\tilde{X}^{(\delta)}$ (in discrete time) converges in distribution to the standard Brownian web as $\delta \to 0$.*

Sketch of proof of Theorem 6.8 We discuss the continuous time case only. The discrete time one is similar. Let us fix a sequence $\delta = \delta_n$ of positive numbers going to zero as $n \to \infty$. We want to verify conditions I_1, B_1 and B_2 of Theorem 6.6. I_1 follows from Donsker's invariance principle for single paths and the fact that the coalescence operation on independent paths

$$(W_{x_1,t_1}, \ldots, W_{x_n,t_n}) \to (\tilde{W}_{x_1,t_1}, \ldots, \tilde{W}_{x_n,t_n}) \tag{6.15}$$

(see (6.10)) is almost surely continuous.

B_1 follows from the equivalence $\hat{\eta}(t_0, t; a, a + \epsilon) \geq 1$ if and only if $X^{(\delta)}_{a,t_0}$ and $X^{(\delta)}_{a+\epsilon_\delta,t_0}$ do not coalesce before $t_0 + t$, where $\epsilon_\delta = \sqrt{\delta}\,\lfloor \delta^{-\frac{1}{2}}\epsilon \rfloor$. This is if and only if $Y^{(\delta)}_\epsilon := X^{(\delta)}_{a+\epsilon,t_0} - X^{(\delta)}_{a,t_0}$ does not hit 0 before $t_0 + t$. But $Y^{(\delta)}_\epsilon$ is a random walk itself (with rate 2 Poisson clocks), whose distribution is independent of a, t_0. By the invariance principle for $Y^{(\delta)}_\epsilon$, and since $\epsilon_\delta \to \epsilon$ as $\delta \to 0$, we have that the limsup in B_1 equals the probability that a standard Brownian motion multiplied by $\sqrt{2}$ does not hit ϵ by time t. This is well known to be $O(\epsilon)$, and B_1 is established.

For B_2, we use B_1 and an inequality to the effect that

$$\mu_n(\hat{\eta}(t_0, t; a, a + \epsilon) \geq 2) \leq [\mu_n(\hat{\eta}(t_0, t; a, a + \epsilon) \geq 1)]^2 \qquad (6.16)$$

for every n, a, ϵ, t_0, t. This holds for the distribution μ_n of the coalescing random walks (rescaled by δ_n) and follows from the FKG inequality. An argument for that can be found in the proof of Proposition 4.1 of [27]. We conclude from (6.16) and B_1 that the limsup in B_2 is $O(\epsilon^2)$, and the result follows.

Remark 6.9 *In the above example, the simplicity of the random walks, with jumps of length 1 only, is crucial for the property of noncrossing of the paths, so Theorem 6.6 can be applied. For the case, e.g., of coalescing random walks with jumps of length 1 and 2, that property no longer holds, and Theorem 6.6 cannot be applied. This case should be treated with a more general result, namely Theorem 6.3 of [27].*

Remark 6.10 *In [23], another system of one dimensional coalescing/non-crossing paths, the* Poisson tree *is analysed and proved to converge to the Brownian web in the diffusive scaling limit. That system can be seen as coalescing random walks in continuum space with short range dependence.*

6.2 Dual and double Brownian webs

We already introduced the *Dual Brownian web* and the *Double Brownian web* informally in Section 5.2. As pointed in that section, it arises as the continuum counterpart of the duality in the voter model of the backward coalescing random walks providing color ancestry, on one side, and the forward coalescing random walks of boundaries, one at each site of $\mathbb{Z}^*$ initially, on the other side. We say that these two families of coalescing random walks are *dual* to/of one another. The interaction between a path in the former family and a path in the latter one is by reflection (whereas inside the families, the interaction is by coalescence). Incidentally, a duality of backward and forward families of coalescing random walks exists also in discrete time. See Figure 6.3.

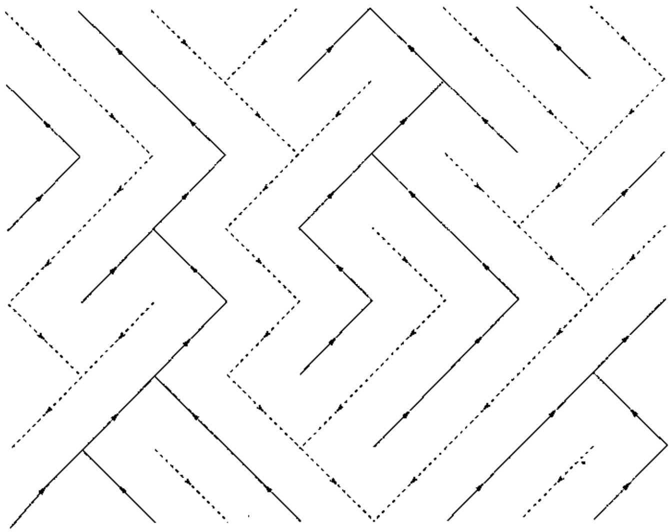

Figure 6.3: Forward coalescing random walks (full lines) in discrete time and their dual backward walks (dashed lines).

In the continuum, we have a family of forward coalescing Brownian paths $\bar{\mathcal{W}}$ and a dual family of backward coalescing Brownian paths $\bar{\mathcal{W}}^b$. Paths of different families interact by reflection. This duality can be very useful to understand what is going on within any single family, as we saw in Section 5.2 in the calculation of the scaling limit of the persistence function. Taking together the Brownian web and its dual Brownian web, we have the *Double Brownian web*, denoted $\bar{\mathcal{W}}^D = (\bar{\mathcal{W}}, \bar{\mathcal{W}}^b)$.

As we mentioned before, in each single Brownian web we have multiple points, which are starting points of more than one path. Based on this, we introduce next *point types*, depending on how many paths start or go through a given point. An analysis of the Double Brownian web allows for a complete classification of possible point types (almost surely) and even an evaluation of the Hausdorff dimension of each type. We discuss all that briefly and informally, and refer to Section 5 of [27] for more details. Part of the results described below were first derived in [55] (for a variant of the Brownian web).

For a point (x_0, t_0) of $\mathbb{R}^2$, let $m_{\text{in}}(x_0, t_0)$ be the number of paths in the forwards web $\bar{\mathcal{W}}$ that go through (x_0, t_0) (they must start before t_0 and be disjoint at all times before t_0), and let $m_{\text{out}}(x_0, t_0)$ be the number of paths in $\bar{\mathcal{W}}$ that start at (x_0, t_0). We define $m^b_{\text{in}}(x_0, t_0)$ and $m^b_{\text{out}}(x_0, t_0)$ similarly for the backwards web $\bar{\mathcal{W}}^b$.

Definition 6.11 *The type of* (x_0, t_0) *is the pair* $(m_{\text{in}}, m_{\text{out}})$*—see Figure 6.4. We denote by* $S_{i,j}$ *the set of points of* $\mathbb{R}^2$ *that are of type* (i, j)*, and by* $\bar{S}_{i,j}$ *the set of points of* $\mathbb{R}^2$ *that are of type* (k, l) *with* $k \geq i$, $l \geq j$.

Remark 6.12 *Using the translation and scale invariance properties of the Brownian web distribution, it can be shown that for any* i, j*, whenever* $S_{i,j}$ *is nonempty (with positive probability), it must be dense in* $\mathbb{R}^2$ *(almost surely). The same can be said of* $S_{i,j} \cap \mathbb{R} \times \{t\}$ *for deterministic* t.

Proposition 6.13 *For almost every realization of the double Brownian web, for* every $(x_0, t_0) \in \mathbb{R}^2$, $m^b_{\text{in}}(x_0, t_0) = m_{\text{out}}(x_0, t_0) - 1$ *and* $m^b_{\text{out}}(x_0, t_0) = m_{\text{in}}(x_0, t_0) + 1$*. See Figure 6.4.*

Theorem 6.14 *For the (double) Brownian web, almost surely, every* (x, t) *has one of the following types, all of which occur:* $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 1)$.

Remark 6.15 *Points of type* $(1, 2)$ *are particularly interesting in that the single incident path continues along exactly one of the two outward paths — with the choice determined intrinsically rather than by some convention. See Figure 6.4 for a schematic diagram of a "left-handed" continuation. An* (x_0, t_0) *is of type* $(1, 2)$ *precisely if both a forward and a backward path pass through* (x_0, t_0)*. It is either left-handed or right-handed according to whether the forward path is to the left or the right of the backward path near* (x_0, t_0)*. Both varieties occur and the Hausdorff dimension of* 1 *applies separately to each of the two varieties (see Theorem 6.16 below).*

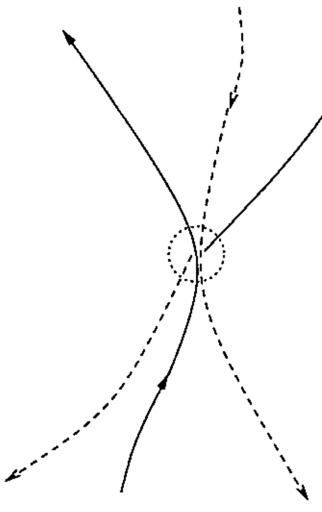

Figure 6.4: A schematic diagram of a point (x_0, t_0) of type $(m_{in}, m_{out}) = (1, 2)$, with necessarily also $(m^b_{in}, m^b_{out}) = (1, 2)$. In this example the incoming forward path connects to the leftmost outgoing path (with a corresponding dual connectivity for the backward paths); at some of the other points of type $(1, 2)$ it will connect to the rightmost path.

Theorem 6.16 *Almost surely, $S_{0,1}$ has full Lebesgue measure in $\mathbb{R}^2$, $S_{1,1}$ and $S_{0,2}$ have Hausdorff dimension $3/2$ each, $S_{1,2}$ has Hausdorff dimension 1, and $S_{2,1}$ and $S_{0,3}$ are both countable and dense in $\mathbb{R}^2$.*

Theorem 6.17 *Almost surely: for every t*

a) $S_{0,1} \cap \mathbb{R} \times \{t\}$ has full Lebesgue measure in $\mathbb{R} \times \{t\}$;

b) $S_{1,1} \cap \mathbb{R} \times \{t\}$ and $S_{0,2} \cap \mathbb{R} \times \{t\}$ are both countable and dense in $\mathbb{R} \times \{t\}$;

c) $S_{1,2} \cap \mathbb{R} \times \{t\}$, $S_{2,1} \cap \mathbb{R} \times \{t\}$ and $S_{0,3} \cap \mathbb{R} \times \{t\}$ have all cardinality at most 1.

For every deterministic t, $S_{1,2} \cap \mathbb{R} \times \{t\}$, $S_{2,1} \cap \mathbb{R} \times \{t\}$ and $S_{0,3} \cap \mathbb{R} \times \{t\}$ are almost surely empty.

Even though Theorems 6.14, 6.16 and 6.17 are about the single Brownian web, the arguments for the proofs are made particularly simple by the use of duality, through Proposition 6.13. We do not give much detail here (rather referring to Section 5 in [27] for full arguments), but only discuss a few cases, and informally.

For example, Property (o) of Theorem 6.1 (see Proposition 6.4 d.) implies that Lebesgue-almost every point of $\mathbb{R}^2$ is of type $(0,1)$.

Points in $\bar{S}_{1,1}$ are *continuation points*, i.e., points in the middle of paths. By Proposition 6.4 e., they belong to paths in the skeleton, which are countably infinite, and each one has Hausdorff dimension 3/2 almost surely (since they are Brownian paths). We also have that $S_{1,1}$ has this same dimension (see [27]). By duality, points of type $(1,1)$ correspond to points of type $(0,2)$ of the dual web, so $S_{0,2}$ inherit the same properties of $S_{1,1}$.

Points of type $(0,3)$ correspond to those of type $(2,1)$ of the dual web. $\bar{S}_{2,1}$ consists of coalescence points, which are in the web skeleton, and we have one of them for every pair of skeleton paths. Since these are countable, we get that $\bar{S}_{2,1}$ is countable, and so must be $S_{2,1}$.

$\bar{S}_{3,1}$ are points in the skeleton where three paths coalesce. Since for any triple of independent Brownian paths starting at distinct space time points, the probability of meeting all at the same point and time is zero, we get that $\bar{S}_{3,1} = \emptyset$ almost surely.

Remark 6.18 *One consequence of Proposition 6.13 we used above and which we will use again in next section is the correspondence of points of type* $(0,2)$, *which we call from now on* double points[1], *on one web to those of type* $(1,1)$, *which are continuation points, on the dual web. This shows that double points and continuation points are dual to each other. See Figure 6.5.*

6.3 Spin dynamics at very low temperature

In this section, we analyse a dynamics at positive temperature, which is a modification of the voter dynamics studied above, and corresponds as well to the Glauber heat-bath dynamics at positive temperature. Then we will take its scaling limit, and thence obtain an aging result for a two-point function (see (6.34) below), as we did in Chapter 5 for the zero temperature case (see (5.23)). In order to have have a nontrivial result, the temperature will

[1] We call the paths issuing out from double points *double paths*.

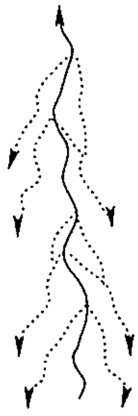

Figure 6.5: All along the full path there starts dual dotted double paths, one path of each pair on each side of the full path, making all the points on the full path (but the starting one) double points of the dual web. The dotted paths reflect off the full one and coalesce with each other.

be also rescaled, in such a way as to go to zero with the scalė. In this context, we refer to very low temperature.

We will consider below a mixture of a zero temperature dynamics, namely the voter model we studied in previous sections and chapters, and an infinite temperature dynamics. In the latter dynamics, the spin updates are made with spins taken successively from an i.i.d. sequence of random variables that are uniformly distributed in $\{1, \ldots, q\}$, where $q \geq 2$ is a parameter of the model, which can take the value ∞ (in which case, the i.i.d. sequence of random variables consists of spins that are all distinct from each other). We describe this dynamics in detail now. We start with the same framework of Poisson clocks of rate 1, one for each site of $\mathbb{Z}$. At each site, when the clock rings, a coin with probability p of coming up heads is tossed. The parameter p may depend on time. If the coin comes up heads, then the infinite temperature dynamics is performed, that is, the update is made using a spin chosen uniformly at random among the q possibilities. If it comes up tails, then the voter dynamics is performed.

We will call this the *marked voter dynamics* (MVD) with rate p and q states. The marks are the Poisson times where the infinite temperature dynamics is performed. They form themselves Poisson point processes of rate p.

We remark that, with the appropriate choice of p, the marked voter dynamics with 2 states is equivalent to a Glauber dynamics for the Ising model, namely the heat-bath dynamics, at inverse temperature β. This choice is $p = 2/(1+e^{\beta})$, with the proper representation of the Hamiltonian. Here, of course, p and β are constant. By analogy, we call $\beta = \log \frac{2-p}{p}$ the temperature (profile) of the MVD for all p and q (p not necessarily constant in time).

The dual model for the marked voter dynamics is the *marked coalescing random walks* (MCRW), which are backward (in time) coalescing random walks starting from every space-time point in $\mathbb{Z} \times \mathbb{R}$ on whose paths the marks of (i.i.d.) Poisson point processes in $\mathbb{Z} \times \mathbb{R}$, denoted

$$\mathcal{E} = \{e_i = (k_i, t_i),\ i = 1, 2, \ldots\},$$

are superimposed. $\mathcal{E}$ is independent of the random walk holding times and jumps.

Let $X_{x_0,t_0}(t)$ denote the position at time t of a backward random walk started at position x_0 and time $t_0 \geq t$. The spin at $(x,t) \in \mathbb{Z} \times \mathbb{R}^+$, denoted $\sigma_t(x)$, under the MVD starting with a product of uniforms in $\{1, \ldots, q\}$ initial condition, denoted by $\eta = \{\eta(x),\ x \in \mathbb{Z}\}$, can be expressed in terms of the dual model as follows. Let the backward random walks in the MCRW have rate $1-p$ and let $\mathcal{E}$ have rate p. Let $\zeta = \{\zeta_i,\ i = 1, 2, \ldots\}$ denote another i.i.d. family of uniforms in $\{1, \ldots, q\}$, independent of everything else. In the case $q = \infty$, then a different spin is assigned to each η_i and ζ_i (so that η and ζ consist of distinct spins altogether). For $t \in \mathbb{R}$ and $x \in \mathbb{Z}$, let

$$T_{x,t} = \sup\{s \leq t : X_{x,t}(s) \in \mathcal{E}\} \tag{6.17}$$

and

$$I_{x,t} = i \text{ such that } e_i = (X_{x,t}(T_{x,t}), T_{x,t}). \tag{6.18}$$

Then,

$$\sigma_t(x) = \begin{cases} \zeta_{I_{x,t}}, & \text{if } T_{x,t} < 0; \\ \eta_{X_{x,t}(0)}, & \text{otherwise,} \end{cases} \tag{6.19}$$

for all $x \in \mathbb{Z}$, $t \in \mathbb{R}^+$. In other words, $\{\sigma_t(x);\ x \in \mathbb{Z},\ t \in \mathbb{R}^+\}$ can be realized by running the MCRW from t to 0 and assigning to $\sigma_t(x)$ the spin assigned to either $e_{I_{x,t}}$ or $\sigma_{X_{x,t}(0)}(0)$, depending on whether $T_{x,t} > 0$ or not, respectively.

Now we want to rescale space-time with the aim of establishing a scaling limit result for the MVD. For that, we will use the duality relation with the

MCRW, and get the result for the former dynamics in terms of the scaling limit for the latter one.

The scaling limit of the MCRW is going to be given by the *marked Brownian web* (MBW). This object can be described as follows. Begin with independent backward (in time) Brownian paths starting from any fixed (say, deterministic) dense countable subset $\mathcal{D}$ of $\mathbb{Z}^2$. Mark each Brownian path with marks from a Poisson point process of rate λ (possibly time-dependent). One way to do this marking is to consider the set of backward Brownian paths $\mathcal{W} := \{W_{x,t}, (x,t) \in \mathcal{D}\}$, where $W_{x,t}$ denotes the (backward) path starting at (x,t), and an independent i.i.d. family $\mathcal{N} := \{N_{x,t}, (x,t) \in \mathcal{D}\}$ of Poisson point process in $\mathbb{R}$ of rate λ. Now mark the path $W_{x,t} = (f(s), s)_{s \le t}$ at the points $(f(S_i), S_i)_{i \ge 1}$, where $S_1, S_2, \dots$ are the succesive event times of $N_{x,t}$ down from t. Let us denote the marked path thus obtained $W^*_{x,t}$ and the set of marked paths $\mathcal{W}^* := \{W^*_{x,t}, (x,t) \in \mathcal{D}\}$. Now introduce the set of coalescing marked paths $\tilde{\mathcal{W}}^* := \{\tilde{W}^*_{x,t}, (x,t) \in \mathcal{D}\}$, as in Section 6.1 (see (6.10)), by introducing a precedence relation on the set of marked paths. The first coalescing marked path of $\tilde{\mathcal{W}}^*$ is the first marked path of $\mathcal{W}^*$. The $n+1$-st coalescing marked path of $\tilde{\mathcal{W}}^*$ is formed first with the portion of the $n+1$-st marked path of $\mathcal{W}^*$ down until it first hits any of the n first coalescing marked paths of $\tilde{\mathcal{W}}^*$; from then on, it follows that marked path (the one it has first hit).

We will call the set of the first n coalescing marked paths obtained in this way *coalescing marked Brownian motions starting at* $\{(x_1,t_1), \dots, (x_n,t_n)\}$, denoted

$$\tilde{\mathcal{W}}^*((x_1,t_1), \dots, (x_n,t_n)), \tag{6.20}$$

where $\{(x_1,t_1), \dots, (x_n,t_n)\}$ is the initial points of the paths. The probability distribution of (6.20) depends of course on $\{(x_1,t_1), \dots, (x_n,t_n)\}$, but not on its ordering.

Turning back to the construction of the MBW, we have just constructed $\tilde{\mathcal{W}}^*$ inductively. We will sometimes call it the MBW *skeleton*. We finally get the MBW as the closure of $\tilde{\mathcal{W}}^*$ in the appropriate metric space, and denote it by $\bar{\mathcal{W}}^*$.

One way to do that (more) explicitly is to take the (unmarked) backward *Brownian web* $\bar{\mathcal{W}}$ obtained as the closure in the appropriate metric space of the (unmarked) backward Brownian web skeleton $\tilde{\mathcal{W}}$ and to have each path of $\bar{\mathcal{W}}$ marked as follows. We will use the fact that for all $\delta > 0$, each path $\bar{\pi}$ of $\bar{\mathcal{W}}$ (starting at time t, say) coincides with a path $\tilde{\pi}$ in $\tilde{\mathcal{W}}$ from time $t-\delta$

down (see Proposition 6.4 e.). Then, naturally, we mark $\bar{\pi}$ with the marks that $\bar{\pi}$ got as an element of $\tilde{\mathcal{W}}^*$. By doing this for a sequence of δ's going to 0, we get $\bar{\pi}$ all marked, except at the initial point. This will be marked if it coincides with any mark in $\tilde{\mathcal{W}}^*$; in this case, it takes that mark; otherwise, it takes no mark.

The object thus described/constructed, the MBW with rate λ, has the following properties. The probability distribution of $\tilde{\mathcal{W}}^*$ does not depend on $\mathcal{D}$. Also, the probability distribution of the subset of $\tilde{\mathcal{W}}^*$ consisting of paths which start at any deterministic set $\{(x_1,t_1),\ldots,(x_k,t_k)\}$, not necessarily in $\mathcal{D}$, where $k \geq 1$ is an integer, is the same as that of $\tilde{\mathcal{W}}^*((x_1,t_1),\ldots,(x_k,t_k))$ obtained in a construction where $(x_1,t_1),\ldots,(x_k,t_k)$ are the first points of the initial dense countable subset of $\mathbb{Z}^2$.

6.3.1 Rescaling the MVD.

We now rescale the MVD in the usual diffusive way: given $\epsilon > 0$, we will make $t \to \epsilon^{-1}t$ and $x \to \epsilon^{-1/2}x$. We will also rescale the temperature, making $\beta \to \log(\epsilon\lambda \wedge 1)^{-1}$, where λ is a nonnegative function, the *scaled temperature profile* of the model; in this way, $p \sim \epsilon\lambda$ as $\epsilon \to 0$ and the marks of the marked voter dynamics (at the origin, say) converge weakly as $\epsilon \to 0$ (under the above time rescaling) to a (possibly inhomogeneous) Poisson point process of rate λ as $\epsilon \to 0$.

Under this rescaling, we conclude, by looking at the simultaneously rescaled (in the same way) MCRW, that for all $\ell \geq 1$ fixed, given

$$x_1,\ldots,x_\ell \in \mathbb{R},\ \ t_1,\ldots,t_\ell \in \mathbb{R}^+$$
$$x_1^{(\epsilon)},\ldots,x_\ell^{(\epsilon)} \in \mathbb{Z},\ \ t_1^{(\epsilon)},\ldots,t_\ell^{(\epsilon)} \in \mathbb{R}^+$$

such that

$$(\epsilon^{1/2}x_1^{(\epsilon)},\ldots,\epsilon^{1/2}x_\ell^{(\epsilon)}) \ \to\ (x_1,\ldots,x_\ell), \tag{6.21}$$
$$(\epsilon t_1^{(\epsilon)},\ldots,\epsilon t_\ell^{(\epsilon)}) \ \to\ (t_1,\ldots,t_\ell) \tag{6.22}$$

as $\epsilon \to 0$, we have that

$$\left\{\sigma_{t_i^{(\epsilon)}}\left(x_i^{(\epsilon)}\right);\ 1 \leq i \leq \ell\right\}, \tag{6.23}$$

with $p = \{2/[1+(\epsilon\lambda)^{-1}]\} \wedge 1$, converges weakly to the following function of $\tilde{\mathcal{W}}^*((x_1,t_1),\ldots,(x_\ell,t_\ell))$. First, let us introduce further marks on the paths of $\tilde{\mathcal{W}}^*((x_1,t_1),\dots,(x_\ell,t_\ell))$ at time 0 (one further mark on each path). Assign to each mark (the extra ones included) a spin taken from an i.i.d. family of random variables which are uniformly distributed in $\{1,\dots,q\}$, one different copy for each mark. (If $q=\infty$, each mark gets a different spin.) Let $s_{x,t}$ denote the spin at mark (x,t), if any. Now, for $1\le i\le \ell$, let T_i be the first time (down from t_i) when $\tilde{W}^*_{x_i,t_i}$ hits one of its marks (the extra ones included). Denoting $\tilde{W}^*_{x_i,t_i}(s)=:(f_i(s),s)$ for $s\le t_i$, we have finally that, as $\epsilon\to 0$, (6.23) converges weakly to

$$\left\{s_{f_i(T_i),T_i};\ 1\le i\le \ell\right\}. \tag{6.24}$$

We want to find the form of the spin-spin correlation

$$\tilde{C}_q=[q/(q-1)](\tilde{G}_q-1/q) \tag{6.25}$$

in the above scaling limit, where $\tilde{G}_q=\tilde{G}_q(x,y;t_w,t_w+t)$ is the probability in the MVD with q states that $\sigma_{t_w}(x)=\sigma_{t_w+t}(y)$. The proper scaling of x,y,t,p as $t_w\to\infty$ is, as before,

$$t=\theta t_w;\quad x=\lfloor v\sqrt{t_w}\rfloor;\quad y=\lfloor z\sqrt{t_w}\rfloor;\quad p=\frac{2}{1+\lambda^{-1}t_w}\wedge 1 \tag{6.26}$$

where $v,z\in\mathbb{R}$ and $\theta\in\mathbb{R}^+$. We start with the (simpler) case of $q=\infty$. Under the above scaling, as $t_w\to\infty$, we then get

$$\tilde{C}_\infty(x,y;t_w,t_w+t)\to\tilde{g}(|z-v|;\theta), \tag{6.27}$$

where $\tilde{g}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is the probability under the MBW that $\tilde{W}^*_{v,1}$ and $\tilde{W}^*_{z,1+\theta}$ coalesce by time 0 and do not cross any mark before coalescing.

To get an expression for $\tilde{g}$, let

$$\begin{aligned} T_{z,\theta} &= \sup\{t\le 1:\ \tilde{W}^*_{0,1}\text{ and }\tilde{W}^*_{z,1+\theta}\text{ coincide}\} && (6.28)\\ &= \sup\{t\le 1:\ \tilde{W}_{0,1}\text{ and }\tilde{W}_{z,1+\theta}\text{ coincide}\}. && (6.29)\end{aligned}$$

Then, clearly,

$$T_{z,\theta}=\sup\{t\le 1:\ W_{0,1}\text{ and }W_{z,1+\theta}\text{ coincide}\}, \tag{6.30}$$

where, we recall, $W_{0,1}$ and $W_{z,1+\theta}$ are two independent (backward) Brownian paths starting at $(0,1)$ and $(z,1+\theta)$, respectively. The distribution of this was computed before. Actually we get an expression for $P_p(T_{z,\theta} \geq 0)$, namely $g(z, \theta)$, in Section 5 (see (5.23)). By the Brownian scaling, we have

$$P_p(T_{z,\theta} \geq 1 - s) = g(z/\sqrt{s}, \theta/s), \tag{6.31}$$

for $s \geq 0$. For the case that z and θ do not simultaneously vanish (otherwise, we have a trivial case), let

$$f_{z,\theta}(s) = \frac{d}{ds} g(z/\sqrt{s}, \theta/s) \tag{6.32}$$

be the probability density function of $1 - T_{z,\theta}$. Given $0 \leq s \leq 1$ and $T_{z,\theta} = 1 - s$, the probability that $\tilde{W}^*_{0,1}$ and $\tilde{W}^*_{z,1+\theta}$ do not cross any mark before $T_{z,\theta}$ is

$$e^{-\int_{1-s}^{1} \lambda(t)\, dt} e^{-\int_{1-s}^{1+\theta} \lambda(t)\, dt}, \tag{6.33}$$

by the independence of the Poisson marks in disjoint paths. Integrating against $f_{z,\theta}$, we get

$$\tilde{g}(z, \theta) = e^{-\int_1^{1+\theta} \lambda(t)\, dt} \int_0^1 e^{-2\int_{1-s}^{1} \lambda(t)\, dt} f_{z,\theta}(s)\, ds. \tag{6.34}$$

The case $q < \infty$ can be analysed as in Chapter 5 (see (5.6)), and we conclude that the limiting expression for $\tilde{C}_q$ equals $\tilde{g}(z, \theta)$ for all q. As in the zero temperature case, the spin-spin correlation does not depend on q in the scaling limit.

6.3.2 The MBW as a continuum space spin dynamics

The discussion of last subsection allows for the definition of a continuum space spin dynamics, by means of the MBW, as follows. Let

$$\xi : \mathbb{R} \to \{1, \dots, q\}$$

be an arbitrary initial (continuum space) spin configuration. Now, as in the discussion in the second paragraph of Subsection 6.3.1, after introducing further marks to the MBW, assign to each mark *that is not at time* 0 a spin taken from an i.i.d. family of random variables which are uniformly distributed in $\{1, \dots, q\}$. For a mark located at $(x, 0)$, assign spin $\xi(x)$. Now, let $T_{x,t}$ be the first time (down from t) when the rightmost path in $\tilde{\mathcal{W}}^*$ starting in (x, t) (typically, there will be only one, by Theorem 6.16) hits one of its marks and let $s_{x,t}$ be the spin assigned to that mark. The spin dynamics $(\sigma_t)_{t\geq 0}$ is then defined as

$$\sigma_0 = \xi \tag{6.35}$$
$$\sigma_t(x) = s_{x,t},\ x\in\mathbb{R},\ t>0. \tag{6.36}$$

We will call this the *MBW (continuum space) spin dynamics with rate* λ *and* q *states.*

We can get equilibrium measures for this dynamics from the scaling limit of equilibrium measures for the discrete dynamics. This of course only makes sense in general for λ constant. In the case that $q=2$, the Gibbs measure for the Glauber dynamics which is equivalent to the MVD is of course invariant for the MVD[2]. This $1D$ Gibbs measure is a Markov chain which is symmetric under spin flip and whose flip transition probability is $\sim\sqrt{p}$ as $p\to 0$. Under the space-temperature rescaling we used for the MVD (with λ constant), it thus converges to a continuum space spin model supported on spin configurations which are constant by parts, in which the spin flip points form a Poisson process of rate $\sqrt{\lambda}$, and such that the distribution of the spin at the origin is uniform in $\{1,2\}$. We will call this the *continuum Ising model with rate* $\sqrt{\lambda}$. From the above, we conclude that the continuum Ising model with rate $\sqrt{\lambda}$ is invariant for the MBW spin dynamics with rate λ and 2 states.

Direct description of the continuum dynamics

The MBW as defined above is a dual/backwards dynamics, in the sense that it is the continuum version of a dynamics, namely the MCRW, which is dual to the MVD and runs backwards in time. In this way we get an indirect/dual/backwards description of the continuum version of the MVD. It is natural then to wonder whether we can get a direct/forward description of that model, in the same way as we did for the zero temperature case, via the restricted Brownian web of paths starting at $\mathbb{R}\times\{0\}$ only (see Remark 5.2). The answer is that we can; however, in this case we will need to take into account not only the full forward web, but also the full dual web. The latter web is needed in order to get the marks placed. (Notice that the marks go on the dual paths.) Once the marks are in place, we can forget about the dual

[2]For $q>2$, the Gibbs measure (associated to the Potts model with q states) is *not* invariant under the MVD.

paths, and consider the paths of the direct/primal web starting at $\mathbb{R} \times \{0\}$ *and* at the marks on the upper half plane. We note that, since the marks were on paths of the dual, each one is a double point of the forward web (see Remark 6.18), and thus originates a color cluster (of the color it was assigned). In the same way, the double points at $\mathbb{R} \times \{0\}$ originate clusters of the color they were assigned. We note that color clusters will occur inside color clusters, with the color of the inside one prevailing.

Let us look at these color clusters, whether starting at a mark or at a double point on $\mathbb{R} \times \{0\}$, as space-time clusters, consisting of the open region of $\mathbb{R}^2$ enclosed by the two paths starting at that mark or double point on $\mathbb{R} \times \{0\}$.[3] In this situation, we call the mark or double point on $\mathbb{R} \times \{0\}$ the *lower tip* of the space-time cluster. We will call the space time point where a space-time cluster ends, i.e., the space time point above the lower tip where the cluster boundaries meet, the *upper tip* of that cluster. See Figure 6.6.

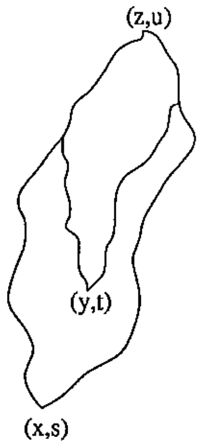

Figure 6.6: Space-time clusters with lower tips at (x,s) and (y,t), and common upper tip at (z,u).

Notice that almost surely there are countably many lower tips, since almost surely there are countably many marks and countably many double points on $\mathbb{R} \times \{0\}$ (see Theorem 6.17). Apart from space-time points at

[3]It can be argued that almost surely no mark and no point of $\mathbb{R} \times \{0\}$ is a triple point (that is, a point originating three forward paths). The latter statement is actually made in Theorem 6.17.

the boundaries of the clusters, which altogether form a set of Hausdorff dimension 3/2 by Theorem 6.16, every other space-time point belongs to a definite cluster, and can thus be assigned the color of that cluster. (Perhaps the simplest way to determine this is to notice by the backward description above that the points off cluster boundaries are of type either $(0,1)$, $(1,1)$ or $(2,1)$ on the dual web, and thus belong only to the color cluster of the first mark on the unique dual path from that point.) The boundary points can be assigned colors by right-continuity (to be consistent with the rightmost path convention adopted for the dual description at the first paragraph of this subsection; see the remark below for the existence of right color limits). Our direct description is complete.

Remark 6.19 *The above descriptions raise a natural question as to how do the color configurations of the MBW dynamics look like at fixed positive times. By the direct description we know that we have space-time color clusters one inside the other almost surely. It is not difficult to see that each cluster has another cluster inside it, e.g., by concluding from the scaling of the marked random walks that the marks of the MBW are dense in $\mathbb{R}^2$. This might suggest that the latter picture occurs also for fixed positive times, i.e., the color clusters at positive times*[4] *would also be such that each one has another cluster inside it. But this is not the case, as one might suspect from the observation about the invariant measure for the MBW dynamics with 2 states (above). In fact, it can be argued that, even in the case that $q = \infty$, the color configurations of the MBW dynamics at fixed positive times have finitely many clusters in each finite interval almost surely.*

[4]By a color cluster at fixed time, say t, we mean any connected component of the intersection with $\mathbb{R} \times \{t\}$ of a space-time color cluster, say C_0, minus the intersections with $\mathbb{R} \times \{t\}$ of the closures of the space-time color clusters contained in C_0.

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