Geometry, Dynamics and Topology of Foliated Manifolds

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Geometry, Dynamics and Topology of Foliated Manifolds

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PREFACE

These notes correspond to the three-months graduated course "Teoria Geométrica das Folheações" addressed by the first author in the Federal University of Rio de Janeiro in 2001-02. Some (but not so much) background in Algebra, Dynamical Systems, Riemannian Geometry and Topology is required. See the interesting books of [8, 9, 23, 29, 30] for such a background. See also the books [4, 15, 22] for references concerning foliations. I would like to thank Prof. Bruno Scárdua for giving me the opportunity to help the above-mentioned course. I would like to thank Alexandre Teixeira Behague by his valuable help in taking the notes of the first version of this manuscript. I would also thank Ivan Aguilar, Serafin Bautista, Helisson Coutinho, Alexandre Soares and Filipe Iório for their collaboration. This work was partially suported by CNPq, FAPERJ and PRONEX/Dynamical Systems.

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Universidade Federal do Rio de Janeiro
Rio de Janeiro, June of 2003

The Geometric Theory of Foliations is one of the fields in Mathematics that congregates several distinct domains: Topology, Dynamical Systems, Differential Topology and Geometry, among others. Its origin dates from the original works of C. Ehresmann and G. Reeb ([12], [13]) and its great development has allowed a better comprehension of several phenomena of mathematical and physical nature. Theorems, nowadays considered to be classical, like the Reeb Stability Theorem, Haefliger's Theorem, and Novikov Compact leaf Theorem, are now searched for holomorphic foliations. Several authors have began to investigate such phenomena (e.g. C. Camacho, A. Lins Neto, E. Ghys, M. Brunella, R. Moussu, S. Novikov and others). The study of such field presumes a knowledge of results and techniques from the *real case*, and nice familiarity with the classical aspects of Holomorphic Dynamical Systems.

These notes are merely introductory and cover only a minor part of the basic aspects of the rich theory of foliations. In particular, rigorous proofs for some results and extensive information may be searched in the bibliography we give at the end of the text. Nevertheless, we have tried to shed some light on the geometry of some classical results and to provide motivation for further study. Our goal is to highlight this geometrical viewpoint despite the loss of some formalism. We hope that this text may be useful to those who appreciate Mathematics, and specially to students that may be interested in this beautiful and fruitful field of Mathematics.

This text may be divided in two basic parts. In the first part, which corresponds to the first eight chapters, we have an exposition of classical results in the Geometric Theory of (real) foliations. Special attention is paid to the Reeb Stability Theorems, Haefliger's Theorem and Novikov Compact Leaf Theorem. Chapter 9 is dedicated to some possible models of *complex versions* for such classical results. Then we arrive at the following central problem according to this point of view:

What would be a compact leaf theorem for codimension one complex (holomorphic) foliations?

In order to answer to such question we try to construct a parallelism between the real and complex worlds. Our approach motivates therefore the search of alternative proofs of the compact leaf theorems for real foliations. With this motivation we expose in Chapter 10 D. Sullivan's homological proof of Novikov compact leaf theorem, which is based in a mix of topological argumentation and invariant measure theory for foliations. This seems to be an applicable procedure for complex foliations. We leave the suggestion...

I am very grateful to Professor Carlos Morales for his friendship and for coauthoring this work. This work was partially supported by CNPq and Faperj. I want to thank my colleagues from the UFRJ for their every-day cooperation. Very special thanks to my wife Aline for her constant support.

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Rio de Janeiro, June of 2003

To Professor César Camacho on his sixtieth birthday

Contents

Chapter 1

Preliminaries

1.1 Definition of foliation

There are essentially three ways to define foliations. Let M be a m-dimensional manifold, $m \in \mathbb{N} - 0$. Let D^k be the unit ball of $\mathbb{R}^k$ where $k \in \mathbb{N} - 0$. Let $0 \leq n \leq m$ be fixed.

Definition 1.1.1. A C^r *foliation* of codimension $m - n$ of M will be a maximal atlas $\mathcal{F} = \{(U_i, X_j)\}_{i \in I}$ of M satisfying the following properties:

1. $X_i(U_i) = D^n \times D^{m-n}$;

2. For all $i, j \in I$ the map $X_j \circ (X_i)^{-1} : X_i(U_i \cap U_j) \to X_j(U_i \cap U_j)$ is C^r and has the form

$$X_j \circ (X_i)^{-1}(x, y) = (f_{i,j}(x, y), g_{i,j}(y)).$$

The number n is called *the dimension* of $\mathcal{F}$. A *plaque* of $\mathcal{F}$ is a set $\alpha = X_i^{-1}(\{y = C\})$ for some $C \in \mathbb{R}^{m-n}$. The plaques of $\mathcal{F}$ define a relation $\approx$ in M as follows: If $x, y \in M$ then $x \approx y$ if and only if there is a finite collection of plaques $\alpha_1, \cdots, \alpha_k$ such that $x \in \alpha_1, y \in \alpha_k$ and $\alpha_i \cap \alpha_{i+1} \neq \emptyset$ for all $1 \leq i \leq k-1$. Clearly $\approx$ is an equivalence and then we can consider the equivalence class $\mathcal{F}_x$ of $\approx$ containing $x \in M$. A *leaf* of $\mathcal{F}$ is precisely an equivalence class $L = \mathcal{F}_x$ of $\approx$ (for some $x \in M$). One can easily prove that every leaf of $\mathcal{F}$ is an immersed submanifold of M. One will see later that a leaf may self-accumulate, and so, the leaves of $\mathcal{F}$ are not embedded in general. Under the viewpoint of the equivalence $\approx$ one can define $\mathcal{F}$ as a partition of M by immersed submanifolds L such that for all $x \in M$ there is a neighborhood U diffeomorphic to $D^{m-n} \times D^n$ such that the leaves of the partition intersect U in the trivial foliation $\{D^n \times y : y \in D^{m-n}\}$ on $D^{m-n} \times D^n$. This allows us to state the following equivalent definition of foliation.

Definition 1.1.2. A C^r foliation of codimension $m-n$ of M is a partition $\mathcal{F}$ of M formed by immersed C^r submanifolds $\mathcal{F}_x \subset M$ such that every $x \in M$ exhibits a neighborhood U and a C^r diffeomorphism $X : U \to D^n \times D^{m-n}$ such that $\forall y \in D^{m-n}$ $\exists F \in \mathcal{F}$ satisfying

$$X^{-1}(D^n \times y) \subset F.$$

The elements of the partition $\mathcal{F}$ are called *the leaves of* $\mathcal{F}$. The element $\mathcal{F}_x$ of $\mathcal{F}$ containing $x \in M$ is called the leaf of $\mathcal{F}$ containing x.

Warning: Not every partition of M formed by immersed submanifolds with the same dimension is a foliation as shown the partition of $\mathbb{R}^2$ depicted in Figure 1.1 (note that the condition for foliation fails at the point x).

The third definition of foliation uses the so-called distinguished applications. Let $\mathcal{F} = \{(U_i, X_i)\}$ be a foliation of a

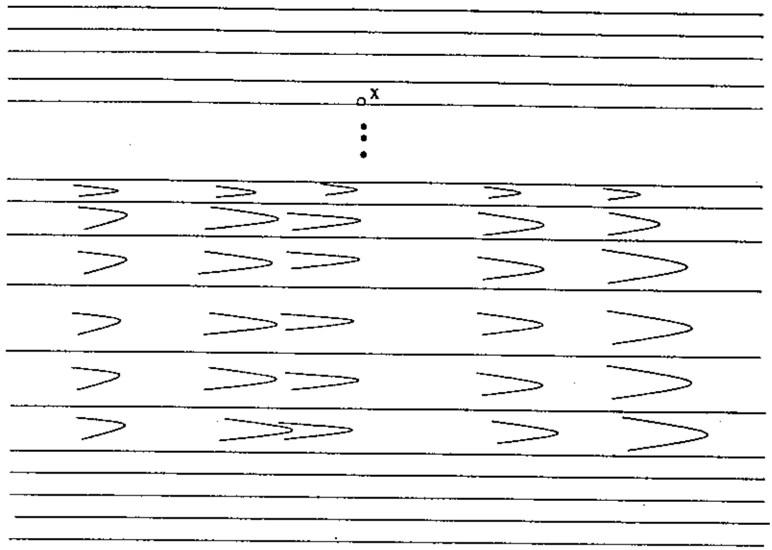


Figure 1.1:

manifold M according to Definition 1.1.1. Then $\forall i, j$ the transition map $X_j \circ (X_i)^{-1}$ has the form

$$X_j \circ (X_i)^{-1}(x, y) = (f_{i,j}(x, y), g_{i,j}(y)).$$

We have that $g_{i,j}$ is a diffeomorphism in its domain. This follows from the fact that the derivative $D(X_j \circ (X_i)^{-1})(x, y)$ has non-zero determinant equals to $\partial_x f_{i,j}(x, y) \cdot g'_{i,j}(y)$. We define for all i the map $g_i = \Pi_2 \circ X_i$, where Π_2 is the projection onto the second coordinate $(x, y) \in D^n \times D^{m-n} \to y$. One has $g_j = g_{i,j} \circ g_i$ as $(\Pi_2 \circ X_j) \circ X_i^{-1} = g_{i,j}$ and then $g_{i,j}^{-1} \circ g_j = X_i \Rightarrow \Pi_2 \circ g_{i,j} \circ g_j = \Pi_2 \circ X_i = g_i \Rightarrow g_j = g_{i,j} \circ g_i$ since Π_2 is the identity in D^{m-n}. We conclude that a C^r foliation $\mathcal{F}$ of codimension $m-n$ of a manifold M^m is equipped with a covering $\{U_i\}$ of M and C^r submersions

$g_i : U_i \to D^{m-n}$ such that for all i, j there is a diffeomorphism $g_{i,j} : D^{m-n} \to D^{m-n}$ satisfying the cocycle relations

$$g_j = g_{i,j} \circ g_i, \quad g_{i,i} = Id.$$

The g_i's are the *distinguished applications* of $\mathcal{F}$.

Conversely, suppose that M^m is equipped with a covering $\{U_i\}$ such that for all i there are C^r submersions $g_i : U_i \to D^{m-n}$ such that for all i, j there is a diffeomorphism $g_{i,j}$ satisfying the cocycle relations above. By the Local Form of the Submersions we can assume that for all i there is a C^r diffeomorphism $X_i : U_i \to D^n \times D^{m-n}$ such that

$$g_i = \Pi_2 \circ X_i.$$

since

$$\Pi_2 \circ X_j \circ (X_i)^{-1}) = g_j \circ (X_i)^{-1} = g_{i,j} \circ g_i \circ (X_i)^{-1} = g_{i,j} \circ \Pi_2,$$

we have that the atlas

$$\mathcal{F} = \{(U_i, X_i)\}$$

defines a foliation of class C^r and codimension $m-n$ of M. The above suggests the following equivalent definition of foliation.

Definition 1.1.3. A C^r foliation of codimension $m-n$ of M is a covering $\{U_i : i \in I\}$ of M such that $\forall i \in I$ there is a C^r submersion $g_i : U_i \to D^{m-n}$ such that $\forall i, j \in I$ there is a diffeomorphism $g_{i,j} : D^{m-n} \to D^{m-n}$ satisfying the cocycle relations

$$g_j = g_{i,j} \circ g_i, \quad g_{i,i} = Id.$$

The g_i's are the *distinguished applications* of $\mathcal{F}$.

The last definition leads to several interesting definitions. For instance, a foliation $\mathcal{F}$ of M is said to be transversely orientable or transversely affine depending on whether its distinguished applications $g_{i,j}$ are orientation preserving or affine. An equivalent definition will be given in Section 2.2. To distinguish foliations we shall use the following definition.

Definition 1.1.4. Two foliations $\mathcal{F}, \mathcal{F}'$ defined on M, M' respectively are *equivalent* if there is a homeomorphism $h\colon M \to M'$ sending the leaves of $\mathcal{F}$ into leaves of $\mathcal{F}'$. In other words,

$$h(\mathcal{F}_x) = \mathcal{F}'_{h(x)}, \ \forall x \in M.$$

The above relation defines an equivalence in the space of foliations. To illustrate it we can observe that the foliations F_1, F_2 in the band $I \times \mathbb{R}$ in Figure 1.2 are not equivalent.

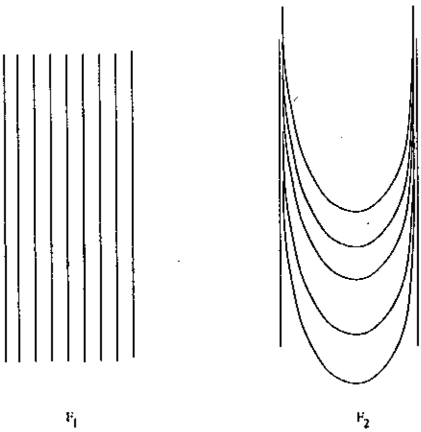


Figure 1.2: Non equivalent foliations.

1.2 Examples of foliations

1.2.1 Foliations derived from submersions

A *submersion* between two manifolds is a smooth map whose derivative has maximal rank everywhere. Submersions can be used to construct foliations in the following way.

Theorem 1.2.1. *Let $f : M^m \to N^n$ be a C^r submersion between the differentiable manifolds M, N. Then the level curves*

$$L_c = f^{-1}(c), \quad c \in N$$

are the leaves of a C^r foliation of codimension n of M.

Proof. By the Local Form of the Submersions [9] there are atlas $\{(U, X)\}, \{(V, Y)\}$ of M, N respectively such that

1. $X(U) = D^n \times D^{m-n}$.
2. $Y(V) = D^{m-n}$.
3. $Y \circ f \circ X^{-1} = \Pi_2$ (see Figure 1.3).

We claim that the collection $\mathcal{F} = \{(U, X)\}$ defines a foliation of M. In fact, let $(U, X), (U^*, X^*)$ be two elements of the covering. Then,

$$\Pi_2 \circ X^* \circ X^{-1} = Y^* \circ f \circ (X^*)^{-1} \circ X^* \circ X^1 =$$

$$= Y^* \circ f \circ X^{-1} = Y^* \circ Y^{-1} \circ Y \circ f \circ X^{-1} = Y^* \circ Y^{-1} \circ \Pi_2.$$

Hence $\Pi_2 \circ (X^* \circ X^{-1}) = (Y^* \circ Y^{-1}) \circ \Pi_2$ does not depend on $x \in D^n$. This proves that $\mathcal{F}$ is a foliation of class C^r and codimension $m - n$ of M. It is clear by definition that the plaques of $\mathcal{F}$ are contained in the level sets of f. This proves that the leaves of $\mathcal{F}$ are precisely the level sets of f and the result follows. □

Let us present some examples to illustrate the above result.

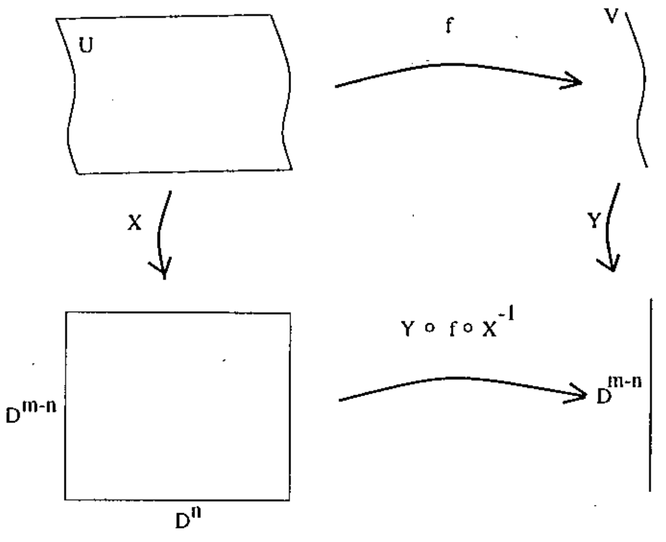


Figure 1.3: Foliation and submersion.

Example 1.2.1. Let $M = \mathbb{R}^2$ anf $f(x,y) = x$ the projection onto the x-axis. Clearly f is a C^ω submersion. Since $\dim(N) = \dim(M) = 1$ in this case we have that f defines a codimension one C^ω-foliation of M whose leaves are the vertical lines in $\mathbb{R}^2$.

Example 1.2.2. Let $M = \mathbb{R}^2$ and $f(x,y) = y - \alpha \cdot x$, where $\alpha \in \mathbb{R}$. The level curves of f define a foliation $\mathcal{F}_\alpha$ in M whose leaves are the straight-lines $y = \alpha \cdot x + c$, $c \in R$. Observe that $\mathcal{F}_\alpha$ is invariant by the translations $(x,y) \to (x+k, y+l)$, $(k,l) \in \mathbb{Z}^2$. Indeed, if $y = \alpha \cdot x + c$ then $y + l = \alpha \cdot x + c + l = \alpha \cdot (x+k) + c'$, where $c' = c - \alpha \cdot k$ proving the invariance. It follows that $\mathcal{F}_\alpha$ projects into a foliation of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ still denoted by $\mathcal{F}_\alpha$. See Figure 1.4. When α is irrational then all the leaves of the induced foliation are lines, and if α is rational then all the leaves are circles. We shall call this example as the *linear foliation* in T^2.

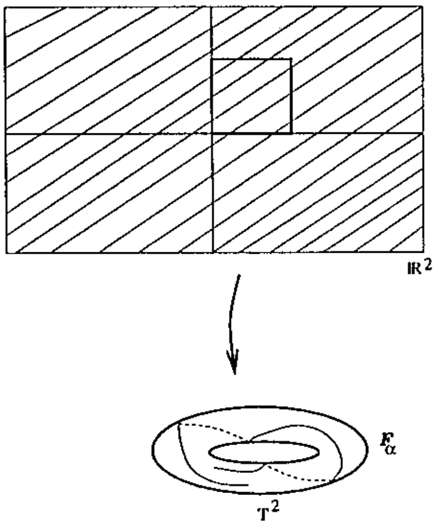


Figure 1.4: Linear foliation on T^2.

Example 1.2.3. Let $M = \mathbb{R}^3$ and $f(x, y, z) = \alpha(r^2)e^z$, where $r^2 = x^2 + y^2$ and α is a C^∞ function such that $\alpha(0) = 1, \alpha(1) = 0$ and $\alpha'(t) < 0$ for all $t > 0$ (see Figure 1.5).

The map f is a submersion since

$$\nabla f(x, y, z) = (2\alpha'(r^2)xe^z, 2\alpha'(r^2)ye^z, \alpha(r^2)e^z) = (0, 0, 0)$$

$\Rightarrow x = y = 0$ and $\alpha(r^2) = 0 \Rightarrow x = y = 0,\ x^2 + y^2 = 1$ contradiction. Hence $\nabla f(x, y, z)$ does not vanish and so f is a submersion. It follows from Theorem 1.2.1 that the level curves

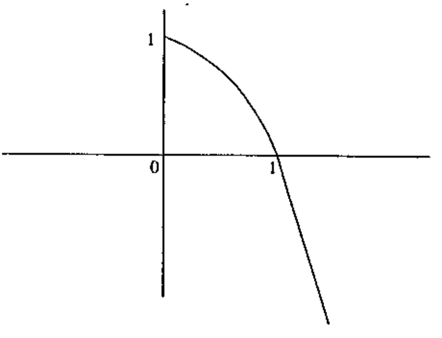


Figure 1.5: Graph of α.

$f^{-1}(c)$ form a foliation of class C^∞ and codimension 1 of M. The leaves of this foliation (i.e. the level curves of f) can be described as follows.

$$f(x, y, z) = c \Leftrightarrow \alpha(r^2)e^z = c.$$

If $c = 0$ then $\alpha(r^2) = 0 \Rightarrow x^2 + y^2 = 1$. Hence the level curve corresponding to $c = 0$ is the cylinder $x^2, y^2 = 1$ in M. If $c > 0$ then

$$\alpha(r^2)e^z = c \Rightarrow \alpha(r^2) > 0.$$

Moreover,

$$z = K - ln(\alpha(r^2)),$$

$(K = ln(c))$. When $c = 1$ we have

$$z = -ln(\alpha(r^2)).$$

The graph of the above curve in the plane $y = 0$ is given by

$$z = -ln(\alpha(x^2)).$$

We have

$$z' = -\frac{2\alpha(x^2))}{\alpha(x^2))} \cdot x = 0 \Rightarrow x = 0.$$

Hence $x = 0$ is the solely critical point of z. We have that $z \to \infty$ as $x \to x \to 1^+$ or 1^-. The graph of z is a parabola-like curve. The graph of the leaves of $\mathcal{F}$ is depicted in Figure 1.6.

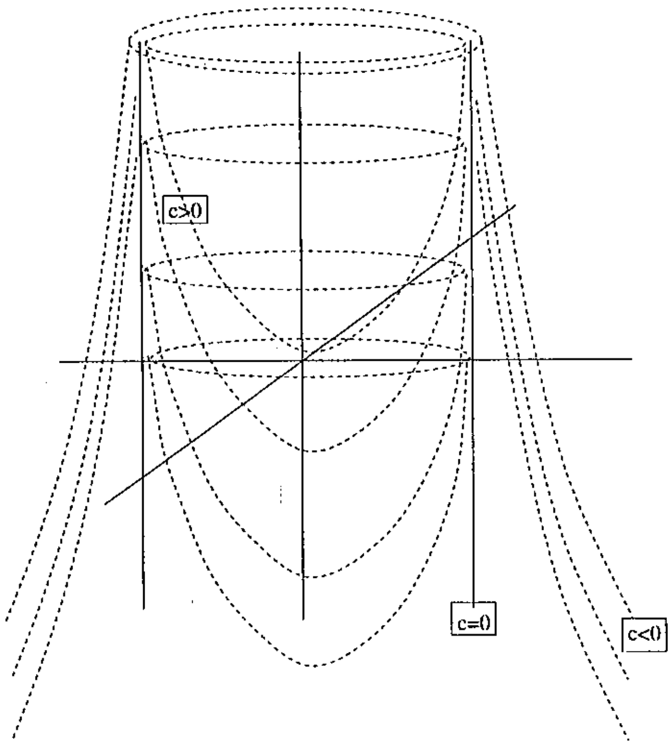


Figure 1.6:

Example 1.2.4 (Fibrations). Let E, B, F be smooth manifolds. We say that E is a fiber bundle over B with fibre F if there is a C^∞ onto submersion $\pi : E \to B$ satisfying the following properties:

1. $\pi^{-1}(b)$ is diffeomorphic to F, $\forall b \in B$.

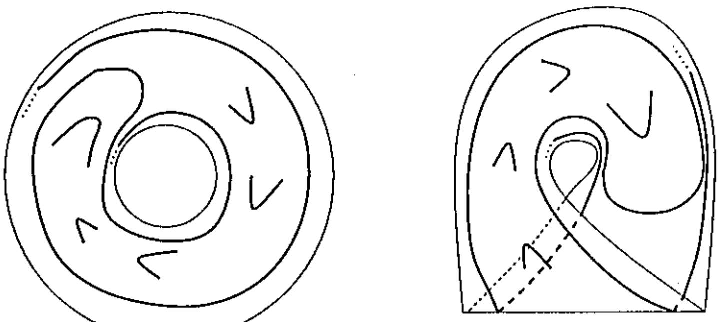
2. $\forall b \in B \; \exists \; U \subset B$ neighborhood of b and $\exists \phi : \pi^{-1} \to U \times F$ diffeomorphism such that $\pi^1 \circ \phi = \pi$, where π^1 is the projection onto the first coordinate in $U \times F$.

Clearly the family $\{\pi^{-1}(b) : b \in B\}$ is a C^∞ codimension $\dim(B)$ foliation of E since π is a C^∞ submersion. Note that the leaves of the resulting foliation are all diffeomorphic to a common manifold F.

1.2.2 Reeb foliations

There are several foliations which can be called Reeb foliations. The first ones are the Reeb foliations in the cylinder and the Moebious band constructed as follows. Define $M = [-1, 1] \times \mathbb{R}$ and let $\mathcal{F}$ be the foliation in M defined by the submersion $g(x, y) = \alpha(x^2)e^y$ where α is decreasing. Let $G: M \to M$ be given by $G(x, y) = (x, y + 2)$. The quotient manifold M/G is the cylinder. Analogously we can replace G by the map $F(x, y) = (-x, y + 2)$. In this case the quotient manifold M/F is the Moebious band. In each case one can see that $\mathcal{F}$ is invariant for G and F. Hence $\mathcal{F}$ induces a foliation $\overline{\mathcal{F}}$ in either M/G or M/F. These are the Reeb foliations in the cylinder and the Moebious band respectively. These foliations are depicted in Figure 1.7.

Consider the foliation $\mathcal{F}$ constructed in the last example of Section 1.2 restricted to the solid cylinder $\{(x, y, z) : x^2 + y^2 \leq$



Reeb foliation in the cylinder

Reeb foliation in the Moebious band

Figure 1.7: : Reeb foliation in the cylinder and Moebious band.

1}. One can easily check that the leaves of this foliation are invariant by the translations $(x, y, z) \to (x, y, z + 1)$. Note that the quotient manifold solid cylinder$/(x, y, z) \to (x, y, z + 1)$ is a solid torus $D^2 \times S^1$. The invariance mentioned above implies that $\mathcal{F}'$ induces a foliation in $D^2 \times S^1$ whose leaves are depicted in Figure 1.8 This foliation is called the *Reeb foliation* in the solid torus $ST = D^2 \times S^1$. The Reeb foliation in the solid torus is used to construct a C^∞ foliation in the 3-sphere S^3 in the following way:

Let ST_1 and ST_2 be two solid torus and denote $\partial ST_1 = T_1$ and $\partial ST_2 = T_2$ the the corresponding boundaries. Consider a diffeomorphism $\varphi \colon T_2 \to T_1$ sending the meridian curves in T_2 into the parallel curves in T_1. For instance we can choose φ by first considering

$$\varphi(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

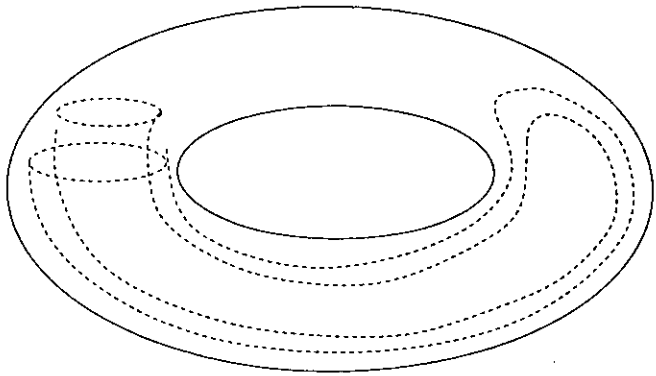


Figure 1.8: Reeb foliation in the solid torus.

Because φ is linear and $\det \varphi = -1 \Rightarrow \varphi(\mathbb{Z}^2) = \mathbb{Z}^2 \Rightarrow \varphi$ defines the desired map.

In $ST_1 \cup ST_2$ we consider the equivalence relation given by $y = \varphi(x)$. In other words we use the identification below.

$$x \sim y \Leftrightarrow \begin{cases} x, y \in \text{ Int } ST_2 \text{ and } x = y, \text{ or} \\ x \in \text{ Int } ST_1, y \in \text{ Int } ST_1 \text{ and } x = y, \text{ or} \\ x \in T_1, \ldots, y \in T_2 \text{ and } \varphi(y) = x \end{cases} \tag{1.1}$$

Consider the quotient manifold $M = (ST_1 \cup ST_2)/ \sim = ST_1 \cup_\varphi ST_2$.

Claim 1.2.2. $M = S^3$.

To prove this claim we use the
Alexander's trick: Let B_1, B_2 two closed 3-balls, $S_1 = \partial B_1$ and $S_2 = \partial B_2$ be the corresponding 2-sphere boundaries. If

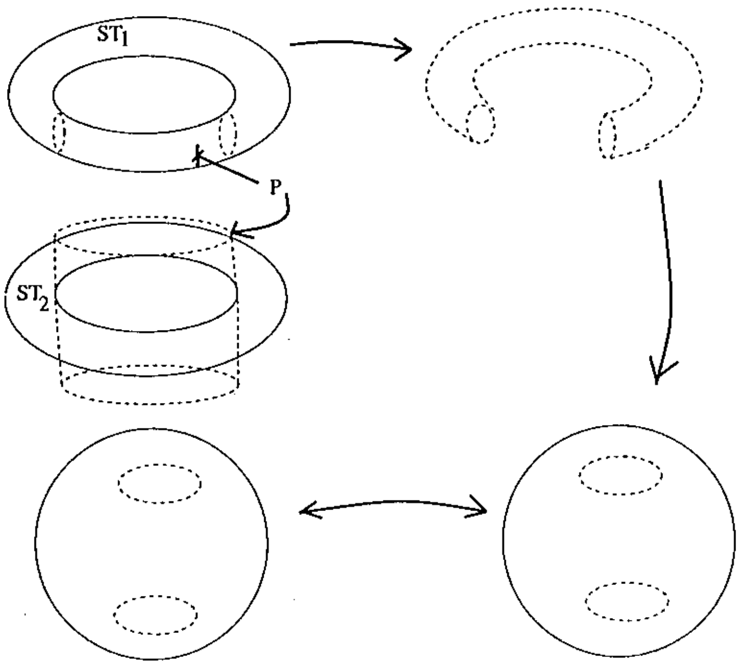


Figure 1.9:

$\varphi\colon S_1 \to S_2$ is a diffeomorphism then $B_1 \cup_\varphi B_2 = S^3$.

Now we return to the proof of the claim. Take a region P in between two meridians of ST_1. Delete P and cap it into the torus hole in $ST2$ as explained in Figure 1.9. With this procedure we obtain two 3-balls whose union along the corresponding boundaries yields S^3 by the Alexander trick. This proves the claim. The *Reeb foliation* in S^3 is precisely the one obtained by the gluing map φ setting inside each solid torus the Reed foliation of the solid torus (see Figure 1.10). The above construction leads to construct foliations with only one compact leaf in any manifold of the form $ST_1 \cup_\varphi ST_2$. For instance if φ were the identity map then the resulting manifold is $S^2 \times S^1$. We have then constructed a foliation with just one compact leaf in $S^2 \times S^1$. The resulting foliation is clearly different to the one obtained by the trivial fibration $\{S^2 \times y : y \in S^1\}$ of $S^2 \times S^1$.

Exercise 1.2.5. Show that the Reeb foliation in S^3 cannot be obtained from a submersion.

Exercise 1.2.6 (Novikov). Show that a vector field transverse to the Reeb foliation in S^3 has a periodic orbit. Observe that there are non-singular C^∞ vector fields in S^3 without periodic orbits (these vector fields are precisely the counterexamples for the Seifert Conjecture). Use the Novikov Theorem to show that all vector field transverse to a codimension one C^2 foliation in a compact manifold with finite fundamental group has a periodic orbit.

Definition 1.2.7. A *Reeb component* of a codimension one foliation $\mathcal{F}$ in M^3 a solid torus $ST \subset M^3$ which is union of leaves of $\mathcal{F}$ such that $\mathcal{F}$ restricted to ST is equivalent to the Reeb foliation in the solid torus $D^2 \times S^1$. A foliation is said to be *Reebless* if it has no Reeb components. We shall see later that compact 3-manifolds supporting Reebless foliations have infinite fundamental group.

1.2.3 Lie group actions

A *Lie group* is a group $(G, \cdot)$ with a differentiable structure making the maps

$$\begin{array}{ccc} G \times G & \to & G \\ x, y & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array}$$

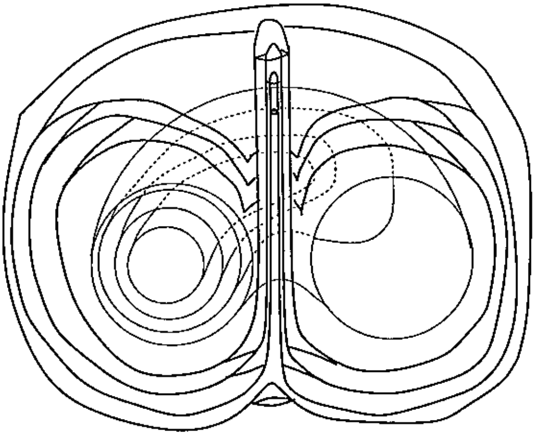


Figure 1.10: Reeb foliation in S^3.

differentiable.

Example 1.2.8. $(\mathbb{R}^n, +)$ is a Lie group. $S^n \setminus \{\text{point}\}$ is a Lie group via stereographic projection. If $\mathbb{C}$ denotes the set of complex numbers (with the complex product), then $\mathbb{C}^* = \mathbb{C} - \{0\}$ with the complex number product is a Lie group. $S^1 \subset \mathbb{C}^*$ is a Lie group when equipped with the product induced by $\mathbb{C}$. Actually it is a Lie subgroup of $\mathbb{C}^*$. $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$ with the product $(z_1, \ldots, z_m)(z_1^1, \ldots, z_m^1) = (z_1 \cdot z_1^1, \ldots, z_n \cdot z_m^1), \quad z_i, z_i^1 \in S^1$ is a compact Lie group.[1] $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}), \det A \neq 0\}$ with the usual matrix product is a Lie group. Note that $\mathrm{GL}_n(\mathbb{R}) \supset \mathbb{R}^n$ and $\mathrm{GL}_n(\mathbb{R}) \not\supset \det^{-1}(0)$. Define $\mathcal{O}(n)$ ={the set of orthonormal matrixes $n \times n$ }. Note that $(v_1, \ldots, v_m) \in \mathcal{O}(n) \Rightarrow ||v_i|| = 1$ and $< v_i, v_j >= \delta_{ij}$ is a compact Lie subgroup

[1] actually the product of Lie groups is a Lie group.

of $\mathrm{GL}_n(\mathbb{R})$.

Definition 1.2.3. An *action* of a Lie group G in M is a map $\varphi : G \times M \to M$ satisfying the following properties:

1. $\varphi(e, x) = x$ for all $x \in M$;
2. $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x))$ for all $x \in M$, $g, h \in G$.

The following notation will be useful: $\varphi(g, x) = g \cdot x$. the orbit of $x \in M$ is the set

$$O_x = \{g \cdot x : g \in G\}.$$

The *isotropy group* of $x \in M$ is the set G_x $g \in G$ fixing x, namely

$$G_x = \{g \in G : g \cdot x = x\}.$$

Clearly G_x is a subgroup of G, for all $x \in M$.

Definition 1.2.4. An action $\varphi : G \times M \to M$ is *locally free* if the isotropy group G_x is discrete $\forall x \in M$. This is equivalent to say that the map

$$\varphi_x : g \in G \to g \cdot x,$$

is an immersion for all $x \in M$ fixed.

Note that if the action $G \times M \to M$ is locally free if and only if the orbit O_x of x is an immersed submanifold of M with constant dimension $\dim(O_x) = \dim(G)$. Since $G_x < G$ is a closed subgroup it is itself a Lie group (Cartan's Theorem) and also the quotient G/G_x has the structure of a differentiable manifold. Actually we have $G_x = G_y$ $\forall\, x, y$ belonging to a some orbit of p and we may introduce the *isotropy subgroup of an orbit* as well.

Given any $x \in M$ we have a natural (diffeomorphism) identification $G/G_x \cong \mathcal{O}_x$ what given an immersed submanifold structure $\mathcal{O}_x \hookrightarrow M$.

Theorem 1.2.5. *The orbits of a C^r locally free action of Lie group of an manifold M^m are the leaves of a C^r foliation of codimension $m - \dim(G)$ of M.*

Proof. Let $\varphi : G \times M^m \to M^m$ be a locally free C^r action. Fix a point $z_0 \in M$ and set $n = \dim(G)$. By assumption $\dim(O_{z_0}) = n$.

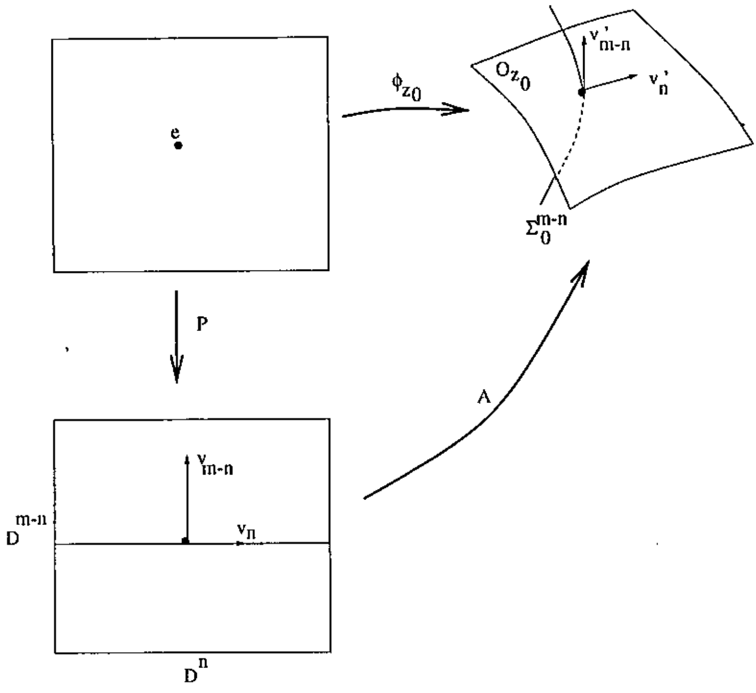


Figure 1.11: Locally free action.

Let Σ_0^{m-n} be a $m - n$ dimensional submanifold transverse to O_{z_0} at z_0. Let (P, U) be a local chart at the unity $e \in G$

such that $P(U) = D^n$. Let $B : (D^{m-n}, 0) \to (\Sigma_O^{m-n}, z_0)$ be a parametrization of Σ_0^{m-n}. Recalling the notation $\varphi(g, x) = g \cdot x$ we define the map $A : D^n \times D^{m-n} \to M$ by

$$A(x, y) = P^{-1}(x) \cdot B(y).$$

The derivative $DA(x, y)$ is given by the expression below:

$$\partial_g \varphi(P^{-1}(x), B(y)) \cdot DP^{-1}(x) + \partial_z \varphi(P^{-1}(x), B(y)) \cdot DB(y).$$

Replacing by $(x, y) = (0, 0)$ one has

$$\begin{aligned} DA(0,0) &= D\varphi_{z_0}(e) \cdot DP^{-1}(0) + \partial_z \varphi(e, z_0) \cdot B(0) \\ &= D\varphi_{z_0}(e) \cdot DP^{-1}(0) + DB(0). \end{aligned}$$

Let us write $v = v_n \oplus v_{m-n}$ for a tangent vector v of the product $D^n \times D^{m-n}$ at $(0, 0)$. Hence

$$DA(0,0) \cdot v = D\varphi_{z_0}(e) \cdot DP^{-1}(0) \cdot v_n + DB(0) \cdot v_{m-n}.$$

Hence $DA(0, 0) \neq 0$ if $v \neq 0$ for the vectors $v'_n = D\varphi_{z_0}(e) \cdot DP^{-1}(0)$ and $v'_{m-n} = DB(0) \cdot v_{m-n}$ are linearly independent in $T_{z_0}M$. We conclude that A is a local diffeomorphism. By the Inverse Function Theorem the inverse $X = A^{-1}$ of A is well defined in a neighborhood U of z_0. This defines an atlas

$$\mathcal{F} = \{(X, U)\}$$

of M. Note that for all fixed $y_0 \in D^{m-n}$ one has

$$A(\{(x, y_0) : x \in D^n\}) = \{P^{-1}(x) \cdot B(y_0) : x \in D^n\}.$$

Hence $A(\{(x, y_0) : x \in D^n\})$ is contained in the orbit $O_{B(y_0)}$. This proves that the plaques of $\mathcal{F}$ are contained in the orbits of the action. As the orbits are pairwise disjoint we conclude that $\mathcal{F}$ is a foliation of M. □

In a similar way one can prove the following.

Theorem 1.2.6. *The orbits O_x of a C^r action on a manifold M are the leaves of a C^r foliation if and only if the map $x \in M \to \dim(O_x)$ is constant. In this case the action is called* foliated action *on M.*

Example 1.2.9. If G is a Lie group and H is a Lie subgroup of G then there is a natural action $H \times G \to G$ by left multiplication. This action is foliated since the left translation is a diffeomorphism and so the orbits of the action form a foliation of G. Note that the action is locally free $\Leftrightarrow$ H is discrete.

Example 1.2.10. A C^r *flow* on a manifold M is an action X of the additive Lie group $\mathbb{R}$ in M. Note that X is non-singular $\Leftrightarrow$ X is locally free. The orbits of X are either circles or lines. In the first case the orbit is called *periodic* and the flow is *periodic* if all its orbits are periodic. A manifold is called *Seifert* if it supports periodic flows.

Theorem 1.2.11 (Hopf's Theorem). *S^3 is Seifert.*

Indeed, note that $S^3 = \{(z_1, z_2) \subset \mathbb{R}^2 \times \mathbb{R}^2 : ||z_1||^2 + ||z_2||^2 = 1\}$. Define $Q\colon S^1 \times S^3 \to S^3$ by $Q(x, (z_1, z_2)) = (x \cdot z_1, x \cdot z_2)$. One sees that Q is an action of S^1 in S^3. The orbits of Q define a foliation by circles in S^3 proving that S^3 is Seifert. The resulting flow is called *Hopf Fibration* of S^3. Seifert manifolds are important in 3-manifold topology since the ones with infinite fundamental group can be described by their fundamental groups. This fact was discovered by Scott. The fundamental group classification of Seifert manifolds with finite fundamental group is false because there are homotopy equivalent lens spaces which are not homeomorphic.

1.2.4 $\mathbb{R}^n$ actions

We shall describe the actions of the additive group $\mathbb{R}^n$ on a manifold M. Let $Q\colon \mathbb{R}^n \times M \to M$ an action and $\{e_1, \ldots, e_n\}$ be a base of $\mathbb{R}^n$. Fix $i = 1, \ldots, n$ and consider a map

$$\mathbb{R} \times M \xrightarrow{X^i} M$$
$$(t, x) \mapsto Q(te_i, x)$$

This map defines an action of $\mathbb{R}$ on M. In fact $X^i(t,x) = Q(t \cdot e_i, x)$ is a flow in M. We still denote by X^i the vector field induced by X^i, namely

$$X^i(X^i_t(x)) = \frac{d}{dt}(X^i_t(x)) \quad \text{and} \quad X^i(x) = \frac{d}{dt}\bigg|_{t=0} Q(t \cdot e_i, x).$$

In this way we have n-vector fields $X^1, \ldots, X^n$ in M such that $X^i_t(x) = Q(t \cdot e_i, x), \quad \forall t \in \mathbb{R}$.

Now, let $v = \sum\limits_{i=1}^{n} t_i e_i \in \mathbb{R}^n$ and $x \in M$ be fixed. Then

$$Q(v,x) = Q\left(\sum_{n=1}^{n} t_i e_i, x\right) = Q\left(\sum_{i=1}^{n-1} t_i e_i, Q(t_n e_m, x)\right)$$

$$= Q\left(\sum_{i=1}^{n-1} t_i e_i, X^n_{t_n}(x)\right) = X^1_{t_1} \circ X^2_{t_2} \circ \cdots \circ X^n_{t_n}(x), \forall x \in M.$$

Hence, for all action $Q\colon \mathbb{R}^n \times M \to M$, there are vetor fields $X^1, \ldots, X^m$ on M such that $Q(v,x) = X^1_{t_1(v)} \circ \cdots \circ X^n_{t_n(v)}(x)$, $\forall x \in M, \quad \forall v \in \mathbb{R}^n$.

Note that $X^1, \ldots, X^n$ pairwise commute, namely $X^i_t \circ X^j_s = X^j_x \circ X^i_t$. In fact, we have $(X^i_t \circ X^j_s)(x) = Q(te_i, X^j_s(x)) =$

$Q(t\,e_i, Q(s\,e_j, x)) = Q(t\,e_i + s\,e_j, x) = Q(s\,e_j + t\,e_i, x) = X^j_s \circ X^i_t(x)$.

It is known that X, Y commute $\Leftrightarrow [X, Y] = 0$.

Conversely, if $X^1, \ldots, X^n$ are pairwise commuting vector fields, then they define an action Q given by

$$Q(v, x) := X_{t_1(v)}(x) \circ \cdots \circ X_{t_n(v)}(x).$$

Definition 1.2.12. The *rank* of a closed manifold M is the maximal number of pairwise commuting linearly independent vector fields defined in M.

Clearly M has rank ≥ 1. Equivalently, the rank of M is the maximal n such that $\mathbb{R}^n$ acts freely in M.

Theorem 1.2.13 (Rank's Theorem). *Rank of* $S^3 = 1$, *Rank of* $S^2 \times S^1 = 1$, *Rank of* $T^3 = 3$.

We shall return to this result in Chapter 7.

1.2.5 Turbulization

Let $\mathcal{F}$ a codimension one foliation on a 3-manifold and γ be a closed curve such that $\gamma \pitchfork \mathcal{F}$. we assume that γ is orientable (i.e. it has a solid torus tubular neighborhood). We modify $\mathcal{F}$ along γ as follows. Pick a neighborhood U of γ and suppose that U is diffeomorphic to a solid torus $S^1 \times D^2$. Since γ is transverse to $\mathcal{F}$ we can assume that $\mathcal{F}$ intersects the solid torus in the trivial foliation by discs $\theta \times D^2$, $\theta \in S^1$.

We consider the Reeb foliation $\mathcal{F}_R$ in $S^1 \times D^2$. We replace (by surgery) the foliation $\mathcal{F} \cap U$ by $\mathcal{F}_R$ in U to obtain a foliation $\mathcal{F}_\gamma$ as in Figure 1.12.

The resulting foliation $\mathcal{F}_\gamma$ is said to be obtained by turbulization of $\mathcal{F}$ along γ. Note that $\mathcal{F}_\gamma$ is C^r if $\mathcal{F}$ is, $0 \leq r \leq \infty$,

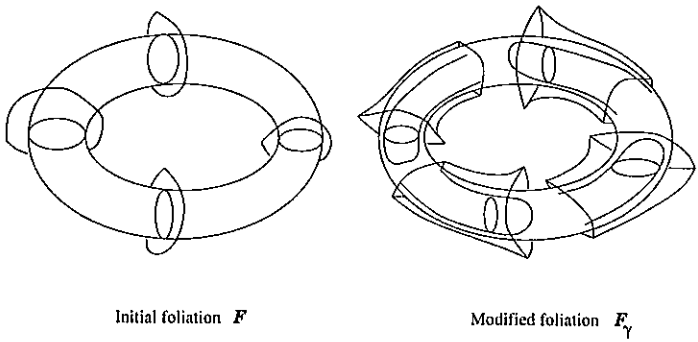


Figure 1.12: Turbulization.

$r \neq w$. The number of Reeb components of the new foliation $\mathcal{F}_\gamma$ is $\geq$ to the number of components of $\mathcal{F}$.

Example 1.2.14. Consider the foliation by discs $\theta \times D^2$ in $S^1 \times D^2$, and let $\gamma = S^1 \times \{0\}$ be the curve in the middle of $S^1 \times D^2$. We have that $\gamma \pitchfork \mathcal{F}$, and so, we can modify $\mathcal{F}$ by turbulization along γ. The compact leaves of the new foliation are T^2 and the non-compact ones are all either planes $\mathbb{R}^2$ (inside the Reeb component) or punctured discs $D^2 \backslash \{\text{point}\}$ (outside the Reeb component).

Example 1.2.15. Consider $\mathcal{F}$ as before and γ as in Figure 1.13.

Note that $\gamma \pitchfork \mathcal{F}$ and so we can modify $\mathcal{F}$ by turbulization along γ. In this case, $\mathcal{F}_\gamma$ is a foliation of $S^1 \times D^2$ whose leaves are T^2, $\mathbb{R}^2$ and $D^2 \backslash \{2 \text{ points}\}$. Analogously, it is easy to construct a foliation of $S^1 \times D^2$ whose leaves are T^2, $\mathbb{R}^2$ and $D^2 - \{n\ points\}$.

Exercise 1.2.16. Show that the Reeb foliation in S^3 satisfies: # Reeb components of $\mathcal{F}_\gamma$ = # Reeb components of $\mathcal{F}$, $\forall \gamma$ curve, $\gamma \pitchfork \mathcal{F}$.

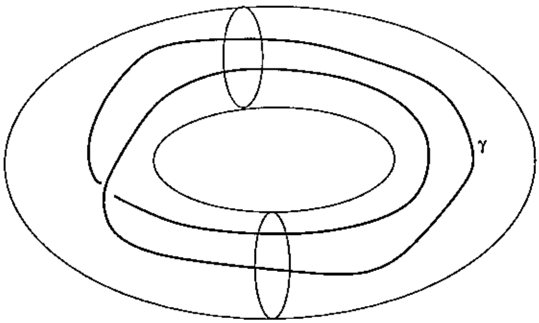


Figure 1.13:

Turbulization can be used to prove the following

Theorem 1.2.17. *All closed 3-manifolds support C^∞ codimension one foliations. On the contrary, the solely closed surfaces supporting codimension one foliations are T^2 and K^2.*

1.2.6 Suspensions

A *representation* of a group G in a group H is a homomorphism

$$Q\colon G \to H.$$

We shall be interested in the case $G = \pi_1(B)$ and $H = \operatorname{Diff}^r(F)$ where B, F are manifolds and $\operatorname{Diff}^r(F)$ is the group of class C^r diffeomorphisms in F endowed with the composition operation.

Suppose that $Q\colon \pi_1(B) \to \operatorname{Diff}^r(F)$ is a representation of $\pi_1(B)$ in $\operatorname{Diff}^r(F)$. Let $\widetilde{B} \xrightarrow{\pi} B$ be the universal covering of B. Recall that $\pi_1(B)$ acts in $\widetilde{B}$ by deck transformations: $\alpha \in \pi_1(B)$, $\tilde{b} \in \widetilde{B}$, $b = \pi(\tilde{b})$, we have $\tilde{\alpha}$ the lift of α. Define $\alpha \cdot \tilde{b} = \tilde{\alpha}(1)$ as the action of $\pi_1(B)$ in $\widetilde{B}$. With this action one has $\widetilde{B}/\pi_1(B) \simeq B$. $\pi_1(B)$ also acts in $\widetilde{B} \times F$ via Q in the following way: define

$$A\colon \pi_1(B) \times (\widetilde{B} \times F) \to \widetilde{B} \times F$$

by setting

$$A(\alpha, (\tilde{b}, x)) = (\alpha \cdot \tilde{b}, Q(\alpha)(x)).$$

$B \times_Q F = (\widetilde{B} \times F)/A \to B \times_Q F$ is a manifold.

Definition 1.2.7. The orbit space of A,

$$B \times_Q F = (\widetilde{B} \times F)/A$$

is called the suspension of Q.

Example 1.2.18. Suppose that $Q(g) = Id_F$ (the identity in F) for all $g \in \pi_1(B)$. Then $B \times_Q F$ is precisely the cartesian product $B \times F$.

Example 1.2.19. Suppose $B = S^1$. In this case $\widetilde{B} = \mathbb{R}$ and $\pi_1(B) = \mathbb{Z}$. Let $Q\colon \pi_1(B) \to \mathrm{Diff}^r(F)$ be a representation. Then $Q(n) = (f^{-1})^n = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}$, with $f\colon F \to F$ being a diffeomorphism of F. By definition $A(n, (\tilde{b}, x)) = (\tilde{b}+n, f^{-n}(x))$. Note that A identifies $(0, x)$ with $(0+1, f^{-1}(x)) = (1, f^{-1}(x))$ by replacing x by $f(x)$, we have that A identifies $(0, f(x))$ with $(1, x)$.

Note that $S^1 \times_Q F$ exhibits a flow given by projecting the constant flow $\dfrac{\partial}{\partial t}$ onto $\mathbb{R} \times F$. See Figure 1.14.

The suspension $B \times_Q F$ is equipped with two foliations $\mathcal{F}_Q$, $\mathcal{F}'_Q$ defined as follows: The action A leads invariant the horizontal and vertical foliations in $\widetilde{B} \times F$ given by

$$\widetilde{\mathcal{F}} = \{\widetilde{B} \times f : f \in F\}, \quad \widetilde{\mathcal{F}}' = \{\tilde{b} \times F : \tilde{b} \in \widetilde{B}\}.$$

Hence the action A induces a pair of foliations $\mathcal{F}_Q, \mathcal{F}'_Q$ in $B \times_Q F$ whose leaves L, L' satisfy

$$L = \pi(\text{leaf of } \widetilde{\mathcal{F}}), \quad L' = \pi(\text{leaf of } \widetilde{\mathcal{F}}'),$$

where $\pi : \widetilde{B} \times F \to B \times_Q F$ is the quotient map. Note that the foliations $\mathcal{F}_Q, \mathcal{F}'_Q$ in $\widetilde{B} \times F$ are transverse. We shall discuss more properties of these foliations later on.

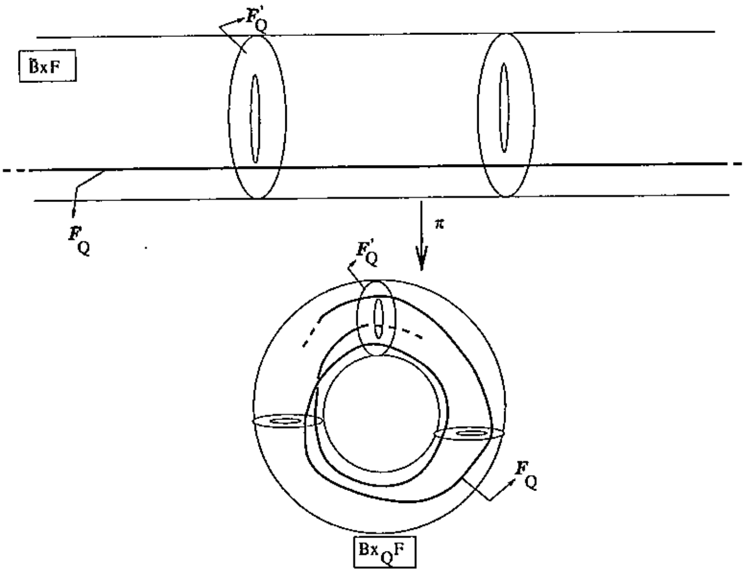


Figure 1.14:

Example 1.2.20. Let B be the bitorus i.e. the genus two orientable closed surface. Then $\pi_1(B)$ has the following presentation:

$$\pi_1(B) = < a, b, c, d : aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 > .$$

Fix $f, g \in \text{Diff}^r(S^1)$ and define the presentation $Q\colon \pi_1(B) \to \text{Diff}^r(S^1)$ by setting

$$Q(a) = f, \quad Q(c) = g, \quad Q(b) = Q(d) = Id$$

and extending linearly. Q is well defined since

$$Q(aba^{-1}b^{-1}cdc^{-1}d^{-1}) = 1.$$

Let us describe the suspension $B \times_Q S^1$ of Q. On one hand consider the subgroup G of $\pi_1(B)$ generated by b and d, i.e. $G =< b, d >$. On the other hand observe that the universal covering of B, $\tilde{B}$, is the Poincaré disc. Let $A_G : G \times \tilde{B} \times S^1 \to \tilde{B} \times S^1$ the action A restricted to G, namely $A_G(\tilde{x}, \theta) = (g \cdot x, Q(g)(\theta)$ Clearly $Q = Id$ in G and so

$$(\tilde{B} \times S^1)/A_G = \tilde{B}/G \times S^1.$$

Consider S^1 as the unit interval $[0, 1]$ with $0 \approx 1$. Figure 1.15 describes the orbit space $(\tilde{B} \times S^1)/A_G$ of A_G.

The internal surface $\tilde{B}/G \times 0$ in the figure is identified with the external one $\tilde{B}/G \times 1$. To obtain $B \times_Q S^1$ we identify the intermediate curves $a \times g(\theta), a^{-1} \times \theta, c \times f(\theta), c^{-1} \times \theta$ according to Figure 1.16.

The leaves the resulting foliation $\mathcal{F}_Q$ spirals around the suspended manifold according to the maps f, g. The another foliation $\mathcal{F}'_Q$ yields a foliation by circles of $B \times_Q S^1$, and so, $B \times_Q S^1$ is Seifert. We will be back to this example later on.

1.2.7 Foliations transverse to the fibers of a fibre bundle

In this section we discuss an important class of foliations given by suspensions, the class of foliations transverse to the fibers of fibre bundles. Let us first recall some basic definitions:

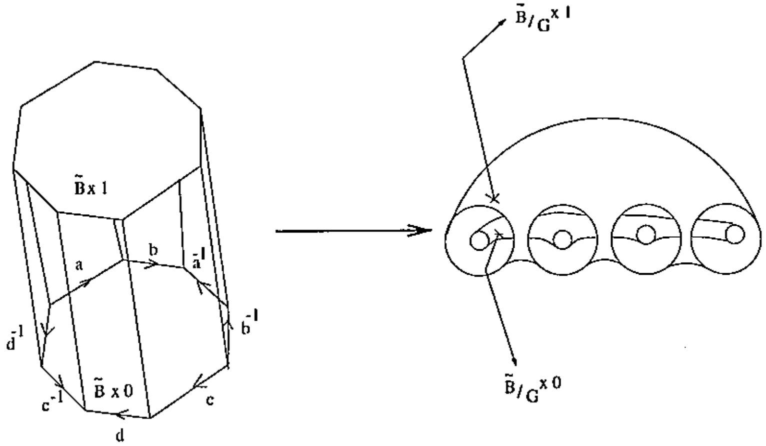


Figure 1.15:

Example 1.2.21 (Fibre bundles). A (differentiable) *fibre bundle* over a manifold M is given by a differentiable map $\pi: E \longrightarrow M$ from a manifold E, called *total space*, which is (the map) a submersion having the following *local triviality property*: for any $p \in M$ there exist a neighborhood $p \in U \subset M$ and a diffeomorphism $\varphi_U: \pi^{-1}(U) \subset E \underset{\sim}{\longrightarrow} U \times F$, where F is fixed manifold called *typical fiber* of the bundle, such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times F \\ \downarrow \pi & \swarrow \pi_1 & \\ U & & \end{array}$$

where $\pi_1: U \times F \to U$ is the first coordinate projection $\pi_1(x, f) = x$. In other words φ_U is of the form $\varphi_U(\tilde{x}) = (\pi(\tilde{x}), \dots)$. Such a diffeomorphism φ_U is called a *local trivialization* of the bundle and U is a *distinguished neighborhood* of $p \in M$. Given

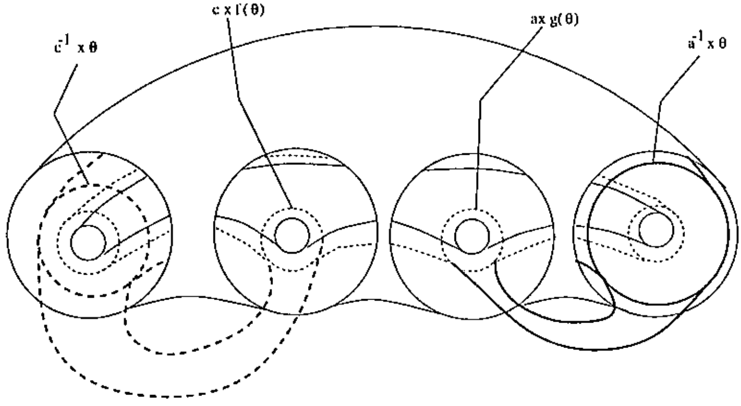


Figure 1.16:

$p \in M$ the *fiber over* p is $\pi^{-1}(p) \subset E$ and by the local trivialization each fiber is an embedded submanifold diffeomorphic to F.

According to Ehresmann Theorem (Theorem 1.2.38) any C^2 proper submersion defines a fibre bundle as above. Let us motivate our next definition with an example.

Example 1.2.22 (Suspension of a foliation by a group of diffeomorphisms). A well known way of constructing transversely homogeneous foliations on fibred spaces, having a prescribed holonomy group is the *suspension* of a foliation by a group of diffeomorphisms. This construction is briefly described below: Let G be a group of C^r diffeomorphisms of a differentiable manifold N. We can regard G as the image of a representation $h\colon \pi_1(M) \to \mathrm{Diff}^r(N)$ of the fundamental group of a complex (connected) manifold M. Considering the differentiable universal covering of M, $\pi\colon \widetilde{M} \to M$ we have a natural free action

$\pi_1 \colon \pi_1(M) \times \widetilde{M} \to \widetilde{M}$, i.e., $\pi_1(M) \subset \mathrm{Diff}^r(\widetilde{M})$ in a natural way. Using this we define an action $H \colon \pi_1(M) \times \widetilde{M} \times N \to \widetilde{M} \times N$ in the natural way: $H = (\pi_1, h)$. The quotient manifold $\frac{\widetilde{M}\times N}{H} = M_h$ is called the *suspension manifold* of the representation h. The group G appears as the *global holonomy* of a natural foliation $\mathcal{F}_h$ on M_h (see [4]). We shall explain this construction in more details. Let M and N be differentiable manifolds of class C^r. Denote by $\mathrm{Diff}^r(N)$ the group of C^r diffeomorphisms of N. Given a representation of he fundamental group of M in $\mathrm{Diff}^r(N)$, say $h \colon \pi_1(M) \to \mathrm{Diff}^r(N)$, we will construct a differentiable fiber bundle M_h, with base M, fiber N, and projection $P \colon M_h \to M$, and a C^r foliation $\mathcal{F}_h$ on M_h, such that the leaves of $\mathcal{F}$ are transverse to the fibers of P and if L is a leaf of $\mathcal{F}$ then $P\,|_L \colon L \to M$ is a covering map. We will use the notation $G = h(\pi_1(M)) \subset Aut(N)$.

Let $\pi \colon \widetilde{M} \to M$ be the C^r universal covering of M. A covering automorphism of $\widetilde{M}$ is a diffeomorphisms f of $\widetilde{M}$ that satisfies $\pi \circ f = \pi$. If we consider the natural representation $g \colon \pi_1(M) \to Aut(\widetilde{M})$ (see [29]) then we know that:

(a)g is injective. In particular $g(\pi_1(M))$ is isomorphic to $\pi_1(M)$.

(b)g is properly discontinuous (see [29]).

We can therefore define an action $H \colon \pi_1(M) \times \widetilde{M} \times N \to \widetilde{M} \times N$ in a natural way:

If $\alpha \in \pi_1(M)$, $\tilde{m} \in \widetilde{M}$ e $n \in N$ then

$$H(\alpha, \tilde{m}, n) = (g(\alpha)(\tilde{m}), h(\alpha)(n)).$$

Using (b) it is not difficult to see that H is properly discontinuous. Thus, the orbits of H define an equivalence relation in $\widetilde{M} \times N$, whose corresponding quotient space is a differentiable manifold of class C^r.

Definition 1.2.23. The manifold $\frac{\widetilde{M}\times N}{H} = M_h$ is called the *suspension manifold* of the representation h.

Notice that M_h is a C^r fiber bundle with base M and fiber N, whose projection $P\colon M_h \to M$ is defined by

$$P(O(\tilde{m}, n)) = \pi(\tilde{m})$$

where $O(\tilde{m}, n)$ denoted the orbit of $(\tilde{m}, n)$ by H.

Let us see how to construct the foliation $\mathcal{F}_h$. Consider the product foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M} \times N$ whose leaves are of the form $\widetilde{M} \times \{n\}$, $n \in N$. It is not difficult to see that $\widetilde{\mathcal{F}}$ is H-invariant and therefore it induces a foliation of class C^r and codimension $q = \dim(N)$, $\mathcal{F}_h$ on M_h, whose leaves are of the form $P(\widetilde{L})$, where $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}$.

Definition 1.2.24. $\mathcal{F}_h$ is called the *suspension foliation* of $\mathcal{F}$ by h.

The most remarkable properties of this construction are summarized in the proposition below (see [15], [4]):

Proposition 1.2.25. *Let $\mathcal{F}_h$ be the suspension foliation of a representation $h\colon \pi_1(M) \to \mathrm{Diff}^r(N)$. Then:*

(i) *$\mathcal{F}_h$ is transverse to fibers of $P\colon M_h \to M$. Moreover, each fiber of P cuts all the leaves of $\mathcal{F}_h$.*

(ii) *The leaves of $\mathcal{F}_h$ correspond to the orbits of h in N in a 1-to-1 correspondence.*

(iii) [2] *If L is a leaf of $\mathcal{F}_h$ corresponding to the orbit of a point $p \in N$, then $P|_L\colon L \to M$ is a covering map (here L is equipped with its natural intrinsic structure).*

[2]Due to (iii) we call G the *global holonomy* of the suspension foliation $\mathcal{F}_h$.

This implies that fixed a point $p \in M$ and its fiber $N_p = P^{-1}(p)$, we obtain by lifting of paths in $\pi_1(M,p)$, to the leaves of $\mathcal{F}_h$, a group $G_p \subset \mathrm{Diff}^r(N_p)$, which is conjugate to G.

(iv) *There exists a collection $\{y_i \colon U_i \to N\}_{i\in I}$ of submersions defined in open subsets U_i of M_h such that*

(a) $M_h = \bigcup_{i\in I} U_i$

(b) $\mathcal{F}_h\big|_{U_i}$ *is given by* $y_i \colon U_i \to N$.

(c) *if* $U_i \cap U_j \neq \phi$ *then* $y_i = f_{ij} \circ y_j$ *for some* $f_{ij} \in G$.

(d) *if L is the leaf of $\mathcal{F}_h$ through the point $q \in N_p$, then the holonomy group of L is conjugate to the subgroup of germs at q of elements of the group $G = h(\pi_1(M,p))$ that fix the point q.*

Conditions (i) and (ii) above motivate the following definition:

Definition 1.2.26. Let $\xi \colon = (\pi \colon E \xrightarrow{F} B)$ be a fibre bundle. A foliation $\mathcal{F}$ on E is said to be *transverse to the fibration* $\pi \colon E \xrightarrow{F} B$ if:
(1) $\mathcal{F}$ is transverse to each fiber of π
(2) $\dim \mathcal{F} + \dim F = \dim E$
(3) For each leaf $L \in \mathcal{F}$ the restriction $\pi\big|_L \colon L \to B$ is a covering map.
In this case $\mathcal{F}$ is conjugate to the *suspension* of the *global holonomy representation* $\varphi \colon \pi_1(B) \to \mathrm{Aut}(F)$ of $\mathcal{F}$. According to Ehresmann ([4]) conditions (1) and (2) imply (3) when the fiber F is compact.

Using the holonomy lifting paths given by condition (3) below we can easily prove:

Theorem 1.2.27. *Let $\mathcal{F}$ be a foliation of class C^r transverse to the fibers of a fibre bundle $\xi\colon = (\pi\colon E \xrightarrow{F} B)$. Then $\mathcal{F}$ is conjugate to the suspension of a representation $\varphi\colon \pi_1(B) \to$ $\mathrm{Aut}(F)$, indeed the global holonomy of $\mathcal{F}$ is naturally conjugate to the image $\varphi(\pi_1(B))$. Conversely if $\mathcal{F}$ is the suspension of a representation $\varphi\colon \pi_1(B) \to \mathrm{Aut}(F)$ for some base manifold B and some fiber manifold F then there is a fibre bundle space $\xi\colon =$ $(\pi\colon E \xrightarrow{F} B)$ such that $\mathcal{F}$ is transverse to the fibers of ξ and the global holonomy of $\mathcal{F}$ is conjugate to the image $\varphi(\pi_1(B)) <$ $\mathrm{Aut}(F)$.*

Recall that a discrete finitely generated group is always conjugate to the fundamental group of a manifold. Thus, suspensions of group presentations and foliations transverse to fibre bundles are in natural bijection. As a natural complement to the above results we have:

Theorem 1.2.28. *Two representations $\varphi\colon \pi_1(B) \to \mathrm{Aut}(F)$ and $\tilde{\varphi}\colon \pi_1(\tilde{B}) \to \mathrm{Aut}(\tilde{F})$ are conjugate if, and only if, there is a fibered diffeomorphism $\Theta\colon E \to \tilde{E}$ (i.e, Θ is the lift of a diffeomorphism $\theta\colon B \to \tilde{B}$ such that $\tilde{\pi} \circ \Theta = \theta \circ \pi$ for the projections $\pi\colon E \to B$ and $\tilde{\pi}\colon \tilde{E} \to \tilde{B}$), with the property that Θ is a conjugacy between the suspension foliations in E and $\tilde{E}$ of φ and $\tilde{\varphi}$ respectively.*

1.2.8 Transversely homogeneous foliations

G be a Lie group and denote by $\mathcal{G}$ the Lie algebra of G. The *Maurer-Cartan* form over G is the unique 1-form $w\colon TG \to \mathcal{G}$ satisfying:

- i) $w(X) = X, \forall X \in \mathcal{G}$

- ii) $Lg^*w = w$, $\forall g \in G$; where $Lg\colon G \hookrightarrow G$ is the left-translation $x \in G \mapsto gx \in G$, $g \in G$ fixed.

The 1-form w satisfies the *Maurer-Cartan formula* $dw + \frac{1}{2}[w,w] = 0$. In fact, given $X, Y \in \mathcal{G}$ we have

$$dw(X,Y) = X.w(Y) - Y.w(X) - w([X,Y]) = -[X,Y].$$

But

$$[w,w](X,Y) = [w(X),w(Y)] - [w(Y),w(X)] = 2[X,Y]$$

because X and Y belong to $\mathcal{G}$ and $w(X) = X$, $\forall X \in \mathcal{G}$.
Thus we have $dw(X,Y) + \frac{1}{2}[w,w](X,Y) = 0$, $\forall X, Y \in \mathcal{G}$ which proves the Maurer-Cartan formula.
Let now $\{X_1, \ldots, X_n\}$ be a basis of $\mathcal{G}$. We have $[X_i, X_j] = \sum_k c_{ij}^k X_k$ for some constants $c_{ij}^k \in \mathbb{C}$, skew-symmetric in (i,j). The c_{ij}^k's are the *structure constants* of G in the basis $\{X_1, \ldots, X_n\}$. Let now $\{w_1, \ldots, w_n\}$ be the dual basis to $\{X_1, \ldots, X_n\}$, with w_j left-invariant. We have $dw_k = -\frac{1}{2}\sum_{i,j} c_{ij}^k w_i \wedge w_j$ and then it is easy to see that $w = \sum_k w_k X_k$ is the Maurer-Cartan form of G.

We recall the following theorem of Darboux and Lie:

Theorem 1.2.29 ([15] pag. 230). *Let α be a differentiable* 1-*form on a manifold M taking values on the Lie algebra $\mathcal{G}$ of G. Suppose α satisfies the Maurer-Cartan formula $d\alpha + \frac{1}{2}[\alpha,\alpha] = 0$. Then α is locally the pull-back of the Maurer-Cartan form of G by a differentiable map. Moreover the pull-back is globally defined if M is simply-connected; and two such local maps coincide up to a left translation of G.*

As an immediate corollary we have:

Corollary 1.2.30. *Let $\alpha_1, \dots, \alpha_n$ be linearly independent differentiable 1-forms on a manifold M. Assume that we have $d\alpha_k = -\frac{1}{2}\sum_{i,j} c^k_{ij}\alpha_i \wedge \alpha_j$ where the c^k_{ij}'s are the structure constants of a Lie group G in the basis $\{X_1, \dots, X_n\}$. Then, locally, there exist differentiable maps $\pi\colon U \subset M \to G$ such that $\alpha_j = \pi^* w_j$, $\forall\, j$ where $\{w_1, \dots, w_n\}$ is the dual (left-invariant) basis of $\{X_1, \dots, X_n\}$. Moreover if M is simply-connected then we can take $U = M$ and if $\pi\colon U \to G$, $\overline{\pi}\colon \overline{U} \to G$ are two such maps with $U \cap \overline{U} \neq \phi$ and connected then we have $\overline{\pi} = Lg \circ \pi$ for some left-translation Lg of G.*

This way we may construct foliated actions of Lie groups on manifolds by defining suitable integrable systems of 1-forms on the manifold. This gives rise to the notion of transversely homogeneous foliations which is a very important notion in the theory.

Definition 1.2.31 (Transversely homogeneous foliation). A foliation $\mathcal{F}$ has a *homogeneous transverse structure* if there are a complex Lie group G, a connected closed subgroup $H < G$ such that $\mathcal{F}$ admits an atlas of submersions $y_j\colon U_j \subset M \to G/H$ satisfying $y_i = g_{ij} \circ y_j$ for some locally constant map $g_{ij}\colon U_i \cap U_j \to G$ for each $U_i \cap U_j \neq \emptyset$. In other words, a suitable atlas of submersions for $\mathcal{F}$ has transition maps given by left translations on G and submersions taking values on the homogeneous space G/H. We shall say that $\mathcal{F}$ is transversely homogeneous *of model* G/H.

Example 1.2.32. Let $F = G/H$ be an homogeneous space of a complex Lie group G ($H \triangleleft G$ is a closed Lie subgroup). Any homomorphism representation $\varphi\colon \pi_1(N) \to \mathrm{Diff}(F)$ gives rise to a foliation $\mathcal{F}_\varphi$ on $(\widetilde{N} \times F)/\Phi = M_\varphi$ which is transversely homogeneous of model G/H.

Example 1.2.33. $G = \mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n$ acts on $\mathbb{R}^n$ by $(A, B), X \mapsto AX + B$ and the isotropy subgroup of $0 \in \mathbb{R}^n$ is $\mathrm{GL}_n(\mathbb{R}) \times 0 = H$ so that $G/H \cong \mathbb{R}^n$ and then the transversely homogeneous foliations of type $\mathrm{Aff}(\mathbb{R}^n)/\mathrm{GL}_n(\mathbb{R})$ are the *transversely affine* foliations.

Example 1.2.34. The real projective unimodular group $G = \mathrm{PSL}\,(2, \mathbb{R})$ acts on $\mathbb{R}P(1)$ by

$$\left(\begin{pmatrix} x & u \\ y & v \end{pmatrix}, z\right) \longmapsto \frac{xz + u}{yz + v}$$

and the isotropy subgroup of $0 \in \mathbb{R}$ is naturally identified with $H = \mathrm{Aff}(\mathbb{R})$, so that $G/H \cong \mathbb{R}P(1)$ and then the transversely homogeneous foliations of type $\mathrm{PSL}(2, \mathbb{R})/\mathrm{Aff}(\mathbb{R})$ are the transversely projective foliations.

Now we introduce the concept of development of a transversely homogeneous foliation which is a basic tool in the study of these foliations:

Proposition 1.2.35. *Let $\mathcal{F}$ be a transversely homogeneous foliation of model G/H on M. Then there exist a homomorphism $h\colon \pi_1(M) \to G$, a transitive covering space $p\colon P \to M$ corresponding to the kernel $H = \mathrm{Ker}(h) \subset \pi_1(M)$ and a submersion $\Phi\colon P \to G/H$ satisfying:*

i) *Φ is h-equivariant which means that $\Phi(\alpha \circ x) = h(x) \circ \Phi$, $\forall x \in P$, $\forall \alpha \in \pi_1(M)$.*

ii) *The foliation $p^*\mathcal{F}$ coincides with the foliation defined by the submersion Φ.*

Such a construction is called a *development of the foliation $\mathcal{F}$* (see [13] pag. 209 for a detailed definition).

We will give an idea of the proof of the above Proposition 1.2.35 according to [13]:
Let $\{y_i \colon U_i \to G\}_{i \in I}$ be a homogeneous transverse structure for $\mathcal{F}$ in M. Denote by f_{ij} the transformation $f_{ij} \colon G/H \to G/H$ such that $y_i = f_{ij} \circ y_j$ in $U_i \cap U_j \neq \phi$.
We can identify f_{ij} in a natural way as an element of G. Now let E be the space obtained as the sum of the $U_i \times G$, $i \in I$. Denote by G_1 the subgroup of G generated by the f_{ij}'s. Consider in E the equivalence relation identifying $(x, y) \in U_i \times G$, where $x \in U_i \cap U_j$, with $(x, f_{ij} \circ g) \in U_j \times G$.
Denote by P the quotient space $E/\sim$. Then P is a principal fibre bundle $p \colon P \to M$ having a discrete structural group $G_1 \subset G$, P being defined by the cocycle (U_i, f_{ij}). The transitive covering space $p \colon P \to M$ has G_1 as group of automorphisms so that there is a natural homomorphism $h \colon \pi_1(M) \to G_1 \subset G$.
Now in each $U_i \times G$ we can construct a holomorphic submersion $\Phi_i \colon U_i \times G \to G/H$ by $\Phi_i(x, g) = g(y_i(x))$. The submersion $\Phi \colon P \to G/H$ is constructed by gluing the submersions Φ_i. Finally we remark that if P is not connected we can replace this space by one of its connected components. □

Corollary 1.2.36. *Let $\mathcal{F}$ be a non-singular transversely homonegeous foliation on a simply-connected manifold M. Then $\mathcal{F}$ is given by a smooth submersion $f \colon M \to G/H$.*

Proof. This corollary is a straightforward consequence of the Darboux-Lie Theorem above but can also be proved by the use of Proposition 1.2.35: In fact, if M is simply connected in Proposition 1.2.35 then we have $H = \operatorname{Ker}(h) \lhd \pi_1(M) = 0$ so that $H = 0$ and then $P = M$. Thus Corollary 1.2.36 follows from ii) of this same proposition. □

Remark 1.2.37. (i) $\alpha \in \pi_1(M)$ acts over P in the following

way: Given $x \in P$ we define $\alpha \cdot x$ as the end-point of the lifting $\tilde{\alpha}_x$ of the path α_x based at the point $p(x)$.
(ii) Conditions (i) and (ii) in the statement of Proposition 1.2.35 (equivariance conditions) are essential in the theory of transversely homogeneous foliations.

In Section 1.3 we shall return to examples of transversely homogeneous foliations.

1.2.9 Fibrations and Ehresmann's Theorem

The fibers of the bundle are the leaves of a foliation on E. Such a foliation is also called a *fibration*. This situation is quite usual as shows the following result:

Theorem 1.2.38 (Ehresmann). *Let $f\colon M \to N$ be a C^2 submersion which is a proper map (i.e., $f^{-1}(K) \subset M$ is compact $\forall\, K \subset N$ compact). Then f defines a fibre bundle over N.*

Proof. The proof is based in the construction of suitable compactly supported vector fields. Let $q \in M$ be given and let $F := \pi^{-1}(q) \subset E$. Then F is a compact submanifold of E. Choose local coordinates $(t_1, \ldots, t_m)$ in a neighborhood U of q in M, with $t_j(q) = 0,\ \ j = 1, \ldots, m$. We take U small enough so that we have:

(i) $\pi^{-1}(U)$ is relatively compact (recall that π is proper) in E.

(ii) There exist smooth vector fields $X_1, \ldots, X_m$ in $\pi^{-1}(U)$ such that $\pi_*(X_j) = \dfrac{\partial}{\partial t_j}$

Claim 1.2.8. *We have $\pi(\Psi(y,p)) = y \quad \forall\,(y,p) \in V_1 \times F$.*

Proof. Given $y \in V_1$ and $p \in F$ denote by $\gamma(z)$ the solution of the ordinary differential equation $\gamma'(z) = Z_y(\gamma(z))$ with initial condition $\gamma(0) = p$ which is defined for all $z \in \mathbb{D}(2)$. Then $\Psi(y, p) = \gamma(1)$ by definition. We have

$$\gamma'(z) = Z_y(\gamma(z)) = \sum_{j=1}^{m} t_j(y) \cdot X_j(\gamma(z)).$$

Therefore

$$\pi_*(\gamma'(z)) = \sum_{j=1}^{m} t_j(y) \cdot \frac{\partial}{\partial t_j}, \quad \text{that is,}$$

$$\frac{d}{dz}\left((\pi \circ \gamma)(z)\right) = \sum_{j=1}^{m} t_j(y) \cdot \frac{\partial}{\partial t_j} \quad \text{in} \quad \mathbb{R}^m.$$

Therefore, $(\pi \circ \gamma)(z) = (\pi \circ \gamma)(0) = z \cdot (t_1(y), \ldots, t_m(y))$ and then

$(\pi \circ \gamma)(1) = \pi(p) + (t_1(y), \ldots, t_m(y)) \Rightarrow$ (since $\pi(p)$ corresponds to the origin and $(t_1(y), \ldots, t_m(y))$ to y in the local chart $(t_1, \ldots, t_m)$)

$\pi(\gamma(1)) = y$ and therefore quad $\pi(\Psi(y, p)) = y$. □

It remains to prove that $\Psi(V_1 \times F) = \pi^{-1}(V_1)$ for sufficiently small $V_1 \ni q$. Since $\pi \circ \Psi = \pi_2$ we have $\pi(\Psi(V_1 \times F)) \subseteq V_1$ so that $\Psi(V_1 \times F) \subseteq \pi^{-1}(V_1)$. If we do not have equality for sufficiently small V_1 then we obtain a sequence $q_n \in U$ with $q_n \to q$ and such that $\pi^{-1}(q_n)$ contains some point p_n which does not belong to the image of Ψ and in fact $\{p_n\}$ avoids some neighborhood W of F in E. Therefore, since Ψ is proper, $\{p_n\}$ has some convergent subsequence say, $p_{n_j} \xrightarrow[j \to \infty]{} p$. But this implies $\pi(p_{n_j}) \xrightarrow[j \to \infty]{} \pi(p)$.

For any point $y \in U$ we consider the vector field $Z_y := t_1(y).X_1 + \cdots + t_m(y).X_m$, defined in $\pi^{-1}(U)$. In particular

$Z_q = 0$ and its flow is complete (defined for all real time). Since Z_y depends differentiably on $y \in U$ we have the following:

Lemma 1.2.39. *There exists a neighborhood $q \in V \subset U$ such that:*
(i) *for each $y \in V$, the flow of Z_y is defined in*

$$\mathbb{D}(2) \times \pi^{-1}(V) \quad (\textit{where } \mathbb{D}(2) = \{z \in \mathbb{C}\,; |z| < 2\}),$$

giving a smooth map

$$\begin{aligned} \varphi^y \colon \mathbb{D}(2) \times \pi^{-1}(V) &\to \pi^{-1}(U) \\ (t,p) &\mapsto \varphi^y(t,p) \end{aligned} \quad (\textit{where } t \textit{ is the real time})$$

with $\varphi^y(0,p) = p, \quad \forall\, p \in \pi^{-1}(V)$,

$$\frac{\partial}{\partial t}\varphi^y(t,p)\bigg|_{(t=0)} = Z_y(\varphi^y(t_0,p).$$

(ii) *For some neighborhood $q \in V_1 \subset V$ we have $\varphi^y(t,p) \in V$, $\forall\, p \in V_1, \quad \forall\, t$ with $|t| \leq 1$.*

Now we may consider the time one flow map

$$\Psi \colon V_1 \times F \to E, \quad \Psi(y,p) := \varphi^y(1,p) \in U.$$

Then ψ is holomorphic and we have an inverse for ψ, which is given by

$$\Psi^{-1} \colon \Psi(V_1 \times F) \to V_1 \times F, \quad \Psi^{-1}(p) := \varphi^y(-1,p).$$

This inverse is well-defined because of (i) and (ii) above so that $q_{n_j} \to \pi(p)$ and $\pi(p) = q$. Thus $p \in F$ what is not possible for $p_n \in E \backslash W, \ \forall n$. This contradiction show that we must have

$\Psi(V_1 \times F) = \pi^{-1}(V_1)$ for every sufficiently small neighborhood V_1 of q in M. $\square$

This is the case if M is compact for instance. One very important result concerned with this framework is due to Tischler.

Theorem 1.2.40 (Tischler). *A compact (connected) manifold M fibers over the circle S^1 if, and only if, M supports a closed non-singular 1-form.*

This is the case if M admits a codimension one foliation $\mathcal{F}$ which is invariant by the flow of some non-singular transverse vector field X on M as we will see in Chapter 8.

1.3 Holomorphic Foliations

A (real) manifold M^{2n} is a *complex manifold* if it admits a differentiable atlas $\{\varphi_j \colon U_j \subset M \to \mathbb{R}^{2n}\}_{j\in J}$ whose corresponding changes of coordinates are holomorphic maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_1(U_i \cap U_j) \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n \to \varphi(U_i \cap U_j) \supset \mathbb{R}^{2n} \simeq \mathbb{C}^n$. Such an atlas is called *holomorphic*.

In this case all the basic concepts of differentiable manifolds (as tangent space, tangent bundle, etc...) can be introduced in this complex setting. This is the case of the concept of foliation:

Definition 1.3.1. A *holomorphic foliation* $\mathcal{F}$ of (complex) dimension k an a complex manifold M is given by a *holomorphic atlas* $\{\varphi_j \colon U_j \subset M \to V_j \subset \mathbb{C}^n\}_{j\in J}$ with the *compatibility property*: Given any intersection $U_i \cap U_j \neq \emptyset$ the change of coordinates $\varphi_j \circ \varphi_i^{-1}$ preserves the horizontal fibration on $\mathbb{C}^n \simeq \mathbb{C}^k \times \mathbb{C}^{n-k}$.

Examples of such foliations are, like in the "real" case, given by non-singular holomorphic vector-fields, holomorphic submersions, holomorphic fibrations and locally free holomorphic complex Lie group actions on complex manifolds.

Remark 1.3.2. (i) As in the "real" case, the study of holomorphic foliations may be very useful in the classification Theory of complex manifolds.

(ii) In a certain sense, the "holomorphic case" is more close to the "algebraic case" than the case of real foliations.

1.3.1 Holomorphic foliations with singularities

One of the most common compactifications of the complex affine space $\mathbb{C}^n$ is the complex projective space $\mathbb{C}P(n)$. It is well-known that any foliation (holomorphic) of codimension $k \geq 1$ on $\mathbb{C}P(n)$ must have some *singularity* (in other words, $\mathbb{C}P(n)$, for $n \geq 2$, exhibits no holomorphic foliation in the sense we have considered up to now.) Thus one may consider such objects: *singular holomorphic foliations* as part of the zoology. Thus one may have consider "singular foliations" when dealing with complex settings.

Example 1.3.3 (Polynomial vector fields on $\mathbb{C}^2$). Let $X = P(x,y)(\partial/\partial x) + Q(x,y)(\partial/\partial y) = (P,Q)$ be a polynomial vector field on $\mathbb{C}^2$. We have an ordinary differential equations:

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases}$$

We have local solutions given by Picard Theorem(the existence and uniqueness theorem of ordinary differential equations):

$$\varphi(z) = (x(z), y(z))$$

$$\frac{d\varphi}{dz} = \dot{\varphi}(z) = X(\varphi(z))$$

Gluing the images of these unique local solutions, we can introduce the *orbits* of X on $\mathbb{C}^2$. The orbits are immersed Riemann surfaces on $\mathbb{C}^2$, which are locally given by the solutions of X.

Now we may be interested in what occurs these orbits in "a neighborhood of the infinity L_∞". We may for instance compactify $\mathbb{C}^2$ as the projective plane $\mathbb{C}P(2) = \mathbb{C}^2 \cup L_\infty$, $L_\infty \cong \mathbb{C}P(1)$.

- What happens to X in a neighborhood of L_∞?
- Is it still possible to consider its orbits around L_∞?

We may rewrite X as the coordinate system $(u, v) : X(u, v) = \frac{1}{u^m} Y(u, v)$, $m \in \mathbb{N} \cup \backslash 0$ where Y is a polynomial vector field. The exterior product of X and Y is zero in common domain $U : X \wedge Y = 0$. So, orbits of Y (or X) are orbits of X (or Y), respectively in U. Then the orbits of X *extend* to the (u, v)-plane as the corresponding orbits of Y along L_∞. This same way, we may consider in the (r, s) coordinate system. These extensions are called *leaves* of a foliation induced by X on $\mathbb{C}P(2)$. We obtain this way: A decomposition of $\mathbb{C}P(2)$ into immersed complex curves which are locally arrayed, as the orbits (solutions) of a complex vector field. This is a holomorphic foliation $\mathcal{F}$ with singularities of dimension one on $\mathbb{C}P(2)$.

Definition 1.3.4. Let M be a complex manifold. A *singular holomorphic foliation* of codimension one $\mathcal{F}$ on M is given by an open cover $M = \bigcup_{j \in J} U_j$ and holomorphic integrable 1-forms $\omega_j \in \bigwedge^1(U_j)$ such that if $U_j \cap U_j \neq \emptyset$, then $\omega_i = g_{ij}\omega_j$ in $U_i \cap U_j$, for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. We put $\operatorname{sing}(\mathcal{F}) \cap U_j = \{p \in U_j ; \omega_j(p) = 0\}$ to obtain $\operatorname{sing}(\mathcal{F}) \subset M$, a well-defined analytic subset of M, called singular set of $\mathcal{F}$. $M \setminus \operatorname{sing}(\mathcal{F})$ is foliated by a holomorphic codimension one (regular) foliation $\mathcal{F}^1$.

Remark 1.3.5. We may always assume that $\text{sing}(\mathcal{F}) \subset M$ has codimension ≥ 2. If $(f_j = 0)$ is an equation of codimension one component of $\text{sing}(\mathcal{F}) \cap U_j$, then we get $\omega_j = f_j^n \bar{\omega}_j$ where $\bar{\omega}_j$ is a holomorphic 1-form and $\text{sing}(\bar{\omega}_j)$ does not contain $(f_j = 0)$.

Using this we may also reformulate the definition above as follows:

Definition 1.3.6. A *singular holomorphic foliation* $\mathcal{F}$ of codimension one on M is given by a pair $\mathcal{F} = (\mathcal{F}^1, \text{sing}(\mathcal{F}))$ where $\text{sing}(\mathcal{F}) \subset M$ is analytic of codimension ≥ 2. $\mathcal{F}$ is a regular holomorphic foliation of codimension one on $M^1 = M \setminus \text{sing}(\mathcal{F})$.

Remark 1.3.7. Assume that we have a holomorphic regular foliation $\mathcal{F}^1$ on $U - \backslash 0$, $0 \in \mathbb{C}^2$, $U \cap \text{sing}(\mathcal{F}) = \backslash 0$. Choose local coordinates (x, y) centered at 0 and define a meromorphic function $f : U - \backslash 0 \to \overline{\mathbb{C}}$, $p \in U - \backslash 0$, as $f(p) =$ inclination of the tangent to the leaf L_p of $\mathcal{F}^1$. By Hartogs' Extension Theorem [64],[17] f extends to a meromorphic function $f : U \cdot\!\!\to \overline{\mathbb{C}}$. We may write $f(x, y) = \frac{a(x,y)}{b(x,y)}$, $a, b \in \mathcal{O}(U)$ and define

$$\frac{dy}{dx} = f(x, y) = \frac{b(x, y)}{a(x, y)},$$

that is,

$$\begin{Bmatrix} \dot{x} = a(x, y) \\ \dot{y} = b(x, y) \end{Bmatrix}.$$

Therefore, $\mathcal{F}$ is defined by a holomorphic 1-form $\omega = a(x, y)\, dy - b(x, y)\, dx$ in U.

Example 1.3.8. Let $f : M \to \overline{\mathbb{C}}$ be a meromorphic function on the complex manifold M. Then $\omega = df$ defines a holomorphic foliation of codimension one with singularities on M. The leaves are the connected components of the levels $\{f = \text{const.}\}$.

Example 1.3.9. Let G be a complex Lie group and $\varphi : G \times M \to M$ a holomorphic action of G on M. The action is foliated if all its orbits have a same fixed dimension. In this case there exists a holomorphic regular foliation $\mathcal{F}$ on M, whose leaves are orbits of φ. However, usually, actions are not foliated, though they may define singular holomorphic foliations. For instance, an action φ of $G = (\mathbb{C}, +)$ on M, $\varphi : \mathbb{C} \times M \to M$ is a holomorphic flows. We have a holomorphic complete vector field $X = \frac{\partial \phi}{\partial t}|_{t=0}$ on M. The singular set of X may be assumed to be of codimension ≥ 2 and we obtain a holomorphic singular foliation of dimension one $\mathcal{F}$ on M whose leaves are orbits of X, or equivalently, of φ.

Problem 1.3.10. *Study and classify actions of complex Lie groups G on a given compact complex M.*

The general problem above may be therefore regarded under the stand-point of singular holomorphic foliations theory.

Example 1.3.11 (Darboux foliations). Let M be a complex manifold and let $f_j \colon M \to \overline{\mathbb{C}}$ be meromorphic functions and $\lambda_j \in \mathbb{C}^*$ complex numbers, $j = 1, \ldots, r$. The meromorphic integrable 1-form $\omega = \prod_{j=1}^{r} f_j \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i}$ defines a *Darboux foliation* $\mathcal{F} = \mathcal{F}(w)$ on M. The foliation $\mathcal{F}$ has $f = \prod_{j=1}^{r} f_j^{\lambda_j}$ as a *logarithmic* first integral.

Example 1.3.12 (Riccati foliations). A *Riccati Foliation* on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is given in some affine chart $(x, y) \in \mathbb{C} \times \mathbb{C}$ by a polynomial 1-form $\omega = p(x)dy - (y^2c(x) - yb(x) - a(x))dx$. Such a foliation is transverse to the fibration $\overline{\mathbb{C}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $(x, y) \mapsto x$, except for a finite number of invariant fibers given in the affine part by $p(x) = 0$. This transversality allows to define a *global holonomy* of the horizontal projective line $\Lambda_0 = \overline{(y = 0)}$ which gives us a group of Moebius transformations $G \subset \mathrm{SL}(2;\mathbb{C})$ of a non-invariant vertical fiber. If $a \equiv 0$ then Λ_0 is an invariant divisor and G is the usual holonomy of the leaf $\Lambda_0 \backslash \operatorname{sing} \mathcal{F}$ as defined above. In this case the elements of G are of the form $f(z) = \frac{\alpha z}{1+\beta z}$.Thus the holonomy of the leaf $\Lambda_0 \backslash \operatorname{sing} \mathcal{F}$ is solvable. In fact, the elements of G are affine maps after the change of coordinates $Z = \frac{1}{z}$ on $\overline{\mathbb{C}}$. Using this remark it is easy to see that the foliation is transversely affine outside the invariant set S given by the union of Λ_0 and the invariant vertical fibers given by the zeros of $p(x)$. If $a \not\equiv 0$ then $\mathcal{F}(\omega)$ is transversely projective outside $S = \bigcup_{p(x)=0} \{x\} \times \overline{\mathbb{C}}$, which is also invariant. We may induce a foliation on $\mathbb{C}P(2)$ with similar properties.

Chapter 2

Plane fields and foliations

2.1 Definition, examples and integrability

A *k-plane field* on a manifold M^m, $1 \leq k \leq m$, is a map $x \in M \to P(x)$, such that $P(x)$ is a k-dimensional subspace of T_xM. When $k = 1$, P is called *line field.* A k-plane field P is C^r if all $x \in M$ exhibits a neighborhood U on which there are defined k linearly independent vector fields $X^1, \dots, X^k \colon U \xrightarrow{C^r} TU$ generating P in U, namely $P(x) = \text{Span}(X^1(x), \dots, X^k(x))$. In this case we say that $X^1, \dots, X^k$ generate P in U.

Example 2.1.1. A C^r foliation $\mathcal{F}$ of dimension k defines the plane field $T\mathcal{F}(x)$ of class C^{r-1} given by

$$T\mathcal{F}(x) = T_x\mathcal{F}_x.$$

The plane field $N\mathcal{F}$ given by

$$N\mathcal{F}(x) = T_xM/T_x\mathcal{F}_x$$

is called the normal plane field of $\mathcal{F}$.

Question 2.1.2. *Is any plane field P of the form $P = T\mathcal{F}$ for some foliation $\mathcal{F}$? Locally the answer is yes but in general the answer is no. This question suggests the following definition.*

Definition 2.1.1. A k-plane field P of class C^r is *integrable* if $P = T\mathcal{F}$ for some C^{r+1} foliation $\mathcal{F}$.

2.1.1 Frobenius Theorem

Let X, Y two vector fields in a manifold M and $p \in M$ be fixed. Denote by X_t the flow of X and similarly Y_t. $X, Y \in C^r$, $r \geq 2$. We define $X_t^*(Y)(p) = DX_{-t}(X_t(p)) \cdot Y(X_t(p))$. Note that $X_t^*(X)(p) = X(p), \quad \forall t$.

Definition 2.1.3. The *Lie bracket* of X, Y is the vector field $[X, Y]$ defined by

$$L_X(Y)(p) = [X, Y](p) = \frac{d}{dt}\Big|_{t=0}(X_t^*(Y)(p)) \quad X, Y \in C^r, \ r \geq 2.$$

In coordinates, $[X, Y]$ has the following form: Writing

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_i b_i \frac{\partial}{\partial x_i}$$

one has

$$[X, Y] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

When X and Y are defined in an open set of $\mathbb{R}^m$, the formula above yields

$$[X, Y] = DY(p) \cdot X(p) - DX(p) \cdot Y(p).$$

A vector field X is tangent to a plane field P (denoted by $X \in P$) if $X(x) \in P(x)$ for all $x \in M$.

Definition 2.1.4. A plane field P is *involutive* if $X, Y \in P \Rightarrow [X, Y] \in P$.

Lemma 2.1.2. *If $\mathcal{F}$ is a foliation, then its associated plane field $T\mathcal{F}$ is involutive.*

Proof. Let X, Y be two vector fields tangent to $T\mathcal{F}$. By using local coordinates defining $\mathcal{F}$ one can assume that X, Y are defined in an open set of $\mathbb{R}^m$ and if $\dim \mathcal{F} = k$, then

$$\begin{aligned} X(x,y) &= (f(x,y),0), \;\; Y(x,y) = (f(x,y),0) \\ [X,Y] &= \begin{pmatrix} \partial_x g & \partial_y g \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} - \begin{pmatrix} \partial_x f & \partial_y f \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ 0 \end{pmatrix} \\ &= (f \cdot \partial_x f - f \cdot \partial_x f, 0). \end{aligned}$$

Hence $[X, Y] \in T\mathcal{F}$ and the proof follows. □

Theorem 2.1.5 (Frobenius' Theorem). *Involutive plane fields are integrable.*

The converse holds by the previous lemma. Hence P is integrable $\Leftrightarrow P = T\mathcal{F} \Leftrightarrow P$ is involutive (i.e. $X, Y \in P \Rightarrow [X, Y] \in P$).

Since all line fields are involutive one has

Corollary 2.1.6. *All line fields are integrable.*

Example 2.1.7. Define $M = \mathbb{R}^3$ and let P the map given by $P(x, y, z) = \mathrm{Span}(X, Y)$, where X, Y are the vector fields defined by $X(x, y, z) = (1 + y, y, z)$ and $Y(x, y, z) = (-y_1, 1 + y, 0)$. As X and Y are orthogonals and non-zero everywhere one has that

P is a plane field of class C^w. Let us use the Frobenius Theorem to show that P is not integrable. Easy computations yield

$$DY = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad DX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So,

$$DY \cdot X = (-y, y, 0), \; DX \cdot Y = (1_y, 1 + y, 0).$$

Hence

$$[X,Y](x,y,z) = DY(x,y,z) \cdot X(x,y,z) - DX(x,y,z) \cdot Y(x,y,z)$$
$$= (-1 - 2y, -1, 0).$$

So $[X,Y] \in P \Leftrightarrow [X,Y] = \alpha X + \beta Y$, for some $\alpha, \beta \in \mathbb{R}$. But

$$\begin{aligned} [X,Y] = \alpha X + \beta Y &\Leftrightarrow (-1 - 1y, -1, 0) \\ &= \alpha(1 + y, y, z) + \beta(-y, 1 + y, 0) \\ &\Leftrightarrow \begin{cases} -1 - 2y = \alpha(1 + y) - \beta y \\ -1 = \alpha y + \beta(1 + y) \\ 0 = \alpha z \end{cases} \end{aligned}$$

Replacing by $(x,y,z) = (1,0,1)$ one has $\alpha = 0, \quad \beta = -1$, $0 = 1$, a contradiction. We conclude that $[X,Y](1,0,1) \notin P(1,0,1)$ and then P is not integrable by Frobenious's.

Theorem 2.1.8 (Thurston's Theorem). *Every $(m-1)$-plane field in a m-manifold M^m is homotopic to an integrable plane field $T\mathcal{F}$, where $\mathcal{F}$ is a C^∞ codimension one foliation of M.*

Example 2.1.9 (integrable systems of differential forms). Let $\omega_1, ..., \omega_r$ be differential 1-forms of class C^r on a manifold M and assume that they are linearly independent at each point $p \in M^n$. We may consider the distribution Δ of $(n-r)$-dimensional planes defined by $\Delta(p) \subset T_pM$ is

$$\Delta(p) = \{v \in T_pM, \omega_j(p) \cdot v = 0, j = 1, ..., r\}.$$

This distribution is called *integrable* if it is tangent to a $-r$ dimensional foliation $\mathcal{F}$ on M. According to Frobenius Integrability Theorem (see also [4]) this occurs if and only if the system of 1-forms is *integrable* what means that we have $d\omega_j \wedge \omega_1 \wedge ... \wedge \omega_r = 0$ for all $j = 1, ..., r$. This occurs for instance if we have a closed 1-form ω with $\omega(p) \neq 0, \forall p \in M$. In this case we have a codimension one foliation $\mathcal{F}$ on M which is defined by the Pffafian equation $\omega = 0$. The leaves of $\mathcal{F}$ are locally given by $f = cte$, where f is a local primitive for ω.

2.2 Orientability

Remember that a k-form w in M is a map

$$\begin{aligned} w\colon M &\to \Lambda^k(TM) \\ p &\mapsto w(p)\colon T_pM \times \cdots \times T_pM \to \mathbb{R} \end{aligned}$$

Remark 2.2.1 (Criterium for orientability of manifolds). A manifold M^m is orientable $\Leftrightarrow$ M^m has a volume form w (i.e. a m-form w such that $w(p) \neq 0$, $\forall\, p \in M$).

Definition 2.2.1. A k-plane field P in M^m, $\quad 1 \leq k \leq m$ is *orientable* if there is a covering $\{U_i\}$ of M and k continuous linearly independent vector fields $X^{1,i}, \ldots, X^{k,i}\colon U_i \to TU_i$ so that

1) $P(x) = \text{span}(X^{1,i}(x), \ldots, X^{k,i}(x)), \quad \forall x \in U_i$

2) $\text{Det}\begin{pmatrix} X^{1,i}(x) \ldots X^{k,i}(x) \\ X^{1,j}(x) \ldots X^{k,j}(x) \end{pmatrix} > 0, \ \forall x \in U_i \cap U_j$.

We say that P is *transversely orientable* if there is an orientable plane field P' on M such that $TM = P \oplus P'$.

Proposition 2.2.2. *A line field P in M is orientable $\Leftrightarrow$ $P(x) =$ Span$(X(x))$ for some continuous everywhere non-vanishing vector field X on M.*

Proof. ($\Leftarrow$) is obvious by taking the trivial covering $\{U_i\} = \{M\}$ of M and $X^{1,i} = X$. ($\Rightarrow$) P orientable $\Rightarrow \exists \{U_i\}$ covering of M and k vector fields $X^i = X^{1,i} \colon U_i \to TU_i$ such that

1) $P(x) = \text{Span}(X^i(x)), \quad \forall x \in U_i$.

2) $\det\begin{pmatrix} X^i(x) \\ X^j(x) \end{pmatrix} > 0, \quad \forall x \in U_i \cap U_j$, i.e. $X^i(x) = a_{ij}(x)$, $X^i(x)$, $\forall x \in U_i \cap U_j$, $a_{ij}(x) > 0$.

Define $X(x) = X^i(x)/||X^i(x)||, \quad \forall x \in U_i$. Then X is well defined since

$$\frac{X^i(x)}{||X^i(x)||} = \frac{X^j(x)}{||X^j(x)||}, \quad \forall x \in U_i \cap U_j \Leftrightarrow$$
$$\frac{X^i(x)}{||X^i(x)||} = \frac{a_{ij}(x) X^j(x)}{||a_{ij}(x)|| \, ||X^j(x)||} = \frac{X^j(x)}{||X^j(x)||}$$

as $a_{i,j}(x) > 0$ for all x. Since X^i generates P in U_i the result follows. □

Example 2.2.3. Choose $M = \mathbb{R}^2 - \{0\}$. None of the line fields induced by the foliations $\mathcal{F}_1, \mathcal{F}_2$ in M at Figure 2.1 is orientable. One can see this by observing how the tangent vector varies along the curve indicated at $\mathcal{F}_1$.

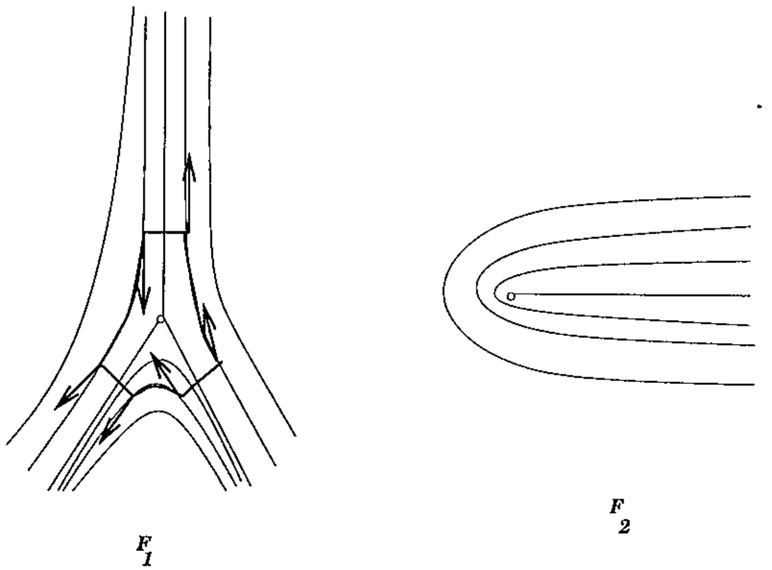


Figure 2.1:

Proposition 2.2.4. *M is orientable (as a manifold) $\Leftrightarrow$ the plane field $P(x) = T_xM$ is orientable.*

Proof. Because P is orientable there are a covering $\{U_i\}$ and vector fields $X^{1,i}, \ldots, X^{m,i}\colon U_i \to TU_i$ such that

$$T_xM = \text{Span}(X^{1,i}(x), \ldots, X^{m,i}(x))\,\forall\, x \in U_i$$

and $\det(X^{n,k}(x))_{1\leq n\leq m}\ k = i, j > 0,\ \forall\, x \in U_i \cap U_j$. For each U_i one choose an m-form W_{U_i} such that if $\forall\, x \in U_i$ then

$\{v_1, \ldots, v_m\}$ is a base of T_xM with

$$\det\begin{pmatrix} v_1, \ldots, v_m \\ X^{1,i}(x), \ldots X^{m,i}(x) \end{pmatrix} > 0 \Leftrightarrow W_{U_i}(x)(v_1, \ldots, v_m) > 0.$$

Let $\{\phi_i\}$ be a partition of the unity subordinate to the covering $\{U_i\}$. Define $W = \sum_i \phi_i, W_{U_i}$. Then w is a m-form with $w(x) \neq 0, \ \ \forall x$. In fact, for $x \subset M$, $w(x) = \sum_{\{i; x \in U_i\}} \phi_i(x) W_{U_i}(x)$. Let i be such that $x \in U_i$ and $\{v_1, \ldots, v_m\}$ is a base T_xM satisfying

$$\det\begin{pmatrix} vr_1, \ldots, v_m \\ X^{1,i}, \ldots, X^{m,1}(x) \end{pmatrix} > 0.$$

Let j be such that $x \in U_j$. Note that

$$\begin{pmatrix} v_1, \ldots, v_m \\ X^{1,j}(x), \ldots, X^{m,j}(x) \end{pmatrix} =$$

$$\begin{pmatrix} X^{1,i}(x) \ldots X^{m,i}(x) \\ X^{1,j}(x) \ldots X^{m,j}(x) \end{pmatrix} \cdot \begin{pmatrix} v_1, \ldots v_m \\ X^{1,i}(x) \ldots X^{m,i}(x) \end{pmatrix}.$$

Because P is orientable one has $\det\begin{pmatrix} vr_1 \ldots v_m \\ X^{1,i} \ldots X^{m,1}(x) \end{pmatrix} > 0$, So $W_{U_i}(x)(v_1, \ldots, v_m) > 0$. Henceforth $\phi_i(x) \cdot W_{U_i}(x)(v_1, \ldots, v_m) \geq 0, \ \ \forall i$. Then

$$W(x)(v_1, \ldots, v_m) = \sum_{\{x; x \in U_i\}} \phi_i(x) \cdot W_{U_i}(x)(v_1, \ldots, v_m) > 0$$

and so there is i such that $\phi_i(x) = 1$. It follows that, $w(x) \neq 0$, $\forall x \in M \Rightarrow M$ is orientable. □

Notation: For $x \in M$ we denote

$$\begin{pmatrix} X^{s,i}(x) \\ X^{s,j}(x) \end{pmatrix} = \begin{pmatrix} X^{1,i}(x) \ldots X^{k,i}(x) \\ X^{1,j}(x) \ldots X^{k,j}(x) \end{pmatrix}$$

and
$$\begin{pmatrix} -X^{s,i}(x) \\ X^{s,j}(x) \end{pmatrix} = \begin{pmatrix} -X^{1,i}(x)X^{2,i}(x)\dots X^{k,i}(x) \\ X^{1,j}(x)X^{2,j}(x)\dots X^{k,j}(x) \end{pmatrix}.$$

Corollary 2.2.5. *Let P and $\overline{P}$ be two plane fields in a manifold M such that*

a) $TM = P \oplus \overline{P}$ (i.e. $T_xM = P(x) \oplus \overline{P}(x)$, $\forall x \in M$);

b) P and $\overline{P}$ are orientable.

Then, M is orientable.

Proof. Exercise.

Example 2.2.6. Let $\overline{\mathcal{F}}$ be the Reeb foliation in the Moebious band (see Section 1.3). Then $T\overline{\mathcal{F}}$ is not orientable. To see this we let $M_\varepsilon = [-1+\varepsilon, 1-\varepsilon] \times \mathbb{R}$, $\varepsilon > 0$ and M_ε/F be the Moebius band. If $T\overline{\mathcal{F}}$ were orientable, then $T\overline{\mathcal{F}}/(M_\varepsilon/F)$ would be orientable. There is a line field P in M_ε/F which is orientable. In fact: it suffices to choose $P(x)$ as $T\mathcal{F}_1$ where $\mathcal{F}_1$ is the projection of the vertical foliation in M_ε/F. P is induced by the vertical vector field $X(x,y) = (0,1)$. Note that X induces a vector field in M_ε/F since

$$DF(0,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

hence P is orientable. But then $T(M_\varepsilon/F) = P \oplus (T\overline{\mathcal{F}}/(M_\varepsilon/F))$ would be orientable, a contradiction. The result follows.

Definition 2.2.2. A foliation $\mathcal{F}$ is *orientable* (resp. *transversely orientable*) if its associated plane field $T\mathcal{F}$ is.

Note that if M supports a foliation $\mathcal{F}$ which is both orientable and transversely orientable, then M is orientable (as a manifold). If M is orientable, then a foliation $\mathcal{F}$ in M is transversely orientable $\Leftrightarrow$ $\mathcal{F}$ is orientable. If $\mathcal{F}$ is a codimension one transversely orientable foliation, then there is a vector field X in M such that $X \pitchfork \mathcal{F}$. Warning: The above does not implies that the time-t map of X preserves $\mathcal{F}$ (i.e. $X_t(\mathcal{F}_x)$ is a leaf for every leaf $\mathcal{F}_x$, $x \in M$).

2.3 Orientability of singular foliations

A C^r *singular foliation on a surface* S is a C^r-foliation $\mathcal{F}$ in the complement $S \setminus \text{sing}(\mathcal{F})$ of a finite set $\text{sing}(\mathcal{F})$ in the interior of S which is transverse to the boundary of S. We denote by $\mathcal{F}_x$ the leaf of $\mathcal{F}$ containing $x \in S \setminus \text{sing}(\mathcal{F})$. One says that $\mathcal{F}$ is C^r-*locally orientable* if there is an open covering $\{U_i : i \in I\}$ of S and a C^r vector fields Y_i in U_i such that $\text{sing}(Y_i) = U \cap \text{sing}(\mathcal{F})$ and $T_x\mathcal{F}_x = \text{Span}(Y_i(x)), \forall x \in U_i \setminus \text{sing}(Y_i)$, where $\text{sing}(Y_i)$ denotes the set of zeroes of Y_i. One says that $\mathcal{F}$ is C^r orientable if the covering $\{U_i : i \in I\}$ above can be chosen with a single element $U_1 = S$. This notion of orientability differs from the corresponding one for non-singular foliations due to the presence of the singularities. One can easily construct singular foliations in D^2 which are not locally orientable (it suffices to complete the ones described in Figure 2.1 to the whole D^2). Clearly a C^r locally orientable singular foliation is C^r orientable. The converse is false in general but true when $S = D^2$, the 2-disc in $\mathbb{R}^2$. Indeed, let $\mathcal{F}$ be a C^r singular foliation in D^2. For an open set U of D^2 one defines $\mathcal{X}^r_{\mathcal{F}}(U)$ as the space of C^r vector fields in U such that $\text{sing}(Y) = U \cap \text{sing}(\mathcal{F})$ and $T_x\mathcal{F}_x = \text{Span}(Y(x))$, $\forall x \in \setminus \text{sing}(Y)$. A finite family $U_1, \cdots, U_k$ of open sets in D^2 is a chain whenever $U_i \cap U_{i+1} \neq \emptyset$ is connected for all $1 \leq i \leq k-1$. Given a chain U_1, U_2 and $Y_1, Y_2 \in \mathcal{X}^r_{\mathcal{F}}(U_1), \mathcal{X}^r_{\mathcal{F}}(U_2)$ we define $\Phi_{U_1,U_2}(Y_1, Y_2)$ to be either Y_2 (if Y_1, Y_2 have the same orientation in $U_1 \cap U_2$) or

$-Y_2$ (otherwise). This definition makes sense because $U_1 \cap U_2 \neq \emptyset$ is connected. Clearly $\Phi_{U_1,U_2}(Y_1, Y_2) \in \mathcal{X}^r_{\mathcal{F}}(U_2)$ and both Y_1 and $\Phi_{U_1,U_2}(Y_2)$ have the same orientation in $U_i \cap U_2$. For general chains $U_1, \cdots, U_k$ and $Y_i \in \mathcal{X}^r_{\mathcal{F}}(U_i)$ $(i = 1, \cdots, k)$ we define $Z_1 = Y_1$, $Z_{i+1} = \Phi_{U_i,U_{i+1}}(Z_i, Y_{i+1})$ and $\Phi_{U_1,\cdots,U_k}(Y_1, \cdots, Y_k) = Z_k$. Under this definition one has

$$\Phi_{U_1,\cdots,U_k}(Y_1, \cdots, Y_k) = \Phi_{U_{k-1},U_k}(\Phi_{U_1,\cdots,U_{k-1}}(Y_1, \cdots, Y_{k-1}), Y_k). \tag{2.1}$$

Now let us assume that $\mathcal{F}$ is C^r locally orientable and let $\{Y^i \in \mathcal{X}^r_{\mathcal{F}}(U_i) : i = 1, \cdots, r\}$ be a fixed C^r local orientation of $\mathcal{F}$. We can assume that all the U_i's are balls, and so, $U_i \cap U_j$ is either empty or connected for all i, j. Define $\tilde{Y}^1 = Y^1$ and for $i = 2, \cdots, r$ we define $\tilde{Y}^i = \Phi_{U_{i_1},\cdots U_{i_k}}(Y_{i_1}, \cdots, Y_{i_k})$, for some chain $U_{i_1}, \cdots, U_{i_k}$ with $i_1 = 1$ and $i_k = i$. The simply connecteness of D^2 implies that the value of $\tilde{Y}_i$ does not depend on the chosen chain $U_{i_1}, \cdots, U_{i_k}$. Let us prove that if $U_i \cap U_j \neq \emptyset$, then $\tilde{Y}_i$ and $\tilde{Y}_j$ have the same orientation in $U_i \cap U_j$. In fact, let $U_{i_1}, \cdots, U_{i_k}$ and $U_{j_1}, \cdots, U_{j_s}$ be two chains realizing $\tilde{Y}_i$ and $\tilde{Y}_j$ respectively. Hence $U_{i_1}, \cdots, U_{i_k}, U_j$ is a chain, and so, the invariance of Φ with respect to the chains implies

$$\tilde{Y}_j = \Phi_{U_{i_1},\cdots,U_{i_k},U_j}(Y_{i_1}, \cdots, Y_{i_k}, Y_j).$$

Then Eq.(2.1) and $i_k = i$ implies

$$\tilde{Y}_j = \Phi_{U_i,U_j}(\Phi_{U_{i_1},\cdots,U_{i_k}}(Y_{i_1}, \cdots, Y_{i_k}), Y_j) = \Phi_{U_i,U_j}(\tilde{Y}_i, Y_j)$$

proving that $\tilde{Y}_i$ and $\tilde{Y}_j$ have the same orientation in $U_i \cap U_j$ as desired. Next we consider a C^∞ partition of the unity $\{Q_1, \ldots, Q_r\}$ of the covering $U_1, \cdots, U_r$ and define

$$Y = \sum_{i=1}^{k} Q_i \tilde{Y}_i.$$

This vector field yields a C^r orientation of $\mathcal{F}$.

2.4 Orientable double covering

Let P be a k-plane field in a manifold M^m. Define $B_x(M)$ the set of ordered basis of $P(x)$ when $x \in M$. Note that $\{v_1, \ldots, v_k\} \neq \{v_2, v_1, \ldots, v_k\} \in B_x(M)$. Define the following relation in $B_x(M)$: $(v_1, \ldots, v_k) \approx_x (w_1, \ldots, w_k) \Leftrightarrow \det \begin{pmatrix} v_s \\ w_s \end{pmatrix} > 0$. Remember that

$$\begin{pmatrix} v_s \\ w_s \end{pmatrix} = \begin{pmatrix} v_1, \ldots, v_k \\ w_1, \ldots, w_k \end{pmatrix}, \quad v_i = \sum_{i<1}^{k} a_{ij} w_j \, .$$

Note that if $(v_1, \ldots, v_k) \approx_x (w_1, \ldots, w_k)$, then $(-v_1, v_2, \ldots, v_k) \not\approx_x (w_1, \ldots, w_k)$. The relation $\approx_x$ is an equivalence since $\det AB = \det A \cdot \det B$. Define $O_x(M) = B_x(M)/ \approx_x$ with projection

$$\pi_x \colon B_x(M) \to O_x(M).$$

Note that

$$O_x(M) = \{O_x, O'_x\}$$

has just two elements. Sometimes we use the notation $-O_x = O'_x$. Finally we define

$$\widetilde{M} = \{(x, O_x) : x \in M, O_x \in O_x(M)\}.$$

At first $\widetilde{M}$ depends on the plane field P. When $P(x) = T_x M$, $\forall\, x \in M$, $\widetilde{M}$ is called *the orientable double covering of* M. In general we call it *the orientable double covering of* P.

Proposition 2.4.1. *$\widetilde{M}$ is a m-dimensional manifold.*

Proof. Fix $(x_0, O_{x_0}) \in \widetilde{M}$. By the definition of plane field P, there is a neighborhood U of x_0 and k smooth vector fields $X^1, \dots, X^k \colon U \xrightarrow{C^r} TU$ such that

$$P(x) = \mathrm{Span}(X^1(x), \dots, X^k(x)), \ \forall x \in U.$$

Define $\widetilde{U}$ as:

$$\widetilde{U} = \{(x, \pi_x(X^1(x), \dots, X^k(x)); x \in U\},$$

if $\pi_{x_0}(X^1(x_0), \cdots, X^k(x_0)) = O_{x_0}$ and

$$\widetilde{U} = \{(x, \pi_x(-X^1(x), \dots, X^k(x)); x \in U\},$$

if $\pi_{x_0}(X^1(x_0), \cdots, X^k(x_0)) = O'_{x_0}$.

We consider the projection

$$\pi \colon \widetilde{M} \to M; \ \pi(x, O_x) = x.$$

We can assume that U is the domain of a local chart (U, Q) around x_0. Define the chart $(\widetilde{U}, \widetilde{Q})$ by $\widetilde{Q} = Q \circ \pi$.

The family $\{(\widetilde{U}, \widetilde{Q})\}$ is an atlas of $\widetilde{M}$. In fact, let (U, Q), (V, ψ) be two such charts with $U \cap V \neq \emptyset$. Then $\widetilde{\psi} \circ \widetilde{Q}^{-1} \colon \widetilde{Q}(\widetilde{U} \cap \widetilde{V}) \to \widetilde{\psi}(\widetilde{U} \cap \widetilde{V})$ satisfies $\widetilde{\psi} \circ \widetilde{Q}^{-1}(y) = \psi \circ Q^{-1}(y)$. Hence $\widetilde{M}$ is a manifold of class C^r and dimension m. Moreover, $\pi \colon \widetilde{M} \to M$ is differentiable and even a local diffeomorphism. Note that $T_{(x,O_x)}\widetilde{M} \simeq T_x M$. Define also a k-plane field $\widetilde{P}$ in $\widetilde{M}$ given by

$$\widetilde{P}(x, O_2) = a(\pi/\widetilde{U})^{-1}(x) \ (P(x)).$$

$\widetilde{M} = (\widetilde{M}, \pi, \widetilde{P})$ is the orientable double covering of P. □

Theorem 2.4.1. *Let M be a manifold and P a plane field in M with orientable double covering $(\widetilde{M}, \widetilde{P})$. Then, $\widetilde{M}$ is connected $\Leftrightarrow$ P is* not *orientable.*

Proof. Proof of ($\Rightarrow$) We fix $(x_0, O_{x_0}) \in \widetilde{M}$. Because $\widetilde{M}$ is connected we can fix a curve $\widetilde{c} \subset \widetilde{M}$ joining (x_0, O_{x_0}) with (x_0, O'_{x_0}) The curve $c = \pi \circ \widetilde{c}$ in M is closed and contains x_0. Suppose by contradiction that P is orientable. Let $\{U_i\}$ the covering of M and $\{X^{1,i}, \cdots, X^{k,i}\}$ be the vector fields in U_i generating P such that

$$det \begin{pmatrix} X^{s,i}(x) \\ X^{s,j}(x) \end{pmatrix} > 0. \tag{2.2}$$

Because c is compact we can suppose that $U_1, \cdots, U_r$ is a covering of c. We can further suppose that $x_0 \in U_1$ and $O_{x_0} = \pi_{x_0}(X^{1,1}(x_0), \cdots, X^{k,1}(x_0))$. Define a new curve $\hat{c} \subset \widetilde{W}$ given by

$$\hat{c}(t) = (c(t), \pi_{c_t}(X^{1,i}(c(t)), \cdots, X^{k,i}(c(t))), \text{ if } c(t) \in U_i,$$

and $i = 1, \cdots, k$.

Note that $\hat{c}$ is well defined by Eq.(2.2). In addition, $\hat{c}$ is continuous because both $c(t)$ and $t \to \pi_{c(t)}(X^{1,i}(c(t)), \cdots, X^{k,i}(x_0))$ are. Define

$$B = \{t \in [0,1] : \hat{c}(t) = \widetilde{c}(t)\}.$$

We have that $B \neq \emptyset$ because $x_0 \in B$. Moreover, B is closed because $\hat{c}$ and $\widetilde{c}$ are continuous. Let us prove that B is open. In fact, if $t_0 \in B$ then $c(t_0) \in U_{i_0}$ for some i_0. If $t_0 \notin \text{Int}(B)$ then there is a sequence $t_n \to t_0$ in $[0,1]$ such that $\hat{c}(t_n) \neq \widetilde{c}(t_n)$ for all n. Because c is continuous and $c(t-0) \in U_{i_0}$ we can suppose that $c(t_n) \in U_{i_0}$ for all n yielding

$$\hat{c}(t_n) = (c(t_n), \pi_{c(t_n)}((X^{1,i_0}(c(t_n)), \cdots, X^{k,i_0}(c(t_n))).$$

Write $\hat{c}(t) = (c(t), \gamma(t))$, where $\gamma(t) \in a(c(t))$ is continuous. Because $\hat{c}(t_n) = \widetilde{c}(t_n)$ one has

$$\pi_{c(t_n)}((X^{1,i_0}(c(t_n)), \cdots, X^{k,i_0}(c(t_n))) = -\gamma(t_n).$$

By taking limits the last expression yields

$$\pi_{c(t_0)}(X^{1,i_0}(c(t_0)), \cdots, X^{k,i_0}(c(t_0))) = -\gamma(t_0)$$

contradicting $\hat{c}(t_0) = \widetilde{c}(t_0)$. This contradiction shows that $\hat{c}(t) = \widetilde{c}(t)$ for all t and then $\hat{c}(1) = \widetilde{c}(1)$. This would imply $O_{x_0} = -O_{x_0}$ which is absurd. This proves ($\Rightarrow$).

Proof of ($\Leftarrow$). Indeed we shall prove that $\widetilde{M}$ not connected $\Rightarrow$ P not orientable. Remember the projection $\pi : \widetilde{M} \to M$ given by $\pi(x, O_x) = x$. Let $\widetilde{M}'$ be a connected component of $\widetilde{M}$. Observe that $\pi(\widetilde{M}') = M$. In fact, since π is a local diffeomorphism we have that $\pi(\widetilde{M}')$ is open in M. Let us prove that $\pi(\widetilde{M})$ is closed in M. Choose $x_n \in \pi(\widetilde{M}') \to x \in M$. By definition there is a neighborhood U of x and k vector fields $X^1, \cdots, X^k$ generating P in U. Obviously there is $\widetilde{y}'_n \in \widetilde{M}'$ such that $\pi(\widetilde{y}'_n) = x_n$. Note that $\widetilde{y}'_n = (x_n, O_{x_n})$ for some $O_{x_n} \in a(x_n)$. Without loss of generality we can assume that $O_{x_n} = \pi_{x_n}(X^1(x_n), \cdots, X^k, (x_n))$ for all n. Passing to the limit the last expression yields $O_{x_n} \to \pi_x(X^1(x), \cdots, X^k, (x)) = O_x$. Hence $\widetilde{y}'_n \to (x, O_x)$. Since $\widetilde{M}'$ is closed in $\widetilde{M}$ we conclude that $(x, O_x) \in \widetilde{M}'$. So, $x = \pi(x, O_x) \in \pi(\widetilde{M}')$ proving that $\pi(\widetilde{M}')$ is closed. Because M is connected we conclude that $\pi(\widetilde{M}') = M$ as desired.

On the other hand, since $\pi^{-1}(x)$ has two elements for all $x \in M$ we conclude that $\widetilde{M}$ has two connected components which we denote by $\widetilde{M}_1, \widetilde{M}_2$. This implies that $\forall x \in M$ and $\forall O_x \in O(x)$ if $(x, O_x) \in \widetilde{M}_1 \Leftrightarrow (x, -O_x) \in \widetilde{M}_2)$. It follows that $\pi/\widetilde{M}_1$ is one-to-one. Hence $\pi : \widetilde{M}_1 \to M$ is a diffeomorphism. Since $\widetilde{P}$ is orientable, and $P = \pi^{-1}(P)$ we would have that P is orientable, a contradiction. This proves the theorem. □

Corollary 2.4.2. *Every plane field on a simply connected manifold is orientable and transversely orientable. In particular, all simply connected manifolds are orientable.*

Proof. Let P be a plane field on a simply connected manifold M. If P were not orientable then its double covering $\pi : \widetilde{M} \to M$ is connected. Being M simply connected we would have $M = \hat{M}$, where $\hat{M} \to M$ is the universal covering of M. This would imply that $\widetilde{M} \to \hat{M}$ is non-trivial covering, a contradiction. This contradiction proves that P is orientable. That P is transversely orientable follows applying the previous result to a complementary plane field of P. The result follows. □

Example 2.4.2. The Reeb foliation in S^3 is orientable and transversely orientable (because $\pi_1(S^3) = 1$).

Exercise 2.4.3. Show that all foliation in the solid torus $D^2 \times S^1$ tangent to the boundary is orientable.

2.5 Foliations and differentiable forms

Remember that a differential k-form w of M is a multilinear map associating to each point $p \in M$ a linear k-form in T_pM, that is is, $w(p) \in \Lambda^k(T_pM)$, where $\Lambda^k(E)$ denotes the space of k-forms in a vector space E. The space of all k-forms in M is denoted by $\Lambda^k(M)$. If $w \in \Lambda^k(M)$ and $\eta \in \Lambda^\ell(M)$, then the alternating product $w \wedge \eta \in \Lambda^{k+\ell}(M)$. If w is a k-form, then there is a derivative dw of w, $d = d_k \colon \Lambda^k(M) \to \Lambda^{k+1}(M)$. w is said to be closed if $dw = 0$, and w is exact if $w = d\eta$. **Poincaré's lemma**: $d(dw) = d^2(w) = 0, \quad \forall w \in \Lambda^k(M)$. Denote by $Z^k(M) = \mathrm{Ker}(d_k)$ is the set of closed k-forms and by $B^k(M) = d_{k-1}(\Lambda^{k-1}(M))$ is the set of exact k-forms. Poincaré's lemma implies that exact forms are closed, namely $B^k(M) \subset 1^k(M)$. The quotient space $H^k(M) = Z^k(M)/B^k(M)$ is called the (de Rham) cohomology k-group of M.

Let $w \in \Lambda^1(M^n)$ be a non-singular 1-form namely, $w(p) \neq 0$,

$\forall p \in M$. The map $p \mapsto \mathrm{Ker}(w(p))$ defines a $(n-1)$-plane field in M.

Proposition 2.5.1. *$w \in \Lambda^1(M)$ is integrable $\Leftrightarrow dw \wedge w = 0 \Leftrightarrow dw = w \wedge \eta$ for some $\eta \subset \Lambda^1(M)$.*

Example 2.5.2. If w is closed, then w is integrable and so w defines a foliation F_w. This remark apply to the following. Define $M = \mathbb{R}^2$, $w = adx + bdy$, $a, b \in \mathbb{R}$. Then w is closed $\Rightarrow w$ induces a foliation F_w. The leaves of this foliation are given by the solution of the differential equation

$$adx + bdy = 0 \Rightarrow y' = -b/a.$$

The general solution of this equation is the straight-line family $y = (-b/a) \cdot x + K$, $K \in \mathbb{R}$. This gives a foliation of $\mathbb{R}^2$ by these straight-lines.

Example 2.5.3 (Thurston). Let L be a closed manifold with $H^1(L) \neq 0$. Let α be a closed non-exact 1-form of L and $f : S^1 \to \mathbb{R}$ be a differentiable map. Denote by $d\theta$ the standard 1-form of S^1. We define the 1-form w in the product $M = L \times S^1$ given by

$$w = d\theta + f(\theta)\alpha.$$

Note that

$$dw = d(d\theta + f(\theta)\alpha) = d(f(\theta)\alpha) = f'(\theta) \cdot d\theta \wedge \alpha + f(\theta) \cdot d\alpha$$

and $d\alpha = 0$. It follows that

$$dw \wedge w = (f'(\theta) \cdot d\theta \wedge \alpha) \wedge (d\theta + f(\theta)\alpha)$$

$$= f'(\theta) \cdot d\theta \wedge \alpha \wedge d\theta + f'(\theta) \cdot d\theta \wedge \alpha \wedge f(\theta) \cdot \alpha = 0.$$

It follows that w is integrable, i.e. $Ker(w)$ is tangent to a C^1 codimension one foliation $\mathcal{F}_w$ in M. Note that the sets $L_{\theta_0} = \{(x, \theta) \in M : \theta = \theta_0\}$, where $\theta_0 \in f^{-1}(0)$ are compact leaves of $\mathcal{F}_w$. In fact, fix $\theta_0 \in f^{-1}(0)$ and $(x, \theta) \in L_{\theta_0}$. If $v_{(x,\theta)} \in T_{(x,\theta)} L_{\theta_0}$ $\Rightarrow v_{(x,\theta)} = (v_x, 0)$ and $\theta = \theta_0$. Hence

$$w(v_{(x,\theta)}) = w(v_x, 0) = d\theta(0) + f(\theta_0)\alpha(v_x) = 0 + 0 \cdot \alpha(v_x) = 0$$

proving that L_{θ_0} is a leaf of λ_w. Clearly L_{θ_0} is diffeomorphic to L and so L_{θ_0} is a compact leaf of $\mathcal{F}_w$ (recall that L is closed).

Chapter 3

Topology of the leaves

3.1 Space of leaves

Let $\mathcal{F}$ be a foliation of a manifold M. The relation $x, y \in M$, $x \sim y \Leftrightarrow x \in \mathcal{F}_y \Leftrightarrow y \in \mathcal{F}_x$ is an equivalence. The quotient space $O_{\mathcal{F}} = M/N$ is called the *space of leaves of* $\mathcal{F}$. Denote by $\pi\colon M \to O_{\mathcal{F}}$ the projection. We set in $O_{\mathcal{F}}$ the topology making π continuous, namely $V \subseteq O_{\mathcal{F}}$ is open $\Leftrightarrow \pi^{-1}(V) \subset M$ is. If $A \subseteq M$ we define $\mathcal{F}(A) = \operatorname{Sat}(A) = \bigcup_{x \in A} \mathcal{F}_x$ and call it the saturated of A. This set is formed by those x such that $\mathcal{F}_x$ meets A.

Example 3.1.1. The leaf space of the foliation F_2 in Figure 1.2 is not Hausdorff. In fact the vertical boundaries of $I \times \mathbb{R}$ correspond to elements in the leaf space which cannot be separated by open sets.

Proposition 3.1.2. *If $A \subset M$ is open, then $\mathcal{F}(A)$ also is.*

Proof. Choose $x \in \mathcal{F}(A)$. By definition, $\mathcal{F}_x \cap A \neq \emptyset$, hence there is $y \in \mathcal{F}_x \cap A$. There there exists a finite collection of plaques

$\alpha_1, \alpha_2, \ldots, \alpha_k$ of $\mathcal{F}$ in $\mathcal{F}_x$ such that $\alpha_i \cap \alpha_{i+1} \neq \emptyset$, $x \in \alpha_k$ $y \in \alpha_1$. Let U_i be the domain of the chart of $\mathcal{F}$ defining α_i. Because $y \in A$, A is open, we can suppose $U_i \subseteq A$. Projects U_i into an open set of $\mathcal{F}(A)$ containing x as follows: Consider $U_1 \cap U_2$ which is non-empty since $\emptyset \neq \alpha_1 \cap \alpha_2 \subset U_1 \cap U_2$. Let $\widetilde{U}_1 = U_1$, $\widetilde{U}_2 = U\{\alpha; \alpha$ be a plaque of U_2 with $\alpha \cap \widetilde{U} \neq \emptyset\}$. Note that $\alpha_2 \subset \widetilde{U}_2$. We have $\widetilde{U}_2$ is open and $\widetilde{U}_2 \subseteq \mathcal{F}(A)$ (since, a plaque, $\alpha \cap \widetilde{U}_2 \neq \emptyset \Rightarrow \alpha \cap \widetilde{U}_1 \neq \emptyset \Rightarrow \alpha \cap \mathcal{F}(A) \neq \emptyset \Rightarrow \alpha \subset \mathcal{F}(a)$). Also, $\alpha_2 \cap \alpha_3 \neq \emptyset$. Define $\widetilde{U}_3 = U\{\alpha, \alpha$ is a plaque of U_3 with $\alpha \cap \widetilde{U}_2 \neq \emptyset\}$. As $\alpha_2 \cap \alpha_3 \neq \emptyset$ and $\alpha_2 \subset \widetilde{U}_2$, we have that $\alpha_1 \subset \widetilde{U}_3$. Hence $\widetilde{U}_3$ is open and $\widetilde{U}_3 \subseteq \mathcal{F}(A)$. Inductively we have $\widetilde{U}_i$, $\forall i = 1, \ldots, k$, such that $\alpha_i \subset \widetilde{U}_i$, $\widetilde{U}_i$ is open and $\widetilde{U}_i \subset \mathcal{F}(A)$. Hence $x \in \alpha_k \subset \widetilde{U}_k$ and $\mathcal{F}(A)$ is open. □

Corollary 3.1.3. *The projection $\pi\colon M \to O_{\mathcal{F}}$ is open (i.e. it sends open sets into open sets)*

Proof. Let $A \subset M$ be open.

$$x \in \mathcal{F}(A) \Leftrightarrow \mathcal{F}_x \cap A \neq \emptyset \Leftrightarrow \exists\, y \in \mathcal{F}_x \cap A \Leftrightarrow \mathcal{F}_x = \mathcal{F}_y$$

and

$$y \in A \Leftrightarrow \pi(x) = \pi(y) \in \pi(A) \Leftrightarrow \pi(x) \in \pi(A) \Leftrightarrow x \in \pi^{-1}(\pi(A))$$

$$\therefore \mathcal{F}(A) = \pi^{-1}(\pi(A)).$$

Because $\mathcal{F}(A)$ is open we have that $\pi(A)$ is open with respect to the quotient topology. □

Warning: Not every projection is open. For instance, consider the projection of a parabola in $\mathbb{R}^2$ into the x-axis.

Definition 3.1.4. We say that $A \subset M$ is *invariant* for $\mathcal{F}$ (or *$\mathcal{F}$-invariant*) if $A = \mathcal{F}(A) = \operatorname{Sat}(A)$.

Lemma 3.1.5. *If A is $\mathcal{F}$-invariant, then $\partial A, \operatorname{Int}(A)$ and $\overline{A}$ are invariant.*

Proof: We have $\operatorname{Int}(A)$ is open $\Rightarrow \mathcal{F}(\operatorname{Int}(A))$ is open $\Rightarrow \operatorname{Int}(A) \subseteq \mathcal{F}(\operatorname{Int}(A)) \subset \mathcal{F}(A) = A$ and $\operatorname{Int}(A)$ is the biggest open set contained in $A \Rightarrow \mathcal{F}(\operatorname{Int}(A)) = \operatorname{Int}(A)$. A is $\mathcal{F}$-invariant $\Rightarrow M \backslash A$ also is $\Rightarrow \operatorname{Int}(M \backslash A)$ also is and $\operatorname{Int}(M \backslash A) = M \backslash \overline{A} \Rightarrow M \backslash \overline{A}$ is $\mathcal{F}$-invariant $\Rightarrow \overline{A}$ is $\mathcal{F}$-invariant. To finish, $\partial A = \overline{A} \backslash \operatorname{Int}(A)$ with $\overline{A}$ and $\operatorname{Int}(A)$ $\mathcal{F}$-invariant $\Rightarrow \partial A$ also is. □

Theorem 3.1.1. *Let F a leaf of a foliation $\mathcal{F}$ and Σ be a transverse of $\mathcal{F}$ intersecting F. Then, one of the following alternatives hold:*

1. *$F \cap \Sigma$ is discrete.*
2. *$\overline{F \cap \Sigma}$ has non-empty interior in Σ.*
3. *$\overline{F \cap \Sigma}$ is a perfect set (i.e. without isolated points) with empty interior.*

Proof. It suffices to prove $F \cap \Sigma$ not discrete $\Rightarrow \overline{F \cap \Sigma}$ is perfect. Suppose by contradiction that $\overline{F \cap \Sigma}$ is not perfect, i.e. $\overline{F \cap \Sigma}$ has an isolated point x_0. Because x_0 is isolated in $\overline{F \cap \Sigma}$ we have $x_0 \in F$. Because $F \cap \Sigma$ is not discrete, there is $x^* \in F \cap \Sigma$ which is an accumulation point of $\{x_n\} \subset F \cap \Sigma$, $\quad x_n \neq x^*$. Because $x_n \in F$ we have that F passes arbitrarily close to x^*. Using a suitable plaque sequence we can see that F passes close to x_0 (see Figure 3.1). This is a contradiction and the proof follows. □

3.2 Minimal sets

Let $\mathcal{F}$ be a foliation in M. A subset $\mu \subseteq M$ is called *minimal* for $\mathcal{F}$ if

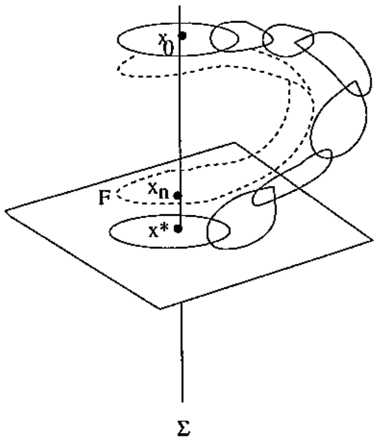


Figure 3.1:

1) μ is closed and $\mathcal{F}$-invariant.
2) if $\emptyset \neq \mu' \subseteq \mu$ satisfies (1), then $\mu' = \mu$.

Remark 3.2.1. Zorn's lemma applied to the set of closed invariant subsets of M (ordered by inclusion) implies that there is at least one minimal subset. Every closed leave is a minimal set. If μ is minimal and F is a leaf of $\mathcal{F}$ in μ, then $\overline{F} = \mu$. If μ is minimal and meets a closed leaf F, then $\mu = F$. In general the set of minimal sets is pairwise disjoint. The Reeb foliation in S^3 has a unique minimal set which is its compact toral leaf. The irrational foliation in T^2 has T^2 as its unique minimal set. As we shall see later on there is no minimal foliation (i.e. the whole manifold is minimal) in S^3 of codimension 1 (by Novikov Compact Leaf Theorem). Analogously there are no minimal foliations on compact manifolds with finite fundamental group. All foliation in the Klein bottle has a compact leave. Hence there is no minimal foliation in the Klein bottle. A foliation is transitive if it has a dense leaf. Minimal foliations are transitive but not conversely. There is no transitive codimension one foliations on compact 3-manifolds with finite fundamental group. In particular S^3 does not support transitive codimension one foliations. Minimal foliations have no compact leaves. As we shall see later transitive codimension one foliation on a compact 3-manifold have no compact leaves as well.

Exercise 3.2.2. Find non-minimal transitive codimension one foliation on M^3 compact.

Lemma 3.2.3. *Suppose that $\mathcal{F}$ is a foliation in M and μ is minimal. Then,* $\mathrm{Int}(\mu) \neq \emptyset \Leftrightarrow \mu = M$.

Proof. Clearly $\mu = M \Rightarrow \mathrm{Int}(\mu) \neq \emptyset$.

Conversely, let μ be minimal with $\mathrm{Int}(\mu) \neq \emptyset$. On one hand μ is closed by definition. On the other hand, μ is open since $\mu = \mathrm{Int}(\mu)$ as $x \in \mu \Rightarrow \overline{\mathcal{F}}_x = \mu \Rightarrow \overline{\mathcal{F}}_x \cap \mathrm{Int}(\mu) \neq \emptyset \Rightarrow x \in \mathcal{F}(\mathrm{Int}(\mu)) = \mathrm{Int}(\mu))$. Since M is connected we conclude that $\mu = M$ and the proof follows. □

Proposition 3.2.4. *Suppose that Σ is a p-disc ($p = \mathrm{cod}\mathcal{F}$), μ is a minimal subset of $\mathcal{F}$ with $\mu \cap \Sigma \neq \emptyset$ and $\mu \cap \partial\Sigma = \emptyset$. Then, μ is not a closed leaf $\Rightarrow \mu \cap \Sigma$ is perfect.*

Proof. Assume that μ is not a closed leaf and prove that $\mu \cap \Sigma$ is a perfect set. For this we proceed as follows. Observe that Σ is compact since it is a p-disc. Let $F \subset \mu$ be a leaf of $\mathcal{F}$. It suffices to prove that $\overline{F \cap \Sigma}$ is perfect. By contradiction suppose that it is not so. Then either $F \cap \Sigma$ is discrete or $\overline{F \cap \Sigma}$ has non-empty interior by Theorem 3.1.1. In the later case $\mu \cap \Sigma$ has non-empty interior (in Σ) since F is dense and Σ is compact. This would imply that μ has non-empty interior and then $\mu = M$ by the previous lemma. This is a contradiction because $\mu \cap \partial\Sigma = \emptyset$. We conclude that $F \cap \Sigma$ is discrete, and so, $F \cap \Sigma$ is finite. If F were not closed then we could find $x \in \overline{F} \setminus F$. Because $F \subset \mu$ and $x \in \overline{F}$ we have $x \in \mu$. Because μ is minimal the leaf F_x of $\mathcal{F}$ containing x is dense in μ. In particular $F_x \cap \mathrm{Int}(\Sigma) \neq \emptyset$ (recall $\mu \cap \partial\Sigma = \emptyset$). By applying the argument described in Figure 3.1 we would have that F intersects Σ infinitely many times, a contradiction. This contradiction proves that F is a closed leaf. Since F is dense in μ we would have that $\mu = F$ is a closed leaf which is impossible. This contradiction proves the result. □

Definition 3.2.1. A minimal set of a foliation on a manifold M is *exceptional* if it is neither a closed leaf nor M.

Problem 3.2.5. *Find necessary and sufficient conditions for the existence of exceptional minimal sets.*

Example 3.2.6. The irrational foliation in T^2 is minimal, and so, it has no exceptional minimal sets. In fact a foliation arising from a C^2 vector field on a closed surface has no exceptional minimal sets. This is false for C^1 vector fields by the Denjoy's counterexample. The Reeb foliation in S^3 has no exceptional minimal sets.

Example 3.2.7. Let B be the bitorus and consider the representation $Q : \pi_(B) \to \mathrm{Diff}(S^1)$ as described in Section 1.4. Recall that the behavior of the suspended foliation $\mathcal{F}_Q$ depends on the maps $f, g \in \mathrm{Diff}(S^1)$ used in the construction of Q. By suitable choice of f, g we have that $\mathcal{F}_Q$ is a C^∞ foliation in some closed Seifert 3-manifold exhibiting an exceptional minimal set. This example (due to Saksteder) gives a counterexample for a possible version of the Denjoy's Theorem

Exercise 3.2.8. Are there transitive codimension one foliations with exceptional minimal sets?

Chapter 4

Holonomy and stability

4.1 Definition and examples

An important tool for the study of foliations is the so-called holonomy group defined as follows. Let $\mathcal{F}$ be a foliation on a manifold M. Let U_i, U_j be two charts of $\mathcal{F}$. Denote by $\pi_i : U_i \to \Sigma_i$ and $\pi_j : U_j \to \Sigma_j$ the projection along the plaques. Suppose that every plaque (of $\mathcal{F}$) in U_i intersects at most one plaque in U_j. Then we can define

$$f_{i,j}(x) = \pi_j(\alpha_x),$$

where $x \in \Sigma_i$ and α_x is the unique plaque of U_i containing $x \in U_i$. The resulting map

$$f_{i,j} : \mathrm{Dom}(f_{i,j}) \subset \Sigma_i \to \Sigma_j$$

is called the holonomy map induced by the two foliated charts $(U_i, X_i), (U_j, X_j)$. Let $U_1, \cdots, U_r$ be a finite family of foliated charts such that every plaque of U_i intersects at most one plaque of U_j (for all i, j). We can define the so-called holonomy map

$f_{1,\cdots,r} : \mathrm{Dom}(f_{1,\cdots,r}) \subset \Sigma_1 \to \Sigma_r$ by

$$f_{1,\cdots,r} = f_{r-1,r} \circ f_{r-2,r-1} \circ \cdots \circ f_{1,2}.$$

Now, let L be a leaf of $\mathcal{F}$ and $x, y \in L$. Clearly L is connected (by definition) and so there is a curve $c : [0,1] \to L$ joining x and y. This curve can covered by a finite family of foliated charts $U_1, \cdots, U_r$ such that $x \in U_1, y \in U_r$ and every plaque of U_i intersects at most one plaque of U_j (for all i, j). Without loss of generality we can assume that $x \in \Sigma_1, y \in \Sigma_r$. The map

$$f_c = f_{1,\cdots,r}$$

is the holonomy induced by the curve c. Note that $f_c(x) = y$ by definition. One can easily prove that f_c does not depend on the foliated covering $U_1, \cdots, U_r$. Moreover, f_c does not depend on the homotopy type of c namely if $c, c' \subset L$ are homotopic in L (with fixed end points) then $f_c = f_{c'}$ in an open subset of Σ_1 containing x. When $x = y$ we obtain a representation

$$\Phi : \pi_1(L) \to \mathrm{Germ}(\Sigma)$$

given by

$$\Phi(\gamma) = [f_c],$$

where c is a representant of $\gamma \in \pi_1(L)$, Σ is a transverse of $\mathcal{F}$ containing $x \in L$ ($\Sigma \approx \Sigma_1$) and

$$\mathrm{Germ}(\Sigma) = \{f : \mathrm{Dom}(f) \subset \Sigma \to \Sigma : f(x) = x\}/\approx$$

is the germ of C^r maps f induced by the equivalence relation $f \approx g$ iff f and g coincides in a neighborhood of x.

Definition 4.1.1. The image $\mathrm{Hol}(L, x) = \Phi(\pi_1(L))$ of Φ is called the *holonomy group of L*.

The group $\mathrm{Hol}(L, x)$ does not depend on $x \in L$. This allows us to define $\mathrm{Hol}(L) = \mathrm{Hol}(L, x)$ for some x. The leaf L has or has no holonomy depending on whether $\mathrm{Hol}(L) \neq 0$ or $= 0$. A foliation without holonomy is a foliation whose leaves have no holonomy.

Example 4.1.1. A simply connected leaf has trivial holonomy. In particular a foliation by planes $\mathbb{R}^2$ is without holonomy.

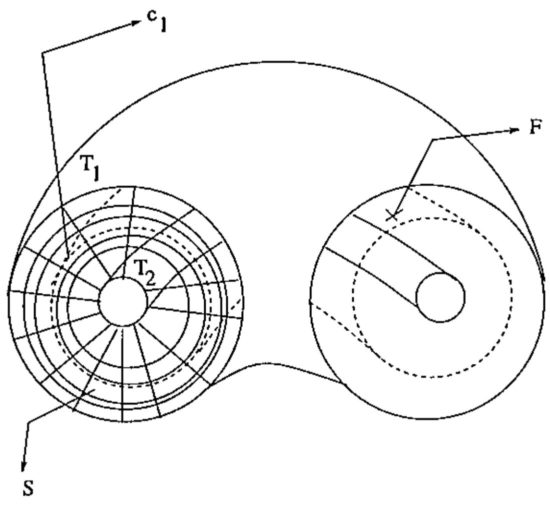


Figure 4.1:

Example 4.1.2. There are non-simply connected leaves without holonomy. Indeed, define $M_0 = I \times T^2$ where I is a compact interval. Then M_0 has a boundary formed by two torus T_1 (external one) and T_2 (internal one), see Figure 4.1. Gluing T_1 with T_2 with a diffeomorphism $\varphi : T_1 \to T_2$ we obtain a manifold M.

The trivial foliation of M_0 formed by concentric torus $* \times T^2$, $* \in I$ defines a foliation $\mathcal{F}$ of M. Any leaf of $\mathcal{F}$ is a torus. All torus bundles over S^1 can be obtained in this way. We observe that $\mathcal{F}$ is a foliation without holonomy. This can be seen as follows. Let F be a leaf of $\mathcal{F}$. Then $\pi_1(F) = \mathbb{Z}^2$ is the free abelian group with two generators $[c_1], [c_2]$, where c_1, c_2 are the meridian curve and the parallel curve in T^2 respectively. The generator c_1 is depicted at Figure 4.1. Consider the transverse surface $S = I \times c_1$ in Figure 4.1. Note that $\mathcal{F}$ intersect S in a circle foliation. The holonomy induced by c_1 in S is precisely the first return induced by this circle foliation. Since this return map is the identity one has $\Phi([c_1]) = Id$. A similar argument shows that $\Phi([c_2]) = Id$. Since $[c_1], [c_2]$ are the generators of $\pi_1(F)$ we conclude that $\text{Hol}(F) = \Phi(\pi_1(F)) = 0$. This proves that $\mathcal{F}$ has no holonomy as desired. With similar arguments we can prove that all foliation arising from a surface bundle over S^1 are without holonomy.

Example 4.1.3 (Holonomy of the Reeb foliation). Let $\mathcal{F}$ be the Reeb foliation in S^3 described in Chapter 1. Then $\mathcal{F}$ has a torus leaf T and all remaining leaves are planes (and so they have in the holonomy). To calculate $\text{Hol}(T)$ we proceed as in the previous example. Indeed, as before $\pi_1(T)$ is generated by the meridian curve and the parallel curve c_1, c_2.

If S is a transverse annulus as in Figure 4.3, Σ is a transverse interval in S centered at $x_0 \in \Sigma \cap T$ then the foliation induces a flow on it whose return map f is as in the right-hand side figure at Figure 4.3.

This map is precisely the holonomy of c_2. Analogously c_2 produces a holonomy having the graph depicted in the left-hand side figure at Figure 4.3. Now $\text{Hol}(T)$ is generated by the classes of there two maps. Note that $\text{Hol}(T)$ is abelian since it is the

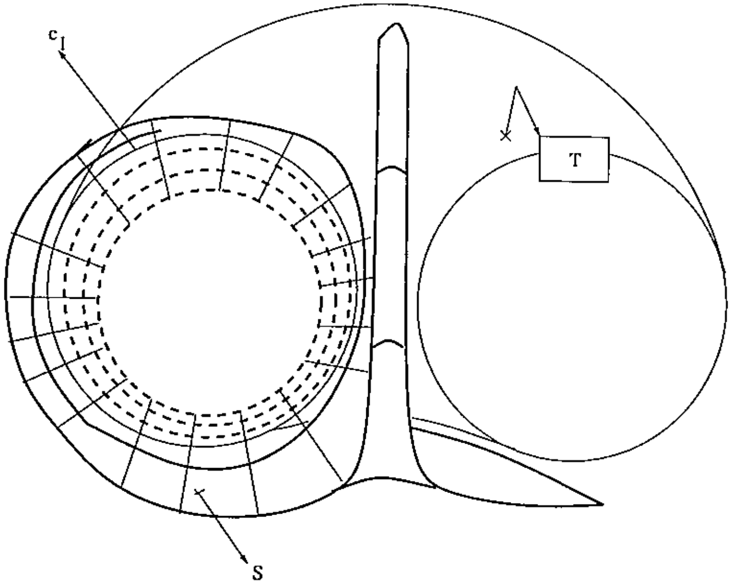


Figure 4.2:

homomorphic image of $\mathbb{Z}^2$ (which is abelian). Because Hol(T) is torsion free we conclude that Hol(T) $= \mathbb{Z}^2$.

Example 4.1.4. A foliation $\mathcal{F}$ tangent to a closed non-singular C^∞ 1-form w in a manifold M has trivial holonomy. Indeed, let X be the gradient of w defined by

$$w_p(v_p) = \langle X(p), v_p \rangle,$$

for all $p \in M$ and $v_p \in T_pM$. Clearly X is non-singular since w is. In addition $\mathcal{F}$ is transverse to $\mathcal{F}$. Let F be a leaf of $\mathcal{F}$ and c a closed curve in F. We can assume that $c : S^1 \to F$ is an immersion. Set $I = [-1, 1]$ and define the map $\phi : S^1 \times I \to S =$

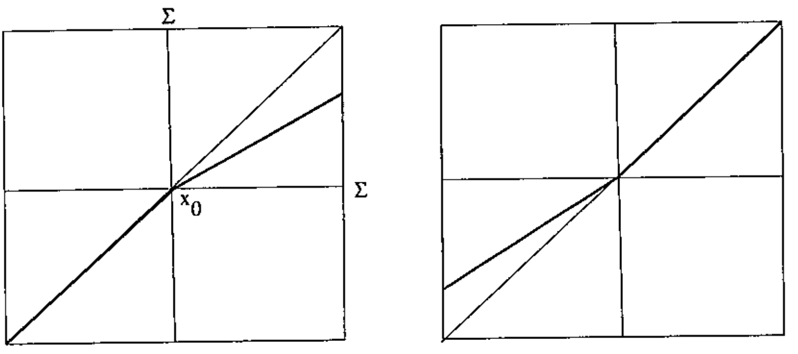


Figure 4.3: Holonomy of the Reeb foliation

$Im(\phi)$ by

$$\phi(\theta, t) = X_f(c(\theta)).$$

It is clear that ϕ is an immersion of class C^2 at least. Then $w^* = \phi^*(w)$ is a well defined 1-form in $S^1 \times I$. Because $dw^* = d\phi(w^*) = \phi^*(dw) = \phi^*(0) = 0$ we have that w^* is closed. Hence w^* defines a foliation $\mathcal{F}^*$ in $S^1 \times I$. Note that $\mathcal{F}^*$ is conjugated to $\mathcal{F} \cap S$. It follows that the curves $c^* = S^1 \times 0$ and c hâve the same holonomy. Let us calculate the holonomy of c^*. Fix $\theta^*, 0) \in c^*$ and $\Sigma^* = \theta^* \times$. Clearly Σ^* is a transverse of $\mathcal{F}^*$. Let $f^* : \text{Dom}(f^*) \subset \Sigma^* \to \Sigma^*$ be the holonomy of c^*, $p \in \text{Dom}(f^*)$ and $q = f^*(p)$. Let α be the arc in Σ^* joining p and q.

Let l be the arc in $\mathcal{F}^*$ joining p, q. Let R be the closed region bounded by the curves c^*, l and α^*. Because

$$0 = \int_R dw^* = \int_{\partial R} w^* = \int_l w^* + \int_\alpha w^* = 0 + 0 + \int_\alpha w^*$$

one has

$$\int_\alpha w^* = 0.$$

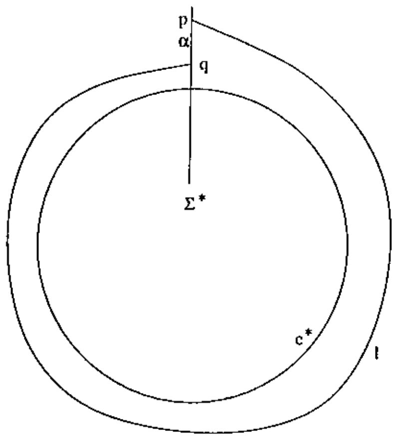


Figure 4.4:

This equality implies that α is trivial and so $p = q = f^*(q)$. We conclude that c^* has trivial holonomy. Hence c has trivial holonomy and the proof follows.

Example 4.1.5 (Holonomy of suspended foliations). We have the following result.

Theorem 4.1.2. *Let $B \times_Q F$ be the suspension of a representation*

$$Q : \pi_1(B) \to \mathrm{Diff}^r(F)$$

and $\mathcal{F}_Q, \mathcal{F}'_Q$ the corresponding foliations. Then,

1. *$\mathcal{F}'_Q$ is a foliation without holonomy.*

2. $\mathrm{Hol}(L) \approx Q(\pi_1(B))$, *$\forall$ leaf L of $\mathcal{F}_Q$.*

Proof. Let B, F smooth manifolds and $M = B \times_Q F$ be the suspension of a representation $Q : \pi_1(B) \to \mathrm{Diff}(F)$. Recall that M is equipped with two foliations $\mathcal{F}_Q$ and $\mathcal{F}'_Q$ which are the projection over $B \times_Q F$ of the trivial foliations $\{\widetilde{B} \times *\}$ and $\{* \times F\}$ on $\widetilde{B} \times F$ respectively. Because the foliation $\mathcal{F}'_Q$ is induced by a fibration (with fibre F) we can see that $\mathcal{F}'_Q$ has no holonomy. So, we restrict ourself to the study of the holonomy of $\mathcal{F}_Q$. For this we fix a leaf L of $\mathcal{F}_Q$ and choose $x_0 \in L$. Fix $(\widetilde{b}_0, f_0) \in \pi^{-1}(x_0)$, where $\pi : \widetilde{B} \times F \to B \times_Q F$ is the natural projection. It follows from the definition of $\mathcal{F}_Q$ that $L = \pi(\widetilde{B} \times f_0)$. Let c_0 be a closed curve containing x_0. We want to calculate the holonomy $h_0 : \mathrm{Dom}(h_0) \subset \Sigma_0 \to \Sigma_0$ of c_0 in L, where Σ_0 is a suitable transverse containing x_0. For this purpose we choose $\Sigma_0 = \pi(\widetilde{b}_0 \times F)$.

Remark 4.1.6. $\pi\big|_{(\widetilde{B} \times f_0)} : \widetilde{B} \times f_0 \to L$ is a covering map.

In fact observe that

$$\pi(\widetilde{b}, f_0) = \pi(\widetilde{d}, f_0) \Rightarrow \{(g\widetilde{b}, Q(g)f_0) : g \in \pi_1(B)\} =$$

$$= \{(g\widetilde{d}, Q(g)f_0) : g \in \pi_1(B)\}$$

$$\Rightarrow \{g\widetilde{b} : g \in \pi_1(B)\} = \{g\widetilde{d} : g \in \pi_1(B)\}.$$

Hence if $P : \widetilde{B} \to B$ is the universal covering of B then $P(\widetilde{b}) = P(\widetilde{d})$ proving the result.

By the previous remark we can consider the lift $\widetilde{c}_0$ of c_0 in $\widetilde{B} \times f_0$ with $\widetilde{c}_0(0) = (\widetilde{b}_0, f_0)$. We can write $\widetilde{c}_0(t) = (\widetilde{\gamma}(t), f_0)$. Define $(\widetilde{b}_1, f_0) = \widetilde{c}_0(1)$. Fix $x \in \Sigma_0$ and

$$(\widetilde{b}_0, f) \in \pi^{-1}(x).$$

The curve

$$c(t) = \pi(\widetilde{\gamma}(t), f)$$

is the lift of c_0 to the leaf $\mathcal{F}_x$ of $\mathcal{F}$ containing x. Hence

$$h_0(x) = \pi(\widetilde{b}_1, f).$$

On the other hand, observe that $\widetilde{b}_1 = \widetilde{\gamma}(1)$ and $\widetilde{\gamma}$ lies in $\widetilde{B}$. It follows that $\widetilde{b}_1 = g_0\widetilde{b}_0$ for some $g_0 \in \pi_1(B)$. Hence

$$h_0(x) = \pi(\widetilde{b}_1, f) = \pi(g_0\widetilde{b}_0, f) = \pi(\widetilde{b}_0, Q(g_0^{-1})f).$$

Since $x = \pi(\widetilde{b}_0, f)$ one has

$$h_0 \circ (\pi \circ i) = (\pi \circ i) \circ Q(g_0^{-1}),$$

where $i : F \to \widetilde{b}_0 \times F$ is the natural inclusion. Then,

$$h_0 = \Phi(Q(g)),$$

where $g = g_0^{-1}$ and $\Phi : \mathrm{Hol}(L) \to Q(\pi_1(B)$ is the map defined by

$$\Phi(Q(g)) = (\pi \circ i) \circ Q(g_0^{-1}) \circ (\pi \circ i)^{-1}.$$

One can prove without difficulty that Φ is an isomorphism. This proves the result. □

4.2 Stability

In this section we denote by $\mathcal{F}$ a foliation of class C^1 of a manifold M.

Definition 4.2.1. A subset $B \subset M$ is *stable* (for $\mathcal{F}$) if $\forall$ neighborhood W of B in M $\exists$ a neighborhood $W' \subset W$ of B in M such that every leaf of $\mathcal{F}$ intersecting W' is contained in W.

Problem 4.2.1. *Find necessary and sufficient conditions for $B \subset M$ to be stable.*

Exercise 4.2.2. Prove that if M is compact then all stable sets of $\mathcal{F}$ are $\mathcal{F}$-invariant.

Exercise 4.2.3. Prove that W' in the definition of stable set can be taken invariant.

Example 4.2.4. Let w be the 1-form in $M = \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $w = xdx + ydy$. Clearly $w = df$ where $f(x,y) = \frac{x^2+y^2}{2}$ and so w is exact. Hence w is tangent to a foliation $\mathcal{F}$ whose leaves are concentric circles around $(0,0)$. Clearly every leaf of $\mathcal{F}$ is stable, has infinite fundamental group and finite holonomy group.

Example 4.2.5. The compact leaf of the Reeb foliation in S^3 is not stable and has infinite holonomy group.

The above example shows the relation between stability and the finiteness of the holonomy group. This relation is the content of the following result.

Theorem 4.2.2. *A compact leaf with finite holonomy group is stable.*

Proof. Let F be a compact leaf of a C^1 foliation $\mathcal{F}$ with finite holonomy group $\mathrm{Hol}(F)$. We fix $x_0 \in F$ a base point and a transverse Σ_0 of $\mathcal{F}$ with $\Sigma_0 \cap F = \{x_0\}$. By assumption there are closed curves $\beta_1, \cdots, \beta_k$ containing x_0 such that

$$\mathrm{Hol}(F) = \{[f_{\beta_1}], \cdots, [f_{\beta_k}]\},$$

where $f_{\beta_i} : \mathrm{Dom}(f_{\beta_i}) \subset \Sigma_0 \to \Sigma_0$ is the holonomy of β_i and $[\cdot]$ denotes the class in the space of germs of Σ at x_0.

Fix a finite covering $\mathcal{U} = \{U_1, \cdots, U_r\}$ of M by charts of $\mathcal{F}$ such that

1. $U_i \cap F$ is a plaque α_i, $\forall i$;

2. $U_i \cap U_j \neq \emptyset \Leftrightarrow \alpha_i \cap \alpha_j \neq \emptyset$, $\forall i, j$;

3. each plaque of U_i intersects at most one plaque of U_j, $\forall i, j$.

The following notation will be useful:

- Σ_i =space of plaques of U_i.
- $\pi_i : U_i \to \Sigma_i$ the plaque projection.
- $x_i = \pi_i(\alpha_i)$. We can suppose that x_i is a point of α_i.

In the case $U_i \cap U_j \neq \emptyset$ ($\Leftrightarrow \alpha_i \cap \alpha_j \neq \emptyset$) we let $\overline{c}_{i,j} \subset \alpha_i \cup \alpha_j$ be a curve joining x_i and x_j. Also in this case we let $\gamma_{i,j}$: $\mathrm{Dom}(\gamma_{i,j}) \subset \Sigma_i \to \Sigma_j$ be the holonomy along the plaques. For all $i = 2, \cdots, r$ we let $c_i \subset F$ be a curve joining x_1 with x_i. The curve c_i induces a holonomy $h_i : \mathrm{Dom}(h_i) \subset \Sigma_1 \to \Sigma_i$.

To prove the stability of the leaf F we fix a neighborhood W of F. Without loss of generality we can assume: $\cup_1^r U_i \subset W$, $\Sigma_1 \subset \Sigma_0$, $x_1 = x_0$ and $D := \cap_1^k \mathrm{Dom}(f_{\beta_i})$ to be a neighborhood of x_0 in Σ_1. The closed curves of the form

$$c_{j,l} = c_l \cup \overline{c}_{l,j} \cup c_j.$$

induce a holonomy map

$$h_{j,l} : \mathrm{Dom}(h_{j,l}) \subset \Sigma_1 \to \Sigma_1.$$

Since $\mathrm{Hol}(F)$ is the union of $[f_{\beta_i}]$'s we have that $h_{l,j} = f_{\beta_p}$ in a neighborhood $D_{j,l} \subset \Sigma_1$ of x_1($= x_0$ the base point). Define

$$D^* = \cap_{j,l} D_{j,l}.$$

Then D^* is an open neighborhood of x_0 in Σ_1. Of course $D^* \subset D$ and $h_{j,l} = f_{\beta_p}$ in D^*. Let $D' \subset D^*$ be a neighborhood of x_1 such that $y' \in D' \Rightarrow f_{\beta_i}(y) \in D$, $\forall i$.

For every $y \in D'$ we define

$$C_y^* = \{\pi_j^{-1}(h_j(f_{\beta_i}(y))) : 1 \leq i \leq k, 1 \leq j \leq r\},$$

where h_1 =Identity by definition.

Claim 4.2.3. *If P is a plaque of U_l (for some $l = 1, \cdots, r$) and $P \cap L \neq \emptyset$ for some $L \in C_y^*$, then $P \in C_y^*$.*

Proof of Claim 4.2.3: By hypothesis there is $L = \pi_j^{-1}(h_j(f_{\beta_i}(y)))$ such that $P \cap L \neq \emptyset$. Because $L \subset U_j$ (as it is a plaque of U_j) and $P \subset U_l$ we have that $U_j \cap U_l \neq \emptyset$. This implies that $c_{j,l}$ is well defined. (recall that this is a curve joining x_i with x_j in $\alpha_i \cap \alpha_j$). By the definition of the holonomy $\gamma_{j,l}$ we have

$$\gamma_{j,l}(h_j(f_{\beta_i}(y))) = \pi_l(P).$$

Hence

$$h_l^{-1} \circ \gamma_{j,l} \circ h_j(f_{\beta_i}(y)) = h_l^{-1}(\pi_l(P)).$$

But $h_l^{-1} \circ \gamma_{j,l} \circ h_j$ is precisely the holonomy $h_{j,l}$ of the curve $c_{j,l}$ by definition. Hence

$$h_{j,l}(f_{\beta_i}(y)) = h_l^{-1}(\pi_l(P)).$$

Because $h_{j,l} = f_{\beta_p}$ one has

$$f_{\beta_p} \circ f_{\beta_i}(y) = h_l^{-1}(\pi_l(P)).$$

As Hol(F) is a group one has $f_{\beta_p} \circ f_{\beta_i} = f_{\beta_{i'}}$ for some i'. Hence

$$f_{\beta,i'}(y) = h_l^1(\pi_l(P)) \Rightarrow \pi_l^{-1}(h_l(f_{\beta_{i'}}(y))) = P,$$

proving $P \in \mathcal{C}_y^*$ as desired. This proves Claim 4.2.3. □

Now, we let

$$L_y^* = \cup\{L : L \in \mathcal{C}_y^*\}.$$

We have the following properties: $L_y^* \subset \mathcal{F}_y$(=the leaf of $\mathcal{F}$ containing y); L_y^* is open in $\mathcal{F}_y$ (since it is union of plaques of $\mathcal{F}_y$); L_y^* is close in M (this is Claim 4.2.3); $L_y^* = \mathcal{F}_y$ (because $\mathcal{F}_y$ is connected); $L_y^* \subset W$ (because the plaques forming $\mathcal{C}_y^*$ are contained in $\cup_1^r U_i \subset W$). The last two properties above imply that $\mathcal{F}_y \subset W$, $\forall y \in D'$. Defining W' as the set of leaves intersecting D' we have that $W' \subset W$ is a neighborhood of F such that every leaf of $\mathcal{F}$ intersecting W' is contained in W. Since W is arbitrary the result follows. □

Exercise 4.2.6. (Prove or disprove) A compact invariant sets whose leaves have finite holonomy group is stable.

4.3 Reeb Stability Theorems

Theorem 4.3.1 (Reeb Local Stability Theorem). *Let F be a compact leaf with finite holonomy group of a C^r foliation $\mathcal{F}$ in a manifold M. Then $\forall$ neighborhood W of F there is a C^r $\mathcal{F}$-invariant tubular neighborhood $\pi : W' \subset W \to F$ of F with the following properties:*

1. *Every leaf $F' \subset W'$ is compact with finite holonomy group.*
2. *If $F' \subset W'$ is a leaf then the restriction $\pi/F' : F' \to F$ is a finite covering map.*
3. *If $x \in F$ then $\pi^{-1}(x)$ is a transverse of $\mathcal{F}$.*

Proof. Let W be a fixed neighborhood of F. We can assume that W is the domain of a C^r tubular neighborhood $\pi_0 : W \to$

F. Because F is compact we can further assume that the fibre $\pi_0^{-1}(x)$ is transverse to $\mathcal{F}$, $\forall x \in F$. Let $W' \subset W$ be given by Theorem 4.2.2. It follows from the proof of Theorem 4.2.2 that all leaves F' in W' are compact (all of them have the form $F' = L_y^*$ for some $y \in D'$ and L_y^* is compact). Define $\pi = \pi_0/W'$. Then $\pi : W' \to F$ is a tubular neighborhood which is invariant and satisfies (3). By shrinking W' if necessary we can assume that F' is transverse to the fibre $\pi^{-1}(x)$, $\forall x \in F$. Since all leaf $F' \subset W'$ is compact we have that F' intersect each fibre finitely many times. The same argument shows that every leaf $F' \subset W'$ has finite holonomy group. This proves (1) and (2). The theorem is proved. □

Next we state two useful lemmas.

Lemma 4.3.2. *Let $Hom(\mathbb{R}, 0)$ be the germ of homeomorphisms in $\mathbb{R}$ fixing 0. If G is a finite subgroup of $Hom(\mathbb{R}, 0)$ then G has at most two elements. If all the elements of G are represented by orientation-preserving maps, then $G = \{[Id]\}$.*

Proof. Suppose that there is $[f] \in G - \{[Id]\}$ represented by a local orientation-preserving homeomorphims f fixing 0. On one hand, there are $n_0 \in \mathbb{N}$ and a neighborhood $U \subset \mathbb{R}$ of 0 such that $f^{n_0}(x) = x$ for all $x \in U$ because $[f]$ has finite order in G (as G is finite). On the other hand, there is $x_0 \in U$ such that $f(x_0) \neq x_0$ because $[f] \neq [Id]$. We can suppose that $0 < x_0$ and that $[0, x_0] \subset U$ without loss of generality. Because f is orientation-preserving one has $0 < f^n(x_0) < f^{n-1}(x_0) < \cdots < f^{(}x_0) < x_0$ for all $n \in \mathbb{N}$ Clearly $f^n(x_0) \in [0, x_0]$ for all n as $[0, x_0] \subset U$. The last applied to $n = n_0$ yields $f^{n_0}(x_0) = x_0$ and so $x_0 < x_0$, a contradiction. This contradiction shows that $[f] = [Id]$ for all element $[f] \in G$ represented by a local orientation-preserving homeomorphism f fixing 0. Let $[g], [g'] \in G$ be represented by orientation-reversing local homeomorphisms fixing 0. Hence $[g] \cdot$

$[g']^{-1}$ is represented by $g \circ (g')^{-1}$ which is orientation-preserving. It follows that $[g] = [g']$ and so there is only one element of G represented by an orientation-reversing map. This proves that G has at most two elements and the proof follows. $\square$

Lemma 4.3.3. *Let F be a compact leaf of a codimension one foliation $\mathcal{F}$ defined on a manifold M. Let F_n be a sequence of compact leaf of $\mathcal{F}$ accumulating to a point in F. Then $\forall$ neighborhood $W \subset M$ one has $F_n \subset W$ for all n large.*

Proof. Let $U_1, \cdots, U_k \subset W$ be covering of F with charts of $\mathcal{F}$ such that $U_i \cap F$ is a single plaque α_i of U_i, $\forall i$. For each i we denote by Σ_i, the space of plaques of U_i, and by $\pi_i : U_i \to \Sigma_i$ the projection along the plaques. Because F_n accumulates a point of F we can assume that $F_n \cap U_1$ contains a plaque arbitrarily close to α_1. From this we can assume that $F_n \cap U_i \neq \emptyset$ for all n, i. Clearly $F_n \cap U_i$ contains a finite number of plaques as F_n is compact. Let $P^{n,i}$ and $p^{n,i}$ be the maximum and the minimum of such plaques with respect to the natural order of Σ_i (=interval). Define $R_n = \cup_{i=1}^k (P^{n,i} \cup p^{n,i})$. Clearly $R_n \subset F_n$ is open in F_n (as it is union of plaques). Let us prove that R_n is closed in F_n. In fact, fix n and choose a sequence $x_j \in R_n$ converging to $x \in F$. We can assume that all the x_j's are in a single plaque $P^{n,io}$ of R_n. Clearly $x \in U_r$ for some $1 \leq r \leq k$ by the definition of R_n. Hence the plaque $\alpha_r(x) \subset U_r$ containing x is well defined. Clearly $P^{n,io} \cap U_r \neq \emptyset$ and so $P^{n,io} \cap \alpha_r(x) \neq \emptyset$. Thus $\alpha_r(x)$ is a plaque of $F_n \cap U_r$. Since $P^{n,io}$ is the maximum of the plaques of $F_n \cap U_{i_0}$ one has that $\alpha_r(x)$ is the maximum of the plaques of $F_n \cap U_r$. In other words $\alpha_r(x) = P^{n,r}$ proving $x \in R_n$. We conclude that R_n is closed in F_n. Since F_n is connected we conclude that $R_n = F_n$. Since $R_n \subset \cup_i U_i \subset W$ we conclude that $F_n \subset W$. The lemma is proved. $\square$

Remark 4.3.1. The conclusion of the lemma above is false for foliations with codimension > 1.

Theorem 4.3.4 (Reeb Global Stability Theorem). *Let $\mathcal{F}$ be a C^1 codimension one foliation of a closed manifold M. If $\mathcal{F}$ contains a compact leaf F with finite fundamental group then all the leaves of $\mathcal{F}$ are compact with finite fundamental group. If $\mathcal{F}$ is transversely orientable then every leaf of $\mathcal{F}$ is diffeomorphic to F; M is the total space of a fibration $f : M \to S^1$ over S^1 with fibre F; and $\mathcal{F}$ is the fibre foliation $\{f^{-1}(\theta) : \theta \in S^1\}$.*

Proof. Denote by $\mathcal{F}_x$ the leaf of $\mathcal{F}$ containing $x \in M$ and define

$$\hat{M} = \{x \in M : \mathcal{F}_x \text{ is compact with } \pi_1 \text{ finite}\}.$$

Note that W is not $\emptyset$ by hypothesis. The Local Stability Theorem implies that $\hat{M}$ is open. We can assume that $\hat{M}$ is connected (otherwise we replace it by a connected component). Let us prove that $\hat{M}$ is closed. For this it suffices to prove that $\partial\hat{M} = \emptyset$. Suppose by contradiction that $\exists x_0 \in \partial\hat{M}$. Let U be a chart of $\mathcal{F}$ containing x_0, Σ the space of plaques of U and $\pi : U \to \Sigma$ be the projection along the plaques. Note that Σ is an interval (as $\mathcal{F}$ has codimension one) and $\hat{M} \cap U$ is union of plaques of $\mathcal{F}$. It follows that $\pi(\hat{M} \cap V) \subset \Sigma$ is a countable family of open intervals. Let J one of these intervals and $\mathcal{F}(J)$ be the union of the leaves of $\mathcal{F}$ intersecting J. Because J is open and contained in $\hat{M}$ we have that $\mathcal{F}(J)$ is open in $\hat{M}$. We claim that $\mathcal{F}(J)$ is closed in $\hat{M}$. Indeed, consider a sequence $x_n \to \mathcal{F}(J) \to x \in \hat{M}$. Assume by contradiction that $x \notin \mathcal{F}(J)$. Since $x \in \hat{M}$ the Local Stability Theorem implies that there is a neighborhood R of $\mathcal{F}_x$ such that every leaf intersecting R is compact with π_1 finite and contained in R. On one hand we can choose R such that $R \cap J = \emptyset$. On the other hand we observe that $\mathcal{F}_{x_n} \cap R \neq \emptyset$ for large n because $x_n \to x$. Hence $\mathcal{F}_{x_n} \subset R$ and also $\mathcal{F}_{x_n} \cap J \neq \emptyset$ by the definition of $\mathcal{F}(J)$. Hence $R \cap J \neq \emptyset$

a contradiction. This contradiction shows that $x \in \mathcal{F}(J)$, i.e. $\mathcal{F}(J)$ is closed. The claim follows. By connecteness we conclude that $\mathcal{F}(J) = \hat{M}$. It follows that every leaf of $\mathcal{F}$ in $\hat{M}$ intersect J. Since every leaf of $\mathcal{F}$ in $\hat{M}$ is compact we conclude that $\pi(\hat{M} \cap U)$ is a finite union of open intervals in Σ. We conclude that x is a boundary point of one of these intervals. Now we claim that the leaf $\mathcal{F}_x$ is closed. Otherwise the above argument would imply that there exist a foliated chart U such that $\hat{M} \cap U$ contains infinitely many connected components, a contradiction. Because M is compact by assumption we conclude that $\mathcal{F}_x$ is compact. Hence there is a tubular neighborhood $P : W \to F$ of F whose fibers $P^{-1}(y)$, $y \in F$ are transverse to $\mathcal{F}$. Because $x \in \partial\hat{M}$ there is a sequence of compact leaves F_n with finite π_1 accumulating on x. By Lemma 4.3.3 we can assume that $F_n \subset W$. The restriction $P/F_n : F_n \to F$ is a finite covering of F. Because F_n has finite fundamental group and $P/F_n : F_n \to F$ is a finite covering we conclude that F has finite π_1. We conclude that $x \in \hat{M}$ contradicting $x \in \partial\hat{M}$. This contradiction proves that $\partial\hat{M} = \emptyset$ proving that $\hat{M}$ is closed. By connectness reasons we conclude that $M = \hat{M}$. Hence all leaves of $\mathcal{F}$ are compact with finite π_1.

Now suppose that $\mathcal{F}$ is transversely orientable. We already know that each leaf L of $\mathcal{F}$ is compact with finite π_1. Being $\mathcal{F}$ transversely orientable we have that the holonomy group of L is represented by an orientation-preserving homomorphism. In other words the subgroup $G = \mathrm{Hol}(L)$ of $Hom(\mathbb{R}, 0)$ is formed by orientation-preserving maps. Then Lemma 4.3.2 implies that $\mathcal{F}$ is a foliation without holonomy. It follows from the proof of the Local Stability Theorem that $\mathcal{F}$ is *locally a product foliation*, i.e. each leaf L of $\mathcal{F}$ is equipped with an invariant product neighborhood $L \times I$ such that the leaves of $\mathcal{F}$ in this neighborhood have the form $L \times *$, $* \in I$, with $L \times 0$ corresponding to L. On the other hand, M is compact by hypothesis. Hence there is a closed curve c transverse to $\mathcal{F}$ (to find c we simply use the non-wandering set of the transverse vector field associated to $\mathcal{F}$). Moreover c can be chosen to intersect some leaf L_0 of $\mathcal{F}$ in a single point. Observe that c intersect all leaves of $\mathcal{F}$ in a single point. In fact, consider the set $c' = \{x \in c : \mathcal{F}_x \cap c = \{x\}\}$. c' is not empty by the existence of L_0. The fact that $\mathcal{F}$ is a locally product foliation implies that l is open and closed in $c \therefore c' = c$ and so c intersect each leaf of $\mathcal{F}$ in one point at most. Now let $\mathcal{F}(c)$ be the set $x \in M$ such that $\mathcal{F}_x \cap c \neq \emptyset$. It is clear that $\mathcal{F}(c)$ is open. One can prove that $\mathcal{F}(c)$ is closed by using the Local Stability Theorem as before. Hence $\mathcal{F}(c) = M$ proving that all leaves of $\mathcal{F}$ intersect c. To define the desired fibration $f : M \to S^1$ we simply define $f(x)$ to be the intersection point of $\mathcal{F}_x$ with c. The theorem is proved. □

Exercise 4.3.2. Prove that there is no codimension one foliation in the 3-ball B having ∂B as a leaf.

Exercise 4.3.3. (Prove or disprove) Codimension one transitive foliations have no compact leaves.

Exercise 4.3.4. Let $\mathcal{F}$ be a transversely orientable codimension one foliation on a closed orientable 3-manifold M. If there is a leaf F of $\mathcal{F}$ whose universal cover is not the real plane $\mathbb{R}^2$ then $M = S^2 \times S^1$ and $\mathcal{F}$ is the product foliation $S^2 \times *$. What about the case M not orientable?

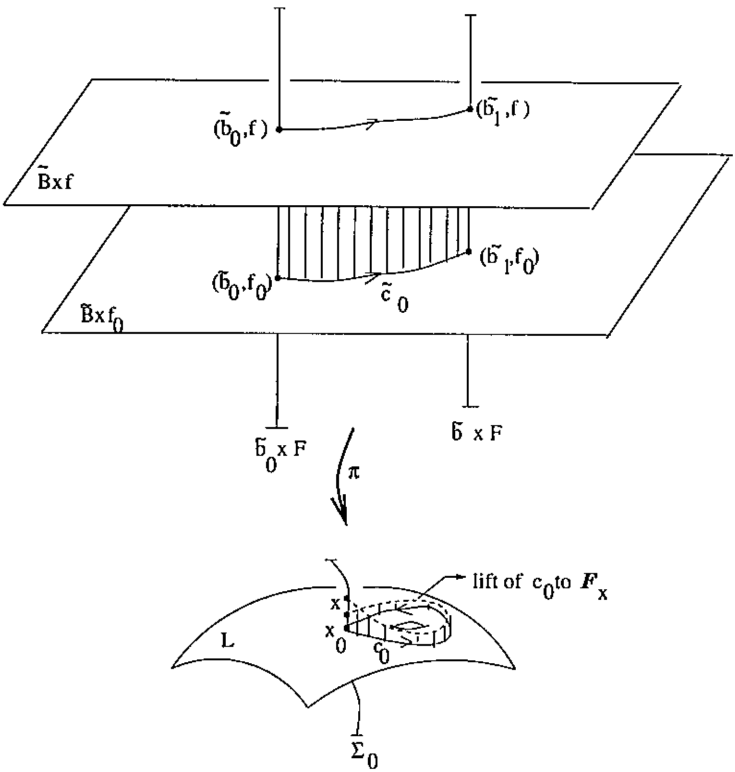

(b̃0,f)
(b̃1,f)
B̃xf
(b̃0,f0)
(b̃1,f0)
c̃ 0
B̃xf0
b̃0 x F
b̃ x F
π
lift of c0 to F x
x
x0
c0
L
Σ0

Figure 4.5:

Chapter 5

Haefliger's Theorem

5.1 Statement

Definition 5.1.1. Let $\mathcal{F}$ be a codimension one foliation on a manifold M. A leaf F of $\mathcal{F}$ has *one-side holonomy* if there are a closed curve $c \subset F$ and $x_0 \in c$ whose holonomy map $f : \mathrm{Dom}(f) \subset \Sigma \to \Sigma$ on a transverse segment Σ intersecting c satisfies the following properties:

1. f is *not* the identity Id in any neighborhood of x_0 in Σ.

2. $f = Id$ in one of the two connected components of $\Sigma \setminus \{x_0\}$.

The graph of f above may be as in Figure 4.3.

Example 5.1.1. A leaf with one-side holonomy cannot be simply connected. The torus fiber of a torus bundle over S^1 is a non-simply connected leaf without one-side holonomy.

Example 5.1.2. The Reeb foliation in S^3 is an example of a codimension one C^∞ foliation on a manifold with finite fundamental group with a one-side holonomy leaf.

Example 5.1.3. Real analytic codimension one foliations cannot have one-side holonomy leaves.

The main result of this section gives a sufficient condition for the existence of one-side holonomy leaves.

Theorem 5.1.2 (Haefliger's Theorem). *Codimension one C^2 foliations with null-homotopic closed transversals have one-side holonomy leaves.*

Corollary 5.1.3. *Codimension one C^2 foliations on compact manifolds with finite fundamental group have one-side holonomy leaves. In particular, there are no real analytic codimension one foliations on manifolds with finite fundamental group.*

In fact, all codimension one foliation on a compact manifold have a closed transverse. If the fundamental group of the manifold is finite then a suitable power of this curve (as element of the fundamental group) yields a null-homotopic closed transverse. Then Haefliger's Theorem applies. The last conclusion of the above corollary applies to the following case:

Corollary 5.1.4. *There is no real analytic codimension one foliations on S^3.*

The proof of Haefliger's Theorem is divided in three parts according to the forthcoming sections.

5.2 Morse theory and foliations

First we explain some classical Morse Theory. Let W be a compact 2-manifold with boundary ∂W (possibly $\emptyset$). Let $f : W \to \mathbb{R}$ a C^r map $r \geq 2$. A point $p \in W$ is a *critical point* of f if

$f'(p) = 0$. We say that p is non-degenerated its second derivative $f''(p)$ is a non-degenerated quadratic form, where

$$f''(p) = \left(\frac{\partial^2 (f \circ x^{-1})(0)}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq 2}$$

for some coordinate system (x_1, x_2) around $p = (0,0)$. We shall use the following lemma due to Morse.

Lemma 5.2.1 (Morse Lemma). *Let p be a non-degenerated critical point of a C^r map $f : W \to \mathbb{R}$, $r \geq 2$. Then there is a coordinate system (x, y) around $p = (0,0)$ such that one of the following alternatives hold:*

1. $f(x,y) = f(0,0) + x^2 + y^2$.
2. $f(x,y) = f(0,0) - x^2 - y^2$.
3. $f(x,y) = f(0,0) + x^2 - y^2$.

The level curves of the three alternatives above are depicted in Figure 5.1.

A *Morse function* is a C^2 map $f : W \to \mathbb{R}$ all of whose critical points are non-degenerated. We denote by $C^r(W, \mathbb{R})$ the set of all C^r functions defined on W endowed with the C^r topology and by $M^r(W, \mathbb{R}) \subset C^r(W, \mathbb{R})$ the set of Morse functions.

Remark 5.2.1. f Morse on W compact $\Rightarrow$ f has finitely many critical points.

The following is a classical result in Morse Theory.

Theorem 5.2.2 (Morse Theorem). *$M^r(W, \mathbb{R})$ is open and dense in $C^r(W, \mathbb{R})$.*

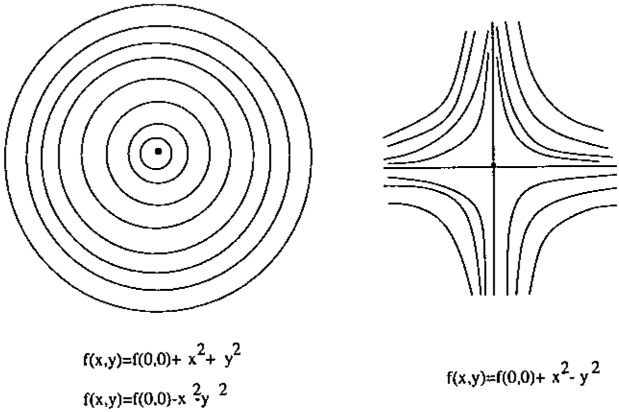


Figure 5.1:

Now we describe the foliated Morse Theory. We let M be a manifold and $\mathcal{F}$ a codimension one foliation of class C^2 in M.

Definition 5.2.3. A C^r map $f : W \to M$ is *Morse with respect to* $\mathcal{F}$, $r \geq 2$, if for all $p \in M$ there is a foliated chart U of $\mathcal{F}$ containing p such that if $\pi : U \to \mathbb{R}$ is the projection along the plaques then $\pi \circ f \in M^r(W, \mathbb{R})$. Morse maps with respect to $\mathcal{F}$ are often said to be *in general position with respect to* $\mathcal{F}$. A *critical point of f with respect to $\mathcal{F}$* is a critical point of $\pi \circ f$ for some foliated chart U.

Theorem 5.2.4. *Let $\mathcal{F}$ be a codimension one foliation of class C^2 on a manifold M. Let W a compact 2-manifold and $A : W \to M$ be a C^r map. Then there is a C^r map $f : W \to M$ arbitrarily C^r close to A such that*

1. *f is Morse with respect to $\mathcal{F}$.*

2. *If p, p' are* different *critical points of f with respect to $\mathcal{F}$ then $f(p)$ and $f(p')$ are in* different *leaves of $\mathcal{F}$.*

Proof. Fix an open covering $Q_1, \cdots, Q_k$ of $A(W)$ by foliated charts of $\mathcal{F}$. We can assume without loos of generality that the chart $\phi^i : Q_i \to D^{n-1} \times D^1$ has the form

$$\phi^i = (\phi^i_1, \cdots, \phi^i_{n-1}, \pi_i),$$

where the last coordinate $\pi_i : Q_i \to \mathbb{R}$ denotes the projection along the plaques.

Define $W_i = A^{-1}(U_i)$ for all i. Hence $W_1, \cdots, W_k$ is an open covering of W. Fix $U_i \subset \overline{U_i} \subset V_i \subset \overline{V}_i \subset W_i$ such that $U_1, \cdots U_k$ is an open covering of W. For each i we fix a C^∞ function $\lambda_i : W \to \mathbb{R}$ satisfying: $\lambda_i \in [0,1]$, $\lambda_i = 1$ in U_i and $\lambda_i = 0$ in $W \setminus V_i$. See Figure 5.2

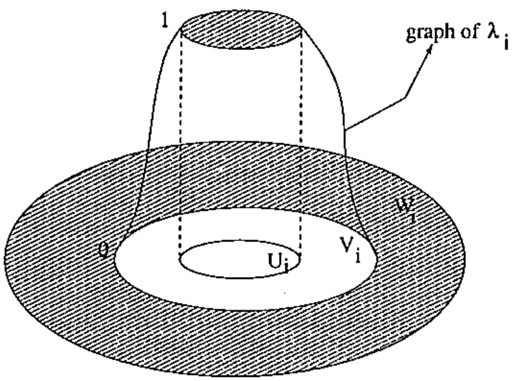


Figure 5.2:

Fix $\epsilon > 0$ and denote by d_r the C^r topology in $C^r(W, M)$. We define inductively a sequence $g_0, g_1, g_2, \cdots, g_k : W \to M$ as follows. First we define $g_0 = A$. For a suitable Morse function $f_1 \in C^r(U_1, \mathbb{R})$ we define $g_1(x)$ as follows:

$$g_1(x) = \left(\phi_1^1(x), \cdots, \phi_{n-1}^1(x), \lambda_1(x) f_1(x) + (1 + \lambda_1(x))(\pi_1 \circ g_0)(x)\right)$$

if $x \in W_1$ and

$$g_1(x) = g_0(x)$$

if $x \in W \setminus W_1$.

It follows that

$$d_r(g_1, g_0) \leq d_r(g_1/W_1, g_0/W_1)(\Sigma_{j=1}^r K_j d_j(\lambda_1, Id/W_1)) \cdot d_r(f_1, \pi_1 \circ g_0),$$

where the constants K_j does not depend on $d_r(f_1, \pi_1 \circ g_0)$. As λ_1 is fixed we have that $d_r(\lambda_1, Id/W_1)$ does not depend on $d_r(f_1, \pi_1 \circ g_0)$. Hence by Morse Theorem we can choose f_1 so that

$$d_r(g_1, g_0) < \epsilon/k.$$

Summarizing g_1 satisfies the following properties:

- $d_r(g_1, g_0) < \epsilon/k$;
- g_1/U_1 is Morse (because $g_1/U_1 = f_1$).

Replacing g_0 by g_1 in the above construction we can find g_2 such that

- $d_r(g_2, g_1) < \epsilon/k$;
- $g2/(U_1 \cup U_2)$ is Morse.

Repeating the argument we can find the sequence $g_0, g_1, \cdots, g_k$. An element g_i of this satisfies:

- $d_r(g_i, g_{i-1}) < \epsilon/k$;

- $g_i/(U_1 \cup \cdots \cup U_i)$ is Morse.

The last map g_k of the sequence is Morse (because $U_1, \cdots, U_k$ is a covering of W). Moreover,

$$d_{C^r}(g_k, A) \leq \Sigma_{i=0}^{k} d_{C^r}(g_i, g_{i+1}) < (\epsilon/k) \cdot k = \epsilon.$$

Hence the last map $g_k : W \to M$ is Morse respect to $\mathcal{F}$ and ϵ-close to A in the C^r topology. It remains to choose f close to g_k satisfying the property (2) of the theorem. To modify g_k to obtain f satisfying (1) and (2).

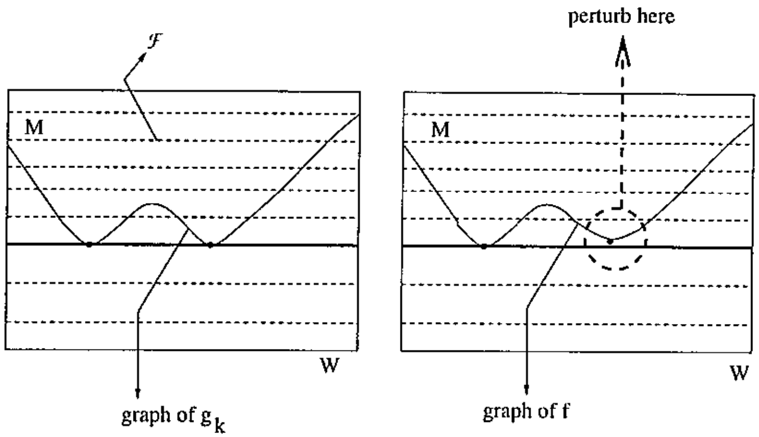


Figure 5.3:

Because the set of Morse function is open we we only have to approximate g_k by a map satisfying the property (2). The last can be attained as in the trivial case described in Figure 5.3 where M is the product foliation $W \times I$ and $\mathcal{F}$ is the trivial foliation $* \times I$. Indeed we only have to perturb around a critical value as indicated in the figure. The theorem is proved. □

5.3 Vector fields on the two-disc

We denote by D^2 the 2-dimensional disc in $\mathbb{R}^2$ and by $\mathfrak{X}^1(D^2)$ the set of C^1 vector fields in D^2 transverse to the boundary ∂D^2 of D^2. The closure of $B \subset D^2$ is denoted by $\overline{B}$. The orbit of $x \in D^2$ is denoted by $O(x)$. Consider $p \in D^2$, $Y \in \mathfrak{X}^1(D^2)$ and denote by $\omega(p)$ the ω-limit set of p. Note that if Σ is an interval transverse to Y and $p \in D^2$ is regular, then $\omega(p)$ intersects Σ at most once. In particular, a periodic orbit of Y intersects Σ once. These facts follow from the trivial topology of the disc D^2. A singularity of Y is called *saddle* or *centre* according to the two portrait face corresponding to Figure 5.1 (left-hand one for centre and right-hand one for saddle). A saddle singularity exhibits two stable separatrixes and two unstable separatrixes. A *graph* of Y is a connected set Γ formed by saddles and saddle's separatrixes in a way that if $s \in \Gamma$ is a saddle then Γ contains at least one stable separatrix and one unstable separatrix of s.

Theorem 5.3.1. *Let $Y \in \mathfrak{X}^1(D^2)$ be such that Y is transverse to ∂D^2 and* $\operatorname{sing}(Y)$ *is a finite set formed by centers and saddles. Suppose that Y has no saddle-connections. Then, there is $x \in D^2$ such that:*

1) *$\overline{O(x)}$ is a closed curve.*

2) *There is an interval δ transverse to Y with the following properties:*

   2.1) $\delta \cap \overline{O(x)} = \delta \cap O(x) = \{x\}$.

   2.2) *The first return map $f : \operatorname{Dom}(f) \subseteq \delta \to \delta$ induced by Y in δ satisfies that: $f = Id$ in a connected component of $\delta \setminus \{x\}$ and $f \neq Id$ in any neighborhood of x in δ.*

Proof. Because Y has no saddle-connection the graphs of Y are as in Figure 5.4. Clearly the complement $D^2 \setminus \Gamma$ of a compact invariant set Γ equals to either a periodic orbit or a graph contains at least one connected components disjoint from ∂D^2. The union of such connected components will be denoted by $R(\Gamma)$. We define an order $<$ on the set formed by periodic orbits and graphs of Y by setting:

$$\Gamma_1 < \Gamma_2 \Leftrightarrow R(\Gamma_1) \subseteq R(\Gamma_2).$$

A *limit cycle* of Y will a compact invariant set L with regular orbits of Y equals to $\omega(p)$ for some $p \notin L$. It is easy to prove that a limit cycle L is either a periodic orbit or a graph. Hence the order $<$ is well defined on the set of limit cycles of Y.

Lemma 5.3.1. *If $\Gamma_1 > \Gamma_2 > \ldots$ is a decreasing sequence of limit cycles of Y, then $\Gamma_\infty = \partial\left(\bigcap_{n=1}^{\infty} R(\Gamma_n)\right)$ is either a periodic orbit or a graph of Y.*

Proof. Because Y is transverse to ∂D^2 we can assume that Y points inward to D^2 in ∂D^2. Clearly Y has finitely many graphs as it has finitely many singularities. Hence we can assume that Γ_n is a periodic orbit, $\forall n$. So, $R(\Gamma_n)$ is a disc and $\partial R(\Gamma_n) = \Gamma_n$, $\forall n$. There is at least one regular point in Γ_∞ because if $s \in \Gamma_\infty \cap sing(Y)$, then s must be saddle and so at least one of the separatrixes of s is accumulated by Γ_n.

First we observe that Γ_∞ cannot contain periodic orbits unless it is a periodic orbit. Indeed this follows from Lemma 4.3.3 but we give a direct proof here. Suppose that Γ_∞ contains a periodic orbit α. Pick $x \in \alpha$ and let Σ_x be a transverse of Y containing x. Clearly for all $n \in \mathbb{N}$ large the set $\Gamma_n \cap \Sigma_x$ consists

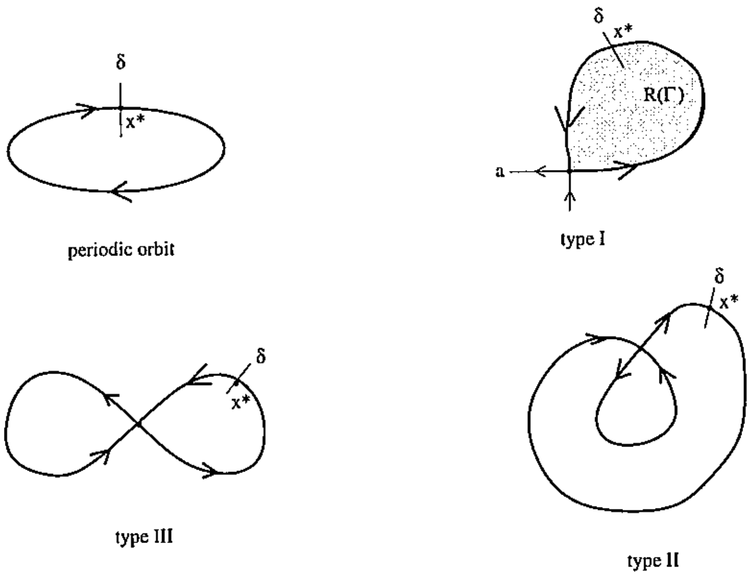


Figure 5.4:

of a single point x_n such that $x_n \to x$ as $n \to \infty$. Let A_n be the annulus in D^2 formed by Γ_n and α. Since $x_n \to x$ the Tubular Flow-Box Theorem implies that $\Gamma_n \to \alpha$ in the Hausdorff topology proving $\Gamma_\infty = \alpha$ as desired.

Second we observe that Γ_∞ cannot contain a graph unless it is a graph. The proof of this is similar to the previous proof. Indeed, let $\Gamma \subset \Gamma_\infty$ be a graph. Hence Γ is one of the three types of graphs in Figure 5.4. If Γ is type II or III then previous argument shows that $\Gamma_\infty = \Gamma$ and we are done. Otherwise Γ is type I. In this case Γ does not contain an unstable separatrix a (say). By the Poincaré Bendixson Theorem we have that $\omega(a)$ is either a singularity or a graph or a periodic orbit. In the later case we have that Γ_∞ is a periodic orbit, a contradiction because it contains the graph Γ. In the former case we have that Γ is one of the graphs in the bottom figures (Y has no saddle connections) again a contradiction. Hence $\omega(p')$ must be a graph which we still denoted by Γ. Γ cannot be of type I for, otherwise, the limit cycle sequence Γ_n must be contained in $R(\Gamma)$ a contradiction since Γ_n is decreasing. This proves that Γ is type II or III and we are done.

Now, fix $p \in \Gamma_\infty$ regular. Clearly Γ_∞ is invariant and so $\omega(p) \subset \Gamma_\infty$. Poincaré-Bendixson implies that $\omega(p)$ is either a singularity or a periodic orbit or a graph. In the last two cases we have that Γ_∞ contains either a periodic orbit or a graph. Hence Γ_∞ is either a periodic orbit or a graph and we are done. We conclude that $\omega(p)$ is a singularity. Analogously $\alpha(p)$ can be assumed to be a singularity Y because points inward to D^2 in ∂D^2. Because Y has no saddle connections we have that the closure $\overline{O(p)}$ of the orbit $O(P)$ is a graph of Y. This graph is evidently contained in Γ_∞. We conclude that Γ_∞ contains a graph and we are done. This proves the lemma. □

Let us finish with the proof of Theorem 5.3.1. Consider the set $\mathcal{R}$ of all compact invariant sets Γ_∞ of Y of the form

$$\Gamma_\infty = \partial \left(\bigcap_{n=0}^{\infty} R(\Gamma_n) \right),$$

for some decreasing sequence of limit cycles Γ_n of Y. Lemma 5.3.1 implies that the order $<$ is defined in $\mathcal{R}$. Lemma 5.3.1 also implies that any decreasing sequence $\Gamma_\infty^1 > \Gamma_\infty^2 > \cdots$ in $\mathcal{R}$ has an infimum in $\mathcal{R}$. The Zorn Lemma implies that there is a minimal element $\Gamma^* = \Gamma_\infty^*$ in $\mathcal{R}$. By Lemma 5.3.1 we have that Γ^* is either a periodic orbit or a graph. In any case we choose $x^* \in \Gamma^*$ as indicated in Figure 5.4. Observe that the closure $\overline{O(x^*)}$ of the orbit $O(x^*)$ is a closed curve. Choose a transverse δ containing

x^* as indicated in the figure. Let $f : \mathrm{Dom}(f) \subset \delta \to \delta$ be the return map induced by Y in δ. Because Γ^* is accumulated by limit cycles of Y one has that $f \neq Id$ in any neighborhood of x^* in δ (for such limit cycles must intersect δ). On the other hand, consider the connected component c of $\delta - \{x^*\}$ contained in Γ^*. Because the number of graphs of Y is finite, we can assume by shrinking δ if necessary that δ does not intersect any graph of Y. In particular c does not intersect any graph of Y. Because $c \subset R(\Gamma^*)$ we conclude that the orbit of any point in c is periodic. Hence $f = Id$ in c. This proves that $x = x^*$ satisfies the properties (1),(2) of the theorem. □

5.4 Proof of Haefliger's Theorem

Let $\mathcal{F}$ be a codimension one C^2 foliation with a null-homotopic closed transverse γ. Because γ is null-homotopic one has that there is a C^∞ map $A : D^2 \to M$ such that $A(\partial(D^2)) = \gamma$. By Theorem 5.2.4 we can assume that $A = f$ is in general position with respect to $\mathcal{F}$. Let $\mathcal{F}^* = f^*(\mathcal{F})$ be the foliation in D^2 induced by f. By definition a leaf L of $\mathcal{F}^*$ is f^{-1}(connected component of $F \cap f(D^2)$) for some leaf F of $\mathcal{F}$. Note that $\mathcal{F}^*$ is a singular foliation of class C^2 in D^2 and a singularity of $\mathcal{F}^*$ is either a center or a saddle. Clearly $\mathcal{F}^*$ is C^2 orientable close to the singularities. Far from the singularities we have that $\mathcal{F}^*$ is C^2 locally orientable by the Tubular Flow-Box Theorem. We conclude that $\mathcal{F}^*$ is C^2 locally orientable. By the results in Section 2.3 we have that $\mathcal{F}^*$ is C^2 orientable, i.e. there is a C^2 vector field Y in D^2 tangent to $\mathcal{F}^*$. Note that Y is transverse to ∂D^2 and contains a finite number of singularities all of them being centers or saddles. Moreover, Y has no saddle connections by Theorem 5.2.4-(2). It follows from Theorem 5.3.1 that there is $x \in D^2$ and a transverse δ satisfying the conclusions (1)-(2) in that theorem. In particular, if $x_0 = f(x)$ then $c = f(\overline{O(x)})$ is a closed curve contained in the leaf $F = \mathcal{F}_{x_0}$ of $\mathcal{F}$ and $\Sigma = f(\delta)$ is a transverse segment of $\mathcal{F}$ intersecting c. The holonomy of c is conjugated to the return map $f : \mathrm{Dom}(f) \subset \delta \to \delta$ induced by Y in δ. One can see that c and F satisfy the properties (1)-(2) of Definition 5.1.1 by using the property (2.2) in Theorem 5.3.1. We conclude that F is a leaf with one-side holonomy of $\mathcal{F}$ proving the theorem. □

Chapter 6

Novikov Compact Leaf Theorem

6.1 Statement

In this chapter we shall prove the celebrated Novikov Compact Leaf Theorem.

Theorem 6.1.1 (Novikov Compact Leaf Theorem). *Codimension one C^2 foliations on compact 3-manifolds with finite fundamental group have compact leaves.*

The proof of this theorem given here is the one in "Séminaire Bourbaki 20e année, 1967-68, Num. 339, p. 433-444". That proof is based on the following definition.

Definition 6.1.2. Let $\mathcal{F}$ be a C^1 codimension one foliation in a manifold M. A *vanishing cycle* of $\mathcal{F}$ is a C^1 map $f : S^1 \times [0, \epsilon] \to M$ (for some $\epsilon > 0$) such that if we denote $f_t(x) = f^x(t) = f(x, t)$, $\forall (x, t) \in S^1 \times [0, \epsilon]$, then the following properties hold:

1. $f_t(S^1)$ is a closed curve contained in a leaf $A(t)$ of $\mathcal{F}$, $\forall t$;

2. $f_t(S^1)$ is null homotopic in $A(t)$ if and only if $t > 0$;

3. $f^x([0, \epsilon])$ is transverse to $\mathcal{F}$, $\forall x$.

Example 6.1.1. Let $\mathcal{F}$ be the Reeb foliation in S^3 and let T be the compact leaf of $\mathcal{F}$. Any generator of $\pi(T)$ is represented by a curve contained in (the image of) a vanishing cycle of $\mathcal{F}$.

Example 6.1.2. A torus fibration over S^1 gives an example of a foliation with compact leaves having no vanishing cycles.

The proof of Novikov Theorem is a direct consequence of the following two preliminary results.

Theorem 6.1.3 (Auxiliary Theorem I). *Codimension one C^2 foliations on compact 3-manifolds with finite fundamental group have vanishing cycles.*

Theorem 6.1.4 (Auxiliary Theorem II). *Codimension one C^1 transversely orientable foliations with vanishing cycles on compact 3-manifolds have compact leaves.*

Proof of Novikov Compact Leaf Theorem using the auxiliary theorems I, II: Let $\mathcal{F}$ be a codimension one C^2 foliation on a compact 3-manifold M. Let $P : \hat{M} \to M$ be a finite covering of M such that the lift $\hat{\mathcal{F}}$ of $\mathcal{F}$ is transversely orientable. $\pi_1(\hat{M})$ is finite since $\pi_1(\hat{M}) < \pi_1(M)$ and $\pi_1(M)$ is finite. By Auxiliary Theorem I we have that $\hat{\mathcal{F}}$ has a vanishing cycle. Hence $\hat{\mathcal{F}}$ has a compact leaf $\hat{F}$ by Auxiliary Theorem II. Then $F = P(\hat{F})$ is a compact leaf of $\mathcal{F}$ proving the result. □

6.2 Proof of Auxiliary Theorem I

Let $\mathcal{F}$ be a codimension one C^2 foliation on a compact 3-manifold M. *It is easy to prove that $\mathcal{F}$ has a closed transverse γ.* In fact, by passing to a finite covering we can assume that $\mathcal{F}$ is transversely orientable, and so, it has a transverse vector field X. Because M is compact we have that X has a non-wandering point x. Hence there is a piece of orbit of X which starts and finishes close to x. By modifying a bit such a piece of orbit nearby x we can construct a closed transverse of $\mathcal{F}$ containing x. This proves the result.

Next we assume that $\pi_1(M)$ is finite and let γ be a closed transverse of $\mathcal{F}$. Because $\pi_1(M)$ is finite we have that there is $n \in \mathbb{N}$ such that the curve γ^n represent a closed null homotopic transverse of $\mathcal{F}$. Without loss of generality we can assume that $n = 1$. Now we proceed as in the proof of Haefliger's Theorem: Because γ is null-homotopic one has that there is a C^∞ map $A : D^2 \to M$ such that $A(\partial(D^2)) = \gamma$. By Theorem 5.2.4 we can assume that $A = f$ is in general position with respect to $\mathcal{F}$. Let $\mathcal{F}^* = f^*(\mathcal{F})$ be the foliation in D^2 induced by f. Note that $\mathcal{F}^*$ is a singular foliation of class C^2 in D^2 and a singularity of $\mathcal{F}^*$ is either a centre or a saddle. Clearly $\mathcal{F}^*$ is C^2 orientable close to the singularities. Far from the singularities we have that $\mathcal{F}^*$ is C^2 locally orientable by the Tubular Flow-Box Theorem. Hence $\mathcal{F}^*$ is C^2 locally orientable. By the last example of Section 2.2 we have that $\mathcal{F}^*$ is C^2 orientable, i.e. there is a C^2 vector field Y in D^2 tangent to $\mathcal{F}^*$. Note that Y is transverse to ∂D^2 and contains a finite number of singularities all of them being centers or saddles. Moreover, Y has no saddle connections by Theorem 5.2.4-(2). It follows from Theorem 5.3.1 that there is $x \in D^2$ and a transverse δ satisfying the conclusions (1)-(2) in that theorem. Let c_0 be the closed curve $c_0 = f(\overline{O(x)})$. Then $c_0 \subset \mathcal{F}_{x_0}$ where $x_0 = f(x)$. Note that c_0 is *not* null-homotopic in $\mathcal{F}_{x_0}$ because its holonomy map is not the identity in any neighborhood of x_0. Moreover, the closed curve $\beta = \overline{O(x)}$ is either a periodic orbit or the closure of a homoclinic loop of Y. These properties motivate us to define *cycle* as a closed curve β in D^2 which is either a periodic orbit or the closure of a homoclinic loop of Y such that $f(\beta)$ is *not* null homotopic in the leaf of $\mathcal{F}$ containing it. As before every cycle β of Y bounds a region $R(\beta)$ which does not intersect ∂D^2. We define the order $<$ in the set of cycles of Y by setting $\beta_1 < \beta_2$ if and only if $R(\beta_1) \subset R(\beta_2)$.

Lemma 6.2.1. *Let $\beta_1 > \beta_2 > \cdots$ be a decreasing sequence of cycles of Y. Then there is a cycle β_∞ of Y such that $\beta_n > \beta_\infty$ for all $n \in \mathbb{N}$.*

Proof. Because the number of homoclinic loops of Y is finite we can assume that β_n is a periodic orbit and that $R(\beta_n)$ is a disc with boundary β_n for all n. The sequence $R(\beta_n)$ is a decreasing sequence of compact sets in D^2. Hence

$$\bigcap_{n=1}^{\infty} R(\beta_n)$$

is a non-empty compact set whose boundary will be denote by β. It is clear that $\beta \neq \emptyset$. Moreover, there is $p \in \beta$ regular because the singularities of Y are centers or saddles (no periodic orbit close to a center of Y can be a cycle of Y). By Poincaré-Bendixson we have that $\omega(p)$ is either a singularity or a periodic orbit or a graph. In the later two cases we have that β contains either a periodic orbit or a graph. Hence, as in the proof of Lemma 5.3.1 in Section 5.3, we have that β itself is either a periodic orbit or a graph of type II or III (see Figure 5.4). If β were a periodic orbit with $f(\beta)$ null homotopic in its leaf then $f(\beta_n)$ would be null homotopic in its leaf for all n large, a contradiction. Hence if β is periodic then $\beta_\infty = \beta$ is the desired cycle. Now suppose that β is a graph of type II or III. The fact that β is surrounded by cycles of Y implies that $f(\beta)$ is not null homotopic in its leaf. Hence one of the two homoclinic loops forming β (say β') satisfies that $f(\beta')$ is not null homotopic in its leaf. Then, $\beta_\infty = \beta'$ is the desired cycle. To finish we assume that $\omega(p)$ is a singularity. In a similar way we can assume that $\alpha(p)$ is a singularity. Hence $\overline{O(p)}$ is a graph contained in β. As previously remarked this implies the existence of the desired cycle β_∞ and the proof follows. □

Let us finish with the proof of Auxiliary Theorem I. By the previous lemma and the Zorn lemma we have that there is a cycle β_∞ of Y which is minimal for the order $<$. By the definition of cycle we have that $f(\beta_\infty)$ is not null homotopic in its leaf. Choose a regular point $x \in \beta_\infty$ and let δ be a transverse of $\mathcal{F}^*$ inside $R(\beta_\infty)$ containing x in its boundary. To simplify the notation we shall assume that $\delta = [0,1]$ with $x \approx 0$. By Poincaré-Bendixson, because the number of graphs of Y is finite, we have that all orbit of Y starting at $y \in \delta \setminus \{x\}$ is periodic with period t_y. Hence the set

$$\mathcal{A} = \overline{O(x)} \cup \{Y_{[0,t_y]} : y \in \delta \setminus \{x\}\}$$

is diffeomorphic to the annulus $S^1 \times [0,1]$. Put a parametrization $P : S^1 \times [0,1] \to \mathcal{A}$ such that: $P_0(S^1) = \overline{O(x)}$; $P_y(S^1) = Y_{[0,t_y]}$ for all $y \in \delta \setminus \{x\}$; and $P^\theta([0,1])$ is transverse to $\mathcal{F}^*$ for all $\theta \in S^1$. Consider the map $g = f \circ P$. Hence $g : S^1 \times [0,1] \to M$ is a limit cycle of $\mathcal{F}$. In fact, g is C^1 since both P and f are. In addition, $g_0(S^1) = f(\overline{O(x)}) \subset \mathcal{F}_{f(x)}$ is not null homotopic in $A(0) = \mathcal{F}_{f(x)}$ because β_∞ is a cycle of Y. Also for $y \neq 0$ we have that $g_y(S^1) = f(a(y))$ is null homotopic in its leaf since β_∞ is minimal with respect to the order $<$. Because $P^\theta([0,1])$ is transverse to $\mathcal{F}^*$ for all $\theta \in S^1$ one has that $g^\theta(S^1) = f(P^\theta([0,1]))$ is transverse to $\mathcal{F}$ for all $\theta \in S^1$. This proves the theorem. □

6.3 Proof of Auxiliary Theorem II

The proof uses the following holonomy lemma. To state it we use some short definitions. Given a codimension one foliation $\mathcal{F}$ we say that a vector field X is *normal for* $\mathcal{F}$ if the trajectories of X are everywhere transverse to $\mathcal{F}$. Clearly a codimension one foliation has a normal vector field if and only if it is transversely orientable. If $\mathcal{F}$ is a transversely orientable foliation in a manifold M we say that a curve $c \subset M$ is *normal to* $\mathcal{F}$ if c is transverse to $\mathcal{F}$ and contained in a solution curve of the normal vector field associated to $\mathcal{F}$. Given a leaf A of $\mathcal{F}$, a compact set K and a C^1 map $g : K \to A$ we say that g *has a normal extension* if there are $\epsilon > 0$ and a C^1 map $G : K \times [0, \epsilon]) \to M$ such that:

1. $G_0/K = g$;
2. $G_t(K) \subset A(t)$ for some leaf $A(t)$ of $\mathcal{F}$ with $A(0) = A$;
3. $\forall x \in K$ the curve $G^x([0, \epsilon])$ is normal to $\mathcal{F}$.

Lemma 6.3.1 (Holonomy Lemma). *Let $\mathcal{F}$ a codimension one transversely orientable foliation, A be a leaf of $\mathcal{F}$ and K be compact set. If $g : K \to A$ is a C^1 map homotopic to constant in A, then g has a normal extension.*

Proof. let $g : K \to A$ of as in the statement and denote by X the normal vector field of $\mathcal{F}$. For all $x \in K$ define the normal curve

$$\Sigma_x = \{X_s(g(x)) : s \in [0, 1]\},$$

where as usual X_t denotes the flow of X and $X_B(A) = \{X_s(z) : (s, z) \in B \times A\}$. By hypothesis $g(K)$ is a compact null homotopic subset of A, and so, it is contained is disc $D \subset A$. Fix $x_0 \in K$ and define

$$G(x, t) = f(X_t(g(x)),$$

where $f = f_x$: $\text{Dom}(f) \subset \Sigma_{x_0} \to \Sigma_x$ is the holonomy induced by a curve $\gamma_x \subset D$ joining $g(x_0)$ to $g(x)$. Since D is contractible we have $G(x,t)$ does not depend on the chosen curve γ_x. Moreover, since $g(K)$ is compact, we can assume that there is $\epsilon > 0$ such that $G(x,t)$ is defined for every $(x,t) \in K \times [0,\epsilon]$. Let us prove that the map $G : K \times [0,\epsilon] \to M$ so obtained is a normal extension of g. It is clear that G is C^1 since g is. First we prove that $G_0/K = g$. In fact, if $x \in K$ then $G_0(x) = G(x,0) = f(X_0(g(x))) = f(g(x)) = g(x)$ by the definition of holonomy and $X_0 = Id$. Second we prove that $G_t(K) \subset A(t)$ for some leaf $A(t)$ of $\mathcal{F}$ with $A(0) = A$. In fact, it is clear that $A(0) = A$. Next we observe that $G_t(x) = G(x,t) = f(X_t(g(x))) \in \mathcal{F}_{X_t(g(x))}$. Since $\mathcal{F}_{X_t(g(x))} = \mathcal{F}_{X_t(g(x_0))}$ by definition of holonomy we have that $A(t) := \mathcal{F}_{X_t(g(x_0)}$ works. Third we prove that $G^x([0,\epsilon])$ is normal to $\mathcal{F}$. In fact, $G^x(t) = G(x,t) = f(X_t(g(x))) \in \Sigma_x$ which is a solution curve of X. The lemma follows. □

Hereafter we let $\mathcal{F}$ be a codimension one C^1 transversely orientable foliation with a vanishing cycle $f : S^1 \times [0,\epsilon] \to M$ on a compact 3-manifold M. We denote by X the vector field in M transverse to $\mathcal{F}$ and by X_t the flow generated by X. This vector field exists because $\mathcal{F}$ is transversely orientable. For simplicity we shall assume that $\epsilon = 1$. A C^1 curve α in a leaf of $\mathcal{F}$ is in *general position* whenever $\#\alpha^{-1}(p) \leq 2$ for all $p \in M$ and if $x, y \in \text{Dom}(\alpha)$ are different points with $\alpha(x) = \alpha(y)$, then $\alpha'(x)$ and $\alpha'(y)$ are not parallel (see Figure 6.1).

Lemma 6.3.2. *We can suppose that the vanishing cycle $f : S^1 \times [0,1] \to M$ of $\mathcal{F}$ satisfies the following additional properties:*

4. *$f_0(S^1)$ is in general position in $A(0)$.*

5. *$f^x([0,1])$ is normal to $\mathcal{F}$.*

6. *If $x, y \in S^1$ and $f^x(0) \neq f^y(0)$, then $f^x([0,1]) \cap f^y([0,1]) = \emptyset$.*

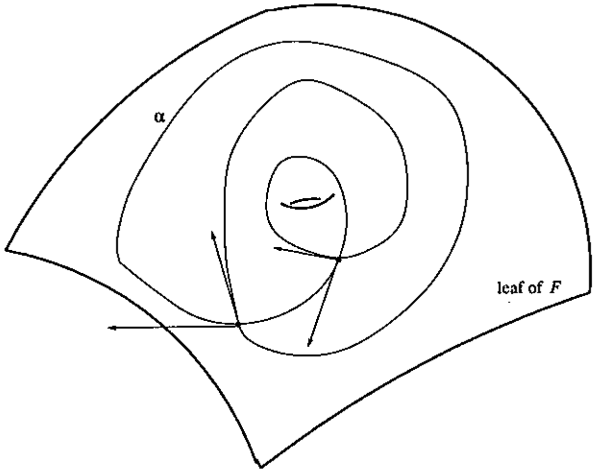


Figure 6.1: General position curve for $\mathcal{F}$

Proof. Clearly (5) implies (6) by ODE reasons. By moving a bit $f_x(S^1)$ we can assume (4). To assume (5) it suffices to project $f^x([0,1])$ to the solution curve of X passing through $f^x(0)$ via holonomy. This is done as follows (see Figure 6.2): For $x \in S^1$ we define $\Sigma = \{X_s(f^x(0)); s \in [0,1]\}$ which is the solution curve of X passing through $f^x(0)$. We define $\Sigma' = f^x([0,1])$. By the definition of vanishing cycle the curve one has $\Sigma' \pitchfork \mathcal{F}$. As $X \pitchfork \mathcal{F}$, we have $\Sigma \pitchfork \mathcal{F}$. Note that $x \in \Sigma \cap \Sigma'$. Hence there is a holonomy map $g : \text{Dom}(g) \subset \Sigma' \to \Sigma$. Define $f^*(x,t) = g(f^x(t))$. By compactness we can assume that $f^*(x,t)$ is defined in $S^1 \times [0,\epsilon]$ for some $\epsilon > 0$. We shall assume that $\epsilon = 1$ for simplicity. Let us prove that f^* is a vanishing cycle of $\mathcal{F}$. First note that $(f^*)_0(x) = g(f^x(0)) = g(x) = x$ (by the definition of g) and so $(f^*)_0(S^1) = f_0(S^1)$. The last implies that $f_0(S^1) \subset A(0)$ is not null-homotopic in $A(0)$. Moreover, $(f^*)_t(x) = f^*(x,t) =$

$g(f^x(t)) \in \mathcal{F}_{f^x(t)} = A(t)$ by the definition of holonomy. Hence $f_t^*(S^1) \subseteq A(t)$, $\forall t$. Now we prove that $F_t^*(S^1)$ is null homotopic in $A(t)$ for all $t > 0$ small. In fact note that $f_t^*(x) = g(f^x(t)) \underset{t\to 0^+}{\longrightarrow} g(f^x(0)) = f_0(x)$ and $f_t(x) \underset{t\to 0^+}{\longrightarrow} f_0(x)$. Hence $d(f_t^*(x), f_t(x) \leq d(f_t^*(x), f_0(x)) + d(f_0(x), f_t(x)) \underset{t\to 0^+}{\longrightarrow} 0$. Thus, $f_t^*(S^1)$ is C^0-close to $f_t(S^1)$ in $A(t)$. Then, $f_t^*(S^1) \simeq 0$ em $A(t)$ as desired. The proof follows. □

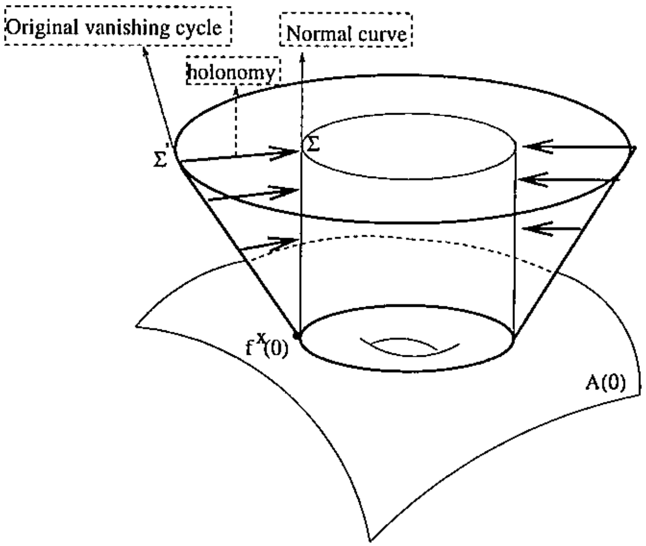


Figure 6.2:

Lemma 6.3.3. *We can assume that the vanishing cycle $f : S^1 \times [0,1] \to M$ of $\mathcal{F}$ satisfies the following additional property:*

7. *The lift $\hat{f}_t : S^1 \to \hat{A}(t)$ of $f_t : S^1 \to A(t)$ to the universal cover $\pi(t) : \hat{A}(t) \to A(t)$ of $A(t)$ is a* simple closed curve, $\forall t > 0$.

Proof. Define $R = \{$ couples (x, y) of different points of S^1 such that $\hat{f}_t(x) = \hat{f}_t(y)$ for some $t > 0\}$, and $r_t = \#R$. We have that $r < \infty$. In fact, for all t we define $B_t = \{p \in f_t(S^1) : \#f^{-1}(p) = 2\}$, $b_t = \#B_t$. $b_0 < \infty$ because f_0 is in general position. Note that $f_t \to f_0$ in the C^1 topology as $t \to 0^+$ and, because f_0 is in general position we have $b_t = b_0$ for all $t \approx 0$. Hence the map $t \to b_t$ is the constant map $t \to b_0$. As $r \leq b_0$ the result follows. On the other hand, $R \neq \emptyset$ for otherwise we are done. Pick $(x, y) \in R$ and define $U = \{t > 0 : \hat{f}_t(x) = \hat{f}_t(y)\}$, $K = \{t \leq 0 : f_t(x) = f_t(y)\}$. The following properties hold:

- K is compact (because f^x and f^y are continuous).
- $U \neq \emptyset$ (because $(x, y) \in R$).
- $U \subset K$ (because $\hat{f}_t(x) = \hat{f}_t(y) \Rightarrow f(t(x) = f_t(y))$.
- U is open: Fix t such that $\hat{f}_t(x) = \hat{f}_t(y)$. Then the curve $(\pi(t) \circ \hat{f}_t)/[x, y]$ is closed, where $[x, y]$ is a suitable arc in S^1 joining x, y. This curve is null homotopic in $A(t)$ because its lift $\hat{f}_t/[x, y]$ in $\hat{A}(t)$ is a closed curve. By the Holonomy Lemma we conclude that $(\pi(s) \circ \hat{f}_s)/[x, y]$ is null homotopic in $A(s)$ for $s \approx t$ proving the result.

Because $U \neq \emptyset$ is open we can fix an interval $(t', t'']$ in U with $t'' \notin U$ and t'' being arbitrarily close to 0. $t' \in K$ because K is closed and $U \subset K$. Hence $f_{t'}(x) = f_{t'}(y)$. We claim that one of the two arcs $[x, y]$ joining x to y in S^1 satisfies that the closed curve $f_{t'}([x, y])$ is *not* null homotopic in $A(t')$. Indeed we have two cases, namely either $t' > 0$ or $t' = 0$. If $t' > 0$ as $t' \notin U$ we have that $\hat{f}_{t'}([x, y])$ is not a closed curve for each arc $[x, y]$. Because the closed curve $f_{t'}([x, y])$ lifts to the non-closed curve $\hat{f}_{t'}([x, y])$ in $\hat{A}(t')$ we conclude that $f_{t'}([x, y])$ is not null homotopic in $A(t)$ for all $[x, y]$ (recall that $\hat{A}(t')$ is the universal cover of $A(t')$). If $t' = 0$ then we let $[x, y], [x, y]'$ be the two possible arcs in S^1 joining x to y. If both $f_0([x, y]), f_0([x, y]')$ were null homotopic in $A(0)$ then $f_0(S^1)$ would be null homotopic in $A(0)$ as it is the product of $f_0([x, y]), f_0([x, y]')$. The last contradicts the definition of vanishing cycle. This proves the claim. Hence we can assume that $f_{t'}([x, y])$ is not null-homotopic in $A(t')$. We not that the closed curve $\hat{f}_{t'}([x, y])$ has less than r multiple points. Moreover the restriction $f/(S^1 \times [t', t''])$: $S^1 \times [t', t''] \to M$ is a vanishing cycle of $\mathcal{F}$ *with* $A(t')$ *close to* $A(0)$. Replacing f by $f/(S^1 \times [t', t''])$ we have less than r multiple points for the new vanishing cycle. Repeating the process we obtain the result. □

Lemma 6.3.4. *Let* $f : S^1 \times [0, 1] \to M$ *be a vanishing cycle of* $\mathcal{F}$ *satisfying the properties* (4)-(6) *of* Lemma 6.3.2 *and* (7) *of* Lemma 6.3.3. *Then, there is an immersion* $F : D^2 \times (0, 1] \to M$ *satisfying the following properties:*

1. $F_t/\partial D^2 = f_t$, $\forall t$.
2. $F(D^2 \times t) \subset A(t)$, $\forall t$.
3. $F^x((0, 1])$ *is normal to* $\mathcal{F}$, $\forall x$.
4. *If* $U = \{x \in D^2 : \lim_{t \to 0^+} F^x(t)$ *exists*$\}$, *then* $\partial D^2 \subset U$; U *is open; and* $D^2 \setminus U \neq \emptyset$.

Proof. Let $\pi(t) : \hat{A}(t) \to A(t)$ be the universal cover of $A(t)$. We have that $\hat{A}(t) = \mathbb{R}^2$ or S^2. The last cannot happen for otherwise the Reeb Global Stability Theorem would imply $M = S^2 \times S^1$ and $\mathcal{F}$ is the trivial foliation $S^2 \times *$, a contradiction since $f_0(S^1)$ is not null homotopic in $A(0)$ (see the last exercise in Section 4.3). On the other hand, (7) of Lemma 6.3.3 says that $\hat{f}_1(S^1)$ is a simple closed curve in $\hat{A}(1)$. By the classical Jordan Theorem we have that there is an embedding $\hat{F} : D^2 \times 1 \to \hat{A}(1)$ with $\hat{F}_1/\partial D^2 = \hat{f}_1$. We define $F : D^2 \times 1 \to A(1)$ by $F = \pi(1) \circ \hat{F}$. Clearly F is an immersion as $\pi(1)$ is a covering and $\hat{F}$ is an embedding. Applying the Holonomy Lemma to F we can extend F to $D^2 \times (t_0, 1]$ for some $t_0 > 0$ satisfying (1)-(3).

We claim that F can be extended to $D^2 \times [t_0, 1]$ still satisfying (1)-(3). In fact, first we show that $\lim_{t \to t_0^+} F(x,t)$ exists for all $x \in D^2$. Because $t_0 > 0$ we have that $\hat{f}_{t_0} : S^1 \to \hat{A}_{t_0}$ is null homotopic. As before there is an embedding $\hat{H}_{t_0} : D^2 \to \hat{A}(t_0)$ with $\hat{H}_{t_0}/\partial D^2 = \hat{f}_{t_0}$. Define $H_{t_0} = \pi(t_0) \circ \hat{H}_{t_0}$. Again by the Holonomy Lemma there are $\delta > 0$ and an immersion $G \colon D^2 \times (t_0 - \delta, t_0 + \delta) \to M$ such that:

a) $G_t(D^2) \subseteq A(t)$;

b) $G_t/\partial D^2 = f_t$;

c) $G^x((t_0 - \delta, t_0 + \delta))$ is normal to $\mathcal{F}$;

d) $G_{t_0} = H_{t_0}$.

Now we fix $t \in (t_0, t_0 - \delta)$ and consider $D := F_t(D^2)$ and $D^1 := G_t(D^2)$. Both D and D^1 are discs contained in $A(t)$ with $\partial D = \partial D^1 = f_t(S^1)$. If $D \neq D^1$ then $A(t)$ would be S^2 a contradiction as before by Reed Stability. Hence $D = D^1$ and so $F_t(D^2) = G_t(D^2)$ for all $t \in (t_0, t_0 - \delta)$. If $x \in D^2$ and $t \in (t_0, t_0 - \delta)$, then $F_t(x) \in F_t(D^2) = G_t(D^2) \Rightarrow F_t(x) = G_t(y(x,t))$ for some $y(x,t) \in D^2$. But $G^{y(x,t)}(t_0 - \delta, t_0 + \delta)$ and $F^x((t_0, 1])$ are both normal to $\mathcal{F}$. Hence $y(x,t) = y(x)$ does not depend on t. Thus, $\lim_{t \to t_0^+} F(x,t) = \lim_{t \to t_0^+} G(y(x),t) = G(y(x),t_0) \subseteq A(t_0)$ proving that $\lim_{t \to t_0^+} F(x,t)$ exists $\forall x \in D^2$. To finish with the claim we simply define $H : D^2 \times [t_0, 1] \to M$ by

$$H(x,t) = \begin{cases} \lim_{t \to t_0^+} F(x,t), & \text{if } t = 0, \\ F(x,t), & \text{if } t \neq t_0. \end{cases}$$

Thus $H/D^2 \times (t_0, 1] = F$, $H_t|\partial D^2 = f_t$, $H_t(D^2) \subset A(t)$, $\forall t$ and $H^x([t_0, 1])$ is normal to $\mathcal{F}$. In other words H is an extension of F to $D^2 \times [t_0, 1]$ satisfying (1)-(3). This proves the claim. If $t_0 > 0$ the the Holonomy Lemma allow us to extend F to $D^2 \times (t_0 - \delta, 1]$ satisfying (1)-(3). Hence we can assume that there is $F : D^2 \times (0, 1] \to M$ satisfying (1)-(3).

Let us prove that F satisfies the property (4) of the lemma. If $x_0 \in U \Rightarrow \exists\, y_0 = \lim\limits_{t \to 0^+} F(x_0, t)$. Let V be a tubular flow-box for X around $y_0 \subseteq$ solution curve of X. Note that $y_0 \in O_X(x_0)$, the orbit of x_0, as X is non-singular ($X \pitchfork \mathcal{F}$). Hence, $X_{t_0}(x_0) = y_0$, for some $t_0 > 0$. By the Tubular Flow-Box Theorem there is a neighborhood B of x_0, such that $X_{t_0}(B) \subseteq V$. As F_1 is continuous there is a neighborhood W of x_1 in D^2 with $x_0 = F_1(x_1)$ such that $F_1(W) \subseteq B$. See Figure 6.3.

Let us prove that $\lim\limits_{t \to 0^+} F(x, t)$ exists $\forall x \in W$. In fact, Consider $x \in W$ and $x' = F_1(x)$. Note that the curve $F^{x'}((0, 1])$ has finite length for, otherwise, it would exist a first exit point z of $F^{x'}((0, 1])$ from V. Clearly $z = F^{x'}(t_z)$ for some $t_z \in [0, 1]$. But $F^x(t_z) = F_{t_z}(x) \in A(t_z)$, $F^{x'}(t_z) = F_{t_z}(x') \in A(t_z)$. As $z \in \partial V$ and $F^x(t_z) \notin \partial V$ (such a point is close to $y \in \mathrm{Int}(V)$) we conclude by the Mean Value Theorem that $A(t_z)$ and $\mathcal{F}_y$ have an intersection point. This intersection point implies $A(t_z) = \mathcal{F}_y$. Because $\dim \mathcal{F} = 2$ we can assume from the beginning that $\mathcal{F}_y \neq A(t_z)$ a contradiction. Hence $F^{x'}((0, 1])$ has finite length $\Rightarrow \exists \lim\limits_{t \to 0^+} F(x', t)$, $\forall x' \in W$. This proves that U is open.

Warning: The last argument proves that if $x \in U$ and $y = \lim_{t \to 0^+} F^x(t)$, then there is a neighborhood W of x in D^2 such that $\lim_{t \to 0^+} F^{x'}(t) = y'$ exists for all $x' \in W$ and $\mathcal{F}_y = \mathcal{F}_{y'}$.

To see $\partial D^2 \subset U$ it suffices to observe that $F^x(t) = f^x(t)$ for all $x \in \partial D^2 = S^1 \Rightarrow \lim_{t \to 0^+} F^x(t) = \lim_{t \to 0^+} f^x(t)$ exists and

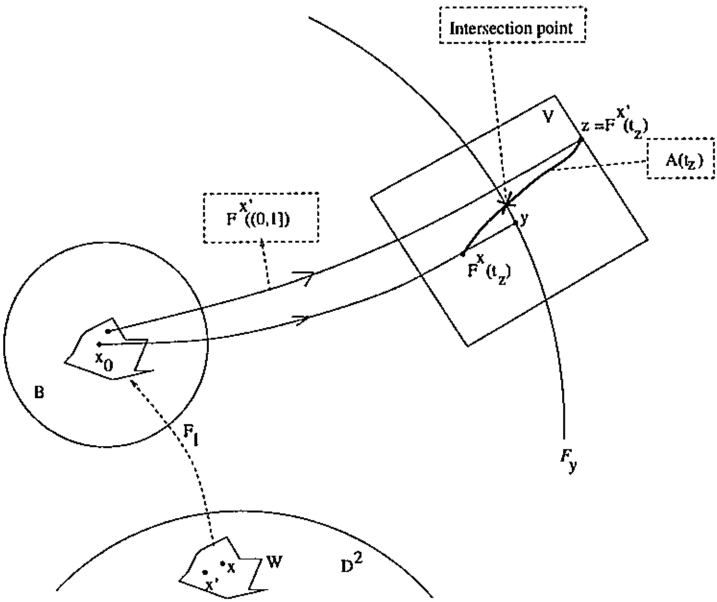


Figure 6.3:

belongs to $A(0)$ for all $x \in S^1$.

Finally we prove that $D^2 \setminus U \neq \emptyset$. If $D^2 = U$ we would have that $F_0 := \lim_{t \to 0^+} F^x(t)$ would exist for all $x \in D^2$. By the Warning above we would have $F_0(x) \in A(0)$ for all $x \in D^2$. The resulting map the map $F_0 : D^2 \to A(0)$ yields a continuous extension of f_0 to D^2, a contradiction since $f_0(S^1)$ is not null homotopic in $A(0)$. This contradiction shows $D^2 \setminus U \neq \emptyset$ and the lemma follows. □

Lemma 6.3.5. *Let F be the immersion in Lemma 6.3.4. Then, $\forall \alpha > 0$ there are $0 < t' < t'' < \alpha$ and an embedding $h : D^2 \to$*

$int(D^2)$ such that

$$F(t'', x) = F(t', x), \quad \forall x \in D^2.$$

Proof. By Lemma 6.3.4 there is $y_0 \in D^2 \setminus U$. Hence the limit $\lim_{t \to 0^+} F^{y_0}(t)$ does not exist. Nevertheless the compactness of M implies that there is a sequence $t_n \to \infty$ such that $F^{y_0}(t_n) \to z$ for some $z \in M$. By using the Tubular Flow-Box Theorem we can assume that $F^{y_0} \in \mathcal{F}_z$ for all n. In addition we can further assume $\mathcal{F}_z \neq A(0)$ because the leaves of $\mathcal{F}$ are two-dimensional. As $F^{y_0}(t_n) \in A(t_n)$ we have $A(t_n) = A(t_m)$ for all n, m. Defining $D(t) = F_t(D^2)$ we have $D(t_n) \subset A(t_n) \subset \mathcal{F}_z$, i.e. $D(t_n) \subset \mathcal{F}_z$ for all n. Note that $z \in A(t_n)$ $\forall n$ large for, otherwise, it would exist $n_k \to \infty$ with $z \notin D(t_{n_k})$. By hypothesis $F^{y_0}(t_{n_k}) = F_{t_{n_k}}(y_0) \in D(t_{n_k})$ converges to z. Because $\partial D(t) = f_t(S^1)$ for all t we conclude that $\exists b_{n_k} \in f_{t_{n_k}}$ sequence converging to z. But the distance $dist(b_{n_k}, f_0(S^1))$ goes to 0 as $k \to \infty$. As $f_0(S^1)$ is compact and $b_{n_k} \to z$ we would have $z \in f_0(S^1) \subset A(0)$ yielding $\mathcal{F}_z = A(0)$ a contradiction. This proves $z \in A(t_n)$ $\forall n$ large. Next we claim that for all $m \in \mathbb{N}$ one has $D(t_m) \subset \text{Int}(D(t_n))$ $\forall n$ large. In fact, note that $\partial D(t_n) = f_{t_n}(S^1)$ hence $\partial D(t_n) \to f_0(S^1)$ uniformly as $n \to \infty$. Clearly we can assume from the beginning that $A(t_n) \neq A(0)$ for all n. Hence $f_{t_n}(S^1) \cap A(0) = \emptyset$ for all n. It follows that for $m \in \mathbb{N}$ fixed one has $\partial D(t_n) \cap \partial D(t_m) = \emptyset$ for all n large. On the other hand, we can assume $z \in D(t_n)$ for all n. From this and $\partial D(t_n) \cap \partial D(t_m) = \emptyset$ one has either $D(t_m) \subset \text{Int}(D(t_n))$ or $D(t_n) \subset \text{Int}(D(t_m))$ for all n large. In the second case we would have $f_0(S^1) \subset D(t_m)$ by taking the limit of the sequence $\partial D(t_n) = f_{t_n}(S^1)$. This would imply $A(t_n) = A(0)$ a contradiction. This contradiction proves $D(t_m) \subset \text{Int}(D(t_n))$ for all n large.

The last claim implies that for $\alpha > 0$ fixed there are $0 < t_n < t_m < \alpha$ such that $D(t_m) \subset D(t_n)$. Choose $t'' = t_m$ and

$t' = t_n$. Clearly $A(t') = A(t'')$. To find the embedding h we let $\hat{F}_{t'} : D^2 \to \hat{A}(t')$ and $\hat{F}_{t''} : D^2 \to \hat{A}(t'')$ be the lift to the universal cover. They exist because D^2 is contractible. Note that $F_{t''}(D^2) \subset \mathrm{Int}(F_{t'}(D^2))$. Hence for a suitable base point one has $\hat{F}_{t''}(D^2) \subset \mathrm{Int}(\hat{F}_{t'}(D^2))$. As both $\hat{F}_{t'}, \hat{F}_{t''}$ are diffeomorphisms we can define

$$h = (\hat{F}_{t'})^{-1} \circ \hat{F}_{t''} : D^2 \to \mathrm{Int}(D^2).$$

Hence h is an embedding satisfying

$$\hat{F}_{t'}(h(x)) = \hat{F}_{t''}(x), \quad \forall x \in D^2.$$

By composition with the projection $\hat{A}(s) \to A(s)$ for $s = t', t''$ one has the desired property. The lemma follows. □

Lemma 6.3.6. *Let $f : S^1 \times [0,1] \to M$ be a vanishing cycle of $\mathcal{F}$ for which there is an embedding $F : D^2 \times [0,1] \to M$ satisfying the conclusion of Lemma 6.3.5. Then, there is in the closed transverse of $\mathcal{F}$ intersecting $A(0)$.*

Proof. Suppose that there is a closed transverse γ of $\mathcal{F}$ intersecting $A(0)$. Modifying a bit γ we can assume that there are $x_0 \in S^1$ and $\alpha > 0$ such that

$$f^{x_0}([0, \alpha]) \subset \gamma,$$

and

$$f^{x}([0, \alpha]) \cap \gamma = \emptyset, \quad \forall x \in S^1 \setminus \{x_0\}.$$

By hypothesis there are $0 < t' < t'' < \alpha$ satisfying the conclusion of Lemma 6.3.5. Let $h : D^2 \to \mathrm{Int}(D^2)$ be the corresponding embedding. In the cylinder $D^2 \times [t', t'']$ we consider the identification $(x, t'') \approx (h(x), t')$. The manifold N obtained from this identification is depicted in Figure 6.4. Note that N is either a solid torus or a solid Klein bottle depending on whether h

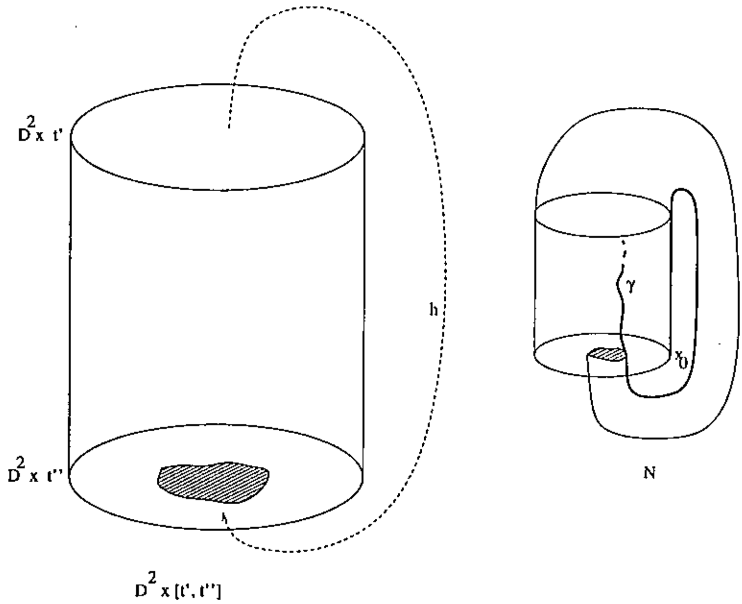


Figure 6.4:

preserves or reverses the orientation in D^2. In any case we let $\Pi : D^2 \times [t', t''] \to N$ be the quotient map. Denote by $P : N \to M$ the map defined by

$$P(z) = F(x,t),$$

where $(x,t) \in \Pi^{-1}(z)$. P is well defined. In fact, if $\Pi(x,t) = \Pi(y,s)$ then $s = t'', t = t'$ and $x = h(y)$. Because $F(y,s) = F(y,t'') = F(h(y),t') = F(x,t') = F(x,t)$ we obtain the result. In addition P is an immersion since F also is. On the other hand, we can assume that γ and the normal vector field X of $\mathcal{F}$ points in the same direction. Now, as γ contain the normal curve $f^{x_0}([0,\alpha])$ (and does not intersect any other normal segment) we have that γ intersects $P(N)$ as in Figure 6.4. Now it suffices to observe that γ cannot exit $P(N)$ because it cannot intersect $P(\Pi(S^1 \times [t', t'']))$. This proves that γ cannot be a closed curve, a contradiction. This contradiction proves the result. □

Proof of Auxiliary Theorem II: The vanishing cycle $f : S^1 \times [0,1] \to M$ of $\mathcal{F}$ can be assumed to exhibit an immersion F satisfying the hypothesis of Lemma 6.3.6. The conclusion of this lemma says that the leaf $A(0)$ cannot intersect a closed transverse of $\mathcal{F}$. And this implies that $A(0)$ is a compact leaf. The result follows. □

6.4 Some corollaries of the Novikov Compact Leaf Theorem

We observe that if M and $\mathcal{F}$ are orientable and transversely orientable then the quotient manifold N in the proof above is a solid torus. In such a case it can be proved without difficulty that N is a Reeb component of $\mathcal{F}$. This remark is summarized in the following result.

Theorem 6.4.1. *A codimension one transversely orientable C^2 foliation on a compact orientable 3-manifold with finite fundamental group has a Reeb component.*

Corollary 6.4.2. *If $\mathcal{F}$ is a codimension one C^2 foliation without compact leaves of a compact 3-manifold M, then the leaves of $\mathcal{F}$ are π_1-injectively immersed in M.*

Proof. Suppose by contradiction that there is a leaf F such that $Ker(\pi_1(F) \to \pi_1(M)) \neq 0$, where $\pi_1(F) \to \pi_1(M)$ is the homomorphism induced by the inclusion $F \to M$. To get the contradiction it suffices by Auxiliary Theorem II to prove that $\mathcal{F}$ has a vanishing cycle. For this we proceed as follows. As $Ker(\pi_1(F) \to \pi_1(M)) \neq 0$ there is a curve $\alpha \subset F$ which is null homotopic in M but not in F. Because α is null homotopic in M we have that there is a map $f : D^2 \to M$ with $\alpha = f(\partial D^2)$. We can assume that f is in general position with respect to $\mathcal{F}$. Hence the induced foliation $\mathcal{F}^*$ in D^2 is a singular foliation tangent to a vector field Y. Note that the singularities of Y are either saddles or centers and there is in the saddle connection for Y. Clearly the closed curve ∂D^2 is a cycle of Y. Hence the set of cycles of Y is not empty. By Lemma 6.2.1 such a set is inductive with respect to the inclusion order. A minimal element in this set produces a vanishing cycle for $\mathcal{F}$ (see the proof of Auxiliary Theorem I in Section 6.2). This yields the desired contradiction and the proof follows. □

Corollary 6.4.3. *Let $\mathcal{F}$ be a codimension one C^2 foliation without compact leaves of a compact 3-manifold M. Then the lift of $\mathcal{F}$ to the universal cover of M is a foliation by planes.*

Proof. Let $\hat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to the universal cover $\pi : \hat{M} \to M$ of M. Suppose by contradiction that there is a non-simply connected leaf $\hat{F}$ of $\hat{\mathcal{F}}$. Hence there is a closed curve $\hat{c} \subset \hat{F}$ which is not null homotopic in $\hat{F}$. Obviously $\hat{c}$ is null homotopic in $\hat{M}$. Hence the closed curve $c = \pi(\hat{c})$ is null homotopic in M. Because $\hat{c}$ is not null homotopic in $\hat{F}$ we have that c is not null homotopic in the leaf $F = \pi(\hat{F})$ of $\mathcal{F}$. This proves that F is *not* π_1-injectively immersed in M. Then $\mathcal{F}$ has a compact leaf by Corollary 6.4.2, a contradiction. This contradiction proves that all the leaves $\hat{F}$ of $\hat{\mathcal{F}}$ are simply connected. Thus $\hat{F} = \mathbb{R}^2$ or S^2. If some $\hat{F}$ is S^2 then $\mathcal{F}$ has a compact leaf with finite fundamental group. By Reeb Global Stability it would follow that all the leaves of $\mathcal{F}$ are compact, a contradiction. This contradiction proves that all the leaves of $\hat{\mathcal{F}}$ are planes as desired. □

Remark 6.4.1. Corollary 6.4.3 shows that closed 3-manifolds supporting codimension one C^2 foliations without compact leaves are *irreducible*, namely every tamely embedded 2-sphere in the manifold bounds a 3-ball. In particular such manifolds are prime, i.e. they are not non-trivial connected sum. We observe that compact 3-manifolds supporting Reebless foliations may be non-irreducible as shown the trivial foliation $\{S^2 \times *\}$ of $S^2 \times S^1$. Nevertheless the 2-sphere bundles over S^1 are the solely closed 3-manifolds which are not irreducible and supports Reebless foliations.

Remark 6.4.2. The results in this section hold true for C^1 foliations.

Chapter 7

Rank of 3-manifolds

The notion of *rank* of a manifold was introduced by J. Milnor, improving original ideas of Hopf, in the search of non-homotopic invariants for manifolds.

Definition 7.0.3. Let M be a differentiable manifold. The *rank* of M is the maximum number $k \in \mathbb{N}$ such that there exist continuous vector fields $X_1, \ldots, X_k$ on M with the property that $[X_i, X_j] = 0$, $\forall i, j$ (i.e., the vector fields commute) and $X_1, \ldots, X_k$ being linearly independent at each point of M.

The Poincaré-Hopf-Euler Theorem states that any continuous tangent vector field on S^2 must have some singularity so that rank $(S^2) = 0$. the following remarkable result is due to E. Lima:

Theorem 7.0.4 ([31]). *The rank of the 3-sphere S^3 is one.*

Notice that, since a C^1 vector field on a *compact* manifold is always complete we may state:

$\{$A compact manifold M has rank $\geq k\} \Leftrightarrow \{M$ admits a locally free action $\varphi\colon \mathbb{R}^k \times M \to M$ of the additive group $(\mathbb{R}^k, +)\}$

Idea of the proof of the Lima Rank Theorem:

First we observe that rank $(S^3) \geq 1$ as it is easily proved by observing that $X(1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ is tangent to S^3 and non-singular (outside the origin $0 \notin S^3$).

Assume by contradiction that rank $(S^3) \geq 2$. By the above remark there exists a locally free action $\varphi\colon \mathbb{R}^2 \times S^3 \to S^3$. The action generates a codimension one foliation $\mathcal{F}$ (assumed to be C^2) on S^3. By Novikov Compact Leaf Theorem $\mathcal{F}$ exhibits some Reeb-component. Thus we are finished once we prove the following:

Lemma 7.0.5. *Given any pair of commuting continuous vector fields X, Y in the solid torus $\overline{D}^2 \times S^1$ such that X and Y are tangent to and linearly independent along $S^1 \times S^1 = \partial(\overline{D}^2 \times S^1)$, then there exists some point $p \in D^2 \times S^1$ whereX and Y are linearly dependent.*

Proof: The boundary torus $\partial(\overline{D}^2 \times S^1)$ has isotropy group of the form $r \cdot \mathbb{Z} + s \cdot \mathbb{Z}$ for some $r, s \in \mathbb{R}^2 \simeq \mathbb{C}$ with $r/s \notin \mathbb{R}$ so that we may re-parameterize φ as $\varphi((rt_1, st_2), \cdot\ )\ (t_1, t_2) \in \mathbb{R}^2$, in such a way that we may assume

$$X|_{S^1 \times S^1} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad Y|_{S^1 \times S^1} = \frac{\partial}{\partial z}$$

for coordinates $(x, y, z) \in \mathbb{R}^3$ with

$$\overline{D}^2 = \{(x, y, z) \in \mathbb{R}^3; \quad z = 0, \quad x^2 + y^2 \leq 1\}$$

We shall therefore prove that any continuous extension $\tilde{\vec{\varepsilon}}_1, \tilde{\vec{\varepsilon}}_2$ of the vector fields $\vec{\varepsilon}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $\vec{\varepsilon}_2 = \frac{\partial}{\partial z}$ On $S^1 \times S^1$ to $\overline{D}^2 \times S^1$ must exhibit some point where $\tilde{\vec{\varepsilon}}_1$ and $\tilde{\vec{\varepsilon}}_2$ are linearly dependent. This is done as follows: we may assume that $\tilde{\vec{\varepsilon}}_1$ and $\tilde{\vec{\varepsilon}}_2$ are orthonormal extensions as it is easy to see.

Such an extension may be regarded as a path homotopy $\tilde{a}\colon \partial\overline{D}^2 \times [0,1] \to G^o_{2,3}$ of the path $a = \tilde{a}(\,\cdot\,,0)\colon \partial\overline{D}^2 \rightsquigarrow G^o_{2,3}$ with a constant; where $G_{2,3}$ is the space of orthonormal oriented pairs of vector on $\mathbb{R}^3$.

By its turn $G^o_{2,3}$ may be identified with the real projective space of dimension 3, $\mathbb{R}P(3)$ as follows: to any element $(v_1, v_2) \in G^o_{2,3}$ we associate a vector $\xi(v_1, v_2) \in \mathbb{R}^3$ as follows.

Denote by $A(v_1, v_2)$ the matrix whose columns one v_1, v_2 and the vectorial product $v_1 \wedge v_2 \in \mathbb{R}^3$. the $A(v_1, v_2)$ is orthogonal and exhibits some eigenvector $\vec{u}(v_1, v_2)$ such that $\pm A(v_1, v_2) \cdot \vec{u}(v_1, v_2) = \vec{u}(v_1, v_2)$. Let $\pi(v_1, v_2)$ be the 2-dimensional subspace of $\mathbb{R}^3$ orthogonal to $\vec{u}(v_1, v_2)$

The restriction $A(v_1, v_2)|_{\pi(v_1,v_2)}$ is an orthogonal linear map of $\mathbb{R}^2$ so that it is a rotation of an angle say $\theta(v_1, v_2) \in [0, \pi]$.

If $\theta = 0$ we define $\xi(v_1, v_2) = 0 \in \mathbb{R}^3$ (in this case $A(v_1, v_2) = Id$). for $\theta(v_1, v_2) \in (0, \pi)$ we choose $\xi(v_1, v_2) \in \pi(v_1, v_2)$ with $|\xi(v_1, v_2)| = \theta$ e same direction and orientation that any $v \wedge A(v_1, v_2) \cdot v$ for $v \in \pi(v_1, v_2) - \{0\}$.

Finally, if $\theta(v_1, v_2) = \pi$. Then $v \wedge A(v_1, v_2) \cdot v = 0$, $\forall v \in \pi(v_1, v_2)$. In this case we cannot define an orientation for $\xi(v_1, v_2)$, what corresponds to identify the vectors $-v$ and $v \in \pi(v_1, v_2)$ having $|v| = \pi$.

Thus ξ above define gives an homeomorphism $\xi\colon G^o_{2,3} \to B^3\overline{(o,\pi)}/\sim$ of $G^o_{2,3}$ on to the quotient space $B^3\overline{(o;\pi)}/\sim$ of the closed ball $\{(x_1, x_2, x_3) \in \mathbb{R}^3;\ \sum_{j=1}^{3} x_j^2 \leq \pi\}$ on $\mathbb{R}^3$ by the equivalence relation that identifies the points v and $-v$ for $v \in \partial B^2\overline{(o,\pi)}$.

Clearly $B^3\overline{(o,\pi)}/\sim$ is homeomorphic to $\mathbb{R}P(3)$ (recall that $\mathbb{R}P(3) \cong \mathbb{R}^3 \cup \mathbb{R}P(2)$).

The path $a = \tilde{a}(\,\cdot\,,o)\colon \partial\overline{D}^2 \to G^o_{2,3}$ is according to this identification, a diameter of $\overline{B^3(0,\pi)}$ parallel to the x_3-axis, from down to up (orientation). Thus this path, once projected into $\mathbb{R}P(3)$, is not homotopic to a constant. This proves the Rank Theorem. Q.E.D.

Remark 7.0.6. The original proof of Lima is from 63 and does not make use of Novikov Compact Leaf Theorem. Actually, the above proof shows:

Theorem 7.0.7 (E. Lima, 63). *A compact simply-connected manifold of dimension three has rank one.*

Remark 7.0.8. The complete solution to the problem of describing the rank of closed 3-manifolds was given by Rosenberg-Roussarie [49] where they prove that a rank two 3-manifold must be a non-trivial fibre bundle over the circle with a torus fiber.

Exercise 7.0.9. Is there any locally free action of the affine group $\mathrm{Aff}(\mathbb{R})$ on the 3-sphere?

Chapter 8

Tischler Fibration Theorem

8.0.1 Preliminaries

Let M be a compact manifold admitting a submersion $f\colon M \xrightarrow{C^2} S^1$. We consider the angle-element 1-form $\theta \in H^1(S^1,\mathbb{R})$ and take $\omega = f^*(\theta)$ its lift to M. We obtain then a closed 1-form, without singularities, of class C^1 in M. Since ω is integrable, it defines a foliation $\mathcal{F}$ of codimension 1, class C^1 in M. Let now $p \in M$ be any point. Since ω is not singular there exist neighborhoods $p \in U_p \subset M$ and C^1 vector-fields X_p in U_p such that $\omega \cdot X_p = 1$ in U_p. Using partition of the unity we obtain finally a global vector-field X in M with the property that $\omega \cdot X = 1$. Since M is compact, X is complete defining therefore a flow $\varphi\colon \mathbb{R} \times M \to M$. From $\omega \cdot X = 1$ we conclude that the flow is transverse to $\mathcal{F}$. Since $d\omega = 0$ we have that $L_X(\omega) = d(\omega \cdot X) + i_X(d\omega) = 0$ so that φ *preserves the foliation* $\mathcal{F}$ (each diffeomorphism $\varphi_t\colon M \to M$ takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}$). We conclude that $\mathcal{F}$ is "invariant by a transverse flow". Tischler

Fibration Theorem states the converse of this fact:

Theorem 8.0.10 (Tischler-1970, [69]). *Let M be a closed differentiable manifold. The following conditions are equivalent:*

(i) *M supports a foliation $\mathcal{F}$, of class C^1 and codimension 1, invariant by a transverse flow C^1.*

(ii) *M supports a closed 1-form of class C^1 without singularities.*

(iii) *M fibers over the circle S^1.*

Taking into account the Theorem of Sacksteder (according to which a foliation of class C^2, codimension 1 and without holonomy is topologically conjugate to a foliation defined by a closed non-singular 1-form (cf. [56]) we obtain in class C^2 the following equivalent condition:

(iv) *M admits a foliation of codimension 1 without holonomy.*

A demonstration of Tischler Fibration Theorem uses strongly the fact that in a closed manifold M we can find closed differentiable 1-forms $\omega_1, \dots, \omega_\ell \in H^1(M, \mathbb{R})$ such that given a base $\gamma_1, \dots, \gamma_\ell$ of the free part of $H_1(M, \mathbb{Z})$ we have $\displaystyle\int_{\gamma_j} \omega_i = \delta_{ij}$ delta of Kronecker. Thus the closed 1-form closed ω in M writes $\omega = \sum_{j=1}^{\ell} \lambda_j \omega_j + df$ for some function $f \colon M \to \mathbb{R}$, where $\{\lambda_1, \dots, \lambda_\ell\}$ generates the *group of periods* $\operatorname{Per}(\omega) < (\mathbb{R}, +)$ of ω. If ω is non-singular then, since $\overline{\mathbb{Q}} = \mathbb{R}$, we can obtain perturbations $\omega' = \sum_{j=1}^{\ell} \lambda_j \omega_j + df$ of ω such that ω' is non-singular and $\operatorname{Per}(\omega') \subset \mathbb{Q}$ and hence for some integral multiple $k \cdot \omega'$

we will $\operatorname{Per}(k\omega') \subset \mathbb{Z}$. Clearly $k\omega' = dg$ for some submersion $g\colon M \to \mathbb{R}/\mathbb{Z} = S^1$. ■

8.0.2 Proof of Tischler Fibration Theorem and generalizations

In this section we state the basic results we need in order to prove the Tischler Fibration Theorem. Throughout this section $\mathcal{F}$ will denote a (non-singular) codimension one smooth foliation on a connected manifold M of dimension $n \geq 2$.

Definition 8.0.11. Let $\varphi\colon \mathbb{C} \times M \to M$ be a smooth flow on M. We say that φ is a flow *transverse* to $\mathcal{F}$ if the vector field $Z = \left.\dfrac{\partial p}{\partial t}\right|_{t=0}$ (where $t \in \mathbb{C}$ is the complex time) is transverse to (the leaves of) $\mathcal{F}$.

We say that $\mathcal{F}$ is *invariant under* the flow φ if each flow map $\varphi_t\colon M \to M$ takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}$.
We shall say that $\mathcal{F}$ is *invariant under the transverse flow* of Z if Z is a complete vector field on M, whose corresponding flow φ is transverse to $\mathcal{F}$ and $\mathcal{F}$ is invariant under φ.

Example 8.0.12. Let M be a n-torus, $M = \mathbb{R}^n/\Lambda$ where $\Lambda \subset \mathbb{R}^n$ is some lattice. Let $\widetilde{\mathcal{F}}$ be the foliation on $\mathbb{R}^n$ by hyperplanes parallel to a given direction $\widetilde{Z} \in \mathbb{R}^n$. Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the quotient $M = \mathbb{R}^n/\Lambda$ which is called a *linear foliation* on the Torus M. Such a foliation is invariant under a transverse flow given by a vector field Z whose lift to $\mathbb{R}^n$ is $\widetilde{Z}$. As it is easily checked, $\mathcal{F}$ is given by a (non-singular) closed smooth 1-form Ω on M, with constant coefficients. The following (classic real) result states the existence of Ω as a general fact:

Proposition 8.0.13. *Let $\mathcal{F}$ be invariant by a transverse smooth flow φ of Z on M. Then $\mathcal{F}$ is given by a (non-singular) closed smooth 1-form Ω characterized by:*

$$\int_{t_1}^{t_2} \Omega(\varphi_t(x)) \cdot Z(\varphi_t(x))\, dt = t_2 - t_1$$

$\forall\, x \in M,\ \ \forall\, t_1, t_2 \in \mathbb{R}.$

Proof. We follow the original construction in [42]. We construct Ω locally as a "time form" for Z. Given any point $p \in M$ choose a distinguished neighborhood $\xi\colon U \subset M \to \mathbb{R}^{n-1} \times \mathbb{R}$ such that ξ takes $\mathcal{F}\big|_U$ into the horizontal foliation on $\mathbb{R}^{n-1} \times \mathbb{R}$. We may also assume that $\xi(p) = 0$ and (most important) $\xi(\varphi_t(p)) \in \mathbb{R}^{n-1} \times \{t\}$, $\forall\, t$ with $\varphi_t(p) \in U$ (here we use the fact that φ is transverse to $\mathcal{F}$ and leaves $\mathcal{F}$ invariant). Define now $\Omega_U := d(\pi \circ \xi)$ where $\pi\colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is the projection $\pi(x, y) = y$.
Given two such distinguished charts $\xi_j\colon U_j \subset M \to \mathbb{R}^{n-1} \times \mathbb{R}$ with U_j connected and having connected intersection $U_1 \cap U_2 \neq \phi$ then if we put $\Omega_j := \Omega_{U_j} = d9\pi \circ \xi_j)$ we obtain in $U_1 \cap U_2$:

$$\begin{aligned} \Omega_1\big|_{U_1\cap U_2} = d(\pi \circ \xi_1)\big|_{U_1\cap U_2} &\overset{(*)}{=} d(\pi \circ \xi_2)\big|_{U_1\cap U_2} \\ &= \Omega_2\big|_{U_1\cap U_2}. \end{aligned}$$

Remark 8.0.14. $\xi_j(\varphi_t(p_n)) \in \mathbb{R}^{n-1} \times \{t\} \Rightarrow \xi_j(\varphi_t(q)) \in \mathbb{R}^{n-1} \times \{t\}\ \ \forall\, q \in L_{p_j} \cap U_j$ where L_{p_j} = leaf of $\mathcal{F}$ through p_j,

$$\begin{aligned} &\therefore \xi_j(\varphi_t(q)) = (a_j(q,t), t), \quad \forall\, q \in L_{p_j} \cap U_j \\ &\therefore (\pi \circ \xi_j)(\varphi_t(q)) = \pi(a_j(q,t), t) = t, \quad \forall\, q \in L_{p_j} \cap U_j, \quad \forall\, t \approx 0 \end{aligned}$$

and finally

$$(\pi \circ \xi_j)(\varphi_t(r)) = t, \qquad \forall\, r \in U_j,\ \forall\, t \approx 0.$$

This way we obtain a well-defined closed one-form Ω on M which satisfies

$$\begin{aligned}\Omega(\varphi_t(p)) \cdot Z(\varphi_t(p)) &= d(\pi \circ \xi)(\varphi_t(p)) \cdot Z(\varphi_t(p)) \\ &= \frac{d}{dt}\left((\pi \circ \xi)(\varphi_t(p))\right) = \\ &= \frac{d}{dt}(t) = 1, \quad \forall p \in M, \ \forall t \in \mathbb{R} \qquad \square\end{aligned}$$

Corollary 8.0.15. *Let $\mathcal{F}$ be a codimension one (non-singular) smooth foliation on a compact (connected) manifold M. The following conditions are equivalent:*
(i) *$\mathcal{F}$ is invariant under some smooth transverse flow.*
(ii) *$\mathcal{F}$ is given by a closed smooth one-form Ω on M.*

Proposition 8.0.16. *Let $\mathcal{F}$, φ, Z, Ω be as in* Proposition 8.0.13 *but assume M is compact. Given any leaf L_0 of $\mathcal{F}$ there exist a differentiable covering*

$$\sigma\colon L_0 \times \mathbb{R} \to M, \qquad \sigma(x,t) = \varphi_t(x);$$

and an exact sequence of groups

$$0 \longrightarrow \pi_1(L_0 \times \mathbb{R}) \xrightarrow{\sigma_\#} \pi_1(M) \longrightarrow A \longrightarrow 0,$$

where A is a finitely generated free abelian group. Moreover, L_0 is compact if, and only if, A is a lattice on $\mathbb{R}$.

Proof. Define

$$H = \left\{[\gamma] \in \pi_1(M); \int_\gamma \Omega = 0\right\}$$

then H is a normal subgroup of $\pi_1(M)$ and it is free because

$$\int_{n\cdot\gamma}\Omega = n\cdot\int_{\gamma}\Omega \quad \forall\gamma\in\pi_1(M),\ \forall n\in\mathbb{Z}.$$

Put $A := \pi_1(M)/H$ then A is finitely generated and also A is abelian because $H \supset [\pi_1(M),\pi_1(M)]$ (the group of commutators) because

$$\int_{\gamma*\delta}\Omega = \int_{\gamma}\Omega + \int_{\sigma}\Omega = \int_{\delta}\Omega + \int_{\gamma}\Omega = \int_{\delta*\gamma}$$

$\forall\,\delta,\gamma\in\pi_1(M)$.
Let $P\colon \widetilde{M}\to M$ be the smooth covering of M, corresponding to H. Let also $\widetilde{\mathcal{F}}$, $\widetilde{\Omega}$ and $\widetilde{\varphi}_t$ be the lifting of $\mathcal{F}$, Ω and φ_t to $\widetilde{M}$ respectively.

Remark 8.0.17. $\widetilde{\mathcal{F}} = P^*(\mathcal{F})$, $\widetilde{\Omega} = P^*(\Omega)$ are usual pull-backs.
$\widetilde{\varphi}\colon \widetilde{M}\times\mathbb{R}\to\widetilde{M}$ is defined by

$$\widetilde{\varphi}_t(\tilde{p}) := \widetilde{\varphi_t(P(\tilde{p}))}, \quad \forall\tilde{p}\in\widetilde{M},\ \forall t\in\mathbb{R},$$

that is, for each $\tilde{\in}\widetilde{M}$, $\widetilde{\varphi}_t(\tilde{p})$ is the lifting by P of the curve $\varphi_t(P(\tilde{p}))$ on M.

This lifting is well-defined because of the following:
Let γ, δ be simple piecewise smooth paths on $\mathbb{R}$ with $\gamma(0) = 0 = \delta(0)$ and $\delta(1) = t = \gamma(1)$.
Put $c = \delta^{-1}*\gamma$ then c is closed. Since Ω is closed and $\varphi_p\colon \mathbb{R}\to M$ is smooth we have that $\int_c \varphi_p^*(\Omega) = 0$. Therefore,

$$\int_{(\varphi_p)_{\#}c}\Omega = 0, \quad \text{that is,} \quad \int_{\varphi_p(c)}\Omega = 0.$$

This says that $\varphi_p(c) \in H$. But $\varphi_p(c) = \varphi_p(\delta)^{-1} * \varphi_p(\gamma)$ so that $\varphi_p(\delta)$ and $\varphi_p(\gamma)$ are paths whose lifts by P exhibit the same final points. Therefore we may define $\widetilde{\varphi}\colon \widetilde{M} \times \mathbb{R} \to \widetilde{M}$ in a natural way. It is now easy to check $\widetilde{\varphi}$ is (locally) smooth in each variable $\tilde{x} \in \widetilde{M}$ and $t \in \mathbb{R}$ separately. By Hartogs' Theorem $\widetilde{\varphi}$ is smooth as a map $\widetilde{M} \times \mathbb{R} \to \widetilde{M}$. Finally we have by construction $P \circ \widetilde{\varphi}(t, \tilde{x}) = \varphi_t(P(\tilde{x}))$ so that

$$\begin{aligned} P \circ \widetilde{\varphi}(t, \widetilde{\varphi}(s, \tilde{x})) &= \varphi_t(P(\widetilde{\varphi}(s, \tilde{x}))) = \varphi_t(\varphi_s(P(\tilde{x}))) \\ &= \varphi_{t+s}(P(\tilde{x})) = P \circ \widetilde{\varphi}(t+s, \tilde{x}). \end{aligned}$$

This implies that $\widetilde{\varphi}(t, \widetilde{\varphi}(s, \tilde{x})) = \widetilde{\varphi}(t+s, \tilde{x})$ so that $\widetilde{\varphi}$ is actually a flow on $\widetilde{M}$.

Let therefore $\widetilde{Z} = \left.\dfrac{\partial \widetilde{\varphi}}{\partial t}\right|_{t=0}$ be the corresponding smooth vector field. It is then clear that $P_* \widetilde{Z} = Z$, that is, $\widetilde{Z}$ is a lift of Z.

By construction if $\widetilde{\gamma} \in \pi_1(\widetilde{M})$ is such that $\int_{\widetilde{\gamma}} \widetilde{\Omega} = 0$ then $\widetilde{\gamma}$ is (homotopic to) the zero element so that $\widetilde{\Omega} = d\tilde{f}$ for some smooth function $\tilde{f}\colon \widetilde{M} \to \mathbb{R}$.

Lemma 8.0.18. *We have* $\tilde{f}(\widetilde{\varphi}_t(\tilde{x})) = t + \tilde{f}(\tilde{x}) \quad \forall t \in \mathbb{R}, \ \forall \tilde{x} \in \widetilde{M}$.

Proof. Indeed,

$$\frac{d}{dt}\left(\tilde{f}(\widetilde{\varphi}_t(\tilde{x}))\right) = d\tilde{f}(\widetilde{\varphi}_t(\tilde{x})) \cdot \widetilde{Z}(\widetilde{\varphi}_t(\tilde{x})) = \widetilde{\Omega}(\widetilde{\varphi}_t(\tilde{x})) \cdot \widetilde{Z}(\widetilde{\varphi}_t(\tilde{x})) = 1$$

and also

$$\tilde{f}(\widetilde{\varphi}_t(\tilde{x}))\big|_{t=0} = \tilde{f}(\tilde{x}). \qquad \square$$

Given any leaf L_0 of $\mathcal{F}$ on M let $\widetilde{L}_0 \subset \widetilde{M}$ be a leaf of $\widetilde{\mathcal{F}}$ such that $P(\widetilde{L}_0) = L_0$. Define the map $g\colon \widetilde{L}_0 \times \mathbb{R} \to \widetilde{M}$ by setting $g(\tilde{x}, t) = \widetilde{\varphi}_t(\tilde{x})$.

Lemma 8.0.19. *g is a smooth diffeomorphism of $\widetilde{L}_0 \times \mathbb{R}$ onto $\widetilde{M}$.*

Proof. We have

$$\frac{\partial g}{\partial t}(\tilde{x}_0, t_0) = \left.\frac{\partial \widetilde{\varphi}_t}{\partial t}\right|_{t=t_0}(\tilde{x}_0) = \widetilde{Z}(\widetilde{\varphi}_{t_0}(\tilde{x}_0)).$$

Also

$$\frac{\partial g}{\partial \tilde{x}}(\tilde{x}_0, t_0) = \left.\frac{\partial}{\partial \tilde{x}}(\widetilde{\varphi}_{t_0}(\tilde{x}))\right|_{\tilde{x}=\tilde{x}_0} = \frac{\partial \widetilde{\varphi}}{\partial \tilde{x}}(\tilde{x}_0, t_0).$$

Since the flow of Z is transverse to $\mathcal{F}$ it follows that the flow $\widetilde{\varphi}$ is transverse to $\widetilde{\mathcal{F}}$ so that g is a local diffeomorphism in $\widetilde{L}_0 \times \mathbb{R}$. Now we notice that if $g(\tilde{x}_1, t_1) = g(\tilde{x}_2, t_2)$ then $\widetilde{\varphi}_{t_1}(\tilde{x}_1) = \widetilde{\varphi}_{t_2}(\tilde{x}_2)$ and $\tilde{f}(\widetilde{\varphi}(t_1, \tilde{x}_1)) = \tilde{f}(\widetilde{\varphi}(t_2, \tilde{x}_2))$ so that $\widetilde{\varphi}_{t_1 - t_2}(\tilde{x}_1) = \tilde{x}_2$ and $t_1 + \tilde{f}(\tilde{x}_1) = t_2 + \tilde{f}(\tilde{x}_2)$.

Now, x_1 and $\tilde{x}_2$ belong to the same leaf $\widetilde{L}_{\tilde{x}_0} = \widetilde{L}_0$ of $\widetilde{\mathcal{F}}$ so that $f(\tilde{x}_1) = f(\tilde{x}_2)$, this implies $t_1 = t_2$ and therefore $\tilde{x}_1 = \tilde{x}_2$. Therefore g is also injective and it is a diffeomorphism of $\widetilde{L}_0 \times \mathbb{R}$ onto its image $g(\widetilde{L}_0 \times \mathbb{R}) \subset \widetilde{M}$. It remains to prove that $g(\widetilde{L}_0 \times \mathbb{R}) = \widetilde{M}$.

It is enough to prove that this image of g is closed. Take any point $\tilde{x}_1 \in \widetilde{M}$ belonging to the closure of $g(\widetilde{L}_0 \times \mathbb{R})$ in $\widetilde{M}$. Let $\widetilde{B}_1 \ni \tilde{x}_1$ be any open ball in the leaf $\widetilde{L}_1 \ni \tilde{x}_1$. Let $\widetilde{U}$ be the "cylinder" $\widetilde{U} = \bigcup\limits_{t \in \mathbb{R}} \widetilde{\varphi}_t(\widetilde{B}_1)$, and take any $\tilde{x} \in \widetilde{U} \cap g(\widetilde{L}_0 \times \mathbb{R})$.

We have $\tilde{x} \in \widetilde{\varphi}_s(\widetilde{L}_0)$ for some $s \in \mathbb{R}$ and also there exists $r \in \mathbb{R}$ such that $\tilde{x} \in \widetilde{\varphi}_r(\widetilde{B}_1)$. Thus $\tilde{x}_1 \in \widetilde{\varphi}_{s-r}(\widetilde{L}_0)$ and hence $\tilde{x}_1 \in g(\widetilde{L}_0 \times \mathbb{R})$. This proves the lemma. □

Now we may prove:

Lemma 8.0.20. *$P|_{\widetilde{L}_0} : \widetilde{L}_0 \subset \widetilde{M} \to L_0 \subset M$ is a bijection and therefore a diffeomorphism.*

Proof. P is injective, for if $\tilde{x}_1, \tilde{x}_2 \in \widetilde{L}_0$ are such that $P(\tilde{x}_1) = P(\tilde{x}_2)$ then we may take a path $\tilde{\alpha}\colon [0,1] \to L_0$ of class C^1 with $\tilde{\alpha}(0) = \tilde{x}_1$ and $\tilde{\alpha}(1) = \tilde{x}_2$.
This gives a projected path $\alpha = P \circ \tilde{\alpha}\colon [0,1] \to L_0 \subset M$ which is closed, i.e., $\alpha \in \pi_1(L_0)$.
We have

$$\frac{d}{dt}\,(\tilde{f}(\tilde{\alpha}(t))) = \widetilde{\Omega}(\tilde{\alpha}(t)) \cdot \tilde{\alpha}'(t) \quad \text{so that}$$

$$\widetilde{\Omega}(\tilde{\alpha}(t)) \cdot \tilde{\alpha}'(t) = 0, \quad \forall\, t \in [0,1] \quad \text{and therefore}$$
$$\Omega(\alpha(t)) \cdot \alpha'(t) = 0, \quad \forall\, t \in [0,1].$$

This gives
$$0 = \int_0^1 \Omega(\alpha(t)) \cdot \alpha'(t)\, dt = \int_\alpha \Omega$$

and therefore $[\alpha] \in H \subset \pi_1(M)$. This gives $\tilde{\alpha}(0) = \tilde{\alpha}(1)$ in $\widetilde{M}$, i.e. $\tilde{x}_1 = \tilde{x}_2$. □

Let now $\eta_t\colon L_0 \times \mathbb{R} \to L \times \mathbb{R}$ be given by $\eta_t(x,s) := (x, s+t)$. Let also $G\colon L_0 \times \mathbb{R} \to \widetilde{M}$ be defined by

$$G(x,s) := g\Big(\underbrace{\underbrace{(P|_{\widetilde{L}_0})^{-1}(x)}_{\widetilde{L}_0 \ni}, s}_{\widetilde{L}_0 \times \mathbb{R} \ni}\Big)$$

Consider the following diagram

$$\begin{array}{ccccc}
M & \xleftarrow{\;P\;} & \widetilde{M} & \xleftarrow{\;G\;} & L_0 \times \mathbb{R} \\
{\scriptstyle \varphi_t}\big\downarrow & & {\scriptstyle \widetilde{\varphi}_t}\big\downarrow & & \big\downarrow {\scriptstyle \eta_t} \\
M & \xleftarrow{\;P\;} & \widetilde{M} & \xleftarrow{\;G\;} & L_0 \times \mathbb{R}
\end{array}$$

The left side is commutative by construction.
Now we observe that given $(x,s) \in L_0 \times \mathbb{R}$ we have

$$\begin{aligned} G(\eta_t(x,s)) = G(x,s+t) &= g\big(\big(P|_{\widetilde{L}_0}\big)^{-1}(x), s+t\big) \\ &= \widetilde{\varphi}_{s+t}\big(\big(P|_{\widetilde{L}_0}\big)^{-1}(x)\big) \\ &= \widetilde{\varphi}_t\big(\widetilde{\varphi}_s\big(\big(P|_{\widetilde{L}_0}\big)^{-1}(x)\big)\big) \\ \Rightarrow \quad G(M_t(x,s)) = \widetilde{\varphi}_t(G(x,s)) = (\widetilde{\varphi}_t \circ G)(x,s). \end{aligned}$$

Therefore the whole diagram is commutative.
Define now

$$\sigma := P \circ G \colon L_0 \times \mathbb{R} \to M$$

by requiring tht the diagram below is commutative:

$$\begin{array}{ccc} M & \xleftarrow{\ \sigma\ } & L_0 \times \mathbb{R} \\ \varphi_t \big\downarrow & & \big\downarrow \eta_t \\ M & \xleftarrow{\ \sigma\ } & L_0 \times \mathbb{R} \end{array}$$

In other words:

$$\varphi_t \circ \sigma(x,s) = \sigma(x,s+t).$$

Lemma 8.0.21. $\sigma \colon L_0 \times \mathbb{R} \to M$ is a covering map.

Proof. We know that P is a covering map $P \colon \widetilde{M} \to M$. It is therefore enough to show that $G \colon L_0 \times \mathbb{R} \to \widetilde{M}$ is a covering map. Actually $g \colon \widetilde{L}_0 \times \mathbb{R} \to \widetilde{M}$ is a diffeomorphism and so is $P_{\widetilde{L}_0} \colon \widetilde{L}_0 \to L_0$, so that $G \colon L_0 \times \mathbb{R} \to \widetilde{M}$ is a diffeomorphism. □

Clearly $\sigma(x,t) = \varphi_t \circ \sigma(x,0) = \varphi_t(x)$, $\forall\, t \in \mathbb{R}$ $\forall\, x \in L_0$. Therefore σ satisfies the first condition in the statement of Proposition 8.0.27.

If for any $[\gamma] \in \pi_1(M)$ we have $[\gamma] = \sigma_\#([\alpha])$ in $\pi_1(M)$, for some $[\alpha] \in \pi_1(L_0 \times \mathbb{R})$ then

$$\int_\gamma \Omega = \int_{\sigma\circ\alpha} \Omega = \int_{\alpha\subset L_0\times\mathbb{R}} \sigma^*(\Omega) = \int_\alpha (\varphi_t)^*(\Omega) \underset{\substack{\text{for } \alpha\subset L_0\times\mathbb{R} \\ \text{and } \Omega=0 \text{along } L_0}}{=} 0$$

so that $[\gamma] \in H$. Conversely, if $[\gamma] \in H$ then $\gamma = P_\#(\widetilde{\gamma})$ for some $\widetilde{\gamma} \in \pi_1(\widetilde{M})$ and therefore we have $\widetilde{\gamma} = g_\#(\widetilde{\alpha})$ for some $\widetilde{\alpha} \in \pi_1(\widetilde{L}_0 \times \mathbb{R})$ so that $\gamma = P_\#(g_\#(\widetilde{\alpha})) \Rightarrow \gamma = (P \circ G)_\#(\alpha)$ where α" =" $P \circ \widetilde{\alpha} \in \pi_1(L_0 \times \mathbb{R})$ is obtained in a natural way. Therefore we have proved the following:

Lemma 8.0.22. *The sequence below is exact*

$$0 \longrightarrow \pi_1(L_0 \times \mathbb{R}) \xrightarrow{\sigma_\#} \pi_1(M) \longrightarrow A \longrightarrow 0.$$

Remark 8.0.23. Another way of seeing the above equivalence is the following: if $[\gamma] \in H$ then $\gamma \in \pi_1(M)$ is such that $\int_\gamma \Omega = 0$. Therefore we may consider the lifting α of γ by σ to $L_0 \times \mathbb{R}$ obtaining a path such that $\int_\alpha (\varphi_t)^*(\Omega) = 0$ and therefore α is closed that is, $[\gamma] = \sigma_\#([\alpha])$ for $[\alpha] \in \pi_1(L_0 \times \mathbb{R})$.

Assume that A has rank one, $A \approx \mathbb{Z}$. We may take a transformation $T\colon L_0 \times \mathbb{R} \to L_0 \times \mathbb{R}$ which corresponds to a generator of $A = \pi_1(M)/H$ (notice that the covering $\sigma\colon L_0 \times \mathbb{R} \to M$ has group isomorphic to $\pi_1(M)/H$ because of the exact sequence $(0 \to \pi_1(L_0 \times \mathbb{R}) \xrightarrow{\sigma_\#} \pi_1(M) \to \pi_1(M)/H \to 0)$.

The diagram
$$\begin{array}{ccc} L_0 \times \mathbb{R} & \xrightarrow{T} & L_0 \times \mathbb{R} \\ \sigma \searrow & & \swarrow \sigma \\ & M & \end{array}$$
commutes.

Let $T_0 := T\big|_{L_0\times\{0\}}\colon L_0 \times \{0\} \to L_0 \times \mathbb{R}$.

Lemma 8.0.24. $\exists\, t_0 \in \mathbb{R} - \{0\}$ such that $T(L_0 \times \{0\}) = L_0 \times \{t_0\}$.

Proof. Write $T(x,t) = (a(x,t), b(x,t))$ so that $\sigma \circ T = \sigma \Rightarrow \varphi_{b(x,t)}(a(x,t)) = \varphi_t(x) \Rightarrow \varphi_{b(x,0)}(a(x,0)) = 0\ \forall\, x \in L_0$. $(*)$
We have $a(x,0) \in L_0$, $\forall\, x \in L_0$ therefore (since φ_t is transverse to $\mathcal{F}$) we must have from (*) from ($b(x,0)$ is constant $\forall\, x \in L_0$ and therefore if we put $t_0 = b(x,0)$ then

$$\varphi_{t_0}(a(x,0)) = x, \quad \forall\, x \in L_0\,.$$

That by

$$T(L_0 \times \{0\}) \subset L_0 \times \{t_0\}\,. \quad \square$$

If $t_0 = 0$ then $b(x,0) = 0$, $\forall\, x \in L_0$ and $T(x,0) = (a(x,0), 0)$ and also form (*) $a(x,0) = x \quad \forall\, x \in L_0$.
Thus $T(x,0) = (x,0)$, $\forall\, x \in L_0$. This is not possible for T is a non trivial covering transformation. $\square$

Define now a map $f\colon M \to \mathbb{R}/t_0\mathbb{Z} \simeq S^1$ by setting $f(x) := s \pmod{t_0}$ where $x \in \varphi_s(L_0)$. Notice that given any $x_1 \in M$, since $\sigma\colon L_0 \times \mathbb{R} \to M$, $\sigma(x,t) = \varphi_t(x)$ is a covering, it follows that $x_1 \in \varphi_s(L_0)$ for some $s \in \mathbb{R}$.
Now, if $s_1, s_2 \in \mathbb{R}$ are such that $x \in \varphi_{s_j}(L_0)$, $j = 1, 2$, then $\exists\, x_1, x_2 \in L_0$ with $x = \varphi_{s_1}(x_1)$, $x = \varphi_{s_2}(x_2)$ so that $\sigma(x_1, s_1) = x = \sigma(x_2, s_2)$. Since the group of covering maps of σ is generated by T we must have $(x_2, s_2) = T^n(x_1, s_1)$ for some $n \in \mathbb{Z}$ so that $s_2 = s_1 + n \cdot t_0$, so that $s_2 = s_1 \pmod{t_0}$. Therefore $f\colon M \to S^1 = \mathbb{R}/t_0\mathbb{Z}$ is well-defined and clearly smooth.
Since φ_t takes leaves of $\mathcal{F}$ onto leaves, f is constant along the leaves of $\mathcal{F}$. Thus f is a smooth first integral for $\mathcal{F}$.
Assume now that $\operatorname{rank}(A) = 0$. In this case $H = \pi_1(M)$ and $P\colon \widetilde{M} \to M$ is the universal covering of M. Since $A = \{0\}$ we

have a diffeomorphism $M \overset{\sigma}{\simeq} L_0 \times \mathbb{R}$ what is not possible because M is compact.

Conversely, assume now that L_0 is a compact leaf of $\mathcal{F}$. Since $\mathcal{F}$ has trivial holonomy the Stability Theorem of Reeb (see [15]) implies that all the leaves of $\mathcal{F}$ are compact and $\mathcal{F}$ is a Seifert (smooth) fibration. Now, the group A acts on $L_0 \times \mathbb{R}$ taking leaves of $\sigma^*(\mathcal{F})$ onto leaves of $\sigma^*(\mathcal{F})$ in a natural way as in Lemma 8.0.22. Therefore, since $\mathcal{F}$ is a compact foliation, the leaves of $\sigma^*(\gamma)$ are closed on $L_0 \times \mathbb{R}$ and therefore the action of A must be discrete so that indeed, A must correspond to a discrete subgroup of $\mathbb{R}$ and therefore $\operatorname{rank}(A) \leq 1$ as it is well-known (see Remark 8.0.25 below).

Remark 8.0.25. We remark that, according to what we have seen, we have an homomorphism of groups

$$\begin{aligned} \xi\colon \pi_1(M) &\to (\mathbb{R}, +) \\ [\gamma] &\mapsto \int_\gamma \Omega \end{aligned}$$

whose kernel is H so that there exists an injective homomorphism $\overline{\xi}\colon A = \pi_1(M)/H \to \mathbb{R}$, so that A is naturally identified to a certain subgroup of $(\mathbb{R}^2, +)$.

Therefore we have proved that $\operatorname{rank}(A) = 1 \Leftrightarrow L_0$ is compact $\Leftrightarrow$ all leaves of $\mathcal{F}$ are compact. This ends the proof of Proposition 8.0.16. $\square$

Corollary 8.0.26. *Let M be compact, $\mathcal{F}$, φ_t, A as in* Proposition 8.0.16. *Then* $1 \leq \operatorname{rank}(A) \leq \operatorname{rank}(H_1(M, \mathbb{Z}))$. *Moreover, if M is an orientable compact manifold and* $\operatorname{rank}(H_1(M, \mathbb{R})) \leq 1$ *then $\mathcal{F}$ is a foliation by compact leaves.*

Next step is the following:

Proposition 8.0.27. *Let $\mathcal{F}$, φ_t, Ω, A and M be as in* Proposition 8.0.16. *Let* $\mathrm{Per}(L_0) := \{t \in \mathbb{R}; \varphi_t(L_0) = L_0\}$ *and* $\eta_t \colon L_0 \times \mathbb{R} \to L_0 \times \mathbb{R}$ *be given by* $\eta_t(x, s) = (x, s + t)$.
(i) *If* $T \colon L \times \mathbb{R} \to L_0 \times \mathbb{R}$ *is a covering transformation of the covering*

$$\begin{aligned} \sigma \colon L_0 \times \mathbb{R} &\to M \\ (x, t) &\mapsto \varphi_t(x) \end{aligned}$$

then $T(L_0 \times \{t\}) = \eta_{t_0}(L_0 \times \{0\})$ *for some* $t_0 = t_0(T) \in \mathbb{R}$.
(ii) *The correspondence* $T \mapsto t_0(T)$ *defines an isomorphism* $A \to \mathrm{Per}(L_0)$.
(iii) $\mathrm{Per}(L_0)$ *is the group of periods of* Ω.

Proof. As we have seen in the proof of Proposition 8.0.16 above for each covering transformation T of σ we must have $T(L_0 \times \{0\}) = L_0 \times \{t_0(T)\}$ for some $t_0(T) \in \mathbb{R}$. Moreover $t_0(T) = 0$ if, and only if, T is the identity. It is also possible to see that $t_0(T)$ depends only on T, not on the choice of the leaf $L_0 \subset M$. Therefore we have $T(L \times \{0\}) = \eta_{t_0(T)}(L \times \{0\})$ for any leaf L of $\mathcal{F}$.
The mapping $\xi \colon A \to \mathbb{R}$, $T \mapsto t_0(T)$ is therefore such that $\xi(A) \subset \mathrm{Per}(L_0)$: given any $x_0 \in L_0$ we have $T(L_0 \times \{0\}) = L_0 \times \{t_0(T)\} \Rightarrow T(x_0, 0) \in L_0 \times \{t_0(T)\} \Rightarrow$ if we write $T(x, t) = (a(x, t), b(x, t))$ then $T(x, 0) = (a(x, 0), t_0(T))$ and therefore

$$\begin{aligned} x_0 = \varphi_0(x_0) = \sigma(x_0, 0) &= \sigma \circ T(x_0, 0) \\ \sigma(a(x_0, 0), t_0(T)) &= \varphi_{t_0(T)}(a(x_0, 0)) \end{aligned}$$

so that $\varphi_{t_0(T)}(L_0) = L_0$ and then $t_0(T) \in \mathrm{Per}(L_0)$. Thus we have $\xi \colon A \to \mathrm{Per}(L_0) \subset \mathbb{R}$.

Lemma 8.0.28. *ξ is an injective group homomorphism.*

Proof. Given $S, T \in A$ be have $\xi(S \circ T) = t_0(S \circ T)$ and by definition

$$S \circ T(L_0 \times \{0\}) = L_0 \times \{t_0(S \circ T)\}.$$

But, on the other hand,

$$S \circ T(L_0 \times \{0\}) = S(T(L_0 \times \{0\})) = S(L_0 \times \{t_0(T)\})$$

$$\Rightarrow S \circ T(L_0 \times \{0\}) = S(L_0 \times \{t_0(T)\}).$$

Now, for any leaf L of $\mathcal{F}$ we have $S(L \times \{0\}) = L \times \{t_0(S)\}$. Therefore

$$S \circ T(L_0 \times \{0\}) = L_0 \times \{t_0(S) + t_0(T)\}.$$

This implies that $t_0(S \circ T) = t_0(S) + t_0(T)$. The injectivity of ξ we have already checked. $\square$

Finally we claim that ξ is surjective. Indeed, given any $t_0 \in \operatorname{Per}(L_0)$ and any $x_0 \in L_0$ we may consider paths $\alpha := \varphi_{st_0}(x_0)$ in M and β in L_0, joining $l_0 \ni \varphi_{t_0}(x_0)$ to x_0 because $t_0 \in \operatorname{Per}(L_0)$. The homotopy class

$$[\gamma] = [\alpha * \beta] \in \pi_1(M)$$

is such that if $T \in A$ corresponds to $[\gamma]$ then $T(L_0 \times \{0\}) = L_0 \times \{t_0(T)\}$ where $t_0(T)$ is given by

$$\begin{aligned} t_0(T) &= \int_\gamma \Omega = \int_\alpha \Omega + \int_\beta \Omega = \int_\alpha \Omega \\ &= \int_0^1 \Omega(\varphi_{st_0}(x_0)) \cdot \frac{d}{ds}\left(\varphi_{st_0}(x_0)\right) dx = t_0 \int_0^1 1 \cdot ds = t_0\,. \end{aligned}$$

Thus $t_0(T) = t_0$ and ξ is surjective. This shows (i) and (ii). $\square$

Lemma 8.0.29. $\mathrm{Per}(L_0)$ *is the group of periods of* Ω *which is defined by*

$$\mathrm{Per}(\Omega) := \left\{ \int_\gamma \Omega; [\gamma] \in \pi_1(M) \right\} < \mathbb{R}.$$

Proof. Let $t_0 \in \mathrm{Per}(L_0)$ and $[\gamma] = [\alpha * \beta]$ as above, then $\int_{[\gamma]} \Omega = t_0$ so that $t_0 \in \mathrm{Per}(\Omega)$. Conversely, given any period $t_0 \in \mathrm{Per}(\Omega)$ say $t_0 = \int_\gamma \Omega$ for some $[\gamma] \in \pi_1(M)$ we may perform small homotopies so that $[\gamma]$ is of the form $[\gamma] = [\alpha_1 * \beta_1 * \cdots * \alpha_r * \beta_r]$ with α_j segment of orbit of φ_t and β_j contained in a single leaf of $\mathcal{F}$, $\forall j \in \{1, \ldots, r\}$. Using the flow we may obtain a homotopy between $\beta_{r-1} * \alpha_r$ and some path of the form $\alpha * \beta$.
Using the flow: $\beta_{r-1} * \alpha_r$ is homotopic to some path of the form $\alpha * \beta$.
Therefore we may assume that $r = 1$, and $\gamma = \alpha_1 * \beta_1$. Therefore

$$t_0 = \int_\gamma \Omega = \int_{\alpha_1} \Omega \Rightarrow \alpha_1(t_0) = \varphi_{t_0}(x_0)$$

and $\alpha_1(0) = x_0$ belong t_0 a same leaf of $\mathcal{F}$ and therefore $t_0 \in \mathrm{Per}(L_0)$. This proves (iii) and Proposition 8.0.27. □

Corollary 8.0.30. *Let* $\mathcal{F}$, Ω, φ_t, A, $\mathrm{Per}(\Omega)$, *M compact be as above. The leaves of $\mathcal{F}$ are compact if, and only if,* $\mathrm{Per}(\Omega) \subset \mathbb{R}$ *has rank one and defines a lattice on* $\mathbb{R}$. *In any other case the leaves of $\mathcal{F}$ are not closed.*

Proof. We have already proved that $\mathrm{rank}(A) \geq 1$ and also $\mathrm{rank}(A) = 1$ and if, and only if, $\mathcal{F}$ is a compact foliation. Moreover $\mathrm{rank}(A) \geq 2$ implies $\mathrm{Per}(\Omega) \subset \mathbb{R}$ is not discrete in fact it is dense, what implies (see Remark 8.0.31) that A acts in the leaves of $\sigma^*(\mathcal{F})$ in $L_0 \times \mathbb{R}$ with non-discrete dynamics. This implies that the leaves of $\mathcal{F}$ are not closed. □

Remark 8.0.31. Let M be a compact differentiable manifold supporting a non-singular codimension one smooth foliation invariant by a transverse flow. Then $\pi_1(M)$ is not finite, indeed $\text{rank}(H_1(M,\mathbb{Z})) \geq 1$.

Indeed, if $\pi_1(M)$ is finite then the universal covering $\widetilde{M}$ of M is also compact so that the closed one-form Ω lifts into a closed non zero smooth 1-form $\widetilde{\Omega}$ on $\widetilde{M}$ which is exact, $\widetilde{\Omega} = d\tilde{f}$ for some smooth function $\tilde{f} \colon \widetilde{M} \to \mathbb{R}$. Since $\widetilde{M}$ is compact $\tilde{f}$ must exhibit some critical point is constant and Ω has some singularity, contradiction. □

8.0.3 Proof of the Tischler Fibration Theorem

Now we are in conditions to prove the Tischler Fibration Theorem.

Proof of the Tischler Fibration Theorem: We may assume that M is orientable and oriented. According to what we have seen above the foliation $\mathcal{F}$ is given by a non-singular smooth closed one-form Ω in M. We may find a basis $\{w_j = \mu^*(\alpha_j)\}$ of the group the De Rham cohomology group $H^1(M, \Omega^1)$ given by (classes of) closed 1-forms in M such that

for some loops $\gamma_1, \dots, \gamma_r$ corresponding to a basis of the free part of $H^1(M, \mathbb{Z})$ we have

$$\int_\gamma w_i = \delta_{ij}\,.$$

We may therefore write $\Omega = \sum\limits_{j=1}^{r} \lambda_j w_j + df$ for some $\lambda_j \in \mathbb{R}$, and some $f \colon M \to \mathbb{R}$ smooth. Then $\lambda_j = \int_{\gamma_j} \Omega$ so that $\{\lambda_1, \dots, \lambda_r\} \subseteq \operatorname{Per}(\Omega)$. Indeed $\operatorname{Per}(\Omega)$ is generated (as a group) by the λ_j's, $j = 1, \dots, r$, i.e., $\operatorname{Per}(\Omega) = \langle\{\lambda_1, \dots, \lambda_r\}\rangle$. Let now $(\lambda'_1, \dots, \lambda'_r) \in \mathbb{R}^r$ be such that $\Omega' := \sum\limits_{j=1}^{r} \lambda'_j \cdot w_j$ is close enough to Ω so that it is also non singular (recall that Ω is non singular and M is compact) and the subgroup $\langle\{\lambda'_1, \dots, \lambda'_r\}\rangle$ of $\mathbb{R}$ is a rank 2 discrete lattice (it is enough to choose $\{\lambda'_1, \dots, \lambda'_r\} \subseteq \mathbb{Q} + \sqrt{-1}\mathbb{Q}$ of rank 2). Thus Ω' defines a fibration of M over the complex torus $\mathbb{R}/\Lambda'$, $\Lambda' = \langle\{\lambda'_1, \dots, \lambda'_r\}\rangle$. $\square$

Chapter 9

Complex versions of classical results from the Theory of Foliations

9.1 Introduction

The Geometric Theory of Foliations has its origins in the classical works of C. Ehresmann [12] and G. Reeb [45], [46], [47]. Its diversity of applications and richness of techniques, congregating several areas in Mathematics as Topology, Geometry, Analysis and Dynamical Systems, has been fundamental in the development of various problems in Mathematics. We mention for instance the study and classification of real 3-manifolds. In this line we can cite some of the central results, already classic nowadays, of the Geometric Theory of Foliations:

1. "Stability Theorems" local and global, due to Reeb.

2. "Tischler Fibration Theorem".

3. "Haefliger's Theorem" for foliations of codimension one.

4. "Novikov compact leaf theorem" in S^3.

5. The "Rank Theorem" of E. Lima about the rank of S^3.

6. Theorems about foliations with homogeneous transverse structure.

7. Works of J. Plante and W. Thurston on the growth of foliations and groups and the existence of compact leaves.

We shall stop here with the alert that the above list is just one among may possible lists, but which is in the realm of the present text.

The notion of complex (holomorphic foliation) foliation by its turn is officially more recent though it is already implicit in the original works of P. Painlevé [38]. Its great development, specially in the last two decades, is also due to the successful use of modern techniques of Complex Geometry and Several Complex Variables.

In a certain sense a great part of the research in Complex Foliations is centered at local aspects of the theory as, for example, the study of singularities of holomorphic vector fields and forms. Such study is already a hard enough work and has shown to be very useful in general, nevertheless some global aspects of the theory also deserve special attention. The aim of this chapter is to expose what there is and motivate the reader for the study of the global aspects of the Geometric Theory of Complex Foliations, parting from the study and classical problems from the "real case" (cf. List 1,..., 7 above). For the reader interested in this approach we suggest the reading of [60].

9.2 Stability theorems

Let L_0 be a leaf of a foliation $\mathcal{F}$ of a manifold M. How can we relate L_0 and the neighbor leaves? Obviously, we must make some hypothesis. Let us therefore assume that L_0 is compact. From this we know that $\pi_1(L_0)$ and, therefore, the holonomy $\mathrm{Hol}(L_0)$ of L_0, are finitely generate. However, $\mathrm{Hol}(L_0)$ can be wild enough in order to impend any uniformity in the distribution of the leaves of $\mathcal{F}$ which are close to L_0. Thus we ask the following:

- $\mathrm{Hol}(L_0)$ is finite.

We obtain with this the already seen Reeb Stability Theorem, which is fundamental in the theory (cf. Chapter 4 Section 3):

Theorem 9.2.1 (Reeb Local Stability Theorem). *Let $\mathcal{F}$ be a foliation C^1 of codimension q of a manifold M having L_0 as a compact leaf with finite holonomy group. Then there exists a fundamental system of saturated neighborhoods V_j, $(j = 1, ..., \infty)$ of L_0 with the following properties:*

a. $\bigcap V_j = L_0\,, \quad V_{j+1} \subset V_j\,.$

b. *Each neighborhood V_j is a union of compact leaves each leaf having finite holonomy group.*

In fact, as we have seen, we can choose a saturated neighborhood V of L_0 where we can define a retraction $\pi\colon V \to L_0$ such that $\pi^{-1}(p) = D_p$ is a transverse disc to $\mathcal{F}$, $\forall\, p \in L_0$. Moreover, for each leaf $L \subset V$ the restriction $\pi|_L\colon L \to L_0$ is a finite covering map whose group of transformations corresponds to a subgroup of $\pi_1(L_0)$ and isomorphic to a subgroup of $\mathrm{Hol}(L_0)$.

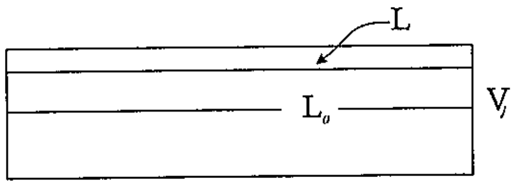


Figure 9.1: *neighborhood saturated by compact leaves*

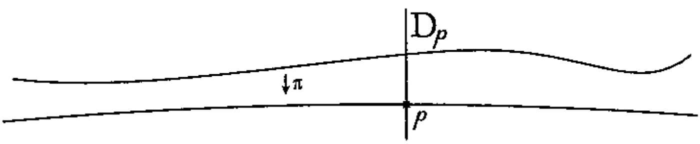


Figure 9.2: *covering of L_0 by neighbor leaves*

As an immediate consequence of this theorem we conclude that the set of compact leaves with finite holonomy is an open subset of the manifold M. Another interesting consequence is the following

Corollary 9.2.2. *Let $\mathcal{F}$ foliation C^1 of codimension q of M having a compact leaf L_0 with finite fundamental group. Then there exists a saturated open neighborhood V of L_0 with all leaves compact having finite fundamental group.*

In codimension 1 the Stability Theorem is much more precise as we recall below (cf. Chapter 4 Section 3):

Theorem 9.2.3 (Global Reeb Stability Theorem). *Let $\mathcal{F}$ be a foliation of class C^1 and codimension 1 of a manifold M*

compact and connected. If $\mathcal{F}$ has a compact leaf L_0 with finite fundamental group then all leaves of $\mathcal{F}$ are compact with finite fundamental group. In case $\mathcal{F}$ is moreover transversally orientable then it is enough to assume that L_0 is compact with $H_1(L_0;\mathbb{R}) = 0$ and, in this case, $\mathcal{F}$ is given by a submersion $f\colon M \xrightarrow{C^1} S^1$.

Complements to the statement:

1. In case $\mathcal{F}$ is not transversally orientable there exists a double covering $\sigma\colon \widehat{M} \to M$ with $\widehat{M}$ compact, equipped with a foliation $\widehat{\mathcal{F}} = \sigma^*\mathcal{F}$ of class C^1 and codimension 1, such that $\widehat{\mathcal{F}}$ is transversally orientable and has a leaf $\widehat{L}_0 \subset \sigma^{-1}(L_0)$ compact with finite fundamental group.

2. According to Ehresmann fibration Theorem (Section 1.2.9) a proper submersion of class C^2 $f\colon M \to N$ is a locally trivial fibration, hence if $\mathcal{F}$ in the theorem is of class C^2 then the submersion $f\colon M \to S^1$ defines $\mathcal{F}$ as a fibration of M over S^1.

3. The result for $H_1(L_0,\mathbb{R}) = 0$ is due to W. Thurston (cf. [67]). Clearly we have $\pi_1(L_0)$ finite $\Rightarrow H_1(L_0;\mathbb{R}) = 0$.

4. As a Corollary of the Global Stability Theorem we conclude that if L_0 is a compact leaf with $\pi_1(L_0)$ finite of the foliation $\mathcal{F}$ of class C^1 and codimension 1 then $\#\,\mathrm{Hol}(L_0) \leq 2$, also $\mathrm{Hol}(L_0) = \{\mathrm{Id}\}$ in case $\mathcal{F}$ is transversally orientable, the same holds for any leaf of L of $\mathcal{F}$. Finally, in the transversally orientable case, we conclude that all leaves of $\mathcal{F}$ are diffeomorphic. In particular we can state:

Corollary 9.2.4. *There exists no foliation $\mathcal{F}$ of class C^1 and codimension 1 of the sphere S^n, $n \geq 3$, having a leaf L_0 diffeomorphic to S^{n-1}.*

In fact, we know that does not exist a fibration $S^n \to S^{n-1}$ with fibre S^{n-1}; what follows for the exact sequence of homotopy of a fibration (cf. [65]).

9.2.1 From Real to Complex

Regarding the global stability of *non-singular* holomorphic foliations we have the following result of Brunella:

Theorem 9.2.5 ([2]). *Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension one on a compact connected manifold M. Assume that there exists a compact leaf L of $\mathcal{F}$ with finite holonomy. Then $\mathcal{F}$ is compact and stable (i.e., each leaf of $\mathcal{F}$ is compact with finite holonomy).*

Notice that by the Local Stability Theorem of Reeb if a compact leaf L of $\mathcal{F}$ has finite holonomy then it is stable. The global stability theorem above holds for complex codimension one transversely holomorphic foliations and is not true in general (in [2] one finds a counterexample for complex codimension two transversely holomorphic $\mathcal{F}$ on a 5-manifold M^5 (real)). However if M is compact and *Kähler* we have the following result of Vitório:

Theorem 9.2.6 ([71]). *Let $\mathcal{F}$ be a holomorphic foliation of codimension q in a compact complex Kähler manifold. If $\mathcal{F}$ has a compact leaf with finite holonomy group then $\mathcal{F}$ is compact and stable.*

Problem 9.2.7. *Let $\mathcal{F}$ be a holomorphic foliation of codimension $q \geq 1$ with a leaf compact L_0 with finite fundamental group. Give conditions for which the leaves L of $\mathcal{F}$ neighbors to L_0 are biholomorphically equivalent to L_0.*

9.2.2 Compact foliations and stability

Let $\mathcal{F}$ be a (non-singular) foliation of codimension $q \geq 1$ of a differentiable manifold M^n; we say that $\mathcal{F}$ is *compact* if all its leaves are compact. Regarding compact foliations we have the following result due to Reeb for $q = 1$ and to Epstein for $q \geq 2$ (cf. [48], [54], [15]).

Theorem 9.2.8. *Let $\mathcal{F}$ be a compact foliation on a connected manifold M. The following are equivalent conditions:*

(i) *The holonomy group of each leaf is finite;*

(ii) *Each leaf admits a fundamental system of saturated neighborhoods;*

(iii) *Given any leaf $L \subset M$ of $\mathcal{F}$ and any open neighborhood U of L in M there exists an open neighborhood $L \subset V \subset U$ of L such that V is saturated by $\mathcal{F}$;*

(iv) *The leaf space $\mathfrak{X}_{\mathcal{F}} = M/\mathcal{F}$ is hausdorff;*

(v) *For any compact (respectively closed) subset $K \subset M$ the saturation $Sat_{\mathcal{F}}(K)$ of K by $\mathcal{F}$ is compact (respectively closed);*

(vi) *The projection $M \to \mathfrak{X}_{\mathcal{F}} = M/\mathcal{F}$ is a closed map;*

(v) *There exists a Riemannian metric on M for which the volume of the leaves is locally bounded;*

(vi) *For any Riemannian metric on M the volume of the leaves is locally bounded.*

A compact foliation satisfying one of the (and therefore all the) conditions above is a *stable foliation* (cf. [15]). Examples of stable foliations are given by:

(1.) Compact Riemannian (and in particular real codimension one) foliations (cf. [53]).

(2.) Codimension two compact foliations on compact manifolds (cf. [55]).

(3.) Compact dimension one foliations defined by S^1 actions on connected (not necessarily compact) manifolds (cf. [15]).

(4.) Minimizable compact foliations (of class C^1) on *compact* manifolds are stable in particular holomorphic compact foliations on compact Kähler manifolds are stable (cf. [55]).

In general we have the following sufficient condition due essentially to Rummler-Edwards-Sullivan-Millet (cf. [52]).

(5.) If $\mathcal{F}$ is compact of class C^1 on M compact then $\mathcal{F}$ is stable provided that there exists some p-form ω on M ($p = \dim \mathcal{F}$) which is closed relatively to $\mathcal{F}$ and such that $\int_L \omega > 0$, $\forall$ leaf L of $\mathcal{F}$.

For the complex case we have the contributions of Holmann, Kaup, Miller and others:

(6.) A compact holomorphic foliation of complex dimension one and defined by a locally free action of $\mathbb{C}$ on a *compact* manifold M is stable (cf. [26]). If we drop the compactness condition for M then the result is not true.

(7.) A non-singular codimension one compact holomorphic foliation on a complex manifold M (compact or not) is stable (cf. [26]).

9.2.3 From Real to Complex

According to the terminology introduced in [58] a leaf L_0 of a singular holomorphic foliation $\mathcal{F}$ on a manifold M is a *compact singular leaf* if $\overline{L_o} \subset L_o \cup \text{sing}(\mathcal{F})$ and (therefore by Remmert-Stein theorem [18],[19]) $\overline{L_o}$ is compact analytic. The foliation $\mathcal{F}$ is a *compact singular foliation* if each leaf L_o of $\mathcal{F}$ is either compact or compact singular. We shall say that a leaf L of a compact singular foliation is *stable* if it admits a fundamental system of saturated neighborhoods in M.

Problem 9.2.9. *Study the stability of complex foliations with singularities with compact leaves with singularities.*

Example 9.2.10. Choose affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ and let $\mathcal{F}|_{\mathbb{C}^2}$ be given by $n\,xdy + my\,dx = 0; \quad m, n \in \mathbb{N} - \{0\}$. Then $\mathcal{F}$ has the rational first integral $R\colon \mathbb{C}P(2) \dashrightarrow \overline{\mathbb{C}}$ given by $R(x, y) = x^m y^n$. We have the following affine expressions for $\mathcal{F}$ on $\mathbb{C}P(2)$: $x^m y^n = cte$, $v^n = cte.u^{n+m}$, $s^m = cte.r^{n+m}$.

Hence $\mathcal{F}$ is clearly a compact singular foliation, $\mathbb{C}P(2)$ is Kählerian each leaf of $\mathcal{F}$ has a finite virtual holonomy group but $\mathcal{F}$ is *not* stable in the usual sense. On the other hand $\mathcal{F}$ is *not* locally integrable because some of its singularities are dicritical, admit meromorphic but no open holomorphic (local) first integral.

Remark 9.2.11. Let $C \subset \mathbb{C}P(2)$ be an algebraic curve invariant by some algebraic foliation $\mathcal{F}$ on $\mathbb{C}P(2)$; by the Index Theorem [32] $\mathcal{F}$ is not locally integrable at all the singularities $p \in \text{sing}(\mathcal{F}) \cap C$: indeed, the Camacho-Sad indexes associated to C and such a singularity are negative rational numbers while the self-intersection $C \cdot C$ is a positive integer number.

Example 9.2.12. On $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ we define $\mathcal{F}$ by $\mathcal{F}|_{\mathbb{C}^2} : x^2 + y^2 = \text{cte}$ for $n, m \in \mathbb{N} - \{0\}$. After an analysis of the singularities of $\mathcal{F}$

one sees that, as in Example 9.2.10, $\mathcal{F}$ is compact singular, the ambient manifold $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is Kählerian, but no leaf is stable, and $\mathcal{F}$ is not locally integrable.

Problem 9.2.13. *Characterize the stable compact singular holomorphic foliations on the complex projective space* $\mathbb{C}P(n)$.

9.3 Tischler Fibration Theorem and fibering of complex manifolds

The well-known Tischler Fibration Theorem [69] states that a closed real differentiable manifold M fibers overs the circle S^1 iff M supports a closed non-singular differentiable one form Ω. Such a one form defines a codimension one foliation $\mathcal{F}$, without holonomy, which is invariant under the transverse flow φ_t of a vector field X defined on M and satisfying $\Omega \cdot X = 1$. Conversely, any foliation of codimension one invariant under some transverse flow φ_t (of X) is given by a closed one form Ω with $\Omega \cdot X = 1$ (see [15] pages 45-47 and [42] Proposition 2.3 pages 737-738). As a consequence, the foliation $\mathcal{F}$ is either a compact foliation or has all leaves dense in M (cf. [42] Proposition 2.7 page 741). In [42] the author applies these techniques in the study of differentiable Anosov flows. It is proved therein that any *jointly integrable* Anosov flow in M admits a smooth section and is topologically conjugate to a suspension of some Anosov diffeomorphism, which is a total automorphism in the codimension one case (see Theorem 3.1 page 744 and Theorem 3.7 page 746). We recall that an Anosov flow $\varphi_t \colon M \to M$ with corresponding splitting $TM = E^u \oplus E^s \oplus E^T$ is *jointly integrable* if the bundle $E^u \oplus E^s$ is integrable, generating therefore a codimension one foliation $\mathcal{F}$ which is invariant under the transverse flow φ_t; giving this way the link Tischler's Theorem above.

Finally, in [14] the author states the definition of holomorphic Anosov flow on a complex manifold in terms of actions of the multiplicative group $\mathbb{C}^*$ (cf. [14] page 586). In this same work, Anosov flows on compact complex 3-manifolds are classified in holomorphic way. This is done using strongly the fact that the stable and unstable foliations, $\mathcal{F}^s$ and $\mathcal{F}^u$, are holomorphic foliations with projective transverse structures. The holomorphy of $\mathcal{F}^s$ is a consequence of the dimension 3 assumption.

9.3.1 From Real to Complex

9.3.2 Motivation

Let M be a complex manifold equipped with a closed homomorphic one-form Ω. Then we may write $\Omega = \alpha + i\beta$ for some closed real analytic one-forms α and β in M. Indeed, assume for simplicity that M has dimension two and take local coordinates (x, y) in M. We may write locally $\Omega = df$ for some holomorphic function $f(x, y)$. Write now $f = u + iv$ with $u(x, y)$, $v(x, y)$ real functions and write also $x = x_1 + ix_2$, $y = y_1 + iy_2$, where $i^2 = -1$. Then $\Omega = du + idv = (u_\alpha\, dx + u_y\, dy) + i(v_x\, dx + v_y\, dy)$. Now $u_x = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x_i} - i\,\dfrac{\partial u}{\partial x_2}\right)$ and $dx = dx_1 + i\, dx_2$ so that

$$u_x\, dx = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x_1}\, dx_1 + \frac{\partial u}{\partial x_2}\, dx_2\right) + i\left(\frac{\partial u}{\partial x_1}\, dx_2 - \frac{\partial u}{\partial x_2}\, dx_1\right)\right].$$

So that

$$\begin{aligned}
\Omega = \frac{1}{2}\Bigg(&\left(\frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2\right) + i\left(\frac{\partial u}{\partial x_1}\,dx_2 - \frac{\partial u}{\partial x_2}\,dx_1\right)\Bigg] + \\
+ \frac{1}{2}\Bigg[&\left(\frac{\partial u}{\partial y_1}\,dy_1 + \frac{\partial u}{\partial y_2}\,dy_2\right) + i\left(\frac{\partial u}{\partial y_1}\,dy_2 - \frac{\partial u}{\partial y_2}\,dy_1\right)\Bigg] + \\
+ \frac{1}{2}i\Bigg[&\left(\frac{\partial v}{\partial x_1}\,dx_1 + \frac{\partial v}{\partial x_2}\,dx_2\right) + i\left(\frac{\partial v}{\partial x_1}\,dx_2 - \frac{\partial v}{\partial x_2}\,dx_1\right)\Bigg] + \\
+ \frac{1}{2}i\Bigg[&\left(\frac{\partial v}{\partial y_1}\,dy_1 + \frac{\partial v}{\partial y_2}\,dy_2\right) + i\left(\frac{\partial v}{\partial y_1}\,dy_2 - \frac{\partial v}{\partial y_2}\,dy_1\right)\Bigg]
\end{aligned}$$

Thus we may take

$$\begin{aligned}
\alpha = \frac{1}{2}\Bigg\{&\Bigg[\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2}\right)dx_1 + \left(\frac{\partial u}{\partial x_2} - \frac{\partial v}{\partial x_1}\right)dx_2\Bigg] + \\
&+ \Bigg[\left(\frac{\partial u}{\partial y_1} + \frac{\partial v}{\partial y_2}\right)dy_1 + \left(\frac{\partial u}{\partial y_2} - \frac{\partial v}{\partial y_1}\right)dy_2\Bigg]\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
\beta = \frac{1}{2}\Bigg\{&\Bigg[\left(-\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)dx_1 + \left(\frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_2}\right)dx_2\Bigg] + \\
&+ \Bigg[\left(-\frac{\partial u}{\partial y_2} + \frac{\partial v}{\partial y_1}\right)dy_1 + \left(\frac{\partial u}{\partial y_1} - \frac{\partial v}{\partial y_2}\right)dy_2\Bigg]\Bigg\}
\end{aligned}$$

Since $f = u + iv$ is holomorphic, u and v are harmonic and satisfy the Cauchy-Riemann equations. The harmonicity of u and v implies that α and β are closed.

Using the Cauchy-Riemann equations we obtain that $\alpha = du(x_1, x_2, y_1, y_2)$ and $\beta = dv(x_1, x_2, y_1, y_2)$ in real coordinates. Now, for any other pair α', β' with $\Omega = \alpha' + i\beta'$ we have $(\alpha - \alpha') + i(\beta - \beta') = 0$ so that $\alpha = \alpha'$ and $\beta = \beta'$. Therefore α and β are globally well-defined. We may write $\alpha = \mathrm{Re}(\Omega)$ and

$\beta = \text{Im}(\Omega)$ and refer to these as the *real part* and the *imaginary part* of Ω in M. Assume now that Ω is non-singular. Then α and β are non-singular as it is easily checked from the expressions above. Now, according to Tischler's Theorem [15], M fibers over the circle S^1 provided that M is compact.

The holomorphic foliation $\mathcal{F}$ given by $\Omega = 0$ may be regarded as the intersection of the two real analytic foliations $\text{Re}(\mathcal{F})$: $\alpha = 0$ and $\text{Im}(\mathcal{F})$: $\beta = 0$, which are codimension one real foliations defined by closed one forms.

Assume that M is compact, then we can find complete real vector fields X and Y over M satisfying $\text{Re}(\Omega) \cdot X = 1$ and $\text{Im}(\Omega) \cdot Y = 1$. Therefore $\text{Re}(\mathcal{F})$ is invariant under the transverse flow X_t of X and $\text{Im}(\mathcal{F})$ is invariant under the transverse flow Y_t of Y.

If X and Y commute then we may define an action of $\mathbb{R}^2$ on M by setting $(s,t), x \mapsto X_s \circ Y_t(x)$ for all $(s,t) \in \mathbb{R}^2$ and $x \in M$. Therefore we have a differentiable action of $\mathbb{C}$ on M defined by $(s+it), x \mapsto X_s \circ Y_t(x)$ and we shall denote by Z the "complex" vector field defined by this action. We have $Z = X + iY$. Denote by J the *complex structure* tensor field of M.

Lemma 9.3.1. *$X + iY$ defines a holomorphic vector field on M if and only if $[X,Y] = 0$, $J(X) = -Y$ and $J(Y) = X$.*

We may always define a meromorphic vector field $Z = X + iY$ in M with $\Omega \cdot Z = 1$ (Lemma). If Z is complete and holomorphic then $\mathcal{F}$ is invariant by the holomorphic transverse flow of $Z = X + iY$. Nevertheless since X and Y are not necessarily unique this is not the unique case. In general, however we do not have even an $\mathbb{R}^2$-action, which is differentiable, transverse to $\mathcal{F}$ and leaves $\mathcal{F}$ invariant.

Since $\text{Re}(\mathcal{F})$ and $\text{Im}(\mathcal{F})$ are transverse and M is compact (the proof of) Tischler's Theorem implies that we have two independent fibrations $\pi_j \colon M \to S^1$ $(j = 1, 2)$ so that M fibers over the Torus $S^1 \times S^1$. In particular we have $\operatorname{rank} H_1(M, \mathbb{Z}) \geq 2$. If $\operatorname{rank} H_1(M, \mathbb{Z}) = 2$ then if we denote by ${}_j$ $(j = 1, 2)$ the group of periods of $\Omega_1 = \operatorname{Re}(\Omega)$, $\Omega_2 = \operatorname{Im}(\Omega)$, we obtain $\operatorname{rank}(A_1) + \operatorname{rank}(A_2) = 2$ and since $A_1 \neq \{0\}$ necessarily we have $\operatorname{rank}(A_1) = \operatorname{rank}(A_2) = 1$.

Since A_j is free we obtain $A_1 \simeq \mathbb{Z} \simeq A_2$ and therefore Ω_j defines already a fibration $\pi_j \colon M \to S^1$, $j = 1, 2$. In particular $\mathcal{F}_j : \Omega_j = 0$ is a foliation by compact leaves and $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is also a compact foliation. This is the case if $\operatorname{rank} H_1(M, \mathbb{R}) = 2$. Thus we have proved:

Theorem 9.3.2. *Let $\mathcal{F}$ be a holomorphic foliation given by a closed (non-singular) holomorphic 1-form Ω on a connected manifold M. Then:*

(i) *$\mathcal{F}$ is the intersection of two transverse codimension one foliations $\mathcal{F}_1$, $\mathcal{F}_2$ given by closed real 1-forms Ω_1, Ω_2 on M.*

(ii) *Assume that M is compact. The manifold M fibers over the Torus $S^1 \times S^1$ in a C^∞-fibration. We have* $\operatorname{rank} H_1(M, \mathbb{Z}) \geq 2$. *if* $\operatorname{rank} H_1(M, \mathbb{Z}) = 2$ *then $\mathcal{F}$ is a compact foliation.*

Question 9.3.3. *Is the fibration $M \to S^1 \times S^1$ in* (ii) *above, holomorphic for some complex structure in the Torus $S^1 \times S^1$?*

Problem 9.3.4. *Study the holomorphic foliations invariant by holomorphic flow, more generally, transverse actions of Lie group actions, in compact complex manifolds.*

Preliminary results in this direction are obtained in [59] together with some applications which are given to the study of holomorphic Anosov flows, inspired in [42]. In particular we have the following questions:

Question 9.3.5. (i) Let M be a compact complex manifold supporting a non-singular closed holomorphic 1-form Ω. Under which conditions M fibers over a complex 1-torus? Denote by $\mathcal{F}$ the codimension one foliation defined by Ω. What is the structure of the leaves of $\mathcal{F}$? (ii) Consider now M a compact complex manifold admitting a codimension one holomorphic foliation $\mathcal{F}$ invariant under some holomorphic transverse flow φ_t. What is the transverse dynamics of $\mathcal{F}$?

9.3.3 Examples

We consider a compact complex manifold M of dimension n and a complex Lie group G of dimension k. Assume we have a (proper) holomorphic map $\pi\colon M \to G/H$ from M onto an homogeneous space G/H where H is a closed Lie subgroup of G, then G/H is compact. We also assume that π is a submersion. Denote by $\mathcal{F}$ the codimension k holomorphic foliation on M whose leaves are the fibers of π. By the Theorem of Ehresmann (Theorem 1.2.38) $\mathcal{F}$ is a C^∞-locally trivial foliation, nevertheless it may be non holomorphically locally trivial, actually the leaves of $\mathcal{F}$ may be non holomorphically equivalent. If these leaves are holomorphically equivalent then we may ask for the existence of a holomorphic action $\varphi\colon G \times M \to M$ of G on M, such that the orbits of φ are the leaves of $\mathcal{F}$. This is the situation we will mainly refer to in this section. Given a foliation $\mathcal{F}$ of codimension k on M and a Lie group action $\varphi\colon G \times M \to M$ on M we shall say that *$\mathcal{F}$ is invariant under the transverse action φ* if:

(i) $\varphi_g^*(\mathcal{F}) = \mathcal{F}, \quad \forall \varphi_g\colon M \to M$ given by $\varphi_g(p) = \varphi(g)$.

(ii) $\mathcal{F}$ is transverse to the orbits of φ.

In other words each automorphism φ_g, induced by the action, takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}$ and moreover the Lie algebra of φ is everywhere transverse to the leaves of $\mathcal{F}$. Let us see some examples.

1. The first obvious example of such situation is the product action $\varphi\colon G\times(G\times N)\to G\times N$ of a Lie group G on $M=G\times N$ where N is any manifold. The foliation $\mathcal{F}$ on M whose leaves are of the form $\{g\}\times N,\quad g\in G$, is invariant under the transverse action φ of G on $G\times N$. The manifold $M=G\times N$ fibers (holomorphically trivially) over G with fibers diffeomorphic to N and therefore to the leaves of $\mathcal{F}$.

2. A less trivial example is given by a closed (normal) subgroup $H\lhd G$ of a Lie group G. Consider the natural action $\varphi\colon H\times G\to G$ and the canonical fibration $\pi\colon G\to G/H$ whose corresponding foliation we denote by $\mathcal{F}$. Clearly $\mathcal{F}$ is invariant under the transverse action φ. More precisely we consider the following situation: G is a simply-connected Lie group, $H<G$ is a discrete subgroup such that G/H is compact. We consider the natural representation $\varphi\colon H\to \operatorname{Diff}(G)$ given by left translations $\varphi(h)=L_h\colon G\to G$. Notice that $\pi_1(G/H)\simeq H$ and the universal covering of G/H is given by the canonical projection $\pi\colon G\to G/H$. Thus we actually have a representation

$$\varphi\colon \pi_1(G/H)\to \operatorname{Diff}(G)\quad\text{and}\quad (\widetilde{G}/H)=G.$$

The natural action, of $\pi_1(G/H)$ on $\widetilde{G}/H$ is therefore given by φ above. We may therefore proceed constructing the suspension of φ as in the classical framework ([15] page 14). The foliation $\mathcal{F}$ is clearly invariant by the natural. There is a natural action of $H\cong\pi_1(G/H)$ on $(\widetilde{G}/H)\times G$ given by $\widetilde{\varphi}\colon H\times(G\times G)\to G\times G,\ h,(g_1,g_2)\mapsto(L_h(g_1),L_h(g_2))$. Since H is discrete this action is properly discontinuous and we obtain a quotient manifold $\frac{G\times G}{\widetilde{\varphi}}=: M$ with the following properties:

(i) M admits a fibration $\xi\colon M \to G/H$ induced by the projection $\pi\colon G \to G/H$, with fiber G and structure group isomorphic to the image $\varphi(H) < \mathrm{Diff}(G)$.
(ii) If we consider the foliation $\mathcal{F}_0$ of G whose leaves are the lateral classes of H in G (i.e., $\mathcal{F}_0$ is given by the projection $\pi\colon G \to G/H$) then $\mathcal{F}_0$ is invariant by the action φ of $\pi_1(G/H)$ on G and therefore the product foliation $G \times \mathcal{F}_0$ on $G \times G = (\widetilde{G}/H) \times G$, is invariant by the action $\widetilde{\varphi}$ of H on $G \times G$ and it induces a foliation $\mathcal{F}$ on M, called the *suspension* of $\mathcal{F}_0$ by $\varphi\colon H \to \mathrm{Diff}(G)$, which is transverse to the fibration $\xi\colon M \to G/H$.
The foliation $\mathcal{F}$ is clearly invariant by the natural action of G on M which is transverse to $\mathcal{F}$. Finally, $\mathcal{F}$ is given by a fibration $\eta\colon M \to G/H$ which is given by the second coordinate projection $\pi_2\colon G \times G \to G$.

3. Finally we may construct examples with H, G like above and with representations $\varphi\colon H \to \mathrm{Diff}(F)$ of H on the group of diffeomorphisms of a compact complex manifold F. This gives fibrations $M \xrightarrow{F} G/H$.

Problem 9.3.6. *Let M be a compact complex manifold equipped with a holomorphic foliation invariant under the transverse action of a complex Lie group G in M. Describe the dynamics of $\mathcal{F}$. Does M fibers over some quotient G/H ?*

9.4 Transverse sections and Haefliger's Theorem

In 1958 A. Haefliger proved the following result (cf. Chapter 5):

Theorem 9.4.1. *Let $\mathcal{F}$ foliation C^2 of codimension 1 of the manifold M admitting a closed curve $\gamma \in \pi_1(M)$ with the following properties:*

(i) *γ is transverse to $\mathcal{F}$.*

(ii) *γ is homotopic to a point (free homotopy) in M.*

Then there exist a leaf $L_0 \subset M$ of $\mathcal{F}$ and a closed path $\alpha \in \pi_1(L_{0;0})$, with base in a point $p_0 \in L_0$, whose corresponding element holonomy group $\mathrm{Hol}(L_0, \Sigma_0)$ relative to small transverse segment $\Sigma_0 \pitchfork L_0$, $\Sigma_0 \cap L_0 \ni p_0$ is the identity in one of the components of $\Sigma_0 - \{p_0\}$ and a contraction (different of the identity) in the other.

As an immediate consequence we obtain:

Corollary 9.4.2. *Let $\mathcal{F}$ be an analytic foliation of codimension 1 in M. Then the leaves of $\mathcal{F}$ are transverse only to non-trivial elements $\gamma \in \pi_1(M)$.*

Figure 9.3 below gives us a pictorial description/outline of the situation prescribed by Haefliger's Theorem:

The very rough (but useful) idea regarding the proof of Haefliger's Theorem is the following: We can assume that the curve $\gamma\colon S^1 \to M$ is of class C^2 and transverse a $\mathcal{F}$; where γ is homotopic to a point and γ is the boundary of a "deformed disc" $D \subset M$ (see Figure 9.4).

We can then intersect the leaves of $\mathcal{F}$ with D obtaining a singular foliation real curves in D. In a more formal way, there exists an embedding of class C^2 $f\colon \overline{\mathbb{D}} \to M$, where $\overline{\mathbb{D}} = \{(x,y) \in \mathbb{R}^2;\, x^2 + y^2 \leq 1\}$ is the closed unit disc with boundary $\partial\overline{\mathbb{D}} = S^1$, such that:

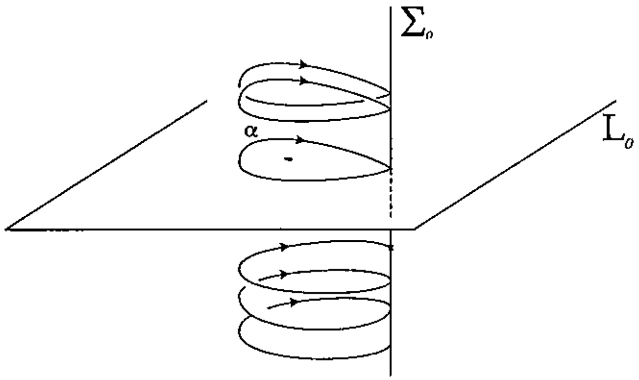


Figure 9.3:

a. $f(\partial\overline{\mathbb{D}}) = \gamma$

b. $f^*\mathcal{F}$ is a foliation of dimension 1 with singularities of "Morse" type.

Such singularities, unavoidable, come from tangency points of $\mathcal{F}$ with f along $\mathbb{D} = \overline{\mathbb{D}}\backslash S^1$.

The picture above illustrates such a tangency. We obtain then a field of vectors X of class C^1 in a neighborhood U of $\overline{\mathbb{D}}$ in $\mathbb{R}^2$ with the following properties:

a'. X is transverse and points inwards $\overline{\mathbb{D}}$ along $S^1 = \partial\overline{\mathbb{D}}$.

b'. The singularities of X in $\mathbb{D}$ corresponds to centers ($d(x^2 + y^2) = 0$) and saddles ($d(x^2 - y^2) = 0$).

Arguments from Differential Topology permit to disturb the application f in a way that we obtain the singular points (i.e.,

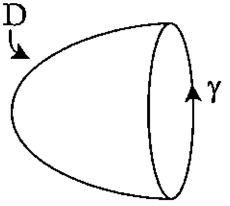


Figure 9.4:

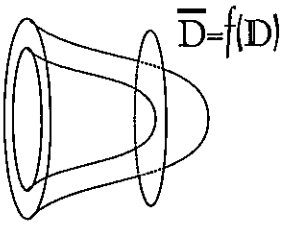


Figure 9.5:

the tangency points of f with $\mathcal{F}$) belong to distinct leaves of $\mathcal{F}$ and applying the Poincaré-Bendixson theorem to $X|_{\mathbb{D}}$ we obtain a limit cycle Γ which is a graph and which, by the absence of saddle-connections, must correspond to one of the above illustrated situations.

Ordering the set of such limit cycles Γ by the natural order defined by the inclusion we obtain, via the Lema of Zorn, a closed orbit α with the following properties (see Fig. 9.10):

c. α is interior limit limit cycles $\mathcal{F}_n \searrow \alpha$

d. α is boundary of a region R in $\mathbb{D}$ which contains some singularity of type center $p_0 \in R$

Figure 9.6:

e. By the minimality of α the orbits concentric to p_0 extend to α.

Drawing a segment Σ_0 transverse to α by a point $a_0 \in \Sigma_0 \cap \alpha$ we obtain that the first return map corresponding to $\pi\colon \Sigma_0, a_0 \to \Sigma_0, a_0$ satisfies $\pi|_{\Sigma_0 \cap R} =$ Id and $\pi|_{\Sigma_0 \cap R^c}$ is a contraction (Figure 9.11). ■

Some consequences of Haefliger's Theorem may be listed as follows:

1. There exists no analytic foliation of codimension one in a compact manifold t M with finite fundamental group. In particular, S^n does not admit a codimension one analytic foliation for $n \geq 2$.

2. If $\mathcal{F}$ is an analytic foliation of codimension one of a manifold M with finite fundamental group then each leaf of $\mathcal{F}$ is closed in M. In particular, the space of leaves $M/\mathcal{F} = \mathfrak{X}$ is a 1-manifold analytic orientable (in general non-separable) and the canonical projection $M \to \mathfrak{X}$ is an analytic submersion.

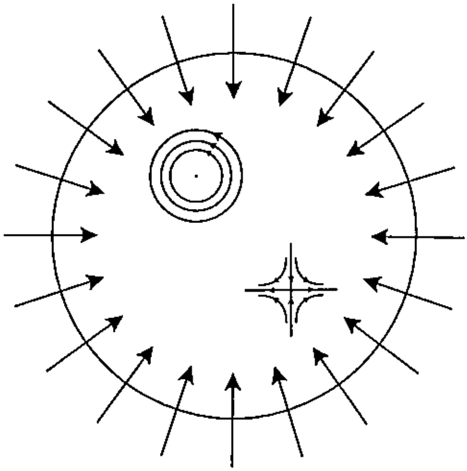


Figure 9.7:

Proof (essentially): 1. and 2. above follow immediately from the following

Lemma 9.4.3. *Let $\mathcal{F}$ be a foliation of codimension 1 of M. If $\mathcal{F}$ has a leaf non-compact then there exists a closed curve $\gamma\colon S^1 \to M$ transverse to $\mathcal{F}$.*

Proof of the Lemma: Let L_0 be a leaf not compact of $\mathcal{F}$. Since M is compact there exists $p \in \overline{L}_0 \backslash L_0$. We denote by L_p a leaf of $\mathcal{F}$ by $p \in M$. We trivialize $\mathcal{F}$ in a neighborhood U of p in M (Fig. 9.12).

Let Σ be a section transverse to $\mathcal{F}$ in U with $p \in \Sigma \cap L_p$. Then $\Sigma \cap L_0$ accumulates in p along a sequence $\{p_n\}$, $p_n \to p$. We

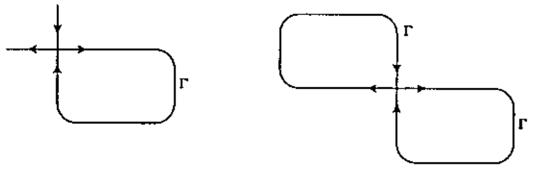


Figure 9.8:

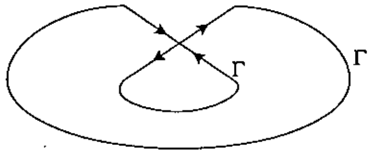


Figure 9.9:

take two sufficiently close terms p_n and p_m (Fig. 9.13).

Join now p_n and p_m by a suitably chosen path $a\colon [0,1] \to L_0$ obtaining, together with the arc of Σ between p_n and p_m, a continuous curve and piecewise C^∞ C, closed, and such that as in the figure below (simplified).

Take now points in the local sections defined by Σ in p_n and p_m and "replace" a by a path $\tilde{a}$ transverse to $\mathcal{F}$ along a fibration by segments of line with base in the trace of α and which contains (the fibration) the parts of Σ by p_n and p_m.

We then make smooth the final curve in order to obtain a smooth closed transverse section closed of $\mathcal{F}$ ■

Remark 9.4.4. In case $\mathcal{F}$ has all its leaves compact it can be

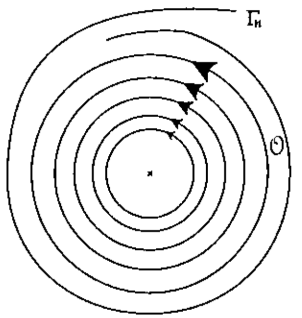


Figure 9.10:

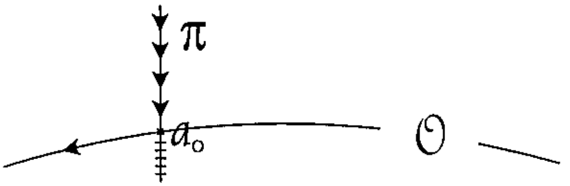


Figure 9.11:

easily proved the existence of a closed transverse curve/section (exercise!).

9.4.1 From Real to Complex

9.4.2 Introduction and Motivation

Haefliger's result has been extended by Plante and Thurston as follows:

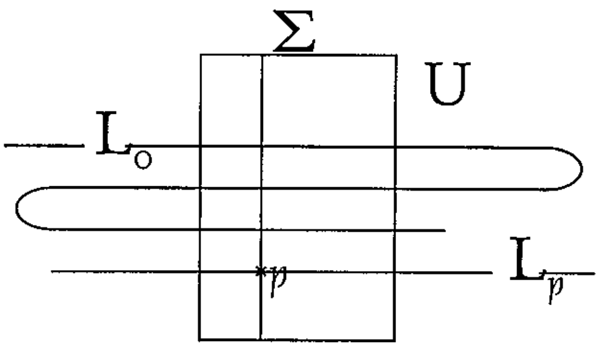


Figure 9.12:

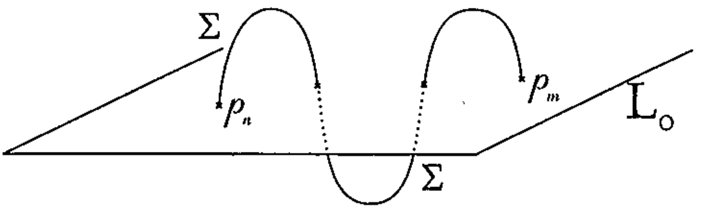


Figure 9.13:

Theorem 9.4.5 (Plante-Thurston). *Let M be a real compact manifold such that $\pi_1(M)$ has polynomial growth. If M admits a real regular codimension one analytic transversely oriented foliation then $H^1(M, \mathbb{R}) \neq 0$.*

Problem 9.4.6. *Is there any kind of relation between the topology of the ambient manifold and the obstruction to the existence of non-singular codimension one foliations also in the complex case though no result like Haefliger's theorém is known yet?*

For instance we have the following question:

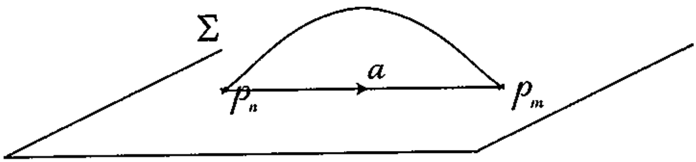


Figure 9.14:

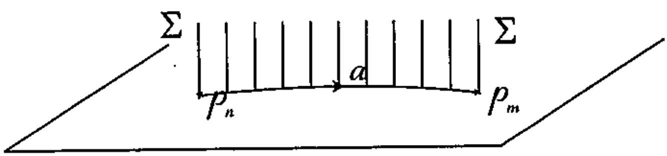


Figure 9.15:

Problem 9.4.7. *Let M be a compact complex surface. Is there any holomorphic regular foliation $\mathcal{F}$ by curves on M?*

In the course of the proof of Haefliger's Theorem one is led to consider real vector fields in a neighborhood of the closed disc $\overline{D^2} \subset \mathbb{R}^2$ which are transverse to the boundary $\partial D^2 \simeq S^1$. The use of Poincaré-Bendixson Theorem shows the existence of some unilateral hyperbolicity, for some closed orbit $\gamma \subset D^2$, what is not compatible with the analytic behavior. These ideas are of capital importance in the study of the theory of codimension one real foliations what, by its turn, is a very useful tool in the classification theory of differentiable manifolds of dimension ≥ 3.

Unfortunately, there is no feature like the classical Poincaré-Bendixson Theorem in the case of holomorphic vector fields. In particular it is not know whether a leaf of a holomorphic foliation by curves on the complex projective plane $\mathbb{C}P(2)$, such a foliation always comes from the extension of one induced by a

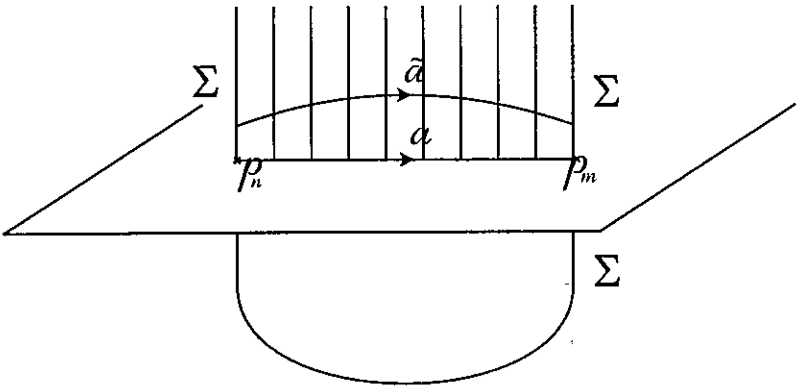


Figure 9.16:

polynomial vector field on $\mathbb{C}^2$, may be *exceptional* in the sense that it accumulates no singularity of the foliation.

The facts above are the very first motivation for the first basic question below:

Question 9.4.8. *What can be said of a holomorphic vector field Z, in a complex manifold M^n, that is transverse to the boundary $\partial\Omega$ of some simply-connected regular domain $\Omega \subset\subset M$?*

At this level of generality, this seems to be a very hard question. Several difficulties arise. First of all the (already mentioned) absence of a holomorphic version of Poincaré-Bendixson Theorem. A second difficulty is the existence of domains $\Omega \subset M$ as above, non-diffeomorphic (even C^1) to each other and, in particular, non-diffeomorphic to the ball $B^4(1) \subset \mathbb{R}^4$. Moreover, we have, in the complex setting, natural domains to be considered which are not regular (at the boundary), as polydiscs for instance $\Delta^2 \subset \mathbb{C}^2$. Therefore, Question 9.4.8 should be somehow extended to such domains. Finally, we should extend this problem to the codimension one case. We obtain therefore the very general question below:

Question 9.4.9. *Let $\mathcal{F}$ be a holomorphic foliation on M^n and assume that $\mathcal{F}$ is transverse to some real closed submanifold $N_{\mathbb{R}} \subset M$. Then, what can be said about $\mathcal{F}$ and $N_{\mathbb{R}}$?*

Numerous examples indicate that the difficulties mentioned above may be overcame in the case we make some further restrictions on Ω or $N_{\mathbb{R}}$. For instance, in [27] it is proved that if a holomorphic vector field Z in a neighborhood of the closed ball $\overline{B^{2n}}(R) = \{z \in \mathbb{C}^n;\ |z| \leq R\}$, is transverse to the sphere $S^{2n-1}(R) = \partial\overline{B^{2n}}(R) = \{z \in \mathbb{C}^n;\ |z| = R\}$, then such vector field exhibits only one singularity $o \in B^{2n}(R)$, which is in the Poincaré-domain. In the sake of generalizations of this result, we consider in §1 the following situation: $\mathcal{F}$ is a codimension-one foliation on a neighborhood U of the closed ball $\overline{B^{2n}(1)} \subset \mathbb{C}^n$; and we investigate the transversality of $\mathcal{F}$ with the sphere $S^{2n-1}(1)$. We *do not know* whether we may have $\mathcal{F} \pitchfork S^{2n-1}(1)$ with $n \geq 3$

We consider now the following problem:

Problem 9.4.10. (i) *Let $\mathcal{F}_{\mathbb{C}}$ be a holomorphic foliation of complex codimension one in the (compact or not) complex manifold $M_{\mathbb{C}}$. Is it possible to have a compact complex submanifold $R_{\mathbb{C}}$ of complex dimension 1 $R_{\mathbb{C}} \subset M$ such that $R_{\mathbb{C}}$ is transverse to $\mathcal{F}$ in M? Are there restrictions in the topology of $M_{\mathbb{C}}$?*
(ii) *Let now $\mathcal{F}_{\mathbb{C}}$ be holomorphic of codimension 1 in $M_{\mathbb{C}}$ and $N_{\mathbb{R}} \subset M_{\mathbb{R}}$ a closed real submanifold real (compact without boundary). Is it possible that we have $\mathcal{F}_{\mathbb{C}} \pitchfork N_{\mathbb{R}}$?*

Example 9.4.11. *Let X_λ be a linear complex vector field in $\mathbb{C}^2$ of the form $X_\lambda(x,y) = x\,\dfrac{\partial}{\partial x} + \lambda y\,\dfrac{\partial}{\partial y}\cdot$. Then if $\lambda \notin \mathbb{R}_-$ it is easy* *to verify that the foliation $\mathcal{F}_{X_\lambda}$ induced by X in $\mathbb{C}^2 - \{(0,0)\}$ is transverse to the spheres $S^3(\varepsilon) = \{(x,y) \in \mathbb{C}^2; |x|^2 + |y|^2 = \varepsilon^2\}$, $\varepsilon > 0$.*

Indeed, it is possible to prove that if a holomorphic vector field $X(x,y)$ in a neighborhood of the origin $(0,0) \in \mathbb{C}^2$ satisfies $\underset{(1^{st}\text{ jet })}{j^1_{(0,0)}} X = X_\lambda$ then in fact $\mathcal{F}_X \pitchfork S^3(\varepsilon)$ for all $\varepsilon > 0$ small enough. The converse of this fact is due to T. Ito and A. Douady (cf. [27]).

Theorem 9.4.12. *Let X be a holomorphic vector field in the open set $U \subset \mathbb{C}^n$, $n \geq 2$. Suppose that there exists a sphere $S^{2n-1}_{(p;R)} \Subset U$ such that $\mathcal{F}_X \pitchfork S^{n-1}(p;R)$ then:*

(i) $\mathcal{F}_X \pitchfork S^{2n-1}(p;R') \quad \forall 0 < R' < R$

(ii) $j^1_p X$ *belongs to Poincaré domain*

(iii) $\mathcal{F}_X|_{B^{2n}(p;R)-\{p\}}$ *is C^∞ equivalent to the product foliation* $\mathcal{F}_X|_{S^{2n-1}_{(p;R)}} \times [0,1)$.

The original theorem is more precise, it allows to understand the dynamics of $\mathcal{F}_X$ in $B^{2n}(p;R)$. In the above case $\mathcal{F}$ is a foliation of complex dimension 1; the codimension one case is studied in [28] where several obstruction results are proved for the existence of such transverse spheres. The results in [27] and [28] enforce the following conjecture:

Problem 9.4.13. *Let $\mathcal{F}$ be a holomorphic foliation of codimension in $\mathbb{C}^n$, $n \geq 3$. Show that $\mathcal{F}$ is not transverse to the sphere $S^{2n-1}(0;R)$.*

9.5 Growth and compact leaves

9.5.1 Novikov Compact Leaf Theorem revisited

The celebrated Novikov Compact Leaf Theorem states that every foliation of codimension 1 and class C^1 of a real compact 3-manifold real admitting a *vanishing cycle* has a compact leaf (cf. Chapter 6). In particular we have (see Chapter 6)

Theorem 9.5.1 (Novikov, 1964). *Let $\mathcal{F}$ be a foliation of class C^2 and codimension one of a 3-manifold compact M^3 with finite fundamental group. Then $\mathcal{F}$ has a compact leaf.*

In fact $\mathcal{F}$ has a "Reeb component", as we explain in what follows:

Given the submersion $f\colon \mathbb{R}^3 \xrightarrow{C^\infty} \mathbb{R}$ na form $f(x,y,z) = \xi(r^2)\cdot e^z$ where $r^2 = x^2 + y^2$ and $\xi\colon \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$ is as in the figure

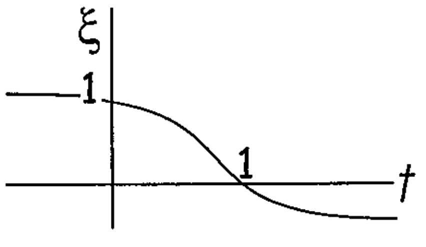


Figure 9.17:

Let $C = \{(x,y,z) \in \mathbb{R}^3;\ x^2 + y^2 \leq 1\}$ then the levels of f interior to C are diffeomorphic to $\mathbb{R}^2$ and the boundary ∂C is also a level of f. By their turn, the levels of f exterior to C are cylinders. Restricting now f to the product $\overline{\mathbb{D}}^2 \times \mathbb{R} \xrightarrow{f} \mathbb{R}$ we obtain then a foliation (by levels) of codimension 1 and, if we identify trivially the boundaries superior and inferior of $\overline{\mathbb{D}}^2 \times [0,1]$, we obtain a foliation $\mathcal{R}$ in the solid torus $\overline{\mathbb{D}}^2 \times X^1$ with the following properties:

(i) the boundary $\partial(\overline{\mathbb{D}}^2 \times X^1) = S^1 \times S^1$ is a leaf

(ii) the interior leaves (i.e., in $\mathbb{D}^2 \times S^1$) are diffeomorphic a $\mathbb{R}^2$.

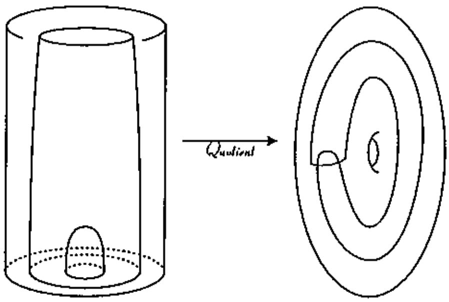


Figure 9.18:

Definition 9.5.2. We shall call $\mathcal{R}$ *Reeb foliation* of $\overline{\mathbb{D}}^2 \times S^1$.

Gluing then two copies of solid tori $\overline{\mathbb{D}} \times S^1$ by the boundary $S^1 \times S^1$ we obtain a C^∞ foliation of codimension 1 in S^3 (this answers to an ancient question of Hopf) and which has a Reeb component. A *Reeb component* of a foliation $\mathcal{F}$ in M^3 is then a foliated portion $\mathcal{R}_1$ of M^3, homeomorphic to $(\overline{\mathbb{D}}^2 \times S^1, \mathcal{R})$.

Novikov Compact Leaf Theorem strongly relies in the notion of *vanishing cycle*:

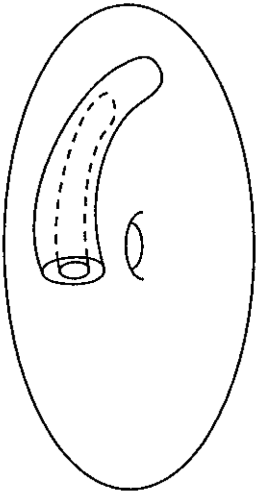


Figure 9.19:

Definition 9.5.3. a loop α in a leaf L_0 of a foliation $\mathcal{F}$ of codimension 1 of M is a *vanishing cycle* if there exists a deformation continuous $\{\alpha_s\}_{s\in[0,1]}$ of $\alpha_0 = \alpha\colon [0,1] \to L_0$ such that:

(i) For each $s \in [0,1]$, $\alpha_s\colon [0,1] \to M$ is a loop in a leaf L_s of $\mathcal{F}$.

(ii) For all $0 < s < 1$, α_s is homotopic to zero in the leaf L_s.

(iii) $\alpha = \alpha_0$ not is homotopic to zero in the leaf L_0.

(iv) For each $t \in [0,1]$, $s \mapsto \alpha_s(t)$ defines a curve transverse to $\mathcal{F}$.

Example 9.5.4. *A Reeb foliation $\mathcal{R}$ of $\overline{\mathbb{D}}^2 \times S^1$ exhibits a vanishing cycle α.*

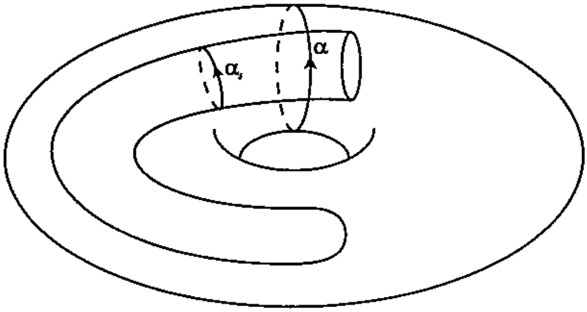


Figure 9.20:

The extremely ingenious proof given by Novikov involves the following sequence of steps:

1. A foliation of codimension one and class C^2 in a compact 3-manifold M with finite $\pi_1(M)$ always has a vanishing cycle (at this point we use ideas already present in the proof of Haefliger's Theorem).

2. Let $\mathcal{F}$ be a foliation of codimension one and class C^1 of a compact manifold M *of dimension* 3. A leaf L_0 containing a vanishing cycle is necessarily compact.

Remark 9.5.5. Some conditions which imply the existence of vanishing cycles as in **2.** above are:

a. $\mathcal{F}$ has a closed transverse γ of finite order in $\pi_1(M)$.

b. M is compact (without boundary) with finite $\pi_1(M)$. In fact, in this case a. is verified.

c. $\mathcal{F}$ has a leaf L_0 such that the canonical homomorphism $i_{\#}\colon \pi_1(L_0) \to \pi_1(M)$ is not injective.

d. If $\mathcal{F}$ is of class C^2 and $\pi_2(M) \neq 0$ and $\pi_2(L) \neq 0$ for each leaf L^{n-1} of $\mathcal{F}$ then $\mathcal{F}$ has a vanishing cycle.

Hence, Novikov Compact Leaf Theorem also holds for M^3 compact with $\pi_2(M^2) \neq 0$.

9.5.2 Growth of foliations and existence of compact leaves

In opposition to a n-dimensional version of the Theorem of Novikov there are examples of foliations C^2 of codimension one of S^n, $n \geq 4$ without compact leaves. In particular, the minimal set of such a foliation is exceptional (cf. [56]). It was J. Plante, in an outstanding work, who initiated the modern comprehension of such facts relating the concepts of growth of leaves and existence of exceptional minimal sets (cf. [39]). Let us recall such concepts:

Growth of Riemmanian manifolds

Let (M, g) be a connected oriented Riemannian manifold of class $C^r, r \geq 1$. Given any point $x \in M$ the *growth function* of M at x is defined by $\gamma_x(r) :=$ volume of the closed metric ball $B[x;r]$. The growth type of γ_x does not depend on the choice of $x \in M$. This way we may introduce of *polynomial growth, exponential growth,* ... for (M, g). If M is compact then it has polynomial growth of degree zero. In the case $r = \infty$ we have the following:

Proposition 9.5.6 (Moussu, Pelletier [15]). *For any $r \geq 0$and any $x \in M$ the closed ball $B[x;r]$ is a standard Whitney domain: the boundary $\partial B[x;r]$ contains a compact subset K*

with zero $(m-1)$-dimensional measure such that $\partial B[x;r] - K$ is a submanifold with boundary of M. Moreover, the function $\gamma_x(r) = \mathrm{vol}(B[x;r])$ is differentiable with respect to r and its derivative at the point r_o is the volume of the $(m-1)$-dimensional sphere $\partial B[x;r_o]$.

It follows from the above result that *if* $\lim\inf_{r\to+\infty} \frac{\mathrm{vol}(\partial B[x;r])}{\mathrm{vol}(B[x;r])} > 0$ *then M has exponential growth.*

Growth of leaves

The notion of growth for the leaves of a foliation on a *compact* manifold may be introduced in a geometric way regarding the growth of the volume of the balls in the leaves, and it will be related to the growth of the orbits of the holonomy pseudogroup of the foliation, as we will see. The main remark is the following:

Proposition 9.5.7 ([15]). *Given two Riemmanian metrics in a compact manifold M equipped with a C^1 regular foliation $\mathcal{F}$, the metrics induce on each leaf L of $\mathcal{F}$, complete quasi-isometric metrics. Therefore, the growth type of the leaf L does not depend on the choice of the ambient metric.*

In the non-compact case however, we may fix the metric and consider the growth type of the leaves with respect to this fixed metric. Let (M,g) be a Riemannian manifold, perhaps non-compact, and let $\mathcal{F}$ be a (regular) C^1 foliation of codimension k on M. Assume that M is oriented and $\mathcal{F}$ is transversely oriented. For each $x \in M$ denote by L_x the leaf of $\mathcal{F}$ through x. The metric on M induces a metric g_x along the (immersed) leaf L_x.

Definition 9.5.8. The *growth type* of the leaf L_x with respect to the metric g is the growth type of the Riemmanian manifold (L_x, g_x).

Therefore, compact leaves have polynomial growth of degree zero.

Growth of orbits

Let X be a hausdorff topological space and Γ a collection of homeomorphisms $g: U \to V$, where U, V are open subsets of X. Denote by $\mathrm{Dom}(g)$ and $\mathrm{Range}(g)$ the domain and the range of $g \in \Gamma$ respectively.

Definition 9.5.9 ([42]). Γ is a *pseudo-group* of local homeomorphisms of X if:
(i) For any $g \in \Gamma$ we have $g^{-1} \in \Gamma$ and $\mathrm{Dom}(g) = \mathrm{Range}(g^{-1})$, and $\mathrm{Dom}(g^{-1}) = \mathrm{Range}(g)$.
(ii) If $g_1, g_2 \in \Gamma$ and $g\colon \mathrm{Dom}(g_1) \cup \mathrm{Dom}(g_2) \to \mathrm{Range}(g_1) \cup \mathrm{Range}(g_2)$ is a homeomorphism such that $g\big|_{\mathrm{Dom}(g_i)} = g_i, i = 1, 2$, then $g \in \Gamma$.
(iii) $Id\colon X \to X$ belongs to Γ.
(iv) If $g_1, g_2 \in \Gamma$ then $g_1 \circ g_2 \in \Gamma$, with $\mathrm{Dom}(g_1 \circ g_2) \subset g_1^{-1}(R(g_2)) \cap \mathrm{Dom}(g_1)$.
(v) If $g \in \Gamma$ and $U \subset \mathrm{Dom}(g)$ is an open subset then $g\big|_U \in \Gamma$.

The *orbit* of x in the pseudogroup Γ is defined by $\Gamma(x) := \{g(x) \in X, g \in \Gamma, x \in \mathrm{Dom}(g)\}$. Assume now that Γ is finitely generated by a (symmetric) finite subset $\Gamma^o \subset \Gamma$.

Definition 9.5.10. For $x \in X$ and $n \in \mathbb{N}$ we define $\Gamma_n(x) := \{y \in X, y = g_{\alpha_1} \circ \ldots \circ g_{\alpha_k}(x), k \leq n, g_{\alpha_j} \in \Gamma^o, j = 1, \ldots, k\}$. The *growth type* of the orbit of x in Γ is the growth type of the function $\gamma_x(n) := \sharp\Gamma_n(x)$ as $n \in \mathbb{N}$.

Combinatorial growth of leaves

Let M be a compact manifold and $\mathcal{F}$ a foliation of codimension k on M. Given a finite covering $\mathcal{U} = \{U_1, \ldots, U_r\}$ of M by distinguished neighborhoods for $\mathcal{F}$ we denote by $\Gamma_{\mathcal{U}}$ the holonomy pseudogroup associated to this covering. Then $\Gamma_{\mathcal{U}}$ is finitely generated. Given a point $x \in U_1$ and $n \in \mathbb{N}$ the value of the growth function $\gamma_{\mathcal{U},x}(n)$ is equal to the number of plaques of $\mathcal{U}$ that can be joined to the plaque $P_{1,x}$ of U_1, by a chain of plaques with at most n plaques.

Proposition 9.5.11 ([15]). *Let $\mathcal{U}$ and $\mathcal{V}$ be finite coverings by distinguished neighborhoods of the manifold M. The growth type of the functions $\gamma_{x,\mathcal{U}}$ and $\gamma_{x,\mathcal{V}}$ is the same. If $\mathcal{F}$ is of class C^1 then the growth type of the functions $\gamma_{x,\mathcal{U}}$ is the same for all points x is a same fixed leaf of $\mathcal{F}$.*

We define therefore the *combinatorial growth type* of a leaf L of a C^1 foliation $\mathcal{F}$ on a compact manifold M as the growth type of the function $\gamma_{x,\mathcal{U}}$ where $x \in L$ is any point and $\mathcal{U}$ is any finite covering of M by distinguished neighborhoods. According to what we have seen, in this compact case, the growth type of a leaf L of $\mathcal{F}$ is equal to the growth type of the orbit of any point $x \in L$ in the holonomy pseudogroup $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. In case the manifold is compact we also have:

Proposition 9.5.12 ([15]). *Let $\mathcal{F}$ be a C^1 transversely oriented foliation on a oriented compact Riemannian manifold M. The (geometric) growth type of any leaf L of M is equal to the combinatorial growth type of L.*

Growth of groups

Let G be a finitely generated abstract group. There exists therefore a subset $G^o \subset G$ such that any element $g \in G$ writes as $g = \prod_{\alpha \in A} g_\alpha^{n_\alpha}$ where A is finite set, $n_\alpha \in \mathbb{Z}$ and $g_\alpha \in G^o, \forall \alpha \in A$. The *set of generators* G^o is *symmetric* if $g \in G^o$ then $g^{-1} \in G^o$.

We will assume that G^o is symmetric. For any $n \in \mathbb{N}$ we define the subset G_n of elements of G which can be expressed as a word of length at most n in the generators. That is,

$$G_n = \{g \in G, g = g_{\alpha_1} \circ ... \circ g_{\alpha_k}, k \leq n, \ g_{\alpha_j} \in G^o, \forall j = 1, ..., k\}.$$

The *growth function* of G is $g(n) = \sharp G_n, n \in \mathbb{N}$. This definition can be extended as follows [44]: Let d be any left-invariant metric on G. We assume that G is *discrete* so that for any $g \in G$ there exists $\epsilon_g > 0$ such that the metric ball $B_G(g; \epsilon_g) \subset G$ contains only the element g. The *growth function* of the pair (G, d) is therefore defined as: for $t > 0$, $\gamma(t) = \sharp B_G(e; t)$ where $e \in G$ is the identity. If $\gamma(t) < \infty, \forall t \geq 0$ then we say that γ is the growth function of (G, d). In case G has a symmetric set of generators S we may consider any function $n_1 \colon S \to \mathbb{R}^+$ such that $n_1^{-1}(0, r] < \infty, \forall r \geq 0$. For any $g \in G$ we define the function $n(g) := \min\{\sum\limits_{i=1}^{\ell} n_1(s_i), \ g = \prod\limits_{i=1}^{\ell} s_i, \ s_i \in S\}$. Clearly $n(.)$ defines a left-invariant metric d on G by setting $d(g, h) := n(g^{-1}h)$. If $S = G$ then any left invariant metric on G is obtained this way. Let us precise our main definition:

Definition 9.5.13 ([44]). The pair (G, d) has *polynomial growth of degree k* if there exists a polynomial $p(x)$ of degree k such that $\gamma(t) \leq p(t), \forall t \geq 0$, where $\gamma(.)$ is the growth function of (G, d). We may also consider polynomials of the form ax^λ, $\lambda \geq 0$, $\lambda \in \mathbb{R}$.

Proposition 9.5.14 ([44]). *Let S be a finite symmetric set of generators of G and $n_1 \colon S \to \mathbb{R}^+$, $n_1 \equiv 1$. Denote by $n(.)$ the metric above corresponding to $n_1 \equiv 1$. The growth of G is polynomial with respect to some left-invariant metric d if and only if, (G, n) has polynomial growth.*

Example 9.5.15. • A finitely generated abelian group has polynomial growth.

• A non-cyclic free group has exponential growth once we have $\sigma(n) = 2m \sum_{k=0}^{n-1}(2m-1)^k = \dfrac{m(2m-1)^n - 1}{m-1}$.
• Let M^2 be a closed surface orientable of genus $g \geq 2$ then $\pi_1(M)$ has exponential growth.

A situation of particular interest is the case of groups of polynomial growth, on which we have essential contributions of J. Plante, J. Wolf and J. Milnor (cf. [72], [36], [41]).
We have:

1. A nilpotent group of finite type has polynomial growth.

2. A solvable group of finite type G which *does not* have a nilpotente subgroup of finite index has exponential growth. In case G has polynomial growth then G is *polycyclic* (G is *polycyclic* if $G = G_k \rhd G_{k-1} \rhd \cdots \rhd G_0 = \{e\}$ with G_k/G_{k-1} cyclic).

3. J. Tits has shown ([70]) that the converse of 1. is true: a group of finite type having polynomial growth has a nilpotente subgroup of finite index.

Holonomy invariant measures

Let X be a hausdorff topological space and Γ a pseudogroup of local homeormorphisms of X. Denote by $\sigma_c(X)$ the ring of subsets of X generated by the compact sets. A measure μ on $\sigma_c(X)$ is Γ-invariant if:
(i) μ is non-negative, finitely additive, finite on compact sets.
(ii) $\forall g \in \Gamma$ and any measurable set $A \subset \mathrm{Dom}(g)$ we have $\mu(g(A)) = \mu(A)$.

Consider now the case M is a C^∞ manifold and $\mathcal{F}$ is a (regular) foliation of codimension $k \geq 1$ on M, assumed to be transversely oriented, of class C^∞. The *holonomy pseudogroup* defined by $\mathcal{F}$ will be denoted by $\Gamma(\mathcal{F})$.

Definition 9.5.16 (Plante, [40]). A foliation $\mathcal{F}$ is said to have a *measure preserving holonomy* if its holonomy pseudogroup has a non-trivial invariant measure which is finite on compact sets. The *support* of an $\mathcal{F}$-invariant measure μ is the set of points $x \in M$ such that: given any k-dimensional disk transverse to $\mathcal{F}$, $D^k \pitchfork \mathcal{F}$, with $x \in \operatorname{Int}(D^k)$, we have $\mu(D^k) > 0$. Since μ is $\mathcal{F}$-invariant, $\operatorname{supp}(\mu)$ is closed and $\mathcal{F}$-invariant.

Example 9.5.17. Assume $\mathcal{F}$ has a closed leaf L on M. Given any transverse section $\Sigma \subset M$ transverse to $\mathcal{F}$ we define a measure on Σ as follows: $\forall A \subset \Sigma$, $\mu(A) := \sharp\{A \cap L\}$. This defines an holonomy invariant measure. Let now $\mathcal{F}$ be given by a closed holomorphic 1-form Ω on a complex manifold M. The holonomy pseudogroup is naturally a pseudogroup of translations $\Gamma(\mathcal{F}) \subset (\mathbb{C}, +)$ and any leaf of $\mathcal{F}$ has trivial holonomy. Any Borel measure on $\mathbb{C}$ which is invariant by translations is also $\Gamma(\mathcal{F})$-invariant. Another situation comes when $\mathcal{F}$ is real given by a closed C^∞ k-form Ω on M. In this case, given any transverse section $\Sigma \subset M$, transverse to $\mathcal{F}$, the restriction $\Omega\big|_\Sigma$ is a volume element ($\mathcal{F}$ is transversely oriented) which is positive on open sets. The fact that Ω is closed implies that the induced transverse measure is $\mathcal{F}$-invariant. Assume now that $M, \mathcal{F}, \Omega$ are holomorphic. Using the complex structure $J\colon TM \to TM$ we may consider the real part and the imaginary part $\operatorname{Re}(\Omega), \operatorname{Im}(\Omega)$ of Ω. Take $\omega = \operatorname{Re}(\Omega) \wedge \operatorname{Im}(\Omega)$, this is a $2k$-form real form, which is closed and defines $\mathcal{F}$ as a real codimension $2k$ foliation. Therefore the restriction $\omega\big|_{\Sigma^{2k}}$, where $\Sigma = \Sigma^{2k}$ is regarded as a $2k$-dimensional real submanifold transverse to $\mathcal{F}$, is positive on open sets and defines an $\mathcal{F}$-invariant transverse measure.

Example 9.5.18. It is a fairly well-known fact that a compact manifold M supporting a *codimension one* C^1 Anosov flow $\varphi_t\colon M \to M$ has fundamental group with exponential growth

[68]. Such a result is not stated for codimension one holomorphic foliations (see [14]). We shall consider an example of such a situation [61]. Let M be a compact complex manifold of dimension n equipped with a closed holomorphic 1-form ω and $f\colon M \to M$ an automorphism such that $f^*\omega = \lambda.\omega$ for some $\lambda \in \mathbb{C} \setminus S^1$. We put $\Omega(x,t) := t.\omega(x)$ on $M \times \mathbb{C}^*$ so that $d\Omega = \eta \wedge \Omega$ for $\eta = \frac{dt}{t}$. The 1-form η is closed and holomorphic in $M \times \mathbb{C}^*$ so that according to [61] $\Omega = 0$ defines a codimension one holomorphic foliation $\tilde{\mathcal{F}}$ on $M \times \mathbb{C}^*$ which is transversely affine. $\tilde{\mathcal{F}}$ is non-singular provided that ω is non-singular on M. The action

$$\tilde{\varphi}\colon \mathbb{Z} \times (M \times \mathbb{C}^*) \to M \times \mathbb{C}^*,\ \tilde{\varphi}(n,(x,t)) = (f^n(x), \lambda^{-n}.t)$$

where $n \in \mathbb{Z}$, $(x,t) \in M \times \mathbb{C}^*$, is a locally free action generated by the automorphism $\varphi\colon M \times \mathbb{C}^* \to M \times \mathbb{C}^*$, $\varphi(x,t) = (f(x), \lambda^{-1}.t)$. Since $\varphi^*\Omega(x,t) = \Omega(x,t)$ and $\varphi^*\eta = \eta$ it follows that $\tilde{\mathcal{F}}$ induces a codimension one transversely affine holomorphic foliation $\mathcal{F}$ of the quotient manifold $V^{n+1} = \tilde{M}/\tilde{\varphi} = \tilde{M}/\mathbb{Z}$. We apply this construction in a concrete situation:

Take $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ as the linear automorphism given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then A has eigenvalues $\lambda_s = \frac{3-\sqrt{5}}{2}$ and $\lambda_u = \frac{3+\sqrt{5}}{2}$. The corresponding eigen-spaces are generated by $v_s = (2, 1-\sqrt{5})$ and $v_u = (2, 1+\sqrt{5})$ respectively. The stable linear foliation and the instable linear foliation are given by the 1-forms $\tilde{\tilde{\omega}}_s = 2dx + (1+\sqrt{5})dy$ and $\tilde{\tilde{\omega}}_u = (1+\sqrt{5})dx - 2dy$ respectively. Take $\tilde{\tilde{\mathcal{F}}}_u : \tilde{\tilde{\omega}}_u = 0$ on $\mathbb{C}^2$. We consider the action of the integer lattice $\mathbb{Z}^2$ on $\mathbb{C}^2$ obtained in the natural way and put $\tilde{M} = \mathbb{C}^2/\mathbb{Z}^2 = \mathbb{C}^* \times \mathbb{C}^*$. The map A leaves $\mathbb{Z}^2$ invariant so that it induces an automorphism $F\colon \tilde{M} \to \tilde{M}$, which is indeed given by $F(z,w) = (zw, w^2)$ for coordinates $z = e^{2\pi i x}, w = e^{2\pi i y}$ on $\mathbb{C}^* \times \mathbb{C}^*$.

Now we consider the $\mathbb{Z}$-action on $\mathbb{C}^*$ given by $\psi\colon \mathbb{Z}\times\mathbb{C}^* \to \mathbb{C}^*, (n,t) \mapsto \lambda_s^{-n}.t$. Then $M = \mathbb{C}^*/\mathbb{Z}\times\mathbb{C}^*/\mathbb{Z} = \tilde{M}/\mathbb{Z}$ obtained this way is a compact surface equipped with an automorphism $f\colon M \to M$ induced by $F\colon \tilde{M}\to\tilde{M}$ indeed, $F(\lambda_s^{-1}.z, \lambda_s^{-1}.w) = \lambda_s^{-2}.(zw, w^2)$.

Now, the 1-form $\tilde{\tilde{\omega}} = \tilde{\tilde{\omega}}_u$ satisfies $A^*(\tilde{\tilde{\omega}}) = \lambda_s^{-1}.\tilde{\tilde{\omega}}$ and corresponds to a Darboux 1-form $\tilde{\omega} = (1+\sqrt{5})\frac{dz}{z} - 2\frac{dw}{w}$ on $\tilde{M} = \mathbb{C}^*\times\mathbb{C}^*$. Therefore, we have $F^*\tilde{\omega} = \lambda_s^{-1}.\omega$ and finally since $\psi^*\tilde{\omega} = \tilde{\omega}$ it follows that $\tilde{\omega}$ induces a closed holomorphic 1-form ω on M with the property that $f^*(\omega) = \lambda_s^{-1}.\omega$. Thus, according to the above construction, the manifold $V^3 = M\times\mathbb{C}^*/\mathbb{Z}$ obtained by quotieting $M\times\mathbb{C}^*$ with the action of $\mathbb{Z}$ given by the action of f on M and of the homoteties $t\mapsto\lambda_s.t$ on $\mathbb{C}^*$, is a compact complex 3-manifold equipped with a transversely affine codimension one holomorphic foliation $\mathcal{F}$ coming from the linear unstable foliation $\mathcal{F}_u$ on $\mathbb{C}^2$. The foliation $\mathcal{F}$ exhibits exponential growth (for any metric on the compact manifold V^3) because A^n expands $v_u.\mathbb{C}$ by a factor λ_u^n. On the other hand, [61], the leaves of $\mathcal{F}$ on V are dense, biholomorphic to $\mathbb{C}^*\times\mathbb{C}^*$ or to $(\mathbb{C}^*/\mathbb{Z})\times\mathbb{C}^*$.

Example 9.5.19. Let G be a Lie group which has polynomial growth in some left invariant metric. Let $\Phi\colon G\times M\to M$ be a locally free smooth action on a manifold M. There exists a Riemannian metric on M which restricts to the Φ-orbits as the induced metric coming from G. Thus Φ defines a foliation $\mathcal{F}$ on M, whose leaves have polynomial growth for this metric. For instance we may take any locally free holomorphic action $\Phi\colon\mathbb{C}^n\times M\to M$ where M is a complex manifold and the euclidian metric on $\mathbb{C}^n$. The foliation by Φ-orbits on M has polynomial growth for a suitable metric on M. For $n=1$ we have a *holomorphic flow* whose orbits have polynomial growth for a given metric on M.

Example 9.5.20. Here we complexify an original example in [41]. Let G be a simply-connected complex Lie group and $H < G$ a closed (Lie) subgroup of (complex) codimension one. Given any discrete subgroup $\Gamma < G$ the group H acts on the quotient G/Γ by left translations generating a foliation $\mathcal{F}$ of codimension one. The leaves of $\mathcal{F}$ are the orbits of the above action. Since G is simply-connected the universal covering $G \to G/\Gamma$ lifts $\mathcal{F}$ into a foliation $\tilde{\mathcal{F}}$ on G whose leaf space is the Riemann surface $H \backslash G$. The exact homotopy sequence of the fibration $G \underset{H}{\to} H \backslash G$ shows that (for H connected) the manifold $H \backslash G$ is simply-connected since G is simply-connected. Therefore, $H \backslash G$ is either diffeomorphic to $\mathbb{C}P(1), \mathbb{C}$ or $\mathbb{D}$. Therefore the action of Γ on $H \backslash G$ defines a global holonomy of $\mathcal{F}$ as a subgroup of $\text{Diff}(N)$ for $N \in \{\mathbb{C}P(1), \mathbb{C}, \mathbb{D}\}$, so that this global holonomy group is either a subgroup of Moebius maps, affine maps or $\text{SL}(2, \mathbb{R})$. If Γ is uniform, that is, the quotient G/Γ is compact, then G is unimodular and the action of G on $H \backslash G$ has an invariant measure iff H is unimodular iff there exists a Γ-invariant measure. Therefore, when G/Γ is compact $\mathcal{F}$ admits an invariant measure iff H is unimodular.

The existence of holonomy invariant measures is a consequence of subexponential growth for the leaves as stated below:

Theorem 9.5.21 (Plante, [40]). *Let $\mathcal{F}$ be a C^2 foliation of codimension $k \geq 1$ on the compact manifold M. Assume that $\mathcal{F}$ exhibits a leaf L having subexponential growth. Then there exists a nontrivial holonomy invariant measure μ for $\mathcal{F}$ which is finite on compact sets and which has support contained in the closure $\overline{L} \subset M$ of L.*

It is also known that if a *codimension one* (real) foliation of class C^2 on a compact manifold admits a non-trivial holonomy invariant measure then $\mathcal{F}$ has a leaf with polynomial growth [40]. *It this also true for complex foliations?*

9.5.3 The complex case

The techniques and features developed above may be useful in a more general situation than the compact case.

Definition 9.5.22. Let $\mathcal{F}$ be a foliation of codimension k on a manifold M (perhaps non-compact). A *compact total transverse section* of $\mathcal{F}$ is a compact k-manifold $\Sigma \subset M$ (possibly with boundary) such that every leaf of $\mathcal{F}$ intersects the interior of Σ.

This condition is fulfilled for holomorphic foliations on complex projective spaces (Exercise!). Let $\mathcal{F}$ be a C^2 foliation on a manifold M such that $\mathcal{F}$ admits a compact total transverse section. Then the holonomy pseudogroup of $\mathcal{F}$ is finitely generated. Plante's result Theorem 9.5.21 above rewrites as follows:

Theorem 9.5.23 (Plante, [40]). *$\mathcal{F}$ be a C^2 foliation of codimension $k \geq 1$ on the Riemannian manifold (M, g). Assume that:*
(i) *$\mathcal{F}$ exhibits a leaf L having subexponential growth with respect to the induced metric.*
(ii) *$\mathcal{F}$ admits a compact total transverse section.*

Then there exists a non-trivial holonomy invariant measure μ for $\mathcal{F}$ which is finite on compact sets and which has support contained in the closure $\overline{L} \subset M$ of L.

Using these notions we also have the following results.

Theorem 9.5.24 (Plante, [39] 1973). *Let $\mathcal{F}$ foliation of class C^2 and codimension 1 of a compact manifold M. Suppose that the leaves of $\mathcal{F}$ have subexponential growth. Then $\mathcal{F}$ has no exceptional minimal set, indeed each leaf of an exceptional minimal* *of a foliation of codimension one in closed manifold closed has exponential growth.*

The theorem above may be interpreted has a kind of Poincaré-Bendixson theorem for foliations of codimension 1 and motivates a series of questions in the complex case. J. Plante also proved the following:

Theorem 9.5.25 (Plante, [43] 1975). *Let $\mathcal{F}$ foliation of class C^2 and codimension q of a compact manifold M. Suppose that $\mathcal{F}$ has a leaf L_0 with subexponential growth. Then $\mathcal{F}$ admits an invariant transverse measure (non trivial) with support contained in $\overline{L}$. If $q = 1$ then the existence of an invariant transverse measure implies that there exists a leaf with subexponential growth and, hence, of a leaf with polynomial growth of degree* $\max\{0, b_1(M)\}$. *In particular if* $\dim H_1(M, \mathbb{R}) \leq 1$ *then $\mathcal{F}$ has a compact leaf.*

9.5.4 From Real to Complex

We shall begin this section with some examples:

Example 9.5.26. Let $R\colon \mathbb{C}P(2) \to \mathbb{C}P(1)$ be a (nonconstant) rational function. The levels of R define a foliation $\mathcal{F}$ by algebraic curves in $\mathbb{C}P(2)$. Given any affine space $\mathbb{C}^2 \subset \mathbb{C}P(2)$ there exists a polynomial vector field whose orbits are punctured leaves of $\mathcal{F}$ in $\mathbb{C}^2$. Since the closure of a leaf is an algebraic curve on $\mathbb{C}P(2)$ the leaves have polynomial growth.

Example 9.5.27 (Darboux foliations). A codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{C}P(n)$ is called a *Darboux foliation* if there is a rational map $\pi\colon \mathbb{C}P(n) \to \mathbb{C}P(m)$ such that $\mathcal{F} =$

$\pi^*(\mathcal{L})$ where $\mathcal{L}$ is the linear Darboux foliation on $\mathbb{C}P(m)$ given by $\mathcal{D} = \big(\prod_{i=1}^{n} x_i\big) . \sum_{j=1}^{n} \lambda_j \frac{dx_j}{x_j} = 0$ in some affine chart $(x_1, \dots, x_m) \in \mathbb{C}^m \hookrightarrow \mathbb{C}P(m)$. If we consider homogeneous coordinates say $(x_o, ..., x_n) \in \mathbb{C}^{n+1}$ then the foliation $\mathcal{F}$ is given by a *Darboux* 1-form (also called *logarithmic 1-form*) in [60] as

$$\Omega = \sum_{j=1}^{k+1} \lambda_j f_1 \dots \hat{f}_j \dots f_{k+1}\, df_j\,, \quad \lambda_j \in \mathbb{C}$$

where the f_j's are homogeneous polynomials in $n+1$ complex variables. Clearly Ω admits the integrating factor $1/f$ where $f = f_1 \dots f_{k+1}$, i.e., the linear 1-form $\frac{1}{f}\Omega$ is closed. We have $\Lambda = \bigcup_j \overline{\{f_j = 0\}}$ and the hypersurfaces $\overline{\{f_j = 0\}}$ are the algebraic leaves of $\mathcal{F}$; they have linearizable holonomy and any other leaf has trivial holonomy.

Let now us apply the ideas of Example 9.5.17. According to what we have seen above $\mathcal{F}\big|_{\mathbb{C}P(n)\setminus(\Omega)_\infty}$ exhibits an holonomy invariant transverse measure induced by $\alpha = \operatorname{Re}(\Omega) \wedge \operatorname{Im}(\Omega)$. Notice that, locally on $\mathbb{C}P(n) \setminus (\Omega)_\infty$, given any transverse disk $\mathbb{D} \approx \Sigma \subset \mathbb{C}P(n) \setminus (\Omega)_\infty$, $\Sigma \pitchfork \mathcal{F}$, we may choose local coordinates $(x_1, ..., x_n, y)$ such that $\Sigma : \{x_1 = ... = x_n = 0\}$ and $\Omega(x_1, ..., x_n, y) = dy$. Thus $\Omega\big|_\Sigma = dy$ so that if we write $y = u + iv$ $(i^2 = -1)$, (u, v) real coordinates, then $\operatorname{Re}\Omega\big|_\Sigma = du$, $\operatorname{Im}(\Omega)\big|_\Sigma = dv$ so that $\alpha\big|_\Sigma = du \wedge dv$. The induced measure is therefore the Lebesgue measure in these local coordinates. Let us see what happens around the algebraic leaves $L_j \subset \pi(\{f_j = 0\}) \subset \mathbb{C}P(n)$. Given a generic point $p \in L_j$ and a transverse disk $\mathbb{D} \approx \Sigma \subset \mathbb{C}P(n)$, $\Sigma \pitchfork \mathcal{F}, \Sigma \cap l_j = \{p\}$ we may choose local holomorphic coordinates $(x_1, ..., x_n, y) \in U$ around p such that $(\Omega)_\infty U = L_j \cap U = \{y = 0\}$ and $\Omega\big|_U = \lambda_j \frac{dy}{y}$. Thus, if we write

$y = u + iv$ as above then we obtain $\alpha\big|_{\Sigma}(u,v) = \frac{|\lambda_j|^2}{u^2+v^2} du \wedge dv$. In other words $\alpha(y) = \frac{|\lambda_j|^2}{|y|^2} dV$; where dV is the volume element induced by $y = (u,v)$ on Σ. For $(y = 0)$ over the leaf L_j the measure is not defined: any disk $D \subset \Sigma$ containing the origin would have an infinite area. On the other hand, if $\lambda_i/\lambda_j \notin \mathbb{R}$ for $i \neq j$ the holonomy group $\mathrm{Hol}(L_j)$ of the leaf L_j contains linearizable attractors $f_j(y) = \exp(2\pi i \frac{\lambda_i}{\lambda_j}).y$ so that any holonomy invariant measure around L_j must be supported on $\overline{L_j}$, that is, must be the Dirac measure with center at L_j.

Problem 9.5.28. *Let $\mathcal{F}_{\mathbb{C}}$ be a holomorphic foliation of complex codimension 1 in the manifold (complex) $M_{\mathbb{C}}$ compact. Assume that $\mathcal{F}_{\mathbb{C}}$, has a leaf with subexponential growth. Does $\mathcal{F}_{\mathbb{C}}$ has a compact leaf?*

In the case foliations in complex projective spaces we always have singularities, we may however ask:

Problem 9.5.29. *Let $\mathcal{F}_{\mathbb{C}}$ be a holomorphic foliation with singularities in $\mathbb{C}P(n)$ = complex projective space of dimension n. Suppose that $\mathcal{F}_{\mathbb{C}}$ has leaves with subexponential growth for the Fubini-Study metric in $\mathbb{C}P(n)$ (the leaves are considered in $\mathbb{C}P(n) -$ Sing$(\mathcal{F})$). Does there exists a leaf L_0 of $\mathcal{F}_{\mathbb{C}}$ whose closure $\overline{L}_0$ in $\mathbb{C}P(n)$ is algebraic of codimension one?*

We recall that according to the Theorem of Chow ([18]) a leaf L_0 of $\mathcal{F}_{\mathbb{C}}$ closed in $\mathbb{C}P(n) -$ Sing$(\mathcal{F})$ will be analytic in $\mathbb{C}P(n)$ and therefore algebraic, provided that $\dim L_0 - \dim\ \mathrm{Sing}(\mathcal{F}) \geq 1$. In this direction we have the following result:

Theorem 9.5.30 ([58]). *Let $\mathcal{F}_{\mathbb{C}}$ be a holomorphic foliation with singularities in $\mathbb{C}P(2)$. Suppose that the singularities of $\mathcal{F}_{\mathbb{C}}$ are* *all hyperbolic and that $\mathcal{F}_{\mathbb{C}}$ has a non-algebraic leaf with subexponential growth. Then $\mathcal{F}_{\mathbb{C}}$ is linear hyperbolic: there exist affine coordinates $(x, y) \in \mathbb{C}^2 \hookrightarrow \mathbb{C}P(2)$ such that $\mathcal{F}_{\mathbb{C}}$ is given by the linear vector field $X(x, y) = x \dfrac{\partial}{\partial x} + \lambda y \dfrac{\partial}{\partial y}$, $\lambda \in \mathbb{C} \backslash \mathbb{R}$. In particular the limit set of $\mathcal{F}_{\mathbb{C}}$ is a union of 3 projective lines.*

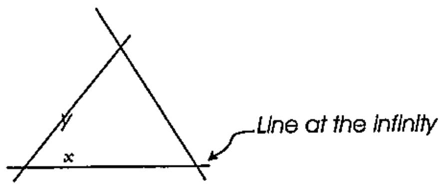


Figure 9.21:

Growth of groups of complex diffeomorphisms

The growth of the leaves of a foliation is naturally related to the growth of its holonomy group (of the leaves). Let us recall briefly these notions. Let G a group finitely generated say $G = \langle g_1, \dots, g_m \rangle$ where we can assume that $g_j^{-1} \in \{g_1, \dots, g_m\}$, $\forall j = 1, \dots, m$. Given $n \in \mathbb{N}$ denote by $\sigma(n)$ the cardinal of the set of elements of g of the form $g_{j_1}^{\alpha_1} \dots g_{j_r}^{\alpha_r}$ where $j_1, \dots, j_r \in \{1, \dots, m\}$, $\alpha_1, \dots, \alpha_r > 0$ and $\sum\limits_{k=1}^{r} \alpha_k = n$. In other words, $\sigma(n)$ is the number of "words" of G that we obtain with n combinations in the generators g_j. The type of growth of the function $\sigma \colon \mathbb{N} \to \mathbb{N}$ does not depend on the choice of the generators g_j

and is called *type of growth* of G.

Clearly every finitely generated abelian group has polynomial growth. According to Wolf [72] a finitely generated group has polynomial growth provided that it has a nilpotent subgroup of finite index. This result is an equivalence for certain classes of groups as we will see below.

Using these results J. Plante and W. Thurston studied foliations of codimension 1 with holonomy group of leaves with polynomial growth or exponential growth (cf. [44], [40]). We are led to the following question:

Problem 9.5.31. *Let $G < Bih(\mathbb{C}^q, 0)$ be a finitely generated subgroup, of the group $Bih(\mathbb{C}^q, 0)$ of germs of holomorphic diffeomorphisms fixing the origin $0 \in \mathbb{C}^q$, $q \geq 1$. Suppose that G has a polynomial growth or subexponential growth. Then what can we say about G? Is G solvable?*

In this standpoint we have:

Proposition 9.5.32 ([6]). *Let $G < Bih(\mathbb{C}, 0)$ be a finitely generated subgroup with polynomial growth. Then G is solvable.*

We believe that "G non-solvable $\Rightarrow$ G has exponential growth (for $G < \mathrm{Bih}(\mathbb{C}, 0)$)".

Chapter 10

Currents, Distributions, Foliation Cycles and Transverse Measures

10.1 Introduction

This second part of the text is dedicated to some other topics in the Global Theory of Foliations. Special attention is paid to the consequences of the Theory of Currents on foliated manifolds. We will therefore exploit aspects, already mentioned in the first part, of growth of leaves and of groups as well as the existence of invariant transverse measures or of foliation cycles for a given foliation. Despite its certain informality our approach and exposition aim to clear the key-points of some central results of the classical theory (e.g. the bijection between transverse invariant measures and foliation cycles and homological versions of Novikov Compact Leaf Theorem) allowing this way the link between the classical real framework and the so called

"Complex World", where the foliations are frequently singular and therefore the ambient manifold may not be compact. After constructing the bases of the theory of currents and foliation cycles in the real case we address the problem of giving a non-geometrical (?) proof of Novikov Compact Leaf Theorem. The central idea/philosophy is that such a proof may be somehow adapted to the complex setting. References for these two parts should be essentially contained in the works of J. Plante, D. Sullivan, S. Schwartzmann, D. Ruelle, A. Haefliger (for the real classic part) and M. McQuillan, M. Brunella, L.G. Mendes [35], for the existing complex part; and may be found in the end of this text ([33], [34], [2], [3], [7]).

10.2 Currents

This section is inspired in the expositions of [63], [11] and [16]. The study of currents associated to foliations has proved to be very useful in the comprehension of and topological dynamical phenomena related to foliations (cf. [66], [62], [43], [20] et al). In this chapter we try to illustrate some of these applications. We shall begin with the basic definitions which are involved, with motivations coming from particular situations already well-known. The first step is to introduce the concept of *current*. We denote by $C_c^\infty(\mathbb{R}^n)$ the vector space of the functions C^∞ of compact support $f\colon \mathbb{R}^n \to \mathbb{R}$. Endow $C_c^\infty(\mathbb{R}^n)$, as usual, with the topology of the uniform convergence in compact sets (for f and its derivatives of all orders). A *distribution* in $\mathbb{R}^n$ is then a linear functional $T \in (C_c^\infty(\mathbb{R}^n))^*$, that is, a linear application $T\colon C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is continuous in the C^∞ topology in $C_c^\infty(\mathbb{R}^n)$. We denote now by $A_c^p(\mathbb{R}^n)$ the $\mathbb{R}$-vector space of differential p-forms of class C^∞ and compact support in $\mathbb{R}^n$, equipped with topology inherited from $C_c(\mathbb{R}^n)$ in the natural way. Then $A_c^p(\mathbb{R}^n)$ is complete and we can consider its topological dual $\mathcal{D}^{n-p}(\mathbb{R}^n)$. In what follows we take $p + q = n$.

Definition 10.2.1. A *current* of *degree* q on $\mathbb{R}^n$ is an element $C \in \mathcal{D}^q(\mathbb{R}^n)$. Thus, a current of degree q on $\mathbb{R}^n$ is a linear continuous form on the space of differential forms of class C^∞ and degree $p = n - q$ having compact support in $\mathbb{R}^n$. Also we shall say that C is a current of *dimension* p.

10.2.1 Examples

1. A current of degree n in $\mathbb{R}^n$ is simply a distribution in $\mathbb{R}^n$.
2. Let $N^p \subset \mathbb{R}^n$ be an oriented submanifold of $\mathbb{R}^n$. The integration along N^p defines a current $C(\varphi) := \int_N \varphi, \quad \varphi \in A_c^p(\mathbb{R}^n)$ of dimension of dimension p.

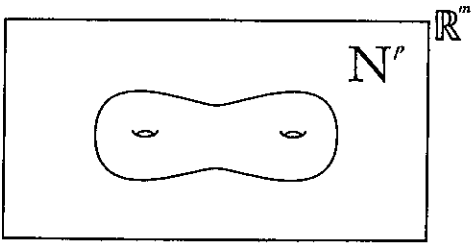


Figure 10.1:

3. Let $\psi = \sum_j \psi_j \, dx_j$ (in affine coordinates $(x_1, \ldots, x_n) \in$

$\mathbb{R}^n$) be a differential q-form with locally integrable coefficients ($\psi_J \in L^1_{\text{loc}}(\mathbb{R}^n)$). To ψ we can associate a current of degree q (and dimension p) $C(\varphi) := \int_{\mathbb{R}^n} \varphi \wedge \psi;\quad \forall\, \varphi \in A^p_c(\mathbb{R}^n)$.

4. Given a singular p-chain $\alpha = \sum_{j=1}^{r} a_j \cdot N_j$ in $\mathbb{R}^n$ we can (as in 2. above) define an integration current by setting

$$C(\varphi) := \int_\alpha \varphi, \quad \forall\, \varphi \in A^p_c(\mathbb{R}^n).$$

In $\{A^p_c(\mathbb{R}^n)\}$ we consider the exterior derivation of forms

$$\begin{aligned} d\colon A^p_c(\mathbb{R}^n) &\to A^{p+1}_c(\mathbb{R}^n) \\ \varphi &\mapsto d\varphi \end{aligned}$$

and induzimos, in natural way, operador derivation in $\mathcal{D}^q(\mathbb{R}^n)$:

$$\begin{aligned} d\colon \mathcal{D}^q(\mathbb{R}^n) &\to \mathcal{D}^{q+1}(\mathbb{R}^n) \\ C &\mapsto dC \end{aligned}$$

$dC(\varphi) := C(d\varphi), \quad \forall\, \varphi \in A^{q+1}_c(\mathbb{R}^n)$.
In a natural way we obtain a complex of cochains $\{d\colon \mathcal{D}^p(\mathbb{R}^n) \to \mathcal{D}^{p+1}(\mathbb{R}^n)\}$ (naturally) associated to the complex of De Rham with compact support of $\mathbb{R}^n$

$$\{d\colon A^p_c(\mathbb{R}^n) \to A^{p+1}_c(\mathbb{R}^n)\}.$$

In particular, $d(dC) = 0$ for every current C in $\mathbb{R}^n$.

We can "localize" the notions above in an obvious way: given open subset $U \subset \mathbb{R}^n$ we introduce the spaces $A^p_c(U)$ and $\mathcal{D}^q(U) := (A^p_c(U))^*$ where the topology we consider is the natural inherited from the topology of uniform convergence in compact parts (for functions and its derivatives of all orders) in $C^\infty_c(U)$. Given a diffeomorphism C^∞ $F\colon U \to V$ between open subsets of $\mathbb{R}^n$ we have a natural homeomorphism $F^*\colon A^p_c(V) \to A^p_c(U)$ which is also linear. Thus, we can introduce the spaces of of currents $\mathcal{D}^p(M^n)$ in a differentiable manifold M^n. Let us see properties of the corresponding complexes *of currents* $\{d\colon \mathcal{D}^q(M) \to \mathcal{D}^{q+1}(M)\}$ and of *De Rham* $\{d\colon A^p_c(M) \to A^{p+1}_c(M)\}$ in M. We recall:

A *complex of cochains* is a collection $\{d_k\colon A_k \to A_{k+1}\}_{k\in\mathbb{Z}}$ of abelian groups A_k and group homomorphisms $d_k\colon A_k \to A_k$ with the property that $d_{k+1} \circ d_k = 0$. In particular we can consider the quotient groups

$$H^k := \frac{\operatorname{Ker}(d_k\colon A_k \to A_{k+1})}{\operatorname{Im}(d_{k-1}\colon A_{k-1} \to A_k)}$$

called the *Cohomology groups* of the complex considered. The *cohomology groups of De Rham with compact support* of M (differentiable manifold) denoted $H^k_{c,DR}(M)$ are defined this way (from $\{d\colon A^k_c(M) \to A^{k+1}_c(M)\}$) for $k \geq 1$ recalling that, by definition,

$$H^0_{c,DR}(M) := \{f\colon M \xrightarrow{C^\infty} \mathbb{R};\ f \text{ has compact support and } df = 0\}$$

is the number of compact connected components of M; also we have $H^k_{c,DR}(M) = 0, \quad \forall\, k \geq n+1$ ($n = \dim M$) and we have the following:

$$H^n_{c,DR}(M;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } M \text{ is orientable} \\ 0 & \text{if } M \text{ is non-orientable.} \end{cases}$$

Remark 10.2.2. We can also work with general differential forms (not necessarily with compact support) of class C^∞ in M obtaining the *De Rham complex* of M, whose cohomology is denoted by $H^k_{DR}(M)$, and the maximal order cohomology is given by:

$$H^n_{DR}(M) = \begin{cases} 0 & \text{if} \quad M \quad \text{is not compact ou not-orientable} \\ \mathbb{R} & \text{if} \quad M \quad \text{is compact and orientable.} \end{cases}$$

Let us return to the currents in M. As we have seen in 3. from the above examples there exists a natural inclusion of the space of q-forms of class C^∞ in M in the space of currents of degree q in M

$$\psi \mapsto C_\psi(\varphi) := \int_M \psi \wedge \varphi, \quad \forall \varphi \in A^p_c(M)$$

$\psi \in A^q(M)$ q-form C^∞ in M.
Such inclusion gives indeed a homomorphism of complexes $\{i_p \colon A^q(M) \to \mathcal{D}^q(M)\}$ that induces by its turn a homomorphism in the cohomology groups

$$i_{p\#} : H^q_{DR}(M) \to H^q(\mathcal{D}^*(M))$$

where $H^q(\mathcal{D}(M))$ denotes the order q cohomology group of the complex of currents $\mathcal{D}^*(M)$ of M.

Theorem 10.2.3 (Theorem of De Rham, [10], [11]). *Given a differentiable oriented manifold M we have natural isomorphisms between the singular cohomology singular groups of Rham and of currents in M:*

$$H^q_{\text{sing}}(M, \mathbb{R}) \simeq H^q_{DR}(M) \simeq H^q(\mathcal{D}^*(M)).$$

10.3 Invariant measures

Let $\mathcal{F}$ be a foliation of class C^∞ dimension p and codimension q of a manifold M^n. There exists a covering $\mathcal{U} = \{U_j\}_{j\in\mathbb{N}}$ of M with the following properties:

1. $\mathcal{U}$ is locally finite: given a compact $K \subset M$ we have $\#\{j \in \mathbb{N};\ U_j \cap K \neq \emptyset\} < \infty$.

2. U_j is connected and $\mathcal{F}|_{U_j}$ is trivial: there exists a diffeomorphism $\varphi_j \colon U_j \to \varphi_j(U_j) \subset \mathbb{R}^n$ such that φ_j takes $\mathcal{F}$ onto the horizontal foliation in $\mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$.

3. In each U_j we have an embedded disc $D^q \simeq \Sigma_j \subset U_j$ which is transverse to the plaques of $\mathcal{F}$ in U_j and parametrizes this space of plaques.

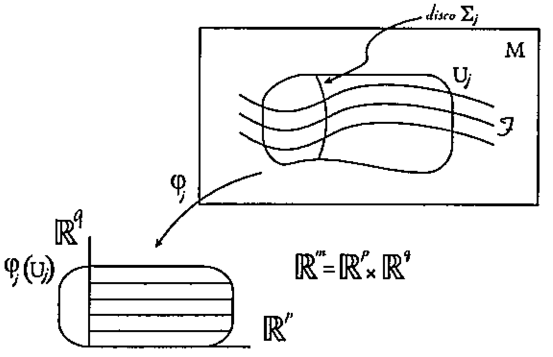


Figure 10.2:

We shall call $\mathcal{U}$ a *regular* covering of M for the foliation $\mathcal{F}$. We also assume, with no loss of generality, that $\varphi_j(U_j) = \mathbb{R}^n$ and that $M = \bigcup\limits_{j\in\mathbb{N}} \varphi_j^{-1}((-1,1)^n)$ and we can then take $\Sigma_j \subset$

$\varphi_j^{-1}((-1,1)^n)$ and also rename $U_j = \varphi_j^{-1}[(-1,1)^n]$ in a way that:

4. Each leaf of $\mathcal{F}$ cuts some transverse disc T_j ; and if $U_i \cap U_j \neq \emptyset$ then each plaque of $\mathcal{F}|_{U_i}$ meets at most one plaque of $\mathcal{F}|_{U_j}$ defining local diffeomorphisms C^∞ say $g_{ij}\colon \Sigma_i \to \Sigma_j$ with the property that in $U_i \cap U_j$ we have $y_j = g_{ij} \circ y_i$ where $y_j =$ is the projection of U_j onto Σ_j (via the chart φ_j).

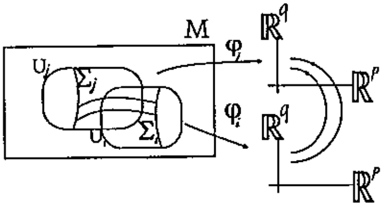


Figure 10.3:

Clearly we have the following condition of cocycle:

5. $U_i \cap U_j \neq \emptyset \;\Rightarrow\; g_{ij} = g_{ji}^{-1}$ and

$$U_i \cap U_j \cap U_k \neq \emptyset \;\Rightarrow\; g_{ij} \circ g_{jk} = g_{ik}$$

in the corresponding domains.

Definition 10.3.1. The *holonomy pseudogroup* of $\mathcal{F}$ for a regular covering $\mathcal{U}$ is the pseudogroup $\Gamma_{\mathcal{U}}$ of local diffeomorphisms C^∞ of the manifold $\Sigma_{\mathcal{U}}$, disjoint sum of transverse discs Σ_j, generated by the local diffeomorphisms g_{ij}.

We recall the following definition:

Definition 10.3.2. Let X be a topological space hausdorff and Γ a collection of local homeomorphisms $g\colon U \to V$ where $U, V \subset$

X are open subsets of X. Let us denote by $\text{Dom}(g)$ and $\text{Im}(g)$ the domain and the image of $g \in \Gamma$ respectively. We say then that Γ is a *pseudogroup* of local homeomorphisms of X if:

(i) $\forall g \in \Gamma$ we have $g^{-1} \in \Gamma$, $\text{Dom}(g) = \text{Im}(g^{-1})$ and $\text{Im}(g) = \text{Dom}(g^{-1})$;

(ii) If $g_1, g_2 \in \Gamma$ and $g\colon \text{Dom}(g_1) \cup \text{Dom}(g_2) \to \text{Im}(g_1) \cup \text{Im}(g_2)$ is a homeomorphism such that $g|_{\text{Dom}(g_j)} = g_j$, $j = 1, 2$ then $g \in \Gamma$.

(iii) $\text{Id}\colon X \to X$ belongs to Γ.

(iv) If $g_1, g_2 \in \Gamma$ then $g_1 \circ g_2 \in \Gamma$ com $\text{Dom}(g_1 \circ g_2) \subset g_1^{-1}(\text{Im}(g_2)) \cap \text{Dom}(g_1)$.

(v) If $g \in \Gamma$ and $U \subset \text{Dom}(g)$ is open then $g|_U \in \Gamma$.

Under these conditions we define the *orbit* of a point $x \in X$ in the pseudogroup Γ by $\Gamma(x) := \{g(x) \in X, g \in \Gamma \text{ and } x \in \text{Dom}(g)\}$.
We denote by $\sigma_c(X)$ the ring of subsets of X generated by the compact sets. A measure μ in $\sigma_c(X)$ is said to be Γ-*invariant* if:

(vi) μ is non-negative, finitely additive, and finite in compact sets.

(vii) $\forall g \in \Gamma$ and any measurable subset $A \subset \text{Dom}(g)$ we have $\mu(g(A)) = \mu(A)$.

In the above case, of the holonomy pseudogroup of the foliation $\mathcal{F}$ relative to the regular covering $\mathcal{U}$ we conclude that , in fact , $\Gamma_{\mathcal{U}}$ is a pseudogroup of local diffeomorphisms C^∞ of $\Sigma_{\mathcal{U}}$. In case we have another regular covering of M relative to $\mathcal{F}$, say $\widetilde{\mathcal{U}} = \{\widetilde{U}_j\}_{j \in \mathbb{N}}$ if we suppose that $\widetilde{\mathcal{U}}$ is *more thin* than

$\mathcal{U}$ (i.e., for each index $j \in \mathbb{N}$ there exists index $\nu(j) \in \mathbb{N}$ such that $\widetilde{U}_j \subseteq U_{k(j)}$ is also which $\widetilde{U}_j$ is *uniform* in $U_{k(j)}$ (i.e., each plaque of $U_{k(j)}$ meets *at most* one plaque of $\widetilde{U}_j$) then we obtain a natural identification beteween the corresponding holonomy pseudogroups $\Gamma_{\widetilde{\mathcal{U}}} \xrightarrow{\sim} \Gamma_{\mathcal{U}}$. This shows the following (exercise!):

Lemma 10.3.3. *All the holonomy pseudogroups $\Gamma_{\mathcal{U}}$, where $\mathcal{U}$ is covering regular of M for a foliation $\mathcal{F}$, are naturally equivalent.*

We may then introduce the (well-defined whether M is compact or not) *holonomy pseudogroup of the foliation $\mathcal{F}$.* This way we can formalize the following notion:

Definition 10.3.4. A foliation $\mathcal{F}$ of a manifold M is said to admit a *holonomy invariant transverse measure* (or simply *invariant transverse measure*) if its holonomy pseudogroup has some invariant measure (non-trivial) which is finite in compact sets. The *support* of an invariant measure μ is the set of points $x \in M$ such that: given any transverse disc to $\mathcal{F}$ of dimension q = codimension of $\mathcal{F}$, $D^q \subset M$ with $x \in \text{Int}(D^q)$, we have $\mu(D^q) > 0$. The support of μ, denoted by $\text{supp}(\mu)$, is closed and (since μ is invariant) it is saturated (invariant) by $\mathcal{F}$.

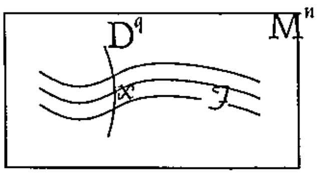


Figure 10.4:

Let us see some examples illustrating the notions above:

10.3.1 Examples

1. Let $\mathcal{F}$ be a foliation of codimension 1 given by a non-singular closed 1-form of class C^∞, Ω in M. Then it is to see from the Poincaré Lemma that the holonomy pseudogroup of $\mathcal{F}$ is naturally a group of translations $\Gamma_\mathcal{F} \subset (\mathbb{R}, +)$ and any leaf of $\mathcal{F}$ has trivial holonomy group (a translation with a finite fixed point finito is the identity). Therefore, any Borel measure in $\mathbb{R}$ invariant by translations is also $\Gamma_\mathcal{F}$-invariant. Suppose now that $\mathcal{F}$ is of codimension k and given by a closed k-form Ω in M. In this case given any transverse k-disc to $\mathcal{F}$ say $D^k \subset M$, the restriction $\Omega|_{D^k}$ is a volume form (assume also that $\mathcal{F}$ is transversely oriented this way) which is positive in open sets. The fact that Ω is closed implies that the transverse measure this way induced is $\mathcal{F}$-invariant.

2. Suppose that $\mathcal{F}$ admits a closed leaf $L_0 \subset M$. Given a transverse disc to $\mathcal{F}$, say $D \subset M$ we define for any $A \subset D$ a measure $\mu(A) := \sharp\{A \cap L\}$. Clearly we obtain this way an invariant transverse measure μ for $\mathcal{F}$; also we have $\operatorname{supp}(\mu) = L_0$.

3. Let $\mathcal{F}$ be a foliation defined by the fibration $M \to B$ of M over a manifold B; then the transverse measures for $\mathcal{F}$ correspond to the measures over B, which are finite on compact sets.

4. Let $\mathcal{F}$ be a foliation of M and $f\colon \widetilde{M} \to M$ a proper application of class C^∞ and transverse to $\mathcal{F}$. Denote by $\widetilde{\mathcal{F}}$ the lift $f^*\mathcal{F}$ to $\widetilde{M}$, if $\mathcal{F}$ admits an invariant transverse measure μ then $\tilde{\mathcal{F}}$ admits an invariant transverse measure $\widetilde{\mu} := f^*(\mu)$ defined naturally by $\tilde{\mu}(\widetilde{D}) := \mu(D)$, where $\widetilde{D} = f^{-1}(D)$ as in the figure below.

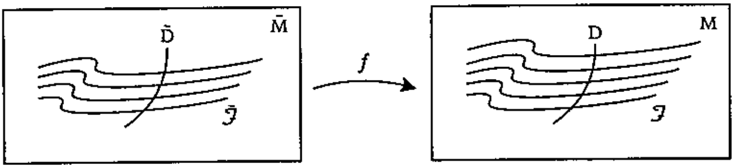


Figure 10.5:

5. According to Joe Plante [43] "if $\mathcal{F}$ is foliation of class C^2 of a *compact* manifold M admitting a leaf L_0 with subexponential growth (geometrical, Riemannian) then $\mathcal{F}$ admits an invariant transverse measure μ, finite in compact sets, whose support $\text{supp}(\mu) \subset \overline{L}_0$."

6. Let us consider now more in details the case of suspensions: Let $\pi\colon E \underset{F}{\longrightarrow} B$ be a fibre bundle of class C^∞ with typical fiber F; base B, projection π and total space E. We say that a foliation $\mathcal{F}$ of E is *transverse to the fibres* of E if:

(a) Given $x \in E$ we have which L_x is transverse to the fibre $F_{\pi(x)} = \pi^{-1}(\pi(x))$ and in fact $\dim \mathcal{F} + \dim F = \dim E$.

(b) The restriction $\pi|_L\colon L \to B$, where L is an arbitrary leaf of $\mathcal{F}$, is a covering map.

Remark 10.3.5. We observe that if the fibre F is compact then (b) from (a); even for B non-compact (see [4] page 94).

Since each restriction $\pi|_L\colon L \to B$ is a covering map we can define a representation $\varphi\colon \pi_1(B) \to \text{Dif}^\infty(F)$ from the fundamental group of the base B into the group of diffeomorphisms

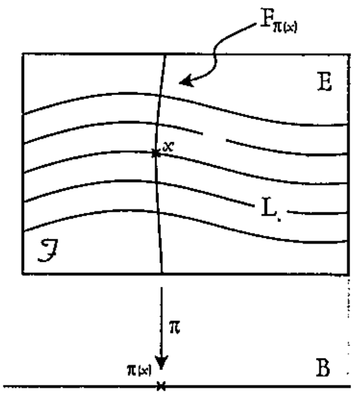


Figure 10.6:

(globais) of class C^∞ (suppose $\mathcal{F}$ of class C^∞), of the fibre F as follows: Fixed base points b_0, $b_0' \in B$. Given a path $\alpha\colon [0,1] \to B$, $\alpha(0) = b_0$, $\alpha(1) = b_0'$ we define, for each $y \in F_{b_0}$, the point $f_\alpha(y) \in F_{b_0}$, as the final point final $\tilde{\alpha}_y(1)$ of the lift $\tilde{\alpha}_y\colon [0,1] \to L_0$ of α, by the covering map $\pi|_{L_y}\colon L_y \to B$, with origin at the point $y = \tilde{\alpha}_y(0)$.

For $b_0 = b_0'$ we identify $F_{b_0} = F_{b_0'} \simeq F$ and we obtain representation

$$\begin{aligned} \varphi\colon \pi_1(B, b_0) &\to \mathrm{Dif}^\infty(F) \\ [\alpha] &\mapsto f_{[\alpha]}\,. \end{aligned}$$

The image of this representation is called *global holonomy of $\mathcal{F}$*. By means of a constructive processo one may prove the following

"I. Let $\mathcal{F}$ and $\mathcal{F}'$ be foliations transverse to the fibres of a fibre bundle $\pi\colon E \xrightarrow[F]{} B$. Then the groups of global holonomy

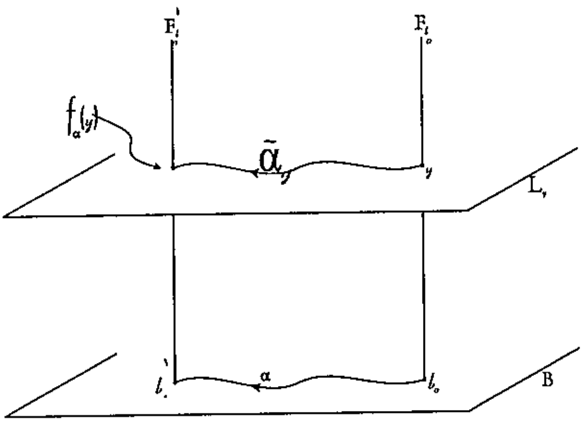


Figure 10.7:

of $\mathcal{F}$ and $\mathcal{F}'$ are conjugate (in $\mathrm{Dif}^\infty(F)$) if and only if $\mathcal{F}$ and $\mathcal{F}'$ are conjugate by a fibred diffeomorphism $\psi\colon E \to E$.

II. Given a fibre bundle space $\pi\colon E \xrightarrow[F]{} B$ there exists a foliation $\mathcal{F}$ transverse to the fibres of the bundle if and only if the structural group of the bundle is discrete.

III. Given a representation $\varphi\colon \pi_1(B) \to \mathrm{Dif}^\infty(F)$ of the fundamental group of a manifold B in the group of diffeomorphisms C^∞ of a manifold F with image $G < \mathrm{Dif}^\infty(F)$ there exist a foliation $\mathcal{F}$ of class C^∞ of the manifold E, a structure of fibre bundle space $\pi\colon E \xrightarrow[F]{} B$ such that $\mathcal{F}$ is transverse to the fibres of the bundle and whose global holonomy (of $\mathcal{F}$) is conjugate to G. By (I) $\mathcal{F}$ is unique up to natural equivalence."

We shall call such a foliation $\mathcal{F}$ the *suspension* of the representation $\varphi\colon \pi_1(B) \to \mathrm{Dif}^\infty(F)$.

We recall that a group G is *amenable* if the space $\mathcal{B}(G) := \{f\colon G \to \mathbb{R}; ||f|| \text{ is bounded}\}$, equipped with the norm of the supreme, admits a positive linear functional $\xi\colon \mathcal{B}(G) \to [0, +\infty)$ with $\xi(1) = 1$ and G-invariant (i.e., $\xi(f \circ L_g) = \xi(f), \quad \forall f \in \mathcal{B}(G)$. Such a functional ξ is called *continuous invariant mean* (cf. [25]). It is proven that if G is a finitely generated group and with subexponential growth then G is *amenable* (cf. [67]) and that solvable finitely generated groups (for example) are amenable.

Let finally $\mathcal{F}$ be a foliation transverse to the fibres of a fibre bundle $\pi\colon E \underset{F}{\longrightarrow} B$ and suppose that the group of global holonomy of $\mathcal{F}$ is amenable then, if a fibre F is compact, $\mathcal{F}$ has invariant transverse measures.

10.4 Current associate to a transverse measure

This section is based in the exposition from [15]. Let $\mathcal{F}$ be a foliation of codimension q in M^n, class C^∞, admitting an invariant transverse measure μ. Let us see how to associate to μ a current C_μ in M; we begin taking regular covering $\mathcal{U} = \{U_j\}_{j\in\mathbb{N}}$ of M relative to $\mathcal{F}$ and considering the holonomy pseudogroup $\Gamma_{\mathcal{U}}$. As before $\Sigma_{\mathcal{U}} = \bigcup_{j\in\mathbb{N}} \Sigma_j$ denotes the space of plaques of $\mathcal{F}$ relatively to $\mathcal{U}$; we can then disintegrate the a measure μ as follows:

- μ defines a Borelian measure over $\Sigma_{\mathcal{U}}$ invariant by $\Gamma_{\mathcal{U}}$.
- Let $\sum_{j\in\mathbb{N}} a_j = 1$ be a partition C^∞ of the unity, strictly

subordinate to the covering $\mathcal{U}$ of M.

- Given $\varphi \in A_c^{n-q}(M)$ form C^∞ of grau $n - q$ ($n = \dim M$) and compact support in M we can consider the product $a_j\varphi \in A_c^{n-q}(\Sigma_j)$ as a continuous function on Σ_j provided that $\mathcal{F}$ is oriented and we consider in U_j (and therefore in Σ_j) the orientation induced by $\mathcal{F}$. In fact, we can consider a function $y \mapsto \int_{P_y} a_j\varphi$

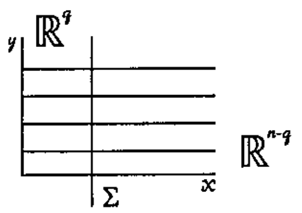


Figure 10.8:

defined in terms of local coordinates (x, y) in U_j that make $\mathcal{F}|_{U_j}\{y = \text{ cte}\}$; the plaques of $\mathcal{F}|_{U_j}$ are of the form $P_y \subset \mathbb{R}^{n-q} \times \{y\}$ and a transverse Σ_j of the form $\Sigma_j \subset \{x = 0\}$.

- We integrate and sum these functions obtaining the value

$$C_\mu(\varphi) := \sum_{j \in \mathbb{N}} \int_{\Sigma_j} \left(\int_{P_y} a_j \varphi(y) \right) d\mu(y).$$

Using the fact that μ is invariant by the local diffeomorphisms $g_{ij}\colon \Sigma_i \to \Sigma_j$ we conclude that in fact the value of $C_\mu(\varphi)$ does not depend on the $\sum\limits_{j \in \mathbb{N}} a_j = 1$ (partition of the unity) neither on the regular covering $\mathcal{U}$ with respect to $\mathcal{F}$ (there is no need to suppose M compact).

Definition 10.4.1. C_μ is the *current associate* to the invariant transverse measure μ for $\mathcal{F}$.

The following result is central in the theory:

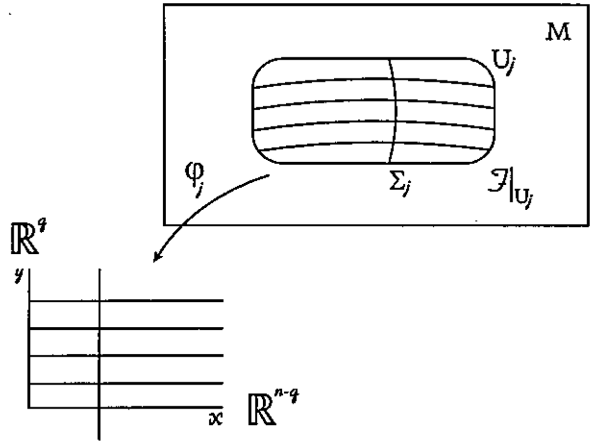


Figure 10.9:

Proposition 10.4.2. *C_μ is a closed current.*

Prova: Using the above notations we have that

$$C_\mu(d\varphi) = \sum_{j\in\mathbb{N}} \int_{\Sigma_j} \left(\int_{P_y} (a_j \, d\varphi)(y) \right) d\mu(y).$$

On the other side, $d\varphi = d\left(\sum_j a_j\varphi\right) = \sum_j d(a_j\varphi)$ so that

$$C_\mu(d\varphi) = \sum_{j\in\mathbb{N}} \int_{\Sigma_j} \left(\int_{P_y} d(a_j \, d\varphi)(y) \right) d\mu(y).$$

Assume now that $a_j\varphi$ has compact support in P_y (not compact) so that, by the Theorem of Stokes, $\int_{P_y} d(a_j\varphi) = 0, \quad \forall y$ and thus

$C_\mu(d\varphi) = 0$. By definition the derivative $dC_\mu \in \mathcal{D}(M)$ is defined by $dC_\mu(\varphi) := C_\mu(d\varphi)$ where $\varphi \in A_c(M)$. Therefore $dC_\mu = 0$, that is, C_μ is a closed current. ■

Let us see some important consequences of this result:

As we have already seen, there exists an isomorphism of (De Rham) cohomology groups

$$H^q_{\text{sing}}(M,\mathbb{R}) \simeq H^q_{DR}(M) \simeq H^q(\mathcal{D}^*(M));$$

therefore each closed current $C \in \mathcal{D}^q(M)$ defines a class $[C]$ in the space $H^q_{DR}(M)$. By the Duality Theorem of of Poincaré, if M is orientable, we have a natural isomorphism $H^q_{DR}(M) \simeq \left(H^p_{c,DR}(M)\right)'$ = topological dual space of the cohomology group (of degree p) with compact support in M, of De Rham. Hence, we can associate to C a class $[[C]]$ in

$$\left(H^p_{c,DR}(M)\right)' \simeq \begin{cases} H_p(M,\mathbb{R}) & \text{if} \quad M \quad \text{is compact} \\ H^q(M,\mathbb{R}) & \text{if} \quad M \quad \text{is orientable.} \end{cases}$$

Thus, for M compact (respectively orientable) we have associate an invariant transverse measure for $\mathcal{F}$, the *homology class* (respectively *class of cohomology*) *of this measure.*
Let us see some examples:

10.4.1 Examples

1. If $N^p \subset M^n$ is an oriented submanifold compact of dimension p invariant by $\mathcal{F}$ then the class of the current of integration corresponding to N is the class $[N]$ of homology of N in $H_p(M,\mathbb{R})$; note that N is a compact leaf of $\mathcal{F}$.
2. Let $\mathcal{F}$ be a foliation of dimension p and codimension q of M^n. Assume that $\mathcal{F}$ and M are oriented and that is $\mathcal{F}$ transversally

orientada. The differential form Ω of grau q in M such that for each transverse disc to $\mathcal{F}$, $D^q \subset M$ we have $\Omega|_{D^q}$ is the form of volume (positive for the induced orientation in D^q) is a *transverse volume form* of $\mathcal{F}$ in M. We can choose a continuous vector field $X_{\mathcal{F}}$ of p-vectors on M such that in each point $x \in M$ we have $T_x\mathcal{F}$ = oriented space generated by $X_{\mathcal{F}}(x)$.

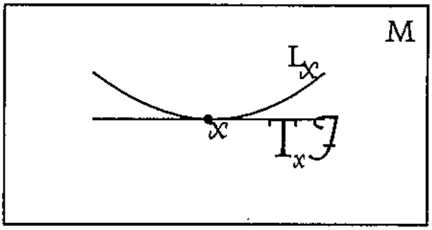


Figure 10.10:

In this case we can obtain a transverse volume form positive $\nu_{\mathcal{F}}$ for $\mathcal{F}$ in M of class C^∞ such that $\nu_{\mathcal{F}}(X_{\mathcal{F}}) = 1$ in M. We shall say that $\nu_{\mathcal{F}}$ is *normalized* for $X_{\mathcal{F}}$. In a general way, given transverse volume q-form Ω for $\mathcal{F}$ in M the associated current to Ω is defined by $C(\varphi) = \displaystyle\int_M \Omega \wedge \varphi$ and the homology class corresponding to C in $H^q(M, \mathbb{R})$ is the corresponding class of Ω in $H^q_{DR}(M)$.

3. Let $\alpha = \sum\limits_{j=1}^{r} a_j N_j$, $a_j \in \mathbb{Z}$; be a singular p-chain in M^n and denote by C the current of integration definida by α in M; if α is closed ($\partial\alpha = 0$) then C is closed ($dC = 0$) as consequence of the Theorem of Stokes. The class of C in $H_p(M, \mathbb{R})$ is the class of α in this same space. In this example we are not necessarily assuming the existence of a foliation in M which leaves α invariant.

4. Let now $\mathcal{F}$ be a foliation transverse to the fibres of the bundle $\pi\colon E \underset{F}{\longrightarrow} B$ in E; given an invariant transverse measure μ we have that μ corresponds (in bijective way) to a Borelian measure μ_0 over the fibre F which is invariant by the global holonomy $\mathrm{Hol}(\mathcal{F}) \subset \mathrm{Dif}(F)$ of $\mathcal{F}$, and finite in compact sets of F. Let C be a current corresponding a μ; then by construction we have which

$$C(\varphi) = \int_B \left(\int_{F_{\pi(y)}} \varphi \, d\mu(y) \right) = \int_B f_\varphi(y) d\mu_0(y)$$

where $f_\varphi\colon B \to \mathbb{R}$ is defined by the integration of φ along the fibres (cf. the figure below).

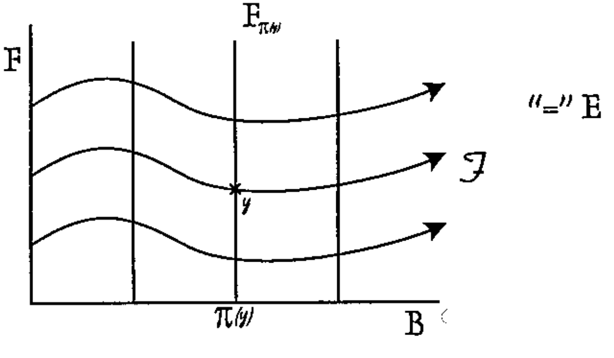


Figure 10.11:

Thus, in order to study the class of C in $H^q(M, \mathbb{R})$ we can report to the (class of the) measure μ_0 in $H^*(B, \mathbb{R})$.

Suppose now that the fibre F is compact and let us study the homology class of (one fibre) $[F_0]$ in $H_q(M, \mathbb{R})$. Take a tubular neighborhood $\Gamma: W \to F_0$ of this fibre in E such that the projection Γ has as fibres (transverse discs) the leaves of $\mathcal{F}|_W$.

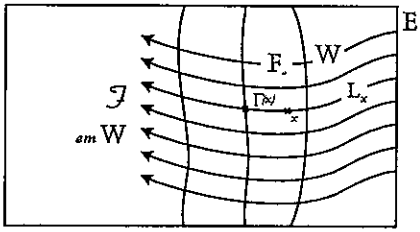


Figure 10.12:

We can assume $\overline{W} \subset M$ compact and using "bump functions" we obtain closed form $\varphi \in A_c^p(M)$ such that supp $\varphi \subset W$ and $\int_{D_x} \varphi = 1, \quad \forall x \in F_0$ where $D_x = r^{-1}(x)$ is a fibre of Γ by $x \in F_0$.

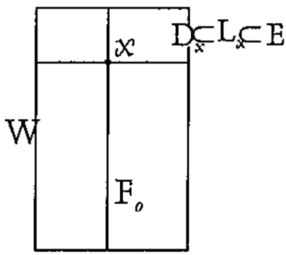


Figure 10.13:

But then we have $C(\varphi) = \mu_0(\pi(W)) > 0$ so that $C(\varphi) \neq 0$.

On the other hand, since φ is closed we have that its class in $H^p_{c,DR}(M,\mathbb{R})$ is dual to the class of F_0 in $H_q(M,\mathbb{R})$ so that if $[F_0] = [0]$ in $H_q(M,\mathbb{R})$ then $[\varphi] = [0]$ in $H^p_{c,DR}(M,\mathbb{R})$ and so $C(\varphi) = 0$ giving a contradiction. This shows that "the class of F_0 is not zero in $H_q(M,\mathbb{R})$" Since F_0 is arbitrary we conclude the same for any fibre of $\pi\colon E \to B$. The same proof gives us:
6. "Let $\mathcal{F}$ and M be oriented and $N^q \subset M^n$ compact submanifold without boundary and transverse to $\mathcal{F}$. If there exists invariant transverse measure μ for $\mathcal{F}$ with supp $\mu \cap N \neq \emptyset$ then $[N] \neq [0]$ in $H_q(M,\mathbb{R})$.

10.5 Cone structures in manifolds

In this section we shall follow [66]. Let $\mathbb{E}$ be a real locally convex topological vector space . Given a convex cone $C \subset \mathbb{E}$ we say that C is a connected convex *compact* if there exists linear functional $\varphi\colon \mathbb{E} \to \mathbb{R}$ such that

1. $\varphi(x) > 0, \quad \forall x \in C\backslash\{0\}$.
2. $\varphi^{-1}(1) \cap C$ is compact; called the *base* of the cone.

We denote by $\overset{\circ}{C}$ the set of radii of of C; $\overset{\circ}{C}$ it is direct identification with its base.

Definition 10.5.1. A *cone structure* in a closed subset F of a manifold C^∞ M is a continuous field of convex compact sets cones, say $\{C_x\}_{x\in F}$, in the vector spaces $\mathfrak{X}_p(x)$ of tangent p-vectors in M (for $x \in F$).

The continuity of the field $\{C_x\}_{x\in F}$ is defined in terms of the movement of its bases $\overset{\circ}{C}_x$ for a suitable metric in the radii

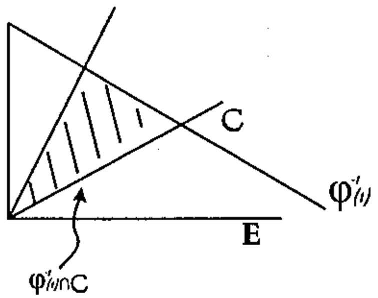


Figure 10.14:

(see [66]). Thus the p-form C^∞ Ω in M is said to be *transverse* to the cone structure $\{C_x\}_{x\in F}$ if $\Omega(x)(v_1, \ldots, v_p) > 0$, $\forall (v_1, \ldots, v_p) \in C_x \subset \mathfrak{X}_p(x)$ non-zero and $\forall\, x \in F$. Such transverse forms can always be constructed and determine currents: given point $x \in M$ we define the *Dirac current* associate to the fixed p-vector $X(p) \in \mathfrak{X}_p(x)$ by $\delta_{X,x}\colon \varphi \mapsto \varphi(X)(x)$; by choosing $X(x)$ in the cone $C_x \subset \mathfrak{X}_p(x)$ we obtain a collection of such currents of Dirac which gives us a closed convex cone of currents called *cone of currents of structure* associate to the cone structure $\{C_x\}_{x\in F}$. When F is compact the cone of currents of structure associate is a compact convex cone. We shall call the *structural cycles* of a cone structure in a manifold the structural currents which are closed (in the sense of currents). It is proven (cf. [66] §2) that if $F \subset M$ is compact then any structural current C writes as $C = \displaystyle\int_F f\, d\mu$ where μ is a measure ≥ 0 in F and f is an integrable function μ-integrable taking values in $\mathfrak{X}_p(M) = \{p\text{-vectors in } m\}$ and such that $f(x) \in C_x$ (cone structure given originally, $\forall x \in F$).

Chapter 11

Foliation cycles: Homological proof of Novikov Compact Leaf Theorem

Let $\mathcal{F}$ be an oriented foliation of class C^∞, dimension p and codimension q in M oriented, $X_\mathcal{F}$ a continuous field of p-vectors generating $T\mathcal{F}$ and $\nu_\mathcal{F}$ transverse volume form normalized for $X_\mathcal{F}$. Clearly $\mathcal{F}$ defines (via $X_\mathcal{F}$) a *foliation current* of dimension p over M; for each $x \in M$ we denote by $C_\mathcal{F}(x)$ the convex cone in T_xM generated by the fields of p-vectors tangent to $\mathcal{F}$ in x and denote by $C_\mathcal{F}$ the cone structure over M obtained this way; an element of the cone of currents of structure associate to $C_\mathcal{F}$ is called a *foliation current* of $\mathcal{F}$. In other words, a foliation current of $\mathcal{F}$ is an element do convex cone closed do space of currents of dimension p over M which is generated by Dirac currents of the form $\delta_{X,x}\colon \varphi \mapsto \varphi(X)(x)$ where $x \in M$ and X is a p-field tangent to $\mathcal{F}$.

Definition 11.0.2. A *foliation cycle* of $\mathcal{F}$ in M is a foliation current of $\mathcal{F}$ which is closed in the sense of currents, that is, a structural cycle of $C_{\mathcal{F}}$.

Owing to our above discussion (cf. §2.4) if μ is an invariant transverse measure for $\mathcal{F}$ in M then a current associate to μ is a foliation cycle of $\mathcal{F}$. The converse in the compact case was proven by D. Sullivan (cf. [66]):
"Let $\mathcal{F}$ be a foliation C^∞ of M compact and suppose $\mathcal{F}$ and M oriented. Then each foliation cycle for $\mathcal{F}$ in M comes (via the construction already presented) of a (unique) invariant transverse measure for $\mathcal{F}$."
We define the support of a current in the obvious way and we can then observe that if C is a foliation cycle for $\mathcal{F}$, coming from an invariant transverse measure μ in M, then $\operatorname{supp}(C) = \operatorname{supp}(\mu) \subset M$; in particular $\operatorname{supp}(C)$ is closed and $\mathcal{F}$-invariant in M.

11.0.1 Examples

1. All examples of currents (foliation) constructed from invariant measures in §2.4 give then examples of foliation cycles.

2. If μ is a measure Boreliana (positive not necessarily $\mathcal{F}$-invariant) over M a current of integration $C_\mu \colon \varphi \mapsto \int_M \varphi(X_{\mathcal{F}})d\mu$ is a foliation current for $\mathcal{F}$; by the Theorem of Sullivan above C_μ is a foliation cycle if, and only if, μ is $\mathcal{F}$-invariant.

3. Let $\mathcal{F}$ foliation transverse to the fibres of the bundle $\pi \colon E \underset{F}{\longrightarrow} B$ with global holonomy $\operatorname{Hol}(\mathcal{F}) \subset \operatorname{Dif}(F)$, then given measure Boreliana μ_0 in B a current C_μ associate to the measure μ defined by μ_0 em E is a foliation current (as in 2. above) which is a foliation cycle if, and only if, μ_0 and Hol($\mathcal{F}$)-invariant.

4. Let $\mathcal{F}$ foliation of codimension 1 in M; according to Haefliger's Theorem (Chapter 5) if $\mathcal{F}$ has a transverse closed curve homotopic to zero in M then there exists leaf L_0 of $\mathcal{F}$ and loop (of holonomy) $\alpha_0 \in \pi_1(L_0)$ with holonomy $f_{\alpha_0} \in \mathrm{Dif}((-\varepsilon,\varepsilon),0)$ such that $f|_{(-\varepsilon,0]}$ is the identity and $f|_{(0,+\varepsilon)}$ is increasing.

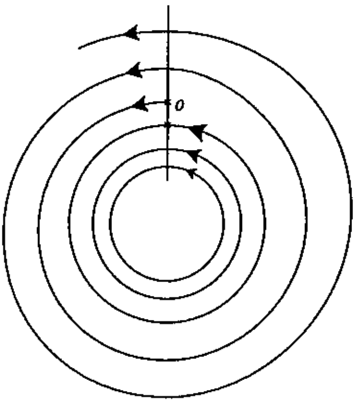


Figure 11.1:

Such a leaf L_0 we will call in general a *ressort leaf.* Let us see how the existence of of invariant measure for $\mathcal{F}$ restricts the existence of ressort leaves. Indeed, given an invariant transverse measure μ for $\mathcal{F}$ let $K = \mathrm{supp}(\mu)$ be the support of μ (not necessarily compact). We claim:

(a) "K is contained in the union of minimal sets of $\mathcal{F}$;

(b) K does not contain ressort leaf."

In order to see (a) it is enough to observe that K is invariant, closed and any leaf contained in K is dense in K. Since the measure μ is finite in compact sets we conclude that K does not contain a ressort leaf and, since $\mathcal{F}$ is of codimension 1 the minimal sets of $\mathcal{F}$ are either closed leaves, or dense, or an uncountable union of exceptional leaves (set of Cantor). This shows (b). ■

11.0.2 Homological proof of Novikov Compact Leaf Theorem

Note that above we strongly use the fact that $\mathcal{F}$ is of codimension 1. Suppose now that $\dim \mathcal{F} = 2$ and $\dim M = 3$ so that $\mathcal{F}$ of codimension 1. We will also assume M compact and that $\mathcal{F}$ has a vanishing cycle, say, in the leaf L_0 of $\mathcal{F}$. We will show how to construct the foliation cycle for $\mathcal{F}$; there is no loss of generality if we assume that vanishing cycle is *simple*: recall that (cf. Chapter 6) a vanishing cycle of $\mathcal{F}$ in the leaf L_0 consists of a lace (closed) $\alpha_0\colon [0,1] \to L_0$ such that it extends to a continuous application $\alpha\colon [0,1] \times [0,1] \xrightarrow{C^0} M$ with the following properties:

(i) Given $t \in [0,1]$ the application $\alpha_t\colon [0,1] \to L_t\,, \quad \alpha_t(s) = \alpha(t,s)$ defines a loop in the leaf L_t of $\mathcal{F}$.

(ii) α_0 is the loop originally given in L_0.

(iii) α_0 is not homotopic to zero in L_0 but α_t is homotopic to zero in $L_t \quad \forall t \in (0,1]$.

(iv) Fixed $s \in [0,1]$ the curve $C_s\colon [0,1] \to M \quad t \mapsto \alpha_t(s)$ is transverse to the foliation $\mathcal{F}$.

The vanishing cycle is called *simple* when also we have

(v) the lift of α_t, denoted by $\hat{\alpha}_t$, to the universal covering $\widetilde{L}_t$ of the leaf L_t is, for each $t \neq 0$, the closed curve (because $\alpha_t \sim 0$

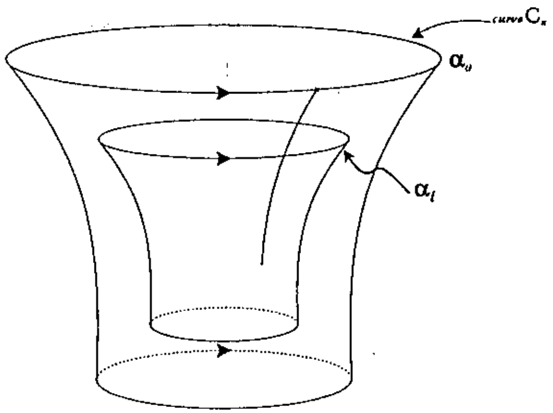


Figure 11.2:

in $\pi_1(L_t)$) which is simple (that is, without self-intersection).

We can approximate continuous functions by functions C^1 so that we can assume that $\alpha\colon S \times [0,1] \to M$ is of class C^1.
Note that the universal covering $\widehat{L_t}$ of L_t is necessarily (diffeomorphic to) $\mathbb{R}^2$ because otherwise $\mathcal{F}$ would have some leaf covered by S^2, this leaf would be compact and being orientable it would be the sphere with $g \geq 0$ aisles; if $g = 1$ the universal covering is $\mathbb{R}^2$ and if $g \geq 2$ then the universal covering (as a Riemman surface) is the unit disc $\mathbb{D} \subset \mathbb{R}^2$ so we must have $g = 0$ and the leaf would be diffeomorphic to S^2.

By the Global Stability Theorem of Reeb $\mathcal{F}$ would be a compact fibration over the circle S^1 with fibres S^2 and in this case it could not have vanishing cycle (all the leaves would be simply-connected).

Now, since each leaf L_t is covered by $\widehat{L_t} \simeq \mathbb{R}^2$ each (simple)

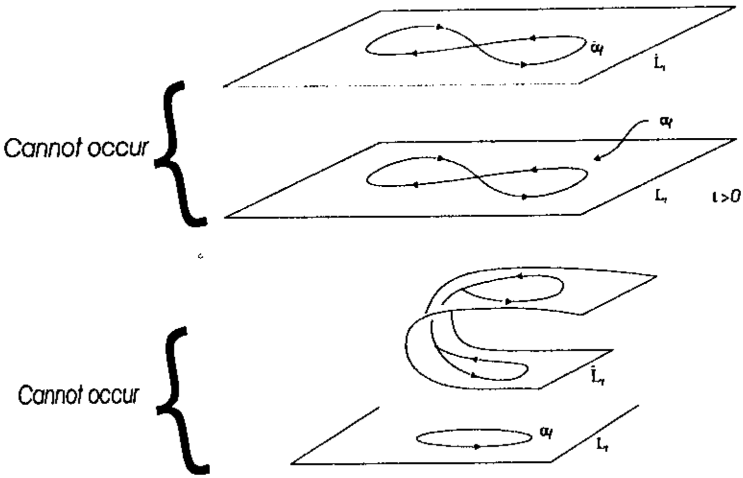


Figure 11.3:

curve $\hat{\alpha}_t$ in $\widehat{L}_t$ is boundary of a disc $\widehat{D}_t \subset \widehat{L}_t$ this allows us to obtain an immersion C^1, $A\colon D^2 \times (0,1] \to M$ of the solid cylinder (not compact) $D^2 \times [0,1]$ in M with the following properties:

(vi) $A_t|_{S^1 = \partial D^2} = \alpha_t\,, \quad \forall t \in (0,1]$ and the image $A_t(D^2) \subset L_t\,, \quad \forall t \in (0,1]$.

(vii) Given an oriented transverse flow $\vec{X} \pitchfork \mathcal{F}$ chosen from the beginning from the transverse orientability of $\mathcal{F}$ in M we have which $A_t\colon D^2 \to L_t$ define, for $0 < \delta \leq t \leq 1$, lift of $A_1\colon D^2 \to L_1$ by the transverse flow $\vec{X}$

Since α_0 not is homotopic to zero in L_0 and since for each $x \in S^1$ the curve $[0,1] \to M, \quad x \mapsto A(x,t)$ has a limit when $t \to 0^+$ we conclude that

(viii) The set $W = \{x \in D^2; t \mapsto A(x,t)$ has a limit when $t \mapsto 0^+\}$ which is an open neighborhood of S^1 in D^2 with $S^1 \subset$

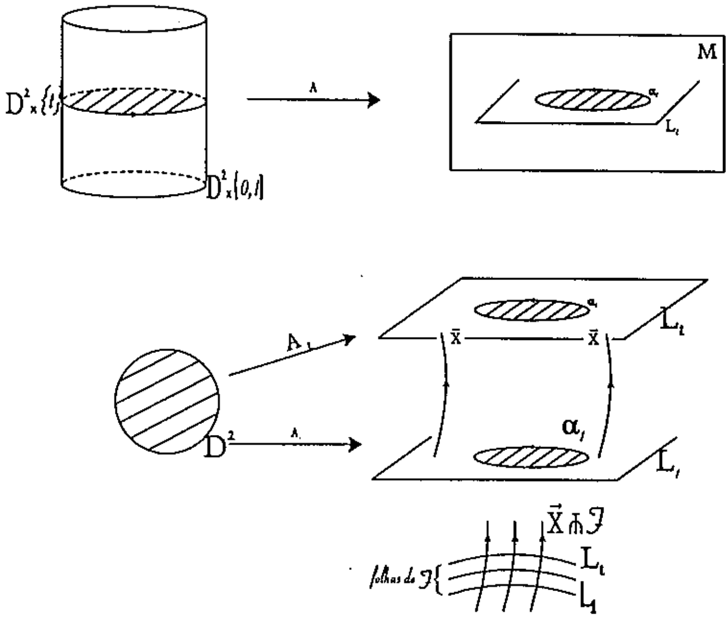


Figure 11.4:

$W \subsetneq D^2$.

We denote by C_t, for each $0 < t \leq 1$, the foliation current of $\mathcal{F}$ defined by the integral

$$C_t(\varphi) := \frac{1}{\mathrm{vol}(D_t)} \int_{D_t} \varphi, \text{ where } \varphi \in A_c^p(M)$$

(note that still we are not using the fact that M is compact, which will be used in what follows), and where $D_t = A_t(D^2) \subset L_t$ (note which $A_t \colon D^2 \to D_t \subset L_t \subset M$ is imersão C^1). We obtain then, using the terminology of [66], a *family of Plante* of foliation currents $\{C_t\}_{t\in(0,1]}$ defined by the properties below which can be easily verified:

(ix) Each C_t has mass 1 so that $\{C_t\}_{t\in(0,1]}$ is a pre-compact family of currents (weak topology).

(x) Each accumulation C of $\{C_t\}_{t\in(0,1]}$ is necessarily a foliation cycle of $\mathcal{F}$: in fact if $t_n \searrow 0$ is such that $C_{t_n} \to C$ then fact that the quotient $\dfrac{\text{length}(\partial D_{t_n})}{\text{Area}(D_{t_n})} \to 0$ implies that the mass of the derivative dC_{t_n} satisfies mass $(dC_{t_n}) \to 0$ and therefore $\text{mass}(dC) = 0$. We obtain then foliation cycles C for $\mathcal{F}$ in M^3, if we suppose M^3 compact, from the existence of a vanishing cycle for a leaf L_0^2 of $\mathcal{F}$, foliation of codimension 1 in M^3.

Such facts have been generalized by D. Sullivan for higher dimension with the notion of vanishing cycle of dimension $= \dim \mathcal{F} - 1$ (cf. [66]).
We can now conclude the following:

$$\left[\begin{array}{c}\mathcal{F} \text{ foliation orientable and transversally orientable } C^2 \text{ of} \\ \text{codimension 1 of } M^3 \text{ compact, } \mathcal{F} \text{ with a leaf } L_0 \\ \text{containing a vanishing cycle}\end{array}\right]$$

$$\Downarrow$$

$$\left[\mathcal{F} \text{ has a foliation cycle } C \text{ whose support contains a leaf } L_0\right]$$

$$\Downarrow$$

$$\left[\begin{array}{c}\mathcal{F} \text{ admits an invariant transverse measure } \mu \text{ whose support} \\ \text{contains } L_0 \text{ (in fact } \text{supp}(\mu) = \text{supp}(C))\end{array}\right]$$

$$\Downarrow$$

since $\mathcal{F}$ is of codimension 1 and M compact we have that $K = \text{supp}(\mu)$ (contains L_0) does not contain ressort leaf and is contained in the union of minimal sets of $\mathcal{F}$ in M. Thus K is a union of compact leaves and hence L_0 is a compact leaf of M that is,

$$\left[M \text{ has } L_0 \text{ as compact leaf.} \right]$$

This is a homological demonstration of Novikov Compact Leaf Theorem.

■

Chapter 12

Miscellaneous exercises

12.1 Exercises for the text

Exercise 12.1.1. Let $\mathcal{F}$ be an orientable foliation with a compact leaf $L \in \mathcal{F}$ homologous to zero. Prove that the Euler characteristic of L is zero.

Exercise 12.1.2. Let $\mathcal{F}$ be a codimension one foliation on S^3 with a compact leaf $L \in \mathcal{F}$ homologous to zero. Show that L is the torus.

Exercise 12.1.3. There exist no analytic foliation of codimension one of the sphere S^n for $n \geq 2$.

Exercise 12.1.4. Let $G < \mathbb{R}^2$ be a discrete subgroup. Show that G is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}^2$. Conclude that the orbits of a locally free action of the affine group $\text{Aff}(\mathbb{R})$ are either planes or cylinders.

Exercise 12.1.5 (Implicit ordinary differential equations). An *algebraic implicit ordinary differential equation* in $n \geq 2$ complex variables is given by expressions:

$$(**)\ f_j(x_1,...,x_n,x_j') = 0$$

where $f_j(x_1,...,x_n,y) \in \mathbb{C}[x_1,...,x_n,y]$ are polynomials and the $(x_1,...,x_n) \in \mathbb{C}^n$ are affine coordinates. Clearly, any polynomial vector field X on $\mathbb{C}^n$ defines such an equation. In general $(**)$ defines a one-dimensional singular foliation in some algebraic variety of dimension n. In order to see it we begin by defining $F_j(x_1,...,x_n,y_2,...,y_n) := f_j(x_1,...,x_n,y_j) \in \mathbb{C}[x_1,...,x_n,y_2,...,y_n]$ polynomials in $n+(n-1) = 2n-1$ variables. Put also $S_j := \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^{n-1}; F_j(x,y) = o\} \simeq \{(x_1,...,x_n,y_j) \in \mathbb{C}^n_x \times \mathbb{C}_{y_j};\ f_j(x_1,...,x_n,y_j) = 0\} \times \mathbb{C}^{n-2} =: \Lambda_j \times \mathbb{C}^{n-2}_{(y_2,...,\hat{y}_j,...,y_n)}$.

We consider the projectivizations $\overline{S_j} \subset \mathbb{C}P(2n-1)$ and the complete intersection subvariety $S := \overline{S_2} \cap ... \cap \overline{S_n} \subset \mathbb{C}P(2n-1)$. Given by the differential forms $\omega_j := y_j dx_1 - dx_j \quad (j = 2,...,n)$ on $\mathbb{C}^n \times \mathbb{C}^{n-1}$. Prove that $\{\omega_j = 0,\ j = 2,...,n\}$ defines an integrable system on S. We say that the implicit differential equation $(*)$ is *normal* if S admits a normalization (desingularization) by blow-ups $\sigma\colon \hat{S} \to S$. In particular we obtain in general a singular foliation $\mathcal{F}(**)$ of dimension one on the algebraic n-dimensional subvariety $S \subset \mathbb{C}P(2n-1)$. Denote by $f_1\colon S \cap \mathbb{C}^n \to \mathbb{C}^1$ the projection in the first coordinate $f_1(x_1,...,x_n,y_2,...,y_n) = x_1$, and extend it to a holomorphic proper mapping $f_1\colon \overline{S} \to \mathbb{C}P(1)$. Assume now that S admits a normalization $\sigma\colon \hat{S} \to S$. Show that the foliation $\mathcal{F}(**)$ lifts to a foliation by curves $\hat{\mathcal{F}}(**)$ on $\hat{S}$ and $\hat{f}_1 = f_1 \circ \sigma$ defines a holomorphic proper mapping from $\hat{\overline{S}}$ over $\mathbb{C}P(1)$. Finally, using Stein Fatorization Theorem find a splitting $\hat{f}_1\colon \hat{S} \xrightarrow{\hat{f}} B \xrightarrow{\alpha} \mathbb{C}P(1)$ where $\alpha\colon B \to \mathbb{C}P(1)$ is a finite ramified covering and $\hat{f}\colon \hat{S} \to B$ is an extended holomorphic fibration over the compact Riemann surface B such that the following diagram therefore commutes

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\sigma} & \overline{S} \\ \hat{f} \downarrow & & \downarrow f_1 \\ B & \xrightarrow{\alpha} & \mathbb{C}P(1) \end{array}$$

for a map $\hat{f}_1 \colon \hat{S} \to \mathbb{C}P(1)$.

Exercise 12.1.6. Let $\mathcal{F}$ be a foliation on M of codimension q. A differentiable map $f \colon N \to M$ is *transverse* to $\mathcal{F}$ if it is transverse to each leaf $L \in \mathcal{F}$ as an immersed submanifold in M. Show that in this case there is a naturally defined foliation $f^*(\mathcal{F})$ in N of codimension q such that for each leaf $L \in \mathcal{F}$ the inverse image $f^{-1}(L)$ is a union of leaves of $f^*(\mathcal{F})$.

Exercise 12.1.7. Let $\mathcal{F}$ be a codimension one transversely orientable foliation on M given by the integrable differential 1-form ω in M. Show that there is a 1-form η in M such that $d\omega = \eta \wedge \omega$, also show that given any leaf $L \in \mathcal{F}$ the restriction $\eta\big|_L$ is closed and given any closed path $\gamma \in \pi_(L,p)$ with base point $p \in L$ then the linear holonomy of the loop γ is given by $h'_{[\gamma]}(0) = \exp(\int_\gamma \eta\big|_L)$.

Exercise 12.1.8. A closed subset $F \subset \mathbb{R}$ is *perfect* if it has no isolated points. Bendixson's Theorem states that a countable closed subset of the real line must have an isolated point. Prove the following: every closed subset of the line is the reunion of a perfect set and a countable set.

Exercise 12.1.9. Let ω be a C^2 integrable 1-form in a neighborhood of the origin $0 \in \mathbb{R}^n$. We assume that the origin is a singularity of center type for ω so that, up to a linear change of coordinates we have $\omega = d(\frac{1}{2}\sum\limits_{j=1}^{n} x_j^2) + (...)$ where $(...)$ means higher order terms. A classical result due to Reeb states that for

$n \geq 3$ there is a neighborhood of the origin where all the leaves of $\mathcal{F}_\omega : \omega = 0$ are diffeomorphic to the $(n-1)$-sphere. This is proved as follows:
(i) Consider the cylindrical blow-up of the origin given by the map $\sigma: \mathbb{R} \times S^{n-1} \to \mathbb{R}^n$, $\sigma(t,x) = t.x$. Show that $\{0\} \times S^{n-1}$ is a leaf of the lifted foliation $\mathcal{F}^* = \sigma(\mathcal{F}_\omega)$ (hint: show that the 1-form $\Omega^* = \frac{1}{t}\sigma^*(\omega)$ defined in $(\mathbb{R} - \{0\}) \times S^{n-1}$ extends to $\mathbb{R} \times S^{n-1}$ as $\Omega^* = dt$ for $t = 0$ in class C^1. Also show that $\{0\} \times S^{n-1}$ is a leaf of $\sigma^*(\omega)$ and so of Ω^*.
(ii) For $n \geq 3$ use the Local Stability Theorem to conclude.

Exercise 12.1.10. Look (possibly in the literature) for a demonstration of the following analytic version (also due to Reeb) of the above exercise: If ω is a real analytic integrable 1-form in a neighborhood of the origin $0 \in \mathbb{R}^n$ and $n \geq 3$. Suppose that the linear part of ω is non-degenerate and $\omega = df + (...)$ for some quadratic analytic function f. Then there is a neighborhood of the origin where $\mathcal{F}_\omega : \omega = 0$ is analytically conjugate to the linear foliation $df = 0$.

Exercise 12.1.11. Give a demonstration of Darboux-Lie Theorem (Theorem 1.2.29) according to the following suggestion:

Given 1-forms forming a basis $\{\omega_1, ..., \omega_n\}$ of the Lie Algebra of the Lie group G and given 1-forms $\{\Omega_1, ..., \Omega_n\}$ a rank-n system of 1-forms in a manifold M such that $d\Omega_k = \sum\limits_{i,j} c_{ij}^k \Omega_i \wedge \Omega_j$, where the $\{c_{ij}^k\}$ are the structure constants of the Lie Algebra relatively to the given basis, we can define 1-forms $\Theta_j = \Omega_j - \omega_j$, $j = 1, ..., n$; in a natural way in the product manifold $M \times G$. The system $\{\Theta_1, ..., \Theta_n\}$ is integrable and by Frobenius Theorem defines a foliation $\mathcal{F}$ of the product manifold. Given a leaf $L \in \mathcal{F}$ we have that Ω_j and ω_j coincide over L. Using then the natural projections $M \times G \to M$ and $M \times G \to G$ we can obtain local submersions $\pi: U \subset M \to G$ such that $\pi^*\omega_j = \Omega_j, \forall j$.

In order to conclude one has to prove that if a diffeomorphism ξ of G preserves ω_j for all j then ξ is a left translation in G.

Exercise 12.1.12. Show that if G is a simply-connected Lie group and M is a compact manifold of dimension $\dim M = 1 + \dim G$ then for $\dim G \geq 2$ there is no locally free action of G in M.

Exercise 12.1.13 (Double of a foliation). Let $\mathcal{F}$ be a smooth foliation on M. Suppose that we have a relatively compact domain $D \subset M$ with smooth boundary ∂D transverse to $\mathcal{F}$. Consider the manifold with boundary $M_0 = M \backslash D$ and the restriction $\mathcal{F}_0 = \mathcal{F}\big|_{M_0}$. Given two copies M_1 and M_2 of M_0 we can construct a manifold M_d by gluing these copies by the common boundary ∂D and equip it with a smooth foliation $\mathcal{F}_d$ such that $\mathcal{F}_d\big|_{M_j}$ is naturally conjugate to $\mathcal{F}_0$.

Exercise 12.1.14. Let $X_{\lambda,\mu} = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ a complex vector field in a neighborhood od the origin $0 \in \mathbb{C}^2$. Show that $X_{\lambda,\mu}$ is transverse to the 3-spheres $S^3(0, R)$ for $R > 0$ small enough, if and only if, $\lambda/\mu \in \mathbb{C} \setminus \mathbb{R}_-$. Let now X be a polynomial vector field in $\mathbb{C}^2$ and assume that the singularities of the corresponding foliation $\mathcal{F}$ on $\mathbb{C}P(2)$ are of local form $X_{\lambda,\mu}$ with $\lambda/\mu \notin \mathbb{R}$. Choose small balls $\mathbb{B}(p_j)$ around the singularities $p_j \in \operatorname{sing}(\mathcal{F})$ in $\mathbb{C}P(2)$. Show that there is a foliation $\mathcal{F}_d$ in a manifold M_d with the following properties: This is a C^∞ regular codimension-two real foliation $\mathcal{F}_d$ on a compact real 4-manifold M_d, which contains two copies of the foliated pair $(\mathbb{C}P(2) \setminus \bigcup_{j=1}^{r} \overline{\mathbb{B}}(p_j), \mathcal{F}\big|_{\mathbb{C}P(2)\setminus \bigcup_{j=1}^{r} \overline{\mathbb{B}}(p_j)})$.

By Schwarz Reflection Principle the leaves of $\mathcal{F}_d$ have also natural structures of Riemann surfaces. Any Riemannian metric g in $\mathbb{C}P(2)$ induces a C^∞ Riemannian metric g_d in M_d, that can be chosen to be hermitian along the leaves of $\mathcal{F}_d$. Show that the leaves of the non-singular foliation $\mathcal{F}\big|_{CP(2)\setminus\text{sing}(\mathcal{F})}$ have the same growth type than the corresponding leaves of $\mathcal{F}_d$.

12.2 Advanced e xercises

Exercise 12.2.1. Show that if $\mathcal{F}$ is a codimension q smooth foliation of a manifold M and $L \in \mathcal{F}$ is a compact leaf with $\text{Hom}(\pi_1(L), \mathbb{R}) = 1$ and $H^(L, \mathbb{R}) = 0$ then L has trivial holonomy (Thurston).

Exercise 12.2.2. If $\mathcal{F}$ is a transversely orientable codimension one smooth foliation on a manifold M and $L \inf \mathcal{F}$ is a compact leaf with $H^1(L, \mathbb{R}) = 0$ then L has trivial holonomy.

Exercise 12.2.3. Let $M = S^n \times D^m$ be the product manifold of the n-sphere and the closed m-disc. Show that for $n \geq 2$ and $m \geq 3$ there is no foliation $\mathcal{F}$ on M of codimension one which is tangent to the boundary of M.

Exercise 12.2.4. Let $\mathcal{F}$ be a C^2 codimension one transversely oriented foliation on the closed manifold M^3. Let L be a compact leaf of genus $g \neq 0$. Show that there is a closed transversal γ to $\mathcal{F}$ that meets L and that the foliation $\mathcal{F}$ is related to a foliation having all leaves diffeomorphic to tori.

Bibliography

[1] M. Brunella; *A Global Stability Theorem for transversely holomorphic foliations*; Annals of Global Analysis and Geometry 15, 179-186, 1997.

[2] M. Brunella; *Courbes entières et feuilletages holomorphes*, Enseign. Math. 45 (1999), 195-216.

[3] M. Brunella; *Birrational geometry of foliations*, Notas de Curso, First Latin American Congress of Mathematicians, IMPA - Rio of Janeiro (2000).

[4] C. Camacho, A. Lins Neto; *Geometric theory of foliations*, Translated from the Portuguese by Sue E. Goodman. Birkhauser Boston, Inc., Boston, MA (1985).

[5] C. Camacho, P. Sad; *Pontos singulares de equações diferenciais analíticas*, 16^{the} Colóquio Brasileiro of Matemática, Publ. IMPA - Rio de Janeiro, 1987.

[6] J. Canille and B. Scárdua; *On the growth of complex projective foliations*, Int. Journ. Mathematics, Vol.13, No.7 (2002), 695-726.

[7] J.-P. Demailly; *Variétés hyperboliques et équations différentielles algébriques*, Gaz. Math. No. 73 (1997), 3–23.

[8] de Melo, W., Palis, J., *Geometric theory of dynamical systems. An introduction.*, Translated from the Portuguese by A. K. Manning. Springer-Verlag, New York-Berlin (1982).

[9] do Carmo, M., P., *Riemannian geometry*, Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhauser Boston, Inc., Boston, MA (1992).

[10] G. De Rham; *Variétés Différentiables*, Paris: Hermann, 1960.

[11] G. De Rham; *Varietés différentiables. Formes, courantes, formes harmoniques.* Paris: Hermann 1955.

[12] C. Ehresmann; *Sur les sections d'un champ d'éléments of contact dans une variété différentiable.* C. R. Acad. Sci. Paris 224, (1947). 444–445.

[13] C. Ehresmann, G. Reeb; *Sur les champs d'éléments of contact*, C.R. Acad. Sci. Paris, 218, 1944.

[14] E. Ghys; *Holomorphic Anosov Systems*, Inv. Math. 119, 585-614 (1995).

[15] C. Godbillon; *Feuilletages: Études Geométriques*, Birkhäuser, Berlin 1991.

[16] P. Griffiths, J. Harris; *Principles of Algebraic Geometry*, John Wiley & Sons, New-York, 1978.

[17] R.C. Gunning; *Introduction to holomorphic functions of several variables, vol. I*, Wadsworth & Brooks/Cole, California 1990.

[18] R.C. Gunning; *Introduction to Holomorphic Functions of Several Variables*, Vol. II, Local Theory, Wadsworth & Brooks/Cole Advanced Books & Software, California, USA, 1990.

[19] R.C. Gunning, H. Rossi; *Analytic Functions of Several Complex Variables*; Prentice Hall; Englewood Cliffs; N.J. 1965.

[20] A. Haefliger; *Some remarks on foliations with minimal leaves.* J. Differential Geom. 15 (1980), in the. 2, 269–284 (1981).

[21] A. Haefliger; *Structures feuilletées et cohomologie à valeur dans un faisceau of groupoïdes.* Comment. Math. Helv. 32 1958 248–329.

[22] Hector, G., Hirsch, U., *Introduction to the geometry of foliations. Part B. Foliations of codimension one.*, Second edition. Aspects of Mathematics, E3. Friedr. Vieweg & Sohn, Braunschweig (1987).

[23] Herstein, I., N., *Topics in algebra*, Second edition. Xerox College Publishing, Lexington, Mass.-Toronto, Ont. (1975).

[24] M. Hirsch, W. Thurston; *Foliated bundles, invariant measures and flat manifolds.* Ann. Math. (2) 101 (1975), 369–390.

[25] M. Hirsch; *Stability of compact leaves of foliations.* Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 135–153. Academic Press, New York, 1973.

[26] H. Holmann; *On the stability of holomorphic foliatons*; Springer Lect. Notes in Math. 198 (1980), 192-202.

[27] T. Ito; *A Poincaré-Bendixson Type Theorem for Holomorphic Vector Fiels,* RIMS Publications, Kyoto, n$^{\text{the}}$ 878, 1994.

[28] T. Ito; B. Scárdua; *On real transverse section of holomorphic foliations*; preprint Kyoto 2001.

[29] E. L. Lima, *Grupo Fundamental e Espaços de Recobrimento*, Segunda Edição ISBN 85-244086-2 Publications IMPA (1999).

[30] E. L. Lima, *Introduccion a la cohomologia de Rham*, Monografias del IMCA. **18** (2001).

[31] E. Lima; Commuting vector fiels on S^3; Ann. of Math., vol. 81, 1965, p. 70-81.

[32] A. Lins Neto; *Algebraic solutions of polynomial differential equations and foliations in dimension two.* Lect. Notes in Math. n$^{\text{the}}$ 1345, 192–231.

[33] M. McQuillan; *Diophantine approximation and foliations*, Publ. Math. I.H.E.S. 87 (1998), pp. 121-174.

[34] M. McQuillan; *Non-commutative Mori Theory,* Prepublication I.H.E.S./M/01/42, Augsut 2001.

[35] L.G. Mendes; *On algebraic and entire curves invariant by singular holomorphic foliations*, preprint Univ. Dijon, 1999.

[36] J. Milnor; *Growth of finitely generated solvable groups.* J. Differential Geometry 2 1968 447–449.

[37] R. Moussu, F. Pelletier; *Sur le théorème of Poincaré-Bendixson.* Ann. Inst. Fourier (Grenoble) 24 (1974), in the. 1, 131–148.

[38] P. Painlevé; *Leçons sur la théorie analytique des équations différentielles*, Paris, Librairie Scientifique A. Hermann, 1897.

[39] J. Plante; *A Poincaré-Bendixson Theorem*, Topology 12, pp. 177-181, 1973.

[40] J. Plante; *On the existence of exceptional minimal sets in foliations of codimension one.* J. Differential Equations 15 (1974), 178–194.

[41] J. Plante; *Measure preserving pseudogroups and a theorem of Sacksteder.* Ann. Inst. Fourier (Grenoble) 25 (1975), in the. 1, vii, 237–249.

[42] J. Plante; *Anosov Flows*, Am. Journal of Mathematics, 94 (1972), pp. 729-754.

[43] J. Plante; *Foliations with measure preserving holonomy*; Ann. of Math., 102, 1975, pp. 327-361.

[44] J. Plante, W. Thurston; *Polynomial growth in holonomy groups of foliations.* Comment. Math. Helv. 51 (1976), in the. 4, 567–584.

[45] G. Reeb; *Variétés feuilletées, feuilles voisines.* C. R. Acad. Sci. Paris 224, (1947). 1613–1614.

[46] G. Reeb; *Sur les variétés intégrales des champs d'éléments of contact complètement intégrables.* C. R. Acad. Sci. Paris 220, (1945). 236–237.

[47] G. Reeb; *Remarque sur les variétés feuilletées contenant une feuille compacte to the groupe of Poincaré fini.* C. R. Acad. Sci. Paris 226, (1948). 1337–1339

[48] G. Reeb; *Sur certaines propriétés topologiques des variétés feuilletées*; Actualités Sci. Ind., Hermann, Paris, 1952.

[49] H. Rosenberg, R. Roussarie, D. Weil; *A classification of closed orientable 3-manifolds of rank two*, Ann. of Math. 91, no. 2, (1970), 449-464.

[50] D. Ruelle; Statistical Mechanics, New York, Benjamin 1969.

[51] D. Ruelle, D. Sullivan; *Currents, Flows and Diffeomorphisms*, Topology, vol. 14, n$^{\text{the}}$ 4.

[52] H. Rummler; Comment. Math. Helv., 54, 1979.

[53] P. Molino; *Riemannian Foliations*; Progress in Mathematics, Birkhäuser, Boston 1988.

[54] D.B.A. Epstein; *Foliations with all leaves compact*; Ann. Inst. Fourier, Grenoble 26, 1 (1976), 265-282.

[55] R. Edwards, K. Millet, D. Sullivan; *Foliations with all leaves compact*; Topology 16, (1977), 13-32.

[56] R. Sacksteder; *Foliations and pseudogroups.* Amer. J. Math. 87, 1965, 79–102.

[57] R. Sacksteder; *On the existence of exceptional leaves in foliations of codimension one*; Ann. Inst. Fourier, Grenoble, 14, 2, 1969, pp. 224-225.

[58] B. Scárdua;*Polynomial foliations with sub-exponential growth*, Indagationes Mathematicae, N.S., 12 (3), pp.293-302, Sept. 2001.

[59] B. Scárdua; *Holomorphic Anosov Systems, Fibrations and Kähler manifolds*; Preprint IM-UFRJ, 2002.

[60] B. Scárdua; *Some Topics on Geometric Theory of Foliations*; Publications do IMCA (Lima-Perú), 2002.

[61] B. Scárdua, Transversely affine and transversely projective foliations, *Ann. Sc. École Norm. Sup.*, 4^e série, t.**30**, 1997, p.169-204

[62] S. Schwartzmann; *Asymptotic cycles*, Ann. Math., 66, pp. 270-284, (1957)

[63] L. Schwartz; *Théorie des distributions*, Nouvelle Édition, Paris, Hermann, 1966.

[64] Y. Siu, Techniques of Extension of Analytic Object, Lecture Notes in Pure and Appl. Math. 8, Marcel Dekker, Inc., New York, 1974.

[65] N.S. Steenrod; *The Topology of fibre bundles*, Princeton Univ. Press, 1951.

[66] D. Sullivan; *Cycles for the Dynamical study of Foliated Manifolds and Complex Manifolds*, Inventiones Math., 36, pp. 225-255, (1976).

[67] W. Thurston; *A generalization of the Reeb Stability Theorem*, Topology 31, pp. 347-352, 1974.

[68] W. P. Thurston, Anosov Flows and the Fundamental Group, *Topology*, Vol. **11**, pp.147-150. Pergamon Press, 1972.

[69] D. Tischler; *On fibering certain foliated manifolds over* S^1, Topology, vol. 9 (1970), pp. 153-154.

[70] J. Tits; *Free subgroups in linear groups*, J. of Algebra, 20, 1972, pp. 232-270.

[71] J. Vitório; PhD. Thesis, IMPA-Rio de Janeiro, 2001.

[72] J. Wolf; *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, Journ. Diff. Geom., 2, 1968, pp. 421-446.

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