Partial Regularity of Solutions of the 3-D Incompressible Navier-Stokes Equations

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Preface

The main motivation for the idea of realization of these lectures in the 23^{th} Brazilian Colloquium of Mathematics was, of course, the evidence gained by the subject, far beyond the circle of specialists, due to the inclusion of the problem of the regularity of solutions of the three-dimensional incompressible Navier-Stokes equations among the seven "Clay Mathematics Institute Millennium Prize Problems", launched last year. This immediately prompted the first author as a highly singular oportunity to disseminate the interest on this remarkably outstanding problem throughout the Brazilian community of specialists in Partial Differential Equations. Since he himself is not a specialist on incompressible Navier-Stokes equations, he would like to express his gladness for having had the younger second author joining him in this enterprise. This became easier for both to learn enough about the subject and to figure out a plan of exposition which, while being as simple and direct as possible, should, nevertheless, display a truly important aspect of the state of the art on the subject. The latter was the guiding principle for the elaboration of these notes. The best partial regularity result for the 3-D incompressible Navier-Stokes equations as yet is the almost 20 years old theorem of Caffarelli, Kohn and Nirenberg [2] which improved the pioneering estimates for the Haudorff dimension of the singular set of suitable weak solutions put forth by Scheffer [12]-[15] from 5/3 to 1. So we decided to present the theorem of Caffarelli, Khon and Nirenberg in the simplest case: the Cauchy problem without external force. This by itself provides a considerable simplification in the proofs while retaining the essence of the method. Moreover, even in this simplest case the solution of the corresponding regularity problem is worthy the 1 million dollars CMI Prize!

In writing the notes the authors have also drawn from F. Lin's paper [9], specifically for the second part which is the passage from Scheffer's estimate to the actual one of Caffarelli, Kohn and Nirenberg, which provides a slightly more direct proof of the corresponding result. Unfortunately, for the first part, that is, the part giving Scheffer's theorem, Lin's paper contains a flaw that these authors were not able to fix[1]. So, for that part, we stuck to the approach in [2], slightly modified, taking advantage of the simplifications allowed by the considered case. This also seems to highlight yet more the essential points of the strategy at that stage.

In sum, we think that we will have achieved our goal if these notes encourage the readers to study fully the original article by Caffarelli, Kohn and Nirenberg as well as other relevant references.

In concluding this preface the authors would like to acknowledge the support of FAPERJ, proc. E-26/151.890/2000, without which this project would not have been possible.

Rio de Janeiro, July, 2001
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[1]The correction of the referred problem in Lin's paper is contained in [7], which is heavily based on ideas of [9]

Chapter 1

Introduction

1.1 Preliminaries

We consider the Cauchy problem for the incompressible Navier-Stokes equations in three space dimensions:

$$u_t^i + u \cdot \nabla u^i - \Delta u^i + \nabla_i p = 0, \qquad i = 1, 2, 3, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad \nabla \cdot u_0 = 0, \qquad x \in \mathbb{R}^3, \tag{1.3}$$

where ∇_i denotes partial derivation with respect to x_i, $i = 1, 2, 3$, and ∇, Δ are the usual gradient and Laplacian operators in $\mathbb{R}^3$.

Definition 1.1. We say that the pair (u, p) is a suitable weak solution of (1.1)-(1.3), in $\Pi_T = \mathbb{R}^3 \times (0, T)$, if the following conditions are satisfied:

1. u, p are measurable functions, $p \in L^{5/3}(\Pi_T)$, and for some constants $E_0, E_1 < \infty$,

$$\int_{\mathbb{R}^3} |u(x,t)|^2 \, dx \leq E_0, \qquad \text{for a.e. } t \in (0,T) \text{ and} \tag{1.4}$$

$$\iint_{\Pi_T} |\nabla u|^2 \leq E_1; \tag{1.5}$$

2. For every $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\zeta \in C_0^\infty(\mathbb{R}^3 \times (-\infty, T); \mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} u \cdot \nabla\psi \, dx = 0, \qquad \text{for a.e. } t \in (0,T), \tag{1.6}$$

$$\iint\limits_{\Pi_T} u \cdot [\zeta_t + \Delta\zeta] + u \otimes u : \nabla\zeta + p\nabla \cdot \zeta \, dx\, dt + \int\limits_{\mathbb{R}^3} u_0(x) \cdot \zeta(x,0)\, dx = 0, \tag{1.7}$$

3. For every $\phi \in C_0^\infty(\Pi_T)$, $\phi \geq 0$,

$$2 \iint_{\Pi_T} |\nabla u|^2 \phi \leq \iint_{\Pi_T} [|u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p) u \cdot \nabla\phi]\, dx\, dt. \tag{1.8}$$

Here, as usual, given two vectors $u, v \in \mathbb{R}^n$ we denote by $u \otimes v$ the matrix $(u_i v_j)_{ij}$, and if A, B are matrices of the same dimensions we denote $A : B = \sum_{ij} A_{ij} B_{ij}$. Conditions 1 and 2 in the above definition characterize weak solutions of (1.1)-(1.3). The existence of weak solutions was proved by Leray in [8]. The concept of suitable weak solution was introduced by Scheffer in [14], where the existence of such solutions of (1.1)-(1.3) was also proved.

Our purpose is to make an exposition of the partial regularity result of Caffarelli, Kohn and Nirenberg [2], for suitable weak solutions of the Cauchy problem (1.1)-(1.3). The latter improves the pioneering result of Scheffer [14], who began the partial regularity theory of the Navier-Stokes equations in a series of papers [12]-[15]. The partial regularity analysis consists in obtaining estimates for the Hausdorff dimension of the set $\mathcal{S}$ of singular points of a weak solution. A point $(x,t) \in \Pi_T$ is said to be *singular* for the weak solution (u,p) if u is not L^∞_{loc} in any neighborhood of (x,t); the remaining points are called *regular points*. Scheffer's theorem (see Chapter 2) establishes that the Hausdorff dimension of $\mathcal{S}$ is $\leq 5/3$, more precisely $\mathcal{H}^{5/3}(\mathcal{S}) = 0$, while the theorem of Caffarelli, Kohn and Nirenberg (C-K-N, henceforth; see Chapter 3) establishes that $\mathcal{H}^1(\mathcal{S}) = 0$, and so the Hausdorff dimension of $\mathcal{S}$ is ≤ 1. By $\mathcal{H}^k(A)$ we denote the k-dimensional Hausdorff measure of A (see, *e.g.*, [4]). The inequality (1.8) is the basic tool of the partial regularity analysis for suitable weak solutions of (1.1)-(1.3).

We remark that the definition that (x,t) is a regular point meaning only that u is merely bounded nearby is motivated by higher regularity results, in particular the one of Serrin [16] which implies that any weak solution of (1.1) on a cylinder $Q = B \times (a,b)$ satisfying

$$\int_a^b \left(\int_B |u|^q \, dx \right)^{s/q} < \infty \qquad \text{with } \frac{3}{q} + \frac{2}{s} < 1$$

is necessarily C^∞ in the space variables on compact subsets of Q. The effect of the pressure prevents one from proving such a local higher regularity result in the time variable. However, if u is absolutely continuous in time and $u_t \in L^q_{loc}(Q)$, $q > 1$, then the same is true of the space derivatives of u_t on compact subsets of Q.

1.2 Dimensional Analisys

An elementary but fundamental procedure to an understanding of the method is the dimensional analysis of the equations. If $u(x,t)$ and $p(x,t)$ solve (1.1) then, for each $\lambda > 0$,

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

also solve (1.1). This property can be encoded, following [2], by assigning a dimension to each quantity, which we denote by enclosing the quantity in the delimiters $\lfloor\ \rceil$:

$$\begin{array}{ll} \lfloor x_i \rceil = 1, & \lfloor t \rceil = 2 \\ \lfloor u^i \rceil = -1, & \lfloor p \rceil = -2 \\ \lfloor \nabla_i \rceil = -1, & \lfloor \partial_t \rceil = -2, \end{array} \tag{1.9}$$

so that each term of (1.1) has dimension -3. Most of the work in the partial regularity analysis, which we expose in the following chapters, is concerned with local, dimensionless estimates for suitable weak solutions. Since time has dimension 2, these estimates are not expressed in balls but in "parabolic cylinders" instead, such as

$$Q_r(x,t) = \{(y,\tau) \,:\, |y-x| < r,\ t - r^2 < \tau < t\}, \tag{1.10}$$

or also

$$Q^*_r(x,t) = \{(y,\tau) \,:\, |y-x| < r,\ t - \frac{7}{8}r^2 < \tau < t + \frac{1}{8}r^2\}. \tag{1.11}$$

Note that $Q^*_r(x,t) = Q_r(x, t + \frac{1}{8}r^2)$ and that (x,t) is the geometric center of $Q_{r/2}(x, t + \frac{1}{8}r^2)$.

1.3 The Heart of the Matter

Here we outline the main points of the strategy of Scheffer and C-K-N to achieve an estimate of the Hausdorff dimension of the singular set $\mathcal{S}$. These are given by Proposition 1.1, Corollary 1.1 and Proposition 1.2 below, and a covering argument which we sketch subsequently. In this section we make use of the fact that $u \in L^{10/3}(\Pi_T)$, $p \in L^{5/3}(\Pi_T)$, valid for any weak solution (u,p) of the Cauchy problem (1.1)-(1.3) in Π_T, which will be proved further on. By an *absolute constant* we mean a number whose value does not depend on any of the data. In all that follows C denotes a generic absolute constant that may vary from line to line. To simplify the notation, the integration over cylinders $Q_r(x_0,t_0)$ will frequently be replaced by an integration over the corresponding cylinders $Q_r(0,0)$, which are contained in the half-space $t \leq 0$, where a weak solution of the Cauchy problem, in principle, does not need to be defined. But, all the time, we only need the fact that (u,p) is a weak solution of the Navier-Stokes system on Π_T, for some $T > 0$ (*i.e.*, satisfies (1.7) for $\zeta \in C_0^\infty(\Pi_T;\mathbb{R}^3)$), paying no attention to the initial data. So, using that $(u(x+x_0,t+t_0), p(x+x_0,t+t_0))$ is also a weak solution, now over the translated strip, we could if necessary replace (u,p) by its translated, which is surely defined in the translated cylinder. We also remark that even though the statements of the results in this section keep the same local character as in [2], here we are only concerned with weak solutions of the Cauchy problem, which then must be defined in some strip Π_T, for some $T > 0$.

Proposition 1.1. *There are absolute constants ε_1 and $C_1 > 0$ with the following property. Suppose (u,p) is a suitable weak solution of the Navier-Stokes system (1.1)-(1.2) on $Q_1 = Q_1(0,0)$. Suppose further that*

$$\iint_{Q_1} (|u|^{10/3} + |p|^{5/3}) \leq \varepsilon_1. \tag{1.12}$$

Then

$$|u(x,t)| \leq C_1 \tag{1.13}$$

for Lebesgue-almost every $(x,t) \in Q_{1/2}$. In particular, u is regular on $Q_{1/2}$.

Applying Proposition 1.1 to the scaled suitable weak solution (u_λ, p_λ), obtained from (u,p) as above, we obtain the following.

Corollary 1.1. *Suppose (u,p) is a suitable weak solution of the Navier-Stokes system (1.1)-(1.2) on some cylinder $Q_r = Q_r(x,t)$. If*

$$r^{-5/3} \iint_{Q_r} (|u|^{10/3} + |p|^{5/3}) \leq \varepsilon_1 \tag{1.14}$$

then

$$|u| \leq C_1 r^{-1} \tag{1.15}$$

Lebesgue-almost everywhere on $Q_{r/2}(x,t)$.

Proposition 1.1 and its Corollary 1.1 are local versions, proved in [2], of a (slightly modified) result of Scheffer in [14]. They form the basic tools to achieve the conclusion that $\mathcal{H}^{5/3}(\mathcal{S}) = 0$. Indeed, using Corollary 1.1 and a Vitali's type covering lemma (we give details later on), one sees that $\mathcal{S}$ can be covered, for any $\delta > 0$, by a family $\{Q_i^* = Q_{r_i}^*(x_i, t_i)\}$ of parabolic cylinders satisfying

$$r_i < \delta \qquad \text{for each } i, \tag{1.16}$$

$$r_i^{-5/3} \iint_{Q^*_{r_i/5}} (|u|^{10/3} + |p|^{5/3}) > \varepsilon_1, \tag{1.17}$$

$$\{Q^*_{r_i/5}(x_i, t_i)\} \qquad \text{are pairwise disjoint.} \tag{1.18}$$

From (1.17) we get

$$\sum r_i^{5/3} \leq C \iint_{\cup_i Q^*_{r_i/5}} |u|^{10/3} + |p|^{5/3}. \tag{1.19}$$

Since $\delta > 0$ is arbitrary, (1.19) shows immediately that $\mathcal{L}^4(\mathcal{S}) = 0$, where by $\mathcal{L}^k(A)$ we denote the k-dimensional Lebesgue measure of $A \subseteq \mathbb{R}^k$. Further, since $\cup_i Q^*_{r_i/5} \subseteq V_\delta$, where, for each $\delta > 0$, V_δ is a neighborhood of $\mathcal{S}$ such that $\cap_{\delta>0} V_\delta = \mathcal{S}$, we conclude, as $\delta \to 0$, that $\mathcal{H}^{5/3}(\mathcal{S}) = 0$.

The exponent 5/3 in the left side of (1.19) is due to the fact that $\iint(|u|^{10/3} + |p|^{5/3})$ has dimension 5/3, in the sense of (1.9). The idea of C-K-N was then to obtain a result similar to Corollary 1.1, but with a space-time integral of dimension 1. Namely, C-K-N [2] prove the following.

Proposition 1.2. *There is an absolute constant $\varepsilon_2 > 0$ with the following property. If (u, p) is a suitable weak solution of the Navier-Stokes system* (1.1)-(1.2) *and if for some (x, t)*

$$\limsup_{r\to 0} r^{-1} \iint_{Q_r^*(x,t)} |\nabla u|^2 \leq \varepsilon_2, \tag{1.20}$$

then (x, t) is a regular point.

From Proposition 1.2, using the same covering argument as above, one easily shows that $\mathcal{H}^1(\mathcal{S}) = 0$. Indeed, we replace (1.17) by

$$r_i^{-1} \iint_{Q^*_{r_i/5}} |\nabla u|^2 > \varepsilon_2, \tag{1.21}$$

and (1.19) by

$$\sum r_i \le C \iint_{\cup_i Q^*_{r_i/5}} |\nabla u|^2, \tag{1.22}$$

and we conclude that $\mathcal{H}^1(\mathcal{S}) = 0$.

The guess of an estimate like the one provided by Proposition 1.2 is explained by C-K-N as follows. Suppose (x_0, t_0) is a singular point; then, by Proposition 1.1, (1.12) must fail for $Q_r(x,t)$ whenever $(x_0,t_0) \in Q_{r/2}(x,t)$. If we denote by $M(r;x,t)$ the left-hand side of (1.12), then $M(r;x,t) > \varepsilon_1$ for a family of parabolic cylinders $Q_r(x,t)$ shrinking to (x_0,t_0). Because of the relations (1.27)-(1.32) below, intuitively, we may think of p as being quadratic in u. So, heuristically, we are led to the conclusion

$$|u| \ge \frac{C}{r}, \qquad \text{as} \qquad r = (|x-x_0|^2 + |t-t_0|)^{1/2} \to 0,$$

in view of which it is natural to guess that

$$|\nabla u|(x,t) \ge \frac{C}{r^2} \qquad \text{as} \qquad (x,t) \to (x_0,t_0);$$

Proposition 1.2 is then, in a certain sense, a rigorous formulation for this guess.

1.4 Interpolation Inequalities

In this section we state and prove an interpolation inequality which, in particular, shows that $u \in L^{10/3}(\Pi_T)$, for any weak solution of (1.1)-(1.3). We first recall the following well-known Sobolev inequality (for the proof see, *e.g.*, [3], p.141). For an open set $U \subseteq \mathbb{R}^n$, let $W^{1,q}(U)$ denote the Sobolev space of the functions in $L^q(U)$ whose first-order partial derivatives, in the sense of distributions, also belong to $L^q(U)$. For $x \in \mathbb{R}^n$, denote $B(x,r)$ the open ball of center x and radius $r > 0$. If $1 \le q < n$, define

$$q^* \equiv \frac{nq}{n-q}.$$

Lemma 1.1. *For each $1 \le q < n$ there exists a constant C, depending only on q and n, such that, for all $u \in W^{1,q}(B(x,t))$,*

$$\left(⨍_{B(x,r)} |u - (u)_{x,r}|^{q^*} \, dy \right)^{1/q^*} \le Cr \left(⨍_{B(x,r)} |Du|^q \, dy \right)^{1/q} \tag{1.23}$$

for all $B(x,r) \subseteq \mathbb{R}^n$, where $(u)_{x,r} = ⨍_{B(x,r)} u \, dy$ and $⨍_B = \mathcal{L}^n(B)^{-1} \int_B$.

Using the above lemma we get the following interpolation inequality.

Lemma 1.2. *For $u \in W^{1,2}(\mathbb{R}^3)$ we have*

$$\int_{B_r} |u|^q \leq C \left(\int_{B_r} |\nabla u|^2 \right)^a \left(\int_{B_r} |u|^2 \right)^{q/2-a} + \frac{C}{r^{2a}} \left(\int_{B_r} |u|^2 \right)^{q/2}, \tag{1.24}$$

where C is a constant independent of r, B_r is a ball of radius r, and

$$2 \leq q \leq 6, \qquad a = \frac{3}{4}(q-2).$$

If u has mean zero or B_r is replaced by all $\mathbb{R}^3$, then the second term on the right-hand side of (1.24) may be ommited.

Proof. We first notice that, for $n = 3$, $2^* = 6$. If $q = 2(1-\theta) + 6\theta$, $0 \leq \theta \leq 1$, Hölder's inequality gives

$$\|u\|^q_{L^q(B_r)} \leq \|u\|^{2(1-\theta)}_{L^2(B_r)} \|u\|^{6\theta}_{L^6(B_r)}.$$

Setting $a = 3\theta$, we get $a = \frac{3}{4}(q-2)$, $(1-\theta) = q/2 - a$, and from (1.23)

$$\|u\|^{6\theta}_{L^6(B_r)} \leq C \|\nabla u\|^{2a}_{L^2(B_r)} + \frac{C}{r^{2a}} \left(\int_{B_r} |u|^2 \right)^a,$$

which substituting in the former inequality gives the desired result.

□

When u is a weak solution, (1.24) with $q = \frac{10}{3}$, so that $a = 1$, gives

$$\int_{B_r} |u|^{10/3} \leq C \left(\int_{B_r} |u|^2 \right)^{2/3} \int_{B_r} |\nabla u|^2 + C r^{-2} \left(\int_{B_r} |u|^2 \right)^{5/3}. \tag{1.25}$$

In particular, from (1.4), (1.5) and (1.25) it follows

$$\int_0^T \int_{B_r} |u|^{10/3} \leq C(E_0^{2/3} E_1 + r^{-2} E_0^{5/3} T), \tag{1.26}$$

where the second term on the right-hand side may be ommited if B_r is replaced by $\mathbb{R}^3$.

1.5 Relation between u and p

Here we discuss the relation between u and p, when (u,p) is a weak solution of (1.1)-(1.3). We observe first that taking formally the divergence of (1.1) and using (1.2) gives

$$\Delta p = -\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(u^i u^j), \qquad \text{on } \Pi_T. \tag{1.27}$$

Actually, from (1.7) with $\zeta(x,t) = \chi(t)\nabla\phi(x)$, for $\chi \in C_0^\infty(0,T)$, $\phi \in C_0^\infty(\mathbb{R}^3)$, using (1.6), one easily deduces that (1.27) holds in the sense of the distributions in $\mathbb{R}^3$, for almost all $t \in (0,T)$. Equation (1.27) holding for a.e. $t \in (0,T)$, allows one to obtain p explicitly in terms of u:

$$p = \sum_{i,j} -\frac{3}{4\pi}\left(\nabla_{ij}\frac{1}{|x|}\right) * (u^i u^j). \tag{1.28}$$

In the right-hand side of (1.28), the operators

$$T_{ij}(g) = -\frac{3}{4\pi}\left(\nabla_{ij}\frac{1}{|x|}\right) * g$$

should be viewed as singular integral operators (see [17]). Actually, we have

$$T_{ij} = R_i R_j,$$

where R_i is the i-th Riez transform, defined by (*cf.*, [17], p.57)

$$R_i(f)(x) = \lim_{\varepsilon\to 0} \frac{2}{\pi^2}\int_{|y|\geq\varepsilon} \frac{y_i}{|y|^{n+1}} f(x-y)\,dy, \qquad i = 1,2,3.$$

Since the Riez transforms are bounded operators from $L^q(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, for $1 < q < \infty$, which follows from a general result on singular integral operators of Calderón and Zygmund (*cf.*, [17], p.39), we obtain

$$\int_{\mathbb{R}^3} |p|^q \leq C(q)\int_{\mathbb{R}^3} |u|^{2q}\,dx, \qquad 1 < q < \infty. \tag{1.29}$$

In particular, for a weak solution on Π_T, from (1.26) one gets

$$\int_0^T\int_{\mathbb{R}^3} |p|^{5/3} \leq \int_0^T\int_{\mathbb{R}^3} |u|^{10/3} \leq C E_0^{2/3} E_1. \tag{1.30}$$

Hence, the condition $p \in L^{5/3}(\Pi_T)$ in Definition 1.1 is, in fact, redundant in the presence of (1.4), (1.5).

We will also need local L^q estimates for p in terms of u. These can be obtained by decomposing a certain localization of p as follows. Let Ω, $\bar{\Omega}_1$ be open bounded subsets of $\mathbb{R}^3$ with $\bar{\Omega}_1 \subseteq \Omega$, and let $\phi \in C_0^\infty(\Omega)$ with $\phi = 1$ in a neighborhood of $\bar{\Omega}_1$. At any time we have

$$\begin{aligned} \phi(x)p(x,t) &= -\frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\Delta_y(\phi p)\,dy \\ &= -\frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}[p\Delta\phi + 2(\nabla\phi, \nabla p) + \phi\Delta p]\,dy. \end{aligned} \tag{1.31}$$

Using (1.27) to substitute Δp in (1.31) and integrating by parts one obtains different decompositions of ϕp. One which we shall use is the following:

$$\begin{aligned} \phi p &= \tilde{p} + p_3 + p_4, \\ \tilde{p} &= \frac{3}{4\pi}\int \nabla_{y_i y_j}\left(\frac{1}{|x-y|}\right)\phi u^i u^j\,dy, \\ p_3 &= \frac{3}{2\pi}\int \frac{x_i - y_i}{|x-y|^3}(\nabla_{y_j}\phi)u^i u^j + \frac{3}{4\pi}\int \frac{1}{|x-y|}(\nabla_{y_i y_j}\phi)u^i u^j\,dy, \\ p_4 &= \frac{3}{4\pi}\int \frac{1}{|x-y|}p(y)\Delta_y\phi\,dy + \frac{3}{2\pi}\int \frac{x_i-y_i}{|x-y|^3}p(y)\nabla_{y_i}\phi\,dy. \end{aligned} \tag{1.32}$$

Again the integral defining $\tilde{p}$ is to be understood in the same way as (1.28), and (1.32) is valid for a.e. t.

1.6 Weak continuity of u as a function of t

In this section we briefly discuss the weak continuity with respect to t of weak solutions of (1.1)-(1.3). This is a well known fact (see, *e.g.*, [18]) and in the present context means that

$$\int_{\mathbb{R}^3} u(x,t)\cdot w(x)\,dx \to \int_{\mathbb{R}^3} u(x,t_0)\cdot w(x), \qquad \text{for each } w \in L^2(\mathbb{R}^3) \text{ as } t \to t_0. \tag{1.33}$$

This can be seen as follows. First we notice that it suffices to show (1.33) for $w \in C_0^\infty(\mathbb{R}^3)$. Let $0 < t_1 < t_2$ and choose ζ in (1.7) of the form $\zeta(x,t) = \chi^h(t)w(x)$, where $\chi^h \in C_0^\infty(0,\infty)$ satisfies: $\chi^h(t) = 0$ out of (t_1,t_2), $\chi^h(t) = 1$

on (t_1+h, t_2-h), $h < (t_2-t_1)/2$, and χ^h is monotone over each of the intervals (t_1, t_1+h) and (t_2-h, t_2). From (1.7) we get

$$\int_0^T \frac{d\chi^h}{dt}(t) \int_{\mathbb{R}^3} u(x,t)w(x)\,dx\,dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \chi^h(t)R(x,t)\,dx\,dt,$$

for a certain function $R(x,t)$ which is integrable over Π_T. Hence, making $h \to 0$ we get

$$\int_{\mathbb{R}^3} (u(x,t_2) - u(x,t_1))w(x)\,dx\,dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} R(x,t)\,dx\,dt,$$

assuming that t_1 and t_2 are Lebesgue points of $\int_{\mathbb{R}^3} u(x,t)w(x)\,dx$. We then see that the left-hand side of the above equation goes to 0 as t_2 approaches t_1 and this proves (1.33). A slight modification of the preceding argument shows that $u(x,t) \rightharpoonup u_0(x)$ as $t \to 0$. The weak continuity of u with respect to t implies that (1.4) holds for all t. Also, if (u,p) is a suitable weak solution on Π_T, then, for each $t \in (0,T)$ and each $\phi \in C_0^\infty(\Pi_T)$, $\phi \geq 0$ and $\operatorname{supp}\phi \subseteq \Omega \times (a,b)$, $0 \leq a < b \leq T$, we have

$$\begin{aligned} &\int_{\Omega\times\{t\}} |u|^2\phi + 2\int_a^t \int_\Omega |\nabla u|^2\phi \\ &\leq \int_a^t \int_\Omega [|u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi], \end{aligned} \tag{1.34}$$

which follows from (1.8) by an argument similar to the one used above to prove (1.33). Indeed, we replace ϕ in (1.8) by $\phi(x,s)\chi((t-s)/h)$, $h > 0$, where $\chi(s)$ is smooth, $0 \leq \chi \leq 1$, $\chi(s) = 0$ for $s \leq 0$ and $\chi(s) = 1$, for $s \geq 1$. Letting $h \to 0$ gives (1.34) for a.e. t, and by weak continuity we obtain (1.34) for all $t \in (0,T)$.

1.7 The measures $\mathcal{H}^k$ and $\mathcal{P}^k$

The results in [2] are stated in terms of certain measures $\mathcal{P}^k$, $k = 5/3$ for Scheffer's theorem and $k = 1$ for C-K-N's theorem, defined in a manner analogous to the Hausdorff measures $\mathcal{H}^k$, but using the parabolic metric on $\mathbb{R}^3 \times \mathbb{R}$. Both measures are special cases of a construction due to Caratheodory, which may be found in [4]. For any $X \subseteq \mathbb{R}^3 \times \mathbb{R}$ and $k \geq 0$, C-K-N define

$$\mathcal{P}^k(X) = \lim_{\delta\to 0+} \mathcal{P}^k_\delta(X),$$

$$\mathcal{P}^k_\delta(X) = \inf\left\{ \sum_{i=1}^\infty r_i^k \;:\; X \subseteq \cup_i Q_{r_i}, r_i < \delta \right\},$$

where Q_r represents any "parabolic" cylinder, that is, one with radius r in space and r^2 in time. $\mathcal{P}^k$ is then an outer measure, for which all Borel sets are measurable. The Hausdorff measure $\mathcal{H}^k$ is defined in an entirely similar manner, but with Q_{r_i} replaced by an arbitrary closed subset of $\mathbb{R}^3 \times \mathbb{R}$ of diameter at most δ. Actually, one usually normalizes $\mathcal{H}^k$ so that, for integer k, it agrees with the surface area on smooth k-dimensional surfaces. Clearly,

$$\mathcal{H}^k \leq C(k)\mathcal{P}^k.$$

1.8 A covering lemma

Here we explicitly state and present the proof of the Vitali's type covering lemma mentioned in section 1.3 (*cf.* [2], Lemma 6.1). It is in fact the analogue for parabolic cylinders of the well known Vitali lemma for balls (see, *eg.*, [3]).

Lemma 1.3. *Let $\mathcal{T}$ be any family of parabolic cylinders $Q_r(x,t)$ contained in a bounded subset of $\mathbb{R}^3 \times \mathbb{R}$. Then, there exists a finite or denumerable subfamily $\mathcal{T}' = \{\, Q_i = Q_{r_i}(x_i, t_i) \,\}$ such that*

$$Q_i \cap Q_j = \emptyset \qquad \text{for } i \neq j, \tag{1.35}$$

$$\forall Q \in \mathcal{T},\ \exists Q_{r_i}(x_i,t_i) \in \mathcal{T}',\ \text{ such that } Q \subseteq Q_{5r_i}(x_i,t_i). \tag{1.36}$$

Proof. The elements of $\mathcal{T}'$ are chosen inductively, just as in the version for balls in Euclidean spaces. Let $\mathcal{T}_0 = \mathcal{T}$ and choose $Q_1 = Q_{r_1}(x_1,t_1)$ such that $r_1 \geq \frac{2}{3} \sup\limits_{Q_r \in \mathcal{T}} r$; once Q_k, $k = 1, \ldots, n$, are chosen, let

$$\mathcal{T}_n = \{\, Q \in \mathcal{T} \,:\, Q \cap Q_k = \emptyset,\ 1 \leq k \leq n \,\}.$$

If $\mathcal{T}_n \neq \emptyset$, we choose $Q_{n+1} \in \mathcal{T}$ so that

$$\text{for any } Q = Q_r(x,t) \in \mathcal{T}_n, \text{ we have } r \leq \frac{3}{2} r_{n+1}. \tag{1.37}$$

Otherwise the process terminates and $\mathcal{T}'$ is finite. Property (1.35) is clear from the construction. To prove (1.36), note first that if $\mathcal{T}'$ is infinite then $r_n \to 0$ as $n \to \infty$. Hence, given any $Q = Q_r(x,t) \in \mathcal{T} \setminus \mathcal{T}'$, there exists $n \geq 0$ such that $Q \in \mathcal{T}_n$ but $Q \notin \mathcal{T}_{n+1}$. Then $Q \cap Q_{n+1} \neq \emptyset$, and by (1.37), $r \leq \frac{3}{2} r_{n+1}$. It follows that

$$Q \subseteq Q_{5r_{n+1}}(x_{n+1}, t_{n+1}).$$

$\square$

Chapter 2

Scheffer's Theorem: $\mathcal{H}^{5/3}(\mathcal{S}) = 0$

This chapter is devoted to the proof of Proposition 1.1, which, together with Scheffer's existence result (see Chapter 4), gives the following theorem of Scheffer [14].

Theorem 2.1. *There exists a globally defined ($T = \infty$) weak solution of* (1.1)-(1.3) *whose set of singular points $\mathcal{S}$ satisfies $\mathcal{H}^{5/3}(\mathcal{S}) = 0$.*

Actually, the result of Scheffer in [14] is stated asserting only that $\mathcal{H}^2(\mathcal{S}) < +\infty$, but an easy modification of his arguments leads to $\mathcal{H}^{5/3}(\mathcal{S}) = 0$. Also, Scheffer gives more information about the regularity of the suitable weak solution (u, p) out of $\mathcal{S}$. Namely, he establishes that u coincides a.e. in $\mathbb{R}^3 \times \mathbb{R}_+ \backslash \mathcal{S}$ with a continuous function. Further, he proves that $\mathcal{S} \cap \{(x,t) \ : \ t \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. While the latter can follow as a consequence of Proposition 1.1, slightly adapting Scheffer's arguments, to prove the continuity of u out of $\mathcal{S}$, after redefining it on a set of measure zero, Scheffer makes use of some results of his in [13]. We shall not enter into these details here.

2.1 Dimensionless Estimates

In this section we prove estimates involving integral functionals of dimension zero, in the sense of (1.9). These estimates play a decisive role in the proof of

Proposition 1.1. Let

$$Q_\rho = Q_\rho(0,0) = \{(x,t) \,:\, |x| < \rho,\ -\rho^2 < t < 0\},$$

and consider a pair of measurable functions u and p defined on Q_ρ. We define the following quantities, for $r \le \rho$:

$$A(r) = \sup_{-r^2<t<0} r^{-1} \int_{B_r \times \{t\}} |u|^2, \tag{2.1}$$

$$\delta(r) = r^{-1} \iint_{Q_r} |\nabla u|^2, \tag{2.2}$$

$$G(r) = r^{-5/3} \iint_{Q_r} |u|^{10/3}, \tag{2.3}$$

$$D(r) = r^{-5/3} \iint_{Q_r} |p|^{5/3}, \tag{2.4}$$

$$L(r) = r^{-2} \iint_{Q_r} |u||p - \bar{p}_r|. \tag{2.5}$$

Here $Q_r = Q_r(0,0)$, $B_r = \{x \,:\, |x| < r\}$, and

$$\bar{p}_r = \bar{p}_r(t) = \fint_{B_r} p(y,t)\, dy.$$

In this section we use only the fact that the quantities defined by (2.1)-(2.5) are finite and

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial^2}{\partial x^i \partial x^j} u^i u^j, \qquad \nabla \cdot u = 0, \tag{2.6}$$

on $B_\rho \times \{t\}$ for almost every t, $-\rho^2 < t < 0$. Note that each of the quantities (2.1)-(2.5) has dimension zero, in the sense of (1.9). The following two lemmas provide estimates for $G(r)$ and $L(r)$ in terms of A, δ and D.

Lemma 2.1. *For $r \le \rho$,*

$$\begin{aligned} G(r) \;\; &\le \;\; C\left\{ \left(\frac{r}{\rho}\right)^{10/3} A(\rho)^{5/3} + \left(\frac{\rho}{r}\right)^{5/3} A(\rho)^{2/3}\delta(\rho) \right. \\ &+ \;\; \left. \left(\frac{\rho}{r}\right)^{10/3} A(\rho)^{5/6}\delta(\rho)^{5/6} \right\}. \end{aligned} \tag{2.7}$$

In particular,

$$G(r) \leq C\{A(r)^{5/3} + A(r)^{2/3}\delta(r) + A(r)^{5/6}\delta(r)^{5/6}\}. \tag{2.8}$$

Proof. Let $r < \rho$, and $\overline{|u|^2_\rho} = \frac{1}{|B_\rho|}\int |u|^2 dx$. We have

$$\begin{aligned}
\int_{B_r} |u|^2 &\leq \int_{B_r} ||u|^2 - \overline{|u|^2_\rho}| + Cr^3\overline{|u|^2_\rho} \\
&\leq C\rho \int_{B_\rho} |u||\nabla u| + Cr^3\rho^{-2}A(\rho) \\
&\leq C\rho \left(\int_{B_\rho} |u|^2\right)^{1/2} \left(\int_{B_\rho} |\nabla u|^2\right)^{1/2} + Cr^3\rho^{-2}A(\rho) \\
&\leq C\rho^{3/2}A(\rho)^{1/2} \left(\int_{B_\rho} |\nabla u|^2\right)^{1/2} + Cr^3\rho^{-2}A(\rho).
\end{aligned}$$

Replacing in (1.25), we get

$$\begin{aligned}
\int_{B_r} |u|^{10/3} &\leq C\left(\int_{B_r} |u|^2\right)^{2/3} \int_{B_r} |\nabla u|^2 + Cr^{-2}\left(\int_{B_r} |u|^2\right)^{5/3} \\
&\leq C\rho^{2/3}A(\rho)^{2/3}\int_{B_\rho} |\nabla u|^2 \\
&\quad + Cr^{-2}\left[\rho^{5/2}A(\rho)^{5/6}\left(\int_{B_\rho} |\nabla u|^2\right)^{5/6} + \frac{r^5}{\rho^{10/3}}A(\rho)^{5/3}\right] \\
&\leq C\rho^{2/3}A(\rho)^{2/3}\int_{B_\rho} |\nabla u|^2 + C\frac{\rho^{5/2}}{r^2}A(\rho)^{5/6}\left(\int_{B_\rho} |\nabla u|^2\right)^{5/6} \\
&\quad + C\frac{r^3}{\rho^{10/3}}A(\rho)^{5/3}.
\end{aligned}$$

Integrating in t over $(-r^2, 0)$, we obtain

$$\begin{aligned}
\iint_{Q_r} |u|^{10/3} &\leq C\rho^{2/3}A(\rho)^{2/3}\iint_{Q_\rho} |\nabla u|^2 + C\frac{\rho^{5/2}}{r^{5/3}}A(\rho)^{5/6}\left(\iint_{Q_\rho} |\nabla u|^2\right)^{5/6} \\
&\quad + C\frac{r^5}{\rho^{10/3}}A(\rho)^{5/3}.
\end{aligned}$$

Finally, dividing by $r^{5/3}$ we get

$$\begin{aligned} r^{-5/3} \iint\limits_{Q_r} |u|^{10/3} &\leq C\left(\frac{\rho}{r}\right)^{5/3} A(\rho)^{2/3}\delta(\rho) + C\left(\frac{\rho}{r}\right)^{10/3} A(\rho)^{5/6}\delta(\rho)^{5/6} \\ &\quad + C\left(\frac{\rho}{r}\right)^{10/3} A(\rho)^{5/3}, \end{aligned}$$

which gives the desired conclusion.

□

Lemma 2.2. *Let $r \leq \frac{1}{2}\rho$. Then*

$$\begin{aligned} L(r) \leq C\left(\frac{r}{\rho}\right)^{9/5} G(r)^{3/10} D(\rho)^{3/5} + C\left(\frac{r}{\rho}\right)^{9/5} G(r)^{3/10} G(\rho)^{3/5} \\ CG(r)^{3/10} G(2r)^{3/5} + Cr^3 G(r)^{3/10} \sup_{-r^2<t<0} \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4}\, dy. \end{aligned} \tag{2.9}$$

Proof. We use the decomposition (1.32) for p, with the function ϕ chosen so that

$$\begin{aligned} \phi(y) = 1 \quad \text{if} \quad |y| \leq \frac{3}{4}\rho \quad , \qquad &\phi(y) = 0 \quad \text{if} \quad |y| \geq \rho, \\ |\nabla_i \phi| \leq C\rho^{-1}, \qquad &|\nabla_{i,j}\phi| \leq C\rho^{-2}. \end{aligned} \tag{2.10}$$

We decompose $\tilde{p}$ further as $\tilde{p} = p_1 + p_2$,

$$\begin{aligned} p_1 &= \frac{3}{4\pi} \int_{|y|<2r} \nabla_{y_i y_j} \left(\frac{1}{|x-y|}\right) \cdot \phi u^i u^j \, dy, \\ p_2 &= \frac{3}{4\pi} \int_{|y|>2r} \nabla_{y_i y_j} \left(\frac{1}{|x-y|}\right) \cdot \phi u^i u^j \, dy. \end{aligned}$$

We note that

$$|p - \overline{p}_r| \leq \sum_{i=1}^{4} |p_i - \overline{p}_i|, \qquad \overline{p}_i = \fint_{B_r} p_i,$$

and we estimate each of the the four terms.

For p_1 we recall that the operators

$$T_{ij}(\psi) = \left(\nabla_{ij} \frac{1}{|x|}\right) * \psi$$

are bounded from $L^q(\mathbb{R}^3)$ into itself, for $1 < q < \infty$, by the Calderón-Zygmund theorem. Taking $\psi = \phi u^i u^j|_{B_{2r}}$ and $q = \frac{5}{3}$ we conclude that

$$\|p_1\|_{L^{5/3}(B_r)} \leq C \left(\int_{B_{2r}} |u|^{10/3} \right)^{3/5}$$

and so

$$\begin{aligned} \int_{B_r} |u||p_1 - \overline{p}_1| &\leq C r^{3/10} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \left(\int_{B_r} |p_1|^{5/3} \right)^{3/5} \\ &\leq C r^{3/10} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \left(\int_{B_{2r}} |u|^{10/3} \right)^{3/5}. \end{aligned} \tag{2.11}$$

For p_2, p_3 and p_4 we estimate $|p_i - \overline{p}_i|$ uniformly on B_r, using the mean value theorem. Indeed, for $|x| < r$,

$$\begin{aligned} |\nabla p_2(x)| &\leq C \int_{2r<|y|<\rho} \left(\frac{|u|^2}{|y|^4} \right) dy, \\ |\nabla p_3(x)| &\leq C \rho^{-4} \int_{B_\rho} |u|^2, \end{aligned}$$

and

$$|\nabla p_4(x)| \leq C \rho^{-4} \int_{B_\rho} |p|.$$

Estimating for $i = 2, 3$ by

$$\begin{aligned} \int_{B_r} |u||p_i - \overline{p}_i| &\leq C r^{21/10} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \sup_{x \in B_r} |p_i(x) - \overline{p}_i| \\ &\leq C r^{31/10} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \sup_{x \in B_r} |\nabla p_i|, \end{aligned} \tag{2.12}$$

we see that

$$\int_{B_r} |u||p_2 - \overline{p}_2| \leq C r^{31/10} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^4} \, dy, \tag{2.13}$$

$$\begin{aligned} \int_{B_r} |u||p_3 - \overline{p}_3| &\leq C \frac{r^{31/10}}{\rho^4} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \int_{B_\rho} |u|^2 \\ &\leq C r^{3/10} \left(\frac{r}{\rho} \right)^{14/5} \left(\int_{B_r} |u|^{10/3} \right)^{3/10} \left(\int_{B_\rho} |u|^{10/3} \right)^{3/5}. \end{aligned} \tag{2.14}$$

For p_4 we have

$$\begin{aligned}\int_{B_r} |u||p_4 - \overline{p}_4| &\le C \left(\int_{B_r} |u| \right) \sup_{x \in B_r} |p_4(x) - \overline{p}_4| \\ &\le C \frac{r}{\rho^4} \left(\int_{B_r} |u| \right) \left(\int_{B_\rho} |p| \right).\end{aligned} \tag{2.15}$$

Integrating each of the inequalities (2.13)-(2.15) in time and using Hölder's inequality as appropriate, we obtain

$$\begin{aligned}
\iint_{Q_r} |u||p_1 - \overline{p}_1| &\le C r^2 G(r)^{3/10} G(2r)^{3/5}, \\
\iint_{Q_r} |u||p_2 - \overline{p}_2| &\le C r^5 G(r)^{3/10} \sup_{-r^2 < t < 0} \int \frac{|u|^2}{|y|^4}\, dy, \\
\iint_{Q_r} |u||p_3 - \overline{p}_3| &\le C r^2 \left(\frac{r}{\rho} \right)^{9/5} G(r)^{3/10} G(\rho)^{3/5}, \\
\iint_{Q_r} |u||p_4 - \overline{p}_4| &\le C \frac{r}{\rho^4} r^{1/5} \left(\int_{-r^2}^0 (\int_{B_r} |u|)^{10/3} \right)^{3/10} \left(\int_{-r^2}^0 (\int_{B_\rho} |p|)^{5/3} \right)^{3/5} \\
&\le C \frac{r^{33/10}}{\rho^{14/5}} \left(\iint_{Q_r} |u|^{10/3} \right)^{3/10} \left(\iint_{Q_\rho} |p|^{5/3} \right)^{3/5} \\
&= C r^2 \left(\frac{r}{\rho} \right)^{9/5} G(r)^{3/10} D(\rho)^{3/5}.
\end{aligned}$$

Hence, the assertion follows by summing the above inequalities, using

$$\iint_{Q_r} |u||p - \overline{p}_r| \le \sum_{i=1}^{4} \iint_{Q_r} |u||p_i - \overline{p}_i|,$$

and dividing by r^2.

□

2.2 The Inductive Argument

We pass now to the actual proof of Proposition 1.1. We first recall that if $\phi \in C^\infty(Q_1(0,0))$, $\phi \geq 0$, and ϕ vanishes near $\{|x|=1\} \cup \{t=-1\}$, then, for $-1 < s < 0$,

$$\int_{B_1 \times \{s\}} |u|^2 \phi + 2 \int_{-1}^{s} \int_{B_1} |\nabla u|^2 \phi \leq \int_{-1}^{s} \int_{B_1} |u|^2 (\phi_t + \Delta \phi) + \int_{-1}^{s} \int_{B_1} (|u|^2 + 2p) u \cdot \nabla \phi. \tag{2.16}$$

The presence of the factor $\phi_t + \Delta\phi$ in the first integral on the right-hand side of inequality (2.16) prompts one to choose the test function

$$\phi^*(x,t) = \chi(x,t)(s-t)^{-3/2} \exp\{-|x-a|^2/4(s-t)\}$$

defined on $\{(x,t) : t < s\}$, where $0 \leq \chi \leq 1$ is a smooth, compactly supported function equal to one near (a,s). Where $\chi = 1$, ϕ^* is a constant times the fundamental solution of the backward heat equation $\phi_t + \Delta\phi = 0$ with singularity at (a,s). Therefore, using ϕ^* in (2.16) leads formaly to a bound for $|u|^2(a,s)$. However, this approach fails because of the lack of a suitable bound for the last term on the right of (2.16). Scheffer's idea, as pointed out in [2], was to approximate the singular test function ϕ^* by a sequence of smoother ones; that is, to apply (2.16) to a sequence of test functions $\{\phi_n\}_{n=1}^\infty$, where ϕ_n is more or less a smoothing of ϕ^* of order 2^{-n}. The argument becomes an inductive one, in which the estimates from Section 2.1 are used at each stage to bound the right-hand side of (2.16) with $\phi = \phi_{n+1}$ in terms of the left-hand side of (2.16) with $\phi = \phi_k$, $k \leq n$. Roughly speaking, the right-hand side of (2.16) behaves like a $\frac{3}{2}$ power of $\int |u|^2 + \iint |\nabla u|^2$, which follows with the help of the estimates of Section 2.1. Hence, if in a first run we obtain that $\int |u|^2 + \iint |\nabla u|^2$ is small, say $\leq \bar{\varepsilon}^{2/3}$, this smallness is improved (by a power $\frac{3}{2}$) in a second run through (2.16), say, $\leq C\bar{\varepsilon} \leq \bar{\varepsilon}^{2/3}$, for $\bar{\varepsilon}$ sufficiently small, and the inductive iteration can be carried out successfully.

2.2.1 Step 1: Setting up the induction

Our hypothesis is

$$\iint_{Q_1} (|u|^{10/3} + |p|^{5/3}) \leq \varepsilon_1, \tag{2.17}$$

with $Q_1 = Q_1(0,0)$. We shall show, for suitable choice of the constants, that

$$\fint_{|x-a|<r_n} |u|^2(x,s)\,dx \le C_0 \varepsilon_1^{3/5}, \tag{2.18}$$

for each $(a,s) \in Q_{1/2}(0,0)$ and each $n \ge 2$, where

$$r_n = 2^{-n}.$$

Therefore one concludes that

$$|u|^2(a,s) \le C_0 \varepsilon_1^{3/5} = C_1$$

provided that (a,s) is a Lebesgue point for u, hence almost everywhere in $Q_{1/2}(0,0)$.

Let $(a,s) \in Q_{1/2}(0,0)$ be a fixed but arbitrary point. Note that $Q_{1/2}(a,s) \subseteq Q_1(0,0)$, so that

$$\iint\limits_{Q_{1/2}(a,s)} (|u|^{10/3} + |p|^{5/3}) \le \varepsilon_1. \tag{2.19}$$

Let

$$Q^n = Q_{r_n}(a,s), \qquad n = 1, 2, \dots$$

The procedure will consist in proving inductively that, for $n \ge 3$,

$$\left(\fint\!\!\!\fint_{Q^n} |u|^{10/3} \right)^{9/10} + r_n^{1/5} \fint\!\!\!\fint_{Q^n} |u||p - \overline{p}_n| \le \varepsilon_1^{3/5}, \tag{2.20}$$

and, for $n \ge 2$,

$$\sup_{s - r_n^2 < t \le s} \fint_{|x-a|<r_n} |u|^2\,dx + r_n^{-3} \iint\limits_{Q^n} |\nabla u|^2 \le C_0 \varepsilon_1^{3/5}, \tag{2.21}$$

where $\fint\!\!\!\fint$ denotes an average, and

$$\overline{p}_n = \overline{p}_n(t) = \fint_{|x-a|<r_n} p\,dx.$$

To begin with we assume $\varepsilon_1 \le 1$, and we shall impose several further smallness conditions on ε_1 as we proceed. Clearly, (2.21) includes the assertion (2.18); thus once (2.21) is established for all $n \ge 2$ the proof will be complete.

We start the induction by proving $(2.21)_2$, which can be deduced from (2.19), using the energy inequality (2.16). Indeed, choosing a smooth function $\phi \geq 0$ with $\phi \equiv 1$ on Q^2 and $\operatorname{supp} \phi \subseteq Q^1$, we see from (2.16) that the left-hand side of $(2.21)_2$ is bounded by

$$C\{\iint\limits_{Q^1} |u|^2 + \iint\limits_{Q^1} (|u|^3 + |u||p|)\}$$

which, by Hölder's inequality and (2.19) is at most $C_0 \varepsilon_1^{3/5}$, provided ε_1 is small.

2.2.2 Step 2: $(2.21)_k$, $2 \leq k \leq n$, implies $(2.20)_{n+1}$ if $n \geq 2$

To achieve this we use the lemmas of Section 2.1. In terms of the dimensionless quantities $A(r)$ and $\delta(r)$, our inductive hypotesis is

$$A(r_k) + \delta(r_k) \leq C_0 \varepsilon_1^{3/5} r_k^2, \qquad 2 \leq k \leq n, \tag{2.22}$$

and we also know, from (2.19), that

$$G(r_1) + D(r_1) \leq C\varepsilon_1. \tag{2.23}$$

By Lemma 2.1, (2.8), we deduce from (2.22) that

$$r_n^{-5/3} \iint\limits_{Q^n} |u|^{10/3} = G(r_n) \leq C\varepsilon_1 r_n^{10/3} \tag{2.24}$$

so that

$$\left(\fint\!\!\fint_{Q^{n+1}} |u|^{10/3} \right)^{9/10} \leq C \left(\fint\!\!\fint_{Q^n} |u|^{10/3} \right)^{9/10} \leq C^* \varepsilon_1^{9/10}. \tag{2.25}$$

Therefore, if ε_1 is so small that

$$C^* \varepsilon_1^{3/10} \leq 1/2 \tag{2.26}$$

we get

$$\left(\fint\!\!\fint_{Q^{n+1}} |u|^{10/3} \right)^{9/10} \leq \frac{1}{2} \varepsilon_1^{3/5}, \tag{2.27}$$

which is half of $(2.20)_{n+1}$.

For the second half of $(2.20)_{n+1}$ we use Lemma 2.2 with $\rho = 1/4$ and $r = r_n$. We have

$$G(r_{n+1}) \leq CG(r_n) \leq C\varepsilon_1 r_n^{10/3}$$

and

$$A(r_{n+1}) \le CA(r_n) \le C\varepsilon_1^{3/5} r_n^2,$$

by (2.24) and (2.22). On the other hand, $D(\frac{1}{4}) \le C\varepsilon_1$, by (2.23). The terms on the right side of (2.9) may therefore be estimated as follows, assuming $\varepsilon_1 \le 1$:

$$\begin{aligned} r_{n+1}^{9/5} G(r_{n+1})^{3/10} D(\frac{1}{4})^{3/5} &\le C r_n^{14/5} \varepsilon_1^{9/10}, \\ r_{n+1}^{9/5} G(r_{n+1})^{3/10} G(\frac{1}{4})^{3/5} &\le C r_n^{14/5} \varepsilon^{9/10}, \\ G(r_{n+1})^{3/10} G(r_n)^{3/5} &\le C r_n^3 \varepsilon_1^{9/10}, \end{aligned}$$

and

$$\begin{aligned} r_{n+1}^3 G(r_{n+1})^{3/10} \sup_{s - r_{n+1}^2 < t < s} \int_{r_n < |y-a| < 1/4} \frac{|u|^2}{|y|^4} &\le C r_n^4 \varepsilon_1^{3/10} \sum_{k=2}^{n} r_k^{-3} A(r_k) \\ &\le C r_n^4 \varepsilon_1^{9/10} \sum_{k=2}^{n} r_k^{-1} \\ &\le C r_n^3 \varepsilon^{9/10}. \end{aligned}$$

Noting that $r_n \le 1$, we conclude that, for $\varepsilon_1 \le 1$,

$$L(r_{n+1}) \le C r_n^{14/5} \varepsilon_1^{9/10},$$

from which it follows

$$r_{n+1}^{1/5} \iint_{Q^{n+1}} |u||p - \overline{p}_{n+1}| \le C r_n^{-14/5} L(r_{n+1}) \le C^{**} \varepsilon_1^{9/10},$$

for some absolute constant C^{**}. We then require that ε_1 be small enough to satisfy

$$C^{**} \varepsilon_1^{3/10} \le \frac{1}{2}, \tag{2.28}$$

so that

$$r_{n+1}^{1/5} \iint_{Q^{n+1}} |u||p - \overline{p}_{n+1}| \le \frac{1}{2} \varepsilon_1^{3/5}. \tag{2.29}$$

From (2.27) and (2.29) we conclude $(2.20)_{n+1}$.

2.2.3 Step 3: $(2.20)_k$, $3 \le k \le n$, implies $(2.21)_n$ if $n \ge 3$.

We apply the generalized energy inequality (2.16) with a suitable test function $\phi = \phi_n$. To simplify the notation, we shift the coordinates in space-time to center them at the point of interest. So, from now on the origin $x = 0, t = 0$ will represent the point we have denoted so far by (a, s).

We take

$$\phi_n = \chi\psi_n, \tag{2.30}$$

where ψ_n is a constant times the fundamental solution of the backward heat equation $\phi_t + \Delta\phi = 0$ with singularity at $(0, r_n^2)$ and χ is a cutoff function. More precisely,

$$\psi_n = \frac{1}{(r_n^2 - t)^{3/2}} \exp\left\{\frac{-|x|^2}{4(r_n^2 - t)}\right\},$$

and χ is C^∞ on $\{t \le 0\}$ with $0 \le \chi \le 1$ and

$$\begin{aligned} \chi &\equiv 1 \quad \text{on} \quad Q^2 = Q_{1/4}(0,0), \\ \chi &\equiv 0 \quad \text{off} \quad Q_{1/3}(0,0). \end{aligned}$$

One easily verifies that $\phi_n \ge 0$ and

$$\frac{\partial \phi_n}{\partial t} + \Delta\phi_n = 0 \quad \text{on} \quad Q^2, \tag{2.31}$$

$$\left|\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right| \le C \quad \text{everywhere}, \tag{2.32}$$

$$\frac{1}{C} r_n^{-3} \le \phi_n \le C r_n^{-3}, \qquad |\nabla\phi_n| \le C r_n^{-4} \quad \text{on } Q^n \quad n \ge 2, \tag{2.33}$$

$$\phi_n \le C r_k^{-3}, \qquad |\nabla\phi_n| \le C r_k^{-4} \quad \text{on} \quad Q^{k-1} \backslash Q^k, \quad 1 \le k \le n, \tag{2.34}$$

for a suitable absolute constant C, independent of n.

Using ϕ_n as test function in (2.16), and estimating ϕ_n from below by (2.33), we see that

$$\sup_{-r_n^2 < t \le 0} \left(\fint_{|x|<r_n} |u|^2(x,t)\, dx \right) + r_n^{-3} \iint_{Q^n} |\nabla u|^2 \le C(I + II + III),$$

where

$$I = \iint_{Q^1} |u|^2 \left|\frac{\partial \phi_n}{\partial t} + \Delta\phi_n\right|,$$

$$II = \iint_{Q^1} |u|^3 |\nabla \phi_n|,$$

$$III = \int_{-1/4}^{0} \left| \int_{B_1} p(u \cdot \nabla \phi_n) \right|.$$

We must estimate each of the three terms, using $(2.20)_k$, $3 \leq k \leq n$, and (2.19).

The estimate for I follows easily using (2.32), Hölder's inequality and (2.19) to obtain

$$I \leq C \left(\iint_{Q^1} |u|^2 \right) \leq C \left(\iint_{Q^1} |u|^{10/3} \right)^{3/5} \leq C\varepsilon_1^{3/5}.$$

We estimate II using (2.33), (2.34), and $(2.20)_k$, $3 \leq k \leq n$, to get

$$\begin{aligned} II &\leq C \sum_{k=1}^{n} r_k^{-4} \iint_{Q^k} |u|^3 \\ &\leq C \sum_{k=1}^{k} r_k^{-7/2} \left(\iint_{Q^k} |u|^{10/3} \right)^{9/10} \\ &\leq C\varepsilon_1^{3/5} \sum_{k=1}^{n} r_k \leq C\varepsilon_1^{3/5}. \end{aligned}$$

The estimate for III is more delicate, since the available estimates are not good to bound $\iint |u||p||\nabla \phi_n|$. We must instead take advantage of the fact that u is divergence-free, and reduce the problem to one involving the oscillation of p. For each $k \geq 1$ let $0 \leq \chi_k \leq 1$ be a C^∞ function on $Q^1 = Q_{1/2}(0,0)$ such that

$$\chi_k \equiv 1 \quad \text{on} \quad Q_{7r_k/8}(0,0), \quad \chi_k \equiv 0 \quad \text{on} \quad Q^1 \setminus Q_{r_k}(0,0),$$

and

$$|\nabla \chi_k| \leq \frac{C}{r_k}.$$

Then, $\chi_1\phi_n = \phi_n$. Let $B_k = \{|x| < r_k\}$. Thus,

$$
\begin{aligned}
III = \int_{-1/4}^{0} \left| \int_{B_1} p(u \cdot \nabla \phi_n) \right| \leq \sum_{k=1}^{n-1} \int_{-1/4}^{0} \left| \int_{B_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) \right| \\
+ \int_{-1/4}^{0} \left| \int_{B_1} pu \cdot \nabla(\chi_n \phi_n) \right|.
\end{aligned} \tag{2.35}
$$

We estimate the terms on the right-hand side of (2.35) separately. Since $\chi_k - \chi_{k+1}$ is supported on Q^k and u is divergence-free, we have for $k \geq 3$

$$
\int_{-1/4}^{0} \left| \int_{B_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) \right| = \int_{-r_k^2}^{0} \left| \int_{B_k} (p - \overline{p}_k) u \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) \right|.
$$

Similarly,

$$
\int_{-1/4}^{0} \left| \int_{B_1} pu \cdot \nabla(\chi_n \phi_n) \right| = \int_{-r_n^2}^{0} \left| \int_{B_n} (p - \overline{p}_n) u \cdot \nabla(\chi_n \phi_n) \right|
$$

while, for $k = 1, 2$,

$$
\int_{-r_n^2} \left| \int_{B_1} pu \cdot \nabla((\chi_k - \chi_{k+1})\phi_n) \right| \leq C \iint\limits_{Q^1} |p||u|
$$

by (2.34). Therefore,

$$
III \leq C \sum_{k=3}^{n} \iint\limits_{Q^k} |u||p - \overline{p}_k| r_k^{-4} + C \iint\limits_{Q^1} |u||p|.
$$

Using the inductive hypothesis $(2.20)_k$, $3 \leq k \leq n$, and also (2.19) we conclude that

$$
III \leq C \sum_{k=3}^{n} r_k^{4/5} \varepsilon_1^{3/5} + C\varepsilon_1^{3/5} \leq C\varepsilon_1^{3/5}.
$$

This completes the induction process and proves Proposition 1.1.

We emphasize that the various constants "C" appearing in the argument for step 3 are universal constants that do not depend on either n or ε_1. Therefore the constants C^* and C^{**} in the step 2 are independent of ε_1 and n. It follows that (2.26) and (2.28) will indeed hold if ε_1 is small enough, and that ε_1 can be chosen without danger of circular reasoning.

Remark 2.1. Proposition 1.1 implies that if the dimensionless estimate

$$\int \left(\int |u|^s + |p|^{s/2} \, dx \right)^{s'/s} dt < +\infty, \quad \frac{10}{3} < s \le s', \quad \frac{10}{3s} + \frac{20}{9s'} = 1,$$

holds, on some domain $D = \Omega \times (a, b) \subseteq \Pi_T$, then u is regular on D. Indeed, using Hölder's inequality we find that

$$\limsup_{r \to 0} r^{-5/3} \iint_{Q_r^*(x,t)} |u|^{10/3} + |p|^{5/3} = 0.$$

Actually, Proposition 1 in [2] allows to obtain a slightly stronger result, which is the corresponding assertion with 10/3 replaced by 3 and 20/9 replaced by 2. In this connection, see [10].

Chapter 3

C-K-N's Theorem: $\mathcal{H}^1(\mathcal{S}) = 0$

In this chapter we derive the main result of C-K-N's paper:

Theorem 3.1. *Let* (u, p) *be a suitable weak solution of* (1.1)-(1.3) *in* Π_T *and* $\mathcal{S}$ *be the set of its singular points. Then* $\mathcal{H}^1(\mathcal{S}) = 0$.

As was mentioned in section 1.3, Chapter 1, Theorem 3.1 is a simple consequence of Proposition 1.2, which in turn was derived in [2] from a certain "decay estimate" (*cf.* [2], Proposition 2, p.797). In our presentation here we follow F. Lin's paper [9] for the proof of Proposition 1.2 (*cf.* [9], Theorem 3.3), which is slightly more direct than the one in [2].

Let

$$\begin{aligned} B_1 &= \left\{x \in \mathbb{R}^3 \,:\, |x| < 1\right\}, \\ B_\theta &= \left\{x \in \mathbb{R}^3 \,:\, |x| < \theta\right\},\ \theta > 0, \\ Q_1^\star &= \left\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \;\middle|\; x \in B_1,\ t \in (-\tfrac{7}{8}, \tfrac{1}{8})\right\}. \end{aligned}$$

In this chapter, by (u, p) we denote a suitable weak solution defined in $\Pi_1^* = \mathbb{R}^3 \times (-\frac{7}{8}, \frac{1}{8})$.

Lemma 3.1. *Let* (u, p) *be a suitable weak solution of Navier-Stokes system in* Π_1^*. *Then, there exists an absolute constant* $C > 0$ *such that, for any* $\theta \in (0, 1/2)$,

$$\iint\limits_{Q_\theta^\star} |p|^{5/3} \le C \left\{ \iint\limits_{Q_1^\star} |u - \bar{u}|^{10/3} + \theta^3 \iint\limits_{Q_1^\star} |p|^{5/3} \right\}, \tag{3.1}$$

where $\bar{u} = \bar{u}(t)$ is the average of $u(t)$ over the ball B_1.

Proof. The pressure p is in $L^{5/3}(\Pi_1^*)$ and satisfies

$$-\Delta p = \sum_{i,j} \partial_{x_i}\partial_{x_j}(u_i u_j) \qquad \text{in } \mathcal{D}'(\Pi_1^*).$$

Let $0 < \theta < 1/2$. Let $\psi(x) = \chi_{B_1}(x)$. Consider the solution $p_1 \in L^{5/3}(\Pi_1^*)$ of the following equation

$$-\Delta p_1 = \sum_{i,j} \partial_{x_i}\partial_{x_j}((u_i - \bar{u}_i)(u_j - \bar{u}_j)\psi)) \quad \text{in } \mathcal{D}'(\Pi_1^*)\ . \tag{3.2}$$

By the properties of the singular integral solution operator, there is an absolute constant $C > 0$ such that:

$$\begin{aligned} \int_{B_\theta} |p_1|^{5/3} dy &\leq \int_{\mathbb{R}^3} |p_1|^{5/3} dy \leq C \int_{\mathbb{R}^3} |u - \bar{u}|^{10/3} \psi^{10/3} dy \\ &\leq C \int_{B_1} |u - \bar{u}|^{10/3} dy, \quad \text{a.e } t \in (-\tfrac{7}{8}, \tfrac{1}{8}). \end{aligned} \tag{3.3}$$

Let $p_2 = p - p_1$. Then, since u is divergence-free we have:

$$\Delta p_2 = 0 \qquad \text{in } \mathcal{D}'\left(B_1 \times \left(-\frac{7}{8}, \frac{1}{8}\right)\right).$$

By the properties of harmonic functions, for any $x \in B_\theta$

$$p_2(x,t) = \frac{1}{|B_{1/2}(x)|} \int_{B_{1/2}(x)} p_2(y) dy. \quad \text{a.e. } t \in \left(-\frac{7}{8}, \frac{1}{8}\right).$$

Using Jensen inequality with exponent $\frac{5}{3}$ in the above representation formula and integrating in x over B_θ we get a.e. $t \in (-\frac{7}{8}, \frac{1}{8})$

$$\int_{B_\theta(0)} |p_2|^{5/3} \leq C\theta^3 \int_{B_1(0)} |p_2|^{5/3}.$$

Integration in t over the interval $(-\frac{7}{8}\theta^2, \frac{1}{8}\theta^2)$ results in

$$\begin{aligned} \iint_{Q_\theta^\star} |p_2|^{5/3} \leq C\theta^3 \iint_{Q_1^\star} |p - p_1|^{5/3} &\leq C\theta^3 \iint_{Q_1^\star} |p|^{5/3} \\ &+ C\theta^3 \iint_{Q_1^\star} |u - \bar{u}|^{10/3}. \end{aligned} \tag{3.4}$$

Finally, from (3.3) and (3.4) we get (3.1). $\square$

We need a scaled version of (3.1). Let $0 < \rho < 1$, $v = \rho u(\rho^2 t, \rho x)$ and $P = \rho^2 p(\rho^2 t, \rho x)$. (v, P) is a solution of Navier-Stockes system on $Q_1^\star$. In the above inequality written for (v, P) we scale the integration variables and set $r = \theta\rho$, $\bar{u}_\rho$ to be the average of $u(t)$ over B_ρ. We get

$$\iint_{Q_r^\star} |p|^{5/3} \leq C \left\{ \iint_{Q_\rho^\star} |u - \bar{u}_\rho|^{10/3} + \left(\frac{r}{\rho}\right)^3 \iint_{Q_\rho^\star} |p|^{5/3} \right\}, \qquad \forall\, r < \frac{1}{2}\rho, \quad (3.5)$$

We use Lemma 1.2, with $q = 10/3, a = 1$, to get for any $0 < r < \frac{1}{2}\rho$

$$\begin{aligned}
\iint_{Q_r^\star} |p|^{5/3} &\leq C \int_{-7\rho^2/8}^{\rho^2/8} dt \left(\int_{B_\rho} |u|^2 dx \right)^{2/3} \int_{B_\rho} |\nabla u|^2 \\
&+ C \left(\frac{r}{\rho}\right)^3 \iint_{Q_\rho^\star} |p|^{5/3} \\
&\leq C \left(\sup_{-7\rho^2/8 < t < \rho^2/8} \int_{B_\rho \times \{t\}} |u|^2 \right)^{2/3} \iint_{Q_\rho^\star} |\nabla u|^2 \\
&+ C \left(\frac{r}{\rho}\right)^3 \iint_{Q_\rho^\star} |p|^{5/3}. \qquad (3.6)
\end{aligned}$$

We introduce the following dimensionless quantities:

$$D^\star(r) = r^{-5/3} \iint_{Q_r^\star} |p|^{5/3}, \quad G^\star(r) = r^{-5/3} \iint_{Q_r^\star} |u|^{10/3},$$

$$A^\star(r) = \sup_{-7r^2/8 < t < r^2/8} r^{-1} \int_{B_r} |u|^2, \quad \delta^\star(r) = r^{-1} \iint_{Q_r^\star} |\nabla u|^2.$$

Now (3.6) reads

$$D^\star(r) \leq C \left\{ \left(\frac{\rho}{r}\right)^{5/3} A^\star(\rho)^{2/3} \delta^\star(\rho) + \left(\frac{r}{\rho}\right)^{4/3} D^\star(\rho) \right\}, \quad \text{for } r \leq \frac{1}{2}\rho. \quad (3.7)$$

The "starred" version of (2.8) in Lemma 2.1 reads

$$
\begin{aligned}
G^\star(r) \quad &\leq \quad C\left\{\left(\frac{r}{\rho}\right)^{10/3} A^\star(\rho)^{5/3} + \left(\frac{\rho}{r}\right)^{5/3} A^\star(\rho)^{2/3}\delta^\star(\rho)\right. \\
&+ \quad \left.\left(\frac{\rho}{r}\right)^{10/3} A^\star(\rho)^{5/6}\delta^\star(\rho)^{5/6}\right\}, \quad \text{for } \; r \leq \rho. \qquad (3.8)
\end{aligned}
$$

The proof is exactly the same as that of (2.8), only replacing A, D, δ, G by A^*, D^*, G^*, δ^*.

Lemma 3.2. *For any* $r \leq \frac{1}{2}\rho$,

$$
\begin{aligned}
A^\star(r) \quad &\leq \quad C\left\{\left(\frac{\rho}{r}\right) G^\star(\rho)^{3/5} + \left(\frac{\rho}{r}\right) G^\star(\rho)^{3/10} A^\star(\rho)^{1/2}\delta^\star(\rho)^{1/2} + \right. \\
&+ \quad \left.\left(\frac{\rho}{r}\right) G^\star(\rho)^{3/10} D^\star(\rho)^{3/5}\right\}. \qquad (3.9)
\end{aligned}
$$

Proof. In the generalized energy inequality (1.34), we take a test function $\psi(x,t)$ satisfying

$$
\begin{array}{ll}
0 \leq \psi \in C_0^\infty(Q_\rho^\star), & \psi \equiv 1 \quad \text{on } Q^\star_{\frac{3}{4}\rho}, \\
|\nabla\psi| \leq c\rho^{-1}, & |\psi_t + \Delta\psi| \leq c\rho^{-2}, \\
r \leq \rho/2. &
\end{array}
$$

We have

$$
\sup_{-7/8<t<1/8} \int_{B_1\times\{t\}} |u|^2\psi \leq \iint_{Q_1^\star} |u|^2(\psi_t + \Delta\psi) + \iint_{Q_1^\star} u\cdot\nabla\psi(|u|^2 + 2p).
$$

Then, (the term $\overline{|u|^2}(t) = \frac{1}{|B_\rho|}\int_{B_\rho} |u(t)|^2$ can be added since u is divergence-free)

$$
\begin{aligned}
A^\star(r) \quad &\leq \quad Cr^{-1}\left\{\iint_{Q_\rho^\star} \rho^{-2}|u|^2 + \rho^{-1}\left(|u|\left||u|^2 - \overline{|u|^2}\right|\right) + \rho^{-1}|u||p|\right\} \\
&\leq \quad C\left\{\left(\frac{\rho}{r}\right) G^\star(\rho)^{3/5}\right. \\
&+ \quad r^{-1}\rho^{-1}\left(\iint_{Q_\rho^\star} |u|^{10/3}\right)^{3/10}\left(\iint_{Q_\rho^\star} ||u|^2 - \overline{|u|^2}|^{3/2}\right)^{2/3}\rho^{1/6} \\
&+ \quad \left.\left(\frac{\rho}{r}\right) G^\star(\rho)^{3/10} D^\star(\rho)^{3/5}\right\}. \qquad (3.10)
\end{aligned}
$$

Now, by Lemma 1.1 applied to the function $|u|^2$, with $q = 1$, we obtain

$$\iint\limits_{Q^\star_\rho} ||u|^2 - \overline{|u|^2}|^{3/2} \le C \int\limits_{-7\rho^2/8}^{\rho^2/8} \Bigg(\int\limits_{B\rho} |u||\nabla u|\Bigg)^{3/2}$$

$$\le C \int\limits_{-7\rho^2/8}^{\rho^2/8} \Bigg(\int\limits_{B\rho} |u|^2\Bigg)^{3/4} \Bigg(\int\limits_{B_\rho} |\nabla u|^2\Bigg)^{3/4}$$

$$\le C\rho^{3/4} A^\star(\rho)^{3/4} \int\limits_{-7\rho^2/8}^{\rho^2/8} \Bigg(\int\limits_{B_\rho} |\nabla u|^2\Bigg)^{3/4} \le C\rho^2 A^\star(\rho)^{3/4}\delta^\star(\rho)^{3/4}.$$

Inserting this estimate in (3.10) one easly gets (3.9). □

We prove the following

Proposition 3.1. *Let (u, p) be a suitable, weak solution of Navier-Stokes system in Π^*_1. For any $\epsilon_1 > 0$, there exists an absolute constant $\epsilon_2 > 0$, such that if*

$$\overline{\lim_{\rho\to 0+}}\, \delta^\star(\rho) \le \epsilon_2, \tag{3.11}$$

then

$$\underline{\lim}_{\rho\to 0+} \{G^\star(\rho) + D^\star(\rho)\} \le \epsilon_1. \tag{3.12}$$

Proof. We use the following notations

$0 < \theta < 1/4, \quad 0 < \rho < 1,$

$c(\theta)$ is any polynomial in θ^{-1},

$K(\theta, \delta^\star(\rho))$ is any finte linear combination of terms of the form

$\theta^{-\alpha}\delta^\star(\rho)^\beta, \quad \alpha, \beta > 0.$

We write (3.7) with $r = \theta\rho$ and then take it to $\frac{6}{5}$ power

$$\begin{aligned} D^\star(\theta\rho)^{6/5} &\le C\left\{\theta^{8/5} D^\star(\rho)^{6/5} + \theta^{-2} A^\star(\rho)^{4/5}\delta^\star(\rho)^{6/5}\right\} \\ &\le C\left\{\theta D^\star(\rho)^{6/5} + \theta A^\star(\rho) + \theta^{-14}\delta^\star(\rho)^6\right\} \\ &= C\left\{\theta D^\star(\rho)^{6/5} + \theta A^\star(\rho) + K(\theta, \delta^\star(\rho))\right\}. \end{aligned} \tag{3.13}$$

Take now (3.8) with $r=\theta\rho$.

$$\begin{aligned} G^\star(\theta\rho) &\leq C\left\{\theta^{10/3}A^\star(\rho)^{5/3}+\theta^{-5/3}A^\star(\rho)^{2/3}\delta^\star(\rho)\right. \\ &+ \left.\theta^{-10/3}A^\star(\rho)^{5/6}\delta^\star(\rho)^{5/6}\right\} \\ &\leq C\left\{\theta^{10/3}A^\star(\rho)^{5/3}+c(\theta)\delta^\star(\rho)^{5/3}\right\}, \end{aligned}$$

and so

$$G^\star(\theta\rho)^{3/5}\leq C\left\{\theta A^\star(\rho)+K(\theta,\delta^\star(\rho))\right\}. \tag{3.14}$$

Now we estimate $A^\star(\theta\rho)$. Take $r=2\theta\rho$ in (3.7) and (3.8)

$$D^\star(2\theta\rho) \leq C\left\{\theta^{4/3}D^\star(\rho)+\theta^{-5/3}A^\star(\rho)^{2/3}\delta^\star(\rho)\right\}, \tag{3.15}$$

$$\begin{aligned} G^\star(2\theta\rho) &\leq C\left\{\theta^{10/3}A^\star(\rho)^{5/3}+\theta^{-5/3}A^\star(\rho)^{2/3}\delta^\star(\rho)\right. \\ &+ \left.\theta^{-10/3}A^\star(\rho)^{5/6}\delta^\star(\rho)^{5/6}\right\}, \end{aligned} \tag{3.16}$$

and $r=\theta\rho,\ \rho=2\theta\rho$ in (3.9)

$$\begin{aligned} A^\star(\theta\rho) &\leq C\left\{G^\star(2\theta\rho)^{3/5}+G^\star(2\theta\rho)^{3/10}A^\star(2\theta\rho)^{1/2}\delta^\star(2\theta\rho)^{1/2}+\right. \\ &+ \left.G^\star(2\theta\rho)^{3/10}D^\star(2\theta\rho)^{3/5}\right\}. \end{aligned} \tag{3.17}$$

Now

$$A^\star(2\theta\rho)\leq C\theta^{-1}A^\star(\rho),$$

$$G^\star(2\theta\rho)^{3/10}A^\star(2\theta\rho)^{1/2}\delta^\star(2\theta\rho)^{1/2}\leq \theta A^\star(\rho)+C\theta^{-2}G^\star(2\theta\rho)^{3/5}\delta^\star(2\theta\rho).$$

Inserting these inequalities in (3.17) we get

$$\begin{aligned} A^\star(\theta\rho) &\leq C\left\{\left(1+\theta^{-2}\delta^\star(2\theta\rho)\right)G^\star(2\theta\rho)^{3/5}+\theta A^\star(\rho)\right. \\ &+ \left.G^\star(2\theta\rho)^{3/10}D^\star(2\theta\rho)^{3/5}\right\}. \end{aligned} \tag{3.18}$$

Substitute $G(2\theta\rho)$ and $D(2\theta\rho)$ in the last inequality by (3.16) and (3.15)

$$\begin{aligned} A^\star(\theta\rho) &\leq C\left(\theta A^\star(\rho)+\theta^{-1}A^\star(\rho)^{2/5}\delta^\star(\rho)^{3/5}+\theta^{-2}A^\star(\rho)^{1/2}\delta^\star(\rho)^{1/2}\right) \\ &\times \left(1+\theta^{-2}\delta^\star(2\theta\rho)\right) \\ &+ C\left[\theta A^\star(\rho)^{1/2}+\theta^{-1/2}A^\star(\rho)^{1/5}\delta^\star(\rho)^{3/10}+\theta^{-1}A^\star(\rho)^{1/4}\delta^\star(\rho)^{1/4}\right] \\ &\times \left[\theta^{4/5}D^\star(\rho)^{3/5}+\theta^{-1}A^\star(\rho)^{2/5}\delta^\star(\rho)^{3/5}\right]\equiv I_1+I_2. \end{aligned}$$

We estimate I_1(note that $\delta^\star(2\theta\rho) \le C\theta^{-1}\delta^\star(\rho)$).

$$\begin{aligned}
I_1 &\le C\theta A^\star(\rho) + c(\theta)A^\star(\rho)\delta^\star(\rho) \\
&+ c(\theta)\Big(A^\star(\rho)^{2/5}(\delta^\star(\rho)^{8/5} + \delta^\star(\rho)^{3/5}) + A^\star(\rho)^{1/2}(\delta^\star(\rho)^{3/2} \\
&+ \delta^\star(\rho)^{1/2})\Big) \\
&\le C\theta A^\star(\rho) + c(\theta)A^\star(\rho)\delta^\star(\rho) + K(\theta, \delta^\star(\rho)), \qquad (3.19)
\end{aligned}$$

and I_2

$$\begin{aligned}
I_2 &\le C\theta^{9/5}A^\star(\rho)^{1/2}D^\star(\rho)^{3/5} + c(\theta)A^\star(\rho)^{1/5}D^\star(\rho)^{3/5}\delta^\star(\rho)^{3/10} \\
&+ c(\theta)A^\star(\rho)^{1/4}D^\star(\rho)^{3/5}\delta^\star(\rho)^{1/4} \\
&+ c(\theta)\Big\{A^\star(\rho)^{9/10}\delta^\star(\rho)^{3/5} + A^\star(\rho)^{3/5}\delta^\star(\rho)^{9/10} \\
&+ A^\star(\rho)^{13/20}\delta^\star(\rho)^{17/20}\Big\}. \qquad (3.20)
\end{aligned}$$

We bound each term on the right in (3.20).

$$\begin{aligned}
\theta^{9/5}A^\star(\rho)^{1/2}D^\star(\rho)^{3/5} &\le \theta A^\star(\rho) + \theta D^\star(\rho)^{6/5}, \\
c(\theta)A^\star(\rho)^{1/5}D^\star(\rho)^{3/5}\delta^\star(\rho)^{3/10} &\le \theta A^\star(\rho) + c(\theta)D^\star(\rho)^{3/4}\delta^\star(\rho)^{3/8} \\
&\le \theta A^\star(\rho) + \theta D^\star(\rho)^{6/5} + c(\theta)\delta^\star(\rho), \\
c(\theta)A^\star(\rho)^{1/4}D^\star(\rho)^{3/5}\delta^\star(\rho)^{1/4} &\le \theta A^\star(\rho) + c(\theta)D^\star(\rho)^{4/5}\delta^\star(\rho)^{1/3} \\
&\le \theta A^\star(\rho) + \theta D^\star(\rho)^{6/5} + c(\theta)\delta^\star(\rho),
\end{aligned}$$

and also

$$\begin{aligned}
&c(\theta)\Big\{A^\star(\rho)^{9/10}\delta^\star(\rho)^{3/5} + A^\star(\rho)^{3/5}\delta^\star(\rho)^{9/10} + A^\star(\rho)^{13/20}\delta^\star(\rho)^{17/20}\Big\} \\
&\qquad \le \theta A^\star(\rho) + K(\theta, \delta^\star(\rho)).
\end{aligned}$$

We use the above estimates, obtaining

$$I_2 \le C\Big\{\theta A^\star(\rho) + \theta D^\star(\rho)^{6/5} + K(\theta, \delta^\star(\rho))\Big\}. \qquad (3.21)$$

Hence, (3.19) and (3.21) give

$$A^\star(\theta\rho) \le C\left\{\theta A^\star(\rho) + \theta D^\star(\rho)^{6/5} + c(\theta)A^\star(\rho)\delta^\star(\rho) + K(\theta, \delta^\star(\rho))\right\}. \quad (3.22)$$

Inequalities (3.22) and (3.13) together give

$$\begin{aligned} A^\star(\theta\rho) + D^\star(\theta\rho)^{6/5} \quad &\le \quad C\left\{\theta A^\star(\rho) + \theta D^\star(\rho)^{6/5}\right. \\ &+ \quad \left. c(\theta)\delta^\star(\rho)^{1/2}A^\star(\rho) + K(\theta, \delta^\star(\rho))\right\}. \end{aligned} \quad (3.23)$$

Choose θ such that $C\theta < 1/4$. Let $\epsilon > 0$ be fixed and choose $\rho_0 > 0$ such that

$$C\left\{K(\theta, \delta^\star(\rho)) + c(\theta)\delta^\star(\rho)^{1/2}\right\} < \min\{1/4, \epsilon/4\},$$

$$\forall \rho \le \rho_0.$$

This can be achieved by choosing ϵ_2 in (3.11), sufficiently small. We get from (3.23) and (3.14)

$$A^\star(\theta\rho) + D^\star(\theta\rho)^{6/5} \le \frac{1}{2}\left\{A^\star(\rho) + D^\star(\rho)^{6/5}\right\} + \frac{\epsilon}{4},$$

$$G^\star(\theta\rho)^{3/5} \le \frac{1}{2}A^\star(\rho) + \frac{\epsilon}{4},$$

$$0 < \rho \le \rho_0.$$

Applying the last inequality recursively to the sequence of points $\theta^n\rho_0$, we obtain

$$A^\star(\theta^n\rho_0) + D^\star(\theta^n\rho_0)^{6/5} \le \frac{1}{2^n}\{A^\star(\rho_0) + D^\star(\rho_0)\} + \frac{\epsilon}{2},$$

$$G^\star(\theta^n\rho_0)^{3/5} \le \frac{1}{2}A^\star(\theta^{n-1}\rho_0) + \frac{\epsilon}{4}.$$

And consequently,

$$A^\star(\theta^n\rho_0) + D^\star(\theta^n\rho_0)^{6/5} \le \epsilon,$$

$$G^\star(\theta^n\rho_0)^{3/5} \le \epsilon.$$

for sufficiently large n. We then choose $\varepsilon > 0$ satisfying $\epsilon^{5/6} + \epsilon^{5/3} = \epsilon_1$ to obtain (3.12).

□

Proof of Theorem 3.1. In view of Proposition 1.1 and Proposition 3.1 the assertion of Proposition 1.2 is immediate and from the latter the theorem follows as explained in chapter 1, section 1.3. □

The following remarks are taken from [2].

Remark 3.1. Proposition 1.2 implies that if the dimensionless estimate

$$\int \left(\int |\nabla u|^2 \, dx \right)^2 dt < \infty$$

holds, over some domain $D = \Omega \times (a, b) \subseteq \Pi_T$, the u is regular on D. Indeed, Hölder's inequality clearly implies

$$\limsup r^{-1} \iint_{Q_r^*(x,t)} |\nabla u|^2 = 0.$$

Remark 3.2. The set of singular times has Hausdorff $\frac{1}{2}$-dimensional measure zero in $\mathbb{R}$ (*cf.* [12]). This follows from the fact that, for any $X \subseteq \mathbb{R}^3 \times \mathbb{R}$, its projection Σ_X onto the t-axis satisfies $\mathcal{H}^{1/2}(\Sigma_X) \leq C\mathcal{P}^1(X)$. We leave the easy verification as an exercise for the reader.

Remark 3.3. If u has cylindrical symmetry about some axis in space, then singularities can occur only on the axis (*cf.* [6]). In fact, any off-axis singularity would give rise to a circle of singular points, contradicting the fact that $\mathcal{H}^1(\mathcal{S}) = 0$.

Chapter 4

Existence of Suitable Weak Solutions

In this chapter we prove the existence of a suitable weak solution for Navier-Stokes system (see Definition 1.1 in Chapter 1). The proof given below is taken "as it is" from Appendix 1 of [2] for the case $\Omega = \mathbb{R}^3$.

We use the following spaces:

$$H^1\left(\mathbb{R}^3\right) = \text{ closure of } C_0^\infty\left(\mathbb{R}^3;\mathbb{R}^3\right) \text{ in the norm } ||\nabla u||_{L^2(\mathbb{R}^3)}$$

$$H^2\left(\mathbb{R}^3\right) = \text{ closure of } C_0^\infty\left(\mathbb{R}^3;\mathbb{R}^3\right) \text{ in the norm } ||u||_{L^2(\mathbb{R}^3)} + ||\Delta u||_{L^2(\mathbb{R}^3)}$$

$$H^{-1}\left(\mathbb{R}^3\right) = \text{ the dual space of } H^1\left(\mathbb{R}^3\right)$$

$$\mathcal{V} = C_0^\infty\left(\mathbb{R}^3;\mathbb{R}^3\right) \cap \{u \ : \ \operatorname{div} u = 0\}$$

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2\left(\mathbb{R}^3\right)$$

$$V = \text{ closure of } \mathcal{V} \text{ in } H_0^1\left(\mathbb{R}^3\right)$$

$$V' = \text{ the dual space of } V$$

We use the notation:

$$\Pi_T = \mathbb{R}^3 \times (0,T), \quad \Pi_{*T} = \mathbb{R}^3 \times (-\infty, T)$$

$$E_0(u) = \text{ess} \sup_{0<t<T} \int_{\mathbb{R}^3} |u|^2$$

$$E_1(u) = \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2$$

Let $u_0 \in H$.

Theorem 4.1. *There exist a weak solution (u,p) of the Navier-Stokes system on Π_T satisfying:*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{4.1}$$

$$u(t) \to u_0 \text{ weakly in } H \text{ as } t \to 0, \tag{4.2}$$

$$p \in L^{5/3}(\Pi_T), \tag{4.3}$$

$$\text{if } \phi \in C_0^\infty(\Pi_{*T}) \quad \phi \geq 0, \text{ then } \forall\, 0 < t < T$$

$$\int\limits_{\mathbb{R}^3 \times \{t\}} |u|^2\phi + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \leq \int_{\mathbb{R}^3} |u_0|^2 \phi(x,0)$$

$$+\int_0^t \int_{\mathbb{R}^3} |u|^2(\phi_t + \Delta\phi) + \int_0^t \int_{\mathbb{R}^3} (|u|^2 + 2p)u\nabla\phi \tag{4.4}$$

First we establish various properties of the solutions of of linearized Navier-Stokes system.

Lemma 4.1. *Let $u_0 \in H$ and $w \in C^\infty\left(\overline{\Pi}_T\right)$,* $\operatorname{div} w = 0$. *Then there exist unique functions u and p such that*

$$u \in C([0,T],H) \cap L^2(0,T,V), \tag{4.5}$$

$$p \in L^{5/3}\left(\mathbb{R}^3 \times (0,T)\right), \tag{4.6}$$

$$u_t + w \cdot \nabla u - \Delta u + \nabla p = 0 \tag{4.7}$$

in the sense of distributions on Π_T,

$$u(0) = u_0. \tag{4.8}$$

Proof. The proof of Theorem 1.1 in Chapter III in [18], which deals with the case $w = 0$, can be carried over with unsubstential changes to our case. We just emphasize some points of it. The existence of

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{4.9}$$

such that for each $v \in V$

$$\frac{d}{dt}\int_{\mathbb{R}^3} (u,v) + \int_{\mathbb{R}^3} (w\nabla u, v) + \int_{\mathbb{R}^3} (\nabla u, \nabla v) = 0 \tag{4.10}$$

and distribution p, such that (4.7) holds, can be proved using Faedo-Galerkin method. Then, since $w \cdot \nabla u \in L^2\left(\mathbb{R}^3 \times (0,T)\right)$, from (4.10) and Lemma 1.1 in Chapter III, [18] follows

$$\frac{d}{dt} u \in L^2\left(0,T;V'\right).$$

By Lemma 1.2, Chapter III, [18] and the argument right after it

$$u \in C\left([0,T],H\right)$$

and u is the unique solution of (4.7), (4.8). Note also that, by Lemma 1.2, $u \in L^q\left(\Pi_T\right), \quad 2 \le q \le \frac{10}{3}$, and

$$||u||_{L^{10/3}(\Pi_T)} \le C E_1^{3/10}(u) E_0^{1/5}(u). \tag{4.11}$$

Taking divergence of (4.7) we get

$$\Delta p = -\sum_{i,j} \partial_{x_i} \partial_{x_j} (w_i u_j). \tag{4.12}$$

By the properties of the singular integral operator, as in section 1.5, there is a unique $p \in L^{5/3}\left(\mathbb{R}^3 \times (0,T)\right)$ satisfying (4.12) and

$$\int_{\mathbb{R}^3} |p|^{5/3} \le C \int_{\mathbb{R}^3} |w|^{5/3} |u|^{5/3} \tag{4.13}$$

with the appropriate $C > 0$. $\square$

Next we prove the generalized energy equality for the solution of linearized Navier-Stokes system.

Lemma 4.2. *Let $u_0 \in H$ and $w \in C^\infty\left(\overline{\Pi}_T\right)$, $\operatorname{div} w = 0$. Let (u,p) be the solution of (4.7), (4.8). Then, for any $\phi \in C_0^\infty\left(\Pi_{*T}\right)$ and all $0 < t \le T$*

$$\int\limits_{\mathbb{R}^3 \times \{t\}} |u|^2 \phi + 2 \iint\limits_{\Pi_T} |\nabla u|^2 \phi = \int\limits_{\mathbb{R}^3} |u_0|^2 \phi(x,0) + \iint\limits_{\Pi_T} |u|^2 (\phi_t + \Delta \phi)$$
$$+ \iint\limits_{\Pi_T} (|u|^2 w + 2pu) \cdot \nabla \phi. \tag{4.14}$$

Proof. First we consider the case where $\operatorname{supp} \phi \subseteq \Pi_T$. Let f be a function defined on Π_T. Define f by 0 on the complement of Π_T. Let $f_\epsilon = f *$

$\rho_\epsilon(x,t)$, where $\rho_\epsilon(x,t)$ is a standard regularization kernel in $\mathbb{R}^4$. Take $\epsilon < \min\{\text{dist}\,(\operatorname{supp}\phi, \{t=0\}), \text{dist}\,(\operatorname{supp}\phi, \{t=T\})\}$. Then (4.7) can be written as

$$(u_\epsilon)_t + (w\cdot\nabla u)_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = 0 \quad \text{in } \Pi_T \cap \operatorname{supp}\phi \tag{4.15}$$

By the well known properties of the regularization, taking into an account the integrability of (u,p), we have

$$\begin{aligned}
u_\epsilon &\to u && \text{in } L^2\left(\mathbb{R}^3\times(0,T)\right)\\
\nabla u_\epsilon &\to \nabla u && \text{in } L^2\left(\mathbb{R}^3\times(0,T)\right)\\
p_\epsilon &\to p && \text{in } L^{\frac{5}{3}}\left(\mathbb{R}^3\times(0,T)\right)\\
(w\nabla u)_\epsilon &\to w\nabla u && \text{in } L^2\left(\mathbb{R}^3\times(0,T)\right)
\end{aligned}$$

as as $\epsilon\to 0$. Multiplying (4.15) by $u_\epsilon\phi$ and integrating over Π_T we deduce

$$\int\limits_{\mathbb{R}^3\times\{t\}} |u_\epsilon|^2\phi + 2\iint\limits_{\Pi_T} |\nabla u_\epsilon|^2\phi = \iint\limits_{\Pi_T} |u_\epsilon|^2(\phi_t+\Delta\phi) + 2\iint\limits_{\Pi_T} p_\epsilon u_\epsilon\cdot\nabla\phi -$$
$$2\iint\limits_{\Pi_T}(w\cdot\nabla u)_\epsilon\phi.$$

And consequently,

$$\int\limits_{\mathbb{R}^3\times\{t\}} |u|^2\phi + 2\iint\limits_{\Pi_T} |\nabla u|^2\phi = \iint\limits_{\Pi_T} |u|^2(\phi_t+\Delta\phi) + \iint\limits_{\Pi_T}(2pu + w|u|^2)\cdot\nabla\phi. \tag{4.16}$$

In the general case, when $\operatorname{supp}\phi\cap\{t=0\}\neq\emptyset$, we approximate ϕ by the sequence of cutoff functions and proceed as in section 1.6; we leave the easy verification to the reader.

□

We now return to the proof of Theorem 4.1. The idea of the proof is to divide the time interval $[0,T]$ into N parts and to solve on each subinterval the linearized Navier-Stokes system (4.7), taking for w the values of u from the previous subinterval, mollified in a particular way.

Let $\psi(x,t)$ be in C^∞ and satisfy

$$\psi \geq 0 \text{ and } \iint \psi\,dx\,dt = 1 \tag{4.17}$$

$$\operatorname{supp}\psi \subseteq \left\{(x,t):\, |x|^2 < t,\, 1 < t < 2\right\}. \tag{4.18}$$

For $u \in L^2(0,T;V)$, let $\hat{u} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ be

$$\hat{u}(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in \overline{\Pi}_T \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

We set

$$\Psi_\delta(u)(x,t) = \delta^{-4} \iint\limits_{\mathbb{R}^4} \psi\left(\frac{y}{\delta}, \frac{\tau}{\delta}\right) \hat{u}(x-y, t-\tau) dy d\tau.$$

The value of $\Psi_\delta(u)$ at time t clearly depend only on the values of u at times $\tau \in (t-2\delta, t-\delta)$.

Lemma 4.3. *For any $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$,*

$$\operatorname{div} \Psi_\delta(u) = 0, \tag{4.20}$$

$$\sup_{0 \le t \le T} \int\limits_{R^4} |\Psi_\delta(u)|^2(x,t) dx \le C E_0(u), \tag{4.21}$$

$$\iint\limits_{\Pi_T} |\nabla \Psi_\delta|^2 \le C E_1(u), \tag{4.22}$$

where C denotes a universal constant.

Proof. (4.20) is straightforward due to the fact that $\hat{u}$ satisfies $\operatorname{div} \hat{u} = 0$ on $\mathbb{R}^4$. (4.21) and (4.22) can be deduced directly from the definition of $\Psi_\delta(u)$. □

Proof of Theorem 4.1. Let N be an integer and $\delta = T/N$. We define (u_N, p_N) to be the solution of

$$\frac{d}{dt} u_N + \Psi_\delta(u_N) \cdot \nabla u_N - \Delta u_N + \nabla p_N = 0 \tag{4.23}$$

$$u_N \in L^2(0,T;V) \cap C([0,T], H) \tag{4.24}$$

$$u_N(0) = u_0, \tag{4.25}$$

by inductively solving (4.7) on each interval $(m\delta, (m+1)\delta)$, $\quad m \in 1, \dots, N-1$, with initial data $u(m\delta)$. From the generalized energy equality (4.14), with $\phi(x,t) \equiv 1$, follows

$$\int\limits_{\mathbb{R}^3 \times \{t\}} |u_N|^2 + 2 \int\limits_{\Pi_T} |\nabla u_N|^2 = \int\limits_{\mathbb{R}^3 \times \{t\}} |u_0|^2. \tag{4.26}$$

This implies that

$$\{u_N\} \text{ is bounded in } L^2\left(0,T;V\right) \cap L^\infty\left(0,T);H\right)$$

Let $V_2 =$ closure of $\mathcal{V}$ in $H^2\left(\mathbb{R}^3\right)$ norm, and $V_2' =$ the dual of V_2. From (4.23) we deduce

$$\left\{\frac{d}{dt}u_N\right\} \text{ is bounded in } L^2\left(0,T;V_2'\right).$$

By Theorem 2.1, Chapter III in [18] $\{u_N\}$ is pre-compact in $L^2\left(0,T;H\right)$. Extracting subsequence if necessary we conclude the existence of $u_\star$ and $p_\star$ such that

$$\begin{aligned} u_N \to u_\star \quad & \text{strongly in } L^2\left(\Pi_T\right), \\ & \text{weakly in } L^2\left(0,T;V\right), \qquad (4.27)\\ & \text{*-weakly in } L^\infty\left(0,T;H\right), \\ p_N \to p_\star \quad & \text{weakly in } L^{\frac{5}{3}}\left(\Pi_T\right) \qquad (4.28) \end{aligned}$$

Further, since $\{u_N\}$ is bounded in $L^{10/3}\left(\Pi_T\right)$, it also converges to $u_\star$ in any $L^q\left(\Pi_T\right)$ for $2 \le q < \frac{10}{3}$. From the definition of Ψ_δ (note that $\delta = T/N$) we have

$$\Psi_\delta(u_N) \to u_\star \text{ strongly in } L^q\left(\Pi_T\right) \text{ for } 2 \le q < \frac{10}{3} \qquad (4.29)$$

The above convergence results allow us to take the limit as $N \to \infty$ in the weak formulation of (4.23) and conclude that $(u_\star, p_\star)$ is a weak solution of Navier-Stokes system. In particular $u_\star \in C\left([0,T],H\right)$ and $u_\star(0) = u_0$. Suppose $\phi \in C^\infty\left(\Pi_{*T}\right)$, $\phi \ge 0$. By (4.14), for all $0 < t \le T$,

$$\begin{aligned} \int\limits_{\mathbb{R}^3\times\{t\}} |u_N|^2\phi + 2\iint\limits_{\Pi_T} |\nabla u_N|^2\phi = \int_{\mathbb{R}^3} |u_0|^2\phi + \iint\limits_{\Pi_T} |u_N|^2(\phi_t + \Delta\phi) \\ \iint\limits_{\Pi_T} (|u_N|^2\Psi_\delta(u_N) + 2p_N u_N)\cdot\nabla\phi. \end{aligned}$$

By the lower semi-continuity of the functional $2\iint|\nabla u_N|^2\phi$ and the convergence results we obtained above we finally arrive at (4.4).

□

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