Computational Methods in the Local Theory of Curves

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À Maria Lucia (A.H.)

Aos meus pais (M.E.H.)

INTRODUCTION

The aim of this short course is to show how contemporary computational methods may effectively be used to compute many invariants associated to singularities of curves.

The main ingredients we use are analog techniques and algorithms to those developed in the case of ideals in polynomial rings by Buchberger and in the case of subalgebras of polynomial rings by Robbiano-Sweedler, both adapted here to the context of formal power series rings. These algorithms will determine special sets of generators of ideals, subalgebras or modules over subalgebras of formal power series rings, which we will call indistinctly *standard bases*. We don't claim originality on the methods but, it seems that no use of them has been yet made systematically to study singularities of germs of curves.

By using such simple but powerful tools we will show how information as, for example, the monoid of values, Milnor's and Tjurina's numbers and the orders of the differentials, associated to a singularity of an algebraic or an analytic irreducible curve, may be described in an unified way. Many examples will be discussed and several known results will be recovered.

The prerequisites to read these notes are minimums: some very basic knowledge of commutative algebra and of the local theory of curves. However, since this small book is not intended to be treatise on the subject, sometimes we will use some deeper facts without proof, but we will always try to explain the terms involved and will give precise references.

Below, we give a brief description of the content of the book.

In Chapter 1 we introduce the discrete structures we need to develop the theory of standard bases in the various contexts we will place ourselves. Namely, we introduce the monoid, monoideal and monomodule structures and the notion of monoidal order.

In Chapter 2 we introduce the analog of Gröbner bases for ideals - formerly developed in polynomial rings - in the context of formal power series rings, which we call standard bases. This adaptation was carried out essentially by Becker in [Bec1] and [Bec2]. We only took the precaution to present proofs for the theorems that could be easily adapted to the other situations we will consider. The key instrument is the analog to Buchberger's algorithm, which in the case under consideration is not always conclusive, but works in many situations such, for example, in the particular applications we have in mind.

In Chapter 3, by means of the standard bases, we show how to compute codimensions of ideals in formal power series rings, which give Milnor's and Tjurina's numbers. We finally apply Buchberger's algorithm to show through some examples how to study the variation of the codimension of the Jacobian and extended Jacobian ideals attached to a formal power series in two variables, when the power series vary in some family. This in particular allows to stratify families of hypersurfaces according to the value of Milnor's number (μ-constant stratification).

In Chapter 4, we adapt, to the framework of formal power series rings, the theory of Subalgebra Analog to Gröbner Bases for Ideals (SAGBI), developed by Robbiano-Sweedler (see [RS]) in the context of polynomial rings. Here again the algorithms become procedures that will work in the applications we have in mind.

Chapter 5 is dedicated to the study of the monoid of values of an irreducible algebroid curve, and its calculation. We present several examples in space and in the plane. In this later case, we compare our method with several other known methods.

In Chapter 6 we present the theory of standard bases of modules over subalgebras of rings of formal power series, contained in such a ring. Here we generalize results due to Miller (see [M]).

Finally in Chapter 7 we introduce the module of Kähler differentials over the ring of coordinates of an irreducible algebroid curve and define an important notion of equivalence between irreducible algebroid curves. We use the modules of differentials to determine very important numerical invariants of curves, modulo equivalence of curves.

The people interested in learning something about the early history of Gröbner bases, are invited to read the charming Section 15.6 of [Ei]. For recent developments of the subject, we will insert at the beginning of some chapters short historical notes.

Parts of this book have been developed in the PhD thesis of the second author [Her], at the ICMC-USP/São Carlos, under the supervision of the first one.

The references in the text to theorems, propositions, etc., will be numbered in such a way that the first digit on the left means the number of the chapter, where the result is, followed by a dot and finally by the number of the result itself. For example, Theorem 2.3 means Theorem 3 of chapter 1, or Proposition 2 means the second proposition in the same chapter.

We wish to thank the organizers of the 23º Colóquio Brasileiro de Matemática for the opportunity given to teach this course. During the development of this work the first author belonged to PRONEX/ALGA and was partially supported by CNPq, while the second author was partially supported by PICD/CAPES. Finally, the first author wishes to thank the hospitality of the ICMC/USP at São Carlos and FAPESP for partial support during the preparation of this manuscript.

Niterói, May 2001

Contents

Chapter 1

BASIC DISCRETE STRUCTURES

In this chapter we will introduce the discrete structures needed to develop the computational methods we will use in this book. The success of these methods relies on the fact that for the rings and modules we will consider, it is possible to translate several algebraic questions into numerical conditions on subsets of Cartesian powers of the natural numbers. These sets will be naturally endowed with simple algebraic structures as monoids, monoideals and monomodules. The notion of order on monoids, which is also central, will be introduced in the last section.

1.1 Monoids

Let $(S, e, *)$, or shortly S, be a commutative *monoid*, that is, a set S with a binary associative and commutative operation $*$ and a unit element e.

A subset B of S will be called a *generator set* of S if any element of S may be obtained by a finite number of operations on elements of B. In this case we will use the notation $S = \langle B \rangle$.

If there exists a finite generator set for a monoid S, we will say that S is a *finitely generated* monoid.

Given $s, t \in S$, we say that s *divides* t, writing $s|t$, if there exists $s' \in S$ such that $t = s * s'$.

The relation of divisibility is reflexive and transitive, but not always antisymmetric.

A monoid S will be called *positive* when

$$\forall s,t \in S, \quad s*t=e \implies s=t=e.$$

A monoid S will be called *cancelative* when

$$\forall s,t,u \in S, \quad s*u=t*u \implies s=t.$$

In a positive cancelative monoid the divisibility relation is antisymmetric. In fact, suppose that $t|s$ and $s|t$, then there exist $s', t' \in S$ such that $s = t*t'$ and $t = s*s'$. From this it follows that

$$s*t = t*t'*s*s' = s*t*s'*t'.$$

From the cancelative property we have $s'*t' = e$, and from the positivity of S we have that $s' = t' = e$. Therefore, $s = t$.

EXAMPLE 1 $(\mathbb{N}^n, 0, +)$ is a positive cancelative monoid. In this case the divisibility relation reads:

$$s|t \iff t-s \in \mathbb{N}^n.$$

EXAMPLE 2 Let $X_1, \ldots, X_n$ be indeterminates. A *monomial* in these indeterminates is an expession of the form

$$\mathbf{X}^\alpha = X_1^{a_1} \cdots X_n^{a_n}, \quad \alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n.$$

Consider the set $\mathbf{T}^n$ of all such monomials with unit element $1 = X_1^0 \cdots X_n^0$ and the operation of multiplication of polynomials. We have that $(\mathbf{T}^n, 1, .)$ is a positive cancelative monoid. This monoid is isomorphic to the monoid of Example 1, through the isomorphism

$$\begin{array}{rccc} \log: & \mathbf{T}^n & \longrightarrow & \mathbb{N}^n \\ & X_1^{a_1} \cdots X_n^{a_n} & \mapsto & (a_1, \ldots, a_n) \end{array}$$

Therefore, in $\mathbb{N}^n$ and in $\mathbf{T}^n$ the divisibility relation is a partial order.

DEFINITION Let A be a subset of a monoid S. A subset D of A will be called a *set of divisors* of A if

$$\forall s \in A, \exists t \in D \quad \text{such that} \quad t|s.$$

A set D of divisors of A will be called a *minimal set of divisors of* A if

$$\forall t, t' \in D, t|t' \Longrightarrow t = t'.$$

Remark that if $D \subset C \subset A$ and D is a set of divisors of A, then C is also a set of divisors of A. So, a minimal set of divisors of A is a set of divisors of A, minimal with respect to inclusion.

PROPOSITION 1 *Let A be a subset of a positive and cancelative monoid S. If there exists a minimal set of divisors of A, then this set is unique.*

PROOF Let D and D' be two minimal sets of divisors of A. Given $t \in D$, there exists $t' \in D'$ such that $t'|t$. On the other hand, there exists $s \in D$ such that $s|t'$. It then follows that $s|t$ and consequently $s = t$, because D is minimal. So $t|t'$. Since S is positive and cancelative, from the relations $t|t'$ and $t'|t$, it follows that $t = t' \in D'$. Hence we proved that $D \subset D'$. The other inclusion is proved in the same way.

□

THEOREM 1 (DICKSON) *Every non-empty subset of $\mathbb{T}^n$ has a finite minimal set of divisors.*

PROOF Let $\emptyset \neq A \subset \mathbb{T}^n$. Write $A = \{M_1, M_2, \ldots\}$ and consider the chain of ideals of $R = \mathbb{Z}[X_1, \ldots, X_n]$:

$$\langle M_1 \rangle \subset \langle M_1, M_2 \rangle \subset \cdots$$

Since R is noetherian, there exists an integer r such that, if I is the ideal $\langle M_1, \ldots, M_r \rangle$, then $I = \langle M_1, \ldots, M_s \rangle$, for all $s \geq r$. Therefore, $A \subset I$. We will prove that $D = \{M_1, \ldots, M_r\}$ is a set of divisors of A. In fact, let $M \in A \subset I$, then there exist $p_1, \ldots, p_r \in R$ such that

$$M = p_1 M_1 + \cdots + p_r M_r.$$

Therefore, there exist a monomial N and an index i such that $NM_i = M$, hence $M_i|M$. It then follows that D is a finite set of divisors of A and may be taken minimal by excluding some of its elements.

□

COROLLARY *Every non-empty subset of $\mathbb{N}^n$ has a finite minimal set of divisors.*

PROOF The result follows immediately by using the above result and the isomorphism log.

□

A subset S' of a monoid S, containing the unit element and closed under the operation of S, will be called a *submonoid* of S. Any element of $S \setminus S'$ will be called a *gap* of S'.

EXAMPLE 3 HOMOGENEOUS LINEAR DIOPHANTINE SYSTEMS

Let S the set of n-tuples of non-negative integers which are solutions of a system of linear homogeneous diophantine equations

$$a_{i,1}X_1 + \cdots + a_{i,n}X_n = 0; \quad i = 1, \ldots, m, \tag{1.1}$$

where the $a_{i,j}$'s are elements of $\mathbb{Z}$. The set S is an additive submonoid of $\mathbb{N}^n$. If $S \setminus \{0\} \neq \emptyset$, denote by D its minimal set of divisors, which we know to be finite from the corollary of Theorem 1. We will show below that S is generated as an additive monoid by the finite set D.

Indeed, in $\mathbb{N}^n$ the divisibility relation $s|t$ means $t - s \in \mathbb{N}^n$, and we have

$$\forall s, t \in S;\ s|t \Longrightarrow t - s \in S.$$

The result follows now easily by induction on $|(t_1, \ldots, t_n)| = t_1 + \cdots + t_n$.

We will also call D the set of *minimal solutions* of the system (1.1). The set D may be determined by implementing an efficient algorithm that can be found in [CD]. In [CF] there is a Pascal implementation of an algorithm to determine D in the case of one diophantine equation.

EXAMPLE 4 SPECIALIZATION OF LINEAR SYSTEMS

Suppose we give a system like in (1.1) and a subset $\{j_1, \ldots, j_r\}$ of $\{1, \ldots, n\}$. We specialize the system (1.1) by assigning the value zero to the indeterminates X_k for all k not in the set $\{j_1, \ldots, j_r\}$. In such a way we get a new system

$$a_{i,j_1}X_{j_1} + \cdots + a_{i,j_r}X_{j_r} = 0; \quad i = 1, \ldots, m. \tag{1.2}$$

To each solution $(\alpha_{j_1}, \ldots, \alpha_{j_r})$ of (1.2) we associate a solution $(\beta_1, \ldots, \beta_n)$ of (1.1), in the following way:

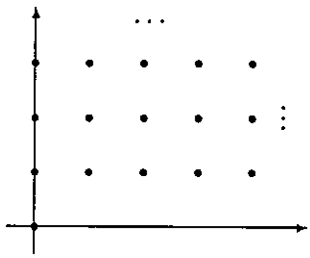
$$\beta_i = \begin{cases} 0, & \text{if } i \notin \{j_1, \ldots, j_r\} \\ \alpha_i & \text{if } i \in \{j_1, \ldots, j_r\}. \end{cases}$$

It is easy to see that every minimal solution of system (1.2) is a minimal solution for the system (1.1). This simple observation will be crucial in Sections 4.3 and 6.2.

The structure of the submonoids of $\mathbb{N}^n$, for $n \geq 2$, is rather complicated. For instance, a submonoid of $\mathbb{N}^n$ is not necessarily finitely generated, if $n \geq 2$.

EXAMPLE 5 $S = \{(a,b) \in \mathbb{N}^2; b \geq 1\} \cup \{(0,0)\}$ is a non-finitely generated submonoid of $\mathbb{N}^2$. Indeed, any set of generators must contain the set

$$\{(a,1);\ a \in \mathbb{N}\}.$$



The set $\{(0,1)\}$ is the minimal set of divisors of $S \setminus \{(0,0)\}$.

EXAMPLE 6 Every submonoid S of $\mathbb{N}$ is finitely generated.
Indeed, put $m_1 = \min(S \setminus \{0\})$, and take

$$m_i = \min(S \setminus m_1\mathbb{N} + \cdots + m_{i-1}\mathbb{N}); \quad i = 2, \ldots$$

From the finiteness of the number of residual classes modulo m_1, we have that $S = m_1\mathbb{N} + \cdots + m_i\mathbb{N}$, for some $i \leq m_1$.

From their construction, the above integers $m_1, \ldots, m_i$ must be contained in any other set of generators of S, and for this reason they are called the *minimal system of generators* of S. The number of elements of the minimal system of generators of a submonoid of $\mathbb{N}$, minus 1, will be called the *genus* of the submonoid.

We will say that an integer c is the *conductor* of the submonoid S of $\mathbb{N}$ if $c - 1 \notin S$ and for all $n \geq c$ we have $n \in S$.

If a submonoid S of $\mathbb{N}$ has a coductor, then S has finitely many gaps.

REMARK 1 A submonoid of $\mathbb{N}$ has a conductor if and only the GCD of all elements in S is 1.

In fact, if GCD(S) = 1, then there are elements $s_1, \ldots, s_n, r_1, \ldots, r_m \in S$ and positive integers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ such that

$$\lambda_1 s_1 + \cdots + \lambda_n s_n = \mu_1 r_1 + \cdots + \mu_m r_m + 1.$$

This implies that there are two consecutive integers, $a = \mu_1 r_1 + \cdots + \mu_m r_m$ and $a+1$, in S. Now, since a and $a+1$ are coprime, the set

$$\{0, a+1, 2(a+1), \ldots, (a-1)(a+1)\}$$

is a complete system of residues modulo a. So, any integer $i \geq (a-1)(a+1)$ is such that $i = j(a+1) + ka$ with $j = 0, \ldots, (a-1)$ and $k \geq 0$, hence $i \in S$, which shows that S has a conductor. The converse is obvious.

DEFINITION A submonoid S of $\mathbb{N}$ with conductor c will be called *symmetric* if

$$\#(\mathbb{N} \setminus S) = \frac{c}{2}.$$

For example we have that $S_1 = \{0, 3, 4, 6, \ldots\}$ is symmetric, while $S_2 = \{0, 3, 4, 5, \ldots\}$ is non-symmetric.

EXAMPLE 7 This is just an example to show that monoids appear in other contexts where they will play a relevant role. For the details see Section 5.1.

Let C be an algebroid irreducible curve defined over an algebraically closed field K, with local ring $\mathcal{O}$. If we denote by v the normalized valuation of the integral closure of $\mathcal{O}$ (which is isomorphic to $K[[T]]$, the ring of power series in one indeterminate T with coefficients in K), we have that $S = v(\mathcal{O})$ is a submonoid of $\mathbb{N}$, called the *monoid of values* of $\mathcal{O}$, or of C. From Example 6, we know that S is finitely generated.

1.2 Monoideals and Monomodules

A non-empty subset Δ of a monoid $S = (S, e, *)$ will be called a *monoideal* of S if

$$\Delta * S := \{\delta * s;\ \ \delta \in \Delta, s \in S\} \subset \Delta.$$

EXAMPLE 8 Let $\mathcal{O}$ and v be as in Example 7. If J is an ideal of $\mathcal{O}$, then $\Delta = v(J)$ is a monoideal of $S = v(\mathcal{O})$.

EXAMPLE 9 Let $S = \mathbf{T}^n$, and denote by $K[[\mathbf{X}]]$ the ring of formal power series $K[[X_1, \ldots, X_n]]$ in the indeterminates $X_1, \ldots, X_n$ and coefficients in the field K. Let J be a monomial ideal (i.e., an ideal generated by monomials) in $K[[X]]$. The set Δ of all monomials in J is a monoideal of S.

DEFINITION A subset B of a monoideal Δ in S is called a *system of generators* of Δ if

$$\Delta = \langle B \rangle,$$

where

$$\langle B \rangle = \{b * s; b \in B, \text{ and } s \in S\}^1.$$

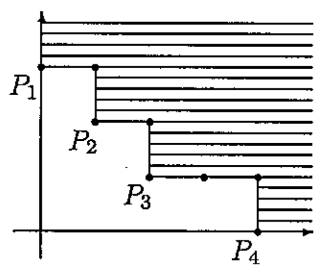
A set of generators B of a monoideal Δ will be said *minimal*, if it is minimal with respect to inclusion.

PROPOSITION 2 *Every monoideal Δ in $\mathbb{N}^n$ or in $\mathbb{T}^n$ has a finite minimal set of generators.*

PROOF This is precisely the minimal set of divisors of $\Delta \setminus \{0\}$, which exists due to Dickson's Theorem.

□

EXAMPLE 10 If Δ is a monoideal of $\mathbb{N}^2$, we have that $(a,b) \in \Delta$ implies $(a,b) + \mathbb{N}^2 \in \Delta$. Therefore, there is associated to Δ a *stair*, closed or not, in $\mathbb{N}^2$.



The set $\{P_1, P_2, P_3, P_4\}$ is a minimal set of generators of Δ.

We generalize the notion of monoideal as follows:

[1]Careful, in this context the notation $\langle B \rangle$ stands for the monoideal generated by B and not for the monoid generated by B.

DEFINITION Let $(S, *, e)$ be a monoid. A set Ω together with an operation

$$\begin{array}{rccc} \circ: & S \times \Omega & \longrightarrow & \Omega \\ & (s, \omega) & \mapsto & s \circ \omega \end{array}$$

will be called an *S-monomodule* if for all $\omega \in \Omega$ and all $s_1, s_2 \in S$, we have
(i) $e \circ \omega = \omega$
(ii) $(s_1 * s_2) \circ \omega = s_1 \circ (s_2 \circ \omega)$.

In particular, every monoideal of a monoid S is an S-monomodule.
A subset B of Ω will be called a *set of generators* of Ω if

$$\Omega = S \circ B.$$

If there is a finite subset B generating Ω we will say that Ω is *finitely generated.*

EXAMPLE 11 Again, as in Example 7, we just want to indicate here the future use of this discrete structure.

Let $\mathcal{O}$ be the local ring of an algebroid irreducible curve and let $\mathcal{O}d\mathcal{O}$ be the module of Kähler differentials of $\mathcal{O}$. The valorization v of $\mathcal{O}$ extends to $\mathcal{O}d\mathcal{O}$. If $\Omega = v(\mathcal{O}d\mathcal{O})$ and $S = v(\mathcal{O})$, then Ω is an S-monomodule.

EXAMPLE 12 NON-HOMOGENEOUS LINEAR DIOPHANTINE SYSTEMS

Let S' be the set of n-tuples of non-negative integers which are solutions of a system of linear non-homogeneous diophantine equations,

$$a_{i,1}X_1 + \cdots + a_{i,n}X_n = b_i; \quad i = 1, \ldots, m,$$

where the $a_{i,j}$'s and the b_i's are elements of $\mathbb{Z}$. In general, the set S' is not an additive submonoid of $\mathbb{N}^n$. Anyway, if $D' = \{w_1, \ldots, w_r\}$ is the minimal set of divisors of $S' \setminus \{0\}$ and if S is the monoid of solutions of the associated homogeneous diophantine system (1.1), with minimal set of divisors D, then any element $v' \in S'$ may be written as

$$v' = \sum_i c_i \beta_i + w; \quad c_i \in \mathbb{N},\ \beta_i \in D \ \text{ and } \ w \in D'.$$

Therefore, S' is an S-monomodule finitely generated by D'.

The sets D and D' above are contained in the set of minimal solutions of a system of homogeneous linear diophantine equations, which can be determined by the same algorithms cited in Example 3.

PROPOSITION 3 *Let $S \neq \{0\}$ be a submonoid of $\mathbb{N}$ and suppose that $\Omega \subset \mathbb{N}$ is an S-monomodule. Then Ω is finitely generated.*

PROOF We already know that S is finitely generated (Example 6). So we have

$$S = \langle v_1, \dots, v_g \rangle = v_1\mathbb{N} + \dots + v_g\mathbb{N}.$$

Let $d = \mathrm{GCD}(S) \neq 0$. Then the monoid

$$S' = \langle \frac{v_1}{d}, \dots, \frac{v_g}{d} \rangle,$$

has a conductor N (Remark 1). So, for every $n \geq N$, we have $dn \in S$.

Consider now the following sequence of integers:

$$\begin{aligned}
e_0 &= \min \Omega \setminus \{0\}, \\
e_1 &= \min \Omega \setminus (e_0 + S) \cup \{0\}, \\
e_2 &= \min \Omega \setminus (e_0 + S) \cup (e_1 + S) \cup \{0\}, \\
&\dots \\
e_i &= \min \Omega \setminus (e_0 + S) \cup \dots \cup (e_{i-1} + S) \cup \{0\}, \\
&\dots
\end{aligned}$$

If Ω were not finitely generated, there would be infinitely many elements in the above sequence congruent to each other modulo d. So there would exist e_i and e_j, with $j > i$, such that $e_j - e_i \geq Nd$ and $e_j \equiv e_i \mod d$. This would imply that $e_j - e_i \in S$, hence $e_j \in e_i + S$, which is a contradiction.

□

1.3 Orders

Let $(S, e, *)$ be a monoid. A *monoidal order* $\leq$ in S is a total order relation having the following properties:

1) $\forall s \in S, \ \ e \leq s$
2) $\forall s, s_1, s_2 \in S, \ \ s_1 \leq s_2 \implies s_1 * s \leq s_2 * s.$

A monoid together with a monoidal order will be called an *ordered monoid.*

REMARK 2 If S is an ordered monoid, then $s|t \Longrightarrow s \leq t$.

In fact, if $s|t$, then there exists $s' \in S$ such that $t = s * s'$. Since $e \leq s'$ we have

$$s = s * e \leq s * s' = t.$$

The following are examples of monoidal orders in $\mathbb{N}^n$.

EXAMPLE 13 THE LEXICOGRAPHICAL ORDER IN $\mathbb{N}^n$

Let α and β be elements in $\mathbb{N}^n$. We will say that $\alpha \leq_{Lex} \beta$ if $\alpha = \beta$ or if the first non-zero coordinate of $\beta - \alpha$ is positive.

As a numerical example, we have

$$(1,2,4) \leq_{Lex} (2,1,0), \quad (1,2,4) \leq_{Lex} (1,2,5)$$

and

$$(0,0,1) \leq_{Lex} (0,1,0) \leq_{Lex} (1,0,0).$$

EXAMPLE 14 THE GRADED LEXICOGRAPHICAL ORDER IN $\mathbb{N}^n$

Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ be elements in $\mathbb{N}^n$. We will say that $\alpha \leq_{GLex} \beta$ if

(i) $a_1 + \cdots + a_n < b_1 + \cdots + b_n$, or

(ii) $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $\alpha \leq_{Lex} \beta$.

As a numerical example, we have

$$(2,1,0) \leq_{GLex} (1,2,4), \quad (1,2,5) \leq_{GLex} (1,3,4)$$

and

$$(0,0,1) \leq_{GLex} (0,1,0) \leq_{GLex} (1,0,0).$$

EXAMPLE 15 THE WEIGHTED ORDER IN $\mathbb{N}^n$

Let $\gamma \in \mathbb{N}^n$, let $\leq$ be a monoidal order in $\mathbb{N}^n$ and $\alpha, \beta \in \mathbb{N}^n$. We say that $\alpha \leq_\gamma \beta$ if

(i) $\gamma.\alpha < \gamma.\beta$, or

(ii) $\gamma.\alpha = \gamma.\beta$ and $\alpha \leq \beta$,

where the dot means the usual inner product in $\mathbb{N}^n$.

Remark that if we take $\gamma = (0, \ldots, 0)$, then $\leq_\gamma$ and $\leq$ coincide. On the other hand, if we take $\gamma = (1, \ldots, 1)$, and $\leq$ equal to $\leq_{Lex}$, then $\leq_\gamma$ and $\leq_{GLex}$ coincide.

PROPOSITION 4 *Every monoidal order in $\mathbb{N}^n$ is a well-order.*

PROOF We have to prove that a subset A of $\mathbb{N}^n$ has a least element. In fact, let D be a finite set of divisors of A (Dickson's Theorem). The least element of D is the least element of A (Remark 2).

□

A monoidal order in $\mathbf{T}^n$ will be called a *monomial order.*

Using the isomorphism log, we have from Proposition 4 that all monomial orders are well-orders. On the other hand, we may transform any monoidal order in $\mathbb{N}^n$ into a monomial order in $\mathbf{T}^n$. In particular, this can be done for the lexicographical, the graded lexicographical and the weighted orders, which will conserve their names. Note that in $\mathbf{T}^n$ we have

$$X_n \leq_{Lex} X_{n-1} \leq_{Lex} \cdots \leq_{Lex} X_1.$$

In particular, when $\mathbf{T}^2 = \{X^\alpha Y^\beta;\ \alpha, \beta \in \mathbb{N}\}$, then $\log(\mathbf{T}^2) = \{(\alpha, \beta) \in \mathbb{N}^2\}$, and therefore, $Y \leq_{Lex} X$ and $Y \leq_{GLex} X$

When we deal with the weighted order on $\mathbf{T}^n$ with respect to $\gamma \in \mathbb{N}^n$, we will define the *weight* of a monomial $\mathbf{X}^\alpha$ as the integer

$$\mathrm{w}_\gamma(\mathbf{X}^\alpha) = \gamma.\alpha.$$

When $\gamma = (1, \ldots, 1)$, then

$$\deg(\mathbf{X}^\alpha) := \mathrm{w}_\gamma(\mathbf{X}^\alpha) = \alpha_1 + \cdots + \alpha_n,$$

called the *degree* of $\mathbf{X}^\alpha$.

Notice that in $\mathbf{T}^1$ there is only one possible monomial order. In fact, given any order $\leq$ in $\mathbf{T}^1$, by definition we have $1 \leq X$, which from the compatibility of orders with respect to multiplication implies

$$1 \leq X \leq X^2 \leq \cdots$$

This proves the result.

In order to deal with infinite families of formal power series, we will be constricted to consider a particular type of monomial orders which have an additional property that we define below.

DEFINITION Let $(S, \leq)$ be an ordered monoid. We will say that the monoidal order $\leq$ has the *finiteness property* if for every $t \in S$,

$$\#\{s \in S;\ s \leq t\} < \infty.$$

EXAMPLE 16 Let $\gamma \in \mathbb{N}^n$ and take the weighted order $\leq_\gamma$ in $\mathbf{T}^n$ with respect to γ and to $\leq_{Lex}$, where we have

$$X_n \leq_{Lex} X_{n-1} \leq_{Lex} \cdots \leq_{Lex} X_1.$$

If all coordinates of γ are positive, then it is easy to verify that $\leq_\gamma$ has the finiteness property.

If one takes $\gamma = (0, \ldots, 0)$, then $\leq_\gamma$ coincides with $\leq_{Lex}$ which doesn't have the finiteness property, if $n \geq 2$, since

$$\{1, X_2, X_2^2, \ldots\} \subset \{s \in \mathrm{T}^n;\ s \leq_{Lex} X_1\}.$$

Chapter 2

STANDARD BASES FOR IDEALS

In this chapter we will present the essentials of the theory of Gröbner bases for ideals in the context of formal power series rings. For more details we recommend the papers [Bec1] and [Bec2].

2.1 Prerequisites on Formal Power Series

Let K be an arbitrary field. Throughout this book we will denote by $K[[\mathbf{X}]]$ the ring of formal power series $K[[X_1, \ldots, X_n]]$ in the indeterminates $X_1, \ldots, X_n$ and coefficients in K.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we will use the notation

$$\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}.$$

Hence, any element $f \in K[[\mathbf{X}]]$ may be represented under the form

$$f = \sum_{\alpha \in I} a_\alpha \mathbf{X}^\alpha, \quad I \subset \mathbb{N}^n.$$

If $f \neq 0$ is given as above, we define the *set of monomials* of f as

$$\mathrm{T}(f) = \{\mathbf{X}^\alpha; \ \alpha \in I, a_\alpha \neq 0\}.$$

If B is a subset of $K[[\mathbf{X}]]$, we will denote by $\langle B \rangle$ the ideal generated by the elements of B. So $\mathcal{M}_\mathbf{X} = \langle X_1, \ldots, X_n \rangle$ will represent the ideal generated by the elements $X_1, \ldots, X_n$.

In the remaining of this section we will state without proof some well known properties of the ring $K[[\mathbf{X}]]$.

First of all, the invertible elements of $K[[\mathbf{X}]]$ are the elements of the form $u = a + g$, where $a \in K \setminus \{0\}$ and $g \in \mathcal{M}_{\mathbf{X}}$. So, u is invertible if and only if $u \notin \mathcal{M}_{\mathbf{X}}$. This in particular shows that $K[[\mathbf{X}]]$ is a local ring with maximal ideal $\mathcal{M}_{\mathbf{X}}$.

Let $\mathcal{M}^i_{\mathbf{X}}$, with $i \in \mathbb{N}$, denote the i-th power of the ideal $\mathcal{M}_{\mathbf{X}}$ and put $\mathcal{M}^0_{\mathbf{X}} = K[[\mathbf{X}]]$. We define the $\mathcal{M}_{\mathbf{X}}$-*adic topology* on $K[[\mathbf{X}]]$ by taking around any element $f \in K[[\mathbf{X}]]$ the sets $f + \mathcal{M}^i_{\mathbf{X}}$, $i \in \mathbb{N}$, as a fundamental neighborhood system of f. This turns $K[[\mathbf{X}]]$ into a topological ring.

Since we have $\cap_{i \in \mathbb{N}} \mathcal{M}^i_{\mathbf{X}} = \{0\}$, the $\mathcal{M}_{\mathbf{X}}$-adic topology on $K[[\mathbf{X}]]$ is Hausdorff. Indeed, it is metrizable, as we show below.

If $f \in K[[\mathbf{X}]]$, we define the *multiplicity* of f as

$$\mathrm{mult}(f) = \sup\{i;\ f \in \mathcal{M}^i_{\mathbf{X}}\}.$$

Let ρ be a real number greater than 1 and put $\rho^{-\infty} = 0$. For $f, h \in K[[\mathbf{X}]]$, the function

$$\mathrm{d}(f, h) = \rho^{-\mathrm{mult}(h-f)},$$

defines a complete metric on $K[[\mathbf{X}]]$.

A family $\{f_\lambda,\ \lambda \in \Lambda\} \subset K[[\mathbf{X}]]$ will be called *summable* if for every $t \in \mathbf{T}^n$ we have

$$\#\{\lambda \in \Lambda;\ t \in \mathbf{T}(f_\lambda)\} < \infty.$$

Since $(K[[\mathbf{X}]], \mathrm{d})$ is a complete metric space, it is easy to see that, for any summable family in $K[[X]]$, the sum $\sum_{\lambda \in \Lambda} f_\lambda$ is meaningful, defining an element in $K[[\mathbf{X}]]$.

All the above properties are elementary and easy to prove (see for example [ZS], Chapter 7). Another result we will need and that we state below is the Weierstrass Preparation Theorem.

THEOREM 1 (WEIERSTRASS PREPARATION THEOREM) *Given $f \in \mathcal{M}_{\mathbf{X}} \subset K[[\mathbf{X}]]$, with $f \neq 0$, there exist an automorphism ϕ and a unit u of $K[[\mathbf{X}]]$ and elements $A_1, \ldots, A_r$ in the maximal ideal of $K[[X_1, \ldots, X_{n-1}]]$ such that*

$$u(f \circ \phi) = X_n^r + A_1 X_n^{r-1} + \cdots + A_r.$$

(For a proof see [ZS], Chapter 7, Corollary 1 of Theorem 5 and Lemma 3).

As a consequence of the above theorem one has the following:

THEOREM 2 $K[[\mathbf{X}]]$ *is a noetherian unique factorization domain.*

(For a proof see for example [ZS], Chapter 7, Theorems 4' and 6).

Since $K[[\mathbf{X}]]$ is noetherian, every ideal in $K[[\mathbf{X}]]$ is finitely generated. This implies in particular that every ideal is a closed set in $K[[\mathbf{X}]]$ (in the $\mathcal{M}_{\mathbf{X}}$-adic topology).

2.2 The Division Algorithm

In what follows we fix once for all a monomial order $\leq$ in $\mathbf{T}^n$.

The *leading power* of $f \in K[[X]]$, with $f \neq 0$, is by definition

$$\mathrm{lp}(f) = \min \mathbf{T}(f),$$

where the minimum is taken with respect to the monomial order $\leq$ we have just fixed. This definition is meaningful because by Proposition 1.4, any monomial order is a well-order.

If $G \subset K[[\mathbf{X}]]$, we will denote by $\mathrm{lp}(G)$ the set of leading powers of all nonzero elements of G.

The *leading term* of $f = \sum_{\alpha \in I} a_\alpha \mathbf{X}^\alpha \neq 0$ is by definition

$$\mathrm{lt}(f) = a_\beta \mathbf{X}^\beta,$$

where $\mathbf{X}^\beta = \mathrm{lp}(f)$.

Since we are going to deal with sets of generators B of ideals in $K[[\mathbf{X}]]$, and all such ideals are finitely generated, we may assume without loss of generality that B is finite. This will avoid some unnecessary technical difficulties.

Let B be a finite subset of $K[[\mathbf{X}]]$. Given two elements $g, r \in K[[\mathbf{X}]]$ with $g \neq 0$, we will say that r is a *reduction* of g modulo B if there exist a monomial $t \in \mathbf{T}^n$, a constant $a \in K$ and an element $f \in B$ such that

$$r = g - atf,$$

and $\mathrm{lp}(r) > \mathrm{lp}(g)$, whenever $r \neq 0$. In this case we write

$$g \xrightarrow{B} r.$$

Note that if $g \xrightarrow{B} r$, then $g - r$ belongs to the ideal generated by B.

Consider a chain (possibly infinite) of reductions

$$g \xrightarrow{B} r_1 \xrightarrow{B} r_2 \xrightarrow{B} \cdots \xrightarrow{B} r_m \xrightarrow{B} \cdots$$

This implies that there exist $t_i \in \mathbf{T}^n$, $a_i \in K$ and $f_i \in B$, $i \geq 1$, such that

$$r_m = g - \sum_{i=1}^{m} a_i t_i f_i,$$

where, because of the definition of reduction,

$$\mathrm{lp}(t_1 f_1) < \mathrm{lp}(t_2 f_2) < \cdots. \tag{2.1}$$

If the chain is infinite, we get a sequence $(\sum_{i=1}^{m} a_i t_i f_i)_{m\geq 1}$ in $K[[\mathbf{X}]]$, which happens to be convergent in $K[[\mathbf{X}]]$ (with respect to the $\mathcal{M}_{\mathbf{X}}$-adic topology). In fact, it is sufficient to verify that the set $\{t_i f_i,\ i \geq 1\}$ is summable. Indeed, if for some $t \in \mathbf{T}^n$,

$$\#\{i;\ t \in \mathbf{T}(t_i f_i)\} = \infty,$$

there would exist $f \in B$ and infinitely many elements $t_i \in \mathbf{T}^n$ such that $t \in \mathbf{T}(t_i f)$ (recall that B is finite). Now, because of (2.1), it follows that the t_i have to be distinct. But this is a contradiction, since $\deg(t_i) \leq \deg(t)$.

We will denote the limit of the sequence $(\sum_{i=1}^{m} a_i t_i f_i)_{m\geq 1}$ by $\sum_{i\geq 1} a_i t_i f_i$. Since all the terms of the above sequence are in $\langle B\rangle$ and any ideal is closed, we have that $\sum_{i\geq 1} a_i t_i f_i$ belongs to the ideal $\langle B\rangle$.

We will now extend the notion of reduction to include all $r_m = g - \sum_{i=1}^{m} a_i t_i f_i$, as above, and their limits. It remains true with this extended definition that if an element r is a reduction of g modulo B, then $g - r \in \langle B\rangle$.

We will say that r is a *final reduction* of g modulo B if r is a reduction of g and r has no further reduction modulo B. That is,

$$r = 0, \text{ or } \ lp(f) \not| \, \mathrm{lp}(r), \ \forall f \in B;$$

writing in this case,

$$g \xrightarrow{B_+} r.$$

If r is a final reduction of g such that $r = 0$ or no element in $\mathbf{T}(r)$ is divisible by any $\mathrm{lp}(f)$ with $f \in B$, then r will be called a *complete reduction* of g modulo B.

If $B = \{f_1, \dots, f_s\}$ is a subset of $K[[\mathbf{X}]]$ and $f \in K[[\mathbf{X}]]$, at least in principle, we may get a complete reduction r of f modulo B and series $q_1, \dots, q_s \in K[[\mathbf{X}]]$, such that

$$g = \sum_{i=1}^{s} q_i f_i + r,$$

by applying the following division algorithm:

THE DIVISION ALGORITHM

INPUT: $f, B = (f_1, \dots, f_s)$;
DEFINE: $q_1 := 0; \dots ; q_s := 0;\ r := 0$;
WHILE $f \neq 0$ DO
    IF THERE EXISTS $\mathrm{lp}(f_i) \mid \mathrm{lp}(f)$
    THEN TAKE THE LEAST
    SUCH INTEGER i, AND DO
        $q_i := q_i + \frac{\mathrm{lt}(f)}{\mathrm{lt}(f_i)}$;
        $f := f - \frac{\mathrm{lt}(f)}{\mathrm{lt}(f_i)} f_i$;
    IF NOT, DO
        $r := r + \mathrm{lt}(f)$;
        $f := f - \mathrm{lt}(f)$;

To obtain merely a final reduction r of f, it is sufficient to replace the last two rows of the above algorithm by

$$r := f;$$

$$f := 0;$$

Since we are dealing with power series, the above procedure may not end after a finite number of steps and in this way we obtain sequences that are approximations (in the $\mathcal{M}_{\mathbf{X}}$-adic topology of $K[[\mathbf{X}]]$) of r and of the q_i's. Below we give such an example.

EXAMPLE 1 Let $f = X, f_1 = Y + X$ and $f_2 = Y + X^2$ in $K[[X, Y]]$ and take the graded lexicographical order in $\mathbf{T}^2$. Applying the division algorithm to

f modulo f_1, f_2, we get

$$\begin{array}{ll}
f = f_1 + r_1 & r_1 = -Y \\
r_1 = -f_2 + s_1 & s_1 = X^2 \\
s_1 = Xf_1 + r_2 & r_2 = -XY \\
r_2 = -Yf_1 + r_3 & r_3 = Y^2 \\
r_3 = Yf_2 + s_2 & s_2 = -X^2Y \\
s_2 = -XYf_1 + r_4 & r_4 = XY^2 \\
r_4 = Y^2f_1 + r_5 & r_5 = -Y^3 \\
r_5 = -Y^2f_2 + s_3 & s_3 = X^2Y^2 \\
\vdots & \vdots \\
s_i = (-1)^{i+1}X^2Y^{i-1}f_1 + r_{2i} & r_{2i} = (-1)^iXY^i \\
r_{2i} = (-1)^iY^if_1 + r_{2i+1} & r_{2i+1} = (-1)^{i+1}Y^{i+1} \\
r_{2i+1} = (-1)^{i+1}Y^if_2 + s_{i+1} & s_{i+1} = (-1)^iX^2Y^i.
\end{array}$$

Although the algorithm doesn't end in this case, nevertheless it allows to easily conclude that 0 is a final reduction of f modulo $\{f_1, f_2\}$.

For a given f, if the division algorithm, with respect to some finite set B, ends after a finite number of steps, we have at hands an algorithm to obtain a final or complete reduction of f modulo B.

Remark also that the algorithm uses an ordination for the elements of B. So, for a given ordination of the elements of B we have that r and the q_i's are uniquely determined. However, for a different ordination of the elements of B, we may get different final or complete reductions of f, modulo B, as shows the following example.

EXAMPLE 2 In $\mathbb{Q}[[X,Y]]$, take $f_1 = Y^4 - 2X^5Y^2 - X^9, f_2 = X^{10}, f_3 = Y^3 - X^5Y - X^7, f_4 = Y^4 - X^5Y^2 - X^7Y + X^{10}$ and $f = Y^6 - X^5Y^4 - X^7Y^3$. Consider $\mathbb{T}^2$ ordered by the graded lexicographical order. From the above algorithm we get that $r = X^9Y^2$ is a complete reduction of f modulo (f_1, f_2, f_3, f_4), while $r = 0$ is a complete reduction of f modulo (f_3, f_2, f_1, f_4).

2.3 Standard Bases for Ideals

In this section we introduce the standard bases for ideals in the context of formal power series rings, one of the main concepts of this course. The name standard bases is due to H. Hironaka, who introduced them in his famous paper [Hi] to study ideals in rings of convergent power series. These objects have been known for a long time in polynomial rings, where they are called Gröbner bases.

DEFINITION A finite subset B of $K[[\mathbf{X}]]$ is a *standard basis for an ideal* if for every $g \in \langle B \rangle$, there exists an element $f \in B$ such that $\mathrm{lp}(f)|\mathrm{lp}(g)$. We will say that B is a *standard basis* for the ideal I, if B is a standard basis for an ideal and $I = \langle B \rangle$.

To say that B is a standard basis for an ideal, is equivalent to say that

$$\mathrm{lp}(\langle B \rangle) = \langle \mathrm{lp}(B) \rangle,$$

where $\langle B \rangle$ denotes the ideal in $K[[\mathbf{X}]]$ generated by B, while $\langle \mathrm{lp}(B) \rangle$ denotes the monoideal generated by $\mathrm{lp}(B)$.

The next theorem will establish the existence of standard bases for ideals in $K[[\mathbf{X}]]$, and will provide other characterizations for them. Before we state and prove the theorem we will need some new concepts.

DEFINITION Given a sum (possibly infinite) in $K[[\mathbf{X}]]$,

$$\sum_{l \in L} f_l, \quad f_l \in K[[\mathbf{X}]],$$

we define the *height* of the sum as

$$\mathrm{ht}(\sum_{l \in L} f_l) = \min\{\mathrm{lp}(f_l),\ l \in L\}.$$

Note that this definition depends upon the representation $\sum_{l \in L} f_l$ as a sum and not upon the element that this sum determines. In fact, we have that

$$\mathrm{ht}(\sum_{l \in L} f_l) \leq \mathrm{lp}(\sum_{l \in L} f_l),$$

where $\leq$ is the monomial order we have fixed.

We will say that f_i *contributes for the height* of $\sum_{l \in L} f_l$, if

$$\mathrm{lp}(f_i) = \mathrm{ht}(\sum_{l \in L} f_l).$$

DEFINITION The *amplitude* of $\sum_{l \in L} f_l$ is defined as the number of summands f_j in $\sum_{l \in L} f_l$ that contribute for its height.

DEFINITION An *S-process* of a pair of non-zero elements $f, g \in K[[\mathbf{X}]]$, is an expression of the form

$$S(f,g) = pf + qg,$$

where $p, q \in K[[\mathbf{X}]]$, such that

$$S(f,g) = 0, \text{ or } \operatorname{lp}(S(f,g)) > \operatorname{ht}(pf + qg).$$

The S-processes in the polynomial context are usually called S-polynomials or critical pairs, while in the power series context they are sometimes called S-series.

THEOREM 3 (GRÖBNER-HIRONAKA-BUCHBERGER)
1) *Every ideal I in $K[[\mathbf{X}]]$ has a standard basis.*
2) *Given a set $B = \{f_1, \ldots, f_s\}$, the following assertions are equivalent:*
(a) *B is a standard basis for an ideal.*
(b) *All final reductions modulo B of any element of $\langle B \rangle$ is zero.*
(c) *Every S-process of any given pair of elements in B has a vanishing final reduction modulo B.*
(d) *Every non-zero S-process of any given pair of elements in B has a representation as a sum $\sum_{i=1}^{s} h_i f_i$, where $h_i \in K[[\mathbf{X}]]$ and $\operatorname{ht}(\sum_{i=1}^{s} h_i f_i)$ is greater than the height of the S-process itself.*

PROOF **Proof of 1:** This is a simple application of Dickson's Theorem. Indeed, let I be an ideal of $K[[\mathbf{X}]]$. Since $\operatorname{lp}(I) \subset \mathbf{T}^n$, then by Dickson's Theorem, there exists a finite set G of divisors of $\operatorname{lp}(I)$. Now, choose a finite subset B of I such that $\operatorname{lp}(B) = G$. It follows that given $g \in I$, there exists $f \in B$ such that $\operatorname{lp}(f) \mid \operatorname{lp}(g)$, showing that B is a standard basis for the ideal I.[1]
Proof of 2: (a)$\Longrightarrow$ (b) Suppose that B is a standard basis for an ideal and let $g \in \langle B \rangle$. If r is a final reduction of g modulo B, then $g - r \in \langle B \rangle$ and therefore we have that $r \in \langle B \rangle$. Suppose $r \neq 0$. Since B is a standard basis, there would exist an element $f \in B$ such that $\operatorname{lp}(f) \mid \operatorname{lp}(r)$, which contradicts the fact that r is a final reduction modulo B. Hence $r = 0$.
(b)$\Longrightarrow$ (c) Since an S-process of elements in B is an element of $\langle B \rangle$, the assertion is obvious.

[1]Remark that the proof of 1 we gave is not constructive since it relies on Dickson's Theorem, which in turn relies on Hilbert's Basis Theorem.

(c)$\Longrightarrow$ (d) Let $h = pf_i + qf_j \neq 0$ be an S-process of a pair f_i, f_j of elements of B. Recall that any reduction r of h is of the form

$$r = h - \sum_{k\geq 1} a_k t_k f_{l_k},$$

where $t_k \in \mathrm{T}^n$, $a_k \in K$ and $f_{l_k} \in B$ are such that $\mathrm{lp}(t_k f_{l_k})$ is a strictly increasing sequence in T^n.

Since from (c) the element h has a vanishing final reduction modulo B, we have that

$$h = \sum_{j\geq 1} a_k t_k f_{l_k}.$$

Now, since the sequence $\mathrm{lp}(t_k f_{l_k})$ is strictly increasing and $h = pf_i + qf_j$ is an S-process, it follows that

$$\mathrm{ht}(\sum_{j\geq 1} a_k t_k f_{l_k}) = \mathrm{lp}(h) > \mathrm{ht}(pf_i + qf_j),$$

concluding the proof of the assertion.

(d)$\Longrightarrow$ (a) Assuming (d), we must show that given a non-zero $g \in \langle B\rangle$, there exsits $f \in B$ such that $\mathrm{lp}(f) \mid \mathrm{lp}(g)$.

In the collection of all representations of g as a sum

$$g = \sum_{i=1}^{s} q_i f_i, \quad q_1, \ldots, q_s \in K[[\mathbf{X}]],$$

we choose among those of maximum height one with the least amplitude, which we denote by $\sum_{i=1}^{s} h_i f_i$. Notice that if the amplitude of $\sum_{i=1}^{s} h_i f_i$ is one, there is nothing to prove because in this case $\mathrm{lp}(g) = \mathrm{lp}(h_i)\mathrm{lp}(f_i)$, for some i.

Suppose that the amplitude of $\sum_{i=1}^{s} h_i f_i$ is greater than one. Without loosing generality we may suppose that $h_1 f_1$ and $h_2 f_2$ contribute to the height of the sum. Therefore, there exists $a \in K$ such that $S = h_1 f_1 + a h_2 f_2$ is an S-process of the pair f_1, f_2. From (d) there exist $g_1, \ldots, g_s \in K[[\mathbf{X}]]$ such that

$$S = \sum_{i=1}^{s} g_i f_i,$$

with $\mathrm{ht}(\sum_{i=1}^{s} g_i f_i) > \mathrm{ht}(h_1 f_1 + a h_2 f_2)$.

Notice that, since

$$h_1 f_1 + h_2 f_2 = (1-a) h_2 f_2 + \sum_{i=1}^{s} g_i f_i,$$

we may write

$$g = (1-a)h_2 f_2 + \sum_{i=1}^{s} g_i f_i + \sum_{i=3}^{s} h_i f_i. \tag{2.2}$$

Suppose that $a \neq 1$. Then the representation of g given in (2.2) will have the same height as the original one, but a smaller amplitude, which is a contradiction.

Suppose that $a = 1$. If the amplitude of the original representation of g were two, then the height of the above representation (2.2) would be greater than the height of the original representation, which is a contradiction. If the amplitude of the original representation of g were greater than two, then the height of the representation (2.2) would be equal to the height of the original representation, but the amplitude would be smaller, which again is a contradiction.

□

REMARK 1 Standard bases solve the ideal membership problem; that is, the problem of deciding whether $f \in I$ or $f \notin I$, where $I = \langle f_1, \ldots, f_r \rangle$ is an ideal of $K[[\mathbf{X}]]$ and $f \in K[[\mathbf{X}]]$. To answer this question we apply the division algorithm to f with $B = (f_1, \ldots, f_r)$. If the final reduction of f modulo B is zero, then certainly $f \in I$. But, if the final reduction of f modulo B is not zero, nothing can be said (cf. Example 2). However, if B is a standard basis for the ideal I, then we have that $f \in I$ if and only if the final reduction of f modulo B is zero.

REMARK 2 When B is a standard basis, then the Division Algorithm modulo B produces the same complete reduction of any element $f \in K[[\mathbf{X}]]$, no matter what ordering we take on B. In fact, let r_1 and r_2 be two complete reductions modulo B of the same element $g \in K[[\mathbf{X}]]$. It follows that we have $r_1 - r_2 \in \langle B \rangle$. Since B is a standard basis, from Theorem 3(b) we have that any final reduction of $r_1 - r_2$ modulo B is zero. Suppose that $r_1 - r_2 \neq 0$, then there would exist an element f in B such that $\mathrm{lp}(f) \mid \mathrm{lp}(r_1 - r_2)$. Since $\mathrm{lp}(r_1 - r_2) \in \mathbf{T}(r_1) \cup \mathbf{T}(r_2)$, we get a contradiction because r_1 and r_2 are complete reductions of g.

REMARK 3 When the condition (c) in Theorem 3 is verified, we say that B is *closed under formation of S-processes.* So, Theorem 3 says that a set B is a standard basis for an ideal if and only if it is closed under formation of S-processes.

REMARK 4 The proof of Theorem 3 was carried out in such a way that it will be possible to adapt it to other situations with minor changes.

2.4 Buchberger's Algorithm

We saw in Theorem 3 that a necessary and sufficient condition for a finite set B to be a standard basis for an ideal is that every S-process of any pair of elements of B has a vanishing final reduction modulo B. However, this condition cannot be checked in practice because every pair of elements has infinitely many S-processes. The next proposition will show that it will be enough to check this on finitely many special S-processes, which we define below.

DEFINITION Let f and g be non-zero elements in $K[[\mathbf{X}]]$ such that $\mathrm{lt}(f) = a\mathbf{X}^{\alpha}$, and $\mathrm{lt}(g) = b\mathbf{X}^{\beta}$. We define the *minimal S-process* of f and g as follows:

$$S_{\min}(f,g) = b\frac{\mathrm{LCM}(\mathbf{X}^{\alpha},\mathbf{X}^{\beta})}{\mathbf{X}^{\alpha}}f - a\frac{\mathrm{LCM}(\mathbf{X}^{\alpha},\mathbf{X}^{\beta})}{\mathbf{X}^{\beta}}g.$$

The Lemma below will be used in the proof of Proposition 1.

LEMMA 1 *Let $\gamma,\delta \in \mathbb{N}^n$ and $f,g \in K[[\mathbf{X}]]$. There exists $\epsilon \in \mathbb{N}^n$ such that*

$$S_{\min}(\mathbf{X}^{\gamma}f,\mathbf{X}^{\delta}g) = \mathbf{X}^{\epsilon}S_{\min}(f,g).$$

Moreover, if

$$\mathrm{ht}(\mathbf{X}^{\gamma}f+\mathbf{X}^{\delta}g) < \mathrm{lp}(\mathbf{X}^{\gamma}f+\mathbf{X}^{\delta}g),$$

then $\mathbf{X}^{\gamma}f+\mathbf{X}^{\delta}g$ is a scalar multiple of the minimal S-process of the pair $\mathbf{X}^{\gamma}f,\mathbf{X}^{\delta}g$.

PROOF Let $a,b \in K$ be such that $\mathrm{lt}(f) = a\mathrm{lp}(f)$ and $\mathrm{lt}(g) = b\mathrm{lp}(g)$. We have that

$$\begin{aligned} S_{\min}\left(\mathbf{X}^{\gamma}f,\mathbf{X}^{\delta}g\right) &= \\ &= b\frac{\mathrm{LCM}(\mathbf{X}^{\gamma}\mathrm{lp}(f),\mathbf{X}^{\delta}\mathrm{lp}(g))}{\mathbf{X}^{\gamma}\mathrm{lp}(f)}\mathbf{X}^{\gamma}f - a\frac{\mathrm{LCM}(\mathbf{X}^{\gamma}\mathrm{lp}(f),\mathbf{X}^{\delta}\mathrm{lp}(g))}{\mathbf{X}^{\delta}\mathrm{lp}(g)}\mathbf{X}^{\delta}g = \\ &= \frac{\mathrm{LCM}(\mathbf{X}^{\gamma}\mathrm{lp}(f),\mathbf{X}^{\delta}\mathrm{lp}(g))}{\mathrm{LCM}(\mathrm{lp}(f),\mathrm{lp}(g))}S_{\min}(f,g). \end{aligned}$$

The second part of the lemma is obvious.

□

PROPOSITION 1 *Let $B = \{f_1,\dots,f_s\}$ be a finite subset of $K[[\mathbf{X}]]$. If every minimal S-process of the pair f_i,f_j has a vanishing final reduction modulo B, then every S-process of f_i,f_j has a representation as a sum $\sum_k g_k f_{l_k}$, with $g_k \in K[[\mathbf{X}]]$, $f_{l_k} \in B$ and such that the height of the sum is greater that the height of the S-process itself.*

PROOF Consider an arbitrary S-process S of the pair f_i, f_j,

$$S = pf_i + qf_j,$$

with $p, q \in K[[\mathbf{X}]]$.

We may rewrite S in the following way:

$$S = \mathrm{lt}(p)f_i + \mathrm{lt}(q)f_j + (p - \mathrm{lt}(p))f_i + (q - \mathrm{lt}(q))f_j. \tag{2.3}$$

Since $\mathrm{ht}(\mathrm{lt}(p)f_i + \mathrm{lt}(q)f_j) < \mathrm{lp}(\mathrm{lt}(p)f_i + \mathrm{lt}(q)f_j)$ we have from Lemma 1 that $\mathrm{lt}(p)f_i + \mathrm{lt}(q)f_j$ is a scalar multiple of the minimal S-process of the pair $\mathrm{lt}(p)f_i, \mathrm{lt}(q)f_j$. Again by Lemma 1, there exist $d \in K$ and $\epsilon \in \mathbb{N}^n$ such that

$$\mathrm{lt}(p)f_i + \mathrm{lt}(q)f_j = d\mathbf{X}^\epsilon S_{\min}(f_i, f_j). \tag{2.4}$$

From the hypothesis we have that $S_{\min}(f_i, f_j) \xrightarrow{B_+} 0$ and using the argumentation we did in the proof of Theorem 3 ((c)$\Longrightarrow$(d)), it follows that

$$S_{\min}(f_i, f_j) = \sum_k h_k f_{l_k}, \tag{2.5}$$

where $h_k \in K[[\mathbf{X}]]$, $f_{l_k} \in B$ and $\mathrm{ht}(\sum_k h_k f_{l_k}) > \mathrm{ht}(S_{\min}(f_i, f_j))$.

Now, substituting (2.5) in (2.4), and substituting the resultant expression in (2.3), we have that

$$S = d\mathbf{X}^\epsilon \sum_k h_k f_{l_k} + (p - \mathrm{lt}(p))f_i + (q - \mathrm{lt}(q))f_j,$$

where the above sum has height greater than $\mathrm{ht}(pf_i + qf_j)$.

□

EXAMPLE 3 Let B be any finite set of monomials in $K[[\mathbf{X}]]$. It is immediate to verify that if $S_{\min}(f, g) = 0$ for all $f, g \in B$, therefore, from Proposition 1 and Theorem 3 ((d) $\Longrightarrow$ (a)), B is a standard basis for an ideal. It follows also easily from Theorem 3(b) that given $f \in \langle B \rangle$, then $\mathbf{T}(f) \subset \langle B \rangle$, showing that the ideal generated by monomials is a monomial ideal.

Although the objects we are dealing with (power series) aren't finite we may, at least theoretically, formulate an algorithm that allows one to find a standard basis of an ideal $I \subset K[[\mathbf{X}]]$, starting with a finite set of generators A of I.

THEOREM 4 (BUCHBERGER'S ALGORITHM). *Suppose that it is possible to determine a final reduction of any given power series modulo any finite ordered subset of* $K[[\mathbf{X}]]$. *Then there exists an algorithm to determine a standard basis for any ideal* I *in* $K[[\mathbf{X}]]$, *starting from any finite set of generators of* I.

PROOF Let A be a finite set of generators of I, ordered in some way. We define the ascending sequence $\{B_i\}_{i \in \mathbb{N}}$ of finite ordered subsets of I in the following way:
1. $B_0 = A$.
2. Suppose that the B_i's are defined for $i \leq m$. If for some pair $f, g \in B_m$ the final reduction r of $S_{\min}(f,g)$ modulo B_m is not zero, define $B_{m+1} = B_m \cup \{r\}$; otherwise put $B_{m+1} = B_m$.

The sequence $\{B_i\}_{i \in \mathbb{N}}$ is stationary. In fact, if it wasn't we would have a sequence $(r_i)_{i \in \mathbb{N}}$, such that for all $i \in \mathbb{N}$, $r_i \in B_{i+1} \setminus B_i$ and $\mathrm{lp}(h)$ doesn't divide $\mathrm{lp}(r_i)$ for all $h \in B_i$. This contradicts Dickson's Theorem, since the set $\cup_{i \in \mathbb{N}}\{\mathrm{lp}(r_i)\} \subset \mathbb{T}^n$ wouldn't admit a finite subset of divisors.

Let $B = B_N$, where $B_m = B_N$ for all $m \geq N$. Since B is closed under formation of minimal S-processes, it follows from Theorem 3 and Proposition 1 that B is a standard basis for I.

□

We present below the algorithm contained in the proof of Theorem 4.

BUCHBERGER'S ALGORITHM

```
INPUT: A;
DEFINE: B_{-1} := ∅, B_0 := A and i := 0;
WHILE B_i ≠ B_{i-1} DO
      IF THERE EXISTS f, g ∈ B_i SUCH THAT
      A FINAL REDUCTION R MODULO B_i
      OF S_min(f, g) IS NONZERO
            THEN
                  B_{i+1} := B_i ∪ {R};
            ELSE
                  B_{i+1} := B_i;
      i := i + 1;
OUTPUT: B_i.
```

During the application of Buchberger's algorithm we have at each stage to decide whether some elements of an ideal I of $K[[\mathbf{X}]]$ (the minimal S-processes) have or not a zero final reduction modulo a finite set. A systematic way to do that is to make use of the division algorithm, which in most cases doesn't end after a finite number of steps. Nevertheless, in several situations, after a finite number of steps in the division algorithm, it is possible to decide whether a final reduction of a given element modulo some finite set is zero or not, turning Buchberger's Algorithm effective.

Buchberger's Algorithm may produce several different standard bases for an ideal I of $K[[\mathbf{X}]]$. This diversity of standard bases depends essentially on the reduction process. However, there is a way to eliminate redundancies, getting a unique reduced basis.

PROPOSITION 2 *Let $B = \{f_1, \ldots, f_s\}$ be a standard basis for an ideal. If there exist i and j with $j \neq i$, such that $\mathrm{lp}(f_j) \mid \mathrm{lp}(f_i)$, then $B' = B \setminus \{f_i\}$ is a standard basis for an ideal and $\langle B' \rangle = \langle B \rangle$.*

PROOF It is clear that $\langle B' \rangle \subset \langle B \rangle$ and that $\langle \mathrm{lp}(B') \rangle = \langle \mathrm{lp}(B) \rangle = \mathrm{lp}(\langle B \rangle)$ (the last equality holding because B is a standard basis).

Therefore it follows that

$$\langle \mathrm{lp}(B') \rangle \subset \mathrm{lp}(\langle B' \rangle) \subset \mathrm{lp}(\langle B \rangle) = \langle \mathrm{lp}(B') \rangle,$$

hence $\mathrm{lp}(\langle B' \rangle) = \langle \mathrm{lp}(B') \rangle$; that is, B' is a standard basis.

Now, let $f \in \langle B \rangle$ and let be r the final reduction of f modulo B'. If $r \neq 0$, then $r \in \langle B \rangle \setminus \langle B' \rangle$, but in this way $\mathrm{lp}(r) \in \langle \mathrm{lp}(B) \rangle = \langle \mathrm{lp}(B') \rangle$, a contradiction. Hence $r = 0$ and therefore $f \in \langle B' \rangle$. This shows that $\langle B \rangle = \langle B' \rangle$.

□

DEFINITION Let B be a standard basis for an ideal. If there are no elements in B whose leading powers divide each other, then we say that B is a *minimal standard basis.*

Remark that Proposition 2 provides a method to determine a minimal standard basis, starting with any standard basis B.

EXAMPLE 4 Let I be an ideal of $K[[\mathbf{X}]]$. The set $\mathrm{lp}(I)$ is a monoideal in $\mathbf{T}^n$. Let A be the minimal set of divisors of the monoideal $\mathrm{lp}(I)$ and let $B \subset I$ be such that $\mathrm{lp}(B) = A$, then B is a standard basis for the ideal I. If

A is the minimal set of generators of the monoideal $\text{lp}(I)$, and if $\#A = \#B$, then B is a minimal standard basis for the ideal I.

DEFINITION A minimal standard basis B for an ideal will be called *reduced* if all elements of B are monic [2] and given $f \in B$, we have

$$\forall g \in B, \forall p \in \mathbf{T}(f) \setminus \{\text{lp}(f)\}; \quad \text{lp}(g) \text{ doesn't divide } p.$$

We have the following result.

PROPOSITION 3 *If $I \subset K[[\mathbf{X}]]$ is an ideal, then I has a unique reduced standard basis.*

PROOF **Existence:** Given a standard basis for an ideal, it is always possible to obtain a minimal standard basis, using the method described in Proposition 2. On the other hand, if B is a minimal standard basis of I, we may assume, without loss of generality, that all elements of B are monic.

For all $f \in B$, let r_f the complete reduction of $f - \text{lt}(f)$ modulo B. Then $B_0 = \{\text{lt}(f) + r_f,\ f \in B\}$ is a reduced standard basis of I.

Unicity: Let B and B' be two reduced standard bases of I.

For each $f \in B'$ there exist $g, h \in B$ such that $\text{lp}(g) \mid \text{lp}(f)$ and $\text{lp}(f) \mid \text{lp}(h)$. Since $\text{lp}(g) \not| \text{lp}(h)$ if $g \neq h$, it follows that $g = h$ and $\text{lp}(f) = \text{lp}(g)$. This establish a bijection between B and B'. In particular, all minimal standard bases of I have the same number of elements.

Take $f \in B'$ and $g \in B$ in such a way that $\text{lp}(f) = \text{lp}(g)$. If $f \neq g$, then $f - g \in I \setminus \{0\}$, hence we have $p = \text{lp}(f - g) > \text{lp}(f) = \text{lp}(g)$, where $p \in \mathbf{T}(f) \cup \mathbf{T}(g)$. Since $f - g \in I \setminus \{0\}$ and B and B' are standard bases for I, from definition of standard bases, there exist an element in B and an element in B' with same leading power which divide p, but this is not possible since B and B' are reduced standard bases. Therefore $B = B'$.

□

The above proposition shows that the reduced standard basis depends only upon the fixed monomial order and the ideal I, but not upon the particular set of generators we started with.

During the proof of the unicity of a reduced standard basis in Proposition 3, we showed that if B and B' are two minimal standard bases (not necessarily reduced) for an ideal I, then we have that

$$\text{lp}(B) = \text{lp}(B').$$

[2] An element $f \in K[[\mathbf{X}]]$ is monic if $\text{lt}(f) = \text{lp}(f)$.

We also have that although possibly $B \neq B'$, they have the same number of elements.

EXAMPLE 5 Let $f = Z^2 - 4X^9Y - X^{13} \in \mathbb{C}[[X,Y,Z]]$, and let $\mathcal{J}(f) = \langle f, f_X, f_Y, f_Z\rangle$. Consider the graded lexicographical order in $\mathbf{T}^3$. Since $f_X = -36X^8Y - 13X^{12}$, $f_Y = -4X^9$ and $f_Z = 2Z$, the minimal S-processes of $B = \{f, f_X, f_Y, f_Z\}$ are:

$$\begin{aligned}
S_{\min}(f, f_X) &= 13X^{12}Z^2 + 144X^{17}Y^2 + 36X^{21}Y\\
S_{\min}(f, f_Y) &= 16X^{18}Y + 4X^{22}\\
S_{\min}(f, f_Z) &= -8X^9Y - 2X^{13}\\
S_{\min}(f_X, f_Y) &= 52X^{13}\\
S_{\min}(f_X, f_Z) &= -26X^{12}Z\\
S_{\min}(f_Y, f_Z) &= 0.
\end{aligned}$$

Applying the division algorithm modulo B, we see that all minimal S-processes have final reduction zero. So, B is a standard basis for $\mathcal{J}(f)$, and $B' = \{X^8Y, X^9, Z\}$ is the reduced standard basis, and $\mathcal{J}(f) = \langle X^8Y, X^9, Z\rangle$.

EXAMPLE 6 Here is an example which shows that minimal standard bases are not unique. Let

$$B = \{f_1, f_2, f_3, f_4\},$$

where $f_1 = Y^4 - 2X^5Y - X^9$,$f_2 = X^{10}$, $f_3 = Y^3 - X^5Y - X^7$ and $f_4 = Y^4 - X^5Y^2 - X^7Y + X^{10}$.

Since there exists $f \in \mathbb{Q}[[X,Y]]$ with two different complete reductions modulo B (see Example 2), it follows from Remark 2 that B is not a standard basis of ideals.

We now apply Buchberger's algorithm to obtain a standard basis for the ideal $\langle B\rangle$.

Fix in $\mathbf{T}^2$ the graded lexicographical order. To obtain the final reduction of an element in $\mathbb{Q}[[X,Y]]$, we will use the division algorithm, fixing the following ordering (f_1, f_2, f_3, f_4) for the elements of B.

Step 1: Computing the minimal S-processes and their final reductions, we find:

$S_{\min}(f_1, f_2) = -2X^{15}Y^2 - X^{19} \xrightarrow{(-2X^5Y^2 - X^9)f_2} 0.$

$S_{\min}(f_2, f_3) = X^{15}Y + X^{17} \xrightarrow{(X^5Y + X^7)f_2} 0.$

$S_{\min}(f_3, f_1) = X^5Y^2 - X^7Y + X^9 =: f_5.$

$S_{\min}(f_1, f_4) = -X^5Y^2 + X^7Y - X^9 - X^{10} =: f_6.$

$S_{\min}(f_2, f_4) = X^{15}Y^2 + X^{17}Y - X^{20} \xrightarrow{(X^5Y^2+X^7Y-X^{10})f_2} 0.$

$S_{\min}(f_3, f_4) = -X^{10} \xrightarrow{-f_2} 0.$

In the next step we may continue in two different ways; namely, with $B_1 = \{f_1, f_2, f_3, f_4, f_5\}$ or with $B_2 = \{f_1, f_2, f_3, f_4, f_6\}$.

If we choose $B_1 = \{f_1, f_2, f_3, f_4, f_5\}$, we have:

Step 2: The minimal S-processes $S_{\min}(f_1, f_2)$, $S_{\min}(f_2, f_3)$, $S_{\min}(f_3, f_1)$, $S_{\min}(f_1, f_4)$, $S_{\min}(f_2, f_4)$, $S_{\min}(f_3, f_4)$ have clearly final reduction zero modulo B_1. The other minimal S-processes, with their final reductions are:

$S_{\min}(f_1, f_5) = X^7Y^3 - X^9Y^2 - 2X^{10}Y^2 - X^{14} \xrightarrow{X^7 f_3} \xrightarrow{-X^4 f_5}$
$\xrightarrow{(-2Y^2-XY+X^2Y+X^3)f_2} 0.$

$S_{\min}(f_2, f_5) = X^{12}Y - X^{14} \xrightarrow{(X^2Y-X^4)f_2} 0.$

$S_{\min}(f_3, f_5) = X^7Y^2 - X^9Y - X^{10}Y - X^{12} \xrightarrow{X^2 f_5} \xrightarrow{(-Y-X-X2)f_2} 0.$

$S_{\min}(f_4, f_5) = X^7Y^3 - X^9Y^2 - X^{10}Y^2 - X^{12}Y + X^{15} \xrightarrow{X^7 f_3} \xrightarrow{-X^4 f_5}$
$\xrightarrow{(-Y^2-XY+X^3+X^4+X^5)f_2} 0.$

Hence, since $B_1 = \{f_1, f_2, f_3, f_4, f_5\}$ is closed under formation of minimal S-processes, it follows from Theorem 3 and Proposition 1 that it is a standard basis for the ideal $\langle B \rangle$.

If instead, we choose B_2, step 2 will become:

Step 2$'$:

$S_{\min}(f_1, f_6) = X^7Y^3 - X^9Y^2 - 3X^{10}Y^2 - X^{14} \xrightarrow{X^7 f_3} \xrightarrow{X^4 f_6}$
$\xrightarrow{(-3Y^2-XY+X^2Y+X^3+X^4)f_2} 0.$

$S_{\min}(f_2, f_6) = X^{12}Y - X^{14} - X^{15} \xrightarrow{(X^2Y-X^4-X^5)f_2} 0.$

$S_{\min}(f_3, f_6) = X^7Y^2 - X^9Y - 2X^{10}Y - X^{12} \xrightarrow{-X^2 f_6} \xrightarrow{(-2Y-X-2X^2)f_2} 0.$

$S_{\min}(f_4, f_6) = X^7Y^3 - X^9Y^2 - 2X^{10}Y^2 - X^{12}Y + X^{15} \xrightarrow{X^7 f_3} \xrightarrow{X^4 f_6}$
$\xrightarrow{(-2Y^2-XY+X^3+2X^4+X^5)f_2} 0.$

Again, in this case, we get that $B_2 = \{f_1, f_2, f_3, f_4, f_6\}$ is a standard basis for $\langle B \rangle$.

By Proposition 2, we have that $B' = \{f_2, f_3, f_5\} \subset B_1$ and $B'' = \{f_2, f_3, f_6\} \subset B_2$ are minimal standard bases for $\langle B \rangle$.

It is easy to see that B' is a reduced standard basis, which by Proposition 3 is the unique such basis.

Chapter 3

CODIMENSION OF IDEALS

In this chapter we will show how we may use a standard basis of an ideal to compute its codimension. This is a relevant information about the ideal. We will also, in some examples, analyze the variation of the codimension of certain ideals associated to families of formal power series in several indeterminates, which play an important role in the theory of singularities.

3.1 The Stair Associated to an Ideal

Let $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$, and let $I \subset K[[\mathbf{X}]]$ be an ideal. The *codimension of the ideal* I is the dimension of the K-vector space

$$Q = \frac{K[[\mathbf{X}]]}{I}.$$

If B is a standard basis for I, then the elements of Q may be represented uniquely under the form $r + I$, where r coincides with its complete reduction modulo B. Therefore, the elements of the form $p + I$, where p is a monomial in $\mathrm{T}^n \setminus \mathrm{lp}(I)$, is a basis of Q as a K-vector space. It then follows that

$$\dim_K Q = \#\left(\mathrm{T}^n \setminus \mathrm{lp}(I)\right).$$

We are going now to give an interpretation of the above equation in terms of Newton diagrams in the case $n = 2$.

Let $I \subset K[[X,Y]]$ be an ideal and $B = \{f_0, \ldots, f_s\}$ be a minimal standard basis for I. From Example 2.4 we have that $\mathrm{lp}(B)$ is the minimal set of generators of the monoideal $\mathrm{lp}(I)$. Let $\mathrm{lp}(f_i) = X^{a_i}Y^{b_i}$, $i = 0, \ldots, s$. Since for $i \neq j$, $\mathrm{lp}(f_i) \not| \, \mathrm{lp}(f_j)$, we may assume, after possibly reordering the elements of B, that we have

$$0 \leq a_0 < a_1 < \cdots < a_s, \text{ and } b_0 > b_1 > \cdots > b_s \geq 0.$$

The *stair* associated to the monomial ideal $\langle \mathrm{lp}(B) \rangle$ is the set

$$E_I = \log(\langle \mathrm{lp}(B) \rangle) = \bigcup_{i=0}^{s} \left[(a_i, b_i) + \mathbb{N}^2 \right] \subset \mathbb{N}^2,$$

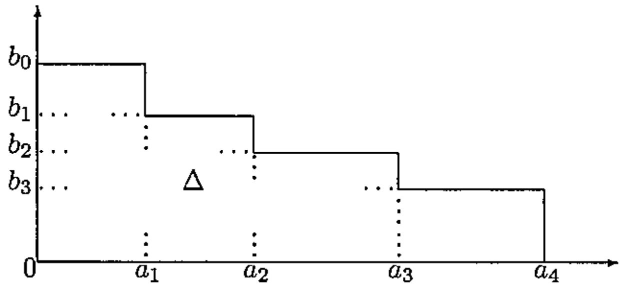
which is independent from the minimal standard basis B, since $\mathrm{lp}(\langle B \rangle) = \mathrm{lp}(I)$.

Now, letting

$$\Delta = \mathbb{N}^2 \setminus E_I,$$

we have from the above discussion that

$$\dim_K Q = \#\Delta.$$

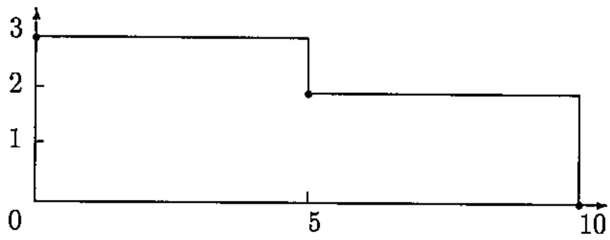


From the above discussion we get immediately the following result:

LEMMA 1 *Let $I \subset K[[X,Y]]$ be an ideal and $E_I = \bigcup_{i=0}^{s} \{(a_i, b_i) + \mathbb{N}^2\}$ its stair. The codimension of I is finite if and only if $a_0 = b_s = 0$. In this case,*

$$\dim_K \frac{K[[X,Y]]}{I} = \sum_{i=1}^{s} a_i(b_{i-1} - b_i) = \sum_{i=1}^{s} b_{i-1}(a_i - a_{i-1}).$$

EXAMPLE 1 In Example 2.6 we saw that $\{X^{10}, Y^3 - X^5Y - X^7, X^5Y^2 - X^7Y + X^9\}$ is the reduced standard basis for the ideal generated by $B = \{Y^4 - 2X^5Y - X^9, X^{10}, Y^3 - X^5Y - X^7, Y^4 - X^5Y^2 - X^7Y + X^{10}\}$. Hence, the stair of $\langle B \rangle$ is



Consequently, from Lemma 1, the codimention of $\langle B \rangle$ is 25.

3.2 Jacobian Ideals

In this section we will use the algorithms so far developed to make explicit computations on Jacobian and extended Jacobian ideals associated to a power series.

Let K be an algebraically closed field of arbitrary characteristic. Let $f \in K[[\mathbf{X}]]$ not a unit. We will assume that f has no multiple factors. We denote by f_{X_i} the partial derivative of f with respect to X_i.

DEFINITION The *Jacobian ideal* associated to f is the ideal

$$J(f) = \langle f_{X_1}, \ldots, f_{X_n} \rangle \subset K[[\mathbf{X}]],$$

and its codimension is called *Milnor's number* of f, denoted by $\mu(f)$.

The *extended Jacobian ideal* associated to f is the ideal

$$\mathcal{J}(f) = \langle f, f_{X_1}, \ldots, f_{X_n} \rangle \subset K[[\mathbf{X}]],$$

and its codimension is called *Tjurina's number* of f, denoted by $\tau(f)$.

These numbers are not necessarily finite, as one can see in Example 2.5. However, it is well known that Milnor's and Tjurina's numbers of f are finite if and only if 0 is an isolated singularity of f; that is, $\sqrt{\mathcal{J}(f)}$ is the maximal ideal of $K[[\mathbf{X}]]$.

When the characteristic of the ground field K is zero, and when $f \in K[[X,Y]]$ has no multiple factors, then these numbers are known to be always finite (see for example [Ri]). These are important invariants in the theory of singularities of algebroid plane curves, as we will see in the subsequent chapters.

To end this chapter, we will discuss some examples, where we describe the variation of $\tau(f)$ and $\mu(f)$ for f varying in some families of power series in $\mathbb{C}[[X,Y]]$.

EXAMPLE 2 We will study the family

$$F_a = Y^n + X^m + \sum_{in+jm>nm} a_{ij}X^iY^j,$$

where $a = (a_{ij})$ with $a_{ij} \in \mathbb{C}$ and $\mathrm{GCD}(n,m) = 1$. This family of formal power series may be thought as a deformation of the polynomial $F_0 = Y^n + X^m$.

Let us compute

$$\mu(F_a) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{\langle (F_a)_X, (F_a)_Y \rangle},$$

using a standard basis for the ideal $\langle (F_a)_X, (F_a)_Y \rangle$, relative to the weighted monomial order in $\mathbf{T}^2$ given by the vector $\gamma = (n,m)$ and the lexicographical order.

Since

$$(F_a)_X = mX^{m-1} + \sum_{(i-1)n+jm>n(m-1)} ia_{ij}X^{i-1}Y^j,$$

and

$$(F_a)_Y = nY^{n-1} + \sum_{in+(j-1)m>(n-1)m} ja_{ij}X^iY^{j-1},$$

the unique minimal S-process, modulo multiplicative constants, to be considered is

$$S_{min}((F_a)_X, (F_a)_Y) = mY^{n-1}(F_a)_X - nX^{m-1}(F_a)_Y,$$

whose leading power has weight greater than $\nu = 2nm - n - m$.

Note that any monomial whose weight is greater than ν has a reduction modulo $\{(F_a)_X, (F_a)_Y\}$. Indeed, suppose that

$$\mathrm{w}_\gamma(X^\alpha Y^\beta) > \nu. \tag{3.1}$$

Two cases may occur:
a) $\alpha \geq m-1$. In this case the monomial has a reduction modulo $(F_a)_X$.
b) $\alpha < m-1$. In this case we may write $\alpha = (m-1-i)$ with $i > 0$. Hence condition 3.1 implies that

$$\beta > (n-1) + i\frac{n}{m} > n-1,$$

and therefore the monomial has a reduction modulo $(F_a)_Y$.

This implies that the S-process has a zero reduction modulo the set $\{(F_a)_X, (F_a)_Y\}$.

From Theorem 2.3 and Proposition 2.1, the set $B = \{(F_a)_X, (F_a)_Y\}$ is a standard basis for $J(F_a)$, and consequently from Lemma 1 it follows that $\mu(F_a) = (n-1)(m-1)$.

This in particular shows that $\mu(F_a)$ is independent from a. Such a family will be called a *μ-constant family.*

EXAMPLE 3 We will now analyze all possible values of

$$\tau(F_a) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{\langle F_a, (F_a)_X, (F_a)_Y \rangle},$$

where

$$F_a = Y^4 + X^9 + a_1X^7Y + a_2X^5Y^2 + a_3X^6Y^2 + a_4X^7Y^2.$$

Note that this is a particular family of the type we considered in Example 2 (with $n = 4$ and $m = 9$).

We have that

$$(F_a)_X = 9X^8 + 7a_1X^6Y + 5a_2X^4Y^2 + 6a_3X^5Y^2 + 7a_4X^6Y^2,$$

and

$$(F_a)_Y = 4Y^3 + a_1X^7 + 2a_2X^5Y + 2a_3X^6Y + 2a_4X^7Y.$$

Applying Buchberger's algorithm, Theorem 2.4, to the extended Jacobian ideal $\langle F_a, (F_a)_X, (F_a)_Y \rangle$, we have:

Step 1: The minimal S-processes of $B_0 = \{F_a, (F_a)_X, (F_a)_Y\}$, as well as their final reductions modulo B_0 are:

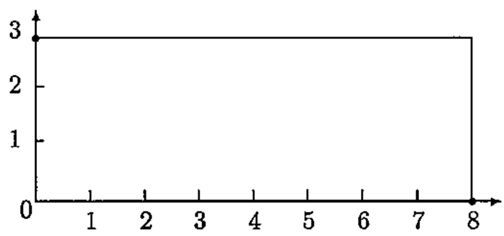
$S(F_a, (F_a)_X) \xrightarrow{\{F_a,(F_a)_X,(F_a)_Y\}} 0$ (this is so because the S-process has weight greater than $\nu = 59$, hence, as we have seen in Example 2 its reducion modulo $\{(F_a)_X, (F_a)_Y\}$ is zero).

$S((F_a)_X, (F_a)_Y) \overset{\{F_a,(F_a)_X,(F_a)_Y\}}{\longrightarrow} 0$ (this follows from Example 2 where we showed that the final reduction of the S-process of the pair $(F_a)_X, (F_a)_Y$ is zero modulo the pair itself).

$S(F_a, (F_a)_Y) = 4F_a - Y(F_a)_Y = 4X^9 + 3a_1X^7Y + 2a_2X^5Y^2 + 2a_3X^6Y^2 + 2a_4X^7Y^2 \overset{\frac{4}{9}X(F_a)_X}{\longrightarrow} F_a' := -\frac{1}{9}a_1X^7Y - \frac{2}{3}a_2X^5Y^2 - \frac{6}{9}a_3X^6Y^2 - \frac{10}{9}a_4X^7Y^2$.

Now to continue, we have several possibilities to consider.

1. If $a_1 = a_2 = a_3 = a_4 = 0$, then $F_a' = 0$ and B_0 is a standard basis for $\langle F_a, (F_a)_X, (F_a)_Y\rangle$ and in this case, $\tau(F_a) = \mu(F_a)$.



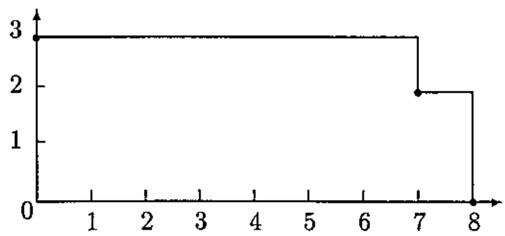
2. If $a_1 = a_2 = a_3 = 0$ and $a_4 \neq 0$, then in the following step we have to consider the set

$$B_1 = \{F_a, (F_a)_X, (F_a)_Y, F_a'\},$$

where in this situation $F_a' = -\frac{10}{9}a_4X^7Y^2$, and the minimal S-processes of B_1 are $S(F_a', F_a), S(F_a', (F_a)_X)$ and $S(F_a', (F_a)_Y)$, which have weights greater than 50.

But, every monomial $X^\alpha Y^\beta$ with $4\alpha + 9\beta > 50$ has a reduction modulo B_1. Indeed, if $\alpha \geq 7$, then the monomial has a reduction modulo $\{(F_a)_X, F_a'\}$. If $\alpha < 7$, i.e., $\alpha = 7 - i$ with $i > 0$, we have $9\beta > 22 + 4i > 26$, then $\beta \geq 3$, and therefore the monomial has a reduction modulo $(F_a)_Y$.

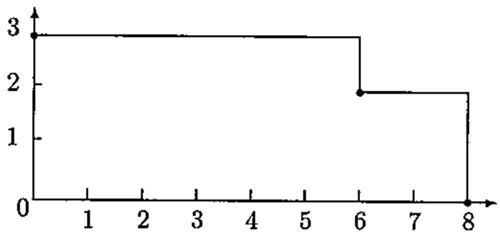
In this case, the algorithm produces the standard basis B_1, and we have $\tau(F_a) = \mu(F_a) - 1$.



3. If $a_1 = a_2 = 0$ and $a_3 \neq 0$, then we have $F_a' = -\frac{2}{3}a_3X^6Y^2 - \frac{10}{9}a_4X^7Y^2$. Since in the next step the minimal S-processes of B_1 are

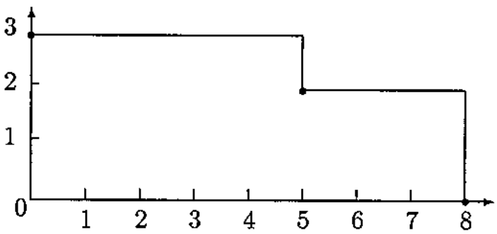
$$S(F_a, F_a'), S((F_a)_X, F_a') \text{ and } S((F_a)_Y, F_a'),$$

which have weight greater than 50, and since every monomial with weight greater than 50 has a reduction modulo $\{F_a', (F_a)_X, (F_a)_Y\}$, it follows that B_1 is a standard basis for the ideal under consideration and $\tau(F_a) = \mu(F_a) - 2$.



4. If $a_1 = 0$ and $a_2 \neq 0$, then $F_a' = -\frac{2}{9}a_2X^5Y^2 - \frac{2}{3}a_3X^6Y^2 - \frac{10}{9}a_4X^7Y^2$.

Since the smallest weight of a minimal S-process involving F_a' is greater than 47 and every monomial whose weight is greater than 47 has a reduction modulo B_1 (This is a computation similar to that we did above), we have that B_1 is a standard basis for $\langle F_a, (F_a)_X, (F_a)_Y \rangle$ and $\tau(F_a) = \mu(F_a) - 3$.



5. It remains to analyze the case $a_1 \neq 0$, that is, when $F_a' = -\frac{1}{9}a_1X^7Y - \frac{2}{9}a_2X^5Y^2 - \frac{2}{3}a_3X^6Y^2 - \frac{10}{9}a_4X^7Y^2$.

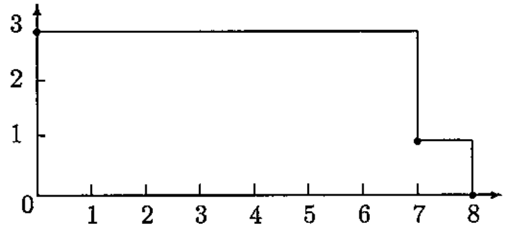
The minimal S-processes $S(F_a, F_a')$ and $S((F_a)_Y, F_a')$ of B_1 have weights greater than $\nu = 56$, so they have zero final reductions modulo $\{(F_a)_X, (F_a)_Y\}$. Hence, it is sufficient to analyze the following S-process of B_1:

$S((F_a)_X, F_a') = a_1Y(F_a)_X - 81XF_a' = (7a_1^2 - 18a_2)X^6Y^2 + 5a_1a_2X^4Y^3 - 54a_3X^7Y^2 + 6a_1a_3X^5Y^3 - 90a_4X^8Y^2 + 7a_1a_4X^6Y^3 =: F_a''$.

We have now the following possibilities:

5.1 If $a_2 = \frac{7a_1^2}{18}$, then the weight of this S-process is ≥ 43. But again, we can show in the same way as above that every monomial of weight greater or equal to 43 has a zero final reduction modulo $\{(F_a)_X, (F_a)_Y, F_a'\}$. So, B_1

is a standard basis for the ideal under consideration and we have $\tau(F_a) = \mu(F_a) - 2$.

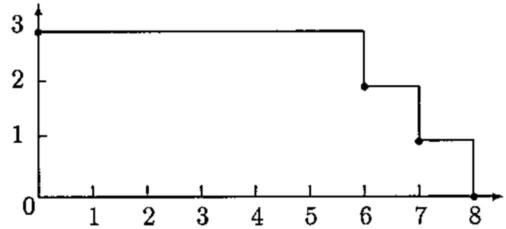


5.2 If $a_2 \neq \frac{7a_1^2}{18}$, then $F_a'' = (7a_1^2 - 18)X^6Y^2 + \cdots$ has to be considered in the next step of the algorithm to produce a standard basis for $\langle F_a, (F_a)_X, (F_a)_Y \rangle$.

Since every S-process involving F_a'' has weight greater than 46, it is easy to verify that it has a zero reduction modulo

$$B_2 = \{F_a, (F_a)_X, (F_a)_Y, F_a', F_a''\},$$

showing that B_2 is a standard basis for the above ideal and in this case we have $\tau(F_a) = \mu(F_a) - 3$.



In the above example we had a family of power series in two indeterminates whose Milnor's number μ was constant along the whole family, and for this family we studied the variation of Tjurina's number τ. In the next example we will have a family with several Milnor's numbers for which we will study the variation of Tjurina's number in each μ-constant stratum.

EXAMPLE 4 Consider the family

$$F_a = (Y^2 - X^3)^3 + \sum_{i \geq 10} a_i X^i \in \mathbb{C}[[X, Y]].$$

We will compute all possible Milnor's and Tjurina's numbers for F_a in terms of a, by using standard bases for the Jacobian and for the extended Jacobian ideals of F_a.

Fix in $\mathbf{T}^2$ the weighted order given by the vector $\gamma = (6, 9)$, together with the lexicographical order. In this way we have:

$$\begin{aligned} F_a &= Y^6 - 3X^3Y^4 + 3X^6Y^2 - X^9 + \textstyle\sum_{i\geq 10} a_i X^i, \\ (F_a)_X &= -9X^2Y^4 + 18X^5Y^2 - 9X^8 + \textstyle\sum_{i\geq 10} i a_i X^{i-1}, \\ (F_a)_Y &= 6Y^5 - 12X^3Y^3 + 6X^6Y. \end{aligned}$$

To compute a standard basis for the ideal $\langle (F_a)_X, (F_a)_Y \rangle$ using Theorem 2.4, we will need to compute final reductions modulo some subsets of $\mathbb{C}[[X,Y]]$, wich we will do by means of the division algorithm.

Step 1: The only minimal S-process of elements in $B_0 = \{(F_a)_X, (F_a)_Y\}$ to be considered is

$$S((F_a)_X, (F_a)_Y) = 2Y(F_a)_X + 3X^2(F_a)_Y = 2\sum_{i\geq 10} i a_i X^{i-1} Y =: F'_a,$$

which coincides with its final reduction modulo B_0.

Note that if $a_i = 0$, for all $i \geq 10$, then F_a has a multiple factor and $\mu(F_a) = \infty$.

Let j, with $j \geq 10$, be the smallest index for which $a_j \neq 0$. In this way, in the next step of the algorithm we have $B_1 = \{(F_a)_X, (F_a)_Y, F'_a\}$.

Step 2: It is sufficient to consider the minimal S-processes $S((F_a)_X, F'_a)$ and $S((F_a)_Y, F'_a)$.

$$S((F_a)_X, F'_a) = 2ja_jX^{j-3}(F_a)_X + 9Y^3F'_a \overset{-18X^3YF'_a}{\longrightarrow}$$
$$-18ja_jX^{j+5} + \boxed{\text{terms of weight } \geq 6j+30} =: F''_a$$

Note that a monomial $X^\alpha Y^\beta$ with weight greater or equal than $6j + 30$, will have a reduction modulo B_1. In fact, supose that $w\gamma(X^\alpha Y^\beta) = 6\alpha + 9\beta \geq 6j + 30$, and consider the following cases:

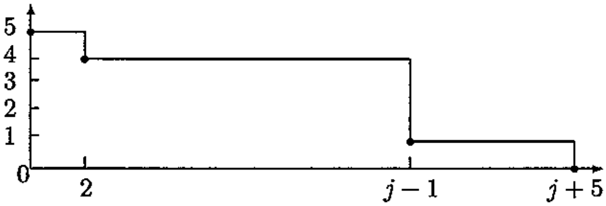
If $\beta \geq 5$, then the monomial has a reduction modulo $\{(F_a)_Y\}$.

If $\beta = 0$, then $\alpha \geq j + 5$ and we use $\{F''_a\}$ to obtain a reduction.

If $0 < \beta < 5$, then $\alpha \geq j + 5 - \frac{3}{2}\beta \geq j - 1$ and $X^\alpha Y^\beta$ has a reduction modulo $\{F'_a\}$.

Since $S((F_a)_Y, F'_a)$ and every S-process involving F''_a have weight greater or equal than $6j + 30$, the set B_1 is a (minimal) standard basis for the Jacobian ideal $J(F_a)$.

The stair associated do $J(F_a)$ is



By Lemma 1, Milnor's number of F_a is $\mu(F_a) = 4(j+1)$.

Now, we will compute a standard basis for the extended Jacobian ideal $\mathcal{J}(F_a) = \langle F_a, (F_a)_X, (F_a)_Y \rangle$.

By the analog of Buchberger's Algorithm (Theorem 2.4), we may start with

$$B_1 \cup \{F_a\} = \{F_a, (F_a)_X, (F_a)_Y, F_a'\}$$

and do the following procedure:

It is sufficient to analyze the minimal S-processes involving F_a. Indeed, $S(F_a, F_a'')$ and $S(F_a, F_a')$ have weight greater than $6j+30$, and we know that such S-processes have zero final reductions modulo B_1.

The other minimal S-processes that we have to consider are $S(F_a, (F_a)_X)$ and $S(F_a, (F_a)_Y)$.

We have

$$S(F_a, (F_a)_Y) = 6F_a - Y(F_a)_Y \quad \xrightarrow{-\frac{2}{3}X(F_a)_X} \quad \frac{2}{3}\sum_{i \geq j}(9-i)a_i X^i =: F_a'''.$$

Note that given a monomial $X^\alpha Y^\beta$, such that $\mathrm{w}\gamma(X^\alpha Y^\beta) = 6\alpha + 9\beta \geq 6j + 18$, it has a reduction modulo $B_2 = B_1 \cup \{F_a, F_a'''\}$. Indeed, this will follow after we analyze the four possibilities below.

1) If $\beta = 0$, then the monomial has a reduction modulo F_a'''.

2) If $0 < \beta < 4$, then $\alpha \geq j-1$ and the monomial has a reduction modulo F_a'.

3) If $\beta = 4$, then $\alpha \geq j-3 \geq 7$ and the monomial has a reduction modulo $(F_a)_X$.

4) If $\beta \geq 5$, then the monomial has a reduction modulo $(F_a)_Y$.

Now, since

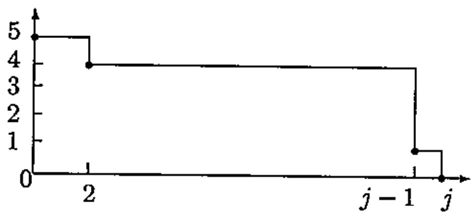
$$S(F_a, (F_a)_X) = 9X^2F_a + Y^2(F_a)_X \quad \xrightarrow{-X^3(F_a)_X} \xrightarrow{-\frac{1}{2}F_a'} \xrightarrow{-\frac{3}{2}F_a'''} \quad 0,$$

and

$$S(F_a', F_a'') = 3jYF_a''' - (9-j)XF_a' \quad \xrightarrow{\frac{-(2j-9)}{ja_j}X^2F_a'} \quad h,$$

with $w_\gamma(h) \geq 6j + 18$ and the other S-processes involving F_a''' have weight greater than $6j + 18$, we have that B_2 is a standard basis for $\mathcal{J}(F_a)$.

In this case, the stair associated to the ideal $\mathcal{J}(F_a)$ is



and by Lemma 1, we have $\tau(F_a) = 4j - 1$.

Chapter 4

STANDARD BASES FOR SUBALGEBRAS

The theory of standard bases and Buchberger's algorithm to determine distinguished sets of generators for ideals in polynomial or formal power series rings may be extended successfully to other algebraic systems. These extensions make this theory widely applicable in several branches of mathematics.

In [RS], Robbiano and Sweedler, introduce the *Subalgebra Analog to Gröbner Bases for Ideals* (SAGBI), as an extension of the previous concept of standard basis to subalgebras of polynomial rings over a field. They also present algorithms to obtain a SAGBI for any finitely generated subalgebra of a polynomial ring over a field.

The main goal of this chapter is to extend the results of [RS] to formal power series rings over a field and to obtain algorithms to compute distinguished bases for complete subalgebras of these rings. Such distinguished bases will be simply called *standard bases.*

4.1 Reduction Process in Algebras

Recall that K is an arbitrary field and that $K[[\mathbf{X}]] = K[[X_1, \dots, X_n]]$, with maximal ideal denoted by $\mathcal{M}_{\mathbf{X}}$. Recall also that we have fixed once for all a monomial order $\leq$ in $\mathbf{T}^n$.

Let $F = \{f_1, \dots, f_m\} \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$ and consider the substitution homomorphism

$$\begin{array}{rccc} S_{(f_1,\dots,f_m)}: & K[[Y_1,\dots,Y_m]] & \longrightarrow & K[[\mathbf{X}]]. \\ & g & \mapsto & g(f_1,\dots,f_m) \end{array}$$

We define the K-subalgebra $K[[F]]$ of $K[[\mathbf{X}]]$ as

$$K[[F]] = \mathcal{S}_{(f_1,\ldots,f_m)}(K[[Y_1,\ldots,Y_m]]).$$

An *F-product* is an element of the form

$$F^\alpha = \prod_{i=1}^{m} f_i^{\alpha_i},$$

where $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m$.

Note that if $a \in K$, then $a = aF^0 = a\prod_{i=1}^{m} f_i^0$, showing that any constant is a scalar multiple of an F-product.

Let G be any nonempty subset of $\mathcal{M}_{\mathbf{X}} \setminus \{0\}$ (even infinite). A *G-product* is an F-product for some finite subset F of G.

We define the K-algebra

$$K[[G]] = \bigcup_{\substack{F\subset G \\ F \text{ finite}}} K[[F]].$$

From now on, we will only call K-subalgebras of $K[[\mathbf{X}]]$ the closed subalgebras of the form $K[[G]]$, for some subset G of $\mathcal{M}_{\mathbf{X}} \setminus \{0\}$. When G is finite, $K[[G]]$ is certainly such a subalgebra.[1].

Given two elements $g, r \in K[[\mathbf{X}]]$, we will say that g *reduces* to r, or that r is *a reduction* of g modulo G, if there exist a G-product F^α and $a \in K$ such that

$$r = g - aF^\alpha, \quad \text{with } \ r = 0 \text{ or } \ \mathrm{lp}(g) < \mathrm{lp}(r).$$

In this case we will write

$$g \xrightarrow{G} r,$$

and we have that $g - r \in K[[G]]$.

Consider a chain (possibly infinite) of reductions

$$g \xrightarrow{G} r_1 \xrightarrow{G} r_2 \xrightarrow{G} \cdots \xrightarrow{G} r_m \xrightarrow{G} \cdots$$

This implies that there exist G-products $F_i^{\alpha^{(i)}}$ and $a_{\alpha^{(i)}} \in K \setminus \{0\}$, such that

$$r_m = g - \sum_{i=1}^{m} a_{\alpha^{(i)}} F_i^{\alpha^{(i)}},$$

[1]Note that we only consider complete subalgebras as subalgebras of $K[[\mathbf{X}]]$. Hence, in our setting, the polynomial ring $K[\mathbf{X}]$, for example, is not to be considered as a K-subalgebra of $K[[\mathbf{X}]]$.

where, because of the definition of a reduction,

$$\operatorname{lp}(a_{\alpha^{(1)}} F_1^{\alpha^{(1)}}) < \operatorname{lp}(a_{\alpha^{(2)}} F_2^{\alpha^{(2)}}) < \cdots. \tag{4.1}$$

If the chain is infinite, we get the following sequence in $K[[\mathbf{X}]]$:

$$s_m = \sum_{i=1}^{m} a_{\alpha^{(i)}} F_i^{\alpha^{(i)}}, \quad m \geq 1. \tag{4.2}$$

When the set G is finite, the sequence (4.2) happens to be convergent in $K[[\mathbf{X}]]$, with respect to the $\mathcal{M}_{\mathbf{X}}$-adic topology. In fact, in this case, the condition (4.1) implies easily that the set $\{a_{\alpha^{(i)}} F_i^{\alpha^{(i)}},\ i \geq 1\}$ is summable, and the sum of its elements is the limit of the sequence.

When G is infinite, this may be not true anymore as we see in the following example.

EXAMPLE 1 Fix the lexicographical order in $\mathbf{T}^2$. Consider the set

$$G = \{X + Y^i,\ i \geq 1\}.$$

Taking $g = \sum_{i \geq 1} Y^i$, we have that

$$g \xrightarrow{G} \sum_{i \geq 2} Y^i - X \xrightarrow{G} \sum_{i \geq 3} Y^i - 2X \xrightarrow{G} \cdots \xrightarrow{G} \sum_{i \geq m} Y^i - (m-1)X \xrightarrow{G} \cdots$$

is an infinite chain of reductions of g, modulo G.

In this case the family $\{X + Y^i,\ i \geq 1\}$ is not summable, since X is present in all sets $\mathbf{T}(X + Y^i)$.

The above example shows that some care has to be taken when the set G is infinite. This difficulty is avoided by fixing an order in $\mathbf{T}^n$ having the finiteness property, as we see in the following result.

LEMMA 1 *If the fixed order in $\mathbf{T}^n$ has the finiteness property, then any family $\{a_{\alpha^{(i)}} F_i^{\alpha^{(i)}},\ i \geq 1\}$, satisfying (4.1) is summable.*

PROOF Let $t \in \mathbf{T}^n$. Since $\operatorname{lp}(F_i^{\alpha^{(i)}})$ is a strictly monotone sequence in $\mathbf{T}^n$, and the order has the finiteness property, there exist finitely many terms of it which are less than t. This implies clearly that $\#\{i,\ t \in \mathbf{T}(a_{\alpha^{(i)}} F_i^{\alpha^{(i)}})\} < \infty$.

□

So, from now on we will assume that the fixed order in $\mathbf{T}^n$ has the finiteness property, whenever we have to reduce modulo infinite sets.

We will denote the limit in $K[[\mathbf{X}]]$ of the sequence $(s_m)_{m\geq 1}$ in (4.2) by s. Since all the terms of the above sequence are in the complete subalgebra $K[[G]]$, we have that $s \in K[[G]]$.

Now, as in the ideal case, we will extend the notion of reduction to include all $r_m = g - s_m$ as above and their limits. It is also true that whenever an element r is a redution of g modulo G, then $g - r \in K[[G]]$.

We will say that r is a *final reduction* of g, modulo G, if r is a reduction of g and r has no further reduction modulo G; that is, $\text{lp}(F^\alpha) \neq \text{lp}(r)$ for all G-product F^α.

In this case we will write $g \xrightarrow{G_+} r$.

If a reduction r of g is zero or $\text{lp}(F^\alpha) \notin \mathbf{T}(r)$ for all G-product F^α, then r will be called a *complete reduction* of g modulo G.

The reduction process corresponds in some way to a division process by the elements of G.

REMARK 1 Notice that we may reduce an element of $K[[\mathbf{X}]]$ in several distinct ways. For instance, let $G = \{f_1 = Y^4, f_2 = Y^3 + \sum_{i=4}^{\infty} X^i\} \subset K[[X, Y]]$ and let the fixed monomial order in $\mathbf{T}^2$ be the graded lexicographical order. We may get two distinct reductions of $f = Y^{12}$ modulo G, as follows:

$$r_1 = f - f_1^3 = 0 \quad \text{and} \quad r_2 = f - f_2^4.$$

For this reason, we will need sometimes to indicate by which G-product we have reduced an element. We will then use a notation like

$$f \xrightarrow{f_1^3} r_1 \quad \text{and} \quad f \xrightarrow{f_2^4} r_2,$$

according to our example.

When the set G is finite, we may give a sort of division algorithm which will systematize the reduction process. In fact, given an element $f \in K[[\mathbf{X}]]$, the G-products that may reduce f modulo G are solutions of the equation $\text{lp}(f) = \text{lp}(G^\alpha)$. Since the later equation is equivalent to a system of n linear Diophantine equations with non negative coefficients in the $\#G$ variables corresponding to the entries of α, and we are interested in non-negative solutions, these will be in finite number. We may then choose, for example, the reduction of f by G^β, with $\beta = \min\{\alpha;\ \text{lp}(G^\alpha) = \text{lp}(f)\}$, where the minimum is taken with respect to a fixed monoidal order in $\mathbb{N}^{\#G}$.

REMARK 2 Note that a necessary and sufficient condition for an element $f \in K[[\mathbf{X}]]$ to have a reduction, modulo $F = \{f_1, \ldots, f_m\} \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$, is that $\mathbf{X}^\alpha = \text{lp}(f)$ belongs to the multiplicative monoid $\langle \mathbf{X}^{\alpha_1}, \ldots, \mathbf{X}^{\alpha_m} \rangle$, where $\mathbf{X}^{\alpha_i} = \text{lp}(f_i)$, for all i. This, in turn, is equivalent to say that α belongs to the additive monoid $\Gamma = \langle \alpha_1, \ldots, \alpha_m \rangle$. In particular, if $\mathbb{N}^n \setminus \Gamma$ is finite, then it is possible to decide after a finite number of steps if a final reduction of f modulo F is zero or not.

4.2 Standard Bases for Subalgebras

We will continue to denote the submonoid generated by a set B by $\langle B \rangle$.

Note that given a subset G of the maximal ideal $\mathcal{M}_{\mathbf{X}}$ of $K[[\mathbf{X}]]$, not containing 0, we always have that $\text{lp}(K[[G]])$ is a submonoid of $\mathbf{T}^n$ and that

$$\langle \text{lp}(G) \rangle \subset \text{lp}(K[[G]]).$$

However, the reverse inclusion is not always true, motivating thus the follownig definition.

DEFINITION We say that a set $G \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$ is a *standard basis of algebras* if

$$\langle \text{lp}(G) \rangle = \text{lp}(K[[G]]).$$

We may rephrase the above definition as follows:
G is standard basis for algebras if for all $f \in K[[G]] \setminus \{0\}$,

$$\text{lp}(f) = \text{lp}(F^\alpha),$$

for some G-product F^α.

A *standard basis for a subalgebra* $A \subset K[[\mathbf{X}]]$ is a standard basis G of algebras, such that $A = K[[G]]$.

In this chapter, standard basis will always mean standard basis of algebras.

EXAMPLE 2 The set $G = \{X_1, \ldots, X_n\}$ is a standard basis for the algebra $K[[X_1, \ldots, X_n]]$ for any monomial order. This is so, since obviously the leading power of any element in $K[[X_1, \ldots, X_n]] \setminus \{0\}$ is a G-product.

EXAMPLE 3 The set $G = \{X^2, \sum_{i=3}^{\infty} X^i\} \subset K[[X]]$ is a standard basis for algebras. Indeed, if $f \in K[[G]] \setminus \{0\}$, then $\text{lp}(f) = 1$ or $\text{lp}(f) = X^\alpha$, for some $\alpha \geq 2$. Hence $\text{lp}(f) \in \langle X^2, X^3 \rangle = \langle \text{lp}(G) \rangle$.

EXAMPLE 4 The set $G = \{X^4, X^6 + X^7\} \subset K[[X]]$ is not a standard basis for algebras. In fact, $2X^{13} + X^{14} = (X^6 + X^7)^2 - (X^4)^3 \in K[[G]]$, but $X^{13} \notin \langle X^4, X^6 \rangle = \langle \mathrm{lp}(G) \rangle$.

With the definition that we gave for standard basis, there will trivially exist a standard basis for every (complete) subalgebra A of $K[[\mathbf{X}]]$. For example, the whole subalgebra is a standard basis for itself. A more interesting question is how to guarantee the existence of finite standard bases? It is obvious that if the subalgebra A is not finitely generated, then there will be no finite standard basis for it. Now, what can be said if the subalgebra is finitely generated? The following example will show that finite standard bases do not always exist, even in this case.

EXAMPLE 5 ([RS]) Let K be any field and consider in $\mathbf{T}^2$ the graded lexicographical order. Let $B = \{Y + X, XY, X^2Y\} \subset K[[X, Y]]$ and $A = K[[B]]$.

Initially notice that $X^kY \in A$ for all $k \geq 1$. Indeed, for $k = 1$ and $k = 2$ this is obvious. Now, if the assertion is true for all $i \leq k$ with $k \geq 2$, then we have

$$X^{k+1}Y = (Y + X)X^kY - (XY)X^{k-1}Y \in A,$$

proving the assertion for all $k \geq 1$.

On the other hand, it is easy to verify that if $\mathrm{lp}(f) = X^i$ for some $i > 0$, then $f \notin A$.

If A had a finite standard basis F, then we could choose a sufficiently large integer j such that X^jY is not the leading power of any element of F. Since $X^jY \in A$ and F is a standard basis for A there would exist elements in F such that their leading powers are $X^{j-i}Y$ and X^i for some $i > 0$. But, as we observed above, no element of A has leading power of the form X^i for $i > 0$. So such a finite standard basis doesn't exist.

An obvious criterion (in general not decidable) for the existence of a finite standard basis for a subalgebra A is the following: the subalgebra A has a finite standard basis as a subalgebra of $K[[\mathbf{X}]]$ if and only if the submonoid $\mathrm{lp}(A)$ of $\mathbf{T}^n$ is a finitely generated.

In particular, since every submonoid of $\mathbf{T}^1$ is finitely generated (Example 1.6), it follows that every subalgebra of $K[[X]]$ has a finite standard basis.

As in the ideal case, we will have the concepts of minimal and reduced standard bases of algebras.

DEFINITION A standard basis of algebras $G \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$ will be called *minimal* if for all $g \in G$, we have $\mathrm{lp}(g) \notin \langle \mathrm{lp}(G \setminus \{g\}) \rangle$.

DEFINITION A minimal standard basis G will be called *reduced* if all $g \in G$ are monic (i.e., $\mathrm{lt}(g) = \mathrm{lp}(g)$) and no element in $\mathrm{T}(g) \setminus \{\mathrm{lp}(g)\}$ is the leading power of a G-product.

For instance, the set G in Example 2 and Example 3, above, are minimal standard bases, while only in Example 2 it is reduced.

The following two results are totally analogous to Propositions 2.2 and 2.3, so we will omit their proofs.

PROPOSITION 1 *Let $G \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$ be a standard basis. If $g \in G$ is such that $\mathrm{lp}(g) \in \langle \mathrm{lp}(G') \rangle$, where $G' = G \setminus \{g\}$, then G' is a standard basis and $K[[G']] = K[[G]]$.*

PROPOSITION 2 *Every subalgebra of $K[[\mathbf{X}]]$ has a unique reduced standard basis.*

Observe that, as in the proof of Proposition 2.3, we have that all minimal standard bases of a subalgebra of $K[[\mathbf{X}]]$ have the same number of elements and the set of values of the elements of any two minimal bases are the same.

In order to give characterizations for standard bases of algebras similar to those of ideals, we need to define the S-processes in the context of algebras.

DEFINITION Let G be any subset of $K[[\mathbf{X}]]$. An *S-process* of G is an element of the form

$$aF^{\alpha} + bF^{\beta},$$

where $a, b \in K \setminus \{0\}$ and F^{α} and F^{β} are G-products, such that

$$\mathrm{lp}(aF^{\alpha} + bF^{\beta}) > \mathrm{ht}(aF^{\alpha} + bF^{\beta}),$$

whenever $aF^{\alpha} + bF^{\beta} \neq 0$.

EXAMPLE 6 Let $G = \{f = X, g = XY^3, h = Y^4 + Y^5, k = Y^6 + Y^7\} \subset K[[X, Y]]$. Consider the graded lexicographical order in T^2. The expressions, $fg - gf$, $h^3 f^4 - g^4$ and $k^2 f^4 - g^4$, are examples of S-processes of G.

The following theorem is the analog for algebras of Theorem 2.3 (Gröbner-Hironaka-Buchberger) and is an adaptation to power series rings of the results of [RS], proved there in the context of polynomial rings.

THEOREM 1 (STANDARD BASES FOR SUBALGEBRAS OF $K[[\mathbf{X}]]$)
1) *Every complete subalgebra of* $K[[\mathbf{X}]]$ *has a standard basis.*
2) *Given* $G \subset \mathcal{M}_{\mathbf{X}} \setminus \{0\}$, *the following assertions are equivalent:*

(a) G *is a standard basis of subalgebras.*
(b) *All final reduction modulo* G *of any element of* $K[[G]]$ *is zero.*
(c) G *is closed under* S*-processes; that is, every* S*-process of* G *has a vanishing final reduction modulo* G.
(d) *Every non-zero* S*-process of* G *has a representation as a sum of scalar multiples of* G*-products with height greater than the height of the* S*-process itself.*

PROOF **Proof of 1:** This is obvious because, as we observed above, any subalgebra of $K[[\mathbf{X}]]$ is a standard basis for itself.
Proof of 2:
(a)$\Longrightarrow$ (b) Suppose that G is a standard basis of algebras and let $f \in K[[G]]$. If r is a final reduction of f modulo G, then from the definition of reduction it follows that $r \in K[[G]]$. Suppose $r \neq 0$. Since G is a standard basis, there exists a G-product F^α such that $\mathrm{lp}(r) = \mathrm{lp}(F^\alpha)$, which contradicts the fact that r is a final reduction modulo G. Hence $r = 0$.
(b)$\Longrightarrow$ (c) Let h be an S-process of G. Since $h \in K[[G]]$, we have that h has a vanishing final reduction modulo G.
(c)$\Longrightarrow$ (d) Let $h = aF^\alpha + bF^\beta \neq 0$ be an S-process of G with a vanishing final reduction modulo G. Recall that any reduction of an element f consists of subtracting from f a sum of scalar multiples of G-products, getting an element which is either zero or has leading power greater than the leading power of f. Therefore, since h has a zero final reduction, we may write

$$h = \sum_\gamma c_\gamma H_\gamma^\gamma,$$

where H_γ^γ are G-products, $c_\gamma \in K \setminus \{0\}$ and the above representation is of amplitude one.

It then follows that $\mathrm{ht}(\sum_\gamma c_\gamma H_\gamma^\gamma) = \mathrm{lp}(h) > \mathrm{ht}(aF^\alpha + bF^\beta)$, proving the assertion.
(d)$\Longrightarrow$ (a) It will be sufficient to prove that every element $f \in K[[G]] \setminus \{0\}$ has a representation as a sum of scalar multiples of G-products with amplitude one. This is equivalent to say that for every element $f \in K[[G]] \setminus \{0\}$ there exits a G-product F^α such that $\mathrm{lp}(f) = \mathrm{lp}(F^\alpha)$, which means that G is standard basis.

Let $f \in K[[G]] \setminus \{0\}$ and take a representation

$$f = \sum_{\delta} a_\delta F_\delta^\delta, \tag{4.3}$$

where $a_\delta \in K \setminus \{0\}$ and F_δ^δ are G-products, of least amplitude λ, among those representations with maximum height.

Suppose by absurd that $\lambda \geq 2$. We may assume that $\mathrm{lp}(a_\alpha F_\alpha^\alpha)$ and $\mathrm{lp}(a_\beta F_\beta^\beta)$ are equal to the height of the sum in (4.3).

Therefore, there exists $b \in K$ such that

$$h := a_\alpha F_\alpha^\alpha + b a_\beta F_\beta^\beta \tag{4.4}$$

is an S-process of G.

Since from hypothesis, there exist $c_\gamma \in K \setminus \{0\}$ and G-products H_γ^γ such that

$$h = \sum_{\gamma} c_\gamma H_\gamma^\gamma, \tag{4.5}$$

with

$$\mathrm{ht}(\sum_{\gamma} c_\gamma H_\gamma^\gamma) = \mathrm{lp}(h) > \mathrm{ht}(a_\alpha F_\alpha^\alpha + b a_\beta F_\beta^\beta).$$

Now, from (4.4) and (4.5), we have

$$a_\alpha F_\alpha^\alpha + a_\beta F_\beta^\beta = (1-b) a_\beta F_\beta^\beta + \sum_{\gamma} c_\gamma H_\gamma^\gamma.$$

So we can write

$$f = (1-b) a_\beta F_\beta^\beta + \sum_{\gamma} c_\gamma H_\gamma^\gamma + \sum_{\delta \neq \alpha,\beta} a_\delta F_\delta^\delta. \tag{4.6}$$

Suppose that $b \neq 1$. Then the representation of f given in (4.6) will have the same height as the one in (4.3), but a smaller amplitude, which is a contradiction.

Suppose that $b = 1$. If the amplitude of the representation of f in (4.3) were two, then the height of the representation (4.6) would be greater than the height of (4.3), which is a contradiction. If the amplitude of the representation (4.3) of f were greater than two, then the height of the representation (4.6) would be equal to the height of (4.3), but the amplitude would be smaller, which again is a contradiction.

□

4.3 The Analog of Buchberger's Algorithm

The characterizations for standard bases given in Theorem 1, although important, they do not suggest any practical algorithm to compute them. We are going now to present a criterion which will lead to an algorithm to determine in finitely many steps a finite standard basis for a subalgebra A, admitting that there exists at least one such basis for A.

To do so we will look more closely to the S-processes of a set $G \subset K[[\mathbf{X}]]$. Define

$$\mathrm{ord}_{X_j}(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = \alpha_j.$$

Since an S-process of G is of the form $S = aF^{\alpha} + bF^{\beta}$, where $a, b \in K$ and $F = \{f_1, \ldots, f_m\}$ is a finite subset of G, it is determined modulo constants by the vectors $\alpha, \beta \in \mathbb{N}^m$, solutions of the following linear system of homogeneous diophantine equations:

$$\left\{ \begin{array}{lcl} \sum_{j=1}^{m} \alpha_j \mathrm{ord}_{X_1}(\mathrm{lp}(f_j)) & = & \sum_{j=1}^{m} \beta_j \mathrm{ord}_{X_1}(\mathrm{lp}(f_j)) \\ \vdots & & \vdots \\ \sum_{j=1}^{m} \alpha_j \mathrm{ord}_{X_n}(\mathrm{lp}(f_j)) & = & \sum_{j=1}^{m} \beta_j \mathrm{ord}_{X_n}(\mathrm{lp}(f_j)). \end{array} \right. \tag{4.7}$$

The set of all solutions of (4.7) is an additive submonoid of $\mathbb{N}^{2m}$, generated by its finite set D of divisors (cf. Example 1.3), called the minimal solutions of the system.

In the case of $n = 1$, that is, in the case of power series in one indeterminate, the system (4.7) reduces to only one linear homogeneous Diophantine equation, and in this case, the algorithm in [CF] is efficient to obtain the set D. Below we present an example of this situation.

EXAMPLE 7 Let $F = \{X^4, X^6 + X^7, 2X^{13} + X^{14}\} \subset \mathbb{C}[[X]]$, the corresponding system (4.7) reduces to the equation

$$4\alpha_1 + 6\alpha_2 + 13\alpha_3 - 4\beta_1 - 6\beta_2 - 13\beta_3 = 0.$$

Using the routine of [CF] we find:

$$\begin{aligned} D = \{ & (1,0,0,1,0,0), (3,0,0,0,2,0), (13,0,0,0,0,4), (0,2,0,3,0,0), \\ & (0,1,0,0,1,0), (0,13,0,0,0,6), (0,0,4,13,0,0), (0,0,6,0,13,0), \\ & (0,0,1,0,0,1), (2,3,0,0,0,2), (8,0,0,0,1,2), (1,0,2,0,5,0), \\ & (0,0,2,2,3,0), (0,1,2,8,0,0), (0,5,0,1,0,2), (5,1,0,0,0,2), \\ & (1,8,0,0,0,4), (0,0,2,5,1,0), (0,0,4,1,8,0)\}. \end{aligned}$$

More generally, when n is arbitrary, the determination of the set D of minimal solutions of (4.7) may be carried out using the algorithm in [CD].

Given a finite subset $F = \{f_1, \dots, f_m\}$ of $K[[\mathbf{X}]]$, the S-processes corresponding to the minimal solutions of (4.7), that is, to the set D, will be called *minimal S-processes* of F.

If G is an arbitrary subset of $K[[\mathbf{X}]]$, then we define a minimal S-processes of G as being a minimal S-process of some finite subset F of G.

REMARK 3 Let $H \subset F$ be finite subsets of G. Every minimal S-process of H is also a minimal S-process of F. This follows from the definition of minimal S-process and from Example 1.4. This shows that the above definition is consistent when G is finite.

The corollary of the next result will show that when standard bases are characterized through properties of S-processes, we only have to consider the minimal S-processes, which are finitely many when we deal with finite sets.

PROPOSITION 3 *Let G be a subset of $\mathcal{M}_{\mathbf{X}} \setminus \{0\}$. If every minimal S-process of G has a vanishing final reduction, then every S-process of G has a representation as a sum of scalar multiples of G-products with height greater than the height of the S-process itself.*

PROOF Fix an S-process of G. Without loosing generality, we may assume that modulo a constant factor, this S-process of G is of the form $F^\alpha + aF^\beta$, for some finite subset F of G and where $a \in K$ is uniquely determined by F, α and β.

Let $S_{\delta,\theta} = F^\delta + a_{\delta,\theta}F^\theta$ represent the minimal S-processes of F. Since we assumed that $S_{\delta,\theta} \xrightarrow{G+} 0$ for some reduction process, we have from what we showed in the proof (c)$\Longrightarrow$(d), in Theorem 1, that we may write

$$S_{\delta,\theta} = \sum_\gamma b_\gamma F_\gamma^\gamma,$$

with amplitude one, where F_γ^γ are G-products and $b_\gamma \in K$.

So

$$F^\delta = -a_{\delta,\theta}F^\theta + \sum_\gamma b_\gamma F_\gamma^\gamma,$$

with $\mathrm{ht}(\sum_\gamma b_\gamma F_\gamma^\gamma) > \mathrm{ht}(F^\delta + a_{\delta,\theta}F^\theta)$.

Since the solution (α, β) of the system (4.7) may be written in the form

$$(\alpha, \beta) = \sum_{\delta,\theta} n_{\delta,\theta}(\delta, \theta),$$

where (δ, θ) are the minimal solutions of the system and $n_{\delta,\theta} \in \mathbb{N}$, we may write

$$F^\alpha = \prod_{\delta,\theta} (F^\delta)^{n_{\delta,\theta}}, \quad \text{and} \quad F^\beta = \prod_{\delta,\theta} (F^\theta)^{n_{\delta,\theta}}.$$

Therefore,

$$F^\alpha = \prod_{\delta,\theta} (F^\delta)^{n_{\delta,\theta}} = \prod_{\delta,\theta} (-a_{\delta,\theta}F^\theta + \sum_\gamma b_\gamma F_\gamma^\gamma)^{n_{\delta,\theta}} =$$

$$= \prod_{\delta,\theta} (-a_{\delta,\theta}F^\theta)^{n_{\delta,\theta}} + \sum_\omega d_\omega F_\omega^\omega =$$

$$= (\prod_{\delta,\theta} (-a_{\delta,\theta})^{n_{\delta,\theta}})F^\beta + \sum_\omega d_\omega F_\omega^\omega,$$

where $\text{ht}(\sum_\omega d_\omega F_\omega^\omega) > \text{ht}(F^\alpha + aF^\beta)$. Since the constant a is uniquely determined, we have that $-\prod_{\delta,\theta}(-a_{\delta,\theta})^{n_{\delta,\theta}} = a$, which implies that the S-process, we started with, has a representation as a sum of scalar multiples of G-products whose height is greater than the height of the S-process itself.

□

From Proposition 3 and from Theorem 1 we get immediatly the following result:

COROLLARY *Let G be any subset of $\mathcal{M}_\mathbf{X} \setminus \{0\}$. Then G is a standard basis for $K[[G]]$ if and only if every minimal S-process of G has some vanishing reduction modulo G.*

EXAMPLE 8 Let $G = \{X^4, X^6 + X^7, 2X^{13} + X^{14}\}$ as in Example 7, where we computed the vectors that determine the minimal S-processes of G. We will refer to the list contained in Example 7.

Notice that the $1st$, $5th$ and the $9th$ elements of that list give us trivial S-processes, hence they may be neglected.

Consider the following pairs in the list: the $2nd$ and the $4th$, the $3rd$ and the $7th$, the $6th$ and the $8th$, the $10th$ and the $13th$, the $11th$ and the $14th$, the $12th$ and the $15th$, the $16th$ and the $18th$, the $17th$ and the $19th$. The given elements in each pair determine, modulo constants, the same S-processes. Hence we have only to consider the following minimal S-processes:
$(X^4)^3 - (X^6 + X^7)^2 = -(2X^{13} + X^{14}) \xrightarrow{F_+} 0$

$16(X^4)^{13} - (2X^{13} + X^{14})^4 = -32X^{53} - 24X^{54} - 8X^{55} - X^{56}$
$64(X^6 + X^7)^{13} - (2X^{13} + X^{14})^6 = 640X^{79} + \cdots$
$4(X^4)^2(X^6 + X^7)^3 - (2X^{13} + X^{14})^2 = 8X^{27} + 11X^{28} + 4X^{29}$
$4(X^4)^8 - (X^6 + X^7)(2X^{13} + X^{14})^2 = -8X^{33} - 5X^{34} - 4X^{35}$
$(X^4)(2X^{13}+X^{14})^2 - 4(X^6+X^7)^5 = -16X^{31} - 39X^{32} - 40X^{33} - 20X^{34} - 4X^{35}$
$4(X^4)^5(X^6 + X^7) - (2X^{13} + X^{14})^2 = -X^{28}$
$16(X^4)(X^6 + X^7)^8 - (2X^{13} + X^{14})^4 = 96X^{53} + \cdots,$

corresponding to the vectors: $(3,0,0,0,2,0), (13,0,0,0,0,4), (0,13,0,0,0,6), (2,3,0,0,0,2), (8,0,0,0,1,2), (1,0,2,0,5,0), (5,1,0,0,0,2), (1,8,0,0,0,4)$.

Since any power of X above 15 may be obtained by means of G-products, we have that every minimal S-process has a vanishing final reduction modulo G. In this way we conclude that G is a standard basis of $\mathbb{C}[[G]]$, furthermore, it is a minimal standard basis.

The following is the analog of Buchberger's Algorithm for algebras.

THEOREM 2 (ANALOG OF BUCHBERGER'S ALGORITHM) *Let B be a subset of $\mathcal{M}_{\mathbf{X}} \setminus \{0\}$. Then we always obtain (theoretically) a standard basis for $K[[B]]$ in the following way:*

```
INPUT: B;
DEFINE: F_0 := B and i := 0;
DO
      S := {s; s is a minimal S-process of F_i};
      R := {r; s --F_i+--> r, s ∈ S and r ≠ 0};
      F_{i+1} := F_i ∪ R;
      i := i + 1;
OUTPUT: G := ⋃_{i≥0} F_i.
```

Moreover, if $K[[B]]$ admits a finite standard basis, then the above procedure will produce one, after finitely many steps.

PROOF The output set $G = \bigcup_{i \geq 0} F_i$ is a standard basis for $K[[B]]$. In fact, from the corollary above, it will be sufficient to verify that every minimal S-process of G has a zero final reduction modulo G. Let s be a given minimal S-process of G. Then s is a minimal S-process of some finite subset F of G. Since F is finite we have that $F \subset F_i$ for some i. Now, because of Remark 3 we have that s is a minimal S-process of F_i. If s reduces to zero modulo F_i, we have nothing to prove. If s doesn't reduce to zero, then by construction of the F_{i+1}, its final reduction is an element of F_{i+1}, so it reduces to zero modulo F_{i+1}, then a fortiori it reduces to zero modulo G. This proves that G is a standard basis for $K[[B]]$.

Suppose now that $K[[B]]$ has a finite standard basis F. We want to show that for some j, the finite set F_j is a standard basis for $K[[B]]$ and this ammounts to show that we have $F_{j+1} = F_j$, for some j. Let

$$F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset G.$$

While $F_i \backslash F_{i-1} \neq \emptyset$, take $r_i \in F_i \backslash F_{i-1}$ which is the final reduction modulo F_{i-1} of a minimal S-process of F_{i-1} of least order . From the construction of the sets F_i, we must have $\text{lp}(r_i) < \text{lp}(r_{i+1})$.

Let $m = \max\{\text{lp}(f); f \in F\}$. So there exists an index j such that the leading power of any element of $E := G \setminus F_j$ is greater than m.

It follows that

$$\langle \text{lp}(F) \rangle = \langle \text{lp}(G) \rangle = \langle \text{lp}(F_j) \cup \text{lp}(E) \rangle.$$

If $f \in F$, then from the above equalities we have that $\text{lp}(f) = \text{lp}(F_j^\alpha H^\beta)$, where F_j^α is an F_j-product and H^β is an E-product. But $\text{lp}(H^\beta) > m$, unless H^β is a constant. The condition $\text{lp}(H^\beta) > m$ cannot hold because $\text{lp}(f) \leq m$. Hence H^β is a constant, implying that $\text{lp}(f)$ is an F_j-product. We have so far shown that $\text{lp}(F) \subset \langle \text{lp}(F_j) \rangle$.

Since F is a standard basis of $K[[B]]$, we have

$$\text{lp}(K[[B]]) = \langle \text{lp}(F) \rangle \subset \langle \text{lp}(F_j) \rangle \subset \text{lp}(K[[B]]),$$

and this shows that F_j is a standard basis for $K[[B]]$.

□

Example 9 Let $A \subset K[[X,Y]]$, the subalgebra generated by $B = \{f_1 = X^2, f_2 = \sum_{i=4}^{\infty} X^i, f_3 = X^7, f_4 = Y^2, f_5 = Y^3 + X^8\}$ and consider the graded lexicographical order on $\mathbf{T}^2$.

Let us apply the algorithm contained in the above theorem to B.

Initially note that every monomial of the form $X^\alpha Y^\beta$ with $\alpha \in \langle 2,7 \rangle$ and $\beta \in \langle 2,3 \rangle$ belongs to A. In this way, all the minimal S-processes of $F_0 = B$ have a zero final reduction, except $f_2 - f_1^2 = \sum_{i=5}^{\infty} X^i$, which coincides with its final reduction, modulo F_0.

In the next step of the algorithm we have

$$F_1 = \{f_1 = X^2, f_2 = \sum_{i=4}^{\infty} X^i, f_3 = X^7, f_4 = Y^2, f_5 = Y^3 + X^8, f_6 = \sum_{i=5}^{\infty} X^i\}.$$

A simple verification shows that every minimal S-process of F_1 reduces to zero modulo F_1. Therefore F_1 is a standard basis for A.

According to Proposition 1, we have that $G = \{f_1, f_4, f_5, f_6\}$ is a minimal standard basis for A. Since $X^5 = f_6 - f_1^3 - f_1 f_6$ and $Y^3 = f_5 - f_1^4$, we have from Proposition 2 that $H = \{X^2, X^5, Y^2, Y^3\}$ is the reduced standard basis for A.

Chapter 5

THE MONOID OF AN IRREDUCIBLE CURVE

In this chapter we will use the theory of standard bases for algebras to study the monoid associated to an irreducible algebroid curve. This is an important invariant for the curve, also called the monoid of values of the curve. We will focus our attention here on the effective determination of such monoids.

The monoid of values of a curve seems to have been first investigated, in the context of irreducible algebroid plane curves, by Apéry, in his 1946 paper [Ap], who discovered some of its important properties, such that the existence of a conductor and its symmetry. Apéry also determined a "standard basis"[1] for the local ring of the curve, seen as a subalgebra of the ring of power series in one variable. Few years later, in his 1952 paper [G], Gorenstein, independently, made the arithmetic of such monoid an important tool for his investigation on the singularities of curves, rediscovering the symmetry, in the plane curve case. It was recognized later that the curves in higher space with symmetric monoids of values had important properties, characterizing the class of the so called *Gorenstein curves* (see [K]).

In the 60's, Zariski, in a famous series of papers [Z2] and [Z3], investigated the singularities of curves, in a purely algebraic context. He recognized that in the case of analytic complex plane curves, the monoid of values is a complete discrete invariant for the topological classification of the curves. This turned the topological equivalence into a purely algebraic concept which could be defined in the algebroid context over any ground field.

In [Az], Azevedo discovered that the monoid of values of an irreducible

[1] The basis consisted of functions whose values are the Apéry sequence of the monoid.

algebroid plane curve is *strongly increasing*, a condition which is also a sufficient for a monoid to be associated to an irreducible algebroid plane curve.

Zariski, Azevedo and Abhyankar-Moh (respectively in [Z4], [Az] and [AM]), determined, as we will see in Section 2, a minimal standard basis for the local ring of an irreducible algebroid plane curve.

In this chapter we will assume that K is an algebraically closed field of arbitrary characteristic.

5.1 Algebroid Curves

In this chapter $K[[\mathbf{X}]]$ will represent the ring $K[[X_1, \ldots, X_n]]$.

DEFINITION An *irreducible algebroid curve*, or a *branch*, in $K[[\mathbf{X}]]$, is a (proper) prime ideal C of $K[[\mathbf{X}]]$ such that the local complete domain

$$\mathcal{O} = \frac{K[[\mathbf{X}]]}{C} \tag{5.1}$$

has Krull dimension one.

The ring $\mathcal{O}$ will be called the *ring of coordinates* or the *ring of functions* of the curve C, whose maximal ideal will be denoted by $\mathcal{M}$.

A set of generators $f_1, \ldots, f_r$ of the ideal C will be called a *Cartesian representation* of C.

For dimension reasons, any Cartesian representation of a branch C in $K[[\mathbf{X}]]$ has at least $n-1$ elements. However, there exist branches in $K[[\mathbf{X}]]$ with no Cartesian representation with $n-1$ elements. Branches in $K[[\mathbf{X}]]$ that have at least one Cartesian representation with exactly $n-1$ elements are called *complete intersections*.

EXAMPLE 1 PLANE BRANCHES

A *plane branch* C is a branch in $K[[X, Y]]$. In this case, it is known that C may be generated by an irreducible power series f; that is, it is a complete intersection. Two generators f and g of C are always associated. This means that there exists a unit $u \in K[[X, Y]]$ such that $g = u.f$. So, a plane branch is simply an equivalence class, modulo associates, of an irreducible power series in $K[[X, Y]]$.

Given a branch C in $K[[\mathbf{X}]]$, it may happen that we can find a homogeneous linear form $\sum_{i=1}^{n} a_i X_i$ contained in the ideal defining the branch[2].

[2] This condition is the algebraic counterpart of the geometric condition, in the complex analytic context, for the existence of a hyperplane that contains the branch.

In this case we will say that C is *degenerate.* It is easy to verify, using Nakayama's Lemma, that C is *non-degenerate* if and only if

$$\dim_K \frac{\mathcal{M}}{\mathcal{M}^2} = n.$$

The dual space

$$\left(\frac{\mathcal{M}}{\mathcal{M}^2}\right)^*$$

is called the *Zariski tangent space* of C. Its dimension, called the *embedding dimension* of C, is the least dimension of a ring $K[[\mathbf{X}]]$, in which the branch may be realized.

From now on we will assume that all branches are non degenerate; that is, the branches will be defined in $K[[X_1, \ldots, X_n]]$ with embedding dimension n.

From general theory it is known that the integral closure $\bar{\mathcal{O}}$ of the domain $\mathcal{O} = K[[x_1, \ldots, x_n]]$, where x_i represents the residual class of X_i modulo C, is isomorphic to the ring $K[[T]]$.

From the inclusion of $\mathcal{O}$ in $\bar{\mathcal{O}}(\simeq K[[T]])$, we have a monomorphism of K-algebras

$$\begin{array}{rccc} \varphi: & \mathcal{O} & \longrightarrow & K[[T]]. \\ & x_i & \mapsto & p_i(T) \end{array} \tag{5.2}$$

The representation

$$\left\{ \begin{array}{ccc} x_1 & = & p_1(T) \\ \vdots & & \vdots \\ x_n & = & p_n(T) \end{array} \right.$$

will be called a *parametrization* of C, where we identified x_i with its image $\varphi(x_i) = p_i(T)$.

We clearly have

1) $\mathcal{O} \simeq \varphi(\mathcal{O}) = K[[p_1(T), \ldots, p_n(T)]] \subset K[[T]]$.

2) The field of fractions of $K[[p_1(T), \ldots, p_n(T)]]$ is equal to $K((T))$ (field of fractions of $K[[T]]$).

EXAMPLE 2 MONOMIAL CURVES

A *monomial curve* in $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$ is a curve which admits a parametrization of the form

$$\begin{cases} x_1 & = & T^{s_1} \\ \vdots & & \vdots \\ x_n & = & T^{s_n}. \end{cases}$$

We will denote by v the normalized valuation of $K[[T]]$, which is obviously the unique monomial order in $K[[T]]$. For $p(T) \in K[[T]] \setminus \{0\}$, write $p(T) = T^m u(T)$, where $u(0) \neq 0$, we have $v(p(T)) = m$, which is called the *value* of $p(T)$.

To any subring B of $K[[T]]$ we may associate the monoid $v(B \setminus \{0\}) \subset \mathbb{N}$, called the *monoid of values* of B. Since any submonoid of $\mathbb{N}$ has a (unique) minimal system of generators (Example 1.6), the number of these generators is determined uniquely by the submonoid. This number minus 1 will be called the *genus* of the submonoid.

Let C be a curve with ring of coordinates $\mathcal{O}$. Given any element $\bar{g}$ in $\mathcal{O} \setminus \{0\}$, where $g \in K[[\mathbf{X}]] \setminus C$, it is possible to assign a value $v(\bar{g})$ to this element, in the following way:

$$v(\bar{g}) = v(\varphi(\bar{g})).$$

The value $v(\bar{g})$ as defined above needs a parametrization of C. However, it may be defined using only the Cartesian representation $f_1, \ldots, f_r$ of the branch. By using some linear algebra, it is possible to show that $v(\varphi(\bar{g}))$ is the codimension in $K[[\mathbf{X}]]$ of the ideal $\langle g, f_1, \ldots, f_r \rangle$. With this formulation, $v(\bar{g})$ may be computed using standard bases of ideals (see Lemma 3.1).

We now define the *monoid of values* of the branch C as

$$v(\mathcal{O} \setminus \{0\}) := v(\varphi(\mathcal{O}) \setminus \{0\}),$$

where $\mathcal{O}$ is the ring of coordinates of C. The genus of this monoid will be called the *genus of* C.

Note that from what we said immediately before the above definition, we have that the monoid of values doesn't depend upon any representation of the curve C, but it is intrinsically associated to it.

There are several classes of curves which are defined according to some property of their monoids of values and conversely, several classes of submonoids of $\mathbb{N}$ are defined by the property of being possibly associated to some particular classes of curves.

DEFINITION We say that a branch is *Gorenstein* if its monoid of values is symmetric (see the definition of a symmetric submonoid of $\mathbb{N}$ in Chapter 1).

Gorenstein curves play an important role in algebraic geometry and are ubiquitous. For example, every complete intersection branch is Gorenstein (see [Ei], Corollary 21.19).

DEFINITION A submonoid $\langle s_1, \ldots, s_r \rangle$ of $\mathbb{N}$ will be called a *monoid of a complete intersection* if the associated monomial curve

$$\begin{cases} x_1 & = & T^{s_1} \\ \vdots & & \vdots \\ x_r & = & T^{s_r} \end{cases}$$

is a complete intersection in $K[[X_1, \ldots, X_r]]$.

It is known that if the monoid of values of a curve is a monoid of a complete intersection, then the curve itself is a complete intersection (see for example [D1]). The converse of this statement is not true, since there are complete intersection curves whose monoids of values are not monoids of a complete intersection. Such an example is due to Herzog and Kunz [HK], and will be discussed in Chapter 7. There also exists an algorithm due to Delorme (see [D1]) that allows to decide whether or not a submonoid of $\mathbb{N}$ is a monoid of a complete intersection.

DEFINITION Let $\Gamma = \langle v_0, \ldots, v_g \rangle$ be a submonoid of $\mathbb{N}$, where $v_0, \ldots, v_g$ is its minimal system of generators. Let $n_0 = 1$, and

$$n_i = \frac{\mathrm{GCD}(v_0, \ldots, v_{i-1})}{\mathrm{GCD}(v_0, \ldots, v_i)}, \quad i = 1, \ldots, g.$$

We say that Γ is *strongly increasing*, if

$$n_i v_i \in \langle v_0, \ldots, v_{i-1} \rangle \text{ and } n_{i-1} v_{i-1} < v_i, \quad i = 1, \ldots, g.$$

Azevedo in [Az] was the first to observe that the monoid of values of a plane branch is strongly increasing. Conversely, it is known that if a submonoid Γ of $\mathbb{N}$ is strongly increasing, then there exists a plane branch whose monoid of values is Γ (see [Bre], [T] or [An]). It is also possible to show, using Delorme's algorithm, quoted above, that any strongly increasing submonoid of $\mathbb{N}$ is the monoid of a complete intersection. It follows that a curve whose monoid of values is strongly increasing, then the curve is a complete intersection.

EXAMPLE 3 Consider the following submonoids of $\mathbb{N}$:

$$\Gamma_1 = \langle 3,4,5\rangle, \;\; \Gamma_2 = \langle 6,8,10,17,19\rangle, \;\; \Gamma_3 = \langle 4,6,7\rangle, \;\; \Gamma_4 = \langle 6,9,19\rangle.$$

The monoid Γ_1 is not symmetric, hence any branch whose monoid of values is Γ_1 is not Gorenstein. The monoid Γ_2 is symmetric but it is not a monoid of a complete intersection. The monoid Γ_3 is a monoid of a complete intersection, but it is not strongly increasing. Finally, Γ_4 is a strongly increasing monoid.

Given a submonoid S of $\mathbb{N}$, the integer $\mathrm{mult}(S) := \min S \setminus \{0\}$ will be called the *multiplicity* of S. The *multiplicity of the branch* C is defined as the multiplicity of its monoid of values, and will be denoted by $\mathrm{mult}(C)$.

Observe that it is not a trivial matter to determine the monoid $\Gamma = v(\mathcal{O} \setminus \{0\})$, since one in principle should compute $v(\bar{g})$ for all $\bar{g} \in \mathcal{O} \setminus \{0\}$. However, since Γ is finitely generated (see Example 1.6), we have that $\mathcal{O}$, more precisely, $\varphi(\mathcal{O})$, admits a finite standard basis as a subalgebra of $K[[T]]$. Now, the values of the elements of a mimimal standard basis of $\mathcal{O}$ form the minimal system of generators of Γ. Theorem 4.2 contains a general algorithm that allows to determine Γ and we will see in the next section how we can improve it in the particular case of branches.

Recall from Section 1.1 that an integer c is the *conductor* of a submonoid Γ of $\mathbb{N}$, if c is the least element (if it exists) of Γ, such that every integer above it belongs to Γ.

An important example of a monoid with conductor is given by a monoid S of values of a branch C. In fact, since the field of fractions of $\mathcal{O}$ is equal to the field of fractions of $K[[T]]$, there are elements f and g in $\mathcal{O}$, such that $f/g = T$. Hence $v(g)$ and $v(g) + 1 = v(f)$ are in S, and we conclude, as in Remark 1.1, that S has a conductor.

Not every set of power series $p_1(T), \ldots, p_n(T)$ has the above property (2), which says that the field of fractions of $K[[p_1(T), \ldots, p_n(T)]]$ is $K((T))$. The set T^2, T^4 is such an example because $K[[T^2, T^4]] \subset K[[T^2]] \subset K((T^2))$.

A set of power series having the above property (2) will be called a *primitive set.*

Conversely, given a primitive set $p_1(T), \ldots, p_n(T)$ of power series, we may associate to it the branch $\ker(\Phi)$, where

$$\begin{array}{rccc} \Phi: & K[[X_1, \ldots, X_n]] & \longrightarrow & K[[T]]. \\ & X_i & \mapsto & p_i(T) \end{array}$$

Notice that in the above situation we have that $\varphi(\mathcal{O})$ is equal to the ring $K[[p_1(T),\dots,p_n(T)]]$, whose field of fractions is $K((T))$.

In general, it may be difficult to verify that a given set of power series is primitive. But if we work with a restricted type of sets, it is possible to get a criterion for deciding primitivity or not, as we will see below.

The K-isomorphisms of $K[[T]]$ onto $K[[T']]$ are of the form

$$\begin{array}{rccc} \tau: & K[[T]] & \longrightarrow & K[[T']], \\ & p(T) & \mapsto & q(T') = p(\tau(T)) \end{array}$$

where $\tau(T) = aT' + \cdots \in K[[T']]$ and $a \neq 0$. Therefore, from our algebraic point of wiew, we will allow to reparametrize power series by means of automorphisms τ. So, given a primitive set $p_1(T),\dots,p_n(T)$, and τ as above, we have that the set $q_1(T') = p_1(\tau(T)),\dots,q_n(T') = p_n(\tau(T))$ is also primitive. It is immediate to verify that the pair of ideals determined in $K[[\mathbf{X}]]$ by this pair of primitive sets are equal, hence they define the same branch.

Let $p_1(T),\dots,p_n(T) \notin K$ be a given primitive set of power series. Without loss of generality, we may suppose that the value m of $p_1(T)$ is minimum among the values of the $p_i(T)$'s. We may write $p_1(T) = T^m u$ where u is a unit in $K[[T]]$. If $\mathrm{char}(K)$ doesn't divide m (this is always the case when $\mathrm{char}(K) = 0$), then it is easy to show that there exists a unit w in $K[[T]]$ such that $u = w^m$. So we may write $p_1(T) = \tau(T)^m = T'^m$, where $T' = \tau(T) = T.w$ with τ an isomorphism from $K[[T]]$ onto $K[[T']]$.

Now we may transform our original primitive set into the primitive set $q_1(T') = p_1(\tau^{-1}(T')) = T'^m$, $q_2(T') = p_2(\tau^{-1}(T'))$, $\dots$, $q_n(T') = p_n(\tau^{-1}(T'))$.

Hence, if $\mathrm{char}(K) = 0$, any primitive set may be reparametrized obtaining a primitive set of the form $q_1(T') = T'^m$, $q_2(T'),\dots,q_n(T')$, called a *Puiseux set.*

A subring of $K[[T]]$ generated by a Puiseux set is called a *Puiseux ring.*

REMARK 1 From what we said above, in characteristic zero the local ring of a branch may be always realized as a Puiseux ring.

The following is a known criterion to decide whether a Puiseux set is primitive or not (see for example [Cam], Proposition 2.1.13). However, our proof is based on a constructive algorithm (Lemma 1, below) which determines two elements in the Puiseux ring generated by the given Puiseux set whose values are consecutive integers. This in particular, gives an upper bound for the conductor of the monoid of values of the Puiseux ring.

PROPOSITION 1 *A set of power series $p_1(T) = T^m$, $p_2(T)$, $\ldots, p_n(T)$ is a Puiseux set if and only if the GCD of the values of all terms of the power series in the set is one.*

PROOF If the GCD of the values of all terms of the series in the given set is not one, then clearly the set is not primitive.

Suppose now that the GCD is one. If we show that there are two elements in $\mathcal{A} = K[[p_1(T), \ldots, p_n(T)]]$ with consecutive values, the proof will be finished since this implies that T belongs to the field of fractions of $\mathcal{A}$.

A consequence of the next Lemma will show that these two elements exist, concluding hereby the proof of the proposition.

□

LEMMA 1 *Given a Puiseux ring $\mathcal{A} = K[[T^m, p_2(T), \ldots, p_n(T)]]$, the following algorithm determines, after a finite number of steps, elements in $\mathcal{A}$ such that the GCD of their values is one.*

```
INPUT: T^m, p_2(T), ..., p_n(T);
DEFINE: {π_1, ..., π_r} = set of prime divisors
        of m and h_0 := T^m;
FOR j FROM 1 TO r DO
     Choose p_i(T) with 2 ≤ i ≤ n
        such that π_j doesn't divide the
        value of some term of p_i(T);
     S_0 := p_i(T) and k := 0;
     WHILE π_j | v(S_k) DO
          S_{k+1} := S_k^α − aT^{βm};
             where α, β ∈ ℕ and a ∈ K* are
             uniquely determined, in such
             a way that α is minimum
             and v(S_{k+1}) > v(S_k^α) = βm;
          k := k + 1;
     h_j := S_k;
OUTPUT: {h_j; j = 0, ..., r} with
          GCD(v(h_0), v(h_1), ..., v(h_r)) = 1.
```

PROOF The existence of $p_i(t)$, such that π_j is coprime with the value of some term of $p_i(t)$, as well the existence of $k < \infty$ such that $\pi_j \nmid v(S_k)$, are guaranteed by the primitivity of the given set of power series.

□

COROLLARY 1 *Given a Puiseux ring $\mathcal{A} = K[[T^m, p_2(T), \dots, p_n(T)]]$, there exist two elements in $\mathcal{A}$ with consecutive values.*

PROOF Take the elements $h_0, h_1, \dots, h_r$ of $\mathcal{A}$ determined in Lemma 1, and let $w_i = v(h_i)$ for $i = 0, \dots, r$. Then $\mathrm{GCD}(w_0, w_1, \dots, w_r) = 1$, and we may find integers α_i for $i = 0, \dots, r$, such that $\sum_{i=0}^{r} \alpha_i w_i = 1$.

Define $P = \{i;\ \alpha_i \geq 0\}$ and $N = \{i;\ \alpha_i < 0\}$. We may write

$$v\left(\prod_{i\in P} h_i^{\alpha_i}\right) = 1 + v\left(\prod_{i\in N} h_i^{-\alpha_i}\right),$$

and the result follows.

□

COROLLARY 2 *The monoid of values of a Puiseux ring has conductor, which is majorated by*

$$\left(v\left(\prod_{i\in N} h_i^{-\alpha_i}\right) - 1\right)\left(v\left(\prod_{i\in N} h_i^{-\alpha_i}\right) + 1\right),$$

where the h_i's are as in Lemma 1.

PROOF This result follows from the above corollary and the argument used in Remark 1.1.

□

5.2 The Monoid of Values of a Branch

Emphasizing what we said in the introduction of the chapter, the importance of the monoid of values of an irreducible plane analytic curve germ was detected by Zariski (see [Z1]), who recognized it as a complete discrete invariant for the topological classification of such germs. He contributed to the proof of the following result: Two plane analytic branches are topologically equivalent if and only if their monoids of values are equal.

In view of this result he proposed the following general definition:

DEFINITION Two branches in $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$ will be said *equisingular* if their monoids of values coincide. An *equisingularity class* is the set of all branches in $K[[\mathbf{X}]]$ that are associated to a given monoid of values.

There are several classical methods to determine the monoid (of values) of a plane branch. For example, the monoid may be determined by a minimal resolution of the singularity of the branch. This is the only known method that works in arbitrary characteristic and may require lengthy computations. In characteristic zero, the monoid may be determined by a Puiseux parametrization of the branch (see [Z4]) or by several other methods (see for example [DS] and [AM]).

What about the case of space branches? In view of the remark we made in the previous section, the set of values of the elements of a minimal standard basis of the ring of coordinates $\mathcal{O}$ of the branch, is the minimal system of generators of its monoid of values. The algorithm in Theorem 4.2, not only gives the monoid, but it also gives the standard basis for the ring $\mathcal{O}$, which is very important in many applications. In the specific case of branches, the algorithm will be shown to be effective and its efficiency will be improved by eliminating several unnecessary verifications.

Recall that in the application of the algorithm of Theorem 4.2, the S-processes are obtained by means of the minimal solutions of a linear homogeneous diophantine equation, which may be determined using, for example, the algorithm in [CF].

All diophantine equations we will have to consider are of the following particular form

$$\sum_{i=1}^{s} a_i W_i = \sum_{i=1}^{s} a_i Z_i.$$

For all $j = 1, \ldots, s$, we have a minimal solution of the form

$$(0, \ldots, 1, \ldots, 0, 0, \ldots, 1, \ldots, 0),$$

where the only nonzero entries are in positions j and $s + j$. These solutions will determine S-processes that are identically zero, hence irrelevant. On the other hand, if (α, β) is a minimal solution, so is (β, α). But these solutions determine, modulo a constant factor, the same S-process. So, it is sufficient to consider only one of them. From now on, when we mention the minimal

S-processes, we will exclude the trivial ones and the redundancies detected above.

The algorithm starts by taking a representation of $\mathcal{O}$ as a subring $K[[F_0]]$ of $K[[T]]$ with F_0 a finite set. In the step i, the algorithm produces a finite set F_i such that $K[[F_0]] = K[[F_i]]$. Suppose that one can get by some mean a majorant c_i for the conductor c of $v(\mathcal{O})$. By Remark 4.2 every minimal S-process of F_i which after finitely many reductions has value greater or equal to c_i will be disregarded, since any final reduction of it modulo F_i will be zero. This in general reduces drastically the number of final reductions of S-processes to be performed in order to produce the finite set F_{i+1}.

In this way we have the following:

ALGORITHM FOR STANDARD BASIS FOR $\mathcal{O}$

INPUT: F_0;
DEFINE: $F_{-1} := \emptyset, i := 0$;
WHILE $F_i \neq F_{i-1}$ DO
  $c_i :=$ majorant for the conductor of $\langle v(F_i)\rangle$;
  $S := \{s;\ s$ is a minimal S-process of F_i, not computed
    in previous step with $v(\mathrm{ht}(s)) < c_i - 1\}$;
  $R := \{r;\ s \overset{F_{i+}}{\longrightarrow} r\ \forall s \in S$ and $r \neq 0\}$;
  $F_{i+1} := F_i \cup R$;
  $i := i + 1$
OUTPUT: $F = F_i$.

The majorants c_i mentioned in the above algorithm ought to be determined, if possible, independently from the algorithm by any non specified way (by inspection, for example). If this is not possible, just put $c_i = \infty$.

If $\mathcal{O}$ has a representation as a Puiseux ring $K[[F_0]]$, with F_0 a Puiseux set, the majorants c_i may be systematically determined using Proposition 1, Lemma 1 and Corollary 2. In many situations, as the examples will show, the c_i so obtained are decreasing, which makes the algorithm even more efficient.

EXAMPLE 4 Let C be the monomial curve

$$\begin{cases} x_1 & = & T^{s_1} \\ \vdots & & \vdots \\ x_n & = & T^{s_n}. \end{cases}$$

Since $G = \{x_1, \ldots, x_n\}$ is closed under formation of S-processes, it follows that G is a standard basis for $\mathcal{O}$.

EXAMPLE 5 Let C be a branch given by the following parametrization

$$\begin{cases} x = T^8 \\ y = T^{10} + T^{13}. \end{cases}$$

Taking $F_0 = \{x, y\}$, the only minimal S-process to be considered is $z = y^4 - x^5$.

Taking its final reduction modulo F_0 we have

$$z \begin{cases} = 4T^{43} + 6T^{46} + 4T^{49} + T^{52} & \text{if } \mathrm{char}(K) \neq 2 \\ \xrightarrow{x^4y^2} \xrightarrow{x^6y} T^{61} & \text{if } \mathrm{char}(K) = 2. \end{cases}$$

Independently from $\mathrm{char}(K)$, a quick analysis shows that there isn't any relevant minimal S-process in the next step of the algorithm. Therefore, $F_1 = \{x, y, z\}$ is a minimal standard basis for $\mathcal{O}$. We give below the monoids of values Γ, with their respective conductors c (obtained possibly by inspection), according to the values of $\mathrm{char}(K)$.

	Γ	c
$\mathrm{char}(K) \neq 2$	$\langle 8, 10, 43 \rangle$	66
$\mathrm{char}(K) = 2$	$\langle 8, 10, 61 \rangle$	84

EXAMPLE 6 Suppose $\mathrm{char}(K) = 0$, and let the branch C given parametrically by

$$\begin{cases} x = T^8 + aT^{13} \\ y = T^{10} + T^{15}. \end{cases}$$

Taking $F_0 = \{x, y\}$, the only minimal S-process to be considered in the first step of the algorithm is $z = y^4 - x^5$. The possible final reductions modulo F_0, according to the values of the parameter a, are

$$z \begin{cases} = (4 - 5a)T^{45} + (6 - 10a^2)T^{50} + (4 - 10a^3)T^{55} + \\ \qquad + (1 - 5a^4)T^{60} - a^5T^{65}, \quad \text{if } a \neq \frac{4}{5} \\ \xrightarrow{\frac{2}{5}y^5} \frac{22}{25}T^{55} + \frac{369}{125}T^{60} + \frac{11576}{3125}T^{65} + 2T^{70} + \frac{2}{5}T^{75}, \\ \qquad \text{if } a = \frac{4}{5}. \end{cases}$$

In any case, there will not exist any relevant minimal S-process to be considered in the next step. In this way, the algorithm ends giving the minimal standard basis $F_1 = \{x, y, z\}$. We list below the possible monoids Γ and their conductors c.

	Γ	c
$a \neq \frac{4}{5}$	$\langle 8, 10, 45 \rangle$	68
$a = \frac{4}{5}$	$\langle 8, 10, 55 \rangle$	78

It is interesting to note that the monoid Γ of the branch C depends on the values of the coefficient a of the parametrization. This may only happen if the parametrization is not Puiseux, since for a Puiseux parametrization, Zariski observed that Γ is independent of its coefficients (see Théorème 3.9 of [Z4]).

EXAMPLE 7 Suppose $\mathrm{char}(K) = 0$ and let C be given by the following Puiseux parametrization

$$\begin{cases} x = T^8 \\ y = T^{10} + T^{13} \\ z = T^{12} + aT^{15}. \end{cases}$$

Taking $F_0 = \{x, y, z\}$ and using the procedure we descibed above to determine an upper bound c_0 for the conductor c, we get $c_0 = 58$, independently from the value of a.

The minimal S-processes and their final reductions, modulo F_0, are respectively:

$$y^4 - x^5 = 4T^{43} + 6T^{46} + 4T^{49} + T^{52}.$$

$$y^2 - xz = \begin{cases} (2-a)T^{23} + T^{26}, & \text{if } a \neq 2 \\ \\ T^{26} \xrightarrow{x^2y} -T^{29}, & \text{if } a = 2. \end{cases}$$

$$z^2 - x^3 = \begin{cases} 2aT^{27} + a^2T^{30}, & \text{if } a \neq 0 \\ \\ 0, & \text{if } a = 0. \end{cases}$$

$$y^2z - x^4 = \begin{cases} (2+a)T^{35} + (1+2a)T^{38} + aT^{41}, & \text{if } a \neq -2 \\ \\ -3T^{38} - 2T^{41} \xrightarrow{-3xy^3} 7T^{41} + 9T^{44} + 3T^{47}, & \text{if } a = -2. \end{cases}$$

$$z^3 - x^2y^2 = \begin{cases} (3a-2)T^{39} + (3a^2-1)T^{42} + a^3T^{45}, & \text{if } a \neq \frac{2}{3} \\ \frac{1}{3}T^{42} + \frac{8}{27}T^{45} \xrightarrow{\frac{1}{3}x^4y} -\frac{1}{27}T^{45}, & \text{if } a = \frac{2}{3}. \end{cases}$$

$$z^4 - y^6 = \begin{cases} 4(a-1)T^{51} + 6(a^2-1)T^{54} + 4(a^3-1)T^{57} + \\ \qquad + (a^4-1)T^{60}, & \text{if } a \neq 1 \\ 0, & \text{if } a = 1. \end{cases}$$

We now analyze all possibilities according to the values of a.

If $a = 0$, then $F_1 = \{T^8, T^{10}+T^{13}, T^{12}, 2T^{23}+T^{26}\}$ is a minimal standard basis for $\mathcal{O}$. In this case, $\Gamma = \langle 8, 10, 12, 23\rangle$ and $c = 38$.

If $a = 2$, then $F_1 = \{T^8, T^{10} + T^{13}, T^{12} + 2T^{15}, 4T^{27} + 4T^{30}, -T^{29}\}$ is a minimal standard basis for $\mathcal{O}$. In this case, $\Gamma = \langle 8, 10, 12, 27, 29\rangle$ and $c = 34$.

If $a \neq 0$ and $a \neq 2$, then

$$F_1 = \{T^8, T^{10} + T^{13}, T^{12} + aT^{15}, (2-a)T^{23} + T^{26}, 2aT^{27} + a^2T^{30}\}$$

is a minimal standard basis for $K[[T^8, T^{10} + T^{13}, T^{12} + aT^{15}]]$. In this case, $\Gamma = \langle 8, 10, 12, 23, 27\rangle$ and $c = 30$.

This example illustrates the known fact that the monoid of values Γ of a space branch doesn't depend exclusively from the terms of a parametrization of the branch, but also depends on the coefficients of the parametrization, even when the parametrization is a Puiseux set, in contrast with the plane case as observed in Example 6.

As we saw in Theorem 4.1, standard bases for algebras solve the membership problem for subalgebras. This is illustrated below.

EXAMPLE 8 Suppose $\text{char}(K) = 0$ and let C be the branch of Example 5 above. In this case, $F = \{x, y, z\}$ is a minimal standard basis for $\mathcal{O}$, where $x = T^8$, $y = T^{10} + T^{13}$ and $z = 4T^{43} + 6T^{46} + 4T^{49} + T^{52}$.

We want to decide whether the series $g = 4T^{43} - 14T^{49} - 17T^{52} - 6T^{55}$ and $h = 4T^{43} - 13T^{49} - 17T^{52} - 6T^{55}$ belong or not to $\mathcal{O}$.

To make such a decision, it is enough to compute a final reduction of g and h modulo the set F. Indeed, we have

$$g \xrightarrow{z} -6T^{46} - 18T^{49} - 18T^{52} - 6T^{55} \xrightarrow{6x^2y^3} 0$$

$$h \xrightarrow{z} -6T^{46} - 17T^{49} - 18T^{52} - 6T^{55} \xrightarrow{6x^2y^3} T^{49}.$$

Therefore, $g \in \mathcal{O}$ and $h \notin \mathcal{O}$.

The examples 4,5,6 and 7 above show how to compute Γ for branches given in parametric form. When the branch is given by a Cartesian representation, it is also possible to make the computations as we will see below.

Let C be a branch in $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$, given by a Cartesian representation $f_1, \ldots, f_r$. As we commented above, the value of an element $\bar{g} \in \mathcal{O}$ is the codimension in $K[[\mathbf{X}]]$ of the ideal $I = \langle g, f_1, \ldots, f_r \rangle$. This may be computed by means of a standard basis for the ideal I.

Let $F = \{\bar{h}_1, \ldots, \bar{h}_s\}$ be a subset of $\mathcal{M} \setminus \{0\}$, and let $H = \{h_1, \ldots, h_s\} \subset K[[\mathbf{X}]]$. Since we know how to compute values of elements in $\mathcal{O}$, we may determine all minimal solutions of the diophantine equation associated to the equality $v(F^\alpha) = v(F^\beta)$, where $\alpha, \beta \in \mathbb{N}^{\#F}$. Now, to produce the minimal S-process of F, associated to a minimal solution (α, β) of the diophantine equation, we must find the unique $a \in K$ such that

$$v(F^\alpha - aF^\beta) > v(F^\alpha) = v(F^\beta).$$

The constant a may be determined computing a standard basis for the ideal $\langle f_1, \ldots, f_r, H^\alpha - aH^\beta \rangle$ and determiming for which value of a one has

$$dim_K \frac{K[[\mathbf{X}]]}{\langle f_1, \ldots, f_r, H^\alpha - aH^\beta \rangle} > dim_K \frac{K[[\mathbf{X}]]}{\langle f_1, \ldots, f_r, H^\alpha \rangle}.$$

In the same way one can perform the reduction of an element of $\mathcal{O}$ modulo F. This is all we need to apply the algorithm to get a standard basis for $\mathcal{O}$.

Notice that since we only can control values of elements, we may eventually determine a minimal standard basis for but not a reduced standard basis for the ring of coordinates of a curve C given by a Cartesian representation.

EXAMPLE 9 Suppose that $\mathrm{char}(K) = 0$ and let C be the plane branch defined by $f(X,Y) = Y^8 - 4X^3Y^6 - 8X^5Y^5 + (6X^6 - 26X^7)Y^4 + (16X^8 - 24X^9)Y^3 + (-4X^9 + 36X^{10} - 20X^{11})Y^2 + (-8X^{11} + 16X^{12} - 8X^{13})Y + X^{12} + 6X^{13} + 21X^{14} - X^{15}$.

If $\bar{g} \in \mathcal{O}$, it is well known (see Theorem 21.19 in [Sei]) that

$$v(\bar{g}) = \dim_K \frac{K[[X,Y]]}{\langle f, g \rangle} = \mathrm{ord}_X(R_Y(f,g)),$$

where $R_Y(f,g)$ is the resultant with respect to Y of f and g.

We apply the algorithm to obtain a standard basis for $\mathcal{O}$, starting with $F_0 = \{x, y\}$. Since $v(x) = 8$ and $v(y) = 12$, we take $c_0 = \infty$.

The only nontrivial minimal S-process in this step is of the form $y^2 - ax^3$. Since

$$R_Y(f, Y^2 - aX^3) = (a-1)^8 X^{24} + (a-1)^4(-52a^2 + 8a + 12)X^{25} +$$
$$+(-40a^5 + 494a^4 - 1256a^3 + 500a^2 + 480a + 78)X^{26} + \cdots,$$

the unique choice for a to get an S-process is $a = 1$, and in this way $v(y^2 - x^3) = 26$.

For next step we take $F_1 = \{x, y, z = y^2 - x^3\}$. Since the monoid $\langle v(F_1)\rangle = \langle 8, 12, 26\rangle \subset v(\mathcal{O})$ doesn't have conductor, we take $c_1 = \infty$, and therefore analyze all the minimal S-processes of F_1. The resultants of each of these S-processes and f are:

$$R_Y(f, Z^2 - a_1X^2Y^3) = (a_1 - 4)^8 X^{52} + (18a_1^8 + 632a_1^7 - 13156a_1^6 + 17248a_1^5 + $$
$$+457280a_1^4 - 10782772a_1^3 - 2153472a_1^2 - 1417216a_1 - 16384)X^{53} + \cdots$$

$$R_Y(f, Z^2 - a_2X^5Y) = (a_2 - 4)^8 X^{52} + (6a_2^8 - 128a_2^7 + 769a_2^6 - 2336a_2^5 +$$
$$+11840a_2^4 - 130048a_2^3 - 384000a_2^2 - 1122304a_2 - 80384)X^{53} + \cdots$$

$$R_Y(f, XZ^2 - a_3Y^5) = (a_3 - 4)^8 X^{60} + (30a_3^8 + 4272a_3^7 - 44260a_3^6 - 149536a_3^5 +$$
$$+2287168a_3^4 - 3451904a_3^3 - 7003136a_3^2 - 1712128a_3 - 16384)X^{61} + \cdots$$

$$R_Y(f, YZ^2 - a_4X^8) = (a_4 - 4)^8 X^{64} + (-120a_4^7 - 9956a_4^6 + 42592a_4^5 +$$
$$+9792a_4^4 - 590848a_4^3 + 1614848a_4^2 - 2138112a_4 + 376832)X^{65} + \cdots$$

$$R_Y(f, Z^4 - a_5XY^8) = (a_5 - 16)^8 X^{104} + \cdots$$

$$R_Y(f, Z^4 - a_6X^{13}) = (a_6 - 16)^8 X^{104} + \cdots$$

$$R_Y(f, Z^6 - a_7Y^{13}) = (a_7 - 64)^8 X^{156} + \cdots$$

Note that in order to the above expressions define S-processes, we have to take $a_1 = a_2 = a_3 = a_4 = 4$, $a_5 = a_6 = 16$ and $a_7 = 64$. But in this way $v(z^2 - 4x^5y) = 53$, and since the conductor of the monoid $\langle 8, 12, 26, 53\rangle$ is 84, we may disregard the last three S-processes because their values exceed 84.

A simple verification shows that in the next step there will not be any non-trivial S-process with height less than 84.

It then follows that $F_2 = \{x, y, z = y^2 - x^3, w = z^2 - 4x^5y, u_1 = z^2 - 4x^2y^3, u_2 = xz^2 - 4y^5, u_3 = yz^2 - 4x^8\}$ is a standard basis for $\mathcal{O}$.

Since $v(u_1) = v(w)$, $v(u_2) = v(xw)$ and $v(u_3) = v(yw)$, we have that $\{x, y, z, w\}$ is a minimal standard basis for $\mathcal{O}$. So the monoid of values of C is $\Gamma = \langle 8, 12, 26, 53\rangle$, and $c = 84$.

The above method may be used without restriction for any branch (not necessarily plane, and in any characteristic).

The minimal standard bases of the local ring $\mathcal{O}$ of a branch C depend upon the Cartesian representation of C we start with, but the set of values of the elements of a minimal standard basis for C doesn't, as we observed immediately after Proposition 4.2. This implies that the monoid of values of C depends only upon C, and not upon the Cartesian representations of C.

5.3 Plane Branches

In this section, C will be a plane branch with local ring of coordinates $\mathcal{O}$ and monoid of values Γ, with conductor c. Since a plane branch may be defined by a single power series $f \in K[[X,Y]]$, it makes sense to talk about Milnor and Tjurina numbers associated to f (cf. Section 3.2). Are these numbers only dependent on C, or depend on the particular Cartesian representation f of C? We will see below that these numbers are intrinsically associated to C.

Let $g = uf$, where u is a unit in $K[[X,Y]]$. The ideals $\langle f_X, f_Y\rangle$ and $\langle g_X, g_Y\rangle$ may be different but nevertheless, we have by a topological result of Milnor [Mil], later proved algebraically, in characteristic zero, by Risler [Ri], that

$$\mu(f) = \dim_K \frac{K[[X,Y]]}{\langle f_X, f_Y\rangle} = c.$$

Since Γ, and hence c, is intrinsically associated to C, then the above equation shows that $\mu(f)$ depends only upon C, and will be called the *Milnor number of* C and denoted by $\mu(C)$.

On the other hand, it is an easy exercise to show that

$$\langle f, f_X, f_Y\rangle = \langle g, g_X, g_Y\rangle,$$

and therefore,

$$\tau(f) = \dim_K \frac{K[[X,Y]]}{\langle f, f_X, f_Y\rangle} = \dim_K \frac{K[[X,Y]]}{\langle g, g_X, g_Y\rangle} = \tau(g),$$

allowing us to define $\tau(C) = \tau(f)$, which will be called the *Tjurina number of* C.

We have seen in the last section that it is possible to determine a minimal standard basis for the ring $\mathcal{O}$. However, in the case of plane branches, and when char(K) = 0, Zariskí and Azevedo and independently Abhyankar and Moh, also described methods to determine minimal standard bases for $\mathcal{O}$. Such methods will be presented below and compared with our algorithm.

From Remark 1, we may look at $\mathcal{O}$ as a Puiseux ring

$$K[[x,y]] = K[[T^{v_0}, \sum_{i \geq v_0} a_i T^i]] \subset K[[T]],$$

where v_0 is the multiplicity of C.

Let us define, for $j \in \mathbb{N}$, the following integers:

$$\begin{aligned} &\beta_0 = e_0 = v_0,\ n_0 = 1, \\ &\beta_j = \min\{i;\ a_i \neq 0, \text{ and } e_{j-1} \not| \, i\}, \\ &e_j = \mathrm{GCD}(e_{j-1}, \beta_j), \\ &n_j = \tfrac{e_{j-1}}{e_j}. \end{aligned}$$

From Proposition 1, it follows that there exists $g \geq 1$, such that $e_g = 1$. The sequence $(\beta_0, \beta_1, \ldots, \beta_g)$ is called the *characteristic sequence* of C.

ZARISKI-AZEVEDO'S METHOD

Let ω be a primitive v_0-th root of 1 and let φ_α the K-automorphism of $K[[T]]$ defined by $\varphi_\alpha(T) = \omega^\alpha T$.

If $y = \sum_{i \geq v_0} a_i T^i$, we define

$$Z_k = \sum_{i < \beta_k} a_i X^{\frac{i}{v_0}},$$

for $k = 2, \ldots, g$. Let

$$P_k(Y) = \prod_{\alpha=1}^{n_0 n_1 \ldots n_{k-1}} (Y - \varphi_\alpha(Z_k)) \in K[[X]][Y].$$

Zariski, in [Z4], and Azevedo, in [Az], showed that $\{X, Y, P_k(Y);\ k = 2, \ldots, g\}$ are series whose values are the minimal system of generators of the monoid $\Gamma = v(\mathcal{O} \setminus \{0\})$ of the branch.

ABHYANKAR-MOH'S METHOD

From the Weierstrass Preparation Theorem (see Theorem 2.1), the branch C may be defined by an irreducible serie of the form

$$f(X,Y) = Y^{v_0} + \sum_{i=1}^{v_0} A_i(X) Y^{v_0 - i},$$

where $A_i(X) \in K[[X]]$, and $A_i(0) = 0$, for $i = 1, \ldots, v_0$.

Let $d \in \mathbb{N}$ be such that $d \mid v_0$. We define the *d-th root's approximation* of f as being the series

$$\sqrt[d]{f} = Y^{\frac{v_0}{d}} + \sum_{i=1}^{\frac{v_0}{d}} C_i(X) Y^{\frac{v_0}{d}-i},$$

where the $C_i(X) \in K[[X]]$ are obtained in a unique way by the relations

$$A_i(X) = dC_i(X) + \sum_{\sum_{j=1}^{i-1} j\delta_j = i} \alpha_{\delta_1,\ldots,\delta_{i-1}} C_1(X)^{\delta_1} \cdot \cdots \cdot C_{i-1}(X)^{\delta_{i-1}},$$

$i = 1, \ldots, \frac{v_0}{d}$, where

$$\alpha_{\delta_1,\ldots,\delta_{i-1}} = \binom{d}{\delta_1 + \cdots + \delta_{i-1}} \frac{(\sum_{j=1}^{i-1} \delta_j)!}{\prod_{j=1}^{i-1} (\delta_j!)}.$$

Abhyankar and Moh, in [AM], proved the existence and the unicity of the root's approximations of f, and that the values of X and of $\sqrt[e_{k-1}]{f}$, $k = 1, \ldots, g$ give the minimal system of generators of the monoid Γ of the branch.

COMPARING METHODS

Although our algorithm, the result of Zariski-Azevedo and that of Abhyankar-Moh give us minimal standard bases for $\mathcal{O}$, these bases are not necessarily the same.

Consider the following example:

$$C : \begin{cases} x = T^8 \\ y = T^{12} + T^{14} + T^{15}. \end{cases}$$

A Cartesian representation of C is

$$f(X,Y) = Y^8 - 4X^3Y^6 - 8X^5Y^5 + (6X^6 - 26X^7)Y^4 + (16X^8 - 24X^9)Y^3 + (-4X^9 + 36X^{10} - 20X^{11})Y^2 + (-8X^{11} + 16X^{12} - 8X^{13})Y + X^{12} + 6X^{13} + 21X^{14} - X^{15}.$$

By Example 9, our algorithm give the minimal standard basis

$$F = \{x, y, z = y^2 - x^3, w = z^2 - 4x^5y = (y^3 - x^3)^2 - 4x^5y\},$$

while the method of Abhyankar-Moh gives

$$F = \{x, y, z = y^2 - x^3, w_1 = z^2 - 4x^5y - 13x^7\}$$

and the method of Zariski-Azevedo,

$$F = \{x, y, z = y^2 - x^3, w_2 = z^2 - 4x^5y - x^7\}.$$

REMARK 2 If one is only interested in determining the monoid of values Γ of a plane branch in characteristic zero, there are other efficient ways to do it.

If the branch is given parametrically, then Zariski in [Z4] shows that the minimal set of generators of Γ is given by $v_0, \ldots, v_g$ is terms of the characteristic sequence $\beta_0, \ldots, \beta_g$, as follows:

$$v_0 = \beta_0 \quad v_{i+1} = n_i v_i + \beta_{i+1} - \beta_i, \;\; i = 0, \ldots, g-1. \tag{5.3}$$

If the branch C is given by a Cartesian representation, then Dickenstein and Sessa in [DS] show that $\beta_0 = v_0 = v(X)$, $\beta_1 = v(f_Y^{(v_0-1)})$, and

$$\beta_{j+1} = v(f_Y^{(e_j-1)}) - v(f_Y^{(e_j)}), \;\; j = 1, \ldots, g-1,$$

where

$$f = Y^{v_0} + \sum_{i=1}^{v_0} A_i(X) Y^{v_0 - i},$$

is a Weierstrass polynomial representing C and $f_Y^{(r)}$ is the rth partial derivative of f with respect to Y. Now, the generators of Γ may be computed by formulas (5.3).

Chapter 6

STANDARD BASES FOR SUBMODULES

In this chapter we will present the theory of standard bases in the following context of modules.

Let A be a complete subalgebra of $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$ and let M be a complete A-submodule of $K[[\mathbf{X}]]$. What is the corresponding notion of standard basis for M as an A-module, and how one may compute such objects?

In the context of polynomial rings and when M is an ideal of A, Miller in [M] introduces the concept of SG-Basis to answer such questions. Here we generalize Miller's result to the case of M a submodule, and extend it to the context of formal power series rings.

From this point on, A will denote a complete subalgebra of $K[[\mathbf{X}]]$ with a given standard basis G, and M a complete A-submodule of $K[[\mathbf{X}]]$ (for example, a finitely generated A-submodule of $K[[\mathbf{X}]]$).

6.1 Standard Bases for Modules

In what follows, we will assume that the monomial order we are working with in $\mathbf{T}^n$ is the monomial order that was used to compute the standard basis G of A. Recall that the monomial order must have the finiteness property when we reduce modulo infinite sets.

In our setting we have that $\mathrm{lp}(M)$ is an $\mathrm{lp}(A)$-monomodule.

DEFINITION Suppose that the module M is generated by a set H. We say that H is a *standard basis* for the A-module M, if the $\mathrm{lp}(A)$-monomodule $\mathrm{lp}(M)$ is generated by $\mathrm{lp}(H)$.

In other words, H is a standard basis for M, if for all $m \in M \setminus \{0\}$ we have

$$\mathrm{lp}(m) = \mathrm{lp}(ag),$$

for some $a \in A$ and some $g \in H$.

Since G is a standard basis for A, we may write any $a \in A$ as $a = \sum_\alpha b_\alpha F^\alpha_\alpha$, where $b_\alpha \in K$, F^α_α is a G-product and the sum has amplitude one (this follows readily from the reduction process of the elements of A modulo a standard basis). If F^α_α is the G-product with minimal order in the above sum, then $\mathrm{lp}(a) = \mathrm{lp}(F^\alpha_\alpha)$. Therefore we may rephrase our definition as follows.

A set H of generators of M over A is a standard basis for M, if for all $m \in M \setminus \{0\}$,

$$\mathrm{lp}(m) = \mathrm{lp}(F^\alpha_\alpha g),$$

for some G-product F^α_α and some $g \in H$.

In analogy with standard bases for ideals and for algebras, we will define a final reduction of an element of $K[[\mathbf{X}]]$ modulo a subset H of M, possibly infinite, which makes sense in the module context.

We say that $h \in K[[\mathbf{X}]]$ *reduces to* r *modulo* (H, G) if there exist $b \in K$, a G-product F^α and $g \in H$ such that

$$r = h - bF^\alpha g,$$

with $\mathrm{lp}(r) > \mathrm{lp}(h)$, whenever $r \neq 0$.

In this case we write

$$h \xrightarrow{(H,G)} r,$$

and say that r is a *reduction* of h modulo (H, G).

When r is obtained from h through a chain (possibly infinite) of reductions, modulo (H, G), and cannot be reduced further, we say that r is a *final reduction* of h modulo (H, G), and will write

$$h \xrightarrow{(H,G)+} r.$$

Note that since M is complete we have that $h - r \in M$, for all r obtained from h by any chain of reductions. Recall that to have a well defined final reduction when H is infinite we must take an order in T^n which has the finiteness property.

As in the previous contexts, a final reduction r will be called a *complete reduction*, if for all $t \in \mathrm{T}(r)$, we cannot reduce t modulo (H, G).

We will now introduce the central notion of S-process in the present context.

DEFINITION Let A be a subalgebra of $K[[\mathbf{X}]]$ with a given standard basis G. An *S-process* of a pair of elements g, h in $K[[\mathbf{X}]]$ over G is an expression of the form

$$S(g,h) = aF^{\alpha}g + bF^{\beta}h,$$

where $a, b \in K$ and F^{α}, F^{β} are G-products, such that

$$\mathrm{lp}(S(g,h)) > \min\{\mathrm{lp}(F^{\alpha}g), \mathrm{lp}(F^{\beta}h)\} = \mathrm{ht}(aF^{\alpha}g + bF^{\beta}h),$$

whenever $S(g,h) \neq 0$.

The following theorem will guarantee the existence of standard bases for modules and at the same time will give several characterizations for them. The proof we will give is very similar in spirit to that of Theorems 2.3 and 4.1, but will have to be adapted due to the differences between the notions of S-processes and reductions in each context. It will not be omitted for the sake of completeness.

THEOREM 1 (STANDARD BASES FOR MODULES IN $K[[\mathbf{X}]]$) *Let A be a complete subalgebra of $K[[\mathbf{X}]]$ with a standard basis G.*
1) *Every complete A-submodule M of $K[[\mathbf{X}]]$ has a standard basis.*
2) *Given a non-empty subset H of $K[[\mathbf{X}]]$, such that the A-submodule M generated by H is complete, the following assertions are equivalent:*
(a) *H is a standard basis for M.*
(b) *All final reduction, modulo (H,G), of any element of M is zero.*
(c) *H is closed under S-processes; that is, every S-process of any pair of elements of H over G has a vanishing final reduction modulo (H,G).*
(d) *Every non-zero S-process of any pair of elements of H over G has a representation of the form $\sum_{\alpha} b_{\alpha}F_{\alpha}^{\alpha}h_{\alpha}$, where $b_{\alpha} \in K$, F_{α}^{α} is a G-product and $h_{\alpha} \in H$, with height greater than the height of the S-process itself.*

PROOF **Proof of 1:** This is obvious because M is a standard basis for itself.
Proof of 2: (a)$\Longrightarrow$ (b) Suppose that H is a standard basis for M. Let $m \in M$ and r a final reduction of m modulo (H,G), then from the definition

of reduction it follows that $r \in M$. If $r \neq 0$, since H is a standard basis, there exist a G-product F^α and $h \in H$ such that $\text{lp}(r) = \text{lp}(F^\alpha h)$, which contradicts the fact that r is a final reduction modulo (H, G). Hence $r = 0$.
(b)$\Longrightarrow$ (c) Let s be an S-process of a pair of elements of H over G. Since $s \in M$, it has a zero final reduction modulo (H, G).
(c)$\Longrightarrow$ (d) Let $s = aF^\alpha h + bF^\beta g \neq 0$ be an S-process of the pair of elements h, g of H over G, with some vanishing final reduction modulo (H, G). Recall that any reduction of an element f consists of subtracting from f a scalar multiple of a G-product times an element of H, getting an element which is either zero or has leading power greater than the leading power of f. Therefore, since s has a zero final reduction, we may write

$$s = \sum_\gamma c_\gamma L_\gamma^\gamma h_\gamma,$$

where the L_γ^γ are G-products, $c_\gamma \in K$ and $h_\gamma \in H$, with amplitude one.

It then follows that

$$\text{ht}(\sum_\gamma c_\gamma L_\gamma^\gamma h_\gamma) = \text{lp}(s) > \text{ht}(aF^\alpha h + bF^\beta g),$$

proving the assertion.
(d)$\Longrightarrow$ (a) It will be sufficient to prove that every element $m \in M \setminus \{0\}$ has a representation of the form $\sum_\alpha b_\alpha F_\alpha^\alpha h_\alpha$, where $b_\alpha \in K$, F_α^α is a G-product and $h_\alpha \in H$, with amplitude one. This is equivalent to say that for every element $m \in M \setminus \{0\}$ there exist a G-product F^α and $h \in H$ such that $\text{lp}(m) = \text{lp}(F^\alpha h)$, which means that H is standard basis for M.

Let $m \in M \setminus \{0\}$ and take a representation

$$m = \sum_\delta a_\delta F_\delta^\delta h_\delta, \tag{6.1}$$

where $a_\delta \in K$, the F_δ^δ are G-products and $h_\delta \in H$, of least amplitude λ, among those representations with maximum height.

Suppose by absurd that $\lambda \geq 2$. Suppose that $\text{lp}(a_\alpha F_\alpha^\alpha h_\alpha)$ and $\text{lp}(a_\beta F_\beta^\beta h_\beta)$ are equal to the height of the sum in (6.1).

Therefore, there exists $b \in K$ such that

$$s := a_\alpha F_\alpha^\alpha h_\alpha + b a_\beta F_\beta^\beta h_\beta \tag{6.2}$$

is an S-process of the pair h_α, h_β in H, over G.

Since by hypothesis, there exist $c_\gamma \in K$, G-products L_γ^γ and elements g_γ in H, such that

$$s = \sum_\gamma c_\gamma L_\gamma^\gamma g_\gamma, \tag{6.3}$$

with

$$\mathrm{ht}(\sum_\gamma c_\gamma L_\gamma^\gamma g_\gamma) = \mathrm{lp}(s) > \mathrm{ht}(a_\alpha F_\alpha^\alpha h_\alpha + b a_\beta F_\beta^\beta h_\beta).$$

Now, from (6.2) and (6.3), we have

$$a_\alpha F_\alpha^\alpha h_\alpha + a_\beta F_\beta^\beta h_\beta = (1-b) a_\beta F_\beta^\beta h_\beta + \sum_\gamma c_\gamma L_\gamma^\gamma g_\gamma.$$

So we can write

$$m = (1-b) a_\beta F_\beta^\beta h_\beta + \sum_\gamma c_\gamma L_\gamma^\gamma g_\gamma + \sum_{\delta \neq \alpha, \beta} a_\delta F_\delta^\delta h_\delta. \tag{6.4}$$

Suppose that $b \neq 1$. Then the representation of m given in (6.4) will have the same height as the representation (6.1), but a smaller amplitude, which is a contradiction.

Suppose that $b = 1$. If the amplitude of the representation (6.1) of m were two, then the height of the representation (6.4) would be greater than the height of the representation (6.1), which is a contradiction. If the amplitude of representation (6.1) of m were greater than two, then the height of the representation (6.4) would be equal to the height of representation (6.1), but the amplitude would be smaller, which again is a contradiction.

□

In the same manner we did for subalgebras and ideals, we will define below minimal and reduced standard bases for modules.

DEFINITION Let A be a subalgebra of $K[[\mathbf{X}]]$ with standard basis G, and let M be a complete A-submodule of $K[[\mathbf{X}]]$ with standard basis H. The Basis H will be called a *minimal standard basis* for M, if for all $h \in H$ we have $\mathrm{lp}(h) \neq \mathrm{lp}(F^\alpha g)$ for all $g \in H \setminus \{h\}$, and all G-product F^α.

DEFINITION A minimal standard basis H of M will be called *reduced* if all element $g \in H$ are monic and for all $t \in \mathbf{T}(g) \setminus \{\mathrm{lp}(g)\}$, we have $t \neq \mathrm{lp}(F^\alpha h)$ for all $h \in H$ and all G-product F^α.

Since the following results are analogous to the corresponding results for subalgebras, with identical proofs (Propositions 2.2 and 2.3), these will be omitted.

PROPOSITION 1 *Let A be a subalgebra of $K[[\mathbf{X}]]$ with standard basis G, and let M be a complete A-submodule of $K[[\mathbf{X}]]$ with standard basis H. If $h \in H$ is such that* $\text{lp}(h) = \text{lp}(F^\alpha g)$ *for some $g \in H$ and some G-product F^α, then $H' = H \setminus \{h\}$ is also a standard basis for M.*

PROPOSITION 2 *Let A be a subalgebra of $K[[\mathbf{X}]]$ with standard basis G. Any complete A-submodule M of $K[[\mathbf{X}]]$ has a unique reduced standard basis*[1].

6.2 The Analog of Buchberger's Algorithm

In this section we will present the analog of Buchberger's algorithm to determine the standard bases for complete A-submodules of $K[[\mathbf{X}]]$, where A is a subalgebra of $K[[\mathbf{X}]]$ with a given standard basis G. For this purpose we are going to describe more explicitely the S-processes of pairs of elements.

An S-process $aF^\alpha g + bF^\beta h$ over G of a pair of elements g, h in $K[[\mathbf{X}]]$, where $F = \{f_1, \ldots, f_s\} \subset G$, is determined, up to a scalar multiple, by a vector $(\alpha, \beta) \in \mathbb{N}^{2s}$, that is a solution of the system

$$\begin{array}{l}\sum_{i=1}^{s} \alpha_i \text{ord}_{X_j}(\text{lp}(f_i)) + \text{ord}_{X_j}(\text{lp}(g)) = \\ \sum_{i=1}^{s} \beta_i \text{ord}_{X_j}(\text{lp}(f_i)) + \text{ord}_{X_j}(\text{lp}(h)); \quad j = 1, \ldots, n.\end{array} \tag{6.5}$$

We denote by D'_F the minimal set of solutions of system (6.5), and by D_F the minimal set of solutions of the associated homogeneous system (see Example 1.12). The S-processes of the pair g, h associated to the elements in D'_F will be called the *minimal S-processes* of the pair g, h, relative to the finite set F.

DEFINITION A *minimal S-process* of the pair $g, h \in K[[\mathbf{X}]]$ over G is a minimal S-process relative to some finite subset F of G.

Here again, as in the case of subalgebras, in virtue of Example 1.4, we have that if $F \subset E \subset G$ and E is finite, then any minimal S-process relative to F is also a minimal S-process relative to E.

The Proposition below will be crucial for the algorithm to compute standard bases for modules.

PROPOSITION 3 *Let A be a given subalgebra of $K[[\mathbf{X}]]$, with a standard basis G. Let M be an A-submodule of $K[[\mathbf{X}]]$ generated by a set H. Suppose that*

[1] Note that the uniqueness depends on fixing a given monomial order in $\mathbf{T}^n$. If we change order, we may find different reduced bases.

for every finite subset F of G, and every pair of elements g, h in H, each minimal S-process of g, h relative to F, has a zero final reduction modulo (H, G). Then every S-process of any pair of elements of H, over G, has a representation of the form $\sum_\lambda b_\lambda F_\lambda^\lambda h_\lambda$, where $b_\lambda \in K$, F_λ^λ is a G-product and $h_\lambda \in H$, with height greater than the height of the S-process itself.

PROOF Modulo a scalar multiple, we may assume that an S-process of a pair $g, h \in H$ over G is given by $F^\alpha g + aF^\beta h$, where F^α and F^β are G-products and $a \in K$ is uniquely determined.

Let $S_{\delta,\theta} = F^\delta g + a_{\delta,\theta} F^\theta h$ be a minimal S-process of the pair $g, h \in H$ relative to F. Since $S_{\delta,\theta}$ has a zero final reduction modulo (H, G), from the proof of Theorem 1, (c) $\Longrightarrow$ (d), we may write

$$S_{\delta,\theta} = \sum_\gamma c_\gamma F_\gamma^\gamma h_\gamma,$$

where $c_\gamma \in K$, F_γ^γ is a G-product, $h_\gamma \in H$ and $\mathrm{ht}(\sum_\gamma c_\gamma F_\gamma^\gamma h_\gamma) > \mathrm{lp}(F^\delta g) = \mathrm{lp}(F^\theta h) = \mathrm{ht}(F^\delta g + aF^\theta h)$.

Let $S = F^\alpha g + aF^\beta h$ be any non-zero S-process of the pair g, h, relative to F. Since the solution (α, β) of the system (6.5) may be written in the form

$$(\alpha, \beta) = (\delta, \theta) + \sum_{\rho,\epsilon} n_{\rho,\epsilon}(\rho, \epsilon),$$

where (δ, θ) is a minimal solution of the system (6.5), (ρ, ϵ) is a minimal solution of the associated homogeneous system and $n_{\rho,\epsilon}$ is a non-negative integer, we may write

$$F^\alpha g = \left(\prod_{\rho,\epsilon} (F^\rho)^{n_{\rho,\epsilon}}\right) F^\delta g,$$

$$F^\beta h = \left(\prod_{\rho,\epsilon} (F^\epsilon)^{n_{\rho,\epsilon}}\right) F^\theta h.$$

Note that there exists a constant $d \in K$, such that

$$\prod_{\rho,\epsilon} (F^\rho)^{n_{\rho,\epsilon}} + d \prod_{\rho,\epsilon} (F^\epsilon)^{n_{\rho,\epsilon}}$$

is an S-process of G. Now, using the same argument we used in the proof of Theorem 4.1 (c) $\Longrightarrow$ (d), we may write

$$\prod_{\rho,\epsilon} (F^\rho)^{n_{\rho,\epsilon}} = -d \prod_{\rho,\epsilon} (F^\epsilon)^{n_{\rho,\epsilon}} + \sum_\nu e_\nu F_\nu^\nu,$$

with $e_\nu \in K$, the F_ν^ν are G-products, and $\mathrm{ht}(\sum_\nu e_\nu F_\nu^\nu) > \mathrm{lp}(\prod_{\rho,\epsilon}(F^\rho)^{n_{\rho,\epsilon}}) = \mathrm{lp}(\prod_{\rho,\epsilon}(F^\epsilon)^{n_{\rho,\epsilon}})$.

Therefore,

$$\begin{aligned} F^\alpha g &= \left(-d\prod_{\rho,\epsilon}(F^\epsilon)^{n_{\rho,\epsilon}} + \sum_\nu e_\nu F_\nu^\nu\right) F^\delta g = \\ &= \left(-d\prod_{\rho,\epsilon}(F^\epsilon)^{n_{\rho,\epsilon}} + \sum_\nu e_\nu F_\nu^\nu\right)\left(-a_{\delta,\theta}F^\theta h + \sum_\gamma c_\gamma F_\gamma^\gamma h_\gamma\right) = \\ &= da_{\delta,\theta}\left(\prod_{\rho,\epsilon}(F^\epsilon)^{n_{\rho,\epsilon}}\right) F^\theta h + \sum_\lambda b_\lambda F_\lambda^\lambda h_\lambda = \\ &= da_{\delta,\theta}F^\beta h + \sum_\lambda b_\lambda F_\lambda^\lambda h_\lambda, \end{aligned}$$

where $\mathrm{ht}(\sum_\lambda b_\lambda F_\lambda^\lambda h_\lambda) > \mathrm{lp}(F^\alpha g) = \mathrm{lp}(F^\beta h)$.

Since the constant a is uniquely determined, it follows that $a = -da_{\delta,\theta}$ and therefore,

$$S = \sum_\lambda b_\lambda F_\lambda^\lambda h_\lambda.$$

□

As a consequence of Proposition 3 and Theorem 1, and analogously to the corollary of Proposition 4.3, we have the following result:

COROLLARY *Let A be a subalgebra of $K[[\mathbf{X}]]$ with standard basis G and let M be a complete A-submodule of $K[[\mathbf{X}]]$ generated by a set H. Then H is a standard basis for M if and only every minimal S-process of H over G has a vanishing final reduction modulo (H, G).*

REMARK 1 Note that if we take the pair $h, h \in H$, a minimal S-process of these elements over G, is not necessarily trivial, because the system (6.5) being homogeneous may have non-trivial minimal solutions. However, a non-trivial S-process of h, h is of the form $S = aF^\alpha h + bF^\beta h = (aF^\alpha + bF^\beta)h$, where $a, b \in K$ and F^α and F^β are G-products.

Since $aF^\alpha + bF^\beta \in A$ and G is a standard basis for A, we have

$$aF^\alpha + bF^\beta \xrightarrow{G+} 0.$$

In this way, if we reduce $aF^\alpha + bF^\beta$ by $c_\gamma F_\gamma^\gamma$, we can reduce S by $c_\gamma F_\gamma^\gamma h$. So, $S \xrightarrow{(H,G)+} 0$.

REMARK 2 If A is a subalgebra of $K[[\mathbf{X}]]$ with standard basis G, then a complete A-module M has a finite standard basis if and only if the $\mathrm{lp}(A)$-monomodule $\mathrm{lp}(M)$ is finitely generated over $\mathrm{lp}(A)$; or more explicitely,

$\text{lp}(M)$ is finitely generated over the monoid $\langle \text{lp}(G) \rangle$. This means that there exist $g_1, \ldots, g_r \in M$, such that for all $m \in M$, we have that $\text{lp}(m) = \text{lp}(F^\alpha g_i)$, for some G-product F^α and some i.

As an application of Remark 2 and from Proposition 1.3, we have that any complete A-module M, where $M \subset K[[X]]$ and A is a K-subalgebra of $K[[X]]$, has a finite standard basis.

In what follows we will present the analog of Buchberger's algorithm for modules.

THEOREM 2 (ANALOG OF BUCHBERGER'S ALGORITHM) *Let A be a subalgebra of $K[[\mathbf{X}]]$ with standard basis G. If M is a complete A-module generated by $B \subset K[[\mathbf{X}]]$, then we always obtain (theoretically) a standard basis H for M by the following algorithm.*

> *INPUT:* G, B;
> *DEFINE:* $H_0 := B$ *and* $i := 0$;
> *DO*
> $S := \{s;\ s$ *is a minimal* S*-process of* H_i *over* $G\}$;
> $R := \{r;\ s \xrightarrow{(H_i,G)+} r$ *and* $r \neq 0,\ \forall s \in S\}$;
> $H_{i+1} := H_i \cup R$;
> $i := i + 1$;
> *OUTPUT:* $H = \cup_{i \geq 0} H_i$.

Moreover, if M has a finite standard basis, then the above procedure will produce one such basis, after finitely many steps.

PROOF Consider $H = \cup_{i \geq 0} H_i$. A minimal S-process of a pair $g, h \in H$ over G is, in particular, a minimal S-process of a pair of elements of H_i for some i.

By the algorithm, this S-process has a vanshing final reduction, modulo (H_{i+1}, G), and consequentely also modulo (H, G). Hence H is a standard basis for M.

Suppose that M has a finite standard basis L. We will show that there exists an index j such that H_j is a standard basis for M.

Let

$$H_0 \subset H_1 \subset \cdots \subset H_i \subset \cdots \subset H.$$

While $H_i \setminus H_{i-1} \neq \emptyset$, we take $r_i \in H_i \setminus H_{i-1}$ wich is the final reduction, modulo (H_{i-1}, G), of least order of a minimal S-process of a pair of elements of H_{i-1} over G. From the algorithm, we have that $\mathrm{lp}(r_i) < \mathrm{lp}(r_{i+1})$.

Let $q = \max\{\mathrm{lp}(h);\ h \in L\}$. There exists an index j such that the leading power of any element of $P := H \setminus H_j$ is greather than q.

If $h \in L$, then $\mathrm{lp}(h) = \mathrm{lp}(F^\alpha g)$ where F^α is a G-product and $g \in H$.

Since $\mathrm{lp}(h) \leq q$, we have that $g \in H \setminus P$, that is, $g \in H_j$. Hence, for all $h \in L$ we have that $\mathrm{lp}(h) = \mathrm{lp}(F^\alpha_\alpha g)$ for some G-product F^α_α and $g \in H_j$.

In this way, given an element $m \in M$, we have that $\mathrm{lp}(m) = \mathrm{lp}(F^\beta_\beta h) = \mathrm{lp}(F^\beta_\beta F^\alpha_\alpha g) = \mathrm{lp}(F^\gamma_\gamma g)$, where $F^\alpha_\alpha, F^\beta_\beta, F^\gamma_\gamma = F^\beta_\beta F^\alpha_\alpha$ are G-products, $h \in L$ and $g \in H_j$.

Hence H_j is a standard basis for M.

□

EXAMPLE 1 Consider the subalgebra $A = K[[X^4, X^9 + X^{15}]] \subset K[[X]]$, whose reduced standard basis is $G = \{X^4, X^9 + X^{15}\}$.

If $a \in A$, then $\mathrm{ord}_X(a) \in \langle 4, 9 \rangle$. Moreover, if $f \in K[[X]]$ is such that $\mathrm{ord}_X(f) \geq 24$, then $f \in A$ (because the conductor of $\langle 4, 9 \rangle$ is precisely 24).

Consider the A-module M generated by $B = \{X^3, X^8 - X^{14}\}$ (hence complete, because B is finite). We already know that M has a finite standard basis which we are going to determine.

Note that given $f \in K[[X]]$, we have:

If $\mathrm{ord}_X(f) \geq 27$, then $\mathrm{lp}(f) = \mathrm{lp}(X^3)\mathrm{lp}(a)$, for some $a \in A$.

If $\mathrm{ord}_X(f) = 26$, then $\mathrm{lp}(f) = \mathrm{lp}(X^8 - X^{14})\mathrm{lp}((X^9 + X^{15})^2)$.

If $\mathrm{ord}_X(f) \in \{23, 24, 25\}$, then $\mathrm{lp}(f) = \mathrm{lp}(X^3)\mathrm{lp}(a)$, for some $a \in A$.

Hence, if $\mathrm{ord}_X(f) \geq 23$, then $f \in M$, because f can be reduced modulo (B, G) to an element of M.

We apply the above algorithm to $G = \{X^4, X^9 + X^{15}\}$ and $H_0 = B = \{X^3, X^8 - X^{14}\}$.

Step 1:

We have that S is the union of the sets of minimal S-processes over G of the pairs (X^3, X^3), $(X^8 - X^{14}, X^8 - X^{14})$ and $(X^3, X^8 - X^{14})$.

From Remark 1 we see that for the first two pairs, the S-processes reduce to zero. For the third pair, modulo multiplicative constants, we have two S-processes. The first is

$$s_1 = (X^4)^8 X^3 - (X^9 + X^{15})^3 (X^8 - X^{14}),$$

with $\mathrm{ord}_X(S_1) > 35$, which obviously will have a zero final reduction modulo (H_0, G).

The second S-process is

$$s_2 = (X^9 + X^{15})X^3 - X^4(X^8 - X^{14}) = 2X^{18}.$$

If $\mathrm{char}(K) = 2$, then the algorithm stops.

If $\mathrm{char}(K) \neq 2$, we proceed to the next step of the algorithm with $H_1 := \{X^3, X^8 - X^{14}, 2X^{18}\}$.

Step 2:

Beside the minimal S-processes computed in the previous step, which will have zero reductions we also have the S-processes over G of the pairs $(2X^{18}, 2X^{18})$, $(X^3, 2X^{18})$ and $(X^8 - X^{14}, 2X^{18})$. The first pair will give no relevant S-processes. The second and third pairs have respectively the following sets of minimal S-processes:

$$\{2(X^4)^6X^3 - (X^9 + X^{15})(2X^{18}), (X^9 + X^{15})^3X^3 - (X^4)^3(2X^{18})\}$$

and

$$\{2(X^4)^7(X^8 - X^{14}) - (X^9 + X^{15})^2(2X^{18}),\\ 2(X^9 + X^{15})^2(X^8 - X^{14}) - (X^4)^2(2X^{18})\}.$$

Since the heights of these S-processes have orders greater than 23, they will have zero final reductions, modulo (H_1, G), and consequently the algorithm terminates.

Therefore the set

$$H := H_1 = \{X^3, X^8 - X^{14}, 2X^{18}\}.$$

is a minimal standard basis of M, in any characteristic.

Chapter 7

EQUIVALENCE OF PLANE BRANCHES

In this chapter we will apply the theory of standard bases for modules, developed in Chapter 6, to obtain numerical invariants with respect to an important equivalence relation among plane branches (finer than equisingularity), which we will define later. In the context of analytic geometry, this equivalence relation is the isomorphism between embedded germs of plane irreducible curves. Our main invariant will be the set of positive integers Λ corresponding to the values of elements of the module of Kähler differentials on the curve. This invariant has been considered by several people and was difficult to compute until the introduction of the methods from the theory of standard bases for modules in formal power series rings that we are presenting here. This set Λ was already considered by Zariski [Z2] and by Berger [Ber]. In 1967, Azevedo in his thesis [Az] uses the set Λ to study the Jacobian ideal of an algebroid irreducible plane curve, stating a conjecture about the maximum value of the Tjurina invariant in an equisingularity class. This conjecture was shown recently by Heinrich (see [He]) to be false. In 1978, Delorme in [D2] presented an algorithm to compute Λ for the generic curve in an equisingularity class associated to a monoid of values of genus one. More recently, Peraire, in [Pe], determined by other methods a set which, after some interpretation (not given in her paper), furnishes the set Λ for the generic curve in an arbitrary equisingularity class, extending Delorme's result. Our contribution in this direction is the development of an algorithm that allows to compute Λ for any given algebroid irreducible curve.

7.1 Kähler Differentials on Branches

Let C be an algebroid irreducible curve in $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$, where K is an algebraically closed field of characteristic zero. So, C is a prime ideal in $K[[\mathbf{X}]]$ such that the local ring $\mathcal{O} = K[[\mathbf{X}]]/C$ has Krull dimension one.

DEFINITION The *module of Kähler differentials* over $\mathcal{O}$ is the $\mathcal{O}$-module

$$\mathcal{O}d\mathcal{O} = \frac{\mathcal{O}^n}{\langle \sum_{i=1}^n e_i f_{X_i};\ f \in C\rangle},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathcal{O}^n$.

We will denote by dx_i the image of e_i in $\mathcal{O}d\mathcal{O}$, for $i = 1, \ldots, n$. Therefore, the elements dx_i, $i = 1, \ldots, n$, are non free generators of $\mathcal{O}d\mathcal{O}$ as $\mathcal{O}$-module. Indeed, we have the following relations

$$\sum_{i=1}^n f_{X_i} dx_i = 0, \ \ \forall f \in C.$$

We have the universal K-derivation map

$$\begin{array}{rccl} d: & \mathcal{O} & \longrightarrow & \mathcal{O}d\mathcal{O}. \\ & g & \mapsto & dg = \sum_{i=1}^n g_{X_i} dx_i \end{array}$$

REMARK 1 If $C = \langle f_1, \ldots, f_r\rangle$, then using the rule of the derivative of a product we have

$$\mathcal{O}d\mathcal{O} = \frac{\mathcal{O}^n}{\langle \sum_{i=1}^n (f_1)_{X_i} e_i, \ldots, \sum_{i=1}^n (f_r)_{X_i} e_i\rangle}.$$

REMARK 2 The $\mathcal{O}$-module $\mathcal{O}d\mathcal{O}$ has, in general, a non trivial torsion submodule $\mathcal{T}$. In fact, there is a famous conjecture due to Berger [Ber] which asserts that the local ring of an algebroid curve is regular if and only if its module of Kähler differentials is torsion free. This conjecture is known to be true when the curve is a complete intersection (and in several other cases), but it remains open in general.

We give below an example of a curve whose torsion submodule $\mathcal{T}$ is non trivial.

EXAMPLE 1 Consider the algebroid plane curve C defined by $f(X,Y) = Y^r - X^s$ with $\min\{r,s\} > 1$ and $\mathrm{GCD}(r,s) = 1$. If $\omega = rxdy - sydx$, we have that $\omega \neq 0$, but

$$\begin{aligned} y^{r-1}\omega &= rxy^{r-1}dy - sy^r dx = \\ &= rxy^{r-1}dy - sx^s dx = \\ &= x(ry^{r-1}dy - sx^{s-1}dx) = \\ &= x(f_Y dy - f_X dx) = 0, \end{aligned}$$

showing that $\omega \in \mathcal{T}$. In fact, in this situation we have that

$$\mathcal{T} = \sum_{\substack{0 \leq i \leq s-2 \\ 0 \leq j \leq r-2}} x^i y^j \omega \mathcal{O}.$$

Let $C = \langle f_1, \ldots, f_r \rangle$ be an algebroid irreducible curve with local ring $\mathcal{O}$ Recall the definition of the monomorphism φ from $\mathcal{O}$ to $K[[T]]$, given in (5.2), which maps x_i into $p_i(T) \in K[[T]]$. Consider the $\mathcal{O}$-modules homomorphism

$$\begin{array}{rccc} \psi : & \mathcal{O}d\mathcal{O} & \longrightarrow & K[[T]], \\ & \sum_{j=1}^n g_j dx_j & \mapsto & \sum_{j=1}^n \varphi(g_j)\frac{dp_j(T)}{dT} \end{array} \tag{7.1}$$

where the structure of $\mathcal{O}$-module on $K[[T]]$ is induced by the map φ.

PROPOSITION 1 *Let $C = \langle f_1, \ldots, f_r \rangle$ be a curve as above. The kernel of the homomorphism ψ is the torsion submodule $\mathcal{T}$ of $\mathcal{O}d\mathcal{O}$.*

PROOF Let $\omega \in \mathcal{T}$. There exists an element $g \in \mathcal{O} \setminus \{0\}$ such that $g\omega = 0$. Therefore,

$$0 = \psi(g\omega) = \varphi(g)\psi(\omega).$$

Now, since φ is injective, $g \neq 0$ and $K[[T]]$ is a domain, it follows that $\psi(\omega) = 0$ and therefore $\omega \in \ker(\psi)$.

Conversely, let $\omega = \sum_{j=1}^n g_j dx_j \in \ker(\psi)$; that is,

$$\sum_{j=1}^n \varphi(g_j)\frac{dp_j(T)}{dT} = 0. \tag{7.2}$$

Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$. We have that the $r \times n$ matrix

$$\Big((f_i)_{X_j}\Big)_{i,j}$$

has rank $n-1$ over $\mathcal{K}$ (see for example [Ma], Remark on page 192, adapted to the power series context, and recalling that $\text{char}(K) = 0$).

Select $n-1$ linearly independent rows of the above matrix, which we will suppose to be the first $n-1$ ones, and form the $n \times n$ matrix M obtained by adjoining the following last row

$$(g_1, \ldots, g_n).$$

Now, note that, for $i = 1, \ldots, r$, we have that

$$\sum_{j=1}^{n} \varphi((f_i)_{X_j}) \frac{dp_j(T)}{dT} = \psi(\sum_{j=1}^{n} (f_i)_{X_j} dx_j) = \psi(0) = 0. \tag{7.3}$$

Since the system of linear equations over $K((T))$ given by (7.2) and (7.3) has the non trivial solution

$$\left(\frac{dp_1(T)}{dT}, \ldots, \frac{dp_n(T)}{dT} \right),$$

and φ is injective, it follows that $\det M = 0$. But as the $n-1$ first rows of M are linearly independent, we have that the last one is a linear combination, over $\mathcal{K}$, of the others. This implies that there exist $h \in \mathcal{O} \setminus \{0\}$ and $h_1, \ldots, h_{n-1} \in \mathcal{O}$ such that

$$h \sum_{j=1}^{n} g_j dx_j = \sum_{i=1}^{n-1} h_i \sum_{j=1}^{n} (f_i)_{X_j} dx_j = 0,$$

proving the result.

□

From the above proposition, we have that

$$\frac{\mathcal{O}d\mathcal{O}}{\mathcal{T}} \cong \operatorname{Im}(\psi).$$

Since the theory of standard bases of modules was developed for submodules of $K[[\mathbf{X}]]$, we will make our computations relative to $\mathcal{O}d\mathcal{O}$ in the $\mathcal{O}$-submodule $\operatorname{Im}(\psi)$ (via φ) of $K[[T]]$.

DEFINITION If $\omega \in \mathcal{O}d\mathcal{O} \setminus \mathcal{T}$, then we define the *value* of ω as

$$v(\omega) = v(\psi(\omega)) + 1,$$

where the second v above is the canonical discrete valuation of $\overline{\mathcal{O}}(= K[[T]])$.

Since we are working in $K[[T]]$ we have that the subalgebra $\varphi(\mathcal{O})$ ($\cong \mathcal{O}$) has a finite standard basis (this follows from the definitions and from Example 1.6). On the other hand, the $\mathcal{O}$-submodule $\mathrm{Im}(\psi)$ of $K[[T]]$ also has a finite standard basis (this follows from Proposition 1.3 and the definitions)

DEFINITION We will say that a differential $\omega \in \mathcal{O}d\mathcal{O}$ is an *exact differential*, if there exists $g \in \mathcal{O}$ such that $\omega = dg$. If this is not the case, we say that ω is a *non exact differential* (NED).

The K-vector space of all exact differentials will be denoted by $d\mathcal{O}$.

REMARK 3 Let Γ be the monoid of values of the curve C and c its conductor. If $\omega \in \mathcal{O}d\mathcal{O}$ is an exact differential, then $v(\omega) \in \Gamma$. Equivalently, if $v(\omega) \notin \Gamma$ (that is, $v(\omega)$ is a gap for Γ), then ω is a NED.

On the other hand, if $v(\omega) \geq c$, then ω is an exact differential. Indeed, since $v(\omega) \geq c$, there exists $h_1 \in \mathcal{O}$ such that $v(\omega) = v(dh_1)$. Hence there exists $c_1 \in K$ such that $v(\omega - c_1 dh_1) > v(\omega)$. In the same way we get recursively a summable family $\{c_i dh_i;\ i \geq 1\}$ such that

$$\omega = \sum_{i \geq 1} c_i dh_i = d(\sum_{i \geq 1} c_i h_i) \in d\mathcal{O}.$$

We define

$$\Lambda = v(\mathcal{O}d\mathcal{O} \setminus \{0\}) := v(\mathrm{Im}(\psi) \setminus \{0\}).$$

Note that for all $h \in \mathcal{O}$ we have that $v(dh) = v(h)$. This in particular implies that $\Gamma \subset \Lambda$.

7.2 Standard Bases for $\mathcal{O}d\mathcal{O}$

In this section we will improve the algorithm, we presented in Chapter 6 (Theorem 6.2), to compute standard bases for the $\mathcal{O}$-submodule $\mathrm{Im}(\psi)$ of $K[[T]]$. Such standard bases will be referred to as standard bases for $\mathcal{O}d\mathcal{O}$. In order to do so, we make some remarks.

Let $B \subset K[[T]]$ and let G be a standard basis of algebras for $\mathcal{O}$. Recall from Remark 6.1 that any minimal S-process over G of a pair g, g in B has a zero reduction modulo (B, G), and doesn't need to be considered in the algorithm of Theorem 6.2.

The algorithm of Theorem 6.2 starts with a generator set of the module for which we want to compute a standard basis. In the particular case of $\mathcal{O}d\mathcal{O}$ we take for example $\{dx_1, \dots, dx_n\}$ as a set of generators.

We may improve the algorithm starting instead with the following set of generators: $\{dh;\ h$ in a minimal standard basis of $\mathcal{O}\}$. This will avoid some unnecessary computations and at the same time will allow more reductions at each step of the algorithm, eliminating eventually some steps.

Besides the above economy in the algorithm, we may use the concept of greatest gap to eliminate some irrelevant S-processes, as we show below.

DEFINITION The greatest integer l such that $l \notin \Lambda$ is called the *greatest gap* of Λ (note that one always has $l \notin \Gamma$ and $l \leq c-1$, where c is the conductor of Γ).

Let l be the greatest gap of Λ. In a given step i of the algorithm of Theorem 6.2, consider the set

$$\Lambda_i = \{v(G^\alpha\omega); \omega \in H_i \text{ and } G^\alpha \text{ is a } G\text{-product}\},$$

and denote by l_i its greatest gap, which is obviously greater or equal than l. Since every minimal S-process over G with height greater or equal to T^{l_i} has a zero final reduction modulo (H_i, G), it can be neglected.

In this way, we get the following improvement of the algorithm to compute standard bases for $\mathcal{O}d\mathcal{O}$, starting with a minimal standard basis G of $\mathcal{O}$.

ALGORITHM FOR STANDARD BASIS FOR $\mathcal{O}d\mathcal{O}$

INPUT: G;
DEFINE: $H_{-1} := \emptyset$;
  $H_0 := \{dh;\ h \in G\}$ and $i := 0$;
WHILE $H_i \neq H_{i-1}$ DO
  $\Lambda_i := \{v(G^\alpha\omega);\ \omega \in H_i$ and G^α is a G-product$\}$;
  $l_i :=$ greatest gap of Λ_i;
  $S := \{s;\ s$ is a non trivial minimal S-process of H_i over G with $v(\mathrm{ht}(s)) < l_i$, not computed in the previous step $\}$;
  $R := \{r;\ s \overset{(H_i,G)+}{\longrightarrow} r\ \ \forall s \in S$ and $r \neq 0\}$;
  $H_{i+1} := H_i \cup R$;
  $i := i+1$;
OUTPUT: $H := H_i, \Lambda := \Lambda_i, l := l_i$.

REMARK 4 The above algorithm computes exclusively NED. The NED belonging to a minimal standard basis of $\mathcal{O}d\mathcal{O}$ will be called *minimal non-exact differentials* , or simply MNED.

REMARK 5 Berger, in [Ber], proved that the length $l(\mathcal{T})$ of the torsion submodule $\mathcal{T}$ of $\mathcal{O}d\mathcal{O}$, for a complete intersection curve C is given by

$$l(\mathcal{T}) = c - \#(\Lambda \setminus \Gamma),$$

or equivalently

$$l(\mathcal{T}) = \frac{c}{2} + \#(\mathbb{N} \setminus \Lambda),$$

where c is the conductor of the monoid Γ of C.

So, our algorithm allows to compute $l(\mathcal{T})$ for complete intersections. On the other hand, Zariski in [Z2], Theorem 1, shows that for a plane branch C, we have

$$l(\mathcal{T}) = \tau(C).$$

REMARK 6 As in the algorithm to compute standard bases for algebras, it will be convenient to establish a uniform way to perform reductions. For example, if it is possible to reduce a given element using $G^\alpha g$ or $G^\beta h$, where $g, h \in H$, we will make the reduction using $G^\alpha g$ when $v(g) < v(h)$. We still use $G^\alpha g$ when $v(g) = v(h)$ and G^α is chosen instead of G^β in the reduction process used for the construction of the standard basis of $\mathcal{O}$ (see Remark 4.1).

EXAMPLE 2 Let C be the monomial curve

$$C : \begin{cases} x_1 &= T^{s_1} \\ \vdots & \vdots \\ x_n &= T^{s_n}. \end{cases}$$

From Example 5.4 we know that $G = \{x_1, \ldots, x_n\}$ is a standard basis for $\mathcal{O}$. Now, it follows immediately that $H = \{dx_1, \ldots, dx_n\}$ is a standard basis for $\mathcal{O}d\mathcal{O}$.

EXAMPLE 3 Let

$$C : \begin{cases} x = T^8 \\ y = T^{10} + T^{13} \\ z = T^{12} + T^{15}. \end{cases}$$

From Example 5.7, we have that $G = \{x = T^8, y = T^{10} + T^{13}, z = T^{12} + T^{15}, u = T^{23} + T^{26}, w = 2T^{27} + T^{30}\}$ is a minimal standard basis for $\mathcal{O}$ and $\Gamma = \langle 8, 10, 12, 23, 27\rangle$ is the monoid of values of C, whose conductor is $c = 30$.

Note that Γ is not symmetric, so C is not Gorenstein and therefore it is not a complete intersection.

We now apply the algorithm to $H_0 = \{dx, dy, dz, du, dw\}$. We have that $l_0 = c - 1 = 29$ is the greatest gap of Λ_0 and the set of minimal S-processes of H_0 over G, to be considered is

$$S := \{4xdy - 5ydx, 4ydy - 5zdx, 6ydy - 5xdz, 2xdz - 3zdx,\\ 5ydz - 6zdy, 2zdz - 3x^2dx, 2x^2dz - 3y^2dx\}.$$

Computing the final reductions of the above S-processes modulo (H_0, G), we have

$$\begin{array}{lll} 4xdy - 5ydx = \omega_1 & \Rightarrow & \psi(\omega_1) = 12T^{20} \\ 5ydz - 6zdy = \omega_2 & \Rightarrow & \psi(\omega_2) = -3T^{24} - 3T^{27}. \end{array}$$

Since $v(\omega_1) = 21$, $v(\omega_2) = 25$ and $v(x\omega_1) = 29$, the greatest gap of H_1 is $l_1 = 19$, which allows to neglect the other S-processes.

A simple analysis shows that there are no extra minimal S-processes to be considered in the next step. So, the algorithm terminates giving the following minimal standard basis

$$H = H_1 = \{dx, dy, dz, du, dw, \omega_1, \omega_2\}$$

for $\mathcal{O}d\mathcal{O}$.

We also get

$$\Lambda = \Lambda_1 = \{0, 8, 10, 12, 16, 18, 20, \ldots\}.$$

EXAMPLE 4 Consider

$$C : \begin{cases} x = T^6 \\ y = T^8 + 2T^9 \\ z = T^{10} + T^{11}. \end{cases}$$

Applying the algorithm to compute standard bases for algebras we get the following minimal standard basis for $\mathcal{O}$:

$$F = \{x, y, z, w = xz - y = -3T^{17} - 4T^{18}, u = y - z - x^3 = 3T^{19} + 2T^{20}\}.$$

The monoid of values of C is then $\Gamma = \langle 6, 8, 10, 17, 19 \rangle$, whose conductor is $c = 22$.

Applying the algorithm above, starting with $H_0 = \{dx, dy, dz, dw, du\}$ whose greatest gap is $l_0 = c - 1 = 21$, we have the following set of minimal S-processes:

$S = \{3xdy - 4ydx, 3ydy - 4x^2dx, 4xdz - 5ydy, 3xdz - 5zdx, 3zdy - 4x^2dx$

$$4ydz - 5zdy, 3ydz - 5x^2dx, 3zdz - 5xydx, 4zdz - 5x^2dy\}.$$

Since

$$3xdy - 4ydx = \omega \;\Rightarrow\; \psi(\omega) = 6T^{14},$$

we have that $v(\omega) = 15$ e $v(x\omega) = 21$. So, the greatest gap of H_1 is $l_1 = 13$, showing that the other S-processes reduce to zero modulo (H_1, G) and also that there are no further S-processes to be analyzed in the following steps. Therefore the algorithm stops giving the following minimal standard basis for $\mathcal{O}d\mathcal{O}$:

$$H = H_1 = \{dx, dy, dz, dw, du, \omega\}.$$

This example is interesting because C is a complete intersection (see [HK]) but Γ is not a complete intersection monoid (see definitions in Chapter 5).

Therefore, by Remark 5 we have

$$l(T) = c - \#(\Lambda \setminus \Gamma) = 22 - \#\{15, 21\} = 20.$$

EXAMPLE 5 Consider

$$C : \begin{cases} x = T^8 \\ y = T^{12} + T^{13}. \end{cases}$$

Using the algorithm to compute standard bases for subalgebras, we get the following minimal standard basis for $\mathcal{O}$:

$$G = \{x, y, z = y^2 - x^3 = 2T^{25} + T^{26}\},$$

whose associated monoid of values is $\Gamma = \langle 8, 12, 25 \rangle$, with conductor $c = 80$.

Applying the above algorithm starting with $H_0 = \{dx, dy, dz\}$, we have

Step 1:

The greatest gap in this step is $l_0 = 79$, and the set of minimal S-processes to be considered is

$S := \{3ydx - 2xdy, 8ydy - 12x^2dx, 8xdz - 25zdx,$
$\quad 12ydz - 25zdy, 25x^2zdx - 8y^2dz, 25yzdy - 12x^3dz\}.$

Computing the final reduction of the elements of S modulo (H_0, G), we have

$3ydx - 2xdy = \omega_1 \;\Rightarrow\; \psi(\omega_1) = -2T^{20}.$
$8ydy - 12x^2dx \xrightarrow{-4dz} 0.$
$8xdz - 25zdx = \omega_2 \;\Rightarrow\; \psi(\omega_2) = 8T^{33}.$
$12ydz - 25zdy = \omega_3 \;\Rightarrow\; \psi(\omega_3) = -38T^{37} - 13T^{38}.$
$25x^2zdx - 8y^2dz \xrightarrow{\frac{202}{25}zdz} \omega_4 \;\Rightarrow\; \psi(\omega_4) = \frac{204}{25}T^{50} + \frac{52}{25}T^{51}.$
$25yzdy - 12x^3dz \xrightarrow{-\frac{619}{50}zdz} \frac{3}{2}\omega_4 \;\Rightarrow\; \psi(\frac{3}{2}\omega_4) = \frac{306}{25}T^{50} + \frac{78}{25}T^{51}.$

Step 2:

Since $v(z\omega_2) = 59$, $v(z\omega_3) = 63$, $v(xz\omega_1) = 67$, $v(z^2\omega_1) = 71$ and $v(xz^2\omega_1) = 79$, the greatest gap of

$$H_1 := \{dx, dy, dz, \omega_1, \omega_2, \omega_3, \omega_4\}$$

is $l_1 = 55$.

The minimal S-processes of H_1, over G, which have not been analyzed in step 1 and whose heights are less than T^{55}, are

$zdx + 8y\omega_1, xdz + 25y\omega_1, zdy + 12x^2\omega_1, ydz + 25x^2\omega_1, yzdx + 8x^3\omega_1,$
$xzdy + 12y^2\omega_1, 2z\omega_1 + y\omega_2, 4x\omega_3 - 38z\omega_1, 19y\omega_2 + 4x\omega_3, 4y\omega_3 + 19x^2\omega_2.$

Computing the final reductions modulo (H_1, G), we get

$zdx + 8y\omega_1 \xrightarrow{\omega_2} 0,$
$xdz + 25y\omega_1 \xrightarrow{3\omega_2} 0,$
$zdy + 12x^2\omega_1 \xrightarrow{\omega_3} 0,$
$ydz + 25x^2\omega_1 \xrightarrow{2\omega_3} 0,$
$yzdx + 8x^3\omega_1 \xrightarrow{6z\omega_1} \omega_5 \;\Rightarrow\; \psi(\omega_5) = -4T^{46},$
$xzdy + 12y^2\omega_1 \xrightarrow{-\frac{5}{2}z\omega_1} \frac{3}{2}\omega_5 \;\Rightarrow\; \psi(\frac{3}{2}\omega_5) = -6T^{46}.$

We have that $v(z\omega_1) = 46, v(\omega_5) = 47, v(xz\omega_1) = 54$ and $v(x\omega_5) = 55$. So, the greatest gap of

$$H_2 := \{dx, dy, dz, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$$

is $l_2 = 43$, and consequently we may neglect the other above S-processes and may go to the next step.

Step 3:

Since in this step we have to compute the minimal S-processes involving only ω_5 and $v(\omega_5) > l_2$, the algorithm stops, giving the standard basis $H := H_2$ for $\mathcal{O}d\mathcal{O}$.

Since C is a complete intersection, from Remark 5 it follows that

$$\begin{aligned} l(\mathcal{T}) &= c - \#(\Lambda \setminus \Gamma) = \\ &= 80 - \#\{21, 29, 34, 38, 42, 46, 47, 51, 54, 55, 59, 63, 67, 71, 79\} = \\ &= 80 - 15 = 65. \end{aligned}$$

The above example is taken from [Az] page 79, where using rudimental methods the NDE are computed, but the existence of a differential with value $v(\omega_4)$ wasn't detected, leaving the example incomplete.

Although $\Gamma \subset \Lambda$, it is in general false that two curves having the same set Λ, will have same monoid of values. Here is an example:

EXAMPLE 6 Let C_1 and C_2 given by

$$C_1 : \begin{cases} x = T^6 \\ y = T^{14} + T^{17} \\ z = T^{39} \end{cases} \qquad C_2 : \begin{cases} x = T^6 \\ y = T^{14} + T^{33} \\ z = T^{23}. \end{cases}$$

Applying the algorithm to compute standard bases for subalgebras we get the following monoids of values:

$$\Gamma(C_1) = \langle 6, 14, 39 \rangle \text{ and } \Gamma(C_2) = \langle 6, 14, 23 \rangle.$$

The algorithm presented in this section gives the standard bases below for the modules of Kähler differentials for the curves under consideration, which have the following unified expression:

$$H = \{dx, dy, dz, \omega\},$$

where

$$\omega = xdy - \frac{14}{6} ydx = \begin{cases} 3T^{22} & \text{for } C_1 \\ 19T^{38} & \text{for } C_2. \end{cases}$$

In this way,

$$\Lambda(C_1) = \Lambda(C_2) = \{0, 6, 12, 14, 18, 20, 23, 24, 26, 28, 29, 30, 32, 34, \ldots\},$$

while $\Gamma(C_1) \neq \Gamma(C_2)$.

7.3 Complete Intersection Curves

In this section we will assume that C is a complete intersection algebroid irreducible curve in $K[[\mathbf{X}]] = K[[X_1, \ldots, X_n]]$; that is, C has a Cartesian representation $f_1, \ldots, f_{n-1}$.

Suppose that $v(X_1) = v_0 = mult(C)$. We have the following relations

$$\sum_{j=1}^{n} (f_i)_{X_j} dx_j = 0, \quad \text{for } i = 1, \ldots, n-1,$$

or equivalently,

$$M_1 \begin{pmatrix} dx_2 \\ \vdots \\ dx_n \end{pmatrix} = - \begin{pmatrix} (f_1)_{X_1} \\ \vdots \\ (f_{n-1})_{X_1} \end{pmatrix} dx_1,$$

where

$$M_1 = \begin{pmatrix} (f_1)_{X_2} & \cdots & (f_1)_{X_n} \\ \vdots & \ddots & \vdots \\ (f_{n-1})_{X_2} & \cdots & (f_{n-1})_{X_n} \end{pmatrix}. \tag{7.4}$$

Carbonne, in [Car] page 376, has shown that $|M_1| = \det(M_1) \in \mathcal{O} \setminus \{0\}$ and $|M_i| = \det(M_i) \in \mathcal{O} \setminus \{0\}$, where

$$M_i = \begin{pmatrix} (f_1)_{X_2} & \cdots & (f_1)_{X_{i-1}} & -(f_1)_{X_1} & (f_1)_{X_{i+1}} & \cdots & (f_1)_{X_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (f_{n-1})_{X_2} & \cdots & (f_{n-1})_{X_{i-1}} & -(f_{n-1})_{X_1} & (f_{n-1})_{X_{i+1}} & \cdots & (f_{n-1})_{X_n} \end{pmatrix} \tag{7.5}$$

for $i = 2, \ldots, n$.

In this way, over the field of fractions $\mathcal{K}$ of $\mathcal{O}$, we have

$$dx_i = \frac{|M_i|}{|M_1|} dx_1 \tag{7.6}$$

for $i = 1, \ldots, n$.

REMARK 7 Herzog and Kunz, in [HK], proved that if C is a complete intersection curve, then

$$v(|M_1|) = c + v_0 - 1,$$

where c is the conductor of the monoid of values of C and v_0 is the multiplicity of C.

So, for $i = 1, \ldots, n$, we have

$$v\left(\frac{|M_i|}{\psi(dx_i)}\right) = c - 1.$$

In the case of plane curves, this was shown by Azevedo in [Az] Proposition 5, page 17 and by Zariski in [Z4], page 11.

Using the relation (7.6), and the fact that

$$v\left(\frac{f}{g}\right) = \dim_K \frac{K[[\mathbf{X}]]}{\langle f_1, \ldots, f_{n-1}, f\rangle} - \dim_K \frac{K[[\mathbf{X}]]}{\langle f_1, \ldots, f_{n-1}, g\rangle}$$

for any hypersurfaces $f, g \in K[[\mathbf{X}]]$, we may apply the algorithm to obtain a standard basis for $\mathcal{O}d\mathcal{O}$ as $\mathcal{O}$-module, even when the curve C is given by a Cartesian representation $f_1, \ldots, f_{n-1}$.

EXAMPLE 7 Let C be an algebroid irreducible plane curve given by

$$f(X,Y) = Y^3 + X^5Y + X^7.$$

Since $\dim_K \frac{K[[X,Y]]}{\langle f,X\rangle} = 3$ and $\dim_K \frac{K[[X,Y]]}{\langle f,Y\rangle} = 7$, we have that $G = \{X, Y\}$ is a minimal standard basis for $\mathcal{O}$, hence $\Gamma = \langle 3, 7\rangle$ is the monoid of values associated to C, with conductor $c = 12$.

As C is a plane curve, and therefore a complete intersection, we have from (7.4) and (7.5) that

$$|M_1| = f_Y \quad \text{and} \quad |M_2| = -f_X.$$

We will now apply the algorithm to compute a standard basis of $\psi(\mathcal{O}d\mathcal{O})$.

Start with $H_0 = \{dx, dy\}$, $\Lambda_0 = \Gamma$, whose greatest gap is $l_0 = c - 1 = 11$. The only minimal S-process to be considered is of the form $S = xdy - aydx$. Using (7.6) which in this case reads $dy = -\frac{f_X}{f_Y}dx$, we have

$$S = \left(-x\frac{f_X}{f_Y} - ay\right)dx = (-xf_X - ayf_Y)\frac{dx}{f_Y}.$$

Since $v(\overline{g}) = \dim_K \frac{K[[X,Y]]}{\langle f,g\rangle} = \text{ord}_X(R_Y(f,g))$ for $g \in K[[X,Y]]$, we may determine a in such a way that S is effectively an S-process; that is, $v(S) > v(xdy) = v(ydx) = 9$.

As

$$R_Y(f, -Xf_X - aYf_Y) = (3a-7)^3X^{21} + (a-2)(2a-5)^2X^{22},$$

the only possible choice for a is $a = \frac{7}{3}$; that is, $S = xdy - \frac{7}{3}ydx$ and hence

$$v(S) = \text{ord}_X \left(R_Y \left(f, -Xf_X - \frac{7}{3}Yf_Y \right) \right) + v\left(\frac{dx}{f_Y}\right) = 22 - 11 = 11.$$

In this way we have $\omega = xdy - \frac{7}{3}ydx$ with $v(\omega) = 11$.

Since in the next step the greatest gap is $l_1 = 8$, there are no further S-processes to be considered. Therefore, the algorithm terminates giving the following minimal standard basis for $\mathcal{O}d\mathcal{O}$:

$$H = H_1 = \{dx, dy, \omega\}.$$

Moreover, $l(\mathcal{T}) = c - \#(\Lambda \setminus \Gamma) = 12 - 1 = 11$.

7.4 Equivalence of Plane Branches

A central question in the theory of plane branches is the classification problem modulo the equivalence relation we define below.

DEFINITION Let C_1 and C_2 be two plane branches, given by the Cartesian representations f_1 and f_2 in $K[[X,Y]]$, respectively. We will say that C_1 is *equivalent* to C_2, writing in this case, $C_1 \sim C_2$, if there exist a unit u and an automorphism Φ of $K[[X,Y]]$, such that

$$\Phi(f) = ug.$$

It is well known, and easy to prove, that two plane branches are equivalent if and only if their rings of coordinates are isomorphic as K-algebras. In the case of analytic plane curves, this equivalence is precisely local analytic isomorphism as embedded germs.

Most of the numerical characters associated to a plane branch are invariant with respect to the above equivalence relation. For example, it is immediate to verify that the multiplicity, that is, the minimal non-zero element in the monoid of values, of a plane branch is such an invariant.

More generally, the monoid Γ of values of C is invariant by equivalence of branches. Indeed, for every unit u and every automorphism Φ of $K[[X,Y]]$ we have that

$$v(\overline{g}) = dim_K \frac{K[[X,Y]]}{\langle f, g\rangle} = dim_K \frac{K[[X,Y]]}{\langle u^{-1}\Phi(f), u^{-1}\Phi(g)\rangle} = v(\overline{u^{-1}\Phi(g)}),$$

and $u^{-1}\Phi(g)$ assume all values in $K[[X,Y]]$ as g varies in $K[[X,Y]]$.

Thus, two equivalent plane branches are equisingular. This, in particular, shows that Milnor's number of C (equal to the conductor of Γ) and the genus of the branch are invariant under equivalence of plane branches.

There are in the literature several works on the classification of plane branches with respect to the equivalence we defined in this section. We list below just few of them.

Ebey in [E] classified all plane branches with multiplicity less than 4 and some equisingularity classes of multiplicity 4, Zariski in [Z4] classified some particular equisingularity classes, Laudal-Pfister in [LaP] classify the equisingularity class of $\Gamma = \langle 5, 11\rangle$, Luengo-Pfister in [LuP] classify a particular family of equisingular classes of genus 2, Hernandes in [Her] classifies all plane branches of multiplicity 4 and Bayer-Hefez in [BH] classify all plane branches with high Tjurina number.

The set Λ introduced in Section 1 is not invariant by equisingularity. Indeed, the curves

$$C_1 : \begin{cases} x = T^4 \\ y = T^9 + T^{10} \end{cases} \qquad C_2 \begin{cases} x = T^4 \\ y = T^9 + T^{15} \end{cases}$$

are such that $\Gamma_1 = \Gamma_2 = \langle 4, 9\rangle$, but $\Lambda_1 \setminus \Gamma_1 = \{14, 23\} \neq \{19, 23\} = \Lambda_2 \setminus \Gamma_2$, showing that $\Lambda_1 \neq \Lambda_2$.

However, we have the following result.

PROPOSITION 2 *The set Λ is invariant by equivalence of plane branches.*

PROOF Let C_1 and C_2 be plane branches with rings of coordinates $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Suppose that C_1 and C_2 are equivalent plane branches. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic as K-algebras. The isomorphism between the rings of coordinates extends to an isomorphism between their fields of fractions which induces an isomorphism $\tilde{\Phi}$ between the integral closures $\overline{\mathcal{O}_1} = K[[T_1]]$ and $\overline{\mathcal{O}_2} = K[[T_2]]$.

Since $\tilde{\Phi}$ preserves values, we have that $\tilde{\Phi}(T_2) = uT_1$, where u is a unit of $K[[T_1]]$.

Consider the homomorphisms φ_j and ψ_j, $j = 1, 2$, as defined in (5.2) and (7.1), respectively.

If $\omega_2 = g_2 dx_2 + h_2 dy_2 \in \mathcal{O}_2 d\mathcal{O}_2$, with $g_2, h_2 \in \mathcal{O}_2$, then

$$\psi_2(\omega_2) = \varphi_2(g_2)\frac{d\varphi_2(x_2)}{dT_2} + \varphi_2(h_2)\frac{d\varphi_2(y_2)}{dT_2} \in K[[T_2]].$$

Since

$$\tilde{\Phi}(\psi_2(\omega_2)) = \tilde{\Phi}(\varphi_2(g_2))\tilde{\Phi}(\frac{d\varphi_2(x_2)}{dT_2}) + \tilde{\Phi}(\varphi_2(h_2))\tilde{\Phi}(\frac{d\varphi_2(y_2)}{dT_2}),$$

and

$$\tilde{\Phi}(\varphi_2(g_2)) = \varphi_1(g_1) \text{ and } \tilde{\Phi}(\varphi_2(h_2)) = \varphi_1(h_1),$$

where $g_1, h_1 \in \mathcal{O}_1$, we have that

$$\tilde{\Phi}(\psi_2(\omega_2)) = \varphi_1(g_1)\tilde{\Phi}(\frac{d\varphi_2(x_2)}{dT_2}) + \varphi_1(h_1)\tilde{\Phi}(\frac{d\varphi_2(y_2)}{dT_2}).$$

Observe now that

$$\frac{d\tilde{\Phi}(\varphi_2(x_2))}{dT_1} = (u + \frac{du}{dT_1}T_1)\tilde{\Phi}(\frac{d\varphi_2(x_2)}{dT_2}),$$

and

$$\frac{d\tilde{\Phi}(\varphi_2(y_2))}{dT_1} = (u + \frac{du}{dT_1}T_1)\tilde{\Phi}(\frac{d\varphi_2(y_2)}{dT_2}).$$

That is,

$$\tilde{\Phi}(\frac{d\varphi_2(x_2)}{dT_2}) = w\frac{d\varphi_1(r_1)}{dT_1},$$

and

$$\tilde{\Phi}(\frac{d\varphi_2(y_2)}{dT_2}) = w\frac{d\varphi_1(s_1)}{dT_1},$$

where w is a unit in $K[[T_1]]$, $r_1, s_1 \in \mathcal{O}_1$, with $v(r_1) = v(x_1)$ and $v(s_1) = v(y_1)$.

So,

$$\tilde{\Phi}(\psi_2(\omega_2)) = \left(\varphi_1(g_1)\frac{d\varphi_1(r_1)}{dT_1} + \varphi_1(h_1)\frac{d\varphi_1(s_1)}{dT_1}\right) w = \psi_1(\omega_1)w,$$

with $\omega_1 \in \mathcal{O}_1 d\mathcal{O}_1$.

Therefore we have,

$$v(\omega_2) = v(\psi_2(\omega_2)) = v(\tilde{\Phi}(\psi_2(\omega_2))) = v(\psi_1(\omega_1)w) = v(\psi_1(\omega_1)) = v(\omega_1);$$

that is, $\Lambda_2 \subset \Lambda_1$.

In the same way we may show that $\Lambda_1 \subset \Lambda_2$ and therefore, $\Lambda_1 = \Lambda_2$.

□

Zariski, in [Z2], used partially the invariance of the set Λ under equivalence. He basicaly only used the invariants $\#(\Lambda \setminus \Gamma)$ and the number

$$\lambda(C) = \min(\Lambda \setminus \Gamma) - \text{mult}(C),$$

called *Zariski's invariant.*

Since from Remark 5 we have that

$$\tau(C) = l(\mathcal{T}) = c - \#(\Lambda \setminus \Gamma),$$

it follows that Tjurina's number is an invariant with respect to equivalence of branches, because it depends only c, Γ and Λ wich are invariants.

Therefore, with the algorithms on pages 25, 69 and 96, we may compute the most important invariants with respect to the equivalence of plane brances. Below we present a summary of the results wich we need to compute the mentioned invariants of a plane branch C.

Invariant	Result
$\Gamma(C)$	Algorithm on page 69.
$\text{mult}(C)$	$\text{mult}(C) = \min(\Gamma(C) \setminus \{0\})$.
$\mu(C)$	$\mu(C)$ is the conductor of $\Gamma(C)$ or the Algorithm on page 25 for the Jacobian ideal.
$g(C)$	$g(C)$ is the number of elements of the minimal system of generators of $\Gamma(C)$.
$\Lambda(C)$	Algorithm on page 96.
$\lambda(C)$	$\lambda(C) = \min(\Lambda(C) \setminus \Gamma(C)) - \text{mult}(C)$.
$\tau(C)$	$\tau(C) = \mu(C) - \#(\Lambda(C) \setminus \Gamma(C))$ or the Algorithm on page 25 for the extended Jacobian ideal.
$l(\mathcal{T})$	$l(\mathcal{T}) = \tau(C)$.

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