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GLOBAL MINIMIZERS OF AUTONOMOUS LAGRANGIANS

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Chapter 1

Introduction.

1-1 Lagrangian Dynamics.

Let M be a boundaryless n-dimensional complete riemannian manifold. An (autonomous) Lagrangian on M is a smooth function $L : TM \to \mathbb{R}$ satisfying the following conditions:

(a) *Convexity:* The Hessian $\dfrac{\partial^2 L}{\partial v_i \partial v_j}(x, v)$, calculated in linear coordinates on the fiber $T_x M$, is uniformly positive definite for all $(x, v) \in TM$, i.e. there is $A > 0$ such that

$$w \cdot L_{vv}(x, v) \cdot w \geq A\, |w|^2 \quad \text{for all } (x, v) \in TM \text{ and } w \in T_x M.$$

(b) *Superlinearity:*

$$\lim_{|v| \to +\infty} \frac{L(x, v)}{|v|} = +\infty, \qquad \text{uniformly on } x \in M,$$

equivalently, for all $A \in \mathbb{R}$ there is $B \in \mathbb{R}$ such that

$$L(x, v) \geq A\, |v| - B \quad \text{for all } (x, v) \in TM.$$

(c) *Boundedness:* For all $r > 0$,

$$\ell(r) = \sup_{\substack{(x,v)\in TM,\\ |v|<r}} L(x,v) < +\infty. \tag{1.1}$$

$$g(r) = \sup_{\substack{|w|=1\\ |(x,v)|\le r}} w \cdot L_{vv}(x,v) \cdot w < +\infty. \tag{1.2}$$

The *Euler-Lagrange equation* associated to a lagrangian L is (in local coordinates)

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x}) = \frac{\partial L}{\partial x}(x,\dot{x}). \tag{E-L}$$

The condition (c) implies that the Euler-Lagrange equation (E-L) defines a complete flow φ_t on TM (proposition 1-3.2), called the *Euler-Lagrange flow*, by setting $\varphi_t(x_0,v_0) = \big(x_v(t), \dot{x}_v(t)\big)$, where $x_v : \mathbb{R} \to M$ is the solution of (E-L) with $x_v(0) = x_0$ and $\dot{x}_v(0) = v_0$.

We shall be interested on coverings $p : N \to M$ of a compact manifold M and the lifted Lagrangian $\mathbb{L} = L \circ dp : TN \to \mathbb{R}$ of a convex superlinear lagrangian L on M. The lagrangian $\mathbb{L}$ then satisfies (a)–(c) and its flow ψ_t is the lift of φ_t.

Observe that when we add a *closed* 1-form ω to the lagrangian L, the new lagrangian $L + \omega$ also satisfies the hypothesis (a)-(c) and has the same Euler-Lagrange equation as L. This can also be seen using the variational interpretation of the Euler-Lagrange equation.

1-2 The Euler-Lagrange equation.

The action of a differential curve $\gamma : [0,T] \to M$ is defined by

$$A(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t))dt \tag{1.3}$$

One of the main problems of Calculus of Variations is to find and to study the curves that minimize the action.

Denote by $C^k(p,q,T)$ the set of $k-$differentiable curves $\gamma : [0,T] \to M$ such that $\gamma(0) = p$ and $\gamma(T) = q$.

1-2.1. Proposition. *Let $(q_1, q_2, ..., q_n)$ a coordinate system in M, then if a curve x in the space $C^k(p,q,T)$ minimize the action among all the curves in $C^k(p,q,T)$, then x satisfies the equation*

$$\frac{d}{dt}L_v(x(t), \dot{x}(t)) = L_x(x(t), \dot{x}(t)) \tag{1.4}$$

This equation is called the Euler-Lagrange equation. Moreover if $(Q_1, Q_2, ..., Q_n)$ is another coordinate system then x satisfies the Euler-Lagrange equation in the coordinates q_i if and only if it satisfies the Euler-Lagrange equation in the coordinates Q_i

Proof Let $h(t)$ a differential curve such that $h(0) = h(T) = 0$, then for every ϵ, sufficiently small the curve $y_\epsilon = x + \epsilon h$ is on $C^k(p,q,T)$ and contained in the coordinate system. Define

$$g(\epsilon) = A_L(y_\epsilon) \tag{1.5}$$

Then g has a minimum in zero and

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{g(\epsilon)-g(0)}{\epsilon} &= \lim_{\epsilon\to 0}\int_0^T \frac{L(x+\epsilon h,\dot{x}+\epsilon\dot{h}))-L(x,\dot{x})}{\epsilon}dt \\
&= \int_0^T \lim_{\epsilon\to 0}\frac{\epsilon L_x h+\epsilon L_v\dot{h}+o(\epsilon)}{\epsilon}dt \\
&= \int_0^T L_x h + L_v\dot{h}dt \\
&= \int_0^T (L_x-\frac{d}{dt}L_v)hdt + L_v h|_0^T \\
&= \int_0^T (L_x-\frac{d}{dt}L_v)hdt
\end{aligned}
$$

Hence

$$
0=\int_0^T (L_x(x(t),\dot{x}(t))-\frac{d}{dt}L_v(x(t),\dot{x}(t)))hdt \tag{1.6}
$$

For any function h, this implies that $x(t)$ satisfies the Euler Lagrange equation [1.4]

This proves shows that $x(t)$ satisfies the Euler Lagrange equation if and only if $x(t)$ is a critical point of the action functional A_L defined in the space $C^k(p,q,T)$, this doesn't depend on the coordinates and proves the second statement. □

The Euler Lagrange equation is a second order differential equation on M, but our hypothesis about the Lagrangian (L_{vv} invertible) implies that this equation can also be seen as a first order differential equation on TM

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= (L_{vv})^{-1}(L_x - L_{vx}v)
\end{aligned}
$$

The associated vector field X_L on TM will be called the Lagrangian vector field and its flow ψ_t the Lagrangian flow.

1-2.2. Remark. It is possible to do the same thing in the space $\mathcal{C}_T(p,q)$, the set of absolutely continuous curves $\gamma : [0,T] \to M$ such that $\gamma(0) = p$ and $\gamma(T) = q$. A priori minimizers do not have to be differentiable and there are examples where they are not, see Ball & Mizel [2]. However when the lagrangian flow is complete (cf. proposition 1-3.2), every absolutely continuous minimizers is C^2 and satisfies the Euler-Lagrange equation. See Mather [39].

1-3 The Energy function.

The *energy function* of the lagrangian L is $E : TM \to \mathbb{R}$, defined by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v)\cdot v - L(x,v). \tag{1.7}$$

Observe that if $x(t)$ is a solution of the Euler-Lagrange equation (E-L), then

$$\tfrac{d}{dt}\, E(x,\dot{x}) = \left(\tfrac{d}{dt}L_v - L_x\right)\cdot \dot{x} = 0.$$

Hence $E : TM \to \mathbb{R}$ is an *integral* (i.e. invariant function[1]) for the lagrangian flow φ_t and its level sets, called *energy levels* are invariant under φ_t. Moreover, the convexity implies that

$$\tfrac{d}{dt}\, E(x,sv)\big|_{s=1} = v\cdot L_{vv}(x,v)\cdot v > 0.$$

Thus

$$\min_{v\in T_xM} E(x,v) = E(x,0) = -L(x,0).$$

Write

$$e_0 := \max_{x\in M} E(x,0) = -\min_{x\in M} L(x,0) > -\infty, \tag{1.8}$$

by the superlinearity, then

$$e_0 = \min\{\, k\in\mathbb{R} \mid \pi : E^{-1}\{k\}\to M \text{ is surjective }\}.$$

By the uniform convexity, and the boundedness condition,

$$A := \inf_{\substack{(x,v)\in TM\\ |w|=1}} w\cdot L_{vv}(x,v)\cdot w > 0,$$

and then using (1.1) and (1.2),

$$\begin{aligned} E(x,v) &= E(x,0) + \int_0^{|v|} \tfrac{d}{ds}\, E\big(x, s\tfrac{v}{|v|}\big)\, ds \\ &\geq -\ell(0) + A\,|v|. \end{aligned} \tag{1.9}$$

[1]The energy is invariant only for *autonomous* (i.e. time-independent) lagrangians.

Similarly,

$$E(x,v) \le e_0 + g(|v|)\,|v|. \tag{1.10}$$

Hence

1-3.1. Remark. If $k \in \mathbb{R}$ and $K \subseteq M$ is compact, then $E^{-1}\{k\} \cap T_K M$ is compact.

1-3.2. Proposition. *The Euler-Lagrange flow is complete.*

Proof: Suppose that $]\alpha,\beta[$ is the maximal interval of definition of $t \mapsto \varphi_t(v)$, and $-\infty < \alpha$ or $\beta < +\infty$. Let $k = E(v)$. Since $E(\varphi_t(v)) \equiv k$, by (1.9), there is $a > 0$ such that $0 \le |\varphi_t(v)| \le a$ for $\alpha \le t \le \beta$. Since $\varphi_t(v)$ is of the form $(\gamma(t), \dot\gamma(t))$, then $\varphi_t(v)$ remains in the interior of the compact set

$$Q := \left\{\, (y,w) \in TM \;\middle|\; d(y,x) \le a\,\big[|\beta - \alpha| + 1\big],\ \ |v| \le a + 1 \,\right\},$$

where $x = \pi(v)$. The Euler-Lagrange vectorfield is uniformly Lipschitz on Q. Then by the theory of ordinary differential equations, we can extend the interval of definition $]\alpha,\beta[$ of $t \mapsto \varphi_t(v)$. $\square$

1-4 Hamiltonian Systems.

Let T^*M be the cotangent bundle of M. Define the *Liouville's 1-form* Θ on T^*M as

$$\Theta_p(\xi) = p\,(d\pi\,\xi) \qquad \text{for } \xi \in T_p(T^*M),$$

where $\pi : T^*M \to M$ is the projection. The *canonical symplectic form* on T^*M is defined as $\omega = d\Theta$.

A local chart $\mathbf{x} = (x_1, \dots, x_n)$ of M induces a local chart $(\mathbf{x}, \mathbf{p}) = (x_1, \dots, x_n; p_1, \dots, p_n)$ of T^*M writing $\mathbf{p} \in T^*M$ as $\mathbf{p} = \Sigma_i\; p_i\; dx_i$. In these coordinates the forms Θ and ω are written

$$\Theta = \mathbf{p} \cdot d\mathbf{x} = \sum_i \; p_i \; dx_i\,,$$
$$\omega = d\mathbf{p} \wedge d\mathbf{x} = \sum_i \; dp_i \wedge dx_i\,.$$

A *hamiltonian* is a smooth function $H : T^*M \to \mathbb{R}$. The *hamiltonian vectorfield* X_H associated to H is defined by

$$\omega(X_H, \cdot\,) = dH.$$

In local charts, the hamiltonian vectorfield defines the differential equation

$$\begin{aligned} \dot{x} &= \;\; H_p\,, \\ \dot{p} &= -H_x\,, \end{aligned} \tag{1.11}$$

where H_x and H_p are the partial derivatives of H with respect to x and p.

We shall be specially interested in hamiltonians obtained by the Frenshel transform of a lagrangian:

$$H(x,p) = \max_{v \in T_xM} \; p\,v - L(x,v).$$

Observe that $H = E \circ \mathcal{L}^{-1}$, where E is the energy function (1.7) and $\mathcal{L}(x,v) = (x, L_v(x,v))$ is the Legendre transform of L. Moreover

1-4.1. Proposition. *The Legendre transform* $\mathcal{L} : TM \to T^*M$, $\mathcal{L}(x,v) = (x, L_v(x,v))$ *is a conjugacy between the lagrangian flow and the hamiltonian flow.*

Proof: By corollary C.2, the convexity and superlinearity hypothesis imply that $L = L^{**} = H^*$. So if $p = L_v(x,v)$ then $v = H_p(x,p)$. With this notation:

$$\begin{aligned} H(x,p) &= v \cdot L_v(x,v) - L(x,v) = E \circ \mathcal{L}^{-1} \\ &= p \cdot H_p(x,p) - L(x, H_p(x,p)). \end{aligned}$$

Thus $H_x = -L_x$, and the Euler-Lagrange equation

$$\begin{aligned} \dot{x} &= \tfrac{d}{dt}\, x &= v &= H_p\,, \\ \dot{p} &= \tfrac{d}{dt}\, L_v &= L_x &= -H_x\,, \end{aligned}$$

is the same as the hamiltonian equations. $\square$

1-5 Examples.

We give here some basic examples of lagrangians.

Riemannian Lagrangians:

Given a riemannian metric $g = \langle \cdot, \cdot \rangle_x$ on TM, the riemannian lagrangian on M is given by the *kinetic energy*

$$L(x, v) = \tfrac{1}{2}\, \|v\|_x^2 . \tag{1.12}$$

Its Euler-Lagrange equation (E-L) is the equation of the geodesics of g:

$$\tfrac{D}{dt}\, \dot{x} \equiv 0, \tag{1.13}$$

and its Euler-Lagrange flow is the geodesic flow. Its corresponding hamiltonian is

$$H(x, p) = \tfrac{1}{2}\, \|p\|_x^2 .$$

Analogous to the riemannian lagrangian is the *Finsler lagrangian*, given also by formula (1.12), but where $\|\cdot\|_x$ is a *Finsler metric*, i.e. $\|\cdot\|_x$ is a (non necessarily symmetric[2]) norm on T_xM which varies smoothly on $x \in M$. The Euler-Lagrange flow of a Finsler lagrangian is called the *geodesic flow* of the Finsler metric $\|\cdot\|_x$.

Mechanic Lagrangians:

The *mechanic lagrangian*, also called *natural lagrangian*, is given by the kinetic energy minus the *potential energy* $U : M \to \mathbb{R}$,

$$L(x, v) = \tfrac{1}{2}\, \|v\|_x^2 - U(x). \tag{1.14}$$

Its Euler-Lagrange equation is

$$\tfrac{D}{dt}\, \dot{x} = -\nabla U(x),$$

[2] i.e. $\|\lambda\, v\|_x = \lambda\, \|v|_x$ only for $\lambda \geq 0$

where ∇U is the gradient of U with respect to the riemannian metric g, i.e.

$$d_xU(v) = \langle \nabla U(x), v\rangle_x \quad \text{for all } (x,v) \in TM.$$

Its energy function and its hamiltonian are given by the kinetic energy plus potential energy:

$$E(x,v) = \tfrac{1}{2}\,\|v\|_x^2 + U(x),$$
$$H(x,p) = \tfrac{1}{2}\,\|p\|_x^2 + U(x).$$

Symmetric Lagrangians.

The symmetric lagrangians is a class of lagrangian systems which includes the riemannian and mechanic lagrangians. These are the lagrangians which satisfy

$$L(x,v) = L(x,-v) \qquad \text{for all } (x,v) \in TM. \tag{1.15}$$

Their Euler-Lagrange flow is *reversible* in the sense that $\varphi_{-t}(v) = -\varphi_t(-v)$.

Magnetic Lagrangians.

If one adds a closed 1-form ω to a lagrangian, $\mathbb{L}(x,v) = L(x,v) + \omega_x(v)$, the Euler-Lagrange flow does not change. This can be seen by first observing that the solutions of the Euler-Lagrange equation are the critical points of the action functional on curves on $C(x,y,T)$ (with fixed time interval and fixed endpoints). Since ω is closed, the action functional of $\mathbb{L}$ and L on $C(x,y,T)$ differ by a constant and hence they have the same critical points.

But adding a non-closed 1-form to a lagrangian does change the Euler-Lagrange flow. We call a *magnetic lagrangian* a lagrangian of the form

$$L(x,v) = \tfrac{1}{2}\;\|v\|_x + \eta_x(v) - U(x), \tag{1.16}$$

where $\|\cdot\|_x$ is a riemannian metric, η is a 1-form on M with $d\eta \neq 0$, and $U : M \to \mathbb{R}$ a smooth function. If $Y : TM \to TM$ is the bundle map such that

$$d\eta(u, v) = \langle Y(u), v\rangle$$

then the Euler-Lagrange equation of (1.16) is

$$\tfrac{D}{dt}\, \dot{x} = Y_x(\dot{x}) - \nabla U(x). \tag{1.17}$$

This models the motion of a particle with unit mass and unit charge under the effect of a magnetic field with Lorentz force Y and potential energy $U(x)$. The energy functional is the same as that of the mechanical lagrangian but its hamiltonian changes because of the change in the Legendre transform:

$$\begin{aligned} E(x, v) &= \tfrac{1}{2}\ \|v\|_x^2 + U(x), \\ H(x, p) &= \tfrac{1}{2}\ \|p - A(x)\|_x^2 + U(x), \end{aligned}$$

where $A : M \to TM$ is the vectorfield given by $\eta_x(v) = \langle A(x), v\rangle_x$.

Twisted geodesic flows.

The twisted geodesic flows correspond to the motion of a particle under the effect of a magnetic field with no potential energy. This can be modeled as the Euler-Lagrange flow of a lagrangian of the form $L(x, v) = \frac{1}{2}\|v\|_x^2 + \eta_x(v)$, where $d\eta \not\equiv 0$. But the Euler-Lagrange equations depend only on the riemannian metric and $d\eta$. A generalization of these flows can be made using a non-zero 2-form Ω instead of $d\eta$ and not requiring Ω to be exact. This is better presented in the hamiltonian setting.

Fix a riemannian metric $\langle\ ,\ \rangle$ and a 2-form Ω on M. Let $K : TTM \to TM$ be the connection map $K\xi = \nabla_{\dot{x}} v$, where $\xi = \frac{d}{dt}\big(x(t), v(t)\big)$. Let $\pi : TM \to M$ be the canonical projection. Let ω_0 be the symplectic form in TM obtained by pulling back the canonical symplectic form via the Legendre transform associated to the riemannian metric, i.e.

$$\omega_0(\xi, \zeta) = \langle d\pi\, \xi\,,\ K\zeta\rangle - \langle d\pi\, \zeta\,,\ K\xi\rangle\,.$$

The coordinates $T_\theta TM \ni \xi \longleftrightarrow (d\pi\,\xi, K\xi) \in T_{\pi(\theta)}M \oplus T_\pi(\theta)M = H(\theta)\oplus V(\theta)$ are the standard way of writing the horizontal and vertical components of a vector $\xi \in T_\theta TM$ for a riemannian manifold M (see Klingenberg [25]).

Define a new symplectic form ω_Ω on TM by

$$\omega_\Omega = \omega_0 + \pi^*\Omega \,.$$

This is called a *twisted symplectic structure* on TM. Let $H : TM \to \mathbb{R}$ be the hamiltonian

$$H(x,v) = \tfrac{1}{2}\,\|v\|_x^2 \;.$$

Consider the hamiltonian vectorfield X_F corresponding to (H,ω_Ω), i.e.

$$\omega_\Omega\big(X_\Omega(\theta),\cdot\big) = dH \,. \tag{1.18}$$

Define $Y : TM \to TM$ as the bundle map such that

$$\Omega_x(u,v) = \langle Y(u), v\rangle_x \,. \tag{1.19}$$

The hamiltonian vectorfield $X_\Omega(\theta) \in T_\theta TM$ is given by $X_\Omega(\theta) = (\theta, Y(\theta)) \in H(\theta)\oplus V(\theta)$. Hence the hamiltonian equation is

$$\tfrac{D}{dt}\,\dot{x} = Y_x(\dot{x}),$$

recovering equation (1.17) with $U \equiv 0$, but where Ω doesn't need to be exact.

If $H^1(M,\mathbb{R}) = 0$, both approaches coincide, and any twisted geodesic flow is the lagrangian flow of a magnetic lagrangian of the form $L(x,v) = \frac{1}{2}\,\|v\|_x^2 + \eta_x(v)$, with $d\eta = \Omega$. For example if N is a compact manifold Ω is a 2-form in N and M is the abelian cover or the universal cover of N; if Ω is not exact, then the corresponding twisted geodesic flow is a lagrangian flow on M but not on N (where it is locally a lagrangian flow). This lagrangian flow on M is actually the lift of the twisted geodesic flow on N.

Embedding flows:

There is a way to embed the flow of any bounded vectorfield on a lagrangian system. Given a smooth bounded vectorfield $F : M \to TM$, let

$$L(x, v) = \tfrac{1}{2} \, \|v - F(x)\|_x^2 \, . \tag{1.20}$$

Since $F(x)$ is bounded, then the lagrangian L is convex, superlinear and satisfies the boundedness condition. The lagrangian L on a fiber T_xM is minimized at $(x, F(x))$, hence the integral curves of the vectorfield, $\dot{x} = F(x)$, are solutions to the Euler-Lagrange equation.

Chapter 2

Mañé's critical value.

2-1 The action potential and the critical value.

We shall be interested on action minimizing curves with *free* time interval. Unless otherwise stated, all the curves will be assumed to be absolutely continuous. For $x, y \in M$, let

$$\mathcal{C}(x,y) = \{\, \gamma : [0,T] \to M \mid T > 0,\ \gamma(0) = x,\ \gamma(T) = y \,\}.$$

For $k \in \mathbb{R}$ define the *action potential* $\Phi_k : M \times M \to \mathbb{R} \cup \{-\infty\}$, by

$$\Phi_k(x,y) = \inf_{\gamma \in \mathcal{C}(x,y)} A_{L+k}(\gamma).$$

Observe that if there exists a closed curve γ on N with negative $L+k$ action, then $\Phi_k(x,y) = -\infty$ for all $x, y \in N$, by going round γ many times.

Define the *critical level* $c = c(L)$ as

$$c(L) = \sup\{\, k \in \mathbb{R} \mid \exists \text{ closed curve } \gamma \text{ with } A_{L+k}(\gamma) < 0 \,\}.$$

Observe that the function $k \mapsto \Phi_k(x,y)$ is increasing. The superlinearity implies that L is bounded below. Hence there is $k \in \mathbb{R}$ such that

$L + k \geq 0$. Thus $c(L) < +\infty$. Since $k \mapsto A_{L+k}(\gamma)$ is increasing for any γ, we have that

$$c(L) = \inf\{\, k \in \mathbb{R} \mid A_{L+k}(\gamma) \geq 0 \;\forall \text{ closed curve } \gamma \,\}.$$

2-1.1. Proposition.

(1) (a) For $k < c(L)$, $\Phi_k(x, y) = -\infty$ for all $x, y \in M$.
(b) For $k \geq c(L)$, $\Phi_k(x, y) \in \mathbb{R}$ for all $x, y \in M$.

(2) For $k \geq c(L)$, $\Phi_k(x, z) \leq \Phi_k(x, y) + \Phi_k(y, z)$, $\forall\, x, y, z \in M$.

(3) $\Phi_k(x, x) = 0$, $\forall\, x \in M$.

(4) $\Phi_k(x, y) + \Phi_k(y, x) \geq 0 \quad \forall\, x, y \in M$.
For $k > c(L)$, $\Phi_k(x, y) + \Phi_k(y, x) > 0$ if $x \neq y$.

(5) For $k \geq c(L)$ the action potential Φ_k is Lipschitz.

2-1.2. Remark. The action potential Φ_k is not symmetric in general, but items (2),(3),(4) imply that

$$d_k(x, y) = \Phi_k(x, y) + \Phi_k(y, x)$$

is a metric for $k > c(L)$ and a pseudo-metric for $k = c(L)$ [i.e. perhaps $d_c(x, y) = 0$ for some $x \neq y$ and $c = c(L)$].

Proof:

(2) We first prove (2) for all $k \in \mathbb{R}$. Since $\Phi_k(x, y) \in \mathbb{R} \cup \{-\infty\}$, the inequality in item (2) makes sense for all $k \in \mathbb{R}$. If $\gamma \in C(x, y)$, $\eta \in C(y, z)$, then $\gamma * \eta \in C(x, z)$ and hence

$$\Phi_k(x, z) \leq A_{L+k}(\gamma * \eta) \leq A_{L+k}(\gamma) + A_{L+k}(\eta).$$

Taking the infimums on $\gamma \in C(x, y)$ and $\eta \in C(y, z)$, we obtain (2).

(1) (a) If γ is a closed curve with $A_{L+k}(\gamma) < 0$ and $\gamma(0) = z$, then

$$\Phi_k(z,z) \leq \lim_{N\to\infty} A_{L+k}(\gamma * \overset{N}{\cdots} * \gamma) = \lim_N N\, A_{L+k}(\gamma) = -\infty.$$

For $x, y \in M$, item (2) implies that

$$\Phi_k(x,y) \leq \Phi_k(x,z) + \Phi_k(z,z) + \Phi_k(z,y) = -\infty.$$

Since the function $k \mapsto \Phi_k(x,y)$ is increasing, then item (1)(a) follows.

(b) Conversely, if $\Phi_k(x,y) = -\infty$ for some $k \in \mathbb{R}$ and $x, y \in M$, then

$$\Phi_k(x,x) \leq \Phi_k(x,y) + \Phi_k(y,x) = -\infty.$$

Thus there is $\gamma \in \mathcal{C}(x,x)$ with $A_{L+k}(\gamma) < 0$. Then $k \leq c(L)$. Observe that the set $\{\, k \in \mathbb{R} \mid A_{L+k}(\gamma) < 0$ for some closed curve $\gamma \,\}$ is open. Hence $\Phi_k(x,y) = -\infty$ actually implies that $k < c(L)$. This proves item (1)(b).

(3) Let $k \in \mathbb{R}$ by the boundedness condition there exists $Q > 0$ be such that

$$|L(x,v) + k| \leq Q \quad \text{for } |v| \leq 2. \tag{2.1}$$

Now let $\gamma : [0,1] \to M$ be a differentiable curve with $|\dot\gamma| \equiv 1$ and $\gamma(0) = x$. Then

$$\begin{aligned}\Phi_k(x,x) &\leq \Phi_k(x,\gamma(\varepsilon)) + \Phi_k(\gamma(\varepsilon),x)\\ &\leq A_{L+k}\left(\gamma|_{[0,\varepsilon]}\right) + A_{L+k}\left(\gamma(t-\varepsilon)|_{[0,\varepsilon]}\right)\\ &\leq 2\,Q\,\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we get that $\Phi_k(x,x) \leq 0$. But the definition of $c(L)$ and the monotonicity of $k \mapsto \Phi_k(x,x)$ imply that $\Phi_k(x,x) \geq 0$ for all $k \geq c(L)$.

(5) Let $k \geq c(L)$. Given $x_1, x_2 \in M$ we have that

$$\Phi_k(x_1, x_2) \leq A_{L+k}(\gamma) \leq Q\ d_M(x_1, x_2),$$

where $\gamma : [0, d(x_1, x_2)] \to N$ is a unit speed minimizing geodesic joining x_1 to x_2 and $Q > 0$ is from (2.1). If $y_1, y_2 \in M$, then the triangle inequality implies that

$$\begin{aligned}\Phi_k(x_1, y_1) - \Phi_k(x_2, y_2) &\leq \Phi_k(x_1, x_2) + \Phi_k(y_2, y_1)\\ &\leq Q\ [d_M(x_1, x_2) + d_M(y_1, y_2)].\end{aligned}$$

Changing the roles of (x_1, y_1) and (x_2, y_2) we get item (5).

(4) The first part of item (4) follows from items (2) and (3). Now suppose that $k > c(L)$, $x \neq y$ and $d_k(x, y) = 0$. Let $\gamma_n : [0, T_n] \to M$, $\gamma_n \in C(x, y)$ be such that $\Phi_k(x, y) = \lim_n A_{L+k}(\gamma_n)$. We claim that T_n is bounded below.

Indeed, suppose that $\lim_n T_n = 0$. Let $A > 0$, from the superlinearity there is $B > 0$ such that $L(x, v) \leq A\,|v| - B$, $\forall\ (x, v) \in TM$. Then

$$\begin{aligned}\Phi_k(x, y) &= \lim_n \int_0^{T_N} L(\gamma_n, \dot\gamma_n) + k\\ &\geq \lim_n\ A \textstyle\int |\dot\gamma| + (k - B)\, T_n\\ &= A\ d_M(x, y)\end{aligned}$$

Letting $A \to +\infty$ we get that $\Phi_k(x, y) = +\infty$ which is false.

Now let $\eta_n : [0, S_n] \to M$, $\eta_n \in C(y, x)$ with $\lim_n A_{L+k}(\eta_n) = \Phi_k(y, x)$. Choose $0 < T < \liminf_n T_n$ and $0 < S < \liminf_n S_n$. Then for $c = c(L) < k$,

$$\begin{aligned}\Phi_c(x, x) &\leq \lim_n A_{L+c}(\gamma_n * \eta_n)\\ &\leq \lim_n A_{L+k}(\gamma_n) + (c - k)T + A_{L+k}(\eta_n) + (c - k)S\\ &\leq \lim_n \Phi_k(x, y) + \Phi_k(y, x) + (c - k)(T + S)\\ &\leq (c - k)(T + S) < 0,\end{aligned}$$

which contradicts item (3). □

2-2 Continuity of the critical value.

2-2.1. Lemma. *The function $C^\infty(M,\mathbb{R}) \ni \psi \mapsto c(L+\psi)$ is continuous in the topology induced by the supremum norm.*

Proof: Suppose that $\psi_n \to \psi$ and let $c_n := c(L+\psi_n)$ and $c := c(L+\psi)$. We will prove that $c_n \to c$.

Fix $\varepsilon > 0$. Since $c - \varepsilon < c$, by the definition of critical value there exists a closed curve $\gamma : [0,T] \to M$ such that $A_{L+\psi+c-\varepsilon}(\gamma) < 0$, hence for all n sufficiently large

$$A_{L+\psi_n+c-\varepsilon}(\gamma) < 0.$$

Therefore for n sufficiently large $c-\varepsilon < c_n$, and thus $c-\varepsilon \leq \liminf_n c_n$. Since ε was arbitrary we have that $c \leq \liminf_n c_n$.

We show now that $\limsup_n c_n \leq c$. Suppose that $c < \limsup_n c_n$. Take ε such that

$$c < c+\varepsilon < \limsup_n c_n. \tag{2.2}$$

Since $\psi_n \to \psi$, there exists n_0 such that for all $n \geq n_0$,

$$-\varepsilon \leq \psi - \psi_n \leq \varepsilon. \tag{2.3}$$

By (2.2), there exists $m \geq n_0$ such that

$$c < c+\varepsilon < c_m.$$

By the definition of critical value there exists a closed curve $\gamma : [0,T] \to M$ such that

$$A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

and hence using (2.3) we have

$$A_{L+\psi+c}(\gamma) \leq A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

which yields a contradiction to the definition of the critical value c.

$\square$

This proof also shows that $L \mapsto c(L)$ is continuous if we endow the set of lagrangians L with the topology induced by the supremum norm.

2-3 Holonomic measures.

Let C^0_ℓ be the set of continuous functions $f : TM \to \mathbb{R}$ having linear growth, i.e.

$$\sup_{(x,v)\in TM} \frac{|f(x,v)|}{1+\|v\|} < +\infty.$$

Let $\mathcal{M}_\ell$ be the set of Borel probabilities μ on TM such that

$$\int_{TM} \|v\| \; d\mu < +\infty,$$

endowed with the topology such that $\lim_n \mu_n = \mu$ if and only if

$$\lim_n \int f \; d\mu_n = \int f \; d\mu$$

for all $f \in C^0_\ell$.

Let $(C^0_\ell)'$ the dual of C^0_ℓ. Then $\mathcal{M}_\ell$ is naturally embedded in $(C^0_\ell)'$ and its topology coincides with that induced by the weak* topology on $(C^0_\ell)'$.

We shall see that this topology is metrizable. Let $\{f_n\}$ be a sequence of functions with compact support on C^0_ℓ which is dense on C^0_ℓ in the topology of uniform convergence on compact sets of TM. Define a metric $d(\cdot,\cdot)$ on $\mathcal{M}_\ell$ by

$$d(\mu_1,\mu_2) = \left|\int |v| \; d\mu_1 - \int |v| \; d\mu_2\right| + \sum_n \frac{1}{2^n}\frac{1}{c_n}\left|\int f_n \; d\mu_1 - \int f_n \; d\mu_2\right| \tag{2.4}$$

where $c_n = \sup_{(x,v)} f_n(x,v)$.

2-3.1. Proposition.

The metric $d(\cdot,\cdot)$ induces the weak topology on $\mathcal{M}_\ell \subset C^0_\ell$.*

Proof: We prove that $d(\cdot,\cdot)$ generates the weak* topology on $\mathcal{M}_\ell$. Suppose that

$$\int f\, d\mu_n \to \int f\, d\mu, \qquad \forall f \in C^0_\ell.$$

Given $\varepsilon > 0$, choose $M > 0$ such that $\sum_{m \geq M} \frac{1}{2^m} \cdot 2 < \varepsilon$, and choose $N > 0$ such that

$$\left| \int f_m\, d\mu_n - \int f_m\, d\mu \right| < \varepsilon, \qquad \text{for } 0 \leq m \leq M, \;\; n \geq N;$$

$$\left| \int |v|\, d\mu_n - \int |v|\, d\mu \right| < \varepsilon, \qquad \text{for } n \geq N.$$

Since $\frac{\|f\|_\infty}{c_n} = 1$, then for $n > N$ we have that

$$d(\mu_n, \mu) \leq \varepsilon + \sum_{m=1}^{M} \tfrac{1}{2^n} \cdot \varepsilon + \sum_{m \geq M+1} \tfrac{1}{2^m} \cdot 2 \cdot \tfrac{\|f_m\|}{c_m} = 3\,\varepsilon.$$

Thus $d(\mu_n, \mu) \to 0$.

Now suppose that $d(\mu_n, \mu) \to 0$. Let K_m be compact sets such that $K_m \subset K_{m+1}$ and that $TM = \cup K_m$. Then

$$\int_{K_m} f\, d\mu_n \longrightarrow \int_{K_m} f\, d\mu, \qquad \forall f \in C^0_\ell, \quad \forall m;$$

$$\int |v|\, d\mu_n \longrightarrow \int |v|\, d\mu.$$

This implies that

$$\lim_{n\to\infty} \int_{TM-K_m} |v|\, d\mu_n = \int_{TM-K_m} |v|\, d\mu, \qquad \forall\, m. \tag{2.5}$$

Given $\varepsilon > 0$, choose $m(\varepsilon) > 0$ such that

$$\int_{TM-K_{m(\varepsilon)}} (1 + |v|)\, d\mu < \frac{\varepsilon}{4},$$

and N such that

$$\int_{TM-K_{m(\varepsilon)}} (1+|v|)\, d\mu_n < \frac{\varepsilon}{2}, \qquad \forall n > N.$$

Fix $f \in C^0_\ell$. Choose $N > 0$ such that

$$\left| \int_{K_{m(\varepsilon)}} f\, d\mu_n - \int_{K_{m(\varepsilon)}} f\, d\mu \right| < \varepsilon, \qquad \forall\, n > N.$$

Then

$$\int_{TM-K_{m(\varepsilon)}} |f|\ d\mu_n \le \|f\|_\ell \int_{TM-K_{m(\varepsilon)}} (1+|v|)\, d\mu_n \le \|f\|_\ell\ \frac{\varepsilon}{2}, \quad \forall\, n > N.$$

Using a similar estimate for μ we obtain that

$$\left| \int f\ d\mu_n - \int f\ d\mu \right| \le \varepsilon + \|f\|_\ell\ \left(\tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{4}\right).$$

$\square$

If $\gamma : [0,T] \to M$ is a closed absolutely continuous curve, let $\mu_\gamma \in \mathcal{M}_\ell$ be defined by

$$\int f\ d\mu_\gamma = \frac{1}{T} \int_0^T f\big(\gamma(t), \dot\gamma(t)\big)\ dt$$

for all $f \in C^0_\ell$. Observe that $\mathcal{C}(M) \subset \mathcal{M}_\ell$ because if γ is absolutely continuous then $\int |\dot\gamma(t)|\ dt < +\infty$. Let $\mathcal{C}(M)$ be the set of such μ_γ's and let $\overline{\mathcal{C}(M)}$ be its closure in $\mathcal{M}_\ell$. Observe that the set $\overline{\mathcal{C}(M)}$ is convex. We call $\overline{\mathcal{C}(M)}$ the set of *holonomic measures* on M.

2-3.2. Theorem (Mañé [27], prop. 1.1, 1.3, 1.2).

1. *$\mathcal{M}(L) \subseteq \overline{\mathcal{C}(M)} \subseteq \mathcal{M}_\ell$.*

2. *If $\mu \in \overline{\mathcal{C}(M)}$ satisfies*

$$A_L(\mu) = \min\{\, A_L(\nu) \mid \nu \in \overline{\mathcal{C}(M)}\,\},$$

then $\mu \in \mathcal{M}(L)$.

3. *If M is compact and $a \in \mathbb{R}$, then the set $\{\, \mu \in \overline{\mathcal{C}(M)} \mid A_L(\mu) \le a\,\}$ is compact.*

The inclusion $\mathcal{M}(L) \subseteq \overline{\mathcal{C}(M)}$ follows from Birkhoff's ergodic theorem and the fact that $\overline{\mathcal{C}(M)}$ is convex.

Proof of item 2-3.2.3:

Since $\overline{\mathcal{C}(M)}$ is closed, it is enough to prove that the set

$$\mathcal{A}(a) := \left\{\, \mu \in \mathcal{M}_\ell \mid A_L(\mu) \le a \,\right\}$$

is compact in $\mathcal{M}_\ell$. First we prove that $\mathcal{A}(a)$ is closed. Let $k > 0$ and define $L_k := \min\{L, k\}$. Let

$$\mathcal{B}_k := \left\{\, \mu \in \mathcal{M}_\ell \mid \textstyle\int L_k \, d\mu \le a \,\right\}.$$

Since $L_k \in C^0_\ell$, then $\mathcal{B}_k$ is closed in $\mathcal{M}_\ell$. Since $\mathcal{A}(a) = \cap_{k>0}\mathcal{B}_k$, then $\mathcal{A}(a)$ is closed.

In order to prove the compactness, consider a sequence $\{\mu_n\} \subset \mathcal{A}(a)$. Applying the Riesz' theorem, taking a subsequence we can assume that there exists a probability μ on the Borel σ-algebra of TM such that

$$\int f_i \, d\mu_n \longrightarrow \int f_i \, d\mu, \tag{2.6}$$

for every f_i in the sequence used for the definition of $d(\cdot,\cdot)$.

Approximating L_k by functions f_i we have that

$$\int L_k \, d\mu = \lim_n L_k \, d\mu_n \leq \liminf_n L \, d\mu_n \leq a.$$

Letting $k \uparrow +\infty$, we get that

$$A_L(\mu) \leq a. \tag{2.7}$$

Let $B > 0$ be such that $|v| < L(x,v) + B$ for all $(x,v) \in TM$. Then

$$\int |v| \, d\mu \leq A_L(\mu) + B \leq a + B < +\infty. \tag{2.8}$$

So that $\mu \in \mathcal{M}_\ell$.

We now prove that $\lim_n \int |v| \, d\mu_n \longrightarrow \int |v| \, d\mu$. Let $\varepsilon > 0$. By adding a constant we may assume that $L > 0$. Choose $r > 0$ such that $L(x,v) > a\,\varepsilon^{-1}\,|v|$ for all $|v| > r$. Then

$$\int_{|v|>r} |v| \, d\mu_n \leq \frac{\varepsilon}{a} \int_{|v|>r} L \, d\mu_n \leq \frac{\varepsilon}{a} \int L \, d\mu_n \leq \varepsilon.$$

Similarly, by (2.7),

$$\int_{|v|>r} |v| \, d\mu \leq \varepsilon.$$

From (2.6) we obtain that there is $N > 0$ such that

$$\left| \int_{|v|>r} |v| \, d\mu - \int_{|v|>r} |v| \, d\mu_n \right| < \varepsilon, \quad \text{for } n > N.$$

Adding these inequalities we get that

$$\left| \int |v| \, d\mu_n - \int |v| \, d\mu \right| \leq 3\,\varepsilon.$$

□

The prove item 2-3.2.2 requires some preliminary results which we present now. Item 2-3.2.2 is proved at the end of the section.

2-3.3. Proposition.

Given $\mu \in \overline{\mathcal{C}(M)}$, there are $\mu_{\eta_n} \in \mathcal{C}(M)$ such that $\mu_{\eta_n} \to \mu$ and

$$\lim_n \int L \, d\mu_{\eta_n} = \int L \, d\mu.$$

2-3.4. Remark. The statement of proposition 2-3.3 is not trivial. It is easy to see that the function $A_L : \overline{\mathcal{C}(M)} \to \mathbb{R}$ is always lower semi-continuous (see the last argument of the proof of 2-3.3), but in general it is not continuous. It is possible to give a sequence $\mu_{\gamma_n} \in \mathcal{C}(M)$ such that $\mu_{\gamma_n} \to \mu$ in $\overline{\mathcal{C}(M)}$ but $\liminf_n A_L(\mu_{\gamma_n}) > A_L(\mu)$ for a quadratic lagrangian L.

This can be made by calibrating the high speeds in γ_n so that $\int_{[|v|>R]} |v| \, d\mu_{\gamma_n} \to 0$ but $a := \liminf_n \int_{[|v|>R]} L \, d\mu_{\gamma_n} > 0$. Then the limit measure μ will have support on $[|v| \le R]$ and "will not see" the remnant a of the action.

Proof: Let $A > 1$ and let $\gamma : [0,T] \to M$ be a closed absolutely continuous curve. We reparametrize γ to a curve $\eta : [0,S] \to M$ such that $\dot\eta = \dot\gamma$ when $|\dot\gamma| < A$ and $\dot\eta = \frac{\dot\gamma}{|\dot\gamma|} A$ when $|\dot\gamma| > A$. So that $|\dot\eta| \le A$. Write $\eta(s(t)) = \gamma(t)$, $w(s) = |\dot\eta(s)|$ and $v(t) = |\dot\gamma(t)|$. We want

$$\int_0^{s(t)} w(s) \, ds = \int_0^t v(t) \, dt,$$

so that

$$s'(t) = \frac{v(t)}{w(s(t))} = \begin{cases} 1 & \text{when } v(t) \le A. \\ \frac{v(t)}{A} & \text{when } v(t) \ge A. \end{cases}$$

Then

$$
\begin{aligned}
S(T) &= \int_{[v(t)\leq A]} dt + \int_{[v(t)\geq A]} \frac{v(t)}{A}\, dt, \\
\frac{S(T)}{T} &= \mu_\gamma([|v|\leq A]) + \int_{[|v|\geq A]} \frac{|v|}{A}\, d\mu_\gamma, \\
\left|\frac{S(T)}{T} - 1\right| &\leq \int_{[|v|>A]} d\mu_\gamma + \int_{[|v|\geq A]} \frac{|v|}{A}\, d\mu_\gamma \\
&\leq 2\int_{[|v|>A]} |v|\, d\mu_\gamma. \qquad (2.9)
\end{aligned}
$$

Suppose that $f : TM \to \mathbb{R}$ is μ_η-integrable. Since $\frac{ds}{dt} = \frac{v(t)}{A}$ when $v(t) > A$ then

$$
\int_{[|\dot\gamma(t(s))|>A]} f(\eta(s), \tfrac{\dot\gamma(t(s))}{|\dot\gamma(t(s))|}A)\, ds = \int_{[|\dot\gamma(t)>A]} f\big(\gamma(t), \tfrac{\dot\gamma(t)}{|\dot\gamma(t)|}A\big)\, \frac{|\dot\gamma(t)|}{A}\, dt.
$$

Then

$$
\begin{aligned}
\int f\, d\mu_\eta &= \frac{1}{S(T)}\int f(\eta(s),\dot\eta(s))\, ds \\
&= \frac{1}{S(T)}\left[\int_{[|\dot\gamma|\leq A]} f(\gamma(t),\dot\gamma(t))\, dt + \int_{[|\dot\gamma(s(t))|>A]} f\big(\eta(s), \tfrac{\dot\gamma(t(s))}{|\dot\gamma(t(s))|}A\big)\, ds\right] \\
&= \frac{T}{S(T)}\left[\int_{[|v|\leq A]} f(v)\, d\mu_\gamma(v) + \int_{[|v|>A]} f\big(\tfrac{v}{|v|}A\big)\, \frac{|v|}{A}\, d\mu_\gamma\right]
\end{aligned}
$$

For $A > 1$ big enough,

$$
\int_{[|v|>A]} |v|\, d\mu_\gamma < \varepsilon < \tfrac{1}{2}. \qquad (2.10)
$$

Define

$$
f_A(v) := \begin{cases} f(v) & \text{if } |v|\leq A, \\ f\big(\frac{v}{|v|}A\big)\,\frac{|v|}{A} & \text{if } |v|>A. \end{cases}
$$

Then

$$\int f\, d\mu_\eta = \tfrac{T}{S(T)} \int f_A\, d\mu_\gamma. \tag{2.11}$$

Observe that from (2.9) and (2.10), we have that

$$\left|\tfrac{T}{S(T)} - 1\right| \le 4\,\varepsilon. \tag{2.12}$$

Then

$$\left|\int f\, d\mu_\eta - \int f_A\, d\mu_\gamma\right| \le \left|\tfrac{T}{S(T)} - 1\right| \int |f_A|\, d\mu_\gamma \le 4\,\varepsilon \int |f_A|\, d\mu_\gamma. \tag{2.13}$$

If $\|f\|_\infty \le 1$, then

$$\begin{aligned}
\left|\int f\, d\mu_\eta - \int f\, d\mu_\gamma\right| &= \left|\tfrac{T}{S(T)} \int f_A\, d\mu_\gamma - \int f\, d\mu_\gamma\right| \\
&\le \left|\tfrac{T}{S(T)} - 1\right| \int |f_A|\, d\mu_\gamma + \int |f - f_A|\, d\mu_\gamma \\
&\le \left|\tfrac{T}{S(T)} - 1\right| + \int_{[|v|>A]} d\mu_\gamma + \int_{[|v|>A]} \frac{|v|}{A}\, d\mu_\gamma \\
&\le \left|\tfrac{T}{S(T)} - 1\right| + 2 \int_{[|v|>A]} |v|\, d\mu_\gamma \\
&\le 4\,\varepsilon + 2\,\varepsilon \le 6\,\varepsilon.
\end{aligned}$$

Also

$$\left|\int |v|\, d\mu_\eta - \int |v|\, d\mu_\gamma\right| = \left|\tfrac{T}{S(T)} - 1\right| \int |v|\, d\mu_\gamma \le 4\,\varepsilon \int |v|\, d\mu_\gamma.$$

Hence

$$d_{\mathcal{C}(M)}(\mu_\eta, \mu_\gamma) \le 6\,\varepsilon \int (|v| + 1)\, d\mu_\gamma. \tag{2.14}$$

Now let $\mu \in \overline{\mathcal{C}(M)}$. Let

$$K := \int (|v|+1)\, d\mu + 1. \tag{2.15}$$

For $R > 0$, define

$$L_R(v) := \begin{cases} L(v) & \text{if } |v| \le R. \\ L\big(\frac{v}{|v|}R\big)\, \frac{|v|}{R} & \text{if } |v| > R. \end{cases}$$

Given $N > 0$, choose $R = R(N) > 0$ such that

$$L(v) > 0 \text{ if } |v| > R \qquad \text{and} \qquad \int_{|v|>R} |v|\, d\mu < \tfrac{1}{N}. \tag{2.16}$$

Choose $\mu_{\gamma_N} \in \mathcal{C}(M)$ such that

$$\begin{aligned} &d_{\mathcal{C}(M)}(\mu_{\gamma_N}, \mu) < \tfrac{1}{N}, \\ &\int L_{R(N)}\, d\mu_{\gamma_N} \le \int L_{R(N)}\, d\mu + \tfrac{1}{N}, \end{aligned} \tag{2.17}$$

and

$$\int_{[|v|\le R(N)]} |v|\, d\mu_{\gamma_N} \ge \int_{[|v|\le R(N)]} |v|\, d\mu - \tfrac{1}{N}. \tag{2.18}$$

Then

$$\int (|v|+1)\, d\mu_{\gamma_N} \le K \qquad \text{from (2.15) and (2.17)}, \tag{2.19}$$

$$\int_{[|v|>R(N)]} |v|\, d\mu_{\gamma_N} < \tfrac{3}{N} \qquad \text{from (2.16), (2.17) and (2.18)}. \tag{2.20}$$

Construct η_N as above for γ_N and $A = R(N)$. Then from (2.14), (2.19) and (2.20), $d_{\mathcal{C}(M)}(\mu_{\eta_N}, \mu_{\gamma_N}) < \frac{18}{N} K$. From (2.17),

$$d_{\mathcal{C}(M)}(\mu_{\eta_N}, \mu) < \tfrac{18}{N} K + \tfrac{1}{N}.$$

Thus $\mu_{\eta_N} \xrightarrow{N} \mu$ in $\overline{C(M)}$. Moreover, from (2.11), (2.12) and (2.10),

$$\int L \, d\mu_{\eta_N} = \tfrac{T_N}{S(T_N)} \int L_{R(N)} \, d\mu_{\gamma_N} \le \tfrac{T_N}{S(T_N)} \left[\int L_{R(N)} \, d\mu + \tfrac{1}{4N} \right]$$
$$\le \left(1 + \tfrac{12}{N}\right) \left[\int L \, d\mu + \tfrac{1}{4N} \right].$$

Hence

$$\limsup_N \int L \, d\mu_{\eta_N} \le \int L \, d\mu.$$

On the other hand, for any fixed $R > 0$ such that $L \ge 0$ on $|v| > R$,

$$\liminf_N \int L \, d\mu_{\eta_N} \ge \lim_N \int L_R \, d\mu_{\eta_N} = \int L_R \, d\mu.$$

Letting $R \uparrow +\infty$, we get that

$$\liminf_N \int L \, d\mu_{\eta_N} \ge \int L \, d\mu.$$

$\square$

Given $x, y \in M$, define

$$S(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,y)} A_L(\gamma).$$

Observe that $S(x, y; T) > -\infty$ because L is bounded below. If $\gamma \in \mathcal{C}^{ac}([0,T], M)$, define

$$S^+(\gamma) := A_L(\gamma) - S(\gamma(0), \gamma(T); T).$$

The absolutely continuous curves γ with $S^+(\gamma) = 0$ are called *Tonelli minimizers.* Observe that a Tonelli minimizer is a solution of (E-L). Given $\gamma_1, \gamma_2 \in \mathcal{C}^{ac}([0,T], M)$, the *absolutely continuous distance* $d_1(\gamma_1, \gamma_2)$ is defined by

$$d_1(\gamma_1, \gamma_2) := \sup_{t \in [0,T]} d\big(\gamma_1(t), \gamma_2(t)\big) + \int_0^T d_{TM}\big([\gamma_1(t), \dot\gamma_1(t)], [\gamma_2(t), \dot\gamma_2(t)]\big) \, dt.$$

2-3.5. Proposition. *Given a compact subset $K \subseteq M$ and given C, $\varepsilon > 0$ there exist $\delta > 0$ such that if $\gamma : [0,T] \to M$ is absolutely continuous and satisfies*

i. $1 \le T \le C$.

ii. $A_L(\gamma) \le C$.

iii. $S^+(\gamma) \le \delta$.

Then either $\gamma([0,T]) \cap K = \varnothing$ or there exists a Tonelli minimizer $\gamma_0 : [0,T] \to M$ such that $d_1(\gamma_0, \gamma) \le \varepsilon$.

Proof: If such δ does not exists then there is a sequence $\gamma_n \in \mathcal{C}^{ac}([0,T_n], M)$ such that $\gamma_n([0,T_n]) \cap K \ne \varnothing$, $1 \le T_n \le C$, $S^+(\gamma_n) \to 0$, $A_L(\gamma_n) \le C$ and $d_1(\gamma_n, \eta) \ge \varepsilon$ for any Tonelli minimizer η.

Adding a constant we can assume that $L > 0$. Let $B > 0$ be such that $L(x,v) > |v| - B$ for all $(x,v) \in TM$. Choose $s_0 \in [0, T_n]$ such that $\gamma_n(s_0) \in K$. Then

$$\begin{aligned} d(K, \gamma_n(t)) \le d(\gamma_n(s_0), \gamma_n(t)) &\le \int_{[s_0,t]} |\dot\gamma_n| \\ &\le \int_{[s_0,t]} \big[\, L(\gamma_n, \dot\gamma_n) + B \,\big] \le C + B\,C. \end{aligned}$$

Let $Q := \{\, y \in M \mid d(y, K) \le C + B\,C \,\}$. Then we have that $\gamma_n([0,T_n]) \subseteq Q$.

We can assume that $T_n \to T$, $\gamma_n(0) \to x \in Q$ and $\gamma_n(T_n) \to y \in Q$. Moreover, we can assume that $T_n \equiv T$, $\gamma_n(0) \equiv x$ and $\gamma_n(T) \equiv y$. By theorem 3-1.2, the set $\mathcal{A}[b] = \big\{ \gamma \in \mathcal{C}_T(x,y) \mid A_L(\gamma) \le b \big\}$ is compact in the d_1-topology. Then we can assume that there is $\gamma_0 \in \mathcal{C}_T(x,y)$ such that $\gamma_n \to \gamma_0$ in the d_1-topology. Let $a := S(x,y;T)$. Then $\gamma_0 \in \cap_n \mathcal{A}[a + S^+(\gamma_n)] = \mathcal{A}[a]$, because $S^+(\gamma_n) \to 0$. Thus γ_0 is a Tonelli minimizer.

□

Let

$$\mathcal{H} := \left\{ h : TM \to \mathbb{R} \,\middle|\, \|f\|_\infty \le 1,\ [h]_{Lip} \le 1,\ h \text{ with compact support} \right\},$$

where

$$[h]_{Lip} = \sup_{(x,v)\neq(y,w)} \frac{|h(x,v) - h(y,w)|}{d_{TM}\big((x,v),(y,w)\big)}$$

is the smallest Lipschitz constant for h.

2-3.6. Corollary.

Given $h \in \mathcal{H}$ and $C > 0$ there exist $\delta = \delta(C,h) > 0$ such that if $\gamma : [0,T] \to M$ satisfies conditions 2-3.5.i, 2-3.5.ii, 2-3.5.iii then

$$\left| \oint_\gamma h - \oint_\gamma h \circ \varphi_1 \right| \le 5. \tag{2.21}$$

Proof: Let $K = \pi\big(\mathrm{supp}(h) \cup \varphi_{-1}(\mathrm{supp}(h))\big)$. Given $C > 0$ and $\varepsilon > 0$ let $\delta = \delta(C,\varepsilon) > 0$ and $A > 0$ be given by proposition 2-3.5 then if $\gamma : [0,T] \to M$ satisfies conditions 2-3.5.i, 2-3.5.ii, 2-3.5.iii we have that either $\gamma([0,T]) \cap K = \varnothing$, or we can take γ_0 minimizing such that $d_1(\gamma_0,\gamma) \le \varepsilon$.

Observe that if $\gamma([0,T]) \cap K = \varnothing$, then $h(\gamma,\dot\gamma) \equiv 0$ and $h \circ \varphi_1(\gamma,\dot\gamma) \equiv 0$. This implies (2.21). Suppose then that $d_1(\gamma_0,\gamma) \le \varepsilon$.

We have that

$$\left| \oint_\gamma h - \oint_{\gamma_0} h \right| \le [h]_{Lip}\, d_1(\gamma,\gamma_0) \le 1 \cdot 1 \cdot \varepsilon,$$

where $[h]_{Lip}$ is the smallest Lipschitz constant of h. Let $Q(h) := \varphi_{-1}(\mathrm{supp}(h))$, then

$$\left| \oint_\gamma h \circ \varphi_1 - \oint_{\gamma_0} h \circ \varphi_1 \right| \le [h]_{Lip}\, [\varphi_1|_{Q(h)}]_{Lip}\, d_1(\gamma,\gamma_0) \le 1 \cdot [\varphi_1|_{Q(h)}]_{Lip} \cdot \varepsilon.$$

Since γ_0 is a solution of (E-L), we have that

$$\left|\oint_{\gamma_0} h - \oint_{\gamma_0} h \circ \varphi_1\right| = \left|\int_0^T h\big(\gamma_0(t), \dot\gamma_0(t)\big) - h\big(\gamma_0(t+1), \dot\gamma_0(t+1)\big)\, dt\right|$$
$$\leq \int_0^1 |h(\gamma_0, \dot\gamma_0)|\, dt + \int_T^{T+1} |h(\gamma_0, \dot\gamma_0)|\, dt \leq 2.$$

Hence

$$\left|\oint_{\gamma} h - \oint_{\gamma} h \circ \varphi_1\right| \leq \varepsilon\big(1 + [\varphi_1|_{Q(h)}]_{Lip}\big) + 2.$$

□

Proof of item 2-3.2.2:

Observe that to prove that μ is invariant it is enough to prove that

$$\int h\, d\mu = \int h\, d(\varphi_1^* \mu) \qquad \text{for all } h \in \mathcal{H}. \tag{2.22}$$

By proposition 2-3.3, there exists a sequence $\mu_{\gamma_n} \in \mathcal{C}(M)$ such that $\mu_{\gamma_n} \to \mu$ and

$$\lim_n A_L(\mu_{\gamma_n}) = A_L(\mu) = \min\left\{\, A_L(\nu) \,|\, \nu \in \overline{\mathcal{C}(M)}\,\right\} =: k. \tag{2.23}$$

Let T_n be a period of the curve $\gamma_n : \mathbb{R} \to M$. Take an integer $N > 0$. By joining a constant curve if necessary, we can assume that every T_n is a multiple of N and that $\lim_{n\to\infty} T_n = +\infty$. Given $C > 0$ let

$$\mathcal{B}_n(C) := \left\{\, j \in \mathbb{N} \,|\, 1 \leq j \leq \tfrac{T_n}{N},\ A_L(\gamma_{n,j}) \geq C\,\right\},$$

where

$$\gamma_{n,j} := \gamma_n|_{[jN,(j+1)N]}.$$

By the superlinearity L is bounded below, adding a constant we can assume that $L > 0$. Then we can assume that

$$A_L(\mu_{\gamma_n}) = \frac{1}{T_n}\int_0^{T_n} L(\gamma_n, \dot\gamma_n)\, dt \leq 2\,k \qquad \forall n.$$

Hence

$$2\,k\,T_n \geq \sum_{j\in\mathcal{B}_n(C)} A_L(\gamma_{n,j}) \geq C\,\#\mathcal{B}_n(C).$$

Thus

$$\frac{\#\mathcal{B}_n(C)}{T_n} \leq \frac{2k}{C}. \tag{2.24}$$

Given $\delta>0$, let

$$\mathcal{B}'_n(\delta) := \big\{\, j\in\mathbb{N} \,|\, 1\leq j\leq \tfrac{T_n}{N}-1,\;\; S^+(\gamma_{n,j})>\delta\,\big\}.$$

Then

$$S^+(\gamma_n) \geq \sum_{j=1}^{(T_n/N)-1} S^+(\gamma_{n,j}) \geq \delta\,\#\mathcal{B}'_n(\delta).$$

Moreover,

$$k \leq \frac{1}{T_n}\,S(\gamma_n(0),\gamma(T_n);T_n) = A_L(\mu_{\gamma_n}) - \frac{1}{T_n}\,S^+(\gamma_n).$$

Hence

$$S^+(\gamma_n) \leq T_n\big(A_L(\mu_{\gamma_n})-k\big).$$

Therefore

$$\frac{\#\mathcal{B}'_n(\delta)}{T_n} \leq \frac{1}{\delta}\,\big(A_L(\mu_{\gamma_n})-k\big). \tag{2.25}$$

Now fix $h\in\mathcal{H}$. Then

$$\left|\int h\,d\mu_{\gamma_n} - \int h\,d(\varphi_1^*\,\mu_{\gamma_n})\right| \leq \frac{1}{T_n}\sum_{j=0}^{(T_n/N)-1}\left|\oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h\circ\varphi_1\right|.$$

Denote $\mathcal{B}''_n := \mathcal{B}_n(C)\cup\mathcal{B}'_n(\delta)$. Since $\sup|h|\leq 1$, then

$$\left|\int h\,d\mu_{\gamma_n} - \int h\,d(\varphi_1^*\,\mu_{\gamma_n})\right| \leq \frac{1}{T_n}\sum_{j\notin\mathcal{B}''_n}\left|\oint_{\gamma_{n,j}} h - \oint_{\gamma_{n,j}} h\circ\varphi_1\right| + \frac{1}{T_n}\,2N\,\#\mathcal{B}''_n.$$

Now choose $C \geq N^2$ and $\delta = \delta(C, h) > 0$ from corollary 2-3.6. Using equations (2.24), (2.25) and corollary 2-3.6 we obtain that

$$\begin{aligned}\left|\int h\, d\mu_{\gamma_n} - \int h\, d(\varphi_1^*\, \mu_{\gamma_n})\right| &\leq \frac{5}{T_n}\,\left(\frac{T_n}{N} - \#\mathcal{B}_n''\right) + \frac{1}{T_n}\,2N\,\#\mathcal{B}_n'' \\ &\leq \frac{5}{N} + 2N\left(\frac{2k}{C} + \frac{1}{\delta}\Big(A_L(\mu_{\gamma_n}) - k\Big)\right).\end{aligned}$$

Now let $n \to \infty$. Using equation (2.23) and that $C \geq N^2$, $\mu_{\gamma_n} \to \mu$ and h, $h \circ \varphi_1 \in C_\ell^0$ (because they have compact support), we obtain that

$$\left|\int h\, d\mu - \int h\, d(\varphi_1^*\, \mu)\right| \leq \frac{5}{N} + \frac{4k}{N}.$$

Since N is arbitrary, this difference is zero and we get (2.22). □

2-4 Ergodic characterization of the critical value.

Given a Borel probability measure μ in TM define its *action* by

$$A_L(\mu) = \int_{TM} L \, d\mu.$$

Since by the superlinearity the lagrangian L is bounded below, this action is well defined.

Let $\mathcal{M}(L)$ be the set of φ_t-invariant probabilities on TM.

2-4.1. Theorem (Mañé [28]). *If M is compact, then*

$$c(L) = -\min\{\, A_L(\mu) \mid \mu \in \mathcal{M}(L) \,\}.$$

We will obtain theorem 2-4.1 from theorem 2-4.2 below, which also applies to the non-compact case.

Recall that if $\gamma : [0, T] \to M$ is a closed absolutely continuous curve, the measure $\mu_\gamma \in \mathcal{M}_\ell$ is defined by

$$\int f \, d\mu_\gamma = \frac{1}{T} \int_0^T f\big(\gamma(t), \dot{\gamma}(t)\big) \, dt$$

for all $f \in C_\ell^0$, and that $\overline{\mathcal{C}(M)}$ is the closure of the set of such μ_γ's in $\mathcal{M}_\ell$.

2-4.2. Theorem.

$$\begin{aligned} c(L) &= -\inf\{\, A_L(\mu) \mid \mu \in \overline{\mathcal{C}(M)} \,\} \\ &= -\inf\{\, A_L(\mu) \mid \mu \in \mathcal{C}(M) \,\}. \end{aligned} \tag{2.26}$$

2-4.3. Remarks.

1. The equality between the two infimums is non-trivial and follows from proposition 2-3.3.

2. theorems 2-4.2 and 2-3.2 imply theorem 2-4.1.

3. If $p : N \to M$ is a covering, M is compact and $\mathbb{L} = L \circ dp$ is the lifted lagrangian, then theorems 2-4.2 and 2-3.2 imply that

$$c(\mathbb{L}) = -\min\{\, A_L(\mu) \mid \mu \in \mathcal{M}(L) \cap \overline{dp_*\mathcal{C}(N)}\,\}, \tag{2.27}$$

   by noticing that $A_L(dp_*\,\nu) = A_{\mathbb{L}}(\nu)$ for $\nu \in \overline{\mathcal{C}(N)}$. Here $dp_*\mathcal{C}(N)$ is the set of probabilities μ_γ on TM where γ is a curve on M whose lifts to N are closed. The compactness property on theorem 2-3.2(3) allows to obtain a minimum on (2.27) instead of the infimum on (2.26) which may not be attained in the non-compact case.

4. The statement for coverings in equation (2.27) allows to obtain minimizing measures which don't appear in the Mather's theory. For example if c_u is the critical value of the universal cover $\widetilde{M}$ of M and c_0 is the critical value of the abelian cover $\widehat{M}$ of M; the minimizing measures on a fixed homology class (corresponding to Mather's theory) all have action $A(\mu) \geq -c_0$, (see equation (2.30) and proposition 2-6.3), while the minimizing measures for $\widetilde{M}$ have action $c_u < c_0$.

   The measures for $\widetilde{M}$ correspond to "minimizing in the zero homotopy class" while the measures for $\widetilde{M}$ are minimizing in the zero homology class.

   The drawback of this approach is that we obtain honest minimizing invariant measures on TM which may not lift to finite measures on the covering TN.

Proof of theorem 2-4.2:

If $\mu_\gamma \in \mathcal{C}(M)$, then $A_{L+c(L)}(\mu_\gamma) \geq 0$. Hence $A_{L+c(L)}(\mu) \geq 0$. Thus

$$-c(L) \leq \inf\{\, A_L(\mu) \mid \mu \in \mathcal{C}(M)\,\} = \inf\{\, A_L(\mu) \mid \mu \in \overline{\mathcal{C}(M)}\,\},$$

where the last equality follows from proposition 2-3.3.

If $k < c(L)$ then there is a closed absolutely continuous curve γ on M such that $A_{L+k}(\gamma) < 0$. Thus $\mu_\gamma \in \mathcal{C}(M)$ and

$$-k > A_L(\mu_\gamma) \geq \inf\{\, A_L(\mu) \mid \mu \in \overline{\mathcal{C}(M)}\}.$$

Now let $k \uparrow c(L)$. $\square$

2-5 The Aubry-Mather Theory.

Through this section we shall assume that M is compact.

2-5.a Homology of measures.

A holonomic probability $\mu \in \overline{\mathcal{C}(M)}$ satisfies $\int_{TM} |v|\, d\mu < +\infty$ and

$$\int_{TM} df\, d\mu = 0 \quad \text{for all } f \in C^\infty(M, \mathbb{R}).$$

Then we can define its *homology class* as $\rho(\mu) \in H_1(M, \mathbb{R}) \approx H^1(M, \mathbb{R})^*$ by

$$\langle\, \rho(\mu)\, , [\omega]\, \rangle = \int_{TM} \omega\, d\mu\, , \tag{2.28}$$

for any closed 1-form ω on M, where $[\omega] \in H^1(M, \mathbb{R})$ is the cohomology class of ω. Here we have used the identification[1] $H_1(M, \mathbb{R}) \approx H^1(M, \mathbb{R})^*$ and equation (2.28) shows how the homology class $\rho(\mu)$ acts on $H^1(M, \mathbb{R})$. Since μ is holonomic, the integral in (2.28) depends only on the cohomology class of ω. The class $\rho(\mu)$ is called the *homology* of μ or the *rotation* of μ by analogy to the twist map theory.

Using a finite basis $\{\, [\omega_1], \dots, [\omega_k]\, \}$ for $H^1(M, \mathbb{R})$ and the topology of $\overline{\mathcal{C}(M)}$, we have that

2-5.1. Lemma. *The map $\rho : \overline{\mathcal{C}(M)} \to H^1(M, \mathbb{R})$ is continuous.*

2-5.b The asymptotic cycle.

Given a differentiable flow φ_t on a compact manifold N and a φ_t-invariant probability μ, the Schwartzman's [56] *asymptotic cycle* of an

[1] In fact, $H^1(M, \mathbb{R}) \approx \hom\big(H_1(M, \mathbb{R}), \mathbb{R}\big) = H_1(M, \mathbb{R})^*$ by the universal coefficient theorem. Since M is compact, then $H_1(M, \mathbb{R})$ is a finite dimensional vector space and hence it is naturally isomorphic to its double dual $H^1(M, \mathbb{R})^*$.

invariant probability μ is defined to be the homology class $A(\mu) \in H_1(N,\mathbb{R}) \approx H^1(N,\mathbb{R})^*$ such that

$$\langle\, A(\mu)\,,[\omega]\,\rangle = \int_N \omega(X)\ d\mu\,,$$

for any closed 1-form ω, where $[\omega] \in H^1(N,\mathbb{R})$ is the cohomology class of ω and X is the vectorfield of φ_t. This integral depends only on the cohomology class of ω because the integral of a coboundary by an invariant measure is zero: in fact, if df is an exact 1-form, then define

$$F(y) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T df\big(X(\varphi_t\, y)\big)\ dt = \lim_{T\to+\infty} \tfrac{1}{T}\ \big[f(\varphi_t\, y) - f(y)\,\big] = 0,$$

by Birkhoff's theorem,

$$\int_N df(X)\ d\mu = \int_N F\ d\mu = 0.$$

If μ is ergodic and $x \in N$ is a *generic point* [2] for μ, then

$$\langle\, A(\mu)\,,[\omega]\,\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \omega\big(X(\varphi_t\, x)\big)\ dt\,.$$

Applying this to a basis $\{\omega_1,\ldots,\omega_k\}$ for $H^1(N,\mathbb{R})$, we get that

$$A(\mu) = \lim_{T\to+\infty} \tfrac{1}{T}\,[\,\gamma_T * \delta_T\,] \in H_1(N,\mathbb{R}),$$

where $\gamma_T(t) = \varphi_t(x)$, $t \in [0,T]$, the curve δ_T is a unit speed geodesic from $\varphi_T(x)$ to x, and the limit is on the finite dimensional vector space $H_1(N,\mathbb{R})$.

In the case of a lagrangian flow, the phase space $N = TM$ is not compact, but it has the same homotopy type as the configuration space M because M is a deformation retract of TM (contracting TM along the

[2] i.e. $\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\varphi_t\, x)\ dt = \int f\ d\mu$ for all $f \in C^0(N,\mathbb{R})$.

fibers to the zero section $M \times 0$). Moreover, the ergodic components of an invariant measure of a lagrangian flow are contained in a unique energy level, which is a compact submanifold of TM by remark 1-3.1.

We see that the homology of an invariant probability and its asymptotic cycle coincide under the identification $H_1(TM, \mathbb{R}) \overset{\pi_*}{\approx} H_1(M, \mathbb{R})$.

2-5.2. Proposition.

$$\pi_*\big(A(\mu)\big) = \rho(\mu) \qquad \textit{for all } \mu \in \mathcal{M}(L),$$

where $\pi_* : H_1(TM, \mathbb{R}) \to H_1(M, \mathbb{R})$ *is the map induced by the projection* $TM \xrightarrow{\pi} M$.

Proof: If ω is a closed 1-form on M, then

$$(\pi^* \omega)(X(x,v)) = \omega\big[d\pi\big(X(x,v)\big)\big] = \omega_x(v)\,,$$

because the lagrangian vectorfield X has the form $X(x,v) = (v, *)$. Then

$$\begin{aligned} \langle\, \pi_* A(\mu)\,, [\omega]\,\rangle = \langle\, A(\mu)\,, \pi^*[\omega]\,\rangle &= \int_{TM} (\pi^* \omega)(X)\ d\mu \\ &= \int_{TM} \omega\ d\mu = \langle\, \rho(\mu)\,, [\omega]\,\rangle. \end{aligned}$$

□

2-5.3. Lemma. *The map* $\rho : \mathcal{M}(L) \to H_1(M, \mathbb{R})$ *is surjective.*

Proof: Let $h \in H_1(M, \mathbb{Z})$ be an integer homology class. Let $\eta : [0,1] \to M$ be a closed curve with homology class h. Let γ be a minimizer of the action of L among the set of absolutely continuous curves $[0,1] \to M$ with the same homotopy class as η. Then by remark 1-2.2, γ is a periodic orbit for the lagrangian flow with period 1. The invariant measure μ_γ satisfies $\rho(\mu_\gamma) = h$.

The map ρ is affine and $\mathcal{M}(L)$ is convex; hence $\rho\big(\mathcal{M}(L)\big)$ is convex and, in particular, it contains the convex hull of $H_1(M, \mathbb{Z})$. Thus, $H_1(M, \mathbb{R}) \subseteq \rho\big(\mathcal{M}(L)\big)$. □

2-5.c The alpha and beta functions.

The action functional $A_L : \mathcal{M}(L) \to \mathbb{R}$ is lower semicontinuous[3] and the sets

$$\mathcal{M}(h) := \{\, \mu \in \mathcal{M}(L) \,|\, \rho(\mu) = h \,\}$$

are closed. Hence we can define the *Mather's beta function* $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) := \min_{\mu \in \mathcal{M}(h)} A_L(\mu).$$

We shall prove below that the β-function is convex. The *Mather's alpha function* $\alpha = \beta^* : H^1(M,\mathbb{R}) \to \mathbb{R}$ is the convex dual of the β-function:

$$\begin{aligned}
\alpha([\omega]) &= \max_{h \in H_1(M,\mathbb{R})} \{\, \langle [\omega]\,, h \rangle - \beta(h) \,\} && \text{by 2-5.3,} \\
&= - \min_{\mu \in \mathcal{M}(L)} \{\, A_L(\mu) - \langle [\omega]\,, \rho(\mu) \rangle \,\} \\
&= - \min_{\mu \in \mathcal{M}(L)} A_{L-\omega}(\mu) \\
&= c(L - \omega)\,, && \text{by 2-4.1.} \qquad (2.29)
\end{aligned}$$

Observe that since $L - \omega$ is also a convex superlinear lagrangian, then $\alpha([\omega])$ is finite.

2-5.4. Theorem. *The α and β functions are convex and superlinear.*

Proof: We first prove that β is convex. Let $h_1, h_2 \in H_1(M,\mathbb{R})$ and $0 \le \lambda \le 1$. Let $\mu_1, \mu_2 \in \mathcal{M}(L)$ be such that $\rho(\mu_i) = h_i$ and $A_L(\mu_i) = \beta(h_i)$ for $i = 1, 2$. The probability $\nu = \lambda\, \mu_1 + (1 - \lambda)\, \mu_2$ satisfies $\rho(\nu) = \lambda\, h_1 + (1 - \lambda)\, h_2$. Hence

$$\beta\big(\lambda\, h_1 + (1-\lambda)\, h_2\big) \le A_L\big(\lambda\, \mu_1 + (1-\lambda)\, \mu_2\big) = \lambda\, \beta(h_1) + (1-\lambda)\, \beta(h_2).$$

[3] A_L is lower semicontinuous iff $\liminf_n A_L(\nu_n) \ge A_L(\mu)$ when $\nu_n \to \mu$.

By proposition 2-5.3, ρ is surjective, and hence β is finite. By proposition C.1, α is superlinear. By C.1, α and β are convex. Formula (2.29), implies that α is finite and then by C.1 β is superlinear. □

For $h \in H_1(M, \mathbb{R})$ and $\omega \in H^1(M, \mathbb{R})$, write

$$\begin{aligned} \mathcal{M}_h(L) &:= \{\, \mu \in \mathcal{M}(L) \mid \rho(\mu) = h,\ A_L(\mu) = \beta(h) \,\}, \\ \mathcal{M}^\omega(L) &:= \{\, \mu \in \mathcal{M}(L) \mid A_{L-\omega}(\mu) = -c(L-\omega) \,\}. \end{aligned}$$

Since the β-function has a supporting hyperplane at each homology class h, if $\omega \in \partial\beta(h)$, then $\mathcal{M}_h(L) \subseteq \mathcal{M}^\omega(L)$. Conversely, since by corollary C.2 $\alpha^* = \beta$, then $\mathcal{M}^\omega(L) \subseteq \mathcal{M}_h(L)$ if $h \in \partial\alpha(\omega)$. Thus

$$\bigcup_{h \in H_1(M,\mathbb{R})} \mathcal{M}_h(L) = \bigcup_{\omega \in H^1(M,\mathbb{R})} \mathcal{M}^\omega(L).$$

We call these measures *Mather minimizing* measures and the set

$$\mathcal{M} := \bigcup_{\omega \in H^1(M,\mathbb{R})} \bigcup_{\mu \in \mathcal{M}^\omega(L)} \operatorname{supp}(\mu) = \bigcup_{h \in H_1(M,\mathbb{R})} \bigcup_{\mu \in \mathcal{M}_h(L)} \operatorname{supp}(\mu)$$

the *Mather set.*

Define the *strict critical value* as

$$\begin{aligned} c_0(L) : &= \min_{\omega \in H^1(M,\mathbb{R})} c(L-\omega) = \min_{\omega \in H^1(M,\mathbb{R})} \alpha(\omega) \\ &= -\beta(0). \end{aligned} \tag{2.30}$$

By corollary 3-6.3 the strict critical value is the lowest energy level which supports Mather minimizing measures and since $c_0(L) = -\beta(0)$, these minimal energy Mather minimizing measures have trivial homology.

2-6 Coverings.

We shall deal mainly with compact manifolds M, but there are some important non-compact cases, for example the coverings of M. Particularly interesting are the abelian cover $\widehat{M}$, the universal cover $\widetilde{M}$ and the finite coverings.

The *abelian cover* $\widehat{M}$ of M is the covering whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M,\mathbb{Z})$. Its deck transformation group is $H_1(M,\mathbb{Z})$ and $H^1(\widetilde{M},\mathbb{Z}) = \{0\}$. When $\pi_1(M)$ is abelian, $\widehat{M} = \widetilde{M}$. A closed curve in $\widehat{M}$ projects to a closed curve in M with trivial homology.

If $M_1 \xrightarrow{p} M$ is a covering, denote by $L_1 := L \circ dp : TM_1 \to \mathbb{R}$ the lifted lagrangian to TM_1.

2-6.1. Lemma. *If $M_1 \xrightarrow{p} M$ is a covering, then $c(L_1) \leq c(L)$.*

Proof: The lemma follows form the fact that closed curves on N project to closed curves on M. □

2-6.2. Proposition. *If M_1 is a finite covering of M_2 then $c(L_1) = c(L_2)$.*

Proof: We know that $c(L_1) \leq c(L_2)$. Suppose that the strict inequality holds and let k be such that $c(L_1) < k < c(L_2)$. Hence there exists a closed curve γ in M_2 with negative $(L_2 + k)$-action. Since M_1 is a finite covering of M_2 some iterate of γ lifts to a closed curve in M_1 with negative $(L_1 + k)$-action which contradicts $c(L_1) < k$. □

2-6.3. Proposition. [52]

$$c_0(L) = c_a(L) = \textit{critical value of the abelian cover.}$$

Then we have

$$c_u(L) \leq c_a(L) = c_0(L) \leq c(L-\omega) \qquad \forall [\omega] \in H^1(M,\mathbb{R}),$$

where c_u is the critical value of the lift of the lagrangian to the universal cover. When $c_u(L) < c_0(L)$, the method in equation (2.27) gives some minimizing measures which are not Mather minimizing. For symmetric lagrangians $c(L) = e_0 = c_0(L) = c_u(L)$. Mañé [32] gives an example in which $e_0 < c_a(L) = c_0(L) < c(L)$. G. Paternain and M. Paternain [52] give an example in which $c_u(L) < c_a(L)$.

Proof: Let ω be a closed form in M. Since $H_1(M,\mathbb{R}) = \{0\}$, the lift $\widehat{\omega}$ of ω to $\widehat{M}$ is exact, then

$$c_a(L) := c(\widehat{L}) = c(\widehat{L} - \widehat{\omega}) \leq c(L - \omega).$$

Hence

$$c_a(L) \leq \min_{\omega \in H^1(M,\mathbb{R})} c(L - \omega) = c_0(L).$$

Moreover,

$$\begin{aligned} -c_a(L) &= \inf\left\{ A_{\widehat{L}}(\mu) \mid \mu \in \overline{\mathcal{C}(\widehat{M})} \right\} \\ &= \inf\{ A_L(\mu) \mid \mu \in \overline{\mathcal{C}(M)},\ \rho(\mu) = 0 \}, \end{aligned}$$

because if $\mu \in \overline{\mathcal{C}(M)}$, by proposition 2-3.3 there are $\mu_{\gamma_n} \in \mathcal{C}(M)$ with $A_{\widehat{L}}(\mu_{\gamma_n}) \to A_{\widehat{L}}(\mu)$. Since γ_n is closed, its projection $p \circ \gamma_n$ has homology $[p \circ \gamma_n] = 0$. Then $p^*(\mu_{\gamma_n}) = \mu_{p\circ\gamma_n} \to p^*\mu$, $A_L(p^*\mu_{\gamma_n}) \to A_L(p^*\mu) = A_{\widehat{L}}(\mu)$ and by proposition 2-5.1 $\rho(p^*\mu) = \lim_n \rho(p^*\mu_{\gamma_n}) = 0$.

Since $\mathcal{M}(L) \subseteq \overline{\mathcal{M}}$, we get that

$$\begin{aligned} -c_a(L) &\leq \min\{ A_L(\mu) \mid \mu \in \mathcal{M}(L),\ \rho(\mu) = 0 \} \\ &= -\beta(0) = c_0(L). \end{aligned}$$

□

The *real abelian cover* is the covering $\check{M}$ of M with $h : \pi(M) \to H_1(M,\mathbb{R})$ is the Hurewicz homomorphism. It is an intermediate covering $\widetilde{M} \to \check{M} \to M$ and the deck transformations of $\widehat{M} \to \check{M}$ are given by the torsion[4] of $H_1(M,\mathbb{Z})$. Hence $\widehat{M} \to \check{M}$ is a finite cover so that they have the same critical value $c_a(L) = c_0(L)$.

[4] i.e. the elements of finite order $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \subseteq H_1(M,\mathbb{Z})$.

Chapter 3

Globally minimizing orbits.

3-1 Tonelli's theorem.

Given x, $y \in M$ and $T > 0$, let

$$\mathcal{C}_T(x,y) := \{\, \gamma \in \mathcal{C}^{ac}([0,T], M) \mid \gamma(0) = x,\ \gamma(T) = y \,\}.$$

We say that $\gamma \in \mathcal{C}_T(x,y)$ is a *Tonelli minimizer* if

$$A_L(\gamma) = \min_{\eta \in \mathcal{C}_T(x,y)} A_L(\eta)\,.$$

In this section we shall prove

3-1.1. Tonelli's Theorem.
For all x, $y \in M$ and $T > 0$ there exists a Tonelli minimizer on $\mathcal{C}_T(x,y)$.

The only difference in the proof of this theorem when M is noncompact is corollary 3-1.7. An independent proof of this corollary is given in remark 3-1.8.

The idea of Tonelli's theorem is to prove that the sets

$$\mathcal{A}(c) := \{\, \gamma \in \mathcal{C}_T(x,y) \mid A_L(\gamma) \le c \,\} \tag{3.1}$$

are compact in the C^0-topology. Then a Tonelli minimizer will be a curve in

$$\bigcap_{c \geq \alpha} \mathcal{A}(c) \neq \varnothing,$$

where $\alpha = \inf_{\eta \in \mathcal{C}_T(x,y)} A_L(\eta) \geq \inf L > -\infty$.

An addendum to Tonelli's theorem due to Mather [39] states that these sets are compact in the topology of absolutely continuous curves. Given γ_1, $\gamma_2 \in C^{ac}([0,T], M)$ define their *absolutely continuous distance* by

$$d_1(\gamma_1, \gamma_2) := \sup_{t \in [0,T]} d_M\big(\gamma_1(t), \gamma_2(t)\big) + \int_0^T d_{TM}\big([\gamma_1(t), \dot{\gamma}_1(t)], [\gamma_2(t), \dot{\gamma}_2(t)]\big)\, dt.$$

3-1.2. Theorem (Mather [39]). *For any x, $y \in M$, $T > 0$, $b \in \mathbb{R}$, the set*

$$\mathcal{A}(b) := \big\{\, \gamma \in \mathcal{C}_T(x,y) \mid A_L(\gamma) \leq b \,\big\}$$

is compact in the d_1-topology.

This theorem follows from the fact that $\mathcal{A}(b)$ is compact in the C^0 topology, which is proved in 3-1.11 and the following proposition (addendum on page 175 of Mather [39]).

3-1.3. Proposition. (**Mather [39]**)

If $N \subseteq M$ is a compact subset and $\gamma_1, \gamma_2, \ldots$ is a sequence in $C^{ac}([a,b], N)$ which converges C^0 to γ and $A_L(\gamma_i)$ converges to $A_L(\gamma)$, then γ_1, $\gamma_2, \ldots$ converges in the d_1-topology to γ.

We shall split the proof of Tonelli's theorem in several parts:

3-1.4. Definition.

A family $\mathcal{F} \subseteq C^0([a,b], M)$ is *absolutely equicontinuous* if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ such that

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{N} d(x_{s_i}, x_{t_i}) < \varepsilon,$$

whenever $]s_1, t_1[, \ldots,]s_N, t_N[$ are disjoint intervals in $[a,b]$.

3-1.5. Remark.

(i) An absolutely equicontinuous family is equicontinuous.

(ii) A uniform limit limit of absolutely equicontinuous functions is absolutely continuous.

3-1.6. Lemma. *For all $c \in \mathbb{R}$ and $T > 0$, the family*

$$\mathcal{F}(c) := \left\{ \gamma \in \mathcal{C}^{ac}([0,T], M) \mid A_L(\gamma) \le c \right\}$$

is absolutely equicontinuous.

Proof: Since by the superlinearity, the lagrangian L is bounded below; by adding a constant we may assume that $L \ge 0$. For $a > 0$ let

$$K(a) := \inf \left\{ \frac{L(x,v)}{|v|} \;\middle|\; (x,v) \in TM,\ |v| \ge a \right\}. \tag{3.2}$$

The superlinearity implies that $\lim_{a \to +\infty} K(a) = +\infty$. Given $\varepsilon > 0$ let $a > 0$ be such that

$$\frac{c}{K(a)} < \frac{\varepsilon}{2}.$$

Let $0 \le s_1 < t_1 \le \cdots \le s_N < t_N \le T$, $J := \cup_{i=1}^N [s_i, t_i]$ and $E := J \cap \big[|\dot{x}| > a\big]$, then $L(x_s, \dot{x}_s) \ge K(a)\,|\dot{x}_s|$ for $s \in E$. We have that

$$\begin{aligned} K(a) \sum_{i=1}^N d(x_s, x_t) &\le K(a) \int_E |\dot{x}| + K(a) \int_{J\setminus E} |\dot{x}| \\ &\le \int_E L(x,\dot{x}) + a \cdot K(a)\, m(J) \\ &\le c + a \cdot K(a)\, m(J), \qquad \text{(because } L \ge 0), \end{aligned}$$

where m is the Lebesgue measure on $[0,T]$

$$d(x_s, x_t) \le \textstyle\int_J |\dot{x}_s| \le \frac{c}{K(a)} + a\, m(J) \le \frac{\varepsilon}{2} + a\, m(J). \tag{3.3}$$

This implies the equicontinuity of $\mathcal{F}(c)$.

□

In order to apply the Arzela-Ascoli theorem we need a compact range, for this we have:

3-1.7. Corollary. *For all $c \in \mathbb{R}$ and $T > 0$ there is $R > 0$ such that for all $x, y \in M$,*

$$\mathcal{A}(c) \subseteq \mathcal{C}^{ac}([0,T], \overline{B}(x,R)),$$

where $\overline{B}(x,R) := \{ z \in M \mid d_M(x,z) \leq R \}$.

Proof: Inequality (3.3) for $N = 1$ and $J = [s,t]$ is $d(x_s, x_t) \leq \frac{\varepsilon}{2} + a\,|t-s|$. It is enough to take $R = \frac{\varepsilon}{2} + a\,T$. □

3-1.8. Remark.

corollary 3-1.7 is the only difference for the proof of Tonelli's theorem when M is non-compact. Another proof for corollary 3-1.7 is the following:

Adding a constant we may assume that $L \geq 0$. There is $B > 0$ such that $L(x,v) \geq |v| - B$ for all $(x,v) \in TM$. Then for $0 \leq s \leq t \leq T$, we have that

$$d(x_s, x_t) \leq \int_s^t |\dot{x}| \leq BT + \int_s^t L(x,\dot{x}) \leq BT + c.$$

□

Recall that

$$\mathcal{F}(c) := \left\{ \gamma \in \mathcal{C}^{ac}([0,T], M) \mid A_L(\gamma) \leq c \right\}$$

3-1.9. Theorem. (Cle)

If $\gamma_n \in \mathcal{F}(c)$ and $\gamma_n \xrightarrow{n} \gamma$ in the uniform topology, then $\gamma \in \mathcal{F}(c)$.

We shall need the following lemma. We may assume that $M = \mathbb{R}^n$,

3-1.10. Lemma. *Given K compact, $a > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that if $x \in K$, $\|x - y\| < \varepsilon$, $|v| \leq a$ and $w \in \mathbb{R}^n$, then*

$$L(y,w) \geq L(x,v) + L_v(x,v)\,(w-v) - \varepsilon. \tag{3.4}$$

Proof of Cle's theorem:

It is immediate from the definition 3-1.4 that a uniform limit of absolutely equicontinuous curves is absolutely continuous. We may assume that $\gamma_n([0,T])$ is contained in a compact neighbourhood K of $\gamma([0,T])$. By the superlinearity we may assume that $L \geq 0$. Let $\varepsilon > 0$ and $E = \big[|\dot\gamma| \leq a\big]$, then by lemma 3-1.10, for n large,

$$\int_{[0,T]\setminus E} \big[L(\dot\gamma) + L_v(\dot\gamma)(\dot\gamma - \dot\gamma_n) - \varepsilon\big] \leq \int_{E^c} L(\dot\gamma_n) \leq c \qquad \text{(since } L \geq 0\text{)}. \tag{3.5}$$

Claim: *$\dot\gamma_n \to \dot\gamma$ in $\mathcal{L}^1$.*

Using the claim, since $L_v(x,v)$ is bounded for $x \in K$ and $|v| \leq a$, then

$$\int_{E^c} L_v(\dot\gamma)\,(\dot\gamma - \dot\gamma_n) \xrightarrow{n} 0.$$

Letting $n \to +\infty$ on (3.5), we have that

$$\int_{E^c} L(\dot\gamma) - \varepsilon\, T \leq c.$$

Since $E^c \uparrow [0,T]$ when $a \to +\infty$, then

$$\int_0^T L(\dot\gamma) = \lim_{a \to +\infty} \int_{E^c} L(\gamma) \leq c + \varepsilon\, T.$$

Now let $\varepsilon \to 0$.

We now prove the claim. By inequality (3.3), for $\varepsilon > 0$ there is $\delta > 0$ such that

$$m(D) < \delta \implies \int_D |\dot\gamma_n| < \varepsilon. \tag{3.6}$$

where m is the Lebesgue measure. If $A = [s,t]$, then

$$\lim_{n\to\infty} \int_A (\dot\gamma_n - \dot\gamma) = \lim_{n\to\infty} (\gamma_n - \gamma)\Big|_s^t = 0. \tag{3.7}$$

Equations (3.6) and (3.7) imply that (3.7) holds for any Borel set A, approximating A by finite unions of intervals. Then

$$\lim_n \int_{[0,T]} (\dot\gamma_n - \dot\gamma)\, \psi = 0,$$

for any $\psi \in \mathcal{L}^\infty$. This implies that $\dot\gamma_n \to \dot\gamma$ in $\mathcal{L}^1$. □

3-1.11. Proof of Tonelli's theorem:

By lemma 3-1.6, the family $\mathcal{A}(c)$ in (3.1) is equicontinuous, and by corollary 3-1.7, the curves in $\mathcal{A}(c)$ have a uniform compact range. By Arzelá-Ascoli's theroem and Cle's theorem, $\mathcal{A}(c)$ is compact. Then

$$\gamma \in \bigcap_{c \geq \inf_{\mathcal{C}_T(x,y)} A_L} \mathcal{A}(c)$$

is a Tonelli's minimizer on $\mathcal{C}_T(x,y)$. □

Proof of lemma 3-1.10:

Let

$$\begin{aligned} C_1 &:= \sup\{\, L_v(x,v) \mid x \in K,\ |v| \leq a \,\} \\ C_2 &:= \sup\{\, L(x,v) - L_v(x,v)\cdot v \mid x \in K,\ |v| \leq a \,\}. \end{aligned}$$

Let $b > 0$ be such that

$$K(b)\cdot b \geq C_2 + C_1\, r \qquad \text{for all } r \geq b,$$

where $K(d)$ is from (3.2). Then if $y \in M$ and $|w| > b$,

$$\begin{aligned} L(y,w) &\geq K(d)\, |w| \\ &\geq C_2 + C_1\, |w| \\ &\geq C_2 + L_v(x,v)\cdot w \\ &\geq L(x,v) + L_v(x,v)\cdot(w - v) \qquad \text{for } |w| \geq d. \end{aligned}$$

Since L is convex,

$$L(x,w) \geq L(x,v) + L_v(x,v)\cdot(w-v) \qquad \forall\, w \in \mathbb{R}^n.$$

Then there is $\delta > 0$ such that for $|x-y| < \delta$, $|v| \leq a$ and $|w| \leq b$ inequality (3.4) holds.

$\square$

3-2 A priori compactness.

The following lemma, due to Mather [39] for Tonelli minimizers in the non-autonomous case, will be very useful. In the autonomous case its proof is very simple.

3-2.1. Lemma.
For $C > 0$ there exists $A = A(C) > 0$ such that if x, $y \in M$ and $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation with $A_L(\gamma) \le C\,T$, then $|\dot{\gamma}(t)| < A$ for all $t \in [0, T]$.

Proof: By the superlinearity there is $D > 0$ such that $L(x, v) \ge |v| - D$ for all $(x, v) \in TM$. Since $A_L(\gamma) \le C\,T$, the mean value theorem implies that there is $t_0 \in]0, T[$ such that

$$|\dot{\gamma}(t_0)| \le D + C.$$

The conservation of the energy and the uniform bounds (1.10) and (1.9) imply that there is $A = A(C) > 0$ such that $|\dot{\gamma}| \le A$.

□

For $k \ge c(L)$ and x, $y \in M$, define

$$\Phi_k(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,t)} A_{L+k}(\gamma).$$

3-2.2. Corollary. *Given $\varepsilon > 0$, there are constants $A(\varepsilon)$, $B(\varepsilon)$, $C(\varepsilon) > 0$ such that if $T \ge \varepsilon$ and $k \ge c(L)$, then*

(i) $\Phi_k(x, y; T) \le C(\varepsilon)\,T$.

(ii) If $\gamma \in \mathcal{C}_T(x, y)$ is a solution of the Euler-Lagrange equation such that $A_{L+k}(\gamma) \le C(\varepsilon)\,T + 1$, then $|\dot{\gamma}| \le A(\varepsilon)$ and $E(\gamma, \dot{\gamma}) \le B(\varepsilon)$.

Proof: Comparing with the action of a geodesic on $C_T(x,y)$, we get (i) with

$$C(\varepsilon) = \sup\{\, L(v) \mid |v| \le \tfrac{d(x,y)}{\varepsilon} \,\}.$$

Using (i) and lemma 3-2.1,we obtain $A(\varepsilon)$. Using $A(\varepsilon)$ and inequality (1.10) we obtain $B(\varepsilon)$. □

3-2.3. Lemma.

There exists $A > 0$ such that if $x, y \in M$ and $\gamma \in C_T(x,y)$ is a solution of the Euler-Lagrange equation with

$$A_{L+c}(\gamma) \le \Phi_c(x,y) + d_M(x,y),$$

then (a) $T > \frac{1}{A}\, d_M(x,y)$.

(b) $|\dot\gamma(t)| < A$ for all $t \in [0,T]$.

Proof: Let $\eta : [0, d(x,y)] \to M$ be a minimal geodesic with $|\dot\eta| \equiv 1$. Let $\ell(r)$ be from (1.1) and $D = \ell(1) + c + 2$. From the superlinearity condition there is $B > 0$ such that

$$L(x,v) + c > D\,|v| - B, \qquad \forall (x,v) \in TM.$$

Then

$$\begin{aligned}
[\ell(1) + c]\; d(x,y) &\ge A_{L+c}(\eta) \ge \Phi_c(x,y) && (3.8)\\
&\ge A_{L+c}(\gamma) - d(x,t) && (3.9)\\
&\ge \int_0^T \big(D\,|\dot\gamma| - B\,\big)\, dt - d(x,y)\\
&\ge D\; d(x,y) - B\,T - d(x,y).
\end{aligned}$$

Hence

$$T \ge \tfrac{A-\ell-c-1}{B}\; d(x,y) \ge \tfrac{1}{B}\; d(x,y).$$

From (3.8) and (3.9), we get that

$$\begin{aligned} A_L(\gamma) &\leq \big[\,\ell(1) + c + 1\,\big]\, d(x,y) - c\,T, \\ &\leq \big\{\, B\,[\,\ell(1) + c + 1\,] - c\,\big\}\, T. \end{aligned}$$

Then lemma 3-2.1 completes the proof. □

3-3 Energy of time-free minimizers.

A curve $\gamma \in \mathcal{C}(x,y)$ is a *global minimizer* or *time free* minimizer for $L+k$ if $k \geq c(L)$ and $A_{L+k}(\gamma) = \Phi_k(x,y)$.

3-3.1. Proposition. *A time-free minimizer for $L+k$ has energy $E \equiv k$.*

We need the following

3-3.2. Lemma. *Let : $x : [0,T] \to M$ be an absolutely continuous curve and $k \in \mathbb{R}$. For $\lambda > 0$, let $x_\lambda(t) := x(\lambda t)$ and $\mathcal{A}(\lambda) := A_{L+k}(x_\lambda)$. Then*

$$\mathcal{A}'(1) = \int_0^T \big[\, E(x,\dot{x}) - k \,\big]\, dt.$$

Proof: Since $\dot{x}_\lambda(t) = \lambda\, \dot{x}(\lambda t)$, then

$$\mathcal{A}(\lambda) = \int_0^{\frac{T}{\lambda}} \big[\, L(x(\lambda t),\, \lambda\, \dot{x}(\lambda t)) + k \,\big]\, dt.$$

Differentiating $\mathcal{A}(\lambda)$ and evaluating at $\lambda = 1$, we have that

$$\mathcal{A}'(1) = -T\,\big[\, L(x(T),\dot{x}(T)) + k \,\big] + \int_0^T \big[\, L_x\, t\, \dot{x} + L_v\, (\dot{x} + t\, \ddot{x}) \,\big]\ dt.$$

Integrating by parts the term $(L_x\, \dot{x} + L_v \ddot{x})\, t = \left(\frac{d}{dt}L\right) t$, we have that

$$\begin{aligned}\mathcal{A}'(1) &= -T\,\big[\, L(x(T),\dot{x}(T)) + k \,\big] + L\, t\,\big|_0^T + \int_0^T \big(\, L_v\, \dot{x} - L \big)\ dt \\ &= -T\, k + \int_0^T E(x,\dot{x})\ dt = \int_0^T \big[\, E(x,\dot{x}) - k \,\big]\, dt.\end{aligned}$$

□

Proof of proposition 3-3.1.

Since γ is a solution of the Euler-Lagrange equation its energy $E(\gamma,\dot{\gamma})$ is constant. Since it minimizes with free time, the derivative in lemma 3-3.2 must be zero. So that $E(\gamma,\dot{\gamma}) \equiv 0$. □

3-3.3. Corollary.

Let $x \in \mathcal{C}^{ac}([0,1], M)$ *and* $k > 0$*. For* $T > 0$*, write* $y_T(t) = x(\frac{t}{T}) : [0,T] \to M$ *and* $\mathcal{B}(T) = A_{L+k}(y_T)$*. Then*

$$\mathcal{B}'(T) = -\frac{1}{T}\int_0^T \left[\, E(y_T, \dot{y}_T) - k \,\right] dt.$$

Proof: Using $\lambda = \frac{T}{S}$ on lemma 3-3.2, we have that $\frac{d}{dS} = -\frac{T}{S^2}\,\frac{d}{d\lambda}$. Thus

$$\frac{d}{dS}\Big|_{S=T}\mathcal{B} = -\frac{1}{T}\,\frac{d}{d\lambda}\Big|_{\lambda=1}\mathcal{A} = -\frac{1}{T}\int_0^T \left[\, E(y_T, \dot{y}_T) - k \,\right] dt.$$

□

3-4 The finite-time potential.

Recall that if $k \geq c(L)$, x, $y \in M$ and $T > 0$,

$$\Phi_k(x,y;T) := \inf_{\gamma \in \mathcal{C}_T(x,t)} A_{L+k}(\gamma).$$

Here we shall prove

3-4.1. Proposition.

For $k \geq c(L)$ and x, $y \in M$, the function $t \mapsto \Phi_k(x,y;t)$ is Lipschitz on $[\varepsilon, +\infty[$ for any $\varepsilon > 0$. Moreover,

$$\lim_{\varepsilon \to 0^+} \Phi_k(x,y;\varepsilon) = +\infty, \qquad \textit{for} \quad k \geq c(L),\ x \neq y.$$

$$\lim_{T \to +\infty} \Phi_k(x,y;T) = +\infty, \qquad \textit{for} \quad k > c(L),\ x \neq y.$$

Proof of proposition 3-4.1

We first compute the limits. Observe that if $k > c(L)$,

$$\lim_{T \to +\infty} \Phi_k(x,y;T) \geq \lim_{T \to +\infty} \big[\, \Phi_c(x,y) + (k-c)\,T \big] = +\infty.$$

Given $A > 0$, let $B > 0$ be such that $L(x,v) > A\,|v| - B$. Then

$$\begin{aligned} \Phi_k(x,y;\varepsilon) = \inf_{\gamma \in \mathcal{C}_T(x,y)} A_{L+k}(\gamma) &\geq \inf_{\gamma} \int_0^{\varepsilon} A\,|\dot{\gamma}| - B + k \\ &\geq A\,d(x,y) + (k-B)\,\varepsilon. \end{aligned}$$

Thus, $\liminf_{\varepsilon \to 0^+} \Phi_k(x,y;\varepsilon) \geq A\,d(x,y)$. Now let $A \to +\infty$.

Fix $\varepsilon > 0$. If $T > \varepsilon$ and $\gamma \in \mathcal{C}_T(x,y)$ is a Tonelli minimizer, from corollary 3-2.2 there exists $C = C(\varepsilon) > 0$ such that $E(\gamma,\dot{\gamma}) - k \leq C(\varepsilon) - k$. Denote $h(s) := \Phi_k(x,y;s)$. If $\gamma_s(t) := \gamma(\frac{s}{T}t)$, then $h(s) \leq$

$A_{L+k}(\gamma_s) =: \mathcal{B}(s)$. Using corollary 3-3.3 we have that

$$\begin{aligned}
f(T) :&= \limsup_{\delta \to 0} \frac{h(T+\delta) - h(T)}{\delta} \\
&\leq \mathcal{B}'(T) = \frac{1}{T} \int_0^T \left[E(\gamma, \dot{\gamma}) - k \right] dt \\
&\leq |C(\varepsilon) - k|.
\end{aligned}$$

If S, $T > \varepsilon$ we have that

$$\begin{aligned}
\Phi_k(x, y; S) &\leq \Phi_k(x, y; T) + \int_T^S f(t)\, dt \\
&\leq \Phi_k(x, y; T) + |C(\varepsilon) - k|\, |T - S|.
\end{aligned}$$

Since we can reverse the roles of S and T, this implies the Lipschitz condition for $T \mapsto \Phi_k(x, y; T)$.

□

3-5 Global Minimizers.

Here we construct curves that realize the action potential.
For $k < c(L)$, $\Phi_k \equiv -\infty$, so there are no minimizers.

3-5.1. Proposition.

If $k > c(L)$ and x, $y \in M$, $x \neq y$, then there is $\gamma \in \mathcal{C}(x,y)$ such that

$$A_{L+k}(\gamma) = \Phi_k(x,y).$$

Moreover, the energy of γ is $E(\gamma,\dot\gamma) \equiv k$.

Proof: Let $f(t) := \Phi_k(x,y;T)$. By proposition 3-4.1, $f(t)$ is continuous and $f(t) \to +\infty$ when $t \to 0^+$ or $t \to +\infty$. Hence it attains its minimum at some $T > 0$. Moreover, $\Phi_k(x,y) = \inf_{t>0} \Phi_k(x,y;t) = \Phi_k(x,y;T)$. Now take a Tonelli minimizer γ on $\mathcal{C}_T(x,y)$. From lemma 3-3.2, the energy of γ is k. □

We now study minimizers at $k = c(L)$. Observe that for $c = c(L)$ and any absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$, we have that

$$A_{L+c}(\gamma) \geq \Phi_c(x,y) \geq -\Phi_c(y,x). \tag{3.10}$$

3-5.2. Definition. Set $c = c(L)$.

An absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$ is said *semistatic* if

$$A_{L+c}(\gamma) = \Phi_c(x,y).$$

An absolutely continuous curve $\gamma \in \mathcal{C}(x,y)$ is said *static* if

$$A_{L+c}(\gamma) = -\Phi_c(y,x).$$

By the triangle inequality for Φ_c the definition of semistatic curve $x : [a,b] \to M$ is equivalent to

$$A_{L+c}\big(x|_{[s,t]}\big) = \Phi_c\big(x(s),x(t)\big), \qquad \forall\, a \leq s \leq t \leq b. \tag{3.11}$$

Inequality (3.10) implies that static curves are semistatic.

Moreover, a curve $\gamma \in C(x,y)$ is *static* if

(a) γ is semistatic, and

(b) $d_c(x,y) = \Phi_c(x,y) + \Phi_c(y,x) = 0$.

3-5.3. Corollary. *Semistatic curves have energy $E \equiv c(L)$.*

3-5.4. Definition.

$$\begin{aligned}
\widetilde{\mathcal{N}} = \Sigma(L) &:= \{\, w \in TM \,|\, x_w : \mathbb{R} \to M \text{ is semistatic } \} \\
\mathcal{A} = \widehat{\Sigma}(L) &:= \{\, w \in TM \,|\, x_w : \mathbb{R} \to M \text{ is static } \} \\
\Sigma^-(L) &:= \{\, w \in TM \,|\, x_w :]-\infty, 0] \to M \text{ is semistatic } \} \\
\Sigma^+(L) &:= \{\, w \in TM \,|\, x_w : [0, +\infty[\to M \text{ is semistatic } \}
\end{aligned}$$

We shall call $\widetilde{\mathcal{N}}$ the *Mañé set*, $\mathcal{P} = \pi(\widehat{\Sigma}(L))$ the *Peierls set*[1] and $\mathcal{A} = \widehat{\Sigma}(L)$ the *Aubry set*.

Using the characterization of minimizing measures 3-6.1 and corollary 3-5.3 we have that[2]

$$\mathcal{M} \subseteq \mathcal{A} \subseteq \widetilde{\mathcal{N}} \subseteq \acute{\mathcal{E}} \,,$$

where $\mathcal{M}$ is the Mather set, $\mathcal{A}$ is the Aubry set, $\widetilde{\mathcal{N}}$ is the Mañé set and $\acute{\mathcal{E}}$ is the energy level $\acute{\mathcal{E}} = [E \equiv c(L)]$. All these inclusions can be made proper constructing examples of embedded flows as in equation (1.20) and adding a properly chosen potential $\phi(x)$.

Denote by $\alpha(v)$ and $\omega(v)$ the α and ω-limits of v under the Euler-Lagrange flow.

3-5.5. Proposition.
A local static is a global static, i.e. if $x_v|_{[a,b]}$ is static then $v \in \widehat{\Sigma}(L)$ (i.e. the whole orbit is static).

[1] The name is justified by proposition 3-7.1.5.
[2] The typographical relationship was observed by Albert Fathi.

Proof: Let $\eta(t) = \pi\,\varphi_t(v)$ and let $\gamma_n \in \mathcal{C}_{T_n}(\eta(b),\eta(a))$ be solutions of (E-L) with

$$A_{L+c}(\gamma_n) < \Phi_c(\eta(b),\eta(a)) + \tfrac{1}{n}.$$

By the apriori bounds 3-2.3 $\dot\gamma_n| < A$. We can assume that $\dot\gamma_n(0) \to w$. Let $\xi(s) = \pi\,\varphi_s(w)$. If $w \neq \dot\eta(b)$ then the curve $\eta|_{[b-\varepsilon,b]} * \xi|_{[0,\varepsilon]}$ is not C^1, and hence by remark 1-2.2, can not be a Tonelli minimizer. Thus

$$\Phi_c(\eta(b-\varepsilon),\xi(\varepsilon)) < A_{L+c}(\eta|_{[b-\varepsilon,b]}) + A_{L+c}(\xi|_{[0,\varepsilon]}).$$

$$\begin{aligned}
\Phi_c(\eta(a),\eta(a)) &\leq \Phi_c(\eta(a),\eta(b-\varepsilon)) + \Phi_c(\eta(b-\varepsilon),\xi(\varepsilon)) + \Phi_c(\xi(\varepsilon),\eta(a)) \\
&< A_{L+c}(\eta_{[a,b-\varepsilon]}) + A_{L+c}(\eta|_{[b-\varepsilon,b]}) + A_{L+c}(\xi|_{[0,\varepsilon]}) + \liminf_n A_{L+c}(\gamma_n|_{[\varepsilon,T_n]}) \\
&\leq A_{L+c}(\eta|_{[a,b]}) + \lim_n \big(\gamma_n|_{[0,\varepsilon]} * \gamma_n|_{[\varepsilon,T_n]}\big) \\
&\leq -\Phi_c(\gamma(b),\gamma(a)) + \Phi_c(\gamma(a),\gamma(b)) = 0,
\end{aligned}$$

which contradicts proposition 2-1.1(3). Thus $w = \dot\eta(b)$ and similarly $\lim_n \dot\gamma_n(T_n) = \dot\eta(a)$.

If $\limsup T_n < +\infty$, we can assume that $\tau = \lim_n T_n > 0$ exists. In this case η is a (semistatic) periodic orbit of period $\tau + b - a$ and then it is static.

Now suppose that $\lim_n T_n = +\infty$. If $s > 0$, we have that

$$\begin{aligned}
&A_{L+c}(\eta|_{[a-s,b+s]}) + \Phi_c(\eta(b+s),\eta(a-s)) \leq \\
&\qquad \leq \lim_n \big\{\, A_{L+c}(\gamma_n|_{[T_n-s,T_n]}) + A_{L+c}(\eta) + A_{L+c}(\gamma_n|_{[0,s]}) \,\big\} \\
&\qquad\qquad\qquad\qquad + \Phi_c(\eta(b+s),\eta(a-s)) \\
&\qquad \leq \Phi_c(\eta(a),\eta(b)) \\
&\qquad\quad + \lim_n \big\{\, A_{L+c}(\gamma_n|_{[0,s]}) + A_{L+c}(\gamma_n|_{[s,T_n-s]}) + A_{L+c}(\gamma_n|_{[T_n-s,T_n]}) \,\big\} \\
&\qquad \leq \Phi_c(\eta(a),\eta(b)) + \Phi_c(\eta(b),\eta(a)) = 0.
\end{aligned}$$

Thus $\eta_{[a-s,b+s]}$ is static. □

3-6 Characterization of minimizing measures.

3-6.1. Theorem (Mañé [32]).
$\mu \in \mathcal{M}(L)$ is a minimizing measure if and only if $\operatorname{supp}(\mu) \subseteq \widehat{\Sigma}(L)$.

Proof: Since $\widehat{\Sigma}$ is closed, it is enough to prove the theorem for ergodic measures. Suppose that $\mu \in \mathcal{M}(L)$ is ergodic and $\operatorname{supp}(\mu) \subseteq \Sigma(L)$. Since μ is finite, by Birkhoff's theorem there is a set of total μ-measure A such that if $\theta \in A$ then $\liminf_{T\to+\infty} d_{TM}(\theta, \varphi_T\theta) = 0$ and

$$\begin{aligned}\int_M L + c\, d\mu &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T L(\varphi_t\,\theta) + c\, dt \\ &\le \liminf_{T\to+\infty} \tfrac{1}{T}\Phi_c\big(\pi(\varphi_T\,\theta), \pi(\theta)\big) = 0.\end{aligned}$$

Now suppose that $\mu \in \mathcal{M}(L)$ is minimizing. By corollary 3-6.5, there is a set A of total μ-measure such that if $\theta \in A$ then there is a sequence $T_n \to +\infty$ such that $d(\theta, \varphi_{T_n}\theta) \to 0$ and

$$\lim_n \int_0^{T_n} L(\varphi_t\,\theta) + c\, dt = 0.$$

Then

$$\begin{aligned}0 \le d_c(\pi\,\theta,\, \pi\varphi_1\,\theta) &= \lim_n \big[\,\Phi_c(\pi\,\theta, \pi\,\varphi_1\,\theta) + \Phi_c(\pi\,\varphi_1\,\theta, \pi\,\varphi_{T_n}\,\theta)\,\big] \\ &\le \lim_n \int_0^{T_n} L(\varphi_t\,\theta) + c\, dt = 0.\end{aligned}$$

This implies that θ is static. Since $\widehat{\Sigma}(L)$ is closed and A is dense in $\operatorname{supp}(\mu)$ then $\operatorname{supp}(\mu) \subseteq \widehat{\Sigma}(L)$. □

It follows from theorem 2-4.1 that

3-6.2. Corollary.
If M is compact then $\widehat{\Sigma}(L) \neq \varnothing$.

Combining theorem 3-6.1 with corollary 3-5.3, we get

3-6.3. Corollary. *If μ is a minimizing measure then it is supported in the energy level $E(\mathrm{supp}(\mu)) = c(L)$.*

3-6.4. Lemma. *Let $(X, \mathfrak{B}, \nu)$ be a probability space, f an ergodic measure preserving map and $F : X \to \mathbb{R}$ an integrable function. Given $A \in \mathfrak{B}$ with $\nu(A) > 0$ denote by $\hat{A}$ the set of point $p \in A$ such that for all $\varepsilon > 0$ there exists an integer $N > 0$ such that $f^N(p) \in A$ and*

$$\left| \sum_{j=0}^{N-1} F(f^j(x)) - N \int F\, d\nu \right| < \varepsilon .$$

Then $\nu(\hat{A}) = \nu(A)$.

Proof: Without loss of generality we can assume that $\int F\, d\nu = 0$. For $p \in X$ denote

$$S_N(p) := \sum_{n=0}^{N-1} F\big(f^n(p)\big) .$$

Let

$$A(\varepsilon) := \big\{\, p \in A \,|\, \exists N > 0 \text{ such that } f^N(p) \in A \text{ and } |S_N F(p)| < \varepsilon \,\big\} .$$

It is enough to prove that $\nu(A(\varepsilon)) = \nu(a)$, because $\hat{A} = \bigcap_n A(\frac{1}{n})$. Let $\hat{X}$ be the set of points for which the Birkhoff's theorem holds for F and the characteristic functions of A and of $A(\varepsilon)$. Take $x \in A \cap \hat{X}$ and let $N_1 < N_2 < \cdots$ be the integers for which $f^{N_i}(x) \in A$. Define $\delta(k)$ by

$$N_k\, \delta(k) = |S_{N_k} F(x)| .$$

Since $x \in \hat{X}$ we have that $\lim_{k \to +\infty} \delta(k) = 0$. Set

$$\begin{aligned} c_k &:= S_{N_k} F(x) , \\ S(k) &:= \big\{\, 1 \le j \le k-1 \,\big|\, \forall \ell > j \, |c_\ell - c_j| \ge \varepsilon \,\big\} . \end{aligned}$$

Then

$$\varepsilon \, \# S(k) \le \delta(k)\, N_k .$$

Moreover if $j \notin S(k)$, then there is an $\ell > j$ such that $|c_\ell - c_j| \le \varepsilon$, so that

$$\left|S_{N_\ell - N_j} F\left(f^{N_j}(x)\right)\right| = |c_\ell - c_j| \le \varepsilon\,.$$

Hence

$$j \notin S(k) \Longrightarrow f^{N_j}(x) \in A(\varepsilon)\,.$$

This implies that

$$\frac{1}{N_k}\,\#\{\,0 \le j < N_k \mid f^j(x) \in A - A(\varepsilon)\,\} \le \frac{1}{N_k}\,\#S(k) \le \frac{\delta(k)}{\varepsilon}\,.$$

Since $\delta(k) \to 0$ when $k \to +\infty$ and $x \in \hat{X}$, , we obtain

$$\nu\big(A - A(\varepsilon)\big) = \lim_{k\to+\infty} \frac{1}{N_k}\#\big\{0 \le j < N_k \mid f^j(x) \in A - A(\varepsilon)\big\} = 0\,,$$

concluding the proof of the lemma. □

3-6.5. Corollary. *If besides the hypothesis of lemma 3-6.4, the space X is compact and metric, and $\mathfrak{B}$ is the Borel σ-algebra, then for a.e. $x \in X$ the following property holds: for all $\varepsilon > 0$ there exists $N > 0$ such that $d(f^N(x), x) < \varepsilon$ and*

$$\left|\sum_{j=0}^{N-1} F\big(f^j(x)\big) - N\int F\,d\nu\right| < \varepsilon$$

Proof: Given $\varepsilon > 0$ let $\{V_n(\varepsilon)\}$ be a countable basis of neighbourhoods with diameter $< \varepsilon$ and let $\hat{V}n$ be associated to V_n as in lemma 3-6.4. Then the full measure subset $\bigcap\limits_m \bigcup\limits_n \hat{V}_n(\frac{1}{m})$ satisfies the required property.

□

3-7 The Peierls barrier.

For $T > 0$ and $x,\ y \in M$ define

$$h_T(x,y) = \Phi_c(x,y;T) := \inf_{\gamma\in\mathcal{C}_T(x,y)} A_{L+c}(\gamma).$$

So that the curves which realize $h_T(x,y)$ are the Tonelli minimizers on $\mathcal{C}_T(x,y)$. Define the *Peierls barrier* as

$$h(x,y) := \liminf_{T\to+\infty} h_T(x,y).$$

The difference between the action potential and the Peierls barrier is that in the Peierls barrier the curves must be defined on large time intervals. Clearly

$$h(x,y) \geq \Phi_c(x,y).$$

3-7.1. Proposition.
If $h : M \times M \to \mathbb{R}$ is finite, then

1. *h is Lipschitz.*
2. *$\forall\ x,\ y \in M,\ h(x,x) \geq \Phi_c(x,y)$, in particular $h(x,x) \geq 0,\ \forall\ x \in M$.*
3. *$h(x,z) \leq h(x,y) + h(y,z), \qquad \forall\, x,\ y,\ z \in M$.*
4. *$h(x,y) \leq \Phi_c(x,p) + h(p,q) + \Phi_c(q,y), \qquad \forall\ x,\ y,\ p,\ q \in M$.*
5. *$h(x,x) = 0 \iff x \in \pi(\widehat{\Sigma}) = \mathcal{P}$.*
6. *If $\widehat{\Sigma} \neq \varnothing$, $h(x,y) \leq \inf_{p\in\pi(\widehat{\Sigma})} \Phi_c(x,p) + \Phi_c(p,y)$.*

Proof: Item *2* is trivial. Observe that for all S, $T > 0$ and $y \in M$,

$$h_{T+S}(x,z) \leq h_T(x,y) + h_S(y,z).$$

Taking $\liminf_{T\to+\infty}$ we get that

$$h(x,z) \leq h(x,y) + h_S(y,z), \qquad \text{for all } S > 0.$$

Taking $\liminf_{S\to+\infty}$, we obtain item *3.*

1. Taking the infimum on $S>0$, we get that

$$\begin{aligned} h(x,z) &\le h(x,y)+\Phi_c(y,z) \qquad \forall\, x,\, y,\, z\in M. \\ &\le h(x,y)+A\, d_M(y,z), \end{aligned}$$

where A is a Lipschitz constant for Φ_c. Changing the roles of x, y, z, we obtain that h is Lipschitz.

4. Observe that

$$\inf_{S>T} h_S(x,y)\le \Phi_c(x,p)+h_T(p,q)+\Phi_c(q,x).$$

Taking $\liminf_{T\to+\infty}$ we get item *4.*

5. We first prove that if $p\in\mathcal{P}=\pi(\widehat{\Sigma})$, then $h(p,p)=0$. Take $v\in\widehat{\Sigma}$ such that $\pi(v)=p$ and $y\in\pi(\omega\text{-limit}(v))$. Let $\gamma(t):=\pi\,\varphi_t(v)$ and choose $t_n\uparrow+\infty$ such that $\gamma(t_n)\to y$. Then

$$\begin{aligned} 0\le h(p,p) &\le h(p,y)+\Phi_c(y,p) \\ &\le \lim_n A_{L+c}(\gamma|_{[0,t_n]})+\Phi_c(y,p) \\ &\le \lim_n -\Phi_c(\gamma(t_n),p)+\Phi_c(y,p)=0. \end{aligned}$$

Conversely, if $h(x,x)=0$, then there exists a Tonelli minimizer $\gamma_n\in\mathcal{C}(x,x;T_n)$ with $T_n\to+\infty$ and $A_{L+c}(\gamma_n)\xrightarrow{n}0$. By lemma 3-2.3, $|\dot\gamma|$ is uniformly bounded. Let v be an accumulation point of $\dot\gamma_n(0)$ and $\eta(t):=\pi\,\varphi_t(v)$. Then if $\dot\gamma_{n_k}(0)\xrightarrow{k}v$, for any $s>0$ we have that

$$\begin{aligned} 0 &\le \Phi_c(x,\pi\,\varphi_s\,v)+\Phi_c(\pi\,\varphi_s\,v,x) \\ &\le A_{L+c}\big(\eta|_{[0,s]}\big)+\Phi_c(\pi\,\varphi_s\,v,x) \\ &\le \lim_k A_{L+c}\big(\gamma_{n_k}|_{[0,s]}\big)+A_{L+c}\big(\gamma_{n_k}|_{[s,T_n]}\big) \\ &=0. \end{aligned}$$

Thus $v\in\widehat{\Sigma}$.

6. Using items *4* and *5*, we get that

$$h(x,y) \leq \inf_{p\in\pi(\widehat{\Sigma})} \left[\, \Phi_c(x,p) + 0 + \Phi_c(p,y) \,\right].$$

□

3-7.2. Proposition. *If M is compact, then*

$$h(x,y) = \inf_{p\in\pi(\widehat{\Sigma})} \left[\Phi_c(x,p) + \Phi_c(p,y)\right].$$

Proof:

From proposition 3-7.1.*6* we have that

$$h(x,y) \leq \inf_{p\in\pi(\widehat{\Sigma})} \left[\, \Phi_c(x,p) + \Phi_c(p,y) \,\right].$$

In particular $h(x,y) < +\infty$ for all x, $y \in M$. Now let $\gamma_n \in C_{T_n}(x,y)$ with $T_n \to +\infty$ and $A_{L+c}(\gamma_n) \to h(x,y) < +\infty$. Then $\frac{1}{T} A_{L+c}(\gamma_n) \to 0$. Let μ be a weak limit of a subsequence of the measures μ_{γ_n}. Then μ is minimizing. Let $q \in \pi\big(\text{supp}(\mu)\big)$ and $q_n \in \gamma_n([0,T_n])$ be such that $\lim_n q_n = q$. Then,

$$\begin{aligned} \Phi_c(x,q) + \Phi_c(q,y) &\leq \Phi_c(x,q_n) + \Phi_c(q_n,y) + 2A\, d(q_n,q) \\ &\leq A_{L+c}(\gamma_n) + 2A\, d(q_n,q). \end{aligned}$$

Letting $n \to \infty$, we get that

$$\Phi_c(x,q) + \Phi_c(q,y) \leq h(x,y).$$

□

3-8 Graph Properties.

In this section we shall see that the projection $\pi : \widehat{\Sigma} \to M$ is injective. We shall call $\mathcal{P} := \pi(\widehat{\Sigma})$ the *Peierls set.* [3] Thus the projection $\pi|_{\widehat{\Sigma}}$ gives an identification $\mathcal{P} \approx \widehat{\Sigma}$.

For $v \in TM$, write $x_v(t) = \pi\, \varphi_t(v)$. Given $\varepsilon > 0$, let

$$\Sigma^\varepsilon := \{ w \in TM \,|\, x_w : [0,\varepsilon) \to M \text{ or } x_w : (-\varepsilon, 0] \to M \text{ is semistatic } \}.$$

3-8.1. Theorem. (Mañé) [32]

For all $p \in \pi(\widehat{\Sigma})$ there exists a unique $\xi(p) \in T_pM$ such that $(p, \xi(p)) \in \Sigma^\varepsilon$, in particular $(p, \xi(p)) \in \widehat{\Sigma}$ and $\widehat{\Sigma} = graph(\xi)$.

Moreover, the map $\xi : \pi(\widehat{\Sigma}) \to \Sigma$ is Lipschitz.

The proofs of the injectivity of π in this book only need that the solutions of the Euler-Lagrange equation are differentiable[4]. The reader may provide those proofs as excercises. The proof of the Lipschitz condition need the following lemma, due to Mather. For the proof see [39] or Mañé [29].

3-8.2. Mather's Crossing lemma. *[39]*

Given $A > 0$ there exists $K > 0$ $\varepsilon_1 > 0$ and $\delta > 0$ with the following property: if $|v_i| < A$, $(p_i, v_i) \in TM$, $i = 1, 2$ satisfy $d(p_1, p_2) < \delta$ and $d((p_1, v_1), (p_2, v_2)) \geq K^{-1} d(p_1, p_2)$ then, if $a \in \mathbb{R}$ and $x_i : \mathbb{R} \to M$, $i = 1, 2$, are the solutions of L with $x_i(a) = p_i$, $\dot{x}_i(p_i) = v_i$, there exist solutions $\gamma_i : [a - \varepsilon, a + \varepsilon] \to M$ of L with $0 < \varepsilon < \varepsilon_1$, satisfying

$$\begin{aligned} \gamma_1(a-\varepsilon) &= x_1(a-\varepsilon) \quad , \quad \gamma_1(a+\varepsilon) = x_2(a+\varepsilon)\,, \\ \gamma_2(a-\varepsilon) &= x_2(a-\varepsilon) \quad , \quad \gamma_2(a+\varepsilon) = x_1(a+\varepsilon)\,, \\ S_L(x_1|_{[a-\varepsilon,a+\varepsilon]}) &+ S_L(x_2|_{[a-\varepsilon,a+\varepsilon]}) > S_L(\gamma_1) + S_L(\gamma_2) \end{aligned}$$

---

[3] This name is justified by proposition 3-7.1(5).

[4] and hence a non-differentiable curve can not be a Tonelli minimizer.

Proof of theorem 3-8.1:

We prove that if $(p,v) \in \widehat{\Sigma}$, $(q,w) \in \Sigma^\varepsilon$, and $d(p,q) < \delta$, then

$$d_{TM}\big((p,v),(q,w)\big) < K\, d_M(p,q)\,.$$

Observe that this implies the theorem. For simplicity, we only prove the case in which $x_v|_{[-\varepsilon,0]}$ is semistatic. Suppose it is false. Then by lemma 3-8.2 there exist α, $\beta : [-\varepsilon,\varepsilon] \to M$ such that

$$\begin{aligned} \alpha(-\varepsilon) &= x_w(-\varepsilon) =: q_{-\varepsilon}\,, & \alpha(0) &= p\,,\\ \beta(-\varepsilon) &= x_v(-\varepsilon) =: p_\varepsilon\,, & \beta(0) &= q\,, \end{aligned}$$

and

$$S_L(\alpha) + S_L(\beta) < S_L(x_w|_{[-\varepsilon,0]}) + S_L(x_v|_{[-\varepsilon,0]}).$$

So

$$\begin{aligned} \Phi_c(q_{-\varepsilon},p) + \Phi_c(p_{-\varepsilon},q) &< \Phi_c(q_{-\varepsilon},q) + \Phi_c(p_{-\varepsilon},p)\\ &= \Phi_c(q_{-\varepsilon},q) - \Phi_c(p,p_{-\varepsilon}) \end{aligned}$$

Thus

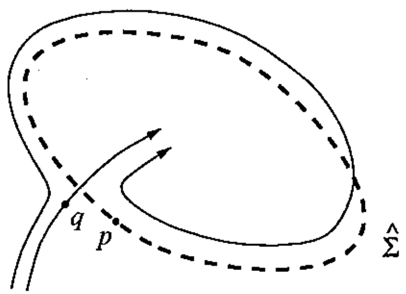

FIG. 1: GRAPH PROPERTY.

$$\Phi_c(q_{-\varepsilon},q) \le \Phi_c(q_{-\varepsilon},p) + \Phi_c(p,p_{-\varepsilon}) + \Phi_c(p_{-\varepsilon},q) < \Phi_c(q_{-\varepsilon},q)$$

which is a contradiction. $\square$

Using the graph property 3-8.1 we can define an equivalence relation on $\widehat{\Sigma}$ by

$$u, v \in \widehat{\Sigma}, \quad u \equiv v \qquad \Longleftrightarrow \qquad d_c(\pi(u), \pi(v)) = 0.$$

The equivalence classes are called *static classes.* The continuity of the pseudometric d_c implies that a static class is closed, and it is invariant by proposition 3-5.5.

For $v \in TM$ denote by $\omega(v)$ its ω-limit. Let Γ be a static class, the set

$$\Gamma^+ = \{\, v \in \Sigma^+(L) \mid \omega(v) \subseteq \Gamma \,\}$$

is called the (forward) *basin* of Γ. Clearly Γ^+ is forward invariant. Let

$$\begin{aligned} \Gamma_0^+ &= \textstyle\bigcup_{t>0} \varphi_t(\Gamma) \\ &= \textstyle\bigcup_{\varepsilon>0} \{\, v \in TM \mid x_v|_{]-\varepsilon,+\infty[} \text{ is semistatic } \}. \end{aligned}$$

The set $\pi(\Gamma^+ \setminus \Gamma_0^+)$ is called the *cut locus* of Γ^+.

3-8.3. Theorem. (Mañé)[32]

For every static class Γ, the projection $\pi : \Gamma_0^+ \to M$ is injective with Lipschitz inverse.

The projection $\pi : \Gamma^+ \to M$ may not be surjective. But when M is compact for generic lagrangians $\pi(\Gamma^+) = \pi(\Gamma^-) = M$ because there is only one static class (cf. theorem 5-0.1.(B)).

Proof: We prove that for K as in lemma 3-8.2, if $v\,, w \in \Gamma_0^+$ then

$$d_{TM}(v, w) \le K\, d_M(\pi(v), \pi(w)). \tag{3.12}$$

Suppose it is false. Then there are v, $w \in \Gamma_0^+$ such that inequality (3.12) does not hold. Let $\varepsilon > 0$ be such that $x_v|_{[-\varepsilon,+\infty[}$ and $x_w|_{[-\varepsilon,+\infty[}$ are semistatic. By lemma 3-8.2, there exist $\alpha \in \mathcal{C}_{2\varepsilon}(x_v(-\varepsilon), x_w(\varepsilon))$ and $\beta \in \mathcal{C}_{2\varepsilon}(x_w(-\varepsilon), x_v(\varepsilon))$ such that

$$A_{L+c}(\alpha) + A_{L+c}(\beta) + \delta < A_{L+c}(x_v|_{[-\varepsilon,\varepsilon]}) + A_{L+c}(x_w|_{[-\varepsilon,\varepsilon]}),$$

for some $\delta > 0$. Let p, $q \in \pi(\Gamma)$ and $s_n\, t_n \to +\infty$ be such that $x_v(s_n) \to +\infty$ and $x_v(s_n) \to p$, $x_w(t_n) \to q$. Then

$$\begin{aligned}
\Phi_c(x_v(-\varepsilon), x_w(t_n)) + \Phi_c(x_w(-\varepsilon), x_v(s_n)) + \delta \\
&\leq A_{L+c}(\alpha * x_w|_{[\varepsilon,t_n]}) + A_{L+c}(\beta * x_v|_{[\varepsilon,s_n]}) + \delta \\
&< A_{L+c}(x_v|_{[-\varepsilon,s_n]}) + A_{L+c}(x_w|_{[-\varepsilon,t_n]}) \\
&= \Phi_c(x_v(-\varepsilon), x_v(s_n)) + \Phi_c(x_w(-\varepsilon), x_w(t_n)).
\end{aligned}$$

Letting $n \to \infty$ and adding $d_c(p,q) = 0$, we have that

$$\begin{aligned}
\Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q) \\
&\leq \Phi_c(x_v(-\varepsilon), q) + \Phi_c(x_w(-\varepsilon), p) + \Phi_c(q,p) + \Phi_c(p,q) \\
&< \Phi_c(x_v(-\varepsilon), p) + \Phi_c(x_w(-\varepsilon), q).
\end{aligned}$$

□

3-9 Coboundary Property.

The coboundary property was first presented by R. Mañé in [31] and further developed in [32] and by A. Fathi.

3-9.1. Theorem. (Mañé) [32]
If $c = c(L)$, then $(L+c)\big|_{\widehat{\Sigma}}$ is a Lipschitz coboundary. More precisely, taking any $p \in M$ and defining $G : \widehat{\Sigma} \to \mathbb{R}$ by

$$G(w) = \Phi_c\big(p, \pi(w)\big),$$

then

$$(L+c)\big|_{\widehat{\Sigma}} = \frac{dG}{df},$$

where

$$\frac{dG}{d\varphi}(w) := \lim_{h\to 0}\ \frac{1}{h}\big[\,G(\varphi_h(w) - G(w)\,\big].$$

Proof: Let $w \in \widehat{\Sigma}$ and define $F_w(v) := \Phi_c(\pi(w), \pi(v))$. We have that

$$\begin{aligned}
\left.\frac{dF_w}{d\varphi}\right|_w &= \lim_{h\to 0}\ \tfrac{1}{h}\left[\,F_w(\varphi_h\, w) - F_w(w)\,\right] \\
&= \lim_{h\to 0}\ \tfrac{1}{h}\left[\,\Phi_c(\pi w, \pi\varphi_h\, w) - \Phi_c(\pi w, \pi w)\,\right] \\
&= \lim_{h\to 0}\ \tfrac{1}{h}\, S_{L+c}\left(x_w|_{[0,h]}\right) \\
&= \lim_{h\to 0}\ \frac{1}{h}\int_0^h \left[L\big(x_w(s), \dot{x}_w(s)\big) + c\right] ds \\
&= L(w) + c.
\end{aligned}$$

We claim that for any $p \in M$ and any $w \in \widehat{\Sigma}$, $h \in \mathbb{R}$,

$$G(\varphi_h\, w) = \Phi_c\big(p, \pi(\varphi_h\, w)\big) = \Phi_c\big(p, \pi(w)\big) + \Phi_c\big(\pi(w), \pi(\varphi_h\, w)\big)$$

$$G(\varphi_h\, w) = \Phi_c\big(p, \pi(w)\big) + F_w\big(\varphi_h(w)\big). \tag{3.13}$$

This is enough to prove the theorem because then

$$\left.\frac{dG}{d\varphi}\right|_w = \left.\frac{d}{dh}F_h(\varphi_h\, w)\right|_{h=0} = \left.\frac{F_w}{d\varphi}\right|_w = L(w) + c,$$

and G is Lipschitz by proposition 2-1.1.

We now prove (3.13). Let $q := \pi(w)$, $x := \pi(\varphi_h\, w)$. We have to prove that

$$\Phi_c(p, x) = \Phi_c(p, q) + \Phi_c(q, x). \tag{3.14}$$

Since the points q and x can be joined by the static curve $x_w|_{[0,h]}$, then

$$\Phi_c(x, q) = -\Phi_c(q, x).$$

Using twice the triangle inequality for Φ_c we get that

$$\Phi_c(p, q) \leq \Phi_c(p, x) + \Phi_c(x, q) = \Phi_c(p, x) - \Phi_c(q, x) \leq \Phi_c(p, q).$$

This implies (3.14). $\square$

3-10 Covering Properties.

3-10.1. Theorem. $\pi(\Sigma^+(L)) = M$.

Proof: First suppose that $\widehat{\Sigma}(L) \neq \varnothing$. Take $p \in \pi(\widehat{\Sigma})$. Given $x \in M \setminus \pi(\widehat{\Sigma})$, take a Tonelli minimizer $\gamma_n \in C_{T_n}(x,p)$ such that

$$A_{L+c}(\gamma_n) < \Phi_c(x,p) + \tfrac{1}{n}.$$

By the apriori bounds 3-2.3, $|\dot{\gamma}_n| < a$ and $T_n > \frac{1}{A}\, d(x,p)$. Let $v = \lim_k \dot{\gamma}_{n_k}(0)$ be an accumulation point of $\langle \dot{\gamma}_n(0) \rangle$. Let $\eta(t) := \pi\, \varphi_t(v)$. Then, if $0 < s < \liminf_k T_{n_k}$, we have that

$$\begin{aligned} A_{L+c}\big(\eta|_{[0,s]}\big) &= \lim_k A_{L+c}\big(\gamma_n|_{[0,s]}\big) \\ &\le \lim_k \big[\, \Phi_c(\gamma_{n_k}(0), \gamma_{n_k}(s) + \tfrac{1}{n_k}\,\big] \\ &= \Phi_c\big(\eta(0), \eta(s)\big). \end{aligned}$$

Then η is semistatic on $[0,S]$, where $S = \liminf_k T_{n_k}$. If $S < +\infty$ then $\eta(S) = \lim_k \gamma_{n_k}(T_k) = p$. Since $x \notin \pi(\widehat{\Sigma})$, this contradicts the graph property 3-8.1; hence $S = +\infty$. Thus $\eta|_{[0,+\infty[}$ is semistatic and $v \in \Sigma^+$.

If $\widehat{\Sigma} = \varnothing$, then by corollary 3-6.2,M is non-compact. Let $x \in M$ and $\langle y_n \rangle \subseteq M$ such that $d_M(x, y_n) \to +\infty$. Let $\gamma_n \in C_{T_n}(x, y_n)$ be a Tonelli minimizer such that

$$A_{L+c}(\gamma_n) < \Phi_c(x, y_n) + \tfrac{1}{n}.$$

Then by lemma 3-2.3, $|\dot{\gamma}_n| < A$, and hence $T_n \to +\infty$. The rest of the proof is similar to the case above. $\square$

3-11 Recurrence Properties.

Let Λ be the set of static classes. Define a reflexive partial order $\preccurlyeq$ in Λ by

(a) $\preccurlyeq$ is reflexive.

(b) $\preccurlyeq$ is transitive.

(c) If there is $v \in \Sigma$ with the α-limit set $\alpha(v) \subseteq \Lambda_i$ and ω-limit set $\omega(v) \subseteq \Lambda_j$, then $\Lambda_i \preccurlyeq \Lambda_j$.

3-11.1. Theorem.
Suppose that M is compact and the number of static classes is finite. Then given Λ_i and Λ_j in Λ, we have that $\Lambda_i \preccurlyeq \Lambda_j$.

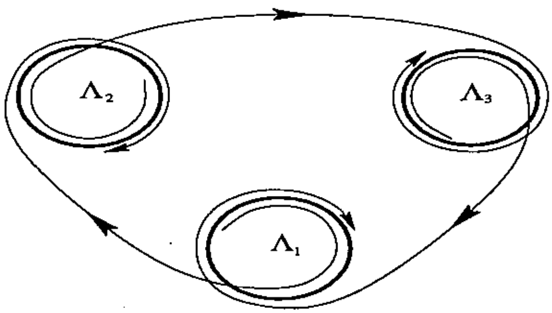

FIG. 2: CONNECTING ORBITS BETWEEN STATIC CLASSES. The three closed curves represent the static classes and the other curves represent semistatic orbits connecting them.

theorem 3-11.1 could be restated by saying that if the cardinality of Λ is finite, then given two static classes Λ_i and Λ_j there exist classes $\Lambda_i = \Lambda_1, \ldots, \Lambda_n = \Lambda_j$ and semistatic vectors $v_1, \ldots, v_{n-1} \in \Sigma$ such that for all $1 \leq k \leq n-1$ we have that $\alpha(v_k) \subseteq \Lambda_k$ and $\omega(v_k) \subseteq \Lambda_{k+1}$. In other words, between two static classes there exists a chain of static classes connected by heteroclinic semistatic orbits (cf. figure 2).

A proof of the following theorem can be found in [6]

3-11.2. Theorem. *If M is compact, then*

1. *$\Sigma(L)$ is chain transitive.*
2. *$\widehat{\Sigma}(L)$ is chain recurrent.*

Now we proceed to prove theorem 3-11.1. Assume for the rest of this section that M is compact.

3-11.3. Proposition.
If $v \in \Sigma$ is semistatic, then $\alpha(v) \subset \widehat{\Sigma}(L)$ and $\omega(v) \subseteq \widehat{\Sigma}(L)$. Moreover $\alpha(v)$ and $\omega(v)$ are each contained in a static class.

Proof: We prove only that $\omega(v) \subset \widehat{\Sigma}$. Let $\gamma(t) = \pi\,\varphi_t(v)$. Suppose that $t_n \to +\infty$ and $\dot\gamma(t_n) \to w \in TM$. Let $\eta(t) = \pi\,\varphi_t(w)$. Since γ and η are solutions the Euler-Lagrange equation, then $\gamma|_{[t_n-s,t_n+s]} \xrightarrow[C^1]{} \eta|_{[-s,s]}$. Then

$$\begin{aligned}
A_{L+c}\big(\eta|_{[-s,s]}\big) + \Phi_c(\eta(s),\eta(-s)) &= \\
&= \lim_n \big\{ A_{L+c}(\gamma|_{[t_n-s,t_n+s]}) + \lim_m A_{L+c}(\gamma|_{[t_n+s,t_m-s]}) \big\} \\
&= \lim_n \lim_m A_{L+c}(\gamma|_{[t_n-s,t_m-s]}) \\
&= \lim_n \lim_m \Phi_c(\gamma(t_n-s),\gamma(t_m-s)) \\
&= \Phi_c(\eta(-s),\eta(-s)) = 0.
\end{aligned}$$

Thus $w \in \widehat{\Sigma}(L)$. Let $u \in \omega(v)$. We may assume that $\gamma(s_n) \to w$ with $t_n < s_n < t_{n+1}$. Then

$$\begin{aligned}
d_c(\pi w, \pi u) &= \Phi_c(\pi w,\pi u) + \Phi_c(\pi u, \pi w) \\
&= \lim_n A_{L+c}(\gamma|_{[t_n,s_n]} + A_{L+c}(\gamma|_{[s_n,t_{n+1}]}) \\
&= \lim_n A_{L+c}(\gamma_n|_{[t_n,t_{n+1}]}) = \Phi_c(\pi w,\pi w) = 0.
\end{aligned}$$

Thus w and u are in the same static class. $\square$

3-11.4. Proposition. *Every static class is connected.*

Proof: Let Λ be a static class and suppose that it is not connected. Let U_1, U_2 be disjoint open sets such that $\Lambda \subseteq U_1 \cup U_2$ and $\Lambda \cap U_i \neq \varnothing$, $i = 1, 2$. Let $p_i \in \pi(U_i \cap \Lambda)$, $i = 1, 2$. Since U_1 and U_2 are disjoint sets we can take a solution $x_{v_n} : [a_n, b_n] \to M$, $a_n < 0 < b_n$ of (E-L) such that $x_{v_n}(0) \notin \pi(U_1 \cup U_2)$, $x_{v_n}(a_n) = p_1$, $x_{v_n}(b_n) = p_2$ and

$$A_{L+c}(x_{v_n}) \leq \Phi_c(p_1, p_2) + \tfrac{1}{n}. \tag{3.15}$$

Let u be a limit point of v_n, then $x_u : \mathbb{R} \to M$ is semistatic (see the proof of claim 2 item (a)). Then, for $a_n \leq s \leq t \leq b_n$,

$$d_c(p_1, p_2) \leq \Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2) + \Phi_c(p_2, p_1),$$

therefore

$$\begin{aligned} d_c(p_1, p_2) &\leq \Phi_c(p_2, p_1) \\ &\qquad + \liminf_n \left[\Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2)\right] \\ &\leq \Phi_c(p_2, p_1) + \liminf_n A_{L+c}(x_{v_n}) \\ &\leq d_c(p_1, p_2) = 0, \end{aligned}$$

where in the last inequality we used (3.15). Hence

$$\Phi_c(p_1, x_u(s)) + \Phi_c(x_u(s), x_u(t)) + \Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) = 0.$$

Combining the last equation with the triangle inequality we obtain

$$\begin{aligned} &d_c(x_u(s), x_u(t)) \leq \\ &\quad \leq \Phi_c(x_u(s), x_u(t)) + [\Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) + \Phi_c(p_1, x_u(s))] = 0. \end{aligned}$$

So that $u \in \widehat{\Sigma}$. Moreover, for $s = 0, t = 1$:

$$\begin{aligned} &d_c(x_u(0), p_1) \leq \\ &\quad \leq \Phi_c(p_1, x_u(0)) + [\Phi_c(x_u(0), x_u(1)) + \Phi_c(x_u(1), p_2) + \Phi_c(p_2, p_1)] = 0. \end{aligned}$$

Hence $x_u(0) \in \pi(\Lambda)$. On the other hand $x_u(0) \notin \pi(U_1 \cup U_2)$. This contradicts the fact that $\Lambda \subseteq U_1 \cup U_2$. □

Proof of theorem 3-11.1.

Given $v \in TM$ denote by $\alpha(v)$ and $\omega(v)$ its α and ω-limits respectively. By proposition 3-11.4 the static classes are connected. Hence if we assume that there are only finitely many of them, the connected components of $\widehat{\Sigma}$ are finite and must coincide with the static classes. For $\varepsilon > 0$, let $\widehat{\Sigma}(\varepsilon)$ be the ε-neighborhood of $\widehat{\Sigma}$, i.e.

$$\widehat{\Sigma}(\varepsilon) := \{\, v \in TM \mid d_{TM}(v, \widehat{\Sigma}) < \varepsilon \,\}.$$

Fix $\varepsilon > 0$ small enough such that the connected components of $\widehat{\Sigma}(\varepsilon)$ are the ε-neighborhoods of the static classes. So that for $0 < \delta < \varepsilon$, $\widehat{\Sigma}(\delta) = \sum_{i=1}^{N(\varepsilon)} \Lambda_i(\delta)$, where $\Lambda_i(\delta)$ are disjoint open sets containing exactly one static class and the number of components $N(\varepsilon)$ is fixed for all $0 < \delta < \varepsilon$.

Now suppose that the theorem is false. Let Λ_i, $\Lambda_k \in \mathbf{\Lambda}$ be such that $\Lambda_i \not\preccurlyeq \Lambda_k$. Let

$$\mathbb{A} := \bigcup_{\{\, \Lambda_j \in \mathbf{\Lambda} \mid \Lambda_i \preccurlyeq \Lambda_j \,\}} \Lambda_j \quad , \quad \mathbb{B} := \bigcup_{\{\, \Lambda_j \in \mathbf{\Lambda} \mid \Lambda_i \not\preccurlyeq \Lambda_j \,\}} \Lambda_j .$$

Given $v \in \Sigma$ with $\alpha(v) \subseteq \mathbb{A}$ and $0 < \delta < \varepsilon$, define inductively

$$\begin{aligned}
s_1(v) &:= \inf\{\, s \in \mathbb{R} \mid f_s(v) \notin \mathbb{A}(\varepsilon) \,\}, \\
t_k(v) &:= \sup\{\, t < s_k(v) \mid f_t(v) \in \mathbb{A}(\delta) \,\}, \\
T_k(v) &:= \inf\{\, t > s_k(v) \mid f_t(v) \in \mathbb{A}(\delta) \,\}, \\
s_{k+1}(v) &:= \inf\{\, s > T_k(v) \mid f_t(v) \notin \mathbb{A}(\varepsilon) \,\}; \\
A_k = A_k(\delta) &:= \sup\{\, |\, T_k(v) - t_k(v) \,|\, v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A} \,\}.
\end{aligned}$$

We split the rest of the proof of theorem 3-11.1 into the following claims:

Claim 1. *$A_k(\delta) < +\infty$ for all $k = 1, 2, \dots$ and all $0 < \delta < \varepsilon$.*

Define

$$\mathbb{M} := \{\, v \mid v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A}\}.$$

Claim 2.

(a) $\overline{\mathbb{M}} \cap \mathbb{B} \neq \varnothing$.

(b) $\lim_k \sup A_k(\delta) = \sup_k A_k(\delta) = +\infty$.

Claim 3. *There exist sequences $v_n \in \Sigma$, $0 < s_n < t_n$ such that $v_n \to u_1 \in \mathbb{A}$, $f_{s_n}(v_n) \to u_2 \notin \mathbb{A}(\varepsilon)$, $f_{t_n}(v_n) \to u_3 \in \mathbb{A}$ and $d_c(\pi u_1, \pi u_3) = 0$.*

We now use claim 3 to complete the proof of theorem 3-11.1. If $u_1 \in \Lambda_j \subseteq \mathbb{A}$, we shall prove that $u_2 \in \Lambda_j \setminus \mathbb{A}(\varepsilon)$, obtaining a contradiction and thus proving theorem 3-11.1. It is enough to show that $d_c(\pi u_1, \pi u_2) = 0$. Indeed

$$\begin{aligned}
d_c(\pi u_1, \pi u_2) &= \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_1) \\
&\leq \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_3) + \Phi_c(\pi u_3, \pi u_1) \\
&\leq \lim_n \left[\Phi_c(\pi v_n, \pi f_{s_n}(v_n)) + \Phi_c(\pi f_{s_n}(v_n), \pi f_{t_n}(v_n))\right] \\
&\qquad\qquad\qquad\qquad\qquad + \Phi_c(\pi u_3, \pi u_1) \\
&= \lim_n \Phi_c(\pi v_n, \pi f_{t_n}(v_n)) + \Phi_c(\pi u_3, \pi u_1) \\
&= d_c(\pi u_1, \pi u_3) = 0,
\end{aligned}$$

where the fourth equation holds because v_n is a semistatic vector.

$\square$

Proof of claim 1:

Suppose that $A_i < +\infty$ for $i = 1, \dots, k-1$ and $A_k = +\infty$. The case $k = 1$ is similar. Then there exists $v_n \in \Sigma$, with $\alpha(v_n) \subset \mathbb{A}$ and $T_k(v_n) - t_k(v_n) \to +\infty$. We can assume that $t_k(v_n) = 0$ and that v_n

converges (Σ is compact). Let $u = \lim_n v_n \in \partial\mathbb{A}(\varepsilon)$. Then

$$m\{\, t < 0 \mid f_t(v_n) \notin \mathbb{A}(\varepsilon) \,\} \leq \sum_{i=1}^{k-1} A_i \,,$$

where m is the Lebesgue measure on $\mathbb{R}$. This implies that

$$m\{\, t < 0 \mid f_t(u) \notin \mathbb{A}(\varepsilon) \,\} \leq \sum_{i=1}^{k-1} A_i$$

and hence $\alpha(u) \subset \mathbb{A}$. Since $f_t(v_n) \notin \mathbb{A}(\varepsilon)$ for $0 < t < T_k(v_n)$ and $T_k(v_n) \to +\infty$, then $f_t(u) \notin \mathbb{A}(\varepsilon)$ for all $t > 0$ and hence $\omega(u) \subseteq \mathbb{B}$. But then the orbit of u contradicts the definition of $\mathbb{B}$.

□

Proof of claim 2:

(a) Let $p \in \pi\mathbb{A}$, $q \in \pi\mathbb{B}$. For $n > 0$, let $x_{v_n} : [a_n, b_n] \to M$ be a solution of (E-L) such that $x_{v_n}(a_n) = p$, $x_{v_n}(b_n) = q$ and

$$A_{L+c}(x_{v_n}) \leq \Phi_c(p,q) + \tfrac{1}{n}.$$

This implies that

$$A_{L+c}\left(x_{v_n}|_{[s,t]}\right) \leq \Phi_c\big(x_{v_n}(s), x_{v_n}(t)\big) + \tfrac{1}{n} \tag{3.16}$$

for all $a_n \leq s \leq t \leq b_n$. We can assume that

$$\inf\{\, s > a_n \mid x_{v_n}(s) \in \mathbb{B}(\delta) \,\} = 0,$$

and that the sequence v_n converges (cf. lemma 3-2.3). Let $u = \lim_n v_n \in \partial\mathbb{B}(\delta)$. Taking limits in (3.16) we obtain that $x_u|_{[s,t]}$ is semistatic for all $\liminf_n a_n \leq s \leq t \leq \limsup_n b_n$.

Any limit point w of $\dot{x}_{v_n}(a_n) = f_{a_n}(v_n)$ satisfies $\pi(w) = p \in \pi\mathbb{A}$, and by the graph property (theorem 3-8.1), $w \in \mathbb{A}$. Similarly, any limit point of $f_{b_n}(v_n)$ is in $\mathbb{B}$. Since $\mathbb{A} \cup \mathbb{B}$ is invariant and $u \notin \mathbb{A} \cup \mathbb{B}$, then $\lim_n a_n = -\infty$, $\lim_n b_n = +\infty$. Hence $u \in \Sigma$. Since $f_t(v_n) \notin \mathbb{B}(\delta)$ for all $a_n \leq t < 0$ and $a_n \to -\infty$, then $f_t(u) \notin \mathbb{B}(\delta)$ for all $t < 0$. Hence

$\alpha(u) \subseteq \mathbb{A}$ and $u \in \mathbb{B}(\delta)$. Thus $u \in \mathbb{M} \cap \mathbb{B}(\delta) \neq \varnothing$. Letting $\delta \to 0$, we obtain that $\overline{\mathbb{M}} \cap \mathbb{B} \neq \varnothing$.

(b) By claim 1 it is enough to show that $\sup_k A_k(\delta) = +\infty$. If $\sup_k A_k(\delta) < T$, then $\mathbb{M} \subseteq \mathbb{M}(\delta, T)$, where

$$\mathbb{M}(\delta, T) = \{\, v \in \Sigma \mid f_{[-T,T]}(v) \cap \mathbb{A}(\delta) \neq \varnothing \,\}.$$

Then $\mathbb{M} \cap \mathbb{B} \subseteq \mathbb{M}(\delta, T) \cap \mathbb{B} = \varnothing$, because $\mathbb{B}$ is invariant and $\mathbb{B} \cap \mathbb{A}(\delta) = \varnothing$. This contradicts item (a).

□

Proof of claim 3:

Given $0 < \delta < \varepsilon$, by claim 2(b) there exists $k > N(\varepsilon)$ such that $A_k(\delta) > 0$. Hence there is $v = v_\delta \in \Sigma$ with $\alpha(v) \subset \mathbb{A}$, such that the orbit of v leaves $\mathbb{A}(\varepsilon)$ and returns to $\mathbb{A}(\delta)$ at least k times. Since $k > N(\varepsilon)$ there is one component $\Lambda_j(\delta) \subseteq \mathbb{A}(\delta)$ with two of these returns, i.e. there exist $\tau_1(\delta) < s(\delta) < \tau_2(\delta)$ with $f_{\tau_1}(v) \in \Lambda_j(\delta)$, $f_s(v) \notin \mathbb{A}(\varepsilon)$ and $f_{\tau_2}(v) \in \Lambda_j(\delta)$. Now, there exists a sequence such that the repeated component $\Lambda_j \subset \Lambda_j(\delta_n)$ is always the same. Let $s_n := s(\delta_n)$, $t_n := \tau_2(\delta_n)$ and choose a subsequence such that $f_{\tau_1(\delta_n)}(v_n)$, $f_{s_n}(v_n)$ and $f_{t_n}(v_n)$ converge. Let $u_1 = \lim_n f_{\tau_1(\delta_n)}(v_n) \in \cap_n \Lambda_j(\delta_n) = \Lambda_j$, $u_3 = \lim_n f_{t_n}(v_n) \in \Lambda_j$ and $u_2 = \lim_n f_{s_n}(v_n) \notin \mathbb{A}(\varepsilon)$. Since $u_1, u_3 \in \Lambda_j$, then $d_c(\pi u_1, \pi u_3) = 0$.

□

Chapter 4

Dynamics on prescribed energy levels.

4-1 The Hamilton-Jacobi equation.

Let ω be the canonical symplectic form on T^*M. A subspace λ of T_pT^*M is called *isotropic* if $\omega(X,Y)=0$ for all X,Y on λ. Since ω is nondegenerate, the isotropic subspaces have dimension $\leq n$, half of the dimension of T^*M. Isotropic spaces of dimension n are called *lagrangian* subspaces We say that a submanifold $W\subset T^*M$ is *lagrangian* if at each point $\theta\in W$, its tangent space $T_\theta W$ is a lagrangian subspace of $T_\theta T^*M$. In particular, $\dim W=\dim M=n$.

4-1.1. Theorem (Hamilton-Jacobi).

If the hamiltonian H is constant on a lagrangian submanifold N, then N is invariant under the hamiltonian flow.

Proof: We only have to show that the hamiltonian vectorfield X is tangent to N. Since H is constant on N, then $dH|_{TN}\equiv 0$. Since $\omega(X,\cdot)=dH$, then $\omega(X,\xi)=0$ for all $\xi\in TN$. Since the tangent spaces to N are lagrangian, they are maximal isotropic subspaces, therefore $X\in TN$. $\square$

Some distinguished n-dimensional manifolds on T^*M are the graph submanifolds, which are of the form

$$G_\eta = \{\ (x, \eta_x) \mid x \in M\ \} \subset T^*M, \tag{4.1}$$

where η_x is a 1-form on M. A *lagrangian graph* is a lagrangian graph submanifold. In fact,

4-1.2. Lemma. *G_η is a lagrangian graph if and only if the form η is closed:*

$$G_\eta \text{ is lagrangian } \iff d\eta \equiv 0$$

Proof: Choose local coordinates $q_1, \ldots, q_n$ of M. Then $\eta(q) = \sum_k p_k(q)\, dq_k$. A basis of the tangent space to the graph G_η is given by $E_i = \left(\frac{\partial}{\partial q_i}, \sum_k \frac{\partial p_k}{\partial q_i} \frac{\partial}{\partial p_k}\right)$. Applying $\omega = dp \wedge dq$,

$$\omega(E_i, E_j) = \frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}.$$

Since

$$d\eta = \sum_{i<j} \left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i}\right)\ dq_j \wedge dq_i\,,$$

then $\omega|_{TG_\eta} \equiv 0 \iff d\eta \equiv 0$.

□

Thus, we can associate a cohomology class $[\eta] \in H^1(M, \mathbb{R})$ to each lagrangian graph G_η. Lagrangian graphs with zero cohomology class are the graphs of the exact 1-forms: G_{df}, with $\eta = df$ and $f : M \to \mathbb{R}$ a smooth function. These are called *exact lagrangian graphs.*

The Hamilton-Jacobi equation for autonomous hamiltonians is

$$H(x, d_x u) = k, \qquad u : M \to \mathbb{R}. \tag{H-J}$$

Thus a smooth solution of the Hamilton-Jacobi equation corresponds to an exact invariant lagrangian graph.

4-2 Subsolutions of the Hamilton-Jacobi equation.

We say that a function $u : M \to \mathbb{R}$ is a *subsolution* of the Hamilton-Jacobi equation if

$$H(x, d_x u) \leq k,$$

i.e if $d_x u$ is an exact lagrangian graph. We shall prove that for $k > c(L)$ there is always a C^∞ subsolution of the Hamilton-Jacobi equation and for $k < c(L)$ there are no differentiable subsolutions. Hence

4-2.1. Theorem. *If M is any covering of the closed manifold N, then*

$$\begin{aligned} c(L) &= \inf_{f \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x f) \\ &= \inf\{k \in \mathbb{R} : \text{ there exists } f \in C^\infty(M,\mathbb{R}) \text{ such that } H(df) < k\}, \end{aligned}$$

where H is the hamiltonian associated with L.

theorem 4-2.1 could be restated by saying that $c(L)$ is the infimum of the values of $k \in \mathbb{R}$ for which $H^{-1}(-\infty, k)$ contains an exact lagrangian graph. This is a very geometric way of describing the critical value.

theorem 4-2.1 will be an immediate consequence of lemma 4-2.2 and proposition 4-2.4 below.

4-2.2. Lemma.
If there exists a C^1 function $f : M \to \mathbb{R}$ such that $H(df) < k$, then $k \geq c(L)$.

Proof: Recall that

$$H(x, p) = \max_{v \in T_x M} \{p(v) - L(x, v)\}.$$

Since $H(df) < k$ it follows that for all $(x, v) \in TM$,

$$d_x f(v) - L(x, v) < k.$$

Therefore, if $\gamma : [0,T] \to M$ is any absolutely continuous closed curve with $T > 0$, we have

$$\int_0^T (L(\gamma,\dot\gamma) + k)\, dt = \int_0^T (L(\gamma,\dot\gamma) + k - d_\gamma f(\dot\gamma))\, dt > 0,$$

and thus $k \geq c(L)$. □

4-2.3. Lemma.
Let $k \geq c(L)$. If $f : M \to \mathbb{R}$ is differentiable at $x \in M$ and satisfies

$$f(y) - f(x) \leq \Phi_k(x,y)$$

for all y in a neighbourhood of x, then $H(x, d_x f) \leq k$.

Proof: Let $\gamma(t)$ be a differentiable curve on M with $(\gamma(0), \dot\gamma(0)) = (x, v)$. Then

$$\limsup_{t\to 0^+} \frac{f(\gamma(t)) - f(x)}{t} \leq \liminf_{t\to 0^+} \frac{1}{t} \int_0^t \left[L(\gamma,\dot\gamma) + k\right] ds.$$

Hence $d_x f(v) \leq L(x,v) + k$ for all $v \in T_x M$ and thus

$$H(x, d_x f) = \max_{v\in T_x M} \{d_x f(v) - L(x,v)\} \leq k.$$

□

4-2.4. Proposition.
For any $k > c(L)$ there exists $f \in C^\infty(M,\mathbb{R})$ such that $H(df) < k$.

Proof: We shall explain first how to prove the proposition in the case of $M = N$ and then we will lift the construction to an arbitrary covering M.

Set $c = c(L)$. Fix $q \in M$ and let $u(x) := \Phi_c(q,x)$. By the triangle inequality, we have that

$$u(y) - u(x) \leq \Phi_c(x,y) \text{ for all } x, y \in M.$$

By the previous lemma, $H(d_x u) \leq c$ at any point $x \in M$ where $u(x)$ is differentiable.

We proceed to regularize u. We can assume that $M \subseteq \mathbb{R}^N$. Let U be a tubular neighbourhood of M in $\mathbb{R}^N$, and $\rho : U \to M$ a C^∞ projection along the normal bundle. Extend $u(x)$ to U by $\overline{u}(z) = u\big(\rho(z)\big)$. Then $\overline{u}(z)$ is also Lipschitz.

Extend the lagrangian to U by

$$\overline{L}(z,v) = L\big(\rho(z), d_z\rho(v)\big) + \tfrac{1}{2}\,|v - d_z\rho(v)|^2\,.$$

Then the corresponding hamiltonian satisfies $\overline{H}(z, p \circ d_z\rho) = H\big(\rho(z), p\big)$ for $p \in T^*_{\rho(z)}M$. At any point of differentiability of $\overline{u}$, we have that $d_z\overline{u} = d_{\rho(z)}u \circ d_z\rho$, and $\overline{H}(d_z\overline{u}) = H(d_{\rho(z)}u) \leq c$.

Let $\varepsilon > 0$ be such that

(a) The 3ε-neighbourhood of M in $\mathbb{R}^N$ is contained in U.

(b) If $x \in M$, $(y,p) \in T^*\mathbb{R}^N = \mathbb{R}^{2N}$, $\overline{H}(y,p) \leq c$ and $d_{\mathbb{R}^N}(x,y) < \varepsilon$ then $\overline{H}(x,p) < k$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\psi(x) \geq 0$, support$(\psi) \subset (-\varepsilon, \varepsilon)$ and $\int_{\mathbb{R}^N} \psi(|x|)\,dx = 1$. Let $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be $K(x,y) = \psi(|x-y|)$. Let N_ε be the ε-neighbourhood of M in $\mathbb{R}^N$. Define $f : N_\varepsilon \to \mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}^N} \overline{u}(y)\; K(x,y)\; dy\,.$$

Then f is C^∞ on N_ε.

Observe that $\partial_x K(x,y) = -\partial_y K(x,y)$. Since $\overline{u}(y)$ is Lipschitz, it is differentiable at Lebesgue almost every point of U (Rademacher's theorem, cf. [14]). Moreover it is weakly differentiable (cf. [14, Section 4.2.3]), that is, for any C^∞ function $\varphi : U \to \mathbb{R}$ with compact support

$$\int_{\mathbb{R}^N} (\varphi\; d\overline{u} + \overline{u}\; d\varphi)\; dx = 0\,.$$

Hence

$$-\int_{\mathbb{R}^N} \overline{u}(y)\, \partial_y K(x,y)\, dy = \int_{\mathbb{R}^N} K(x,y)\, d_y\overline{u}\, dy.$$

Now, since

$$d_x f = \int_{\mathbb{R}^N} \overline{u}(y)\, \partial_x K(x,y)\, dy,$$

we obtain

$$d_x f = \int_{\mathbb{R}^N} K(x,y)\, d_y\overline{u}\, dy.$$

From the choice of $\varepsilon > 0$ we have that $\overline{H}(x, d_y\overline{u}) < k$ for almost every $y \in \operatorname{supp} K(x,\cdot)$ and $x \in M$. Since $K(x,y)\,dy$ is a probability measure, by Jensen's inequality

$$H(d_x f) \le \overline{H}(d_x f) \le \int_{\mathbb{R}^N} \overline{H}(x, d_y\overline{u})\, K(x,y)\, dy < k\,.$$

for all $x \in M$.

Now, suppose that M is a covering of a compact manifold N with covering projection p. Assume that $N \subseteq \mathbb{R}^N$. Fix $q \in M$ and set $u(x) := \Phi_{c(L)}(q,x)$. We can regularize our function u similarly as we shall now explain. For $\widetilde{x} \in M$ let x be the projection of $\widetilde{x}$ to N and let μ_x be the Borel probability measure on N defined by

$$\int_N \varphi\, d\mu_x = \int_{\mathbb{R}^N} (\varphi \circ \rho)(y)\, K(x,y)\, dy,$$

for any continuous function $\varphi : N \to \mathbb{R}$. Then the support of μ_x satisfies

$$\operatorname{supp}(\mu_x) \subset \{y \in N \,:\, d_N(x,y) < \varepsilon\}.$$

Let $\widehat{\mu}_{\widetilde{x}}$ be the Borel probability measure on M uniquely defined by the conditions: $\operatorname{supp}(\widehat{\mu}_{\widetilde{x}}) \subset \{\widehat{y} \in M \,:\, d_M(\widetilde{x},\widehat{y}) < \varepsilon\}$ and $p_*\widehat{\mu}_{\widetilde{x}} = \mu_x$. Then we have

$$\frac{d}{d\widetilde{x}} \int_M \varphi\, d\widehat{\mu}_{\widetilde{x}} = \int_M d_{\widehat{y}}\varphi\, d\widehat{\mu}_{\widetilde{x}}(\widehat{y}),$$

for any weakly differentiable function $\varphi : M \to \mathbb{R}$. The same arguments as above show that

$$f(\widetilde{x}) = \int_M u(\widehat{y})\, d\widehat{\mu}_{\widetilde{x}}(\widehat{y}),$$

satisfies $H(d_{\widetilde{x}} f) < k$. □

4-2.5. Corollary.

$$\alpha(q) = \inf_{[\omega]=q} \sup_{x\in N} \mathbb{H}(x, \omega(x)).$$

Proof: Let us fix a closed one form ω_0 such that $[\omega_0] = q$. By equality (2.29) we have that $\alpha(q) = c(\lambda - \omega_0)$. Hence, it suffices to show that

$$c(\lambda - \omega_0) = \inf_{[\omega]=q} \sup_{x\in N} \mathbb{H}(x, \omega(x)). \tag{4.2}$$

It is straightforward to check that the hamiltonian associated with $\lambda - \omega_0$ is $\mathbb{H}(x, p + \omega_0(x))$. Since all the closed one forms in the class q are given by $\omega_0 + df$ where f ranges among all smooth functions, equality (4.2) is now an immediate consequence of theorem 4-2.1.

□

4-2.6. Corollary. *If $k > c(L)$, then it is possible to see the dynamics of $\phi_t|_{E^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriately chosen Finsler metric on M.*

Proof: If $k > c(L)$, then $H^{-1}(-\infty, k)$ contains an exact lagrangian graph. This means that there exists a smooth function $f : M \to \mathbb{R}$ such that $H(x, d_x f) < k$ for all $x \in M$. Therefore the new hamiltonian $H_{df}(x,p) \stackrel{\text{def}}{=} H(x, p + d_x f)$ is such that $H_{df}^{-1}(-\infty, k)$ contains the zero section of T^*M. Let $\varphi : T^*M \to T^*M$ be the map $\varphi(x,p) = (x, p + d_x f)$. Observe that the hamiltonian flow ϕ_t^* of H and the hamiltonian flow ψ_t of H_{df} are related by $\psi_t \circ \varphi = \varphi \circ \phi_t^*$. Define now a new hamiltonian G on T^*M minus the zero section such that G takes the value one on

$H_{df}^{-1}(k)$ and such that $G(x, \lambda p) = \lambda^2 G(x,p)$ for all positive λ. Since G is positively homogeneous of degree two and convex in p, it follows that the Legendre transform $\mathcal{L}_G$ associated to G is a diffeomorphism from TM minus the zero section to T^*M minus the zero section. Therefore the hamiltonian G induces a Finsler metric on M simply by taking $G \circ \mathcal{L}_G$.

Since by definition $G^{-1}(1) = H_{df}^{-1}(k)$ it follows that the hamiltonian flows of G and $H_{df}^{-1}(k)$ coincide up to reparametrization on the energy level $G^{-1}(1) = H_{df}^{-1}(k)$ and therefore the Euler-Lagrange solutions of L with energy k are reparametrizations of unit speed geodesics of $G \circ \mathcal{L}_G$. □

Given a Finsler metric $\sqrt{F}$ and an absolutely continuous curve γ we can define its Finsler length as

$$l_F(\gamma) = \int \sqrt{F(\dot\gamma)}$$

Observe that since the Finsler metric is homogeneous of degree one, the definition does not depend on the parametrization of the curve. Finally we define the Finsler distance as

$$D_F(x,y) = \inf\{l_F(\gamma)\}$$

where the infimum is taken over all absolutely continuous curves joining x and y.

4-2.7. Corollary. *[24] If $k > c(L)$, then there exists a Finsler metric $\sqrt{F}$ and a C^∞ real valued function f on M such that $\Phi_k(x,y) = D_F(x,y) + f(y) - f(x)$. Moreover if $k > -\inf L$, then we can choose $f = 0$.*

Proof: We begin with the last statement. Note that $L + k > 0$ if and only if $H(x,0) < k$. Indeed

$$H(x,p) = \max_{v \in T_xM} (pv - L(x,v))$$

then

$$H(x,0) = \max_{v \in T_x M}(-L(x,v)) = -\min(L(x,v).$$

So if $k > -\inf L$ then $H^{-1}(-\infty, k)$ contains the zero section of T^*M.

Define as before a Hamiltonian G on T^*M minus the zero section such that G takes the value μ on $H^{-1}(k)$ and such that $G(x,\lambda p) = \lambda^2 G(x,p)$ for all positive λ. Let F be the associated Finsler metric on M by the Legendre transformation.

We claim that for an appropriate choice of μ and if $E(x,v) = k$ then

$$\sqrt{F(x,v)} = L + k.$$

From proposition 3-5.1, for $k > c(L)$ and for any $x, y \in M$ there exists γ such that $A_{L+k}(\gamma) = \Phi_k(x,y)$. Moreover γ is a solution of the Euler-Lagrange equation and has energy k. Also, if $k > c(L)$, every curve can be reparametrized to have energy k and the Finsler length does not depend on the reparametrization. So in the definitions of D_F and Φ_k we may restrict ourselves to curves with energy k and theorem 4-2.7 follows in this case.

Proof of the claim: Since G is homogeneous of degree 2 it follows from Euler's formula that F and G take the same value at Legendre related points.

Define $\lambda(x,p)$ such that $H(x, \frac{p}{\lambda}) = k$, then $G(x,p) = \mu\lambda^2(x,p)$.

We have that

$$\tfrac{\partial H}{\partial p}(x, \tfrac{p}{\lambda})\, \lambda^{-1} - \tfrac{\partial H}{\partial p}(x, \tfrac{p}{\lambda}) \cdot p\, \lambda^{-2}\, \tfrac{\partial \lambda}{\partial p} = 0 \tag{4.3}$$

and

$$\frac{\partial G}{\partial p} = 2\,\mu\,\lambda\,\frac{\partial \lambda}{\partial p}.$$

Then using (4.3) multiplied by λ^2 we get

$$\tfrac{\partial H}{\partial p}(x, \tfrac{p}{\lambda}) \cdot p\, \tfrac{\partial G}{\partial p} = 2\,G(x,p)\, \tfrac{\partial H}{\partial p}(x, \tfrac{p}{\lambda}). \tag{4.4}$$

Suppose now that $E(x,v) = k$ and define $P(x,v) = \frac{\partial L}{\partial v}$ then by definition we have

$$\begin{aligned} \lambda(x, P(x,v)) &= 1, && (4.5)\\ G(x, P(x,v)) &= \mu, && (4.6)\\ \tfrac{\partial H}{\partial p}(x, P(x,v)) &= v, && (4.7) \end{aligned}$$

and so

$$\begin{aligned} \tfrac{\partial H}{\partial p}(x, P(x,v)) \cdot P(x,v) &= v\,\tfrac{\partial L}{\partial v} && (4.8)\\ &= L + k && (4.9)\\ &> 0. && (4.10) \end{aligned}$$

Hence from (4.4) we have

$$\frac{\partial G}{\partial p}(x, P(x,v)) = \frac{2v}{v \cdot P(x,v)}.$$

Then since $\frac{\partial G}{\partial p}$ is homogeneous of degree one and from (4.8) $v \cdot P(x,v)$ is positive we obtain

$$\tfrac{\partial G}{\partial p}(x, [\tfrac{1}{2}v \cdot P(x,v)]\, P(x,v)) = v.$$

So v is related to $\frac{1}{2}v \cdot P(x,v)P(x,v)$ with respect to the Legendre transform of F. Hence

$$\begin{aligned} F(x,v) &= G(x, \tfrac{1}{2}v \cdot P(x,v)P(x,v))\\ &= \frac{(v \cdot P(x,v))^2}{4}\, G(x, P(x,v))\\ &= \frac{(v \cdot P(x,v))^2}{4}\, \mu. \end{aligned}$$

So if $\mu = 4$

$$\sqrt{(F(x,v))} = v \cdot \frac{\partial L}{\partial v}$$

Let now $k > c(L)$, then by proposition 4-2.4 there exists a C^∞ real valued function on M f, such that $H(x, df_x) < k$. Define as before $H_{df}(x,p) = H(x, p + df_x)$. The Legendre transformation L_{df} of H_{df} is

$$\begin{aligned} L_{df}(x,v) &= \max_{p\in T_x^*M}(pv - H_{df}(x,p) \\ &= \max_{p\in T_x^*M}(pv - H(x,p+df_x)) \\ &= \max_{p\in T_x^*M}((p-df_x)v - H(x,p)) \\ &= L(x,v) - df_x v. \end{aligned}$$

It turns out that

$$\begin{aligned} E(L_{df}) &= E(L), \\ c(L_{df}) &= c(L), \\ \Phi_k(L_{df})(x,y) &= \Phi_k(L)(x,y) - f(y) + f(x). \end{aligned}$$

So as the zero section is contained in $H_{df}^{-1}(-\infty, k)$, $L_{df} + k$ is positive and there is a Finsler metric such that

$$\Phi_k(L_{df})(x,y) = D_F(x,y).$$

So

$$\Phi_k(L)(x,y) = D_F(x,y) + f(y) - f(x).$$

□

4-3 Anosov energy levels.

An *Anosov energy level* is a regular energy level on which the flow ϕ_t is an Anosov flow.

4-3.1. Theorem. *If the energy level* $\mathbb{E}^{-1}(k)$ *is Anosov, then*

$$k > c_u(\lambda).$$

Proof: Suppose that the energy level k is Anosov and set $\Sigma \stackrel{\text{def}}{=} \mathbb{H}^{-1}(k)$. Let $\pi : T^*N \to N$ denote the canonical projection. G.P. Paternain and M. Paternain proved in [49] that Σ must project onto the whole manifold N and that the weak stable foliation $\mathcal{W}^s$ of ϕ_t^* is *transverse* to the fibers of the fibration by $(n-1)$-spheres given by

$$\pi|_\Sigma : \Sigma \to N.$$

Let $\widetilde{N}$ be the universal covering of N. Let $\widetilde{\Sigma}$ denote the energy level k of the lifted hamiltonian H. We also have a fibration by $(n-1)$-spheres

$$\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}.$$

Let $\widetilde{\mathcal{W}^s}$ be the lifted foliation which is in turn a weak stable foliation for the hamiltonian flow of H restricted to $\widetilde{\Sigma}$. The foliation $\widetilde{\mathcal{W}^s}$ is also transverse to the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$. Since the fibers are compact a result of Ehresman (cf. [3]) implies that for every $(x,p) \in \widetilde{\Sigma}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)} : \widetilde{\mathcal{W}^s}(x,p) \to \widetilde{N},$$

is a covering map. Since $\widetilde{N}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}^s}(x,p)$ is simply connected. Consequently, $\widetilde{\mathcal{W}^s}(x,p)$ intersects each fiber of the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$ at just one point. In other words, each leaf $\widetilde{\mathcal{W}^s}(x,p)$ is the graph of a one form. On the other hand it is well known that the weak stable leaves of an Anosov energy level are lagrangian submanifolds. Since any closed one form in the universal covering must be exact, it follows that each leaf $\widetilde{\mathcal{W}}^s(x,p)$ is an exact lagrangian graph. The theorem now follows from lemma 4-2.2 and the fact that by structural stability there exists $\varepsilon > 0$ such that for all $k' \in (k-\varepsilon, k+\varepsilon)$ the energy level k' is Anosov. □

For $e \in \mathbb{R}$, let $\mathcal{A}_e$ be the set of $\phi \in C^\infty(M)$ such that the flow of $H+\phi$ is Anosov in $(H+\phi)^{-1}(e)$ and let $\mathcal{B}_e$ be the set of $\phi \in C^\infty(M)$ such that $(H+\phi)^{-1}(e)$ contains no conjugate points. As is well known $\mathcal{A}_e$ is open in C^k topology and $\mathcal{B}_e$ is closed. On the other hand G. and M. Paternain [47] have shown that $\mathcal{A}_e$ is contained in $\mathcal{B}_e$. It is proved in [9] the following

4-3.2. Theorem. *The interior of $\mathcal{B}_e$ in the C^2 topology is $\mathcal{A}_e$.*

This theorem is an extension to the Hamiltonian setting of a result of R. O. Ruggiero for the geodesic flow [55]. Theorems 4-3.2 and 4-3.1 have as corollary:

4-3.3. Corollary. *Given a convex superlinear lagrangian L, $k < c_u(L)$ and $\varepsilon > 0$ there exists a smooth function $\psi : N \to \mathbb{R}$ with $|\psi|_{C^2} < \varepsilon$ and such that the energy level k of $L+\psi$ possesses conjugate points.*

Proof: Suppose now that there exists $\epsilon > 0$ such that for every ψ with $|\psi|_{C^2} < \epsilon$, the energy level k of $\lambda+\psi$ has no conjugate points. The main result in [9] says that in this case the energy level k of λ must be Anosov thus contradicting theorem 4-3.1. □

4-3.4. Proposition. *If k is a regular value of the energy such that $k < e$, then the energy level k has conjugate points.*

Proof: If an orbit does not have conjugate points then there exist along it two subbundles called the *Green subbundles*. They have the following properties: they are invariant, lagrangian and they have dimension $n = \dim N$. Moreover, they are contained in the same energy level as the orbit and they do not intersect the vertical subbundle (cf. [7]). If k is a regular value of the energy with $k < e$, then $\pi(\mathbb{E}^{-1}(k))$ is a manifold with boundary and at the boundary the vertical subspace is completely contained in the energy level. Therefore the orbits that begin at the boundary must have conjugate points, because at the boundary two n-dimensional subspaces contained in the energy level (which is $(2n-1)$-dimensional) must intersect. □

4-4 Weak KAM Solutions.

Given a continuous function $u : M \to \mathbb{R}$, we shall write $u \prec L + c$ whenever $u(x) - u(y) \leq \Phi_c(y, x)$ for all $x, y \in M$. Let us define the sets

$$\Gamma_0^+(u) := \{\, v \in \Sigma^+ \mid u(x_v(t)) - u(x_v(0)) = \Phi_c(x_v(0), x_v(t))\ \forall t > 0 \,\},$$
$$\Gamma^+(u) := \bigcup_{t>0} \phi_t(\Gamma^+(u)),$$

where ϕ_t is the Euler-Lagrange flow on TM.

We shall say, following Fathi, that a continuous function $u : M \to \mathbb{R}$ is a *weak KAM solution* if u satisfies the following three conditions:

1. u is Lipschitz;

2. $u \prec L + c$;

3. $\pi(\Gamma_0^+(u)) = M$.

It is important to point out that using the action potentials it is quite simple to show the existence of a function u that satisfies only properties 1 and 2 above. Take any point $p \in M$ and set $u(x) = \Phi_c(p, x)$. Elementary properties of the action potential (cf. chapter 3) show that u satisfies 1 and 2. This is used in the proof of theorem 4-2.1 which is based on a convolution argument that smoothes out a function u that satisfies 1 and 2.

Fathi shows in [16] that weak KAM solutions exist assuming that M is compact. His proof is based on applying the Banach fixed point theorem to a certain semigroup of operators defined on the space of continuous functions on M divided by the constant functions and, as presented, it cannot be applied when M is not compact. In our next theorem we show the existence of weak KAM solutions for M an arbitrary covering of a compact manifold. We hope that this will clarify the relationship between Fathi's approach and Mañé's.

Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u : M \to \mathbb{R}$ by

$$u(x) = \sup_{t>0} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right]. \tag{4.11}$$

The function u thus defined clearly resembles the Busemann functions from Riemannian geometry. In fact, the supremum in (4.11) is a limit as we shall see in Section 5, thus if ω-limit $\omega(w) \neq \varnothing$, then $u(x) = u(p) - \Phi_c(x, p)$ for all $x \in M$ and any p in $\pi(\omega(w))$.

4-4.1. Theorem. *The function $u(x)$ in* (4.11) *is a weak KAM solution.*

Hence when M is compact, since $\omega(w) \neq \varnothing$ for any semistatic vector w, the function $u(x) = -\Phi_c(x, p)$, where p is any point in the static class of $\pi(\omega(w))$, is a weak KAM solution.

Recall the set

$$\Sigma^+ := \left\{\, v \in TM \mid x_v : [0, +\infty) \to M \text{ is semistatic } \right\}$$

For the proof of theorem 4-4.1 we need the following

4-4.2. Lemma. $\Sigma^+ \neq \varnothing$.

Proof: If M is compact then there exists a minimizing measure μ then it follows by corollary 3-6.2

Assume that M is not compact. Then there is a sequence $\{q_n\} \subset M$ such that $d_M(q_0, q_n) \to +\infty$. Let $x_n : [0, T_n] \to M$ be a Tonelli minimizer such that $x_n(0) = q_0$, $x_n(T_n) = q_n$ and

$$A_{L+c}(x_n) \leq \Phi_c(q_0, q_n) + \tfrac{1}{n} \cdot \tag{4.12}$$

Since for any $x : \mathbb{R} \to M$, the function $\delta(t) = A_{L+c}(x|_{[0,t]}) - \Phi_c(x(0), x(t))$ is non-decreasing, inequality (4.12) implies that

$$A_{L+c}\left(x_n|_{[s,t]}\right) \leq \Phi_c(x_n(s), x_n(t)) + \tfrac{1}{n} \tag{4.13}$$

for all $0 \leq s \leq t \leq T_n$.

By lemma 3-2.1, $|\dot{x}_n(t)| < A$ for all n large enough, $0 \leq t \leq T_n$. Let $v_n = \dot{x}_n(0)$ and v an density point of $\{v_n\}$. We can assume that $v_n \to v$. Since $d_M(q_0, q_n) \to +\infty$, then $T_n \to +\infty$. Since $x_n|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$ for all $t > 0$, from (4.13) we obtain that $x_v : [0, +\infty) \to M$ is semistatic. □

Proof of theorem 4-4.1

Our candidate for a weak KAM solution is defined as follows. Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u : M \to \mathbb{R}$ by

$$u(x) = \sup_{t>0} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right].$$

By the triangle inequality, for $x \in M$ and $t > 0$,

$$\begin{aligned}\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t)) &\leq \Phi_c(\gamma(0), x) + \Phi_c(x, \gamma(t)) - \Phi_c(x, \gamma(t)) \\ &= \Phi_c(\gamma(0), x).\end{aligned}$$

Hence $u(x) \leq \Phi_c(\gamma(0), x) < +\infty$. Moreover, the function $\delta(t) := \Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))$ is increasing in t because if $0 \leq s \leq t$, then

$$\begin{aligned}\delta(t) - \delta(s) &= \Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t)) - \Phi_c(\gamma(0), \gamma(s)) + \Phi_c(x, \gamma(s)) \\ &= \Phi_c(\gamma(0), \gamma(s)) + \Phi_c(\gamma(s), \gamma(t)) - \Phi_c(x, \gamma(t)) \\ &\qquad\qquad - \Phi_c(\gamma(0), \gamma(s)) + \Phi_c(x, \gamma(s)) \\ &\geq 0.\end{aligned}$$

In the last inequality we used the triangle inequality for the triangle $(x, \gamma(s), \gamma(t))$. Hence the supremum in the definition of u is a limit. For $x, y \in M$, we have that

$$\begin{aligned}
u(x) - u(y) &= \lim_{t\to+\infty} \big[\, \Phi_c(\gamma(0),\gamma(t)) - \Phi_c(x,\gamma(t)) \;-\; \Phi_c(\gamma(0),\gamma(t)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Phi_c(y,\gamma(t)) \,\big] \\
&= \lim_{t\to+\infty} [\Phi_c(y,\gamma(t)) - \Phi_c(x,\gamma(t))] \qquad (4.14)\\
&\le \lim_{t\to+\infty} [\Phi_c(y,x) + \Phi_c(x,\gamma(t)) - \Phi_c(x,\gamma(t))] \\
&\le \Phi_c(y,x).
\end{aligned}$$

Hence $u \prec L + c$. This property implies that

$$|u(x) - u(y)| \le \max\{\, |\Phi_c(x,y)|\ ,\ |\Phi_c(y,x)|\,\},$$

and hence $u(x)$ is Lipschitz, with the same Lipschitz constant as Φ_c.

We show now that $M \setminus \pi(\widehat{\Sigma}) \subseteq \pi(\Gamma_0^+(u))$, where $\Gamma_0^+(u)$ was defined in the introduction . Let $x \in M \setminus \pi(\widehat{\Sigma})$ and let $x_{v_n} : [0, T_n] \to M$ be a Tonelli minimizer such that $x_{v_n}(0) = x$, $x_{v_n}(T_n) = \gamma(n)$ and

$$\Phi_c(x,\gamma(n)) \le A_{L+c}\left(x_{v_n}|_{[0,T_n]}\right) \le \Phi_c(x,\gamma(n)) + \tfrac{1}{n}.$$

The same argument as in inequality (4.13) shows that

$$A_{L+c}\left(x_{v_n}|_{[s,T_n]}\right) - \tfrac{1}{n}\cdot \le \Phi_c(x_{v_n}(s),\gamma(n)) \le A_{L+c}\left(x_{v_n}|_{[s,T_n]}\right),$$

for all $0 \le s \le T_n$, and then

$$\left|\Phi_c(x,\gamma(n)) - \Phi_c(x_{v_n}(t),\gamma(n)) - A_{L+c}\left(x_{v_n}|_{[0,t]}\right)\right| \le \tfrac{1}{n}\,\cdot \qquad (4.15)$$

By lemma 3-2.1, $|\dot{x}_{v_n}(t)| < A$ for all n large enough and $0 \le t \le T_n$. We prove below that $T_n \to +\infty$, then the same arguments as in lemma 4-4.2 show that any limit point of $\{v_n\}$ is in Σ^+ so we may assume that $v_n \to v \in \Sigma^+$. Using the triangle inequality we get

$$\begin{aligned}
\Phi_c(x_v(t),\gamma(n)) - \Phi_c(x_v(t),x_{v_n}(t)) &\le \Phi_c(x_{v_n}(t),\gamma(n)) \\
&\le \Phi_c(x_v(t),\gamma(n)) + \Phi_c(x_{v_n}(t),x_v(t))
\end{aligned}$$

and then

$$|\Phi_c(x_{v_n}(t),\gamma(n)) - \Phi_c(x_v(t),\gamma(n))| \le K\, d_M(x_{v_n}(t), x_v(t)), \tag{4.16}$$

where K is the Lipschitz constant of Φ_c. Combining (4.16) and (4.15) we obtain

$$\begin{aligned} \left|\Phi_c(x,\gamma(n)) - \Phi_c(x_v(t),\gamma(n)) - A_{L+c}\left(x_{v_n}|_{[0,t]}\right)\right| \le& \\ \le \tfrac{1}{n} + K\, d_M(x_{v_n}(t), x_v(t)).& \end{aligned} \tag{4.17}$$

Therefore

$$\begin{aligned} u(x_v(t)) - u(x) &= \lim_{n\to+\infty} \left[\Phi_c(x,\gamma(n)) - \Phi_c(x_v(t),\gamma(n))\right] \\ &= \lim_{n\to+\infty} A_{L+c}\left(x_{v_n}|_{[0,t]}\right) \\ &= A_{L+c}\left(x_v|_{[0,t]}\right) \\ &= \Phi_c(x, x_v(t)), \end{aligned}$$

because $x_{v_n}|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$ and x_v is semistatic.

Now we prove that $\lim_n T_n = +\infty$. Suppose that this is not the case. Then there exists a subsequence that we still denote by $\{T_n\}$ such that $\lim_n T_n = T_0 < +\infty$. Hence the speed $|\dot{x}_{v_n}|$ is uniformly bounded in $[0,T_0]$ and therefore we can assume that $\{v_n\}$ converges to a vector v, $\lim \dot\gamma(n) = (p, w_1) \in \widehat{\Sigma}$, $x_{v_n}|_{[0,T_0]} \xrightarrow{C^1} x_v|_{[0,T_0]}$ and that $x_v|_{[0,T_0]}$ is semistatic. Note that $\dot{x}_v(T_0)$ has the form (p, w_2). Since $x_v|_{[0,T_0]}$ is semistatic, then the graph property in theorem 3-8.1 implies that $w_1 = w_2$. Since $\widehat{\Sigma}$ is invariant, then $x \in \pi(\widehat{\Sigma})$. This contradicts the hypothesis $x \in M \setminus \pi(\widehat{\Sigma})$.

Now let $(x,v) \in \widehat{\Sigma}$ and $t > 0$. Let $p = x_v(t)$ and $y \in M$. Since $d_c(x,p) = 0$, then

$$\begin{aligned} \Phi_c(x,y) &= \Phi_c(x,p) + \Phi_c(p,x) + \Phi_c(x,y) \\ &\ge \Phi_c(x,p) + \Phi_c(p,y) \ge \Phi_c(x,y). \end{aligned}$$

Hence $\Phi_c(x,y) = \Phi_c(x,p) + \Phi_c(p,y)$. For $y = \gamma(s)$ (and $p = x_v(t)$), we have that

$$\begin{aligned} u(x_v(t)) - u(x) &= \lim_{s\to+\infty} [\Phi_c(x,\gamma(s)) - \Phi_c(p,\gamma(s))] = \Phi_c(x, x_v(t)) \\ &= A_{L+c}\left(x_v|_{[0,t]}\right). \end{aligned}$$

Chapter 5

Generic Lagrangians.

In [27], Mañé introduced the concept of *generic property of a lagrangian* L. A property P is said to be generic for the lagrangian L if there exists a generic set $\mathcal{O}$ (in the Baire sense) on the set $C^\infty(M,\mathbb{R})$ such that if ψ is in $\mathcal{O}$ then $L+\psi$ has the property P. One of Mañé's objectives was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if we restrict ourselves to generic lagrangians. The main purpose of this chapter is to proof the following

5-0.1. Theorem. *For every lagrangian L there exists a generic set $\mathcal{O} \subseteq C^\infty(M,\mathbb{R})$ such that*

(A) If ψ is in $\mathcal{O}$ then $L+\psi$ has a unique minimizing measure, μ and this measure is uniquely ergodic.

(B)Moreover $supp(\mu) = \hat{\Sigma}(L+\psi) = \Sigma(L+\psi)$.

(C) When μ is supported on a periodic orbit or a fixed point, this orbit (point) Γ is hyperbolic and its stable and unstable manifolds if intersect they do it transversally.

On Mañé [28], it is conjectured that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.

The first statement of this was proved by Mañé in [27]. The second statement is proved in [11] and the third one in [7]. We will prove only the first part and give only the ideas of the proofs of the last part.

5-1 Generic Lagrangians.

Proof of (A)

Given a potential ψ on $C^\infty(M,\mathbb{R})$ define

$$\begin{aligned} m(\psi) &= \min_{\nu\in\overline{C}} \int L+\psi d\nu, \\ M(\psi) &= \{\nu\in\overline{C} : \int L+\psi d\nu = m(\psi)\} \end{aligned}$$

Where $\overline{C}$ is the set of holonomic measures. For $\epsilon > 0$ let

$$\mathcal{O}_\epsilon = \{\psi : \text{diam } M(\psi) < \epsilon\}$$

This set is open, in fact if ν_n is in $M(\psi_n)$ then

$$\int L+\psi d\nu_n \le m(\psi) + 2||\psi-\psi_n||_{C^0} \tag{5.1}$$

So by theorem 2-3.2 in chapter 2 if $\psi_n \to \psi$ then the sequence ν_n is precompact and the limit is in $M(\psi)$. From this follows that $\mathcal{O}_\epsilon$ is open.

It remains to prove that it is also dense.

Given a compact convex set K_0 on $\overline{C}$ and potential ψ on $C^\infty(M,\mathbb{R})$ define

$$\begin{aligned} m_0(\psi) &= \min_{\nu\in K_0} \int \psi d\nu, \\ M_0(\psi) &= \{\nu\in K_0 : \int \psi d\nu = m_0(\psi)\} \end{aligned}$$

5-1.1. Lemma. *Let K_0 as before then if μ is an extremal point of K_0, for all $\epsilon > 0$there exists ψ on $C^\infty(M,\mathbb{R})$ such that*

$$\begin{aligned} \text{diam } M_0(\psi) &< \epsilon \\ d(\mu, M_0(\psi)) &< \epsilon \end{aligned}$$

Proof

Denote by D the diagonal of $K_0 \times K_0$ for each pair (μ,ν) in $K_0 \times K_0 - D$ take a potential $\psi_{(\mu,\nu)}$ such that $\int \psi_{(\mu,\nu)} d\mu \neq \int \psi_{(\mu,\nu)} d\nu$, then there is a neigbourhood $U(\mu,\nu)$ contained on $K_0 \times K_0$ such that $\int \psi_{(\mu,\nu)} d\mu' \neq \int \psi_{(\mu,\nu)} d\nu'$ for every (μ',ν') in $U(\mu,\nu)$.

Take a covering $\{U(\mu_n,\nu_n)\}$ of $K_0 \times K_0 - D$ and set $\psi_n = \psi_{(\mu_n,\nu_n)}$ then if (μ,ν) in $K_0 \times K_0 - D$ there exist n such that

$$\int \psi_n d\mu \neq \int \psi_n d\nu \tag{5.2}$$

Define $T_n : \overline{C} \to \mathbb{R}^n$ as

$$T_n(\mu) = (\int \psi_1 d\mu, ..., \int \psi_n d\mu)$$

Using (5.2) and the compactness of $K_0 \times K_0$ it is easy to see that given ϵ there exist $\delta > 0$ and $n > 0$ such that

$$S \subset \mathbb{R}^n, \text{diam } S < \delta \Rightarrow \text{diam } T^{-1}S < \epsilon \tag{5.3}$$

Let $B = T_n(k_0)$ then B is a compact convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ a linear function such that its minimum restricted to B is attained in only one point p.

Define $\psi = \sum_i \lambda_i \psi_i$ where $f = \sum_i \lambda_i p_i$, then

$$f \circ T_n = \sum_i \lambda_i \int \psi_i$$

and so

$$M_0(\psi) = T_n^{-1}(p)$$

Then by (5.3) we get

$$\text{diam } M_0(\psi) < \epsilon$$

The following lemma proves the density of $\mathcal{O}_\epsilon$ and hence the first part of (A).

5-1.2. Lemma. *If ψ is on $C^\infty(M, \mathbb{R})$ and μ is an extremal point of $M(\psi)$ then for every neighbourhood U of ψ and every $\epsilon > 0$ there exists ψ_1 on U such that*

$$\text{diam } M(\psi_1) = \epsilon$$

Proof

For $K_0 = M(\psi)$ applying the previous lemma, we can find given ϵ a ψ_1 such that $\int \psi_1 d\nu$ attains its minimum, say m_1 for all measures ν on $K_0 = M(\psi)$ on a set $S = M_0(\psi_1)$ such that $d(\mu, S) < \epsilon$. Set

$$\begin{aligned} m_0 &= m(\psi) \\ f_0(\nu) &= \int L + \psi - m_0 d\nu \\ f_1(\nu) &= \int \psi_1 - m_1 d\nu \end{aligned}$$

Then

$$f_1(\nu) = f_0(\nu) \text{ if } \nu \in S \tag{5.4}$$

$$f_1(\nu) \geq 0 \text{ if } \nu \in M(\psi) \tag{5.5}$$

$$\nu \in M(\psi), f_1(\nu) = 0 \Rightarrow \nu \in S \tag{5.6}$$

$$\nu \in \overline{C}, f_0(\nu) = 0 \Rightarrow \nu \in M(\psi) \tag{5.7}$$

For $\lambda > 0$ define

$$f_\lambda = f_0 + \lambda f_1$$

and set

$$\begin{aligned} m(\lambda) &= \min_{\nu \in \overline{C}} f_\lambda(\nu), \\ M(\lambda) &= \{\nu \in \overline{C} : f_\lambda(\nu) = m(\lambda)\}. \end{aligned}$$

We claim that

$$\lim_{\lambda \to 0} \text{diam } (M(\lambda), \{\mu\}) \le \epsilon \tag{5.8}$$

This proves the lemma since

$$M(\lambda) = M(\psi + \lambda \psi_1)$$

Proof of the claim

Suppose otherwise that there exist $\lambda_n \to 0$ and $\mu_{\lambda_n}, \nu_{\lambda_n}$ on $M(\lambda_n) = M(\psi + \lambda_n \psi_1)$ such that $d(\mu_{\lambda_n}, \nu_{\lambda_n}) > \epsilon$. Then by (5.1) $\{\mu_{\lambda_n}\}$ and $\{\nu_{\lambda_n}\}$ are precompact and as in the proof of the open property we may assume that $\mu_{\lambda_n} \to \mu \in M(\psi)$ and $\nu_{\lambda_n} \to \nu \in M(\psi)$. Naturally $d(\mu, \nu) \ge \epsilon$.

Now Because of (5.4) we have that $m(\lambda) \le 0$

$$\begin{aligned} 0 &\ge m(\lambda_n) = \\ &= f_0(\mu_{\lambda_n}) + \lambda f_1(\mu_{\lambda_n}) \\ &\ge \lambda_n f_1(\mu_{\lambda_n}) \end{aligned}$$

So $f_1(\mu_{\lambda_n}) \leq 0$ and hence $f_1(\mu) \leq 0$, since $\mu \in M(\psi)$ by (5.5) $f_1(\mu) = 0$ and then by (5.6) μ is in S. Similarly ν is in S. This is a contradiction with the fact that the diameter of S is less than ϵ.

The fact that μ is uniquely ergodic follows from the fact that ergodic components of a minimizing measure are also minimizing. And the proof of (A) is complete.

It is worth to remark that the proof presented here is a particular case of Mañé's original [31] more general setting:

Let E, F be real convex spaces, K contained on F a metrizable convex subset and $\phi : E \to F'$, $L : F \to \mathbb{R}$ linear maps satisfying

- (a) The map $E \times F \to \mathbb{R}$ defined by $(w, x) \mapsto \phi(w)(x)$ is continuous.
- (b) For any $x \neq y$ in K there exists w in E such that $\phi(w)(x) \neq \phi(w)(y)$.
- (c) For all w in E and c in $\mathbb{R}$ the set

$$\{x \in K : L(x) + \phi(w)(x) \leq c\}$$

is compact.

Denote by

$$m(w) = \min_{x \in K} L(x) + \phi(w)(x)$$

which exists by (c). And

$$M(w) = \{x \in K : L(x) + \phi(w)(x) = m(w)$$

5-1.3. Proposition. *If E is a Frechet space then there exists a residual set $\mathcal{O}$ contained on E such that if w is on E then $M(w)$ has only one element.*

The reader can verify that with the following choices, we get the desired result.

- (1) Let E be the Banach space $C^\infty(M,\mathbb{R})$
- (2) As in section 2-3 let C^0_ℓ be the set of continuous functions f on TM such that $\sup \frac{f(x,v)}{1+|v|} \leq \infty$, and $\overline{C}$ the set of holonomic probabilities. Let K be $\overline{C}$ and F be the subspace of $(C^0_\ell)^*$ spanned by $\overline{C}$.
- (3) Finally let $L : F \to \mathbb{R}$ is the linear map such that if μ is in $\overline{C}$ then $L(\mu) = \int L d\mu$; and for ψ in $C^\infty(M,\mathbb{R})$ $\phi(\psi)$ is the restriction to F of the linear map on $(C^0_\ell)^*$ such that $w \mapsto < w, \psi >$.

This general setting has some other applications see theorems A, C D in [28] and also [10]

Proof of (C)

Let $\mathcal{O}$ be the residual given by (A). Let $\mathcal{A}$ be the subset of $\mathcal{O}$ of potentials ψ for which the measure on $\mathcal{M}(L+\psi)$ is supported on a periodic orbit. Let $\mathbb{B} := \mathcal{O} \setminus \mathcal{A}$ and let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ on which the minimizing periodic orbit is hyperbolic. We prove that $\mathcal{A}_1$ is relatively open on $\mathcal{A}$. For, let $\psi \in \mathcal{A}_1$ and

$$\mathcal{M}(L+\psi) = \{\mu_\gamma\}$$

where μ_γ is the invariant probability measure supported on the hyperbolic periodic orbit γ for the flow of $L+\psi$. We claim that if $\phi_k \in \mathcal{A}$, $\phi_k \to \psi$ and $\mathcal{M}(L+\phi_k) = \{\mu_{\eta_k}\}$, then $\eta_k \to \gamma$. Indeed, since L is superlinear, the velocities in the support of the minimizing measures $\mu_k := \mu_{\eta_k}$ are bounded (cf. corollary 2-4 and inequality 1.9), and hence there exists a subsequence $\mu_k \to \nu$ converging weakly* to a some invariant measure ν for $L+\psi$. Then if $\nu \neq \mu_\gamma$,

$$\lim_k S_{L+\phi_k}(\mu_k) = S_{L+\psi}(\nu) > S_{L+\psi}(\mu_\gamma)\,. \tag{5.9}$$

Thus if δ_k is the analytic continuation of the hyperbolic periodic orbit γ to the flow of $L+\phi_k$ in the original energy level $c(L+\psi)$, since $\lim_k S_{L+\psi_k}(\mu_{\delta_k}) = S_{L+\psi}(\mu_\gamma)$, for k large we have that,

$$S_{L+\phi_k}(\mu_{\delta_k}) < S_{L+\phi_k}(\mu_{\eta_k})\,,$$

which contradicts the choice of η_k. Therefore $\nu = \mu_\gamma$. For energy levels h near to $c(L+\psi)$ and potentials ϕ near to ψ, there exist hyperbolic periodic orbits $\gamma_{\phi,h}$ which are the continuation of γ. Now, on a small neighbourhood of a hyperbolic orbit there exists a unique invariant measure supported on it, and it is in fact supported in the periodic orbit. Thus, since $\eta_k \to \gamma$, then η_k is hyperbolic. Hence $\phi_k \in \mathcal{A}_1$ and $\mathcal{A}_1$ contains a neighbourhood of ϕ in $\mathcal{A}$.

Let $\mathcal{U}$ be an open subset of $C^\infty(M,\mathbb{R})$ such that $\mathcal{A}_1 = \mathcal{U} \cap \mathcal{A}$. We shall prove below that $\mathcal{A}_1$ is dense in $\mathcal{A}$. This implies that $\mathcal{A}_1 \cup \mathbb{B}$ is generic. For, let $\psi := \text{int}\,(C^\infty(M,\mathbb{R}) \setminus \mathcal{U})$, then $\mathcal{U} \cup \psi$ is open and dense in $C^\infty(M,\mathbb{R})$. Moreover, $\psi \cap \mathcal{A} = \varnothing$ because $\mathcal{A} \subseteq \overline{\mathcal{A}_1} \subseteq \overline{\mathcal{U}}$ and $\psi \cap \mathcal{A} \subseteq \mathcal{A} \setminus \overline{\mathcal{U}} = \varnothing$. Since $\mathcal{O} = \mathcal{A} \cup \mathbb{B}$ is generic and

$$\begin{aligned}(\mathcal{U} \cup \psi) \cap (\mathcal{A} \cup \mathbb{B}) &= (\mathcal{U} \cap \mathcal{A}) \cup ((\mathcal{U} \cup \psi) \cap \mathbb{B}) \\ &\subseteq \mathcal{A}_1 \cup \mathbb{B},\end{aligned}$$

then $\mathcal{A}_1 \cup \mathbb{B}$ is generic.

The perturbation to achieve hyperbolicity in a fixed point is easy. Is very much as the mechanic case: $L = \frac{1}{2} < v, v >_x - U(x)$. The reader can verify that if $x_0 = \max U$ then the Dirac measure supported on the point $(x_0, 0)$ is minimizing. And it is well known that this critical point is hyperbolic if and only if the maximum has

non degenerate quadratic form.

In fact, from the Euler-Lagrange equation (E-L) we get that $L_x(x_0, 0) = 0$. Differentiating the energy function (1.7) we see that $(x_0, 0)$ is a singularity of the energy level $c(L)$. Moreover, the minimizing property of μ implies that x_0 is a minimum of the function $x \mapsto L_{xx}(x, 0)$. In particular, $L_{xx}(x_0, 0)$ is positive semidefinite in linear coordinates in $T_{x_0}M$. And it is hyperbolic if and only if it is positive definite. So to achieve hyperbolicity we must just add a small quadratic form.

The perturbation needed in the case of a periodic orbit the same spirit; Because of the graph property the projection of the orbit Γ, $\pi(\Gamma)$

is a simple closed curve. We add a C^∞-small non negative potential ψ, which is zero if and only if x is on $\pi(\Gamma)$ that is nondegenerate in the transversal direction. It follows that Γ is also a minimizing solution of the perturbed lagrangian $L+\psi$.

To prove that it actually is hyperbolic is much more difficult. The reason is that the linearization of the flow (the Jacobi equation) is along the periodic orbit and hence non autonomous as in the case of a singularity.

To explain the idea of the proof we need some definitions. Let H be the associated hamiltonian by the Legendre transformation on T^*M and ψ its flow. Denote by $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $\psi(\theta) = \ker(d\pi)$. Two points θ_1, $\theta_2 \in T^*M$ are said *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(\psi(\theta_1)) \cap \psi(\theta_2) \neq \{0\}$.

A basic property of orbits without conjugate points is given by the following

5-1.4. Proposition. *Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two φ-invariant lagrangian subbundles $\mathbb{E}$, $\mathbb{F} \subset T(T^*M)$ along the orbit of θ given by*

$$\mathbb{E}(\theta) = \lim_{t\to+\infty} d\psi_{-t}\big(\psi(\psi_t(\theta))\big) \ ,$$
$$\mathbb{F}(\theta) = \lim_{t\to+\infty} d\psi_{t}\big(\psi(\psi_{-t}(\theta))\big) \ .$$

Moreover, $\mathbb{E}(\theta)\cup\mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta)\cap\psi(\theta) = \mathbb{F}(\theta)\cap\psi(\theta) = \{0\}$, $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ and $\dim\mathbb{E}(\theta) = \dim\mathbb{F}(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the hamiltonian vectorfield and $\Sigma = H^{-1}\{e\}$.

These bundles where constructed for disconjugate geodesics of riemannian metrics by Green [22] and of Finsler metrics by Foulon [19]. In the general case where constructed in [7]

We will only sketch the proof;

Fix a riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two lagrangian subspaces: the vertical subspace $\psi(\theta) = \ker(d\pi(\theta))$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M \times T^*_{\pi(\theta)}M$, $\xi \mapsto (d\pi(\theta)\,\xi\,,\, \nabla_\theta(\pi\,\xi))$, we can identify $H(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $\psi(\theta) \approx \{0\} \times T^*_{\pi(\theta)}M \approx T_{\pi(\theta)}M$. If we choose local coordinates along $t \mapsto \pi\,\psi_t(\theta)$ such that $t \mapsto \frac{\partial}{\partial q_i}(\pi\,\psi_t(\theta))$ are parallel vectorfields, then this identification becomes $\xi \leftrightarrow (dq(\xi),\, dp(\xi))$. Let $E \subset T_\theta T^*M$ be an n-dimensional subspace such that $E \cap \psi(\theta) = \{0\}$. Then E is a graph of some linear map $S : H(\theta) \to \psi(\theta)$. It can be checked that E is lagrangian if and only if in symplectic coordinates S is symmetric.

Take $\theta \in T^*M$ and $\xi = (h, v) \in T_\theta T^*M = H(\theta) \oplus \psi(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Consider a variation

$$\alpha_s(t) = \big(q_s(t), p_s(t)\big)$$

such that for each $s \in]-\varepsilon, \varepsilon[$, α_s is a solution of the hamiltonian H such that $\alpha_0(0) = \theta$ and $\frac{d}{ds}\alpha_s(0)|_{s=0} = \xi$.

Writing $d\psi_t(\xi) = \big(h(t), v(t)\big)$, we obtain the hamiltonian Jacobi equations

$$\begin{aligned} \dot h &= \quad H_{pq}\, h + H_{pp}\, v\,, \\ \dot v &= -H_{qq}\, h - H_{qp}\, v\,, \end{aligned} \tag{5.10}$$

where the covariant derivatives are evaluated along $\pi(\alpha_o(t))$, and H_{qq}, H_{qp}, H_{pp} and H_{qq} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)$, $\left(\frac{\partial^2 H}{\partial q_i\, \partial p_j}\right)$, $\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right)$ and $\left(\frac{\partial^2 H}{\partial p_i\, \partial q_j}\right)$. Moreover, since the hamiltonian H is convex, then H_{pp} is positive definite.

We derive now the Ricatti equation. Let E be a lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon, \varepsilon[$ we have that

$d\psi_t(E) \cap \psi(\psi_t(\theta)) = \{0\}$. Then we can write $d\psi_t(E) = \text{graph } S(t)$, where $S(t) : H(\psi_t\,\theta) \to \psi(\psi_t\,\theta)$ is a symmetric map. That is, if $\xi \in E$ then

$$d\psi_t(\xi) = \big(h(t), S(t)\,h(t)\big)\,.$$

Using equation (5.10) we have that

$$\dot{S}h + S(H_{pq}h + H_{pp}Sh) = -H_{qq}h - H_{qp}Sh\,.$$

Since this holds for all $h \in H(\psi_t(\theta))$ we obtain the Ricatti equation:

$$\dot{S} + SH_{pp}S + SH_{pq} + H_{qp}S + H_{qq} = 0\,. \tag{5.11}$$

Let $K_c(\theta) : H(\theta) \to \psi(\theta)$ be the symmetric linear map such that $\text{graph}(K_c(\theta)) = d\psi_{-c}(\psi(\psi_c(\theta)))$. Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \succ B$ if $A - B$ is positive definite.

The following proposition based essentially on the convexity of H proves 5-1.4.

5-1.5. Proposition. *For all* $\varepsilon > 0$,

(a) If $d > c > 0$ *then* $K_{-\varepsilon} \succ K_d \succ K_c$.

(b) If $d < c < 0$ *then* $K_\varepsilon \prec K_d \prec K_c$.

(c) $\lim_{d\to+\infty} K_d = \mathbb{S}$, $\quad \lim_{d\to-\infty} K_d = \mathbb{U}$.

(d) $\mathbb{S} \preccurlyeq \mathbb{U}$.

(e) The graph of $\mathbb{S}$ *is the stable green bundle* $\mathbb{E}$ *and the graph of* $\mathbb{U}$ *is the unstable green bundle* $\mathbb{F}$

An example of the relationship between the transversality of the Green subspaces and hyperbolicity appears in the following

5-1.6. Proposition. *Let Γ be a periodic orbit of ψ_t without conjugate points. Then Γ is hyperbolic (on its energy level) if and only if $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta) \rangle$ for some $\theta \in \Gamma$, where $\langle X(\theta) \rangle$ is the 1-dimensional subspace generated by the hamiltonian vectorfield $X(\theta)$. In this case $\mathbb{E}$ and $\mathbb{F}$ are its stable and unstable subspaces.*

This proposition follows ideas of Eberlein [13] and Freire, [20].

It is known that minimizing orbits do not have conjugate points. So by proposition 5-1.4 and 5-1.6 to prove the density of hyperbolicity it is enough to perturb to make the Green bundles transverse. This is done using two formulas for the index. One in the lagrangian setting and another one in the hamiltonian setting.

Let Ω_T be the set of continuous piecewise C^2 vectorfields ξ along $\gamma_{[0,T]}$. Define the *index form* on Ω_T by

$$I(\xi, \eta) = \int_0^T \left(\dot{\xi}\, L_{vv}\, \dot{\eta} + \dot{\xi}\, L_{vx}\, \eta + \xi\, L_{xv}\, \dot{\eta} + \xi\, L_{xx}\, \eta \right) dt\,, \tag{5.12}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s} \in \Omega_T$. For general results on this form see Duistermaat [12].

From this formula it is easy to compare the index of the original and the perturbed lagrangian along the same solution Γ.

Finally we use the following transformation of the index form. It is taken from Hartman [23] and originally due to Clebsch [5] see also [7]. Let $\theta \in T^*M$ and suppose that the orbit of θ, $\psi_t(\theta)$, $0 \le t \le T$ does not have conjugate points. Let $E \subset T_\theta T^*M$ be a lagrangian subspace such that $d\psi_t(E) \cap \psi(\psi_t(\theta)) = \{0\}$ for all $0 \le t \le T$.

Let $E(t) := d\psi_t(E)$ and let $H(t)$, $\psi(t)$ be a matrix solution of the hamiltonian Jacobi equation (5.10) such that $\det H(t) \neq 0$ and $E(t) = \mathrm{Image}\,(H(t), \psi(t)) \subset T_{\psi_t(\theta)}(T^*M)$ is a lagrangian subspace. For $\xi = H\zeta \in \Omega_T$, $\eta = H\rho \in \Omega_T$, we obtain (see [7])

$$I(\xi, \eta) = \int_0^T (H\zeta')^*\, (H_{pp})^{-1}\, (H\rho')\, dt + (H\zeta)^*(V\rho)\big|_0^T. \tag{5.13}$$

Define $\mathrm{N}(\theta) := \{\, w \in T_{\pi(\theta)}M \mid \langle w, \dot\gamma\rangle = 0 \,\}$. Then $\mathrm{N}(\theta)$ is the subspace of $T_{\pi(\theta)}M$ generated by the vectors $\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$. Let $v_o \in \mathrm{N}(\theta)$, $|v_o| = 1$ and let $\xi^T(t) := \widetilde{Z}_T(t)\, v_o$. Denote by $\tilde I_T$ and I_T be the index forms on $[0,T]$ for $\widetilde{L}$ and L respectively. Using the solution $(\widetilde{Z}_T, \widetilde{V}_T)$ on formula (5.13), we obtain that

$$\tilde I_T(\xi^T, \xi^T) = -(\widetilde{Z}_T(0)\, v_o)^* \,(\widetilde{V}_T(0)\, v_o) = -v_o{}^* \,\tilde K_T(0)\, v_o\,. \tag{5.14}$$

Moreover, in the coordinates $(x_1, \dots, x_n; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ on TU we have that

$$\begin{aligned}\tilde I_T(\xi^T, \xi^T) &= \int_0^T \left(\dot\xi^T \,\widetilde{L}_{vv}\, \dot\xi^T + 2\,\dot\xi^T \,\widetilde{L}_{xv}\, \xi^T + \xi^T \,\widetilde{L}_{xx}\, \xi^T\right) dt \\ &= \int_0^T \left(\dot\xi^T \,L_{vv}\, \dot\xi^T + 2\,\dot\xi^T \,L_{xv}\, \xi^T + \xi^T \,L_{xx}\, \xi^T\right) dt \qquad (5.15)\\ &\qquad\qquad + \int_0^T \varepsilon \sum_{i=2}^n \left|\xi_i^T\right|^2 dt\,.\end{aligned}$$

We have that $\widetilde{Z}_T(0) = I$ and for all $t > 0$, $\lim_{t\to\infty} \widetilde{Z}_T(t) = \widetilde{\mathbf{h}}(t)$ with $\widetilde{\mathbf{h}}(t)$ the solution of the Jacobi equation for $\tilde H$ corresponding to the stable Green bundle. Writing $\pi_N(\xi) = (\xi_2, \xi_3, \dots \xi_n)$ then $\left|\pi_N \widetilde{\mathbf{h}}(0)\, v_0\right| = |v_0| = 1$ because $v_0 \in \mathrm{N}(\theta)$. Hence there exists $\lambda > 0$ and $T_0 > 0$ such that $\left|\pi_N \xi^T(t)\right| = \left|\pi_N \widetilde{Z}_T(t)\, v_0\right| > \frac{1}{2}$ for all $0 \le t \le \lambda$ and $T > T_0$. Therefore

$$\tilde I_T(\xi^T, \xi^T) \ge I_T(\xi^T, \xi^T) + \frac{\varepsilon\lambda}{4}\,. \tag{5.16}$$

Let $(\boldsymbol{h}(t), \boldsymbol{v}(t)) = d\psi_t \circ \left(d\pi|_{\mathbb{E}(\theta)}\right)^{-1}$ be the solution of the Jacobi equation for H corresponding to the stable Green subspace $\mathbb{E}$ and let $\mathbb{S}(\psi_t(\theta)) = \boldsymbol{v}(t)\, \boldsymbol{h}(t)^{-1}$ be the corresponding solution of the Ricatti equation, with $\mathrm{graph}[\mathbb{S}(\psi_t(\theta)] = \mathbb{E}(\psi_t(\theta))$. Using formula (5.13), and writing

$\xi^T(t) = h(t)\,\zeta(t)$, we have that

$$I_T(\xi^T, \xi^T) = \int_0^T (h\,\dot\zeta)^*\, H_{pp}^{-1}\, (h\,\dot\zeta)\, dt + 0 - \big(h(0)\,\zeta(0)\big)^*\big(v(0)\,\zeta(0)\big)$$
$$I_T(\xi^T, \xi^T) \geq -v_o^*\,\mathbb{S}(\theta)\, v_o\,. \tag{5.17}$$

From (5.14), (5.16) and (5.17), we get that

$$v_o^*\,\mathbb{S}(\theta)\,v_o \geq v_o^*\,\tilde K_T\, v_o + \frac{\varepsilon\lambda}{4}\,.$$

From proposition 5-1.5, we have that $\lim_{T\to+\infty} \tilde K_T(0) = \widetilde{\mathbb{S}}(\theta)$, where $\operatorname{graph}(\widetilde{\mathbb{S}}(\theta)) = \widetilde{\mathbb{E}}(\theta)$, the stable Green bundle for $\tilde H$. Therefore

$$v_o^*\,\mathbb{S}(\theta)\,v_o \geq v_0^*\,\widetilde{\mathbb{S}}(\theta)\,v_o + \frac{\varepsilon\lambda}{4}\,. \tag{5.18}$$

Similarly, for the unstable Green bundles we obtain that

$$v_o^*\,\mathbb{U}(\theta)\,v_o + \lambda_2 \leq v_o^*\,\widetilde{\mathbb{U}}(\theta)\,v_o \qquad \text{for } v_o \in \mathrm{N}(\theta), |v_o| = 1. \tag{5.19}$$

for some $\lambda_2 > 0$ independent of v_o.

From proposition 5-1.5 we have that $\mathbb{U}(\theta) \succeq \mathbb{S}(\theta)$. From (5.18) and (5.19) we get that $\widetilde{\mathbb{U}}|_{\mathbf{N}} \succ \mathbb{U}|_{\mathbf{N}} \succeq \mathbb{S}|_{\mathbf{N}} \succ \widetilde{\mathbb{S}}|_{\mathbf{N}}$. Since $\widetilde{\mathbb{E}}(\theta) = \operatorname{graph}(\widetilde{\mathbb{S}}(\theta))$ and $\widetilde{\mathbb{F}}(\theta) = \operatorname{graph}(\widetilde{\mathbb{U}}(\theta))$, we get that $\widetilde{\mathbb{E}}(\theta) \cap \widetilde{\mathbb{F}}(\theta) \subseteq \langle \widetilde X(\theta)\rangle$. Then proposition B shows that Γ is a hyperbolic periodic orbit for $L + \phi$.

This proves that $\mathcal{A}_1$ is dense in $\mathcal{A}$.

Let $\mathcal{A}_2$ be the subset of $\mathcal{A}_1$ of potentials ψ for which the minimizing hyperbolic periodic orbit Γ has transversal intersections. The proof that $\mathcal{A}_2$ is open and dense in $\mathcal{A}_1$ is similar to the previous proof, see [7].

5-2 Homoclinic Orbits.

Assume in this section that $\widehat\Sigma$ contains only *one* static class. By theorem 5-0.1.(A), this is true for generic lagrangians. By proposition 3-11.4, the static classes are always connected, thus if we assume that there is only one static class, $\widehat{\Sigma}$ must be connected.

Given $\varepsilon > 0$, let U_ε be the ε-neighbourhood of $\pi(\widehat{\Sigma})$. Since $\widehat{\Sigma}$ is connected, the open set U_ε is connected for ε sufficiently small. Let $H_1(M, U_\varepsilon, \mathbb{R})$ denote the first relative singular homology group of the pair (M, U_ε) with real coefficients.

We shall say that an orbit of L is *homoclinic* to a closed invariant set $K \subset TM$ if its α and ω-limit sets are contained in K.

Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to the set of static orbits $\widehat{\Sigma}$ we can associate a homology class in $H_1(M, U_\varepsilon, \mathbb{R})$. Indeed, since there exists $t_0 > 0$ such that for all t with $|t| \geq t_0$, $x(t) \in U_\varepsilon$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M, U_\varepsilon, \mathbb{R})$. Let us denote by $\mathcal{H}$ the subset of $H_1(M, U_\varepsilon, \mathbb{R})$ given by all the classes corresponding to homoclinic orbits to $\widehat{\Sigma}$.

5-2.1. Theorem. *Suppose that $\widehat{\Sigma}$ contains only one static class. Then for any ε sufficiently small the set $\mathcal{H}$ generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to the set of static orbits $\widehat{\Sigma}$.*

Let U_ε be an ε-neighbourhood of $\operatorname{supp}(\mu)$. From theorems 5-2.1 and 5-0.1 we obtain:

5-2.2. Corollary. *Given a lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the lagrangian $L + \psi$ has a unique minimizing measure μ in $\mathcal{M}^0(L + \psi)$ and this measure is uniquely ergodic. For any ε sufficiently small the set $\mathcal{H}$ of homoclinic orbits to* $\operatorname{supp}(\mu)$ *generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to* $\operatorname{supp}(\mu)$.

To prove theorem 5-2.1 we consider finite coverings M_0 of M whose group of deck transformations is given by the quotient of the torsion free part of $H_1(M, U_\varepsilon, \mathbb{Z})$ by a finite index subgroup. Using that the lifted lagrangian L_0 has the same critical value as L, we conclude that the number of static classes of L_0 must be finite. Hence we can apply theorem 3-11.1 to L_0 to deduce that the group generated by the homoclinic orbits to the set of static orbits of L coincides with $H_1(M, U_\varepsilon, \mathbb{R})$.

We note that the homoclinic orbits that we obtain in theorem 5-2.1 and corollary 5-2.2 have energy c but they are not semistatic orbits of L. However, they are semistatic for lifts of L to suitable finite covers.

Combining corollary 5-2.2, theorem 5-0.1 and lemma 5-2.4, we obtain

5-2.3. Corollary. *Let M be a closed manifold with first Betti number ≥ 2. Given a lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the lagrangian $L+\psi$ has a unique minimizing measure in $\mathcal{M}^0(L+\psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and the stable and unstable manifolds have transverse homoclinic intersections.*

5-2.4. Lemma. *Let M be a closed manifold with first Betti number $b_1(M, \mathbb{R}) \geq 2$. Then if $A \subset M$ is a closed submanifold diffeomorphic to S^1 and U_ε denotes the ε neighborhood of A, we have that $H_1(M, U_\varepsilon, \mathbb{R})$ is non zero for all ε sufficiently small.*

Proof: Since A is diffeomorphic to a circle, the singular homology of the pair (M, U_ε) coincides with the singular homology of the pair (M, A) and therefore the vector space $H_1(M, U_\varepsilon, \mathbb{R})$ must have dimension $\geq b_1(M, \mathbb{R}) - 1 \geq 1$. □

For the proof of theorem 5-2.1 we shall need the following lemma:

5-2.5. Lemma. *Let $p : M_1 \to M_2$ be a covering such that $c(L_1) = c(L_2)$. Then any lift of a semistatic curve of L_2 is a semistatic curve of L_1. Also the projection of a static curve of L_1 is a static curve of L_2. If in addition, p is a finite covering, then any lift of a static curve of L_2 is a static curve of L_1.*

Proof: Observe first that for any $k \in \mathbb{R}$ we have that

$$\Phi_k^1(x,y) \geq \Phi_k^2(px,py),$$

for all x and y in M_1. Hence if we write $c = c(L_1) = c(L_2)$ we have

$$\Phi_c^1(x,y) \geq \Phi_c^2(px,py), \tag{5.20}$$

for all x and y in M_1.

Suppose now that $x_2 : \mathbb{R} \to M_2$ is a semistatic curve of L_2 and let $x_1 : \mathbb{R} \to M_1$ be any lift of x_2 to M_1. Using (5.20) and the fact that x_2 is semistatic we have for $s \leq t$,

$$\begin{aligned}\Phi_c^1(x_1(s), x_1(t)) &\leq A_{L_1+c}(x_1|_{[s,t]}) = A_{L_2+c}(x_2|_{[s,t]}) \\ &= \Phi_c^2(x_2(s), x_2(t)) \leq \Phi_c^1(x_1(s), x_1(t)).\end{aligned}$$

Hence x_1 is semistatic for L_1.

Suppose now that $x_1 : \mathbb{R} \to M_1$ is a static curve of L_1 and let $x_2 : \mathbb{R} \to M_2$ be $p \circ x_1$. Using (5.20) and the fact that x_1 is static we have for $s \leq t$,

$$\begin{aligned}-\Phi_c^1(x_1(t), x_1(s)) &= \Phi_c^1(x_1(s), x_1(t)) = A_{L_1+c}(x_1|_{[s,t]}) = A_{L_2+c}(x_2|_{[s,t]}) \\ &\geq \Phi_c^2(x_2(s), x_2(t)) \geq -\Phi_c^2(x_2(t), x_2(s)) \geq -\Phi_c^1(x_1(t), x_1(s)).\end{aligned}$$

Hence x_2 is static for L_2.

Suppose now that p is a *finite* covering and let $x_2 : \mathbb{R} \to M_2$ be a static curve of L_2. Let $x_1 : \mathbb{R} \to M$ be any lift of x_2 to M_1. Since x_2 is static, given $s \leq t$ and $\varepsilon > 0$, there exists a curve $\alpha : [0,T] \to M_2$ with $\alpha(0) = x_2(t)$, $\alpha(T) = x_2(s)$ such that

$$A_{L_2+c}(x_2|_{[s,t]}) + A_{L_2+c}(\alpha) \leq \varepsilon.$$

Since p is a finite covering, there exists a positive integer n, bounded from above by the number of sheets of the covering, such that the n-th iterate of $x_2|_{[s,t]} * \alpha$ lifts to M_1 as a closed curve. Hence, there exists a curve β joining $x_1(t)$ to $x_1(s)$ such that

$$A_{L_1+c}(x_1|_{[s,t]}) + A_{L_1+c}(\beta) \leq n\varepsilon,$$

and thus x_1 is static for L_1. $\square$

Proof of theorem 5-2.1:

Let $U \stackrel{\text{def}}{=} U_\varepsilon$ denote the ε-neighborhood of $\pi(\widehat{\Sigma}(L))$, where $\widehat{\Sigma}(L)$ is the set of static vectors of L. Since we are assuming that $\widehat{\Sigma}(L)$ contains only one static class, the set U is also connected for small ε. Let $i : U \to M$ be the inclusion map. The vector space $H_1(M, U, \mathbb{R})$ is isomorphic to the quotient of $H_1(M, \mathbb{R})$ by $i_*(H_1(U, \mathbb{R}))$.

Let H denote the torsion free part of $H_1(M, \mathbb{Z})$ and let K denote the torsion free part of $i_*(H_1(U, \mathbb{Z}))$. Let us write $G \stackrel{\text{def}}{=} H/K = \mathbb{Z} \oplus \overset{k}{\dots} \oplus \mathbb{Z}$ where $k = \dim H_1(M, U, \mathbb{R})$. Let J be a finite index subgroup of G. There is a surjective homomorphism $j : G \to G/J$ given by the projection.

If we take the Hurewicz map

$$\pi_1(M) \mapsto H_1(M, \mathbb{Z}),$$

and we compose it with the projections $H_1(M, \mathbb{Z}) \mapsto H$, $H \mapsto G$ and $j : G \to G/J$, we obtain a surjective homomorphism

$$\pi_1(M) \mapsto G/J,$$

whose kernel will be the fundamental group of a finite covering M_0 of M with covering projection map $p : M_0 \to M$ and group of deck transformations given by the finite abelian group G/J.

Since J is a subgroup of $G = H/K$, G/J acts transitively and freely on the set of connected components of $p^{-1}(U)$ which coincides with the set of connected components of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Therefore we have

5-2.6. Lemma. *There is a one to one correspondence between elements in G/J and connected components of $p^{-1}(\pi(\widehat{\Sigma}(L)))$.*

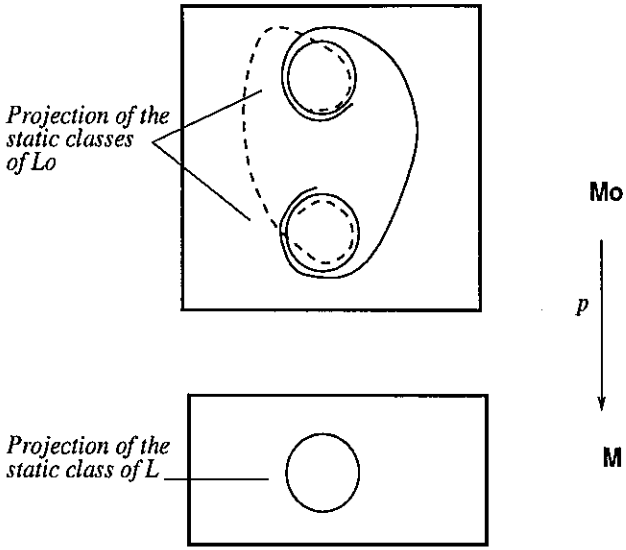

FIG. 1: Creating homoclinic connections with finite coverings.

Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to $\widehat{\Sigma}(L)$ we can associate a homology class in H/K. Indeed, since there exists $t_0 > 0$ such that for all t with $|t| \geq t_0$, $x(t) \in U$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M, U, \mathbb{Z})$. Let us denote by $\mathcal{H}$ the subset of H/K given by all the classes corresponding to homoclinic orbits to $\widehat{\Sigma}(L)$.

5-2.7. Lemma. *For any J as above, the image of $\langle \mathcal{H} \rangle$ under j is precisely G/J.*

Proof: Let L_0 denote the lift of the lagrangian L to M_0. Observe first that by proposition 2-6.2, $c(L) = c(L_0)$ and therefore by lemma 5-2.5 we have

$$\pi_0(\widehat{\Sigma}(L_0)) = p^{-1}(\pi(\widehat{\Sigma}(L))), \tag{5.21}$$

where $\pi_0 : TM_0 \to M_0$ is the canonical projection of the tangent bundle TM_0 to M_0.

Let us prove now that L_0 satisfies the hypothesis of theorem 3-11.1, that is, the number of static classes of L_0 is finite. In fact, we shall show that the projection to M_0 of a static class of L_0 coincides with a connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Using (5.21) and proposition 3-11.4 we see that the projection of a static class of L_0 to M_0 must be contained in a single connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$. Hence, it suffices to show that if x and y belong to a connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$ then $d_c^0(x,y) = 0$. Since we are assuming that $\widehat{\Sigma}(L)$ contains only one static class we have that $d_c(px, py) = 0$. Since $p : M_0 \to M$ is a finite covering there are lifts x_1 of px and y_1 of py such that $d_c^0(x_1, y_1) = 0$. Since static classes are connected x_1 and y_1 must belong to the same connected component of $p^{-1}(\pi(\widehat{\Sigma}(L)))$ and thus there is a covering transformation taking x_1 into x and y_1 into y which implies that $d_c^0(x,y) = 0$ as desired.

Now theorem 3-11.1 and (5.21) imply that every covering transformation in G/J can be written as the composition of covering transformations that arise from elements in $\mathcal{H}$, that is, $j(\langle\mathcal{H}\rangle) = G/J$. □

We shall need the following algebraic lemma.

5-2.8. Lemma. *Let $G = \mathbb{Z} \oplus \overset{k}{\cdots} \oplus \mathbb{Z}$. Given a finite index subgroup $J \subset G$ let us denote by $j : G \to G/J$ the projection homomorphism.*

Let A be a subgroup of G. If A has the property that for all J as above $j(A) = G/J$, then $A = G$.

Proof: The hypothesis readily implies that

$$A/A \cap J \text{ is isomorphic to } G/J \tag{5.22}$$

- If the rank of A is strictly less than the rank of G, one can easily construct a subgroup $J \subset G$ with finite index such that $A \subseteq J$ and $G/J \neq \{0\}$. But this contradicts (5.22) because $A/A \cap J = \{0\}$.

- If the rank of A equals the rank of G, then A has finite index in G and by (5.22) $G/A = \{0\}$ and thus $G = A$.

□

Observe now that any set $\mathcal{H}$ of a free abelian group G of rank k such that the group generated by $\mathcal{H}$ is G must have at least k elements. Therefore if we combine lemma 5-2.7 and lemma 5-2.8 with $\langle\mathcal{H}\rangle = A$ we deduce that the set $\mathcal{H}$ of classes corresponding to homoclinic orbits generates G and must have at least k elements thus concluding the proof of theorem 5-2.1. □

Appendix.

A Absolutely continuous functions.

A.1. Definition. A function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous*, if $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{N} |f(t_i) - f(s_i)| < \varepsilon,$$

whenever $]s_1, t_1[, \ldots,]s_N, t_N[$ are disjoint intervals in $[a, b]$.

A.2. Proposition.

The function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if

(i) The derivative $f'(t)$ exists for a.e. $t \in [a, b]$.

(ii) $f' \in \mathcal{L}^1([a, b])$.

(iii) $f(t) = f(a) + \int_a^t f'(s)\, ds$.

Proof: Define

$$\mu([s, t]) := f(t) - f(s).$$

We claim that μ defines a finite signed Borel measure on $[a, b]$. Indeed, let $\mathcal{A}$ be the algebra of finite unions of intervals. The function μ can be extended to a σ-additive function on $\mathcal{A}$. Moreover, if B is a Borel set and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a family with $A_n \downarrow B$, then $\mu(B) := \lim_n \mu(A_n)$ exists because $\mu(B_n \setminus B_m) \to 0$ when $n, m \to +\infty$.

Observe that the absolute continuity of f implies that $\mu \ll m$, where m is the Lebesgue measure. Let $g = \frac{d\mu}{dm}$ be the Radon-Nikodym derivative. Then $g \in \mathcal{L}^1$ and

$$f(t) - f(a) = \mu([a,t]) = \int_a^t g(s)\, ds.$$

By the Lebesgue differentiation theorem

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} g = g(t) \quad \text{for a.e. } t \in [a,b].$$

Conversely, suppose that (i)-(iii) hold. Using (ii), let $\mu(A) = \int_A f'\, dm$. Then by (iii),

$$\mu([s,t]) = f(t) - f(s) \quad \text{for } s,\, t \in [a,b].$$

Then $\mu \ll m$ implies[1] that f is absolutely continuous. □

The Lebesgue differentiation theorem gives the following characterization.

A.3. Corollary.

The function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if there exists $g \in \mathcal{L}^1([a,b])$ such that $f(t) = f(a) + \int_a^t g'(s)\, ds$.

[1] If μ is finite, then $\mu \ll m$ is equivalent, using the Borel-Cantelli lemma, to

$$\forall \varepsilon > 0 \;\; \exists \delta > 0, \;\; m(A) < \delta \Longrightarrow |\mu|(A) < \varepsilon.$$

B Convex functions.

A function $f : \mathbb{R}^n* \to \mathbb{R}$ is said *convex* if

$$f\big(\lambda\, x + (1-\lambda)\, y\big) \le \lambda\, f(x) + (1-\lambda)\, f(y)$$

for all $0 \le \lambda \le 1$ and $x, y \in \mathbb{R}^n$. Equivalently, if the set $\{\,(x,r) \in \mathbb{R}^n \times \mathbb{R} \,|\, r \ge f(x)\,\}$ is convex.

For $x_0 \in \mathbb{R}^n$ the elements of the set

$$\partial f(x_0) := \{\, p \in \mathbb{R}^{n^*} \,|\, f(x) \ge p\,(x - x_0) + f(x_0)\,\}.$$

are called *subderivatives* of f at x_0, and the planes

$$\{\,(x,r) \in \mathbb{R}^n \times \mathbb{R} \,|\, r = p\,(x - x_0) + f(x_0)\,\}$$

are called *supporting hyperplanes* for f at x_0. The functional $p \in \mathbb{R}^{n^*}$ is called the *slope* of the hyperplane.

For the proof of the following proposition see Rockafellar [54].

B.1. Proposition.

(a) *$\partial f(x) \neq \varnothing$ for every $x \in \mathrm{Dom}(f)$.*

(b) *A finite convex function is continuous and Lebesgue almost everywhere differentiable.*

(c) *If $\partial f(x) = \{p\}$ then f is differentiable at x and $f'(x) = p$.*

C The Frenshel and Legendre Transforms.

Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$ the *Frenshel Transform* (or the *convex dual* of f is the function $f : (\mathbb{R}^n)^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(p) = \max_{x \in \mathbb{R}^n} [\, p\, x - f(x)\,] \tag{C.1}$$

The function f admits a supporting hyperplane with slope $p \in \mathbb{R}^{n^*}$ if and only if $f^*(p) \neq +\infty$. If f is superlinear, then f^* is finite on all $\mathbb{R}^n$.

C.1. Proposition.

1. *If f is convex then f^* is convex.*
2. *If f and f^* are superlinear then $f^{**} = f$.*
3. *f is superlinear if and only if f^* is bounded on balls, more explicitly,*

$$f(x) \geq A\, |x| - B(A)\,, \quad \forall x \in \mathbb{R}^n \iff f^*(p) \leq B(|p|)\,, \quad \forall p \in \mathbb{R}^{n^*}$$

4. *If f is superlinear, the maximum C.1 is attained at some point $x \in \mathbb{R}^n$.*

Proof:

1. Given $0 \leq \lambda \leq 1$ and $p_1, p_2 \in \mathbb{R}^{n^*}$ we have that

$$\begin{aligned} f^*\big(\lambda\, p_1 + (1-\lambda)\, p_2\big) &= \max_{x \in \mathbb{R}^n} \big[\big(\lambda\, p_1 + (1-\lambda)\, p_2\big)\, x - f(x)\big] \\ &\leq \lambda \max_{x \in \mathbb{R}^n} [\, p_1\, x - f(x)\,] + (1-\lambda) \max_{x \in \mathbb{R}^n} [\, p_2\, x - f(x)\,] \\ &= \lambda\, f^*(p_1) + (1-\lambda)\, f^*(p_2). \end{aligned}$$

2. From (C.1) we get that

$$f(x) \geq p\, x - f^*(p) \qquad \text{for all } x \in \mathbb{R}^n,\ p \in (\mathbb{R}^n)^*$$

Hence,

$$f(x) \geq \sup_{p \in \mathbb{R}^{n^*}} [\, p\, x - f^*(p)\,] = f^{**}(x).$$

Let $p_x \in \partial f(x) \neq \varnothing$. Then

$$f(y) \geq f(x) + p_x\,(y - x), \qquad \forall\, y \in \mathbb{R}^n.$$

Hence

$$f^*(p_x) = \max_{y \in \mathbb{R}^n} \big[\, p_x\, y - f(y)\,\big] = p_x\, x - f(x).$$

And

$$f(x) = p_x\, x - f^*(p_x) \leq \max_{p \in \mathbb{R}^{n^*}} \big[\, p\, x - f(x)\,\big] = f^{**}(x).$$

3. We have that

$$\begin{aligned} f_x^*(p) &= \max_{v \in \mathbb{R}^n} [\, p\, v - f_x(v)\,] \\ &\leq \max_{v \in \mathbb{R}^n} [\, p\, v - |p|\, v\,] + B(|p|) \\ &= B(|p|). \end{aligned}$$

Conversely, suppose that $f_x^*(p) \leq B(|p|)$. Given $A \in \mathbb{R}$ and $x \in \mathbb{R}^n$ there exists $p_x \in \mathbb{R}^{n^*}$ such that $|p_x| = A$ and $p_x\, v = |p_x|\,|x| = A\,|x|$. Then

$$\begin{aligned} f(x) &= \max_{p \in \mathbb{R}^{n^*}} [\, p\, x - f^*(p)\,] \\ &\geq p_x\, v - B(|p_x|) = A\,|x| - B(A). \end{aligned}$$

4. By item *3*, f^* is finite. Let $p \in \mathbb{R}^{n^*}$. If $b > 0$ is such that $f(x) > (|p| + 1)\,|x| - b$, then

$$p\, x - f(x) < b - |x| < f^*(p) - 1 \quad \text{ for } |x| > b + 1 - f^*(p).$$

Hence

$$f^*(p) = \max_{|x| \leq b+1-f^*(p)} [\, p\, x - f(x)\,],$$

and the maximum is attained at some interior point x_p in the closed ball $|x| \le b + 1 - f^*(p)$.

$\square$

C.2. Corollary.
If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear then so is $f^* : \mathbb{R}^{n^*} \to \mathbb{R}$. In this case $f^{**} = f$.

Observe that in this case we have

$$f^*(0) = -\min_{x \in \mathbb{R}^n} f(x) \qquad \text{and} \qquad f(0) = -\min_{p \in \mathbb{R}^{n^*}} f^*(p).$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear we define the *Legendre Transform* $\mathcal{L} : \mathbb{R}^n \to 2^{\mathbb{R}^{n^*}}$ of f, by

$$\mathcal{L}(x) = \{\, p \in \mathbb{R}^{n^*} \mid p\, x = f(x) + f^*(p) \,\}, \tag{C.2}$$

C.3. Proposition. *If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and there is $a > 0$, such that*

$$y \cdot f''(x) \cdot y \ge a\,|y|^2 \quad \textit{for all } x,\, y \in \mathbb{R}^n,$$

then the Legendre transform $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^{n^*}$ is a C^1 diffeomorphism given by $\mathcal{L}(x) = d_x f$.

Proof: The function f is convex and it is superlinear because

$$\begin{aligned} f(x) &= f(0) + \int_0^1 f'(sx)\; ds \\ &= f(0) + \int_0^1 \int_0^1 sx\, f''(ts\, x)\, x\; dt\; ds \\ &\ge f(0) + \tfrac{1}{2}\, a\, |x|^2. \end{aligned}$$

From (C.1) we get that

$$p\,x \leq f(x) + f^*(p) \qquad \text{for all } x \in \mathbb{R}^n,\ p \in \mathbb{R}^{n^*}. \tag{C.3}$$

By proposition C.2, f^* is superlinear. Then item *4* in proposition C.2 implies that

$$\mathcal{L}(x) = \arg \max_{p\in\mathbb{R}^{n^*}} \{\, p\,x - f^*(p)\,\} \neq \varnothing.$$

Moreover, from (C.3), if $p \in \mathcal{L}(x)$ then $x = \arg\max_{x\in\mathbb{R}^n}\{\, p\,x - f(x)\,\}$. Thus $p = d_x f = \mathcal{L}(x)$. This proves that $\mathcal{L}$ differentiable and singled valued. Moreover since $d_x\mathcal{L} = f''(x)$ is non-singular, then $\mathcal{L}$ is a local C^1 diffeomorphism.

Since

$$(y-x)\cdot[\,d_y f - d_x f\,] = \int_0^1 f''\big(sx + (1-s)y\big)\, ds > 0,$$

then $x \mapsto d_x f = \mathcal{L}(x)$ is injective. We now prove that $\mathcal{L}$ is surjective. By item *4* in proposition C.1 the maximum

$$f^*(p) = \max_{x\in\mathbb{R}^n}[\, p\,x - f(x)\,]$$

is attained at some $x_p \in \mathbb{R}^n$. Then $p \in \mathcal{L}(x_p)$. □

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