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TOPICS ON WAVE PROPAGATION AND HUYGENS' PRINCIPLE

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Dedicated to my mother,

Lucy

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This work would not have been possible without the help of many people.

First without the inspiring initial guidance of F.A. Grünbaum, who introduced me to many of the subjects discussed herein. Then, in chronological order, to enthusiastic conversations with F. Magri whose collaboration has been instrumental in my view of the subject. M. Pedroni has helped me by sending a copy of Beltrami's article [14] from Milano, and Henrique Bursztyn by bringing me a copy of reference [34] from Berkeley.

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Like as the waves make towards the pebbled shore,
So do our minutes hasten to their end;
Each changing place with that which goes before,
In sequent toil all forwards do contend.
Nativity, once in the main of light,
Crawls to maturity, wherewith being crown'd,
Crooked elipses 'gainst his glory fight,
And Time that gave doth now his gift confound.
Time doth transfix the flourish set on youth
And delves the parallels in beauty's brow,
Feeds on the rarities of nature's truth,
And nothing stands but for his scythe to mow:
And yet to times in hope my verse shall stand,
Praising thy worth, despite his cruel hand.

W. Shakespeare. Sonnet LX

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1. Introduction and Overview

This work is concerned with three apparently unrelated topics. They are:

- Huygens' Property.
- Rational Solutions of the Korteweg-de Vries equation.
- The bispectral problem of Duistermaat and Grünbaum.

Huygens' property is a particularly fascinating property of the solutions to the wave equation in three spatial dimensions, i.e., in the four dimensional space-time. It is in fact a property crucial to the meaningful transmission of information such as music and light. In this text, we are going to be concerned with the Huygens' property in Hadamard's sense. The physical interpretation being that an "instantaneous" signal in three spatial dimensions remains "instantaneous" for every observer at a later time. We quote from Courant and Hilbert,

> "Thus our actual physical world, in which acoustic and electromagnetic signals are the basis of communication, seems to be singled out among other mathematically conceivable models by intrinsic simplicity and harmony" (Courant and Hilbert, Methods of Mathematical Physics, Vol. II, page 765)

The second topic, which is the Korteweg-de Vries (KdV) equation, belongs to the realm of nonlinear wave propagation. The KdV equation appears as a sort of universal model for the (unidirectional) propagation of waves in the presence of weak nonlinearity and dispersion. See [79]. The rational solutions of KdV are limiting cases of the now celebrated soliton solutions [3, 4, 2]. The poles (in the space variable) of the rational solutions of KdV evolve according to very special dynamical systems. More precisely, their evolution under KdV corresponds to the dynamics of conserved quantities of classical completely integrable Calogero-Moser systems.

The third topic, which we call the bispectral problem, was motivated by questions in signal processing and computerized tomography. It turns out to be natural also in the context of orthogonal polynomials. Let's consider a smooth family of eigenfunctions $\Psi(x, z)$ depending on a spectral parameter z. The *bispectral problem* consists in characterizing the differential operators $L(x, \partial_x)$

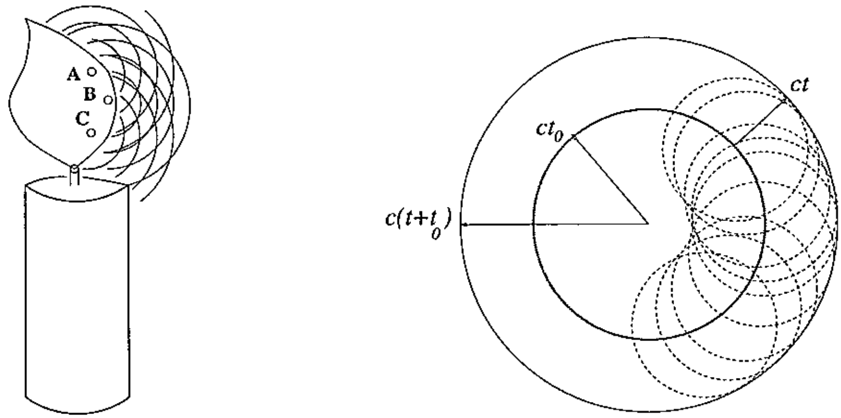

Fig. 1.1. Huygens' Construction.

whose eigenfunctions $\Psi(x, z)$ are also eigenfunctions of an operator in the spectral variable z. The most trivial (and not so interesting) example being[1]

$$\Psi(x, z) = \exp(izx) \ ,$$

which is a joint eigenfunction of $-\partial_x^2$ and $-\partial_z^2$. The bispectral problem has very interesting connections with completely integrable systems, as was shown in the seminal paper of Duistermaat and Grünbaum [33].

1.1 Hadamard's Approach to Huygens' Construction

In this text, as in many PDEs texts, we are assigning a rather special meaning to the "Huygens' property." The classical construction of Huygens' [54], which played a crucial role in the understanding of wave phenomena, consists in determining the wave front at a time $t + t_0$ by drawing the envelope to the circles of radii ct issued from the wave front at time t_0. See Figure 1.1.

The mathematical ideas behind Huygens' construction were subject of investigation by several key figures in the development of science. A few names to cite are Fresnel, Kirchhoff, Poisson, Beltrami and Hadamard.

Hadamard studied the classical construction of Huygens and formulated it in terms of the property of dependence on the initial data only on the intersection of the initial manifold and the light cone. It is now called *Huygens' property* or more precisely *strict Huygens' property.* He also stated the problem of determining the hyperbolic operators that satisfy Huygens' property.

[1] The crucial point in this example is that one is taking functions that are products of x and z, which makes the bispectrality trivial in this case.

Due to its historical reasons we now describe Hadamard's view of Huygens' construction. It was expressed in his classical text "Lectures on the Cauchy Problem" and consisted in dividing the construction in the form of a syllogism. Quoting from [51]:

(A) "Major Premise"

> The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediary instant $t = t'$, i.e., (assuming $0 < t' < t_0$), ...

(B) "Minor Premise"

> If we produce a luminous disturbance localized in a neighborhood of 0, its effect after an elapsed time t_0 will be localized in a neighborhood of the sphere centered at 0 with radius ct_0.

(C) "Conclusion"

> In order to calculate the effect of our initial luminous phenomenon produced at 0 at $t = 0$, we may replace it by a proper system of disturbances taking place at $t = t'$ and distributed over the surface of the sphere with center 0 and radius ct'.

In the above syllogism, statement **(A)**, as Hadamard puts it, is an empirical fact connected to the way we view the world. In modern terms it could be interpreted as a property of the propagator. It is certainly of interest, but it is not the subject of this text.

On the other hand, statement **(B)** is a property deeply related to the solutions of the wave equation in an *odd* number of spatial dimensions. As we shall see in Chapter 2 the solutions of the wave equation in an even number of spatial dimensions have the property that their domain of dependence at a point (t, x) includes the inside region of the characteristic cone issued from (t, x).

Another way of stating such a distinction is by considering the Cauchy problem:

$$\Box G(t,x) \stackrel{\text{def}}{=} (\partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2) G(t,x) = 0 \tag{1.1}$$

$$G(0,x) = 0 \tag{1.2}$$

$$\partial_t G(0,x) = \delta(x) \; . \tag{1.3}$$

For $n = 3$ the solution is given by

$$G(t,x) = \frac{1}{2\pi} \delta(t^2 - |x|^2) \; .$$

On the other hand, for $n = 2$ the solution is given by

$$G(t,x) = \frac{1}{2\pi}\frac{H(t^2-|x|^2)}{\sqrt{t^2-|x|^2}} , \tag{1.4}$$

where H is Heaviside's function defined by

$$H(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

From the physical point of view, Huygens' property is crucial to the transmission of information such as music. In two dimensions, for example, if one considers an observer stationed at a point x_0 listening to a musical note emitted at 0 and that ended at a certain time, then such observer would "hear" this note no matter how much time has elapsed from its end.

Along the lines of Hadamard's question, another very important one is what potentials could be added to the wave operator of equation (1.1) and still preserve Huygens' property. This problem was studied by a number of people. In the sixties, Lagnese and Stellmacher [68, 66, 67] gave a characterization of such potentials.

Chapter 2 of this text starts with some basic background on hyperbolic PDEs and discusses the crucial criterion of Hadamard to characterize Huygens operators. It also briefly reviews the method of Riesz kernels, which plays a role in the modern approach to Huygens property.

1.2 The Darboux Method and Rational Solutions of the KdV

The ingenious construction developed in the work of Lagnese and Stellmacher was a fascinating rediscovery of a method employed by Darboux in connection with the study of surfaces. We quote from the Darboux's monumental treatise:

> Étant donée l'équation différentielle du second ordre
>
> $$\frac{d^2y}{dt^2} = [\varphi(t) + h]y ,$$
>
> supposons qu'on sache l'intégrer pour toutes les valeurs de h. Soit $f(t)$ une solution de cette équation, correspondante à une valeur particulière de h, par example $h = h_1$. On saura aussi intégrer, pour toutes les valeurs de h, l'équation
>
> $$\frac{d^2y}{dt^2} = [f\left(\frac{1}{f}\right)'' + h - h_1]y .$$
>
> (G. Darboux, "Leçons sur la Théorie Générale de Surfaces et les Applications Géométriques du Calcul Infinitésimal, Deuxième Partie", page 210)

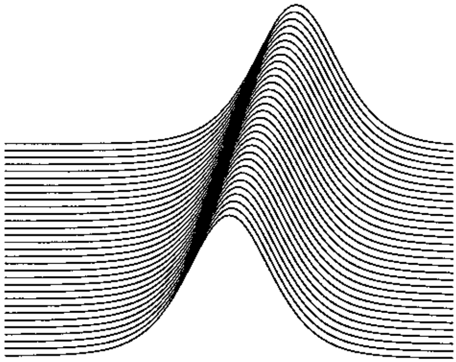

Fig. 1.2. One-soliton solution of KdV.

The above mentioned results of Lagnese turned out to be connected to other problems. One such problem, being the rational solutions of the KdV equation

$$u_t = 6uu_x - u_{xxx} \,. \tag{1.5}$$

The KdV equation models unidirectional nonlinear wave propagation under certain simplifying assumptions. It exhibits the phenomenon of solitons, which in very broad terms, are (nonlinear) traveling waves that preserve their shapes after interaction. Much as if they were like particles. See Figure 1.2 where the one soliton solution of KdV is displayed and the interaction of two solitons in Figure 1.3.

It turns out that the soliton solutions have as limiting case certain classes of rational functions that remain rational by the flow of KdV. These are the now called rational solutions of KdV hierarchy and will play a major role in several parts of these notes.

The rational solutions of KdV were characterized by Adler & Moser [3] in terms of a sequence of polynomials obtained precisely by iterations of the Darboux method. It turns out that the Adler-Moser polynomials were also present in the ealier work of Buchnall and Chaundy [23]. The resulting potentials thus coincide with the Huygens potentials of Lagnese.

Chapter 3 deals with completely integrable systems, the KdV hierarchy, and the Darboux method. It concludes with some remarks one the Painlevé analysis of PDEs, which we believe will play an important role in the understanding of many of the connections between all the subjects mentioned here.

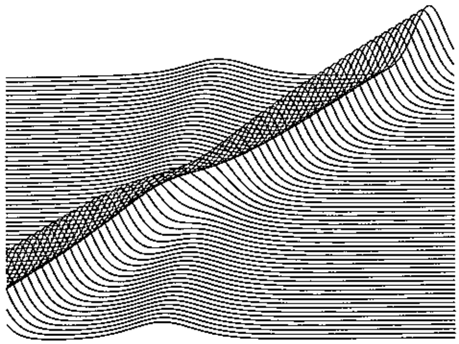

Fig. 1.3. The nonlinear interaction of two solitons.

1.3 Bispectrality

But the miracles do not stop with the relation between Darboux transformations, Huygens potentials and rational solutions of KdV. The rational solutions of KdV are also bispectral potentials! In other words, taking $u(x)$ one of the potentials that remain rational by the KdV flow one obtains a bispectral operator L of the form

$$L = -\partial_x^2 + u(x) \ .$$

In the last few years, many more connections between the three topics mentioned above have been found. It turned out that both the bispectral problem and Huygens' property for a certain class of wave operators are connected very directly with the algebra of Virasoro. This interesting infinite dimensional Lie algebra has been playing an increasingly fundamental role in modern physics (particularly conformal field theory). It can be characterized very simply by means of central extensions of the algebra of vector fields of the circle. The connection between the Virasoro algebra and integrable systems has been known for a while [83]. As for the connection between bispectrality the Virasoro algebra, this was noticed in [110]. Finally, the connection between the Huygens' property and Virasoro algebra appeared in the work of Berest [15]

Chapter 4 starts with a little bit of motivation for the bispectral problem, then focus on the results of [33] for Schrödinger operators, and finally discusses more recent results which connect bispectrality with Virasoro algebras.

We conclude the text in Chapter 5 where we try to link the different topics. We pay special attention to the recent results of Berest, and discuss the topic of iso-Huygens deformations. In order to provide some of the basic tools for the analysis of such deformations we provide a very short appendix on Lie's approach to symmetries of differential equations.

2. Hyperbolic PDEs and Huygens' Principle

2.1 Introduction

In this section we describe Huygens' property from Hadamard's point of view. We shall do it first for the wave operator with constant coefficients for simplicity of exposition.

First a little bit of notation. Throughout this text, when dealing with wave operators, we shall use the convention that we are working in $(n+1)$ dimensions, where n indicates the number of space dimensions and 1 indicates the time dimension. A point of the space-time will be of the form $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ where $x_0 = t$ is the time variable. Further, the wave operator in $(n+1)$-dimensional Lorentz space-time will be written as

$$\Box\psi \stackrel{\text{def}}{=} (\partial_t^2 - c^2\Delta)\psi ,$$

where

$$\Delta\psi = \sum_{i=1}^{n} \partial_{x_i}^2 \psi .$$

The Cauchy problem for the wave equation is

$$\Box\psi = 0 \tag{2.1}$$

$$\psi\Big|_{t=0} = f \tag{2.2}$$

$$\partial_t\psi\Big|_{t=0} = g . \tag{2.3}$$

Let's focus first on our familiar case of $n = 3$. One straightforward way of obtaining the solution to the wave equation in this case is by the classical method of spherical means [62, 27, 28]. More precisely, define the average (see Figure 2.1)

$$\mathcal{M}_r[u] \stackrel{\text{def}}{=} \frac{1}{4\pi r^2}\int_{|x-x'|=r} u(x')\; dS(x') , \tag{2.4}$$

where dS is the surface measure on the sphere of radius r in $\mathbb{R}^n$.

Exercise 2.1 *Check that the solution to the Cauchy problem is given by*

$$\psi = \frac{d}{dt}(t\mathcal{M}_{ct}[f]) + t\mathcal{M}_{ct}[g] \ .$$

As a consequence of the above formula, the solution to the Cauchy problem at the point $(\bar{t}, \bar{x}) \in \mathbb{R}^4$ depends only on the initial data at the points on the sphere in $\mathbb{R}^3$ of radius $c\bar{t}$ centered at $\bar{x}$, as shown in Figure 2.2. In particular, this means that data with support in a ball of radius ϵ of the origin will be "felt" only on a neighborhood of radius ϵ of the light cone with vertex at the origin, i.e., $c^2t^2 = x_1^2 + x_2^2 + x_3^2$. This property is instrumental for the transmission of information in three spatial dimensions. The same property does not hold in two dimensions. More precisely, if one looks at a disturbance caused by a source supported at neighborhood of zero, the effect of such a disturbance, in principle, could be felt at the point (t, x_1, x_2) for all time greater than $(\epsilon + \sqrt{x_1^2 + x_2^2})/c$. Indeed, to see this experimentally, all one needs to do is to throw pebbles on a lake. The proof of this claim is obtained by means of the so called Hadamard's Method of Descent. See Figure 2.2

The Method of Descent consists in using the solution of the given equation in a higher dimensional space to produce the solution in a lower dimensional space by introducing extra "dummy variables" in the equation as well as in the data. In the case at hand, to solve the wave equation in two dimensions by means of the solution of equation (2.1) for $n = 3$, all one has to do is the following: Take the data in $\mathbb{R}^2$ and extend it to $\mathbb{R}^3$ assuming that it is independent of x_3. Due to the invariance of the wave operator by translations in any of the variables, the solution with data independent of x_3 will also be independent of x_3. Hence, one obtains a solution of the wave equation in $\mathbb{R}^2$.

The Method of Descent is a very elementary application of the use of symmetry principles that are extremely important in the study of partial differential equations.

We recall that the light cone passing through a point $\xi = (t, x)$ is defined by the equation

$$C_\xi = \left\{(t', x') | c^2(t' - t)^2 = (x' - x)^2\right\} \ . \tag{2.5}$$

By the interior part of the light cone we mean the set

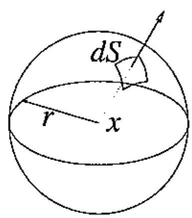

Fig. 2.1. Average of a function over the sphere.

$$D_\xi = \left\{(t', x') | c^2(t'-t)^2 > (x'-x)^2\right\} .$$

The general situation for the Cauchy problem (2.1), (2.2),(2.3) for the wave equation in n spatial dimensions is following:

Theorem 2.2 *For n odd and greater than 1 the solution of the wave equation depends on the initial data on the intersection of the light cone with the initial data manifold $t = 0$.*

For n even the solution depends on the values of the data on the closure intersection of the interior part of the light cone and the initial data manifold $t = 0$.

One proof of the above result could be readily obtained by using the method of spherical means indicated above for n odd, and descending to even n. We leave the details for the enthusiastic reader [62].

2.2 Second Order Hyperbolic Equations

In what follows we shall be concerned with the second order hyperbolic operators. By that we mean operators of the form

$$\mathcal{L}\psi \stackrel{\text{def}}{=} \sum g^{ij}\psi_{x_i x_j} + \sum b^i \psi_{x_i} + u\psi , \tag{2.6}$$

where the matrix g^{ij} defines a quadratic form of signature $(1, n)$, depending smoothly on the point $x \in \Omega \subseteq \mathbb{R}^{n+1}$. [1] From the geometric point of view one can think of $\mathcal{L}$ as a type of Laplace-Beltrami operator on a pseudo-Riemannian manifold.

The operator $\mathcal{L}$ reduces to the standard wave operator, with $c = 1$, when we take $g = \text{diag}[1, -1, -1, -1]$ and $b = u = 0$.

[1] The notion of hyperbolic operators is more general. We shall not delve into it in the present monograph, and refer the interested reader to the comprehensive treatise of Hörmander's [53].

Fig. 2.2. Comparison of the domain of dependence of the solutions of the wave equation in two (left) and three (right) spatial dimensions.

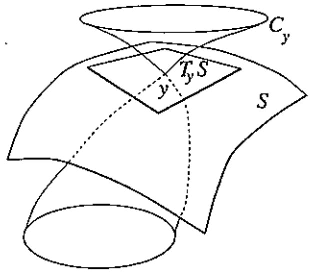

Fig. 2.3. Pictorial description of a space-like submanifold on a pseudo-Riemannian manifold

The principal symbol of the operator $\mathcal{L}$ comes from a pseudo-Riemannian metric g_{ij}, where as usual we use the notation $\sum g^{il}g_{lj} = \delta^i_j$.

Take $X, Y \in \mathbb{R}^{n+1}$. As usual, let's denote by

$$\langle X, Y \rangle_g \stackrel{\text{def}}{=} \sum_{ij} g_{ij} X^i Y^j \ .$$

The metric $\langle , \rangle_g$ induces a geodesic flow on the set Ω. Under suitable hypothesis on a neighborhood of every point ξ one has the notion of a geodesic distance. (although in this case strictly speaking we don't have a distance in the sense of metric space) The square of geodesic distance between two sufficiently close points x and ξ in Ω is given by $\Upsilon(x, \xi)$, where Υ satisfies the partial differential equation

$$\sum_{ij} g^{ij} \Upsilon_{x_i} \Upsilon_{x_j} = 4\Upsilon \tag{2.7}$$

with the further conditions

$$\Upsilon(\xi, \xi) = 0 \ , \ \partial_{x_i} \Upsilon(\xi, \xi) = 0 \ , \ \partial_{x_i} \partial_{x_j} \Upsilon(\xi, \xi) = 2 g_{ij}(\xi) \tag{2.8}$$

For fixed y the set defined by

$$\Upsilon(x, y) = 0 \tag{2.9}$$

defines the characteristic conoid C_y emanating from y. Obviously C_y is a smooth hyper-surface with the exception of the point y We recall that a manifold $S \subseteq \Omega$ of dimension n is called space-like if all its tangent vectors are space-like, i.e., the pseudo-metric $\langle , \rangle_g$ is actually a metric on the tangent space to S. See figure 2.3

As an example, for the wave operator, the associated metric corresponds to the Lorentz metric and the characteristic conoid is the the light cone C_ξ of equation (2.5).

The Cauchy problem for $\mathcal{L}$ is posed by specifying data on a space-like hypersurface S

$$\begin{cases} \mathcal{L}\psi = 0 \\ \psi|_S = f \\ \partial_\nu \psi|_S = g \end{cases} \tag{2.10}$$

where ν denotes the unit normal to the hypersurface S.

In Section 2.3 we are going to construct fundamental solutions to the wave operator following Hadamard's classical construction.

We are now ready to state for curved space-times the notion of and operator to satisfy Huygens' property.

Definition 2.3 *A hyperbolic operator $\mathcal{L}$ as above, is said to satisfy Huygens' principle (in the sense of Hadamard's minor premise, or in the strict sense) if the solution to every (well posed) Cauchy problem (2.10) depends on the initial data f, g only in the intersection of the characteristic conoid and the space-like manifold S.*

We notice a few trivial symmetries for the strict Huygens property.

1. Nonsingular coordinate transformations of the independent variables:

$$x \mapsto \tilde{x} = \tilde{x}(x)$$
$$\det(D\tilde{x}) \neq 0$$

2. Gauge transformations:

$$\psi \mapsto \tilde{\psi} = \lambda^{-1}\psi$$
$$\mathcal{L} \mapsto \tilde{\mathcal{L}} = \lambda^{-1}\mathcal{L}\lambda \,,$$

where $\lambda = \lambda(x) \neq 0$.

3. Multiplication by a scalar function:

$$\mathcal{L} \mapsto \tilde{\mathcal{L}} = \mu\mathcal{L} \,,$$

where $\mu = \mu(x) \neq 0$.

Definition 2.4 *The above transformations 1, 2, and 3, are called trivial symmetries for Huygens' principle. Operators that can be obtained from wave operators in $\mathbb{R}^{n+1}$ by means of trivial transformations are called trivial Huygens operators.*

We are going to assume henceforth that the operator $\mathcal{L}$ as well as the data are analytic functions of the real variable x. By restricting our focus to sufficiently small domains Ω throughout this book we shall only be concerned with the local aspects of (strict) Huygens' property. Notwithstanding, it must be emphasized that global aspects of Huygens' ought to have interesting connections with topological aspects of pseudo-Riemannian metrics in space-time.

As we mentioned in Theorem 2.2 Hadamard proved that the strict Huygens' property holds for the wave operator in n space dimensions if, and only if, n is odd and greater than 1. This prompts the following question:

Problem 2.5 *Determine all hyperbolic operators that satisfy a strict Huygens' principle.*

The lack of examples of strict Huygens' operators in $3+1$ dimensions that are not trivially equivalent to the wave operator, led to the so called "Hadamard conjecture." It became immortalized in Courant and Hilbert's [27]. It surmises that all Huygens' operators are equivalent modulo trivial symmetries to the wave operator $\Box$ in Minkowsky space-time of even dimension ≥ 4. It turned out to be false. The counter-example of Stellmacher mentioned below proved that it could not be the case.

The tale of Hadamard's problem is fascinating. Instead of cluttering the exposition with historical details we refer the reader to the comprehensive book of Günther [49] and references therein. Also, the expository article [50] is an interesting source of information as well as Ibragimov's monograph [57].

E. Hölder in 1938 [52] considered operators of the form

$$\mathcal{L}\psi = \text{div grad } \psi = \sum \frac{1}{\sqrt{\gamma}} \partial_{x_i}(\sqrt{\gamma} g^{ij} \partial_{x_j} \psi) ,$$

where $\gamma = 1/|\det(g)|$, for a 4 dimensional space. He proved that in this case the scalar curvature R of the metric vanishes.

In 1939 M. Mathisson [77] considered operators in 3 spatial dimensions of the form

$$\mathcal{L}\psi = \Box\psi + \sum_{i=0}^{3} a^i \psi_{x_i} + u(x)\psi .$$

Mathisson himself, and independently Asgeirsson [9], showed that any Huygens operators of the above form is trivial.

In 1952 P. Günther expanded the results of Asgeirsson and Mathisson for curved metrics in space-time [50]. Later on, using methods of transformation groups Ibragimov [56, 55] characterized the $3+1$ dimensional pseudo-Riemannian spaces that admit Huygens' operators.

In 1955 K. Stellmacher [96] found the following counter-example to Hadamard's conjecture:

$$\mathcal{L} = \partial_{x_0}^2 + \frac{\mu_0}{x_0^2} - \sum_{i=1}^{n} (\partial_{x_i}^2 + \frac{\mu_i}{x_i^2}) , \tag{2.11}$$

where $\mu_i = -\nu_i(\nu_i + 1)$ with $\nu_i \in \mathbb{Z}_{\geq 0}$ and

$$\sum_{i=0}^{n} \nu_i \leq \frac{n-3}{2} . \tag{2.12}$$

As a particular example one has the wave operator in 5 space dimensions

$$\mathcal{L} = (\partial_{x_0}^2 - \frac{2}{x_0^2}) - \Delta_5 \, . \tag{2.13}$$

Notice that in this case the coefficient u of the general expression (2.6) function is dependent on the time variable x_0.

The example in equation (2.13) is just the tip of an iceberg. In 1967 Lagnese and Stellmacher [68] generalized this by constructing families of Huygens operators that are *not* equivalent to wave operators. Their process of constructing such operators was by means of the factorization technique which the soliton literature calls "Darboux Transformations", although it must be pointed that their work preceded its use in soliton theory. The method of Stelmacher and Lagnese, which will be reviewed in our Section 3. On the other hand, Darboux transformations, allowed Adler and Moser tó generate the rational potentials (decaying at infinity) that remain rational by the KdV flows [3]. The latter are usually referred as rational KdV potentials or rational solutions of KdV. Curiously, the important remark that the fields $u = u(t)$ obtained by the Lagnese-Stelmacher transformation coincides with the rational solutions of the KdV took quite a while. (See for example R. Schimming [88, 89, 90].) They will be the subject of Section 3 and will display a beautiful, albeit unexplored, connection between Integrable Systems and Huygens' principle.

The subject of Integrable Systems is also developed in the Lecture Notes [25] of the course by Magri and Pedroni in the 21st Brazilian Mathematical Colloquium.

2.3 Construction of Fundamental Solutions

In order to discuss Hadamard's conjecture several approaches have been tried out [50, 48]. The approach we shall use here follows closely the one developed in the survey of Berest and Veselov [16]. We start by studying the so called elementary solutions for hyperbolic operators which were instrumental in Hadamard's analysis [51]. His elementary solutions correspond, in a rough sense, to what now we call fundamental solutions, i.e., the solution of the problem $\mathcal{L}F = \delta$. One basic distinction being that the modern theory is concerned with C^∞ coefficients whereas Hadamard's approach was devoted to treating the real analytic case. A very good description and comparison between the methods could be found in Babich's article [12].

Let's consider the following problem for the wave operator [2]

$$\begin{cases} \Box E = \delta_\xi \\ E\Big|_{t<t_0} = 0 \, , \end{cases} \tag{2.14}$$

where $\xi = (t_0, \xi_1, \ldots, \xi_n)$ is an arbitrary point, and δ_ξ means the shifted δ function. We remark that the wave operator $\Box$ admits the following solutions

[2] Henceforth, we set $c = 1$. Such a change is immaterial since one can always recover the more general case by changing the time scale.

depending on the value of n.

$$E(x,\xi) = \begin{cases} c_n \delta^{(n-3)/2}(\Upsilon) & n \text{ odd} \\ c_n \Upsilon_+^{-(n-1)/2} & n \text{ even}, \end{cases} \tag{2.15}$$

where Υ denotes the square of the Minkowski distance, i.e.,

$$\Upsilon = (t-t_0)^2 - \sum_{i=1}^{n}(x_i - \xi_i)^2 ,$$

and $\delta^{(k)}$ denotes the k-th derivative of Dirac's delta function. In formula (2.15) derivative is to be interpreted in the sense of distributions and Υ_+^α means the Heaviside function multiplied by Υ^α. The nonvanishing numerical constant c_n is given by

$$c_n = \frac{1}{2\pi^{(n-1)/2}\Gamma((n-3)/2)} , \qquad \text{even } n,$$

where the function $\Gamma()$ is the celebrated Euler's gamma function, defined by

$$\Gamma(\lambda) \stackrel{\text{def}}{=} \int_0^\infty x^{\lambda-1}e^{-x}dx, \qquad \Re[\lambda] \geq 0 .$$

Exercise 2.6 *Show that the above defined Euler's gamma function extends as a meromorphic function on the complex plane with simple poles at the values of* $\lambda \in \mathbb{Z}_{\leq 0}$. *Furthermore,*

$$\Gamma(n+1) = n! , \qquad n \in \mathbb{Z}_{\geq 0} .$$

The distribution that appears for n even in equation (2.15)

$$S_n(\Upsilon) = \frac{\Upsilon_+^{-(n-1)/2}}{\Gamma((n-3)/2)}$$

plays an important role in what follows. It should be interpreted by evaluating at $x = \Upsilon$ the function

$$S_\nu(x) = \frac{x_+^\nu}{\Gamma(\nu+1)} \stackrel{\text{def}}{=} \begin{cases} \frac{x^\nu}{\Gamma(\nu+1)} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.16}$$

This function in turn has the special property of being homogeneous of degree ν.

We remark that in the case of $n = 2$ equation (2.15) gives back the solution of the wave equation given by formula (1.4).

The formula for the wave equation suggests that in the case of a general second order linear hyperbolic operator $\mathcal{L}$, one should search for solutions in the form

$$\sum_{\nu=0}^{\infty} U_\nu(x,\xi) S_{\nu-p}(\Upsilon) , \tag{2.17}$$

where now Υ is the square of the geodesic in the metric defined by the g_{ij} and

$$p \stackrel{\text{def}}{=} \frac{n-1}{2} . \tag{2.18}$$

It turns out that for even values of the number of spatial dimensions n one must search for solutions $E(x,\xi)$ of the form

$$E(x,\xi) = W_1(x,\xi)\Upsilon^{-p} \tag{2.19}$$

where

$$W_1(x,\xi) = \sum_{j=0}^{\infty} U_j(x,\xi)\Upsilon^j .$$

The case of an odd number n of spatial dimensions takes the form

$$E(x,\xi) = V(x,\xi)\Upsilon^{-p} + W_0(x,\xi)\log\Upsilon + \mathcal{R} , \tag{2.20}$$

where

$$W_0(x,\xi) = \sum_{j=p}^{\infty} U_j(x,\xi)\Upsilon^{j-p} ,$$

$$V(x,\xi) = \sum_{j=0}^{p-1} U_j(x,\xi)\Upsilon^j ,$$

and $\mathcal{R}$ is a smooth function.

Definition 2.7 *A solution $E(x,\xi)$ of the problem $\mathcal{L}E = 0$ for $\Upsilon(x,\xi) \neq 0$ of the form given in equation (2.19) for n odd (resp. in equation (2.20), for n even) is called an elementary solution.*

Hadamard has shown the following:

Theorem 2.8 (Hadamard) *If $\mathcal{L}$ is a second order hyperbolic operator with analytic coefficients, then it admits an elementary solution.*

A classical version of the proof can be found in [51]. (See also [27, 28].) In Section 2.5 we present a more modern approach and indicate the main point in the argument of the proof.

For the time being, we remark that the coefficients of the formal expansion (2.17) can be found by substituting this ansatz into the equation

$$\mathcal{L}E = \sum g^{ij}E_{x_ix_j} + \sum b^iE_{x_i} + uE = 0$$

and matching the behavior of the powers of Υ. This yields for $r = 0, 1, \ldots$

$$2\sum g^{ij}\Upsilon_{x_i}\partial_{x_j}U_r + 4((r-1)-p)U_r + U_r(\mathcal{L}\Upsilon - u\Upsilon) = -\mathcal{L}U_{r-1} , \tag{2.21}$$

where we set $U_{-1} \stackrel{\text{def}}{=} 0$.

Exercise 2.9 *Show that the first summation term in the expression (2.21) can be interpreted as*

$$\sum g^{ij} \Upsilon_{x_i} \partial_{x_j} U_r = 2s \frac{d}{ds} U_r ,$$

where $s = \sqrt{\Upsilon}$ is the geodesic distance in the hyperbolic metric g_{ij}.

Hence, one can interpret the set of equations (2.21) as recursive system of "transport equations." For example, the leading equation ($r = 0$) becomes

$$4s \frac{d}{ds} U_0 + (-2n - 2 + \mathcal{L}\Upsilon - u\Upsilon) U_0 = 0 .$$

Exercise 2.10 *Show that for the perturbed wave operator $\mathcal{L} = \Box + u$ the recursion becomes*

$$\sum_{i=0}^{n} (x_i - \xi_i) \partial_{x_i} U_r + r U_r = \frac{1}{4} \alpha(p) \mathcal{L} U_{r-1} , \tag{2.22}$$

where $\alpha(p) = 1$ and $\alpha(r) = -(r-p)^{-1}$ if $r \neq p$.

2.4 Hadamard's Criterion

The question of determining which hyperbolic operators satisfy a strict Huygens' principle received an indirect characterization in the work of Hadamard. This indirect characterization was obtained in terms of the elementary solutions of the formal adjoint $\mathcal{L}^\dagger$ to the operator $\mathcal{L}$. The operator $\mathcal{L}^\dagger$ is defined as

$$\mathcal{L}^\dagger \psi \stackrel{\text{def}}{=} \sum \partial_{x_i} \partial_{x_j} (g^{ij} \psi) - \sum \partial_{x_i} (b^i \psi) + u\psi .$$

The fact that $\mathcal{L}^\dagger$ is involved is not surprising since the actual solutions to the Cauchy problem posed for the operator $\mathcal{L}$ would be obtained in terms of averages (or "convolutions") with the elementary solutions of $\mathcal{L}^\dagger$.

The criterion found by Hadamard is

Theorem 2.11 (Hadamard's Criterion) *The operator $\mathcal{L}$ satisfies a strict Huygens' principle iff the number of spatial dimensions n is odd, greater than* 1, *and the elementary solution of the adjoint operator $\mathcal{L}^\dagger$ contains no logarithmic term, i.e., $W_0(x, \xi) = 0$ for all ξ and all x in the internal part of the characteristic conoid.*

Hadamard's criterion, although highly nontrivial is not extremely helpful in deciding whether a given operator is strictly Huygens or not, unless one has a way of producing the term W_0.

In Section 3.5 we give an explanation of the connection between Hadamard's criterion and what became known in the literature as Painlevé's property for partial differential operators [100].

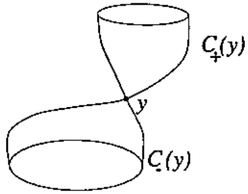

Fig. 2.4. The forward and backward half-conoids issued from a point y

2.5 Riesz Kernels

Marcel Riesz in his fundamental work [87] gave an elegant unified treatment of the expression for the solution to the wave equation for different values of dimension n. His idea is based on extending for λ in the complex plane the expression

$$\Xi(x,t,\lambda) \stackrel{\text{def}}{=} \begin{cases} (x_0^2 - \sum_i^n x_i^2)^\lambda & (\sum_i^n x_i^2)^{1/2} < t \\ 0 & \text{elsewhere.} \end{cases}$$

This last expression is defined for $\Re[\lambda] > 0$ and the meromorphic extension takes values in an appropriate space of distributions.

It turns out that by normalizing Ξ by a suitable meromorphic function (of λ) one gets for non-positive integers derivatives of the δ function, which in turn play a fundamental role in the construction of solutions to wave operators.

This section is devoted to a quick summary of the properties of Riesz kernels. We start with a few technical considerations about causal domains for a general hyperbolic metric, although the thrust of our exposition concerns the wave operator. We then define rigorously the notion of a Riesz kernel and use it to construct he Hadamard series for the elementary solution. The main goal is to give the general idea of the proof of Hadamard's criterion (See Theorem 2.17 below).

Let g be a Lorentzian metric of signature $+,-,-,\ldots,-$. In Section 2.2 we considered the square of the geodesic distance $\Upsilon(x,\xi)$, which is the solution of equation 2.7. The function Υ defines the characteristic conoid $C(\xi)$, by the equation $\Upsilon(x,\xi) = 0$. If we exclude the point ξ from $C(\xi)$, then we obtain two connected components $C_+(\xi)$ and $C_-(\xi)$, which are naturally associated with the forward and backward time. (The decomposition of $C(\xi)$ into two connected components is possible by restricting Ω if necessary.) We define, the corresponding open subsets of Ω as $D_+(\xi)$ and $D_-(\xi)$. See Figure 2.4.

From now on we work with an operator of the form

$$\mathcal{L} = \text{div grad} + \langle a, \nabla \rangle + u \ ,$$

with all coefficients (real) analytic and defined in a *causal domain* Ω. By a *causal domain* we mean that the following two properties hold

1. Any two points x and ξ are joined by a unique geodesic.

2. The set $D_+(x) \cap D_-(\xi)$ is either empty or compact in Ω.

The reader may in this case, base the intuition on the familiar case of the usual Minkowsky metric $\mathrm{diag}(1, -1, \ldots, -1)$.

A modern approach to the construction of fundamental solutions making use of theory of distributions could be developed by the formalism of Riesz kernels.

As usual $\mathcal{D}'(\Omega)$ denotes the space of distributions defined on Ω, i.e., the topological dual of the locally convex space of compact-supported C^∞ functions in Ω with the semi-norms defined by the C^k-norm over compact subsets of Ω.

Definition 2.12 *Let $\mathcal{L}$ be a second order hyperbolic operator, as above, defined on a causal domain Ω. A forward Riesz kernel of the operator $\mathcal{L}$ is a holomorphic mapping*

$$\mathbb{C} \ni \lambda \mapsto \Phi_\lambda^\Omega(\cdot, \xi) \in \mathcal{D}'(\Omega) ,$$

satisfying the following conditions

$$\mathrm{supp}[\Phi_\lambda^\Omega(\cdot, \xi)] \subseteq \overline{D_+(\xi)} \tag{2.23}$$

$$\mathcal{L}[\Phi_\lambda^\Omega(\cdot, \xi)] = \Phi_{\lambda-1}^\Omega(\cdot, \xi) \tag{2.24}$$

$$\Phi_0^\Omega(\cdot, \xi) = \delta_\xi . \tag{2.25}$$

We now discuss briefly the intuitive idea behind this definition. Remark that for $\lambda = 1$ the Riesz kernel gives a distribution $E_+(\cdot, \xi) = \Phi_1^\Omega(\cdot, \xi)$ such that

$$\mathcal{L}E_+(\cdot, \xi) = \delta_\xi$$

and also

$$\mathrm{supp}[E_+(\cdot, \xi)] \subseteq \overline{D_+(\xi)} .$$

Thus, providing us with a fundamental solution to our operator. Notice that the first condition (2.23) concerns the causality of the solution.

Modern treatment of the asymptotic behavior of solutions to hyperbolic problems studies the so called "asymptotics in smoothness"[12] It turns out, however that if the operator $\mathcal{L}$ is analytic, this asymptotic behavior can be replaced by a convergent series. This type of argument has been used extensively in the recent work of Berest [20].

The construction of Riesz kernels is based on the following analytic continuation technique:

Let $g \in \mathcal{D}(\Omega)$ be a test function, and start by defining[3]

$$\langle R_\lambda(x, \xi) | g \rangle = \int_{D_+(\xi)} \frac{\Upsilon(x, \xi)^{\lambda - \frac{n+1}{2}}}{H_{n+1}(\lambda)} g(x) dx ,$$

[3] Here, $\langle | \rangle$ denotes the duality pairing between $\mathcal{D}'$ and $\mathcal{D}$.

for $\Re[\lambda] > \frac{n-1}{2}$ and

$$H_{n+1}(\lambda) \stackrel{\text{def}}{=} 2\pi^{\frac{n-1}{2}} \Gamma(\lambda)\Gamma(\lambda - \frac{n-1}{2}) .$$

Then, extend for all $\lambda \in \mathbb{C}$ the above definition by analytic continuation, using iterates of the (classical) formula

$$\Box R_\lambda = R_{\lambda-1} , \tag{2.26}$$

which is valid for $\Re[\lambda] > (n+3)/2$.

It is clear that the distribution $R_\lambda(x,\xi)$ satisfies

$$\operatorname{supp}[R_\lambda(x,\xi)] \subseteq \overline{D_+(\xi)} ,$$

Exercise 2.13 *Show that the convolution of any two elements R_λ and R_μ is well defined and one has for all λ and μ in $\mathbb{C}$*

$$R_\lambda * R_\mu = R_{\lambda+\mu} . \tag{2.27}$$

Check also that

$$(x - \xi|\partial_x)R_\lambda = (2\lambda - n + 1)R_\lambda , \tag{2.28}$$

where $(|)$ *denotes the product in the Riemannian metric, and that for $\nu \in \mathbb{Z}_{\geq 0}$*

$$\Upsilon^\nu R_\lambda = 4^\nu \frac{\Gamma(\lambda+\nu)\Gamma(\lambda+\nu-(n-1)/2)}{\Gamma(\lambda)\Gamma(\lambda-(n-1)/2)} R_{\lambda+\nu} . \tag{2.29}$$

Recalling that

$$R_0(x,\xi) = \delta_\xi(x) , \tag{2.30}$$

a simple application of the above results gives that

Proposition 2.14 *For an odd number n of spatial dimensions and $\lambda \in \{1, 2, \dots, (n-1)/2\}$ we have*

$$R_\lambda(x,\xi) = \frac{\delta_+^{(\frac{n+1}{2}-\lambda)}(\Upsilon)}{\beta_{n,\lambda}}, \tag{2.31}$$

where $\delta_+(\Upsilon)$ stands for the the Dirac's delta measure concentrated on $\overline{C_+(\xi)} \cap \{\Upsilon = 0\}$, and the numerical constant

$$\beta_{n,\lambda} = 2\pi^{\frac{n+1}{2}-\lambda} 4^{\lambda-1}(\lambda-1)! .$$

Needless to say that once a Riesz kernel for a hyperbolic operator $\mathcal{L}$ is constructed, one has immediately a lot of information on the fundamental solution of $\mathcal{L}$, in particular as far as Huygens' principle is concerned.

For a general operator with C^∞ coefficients in a Lorentzian manifold, one looks for an expansion of the fundamental solution of the form

$$E(x,\xi) \sim \sum_{\nu=0}^{\infty} U_\nu(x,\xi) R_{\lambda+\nu}(x,\xi) . \tag{2.32}$$

As in the case of the Hadamard expansion, if we formally substitute expression (2.32) into the wave equation $\mathcal{L}E = \delta_\xi$ one obtains a set of recursive transport relations of the form

$$(x-\xi|\partial_x)U_\nu(x,\xi) + \nu U_\nu(x,\xi) = -\frac{1}{4}\mathcal{L}[U_{\nu-1}(x,\xi)] . \tag{2.33}$$

The coefficients U_ν are not necessarily the same ones as in the Hadamard expansion. It can be shown [20, 12] that the above recursive system has a unique solution provided one normalizes

$$U_0(x,\xi) = 1 , \tag{2.34}$$

and requires for $r \geq 1$

$$U_r(x,\xi) = \mathcal{O}(1) , \; x \to \xi . \tag{2.35}$$

It is well known ([20, 12, 51]) that if the operator $\mathcal{L}$ has analytic coefficients, then the series 2.33 is uniformly convergent in a sufficiently small neighborhood of $\Upsilon = 0$. In this case, the Riesz kernel for the operator $\mathcal{L}$ can be expanded as

$$\Phi_\lambda^\Omega(x,\xi) = \sum_{\nu=0}^{\infty} 4^\nu \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} U_\nu(x,\xi) R_{\lambda+\nu}(x,\xi) .$$

At this point we can even give the main argument for the proof of Hadamard's result 2.11. Indeed, if the number of spatial dimensions is even then for $\nu = 0, 1, 2, \ldots$ we have

$$\mathrm{supp}[R_{\nu+1}(x,\xi)] = \overline{D_+(\xi)} ,$$

and so Huygens principle does not hold. However, if the number of spatial dimensions n is odd then for $\nu = 0, 1, 2, \ldots, (n-3)/2$ we have

$$\mathrm{supp}[R_{\nu+1}(x,\xi)] = \overline{C_+(\xi)} .$$

Hence, using equation (2.33), and setting as usual $p = (n-1)/2$ one gets Hadamard's classical formula for the fundamental solution for n odd

$$E_+(x,\xi) = \frac{1}{2\pi^p}(V(x,\xi)\delta_+^{(p-1)}(\Upsilon) + W(x,\xi)H_+(\Upsilon)) ,$$

where $H_+(\gamma)$ is the the Heaviside step distribution on the forward region $D_+(\xi)$.

$$\langle H_+(\gamma)|\varphi\rangle = \int_{D_+(\xi)} \varphi(x)dx .$$

Furthermore, the functions $V(x,\xi)$ and $W(x,\xi)$ are analytic in a neighborhood of $x=\xi$ with expansions

$$V(x,\xi)=\sum_{\nu=0}^{p-1} s_\nu U_\nu(x,\xi)\Upsilon^\nu \ ,$$

where $s_\nu = [(1-p)\dots(\nu-p)]^{-1}$ and [4]

$$W(x,\xi)=\sum_{\nu=p}^{\infty}\frac{1}{(\nu-p)!}U_\nu(x,\xi)\Upsilon^{\nu-p} \ .$$

In conclusion, the necessary and sufficient condition for a strict Huygens' principle to hold is the vanishing of the term W.

In fact, following Berest, this condition of the vanishing can be weakened.

Lemma 2.15 *The term $W(x,\xi)$ vanishes iff $U_p(x,\xi)=0$ for x on the surface of the forward light cone $\overline{C_+(\xi)}$.*

The assertion of the lemma follows from the fact that $W(x,\xi)$ is a solution of the characteristic Goursat problem

$$\mathcal{L}[W(x,\xi)]=0 \ ,$$

with a boundary value given on the cone surface $\overline{C_+(\xi)}$. This problem has a unique solution [51]. Hence, $W(x,\xi)\equiv 0$ iff $W(x,\xi)=0$ for $x\in\overline{C_+(\xi)}$.

This makes it natural to introduce the following definition:

Definition 2.16 *The Hadamard series is said to be truncated (or terminated) at level ν_0 if the sequence defined by the recursions (2.33), (2.34), and (2.35) is zero for $\nu\geq\nu_0$.*

As a corollary of Lemma 2.15 we get the following result

Theorem 2.17 *Let $\mathcal{L}$ be a real-analytic formally self-adjoint hyperbolic on a causal domain $\Omega\subset\mathbb{R}^{n+1}$ with $n>1$. Then, $\mathcal{L}$ is strictly Huygens iff n is odd and the Hadamard series for $\mathcal{L}$ is truncated at level $p=(n-1)/2$.*

[4] The W term is usually referred in the literature as the logarithmic term

3. Completely Integrable Systems

3.1 Review of Hamiltonian Mechanics and Poisson Manifolds

This section is concerned with reviewing some of the basic concepts from Classical Mechanics with special emphasis on completely integrable systems. It is not intended to be comprehensive, one can find most of the classical material in excellent books such as [8, 76]. As for the more recent material on soliton equations and bihamiltonian formalism, the reader is referred to the text of Magri and Pedroni's course at the present 21st Brazilian Mathematical Colloquium [25].

We start with the familiar situation of Newtonian mechanics by considering the movement of a particle in $\mathbb{R}^N$ under the action of a conservative field. We denote the potential of the field by $U(x)$, and for simplicity we take the mass equal to 1. In this case, Newton's law gives us

$$\ddot{x}_i = -\frac{\partial U}{\partial x_i}\,, \qquad i = 1, \ldots, N. \tag{3.1}$$

The familiar change of variables $p = \dot{x}$ and $q = x$ transforms the second order system (3.1) into a first order one in the variables (p, q). From long ago we know that the energy

$$H = \frac{1}{2}p^2 + U$$

is conserved by the flow. Newton's equation becomes

$$\begin{cases} \dot{p}_i &= -\frac{\partial H}{\partial q_i}\,, \quad i = 1, \ldots, N \\ \dot{q}_i &= \frac{\partial H}{\partial p_i}\,, \quad i = 1, \ldots, N\,. \end{cases} \tag{3.2}$$

Introducing the phase space variable $z = (p, q)$ the system takes the concise form

$$\dot{z} = J\nabla H\,, \tag{3.3}$$

where J is the skew-symmetric matrix

$$J \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

A careful analysis of the above construction aiming at a coordinate free description of the physical laws leads naturally to the need of considering the right hand side of equation (3.3) as vector field of the form

$$X_H \stackrel{\text{def}}{=} \mathcal{J} dH \ .$$

Here, $\mathcal{J}$ is a skew-symmetric mapping sending 1-forms in vector fields [8]. It also leads naturally to the concept of a symplectic manifold.

Definition 3.1 *A symplectic manifold is a (Banach) manifold M endowed with a nondegenerate closed 2-form ω.*

We recall that a 2-form ω is called *nondegenerate* iff [1]

$$\omega(X,Y) = 0 \ , \forall Y \in T_pM \quad \Rightarrow \quad X = 0 \ .$$

Example 3.2 Take $\mathbb{C}^N = \mathbb{R}^{2N}$ with the coordinates

$$z_j = p_j + iq_j \quad , j = 1, \dots N$$

and

$$\omega = \frac{1}{2i} \left(dz_1 \wedge d\bar{z}_1 + \dots + dz_N \wedge d\bar{z}_N \right) \ . \tag{3.4}$$

Note that the case $N = 1$ gives the usual area element of the complex plane.

The elegant concept of a symplectic form turns out to be extremely helpful in the understanding of classical mechanics. The point being that it can be used to construct vector fields through the following process:

Since ω is nondegenerate it establishes an isomorphism between the tangent and the cotangent spaces at a given point. The isomorphism is given by

$$T_pM \ni X \longmapsto \omega(X, \cdot) \in (T_pM)^* \ . \tag{3.5}$$

Let's call $\mathcal{J}$ the inverse of this map. To produce a vector field out of a given function H we consider

$$H \in C^\infty(M) \longmapsto dH \stackrel{\mathcal{J}}{\longmapsto} X_H \stackrel{\text{def}}{=} \mathcal{J} dH \tag{3.6}$$

Example 3.3 In the case of R^{2N} with the canonical symplectic structure

$$\omega = dp \wedge dq \stackrel{\text{def}}{=} \sum dp_i \wedge dq_i \ ,$$

the above construction gives the following association

[1] We note that if the manifold is infinite dimensional the above definition is of a *weakly* symplectic manifold. In the infinite dimensional context we shall assume even further that ω is strongly nondegenerate, i.e., the mapping in equation (3.5) defines an isomorphism of T_pM onto T_pM^*.

$$H \longmapsto \sum \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) \longmapsto \sum \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) \qquad (3.7)$$

Exercise 3.4 *Show that H is a conserved quantity of the Hamiltonian flow generated by vector field X_H.*

Closely related to the above construction and to the notion of symplectic manifolds is the concept of a Poisson bracket. It could have been taken as our starting point.

Definition 3.5 *A Poisson bracket on a manifold M is a bilinear and skew-symmetric mapping $\{,\}$ from $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the following properties*

1. *It acts as a derivation, i.e.,*

$$\{FG, H\} = F\{G, H\} + \{F, H\} G \qquad (3.8)$$

2. *It satisfies the Jacobi identity*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \qquad (3.9)$$

Example 3.6 The harmonic oscillator (or linearized pendulum): In $\mathbb{R}^2$ with the canonical form, let's consider the Hamiltonian

$$H(p, q) = \frac{1}{2}(p^2 + \kappa^2 q^2) \ .$$

The Hamiltonian vector field X_H is given by

$$X_H = -\kappa^2 q \frac{\partial}{\partial p} + p \frac{\partial}{\partial q} \ .$$

Each orbit, distinct from the equilibrium configuration $(0, 0)$, is an ellipsis. See Figure 3.1.

Given a symplectic structure, one can always construct a Poisson manifold by means of

$$\{F, G\} = \mathcal{J}(dF)G = dG(\mathcal{J}(dF)) \ .$$

For example, in the case of $\mathbb{R}^{2n}$ with the canonical symplectic structure we have

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \ .$$

Definition 3.7 *A Poisson bracket is called degenerated if there exists a non-constant function $F \in C^\infty(M)$ such that for every $G \in C^\infty(M)$*

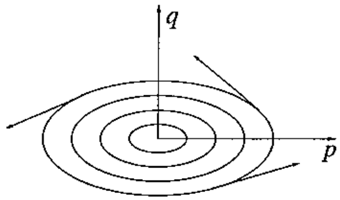

Fig. 3.1. Orbits of the harmonic oscillator of Example 3.6in phase space.

$$\{F, G\} = 0 \ .$$

The reader unfamiliar with symplectic structures may well think locally as the familiar case of the canonical symplectic form of $\mathbb{R}^{2N}$. Indeed, due to the following classical result of Darboux

Theorem 3.8 (Darboux) *Locally, a $2N$-dimensional symplectic manifold is diffeomorphic to $\mathbb{R}^{2N}$ endowed with the canonical symplectic form $dp \wedge dq$.*

For a proof the reader is referred to [8].

Exercise 3.9 *Show that the mapping*

$$\mathcal{J} : dH \longmapsto X_H \stackrel{\text{def}}{=} \mathcal{J} dH$$

is a Lie algebra homomorphism between the Lie algebra $(C^\infty(M), \{,\})$ and the Lie algebra of smooth vector fields $(\mathrm{Vec}(M), -[,])$ where $[,]$ denotes the standard vector field commutator.

Show that two Hamiltonian vector fields X_H and X_F commute iff $\{H, F\}$ is locally constant.

Note that the geometric interpretation of the Poisson bracket $\{H, F\}$ is the directional derivative along the vector field X_H. In other words the rate of change of F along the flow line of the Hamiltonian vector field X_H.

As a consequence of the above remarks, it follows that if $\{H, F\} = 0$, then F is a conserved quantity of the Hamiltonian vector field X_H. In particular, if c is a regular value of F, this means that the flow of X_H restricts itself to the hypersurfaces defined by $F^{-1}(c)$.

Example 3.10 In the harmonic oscillator of example 3.6, note that the change of variables

$$\begin{cases} I = \frac{1}{2\kappa}(p^2 + \kappa^2 q^2) \\ \phi = \tan^{-1}(\frac{\kappa q}{p}) \end{cases} \tag{3.10}$$

transforms the system into

$$\begin{cases} \dot{I} = 0 \\ \dot{\phi} = \kappa \end{cases} \tag{3.11}$$

Exercise 3.11 *Show that the transformation given in equation (3.10) is canonical in the sense that it maps the form $dp \wedge dq$ into $dI \wedge d\phi$.*

Definition 3.12 *Two functions F and H in $C^\infty(M)$ are said to be in involution w.r.t. to the Poisson bracket $\{,\}$ if*

$$\{F, H\} = 0 \ . \tag{3.12}$$

Now, if $\{F, H\} = 0$, then X_F is the infinitesimal generator of a group of symmetries of X_H. Hence, it is natural to expect that if a Hamiltonian is in involution w.r.t. many independent quantities then the associated Hamiltonian vector field possesses many symmetries.[2] These symmetries in turn should enable one to construct coordinates where the given Hamiltonian vector field has a simple form. This is indeed the content of the classical result of Liouville.

Theorem 3.13 *(Liouville [71], Arnold [8]) Let (M, ω) be a $2N$-dimensional symplectic manifold, with N independent quantities $F_1, \ldots, F_N$ in involution. Suppose that each of the Hamiltonian vector fields X_{F_i} is complete at the level set*

$$\mathcal{V}_a \stackrel{\text{def}}{=} \{m \in M | (F_1(m), \ldots, F_N(m)) = a\} \ ,$$

which is assumed nonempty. Then,

1. *$\mathcal{V}_a$ is an invariant submanifold of M w.r.t. any of the Hamiltonian vector fields X_{F_i} and diffeomorphic to the union of cylinders of the form $\mathbb{R}^k \times \mathbb{T}^{n-k}$.*

2. *It is possible to find, at least locally, a system of coordinates*

$$(I_1, \ldots, I_N, \phi_1, \ldots, \phi_N)$$

where the symplectic form ω is given by

$$\omega = \sum dI_i \wedge d\phi_i \ .$$

and furthermore, in these coordinates the flow of the vector field X_{F_i} takes the form

$$\begin{cases} \dot{I} = 0 \\ \dot{\phi} = f_i(I) \ . \end{cases} \tag{3.13}$$

[2]We recall that functions $F_1, \ldots, F_k$ are said to be independent on a manifold if their derivatives $dF_1, \ldots, dF_k$ are linearly independent on all points of the manifold.

For a proof of this result we refer the reader again to [8, 86]. We limit ourselves to a few remarks:

1. The hypothesis of having complete vector fields can be dropped if one knows a priori that the manifolds $\mathcal{V}_c$ are compact and connected.

2. The variables (I, ϕ) given in the above result are called *action-angle variables.* This is due to the fact that in the compact case, the variables $\phi_1, \ldots, \phi_N$ denote a point in the torus $\mathbb{T}^N$, and in the classical construction, the variables $I_1, \ldots, I_N$ are constructed with the same dimensions of the action $\int \mathcal{L}dt$, where $\mathcal{L}$ is the Lagrangian.

3. The classical proof of Liouville's theorem leads to a theoretical construction of the action-angle variables using only algebraic operations, integrations, composition and inversion of functions. For this reason, systems falling under the umbrella of the above theorem were called integrable by quadratures. Unfortunately, the number of cases where such a program could be implemented in practice is extremely small. No wonder that until the early 70's very few examples of (explicitly) integrable systems were known. Nowadays, to perform the process of finding the action-angle variables one counts with effective techniques such as Lax pairs, zero curvature equations, and Lie-Algebraic techniques. For a comprehensive discussion of these techniques see [86].

3.2 KdV, KP, and Friends

In this section we survey some aspects of the theory of the Korteweg-de Vries equation

$$\psi_t + c\psi_x + \alpha\psi_{xxx} + \beta\psi\psi_x = 0 \; . \tag{3.14}$$

Here, c, α, β are constants, which can be scaled out at our will.

The KdV equation, together with its two dimensional generalization, the Kadmontsev-Petviashvilli equation

$$\frac{\partial}{\partial x}\left(u_t - 6uu_x - u_{xxx}\right) = \kappa u_{yy} \tag{3.15}$$

are archetypical examples of infinite dimensional completely integrable systems. They exhibit a tremendous amount of structure in terms of conservation laws and symmetries.

The importance of the KdV rests not only on its simplicity and beautiful mathematical structure. It is also due to the fact that it models a number of situations of (unidirectional) wave propagation under a weak nonlinearity and small dispersion. For an argument in favor of this the reader is referred to the introduction of [79].

Notice that a change of dependent and independent variables, which is left as an exercise to the reader, transforms equation (3.14) into the equation

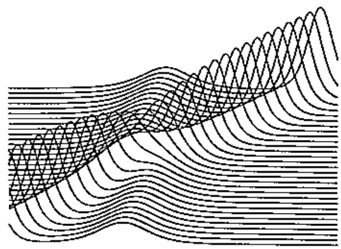

Fig. 3.2. Two-soliton solution of the KdV equation.

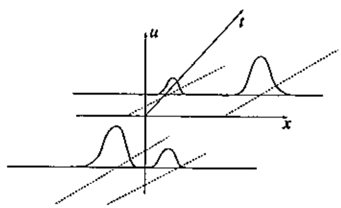

Fig. 3.3. Diagram of the interaction of two solitons.

$$u_t = -u_{xxx} + 6uu_x \ . \tag{3.16}$$

One of the first remarkable features of (3.16) is the presence of solitary wave solutions, i.e., solutions of the form

$$u(x,t) = f(x - ct) \ . \tag{3.17}$$

Exercise 3.14 *Let u be a solution of the form (3.17) to the KdV equation (3.16). Assuming that f is sufficiently smooth, and decays at ∞ together with its derivatives, show that it can be written as*

$$f(x) = -\frac{1}{2}c \ \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c}(x - x_0)\right) \tag{3.18}$$

The solitary wave solution above is called a *soliton.* We shall explain this term shortly. First we remark that the KdV admits nonlinear superposition of N such solitons, which are called N-soliton solutions of KdV. Their asymptotic behavior along certain directions on the (x,t)-plane takes the form of equation (3.18) with appropriate constants. See Figure 3.3.

It can be shown, using the inverse scattering method [37, 79] or the Darboux method described below, that the KdV admits solutions of the form

$$u(x,t) = -2\partial_x^2 \log \det[A(x,t)] \ ,$$

where A is the $N \times N$ matrix whose entries are

$$A_{ij} = \delta_{ij} + \frac{\beta_i}{\kappa_i + \kappa_j} \exp(-(\kappa_i + \kappa_j)x + 8\kappa_i^3 t) \ .$$

The asymptotic behavior along lines of slope $4\kappa_n^2$ on the (t,x)-plane displays interesting particle like interactions, between the different solitons. We refer the reader to [79] for more details

The term *soliton* was coined by M. Kruskal to denote the aspect of a *solitary wave* and a *particle*, whence the greek suffix "ton".

The persistence of the solitonic form in the solutions of the KdV equation was first remarked numerically in a series of classical experiments by Kruskal and Zabusky [103]. Such behavior makes it plausible that the KdV would possess many conservation laws. A few of them are quite evident on physical grounds, they are

$$H_{-1} = \int u \ dx \tag{3.19}$$

$$H_0 = \int \frac{1}{2}u^2 \ dx \tag{3.20}$$

$$H_1 = \int (\frac{1}{2}u_x^2 + u^3) \ dx \tag{3.21}$$

The strange labeling of the conservation laws will become clear soon.

For a short while, it was an open problem whether the KdV had infinitely many independent conserved quantities. See Chapter 1 of [78] for an illustrative historical survey.

The problem of obtaining infinitely many conserved quantities was settled by the inverse scattering method [37]. The construction initially looked like an obscure trick. It involved introducing the Schrödinger operator

$$L = -\partial_x^2 + u \ , \tag{3.22}$$

and looking into its spectral theory. In particular, looking at the time evolution of the eigenvalues and the scattering parameters as the potential evolved according to the KdV equation. All became more apparent with the development of the Lax pair formalism [69], and later with the ZS-AKNS method [1, 104]. Unfortunately, will not have time to touch this subject here. The interested reader could consult [79] and references therein.

It is now known that the KdV is a completely integrable system in the following sense:

There exists an infinite sequence of conservation laws in involution with respect to a certain Poisson bracket. Associated to these conserved quantities, the first of which were given above, there exists a sequence of angle variables. Furthermore, the system becomes linearized in these infinite set of angle variables.

One choice of Poisson bracket w.r.t. which the conserved quantities of the KdV are in involution is given by

$$\{H, F\} \stackrel{\text{def}}{=} \int \frac{\delta H}{\delta u} \partial_x \frac{\delta F}{\delta u} \quad dx , \tag{3.23}$$

where the $\delta H/\delta u$ denotes the variational derivative of the quantity $H[u]$.

We recall the for a functional $H : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ the variational derivative $\delta H/\delta u$ is defined as the element in $\mathcal{S}'(\mathbb{R})$ that realizes

$$dH[u](\delta u) = \int \frac{\delta H}{\delta u} \quad \delta u \quad dx$$

where $dH[u]$ is the linearization of H at u given by

$$H[u + \delta u] = H[u] + dH[u](\delta u) + \mathcal{O}((\delta u)^2) .$$

Exercise 3.15 *Suppose that the functional $H[u]$ is given by*

$$H[u] = \int h(u, u^{(1)}, \dots, u^{(k)}) \quad dx .$$

Show that,

$$\frac{\delta H}{\delta u} = \frac{\partial h}{\partial u} - \frac{d}{dx}\frac{\partial h}{\partial u^{(1)}} + \frac{d^2}{dx^2}\frac{\partial h}{\partial u^{(2)}} + \dots + (-1)^k \frac{d^k}{dx^k}\frac{\partial h}{\partial u^{(k)}} .$$

In the Hamiltonian context we mentioned before, the KdV takes the form

$$u_t = \partial_x \frac{\delta H_1}{\delta u} ,$$

whereas the vector field generated by H_0 is nothing more than

$$X_{H_0}[u] = u_x .$$

It generates the translation flow, which obviously commutes with the KdV flow, since the KdV equation does not have any x-dependent coefficient.

The existence of infinitely many conserved quantities in involution leads to a full hierarchy of vector fields commuting with one another and with the KdV flow. These flows can be constructed directly with the formalism of Lax pairs. However, we will refrain to do it like that since it is the traditional approach and can be found in most introductory texts on solitons. We will use instead the formalism of recursion operators, which is directly linked with the "bihamiltonian structure of KdV." The key point being that the KdV hierarchy can be written also as hamiltonian system with respect to a *second* Poisson structure. This second structure being compatible with the first one in the following sense:

Any linear combination of the two Poisson structures is also a Poisson structure.

Exercise 3.16 *Let*

$$\mathsf{K}_u\varphi \stackrel{\text{def}}{=} -\partial_x^3\varphi + 4u\partial_x\varphi + 2u_x\varphi \ . \tag{3.24}$$

Show that

$$\{F,G\}_2 \stackrel{\text{def}}{=} \int \frac{\delta F}{\delta u}\mathsf{K}_u\frac{\delta G}{\delta u} \quad dx \tag{3.25}$$

defines a Poisson bracket on the space $\mathcal{S}(\mathbb{R})$ (or $C^\infty(\mathbb{T})$). Show that one can write the KdV equation (3.16) as

$$u_t = \mathsf{K}_u\frac{\delta H_0}{\delta u} \ , \tag{3.26}$$

where H_0 was defined in Equation (3.20).

The bihamiltonian structure formalism of integrable equations was studied extensively by F. Magri and collaborators [72, 73, 65, 74, 26, 24, 75]. It is the subject of the lecture notes [25]. For this reason we shall not extend more on the subject referring the reader directly to the inspiring source.

3.3 Recursion Operators and Master Symmetries

This section is concerned with the construction of the KdV hierarchy from the point of view of recursion operators. The exposition in the next two sections follows closely our joint work with F. Magri [75, 110].

We think of the KdV equation

$$u_t = -u_{xxx} + 6u_xu \stackrel{\text{def}}{=} X_1(u) \ . \tag{3.27}$$

as a vector field on a (linear) manifold we denote by $\mathcal{U}$. There are several spaces that could be used for such studies. For instance, the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions on the line, or the smooth functions on the circle $C^\infty(\mathbb{T})$. For our application to the rational solutions of the KdV hierarchy, we shall make use of yet another space. We postpone this discussion to the end of the next section. The crucial point being that the construction is algebraic and only depends on certain algebraic relations.

The Nijenhuis Tensor

We start with two important remarks. The first one is that we can write

$$X_1(u) = \mathsf{N}_uX_0(u),$$

where $X_0(u) \stackrel{\text{def}}{=} u_x$, and N_u is a linear map, depending on $u \in \mathcal{U}$, defined by

$$\mathsf{N}_u\psi = -\partial_x^2\psi + 4u\psi + 2u_x\partial_x^{-1}\psi \tag{3.28}$$

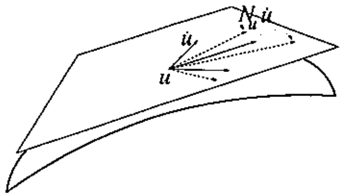

Fig. 3.4. Pictorial description of a tensor of type $(1,1)$ sending vector fields into vector fields.

for $\psi \in \mathcal{U}$. Here, by $\partial_x^{-1}\psi$ we mean a formal antiderivative of ψ.

We think of N_u as a tensor field of type $(1,1)$ on $\mathcal{U}$. This means that N_u sends vector fields into vector fields. See Figure 3.4.

The next ingredient is the Nijenhuis torsion of a tensor field.

Definition 3.17 *If X and Y are vector fields on a manifold and G is a tensor field of type $(1,1)$, the Nijenhuis torsion of G is a vector valued 2-form defined by*

$$\mathsf{T}_u(X,Y) \stackrel{\text{def}}{=} [GX,GY] - G[X,GY] - G[GX,Y] + G^2[X,Y]\ , \tag{3.29}$$

where the bracket $[\cdot,\cdot]$ denotes the vector field commutator

$$[X,Y](u) = Y'_u X(u) - X'_u Y(u)\ . \tag{3.30}$$

In the case of linear manifold, we can compute the above derivative as follows. For a constant vector field φ and an arbitrary vector field $X(u)$

$$X'_u(\varphi) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} X(u+\epsilon\varphi)$$

The derivative of a tensor field N_u on the other hand can be computed as

$$\mathsf{N}'_u(\varphi;\psi) \stackrel{\text{def}}{=} \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathsf{N}_{u+\epsilon\psi}\varphi\ ,$$

where φ and ψ are constant vector fields.

Exercise 3.18 *For the tensor field N_u defined in (3.28) we have explicitly*

$$\mathsf{N}'_u(\varphi;\psi) = 4\varphi\psi + 2\psi_x\partial_x^{-1}\varphi. \tag{3.31}$$

Hence, the Nijenhuis torsion of N_u is the vector valued 2-form given by

$$\mathsf{T}_u(\varphi,\psi) = \mathsf{N}_u'(\psi,\mathsf{N}_u\varphi) - \mathsf{N}_u'(\varphi,\mathsf{N}_u\psi) + \mathsf{N}_u\mathsf{N}_u'(\varphi,\psi) - \mathsf{N}_u\mathsf{N}_u'(\psi,\varphi)\ . \qquad (3.32)$$

Notice that T_u is skew-symmetric in (φ,ψ).

Exercise 3.19 *Verify that for* N_u *given by (3.28), the torsion*

$$\mathsf{T}_u(\varphi,\psi) = 0.$$

Recall that the Lie derivative of the $(1,1)$ tensor N_u along the vector field $X(u)$ is a tensor field of the same type defined by

$$\mathcal{L}_X(\mathsf{N}_u)\varphi = \mathsf{N}_u'(\varphi;X) + \mathsf{N}_u X_u'\varphi - X_u'\mathsf{N}_u\varphi.$$

The second important remark is that for N_u given by (3.28) and $X_0(u) = u_x$, we have that

$$\mathcal{L}_{X_0}(\mathsf{N}_u) = 0.$$

So, we can say that X_0 is an infinitesimal symmetry of N_u.

The consequence of the two remarks above is the following result:

Proposition 3.20 *The iterated vector fields*

$$X_j \stackrel{\text{def}}{=} \mathsf{N}_u^j X_0$$

satisfy

$$[X_j, X_l] = 0\ .$$

The vector fields X_j generate the higher order KdV flows [82, 73, 65]

$$\partial_{t_j} u = X_j(u).$$

These flows can also be obtained via the Lax pair formalism as we mentioned before, and by fractional powers of differential operators [38, 39, 69, 91].

Master Symmetries

We are ready to construct the hierarchy of master symmetries for the KdV equation. They have the important property of producing by means of commutators higher order vector fields of the KdV hierarchy. Thus, generating symmetries for the KdV, since the higher order KdV flows commute with the KdV flow.

The concept of master symmetries was introduced by Fuchssteiner and was applied to a number of important examples such as the Benjamin-Ono equation and the K.P. hierarchy [35, 36, 80, 81].

Consider the vector field

$$\tau_0(u) = \frac{1}{2}xu_x + u.$$

If we compute the Lie derivative of N_u, with respect to τ_0, we find

$$\mathcal{L}_{\tau_0}(\mathsf{N}_u) = \mathsf{N}_u \ . \tag{3.33}$$

Leibniz rule gives

$$\mathcal{L}_{\tau_0}(\mathsf{N}_u^j) = j\mathsf{N}_u^j \ . \tag{3.34}$$

Let's now consider the hierarchy $\{\tau_j\}_{j=0}^{\infty}$ obtained by defining

$$\tau_j = \mathsf{N}_u^j \tau_0 \ .$$

The vector fields τ_j, for $j \geq 0$, do not commute with one another, since τ_0 is not a symmetry of N_u. However, they do verify the following remarkable commutation relation

$$[\tau_j, \tau_l] = (l-j)\tau_{j+l} \ , \tag{3.35}$$

as will be shown bellow.

We digress a little to remark that the reader familiar with Virasoro algebras will recognize equation (3.35) as defining a Virasoro algebra of zero central charge. This algebra plays a central role in soliton theory [83] and conformal field theory [13]. It was introduced by M. A. Virasoro in [97]. It can obtained by first considering the complexification of the algebra of vector fields of the circle and then looking into its central extensions.

Now, a (complex) vector field of $\mathbb{S}^1$ can be thought of as

$$A = \sum_k a_k z^{k+1} \frac{d}{dz} \ . \tag{3.36}$$

Thus, it is natural to think of the vector fields of the circle as generated by

$$\left\{ \ V_k \stackrel{\text{def}}{=} z^{k+1} \frac{d}{dz} \ \right\}_{k=-\infty}^{\infty} \ . \tag{3.37}$$

A simple computation gives

$$[V_k, V_m] = (m-k)V_{k+m} \ .$$

Gelfand and Fuchs characterized the central extensions of the algebra of vector fields of the circle. They turn out to be parametrized by a value c, called the charge. For each c one has a central extension of $\mathcal{V}$. It is generated by the elements V_k above and a central element $\mathbb{1}$ so that

$$[V_k, V_m] = (m-k)V_{k+m} + \frac{c}{12}\delta_{m,-k}(k^3-k)\mathbb{1} \ .$$

Back to the master symmetries of KdV, we now prove equation (3.35).

We begin by noting that the vanishing of the torsion of N_u yields

$$\mathcal{L}_{\mathsf{N}_u X}(\mathsf{N}_u) = \mathsf{N}_u \mathcal{L}_X(\mathsf{N}_u) \tag{3.38}$$

and hence

$$\mathcal{L}_{\mathsf{N}_u^j X}(\mathsf{N}_u) = \mathsf{N}_u^j \mathcal{L}_X(\mathsf{N}_u) \ . \tag{3.39}$$

Indeed, from the vanishing of the torsion and by Leibniz rule

$$\begin{aligned} 0 &= [\mathsf{N}_u X, \mathsf{N}_u Y] - \mathsf{N}_u[\mathsf{N}_u X, Y] - \mathsf{N}_u[X, \mathsf{N}_u Y] + \mathsf{N}_u^2[X, Y] \\ &= \mathcal{L}_{\mathsf{N}_u X}(\mathsf{N}_u Y) - \mathsf{N}_u \mathcal{L}_{\mathsf{N}_u X}(Y) - \mathsf{N}_u \mathcal{L}_X(\mathsf{N}_u Y) + \mathsf{N}_u^2 \mathcal{L}_X(Y) \\ &= (\mathcal{L}_{\mathsf{N}_u X}(\mathsf{N}_u) - \mathsf{N}_u \mathcal{L}_X(\mathsf{N}_u))Y \ . \end{aligned}$$

Now, we compute

$$\begin{aligned} [\tau_j, \tau_l] &= \mathcal{L}_{\tau_j}(\mathsf{N}_u^l \tau_0) \\ &= \mathcal{L}_{\tau_j}(\mathsf{N}_u^l)\tau_0 + \mathsf{N}_u^l \mathcal{L}_{\tau_j}(\tau_0) \\ &= \mathcal{L}_{\mathsf{N}_u^j \tau_0}(\mathsf{N}_u^l)\tau_0 - \mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j \tau_0) \\ &= \mathcal{L}_{\mathsf{N}_u^j \tau_0}(\mathsf{N}_u^l)\tau_0 - \mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j)\tau_0 - \mathsf{N}_u^{l+j} \mathcal{L}_{\tau_0}\tau_0 \\ &= \mathcal{L}_{\mathsf{N}_u^j \tau_0}(\mathsf{N}_u^l)\tau_0 - \mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j)\tau_0 \ . \end{aligned}$$

From equations (3.39) and (3.34) we get

$$\begin{aligned} [\tau_j, \tau_l] &= \mathsf{N}_u^j \mathcal{L}_{\tau_0}(\mathsf{N}_u^l)\tau_0 - \mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j)\tau_0 \\ &= (\mathsf{N}_u^j l \mathsf{N}_u^l - \mathsf{N}_u^l j \mathsf{N}_u^j)\tau_0 \\ &= (l - j)\tau_{j+l} \ . \end{aligned}$$

The first two nonlinear master symmetry fields constructed above are given by

$$\begin{aligned} \tau_1(u) &= -\frac{x}{2}(u_{xxx} - 6uu_x) - 2u_{xx} + u_x\partial_x^{-1}u + 4u^2 \\ \tau_2(u) &= \frac{x}{2}(u_{5x} - 10uu_{3x} - 18u_xu_{xx} + 24u^2u_x) + 3u_{4x} - u_{3x}\partial_x^{-1}u \\ &\quad -24uu_{xx} - 15u_x^2 + u_x(4u\partial_x^{-1}u + 2\partial_x^{-1}\tau_1(u)) + 16u^3 \ . \end{aligned}$$

Exercise 3.21 *(Messy!) Prove equation (3.35) for* $0 \le l, j \le 2$ *directly.*

We now describe explicitly how the master symmetry vector fields are connected to the KdV flows. To do this we first compute the commutator of X_0 and τ_0, which gives

$$[X_0, \tau_0] = -\frac{1}{2}X_0 \ . \tag{3.40}$$

Thus, by repeating the argument given above for $[\tau_j, \tau_l]$ we find that

$$[X_j, \tau_l] = \mathcal{L}_{\mathsf{N}_u^j X_0}(\mathsf{N}_u^l \tau_0) \tag{3.41}$$

$$= \mathcal{L}_{\mathsf{N}_u^j X_0}(\mathsf{N}_u^l)\tau_0 + \mathsf{N}_u^l \mathcal{L}_{\mathsf{N}_u^j X_0}(\tau_0) \tag{3.42}$$

$$= \mathsf{N}_u^j \mathcal{L}_{X_0}(\mathsf{N}_u^l)\tau_0 - \mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j)X_0 - \mathsf{N}_u^{l+j}\mathcal{L}_{\tau_0}(X_0) \tag{3.43}$$

$$= -\mathsf{N}_u^l \mathcal{L}_{\tau_0}(\mathsf{N}_u^j)X_0 - \mathsf{N}_u^{l+j}\mathcal{L}_{\tau_0}(X_0) \tag{3.44}$$

$$= -j\mathsf{N}_u^{l+j}X_0 - \frac{1}{2}\mathsf{N}_u^{l+j}X_0 \tag{3.45}$$

$$= (-j - \frac{1}{2})X_{l+j} \ . \tag{3.46}$$

In going from equation (3.44) to (3.45) we used equations (3.34) and (3.40). The conclusion from equation (3.46) is that the master symmetry fields can be used to generate, via commutators, the fields in the KdV hierarchy. We summarize the results thus far in the following:

Theorem 3.22 *The hierarchies* $\{X_k\}_{k=0}^{\infty}$ *and* $\{\tau_j\}_{j=0}^{\infty}$ *satisfy*

$$[X_j, X_k] = 0 \ ,$$

$$[\tau_j, \tau_l] = (l-j)\tau_{j+l} \ ,$$

and

$$[X_j, \tau_l] = -(j+\frac{1}{2})X_{l+j} \ .$$

A Few Technical Remarks

We end this section with a few remarks about the linear manifold $\mathcal{U}$ of functions under consideration.

The goal of the next sections is to produce integral curves of the KdV flows and of the master symmetry hierarchy starting from certain potentials of the form c/x^2 and then applying the Darboux method. We will see that the method produces a sequence of functions that are at worst rational in $x^{1/2}$ and $\log x$. We choose a branch of the logarithm with a cut coinciding with the *positive real axis*. Hence, for our purposes, it will be convenient to take the manifold $\mathcal{U}$ to be a space of functions $u(x)$ satisfying the following two properties:

1. The function $u(x)$ is holomorphic in an open sector S containing an unbounded subinterval of the negative real axis. (The sector S depends on the function u.)

2. The asymptotic behavior of $u(x)$, as $x \to \infty$, is

$$u(x) = \mathcal{O}(\frac{1}{x^2}) \ , \ x \in S.$$

In this context the symbol ∂_x^{-1} becomes

$$\partial_x^{-1}\psi = \int_{\infty}^{x} \psi(s)ds,$$

where the path of integration connecting ∞ to x lies in S.

Exercise 3.23 *Given* $u \in \mathcal{U}$*, show that* $u_x = \mathcal{O}(1/x^3)$ *and* $\partial_x^{-1}u = \mathcal{O}(1/x)$ *for* x *in any sub-sector whose rays are in the interior of* S*. See [98].*

We close by remarking that the vector fields τ_j are well defined in the space $\mathcal{U}$, since $\tau_0(\mathcal{U}) \subset \mathcal{U}$ and $\mathsf{N}_u(\mathcal{U}) \subset \mathcal{U}$.

3.4 Darboux Method

In this section we review the background on classical Darboux transformations. For second order operators, they were used by G. Darboux in connection with problems in differential geometry [30]. For instances where the method plays a role we refer the reader to [22, 29, 32, 33, 60, 61, 110] and references therein.

Suppose we know how to factor an operator L as the product of two operators P and Q

$$L = PQ \; . \tag{3.47}$$

Let's consider now the operator

$$\widetilde{L} = QP \; . \tag{3.48}$$

Obviously, we have that Q intertwines L and $\widetilde{L}$, i.e.,

$$\widetilde{L}Q = QL \; .$$

Hence, if ψ is an eigenfunction of L, not in the kernel of Q, it follows that $\widetilde{\psi} \stackrel{\text{def}}{=} Q\psi$ is also an eigenfunction of $\widetilde{L}$.

Exercise 3.24 *If we take $L = -\partial_x^2 + u$ and impose that P and Q are first order operators of the form $P = -\partial_x - s$ and $Q = \partial_x - v$, show that $L = PQ$ implies that $s = v$ and that v satisfies the Riccati equation*

$$v_x + v^2 = u \; . \tag{3.49}$$

As a particular case, we consider the second order differential operator

$$L = -\partial_x^2 + u \; ,$$

which was the case studied by Darboux. Let's assume we know a solution of $L\phi = 0$. Then, we can factorize L as

$$L = (-\partial_x - v)(\partial_x - v) \; , \tag{3.50}$$

by taking $v = \partial_x \log(\phi)$. Indeed, the necessary and sufficient condition for one to be able to write L in the form (3.50) is that v satisfies (3.49). The solutions of the Riccati equation (3.49) are given by $v = \partial_x \log(\phi)$, where ϕ is in the kernel of L.

We remark in passing that (3.49) defines the now celebrated Miura transformation, which relates the KdV equation to the modified KdV one.

The new potential is given by

$$\widetilde{u} = u - 2v_x \; . \tag{3.51}$$

As a consequence of the previous line of reasoning it follows that

$$\widetilde{\psi} = (\partial_x - v)\psi$$

is a solution of

$$\widetilde{L}\widetilde{\psi} = \lambda\widetilde{\psi} \ ,$$

whenever ψ satisfies $L\psi = \lambda\psi$.

At this point it is natural to ask how general are the operators obtained by Darboux transformations.

Proposition 3.25 *Let L and $\widetilde{L}$ be n-th order differential operators. Then, the most general monic first order differential operator Q such that*

$$\widetilde{L}Q = QL \ ,$$

is of the form

$$Q = \partial_x - (\log\phi)_x \ , \tag{3.52}$$

where $\phi \in \ker(L - \lambda_0 I)$ and λ_0 is an arbitrary constant.

Proof: Since Q is first order, we can look at its kernel as $\operatorname{span}\{\phi\}$ where $\phi \neq 0$. We can also write $Q = \partial_x - (\log\phi)_x$. The fact that $\ker Q$ is invariant by the linear operator L implies that there exists λ_0 such that $L\phi = \lambda_0\phi$. *Q. E. D.*

In the case of the Schrödinger operator, the above result implies the following:

Corollary 3.26 *Let $L = -\partial_x^2 + u$ and $\widetilde{L} = -\partial_x^2 + \widetilde{u}$. The most general monic first order differential operator intertwining L and $\widetilde{L}$ in the sense that $\widetilde{L}Q = QL$ is of the form $Q = \partial_x - v$, where v satisfies*

$$v_x + v^2 = u + \lambda_0 \ , \tag{3.53}$$

for some constant λ_0. Furthermore, in this case, the potential $\widetilde{u}$ is given by equation (3.51).

The two previous results give necessary conditions for a monic first order operator Q to intertwine L and $\widetilde{L}$. It is easy to see that if $\phi \in \ker(L - \lambda_0 I)$ then we can define Q by equation (3.52) and factorize $L - \lambda_0 I$ as $L - \lambda_0 I = PQ$. Then, by defining $\widetilde{L} - \lambda_0 I = QP$ we get that $\widetilde{L}Q = QL$. As a particular case we have:

Proposition 3.27 *Suppose that v satisfies the Riccati equation (3.53). If we define $\widetilde{L} = -\partial_x^2 + \widetilde{u}$ with $\widetilde{u}$ as in equation (3.51), then $Q = \partial_x - v$ intertwines L and $\widetilde{L}$.*

Iterating Darboux Transformations

The goal now of is to describe the potentials u_n obtained by n successive applications of the Darboux transformations to

$$L_0 = -\partial_x^2 + u_0 \ , \tag{3.54}$$

where u_0 is a given starting point. Typically we will be thinking of u_0 as 0 or as $u_0 = (l^2 - 1/4)/x^2$ and $l \in \mathbb{Z}_{\geq 0}$. As we shall see, each Darboux transformation introduces a new complex parameter in the resulting potential. By introducing such parameters in a suitable way we shall construct an n-dimensional complex manifold M_n, given by

$$M_n \stackrel{\text{def}}{=} \{u_n(\cdot; t_0, \cdots, t_{n-1}) | (t_0, \cdots, t_{n-1}) \in \mathbb{C}^n\} \ .$$

The choice of $u_0 = 0$ yields the Adler-Moser potentials.

On the other hand, if one starts with

$$u_0 = \frac{l^2 - 1/4}{x^2} \ ,$$

one obtains a sequence of rational potentials only up to a certain number of Darboux transformations and with specific choices of the parameter. In this case one obtains a class of rational potentials that are not rational solutions of KdV. However, they do possess the bispectral property. These potentials appeared for the first time in the work of Duistermaat and Grünbaum [33] in connection with the bispectral problem.

Let A denote the first-order differential operator

$$A = \partial_x - v \tag{3.55}$$

and $A^\dagger$ its formal adjoint

$$A^\dagger = -\partial_x - v \ . \tag{3.56}$$

If we take

$$v = \chi'/\chi \ , \tag{3.57}$$

where χ is a solution of

$$L\chi = (-\partial_x^2 + u)\chi = 0 \ ,$$

then

$$L = A^\dagger A \ . \tag{3.58}$$

We start from L_0 as in equation (3.54), and χ_0 a nonzero element in $\ker L_0$. This gives a factorization

$$L_0 = A_0^\dagger A_0$$

with A_0, $A_0^\dagger$ and v as in equations (3.55), (3.56), and (3.57). Take

$$L_1 \stackrel{\text{def}}{=} A_0 A_0^\dagger \ .$$

A simple computation shows that $L_1 = -\partial_x^2 + u_1$, with

$$u_1 = u_0 - 2\partial_x^2 \log \chi_0 \ .$$

Suppose we have found pairs $(\chi_0, L_0), \cdots, (\chi_{n-2}, L_{n-2})$ such that

$$L_j = A_j^\dagger A_j \ , \text{ for } 0 \le j \le n-2 \ , \tag{3.59}$$

and

$$L_j = A_{j-1} A_{j-1}^\dagger \ , \text{ for } 0 < j \le n-1 \ . \tag{3.60}$$

We take $\chi_{n-1} \in \ker L_{n-1} \backslash \{0\}$, set

$$A_{n-1} \stackrel{\text{def}}{=} \partial_x - v_{n-1} \ , \tag{3.61}$$

$$v_{n-1} \stackrel{\text{def}}{=} \chi'_{n-1}/\chi_{n-1} \ , \tag{3.62}$$

and

$$L_n \stackrel{\text{def}}{=} A_{n-1} A_{n-1}^\dagger \ . \tag{3.63}$$

Notice that our choice of $\chi_{n-1} \in \ker L_{n-1}$ gives $L_{n-1} = A_{n-1}^\dagger A_{n-1}$. We can illustrate this process via the following diagram:

$$\begin{array}{c} L_0 \to \chi_0 \to A_0 \\ \swarrow \\ L_1 \to \chi_1 \to A_1 \\ \swarrow \\ \vdots \\ L_{n-2} \to \chi_{n-2} \to A_{n-2} \\ \swarrow \\ L_{n-1} \to \chi_{n-1} \to A_{n-1} \end{array}$$

We define a sequence of operators $\{U_n\}_{n=0}^\infty$ as follows:

$$U_0 = 1$$

and, for $n > 0$,

$$U_n \stackrel{\text{def}}{=} A_{n-1} U_{n-1} \ . \tag{3.64}$$

The operator L_n is related to L_0 by means of U_n

$$L_n U_n = U_n L_0 \ . \tag{3.65}$$

So, U_n intertwines the operators L_n and L_0. We remark that this formalism follows very closely that of [3], which starts from $-\partial_x^2$ and generates the potentials in the manifold of rational solutions of KdV.

The attentive reader might have noticed that the previous construction of L_n is not complete if we do not have a mechanism for generating the elements in the kernel of L_{n-1}. Fortunately, such mechanism is given by the formula

$$\chi_{n-1} = \frac{\alpha}{\chi_{n-2}} + \frac{\beta}{\chi_{n-2}} \int^x \chi_{n-2}^2(s) ds \ , \tag{3.66}$$

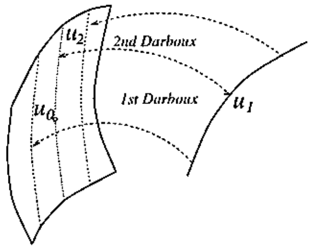

Fig. 3.5. Darboux's method depicted, introducing one parameter on the first iteration and another one on the second. Notice after the first iteration we have a one dimensional manifold and after the second we get a two dimensional one.

where

$$\chi_{n-2} \in \ker L_{n-2} \ ,$$

and $\alpha, \beta \in \mathbb{C}$. The proof of this is elementary. Indeed, from $L_{n-1} = A_{n-2}A_{n-2}^{\dagger}$ it follows that $1/\chi_{n-2} \in \ker L_{n-1}$. Equation (3.66) is just the requirement that the Wronskian of χ_{n-1} and $1/\chi_{n-2}$ be a constant.

Since

$$v_{n-1} = \frac{\chi_{n-1}'}{\chi_{n-1}} \tag{3.67}$$

and

$$u_n = v_{n-1}^2 - v_{n-1}' \ , \tag{3.68}$$

it follows that each step of the Darboux method introduces a new parameter, which is essentially the ratio β/α in equation (3.66). In Figure 3.5 we depict this process of introducing extra parameters in the potential through Darboux transformations. In the forthcoming construction, instead of using (3.66) we shall actually exhibit elements in the kernel of L_{n-1}.

A Few Facts

Let $U_n^{\dagger}$ denote the formal adjoint of the operator U_n defined in (3.64). Then, we have the following elementary fact

Lemma 3.28 *For* $n \geq 0$,

$$L_0^n = U_n^{\dagger} U_n \tag{3.69}$$

and

$$L_n^n = U_n U_n^{\dagger} \ . \tag{3.70}$$

Proof: We first prove (3.70) by induction. Suppose that

$$L_{n-1}^{n-1} = U_{n-1} U_{n-1}^\dagger ,$$

then

$$\begin{aligned} L_n^n &= (A_{n-1}A_{n-1}^\dagger)\cdots(A_{n-1}A_{n-1}^\dagger) \\ &= A_{n-1}(A_{n-1}^\dagger A_{n-1})\cdots(A_{n-1}^\dagger A_{n-1})A_{n-1} \\ &= A_{n-1}U_{n-1}U_{n-1}^\dagger A_{n-1}^\dagger \\ &= U_n U_n^\dagger . \end{aligned}$$

To prove (3.69), equation (3.65) implies that $L_n^n U_n = U_n L_0^n$, and hence $U_n U_n^\dagger U_n = U_n L_0^n$. But since U_n is a monic differential operator and the cancellation law holds for such operators [60], it follows that $U_n^\dagger U_n = L_0^n$. Q. E. D.

Lemma 3.28 implies that the process of applying n Darboux transformations leads to a factorization of L_0^n. It also implies that $\ker U_n \subset \ker L_0^n$. From equation (3.65) it follows that $\ker U_n$ is invariant by L_0, and in fact is nilpotent of degree less than n in this space.

The intertwining operator U_n of equation (3.64) is monic and therefore can be written as [60]

$$U_n \varphi = \frac{\mathrm{Wr}[\varphi_1, \cdots, \varphi_n, \varphi]}{\mathrm{Wr}[\varphi_1, \cdots, \varphi_n]} , \tag{3.71}$$

where $\mathrm{span}\{\varphi_1, \cdots, \varphi_n\} = \ker U_n$ and Wr denotes the Wronskian determinant. If we compare both sides of the equation $L_n U_n = U_n L_0$, we obtain that

$$u_n = u_0 - 2\partial_x^2 \log W_n , \tag{3.72}$$

where $W_n = \mathrm{Wr}[\varphi_1, \cdots, \varphi_n]$. Here, $\varphi_i \in \ker U_n \subset \ker L_0^n$.

The next two subsections deal with the particularization of the the Darboux method to two situations that are directly relevant to the bispectral problem. The first one turns out to be in direct connection with the rational solutions of the KdV hierarchy. The link of the second one with the KdV hierarchy became more transparent in [110].

The Adler-Moser Polynomials

If one iterates the Darboux process starting from $u_0 = 0$ one obtains from the results of the previous section that $u_n = -2\partial_x^2 \log W_n$. The polynomial W_n, after a suitable choice of parameters and normalization, is nothing more than the n-th Adler-Moser polynomial. We shall now describe a few more results from [3] about this remarkable family of polynomials. The first remark is that the choice of parameters in the sequence of Darboux transformations can be made so that

$$u_n = -2\partial_x^2 \log \vartheta_n(x + t_1, t_3, ..., t_{2n-1}) ,$$

satisfies the flows of the KdV hierarchy, i.e.,

$$\partial_{t_k} u_n = X_k[u_n] .$$

The sequence ϑ_n satisfies the following properties:

1. The recursion relation

$$\vartheta'_{n+1}\vartheta_{n-1} - \vartheta'_{n-1}\vartheta_{n+1} = (2n+1)\vartheta_n^2 \ . \tag{3.73}$$

2. As a polynomial in x, ϑ_n is monic and of degree $\delta_n = \frac{n(n+1)}{2}$.

3. The operator $L_n = -\partial_x^2 + u_n$ can be factored as

$$L_n = A_n^\dagger A_n \ , \tag{3.74}$$

where

$$A_n = \partial_x - v_n \ , \tag{3.75}$$

$$v_n = \partial_x \log \phi_n \ , \tag{3.76}$$

and

$$\phi_n = \frac{\vartheta_{n+1}}{\vartheta_n} \ . \tag{3.77}$$

Exercise 3.29 *The Adler-Moser polynomials satisfy*

$$\frac{\vartheta''_{j+1}}{\vartheta_{j+1}} + \frac{\vartheta''_j}{\vartheta_j} - 2\frac{\vartheta'_j}{\vartheta_j}\frac{\vartheta'_{j+1}}{\vartheta_{j+1}} = 0 \ . \tag{3.78}$$

The Even Family

We conclude this section by particularizing to the case

$$u_0 = \frac{l^2 - 1/4}{x^2} \ .$$

They were first studied in connection with the bispectral problem by Duistermaat and Grünbaum. In this case, it turns out that only suitable Darboux transformations give rational functions. They are, however, even functions of x.

The main goal is to give part of the argument leading to the following result:

Proposition 3.30 *The potential u_n obtained by applying n Darboux transformations to u_0, using at each step a dominant solution can be written for any $n \geq 1$ as*

$$u_n = \frac{l^2 - 1/4}{x^2} - 2\partial_x^2 \log W_n \tag{3.79}$$

with

$$W_n = \mathrm{Wr}[\psi_0^- + t_0\psi_0^+, \cdots, \psi_{n-1}^- + t_0\psi_{n-1}^+ + \cdots + t_{n-1}\psi_0^+] \ , \tag{3.80}$$

where the functions $\psi_i^\pm$ are polynomials in $x^{1/2}$ and $\log(x)$. Furthermore, the asymptotic behavior of u_n at $x = \infty$, in a sector of angle smaller than 2π near to infinity, is $u_n = \mathcal{O}(1/x^2)$.

We start by exhibiting sequences $\{\psi_j^{\pm}\}_{j=0}^{l-1}$ of functions satisfying

$$L_0\psi_j^{\pm} = \psi_{j-1}^{\pm} \tag{3.81}$$

with

$$\psi_0^{\pm} = x^{\pm l+1/2} \in \ker L_0 \ .$$

For $j \leq l$, the trick lies in noticing that[3]

$$L_0 x^{2j\pm l+1/2} = [(2j \pm l)^2 - l^2]x^{2(j-1)\pm l+1/2} \ .$$

We define $\mu_0^{\pm} = 1$ and

$$\mu_j^{\pm} = \frac{1}{(2j \pm l)^2 - l^2}\mu_{j-1}^{\pm} \ , \tag{3.82}$$

for $1 \leq j \leq l-1$. Thus, we have that

$$\psi_j^{\pm} \stackrel{\text{def}}{=} \mu_j^{\pm} x^{2j\pm l+1/2} \ , \tag{3.83}$$

satisfies equation (3.81) for $j \leq l-1$. To extend this for $j \geq l$ we have to look at the "+" and the "−" cases separately. In the "+" case, equations (3.82) and (3.83) can be used to define ψ_j^+ for every $j \geq 1$. To extend the definition of ψ_j^-, for $j \geq l$, firstly we define α_i and β_i recursively by setting $\alpha_0 = \mu_{l-1}^-/(-2l)$, $\beta_0 = 0$, and for $i \geq 1$:

$$\begin{aligned} \alpha_i &= \frac{-1}{4i(l+i)}\alpha_{i-1} \\ \beta_i &= \frac{-1}{4i(l+i)}\beta_{i-1} - \frac{(l+2i)}{2i(l+i)}\alpha_i \ . \end{aligned}$$

Secondly, we take, for $i \geq 0$,

$$\psi_{l+i}^- \stackrel{\text{def}}{=} \alpha_i x^{l+2i+1/2}\log x + \beta_i x^{l+2i+1/2} \ .$$

It is easy to check that

$$L_0\psi_{l+i}^- = \psi_{l+i-1}^- \ . \tag{3.84}$$

Using equations (3.81) and (3.84), it is straightforward to show that $\{\psi_l^{\pm}\}_{l=0}^{n-1}$ is a basis for $\ker L_0^n$.

3.5 Painlevé Property

The problem of constructing, at least theoretically, solutions to systems of equations of the form

$$y^{(m)} = F(z, y^{(0)}, \ldots, y^{(m-1)}) \tag{3.85}$$

[3] This remark is due to P. Wright [102]

with F an analytic function far from a singularity is settled by Cauchy's Method of Majorants [60]. If F is linear on $(y^{(0)}, \dots, y^{(m-1)})$, then the solutions to the Cauchy problem with initial data

$$\begin{cases} y(z_0) = \eta_0 \\ \vdots \\ y^{(m-1)}(z_0) = \eta_{m-1} \end{cases} \tag{3.86}$$

can be extended in a radius as big as the distance from z_0 to the closest singularity of $F(z, y^0, \dots, y^{(m-1)})$ *independently* of what the initial data $(\eta_0, \dots, \eta_{n-1})$. However, the picture changes dramatically in the nonlinear case. Even extremely simple equations such as

$$y' = -y^2 \tag{3.87}$$

have singularities depending on the initial data. Indeed, its general solution is given by

$$y(z) = \frac{1}{z+c},$$

where $c = y_0^{-1} - z_0$ and $y_0 \in \mathbb{C}$. Obviously, the singularities develop at $z = -c$

Definition 3.31 *A singularity of family of solution $y(z, z_0)$ of the system of equations (3.85) with initial data (3.86) is called movable if its location depends on the initial value $(y(z_0) = \eta_0, \dots y(z_{n-1}) = \eta_{n-1})$.*

Example 3.32 The singularities of the equation

$$\frac{dy}{dz} = -\frac{z}{y} \tag{3.88}$$

are movable.

Given the fact that for nonlinear equations on the complex domain, *movable* singularities are a fact of life, it is natural to ask:

Which equations have only the nicest singularities? i.e.: Which ones have only singularities that are of the form of poles (and not weirder things as essential singularities or branch cuts)?

This theme was an important one in the end of last century, when a great deal of research about it was developed. For a comprehensive set of references see the masterpiece [60]. Unfortunately, a lot of it was almost forgotten. Among the contributors to the field, the name of Painlevé stands out in his characterization of second order equations whose movable singularities are only poles. This leads to the definition:

Definition 3.33 *An analytic ODE of the form (3.85) is said to possess the Painlevé property if the only movable singularities of its solutions are poles (and not branch cuts or essential singularities).*

Example 3.34 Equation (3.87) obviously has the Painlevé property, whereas equation (3.88) does not.

Painlevé (See [85, 60] and references therein) studied second order equations that possess the above property of the form

$$y'' = F(z, y, y') ,$$

where F is rational in y and y' and analytic in z. His result was a long list of equations, most of them transformable to equations which are integrable in terms of "known" functions. However for six of those equations this was not the case, they form the so called Painlevé transcendents.

The theory of integrable systems saw a revival of the interest on this subject due to the fact that self-similarity solutions of completely integrable partial differential equations led to Painlevé equations.

Example 3.35 Take the modified-KdV equation

$$v_t = 6v^2 v_x - v_{xxx} . \tag{3.89}$$

Search for a solution of the form

$$v(x,t) = \frac{1}{(3t)^{1/3}} w(\frac{x}{(3t)^{1/3}}) . \tag{3.90}$$

Then, w must satisfy

$$w_{zz} = 2w^3 + zw + \alpha , \tag{3.91}$$

where α is an integration constant. Equation 3.91 is exactly the second Painlevé transcendent, as can be found for example on page 345 of [60].

A vast amount of experimental evidence suggested a strong correlation between complete integrability of a certain PDEs and the Painlevé property of its self-similar reductions. In fact, many interesting results were proved by Ablowitz, Fokas, and collaborators. See [100] and references therein. The need then arose to a generalization of the notion of the Painlevé property that would be suited to PDEs.

The effort to understand the link between completely integrable systems and the Painlevé property led Weiss, Tabor, and Carnevale to propose a definition that would be independent of the reduction considered.

Now, in several dimensions, singularities of analytic functions are determined by (complex) hypersurfaces defined by equations of the form

$$M_\phi = \{(x_0, \ldots, x_n) \mid \phi(x_0, x_1, \ldots, x_n) = 0\} , \tag{3.92}$$

where 0 is a regular value of ϕ. See [84].

In analogy with the ODE case, Weiss et al requested that the solutions with singularities along the surface M_ϕ be single valued. More precisely, they asked that solutions with singularities along (3.92) to have the form

$$\begin{cases} U = \phi^\alpha \sum_{k=0}^{\infty} U_k \phi^k \\ \alpha \in \mathbb{Z} \\ U_k \text{ analytic.} \end{cases} \tag{3.93}$$

Definition 3.36 *A PDE is said to possess the Painlevé property if for every regular hypersurface M_ϕ defined by (3.92) and for every u with singularities on M_ϕ, the solution u is of the form (3.93).*

Example 3.37 Burgers' equation

$$U_t + UU_x = \sigma U_{xx} \ . \tag{3.94}$$

Indeed, following [100], if one imposes u to be of the form

$$U = \phi^\alpha \sum_{k=0}^{\infty} U_k \phi^k \ , \tag{3.95}$$

with ϕ, U_j analytic near $M = \{(t,x) | \phi(t,x) = 0\}$, then matching the asymptotic behavior of the leading order term gives $\alpha = -1$.

Exercise 3.38 *Show that the coefficients U_j in this case satisfy the recursion relation*

$$\begin{aligned} U_{j-2,t} + (j-2)U_{j-1}\phi_t \quad &+ \quad \sum_{m=0}^{j} U_{j-m}[U_{m-1,x} + (m-1)\phi_x U_m] \\ = \sigma[U_{j-2,xx} + 2(j-2)U_{j-1,x}\phi_x \quad &+ \quad (j-2)U_{j-1}\phi_{xx} + (j-1)(j-2)U_j\phi_x^2] \ . \end{aligned}$$

Conclude that one has the recursion relation for $j = 0, 1, 2 \ldots$

$$\sigma\phi_x^2(j-2)(j+1)U_j = R_j \ ,$$

where R_j depends on ϕ, U_k for $k < j$ and their derivatives.

The multidimensional sine-Gordon equation was studied in [99], and it was pointed out the need of restricting the singular manifold only to certain families. This led to the following notion of *relative* Painlevé property:

Definition 3.39 *A PDE is said to possess the Painlevé property w.r.t. a certain family $\mathcal{M}$ of hypersurfaces if $\forall M \in \mathcal{M}$ and every u with singularities in M, the solution is single valued of the form (3.93).*

Recall that in the case of a hyperbolic equation, the singularities are transmitted along characteristic manifolds, which in this case are of the form

$$\Gamma(x, \xi) = 0 \ .$$

At the light of the definition of the relative Painlevé property one can restate Hadamard's criterion (Theorem 2.11) in the form of the following result:

Theorem 3.40 *A hyperbolic operator $\mathcal{L}$ satisfies a strict Huygens' principle iff its adjoint $\mathcal{L}^{\dagger}$ possesses a Painlevé property relative to the family of characteristic conoids.*

4. Bispectrality

In this chapter we discuss in more detail the bispectral problem, which was initially posed by A. Grünbaum. We start with a little bit of motivation, although much more could be said about it. For the early history see [40, 41, 43, 45, 46].

4.1 Motivation to Bispectrality: Band and Time Limiting

The analysis of the reconstruction problem in limited angle tomography leads to the analysis of certain integral operators. To obtain information about the accuracy in the reconstruction it is necessary to obtain the eigenvalues, and in some cases also the eigenfunctions, of such operators. Even numerically such problems could become intractable due to the size of the (full) matrices involved. It turns out, however, that in some cases it is possible to find a differential operator of simple spectrum with such the original operator. In this case, one is capable of using sparse matrix techniques to handle the eigenvalue problem of the discretized problem. For an account of the tomographic problem see [44, 42, 31].

To illustrate the point of the previous paragraph, we shall discuss a beautiful analysis developed in a series of papers by Slepian and collaborators[95, 92, 93, 94] The motivation comes from Communications Theory, which was the underlying application Slepian et al had in mind [94].

Communications Theory uses as a main tool the Fourier transform. Given a signal $g(t) \in L^2(\mathbb{R})$ its Fourier transform is given by[1]

$$\hat{g}(\omega) = \mathcal{F}[g](\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) e^{i\omega t}\, dt \,.$$

The physical interpretation of $|\hat{g}(\omega)|^2$ being the density of energy at the (angular) frequency ω.

When signals are transmitted along physical channels, even under fairly ideal situations, they are forced into some kind of band limiting. Let's suppose that the channel does not allow frequencies above the frequency Ω. This amounts to a multiplication by the characteristic function $\chi_{[-\Omega,\Omega]}$ of the interval $[-\Omega, \Omega]$.

[1]This integral being taken in the sense of an L^2 limit.

It turns out that such signals are also limited by a finite duration. This amounts for a multiplication by a characteristic function of the interval $[-T, T]$. Hence, it is a natural question to ask how good is the reproduction after the following process:

$$g \mapsto \chi_{[-T,T]}g \mapsto \mathcal{F}[\chi_{[-T,T]}g] \mapsto \chi_{[-\Omega,\Omega]}\mathcal{F}[\chi_{[-T,T]}g] \ . \tag{4.1}$$

The corresponding integral operator is given (modulo a multiplicative constant) by

$$(\mathsf{A}g)(\lambda) = \begin{cases} \int_{-T}^{T} e^{i\lambda t} g(t)dt & \lambda \in [-\Omega, \Omega] \\ 0 & \lambda \in \mathbb{R} \setminus [-\Omega, \Omega] \end{cases}$$

The problem of analyzing the quality of the reconstruction of the signal then boils down to studying the singular value decomposition of A, i.e., studying the eigenvalues and eigenvectors of the operator A^*A.

Exercise 4.1 *Show that*

$$(\mathsf{A}^*\mathsf{A}g)(t) = \begin{cases} \int_{-T}^{T} \frac{\sin \Omega(t-s)}{t-s} g(s)ds & t \in [-T, T] \\ 0 & t \in \mathbb{R} \setminus [-T, T] \end{cases} \tag{4.2}$$

and that $\mathsf{T} \stackrel{\text{def}}{=} \mathsf{A}^*\mathsf{A}$ *defines a compact self-adjoint operator on* $L^2(\mathbb{R})$.

We remark that numerically the problem of computing the eigenvalues of $\mathsf{T} \stackrel{\text{def}}{=} \mathsf{A}^*\mathsf{A}$ is not a trivial one, since this operator when discretized gives a full matrix.

At this point a "miracle" occurs. Slepian, Landau, and Pollak found a *differential operator* L, with simple spectrum, that commutes with T. Hence one can compute the eigenfunctions (and eigenvalues) of T by computing the corresponding eigenfunction of L. More specifically, the differential operator

$$\mathsf{L} : g \mapsto \frac{d}{dt}\left((T^2 - t^2)\frac{d}{dt}g\right) - \Omega^2 t^2 g \ .$$

has the property that

$$[\mathsf{L}, \mathsf{T}] = 0 \ .$$

Furthermore, L has simple spectrum.

Exercise 4.2 *Show that if an operator* L *with simple point spectrum at* $\lambda \neq 0$ *commutes with* T, *then it shares the* λ*-eigenspace with* T.

In the references [92, 93] different extensions of this were found in the case of discretizations of time and frequency.

Obviously, the above situation is quite exceptional. However, its consequences in communications theory were interesting enough to justify the question of what are the most general kernel one can have with the "commuting property."

Grünbaum studied several instances where generalized Fourier transformations yielded the commuting property.[2]

For instance, take the familiar harmonic oscilator operator of quantum mechanics

$$\mathsf{H} = -\frac{d^2}{dx^2} + x^2 \ .$$

It leads to a decomposition of $L^2(\mathbb{R})$ in terms of products of Hermite polynomials by $\exp(-x^2/2)$. More precisely, we recall that the k-th Hermite polynomial given by

$$H_k(x) \stackrel{\text{def}}{=} (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \quad k = 0, 1, 2, \dots \ ,$$

and

$$h_k \stackrel{\text{def}}{=} H_k(x) e^{-x^2/2} \ .$$

The function h_k satisfies,

$$\mathsf{H} h_k = (2k+1) h_k \ .$$

For such results, a good reference is [70].

Exercise 4.3 *Show that the Hermite polynomials satisfy the three-term recursion relation*

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \qquad n = 1, 2 \dots \tag{4.3}$$

The Fourier analysis corresponds, in this case, to decomposing the given function in the basis of $L^2(\mathbb{R})$ formed by the orthonormal set $\{h_k\}_{k=0}^{\infty}$. So we would write an arbitrary function $g \in L^2(\mathbb{R})$ as

$$g = \sum_{k=0}^{\infty} b_k h_k \ .$$

As before, the physical interpretation of $|b_k|^2$ is that of the energy at level k. In this case the "frequency" space is discrete.

The analog of the kernel $\sin(\Omega(x-y))/(x-y)$, which appeared in equation (4.2), is given by the kernel

$$h(x,y) \stackrel{\text{def}}{=} \sum_{k=0}^{N} h_k(x) h_k(y) \ .$$

F.A. Grünbaum has shown the following result:

Theorem 4.4 *(F.A. Grünbaum [46]) For every N there exists a differential operator commuting with the integral operator with kernel $h(x,y)$.*

For every T there exists a tridiagonal matrix (with simple spectrum) commuting with the matrix

[2] We substitute the usage of t for x from now on to conform with the standard practice.

$$(G_{ij}(T))_{1\le i,j\le N} \ ,$$

where $G_{ij}(T) = \int_{-\infty}^{T} h_i(\xi) h_j(\xi) d\xi$.

The proof of this result (though not given here) turns out to be related to the fact that the Hermite polynomials satisfy a *three-term recursion relation.* Grünbaum also verified that the above mentioned property of commutativity is true whenever one works with one of the so called classical orthogonal polynomials. (Jacobi, Laguerre, Bessel, Hahn, etc.) See [46] and references therein.

In conclusion, the instances where the "commuting property" held were associated to the fact the family of eigenfunctions satisfied both an eigenvalue problem of the form

$$L(x)\varphi = \lambda\varphi \ ,$$

and another one of the form

$$B(\lambda)\varphi = \Theta(x)\varphi \ .$$

Here, L and B are either differential operators or difference operators, depending on whether we are in a continuous-continuous situation (s.a. the classical Fourier transform mentioned first), or a continuous-discrete situation, like the situation discussed in Theorem 4.4.

The simplest case seems to be the continuous-continuous one. In fact, it is the best understood so far. From now on we shall concentrate in it. The reader interested in the discrete cases should consult the recent work [47] and references therein.

In the continuous case, the *bispectral problem* could be rephrased as follows:

Problem *Find all instances of differential operators* $L(x, \partial_x)$ *such that there exists a family of eigenfunctions* $\varphi(x, \lambda)$ *satisfying simultaneously the equation*

$$L(x, \partial_x)\varphi = \lambda\varphi \ .$$

and a differential equation in the spectral parameter of the form

$$B(\lambda, \partial_\lambda)\varphi = \Theta(x)\varphi \ .$$

It is crucial that one normalizes the operators somehow. Otherwise, one could produce repetitious classes of bispectral operators just by multiplying the above examples, say, by functions of x alone.

This could be done for example, requiring that the differential operator L be constant and its second highest coefficient vanish. This could always be accomplished by Liouville's change of variables. If we are dealing with second order operators the final result is an operator in Schrödinger's form:

$$L = -\partial_x^2 + u(x) \ .$$

Definition 4.5 *Let $L(x, \partial_x)$ and $B(x, \partial_k)$ be differential operators of positive order and $\Theta(x)$ a smooth function independent of k. The triple (L, B, Θ) is called bispectral if there exists a family $\{\varphi(x, k)\}$ such that*

$$L\varphi = \lambda(k)\varphi \, , \tag{4.4}$$

for some non-constant smooth function $\lambda(k)$ and

$$B\varphi = \Theta(x)\varphi \, . \tag{4.5}$$

A few examples are in order.

Example 4.6 The simplest one is given by

$$\varphi(x, k) = \exp(ikx) \, .$$

In this case one obviously has

$$-\partial_x^2\varphi = k^2\varphi \, .$$

and a similar equation swapping the roles of x and k.

This example, is a particular case of a more general one given by the next example.

Example 4.7 Take $f(z)$ a solution of Bessel's equation

$$f_{zz} + \frac{c}{z^2}f = f \, ,$$

and set

$$\varphi(x, k) = f(xk) \, .$$

Then, obviously φ is bispectral.

Yet another trivial example is supplied by taking $f(z)$ a solution of Airy's equation

$$f_{zz} = zf \, ,$$

and taking

$$\varphi(x, k) = f(x + k) \, .$$

So far all the above examples are rather trivial and mathematically uninteresting since they were obtained by means of switching the roles of x and k. One of the first miracles in the bispectral problem is that besides the potentials described in the above trivial examples, one also has the rational solutions of KdV, i.e., if $u(x) = -2\partial_x^2 \log \vartheta_n(x)$, then u is bispectral.

The maximal dimension of the space of common eigenfunctions of equations (4.4) and (4.5) is called the *rank* of the bispectral triple (L, B, Θ). All the above examples led to rank 2 triples. It was shown in [33] that generic KdV potentials have rank 1.

We close this section with a lemma that played a crucial role in the work of [33]. We recall the definition of the ad-operator from Lie Algebra theory.

$$\mathrm{ad}_L(M) \stackrel{\text{def}}{=} [L, M] \ .$$

Lemma 4.8 *(Duistermaat-Grünbaum[33]) If (L, B, Θ) is a bispectral triple with $B(k, \partial_k)$ a differential operator of order m then*

$$(\mathrm{ad}_L)^{m+1}\, \Theta = 0 \ .$$

The proof of this result is a direct consequence of the fact that the compatibility condition of (4.4) and (4.5) is given by

$$[L, \Theta]\varphi = -[\lambda, B]\varphi \ .$$

Hence,

$$(\mathrm{ad}_L)^j(\Theta)\varphi = (-1)^j(\mathrm{ad}_\lambda)^j(B)\varphi \ . \tag{4.6}$$

Without loss of generality, we can assume that we have performed a change of variables so that $B = B(\lambda, \partial_\lambda)$. In this case, the order of the an operator in the variable λ decreases by one each time one performs the operation ad_λ. Since $B(\lambda, \partial_\lambda)$ is of order m, when $j = m + 1$ the RHS of equation (4.6) vanishes. Since, $(\mathrm{ad}_L)^{m+1}\, \Theta$ is a differential operator independent of λ the result follows. *Q. E. D.*

Exercise 4.9 *Let's consider the symbol of the operator $B(\lambda, \partial_\lambda)$, i.e., the polynomial*

$$B(\lambda, \xi) = b_0(\lambda)\xi^m + \ldots + b_m(\lambda)\xi^0 \ .$$

Show that the operator on the LHS of equation (4.6) is given by

$$(\mathrm{ad}_\lambda)^j(B) = \left.\frac{\partial^j}{\partial \xi^j}\right|_{\xi=\partial_\lambda} B(\lambda, \xi) \ .$$

Conclude from the previous exercise that if L is bispectral then (at least formally) it satisfies a *generalized string equation*, i.e., an equation of the form [3]

$$[L, P] = f(L) \ , \tag{4.7}$$

where $P = (\mathrm{ad}_L)^{m-1}(\Theta)$, and $f(\cdot)$ is proportional to $b_0(\cdot)$.

Using Lemma 4.8 it can be shown that if $u(x)$ is bispectral, then it must be a rational function of x.

[3]The author is indebted to M. Pedroni and T. Shiota for a nice discussion leading to this remark.

4.2 The Results of Duistermaat and Grünbaum

In this section we focus on the bispectral problem for Schrödinger operators. We write one of these operators as

$$L = -\partial_x^2 + u \; . \tag{4.8}$$

We shall say that $u(x)$ is a *bispectral potential* if the operator L of equation (4.8) is bispectral.

Our first goal in this section is to report the full description of the class of bispectral Schrödinger operators. It was first given in [33] with the help of Darboux transformations, which were described in Section 3.4. Before we discuss this description we should make two remarks.

The first remark concerns the fact that one can show that the bispectral potentials are necessarily rational functions of x. This results comes as a byproduct of a detailed analysis of the coefficients of the operator in Lemma 4.8. See [33] for details.

The second remark concerns the trivial symmetries possessed by the class of bispectral potentials.

- Translations in x, i.e., transformations of the form

  $$u(x) \mapsto u(x+c) \; ,$$

  where c is a constant.

- Scaling in the variable x, i.e., transformations on the variable x of the form

  $$x \mapsto \varepsilon x \; ,$$

  which induces the transformation in u of the form

  $$u(x) \mapsto \varepsilon^2 u(\varepsilon x) \; .$$

- Addition of a constant to the potential

  $$u(x) \mapsto u(x) + c \; .$$

We are now ready to state the main result in [33].

Theorem 4.10 *(Duistermaat and Grünbaum [33]) Modulo the trivial symmetries mentioned above, the bispectral potentials are*

$$u(x) = x \; , \tag{4.9}$$

or

$$u(x) = \frac{c}{x^2} \; , \tag{4.10}$$

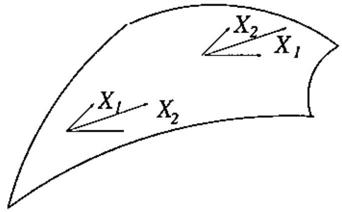

Fig. 4.1. Flows of the KdV hierarchy tangent to the rank one bispectral potentials.

or those obtained from $u(x) = 0$ *or from*

$$u(x) = -\frac{1}{4x^2} \tag{4.11}$$

by means of finitely many rational Darboux transformations.

The proof of this result is fairly technical, and goes beyond the scope of the present work. We refer the reader to [33] for the original proof. For more recent approaches and extensions see [107, 101].

As stated above the most interesting classes of bispectral potentials are not explicit. They are exactly the ones obtained by applying Darboux transformations. We shall now make a few comments about these.

The first class of nontrivial bispectral potentials is exactly the rational solutions of the KdV hierarchy. Indeed, they are obtained by applying Darboux transformations to $u = 0$ and because of the results of Section 3.4 they are composed of potentials of the form

$$u = -2\partial_x^2 \log \vartheta_n ,$$

where ϑ_n is the n-th Adler-Moser potential.

From a geometric viewpoint, they are integral surfaces of the first n flows of the KdV hierarchy, and stationary solutions of all the higher order ones. See Figure 4.1. They have in their closure potentials of the form given in equation (4.10) with $c = \nu(\nu - 1)$ where $\nu \in \mathbb{Z}_{\geq 1}$. However, except for lower dimensional sets in the space of parameters, they are not of the form (4.10). For this reason the potentials in this class are called the KdV family.

From the point of view of bispectral operators, the potentials in the KdV family can be singled out by the fact that they are generically of rank 1, in fact except for the situations when they degenerate into Bessel potentials, they have rank 1. The proof can be found in [33]. One could then say that the bispectral property is preserved by the KdV flows when we restrict ourselves to rank 1 bispectral potentials.

On the other hand, the potentials in the orbit of rational Darboux transformations issued from u_0 as in equation (4.11) is much more mysterious. As we

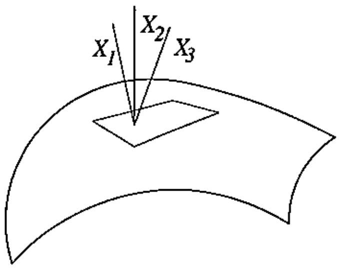

Fig. 4.2. Flows of the KdV hierarchy transversal to the rank 2 bispectral potentials.

mentioned before they were found in connection with the bispectral problem in [33].

They turn out to be always of *rank two* and even in x. Because of the latter property they are called the even family.

Potentials in the even family do not remain rational by the flows of KdV hierarchy. Hence, the KdV hierarchy does not preserve bispectrality in this context. See Figure 4.2. The following question, first posed by F. A. Grünbaum, becomes natural:

Problem 4.11 *Is there any interesting hierarchy of nonlinear evolution equations whose integral curves are the even-family bispectral potentials?*

The answer to this problem appeared in our joint work with F. Magri[110], which is described in the next section.

4.3 Virasoro Flows and Bispectral Potentials

The subject of the present section is the relationship between the family of master symmetries for the KdV hierarchy and the bispectral problem. In Section 3.3, using the Nijenhuis tensor N_u, we constructed the hierarchy of vector fields

$$\tau_j \stackrel{\text{def}}{=} \mathsf{N}_u^j \tau_0 , \tag{4.12}$$

where

$$\tau_0(u) = \frac{1}{2} x u_x + u.$$

We showed that this hierarchy satisfies a Virasoro type relation, and therefore it is natural to also call it the (non-negative) Virasoro hierarchy.

It turns out that the vector field τ_0 is clearly playing a role in the bispectral problem.

Fig. 4.3. On the left: one of the even family manifolds, with the KdV flows transversal. On the right: one of the KdV family manifolds, with the KdV flows in the tangent space. In both cases the Virasoro flows are tangential.

Exercise 4.12 *Show that the infinitesimal generator of the group of symmetries of scaling transformations is proportional to* $\tau_0(u)$.

What is not at all evident is that the full hierarchy $\{\tau_j\}_{k=0}^{\infty}$ plays a crucial role in the bispectral problem.

The following result answers the question posed in Problem 4.11

Theorem 4.13 *[110] The flows of the master symmetry hierarchy* $\{\tau_j\}_{k=0}^{\infty}$ *are tangent to the even-family bispectral potentials.*

In fact, it follows as a corollary of the proof presented in [110] that all the bispectral potentials decaying at infinity. We shall not repeat the proof of this result here. The interested reader can find it in [110].

In conclusion one can interpret the above result by saying that the bispectral potentials (decaying at infinity) are organized into nice finite dimensional manifolds which are invariant by the scaling vector field τ_0 and the translation one X_0. The iterates of the Nijenhuis tensor N_u applied to τ_0 yields a family of tangent vectors to these manifolds. On the other hand, the iterates of N_u to X_0 are tangent to the bispectral manifold only in the case of the KdV family. See Figure 4.3.

5. Connecting the Different Topics

5.1 Rational Solutions of KdV are Huygens' Potentials

In this section we describe the family of examples constructed by Lagnese and Stellmacher of strict Huygens's operators that are not trivially equivalent to wave operators. We shall not give all the proofs of their results, since their works are readily available, and very readable. Instead, we shall give a few proofs based on a point of view developed in very recent works by Yu. Yu. Berest [20]. We believe that the approach put forward in the latter is very promising.

The simplest example of the family of examples discovered by Lagnese and Stellmacher looks like

$$\mathcal{L} = \Box + u(x_0) \ , \tag{5.1}$$

where the potential u belongs to the class of rational potentials that remain rational by the flows of the KdV hierarchy and $\Box$ is the D'Alembertian operator in a suitable number of odd spatial dimension.

More precisely, u is a potential of the form

$$u_k(x_0) = 2\partial_{x_0}^2 \log \vartheta_k(x_0) \ , \tag{5.2}$$

where ϑ_k is the k-th Adler-Moser polynomial. The Adler-Moser polynomials are defined (Section 3) by the relations

$$\begin{aligned} \vartheta_0 &= 1 \\ \vartheta_1 &= x \\ \vartheta'_{k+1}\vartheta_{k-1} - \vartheta'_{k-1}\vartheta_{k+1} &= (2k+1)\vartheta_k^2 \ . \end{aligned}$$

The key step in the construction of such families of potentials was the Darboux method, which we described in Chapter 3. We emphasize, that in the context of wave operators the variable x_0 is the time variable. In fact, Lagnese and Stellmacher's construction also allows for more general potentials, that are a sum of Adler and Moser polynomials in the other spatial variables. Recently Berest and Winternitz generalized this result even further [19].

The main goal of this section is to give a general idea of the proof of the next result. Although the proof herein is not self-contained we hope to give a flavor of the main ideas behind it and to entice the reader to look into the literature. We recommend special attention to the elegant results of [20], from where most of the ideas in the present exposition have been transcribed.

Theorem 5.1 *(Lagnese and Stelmacher [68]) Let ϑ_k be the k-th Adler-Moser polynomial and u_k as in equation (5.2). If the number n of spatial dimensions is odd and $n \geq 3 + 2k$, then, the operator*

$$\mathcal{L} = \Box + u_k(x_0) \ , \tag{5.3}$$

is a strict Huygens operator.

Furthermore, Lagnese has shown that

Theorem 5.2 *(Lagnese [66]) If $\mathcal{L}$ is a strict Huygens operator of the form (5.1), and $u(x_0)$ is an analytic potential, then $\partial_{x_0}^2 + u$ is obtainable from $\partial_{x_0}^2$ by sequence of rational Darboux transformations.*

In other words, the Huygens potential u of Theorem 5.2 is typically of the form (5.2).

The proof of Theorem 5.1, given by Berest[20], hinges upon the introduction of a "quantum" (or non-commutative) version of the "ad" operator, which plays a fundamental role in Lie Algebra theory and was instrumental in the arguments of Duistermaat and Grünbaum. For any three (formal) operators L, L_0, and Θ we put [1]

$$\mathrm{ad}_{L,L_0}\Theta \stackrel{\mathrm{def}}{=} L \circ \Theta - \Theta \circ L_0 \ , \tag{5.4}$$

and its iterates are defined recursively by

$$\mathrm{ad}^k_{L,L_0}\Theta \stackrel{\mathrm{def}}{=} \mathrm{ad}_{L_0,L}[\mathrm{ad}^{k-1}_{L_0,L}\Theta] \tag{5.5}$$

for $k \geq 1$. Notice that if $L = L_0$ one obtains the usual "ad" operator of Lie algebra theory. Also, if $L = \sum_j^m a_j \partial_x^j$ is an ordinary differential operator and Θ is a nonconstant polynomial we obtain a necessary (and under certain conditions also sufficient) condition for L to be bispectral, which is

$$\mathrm{ad}^{m+1}_{L,L}\Theta = 0 \ .$$

Exercise 5.3 *Verify that (5.5) implies*

$$\mathrm{ad}^N_{L,L_0}\Theta = \sum_{k=0}^{N}(-1)^k \binom{N}{k} L^{N-k}\Theta L_0^k \ .$$

The following result sheds some light on the connection between strict Huygens operators and the Darboux transformation technique.

Lemma 5.4 *(Berest[20]) Let $\mathcal{L}_0$ be a terminating operator of the form*

[1] We use $\circ$ here to denote the composition of operators for the last time, henceforth the composition of operators should be assumed unless otherwise noted.

$$\mathcal{L}_0 = \Box + \langle a_0(x), \nabla \rangle + u_0$$

and $\mathcal{L}$ another operator of the same form, both of them defined on the same causal domain. Suppose that there exists a differential operator

$$\Theta(x,\xi) = \sum_{|\alpha| \leq m} b_\alpha(x,\xi)\partial_x^\alpha \tag{5.6}$$

such that for some $N \geq 1$

$$\mathrm{ad}^N_{\mathcal{L},\mathcal{L}_0}\Theta = 0 \ ,$$

and

$$\Theta[\delta_\xi] = \delta_\xi \ ,$$

for all $\xi \in \Omega$. Then, the operator $\mathcal{L}$ is also terminating.

Furthermore, if $\mathcal{L}_0$ is is p-terminating then $\mathcal{L}$ is $(p + N - 1)$-terminating.

We recall, from definition 2.16 that an operator $\mathcal{L}$ is said to be *terminating (at level ν_0)* if it admits an elementary solution whose Hadamard's coefficients of series (2.17) are all zero for $\nu \geq \nu_0$. Also, as before, δ_ξ denotes Dirac's delta measure at the point ξ.

For a proof of Lemma 5.4 we refer the reader to [20]. This result makes it natural to think of a further "discrete symmetry" of strict Huygens operators. Following Berest (oppus cit), we introduce the following:

Definition 5.5 *An operator $\mathcal{L}$ is said to be N-gauge related to $\mathcal{L}_0$ if there exists a* **nevervanishing** *function $\theta(x) \in C^\infty(\Omega)$, such that*

$$\mathrm{ad}^N_{\mathcal{L},\mathcal{L}_0}\theta = 0 \ , \tag{5.7}$$

for some $N \geq 1$.

We remark that if both $\mathcal{L}$ and $\mathcal{L}_0$ are formally self-adjoint operators, then $\mathcal{L}$ is N-gauge to related to $\mathcal{L}_0$ iff $\mathcal{L}_0$ is N-gauge to related to $\mathcal{L}$. Therefore, we have a symmetric and (obviously) reflexive relation in the class of formally self-adjoint operators.

Exercise 5.6 *Show that the property of being N-gauge related is also transitive in the above mentioned class and therefore defines an equivalence relation.*

We now give an idea of the proof of Theorem 5.1. To do that we use the following Lemma due to Berest, whose ingenious proof can be found in [20].

Lemma 5.7 *(Berest [20]) Let ϑ_k be the k-th Adler-Moser polynomial, and set*

$$N = \deg(\vartheta_k) + 1 \ .$$

Then,

$$\mathcal{L} = \mathcal{L}_0 + u_k(x_0) \ ,$$

is N-related to $\mathcal{L}_0$.

Exercise 5.8 *Show that the k-th Adler-Moser polynomial has degree* $k(k+1)/2$.

Now, once we have the result 5.7, we take

$$\mathcal{L}_0 = \partial_{x_0}^2 - \sum_{i=1}^{3} \partial_{x_i}^2 \ ,$$

which is 1-terminating. It follows from Lemma 5.4 that the operator

$$\mathcal{L}_k = \partial_{x_0}^2 - \sum_{i=1}^{n} \partial_{x_i}^2 + u_k(x_0) \ ,$$

is p-terminating, with $p = (n-1)/2$, provided $n \geq 3 + 2k$ is odd. This ends the proof of Theorem 5.1.

5.2 Iso-Huygens Deformations

In this section we shall discuss some of the results of Berest[15] on deformations preserving a strict Huygens' property of differential operators. We shall refer to such deformations as iso-Huygens deformations.

Here we shall make use of the theory of transformation groups, which is briefly reviewed in Appendix A. The goal is to describe the result of Berest that characterizes the algebra of iso-Huygens symmetries. As remarked in Section 4 this algebra coincides with the one that preserves the bispectral property of Duistermaat and Grünbaum.

Let's take $\mathcal{L} = \Box + u$ and consider the augmented system

$$\begin{cases} \mathcal{L}\psi = 0 \\ \partial_{x_i} u = 0 \quad \text{for } i = 1, \ldots, n \end{cases} . \tag{5.8}$$

The reason for adding the equations of the form $\partial_{x_i} u = 0$ is to force the deformed potentials to depend only on the time variable $x_0 = t$ and not on the spatial variables.

The first step in Berest's approach is to study the trivial point transformations of system (5.8). The family of trivial transformations of Definition 2.4 leads to vector fields of the form

$$X = \xi^0 \frac{\partial}{\partial t} + \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x_i} + \lambda(x,t)\psi \frac{\partial}{\partial \psi} + \gamma(x,t,u) \frac{\partial}{\partial u} \ ,$$

which must satisfy

$$X^{(2)} \mathcal{L}\psi = \mu \mathcal{L} \ , \tag{5.9}$$

where $X^{(2)}$ denotes the second extension of the vector field X, as explained in the Appendix A

Let's call the algebra of trivial iso-Huygens deformations g_0. Then, g_0 is generated by $\mathcal{Z}_0, \mathcal{X}_0, \mathcal{E}_0^i, \mathcal{T}_0, \Lambda_0^{ij}$ with these generators given as follows

$$\mathcal{Z}_0 = \psi\frac{\partial}{\partial\psi} \tag{5.10}$$

$$\mathcal{X}_0 = \frac{\partial}{\partial t} \tag{5.11}$$

$$\mathcal{E}_0^i = \frac{\partial}{\partial x_i} \quad i = 1, \dots n \tag{5.12}$$

$$\mathcal{T}_0 = \frac{1}{2}t\frac{\partial}{\partial t} + \frac{1}{2}\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} - \frac{1}{4}\psi\frac{\partial}{\partial\psi} + u\frac{\partial}{\partial u} \tag{5.13}$$

$$\Lambda_0^{ij} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} \quad 1 \le i < j \le n \tag{5.14}$$

Although the algebra g_0 above seems rather uninteresting, it is the crucial step in the construction of the full algebra g of iso-Huygens deformations. Indeed, Berest was able to construct a recursion operator, whose iterates give such algebra. In order to obtain g one has to allow more general transformations then the point transformations defining the trivial symmetries. In other words, one has to allow Lie-Bäcklund transformations. We briefly review the definition of the latter in the Appendix A.

Henceforth, in this section, we use the notation

$$\psi_i \stackrel{\text{def}}{=} \frac{D}{Dx_i}\psi\ , \qquad i = 1, \dots n\ ,$$

and

$$\psi_t \stackrel{\text{def}}{=} \frac{D}{Dt}\psi\ ,$$

where the D/Dx_i denotes the total derivative, as defined in the Appendix A.

We then have

Lemma 5.9 (Berest[15]) *The most general form of the canonical Lie-Bäcklund operator admitted by the system (5.8) is of the form*

$$X = (\widehat{A}\psi - \widehat{B}\psi_t - \sum_{i=1}^{n}\widehat{C}^i\psi_i)\frac{\partial}{\partial\psi} + \mu\frac{\partial}{\partial u}\ , \tag{5.15}$$

where

$$\widehat{A} = \sum_{k=0}^{\infty} a_k(x, t, u, u_t, \dots)\Delta^k\ , \tag{5.16}$$

with similar series for $\widehat{B}$ and $\widehat{C}$, and $\mu = \mu(x, t, u, u_t, \dots)$.

It turns out that the algebra g_0 coincides with the algebra generated by [2]

[2] Attention we are renaming the generators.

$$\mathcal{Z}_0 = \psi\frac{\partial}{\partial\psi} \tag{5.17}$$

$$\mathcal{X}_0 = \psi_t\frac{\partial}{\partial\psi} + u_t\frac{\partial}{\partial u} \tag{5.18}$$

$$\mathcal{E}_0^i = \psi_i\frac{\partial}{\partial\psi} \quad i = 1, \dots n \tag{5.19}$$

$$\mathcal{T}_0 = (\frac{1}{2}t\psi_t + \frac{1}{2}\sum_{i=1}^{n} x_i\psi_i - \frac{1}{4}\psi)\frac{\partial}{\partial\psi} + (u + \frac{1}{2}tu_t)\frac{\partial}{\partial u} \tag{5.20}$$

$$\Lambda_0^{ij} = (x_i\psi_j - x_j\psi_i)\frac{\partial}{\partial\psi} \quad 1 \le i < j \le n \tag{5.21}$$

Let's define the recursion operator R as the 2×2 block operator

$$\mathsf{R} \stackrel{\text{def}}{=} \begin{pmatrix} -\frac{1}{4}(\partial_t^2 + 4u + 2u_t\partial_t^{-1}) & 0 \\ \frac{1}{4}\psi - \frac{1}{2}\psi_t\partial_t^{-1} & -\Delta \end{pmatrix} . \tag{5.22}$$

We remark in passing that the $(1,1)$ block of R coincides, except for an unimportant constant, with the recursion operator of soliton theory.

Now, we think of R as acting on column vector fields through the identification

$$V = v^{(u)}\frac{\partial}{\partial u} + v^{(\psi)}\frac{\partial}{\partial\psi} \equiv \begin{pmatrix} v^{(u)} \\ v^{(\psi)} \end{pmatrix} \tag{5.23}$$

We are now ready to state the nice result of [15]

Theorem 5.10 *The effect of the recursion operator* R *on the vector fields (5.17)–(5.21) is to produce a hierarchy of vector fields that generates the full algebra* g *of iso-Huygens deformations of (5.8) by means of*

$$\mathcal{Z}_{k+1} = \mathsf{R}\mathcal{Z}_k \tag{5.24}$$

$$\mathcal{X}_{k+1} = \mathsf{R}\mathcal{X}_k \tag{5.25}$$

$$\mathcal{E}_{k+1}^i = \mathsf{R}\mathcal{E}_k^i \tag{5.26}$$

$$\mathcal{T}_{k+1} = \mathsf{R}\mathcal{T}_k \tag{5.27}$$

$$\Lambda_{k+1}^{ij} = \mathsf{R}\Lambda_k^{ij} , \tag{5.28}$$

for $k \in \mathbb{Z}_{\geq 0}$.

Exercise 5.11 *The vector fields obtained by means of the recursion operator* R *satisfy the following Virasoro-type relations (among many others [15])*

$$[\mathcal{T}_l, \mathcal{T}_m] = (m - l)\mathcal{T}_{m+l} \tag{5.29}$$

and

$$[\mathcal{T}_l, \mathcal{X}_m] = (m + \frac{1}{2})\mathcal{X}_{l+m} \tag{5.30}$$

The relations (5.29) and (5.30) are the same satisfied by the corresponding ones in Theorem 3.22.

5.3 Conclusion

In this section we shall highlight the connections between the different topics discussed throughout these notes. We shall also try to point out to some topics we believe will be of increasing interest in the near future.

At a first glance, one might say that the main thread permeating the different subjects in these notes is the class of potentials known as rational solutions of the KdV, which were obtained by means of Darboux transformations in Section 3.4. These are, in turn, the potentials that added to a suitable D'Alembertian yield strict Huygens operators. They are also bispectral potentials, as discussed in Section 4.2.

However, the bispectral problem for Schrödinger operators admits another class of solutions, the even family potentials of Section 3.4. Initially, these potentials seemed to be out of place in the general picture which came with the characterization of bispectral Schrödinger operators given by Duistermaat and Grünbaum. A better understanding came about in [110] with the crucial remark that all the bispectral Schrödinger potentials (decaying at infinity) are invariant by the (nonnegative) Virasoro flows. These flows were described in Section 3.3 with the aid of recursion operators. This furthered the link with the theory of integrable systems, and in particular with the KP hierarchy. After all, the Virasoro algebra, and more generally, the so called $\mathcal{W}_\infty$ algebras are at the heart of present day understanding of KP symmetries.

The full characterization of bispectral operators of order greater than two is still open, despite recent advances. These advances have taken place in different fronts, and trying to survey them would take us too far afield. We hope to have enticed the reader into browsing at the recent literature [47, 46, 45, 105, 108, 106]. We content ourselves in remarking that the rational solutions of the KP hierarchy also yield bispectral operators. More precisely, it was shown in [107] that Sato's polynomial τ-functions for the KP hierarchy produce bispectral operators. [3] Using very elegant methods of algebraic geometry, G. Wilson [101] characterized all the bispectral operators associated to commutative rings of differential operators of rank one. These in turn correspond exactly to polynomial τ-functions for KP.

Wilson's beautiful characterization, however, calls for an extension to higher rank, which to this date has not been completed. One should not forget to mention, though, that substantial progress has been made in the work of A. Kasman [63, 64] concerning the higher rank case and its connection to Calogero-Moser systems.

The link between the strict Huygens' property and bispectrality becomes even stronger when investigated at the light of deformations preserving such properties. Indeed, the main result of Berest in [15], as we saw in Section 5.2 is the existence of a recursion operator in the class of Lie-Bäcklund symmetries of

[3] The result stated in [107] was only about the polynomial τ-functions obtained from Schur functions, however the method used apply equally as well for the more general polynomial τ-functions. See [109]

Huygens operators $\mathcal{L}$ of the form

$$\mathcal{L} = \Box + u(x_0) \; .$$

These symmetries when acting on the potential u coincide with the symmetries of the bispectral problem. The latter in turn, are the (nonnegative) Virasoro flows, which were studied in Section 3.3.

Another way of phrasing the last remark is the following: Strict Huygens potentials and bispectral potentials share the same family of symmetries, as can be seen from comparing the results of the commutation relations from obtained from Theorems 5.10 (Sec. 5.2) and 3.22 (Sec. 3.3).

Iso-Huygens Deformations	KdV Hierarchy/Master Symmetries
$\begin{cases} [\mathcal{T}_j, \mathcal{T}_l] = (l-j)\mathcal{T}_{j+l} \; , \\ [\mathcal{T}_j, \mathcal{X}_l] = (j+\frac{1}{2})\mathcal{X}_{l+j} \; . \end{cases}$	$\begin{cases} [\tau_j, \tau_l] = (l-j)\tau_{j+l} \; , \\ [\tau_l, X_j] = (j+\frac{1}{2})X_{l+j} \; . \end{cases}$

The full implications of the above coincidence are still not understood. Certainly the rational solutions of KdV are invariant by the (nonnegative) Virasoro flows and by the KdV flows. Furthermore, they are bispectral and enjoy a strict Huygens property. On the other hand, Lagnese and Stellmacher have shown that the only potentials $u = u(x_0)$ one can add to the D'Alembertian (in a sufficiently high odd number of space dimensions) and still get a Huygens operator are the ones in the class of rational solutions of KdV. The ones in the even family are therefore ruled out, in apparent contradiction with the iso-Huygens deformation result. The plausible hint to this apparent contradiction comes from the remark that the deformations allowed by Berest are in fact leaving room for the action of the trivial group of deformations. The latter would obviously change the form of the operator $\mathcal{L}$. This subject certainly deserves further investigation.

On the front of generalizations to several dimensions, progress has been made both for Huygens property and more recently for bispectrality, see [20, 19, 16, 15, 17, 18].

One final subject not touched here, among many others, is the study of lacunæ for higher order linear hyperbolic operators with constant coefficients. Once again the prospects of interesting connections are very encouraging [10, 11, 34, 17]

A. Symmetries of Differential Equations

In this appendix we discuss a few ideas behind the use of symmetries in the study of differential equations, with particular aim at Huygens's property. The use of symmetries in differential equations, and in science in general is fairly old. However, the systematic use of these ideas is due to Sophus Lie, whose monumental work is often forgotten. We shall not give here a due account of such theory, we refer the interested reader to the literature [5, 7, 21, 57, 59, 82]

Symmetries of Differential Equations

When studying systematically symmetries of differential equations, it is natural to think of the equation as defining a manifold in a sufficiently high dimensional space. The solutions of the equation led to parametrizations of such manifold. This is the origin of the concept of *skeleton* of a differential equation. More precisely, let's consider the equation of the form

$$F(x, u, \partial u, \ldots, \partial^\alpha u) = 0 , \tag{A.1}$$

where as usual $x \in \mathbb{R}^n$ is the independent variable and u is the dependent one. The symbols $\partial u \ldots \partial^\alpha u$ denote the derivatives of u with respect to x.

Definition A.1 *The skeleton of the differential equation A.1 is the manifold in $\mathbb{R}^N$ defined by the equation*

$$F(x, y) = 0 , \tag{A.2}$$

where $y \in \mathbb{R}^m$, $N = m + n$, and m is the appropriate dimension to capture all the derivatives of u up to the order of (A.1).

For a motivation and application see the enthralling article of Ibragimov's [58] whence the next example is taken.

Example A.2 The skeleton of the Ricatti equation

$$u' + u^2 = \frac{2}{x^2} , \qquad ' = \frac{d}{dx}$$

is the surface in $\mathbb{R}^3$ defined by the equation

$$\frac{2}{x^2} - y^2 + z = 0 \ .$$

The next ingredient in the systematic study of symmetries is to consider families of transformations preserving the skeleton. These transformations could be either discrete or dependent on a continuous variable. Our focus now is on the continuous ones, which would hopefully be smoothly dependent on the parameters. Each of the corresponding parameters would lead to a one parameter group of transformations. It turns out that the restriction of having a group is too stringent. Furthermore, it is more convenient to handle vector fields (Lie algebras) then their flows (Lie groups). Hence, one is led to consider tangent vector fields to the skeleton.

Since one is concerned with solutions of differential equations, not any vector vector field tangent to the skeleton would be of interest. Generic vector fields on the skeleton might yield one-parameter groups of transformations that do not preserve derivatives. This leads us to the concept of "prolongation" of vector fields.

For example, if one is only concerned with one dependent x and one independent variable u, we would like to preserve

$$du = \frac{du}{dx}\, dx \ . \tag{A.3}$$

Let's take a one parameter group of transformations dependent on the parameter ε. Let's denote these transformations by

$$(x, u) \mapsto (\tilde{x}(x, u; \varepsilon), \tilde{u}(x, u; \varepsilon)) \ , \tag{A.4}$$

and set, as usual,

$$u_1 \stackrel{\text{def}}{=} du/dx \ . \tag{A.5}$$

The condition $d\tilde{u} = \tilde{u}_1 d\tilde{x}$ leads to

$$\frac{\partial \tilde{u}}{\partial x}dx + \frac{\partial \tilde{u}}{\partial u}du = \tilde{u}_1\left(\frac{\partial \tilde{x}}{\partial x}dx + \frac{\partial \tilde{x}}{\partial y}dy\right) \ .$$

After substitution of (A.5) and solving for $\tilde{u}_1$ gives

$$\tilde{u}_1 = \frac{\frac{\partial \tilde{u}}{\partial x} + u_1 \frac{\partial \tilde{u}}{\partial u}}{\frac{\partial \tilde{x}}{\partial x} + u_1 \frac{\partial \tilde{x}}{\partial u}} \ . \tag{A.6}$$

Equation (A.6) becomes more palatable, and well suited for generalizations, if we define the total derivative operator (familiar in calculus of variations):

$$\frac{D}{Dx} \stackrel{\text{def}}{=} \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \dots \ , \tag{A.7}$$

where

$$u_k \stackrel{\text{def}}{=} \frac{d^k u}{dx^k} \ .$$

Using the above definition we get

$$\widetilde{u}_1 = \frac{D\widetilde{u}}{Dx} / \frac{D\widetilde{x}}{Dx} \ .$$

Exercise A.3 *The higher order derivatives of u transform as the following (recursive) equation*

$$\widetilde{u}_k = \frac{D\widetilde{u}_{k-1}}{Dx} / \frac{D\widetilde{x}}{Dx} \ , \ k > 1 \ . \tag{A.8}$$

At this point, it is natural to seek what is the transformation rule for vector fields on the space $(x, u, u_1, \ldots, u_k)$. This is also given in terms of total derivatives. The expression is the result of the next lemma, whose proof is a simple exercise.

Lemma A.4 *Given a vector field on (x, u) space of the form*

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \ ,$$

the corresponding vector field in the extended space $(x, u, u_1, \ldots, u_k)$ obtained by transforming the variables $u_1, \ldots, u_k$ so as to preserve the tangency conditions $du_{k-1} = u_k dx$ is given by

$$X^{(k)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \eta^{(1)} \frac{\partial}{\partial u_1} + \ldots + \eta^{(k)} \frac{\partial}{\partial u_k} \ ,$$

where the vector fields $\eta^{(k)}$ are computed recursively by

$$\eta^{(k)} = \frac{D\eta^{(k-1)}}{Dx} - u_k \frac{D\xi}{Dx} \ . \tag{A.9}$$

The above result can be easily extended for several independent and dependent variables [21]. In the case of n independent variables the total derivative operator has to be generalized to

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots + u_{ii_1 i_2 \ldots} \frac{\partial}{\partial u_{i_1 i_2 \ldots}} + \ldots \ ,$$

with the summation convention fully enforced for ecological reasons.

The extended transformations of a given vector field of the form

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}$$

is given by

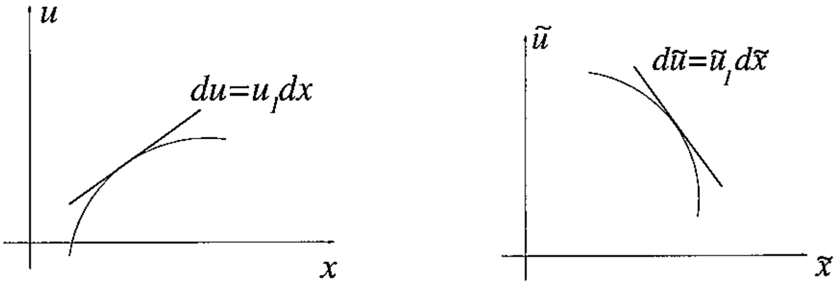

Fig. A.1. Contact Transformation.

$$X^{(k)} = \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \eta_i^{(1)} \frac{\partial}{\partial u_i} + \ldots + \eta^{(k)}_{i_1 i_2 \ldots i_k} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}}$$

with the coefficients obtained from

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j \ ,$$

and from

$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k} \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k} \xi_j) u_{i_1 i_2 \ldots i_{k-1} j} \ .$$

for $k \geq 2$.

Lie-Bäcklund Transformations

In the previous Section we were concerned with transformations of the form

$$(x, u) \mapsto (\tilde{x}(x, u; \varepsilon), \tilde{u}(x, u; \varepsilon)) \ . \tag{A.10}$$

These are called *point transformations* since they do not involve derivatives of the variables.

It turns out that many times it is interesting to have more general transformations where the derivatives would also play a role, for example,

$$(x, u, \underset{1}{u}) \mapsto \left(\tilde{x}(x, u, \underset{1}{u}; \varepsilon), \tilde{u}(x, u, \underset{1}{u}; \varepsilon), \ \underset{1}{\tilde{u}} \ (\tilde{x}(x, u, \underset{1}{u}; \varepsilon) \right) \ , \tag{A.11}$$

where $\underset{1}{u}$ denotes all the first order derivatives of u. As before, it is crucial that such transformations preserve tangencies. See Figure A.1.

This leads to the concept of contact transformations.

Definition A.5 *A contact transformation is a transformation of the form (A.11) that preserves the contact condition*

$$du = \underset{1}{u}\, dx \ .$$

It is clear from the construction of the prolongation of the transformations, that if one starts with point transformations such as (A.10) then the prolongation would be a contact transformation. A natural question would be whether the converse is true. In other words, are there more general contact transformations than the ones obtained by prolongations of contact transformations? The answer to this question is *no*, if the dimension of the space of dependent variables is $m > 1$. More precisely, one has the following important result

Theorem A.6 *If $m > 1$, then the group of contact transformations of the form (A.11) is the first prolongation of the group of point transformations (A.10).*

Contact transformations, due to their high content of geometrical information played an important role in mechanics and optics. As matter of fact, Huygens himself used them in his constructions. (See page 194 of [57]). One example of a contact transformation is Legendre's transformation that relates the Lagrangian formulation with the Hamiltonian one. They also play an important role in the study of first order PDEs, since under those transformations the class of first order PDEs is preserved. Furthermore, solutions are sent into solutions.

A natural generalization of the concept of contact transformations would be to require that higher order of tangency be preserved under the transformations.

More precisely, let's consider transformations of the form:

$$(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) \xmapsto{T_\epsilon} (\tilde{x}, \tilde{u}, \underset{1}{\tilde{u}}, \ldots, \underset{k}{\tilde{u}}) \ , \tag{A.12}$$

where one requests that a tangency condition of order k holds. This means that if we denote by u^j the j-th component of the vector u, $j = 1, \ldots, m$, and define

$$u^j_{i_1,\ldots,i_s} \stackrel{\text{def}}{=} \frac{\partial^s u^j}{\partial x_{i_1} \ldots \partial x_{i_s}}$$

the derivative of order s of u^j. The tangency condition means that the equation

$$du^j_{i_1,\ldots,i_s} = u^j_{i_1,\ldots,i_s,l} dx_l \tag{A.13}$$

is preserved by T_ϵ for all $s \leq k$. See Figure A.2 for a pictorial description of the case of second order tangency.

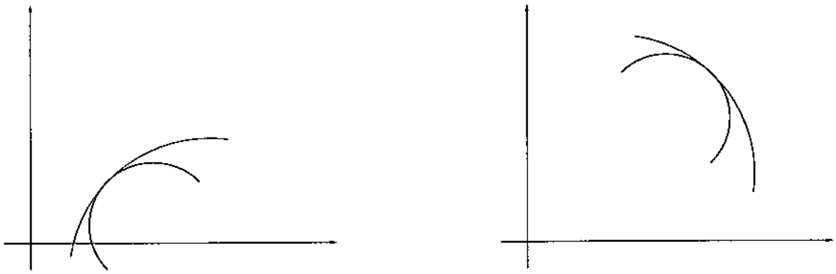

Fig. A.2. A Transformation preserving second order tangency (osculation).

Once again, one is tempted to ask the question whether one can find more general transformations than the ones obtained by prolongations of contact transformations (or point transformations). Lie conjectured a negative answer to this question, and Bäcklund confirmed Lie's conjecture. His result states

Theorem A.7 *(Bäcklund) Every group of tangent transformations of order $k < \infty$ is the prolongation of a group of point transformations for $m > 1$, and of a group of contact transformations for $m = 1$.*

For a proof of this result we refer the reader to [57, 6].

Finally, one is led to consider infinite order tangent transformations, i.e., transformations of the form

$$(x, u, \underset{1}{u}, \ldots, \underset{k}{u}, \ldots) \xmapsto{T_\epsilon} (\tilde{x}, \tilde{u}, \underset{1}{\tilde{u}}, \ldots, \underset{k}{\tilde{u}}, \ldots)\,, \tag{A.14}$$

for which equation (A.13) holds for all orders.

Definition A.8 *A transformation of the form (A.14) which leaves invariant the set of equations (A.13) for all k is called a Lie-Bäcklund transformation.*

The reader is referred to the excellent book ([57, 6]) where one can find a number of examples.

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Errata

A few typos were detected after printing started. Certainly not all of them are listed here, so please send any additional correction to `zubelli@impa.br`

Chapter 2

- Page 9, line 7: Add "of the" to the end of the line to obtain "...closure of the".
- Page 9, line 11: Substitute "even n" by "even $n - 1$".
- Page 9, line 13: Delete "the" from "...with the second order..."
- Page 11, first line of third paragraph: Substitute "and" by "an" to get "...notion of an operator".

Chapter 4

- Page 51, line 12: Change "...spectrum with such the..." to "...spectrum commuting with the..."
- Page 51, lines -11 and -13: Take away the "s" from "Communications Theory" to get "Communication Theory"
- Page 52, lines -3: Idem.
- Page 53, line 7: Add "is" after "polynomial".
- Page 54, line -5 (Last paragraph): Add "leading coefficient of the" between "the" and "differential" to get "...requiring that the leading coefficient of the differential".
- Page 60, line -10: Between the words "that" and "all", insert "the (non-negative) Virasoro flows are tangent to" to get the following: "...that the (non-negative) Virasoro flows are tangent to all the ..."

Chapter 5

- Page 67, line -2: Substitute "for the" by "to" in order to get "...equally as well to more general ..."