21º COLÓQUIO BRASILEIRO DE MATEMÁTICA

INTRODUCTION TO INTEGRAL GEOMETRY

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A geometria integral e um assunto que comencou com Buffon no seculo 18. Acompanhou o nascimento da teoria da medida, o estudo de corpos convexos e o desenvolvimento da topologia.

A escolha dos resultados apresentados aqui e parcial. Deixei de lado ramos importantes para insistir nos balbucios (Buffon e os paradoxos de Bertrand), e nos resultados ligados á topologia, á teoria das folheacoes e á geometria complexa. O ultimo capitulo tem raizes na matematica do seculo passado e apresenta conjeturas e perguntas em geometria conforme.

Sistematicamente deixei de lado as provas tecnicas de um resultado geral para só apresentar as ideas com a prova de um caso particular. Uso e abuso de figuras: no titulo geometria integral tem geometria!

Tenho muito prazer em dar esse curso no Brasil, onde alguns resultados mencionados no texto foram provados.

Agradeco a Daniel Lines e Harold Rosenberg para ter concertado o ingles. Os erros que sobram vem de acrescimentos de ultima hora.

Enfim o estimulo de Manfredo do Carmo chegou no momento certo para que un projeto se transformasse em livro.

Rémi Langevin

Introduction to integral geometry

R. Langevin

Contents

1 Introduction

In 1777 Buffon published his *Essai d'arithmétique morale* [Bu], where he describes the needle experiment.

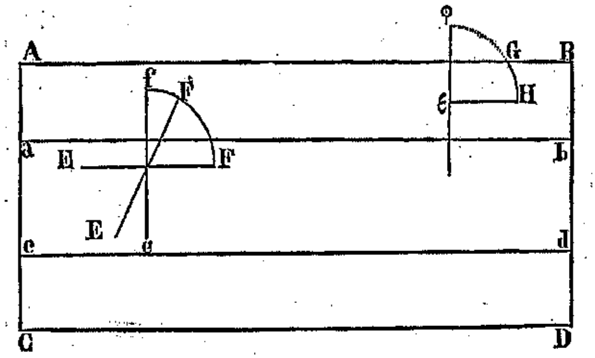


Figure 1: Buffon's calculation

The first paragraph of the essai is:

La mesure des choses incertaines est ici mon objet: je vais tacher de donner quelques règles pour estimer les rapports de vraisemblance, les degrés de probabilité, le poids des témoignages, l'influence des hasards, l'inconvénient des risques, et juger en même temps de la valeur réelle de nos craintes et de nos espérances. After some considerations about a game called "franc-carreau", where the players gamble on the probabilities of a coin falling entirely in a tile or accross some division line, Buffon proves that, when a needle is thrown "at random" on the boards of a parquet, if the length of the needle is equal to the width of the boards, the probability it will lay across two boards is $2/\pi$. He admits without the slightest doubt that the right probability measure on the space of positions of the needle is the measure $\frac{1}{2\pi}|dx \wedge d\theta|$ which we shall consider below.

The appearence of the number π hides a circle. The physicist Paul Langevin described in 1908 a way to visualise a proof of Buffon's observation.

Let us throw thousands of needles and move them using only translations parallel to the boards or perpendicular to them with length an integer multiple of the width of the board. As all relative positions

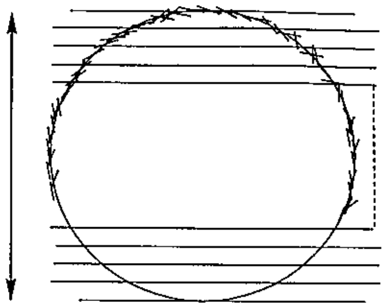


Figure 2: Needles and rearranged Needles

(angle, distance of the needle to the lines boundary of the boards) are equally likely, we can rearrange the needle along a very large circle ase in fig.2 having essentially the same amount, say N, of needles above any point of the circle. The total amount of needles is close to N.L, where L is the length of the circle and the number of needles crossing the lines is close to

$$N.(number\ of\ intersection\ points\ of\ the\ lines\ with\ the\ circle)$$

that is 2N.D, where D is the diameter of the circle. The required probability is then $2N.D/N.L = \frac{2}{\pi}$.

A hundred years will be needed to clarify the notion of probability involved. Before coming to that, let us give a conventional proof confirming Buffon's result. Locate the position of the needle on the floor by the position of its tip and the angle of the needle with the direction of the lines. Using as before translations parallel to the boards, or multiple of the width of the boards, we can suppose that our needle has its tip on the vertical segment AB of fig 3. We assume that AB has length 1. Call x the distance between the tip of the needle and A.

Therefore the set of all possible positions of the needle is $[AB[.S^1$, (or rather $\mathbb{R}/\mathbb{Z} \cdot S^1$). The needle meets the line L_B if $x + sin\theta \geq 1$ and L_A if $x + sin\theta \leq 0$.

The ratio between the dashed area and the area of the rectangle $]0,1].[0,\pi[$ is $2/\pi$.

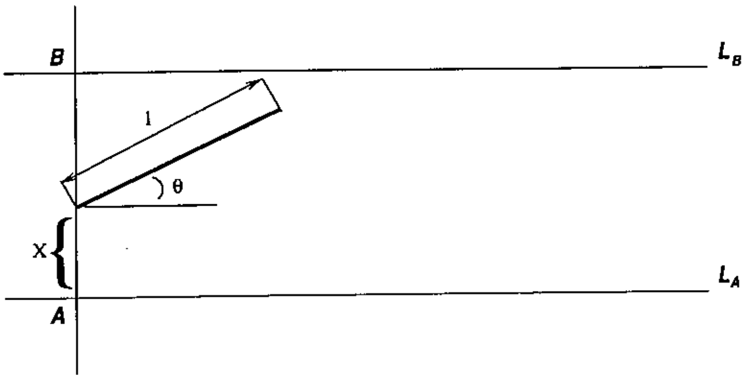


Figure 3: localization of the needle

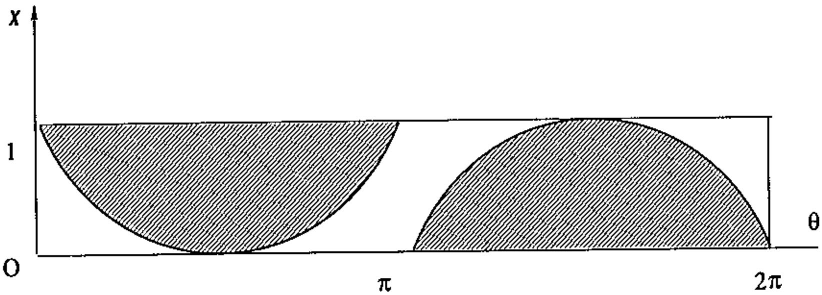


Figure 4: domain corresponding to the needles which lay across two boards

2 The birth of the notion of geometric measure

2.1 Cauchy and Crofton

In 1832, in a communication to the French Academy of Sciences, Cauchy noticed that the length of a convex curve is the average of the lengths of the orthogonal projection of the convex curves on all lines through the origin. More generally, for any rectifiable planar curve C, denote by m(C,L) the "abolute length" of the ortogonal projection of C on the line L, the length of the projection counted with multiplicity. In modern language:

$$m(C, L) = \int card(p^{-1}(y)dy); y \in L$$

Then:

Theorem 2.1.1 *Cauchy formula[Cau]*

$$\int_{-\pi 2}^{\pi/2} m(C, L_\theta)d\theta = 2(length\, of\, C)$$

Cauchy's proof amounted to prove the formula for a segment, and then approximate any curve by inscribed polygons.

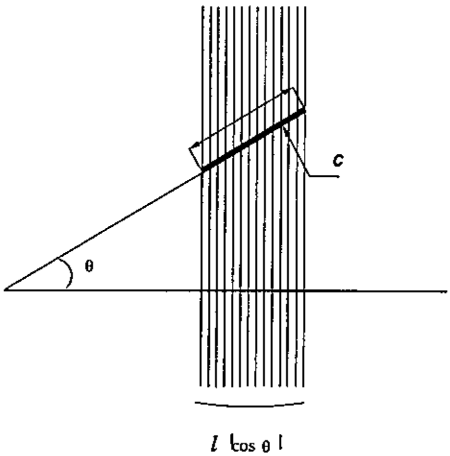


Figure 5: area of the needles crossing the lines in $[0,1[\times[0,\pi[$

From Cauchy's communication to the french academy in 1832 [Cau], to Crofton's mémoire (1868) [Cro] 36 years where needed to clarify the notion of a measure on the set of affine lines. Let us quote Crofton: *The expression "at random" has in common language a very clear and definite meaning; one which cannot be better conveyed than by Mr Wilson's expression "according to no law"... There is always a direct reference to the assemblage of things to which it belongs and from which we take, and not till then, we can proceed to sum up the favorable cases,... But there are several classes or questions in which the totality of cases is not merely infinite, but of an inconceivable nature... We can thus continually suppose variations of the experiment, each variation giving a new infinity of cases.* (then Crofton justifies the choice of the measure on the plane). *What means: an infinity of lines drawn at random on the* *plane, what is the nature of this aggregate? First, since any direction is as likely as the others, as many of the lines are parallel to any direction as to any other. As this infinite system of parallels is drawn at random, they are as thickly disposed along any part of the perpendicular as along any other...*

Crofton did found the right answer as we will see in next section. Nevertheless, at the turn of the century the choice of a measure on a continuum was not obvious, because there were too many possibilities.

2.2 Bertrand's paradoxes

Let us give three different answers proposed by the probabilist Bertrand to the same problem of elementary geometry. At that point, integral geometry was close to desappear. The question is (see pictures below): what is the probability for a chord of a circle taken at random to be longer than the side of an equilateral inscribed circle? The three different answers Bertand proposes will come from three different ways to choose the chord.

1)Chose an arbitrary point A on the circle. Using the rotational symetry of the picture we can forget about A and choose now another point B on the circle, endowed with arc length measure.

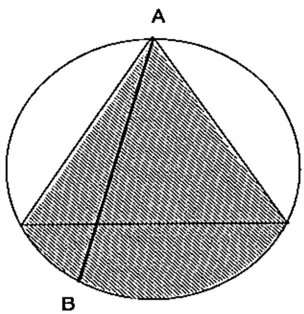


Figure 6: probability 1/3

The chord is then longer than the side of the inscribed equilateral triangle with probability 1/3.

2)Chose at random the affine line supporting the chord. The rotational symetry of the picture allows us to forget about the direction of the line.

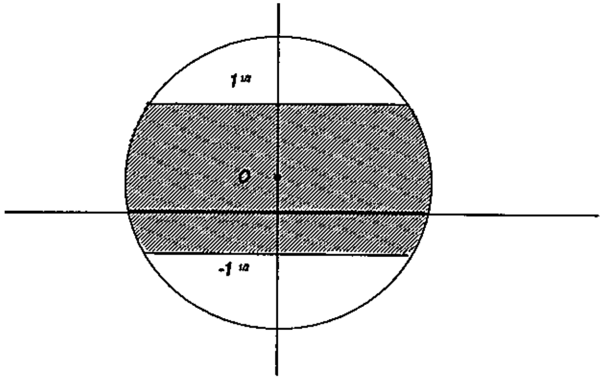


Figure 7: probability 1/2

As $cos(\pi/3) = 1/2$ the probability is now 1/2.

3)Chose at random the middle of the chord in the disc (the measure is the Lebesgue measure on the disc). We ignore chords through the origin, as they form a set of measure zero.

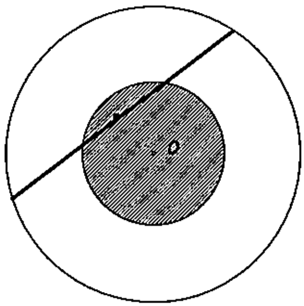


Figure 8: probability 1/4

Then the probability is 1/4.

Poincaré will take integral geometry out of this dead end. For him, (see for example his book published in 1912 [Poin] , the most interesting measure is the one which is invariant under the group of affine isometries of the plane. Only isometries preserving the origin are allowed by presentations 1) and 3). In 2) translations also act on the set of affine lines and preserve our measure. It was also Crofton's answer.

3 The euclidean plane

3.1 Geometric measures on sets of lines

We will start with the Euclidean geometry of the plane. The group of Euclidean motions M acts on the points of $\mathbb{R}^2$. It leaves invariant the Lebesgue measure $dx \wedge dy$. It acts also on the set of affine lines of the plane $\mathcal{A}(2,1)$. The oriented lines through the origin of $\mathbb{R}^2$ form a circle, as any oriented half-line cuts the unit circle in a point. This correspondence defines the topology of the set of oriented lines through the origin. The set of unoriented lines is the quotient of this first circle by the relation $x \sim -x$ We denote this set by $G(2,1)$. We can visualize the latter identifying a line (distinct from the x-axis) with its intersection (different from the origin) with the circle tangent at the origin to the x-axis of next picture.

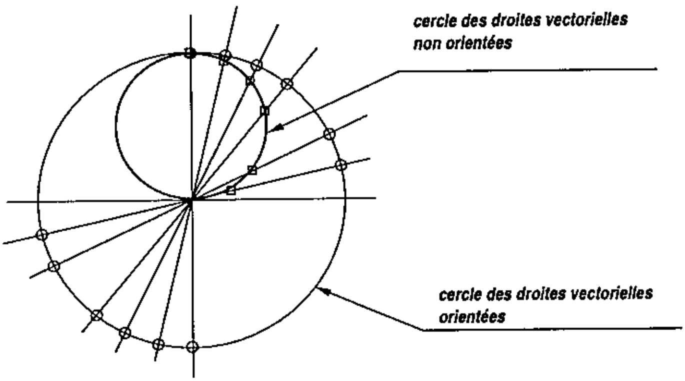


Figure 9: Oriented and non oriented directions.

A non-oriented affine line corresponds to each point t of a direction D; just take Δ_t to be the perpendicular through t to D. Using oriented directions we would get oriented affine lines $\mathcal{A}^+(2,1)$. In that case we consider an oriented direction D^+ and the affine line perpendicular to a point $t \in D^+$ (which can be identifies with its coordinate on the oriented line D^+). Let θ be the oriented angle of the x-axis and D^+. We see that the oriented affine grassmannian $\mathcal{A}^+(2,1)$ is a cylinder $S^1 \times \mathbb{R}$. on which natural coordinates are θ and t. From the angular measure $|d\theta|$ on the unit circle and the Lebesgue measure $|dt|$ on the line D, we get a measure

$|d\theta \wedge dt|$ on the set of oriented affine lines. This measure is invariant by a rotation of center the origin. A translation of vector v moves the line (θ, t) to the line $(\theta, t+ < e^{i\theta}|v >$; it also leaves invariant the measure $|d\theta \wedge dt|$. As an exercise, let us represent on the cylinder $S^1 \times \mathbb{R}$ the oriented affine lines through the extremity O' of the vector v on the picture below.

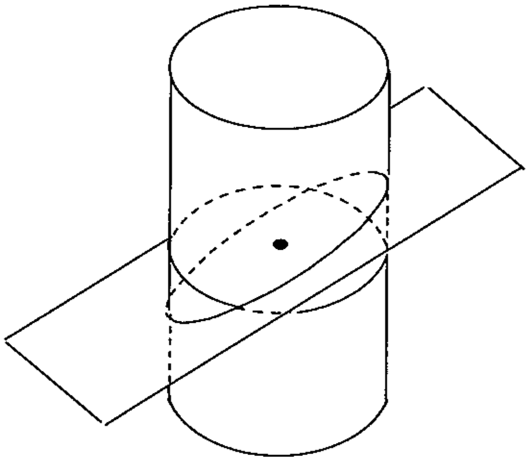


Figure 10: oriented lines through the origine and through another point.

We will call the family of lines through a point, or the family of lines parallel to a given direction a *linear pencil.* The equation of a line of a linear pencil is a linear combination of the equations of any two different lines of the family.

Remark: The oriented affine lines through the point $m = (a, b)$ are the intersection of the cylinder $x^2 + y^2 = 1$ with the plane of equation $z = ax + by$. Parallel lines are the intersection of the cylinder with a vertical plane through the origin.

The projection (forgetting the orientation) of $\mathcal{A}^+$ on $\mathcal{A}$ defines the measure, still denoted $|d\theta \wedge dt|$, on $\mathcal{A}^+(2,1)$. this projection also permits us to recognize that $\mathcal{A}(2,1)$ is the Möbius band obtained from the rectangle $[0, \pi] \times \mathbb{R}$ identifying $(0, t)$ with $(\pi, -t)$. the next picture shows the set of lines corresponding to the small rectangle $[\theta_1, \theta_2].[t_1, t_2]$.

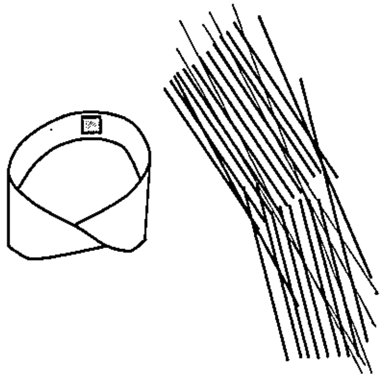


Figure 11: The Möbius band.

3.2 The Gauss map

During this section, curves will be of class C^∞. An essential tool in the study of hypersurfaces of $\mathbb{R}^n$ and first planar curves, is the Gauss map γ which to each point m of an oriented curve C associates its oriented normal, $N(m) = R_\pi/2(T(m))$ where $T(m)$ is the oriented unit tangent at m to the curve.

$$\gamma : C \to S^1$$

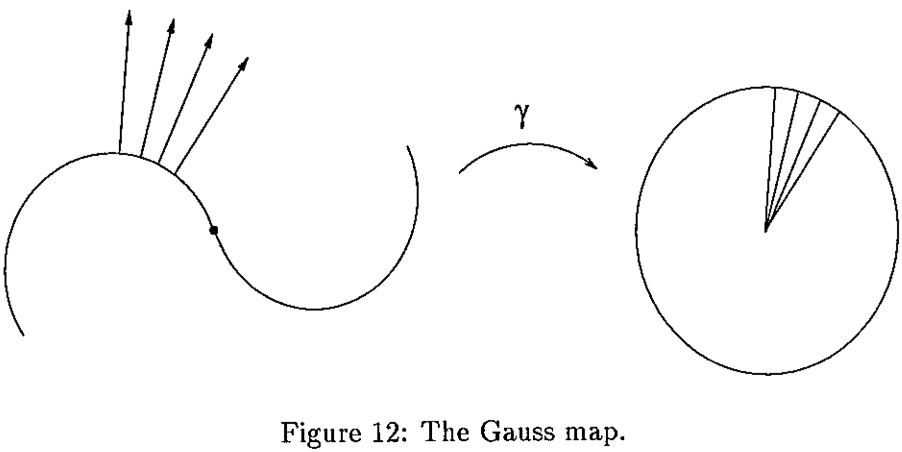


Figure 12: The Gauss map.

The jacobian k(m) of γ at a point $m \in C$ is called the curvature of C at that point. Notice that we can define a Gauss map with value in P_1, forgetting the orientation of $N(m)$. Notice also that the tangent map $T : C \mapsto S^1$ mapping a point $m \in C$ to the oriented unit tangent to C at m, has the same jacobian $k(m)$. We will use the map $C \mapsto \mathcal{A}^+(2,1)$ the paragraph "envelopes" of next section.

Remark: Let $m \in C$ be a noncritical point of the Gauss map. Then the point m is a nondegenerate critical point of the orthogonal projection of C on the oriented line $L(x)$ defined by $N(x)$.

Proof: Locally C has the equation $y = f(x)$ where x is a coordinate on the line generated by $T(x)$ and y on the line $L(x)$. We can choose the euclidean coordinates x, y such that $f(0) = 0$. We also have $f'(0) = 0$. The curvature $k(m)$ is in that case just $f''(0)$. If the curvature is nonzero, the orthogonal projection of C on L has the nondegenerate hessian $f''(0)$. □

For a direction L, denote by $\mu(C, L)$ the number of critical points of the orthogonal projection of C on L. The change of variable theorem implies then that there exists a neighbourhood v of m such that

$$\int_v |k(m)|dm = \int_{P_{n-1}} |\mu|(v, L)dL$$

The result holds globally on C.

Theorem 3.2.1

$$\int_C |k(m)|dm = \int_{P_{n-1}} |\mu|(C, L)dL$$

Proof: The proof relies on Sard's theorem. The set Σ of critical values of γ is of zero measure. Its inverse image $\gamma^{-1}(\Sigma)$ is the union of critical points of γ, where $k = 0$ and noncritical points of γ with image in σ which form a set of measure zero. The complement of $\gamma^{-1}(\Sigma)$ is a denumerable union of open sets of C. Discarding at most a denumerable set of points if necessary, we get a denumerable union $\bigcup_i(U_i)$ of open set of C where the restriction of γ is a diffeomorphism on its image. Using the change of variable theorem and summing on i we get:

$$\int_C |k(m)|dm = \int_{\bigcup_i(U_i)} |k(m)|dm = \sum_i(\gamma(U_i) = \int_{P_{n-1}} |\mu|(C, L)dL$$

□

We can also count "most" of critical points with a sign. Assign to the non degenerate critical points of the orthogonal projection of the oriented curve C on the oriented line L^+ the sign $\epsilon(m) = (-1)^{index(m)}$. When the two unit vectors contained in L are non degenerate values of the Gauss map, we can, at each point m such that $\gamma(m) \subset L$ orient the line L using the normal $N(m)$ to define $\epsilon(m)$. Thus we get:

$$\mu(C, L) = \sum_{\gamma(m) \in L} (\epsilon(m))$$

Theorem 3.2.2 *If one of the integrals*

$$\int_C |k(m)| dm = \int_{P_{n-1}} |\mu|(C, L) dL$$

is finite, then:

$$\int_C k(m) dm = \int_{P_{n-1}} \mu(C, L) dL$$

To prove this last theorem, is is enough to track the signs in the proof of the preceeding one.

A classical theorem for embedded closed planar curve states that:

Theorem 3.2.3

$$|\int_C k(m) dm| = 2\pi$$

It is a consequence of the following fact that we will explain below:

$$\mu(C, L) = 2 \cdot degree(Gauss\ map) = \pm 2$$

when C is a simple closed curve, and when $\mu(C, L)$ makes sense.

As a corollary we get the inequality:

$$\int_C |k(m)| dm \geq 2\pi$$

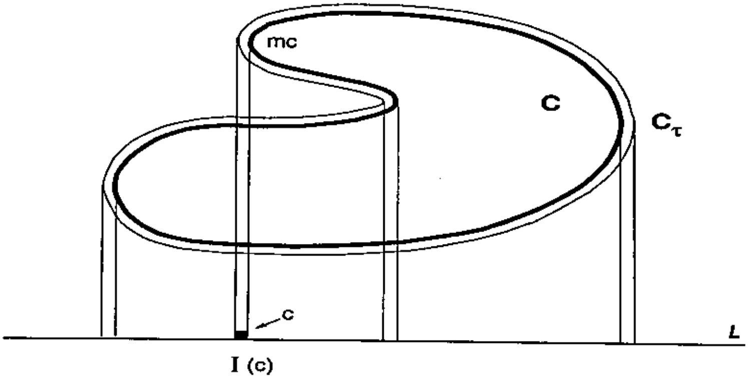


Figure 13: Thickening of a curve.

3.3 Volume of the tube around a curve

We will use the previous definitions to compute the volume of a small tubular neighbourhood of a closed planar curve C, and the volume of the thickening on one side of the curve.

Let C_τ be the curve

$$C_\tau = \{C(t) + \tau N(t)\}$$

The tubular neibourhood of C

$$Tub_r(C) = \{m | d(m, C) \leq r\}$$

is the union

$$Tub_r(c) = \{\cup_{-r \leq \tau \leq r} C_\tau\}$$

The tubular neighbourhood lemma tells us that for r small enough the map:

$$(t, \tau) \mapsto \{C(t) + \tau N(t)\}$$

is diffeomorphism. Let us also define the thickening (on the side of N) of C:

$$Th_r(C) = \{\cup_{0 \leq \tau \leq r} C_\tau\}$$

The volume of $Th_r(C)$ is the integral:

$$vol[Th_r(C)] = \int_{0 \leq \tau \leq r} vol(C_\tau)$$

The projection, "counted with multiplicity" of the curve C_τ on a line L is obtained, modifying the projection of C on intervals of length τ with one extremity a critical value of the orthogonal projection of C on L. See fig.13. To give a formula suppose first that the orthogonal projection π_L on L is a *Morse function*, that is, has only non degenerate critical points, which all have different images. Then a critical value $c \in L$ is the image of one critical point $m_c \in C$. The normal $N(m)$ is parallel to L and allows us to define the interval $I(c) = [c, c + \tau N(m)]$. The projection of C on L defines a function (with integer values) on L:

$$\varphi(C, L)(y) = \sharp\, \pi_L^{-1}(y)$$

Depending on the local position of C , $N(m)$ and the line orthogonal to L in c, we define a sign

$$\epsilon(c) = \epsilon(m_c) = \pm 1$$

(this generically makes sense, as the critical value c will, for almost every line, be the image of a unique critical point; see section 7.3 for more precise statements). then:

$\epsilon(c) =$

$$= +1\, if\, C\, is\, locally\, not\, on\, the\, side\, of\, N(m),\ -1\, if\, C\, is\, locally\, on\, the\, side\, of\, N(m)$$

Remark: To change the orientation of N will change the sign of $\epsilon(c)$.

Proposition 3.3.1 *The function $\varphi(C_\tau, L)$ is equal, when π_L is a Morse function, to*

$$\varphi(C_\tau, L) = \varphi(C, L) + \sum_c \epsilon(c) \cdot 2 \cdot 1_{I(c)}$$

In the formula the summation is over all critical values c of π_L, and $1_{I(c)}$ is the characteristic function of the interval $I(c)$.

Observe that the degree of the Gauss map γ can be computed using any *generic* line L, that is, here, any line such that the projection π_L is a Morse function. This degree is

$$\sum_c \epsilon(c)$$

We now also that the set of non-generic lines is of measure 0. We know that, depending of the orientation of the curve, this degree is ± 1. The proof follows from the definition of the function $\varphi(C, L)$ See fig. Rewrite Cauchy's formula for C_τ using the functions φ:

$$2 \cdot length(C_\tau) = \int_{L \in P_1} \int_L \varphi(C_\tau, L)$$

Using the proposition, the remark on the degree of γ, and permuting the order of integration (this makes sense when the curve is compact smooth arc) one gets the:

Theorem 3.3.2

$$vol(Th_r(C)) = r \cdot length(C) + (-1)^{ind\gamma} \cdot \pi \cdot r^2$$

and, using also the previous remark, we get the corollary:

Corollary 3.3.3 *for r small enough,*

$$vol(Tub_r(C)) = 2r \cdot length(C)$$

In the section **higher dimensional convex bodies**, we will generalise this proof to higher dimensions.

4 Two dimensional convex bodies and translations

4.1 Envelopes

We mentioned in the previous section the map $C \mapsto \mathcal{A}(2,1)$ which associates to each point $m \in C$ its oriented affine tangent. Conversely, to a smooth one-parameter family of affine lines, corresponds in general a curve,which is the envelope of this family of lines. Let $D_t = \{a(t)x+b(t)y+c(t)=0\}$ be a smooth family of lines where $a(t), b(t), c(t)$ are smooth functions of t. The lines D_t and D_{t+h} have an intersection in the plane if they are not parallel. When h goes to zero this intersection point may have a limit $m(t)$. Let us give a sufficient condition for the points $m(t)$ to exist, and belong to a curve C which admits the tangent $D(t)$ at the point $m(t)$.

Theorem 4.1.1 *Let D_t be a smooth family of lines of equations $a(t)x + b(t)y + c(t) = 0; (x,y) \in I\!R^2$. If for all $t \in [\alpha, \beta]$, the determinant det $\begin{pmatrix} a(t) & b(t) \\ a'(t) & b'(t) \end{pmatrix}$ is different from zero, the family envolves a curve C, that is, the curve is the union of the points: $m(t) = D_t \cap D'_t$, where D'_t is the affine line of equation $a'(t)x + b'(t)y + c'(t) = 0$. Moreover if the determinant det $\begin{pmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c't \\ a''(t) & b''(t) & c''(t) \end{pmatrix}$ is also different from zero, the curve is smooth at $m(t)$ and the tangent to C at $m(t)$ is $D(t)$.*

We will note $D''(t)$ the line of equation $a''(t)x + b''(t)y + c''(t) = 0$.

Proof: Let us find the intersection point of D_t and D_{t+h}. We need to solve the linear system:

$$\begin{aligned} a(t)x + b(t)y + c(t) = 0 \\ a(t+h)x + b(t+h)y + c(t+h) = 0 \end{aligned}$$

A first order Taylor expansion of the second equation gives:

$$\begin{aligned} &a(t)x + b(t)y + c(t) = 0 \\ &(a(t) + a't)h + o(h))x + (b(t) + b'(t)h + o(h))y + (c(t) + c'(t)h + o(h)) = 0 \end{aligned}$$

This is equivalent to the system:

$$a(t)x + b(t)y + c(t) = 0$$
$$[a'(t)h + o(h)]x + [b'(t)h + o(h)]y + [c'(t)h + o(h)] = 0$$

If the determinant $det \begin{pmatrix} a(t) & b(t) \\ a'(t) & b'(t) \end{pmatrix} \neq 0$, the limit of the solution, when h goes to zero, is the solution $m(t)$ of the system:

$$a(t)x + b(t)y + c(t) = 0$$
$$a'(t)x + b'(t)y + c'(t) = 0$$

(we shall refer to that system as (*)).

The condition $det \begin{pmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c't \\ a''(t) & b''(t) & c''(t) \end{pmatrix} \neq 0$ guarantees that the three lines D, D' and D'' do not belong to the same linear pencil. Up to terms negligible compared with h the point $m(t+h)$ is the point $D_t \cap D'_{t+h}$, which show that the limit of the line containing the chord $m(t), m(t+h)$ is D_t; See next picture □

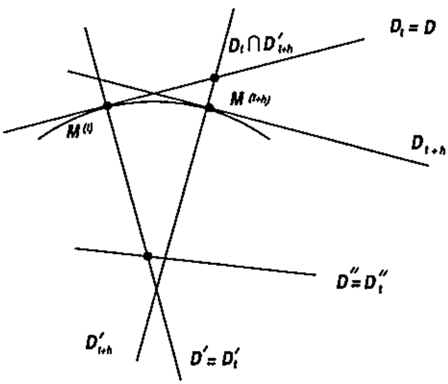


Figure 14: a non degenerate piece of envelope

Linear pencils are in that sense "degenerate" envelopes.

4.2 Support functions and hérissons

The name *hérisson* (french word for hedge hog) has been chosen because the skin of this animal cannot fold to much without inconvenience because of its spikes. We will call *hérisson* the envelope of a family of lines parametrised by their direction. In fact our definition gives oriented affine lines, as we can orient D_u by $R_{\pi/2}(u)$. More precisely, each lines of the family $D_u, u \in S^1$ admits the equation:

$$D_u = \{m| < m|u >= h(u)\}$$

where u is a unit vector, and h(u) a real function. The system(*)which gives the points of the envelope becomes:

$$< m|u >= h(u)$$

$$< m|R_{\pi/2}(u) >= (dh/du)(u)$$

and has automatically a non zero determinant. Let Q be a compact convex body. We can define a function $h(u)$ by :

$$h(u) = sup[< m|u >; m \in Q]$$

The line D_u of equation $< m|u >= h(u)$ is the support line of Q in the oriented direction u.

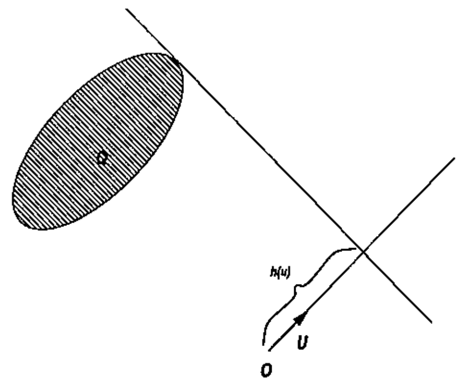


Figure 15: support function

It touches Q and Q stays on one side of D_u. The convex body Q is the intersection of the half spaces $< m|u >\leq h(u)$.

Proposition 4.2.1 *When ∂Q is a smooth curve of nowhere zero curvature, it is the envelope of the family D_u. The radius of curvature of ∂Q at the point where D_u is tangent to the curve is $h(u) + h''(u)$, where h is the support function defining the family D_u. Conversely a bounded smooth support function h such that $h+h''$ is everywhere strictly positive has an envelope which is the boundary of a compact convex body.*

In a generalized sense the boundary ∂Q can always be seen as the envelope of the family D_u. The condition (*) is always satisfied. At a point where ∂Q has a right and a left tangent which are different, the family D_u contains an arc in the pencil of line through that point.

Proof: When it is different from zero, the curvature of the boundary ∂Q, with the (conterclockwise) boundary orientation, is positive. Let us compute the radius of curvature of a hérisson, and prove that, if it is always positive, the hérisson is the boundary of a smooth convex body, whith everywhere positive curvature. The caracteristic point $m(u) = (D_u \cap D'(u))$ satisfies the equations:

$$< m|u >= h(u); < m|R_{\pi/2}(u) >= h'(u)$$

Let θ be the angle $((0,1), u)$.The rotation R_θ sends the vector $(1,0)$ to u and $(0,1)$ to $R_{\pi/2}(u)$. The two equations are equivalent to:

$$R_{-\theta}(m) = (h(u), h'(u))$$

The solution is then:

$$m(u) = R_\theta(h(u), h'(u))$$

Therefore, the derivative of the map $G : u \mapsto m(u)$ is:

$$R_{\theta+\pi/2}(h(u), h'(u)) + R_\theta(h'(u), h''(u))$$

Here we identify the derivative with respect to $u \in S^1$ and the derivative with respect to θ. This vector is just $R_{\theta+\pi/2}(h(u) + h''(u))$. Of course the tangent to the envelope is, at least when $h+h'' \neq 0$, the line D_u. The map G is the inverse of the Gauss map γ, and we have just proved that its jacobian is $h(u) + h''(u)$. The radius of curvature ρ of the envelope is then:

$$\rho(u) = 1/k(m(u)) = h(u) + h''(u)$$

The envelope is locally convex and closed, therefore it is the boundary of a compact convex body. Conversely, the condition $k > 0$ implies that there is only one point $m(u)$ on ∂Q satisfying:

$$< m(u)|u >= h(u) = sup < m|u >; m \in Q$$

Moreover, the Gauss map is invertible because $k \neq 0$. Therefore the tangents to C can be parametrised by $u \in S^1$. Observe that D_u is orthogonal to u. The limits

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

and

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

exist because Q is convex. The point $m(u)$ has to be equal to both:

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

and

$$lim_{\delta \to 0; \delta < 0} D_u \cap D_{u+\delta}$$

as, if any of these limits were different, the tangent at that point would also be D_u, which is impossssible, as γ is a bijection. □

Remark: Using standard arguments in singularity theory, one can check that for a generic support function h, a plane hérisson will have only non degenerate cusps (where $R(u) = 0, R'(u) \neq 0$).

As an example the hérisson defined by the support function $h(\theta) = cos 3\theta$ is pictured below

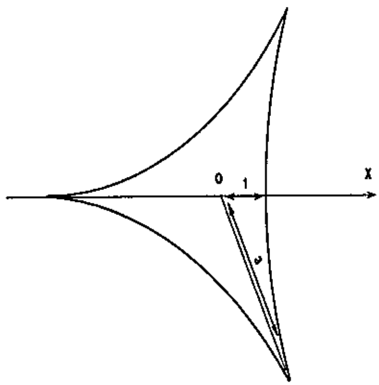


Figure 16: $\cos 3\theta$

4.3 Minkowski sum and mixed volumes

The intersection of a compact convex body with one of its support lines D_u has to be convex, that is has to be a segment. Let us define the *Minkowski sum* of two convex bodies Q_1 and Q_2 by:

$$Q_1 + Q_2 = \{m_1 + m_2 | m_1 \in Q_1, m_2 \in Q_2\}$$

One verifies that the support line of $Q_1 + Q_2$ orthogonal to the vector $u \in S^1$ has the equation:

$$< m|u >= h_1(u) + h_2(u)$$

where $Q_1 + Q_2$ are the support functions of Q_1 and Q_2. In other words, $h_1 + h_2$ is the support function of $Q_1 + Q_2$. Of course scalar multiplication (homothety) is compatible with the Minkowski sum:

$$\lambda Q = \lambda_1 Q + \lambda_2 Q \ when \ \lambda_1 + \lambda_2 = \lambda, \lambda_1 \geq 0, \lambda_2 \geq 0$$

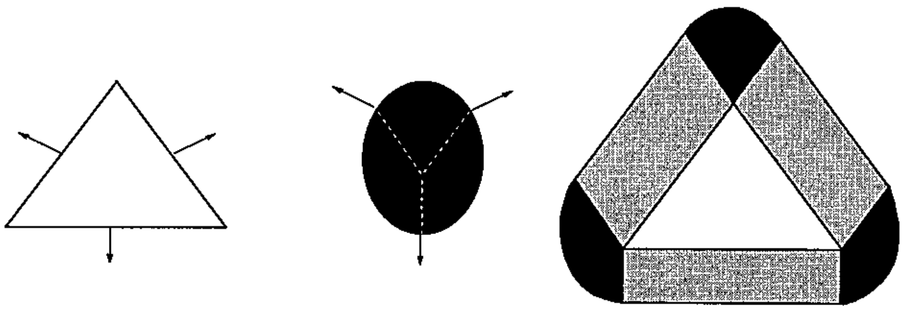


Figure 17: Minkowski sum of a triangle and a convex body of smooth boundary

Remark: When the two convex bodies have at every point of their boundary, a strictly positive curvature, the boundary of $Q_1 + Q_2$ is the set of points $\{m_1(u) + m_2(u), u \in S^1\}$.

Proposition 4.3.1 *The volume of the Minkowski sum $\lambda Q_1 + \mu Q_2$ is an homegeneous polynomial in λ and μ:*

$$vol(\lambda Q_1 + \mu Q_2) = \lambda^2 vol Q_1 + (\lambda \cdot \mu) V(Q_1, Q_2) + \mu^2 vol Q_2$$

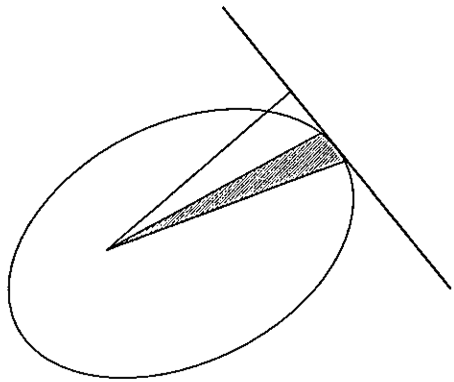


Figure 18: aera of a convex body in terms of the support function

Proof: We will compute the area of a convex body Q in terms of its support function h. Mixing the support function h and the arc length ds of the boundary ∂Q one gets (see fig)

$$vol(Q) = \int_{\partial Q} hds$$

An unambiguous , but heavier notation would be:

$$vol(Q) = \frac{1}{2} \int_{\partial Q} h(N(c(s))ds$$

where $c : S^1_{lenght(\partial Q)} \rightarrow \partial Q$ is a parametrisation by arc length of ∂Q, and $N(c(s))$ is the exterior normal at $c(s) \in \partial Q$. We have computed in terms of the support function the ratio between the arc length and the length swept by the normal:

$$\frac{ds}{du} = R(u) = |jac(G)| = h(u) + h''(u)$$

Here $R(u)$ denotes the radius of curvature of the envelope of the lines $< m|u >= h(u)$ at the caracteristic point $m(u)$. We get:

$$vol(Q) = \frac{1}{2} \int_{S^1} h(h + h'')du$$

Recalling that the support function of the Minkowski sum of $\lambda Q_1 + \mu Q_2$ is: $h_1 + h_2$ (h_1 and h_2 being the support functions of Q_1 and Q_2), we get :

$$vol(\lambda Q_1 + \mu Q_2) = \frac{1}{2}\int_{S^1}(\lambda h_1 + \mu h_2)[(\lambda h_1 + \mu h_2) + (\lambda h_1 + \mu h_2)'']du$$

$$= \frac{1}{2}\int_{S^1}(\lambda h_1)(\lambda h_1 + \lambda h_1'') + \frac{1}{2}\int_{S^1}(\mu h_2)(\mu h_2 + \mu h_2'') +$$

$$+\frac{1}{2}\int_{S^1}(\lambda h_1)(\mu h_2 + \mu h_2'') + \frac{1}{2}\int_{S^1}(\mu h_2)(\lambda h_1 + \lambda h_1'')$$

The first two integrals are respectively $\lambda^2 vol(Q_1)$ and $\mu^2 vol(Q_2)$. The sum of the two last ones is $\lambda\mu$ times an integral mixing the two support functions and the two radii of curvature. This integral $V(Q_1, Q_2)$ is called the *mixed volume* of Q_1 and Q_2. □

We "see" the mixed volume on fig *minkowski sum of a triangle and a convex of smooth boundary* above. It has also interesting interpretations in algebraic geometry see [Tei4].

4.4 Inequalities

Inequalities between functions of length, volume, mixed volume of convex bodies is a very rich topic, including isoperimetric inequalities. The interested reader can consult [Bo-Fe], [Schnei] for example.

5 Grassmann manifolds

5.1 Definition of vectorial and affine Grassmann manifolds

Let us now show that the set $G(n,p)$, called Grassmann manifold, of vectorial subspaces of dimension p of $\mathbb{R}^n$ has a natural structure of a $(n \cdot p)$-dimensional manifold. Consider a p-dimensional subspace h_0 of $\mathbb{R}^n$. Let us denote by $h_0^\perp$ its othogonal subspace ($h_0^\perp$ has dimension n-p). Any p-dimensional subspace h of $\mathbb{R}^n$ transverse to $h_0^\perp$ is the graph of a linear map L_h from h_0 to $h_0^\perp$, and any such graph is a p-dimensional subspace transverse to $h_0^\perp$. Chosing bases in h_0 and $h_0^\perp$ the matrix of that map is a $p \times (n-p)$ matrix. This procedure defines a chart of $G(n,p)$. Using all the p-dimensional subspaces of $\mathbb{R}^n$, we get an atlas of $G(n,p)$. It is, in fact, enough to consider the $\binom{n}{p}$ p-dimensional coordinate subspaces to get an atlas.

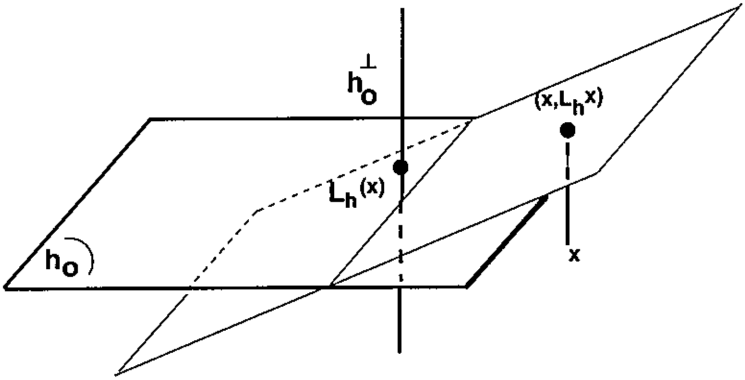


Figure 19: A chart of G(n,p)

Remark: The Grassmann manifold $G(n,1)$, that is the set of lines of $\mathbb{R}^n$, is the projective space $\mathbb{P}_{n-1}$. It is the quotient of the sphere S^{n-1} by the antipodal map, $q \mapsto (-q)$. The Grassmann manifold $G(n,n-1)$ is also diffeomorphic to $\mathbb{P}_{n-1}$ as you can see using the correspondance between a plane and its orthogonal line. Using the same diffeomorphism we can see that the hyperplanes containing a given line form a $\mathbb{P}_{n-2} \subset \mathbb{P}_{n-1}$. After defining a riemannian metric on $G(n,1)$ we shall see that the diffeomorphism $G(n,1) \rightarrow G(n,n-1)$ is also an isometry.

Using the action of the group of linear isometries on $G(n,p)$ we will prove that the Grassmann manifolds are compact.

Lemma 5.1.1 *The Group $O(n)$ of linear isometries of $\mathbb{R}^n$ is compact.*

Proof: The product $S^n \times S^n \times ... \times S^n$ (n times) is compact . The set of orthonormal bases of $\mathbb{R}^n$ is a closed subset of $S^n \times S^n \times ... \times S^n$, defined by the equations $< u_i | u_j >= 0$. It can be identified with the linear map which sends the canonical basis $(e_1, e_2, ..., e_n)$ to the orthogonal basis $(u_1, u_2, ..., u_n)$.The group O(n) is therefore a compact set. □

Theorem 5.1.2 *The Grassmann manifold $G(n,p)$ is homeomorphic to the quotient:*

$$SO(n)/SO(p) \times SO(n-p)$$

Proof: Let us first prove that the two sets are the same. The image by an element g of $O(n)$ of the p first vectors $(e_1, e_2, ..., e_p)$ of the canonical basis generate a p-dimensional subspace h of $\mathbb{R}^n$. Let us call

$$E_p : O(n) \to G(n,p)$$

this map. Let us now consider two isometries, $g_1 \in O(p)$ and $g_2 \in O(n-p)$. They determine an isometry $(g_1 \oplus g_2) \in O(n)$. The image of $g \circ (g_1 \oplus g_2)$ is again h. A subspace h of dimension p admits an orthogonal basis $(u_1, u_2, ..., u_p)$; the orthogonal $h^{\perp}$ admits an orthogonal basis $(u_{(p+1)}, ..., u_n)$ and the basis $(u_1, ..., u_n)$ is the image by a linear map of the form $g \circ (g_1 \oplus g_2)$ of the canonical basis $(e_1, ..., e_n)$. Therefore, the kernel of the map E_p is the subgroup $[(g_1 \oplus g_2)] \in O(n)$. This proves the set equality $G(n,p) = O(n)/[O(p) \times O(n-p)]$. To prove that the topologies coincide, one needs essentially to prove that the map from an orthogonal system $(u_1, ..., u_p)$ to the linear subset h it generates is continuous. This is easy, lengthy and boring, therefore we "leave that proof to the reader". □

Remark: Equally exciting is to prove that the topology on $G(n,p)$ obtained using the Haussdorff distance on the intersections of p-dimensional subspaces with the closed unit ball (or with the unit sphere) is again the same as the manifold topology.

Remark: The orthogonality in $\mathbb{R}^n$ provides a diffeomorphism between $G(n,p)$ and $G(n,(n-p))$. This diffeomorphism is an isometry for the riemannian metrics invariant by the action of the isometries we define below.

The set $\mathcal{A}(n,p)$ of affine p-dimensional subspaces form a fiber space over $G(n,(n-p))$ with fiber $\mathbb{R}^{(n-p)}$. The fibration map associates to a p-dimensional affine subspace H of $\mathbb{R}^n$ its orthogonal complement $h^\perp$. The intersection $H \cap h^\perp$ gives the isomorphism between the fiber and $\mathbb{R}^n$.

5.2 Metrics and measures

The group of linear isometries of $\mathbb{R}^n$ acts on $G(n,p)$. It is natural to look for a metric on $G(n,p)$ which is invariant by this action. to do that, first observe that our charts

$$\{linear\,maps\,h \mapsto h^\perp\}$$

give also the tangent space in h to $G(n,p)$. The euclidean metric of $\mathbb{R}^n$ allows us to choose an orthogonal basis in h and in $h^\perp$. Let us put on the $(p \times (n-p))$ matrix space the natural euclidean norm:

$$|\mathcal{M}|^2 = \sum (squares\,of\,the\,coefficients)$$

This defines on $G(n,p)$ a riemannian metric invariant by the action of the linear isometries. We leave as an exercise for the reader to check that the covering map from $S^{(n-1)}$ to $G(n,1)$ is a local isometry.

The measure associated to this riemannian metric is also invariant by the group of linear isometries.

Remark: The previous results can be rephrased in terms of homogeneous spaces. One then observes that the measure defined above is a quotient of the Haar measure on $O(n)$, and that the metric we defined on $G(n,p)$ is such that the projection $O(n) \to G(n,p)$ is a riemannian submersion.

6 The Gauss map and what can be done in higher codimension

6.1 Gauss map and principal curvatures

We consider first the case of an embedded submanifold M of $\mathbb{R}^n$. It is then oriented (the normal vector $N(m)$ in $m \in M$ points out of the bounded component of $\mathbb{R}^n \setminus M$), we can define the Gauss map:

$$\gamma : M \to S^{n-1}, m \mapsto N(m)$$

We will also consider a projective Gauss map, also denoted by γ when there will be no ambiguity, using the line $L(m)$ normal in m to M:

$$\gamma : M \to \mathbb{P}_{n-1}, m \mapsto L(m)$$

Its critical values are images under the natural projection of the critical values of the (spherical) Gauss map and the critical points of both Gauss maps are the same. The *Gauss (or Gauss-Kronecker) curvature* $K(m)$ in $m \in M$ is the jacobian at m of the (spherical) Gauss map. The eigenvalues of $d\gamma(m)$: $k_1, k_2, ..., k_{n-1}$ (there may be repetitions) are called the *principal curvatures* of M in m. To each corresponds an eigenvector e_i, and these eigenvectors can be chosen to form an orthonormal basis. The *second fundamental form* $II(m)$ is defined by:

$$II(m)(v) = < d\gamma(m)(v)|v >$$

It can be diagonalised in an orthonormal basis, precisely the one we have chosen before to diagonalise $d\gamma$. The *symmetric functions of curvature* are the coefficients of the polynomial

$$det[Id + td\gamma(m)] = \prod(1 + k_1 t)(1 + k_2 t)...(1 + k_{n-1} t) = \sum_0^{n-1} \sigma_i(m) t^i$$

When possible, we shall drop the point m in $\sigma_i(m)$.

Remark: Consider an i-dimensional subspaces $h \subset T_m M$. In a neighbourhood of the point m the intersection $M \cap (h \oplus L(m))$ is an hypersurface of $h \oplus L(m)$. Denote by $K(m, h)$ the Gauss-Kronecker curvature of this last hypersurface, oriented by $N(m)$.

Proposition 6.1.1

$$\sigma_i(m) = const \cdot \int_{G(T_mM,i)} K(m,h)$$

where $G(T_mM,i)$ is the Grassmann manifold of i-dimensional subspaces $h \subset T_mM$ and const is a constant depending only on dimensions.

The proof amounts to compare the integral of the proposition with a trace of $\wedge^i(\gamma)$ acting on the exterior algebra $\bigwedge^i(T_mM)$.("folklore", [Lan5]).

We can locally write an equation:

$$x_n = f(x_1, x_2, ..., x_{n-1})$$

for M, choosing the first (n-1) coordinates to be on axes generated by the vectors e_i and the last on the axis generated by the normal $N(m)$. Then the Hessian of f at m is a diagonal matrix with entries $k_1, k_2, ..., k_{n-1}$. This proves the:

Proposition 6.1.2 *The point m is a degenerate critical point of the orthogonal projection of M on the line $L(m)$ generated by the normal $N(m)$ if and only if m is a critical point of the projective Gauss map.*

Corollary 6.1.3 *The set of lines $L \in \mathbb{P}_{n-1}$ such that the projection p_L on L admits degenerated critical points is of zero measure.*

Proof: By Sard theorem, those lines, which are critical values of the projective Gauss map, form a set of measure zero. □

The proof of the Exchange theorem of section 3 can be copied to get the:

Theorem 6.1.4 *Exchange theorem in codimension 1*

$$\int_M |K(m)|dm = \int_{\mathbb{P}_{n-1}} |\mu|(M,L)dL$$

6.2 Lipschitz-Killing curvature

Suppose now that M is a submanifold of codimension $p > 1$ of $\mathbb{R}^N$. The dimension of M is n. We denote by $\mathcal{N}(M)$ the normal bundle of M and by $\mathcal{N}(m)$ its fiber: $(T_mM)^\perp \subset T_mM$. We can either

- Define a generalised Gauss map from the unit normal bundle $\mathcal{N}^1(M)$ of M to S^{N-1} by $\gamma(m,v) = w$. Denote by $K(m,v)$ its jacobian at the point $(m,v) \in \mathcal{N}^1(M)$. This makes sense as the unit normal bundle has a natural metric, induced by its embedding in $T\mathbb{R}^N$, which makes the bundle projection a riemannian submersion:

$$\tilde{g} \mid_{fiber} = \text{restriction of the ambient euclidean metric}$$

$$\tilde{g} \mid_{horizontal\, space} = \text{pull back of the metric of } M$$

We also define the projective normal bundle $P\mathcal{N}(M)$ as the quotient of $\mathcal{N}^1(M)$ by the antipodal map on each fiber; we denote by $P\mathcal{N}(m)$ the fiber of this bundle.

1)The Lipschitz-Killing curvature of M at m is:

$$K(m) = 1/2 \int_{\mathcal{N}_1(m)} K(m,v)$$

When the dimension of M is even $K(m,v) = K(m,(-v))$ so we can write:

$$K(m) = \int_{\mathbb{P}\mathcal{N}(m)} K(m,v)$$

2)The absolute curvature of M at m is:

$$|K|(m) = \int_{\mathbb{P}\mathcal{N}(m)} |K(m,v)|$$

Notice that in general $|K|(m) \neq |K(m)|$.

- Consider, for each $v \in \mathcal{N}^1(m)$ the orthogonal projection $p_{m,v}$ of a neighbourhood of m on the subspace $T_mM \oplus \mathbb{R} \cdot v$. At m we can compute the Gauss-Kronecker curvature of the hypersurface $p_{m,v}(neighbourhood\, of\, m)$. Let us call it also $K(m,v)$. The Lipschitz-Killing curvature and the absolute curvature are then obtained by the same formula as above:

$$K(m) = 1/2 \int_{\mathcal{N}_1(m)} K(m,v)$$

and:

$$|K|(m) = \int_{\mathbb{P}\mathcal{N}(m)} |K(m,v)|$$

Proposition 6.2.1 *The two definitions of $K(m,v)$ given above coincide.*

Proof: Let us take a point (m,v) of the unit normal bundle. If $K(m,v) \neq 0$, locally, the inverse image by the Gauss map of

$$\mathbb{R} \cdot v \oplus T_m M$$

is an n-dimensional submanifold $\mathcal{V} \in \mathcal{N}^1(M)$ transverse at (m,v) to the fiber $\mathcal{N}^1(m)$. Observe that if (x,w) is a point of $\mathcal{V}$, the vector w is orthogonal to $p_{m,v}(T_x M)$ at $p_{m,v}(x)$. Let $J(x,w)$ be the jacobian of the projection of $T_{x,w}\mathcal{V}$ onto the horizontal space $\mathcal{H}$. Almost by definition of the horizontal space it is also the jacobian of the restriction to $T_{m,v}\mathcal{V}$ of the differential of the projection of the fiber bundle $\mathcal{N}^1(M)$ onto its base space M. Using the splittings :

$$T_{m,v}\mathcal{N}^1(M) = \mathcal{H} \oplus T_{m,v}(\mathcal{N}^1(m))$$

$$\mathbb{R}^N = T_m M \oplus \mathcal{N}(m)$$

the linear map $dG(m,v)$ has the matrix:

$$\begin{pmatrix} (dG(m,v)|\mathcal{H}) & (0) \\ * & Id \end{pmatrix}$$

Therefore, using the first definition of $K(m,v)$:

$$K(m,v) = det(dG(m,v)|\mathcal{H}$$

As

$$G|\mathcal{V}(x,w) = w = \gamma(p_{m,v}(x)$$

One has, using the second definition of $K(m,v)$, which uses the projection $pm,v(M)$:

$$J(m,v)K(m,v) = detd(G|\mathcal{V}(m,v)) = J(m,v).det(dG|\mathcal{H})$$

□

An exchange theorem can now be stated in any dimension and codimention:

Theorem 6.2.2 *General exchange theorem*

$$\int_M |K|(m)|dm = \int_{\mathbb{P}_{n-1}} |\mu|(M, L)dL$$

Proof: Use the change of variable theorem for the map

$$G : \mathcal{N}^1(M) \to S^{N-1},$$

the first definition of the Lipschitz-Killing curvature, and use Sard's theorem as before. □

Example

Let C be a curve in $\mathbb{R}^3$.We will use the Frenet frame (T, N, B), $T(m)$ unit tangent vector to C in m given by the orientation, $N(m) = \frac{dT}{ds}$ and $B(m) = T(m) \wedge N(m)$. Let θ be the angle between a vector $v \in \mathcal{N}(m)$ and the principal normal $N(m)$ in the normal plane oriented by the base $N(m), B(m)$. then $K(m, v) = k(m) \cdot cos\theta$, where k(m) is the curvature of C at m. This proves:

Proposition 6.2.3 *For a space curve $C \subset \mathbb{R}^3$, the absolute curvature satisfies:*

$$|K|(m) = 2k(m)$$

Remark: Using our second viewpoint we can also associate to each projection $p_{m,v}(M)$ a second fundamental form $II_{m,v}$.

6.3 Total curvature of submanifolds

As in the previous section, M is an n-dimensional submanifold of codimension p of $\mathbb{R}^n$.

Definition 6.3.1 *The total curvature of M is :*

$$L_0(M) = \frac{1}{2|\mathbb{P}_{N-1}|} \int_M |K|$$

The constant is chosen in a way that round spheres Σ contained in an affine p-space of $\mathbb{R}^N$ satisfy $L_0(\Sigma) = 1$, extending the choice $L_0(point) = 1$,which one may view as the starting point of integral geometry!

Theorem 6.3.2 *Exchange theorem*

$$L_0(M) = \frac{1}{2|\mathbb{P}_{N-1}|}\int_{\mathbb{P}_{N-1}L_0(M,L)} = \frac{1}{2|\mathbb{P}_{N-1}|}\int_M |K|$$

where $L_0(M,L)$ is the number of critical points of the orthogonal projection of M on L.

Remark: From now on the notation $L_0(M,L)$ is more convenient than the usual one: $|\mu|(M,L)$,as it will give a nicer form to the reproductibility property of the p-length functional (see chapter **Blaschke formulas and kinematic formulas**).

Proof: It reduces to an application to the generalised Gauss map:

$$\gamma : \mathbb{P}\mathcal{N}(M) \to \mathbb{P}_{N-1}$$

of the coarea formula:

$$\int_{\mathbb{P}_{N-1}} \sharp(\gamma^{-1}(L) = \int_{\mathbb{P}\mathcal{N}(M)} |jac\gamma|$$

A point $m \in M$ is a critical point of the orthogonal projection p_L on the line L if and only if L is contained in the normal space at m to M, that is, if and only if $L \in \mathbb{P}(\mathcal{N}(m))$. This shows that the number $\sharp\gamma^{-1}(L)$ is just the number $L_0(M,L)$. Finally observe that for almost all lines L, the orthogonal projection on L is a Morse function (see [Mi2]), which implies that

$$L_0(M,L) = \sharp(critical\, values\, of\, p_L)$$

□

In particular, for curves and surfaces immersed in $\mathbb{R}^3$ we get:

Proposition 6.3.3 *Let C be a curve in $\mathbb{R}^3$, then:*

$$L_0(C) = \frac{1}{2|\mathbb{P}^2|}\int_{\mathbb{P}^2} L_0(C,L)$$

This formula is usually written as:

$$\int_C k = \frac{1}{2}\int_{\mathbb{P}^2} |\mu|(C,L)$$

Proposition 6.3.4 *Let M be a surface immersed in $\mathbb{R}^3$, then:*

$$L_0(M) = \frac{1}{2|\mathbb{P}^2|}\int_{\mathbb{P}^2} L_0(M,L) = \frac{1}{2|\mathbb{P}^2|}\int_M |K|$$

This formula is usually written as :

$$\int_M |K| = \int_{\mathbb{P}^2} |\mu|(M,L)$$

7 Higher dimensional convex bodies and related matters

7.1 Support function

As in the case of $\mathbb{R}^2$ let us consider a function $H : S^{(n-1)} \to \mathbb{R}$. To such a map corresponds a family (parametrised by $S^{(n-1)}$) of hyperplanes of $\mathbb{R}^n$:

$$u \mapsto h = \{x| < x|u >= H(u)\}$$

We have observed in part 4 (Prop 4.2.1) that a support function on S^1 defines the boundary of a convex body if $(h' + h")$ is everywhere strictly positive. A.D. Alexandrov [Al] observed that, if a support function $H : S^{n-1} \to \mathbb{R}$ satisfies $det[hessian(H) + Id \cdot H] > 0$ then the hyperplanes of equation $< m|u >= H(u)$ envelope the boundary of a convex body. The Minkowski sum is defined as in the dimension 2 case:

$$Q_1 + Q_2 = \{m_1 + m_2|m_1 \in Q_1, m_2 \in Q_2$$

In the same way as in the plane case, the mixed volumes $V(p, Q_1, q, Q_2), p+q = n$ appear as coefficient of the homogeneous polynomial $vol(\lambda Q_1 + \mu Q_2)$.

Theorem 7.1.1 *Let Q_1 and Q_2 be two compact convex bodies of $\mathbb{R}^n$. The volume of the convex body $(\lambda Q_1 + \mu Q_2)$ is an homogeneous polynomial of weight n in λ and μ:*

$$vol(\lambda Q_1 + \mu Q_2) =$$

$$= \lambda^n volQ_1 + \lambda^{n-1}\mu V(n-1, Q_1, 1, Q_2) + ... + \lambda^p \mu^q V(p, Q_1, q, Q_2) + ... + \mu^n volQ_2$$

Proof: We need, as before to observe that the support function of the Minkowski sum is the sum of the support functions of the convex bodies λQ_1 and μQ_2, and to use the formula:

$$volQ = \int_{S^{n-1}} H \cdot det[Hess(H) + Id \cdot H]$$

where again H is the support function of Q. □

7.2 Quermassintegrals and Steiner's formula

A particular case is the case where the second convex body is the unit ball $B(0,1)$. The Minkowski sum $Q + rB(0,1)$ is the thickened convex set:

$$Q_r = \{x | d(x, Q) \leq r\}$$

There are two other ways to compute $volQ_r$.

Proposition 7.2.1 *$vol(Q_r)$ is a polynomial in r, the coefficients of which are the symmetric functions of curvature defined in the previous paragraph:*

$$vol(Q_r) = volQ + \sum_{p=0}^{n-1} \frac{r^{p+1}}{p+1} \int_{\partial Q} \sigma_p$$

Proof: Let us consider the map ϕ_t from ∂Q to ∂Q_t defined by:

$$\phi_t : m \mapsto m + tN(m), 0 \leq t \leq r$$

which, for a fixed t, maps ∂Q to ∂Q_t. The reader can check that $T_m \partial Q$ and $T_{m+tN(m)} \partial Q_t$ are parallel. We can compute the jacobian $|det(d\phi_t)$:

$$|det(d\phi_t)| = |det Id + t d\gamma(m)| = \sum_{p=0}^{n-1} \sigma_p t^p$$

Integrating on ∂Q, and for $0 \leq t \leq r$, one gets:

$$volQ_r = volQ + \sum_{p=0}^{n-1} \frac{r^{p+1}}{p+1} \int_{\partial Q} \sigma_p$$

□

To state a second way of computing $volQ_r$ we need first to define the Quermassintegrales of the compact convex body Q.

Definition 7.2.2

$$M_{p+1}(Q) = \int_{G(n,n-1-p)} vol(p_h(Q))$$

where p_h is the orthogonal projection on the (n-1-p)-dimensional space h.

In particular M_1 is the volume of ∂Q. By convention M_n is 2.

Theorem 7.2.3 *Steiner's formula*

$$volQ_r = volQ + \sum_{p=0}^{n-1} \binom{n}{p+1} M_{p+1}(Q) \cdot r^{p+1}$$

Proof: The proof uses induction on the dimension. The convex Q_r is the union of Q and the parallel hypersurfaces $\partial Q_{t_1}, 0 \leq t_1 \leq r$. Therefore:

$$volQ_r = volQ + \int_0^r vol(\partial Q_{t_1}) dt_1$$

Let us compute $vol(\partial Q_{t_1})$ using Cauchy's formula:

$$vol(\partial Q_{t_1}) = \frac{n-1}{\omega_{n-2}} \int_{G(n,n-1)} vol(p_h(\partial Q_{t_1})$$

where ω_{n-2} is the volume of the unit (n-2)-sphere, and h is a hyperplane of $\mathbb{R}^n$. The projection $p_h(\partial Q_{t_1})$ is the Minkowski sum of the projection $p_h(\partial Q)$ and the ball $B(0, t_1)$ of radius t_1 in h. Therefore by the induction hypothesis it is a polynomial in t_1:

$$vol(p_h(\partial Q_{t_1})) = vol(p_h \partial Q) + \sum_{p=0}^{n-2} \binom{n-1}{p+1} M_{p+1}(p_h \partial Q) t_1^{p+1}$$

Integrating the constant term, for $0 \leq t_1 \leq r$ will give the coefficient of r in Steiner's formula. To get the other terms we proceed with the induction. Let $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n$ be a flag of nested subspaces of codimension (q,...,2,1) of $\mathbb{R}^n$. The projection p_{h_q} satisfies:

$$p_{h_q} = p_{h_q} \circ p_{h_{q-1}} \circ ... \circ p_{h_2} \circ p_{h_1}$$

We call *flaf space* the set of $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n$. The natural map $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n \mapsto h_i; 1 \leq i \leq q$ defines , for each i, a fibration of total space the flag space and base the Grassmann manifold $G(n, n-i)$. These fibrations endowed with natural metrics we shall not explicit in general inherit measures invariant by the action of the group of isometries which can be locally decomposed in the product of a measure on the fiber and a measure on the base. This justifies our frequent use of Fubini's theorem, in particular when a given flag space admits two different projections on two different Grassmann manifolds. Integrating on the *flag space*

$$\mathcal{F}(n, n-1, ...n-q) = \{\mathbb{R}^n \supset h_1 \supset h_2 \supset h_{n-q}\}$$

we get :

$$const \cdot \int_{\mathcal{F}(n,n-1,...n-q)} vol(p_{h_q}(\partial Q)) = \int_{G(n,n-q)} vol(p_{h_q}(Q)) = M_q$$

□

Remark: Identifying the coefficient of r^p in the two expressions of $volQ_r$, we get an equality between a quermassintegrale and an integral of a symmetric function of curvature on ∂Q.

7.3 Orthogonal projections, polar varieties, and p-length of an n-dimensional submanifold of $\mathbb{R}^n$

In this paragraph we shall modify the definition of Quermassintegrale so that it can be extended to any submanifold, and will also carry a sign information.

Definition 7.3.1 *Let Γ_h be the set of critical points of the orthogonal projection p_h of M on h, and let $\gamma_h = p_h(\Gamma_h)$ be the critical locus of p_h. We shall call Γ_h a polar variety .*

It is not in general a manifold but is one almost everywhere for almost every h.

In this paragraph we shall often use the word *generically* which means: "up to a suitably chosen measure zero set", the measure should be natural, and the choice is often part of a nontrivial theorem involving sometimes a computation in a jet space. A theorem of Thom [Th] implies that generically (for almost every h) γ_h is almost everywhere a (dim(h)-1)-submanifold of h if $dim(M) \geq dim(h) - 1$. If $dim(M) \leq dim(h) - 1$ then γ_h is just $p_h(M)$ and has generically the same dimension as M. Moreover generically the projection of Γ_h on γ_h is one-to-one and a local diffeomorphism.

Polar varieties will appear again in the study of foliations and of complex singularities.

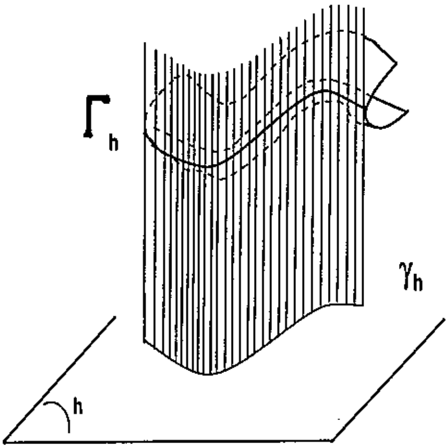


Figure 20: The polar curve Γ_h and its projection γ_h

Instead of proving the affirmation, we shall justify it, by describing Γ_h (and its projection γ_h) in a neighbourhood of a point m where M is not flat.

Proposition 7.3.2 *Let h be a linear subspace of $\mathbb{R}^N$ of dimension n and let $M^n \subset \mathbb{R}^N$ be an n-dimensional submanifold. Let m be a critical point of the orthogonal projection p_h on h. Let H be the affine subspace orthogonal to h and containing m. Let v be a unit vector contained in $(T_mM)^\perp \cap h$ and w be a unit vector contained in $T_mM \cap h^\perp$. Then, if $II_{m,v}(w)$ is different from zero, the polar variety Γ_h is transverse to $T_mM \cap h^\perp$.*

Proof: Choose a local parametrisation Φ of M such that

$$\frac{\partial \Phi}{\partial t_1}(m) = w \in (h^\perp \cap T_mM), \frac{\partial \Phi}{\partial t_i}(m) \in (\mathbb{R}w)^\perp for\, i > 1$$

Then at m,

$$det[p_h(\frac{\partial \Phi}{\partial t_1}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})](m) = 0$$

The derivative at m of that determinant is different from 0:

$$\frac{\partial}{\partial t_1} det[p_h(\frac{\partial \Phi}{\partial t_1}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})](m) =$$

$$= det[p_h(\frac{\partial^2\Phi}{\partial t_1^2}), p_h(\frac{\partial\Phi}{\partial t_2}), ..., p_h(\frac{\partial\Phi}{\partial t_n})](m) +$$

$$+ \sum_{i\geq 1} det[p_h(\frac{\partial\Phi}{\partial t_1}), ..., p_h(\frac{\partial^2\Phi}{(\partial t_i)^2}), ..., p_h(\frac{\partial\Phi}{\partial t_n})]$$

So:

$$\frac{\partial}{\partial t_1} det[p_h(\frac{\partial\Phi}{\partial t_1}), p_h(\frac{\partial\Phi}{\partial t_2}), ..., p_h(\frac{\partial\Phi}{\partial t_n})] =$$

$$= det[p_h(\frac{\partial^2\Phi}{(\partial t_1)^2}), p_h(\frac{\partial\Phi}{\partial t_2}), ..., p_h(\frac{\partial\Phi}{\partial t_n})]$$

as we have chosen the coordinates such that $p_h(\frac{\partial\Phi}{\partial t_1})(m) = 0$. It is not difficult now to check that the component of $p_h(\frac{\partial^2\Phi}{(\partial t_1)^2})$ on w is non zero if and only if $II_{m,v}(w)$ is non zero.

In a similar way, when h is p-dimensional we have the:

Proposition 7.3.3 *Let h be a linear subspace of $\mathbb{R}^N$ of dimension $p \leq n$ and let $M^n \subset \mathbb{R}^N$ be an n-dimensional submanifold. Let m be a critical point of the orthogonal projection p_h on h. Let H be the affine subspace orthogonal to h and containing m. Let v be a unit vector contained in $(T_mM)^\perp \cap h$. Then, if $II_{m,v} \mid T_mM \cap h^\perp$ is non degenerate, the polar variety Γ_h is transverse to $T_mM \cap h^\perp$.*

Proof: As the second fundamental form $II_{m,v} \mid T_mM \cap h^\perp$ is symmetric we can choose a basis $(b_1, ..., b_{n-p+1})$ of $T_mM \cap h^\perp$ made of eigenvectors. The polar variety Γ_h is the intersection of the polar varieties Γ_{h_j} where the n-dimensional spaces h_j are generated by h and all the vectors of the base $(b_1, ..., b_{n-p+1})$ except b_j. Then we can apply the previous proposition to the projections p_{h_j}. □

Definition 7.3.4 *The p-length of M, $L_p(M)$ is:*

$$L_p(M) = C(N, n, p) \int_{G(N,p+1)} |\gamma_h| dh$$

where $|\gamma_h|$ denotes the volume of γ_h (when $p = O$, γ_h is a finite set and $|\gamma_h$ is the number of points $\sharp(\gamma_h)$ of γ_h). The constant $C(N, n, p)$ is *chosen so that if M is the boundary of an ε-tubular neighbourhood of a p-dimensional submanifold C of* $\mathbb{R}^N$, *then:*

$$lim_{\epsilon\to O} L_p(M) = |C|$$

If tM denotes an homothetic image of M by an homothety of ratio $t > 0$ then:

$$L_p(tM) = t^p L_p(M)$$

This motivates the choice of the constant $\frac{1}{2|\mathbb{P}_{N-1}|}$ occuring in the definition of L_0, since a sphere of any dimension (≥ 1) satisfies $|\gamma_L| = 2$ for every line $L \in G(N,1) = \mathbb{P}_{N-1}$, and in particular so does a small sphere of radius ϵ centred at a point p.

The functional L_1 has been applied to measure the ability of an algae to house little mobile marine animals (see [Ja-La]). As an exercise, the reader may check the value of the constant in the definition of L_1 when M is a surface in $\mathbb{R}^3$:

$$L_1(M) = \frac{1}{\pi^2}\int_{G(3,1)} |\gamma_h|$$

Hint: Compare the projections of a round cylinder and of its axis on the plane $h \in G(3,2)$

In section 8 we will show that the functionals L_p satisfy a linear kinematic formula relating them to the functional L_0.

□

7.4 Tubes (2)

The main tool to add a sign information to the varieties γ_h is d'Ocagne's theorem. Let M be an oriented surface of $\mathbb{R}^3$ and let h be a plane. Let m be a critical point of the orthogonal projection of M on h such that $II_m(w) \neq 0$, where w is a unit vector generating $h^\perp$. Then we have seen in the previous subsection that the projection $p_h(\Gamma_h \cap v(m))$ of the critical points of $p_h\,|_M$ contained in a neighbourhood $v(m)$ of m form an oriented curve γ_h in a neighbourhood of $p_h(m)$.

Theorem 7.4.1 *d'Ocagne's theorem* *The Gauss curvature of the surface M at m is related to the normal curvature* $II_m(w)$ *in the direction w and the curvature* k_{γ_h} *of* γ_h *at* $p_h(m)$ *by:*

$$K(m) = II_m(w)\cdot k_{\gamma_h}(p_h(m))$$

Proof: First recall that the orientation of M imposes the choice of the normal vector $N(m)$ used in the definition of the Gauss map and of the second fundamental form. This normal vector $N(m)$ belongs to h, and therefore is normal to γ_h at $p_h(m)$, define the orientation of γ_h. The vector $N(m)$ is also normal at m to the curve $C = M \cap (h^\perp \oplus \mathbb{R}N(m))$. Meusnier's theorem implies in particular that the curvature at m of the curve C is $II_m(w)$. We will now compute $d\gamma(m)$ using at the target the orthogonal basis (w, e), where e is a unit vector tangent to γ_h at $p_h(m)$, and at the source the basis(not orthogonal but of determinant one) (w, ϵ), where ϵ is a tangent vector to Γ_h at m such that $p_h(\epsilon) = e$. The matrix of $d\gamma(m)$ is:

$$\begin{pmatrix} II_m(w) & 0 \\ * & d\tilde{\gamma}(p_h(m)) \end{pmatrix}$$

where $\tilde{\gamma}$ is the Gauss map associated to γ_h. Therefore:

$$K(m) = det \begin{pmatrix} II_m(w) & (0) \\ * & d\tilde{\gamma}(p_h(m)) \end{pmatrix}$$

□

Remark: Suppose m is a point of negative Gauss curvature $K(m) < 0$. If $h^\perp$ is an asymptotic direction of T_mM, then $II_m(w) = 0$ and the critical curve γ_h will have a cusp at $p_h(m)$. As the curvature goes to infinity when a point approches a cuspidal point, this agrees with d'Ocagne's theorem.

The generalisation of this theorem to higher dimension hypersurfaces $M^{n-1} \subset \mathbb{R}^n$ is straigtforward. The subspace h is now p-dimensional, γ_h has generically dimension p-1, and the intersection $C = M \cap (h^\perp \oplus \mathbb{R}N(m))$ is now an hypersurface of the (n-p+1)-dimensional affine space (containing the point $m \in M$), $(h^\perp \oplus \mathbb{R}N(m))$. At m, C is oriented by $N(m)$, so we can compute the Gauss-Kronecker curvature $K(C, N(m), M)$ of C at m. As γ_h is also oriented by $N(m)$ at $p_h(m)$, the Gauss-Kronecker curvature $K(\gamma_h, N(m), p_h(m))$ is also well-defined.

Recall that in the previous paragraph we showed that if the restriction to $(h^\perp \oplus \mathbb{R}N(m))$ of the second fundamental form II_m is non degenerate, that is if $K(C, N(m), m)$ is different from zero, then Γ_h is, in a neighbourhood of m, transverse to $h^\perp$.

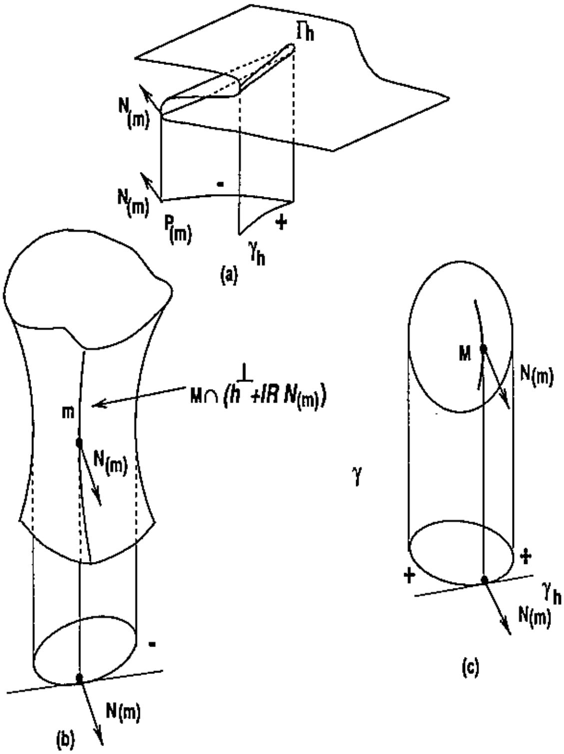


Figure 21: γ_h with a sign

Theorem 7.4.2 *Let h be a p-dimensional subspace of $\mathbb{R}^n$ and M an hypersurface. If $K(C, N(m), m)$ is different from zero, then :*

$$K(m) = K(C, N(m), m) \cdot K(\gamma_h, N(m), p_h(m))$$

Proof: Use at the target an orthonormal basis

$$(e_1, e_2, ..., e_{n-p}, \epsilon_1, ..., \epsilon_{p-1}),$$

split between $h^\perp$ and $T_{p_h(m)}\gamma_h$, and at the source the basis of determinant one: $(e_1, ..., e_{n-p}, \alpha_1, ..., \alpha_{p-1})$ where α_j is a vector tangent to Γ_h at m satisfying $p_h(\alpha_j) = \epsilon_j$, and repeat the previous computation. □

That way we can see γ_h as a weighted variety (or a chain), weighting generically the points $\omega \in \gamma_h$ with the sign $\epsilon(\omega)$ defined below:

Definition 7.4.3

$$\epsilon(\omega) = sign[K(C, N(m), m)]$$

where m is the (generically unique) point in γ_h which projects on ω and where $C = (h^\perp \oplus I\!RN(m)) \cap M$ is the oriented "vertical" intersection considered above.

We need now to define the sign $\epsilon(\omega)$ when M is of codimension higher than one. Each generic projection on a p-dimensional space h determines two varieties Γ_h and γ_h. At a generic point $\omega \in \gamma_h$ a normal line ν is well-defined. When the dimension of $C = M \cap (h^\perp \oplus \nu)$ is even, the sign of the Gauss-Kronecker curvature of the orthogonal projection of C on $T_mC \oplus \nu$ does not depend on the choice of the unit vector generating the line ν. So we can still define

$$\epsilon(\omega) = sign(K(C, m, \nu))$$

when the dimension $dim(C) = n - p + 1$ is even.

A d'Ocagne theorem will still be valid, for generic h and $m \in \Gamma_h$, when the dimension of M (and γ_h) will also be even:

$$K(M, m, \nu) = K(C, m, \nu) \cdot K(\gamma_h, p_h(m), \nu)$$

In particular, when h is a line L, γ_L is generically finite and $\epsilon(\omega)$ is well defined if M is even dimensional, or if M is an oriented hypersurface. Then:

$$\epsilon(\omega) = sign(K(M, m, L)) \, or \, sign(K(M, m, N(m)))$$

In the first case it coincides with $(-1)^{index(m)}$, where the index is the Morse index of the critical point m of the Morse function p_L (its parity does not depend on the orientation of L, as $dim(M)$ is even).

Definition 7.4.4 *We will call γ_h^+ the chain obtained by considering along γ_h the almost everywhere defined weight $\epsilon(\omega)$*

Definition 7.4.5 *We will call $\mid \gamma_h^+ \mid$ the integral:*

$$\mid \gamma_h^+ \mid = \int_{\gamma_h} \epsilon(\omega) d\omega$$

D'Ocagne's theorem implies that the sign $\epsilon(\omega)$ behaves nicely through compositions of projections. Let us consider a flag $h_1 \subset h_2 \subset ... \subset h_k$ of nested linear subspaces of $\mathbb{R}^N$ such that $dim(h_k) < dim(M)$. Let $m \in \Gamma_h$ be a critical point of p_{h_1} such that $K(M \cap (h_1^\perp \oplus \mathbb{R}N(m))$ is not zero. Let $\omega_1 = p_{h_1}(m), \omega_2 = p_{h_2}(m), ..., \omega_k = p_{h_k}(m)$. Suppose also that the projection of $\gamma_{h_{i+1}}$ on γ_{h_i} is such that the curvature $K(\gamma_{h_{i+1}} \cap [(h_i)^\perp \cap h_{i+1} \oplus \mathbb{R}N(m)], N(m))$ is non zero at ω_{i+1}. Then we can define the sign:

$$\epsilon(i+1,i) = sign(K(\gamma_{h_{i+1}} \cap [(h_i)^\perp \cap h_{i+1} \oplus \mathbb{R}N(m)], N(m)))$$

Similarly, projecting γ_j on γ_i for $j > i$, we can define an index

Definition 7.4.6

$$\epsilon(h_j, h_i) = \epsilon(j,i) = sign(K(\gamma_{h_j} \cap [(h_i)^\perp \cap h_j \oplus \mathbb{R}N(m)], N(m)))$$

Proposition 7.4.7 *The signs $\epsilon(j,i)$ multiply in a nice way:*

$$\epsilon(j,i) = \epsilon(j,l) \cdot \epsilon(l,i) \, if \, j < l < i$$

and in particular:

$$\epsilon(\omega) = \prod_{n-1}^{p} (\epsilon(i+1,i)$$

We can now apply Steiner's method to compute the volume of $Tub_r(M)$, and $Th_r(M)$ when M is of codimension 1, replacing the Quermassintegrales by the signed lengths $\mid \gamma_h^+ \mid$. This is what we have already done for plane curves in §3. Let us prove a theorem for compact surfaces in $\mathbb{R}^3$. Its generalisation to $M^n \subset \mathbb{R}^N$ is natural but cumbersome.

Theorem 7.4.8 *The volume of the thickening $Th_r(M)$ of the compact oriented surface M immersed in $\mathbb{R}^3$ is:*

$$volTh_r(M) = r[vol(\partial M) + r\frac{1}{3\pi}\int_{G(3,2)} \mid \gamma_h \mid + \frac{1}{3}r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid$$

Proof: To prove the formula, we have to compare two functions on the plane h defined using the vertical (orthogonal to h) affine lines L_y through points $y \in h$:

$$\varphi_{h,0}(y) = \sharp(L_y \cap M)$$

$$\varphi_{h,t}(y) = \sharp(L_y \cap M_t)$$

where M_t is the surface:

$$M_t = \{m + tN(m), m \in M\}$$

Let us also denote by $\gamma_{h,t}$ the critical locus of the orthogonal projection of M_t on h, and by $\gamma_{h,t}^+$ the corresponding weighted curve.

The discontinuity locus of $\varphi_{h,0}$ is contained in the curve γ_h, the distribution derivative of $\varphi_{h,0}$ is γ_h^+. In the same way, the distribution derivative of $\varphi_{h,t}$ is $\gamma_{h,t}^+$.

Lemma 7.4.9 *For a given h, the difference $\varphi_{h,t} - \varphi_{h,0}$ is:*

$$\varphi_{h,t} - \varphi_{h,0} = \int_0^t \mid \gamma_{h,t}^+ \mid$$

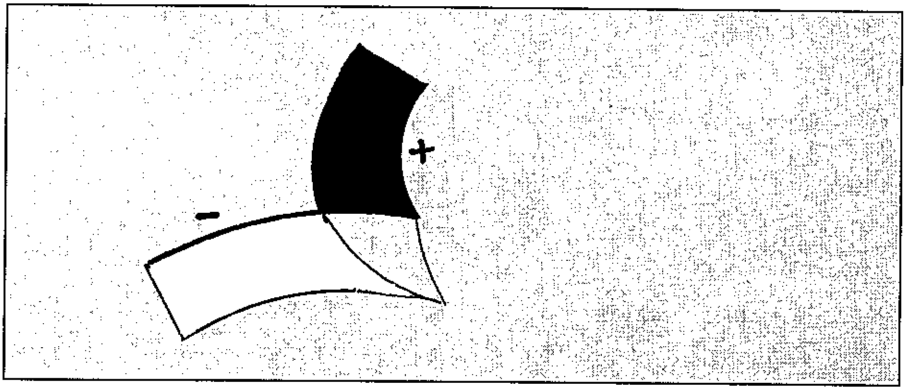


Figure 22: $\int \mid \gamma_{h,t}^+ \mid$

Proof: (of the lemma) The curve $\gamma_{h,t}$ is parallel to γ_h:

$$\gamma_{h,t} = \{\omega + tN(m), \omega \in \gamma_h\}$$

where m is the (generically unique) point of Γ_h which projects on $\omega \in \gamma_h$. For almost every h, almost every ω the curve γ_h is smooth in a neighbourhood of ω. Then so is $\gamma_{h,t}$ in a neighbourhood of $\omega + tN(m)$; the vector $N(m) = N(\omega)$ is orthogonal to all the curves $\gamma_{h,\tau}, 0 \leq \tau \leq t$ at the point $\omega + \tau N(m)$.

It is clear that, out of the union of the curves $\gamma_{h,\tau}, 0 \leq \tau \leq t$, the functions $\varphi_{h,0}$ and $\varphi_{h,t}$ are equal. In a neighbourhood of a small smooth arc α_1 of $\gamma_{h,t}$, itself of the form $\alpha_1 = \{\omega + tN(\omega), \omega \in \alpha \subset \gamma_h\}$, we can take a patch of the form:

$$\{\omega_1 + \theta \cdot N(\omega), \omega_1 = \omega + tN(\omega), \omega \in \alpha\}$$

On this patch the difference $(\varphi_{h,t+t_1} - \varphi_{h,t}$ is $2 \cdot \epsilon(\omega)$. The area of the patch is $\int_t^{t+t_1} |\gamma_{h,\theta}|$. Then the functions $\varphi_{h,t}$ and $\varphi_{h,0}$ may have different values in $y \in h, y \notin \gamma_h, y \notin \gamma_{h,t}$ only if y belongs to some curve $\gamma_{h,\theta}, 0 < \theta < t$; more precisely, if y is not a center of curvature of γ_h, then:

$$\varphi_{h,t}(y) - \varphi_{h,0} = \sum_{a \in A} \epsilon(a)$$

where A is the set:

$$A = \{a \in h \mid y = a + \tau_a N(m_a), \tau_a < t, p_h(m_a) = a \in \gamma_h$$

□

Let $p_{h,L}$ be the projection of the curve γ_h on a line $L \subset h$. We get a function $\varphi_{h,L,0}$ defined by:

$$\varphi_{h,L,0} = \sum_{u \in p_{h,l}^{-1}(z)} \epsilon(z)$$

Cauchy's formula implies that:

$$\mid \gamma_h^+ \mid = \int_{\mathbb{P}_1} \varphi_{h,L,O} dL$$

As the same is true for the curves $\gamma_{h,t}^+$, we need now to compare the functions $\varphi_{h,L,t}$ and $\varphi_{h,L,0}$

Notice that the projection of the cusp of γ_h is not critical for the projection $p_L : M \to L$, as the tangent to γ_h at that point is not orthogonal to L. We can compute the integral on L :

$$\int_L (\varphi_{h,l,t} - \varphi_{h,L,0}) = t[\sum_{critical\, points\, of\, p_L} \epsilon(h, L)(z)] \cdot \epsilon(\omega)$$

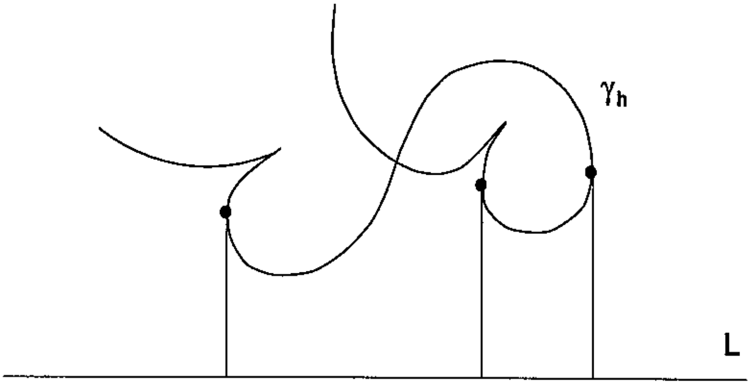


Figure 23: critical points of $p_{h,L}$

where we define the sign $\epsilon_{h,L}$ using the curve γ_h oriented by $p_h(N(m)) = N(m) = N(\omega)$; D'Ocagne's theorem proves that this integral is:

$$\int_L (\varphi_{h,l,t} - \varphi_{h,L,0}) = t[\sum_{critical\, points\, of\, p_l} \epsilon(z)]$$

We can now perform the same induction as for convex bodies to get:

$$volTh_r(M) = r[vol(M) + \frac{1}{\pi} r \int_{G(3,2)} \mid \gamma_h \mid + \frac{1}{3} r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid]$$

and integrating from $-r$ to $+r$

$$volTub_r(M) = 2r[vol(M) + \frac{1}{3} r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid$$

this formula gives the "usual" one:

$$volTub_r(M) = 2rvol(M) + \frac{4\pi}{3} \cdot \chi(M)$$

as $\mid \gamma_L \mid = \mu(M, L) = \chi(M)$. □

The universal constants in the general formulas are more complicated, but we can conclude that, up to universal constants depending only on the dimensions involved, the volume of $Tub_r(M)$ and of $Th_r(M)$ when M is an oriented hypersurface, are polynomials in r whose coefficients are the oriented p-lengthes $L_p^+(M) = \int_{G(N,p+1)} \mid \gamma_h^+ \mid$.

7.5 The localization of the p-lenghts L_p

In 1939 H.Weyl [Wey] has computed the volume of the tube $Tub_r(M)$ in another way, proving of course it is a polynomial in r, the coefficients of which are integrals on M of functions that can be computed from the curvature tensor. From the previous result we get equalities between Weyl's integrals of curvature and the oriented p-lengths.

A natural question is: is it possible to "localize" the (non-oriented) p-lengths $L_p(M)$?

The answer is positive. Let us first define the function $h_1(m)$ on a surface $M \subset \mathbb{R}^3$. In the chapter **The Gauss map and what can be done in higher dimensions**) we expressed the symmetric functions of curvature $\sigma_i(m)$ of an hypersurface as integrals of Gauss curvature of properly chose sections. Now define:

Definition 7.5.1

$$h_1(m) = \frac{1}{vol\mathbb{P}_1}\int_{\mathbb{P}_1(T_mM)} |k(m,l)|$$

Where $\mathbb{P}_1(T_mM) = G(T_mM, 1) = \{lines\ in\ T_mM\}$, *and* $\mid k(m,l) \mid$ *is the absolute value of the curvature at* m *of the curve* $M \cap (l \oplus L(m))$.

For future calculations it is useful to introduce the following notation. Let $p : E \to B$ be a riemannian fibration and $V \subset E$ a submanifold transverse to the fibers $F(y) = p^{-1}(y)$, $y \in B$. Let $\mathcal{H} = \{\mathcal{H}(x)\}$ be the horizontal plane field of the fibration.

The normal bundle $N \to M$ is endowed with a metric turning it into a riemannian fibration. At $x \in N$, T_xN is the orthogonal sum $t_x(N \cap F_{p(x)} \oplus V(x)$ where $V(x)$ is a subspace transverse to the fibers of complementary dimension as $\mathcal{H}(x)$. Denote by $Jacp_{\mathcal{H}(x)}$ the jacobian of the orthogonal projection of $V(x)$ to $\mathcal{H}(x)$. Then the coarea formula ([Bu-Za]) yields:

$$\int_N |Jacp_{\mathcal{H}(x)}|dx = \int_B |F(y) \cap N|dy$$

and more generally, if

$$\phi : M \to E$$

is an immersion transverse to the fibers, $N = \phi(M)$, then:

$$\int_M |Jac\phi||Jacp_{\mathcal{H}(x)}|dx = \int_B |F(y) \cap N|dy.$$

Now we can "localize" $L_1(M)$.

Proposition 7.5.2 *For M a surface in $\mathbb{R}^3$,*

$$L_1(M) = \frac{1}{\pi}\int_M h_1$$

Proof: Let $\pi : E = E(3,2) \to G(3,2) = G$ be the tautological line bundle, $E = \{l \in G, m \in l\}$.

Define also the projective tangent bundle of M: $\mathbb{P}_1(M) = \bigcup_{m\in M} \mathbb{P}(T_m M)$.

Let $\phi : \mathbb{P}_1(M) \to E$ be the map:

$$\phi(m,l) = (h = l^{\perp}, p_h(m))$$

where p_h is the orthogonal projection on the plane h, and let $\phi(\mathbb{P}_1(M)) = N$. We have just recalled that:

$$\int_G |\gamma_h| = \int_{\mathbb{P}_1(M)} |Jac\phi||Jacp_{\mathcal{H}}|,$$

so we compute the jacobians. Let l be a line through m in $T_m M$, $L(m) \subset T_m M$ denotes the line normal to M at m, $h = l^{\perp}$ the subspace of $\mathbb{R}^3$ orthogonal to l and W the orthogonal to $L(m)$ in h; see next picture. We choose a basis of $T_{(m,l)}(\mathbb{P}_1(M))$ as follows:

-U_f is a unit vector tangent to the circle fiber of $\mathbb{P}_1(M)$ at m

-U_Γ is a horizontal lift of a unit vector tangent to the polar curve Γ_h at m.

-U_l is a horizontal lift of a unit vector tangent to $(l \oplus L(m)) \cap M$ at m.

Also, let U_γ be a horizontal lift (in E) of a unit vector tangent to the critical locus γ_h at $y = p_h(m)$.

The volume of the parallelepiped generated by the first three vectors is $|cos\theta|$, where θ is the angle between $T_m\Gamma_h$ and h.

The image $d\phi(U_\Gamma)$ is the vector $\pm cos\theta \cdot U_\gamma$. The vectors $d\phi(U_f)$ and $d\phi(U_l)$ are projected by the differential $d\pi$ of the projection $\pi : E(3,2) \to G(3,2)$ on two orthogonal vectors of $T_{\pi\phi(m)}G(3,2)$, the first unitary and the second of norm $|k(m,l)|$.

Hence

$$|Jac\phi(m)||Jacp_{\mathcal{H}}| = |k(m,l)|,$$

and the proposition follows by integrating over the fibers of $\mathbb{P}_1(M)$.

Remark: A different proof of the proposition can be found in [La-Shi] based on a Meusnier's formula.

□

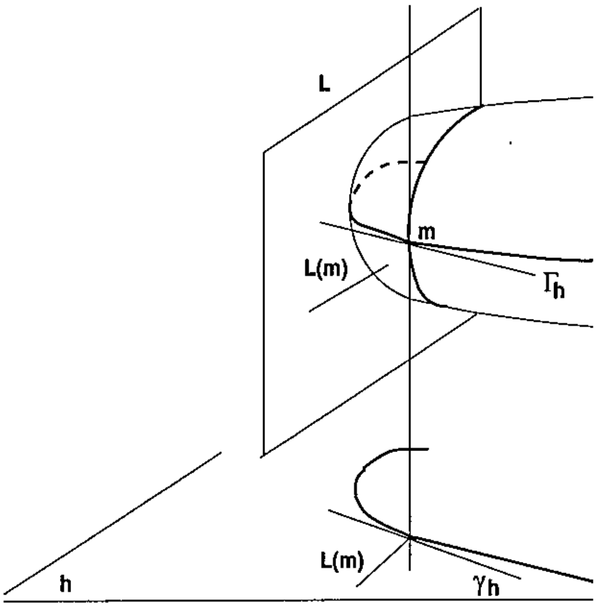


Figure 24: Localization of $L_(m)$

More generally we can define the functions $h_i(m)$ on an hypersurface $M \subset \mathbb{R}^n$. Let h be an i-dimensional subspace of $T_m M$, and $L(m)$ be the normal line to M at m. Denote by $|K|(x, h)$ the absolute value of the Gauss-Kronecker curvature at m of the hypersurface $M \cap (h \oplus L(m))$ of $h \oplus L(m)$.

Definition 7.5.3

$$h_i(m) = \frac{1}{volG(n-1,i)} \int_{G(T_m M, i)} |K|(m,h)dh,$$

where again $G(T_m M, i)$ is the set of i-dimensional subspaces of $T_m M$.

The next proposition is now natural:

Proposition 7.5.4 *The functions $h_{n-i}(m)$ localize the i-length $L_i(M)$; more precisely,*

$$\int_M h_{n-i}(m) = const \cdot L_i(M),$$

where the constant const depends only on dimensions.

The proof can be found in [La-Ro2].

The definitions of the function $h_i(m)$ in higher codimensions can also be found in [La-Ro2].

8 Blaschke's formulas and kinematic formulas

It is not by chance that the name "integral geometry" was used (and probabily invented by) Blaschke.[Bla]. One essential tool will now be a measure on the group of affine isometries invariant by left and right composition by an element of the group. Choosing an origin 0 of the euclidean plane we can write the group of affine isometries as the semidirect product:

$$\mathcal{G} = \mathbb{R}^2| \times SO(2)$$

The invariant measure is then $dg = |dv \wedge d\theta|$, where dv is the volume of $\mathbb{R}^2$, and θ the angle of the rotation. The existence of such an invariant volume on a Lie group is a more general phenomenon; see [Sa2].

8.1 Poincaré's formulas

The first directly generalises Cauchy's:

Theorem 8.1.1 *Poincaré's formula Let C_1 and C_2 be two compact arcs, then:*

$$\int_{\mathcal{G}} \sharp(C_1 \cap C_2) = 4length(C_1) \cdot length(C_2)$$

Proof: Let us consider the map

$$\Phi : C_1 \times C_2 \times S^1 \to \mathcal{G}$$

$$(m_1, m_2, \theta) \mapsto (translation\, m_1 \mapsto m_2) \circ R_\theta$$

to compute the jacobian the choice of the origin is irrelevant, so we can choose m_1, and see that it is: $|sin\phi|$, the angle at m_2 of $(translation\, m_1 \mapsto m_2) \circ R_\theta(C_1)$ and C_2. The coarea formula gives:

$$\int_{\mathcal{G}} \sharp(C_1 \cap C_2) = \int_{C_1 \times C_2 \times S^1} |sin\phi|$$

Integrating the left term on S^1 give the theorem. □

Remark: We can reformulate that proof, saying that the kinematic density satisfies locally

$$|dg| = |sin\theta| ds_1 \wedge ds_2 \wedge d\theta|$$

where θ is the angle at a point $P \in C_1 \cap g(C_2)$ of the two curves.

In the same vein is the:

Theorem 8.1.2 *Let $-\pi \leq \theta < \pi$ be the angle at an intersection point of the oriented curves C_1 and C_2. Then:*

$$\int_{\mathcal{G}} \sum_{C_1 \cap C_2} |\theta| = 2\pi length(C_1) \cdot length(C_2)$$

the only difference with the previous proof is that we need to compute $\int_0^\pi |\theta| \cdot |sin\theta| d\theta$.

8.2 Blaschke formulas

As usual in this book we will presentonly the simplest cases of the theory. A comprehensive reference is Santalò's book [Sa2] . Blaschke formulas compute averages of Euler characteristics of intersections of a compact domain with boundary of $\mathbb{R}^2$ and the image of another by all the isometries. The "miracle" is that averaging the Euler characteristic of the intersection $D_1 \cap D_2$ of two such domains on all affine isomètries, the result can be calculated using only integrals defined separately using D_1 and D_2. Let us attribute weight zero to area, weight one to length and weight two to integrals of curvature along curves. Just observe that the weight is related to their place in the formula giving the volume of a tube or in Steiner's formula. As often it is easier to prove first a formula "with no sign". So, let us first prove a formula for the total curvature of the boundary of a domain:

definition The total curvature of an arc piecewise of class $C^\in$ is:

$$\mathcal{T}(C) = \int_C |k| + \sum |\theta_i|$$

where the angles θ_i at the corners are oriented, defined by the oriented tangents to the two curves.

Then one has the :

Theorem 8.2.1

$$\int_{\mathcal{G}} \mathcal{T}\partial(D_1 \cap g \cdot D_2) = \int_{\mathcal{G}} \mathcal{T}\partial(D_2 \cap g \cdot D_1) =$$

$$= 2\pi[vol(D_1)\mathcal{T}\partial(D_2) + length(\partial D_1) length(\partial D_2) + \mathcal{T}\partial(D_1) vol(D_2)]$$

Proof: Let us first compute:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial(D_1 \cap gD_2)} |k|] dg$$

The map $g \mapsto g^{-1}$ is an isometry of $\mathcal{G}$. So the integral I_1 is equal to:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial(gD_1 \cap D_2)} |k|] dg$$

The integral I_1 is split into two pieces: one taking care of pieces of $\partial(D_1 \cap gD_2)$ images of arcs of ∂D_1, the other taking care of pieces of $\partial(D_1 \cap D_2)$ images of arcs of ∂D_2. For the first piece we use the second expression of I_1, for the second piece, the first expression. The measure of the set of isometries which send an infinitesimal arc ds of ∂D_1 centered in $m_1 \in \partial D_1$ into D_2 is $2\pi vol(D_2)$. In the same way, the measure of the set of isometries which send an infinitesimal arc ds of ∂D_2 centered in $m_2 \in \partial D_2$ into D_1 is $2\pi vol(D_1)$

We then get

$$I_1 = 2\pi [\int_{\partial D_1} |k| \cdot vol(D_2) + vol(D_1) \cdot \int_{\partial D_2} |k|]$$

The angles of $\partial(D_1 \cap D_2)$ are of two kinds:

the angles $\theta_b, b \in B$ between an arc of ∂D_1 and an arc of $g(\partial D_2)$, and the angles $\theta_i^j, j = 1, 2$ of ∂D_1 or $g(\partial D_2)$ (here all angles are between $-\pi$ and π). Let

$$I_2 = \int_{\mathcal{G}} \sum_{i \in I_1} |\theta_i^1|$$

$$I_3 = \int_{\mathcal{G}} \sum_{i \in I_2} |\theta_i^2|$$

$$I_4 = \int_{\mathcal{G}} \sum_{b \in B} |\theta_b|$$

Inverting as above the orders of integration we get:

$$I_2 + I_3 = 2\pi [\sum_{i \in I_1} |\theta_i^1| \cdot vol(D_2) + vol(D_1) \cdot \sum_{i \in I_2} |\theta_i^2|]$$

Summing with I_1 we get :

$$I_1 + I_2 + I_3 = 2\pi [\mathcal{T}\partial(D_1) \cdot vol(D_2) + vol(D_1) \cdot \mathcal{T}\partial(D_2)]$$

The integral I_4 is an avatar of Poincaré's formula (proved in previous subsection) for all the pairs of curves, one contained in ∂D_1 and the other in ∂D_2. We conclude that:

$$I_4 = 2\pi length(\partial D_1) \cdot length(\partial D_2)$$

□

Taking care of the signs of the curvature and the angles in the previous formula we get:

Theorem 8.2.2 ***(Blaschke's formula)*** *The following weighted homogeneous formula holds:*

$$\int_{\mathcal{G}} \chi(D_1 \cap g \cdot D_2) = \int_{\mathcal{G}} \chi(D_2 \cap g \cdot D_1) =$$

$$= 2\pi[vol(D_1)\chi(D_2) + length(\partial D_1)length(\partial D_2) + \chi(D_1)vol(D_2)]$$

Proof: The Gauss-Bonnet theorem for a compact domain D of $\mathbb{R}^2$ with boundary a piecewise C^2 boundary is:

$$\chi(D) = \int_{\partial D} k + \sum \theta_i$$

where the sign of the curvature is defined using the boundary orientation of ∂D and where θ_i are exterior angles at corner points counted with the appropriate sign; see do Carmo's book [dCa].

Let us compute first, exactly as in the previous theorem:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial D_1 \cap g D_2} k] dg$$

We then get

$$I_1 = 2\pi [\int_{\partial D_1} k \cdot vol(D_2) + vol(D_1) \cdot \int_{\partial D_2} k]$$

As in the previous theorem, consider: the angles $\theta_b, b \in B$ between an arc of ∂D_1 and an arc of $g(\partial D_2)$, and the angles $\theta_i^j, j = 1, 2$ of ∂D_1 or $g(\partial D_2)$ (again all angles are between $-\pi$ and π). Let

$$I_2 = \int_{\mathcal{G}} \sum_{i \in I_1} \theta_i^1$$

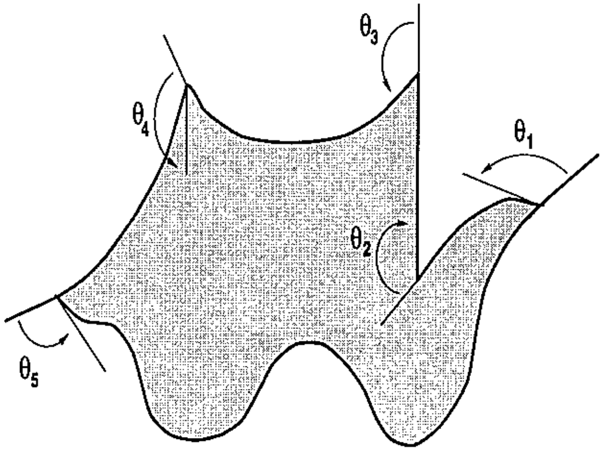


Figure 25: Gauss-Bonnet for a planar domain with boundary(and corner)

$$I_3 = \int_{\mathcal{G}} \sum_{i \in I_2} \theta_i^2$$

$$I_4 = \int_{\mathcal{G}} \sum_{b \in B} \theta_b$$

Exactly as above we get:

$$I_2 + I_3 = 2\pi[\sum_{i \in I_1} \theta_i^1 \cdot vol(D_2) + vol(D_1) \cdot \sum_{i \in I_2} \theta_i^2]$$

Summing with I_1 we get :

$$I_1 + I_2 + I_3 = 2\pi[\chi(D_1) \cdot vol(D_2) + vol(D_1) \cdot \chi(D_2)]$$

Now observe that if we take care simultaneously of the sign of the angles θ_b and of the orientation $\epsilon(F)$ of the frame F made of the tangent vectors to ∂D_1 and ∂D_2 we get to compute:

$$\int_{\mathcal{G}} \sum_{C_1 \cap C_2} \theta \cdot (\epsilon(F) = 2\pi lenth(C_1) \cdot length(C_2)$$

Notice that the density $|sin\theta| ds_1 \cdot ds_2 \cdot d\theta$ coincide with the differential form $sin\theta ds_1 \wedge ds_2 \wedge d\theta$. The integral above is equal to the integral of theorem 8.1.2 as $\theta\epsilon(F) = |\theta|$.

We conclude that:

$$I_4 = 2\pi length(\partial D_1) \cdot length(\partial D_2)$$

□

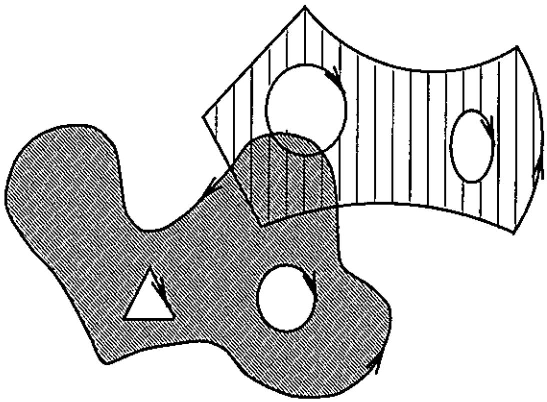


Figure 26: Blaschke formula

8.3 Linear kinematic formulas, variation of a functional

In 1950, at the beginning of his book "multidimensional variation" [Vi] Vitushkin proposes the following general construction: Let M be a compact submanifold of $\mathbb{R}^n$. The intersections of M with almost all affine subspaces are also submanifolds. More precisely, the intersection of M with an affine subspace of dimension $N-p$ achieves its maximal dimension $n+p-N$ and is a transverse intersection on an open subset of $\mathcal{A}(N, N-p)$ if $(n+p \geq N$. It is void on another open subset of $\mathcal{A}(N, N-p)$. the other cases form a subset of measure zero of $\mathcal{A}(N, N-p)$. Consider any functional F defined on submanifolds of euclidean space like

- The Euler characteristic $\chi(M)$
- The total curvature $L_0(M)$
- the number of connected components of M: cal N(M)

Then intersecting M with affine subspaces H such that the dimension of $M \cap H$ is p, and averaging we get:

Definition 8.3.1 *The variation F_p of the functional F is the integral:*

$$\int_{\mathcal{A}(N,N-p)} F(M \cap H)$$

We can now state a theorem relating the variation of the total curvature and integrals of functions locally defined on the manifold:

Theorem 8.3.2 *let M be a compact connected smooth (at least C^2) n-dimensional submanifold of $\mathbb{R}^N$; then:*

$$(L_0)_p(M) = const \cdot L_p(M)$$

Notice that:

$$(L_0)_n(M) = L_n(M) = vol(M)$$

The variations of the Euler characteristic are linked to the symmetric functions of curvature σ_i, the key result is Gauss-Bonnet's formula:

$$\chi(M) = \frac{2}{volS^n} \int_M K$$

when M is an n-dimensional hypersurface. We have observed that Weyl's formula computing the volume of the tube $Tub_r(M)$ implies that:

$$L_p^+(M) = const \cdot \int_M \sigma_{n-p}$$

The following *reproductibility formulas* are equivalent to Chern's linear Kinematic formulas.

Theorem 8.3.3 *(Reproductibility formulas) Let M be a compact connected smooth (at least C^2) n-dimensional submanifold of $\mathbb{R}^N$; then:*

$$L_p^+(M) = const.L_0^+)_p(M)$$

For $n = dim(M)$, for $i = n$, $(L_0^+)_n(M) = vol(M)$

The p^th variation of L_0 is the integral $\int_{\mathcal{A}(N,N-p)} \int_{M \cap H} \sigma_{n-p}$, so we get

Theorem 8.3.4 *(Chern's linear kinematic formulas)*

$$\int_M \sigma_i = const \cdot \int_{\mathcal{A}(N,N-p)} \int_{M \cap H} \sigma_p$$

In this form the name "reproductibility" given to that property of the symmetric functions of curvature becomes clear. Chern was asking if this was a characteristic property of those functions. The theorem concerning the p-length function and the fact they are integrals of the locally defined functions h_{n-p} proves that the functions h_{n-p} also are reproductible.

Proof: (of the reproductibility formulas)

Let $\mathcal{GA}(N, p+1, 1)$be the flag space of all couples $L \subset h$; h a (p+1)dimensional vector subspace of $\mathbb{R}^N$ and L an affine subspace in h. let H be the affine subspace of $\mathbb{R}^N$:

$$H = L \oplus h^{\perp}$$

Lemma 8.3.5 *If the line L is transverse to γ_h, the intersection $L \cap \gamma_h$ is the set of critical values of the orthogonal projection of $(M \cap H)$ on L.*

Proof: A critical point ω of the projection of $(M \cap H$ on L is a critical point of the projection of M on h, as L cannot belong to the image $p_h(T_m(M)$ of the tangent space to M at m by p_h.

Conversely, the projection of the tangent space $T_m(M)$ is orthogonal in h to $(T_m(M)) \cap h$ and is the tangent space at ω to γ_h when γ_h is smooth. If L is transverse to γ_h then:

$$p_n(T_m(M \cap H)) = p_h(h^{\perp} + (T_m(M) \cap h)^{\perp}) = \{0\}$$

which implies that

$$p_h(T_m(M \cap H)) = \{0\}$$

□

Observe now that the flag space $\mathcal{GA}(N, p+1, 1)$ can be identified with the flag space $\mathcal{AG}(N, N-p, 1)$ of vectorial lines contained in affine (N-p)-spaces.

By definition

$$L_p(M) = const \cdot \int_{G(n,p+1} |\gamma_h|$$

Using Cauchy's formula for γ_h we get:

$$L_p(M) = const \cdot \int_{\mathcal{GA}(N,p+1,1)} \sharp(\gamma_h \cap L)$$

Using the lemma and the previous identification between flag spaces we get:

$$L_p(m) = const \cdot \int_{\mathcal{AG}(N,N-p,1)} \sharp(\gamma_L(M \cap H))$$

Integrating on the fibers of the fibration

$$\mathcal{AG}(N, N-p, 1) \to \mathcal{A}(N, N-p)$$

we get the desired equality. □

To get the result concerning signed length it is enough to observe that the sign $\epsilon(\omega)$ is precisely the sign of the Gauss-Kronecker curvature of the projection of $M \cap h$ on $(T_m M \cap H) + L$. This last sign is also equal to $(-1)^{index(m)}$, where $index(m)$ is the Morse index of the projection of $M \cap H$ on the line L , oriented by $N(m)$, if M is an odd dimensional codimension one submanifold.

As an exercise, juggling with flag spaces, the reader can prove that a variation of one of the previous variations is a variation, that is:

Proposition 8.3.6 *For $i < p$, one has:*

$$L_p(M) = const \cdot \int_{\mathcal{A}(N,N-p+i)} L_i(M \cap H)$$

$$L_p^+(M) = const \cdot \int_{\mathcal{A}(N,N-p+i)} L_i^+(M \cap H)$$

8.4 General kinematic formulas

We have described a natural path leading from Blaschke's formula to Chern's kinematic formulas: Consider two objects in $\mathbb{R}^n$, move the second, integrate some curvature function on the intersection, and average on $\mathcal{G}$. The result is a weighted homogeneous polynomial in curvatures integrals on the two initial objects.

Theorem 8.4.1 *Chern's kinematic formulas [Che] . If one of the integrals is absolutely convergent, then both following integrals are finite and equal:*

$$\int_{\mathcal{G}} L_i^+(M_1 \cap g(M_2) = \sum_{p+q=i} const \cdot L_p^+(M_1) \cdot L_q^+(M_2)$$

As before const replaces constants depending only on dimensions.

The reader who needs the constants will find them in Santalò's book [Sa2].

8.5 Pohl's, Banchoff-Pohl's formulas and other formulas involving linking numbers

The ancestor of the linking number is the index $i_C(m)$ of a point m with respect to an oriented closed plane curve C. When the curve is also simple the isoperimetric inequality is:

$$L^2 - 4\pi A \geq 0$$

where L is the length of the curve and A is the area it bounds. Equality holds if and only if the curve is a circle.

For non simple closed curves we have ([Po1] [Ba-Po])

Theorem 8.5.1

$$L^2 - 4\pi \int_{\mathbb{R}^2} (i_C(x))^2 \geq 0$$

Equality holds for a circle, or a multiple circle (a circle traversed several times or several coïncident circles each traversed in the same direction any number of times).

This can be generalized to higher dimensions. For example let C be a closed space curve, then the linking numbers of affine lines with the curve also satisfy an analogous inequality [Ba-Po]

Theorem 8.5.2

$$L^2 - 2 \int_{\mathcal{A}(3,1)} link(C, D) \geq 0$$

Equality holds here only for C a circle, which may be multiple.

Kinematic-like formulas using the linking number of two curves can also be obtained and have been applied to obtain a better estimate of the osmotic pressure of a solution of circle-shaped molecules as a function of the concentration [Po2], [Edw1], [Edw2], [Del], [Dup] .

9 Integral geometry and topology

The development of this chapter of integral geometry really started in 1949, although Fenchel's results [Fe1] $\int_C |k| \geq 2\pi$ was already proved in 1929.

9.1 Integral geometry of polyhedral surfaces in $\mathbb{R}^3$

It is more elementary to first prove results concerning polyhedral surfaces, so we will first present Banchoff's proof of the Gauss-Bonnet theorem;[Ban1] . Let us first define a polyhedral surface in S^3. The basic pieces are the closed plane triangles. Any triangle has in its boundary three edges and three vertices. Triangles, edges and vertices will be called *simplices.* A polyhedral surface is a union of triangles σ_i satisfying the following properties:

1. The interiors of the σ_i are disjoint.
2. The union of the σ_i is connected, and homeomorphic to a closed surface.
3. The intersection of two triangles is a simplex.

As the triangles are usual euclidean triangles, given a vertex $v \in \sigma$ we define the segment $e(v, \sigma)$ as the image of the edge of σ opposite to v by the homotethy of center v and ratio 1/2.

The *link* of v is the union:

$$\mathcal{L}(v) = \cup e(v, \sigma); v \in \sigma$$

If q edges contain the vertex v, the planes containing an edge which contains v form q projective lines in$\mathbb{P}_2$. Let us call $\mathcal{C}$, (for critical) or $\mathcal{C}(v)$ the union of the projective lines defined previously. Any plane through v not belonging to $\mathcal{C}$ cuts $\mathcal{L}(v)$ in a finite number of points. If all the triangles containing the vertex v are in the same plane, for a plane $P \in P_2 \setminus \mathcal{C}$ one has

$$\sharp(\mathcal{L}(v) \cap P) = 2$$

The number $(\mathcal{L}(v) \cap P)$ is always even. It is natural to mesure how "nontrivial" the plane P is with respect to v by:

$$\phi(v, P) = (1/2)[2 - \sharp(\mathcal{L}(v) \cap P)]$$

We can now define the *extrinsic curvature* of v as the integral:

$$k(v) = \int_{P_2} \phi(v, P) dP$$

Intrinsically, that is inside the polyhedral surface $M = \cup \sigma_i$, at each vertex we can compute the intrinsic curvature $k(v)$ as the difference of 2π with the sum of the angles in v of the triangles which contain v.

$$k(v) = 2\pi - \sum_i \alpha(i, v); v \in f_i$$

The ambiguity between the two definitions we gave of $k(v)$ desappears with the following theorem:

Theorem 9.1.1 ***theorema egregium, (remarkable theorem in latin)***
The intrinsic and the extrinsic way of computing $k(v)$ give the same result

Proof: Let us compute the measure of the planes which intersect one side e of $\mathcal{L}(v)$. In P_2 the length of the arc formed by the planes through v, intersecting e and orthogonal to the plane containing v and e is the angle α of the triangle containing v and e at e. The measure of the planes through v that intersect e is then $2\alpha_e$. In fig below we draw the corresponding set of oriented planes in S^2.

psfigfigure=oriented.eps,height=6cm,width=10cm

Figure 27: oriented planes intersecting e as vectors in S^2

Summing on all the edges of $\mathcal{L}(v)$ we get :

$$\int_{P_2} \sharp(\mathcal{L}(v) \cap P)] = 2 \cdot \sum_{e \in \mathcal{L}(v)} \alpha_e$$

or :

$$\int_{P_2} \phi(v, P) = 2\pi - \sum \alpha_e$$

which is the relation we sought after between the extrinsic integral $\int_{P_2} \phi(v, P)$ and the intrinsic defect or excess of angle (compared to a point of the euclidean flat plane): $2\pi - \sum_{e \in \mathcal{L}(v)} \alpha_e$ □

We can now prove the polyhedral version of the Gauss-Bonnet theorem:

Theorem 9.1.2 ***(Polyhedral Gauss-Bonnet theorem)***

Let M be a polyhedral surface embedded (or immersed) in $\mathbb{R}^3$ then its total curvature satisfies:

$$\sum_{v\, vertex\, of\, M} k(v) = 2\pi \cdot \chi(M)$$

Proof: Every triangle (face of M) has three edges, and, as M is a surface, every edge belongs to two faces. Let consider the set $\mathcal{D}$ of all pairs $e \in f$ of an edge contained in a face. There is a map between $\mathcal{D}$ and the set $\mathcal{F}$ of all faces and a map between $\mathcal{D}$ and the set $\mathcal{E}$ of all edges.

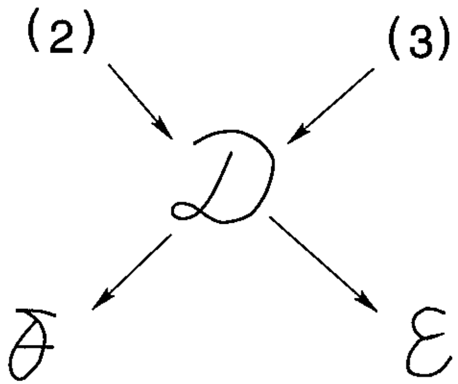


Figure 28: diagramm

By the first map, a face has three inverse images: the pairs formed by one edge of the face and the face itself. By the second an edge has two inverse images: the pairs formed by the edge and one of the two faces which contain it. Then:

$$\sharp\mathcal{D} = 3 \cdot \sharp\mathcal{F} = 2 \cdot \sharp\mathcal{E}$$

Let us denote the set of vertex of M by $\mathcal{V}$. The sum $\sum_{\mathcal{V}} k(v)$ is $2\pi\sharp\mathcal{V}M$ minus the sum of all the angles of the faces of M. It is then equal to $2\sharp\mathcal{V} - \pi \cdot (\sharp\mathcal{F})$. Adding $0 = \pi[3 \cdot \sharp\mathcal{F} - 2 \cdot \sharp\mathcal{E}]$, we get:

$$2\pi \cdot (\sharp\mathcal{F}) - 2\pi \cdot (\sharp\mathcal{E}) + 2\pi\sharp\mathcal{V} = \sum_{\mathcal{V}} k(v)$$

The first term is $2\pi \cdot \chi(M)$. □

9.2 Critical points and Gauss curvature, Chern and Lashoff's theorem

In subsection **The Gauss map**, we have proved an exchange theorem. The same is true, with exactly the same proof, for hypersurfaces in $\mathbb{R}^n$, in particular surfaces in $\mathbb{R}^2$. First suppose that M is an oriented hypersurface of $\mathbb{R}^n$. Let $N(m) \in S^{n-1}$ be the oriented normal in m to M. The Gauss map sends m to $N(m)$, we note $K(m)$ its jacobian at m. The projective Gauss map sends m to the non-oriented normal $L(m) \in \mathbb{P}_{n-1}$. Let us observe that, even if M is not orientable, then the projective Gauss map and the absolute value $|K(m)|$ still make sense. Starting with a line L, let $|\mu|(M,N)$ be the number of critical points of the orthogonal projection p_L of M onto L. If the manifold M is oriented, we can compute the index of each critical point of the orthogonal projection on the line L_z generated and oriented by a vector $z \in S^{n-1}$. Let us define:

$$\mu(M,N) = \sum_{m\ critical} (-1)^{index(m)}$$

If the dimension of M is even, the previous sum does not depend on the orientation of L, and indeed can also be defined without assuming that M is orientable.

Theorem 9.2.1 *(Exchange formula)*

$$\int_M |K(m)|dm = \int_{P_n} |\mu|(M,L)dL$$

When the previous integrals converge, and if either M is oriented, or M is even dimensional, an analogous equality , keeping track of signs, holds.

Theorem 9.2.2

$$\int_M K(m)dm = \int_{P_n} \mu(M,L)dL = \chi(M)$$

Using Morse inequality, the theorem of Chern and Lashoff is now a natural application of the first exchange theorem [Ch-La] :

Theorem 9.2.3 *The total curvature of a surface of genus g embedded or immersed in $\mathbb{R}^3$is bigger or equal to $2\pi(2g+2)$. More generally, if M is a compact hypersurface immersed in $\mathbb{R}^n$, one has:*

$$\int_M |K(m)|dm \geq vol(P_{n-1} \cdot \sum_{i=1,\dots,n-1} \beta_i(M)$$

Where the numbers β_i are the Betti numbers of M.

First we need a lemma:

Lemma 9.2.4 *For almost any line L (that is except for a measure zero set in P_{n-1}), the orthogonal projection of M on L is a Morse function.*

To prove the lemma the reader will need to check that the hessian of a local equation of M as a graph of a function from the tangent plane at m to the normal line at m coincides with the second fundamental form of M in M. Degenerated critical points of the projection on a line are then critical points of the Gauss map, and the critical values in P_{n-1} of the Gauss map form a subset of measure zero.

To prove the theorem we need only to integrate on P_{n-1} the Morse inequality:

$$|\mu|(M,L) \geq \sum_{i=1,\dots,n-1} \beta_i(M)$$

When M is a surface $\sum_{i=1,\dots,n-1} \beta_i(M) = 2g+2$.

9.3 Total curvature of closed curves and knots

Historically the first result in this line is Fenchel's theorem .

Theorem 9.3.1 *The total curvature of a closed curve C immersed in $\mathbb{R}^3$ satisfies:*

$$\int_C |k| \geq 2\pi$$

In 1949, independently, Fary, Fenchel and Milnor proved that "more topology implies more geometry". [Far] [Fe2] [Mil1] .

Theorem 9.3.2 *If the curve C is knotted (that is embedded and not the boudary of an embedded disc) then its total curvature satisfies:*

$$\int_C |k| > 4\pi$$

The proof of the first theorem and of the large inequality $\int_C |k| \geq 4\pi$ are a consequence of an easy topological argument.

Lemma 9.3.3 *The orthogonal projection p_L on the line L of an immersed curve C satisfies, if C is not in a plane orthogonal to L:*

$$|\mu|(C, L) \geq 2$$

If moreover C is knotted, and the projection p_L is a Morse function, then:

$$|\mu|(C, L) \geq 4$$

Proof: For all lines L (except one if the curve is planar) the projection p_L has at least one maximum and one minimum, so $|\mu|(C, L) \geq 2$. Let us now suppose that there exist a direction L such that p_L is a Morse function and such that $|\mu|(C, L) = 2$. Let a and b be the minimal and maximal values of the function p_L; let m_a and m_b be the corresponding critical points of p_L . Any plane P_t orthogonal to L in $a < t < b$ intersects the curve C transversly in exactly two points. Let I_t be the segment joining the two points $C \cap P_t$. the union :

$$x_a \cup \bigcup_{a<t<b} I_t \cup x_b$$

is an embedded disc with boundary C, and C cannot then be knotted. □

9.4 More theorems involving the topology of an immersion or of an embedding

the next question concerns embeddings of surfaces in $\mathbb{R}^3$: Does the topology of the embedding force "more geometry"? In particular do

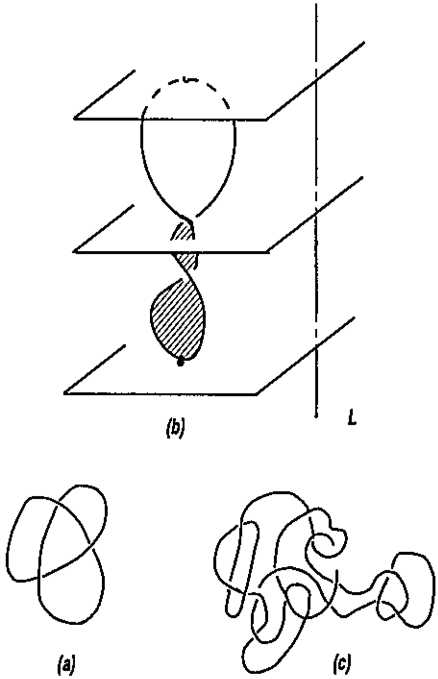


Figure 29:

topological conditions imply a lower bound for the total curvature of a submanifold? The answer is often yes.

Let us state a result of this type about tori in $\mathbb{R}^3$. We will add a point "at infinity" to $\mathbb{R}^3$ to get the compactification S^3. Naturally an embedding in $\mathbb{R}^3$ can then be considered simultaneously as an embedding in S^3.

Theorem 9.4.1 *[La-Ro1]. Let T be a torus embedded in $\mathbb{R}^3$. If T is knotted, that is if the two components of $S^3 \setminus T$ are not both solid tori $D^2 \times S^1$, then*

$$\int_T |K(m)|dm \geq 8\pi$$

Recall that Chern-Lashoff's theorem proves that for any immersed torus one has:

$$\int_T |K(m)|dm \geq 4\pi$$

Using the exchange theorem, we need to prove the following inequality:

Lemma 9.4.2

$$|\mu|(T, L) \leq 6 \Rightarrow T\, is\, not\, knotted$$

Proof: There exists a direction L such that $|\mu|(T, L)$ is 4 or 6. The proof is easier when $|\mu|(T, L)$ is 4. Let T_t be the set $(p_L)^{-1}(-\infty, t]$. Suppose that the four critical values are $a < b < c < d$, and the corresponding critical points m_a, m_b, m_c, m_d. The interior of $T_c \setminus T_b$ is the union of two cylinders C_1 and C_2. In the closure of C_1 we can choose a monotonous arc α_1(for the projection on L), joining m_b to m_c, and in the same way an arc α_2 in C_2. The union of these two arcs is a closed curve α such that its orthogonal projection on L has two critical points. The intersection of T with the plane $P_t = (p_L)^{-1}(t), b < t < c$ is the union of two disjoint circles $C_{t,1}$ and $C_{t,2}$. One, at least, is innermost and therefore bounds a disc D_1 in P. We can then choose an embedded arc c_t joining the two curves $C_{t,1}$ and $C_{t,2}$ and meeting them only at its end points. When t goes from b to c, the arc c_t sweeps a disc D_2 contained in the component of $S^3 \setminus T$ which does not contain D_1.

The boundary of D_1 and the boundary of D_2 intersect in one point. Both then are non zero in $H_1(T)$. So if we cut the component of $S^3 \setminus T$ along D_1 we get a ball B^3, proving that the component was a solid torus $S^1 \times D^2$. Cutting the other component of $S^3 \setminus T$ along D^2 we get another ball B^3 proving that the second component of $S^3 \setminus T$ is also a solid torus.

We need now to find similar discs in the two components of $S^3 \setminus T$ with the weaker hypothesis $|\mu|(T, L) = 6$. Now the Morse function p_L has six critical points $m_1, m_2, .., m_6$. With no loss of generality we can suppose that the critical values are $1, 2, ..., 6$. The intersection $P_{i+1/2} \cap T$ is a disjoint union of closed curves embedded in T. Let $n(i+1/2)$ be the number of connected components of the intersection $P_{i+1/2} \cap T$. There are two possibilities for the sequence $n(i + 1/2), 1 \leq i \leq 5$: $(1, 2, 1, 2, 1)$ and $(1, 2, 3, 2, 1)$.

Let us first consider the case $(1, 2, 1, 2, 1)$. Let $C = P_{3+1/2} \cap T$. Since C is a simple closed curve on T, it separates T into two connected components: A and B. One , say is $A = T \setminus open\, disc$, and the other $B = D^2$ an open disc. Suppose that the level $P_{1+1/2}$ is contained in A, and denote by C_a and C_b the two connected components of $P_{2+1/2} \cap T$. Let α_1 be an arc from a point of $P_{1+1/2}$ to C intersecting C_a which satisfies: $(p_L) \circ \alpha_1)'(t) < 0, 1 + 1/2 < t < 3 + 1/2$. Similarly let α_2 joining $P_{1+1/2}$ to C intersecting C_b. Let α be the union $\alpha = \alpha_1 \cup \alpha_2$. See fig below

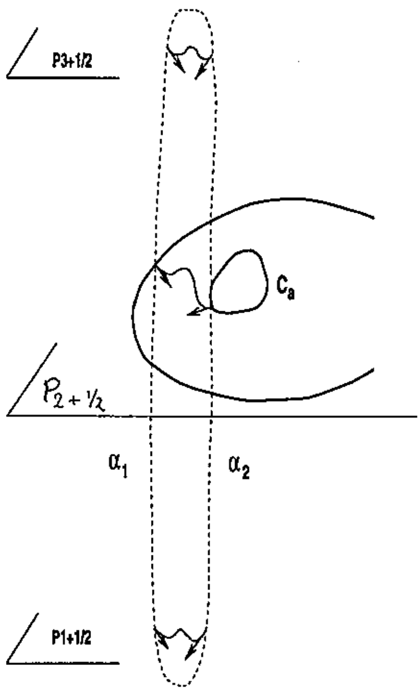


Figure 30: Construction of a monotone arc joining critical points

As in the easy case $|\mu|(T, L) = 4$, we can construct two embedded discs, the interior of which meet just one component of $S^3 \setminus T$. In the plane $P_{2+1/2}$, one, at least, of the curves C_a and C_b bounds a disc D_1. Suppose then $C_a = \partial D_1$. This disc is contained in one of the components of $S^3 \setminus T$. The curve α, following the same proof as in the case $|\mu|(T, L) = 4$, bounds a disc D_2 contained in the other component of $S^3 \setminus T$.

Again, the boundary of D_1 and the boundary of D_2 intersect in one point. Both then are non zero in $H_1(T)$. So if we cut the component of $S^3 \setminus T$ along D_1 we get a ball B^3, proving that the component was a solid torus. Cutting the other component of $S^3 \setminus T$ along D^2 we get another ball B^3 proving that the second component of $S^3 \setminus T$ is also a solid torus.

Let us now consider the case $(1,2,3,2,1)$. Let A be the part of T above $P_{3+1/2}$ and B be the part below. If B were to contain only critical points of index 0, A would contain two critical points of index 1 and one of index 2, to guarantee the connectedness of T. Then T would be a sphere. Therefore,we know that B contains a point of index 1. As $P_{3+1/2} \cap T$ is three closed curves, B has to contain two critical points of index 0 (one is m_1). B has two connected components. If A were not connected, it would contain one critical point of index 1 at most, and inspection will show that T would be one or two spheres. Therefore we know that A is connected and contains two critical points of index 1. Let C_a, C_b, C_c be the three components of $P_{3+1/2} \cap T$, labelled so that C_a and C_b do not bound a disc in B. C_a and C_b are then both generators of $\pi_1(T)$. Let P be the one point compactification of $P_{3+1/2}$. One of the circles C_a, C_b, say C_a, bounds a disc D_1 in P whose interior does not meet $C_b \cup C_c$. Then the connected component of $S^3 \setminus T$ containing D_1 is a solid torus. As before we can construct an arc α such that the restriction of p_L to α has only two critical points, and which meets C_a and C_b in one point. It bounds a disc D_2 which contains an embedded arc joining C_a to C_b in P meeting C_a and C_b only at its endpoints. The disc D_2 is then contained in the other component of $S^3 \setminus T$, bounds the nontrivial curve α on T, so that the other component is also a solid torus, proving that T is unknotted. □

In a similar way we can prove the

Theorem 9.4.3 *Let S be a surface of genus 2, and suppose that one of the Morse projection p_L has six critical points on S, then it is unknotted which means that it is isotopic to the surface of fig below.*

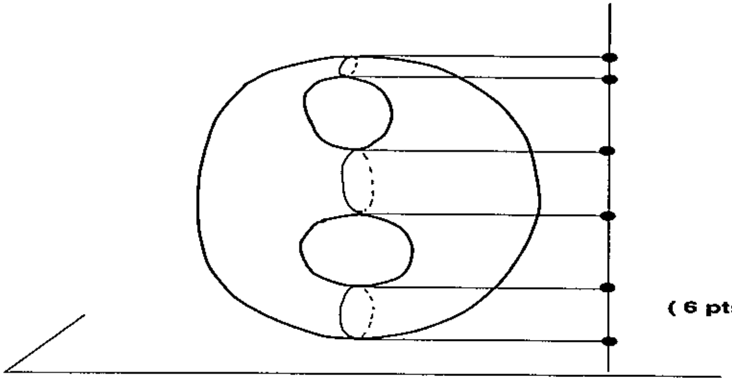


Figure 31: Standard embedding of a surface of genus two

The proof can be found in [La-Ro1] .

9.5 The equality case: tight immersions

The theory of tight immersions started whith N.H.Kuiper's article [Kui1] in 1960. It was followed by many others. Good references are also [Kui2] [Kui3].

We have just seen that the total curvature of a sphere satisfies:

$$\int_M |K| \geq 4\pi$$

Gauss-Bonnet's theorem implies that:

$$\int_M K = 4\pi$$

When the total curvature of the sphere M is 4π, the Gaussian curvature has to be everywhere nonnegative, which implies that M is the boundary of a convex body.

Definition 9.5.1 *Tight immersions are immersions which achieve equality in the theorem of Chern and Lashoff:*

$$\int_M |K| = 2\pi(2g+2)$$

where g is the genus of the oriented surface M.

To avoid heavy notation we will denote by M the surface and its image by the immersion. To feel more comfortable the reader may suppose M is embedded. Let us denote $\mathcal{H}(M)$ the *convex hull* of M:

$$\mathcal{H}(M) = \{\lambda_1 m_1 + \lambda_2 m_2 \,;\, lambda_i \geq 0 \,;\, \lambda_1 + \lambda_2 = 1\}$$

Let us call the *convex envelope* of M the boundary $\partial\mathcal{H}(M)$ of the convex hull of M.

Let $z \in S^2$ be a unit vector, and let p_z be the orthogonal projection of M on the oriented line generated by z.

Definition 9.5.2 *The topset $Top(M, z)$ of M in the direction z is the intersection of M with the plane of equation:*

$$< z|m >= max_M(< z|m >)$$

Let us discriminate the indices of the critical points of a projection p_z.

Definition 9.5.3

$$\mu_2(z) = \sharp\{critical\, points\, of\, index\, 2\, of\, p_z\}$$

$$\mu_{0,2}(z) = \sharp\{critical\, points\, of\, index\, 0\, or\, 2\, of\, p_z\}$$

$$\mu_1(z) = \sharp\{critical\, points\, of\, index\, 1\, of\, p_z\}$$

When we need to specify the surface m, or nonzero measure subset v of a surface where we count critical points we write:

$$\mu_{0,2}(M, z),\, \mu_{0,2}(v, z),\, \mu_1(M, z)\, or\, \mu_1(v, z)$$

When a point $m \in M$ has positive curvature, the Gauss map is a diffeomorphism from a neighbourhood v of m on its image $\gamma(v) \subset S^2$. Moreover:

$$\int_{\gamma(v)} \mu_{0,2}(v) = \int_v K = \int_v |K|$$

Similarly we get in a small enough neighbourhood of a point of negative curvature:

$$\int_{\gamma(v)} \mu_1(v) = -\int_v K = \int_v |K|$$

Remark: As by hypothesis M is tight, one has:

$$\frac{1}{2}\int_{S^2}\mu_{0,2}(z)=\int_M K=\int_v |K|=4\pi$$

where if $K>0; K^+=K$, if $K\leq 0; K^+=0$ and:

$$\frac{1}{2}\int_{S^2}\mu_1(z)=4\pi g$$

Proof: Rephrasing the theorem of Chern and Lashoff one gets:

$$\frac{1}{2}\int_{S^2}\mu_{0,2}(z)=\int_M K^+$$

and:

$$\frac{1}{2}\int_{S^2}\mu_1(z)=\int_M K^-$$

where if $K<0; K^-=-K$, if $K\geq 0; K^-=0$

We know that the projection p_z should have at least a maximum and a minimum, which have to belong to the convex envelope $\partial H(M)$. There cannot exist more than two critical points of p_z where the Gauss curvature is positive, otherwise:

$$\int_{S^2}\mu_{0,2}(z)>4\pi$$

which will contradict tightness.

It follows that the point $m\in M$ where $K>0$ must belong to the intersection of the envelope of M and M as:

$$\int_{\partial H(M)}|K|=4\pi$$

At a point m of $\partial H(M)$ the Gauss curvature has to be nonnegative, as it is a maximum of the function $p_{N(m)}$. □

Lemma 9.5.4 *For almost every $z\in S^2$, the topset $Top(M,z)$ is a point, the only point $m\in M$ where $N(m)=z$*

Proof: If that is not the case the function $\mu_{0,2}(z)$ would be ≥ 3 for a non zero measure set of S^2 . This contradicts tightness, as we can see using the exchange theorem. □

Lemma 9.5.5 *Let $Top(M, z)$ be the topset of the immersed surface in the direction z, and let γ be the maximum of the orthogonal projection p_z on the oriented axis defined by z ($\gamma = p_z(Top(M, z))$.*

Let W be a compact "isolated" subset of the topset in the direction z of an immersion of a compact surface M, that is a piece of $Top(M, z)$ which admits an open neighbourhood U such that, for a positive ϵ,

$$sup_{\partial U} p_z \geq \gamma - 3\epsilon$$

Then we can follow the piece W in U when we move z' in a neighbourhood of z. More precisely there exists a neighbourhood $v(z)$ of $z \in S^2$ such that for almost any $z' \in v(z)$

$$\mu_2(v(z), z) \geq 1$$

Proof: The function $p_z(m)$ is continuous on $M \times S^2$, so if we choose $m \in \overline{U}$, there exists a neigbourhood $v(z) \subset S^2$ such that, for $z' \in v(z)$

$$|p_z(m) - p_{z'}(m)| < \epsilon$$

in particular for $m \in W$,

$$p'_z(m) \geq p_z(m) - \epsilon = \gamma - \epsilon$$

and for $m \in \partial U$,

$$p_{z'}(m) \leq p_z(m) + \epsilon \leq \gamma - 3\epsilon + \epsilon = \gamma - 2\epsilon$$

This implies that the point in $\overline{U}$ where $p_{z'}$ takes its maximum value does not belong to the boundary, but to the interior. The conclusion follows now from the fact that for almost all $z \in S^2$ all critical points of p_z are non degenerate. □

Corollary 9.5.6 *Any topset $Top(M, z)$ of a tight immersed surface is connected.*

The next step of the proof is motivated by the idea that in some sense a topset of a tight immersion has to be tight, in fact a point, a disc, a plane convex curve or a planar domain bounded by convex curves.

To prove such a result it is natural to consider the topset of a topset.

Let $Top(M, z_1)$ be the topset of the immersion M in the direction z_1. It is contained in a plane orthogonal to z_1. We can construct the toptopset $Top((Top(M, z_1)z_2)$.

Lemma 9.5.7 *If M is a tight immersed surface, then the toptopset associated to two orthogonal vectors (z_1, z_2), $Top((Top(M, z_1)z_2)$ is connected.*

Proof: Let again γ be the value $p_{z_1}(Top(M, z_1)$ and let γ^* be the value $p_{z_2}(Top(Top(M, z_1), z_2))$.

Suppose that $Top((Top(M, z_1)z_2)$ is the union of two disjoint closed sets W_1 and W_2. Choose two open neigbourhoods U_1 and U_2 of W_1 and W_2 in M, with disjoint closure.

A point $m \in \partial U_i$ is in $Top(M, z_1) = W_1 \cup W_2$ if and only if $p_{z_1}(m) = \gamma$. As $Top(Top(M, z_1), z_2)$ is in the open set $U_1 \cup U_2$ it does not contain any point of $\partial U_1 \cup \partial U_2$, so a point m of $\partial U_i \cap Top(M, z_1)$ satisfies $p_{z_2}(m) < \gamma^*$. This implies that the function p_{z_1} does not take the value γ on any of the closed sets

$$\partial U_i \cap \{m | p_{z_2}(m) \geq \gamma^*\}$$

(see picture below)

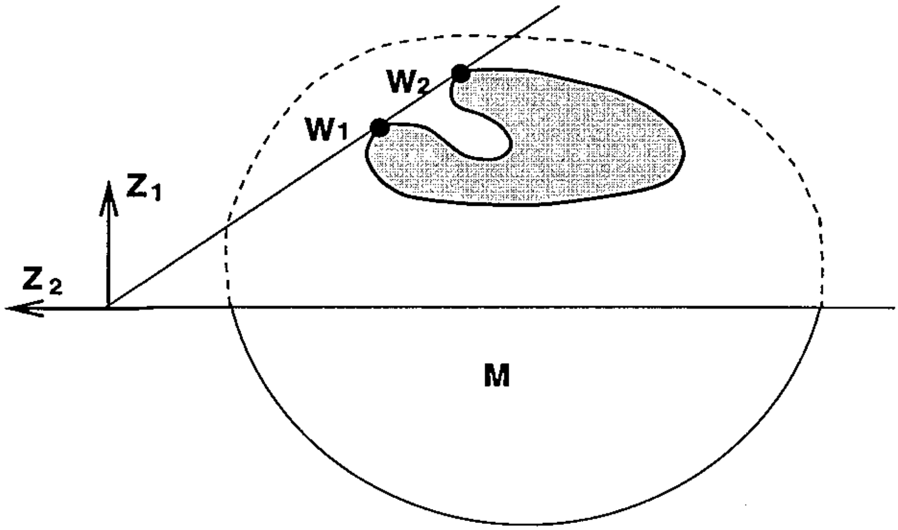


Figure 32: toptopset, z_1, z_2

Hence the function p_{z_1} achieves on $(\partial U_1 \cup \partial U_2) \cap \{m | p_{z_2}(m) \geq \gamma^*\}$ a maximal value, strictly smaller than γ, which we will note $\gamma - 3\epsilon; \epsilon > 0$.

Let us now tilt the $z_1 - axis$ by a very small angle $0 < \eta < \pi/2$ in the direction z_2. We observe that the function p_z, where $z = z_1 cos\eta + z_2 sin\eta$, has for small enough η two local maxima . We choose η small enough to have, for any $m \in M$, $|p_z(m) - p_{z_1}(m)| < \epsilon$.

Let us now study the function p_z on the open set:

$$U_i^* = \{m | p_{z_2}(m) > \gamma^*\} \cap U_i$$

and on its closure $\overline{U}_i^*$.

Claim The function p_z takes its maximal value on $\overline{U}_i^*$ in the interior U_i^*.

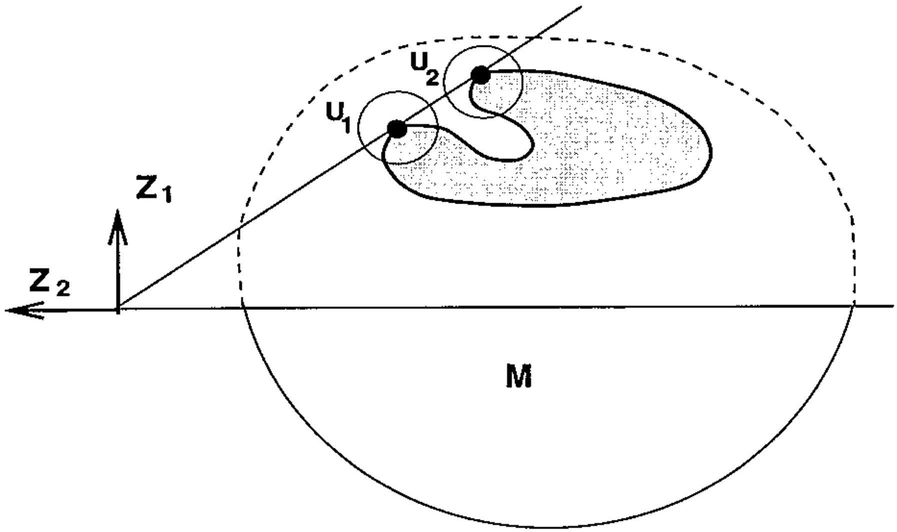


Figure 33:

Fig. the domains U_i^*

Let us now take a point $m \in \partial U_i^* \cap \{m | p_{z_2}(m) \geq \gamma^*\}$, we have:

$$p_z(m) \leq p_{z_1}(m) + \epsilon \leq \gamma - 3\epsilon + \epsilon = \gamma - 2\epsilon$$

The value of p_z at a point $w \in Top(Top(M, z_1), z_2)$ is $cos\eta\gamma + sin\eta\gamma^*$ (as $p_{z_1}(w) = \gamma$ and $p_{z_2}(w) = \gamma^*$). Then for a point $m \in \partial U_i^* \cap \{m | p_{z_2}(m) \geq \gamma^*\}$ we have, as $p_{z_2}(m) \leq \gamma^*$ and $p_{z_1}(m) \leq \gamma$:

$$p_z(m) \leq p_z(w)$$

Hence the restriction to ∂U_i^* of p_z takes its maximal values at the points of $W_1 \cup W_2$. Take again a point $w \in W_i$, we can construct a differentiable curve $c(t), c(0) = w, \forall t \in [0,1]\, c(t) \in \overline{U}_i^*$ starting at w tangent to z_2); we can suppose $\frac{dc}{dt}(0) = z_2$. The curve is normal at w to $Top(Top(M, z_1), z_2)$ and tangent to the plane containing $Top(M, z_1)$. As $\frac{dp_{z_2}(c)}{dt} = 1$, we can compute:

$$\frac{dp_z(c)}{dt} = (sin\eta) \cdot 1 + (cos\eta) \cdot 0 = sin\eta > 0$$

As the curve $c(t)$ goes from $w = c(o)$ to the interior of $\overline{U}_i^*$ (which is also the interior of U_i^*), and as the function p_z is strictly increasing along that curve, the function p_z has in U_i^* values which are greater than the maximal value $p_z(w)$ achieved on $\partial\overline{U}_i^*$. therefore the restriction of p_z to U_i^* has a topset in the interior of U_i^* (for i=1,2). This imples that:

$$\mu_2(z) \geq 2\, and\, \mu_{2,0} \geq 3$$

and again contradicts tightness. □

Corollary 9.5.8 *For any tight immersion of a surface in $\mathbb{R}^3$ the topset in any top plane contains its convex envelope (the boundary of its convex hull) in this plane.*

Using local maxima of the restriction to $Top(M, z_1)$ of p_{z_2} we prove the same way the:

Corollary 9.5.9 *The topset $Top(M, z_1)$ is either a point, a convex closed curve or a planar domain with boundary convex closed curves.*

When the topset $Top(M, z_1)$ is not a point let us call *top 1-cycle* the outer convex curve in $\partial Top(M, z_1)$. If the topset is a disc, we will say that the top-cycle is *simple*

Lemma 9.5.10 *Let M be a tight surface and let $U \subset M$ be a topological disc; we denote by $M \setminus U$ the complement of U in M. Suppose that the boundary $\gamma = \partial U$ is a top 1-cycle associated to the topset $Top(M, z_1)$. Then either U or $M \setminus U$ is the plane interior $Int(conv(\gamma))$ of the plane disc bounded by γ.*

Proof: Let us suppose that U is a topological disc. If U is $Int(conv(\gamma))$, then:

$$\int_U |K| = 0$$

If not, for z_1 or $-z_1$, U has a topset contained in its interior, providing an open set of direction z such that $Top(U, z)$ is a point contained in the interior of U. Then:

$$\int_U |K| > 0$$

Replacing U by the plane disc with boundary γ will then decrease stricly the total curvature of M. (the new immersion is a priori only C^1 but we can smooth it increasing as little as we want the total curvature , in particular the increase of the total curvature is smaller than $\frac{1}{2}\int_U |K|$, and keep the contradiction, even in the smooth case). □

We have proved the:

Theorem 9.5.11 *An immersed tight orientable surface is obtained from the boundary N of a convex body by replacing a finite (≥ 2) number of convex plane discs by surfaces of negative curvature contained in the convex hull of N with boundary the convex plane curves boundaries of the previous discs.*

Remark: A torus of revolution is a tight embedded torus.

One can also construct immersed and non embedded tight tori; [Lan5]. The idea is to construct a ruled surface (with double points) spanned by segments, the extremities of which belong to two plane convex curves situated in parallel planes. The end points of the segments are chosen using the two Gauss maps of the curves to spin them properly.

With more topology one can prove that the projective plane and the Klein bottle do not admit tight immersions in $\mathbb{R}^3$; [Kuil].

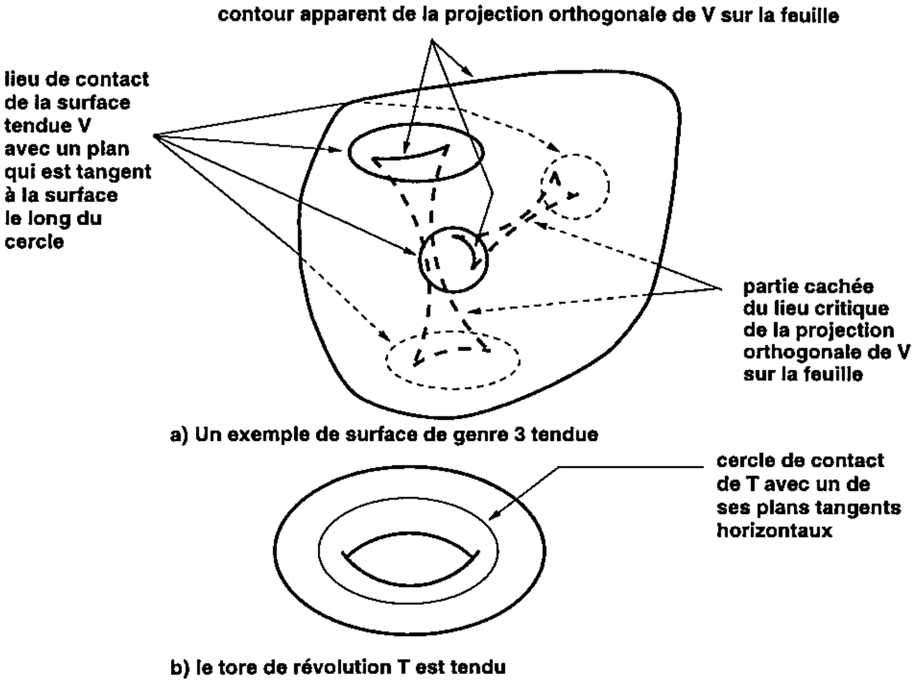


Figure 34: A tight embedded surface

remorse We have not said much about polyhedral surfaces. An important difference with smooth surfaces is the fact tightness is not equivalent to the *two piece property.*

Definition 9.5.12 *A compact subset A in* $\mathbb{R}^N$ *satisfies the two piece property if any affine hyperplane separates A in at most two connected components.*

A good reference to start the study of polyhedral surfaces is Banchoff's article [Ban1].

10 Integral geometry in spheres

The results of this paragraph come from [La-Ro2]. When C is a submanifold of dimension p of S^N,we shall use the notation $|C|$ for the p-volume of C . We sometimes for aesthetic reasons shall use the notation $L_p(M)$.

10.1 The spherical formula of Cauchy and Crofton

We shall prove it for surfaces $M \in S^3$, the proof for hypersurfaces of S^n is identical. The proof for higher codimension submanifolds is more technical; see [Sa2] [La-Ro2]. We denote by $L_2(M)$ the area of the surface $M \subset S^3$.

Theorem 10.1.1

$$L_2(M) = \frac{1}{\pi} \int_{G(4,2)} |M \cap l| dl,$$

where l is a geodesic circle of S^3 which we can think of as a 2-plane through the origin of $\mathbb{R}^4$; $|M \cap l|$ is the number of points of $M \cap l$.

Proof: Denote by $P(E)$ the projective space of vectorial lines of the vector space E. From the restriction to M of the tangent bundle to S^3 we construct the fiber bundle $\mathbb{P}(TS^3|_M)$ replacing the fibers $\mathbb{R}^3$ by projective planes $\mathbb{P}_2$. Denote by $\mathbb{P}_m(TS^3|_M)$ its fiber above the point $m \in M$; it is a riemannian fiber bundle on M. Consider the map

$$\phi : \mathbb{P}(TS^3|_M) \to G(4,2), \ \phi(m, L) = l$$

where l is the geodesic circle whose tangent at m is $L \in \mathbb{P}(T_m M)$.

Write the tangent space to $G(4,2)$ at l_0 as an orthogonal sum:

$$T_{l_0} G(4,2) = T_{l_0}\{l | m \in l\} \oplus T_{l_0}\{l \perp \Sigma_{l_0,m}\},$$

where $\Sigma_{l_0,m}$ is the geodesic 2-sphere orthogonal to l at m.

Write $T_{(m,L)}(\mathbb{P}S^3|_M) = V \oplus H$, where V is the tangent space to the fiber and $H = V^\perp$. Then $d\phi$ is given by the matrix:

$$\begin{pmatrix} Id & * \\ 0 & p_{L^\perp} \end{pmatrix},$$

where $p_{L^\perp}$ is the orthogonal projection of T_mM to $T_m(\Sigma_{l,m}) = L^\perp$. Then:

$$\int_{L\in\mathbb{P}_m(TS^3|_M)} |Jac(d\phi)| = \int_{\mathbb{P}_2} |cos(angle(L^\perp, T_mM))| = \pi$$

Since

$$\int_{G(4,2)} |(\phi)^{-1}(l)| = \int_{G(4,2)} |l \cap M|,$$

we have:

$$\int_{G(4,2)} |l \cap M| = \pi |M| = \pi L_2(M)$$

□

10.2 Flags

A flag in a vector space is a nested sequence of subspaces

$$(h_1 \subset h_2 \subset ... \subset h_k)$$

We call it complete if it contains a subspace in each dimension.

Let us denote by $|\mu|(M, \mathcal{F})$ the number of contact points of the submanifold M and the codimension one foliation $\mathcal{F}$. The notion makes sense even if the foliation admits a singular locus, as far as it is of codimension higher than one.

In S^2 a complete flag is just a pair $\Sigma_0 \subset \Sigma_1$, where σ_0 is a pair of antipodal points $(x, -x)$, intersection of S^2 with a vectorial line and Σ_1 a geodesic circle intersection of S^2 with a vectorial plane.

In S^3 a complete flag is a a sequence

$$\Sigma_0 \subset \Sigma_1 \subset \Sigma_2$$

of spheres, intersection of S^3 with vectorial subspaces of $\mathbb{R}^4$ of dimension 1,2,3. Replacing 1,2,3 by 1,2,...k and $\mathbb{R}^4$ by $\mathbb{R}^{k+1}$, we get the definition of a complete flag of S^k.

Definition 10.2.1 *We denote by C_k the set of complete flags of S^k*

We start with curves $C \subset S^2$ to give the flavour of the proofs, although the significant results start in S^3. We can define the total number of contact points of a curve C with the foliations associated to a complete flag Δ:

$$Geom(C, \Delta) = \sharp(C \cap \Sigma_1) + |\mu|(C, \mathcal{F}(\Sigma_0)$$

In Bourbaki style, the first number would be the number of contact points of $(C \cap \Sigma_1)$ with the point foliation of Σ_1. We can now define:

Definition 10.2.2

$$Geom(C) = \frac{1}{vol(\mathcal{C}_2)} \int_{\mathcal{C}_2} Geom(C, \Delta)$$

The number $Geom(C, \Delta)$ plays the same role as the total number of critical points of the orthogonal projection of the curve on a line in plane geometry. Let us first construct a sequence of foliations by curves in S^2 associated to a complete flag Δ, which will better and better follow the foliations $\mathcal{F}_0$ of Σ_1 by points and the foliation $\mathcal{F}(\Sigma_0)$ of S^2.

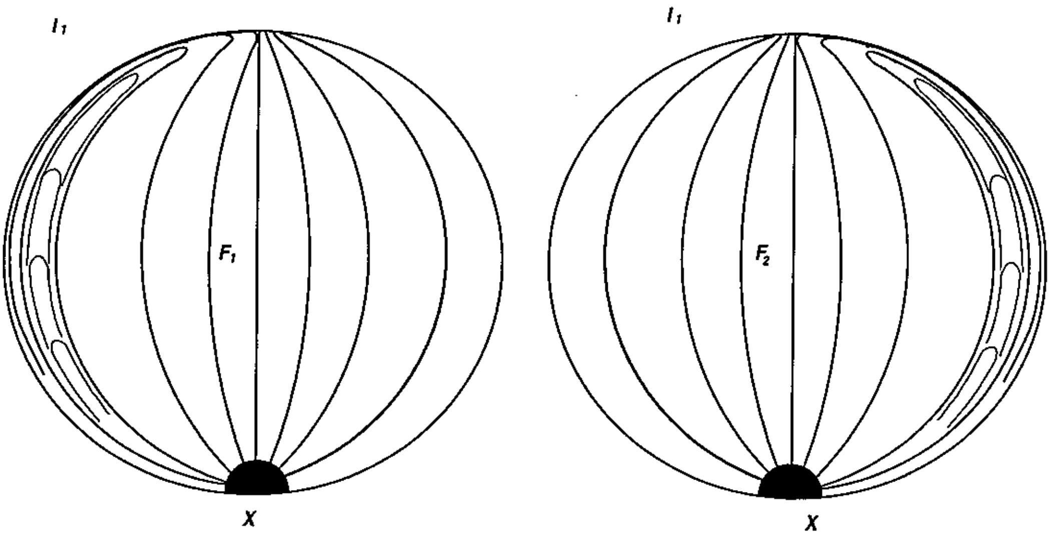


Figure 35: foliation

Choose a point $x \in \Sigma_0$ and delete from S^2 a small disc $B(x, \epsilon)$ of radius ϵ centered at x. The circle Σ_1 is divided in two arcs of length π, δ_1 and δ_2 by the two antipodal points $(x, -x)$ of Σ_O Now follow, starting near x, δ_1 with very thin nested arcs with boudary on the boundary of the small disc up to the $(-x) \in \Sigma_0$. Then continue the construction of the foliation with arcs, the left side of which will sneak along δ_1 from $(-x)$ to x and the right part of which will sweep half of the sphere Σ_2 by curves mostly equal to arcs (geodesic arcs) of the foliation $\mathcal{F}(\Sigma_0)$. The last leaf is $\Sigma_1 \cap (\textit{complement of the small disc})$. Proceed symetrically to fill up the other half of Σ_2. We shall call $\mathcal{F}_\epsilon$ the foliations associated to Δ. Do not ask the author what exactly means ϵ in the construction!

Observe that the foliation we have constructed is a product foliation by intervals of $S^2 \setminus B(x, \epsilon)$. This gives a diffeomorphism sending $S^2 \setminus B(x, \epsilon)$ to the plane, the leaves of the foliation to the horizontal affine lines and C to another closed curve.

As the projection of this image curve on the vertical has at least two critical points, we know that C has at least two points of contact with the foliation.

Corollary 10.2.3 *Any closed curve in S^2 satisfies:*

$$Geom(C) \geq 2$$

In the sphere S^3, $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2$ allows us to define a pencil of circles $\mathcal{F}(\Sigma_0)$ in Σ_2: the circles of Σ_2 which contain Σ_0 In the same way $\Sigma_1 \subset \Sigma_2 \subset S^3$ allows us to define a pencil of 2-dimensional geodesic spheres $\mathcal{F}(\Sigma_1)$: the geodesic spheres which contain Σ_2.

We define:

Definition 10.2.4

$$Geom(M, \Delta) = \sharp(M \cap \Sigma_1) + |\mu|((M \cap \Sigma_2), \mathcal{F}(\Sigma_0)) + |\mu|(M, \mathcal{F}(\Sigma_1))$$

and:

$$Geom(M) = \frac{1}{vol(C_3)} \int_{C_3} Geom(M, \Delta)$$

Let us now construct the foliations $\mathcal{F}_\epsilon$ approximating the foliations defined by the complete flag Δ. The point x is disjoint from M and we choose ϵ such that the ball $B(x, \epsilon)$ does not meet M. Let H_1 and H_2 be the hemispheres of Σ_2 bounded by Σ_1. Let $\mathcal{F}_\epsilon^2$ be the one dimensional foliation of $\Sigma_2 \setminus (B(x, \epsilon) \cap \Sigma_2)$ defined above. The trace on Σ_2 of $\mathcal{F}_\epsilon$ will be $\mathcal{F}_\epsilon^2$. Each leaf α of $\mathcal{F}_\epsilon^2$ (more precisely, each leaf α of $\mathcal{F}_\epsilon^2$, together with an arc (we choose one of length $\leq \pi \cdot \epsilon$) on $\partial B(x, \epsilon) \cap \Sigma_2$ joining the extremities of α) bounds a disc in Σ_2 Let $D(\alpha)$ be the "small" one; there will be only one ambiguous case: when α is an arc of σ_1. Starting with the small arcs α emerging near x which sneak along δ_1 we obtain discs $D(\alpha)$ which are thin flat tongues. Now inflate those to obtain thin glove fingers following δ_1. When the discs $D(\alpha)$ spead over H_1, inflating them slightly provides thin pancakes, foliating a thickening of H_1. Next step fills one of the half spheres , say B_1 of boundary Σ_2, inflating the last pancake of the previous step dissimetrically. One of the sides will sweep B_1 following the pencil of geodesic spheres $\mathcal{F}(\Sigma_1)$, the other side will just move slightly . We are in fact sweeping the ball B_1 exactly as we swept a disc of S^2, bounded by a geodesic circle Σ_1 . We proceed symmetrically to fill the other half of S^3. The foliations $\mathcal{F}^3_\epsilon$ we have constructed prove the following lemma:

Lemma 10.2.5 *For any flag Δ in general position with respect to M, there exists a sequence of foliations $\mathcal{F}^3_\epsilon$ by discs of $S^3 \setminus B(x,\epsilon)$ such that:*

$$lim_{\epsilon\to 0}|\mu|(M,\mathcal{F}^3_\epsilon) = Geom(M,\Delta)$$

Moreover the foliations $\mathcal{F}^3_\epsilon$ are product foliations defining a diffeomorphism

$$\Phi_\epsilon : S^3 \setminus B(x,\epsilon) \to \mathbb{R}^3$$

Proof: The reader should to check that the contact points of $\mathcal{F}^3_\epsilon$ and M, for ϵ small enough, correspond to points counted in $Geom(M,\Delta)$. □

Morse theory applied to the $\mathbb{R}$-valued function defined by the foliation $\mathcal{F}^3_\epsilon$ implies that

$$|\mu|(M,\mathcal{F}^3_\epsilon) \geq 2g+2,$$

so we get,using the considerations of the chapter **Integral geometry and topology** the theorem :

Theorem 10.2.6 *Let M be a surface embedded in S^3, then*

$$Geom(M) \geq 2g+2$$

If M is a knotted torus, then

$$Geom(M) \geq 8$$

and if M is a knotted (oriented) surface of genus g then:

$$Geom(M) \geq 2g+4$$

Instead of integrating $Geom(M, \Delta)$ we could have integrated separately the different terms

$$\sharp(M \cap \Sigma_1),\ |\mu|((M \cap \Sigma_2), \mathcal{F}(\Sigma_0)),\ |\mu|(M, \mathcal{F}(\Sigma_1)).$$

Integrating on $\mathcal{C}_3$ a geometric term which depends only on one of the constituents of the complete flags Δ just multiply by a constant depending only on dimensions the corresponding integral on the set of geodesic k-spheres of S^3.

We can now recognize spherical versions of the p-lengths defined in section **higher dimensional convex bodies and related matters:**

$$L_p(M) = C(N, n, p) \int_{G(N,p+1)} |\gamma_h| dh$$

where $|\gamma_h|$ denotes the volume of γ_h (when $p = O$, γ_h is a finite set and $|\gamma_h|$ is the number of points $\sharp(\gamma_h)$ of γ_h). Recall that the constant $C(N, n, p)$ has been chosen so that if M is the boundary of an ϵ-tubular neighbourhood of a p-dimensional submanifold C of $\mathbb{R}^N$, then:

$$lim_{\epsilon \to O} L_p(M) = |C|$$

First observe that the set of antipodal pairs in S^3 is the Grassmann manifold $G(4, 1)$, the set of geodesic circles is $G(4, 2)$ and the set of geodesic spheres is $G(4, 3)$.

The reader will easily believe that the integral:

$$\int_{G(4,2)} \sharp(M \cap \Sigma_1)$$

is proportional to the area of M. Define in S^3:

Definition 10.2.7

$$L_2(M) = \frac{1}{\pi} \int_{G(4,2)} \sharp(\Sigma_1 \cap M)$$

To unify notations we will note:

$$|\Sigma_1 \cap M| = \sharp(\Sigma_1 \cap M)$$

A pencil $\mathcal{F}(\Sigma_1)$ of geodesic 2-spheres of axis a geodesic circle Σ_1 defines a projection $p_{\mathcal{F}(\Sigma_1)}$ of $S^3 \setminus \Sigma_1$ on the set $\{leaves\ of(\mathcal{F}(\Sigma_1))\}$ which is a circle. Restricted to $M \setminus (M \cap \Sigma_1)$ this projection has in general a discrete critical locus γ_{Σ_1} and a finite number of critical values $|\gamma_{\Sigma_1}|$. Define:

Definition 10.2.8

$$L_0(M) = \frac{1}{2volG(4,2)} \int_{G(4,2)} |\gamma_{\Sigma_1}|$$

As the function $p_{\mathcal{F}(\Sigma_1)}$ is generically a Morse function on $M \setminus (M \cap \Sigma_1)$ the number $|\gamma_{\Sigma_1}|$ is generically equal to the number $|\mu|(M, \mathcal{F}(\Sigma_1))$. So the integral of the last term of $Geom(M, \Delta)$ is proportional to $L_0(M)$.

To define the 1-lenth $L_1(M)$, project M on a geodesic sphere Σ_2 following the geodesic arcs orthogonal to it. These arcs are contained in the geodesic circles containing the two points $h^{bot} \cap S^3 = (x, -x)$ where h is the subspace of $\mathbb{R}^4$ such that $h \cap S^3 = \Sigma_2$. We say that the points $(x, -x) = h^{bot} \cap S^3$ are conjugate to Σ_2. The arcs are of the form $\Sigma_1 \setminus (x, -x); (x, -x) \subset \Sigma_1$. Loosing only a measure zero set of spheres,we can suppose that none of the conjugate points $x, -x$ to geodesic spheres Σ_2 are on M. Denote by p_{Σ_2} this projection on Σ_2 and by γ_{Σ_2} its critical locus.

Definition 10.2.9

$$L_1(M) = \frac{1}{\pi^2} \int_{G(4,3)} |\gamma_{\Sigma_2}|$$

It is also true, but less straitforward to prove, that the integral of the middle term of $Geom(M, \Delta)$ is proportional to $L_1(M)$. This last result is a consequence of the following kinematic-type formula:

Theorem 10.2.10 *Let M be a surface in S^3 Then:*

$$L_1(M) = \frac{1}{\pi} \int_{G(4,3)} L_0(M \cap \Sigma)$$

where Σ runs over the set of all geodesic 2-spheres of S^3.

Proof: First observe that the constant is obtained considering small spheres of geodesic radius t. Then $L_1(S_t) \approx 4t$ and $\int_{G(4,2)} L_o(S_t \cap \Sigma) \approx 4\pi t$. Recall that by definition

$$L_1(M) = \frac{1}{2\pi^2} \int_{G(4,3)} |\gamma_\Sigma|.$$

The Cauchy-Crofton formula in S^2 says:

$$|\gamma_\Sigma| = \frac{1}{2}\int_{G(3,2)} |\gamma_\Sigma \cap l|$$

where l runs over the set of geodesic circles in Σ.

The inverse image of the orthogonal projection onto Σ of the geodesic circle l is a sphere Σ_l. the points of $\gamma_\Sigma \cap l$ are the critical points of the orthogonal projection of $\Sigma_l \cap M$ onto l. The reader is invited to compare this argument with the argument proving the linear reproductibility formula in section **Blashke's formulas and kinematic formulas.** Hence:

$$L_1(M) = \frac{1}{4\pi^2}\int_{G(4,3)}\int_{G(3,2)} |\gamma_\Sigma \cap l| = \frac{1}{4\pi^2}\int_{D(4,3,2)} |\mu|(\Sigma_l \cap M, \mathcal{F}(l)),$$

where $\mathcal{F}(l)$ is the (singular) foliation of the 2-sphere Σ_l by geodesic circles orthogonal to l. Here $D = D(4,3,2)$ is the space of flags $(\Sigma, l), \Sigma \supset l$. The flag space D fibers over $G(4,3)$ and over $G(4,2)$, so using Fubini's theorem for both fibrations, we get:

$$L_1(M) = \frac{1}{4\pi^2}\int_{G(4,3)} 4\pi L_0(\Sigma \cap M) = \int_{G(4,3)} L_0(\Sigma \cap M)$$

□

Gathering our results we can express $Geom(M)$ in terms of the p-lengths or of integrals of the functions h_i.

Theorem 10.2.11 *[La-Ro2] Let M be a compact surface in S^3; then:*

$$Geom(M) = \pi^2 L_2(M) + 4\pi^3 L_1(M) + 2\pi^2 volG(4,2) L_0(M)$$

$$Geom(M) = \int_M [\pi^3 + 2\pi h_1 + \frac{\pi}{2} volG(4,2)|K|]$$

10.3 Functions h_i

In this subsection we construct functions on M the integral of which are the spherical p-lengths $L_p(M)$ analogous to the euclidean p-lengths defined in section **higher dimensional convex bodies and related matter** and define the functions which localize them.

Let Σ_{p+1} be a (p+1)-dimensional geodesic sphere of S^N; it is the intersection of a (p+2) plane h_1 of $\mathbb{R}^{n+1}$ with S^n. The intersection $(h_1)^\perp \cap S^N$ is called the (geodesic) sphere conjugate to Σ_{p+1}; we denote it by Σ^*_{p+1}. The set of geodesic spheres of dimension (N-p-1) containing the (N-p-2) gedesic sphere Σ^*_{p+1} foliate $S^N \setminus \Sigma^*_{p+1}$. Moreover each leaf of the foliations meets Σ_{p+1} in two antipodal points. The foliation then defines a projection $p_{\Sigma_{p+1}}$ of $S^N \setminus \Sigma^*_{p+1}$ on $\mathbb{P}_{p+1}$. Consider the restriction of this projection to $M \setminus (M \cap \Sigma^*_{p+1})$.

Definition 10.3.1 *The polar variety* $\Gamma_{\Sigma_{p+1}}$ *is the closure of the set of critical point of the restriction* $p_{\Sigma_{p+1}}|_{M\setminus(M\cap\Sigma^*_{p+1})}$.

The critical locus $\gamma_{\Sigma_{p+1}}$ *is the closure of the inverse image by the covering map*

$$\pi : S^{p+1} \to \mathbb{P}^{p+1}$$

of the critical locus of $p_{\Sigma_{p+1}}|_{M\setminus(M\cap\Sigma^*_{p+1})}$.

To define the p-lenth we need just to integrate the p-volume $|\gamma_{\Sigma_{p+1}}|$ of $\gamma_{\Sigma_{p+1}}$.

Definition 10.3.2

$$L_p(M) = const \cdot \int_{G(N+1,p+2)} |\gamma_{\Sigma_{p+1}}|$$

where the constant depends only on the dimensions involved and is chosen in such a way that:

$$lim_{r\to 0} L_p(Tub_r(M)) = p - volume(M)$$

if M *is p-dimensional.*

When M is of codimension 1 the functions $h_i(m)$ are defined exactly as in the euclidean case using the second fundamental form of $M \subset S^n$. The numbers $|k(m,h)|$ are absolute values of the determinant of the restriction of this second fundamental form to $h \subset T_m M$, expressed in an orthonormal basis.

Remark: The inverse image $(exp_m)^{-1}(M) \subset T_m S^n$ has at $m \in (exp_m)^{-1}(M)$ the same fundamental form as $M \subset S^n$ at $m \in M$

We can know state a localization theorem:

Theorem 10.3.3 *Let M be a codimension 1 submanifold of S^n. The functions $h_{n-i}(m)$ localize the i-lenghts $L_i(M)$; more precisely:*

$$\int_M h_{n-1-i} = const \cdot L_i(M)$$

11 Integral geometry of foliations

A foliation $\mathcal{F}$ of a manifold M is a partition of M by connected submanifolds called *leaves* in a way such that locally the connected components of the intersection of a leaf with open sets of a suitable family, the *distinguished charts*, have a product structure. See [Ca-Li] for a rigorous definition and basic properties of foliations; another more riemannian viewpoint can be found in [To], a very complete reference is [Go] .

We will soon need to relax a little the definition, accepting a singular locus Σ, a stratified set of codimension bigger than one . The foliated manifold in this case is $M \setminus \Sigma$.

Many results will still be valid if we suppose only the existence of a p-plane field, dropping the *integrability* condition, (a plane field P is integrable if there exists a foliation such that it is tangent to it).

11.1 Codimension 1 foliations of a domain of $\mathbb{R}^{n+1}$

Let $W \subset \mathbb{R}^n$ be an open subset, and let $\mathcal{F}$ be a codimension 1 orientable foliation of W. As $\mathcal{F}$ is orientable, a unit normal $N(m)$ is defined at each point $m \in W$. **Symmetric functions of curvature associated to a foliation**

As through every point m of the foliated space passes a leaf L_m of $\mathcal{F}$, the symmetric functions of curvature of the leaf L_m at the point m defined by:

$$det Id + t(d\gamma)(m) = \sum t^i \cdot \sigma_i^+$$

give functions σ_i^+ defined on W. the first result computing the integrals $\int \sigma_i^+$ where obtained by D. Asimov :

Theorem 11.1.1 *[Asi] Let $\mathcal{F}$ be an oriented codimension 1 foliation of the flat torus T^{n+1}. The the integrals of symmetric functions of curvature satisfy:*

$$\int_{T^{n+1}} \sigma_i^+ = O, i \geq 1$$

Proof: Note $N(m)$ the unit vector normal in m to the leaf of $\mathcal{F}$ through m, defined by the orientation of $\mathcal{F}$. (The torus T is the quotient $\mathbb{R}^{n=1}/\mathbb{Z}^{n+1}$ or $\mathbb{R}^{n+1}/\Lambda$ for an (n+1)-dimensional lattice Λ, so vectors define an element of the quotient torus). There exists a fundamental domain $W \subset \mathbb{R}^{n+1}$, the unit cube for the "square" torus $\mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ of the projection $\mathbb{R}^{n+1} \to T$. So we can identify the normal at m to L_m and the normal at $\tilde{m} \in \mathbb{R}^{n+1}$ to $\tilde{L_m}$. Consider the map

$$m \mapsto m + tN(m)$$

When t is small enough this map is a diffeomorphism. Its differential computed using an orthonormal basis split between $T_m\mathcal{F}$ and $T_m\mathcal{F}^\perp$ is:

$$\begin{pmatrix} t \cdot d\gamma(m) + Id & O \\ * & 1 \end{pmatrix}$$

Its jacobian is

$$det(Id + t(d\gamma)(m)) = \sum t^i \cdot \sigma_i$$

The integral:

$$\int_T det(Id + t(d\gamma)(m) = \int_T 1 + \sum \sigma_i^+ t^i$$

is equal to the volume of the torus. Therefore the integrals of the coefficients of the monomials $t^i, 1 \leq i \leq n$ are all zero. □

Asimov, and then Brito Langevin and Rosenberg then computed integrals of curvature associated to foliations of compact manifolds of constant curvature using carefully chosen differential forms.[B-L-R].

Here we will prove first euclidean results and sketch their extensions to constant curvature spaces using again an exchange theorem.

Contacts with affine hyperplanes and the exchange theorem

Let H be an affine hyperplane of $\mathbb{R}^{n+1}$. The traces $\mathcal{F}|_H$ of $\mathcal{F}$ on H is generically a foliation of $(W \cap H)$ with only isolated singularities forming a set $\Sigma(\mathcal{F}|_H)$. In fact generically those singularities are hyperbolic,that is, here, of one of the two types: center or saddle. We attribute signs to those singular points:

$$\epsilon(saddle) = -1 \, and \, \epsilon(center) = +1$$

Definition 11.1.2 *the number $|\mu|(\mathcal{F}, H)$ is the number of singular points of $\mathcal{F}|_H$ when $|\mu|(\mathcal{F}, H)$ is finite,and the singularities are all hyperbolic, the number $\mu(\mathcal{F}, H)$ is:*

$$\mu^+(\mathcal{F}, H) = \sum_{m \in \Sigma(\mathcal{F}|_H)} \epsilon(m)$$

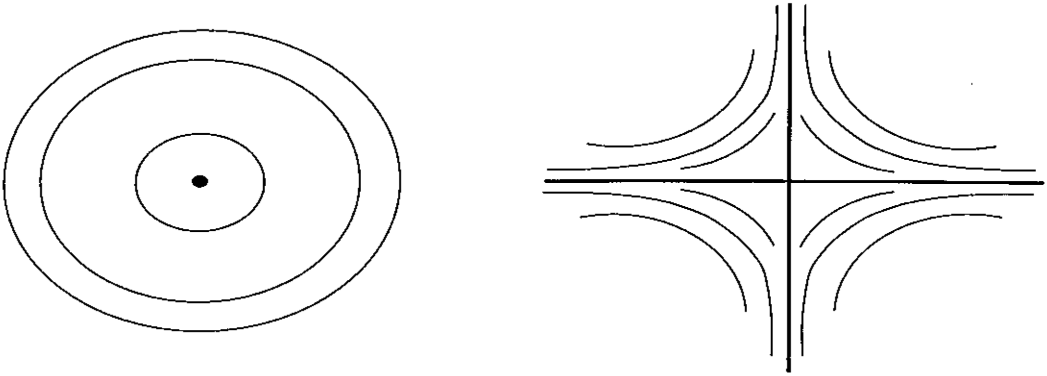


Figure 36: center and saddle

Remark: The singular point m of $\mathcal{F}|_H$ is a point where the leaf L_m is tangent to H. We can also locally project L_m on the normal in m to H (and L_m). We get a function which is in general a Morse function, for which the Morse index of m satisfies:

$$(-1)^{Morse\,index\,of\,m} = \epsilon(m)$$

The sign $\epsilon(m)$ is, when the dimension of the leaves of $\mathcal{F}$ is even, the sign of the Gauss curvature of L_m at m.

Analogously with the manifold case we will call the integral $\int_W |K|$ (or $\int_W |k|$ when W is of dimension 2) the *total curvature of $\mathcal{F}$*

Theorem 11.1.3 ***foliated exchange theorem***

$$\int_W |K| = \int_{\mathcal{A}(3,2)} |\mu|(\mathcal{F}, H)$$

Moreover, if one of the previous integrals are finite:

$$\int_W K = \int_{\mathcal{A}(3,2)} \mu^+(\mathcal{F}, H)$$

To prove this theorem, we will define the polar curves of the foliation and a foliated Gauss map.

Polar curves

The critical points of the orthogonal projection of a leaf L of $\mathcal{F}$ on a line δ are in general isolated on L.

Definition 11.1.4 *The closure of the union of those critical points :*

$$\Gamma(\mathcal{F},\delta) = \overline{\bigcup_L crit(p_\delta|_L)}$$

is generically almost everywhere a smooth curve (it may have singular points).

Proposition 11.1.5 *[Th2] Generically the polar curve $\Gamma(\mathcal{F},\delta)$ is transverse to $\delta^\perp$*

Remark: When $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is tangent to $T_m\mathcal{F}$ the Gauss curvature of the leaf L_m is zero, as, in that case, the differential of the Gauss map of the leaf L_m restricted to $T_m\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is zero.

To prove the foliated exchange theorem we need to introduce a foliated Gauss map with values in $\mathcal{A}(3,2)$:

Definition 11.1.6

$$\gamma_{\mathcal{F}}(m) = the\ affine\ plane\ tangent\ at\ m\ to\ \mathcal{F}$$

Proof: To compute the jacobian of the foliated Gauss map $\gamma_{\mathcal{F}}$ at a point $m \in W$ we will use, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$ is transverse to $T_m\mathcal{F}$, in the domain, the frame $u_1, u_2, ..., u_n$, $u_1, u_2, ..., u_n$ orthogonal basis of $T_m\mathcal{F}$, u_n unit vector tangent at m to $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$. In $\mathcal{A}(3,2)$ we use at $\gamma_{\mathcal{F}}(m)$ the frame v_1, v_2, v_3, where v_1, v_2 form an orthogonal basis of the horizontal space at $\gamma_{\mathcal{F}}(m)$ of the riemannian fiber bundle $\mathcal{A}(3,2) \to \mathbb{P}_2$, and where v_3 is a unit vector tangent to the fiber of $\mathcal{A}(3,2) \to \mathbb{P}_2$. In these bases, the matrix of $d\gamma_{\mathcal{F}}$ is:

$$\begin{pmatrix} d\gamma_{\mathcal{F}}|_{L_m} & 0 \\ * & |cos\phi| \end{pmatrix}$$

where ϕ is the angle between $T_m\Gamma_{\mathcal{F}}$ and $T_m\mathcal{F}^\perp$

As the volume of the parallelogram determined by the frame u_1, u_2, u_n is also $|cos\phi|$, and as the map $d\gamma_{\mathcal{F}}|_{L_m}$ is just the Gauss-Kronecker map of the leaf L_m, the jacobian we are looking for is just $|K|$.

On the one hand, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$ is tangent to $T_m\mathcal{F}$ the Gauss-Kronecker curvature K is zero. On the other hand using a frame split

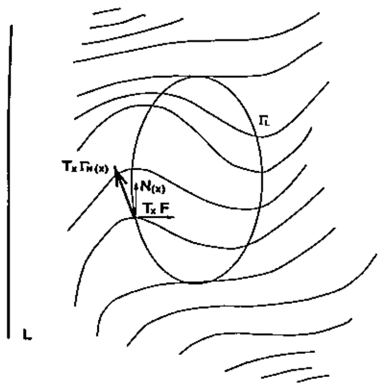


Figure 37: the computation of the jacobian of $\gamma_{\mathcal{F}}$

between $T_m\mathcal{F}$ and $T_m\mathcal{F}^{\perp}$ we see that at such a point the matrix of $d\gamma_{\mathcal{F}}$ is:

$$\begin{pmatrix} d\gamma(m) & * \\ 0 & 1 \end{pmatrix}$$

where in the formula $d\gamma$ is the Gauss map of the leaf L_m. As the rank of $d\gamma(m)$ is one the point m is critical for $\gamma_{\mathcal{F}}$, by Sard's theorem the measure of the images by $\gamma_{\mathcal{F}}$ of these points is zero. □

Let us first give some applications of the foliated exchange theorem in dimension 2. We note $|k|(m)$ the absolute value of the curvature of the leaf L_m of $\mathcal{F}$ through m.

Theorem 11.1.7 *[La-Le2] Let $D \in \mathbb{R}^2$ be the unit disc and $\mathcal{F}$ be an orientable foliation with isolated singularities, tangent to ∂D. Then:*

$$\int_D |k| \geq 4\pi - 2$$

the minimal value is achieved by the foliation (a) of the next picture.

Proof: Let us choose an orientation of $\mathcal{F}$; that induces an orientation of $\partial D \setminus sing(\mathcal{F})$. Among the singularities of $\mathcal{F}$ on ∂D let A be those where the orientation of ∂D changes. The set A is finite and has an even number of points $A = a_1, a_2, ..., a_{2n}$. Let G_e be the set of lines which meet D, do not meet A, and cut A in two subsets containing an even number of points; let G_o be the similar set of lines cutting A in two subsets of odd cardinality. Cauchy's formula implies that the sum

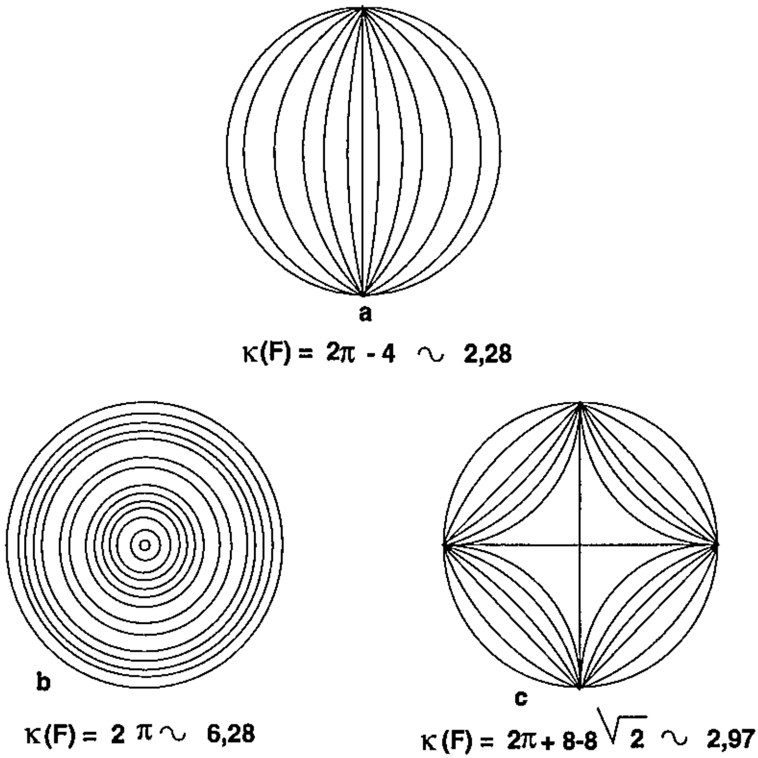


Figure 38: 3 exemples of foliations of the disc

of the measures of G_e and G_o is 2π (the length of ∂D). If a line L is in G_e, then, if it contains no singularity of $\mathcal{F}$, $|\mu|(\mathcal{F}, L) \geq 1$ (see next picture)

Using the exchange theorem, we get the inequality:

$$\int_D |k| \geq measure(G_e) = 2\pi - measure(G_o)$$

In order to finish the proof we need a lemma:

Lemma 11.1.8 *for any finite subset A of the unit circle ∂D the measure of the set G_o of lines cutting A in two odd subsets satisfies:*

$$measure(G_o) \leq 4$$

□

Remark: When $A = a, -a$ is made of two opposite points, $measure(G_o) = 4$, when $A = \emptyset$, $measure(G_o) = 0$, when A is the union of the vertices of a regular 2n-gon, $measure(G_o$ goes to π when n goes to infinity.

The proof of the lemma is elementary but technical and can be found in [La-Le2] .

Let now $D \subset \mathbb{R}^2$ be a domain homeomorphic to a disc and with a piecewise C^2 boundary ∂D.

Definition 11.1.9 *The internal distance $d(m_1, m_2)$ of two points m_1 and m_2 is:*

$d(m_1, m_2) =$

$$= inf\{length(\gamma)|\gamma : [a, b] \to D \text{ a regular curve }, \gamma(a) = m_1, \gamma(b) = m_2$$

where $length(\gamma)$ is the length of the curve γ

We get that way a metric on D. In fact the assumptions made on D imply that given the two end points there exists exactly one minimising curve joining them. Such a curve will be called a *geodesic of D.*

Definition 11.1.10 *The diameter of D is defined as:*

$$d = sup\{d(m_1, m_2)|m_1 \in D, m_2 \in D\}$$

Theorem 11.1.11 *[La-Po] Let $\mathcal{F}$ be a foliation (by curves) of D, tangent to ∂D, with isolated singularities of positive index, not necessarily orientable. Then:*

$$\int_D |K| \geq length(\partial D) - 2d$$

Definition 11.1.12 *the index of an isolated singularity m of a non-orientable foliation is a half integer $\iota(m) \in \frac{1}{2}\mathbb{Z}$ which is half of the degree of the map*

$$\Phi_\epsilon : S_\epsilon(m) \to \mathbb{P}_1$$

associating to a point q of a small enough circle centered at m the direction of the line $T_q\mathcal{F}$. (if the singularity is orientable, the index is the usual one).

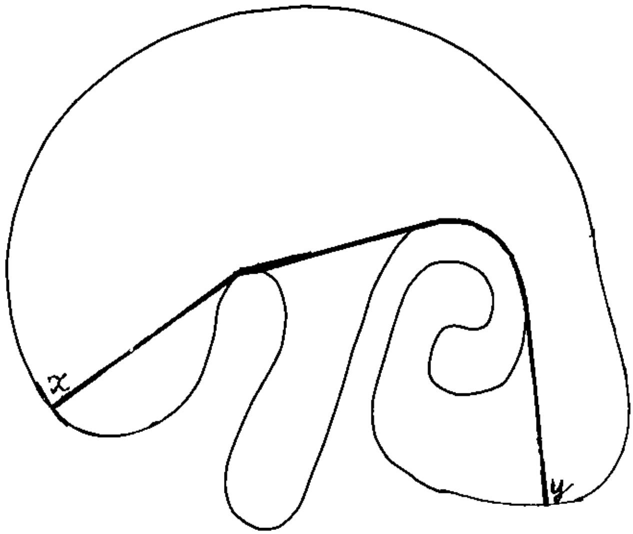


Figure 39: diameter of a topological disc

Proof: Let us first show that we can eliminate the case when $\mathcal{F}$ has a singularity of index one, studying only the case where $\mathcal{F}$ has two singularities of index $\frac{1}{2}$. which are of sunset type (see next picture).

All singularities can be substituted by a source/sink or a sunset singularity without increasing the total curvature of the foliation by more than a given ϵ. This can be done by considering on the boundary of a small disc D_r of radius r an homotopy between the "angle" function determined by $\mathcal{F}$ and the "angle" function of one of the models of the next picture.

A source/sink can be replaced by two sunsets using the modification indicated in the next figure:

Let P and Q be two sunsets of $\mathcal{F}$, and γ be a geodesic of D joining P to Q. we need to estimate the number of contact points of $\mathcal{F}$ with an affine line L. all lines, but a measure zero set, meet the disc D in a finite number of segments.

Let $[a, b]$ be a connected component of $L \cap D$ such that $[a, b] \cap \gamma = \emptyset$. Then $[a, b]$ divides D into two discs, one of them containing P and Q.In the other disc, $\mathcal{F}$ is orientable, and therefore there is at least one point of contact between $\mathcal{F}$ and the segment $[a, b]$. See next figure:

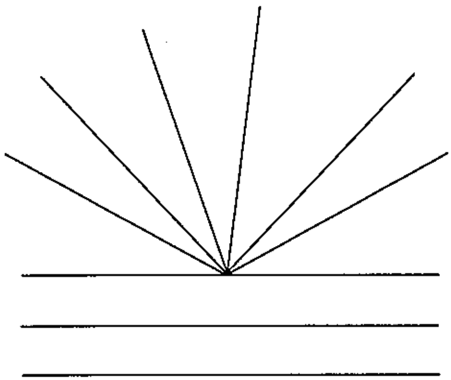


Figure 40: sunset

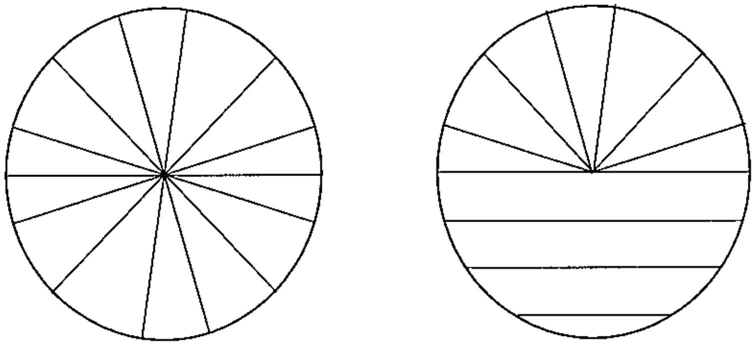


Figure 41: source/sink and sunset

Let $n(L)$ be the number of segments of $L \cap D$ in which L meets γ, and $c(L)$ the number of segments of $D \cap L$ which do not. then we have:

$$|\mu|(\mathcal{F}, L) \geq c(L)$$

Cauchy's formula yields:

$$\int_{\mathcal{A}(2,1)} \sharp\{(components\, of L \cap D\} = \frac{1}{2} \int_{\mathcal{A}(2,1)} \sharp\{L \cap \partial D\} = length(\partial D)$$

Applying Cauchy's formula to the arc γ we get $length(\gamma) = \frac{1}{2} \int_{\mathcal{A}(2,1)} \sharp(L \cap \gamma)$. Then we have:

$$length(\partial D) = \int_{\mathcal{A}(2,1)} n(L) + c(L) = \int_{\mathcal{A}(2,1)} \sharp(L \cap \gamma) + \int_{\mathcal{A}(2,1)} c(L)$$

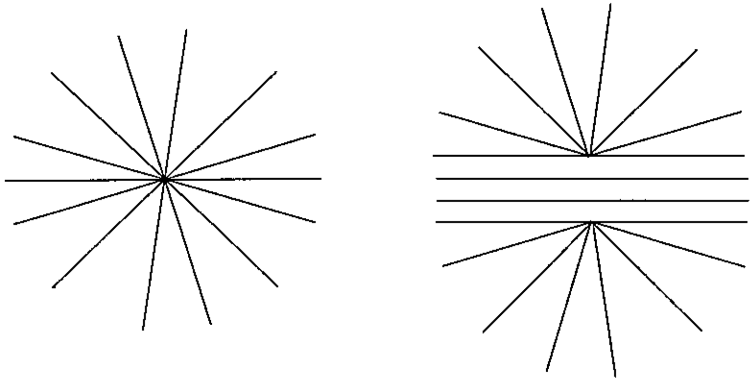


Figure 42: transformation of a source/sink into two sunsets

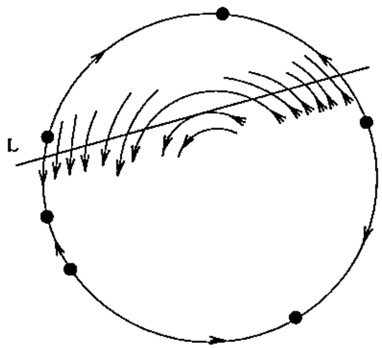


Figure 43: forced

using the exchange theorem and the inequality on $|\mu|(\mathcal{F}, L)$ we get:

$$length(\partial\gamma) \leq 2 \cdot length(\gamma) + \int_{\alpha} (2,1)|\mu|(\mathcal{F}, L) = 2 \cdot length(\gamma) + \int_{D} |k|$$

□

With the same techniques, one can obtain inequalities for foliations of a compact flat annulus, and for foliations of a disc extending a given line field defined on the boundary. In the second case a sort of "length" of the envelope of the one parameter family of affine lines defined by the boundary condition will play a role [La-Po].

When the disc D is not convex we can show there do not exists *tight* foliations tangent to ∂D with singulaties of positive index. This comes from the fact that if $P \in \partial D$ is a point of inflexion, and a regular point of $\mathcal{F}$, then there is an open set of affine lines which have more than one contact point with $\mathcal{F}$ in a neighbourhood of P. But we can exhibit a sequence $\mathcal{F}_n$ of foliations of D satisfying the hypothesis of our theorem such that:

$$lim_{n\to\infty} \int_D |k| = length(\partial D) - 2d$$

We can think of the limit of this sequence of foliations as a foliation all leaves of which have corners along ∂D, in order to force on ∂D all the critical points of the orthogonal projections of the leaves on lines; see next picture.

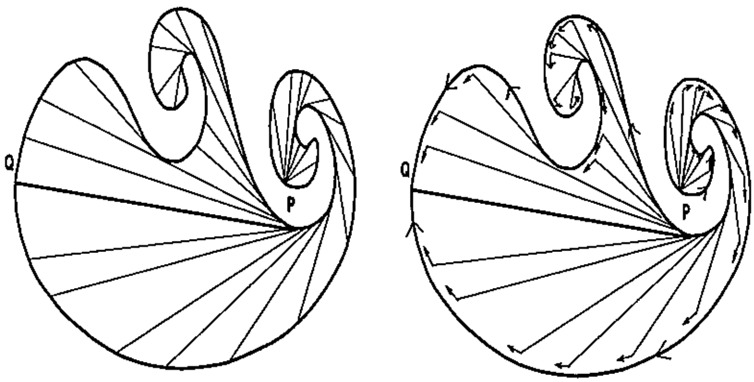


Figure 44: A tight singular foliation $\mathcal{F}$; non-singular $\mathcal{F}_n$ close to $\mathcal{F}$

11.2 Codimension one foliations of spaces of constant curvature

When the foliated space is a domain W in S^n or $\mathbb{H}^n$, one can also prove an exchange theorem, replacing Gauss-Kronecker curvature by the determinant of the second fundamental form obtained from the normal vector given by the orientation (in an orthonormal basis) that we will still denote by K, and replacing the euclidean affine hyperplanes by codimension one totally geodesic subspaces $H \in \mathcal{A}$. The form of the theorem is the same for $W \subset \mathbb{H}^{n+1}$, $W \subset \mathbb{R}^{n+1}$, $W \subset S^{n+1}$. In each case the set $\mathcal{A}$ admits a measure invariant by the action of the isometries of the space [Sa2] p.28 and 307. In dimension 2, we can choose (local in the case of S^2) coordinates $(r, \theta); r \in \mathbb{R}^+, \theta \in \mathbb{P}_1$ on a neighbourhood of a geodesic γ_0. Choose a point $m \in \gamma_O$; the geodesics rays through m form a circle S^1, identifying them with their unit tangent vectors at m. A geodesic γ of $\mathbb{H}^2$ or $\mathbb{R}^2$ which does not contain m is orthogonal to exactly one geodesic ray starting at m and intersects it at a point q. This is true for all geodesics of S^2 different from the "equator" conjugated to m, and not containing m. This defines the coordinates $\theta(\gamma), r(\gamma = d(m, y)$. the measures are:

- $m = |dr \wedge d\theta|$ if $W \subset \mathbb{R}^2$
- $m = |cosr \cdot dr \wedge d\theta|$ if $W \subset S^2$
- $m = |coshr \cdot dr \wedge d\theta|$ if $W \subset \mathbb{H}^2$

We have seen the first measure in the chapter **the euclidean plane**; for the other two see [Sa2]. the (natural) formulas for the measures on the set $\mathcal{A}$ of totally geodesic hypersurfaces in $\mathbb{R}^{n+1}$, $S^{n=1}$ and $\mathbb{H}^{n+1}$ can also be found in [Sa2] .

Theorem 11.2.1

$$\int_W |K| = \int_{\mathcal{A}} |\mu|(\mathcal{F}, H)$$

Proof: We need to replace the orthogonal projections on lines. A geodesic L defines a one-parameter family, called a *pencil* $\mathcal{P}_{\mathrm{L}}$ of totally geodesic hypersurfaces: those orthogonal to it. In $\mathbb{H}^{n+1}$ a pencil is a foliation and defines a projection on the geodesic L ; In S^{n+1} a pencil defines a foliation of $S^{n+1} \setminus S^n$ and a projection of $S^{n+1} \setminus S^n$ on $\mathbb{P}_1$.

Definition 11.2.2 *The polar curve $\Gamma_{\mathcal{P}}$ is the closure of the set of points where a hypersurface of the pencil $\mathcal{P}$ is tangent to the foliation.*

Remark: as in the euclidean case, $\Gamma_{\mathcal{P}}$ is for almost all $\mathcal{P}$, almost everywhere a smooth curve.

Definition 11.2.3 *The foliated Gauss map $\gamma_{\mathcal{F}} : W \to \mathcal{A}$ associates to a point $m \in W$ the totally geodesic hypersurface tangent at m to the leaf L_m of $\mathcal{F}$ through m.*

The computation of the jacobian of $\gamma_{\mathcal{F}}$ is the same as in the euclidean case, observing that the totally geodesic hypersurfaces orthogonal to the geodesic L(m) through m orthogonal to L_m, and the totally geodesic hypersurfaces through m, form two submanifolds of $\mathcal{A}$ othogonal in $\mathcal{A}$ for the natural riemannian metric of $\mathcal{A}$. □

The following theorem is now a consequence of the fact that the intersection of a foliation of S^3 with a generic totally geodesic S^2 has at least two singular points.

Theorem 11.2.4 *Let $\mathcal{F}$ be a foliation of S^3 having a finite number of singularities, then*

$$\int_{S^3} |K| \geq 2\pi^2$$

Using the Poincaré-Hopf theorem on all the generic S^2's we prove also the following theorem:

Theorem 11.2.5 *If one of the previous integrals is finite,then:*

$$\int_{S^3} K = 2\pi^2$$

Let us now state a theorem for foliations with only saddle-like singularities of compact surfaces of constant curvature (-1) [La-Le1]. It is similar to the result of $La - Ro1$ in the sense that it translates in terms of total curvature a topological property of those foliations.

Theorem 11.2.6 *Let M be a compact surface without boundary endowed with a hyperbolic metric (that is a metric of constant curvature (-1)) and $\mathcal{F}$ a foliation the only singularities of which are saddles. The total curvature of $\mathcal{F}$ satisfies:*

$$\int_M |k| \geq (12Log2 - 6Log3)|\chi(M)|$$

Remark:

- We will give below examples of foliations which achieve the minimal value given by the theorem.
- If all the saddles have an even number of separatrices (in particular if $\mathcal{F}$ is orientable), one can show that the total curvature of $\mathcal{F}$ satisfies:

$$\int_M |K| \geq 4Log2 \cdot |\chi(M)|$$

• It is hopeless to look for a generalisation to all surfaces; see [La-Le1]

We will need a few facts from hyperbolic geometry. the hyperbolic plane $\mathbb{H}^2$ is identified with the interior of the unit disc (Poincaré's model). The boundary S_∞ of this disc is the *circle at infinity* of $\mathbb{H}^2$. The geodesics of $\mathbb{H}^2$ are the arcs of circles orthogonal to S_∞ contained in $\mathbb{H}^2$. Recall that by analogy with the notation $\mathcal{A}(3,1)$ used for the set of affine lines of $\mathbb{R}^3$we denote by $\mathcal{A}$ the set of all geodesics of $\mathbb{H}^2$. It has a measure invariant by the action of the hyperbolic isometries Two distinct points m and m' of $\mathbb{H}^2$ are "joined" by a unique geodesic; it is also the case if m and m' are in S_∞; in that case we say that the points are the *points at infinity* of the geodesic. Three distinct points of S_∞ define that way an *asymptotic triangle* and all asymptotic triangles are isometric (there is a global isometry of $\mathbb{H}^2$ sending one on the other). An asymptotic triangle has, as one can check using the Gauss-Bonnet theorem, area π. Let $p : \mathbb{H}^2 \to M$ be the universal covering map. If the restriction of p to the interior of an asymptotic triangle is injective, we will also call its image in M an asymptotic triangle. In order to get foliations minimising total curvature, we need first to construct a foliation $\mathcal{F}_a$ on an asymptotic triangle $\mathcal{T}$ (see next picture)

Let b be the center of symmetry of $\mathcal{T}$. The foliation $\mathcal{F}_a$ has just one singularity, at b, a three prong saddle. The separatrices starting at b are geodesic rays joining b to the points at infinity of $\mathcal{T}$; they intersect in b in equal angles (equal to $2\pi/3$). To get $\mathcal{F}_a$ just fill each sector with geodesically convex curves, in such a way that the boundary of $\mathcal{T}$ is the union of three leaves. If the projection p is injective on $\mathcal{T}$, we can project $\mathcal{F}_a$ on M; see next picture

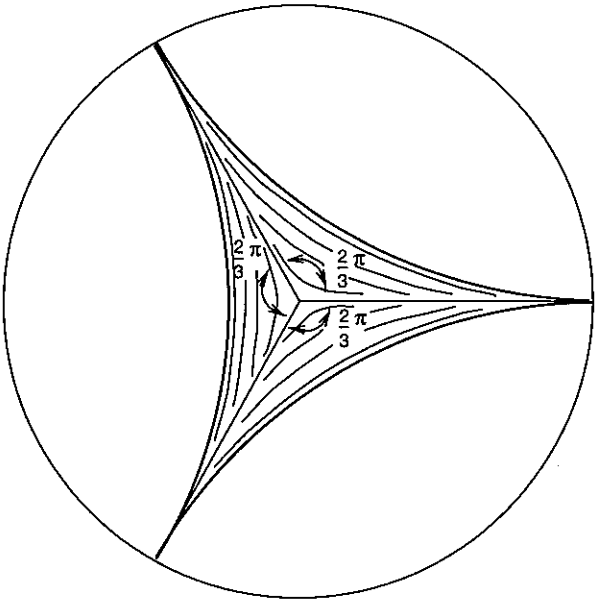


Figure 45: foliation of an asymptotic triangle

Figure How to fit an asymptotic triangle on a hyperbolic pair of pants

The total curvature of that foliation of the asymptotic triangle is 2Log2-Log3) as we will see below. Let now M be a closed orientable hyperbolic surface of genus g. Choose on M a family of $3g-3$ compact disjoint geodesics slicing M into g pairs of pants (each pair of pants is topologically a disc with two holes). Choose in each pair of pants three disjoint geodesics spiraling towards the boundary (see the picture above).

We can then fill the surfaces with copies of the model foliation constructed above, achieving the lower bound given by the theorem. Using Whitehead transformations we can split the saddles with more than two separatrices into three prong saddles without increasing the curvature by more than ϵ, see picture below and [F-L-P] for a carefull construction.

As the singularities of $\mathcal{H}$ are all saddles, one cannot find in $\mathcal{H}$ Whitehead discs, that is discs with boundary made either of a finite number of arcs of leaves, or of a finite number of arcs of leaves and one arc

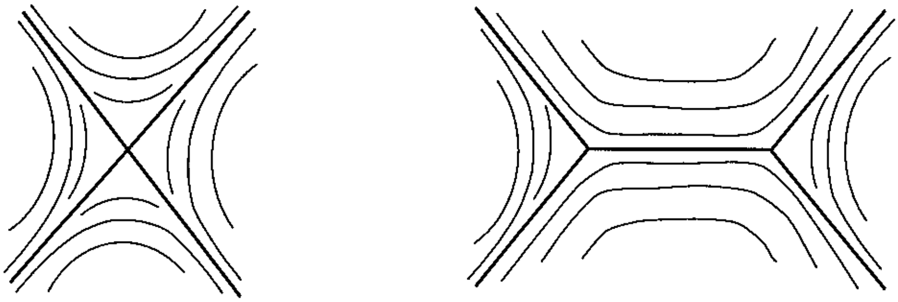


Figure 46: a Whitehead transformation

transverse to $\mathcal{F}$.

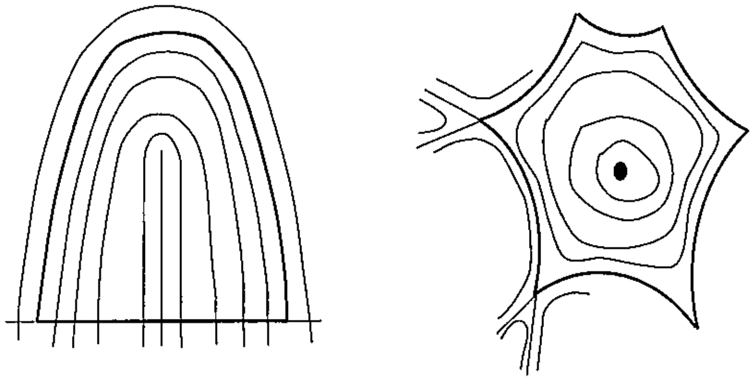


Figure 47: Whitehead discs

We can also, without increasing the total curvature by more than ϵ, brake all the saddle connections. The foliation $\mathcal{F}$ of M lifts to a foliation $\mathcal{H}$ of $\mathbb{H}^2$.

Affirmation We can associate to each saddle s of $\mathcal{H}$ a set of geodesics A_s of measure bigger or equal to (6Log2-3Log3), and an injection of A_s in $\mathbb{H}^2$ sending each geodesic to a point where it is tangent to $\mathcal{H}$. Moreover the respective images $B_s \subset \mathbb{H}^2$ and $B_{s'} \subset \mathbb{H}^2$ of the sets of geodesics A_s and $A_{s'}$ associated to different saddles are disjoint.

The fact that $\mathcal{F}$ has $2|\chi(M)|$ saddles, and a carefull application of the foliated exchange theorem will end the proof of our result about hyperbolic surfaces.

Lemma 11.2.7 *any half-leaf δ of $\mathcal{H}$ which does not end at a saddle goes to a point of the circle at infinity S_∞*

Proof: First observe that the behaviour at infinity of the half leaves of $\mathcal{H}$ does not change if we change $\mathcal{F}$ by an isotopy (if $\tilde{\phi}$ is an homeomorphism of $\mathbb{H}^2$ lifting of a homeomorphism of M isotopic to the identity, then $sup_{m\in\mathbb{H}^2}[d(m,\tilde{\phi}(m))]$ is finite). This proves the lemma if the half leaf $p(\delta)$ of $\mathcal{F}$ is compact or spirals towards a compact leaf: a compact leaf of $\mathcal{F}$ cannot be null-homotopic in M , as it cannot bound a disc, and then is (free)homotopic to a closed geodesic. If the closure $\overline{p}(\delta)$ does not contain a compact leaf, we can choose a leaf $\delta_1 \in \overline{p}(\delta)$ and a closed curve c transverse to $\mathcal{F}$ and intersecting δ_1. The curve C meets δ infinitely many times, as it cannot bound a foliated disc, it is also homotopically not null-homotopic, so its lift to $\mathbb{H}^2$ will stay at bounded distance from the closed geodesic in the same free homotopy class. As the foliation $\mathcal{H}$ of $\mathbb{H}^2$ does not admit Whitehead discs, the half-leaf δ meets a component of $p^{-1}(C)$ in at most one point. The intersection in $\mathbb{H}^2 \cup S_\infty$ of the sequence of nested half-spaces which δ enters (see next picture) is exactly one point of S_∞, because it cannot contain any point of $\mathbb{H}^2$, as the distance between two different lifts of C is bounded below (it cannot contain two points of S_∞ without containing the geodesic joining them). □

Remark: Two separatrices δ and δ' starting at the same saddle s of $\mathcal{H}$ converge to distinct points of S_∞.

Proof: This is true when the union $p(\delta) \cup p(\delta')$ meets at least twice a closed simple curve C transverse to the foliation $\mathcal{F}$, as, again, $\mathcal{H}$ has no Whitehead discs, so any component $p^{-1}(C)$ meeting δ or δ' separates the points at infinity of δ and δ'. If such a curve C does not exist, then $p(\delta)$ and $p(\delta')$ spiral towards compact leaves δ_0 and δ_0' of $\mathcal{F}$. If δ and δ' where isotopic, the compact leaves δ_0 and δ_0' should also be, as two geodesics which have compact projections cannot share a point at infinity if they do not coincide. If $\delta_0=\delta_0'$ the union of two arcs starting at s of respectively $p(\delta)$ and $p(\delta')$, with an arc transverse to $\mathcal{F}$ joining their endpoints, will bound a Whitehead disc, providing a contradiction. If δ_0 and δ_0' were distinct , they should bound an anulus. This annulus cannot contain singularities of $\mathcal{F}$ because the singulaties of $\mathcal{F}$, all saddles, will give to the anulus a negative Euler characteristic. Looking at the same time at $\mathcal{H}$ and $\mathcal{F}$ the reader will check that the only remaining possibility is that $p(\delta)$ and $p(\delta')$ are spiraling toward the same leaf of $\mathcal{F}$, on the same side, which again will allow the construction of a Whitehead disc.

□

so we can associate to each saddle of $\mathcal{H}$ three points of S_∞ which define an asymptotic triangle Δ_s (see next picture).

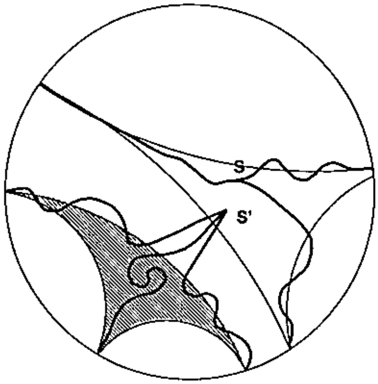


Figure 48: Asymptotic triangle associated to a saddle s

We will call the three geodesics joining these points at infinity the *asymptotes* of s. Two asymptotes starting at distinct saddles cannot intersect in $\mathbb{H}^2$ (as it will force an intersection of some of the separatrices), so the asymptotic triangles associated to distinct saddles have disjoint interiors.

Fix now a geodesic L of $\mathbb{H}^2$ which does not contain any saddle of $\mathcal{F}H$, is not asymptotic to any separatix of $\mathcal{H}$, and is not tangent to any separatrix of $\mathcal{H}$ (these conditions are generic).

Definition 11.2.8 *Given a generic geodesic L, the couple (s, D), s a saddle of $\mathcal{H}$, and D one of its three asymptotes , is called admissible if it satisfies the following conditions:*

-$s \notin D$

-s and Δ_s are on the same side of D

-L does not intersect D and separates s from D

To each L-admissible couple (s, D) we will associate a compact domain $T_{s,D}$ see picture below).

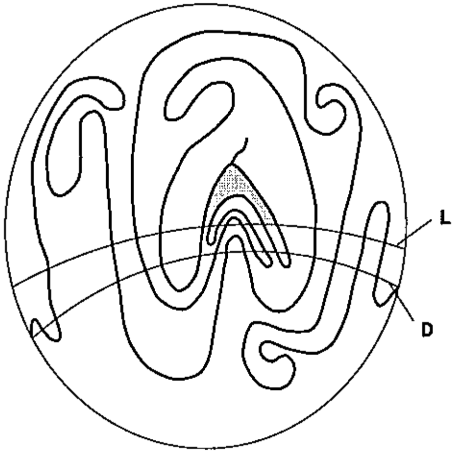


Figure 49: the domain $T_{s,d}$

The two separatrices starting at s and asymptotic to D cut $\mathbb{H}^2$ into two domains. We will call $\mathcal{D}_{s,D}$ the closure of the one which does not contain the points at infinity of L. Let us call $H^+(\mathrm{L}, s)$ the closed half plane of boundary L which contains s, and $T_{s,D}$ the connected component of $H^+(\mathrm{L}, s) \cap \mathcal{D}_{s,D}$ which contains s. The domain $T_{s,D}$ is compact and homeomorphic to a disc (see picture above).

If (s, D) and (s', D') are two L-admissible couples, only the four following situations are possible:

-$T_{s,D}$ is contained in $T_{s',D'}$

-$T_{s',D'}$ is contained in $T_{s,D}$

-$T_{s,D}$ and $T_{s',D'}$ are disjoint

-$T_{s,D}$ and $T_{s',D'}$ have disjoint interiors and $s = s'$

in particular the situation of the next picture is impossible.

Lemma 11.2.9 *For any L-admissible couple (s, D), the collections of arcs $L \cap T_{s,D}$ is tangent to $\mathcal{H}$ at at least one point.*

Proof: the compacity of $T_{s,D}$ and the fact that the set of saddles of $\mathcal{H}$ is discrete implies that $T_{s,D}$ can contain at most a finite number of domains $T_{s',D'}$. It is then enough to prove the lemma for a minimal (for the inclusion) domain $T_{s,D}$ (see next picture).

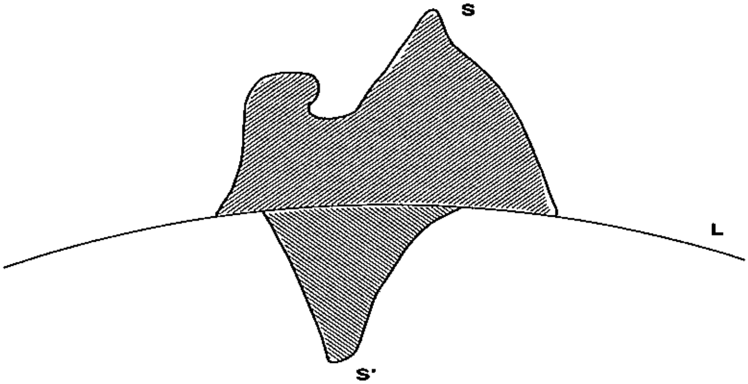


Figure 50: impossible position of the two domains $T_{s,d}$ and $T[s', d']$

If the lemma is false, $T_{s,D}$ is a disc which does not contain in its interior any singularity of $\mathcal{H}$ and the boundary of which is made alternatively of arcs of leaves of $\mathcal{H}$ and arcs transverse to $\mathcal{H}$. Moreover, the definiton of an L-admissible couple implies that, in a neighbourhood of s the third separatrix starting at s (the one which is not asymptotic to D) is not contained in $T_{s,D}$ (see next picture)

The only possibility for $T_{s,D}$ is to be a "rectangle" (see picture above). The arc of the leaf between the points 2 and 3 on the above picture belongs to one of the separatrices starting at s and asymptotic to D, say the one which contains the point 1. Let us now consider the arc of separatix joining 1 to 2. this arc does not meet the segment of L of extremities 1 and 2, and, with this segment, bounds a Whitehead disc, providing a contradiction. □

We will call *strongly L-admissible* a couple (s, D) if it is L-admissible and if L meets Δ_s (and then the two sides of Δ_s different from D). Then, given L, a saddle s cannot belong to more than one couple strongly L-admissible, and the domains $T_{s,D}$ corresponding to different couples strongly L-admissible are disjoint.

To a saddle s of $\mathcal{H}$ let us now associate the set A_s of geodesics L such that there exists an asymptote D of s such that the couple (s, D) is strongly L-admissible. We obtain the required injection $i_s : A_s \to \mathbb{H}^2$ associating to a geodesic L one of the points of $L \cap T_{s,D}$ where L is tangent to $\mathcal{H}$ (see lemma 2). We can choose the injection i_s in an equivariant

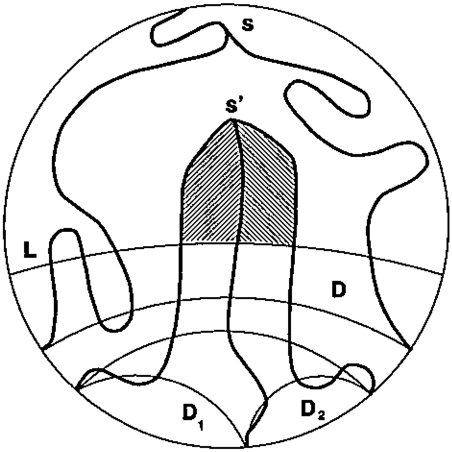


Figure 51: domain $T_{s,d}$ which is not minimal

way, that is, if σ is an automorphism of the universal covering $\mathbb{H}^2 \to M$, and σ^* the induced transformation on the set of geodesics $\mathcal{A}$, then, for all saddles s of $\mathcal{H}$, $i_{\sigma(s)} \circ \sigma^* = \sigma \circ i_s$. Let us call B_s the image $i_s(A_s) \subset \mathbb{H}^2$. As, for a fixed L, the domains $T_{s,D}$ corresponding to distinct strongly L-admissible couples are disjoint, we have: $B_s \cap B_{s'} = \emptyset$ if $s \neq s'$. To prove the affirmation stated above, we need now to check the inequality $m(A_s) \geq (6Log2 - 3Log3)$. Let us first proof a lemma of hyperbolic geometry.

Lemma 11.2.10 *Let $0 < \alpha \leq \pi$ be the angle between two geodesic rays starting at a point $s \in \mathbb{H}^2$ and asymptotic to a geodesic D (see picture below), and let $f(\alpha)$ be the measure of the set of geodesics which do not intersect D but separate D and s. Then:*

a) $f(\alpha) = -2Log\, sin(\alpha/2)$

b)if $0 < \alpha \leq \pi$, $0 < \beta \leq \pi$ *and* $0 < \gamma \leq \pi$ *are three angles such that* $\alpha + \beta + \gamma = 2\pi$, *then*

$$f(\alpha) + f(\beta) + f(\gamma) = 6Log2 - 3Log3$$

Proof: a) As $f(\pi) = 0$ it is enough to prove that $f'(\alpha) = -cotg(\alpha/2)$. Let $h(\alpha)$ be the (hyperbolic) distance between s and D, the quantity $f'(\alpha) \cdot d\alpha$ is equal to the measure of the set of geodesics intersecting a geodesic segment of infinitesimal length $dh = h'(\alpha) \cdot d\alpha$ with

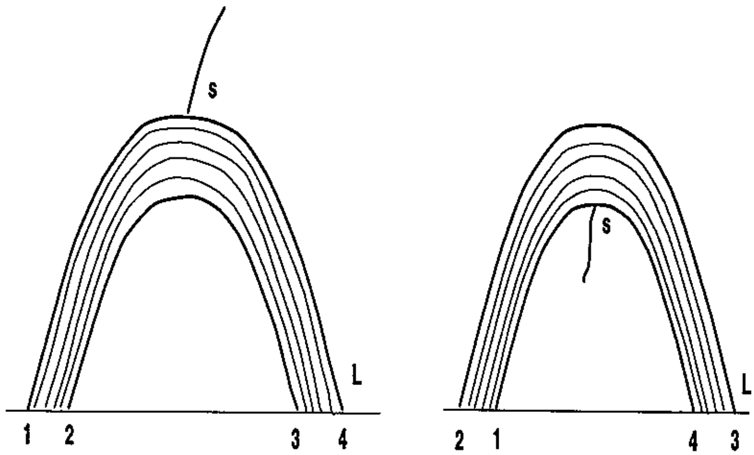


Figure 52: position of the third separatrix

an angle bigger or equal to $\alpha/2$. This measure is proportional to dh and the coefficient $(2cos(\alpha/2)$, can be computed using the "euclidean" formula, tangent to the hyperbolic one if the origin is in dh. Then $f'(\alpha) = 2cos(\alpha/2) \cdot h'(\alpha)$. Hyperbolic trigonometry provides the formula $cosh(h(\alpha) = \frac{1}{sin(\alpha/2}$ (see for example [Thu2] formula 2.6.12). After checking that $h'(\alpha) = 1/2\, sin(\alpha/2)$, we get the required formula $f'(\alpha) = -cotg(\alpha/2)$.

b) Triples (α, β, γ) of angles between 0 and π parametrise the vertices of an asymptotic triangle. For example, if the point s is on the boundary of the triangle, one of the angles, say $\gamma = \pi$, and

$$f(\alpha) + f(\beta) + f(\gamma) = -2Log[sin(\alpha 2) \cdot sin((\pi - \alpha)2)] = 2Log(2/sin\alpha)$$

Then:

$$f(\alpha) + f(\beta) + f(\gamma) \geq 2Log2 > 6Log2 - 3Log3$$

If the point s tends to a vertex of the asymptotic triangle, then one of the angles goes to 0 and $f(\alpha) + f(\beta) + f(\gamma)$ goes to $+\infty$. to prove assertion (b) it is enough to check that the only extremum of $f(\alpha)+f(\beta)+f(\gamma)$ in the triangle is achieved when s is a center of symmetry and $\alpha = \beta = \gamma = 2\pi/3$. This is true, as the differential of the function $f(\alpha) + f(\beta) + f(\gamma)$ is $-cotg(\alpha/2) \cdot d\alpha$ is zero only if $cotg(\alpha/2) = cotg(\beta/2) = cotg(\gamma/2)$, that is if $\alpha = \beta = \gamma = 1\pi/3$ □

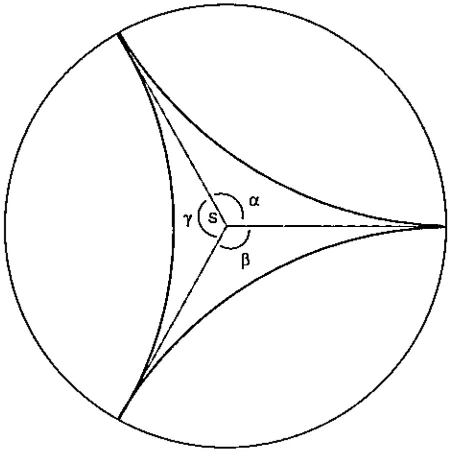


Figure 53: The only extremum of $f(\alpha)+f(\beta)+f(\gamma)$

Coming back to a saddle s of $\mathcal{H}$, two cases are possible:

-s belongs to the asymptotic triangle Δ_s (or to its boundary), then the previous lemma implies that:

$$m(A_s) \geq 6Log2 - 3Log3$$

-s is exterior to Δ_s (see next picture) then the couple (s, D_i), $(i = 1, 2)$ is strongly L-admissible for m-almost all geodesics L which does not intersect D_i and separating the point $t \in \partial\delta_s$ (see picture below) from D_i.

Then

$$m(A_s) > f(\alpha) + f(\beta) = f(\alpha) + f(\beta) + f(\pi) \geq 6Log2 - 3Log3$$

We proved the affirmation; let us now deduce the theorem from the affirmation. For each saddle $\overline{s}_i$ of $\mathcal{F}$ we choose a lift s_i in $\mathbb{H}^2$. Recall that the number of (three prong) saddles of $\mathcal{F}$ is $h = 2\chi(M)$. Let B be the disjoint union of the sets B_{s_i}. As for any automorphism σ of the covering, we have: $\sigma B = \cup_{i=1}^{h} A_{\sigma s_i}$ the sets B and σB are disjoint if $\sigma \neq Id$, and this implies that the restriction to B of the covering projection p is injective.

Suppose first there exists a neighbourhood U of B such that the restriction of p to U is also injective. Then the total curvature of $\mathcal{F}$ is

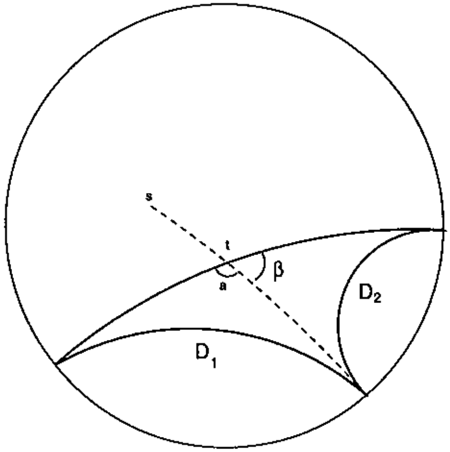


Figure 54: outside of the asymptotic triangle δ_s

bigger than or equal to $\sum_{i=1}^{h} m(A_{s_i}) \geq 2 \cdot |\chi(M)| \cdot (6Log2 - 3Log3)$. If such a neighbourhood U would exist, the theorem would be proven.

In general it is impossible to find the neighbourhood U of B, but we will construct, for each small $\epsilon > 0$, subsets $A^{\epsilon}_{s_i}$such that $m(A_{s_i} \setminus A^{\epsilon}_{s_i})$ goes to 0 with ϵ, and such that we can find an open neighbourhood U^{ϵ} of the corresponding set B^{ϵ} to which the restriction of p is injective. The foliated exchange theorem implies that the theorem is a consequence of the existence of the sets $A^{\epsilon}_{s_i}$

Let us fix $\epsilon > 0$ and let L be a geodesic of A_{s_i}. There exists then an asymptote D of s_i such that the couple (s_i, D) is strongly L-admissible. From s let us consider the geodesic ray orthogonal to D. It intersects D at a point t. The geodesic D' is orthogonal to that ray at a point situated between s and t, at distance ϵ from t (see the picture below).

We can suppose that D' is transverse to the two separatrices starting at s_i and asymptotic to D and define as with s, D and L a compact domain $T^{\epsilon}_{s_i,D'} = \mathcal{D}_{s,D} \cap H^{+}_{s,D}$ (shaded on previous picture). Let n be the number of saddles contained in $T^{\epsilon}_{s_i,D'}$; we can choose a neighbourhood v_{ϵ} of the boundary $\partial T^{\epsilon}_{s_i,D'}$ of the domain $T^{\epsilon}_{s_i,D'}$, such that the total curvature of $\mathcal{H}|_{v_{\epsilon}}$ is bounded by ϵ.

We keep in $A^{\epsilon}_{s_i}$ a geodesic $\mathrm{L} \in A_{s_i}$if and only if:

i)L does not intersecct D' and separates s_i from D'

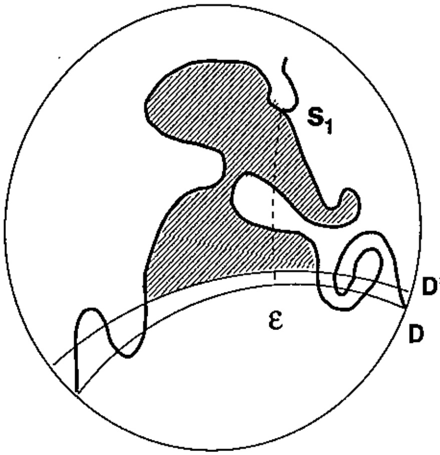


Figure 55: construction of A^ϵ

ii)the distance from L to each saddle $s \in T^\epsilon_{s_i,D'}$ is at least ϵ/n

iii)L is transverse to $\mathcal{H}$ in the neighbourhood v_ϵ of $\partial T^\epsilon_{s_i,D'}$

The exchange theorem and the definition of v_ϵ show that the measure $m(A_{s_i} \setminus A^\epsilon_{s_i})$ goes to zero with ϵ. Let $B^\epsilon_{s_i} \subset B_{s_i}$ be the image of $A^\epsilon_{s_i}$ in $\mathbb{H}^2$, and let $B^\epsilon = \cup_{i=0}^h B^\epsilon_{s_i}$.

To finish the proof we will show that for fixed ϵ, i and j, the distance from $B^\epsilon_{s_i}$ to the union of the conjugates of $B^\epsilon_{s_j}$ is strictly positive (if $i = j$ we use only conjugation of the covering different from the identity). Let then $Q \in B^\epsilon_{s_i}$ and $Q' \in B^\epsilon_{s_j}$ be such that Q and $\sigma Q'$ are very close (supposing again that σ is not the identity if $i = j$). The condition (iii) above implies that $\sigma Q'$ is in $T^\epsilon_{s_i,D}$ (see next picture)

The asymptotic triangle associated to the saddle σs_j is then on the side of D which does not contain s_i (the analogous condition interverting the roles of s_i and s_j may also happen). The geodesic L$'$ tangent to $\mathcal{H}$ at $\sigma Q'$ should then meet D and D'.

One cannot define a reasonable metric on the set of all geodesics of $\mathbb{R}^2$ or $\mathbb{H}^2$. Two geodesics intersecting with a small angle should be close. Then it is impossible to separate parallel geodesics ($\mathbb{R}^2$) or asymptotic geodesics (geodesics with one point at infinity in common, in the $\mathbb{H}^2$ case). But it is possible to define a distance on the set of geodesics

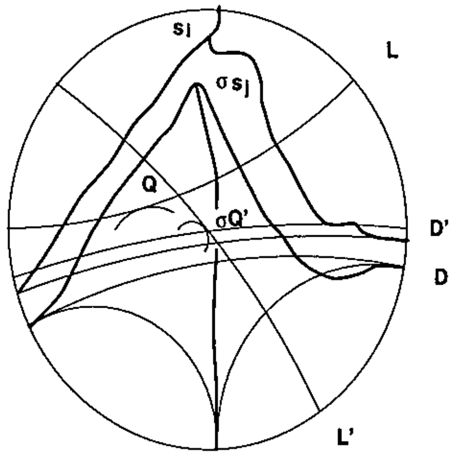


Figure 56: position of $B^{\epsilon}_{s_i}$ and $B^{\epsilon}_{s_j}$

which intersect a given compact $K \subset \mathbb{H}^2$ by:

$$d_K(\mathrm{L}, \mathrm{L}') = sup[(inf_{x \in \mathrm{L} \cap K, y \in \mathrm{L}' \cap K} d(x, y) \, ; \, angle(\mathrm{L}, \mathrm{L}')]$$

if $\mathrm{L} \cap \mathrm{L}' \cap K = \emptyset$ (just forget the angle term if $\mathrm{L} \cap \mathrm{L}' = \emptyset$)

$$= angle(\mathrm{L}, \mathrm{L}') \, if \, \mathrm{L} \cap \mathrm{L}' = m \in K$$

The geodesics L and L′ constructed above satisfy $d_K(\mathrm{L}, \mathrm{L}') \geq \eta > 0$ taking $K = T^{\epsilon}_{s_i, D'}$, where η does not depend on Q, Q' and σ. If L and L′ do not intersect, or intersect far from K, they cannot be close in K and satisfy the required conditions. Otherwise , as our conditions (ii) garantees L does not pass by to close to the saddles, this implies the distance between Q and $\sigma Q'$ is bounded below by a positive constant independant of Q, Q' and σ. We use the following fact: given $\theta > 0$ and a compact $K \subset M$ containing no saddle of $\mathcal{F}$, there exist $\alpha > 0$ such that, if two geodesics L_1 and L_2 tangent to $\mathcal{H}$ at two points a_1 and a_2 belonging to $p^{-1}(K)$ intersect at an angle bigger than θ, then the distance between a_1 and a_2 is at least α

In [La-Le1] the reader can find an application of the foliated exchange theorem to pairs of orthogonal foliations of S^2.

11.3 Tight foliations

We have seen that the foliated exchange theorem and some topological analysis of the foliation provide inequalities. Do there exist foliations achieving the equality case? We will call *tight* such foliations. An example of a positive result is the following:

Theorem 11.3.1 *Let A be a plane annulus limited by two convex curves C_1 of length δ_1 and C_2 of length δ_2. We suppose that C_2 is the "inner" one (Cauchy's formula implies that $\delta_1 > \delta_2$). Then the leaves of the tight foliation of the annulus (tangent to the boundary) are either closed convex curves isotopic in A to C_1 (and C_2) or locally convex curves spiraling towards convex curves isotopic to C_1. (see picture below). the total curvature of the foliation is, in that case:*

$$\int_A |k| = \delta_1 - \delta_2$$

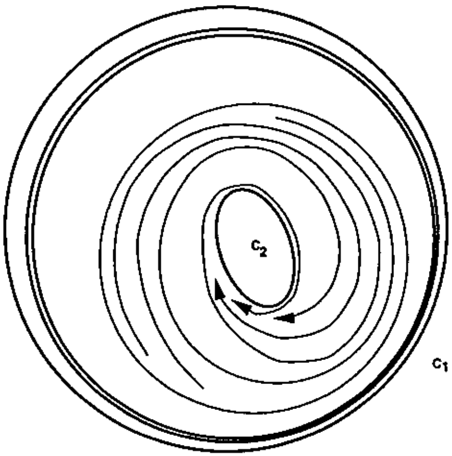


Figure 57: Tight foliation of a plane annulus with convex boundary curves

Proof: Using Cauchy's formula we know that the set B of affine lines intersecting C_1 and not intersecting C_2 has measure $\delta_1 - \delta_2$. Such a line L intersect the annulus in a segment I. The foliation $\mathcal{F}$ is not transverse to the interior of I, otherwise the boundary of C_1 and I

would form a Whitehead disc for $\mathcal{F}$, which is impossible as $\mathcal{F}$ has no singularity. Then

$$|\mu|(\mathcal{F},\mathrm{L}) \geq 1$$

so the total curvature of $\mathcal{F}$ is bigger or equal than the measure of B. The equality is achieved for the foliations described in the theorem, as they satisfy:

$$\mathrm{L} \in B \Rightarrow |\mu|(\mathcal{F},\mathrm{L}) = 1$$

$$\mathrm{L} \notin B \Rightarrow |\mu|(\mathcal{F},\mathrm{L}) = 0$$

□

In [Lan2] the reader will find a study of tight (in their isotopy class) foliations of the torus T^2.

Let us now consider the same question for (nonsingular) foliations of S^3.

Theorem 11.3.2 *Their does not exist any tight foliation of the sphere S^3.*

Proof: We have seen before that the total curvature of a foliation $\mathcal{F}$ of S^3 satisfies:

$$\int_{S^3} |K| \geq 2\pi^2$$

because for a generic totally geodesic sphere $\Sigma \subset S^3$ one has $|\mu|(\mathcal{F},\Sigma) \geq 2$. We have also seen that

$$\int_{S^3} K = 2\pi^2$$

If a foliation $\mathcal{F}$ of S^3 satisfy $\int_{S^3} |K| = \int_{S^3} K$ then the curvature function should satisfy $K \geq 0$. In S^3 the intrinsic curvature K_e of an embedded surface satisfy $K_e = K + 1$ (one can perform the computation using the exponential map (see [Spi]). Novikov's theorem states that the foliation has a Reeb component ([Ca-Li]) with boundary a torus leaf L. The Gauss-Bonnet theorem applied to L states that $\int_L K_e = 0$. Then $\int_L K = -vol(L) < 0$ so the leaf has a point of negative (extrinsic) curvature K, contradicting the hypothesis. The theorem will then be proved if we can show that:

$$inf \int_{S^3} |K| = 2\pi^2$$

Let us consider the singular foliation $\mathcal{P}$ of S^3 defined by a pencil of geodesic 2-spheres. It has a one dimensional singular locus: a geodesic circle C. The trace of $\mathcal{P}$ on a geodesic sphere Σ transverse to C is a foliation with two singular points of index 1 (of type sink/source). The next object we need is the model Reeb foliation of the thick torus $D^2 \times S^1$. To obtain it we will construct a foliation of $D^2 \times \mathbb{R}$ invariant by unit translations in $\mathbb{R}$ (we can visualise $D^2 \times \mathbb{R}$ as a vertical thick cylinder). In the vertical band $[-1, 1] \times \mathbb{R}$ of the $(x, z) - plane$ consider a convex curve asymptotic to both sides of the band. The equation $z = tg(\pi/2)x^2$ should provide such a curve. by revolution around the $z - axis$ we obtain a convex surface asymptotic to the boundary of the cylinder (on the $z \to +\infty$ side. Translating it vertically, we foliate the thick cylinder. By construction the foliation is invariant by vertical translation and then gives a foliation of the thick torus $T = (D^2 \times \mathbb{R}/(2\pi \cdot \mathbb{Z})$. (see picture below)

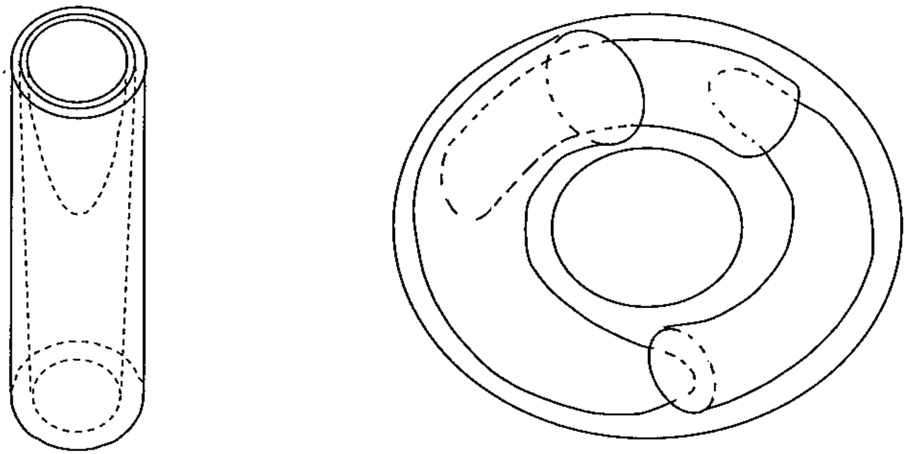


Figure 58: Reeb component

We will now shadow the foliation $\mathcal{P}$ by non singular ones, introducing a very thin Reeb component in a tubular neighbourhood of C.

To construct the foliation in a tubular neigbourhood $Tub_{2r}(C)$ of radius 2r of C we will first construct a model in the cylinder $D^2_{2r} \times \mathbb{R}$, invariant by vertical translations. In the cylinder $D^2_r \times \mathbb{R}$ just put a Reeb component defined as above. In the anulus $D^2_{2r} \setminus D^2_r$, seen as a subset of

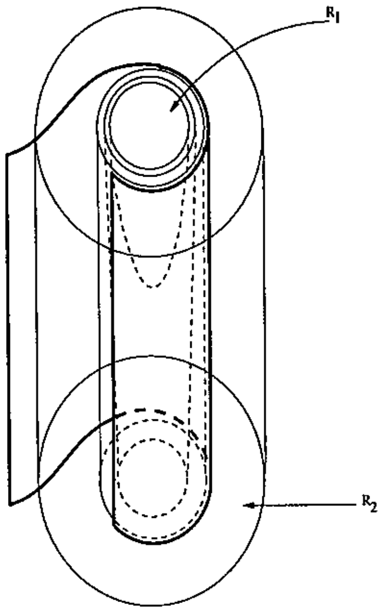


Figure 59: A piece of a thin Reeb component and how the other leaves wrap around it

the $(x, y) - plane$, consider a curve entering, normally to the boundary, into D^2_{2r} and spiraling towards the circle ∂D^2_r (see picture below).

The product of that curve by the vertical line is a surface of $\mathbb{R}^3$ entering normally the cylinder $D^2_r \times \mathbb{R}$ and spiraling toward the inner cylinder $D^2_r \times \mathbb{R}$. By rotation around the z-axis we foliate the set $(D^2_{2r} \setminus D^2_r) \times \mathbb{R}$. So we get the desired foliation of the thick cylinder $D^2_r \times \mathbb{R}$.

The quotient by the vertical translations by vectors of lenth 2π is a foliation of $D^2_{2r} \times S^1$. Let us now map $D^2_{2r} \times S^1$ to the tubular neighbourhood of (geodesic) radius 2r of C, mapping isometrically S^1 on C and using the exponential map to map the discs D^2_{2r} centered on points $(0, 0, z) \in S^1$ onto totally geodesic discs normal to C. that way we obtain a foliation $\mathcal{F}_r$ which fits with $\mathcal{P}|_{S^3 \setminus Tub_{2r}(C)}$. The reader will now believe that

-the geodesic spheres Σ satisfy $|\mu|(\mathcal{F}_r, \Sigma) = 2$ if Σ intersects C with not too small an angle.

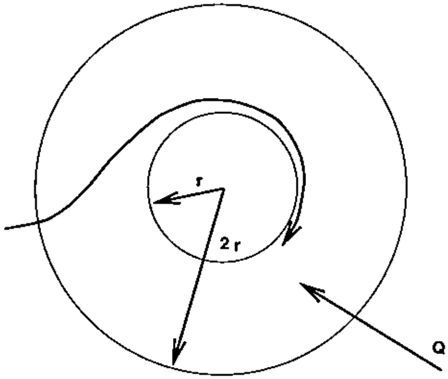


Figure 60: Horizontal section of the foliation $(D_{2r}^2 \setminus D_r^2) \times \mathbb{R}$

-There exists a uniform bound, independent of r, for the number $|\mu|(\mathcal{F}_r, \Sigma)$ when it is finite.

As the measure of the geodesic spheres which intersect C with an angle smaller than ϵ goes to zero with ϵ, we proved, using the foliated exchange theorem, that

$$lim_{r\to 0} \int_{S^3} |K| = 2\pi^2$$

where $|K|$ is the curvature function defined by the leaves of $\mathcal{F}_r$. □

11.4 Codimension higher than one, diverging integrals and conformal results

We will present here without proofs particular cases of the results of [La-Ni]

Theorem 11.4.1 *Let $\mathcal{F}$ be a smooth foliation by curves of a domain $W \subset \mathbb{R}^3$. Let C_H be the contact set (in general a curve) of $\mathcal{F}$ and the affine hyperplane H:*

$$C_H = \{m \in W | T_m\mathcal{F} \subset H\}$$

Then

$$\int_W |k| = const \int_{\mathcal{A}(3,1)} \int_{C_H} |sin\varphi|$$

Where at a smooth point $m \in C_H$, φ is the angle between C_H and the leaf of $\mathcal{F}$ through m.

Conformal integral geometry of foliations

Let now $\mathcal{F}$ be a codimension 1 foliation of a domain $W \subset \mathbb{R}^3$. The number $N^-(\Sigma)$ of negative contact points of Σ with $\mathcal{F}$ is the number of saddle tangencies of Σ and $\mathcal{F}$. It is clear that the number $N^-(\Sigma)$ is conformally well-defined.

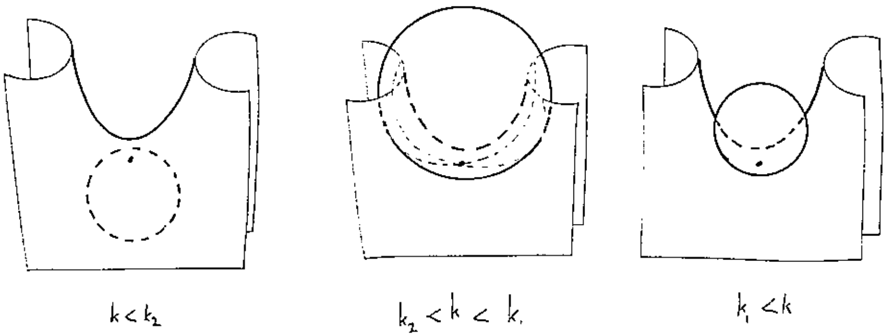


Figure. possible generic contacts of a sphere and a foliation

A measure on the set $\mathcal{S}$ of spheres of $\mathbb{R}^3$, considered as a subset of the set of spheres of S^3 is constructed in the chapter **The space of spheres.** Using that measure we have the

Theorem 11.4.2 *Let $\mathcal{F}$ be a smoothe foliation of a domain $W \subset \mathbb{R}^3$. Then*

$$\frac{1}{6}\int_W |k_1 - k_2|^3 = \int_S N^-(\Sigma)dm(\Sigma)$$

where k_i are the principal curvatures of the leaves.

Remark: We could have stated the theorem in S^3 as the form $|k_1 - k_2|^3 dv$, where dv is the volume element of W, is a conformal invariant.

12 Complex integral geometry

The n-dimensional complex space $\mathbb{C}^n$ has a natural hermitian structure, and an associated scalar product:

$$< u|v >_e = \mathcal{R}e(< u|v >)$$

The n-dimensional complex space $\mathbb{C}^n$ endowed with the quadratic form $|v|^2 = \mathcal{R}e(< v|v >$ is just a euclidean space of dimension 2n. But, among the real euclidean planes, some have an extra property: they are globally invariant by multiplication by complex numbers. The complex integral geometry will deal with those particular real planes: the complex lines. To compensate the relatively few partial datas given by the projections on the complex lines and complex subspaces only, and by the section by the affine complex subspaces only, we need to suppose that the submanifolds studied have some extra structure. So in this chapter the submanifolds are local images of $\mathbb{C}^p$ by a locally defined holomorphic map.

12.1 Critical points of projections on complex lines

The orthogonal projection of $\mathbb{C}^n$ onto a complex line of $\mathbb{C}^n$ is a holomorphic map.

Many interesting consequences can be deduced from the properties of the complex curve C of equation $y = ax^2$ in a neighbourhood of the origin.

The tangent space to C at $(0,0)$ is the x-line and the normal space at $(0,0)$ is the y-line. Let D_θ be the oriented real line of the y complex line making the angle θ with the oriented real axis. The orthogonal projection C_θ of C on the sum :

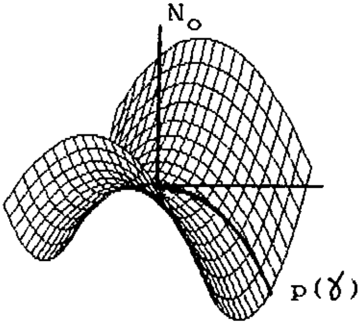
$$E_\theta = (x\ complex\ line) \oplus D_\theta$$

has equation :

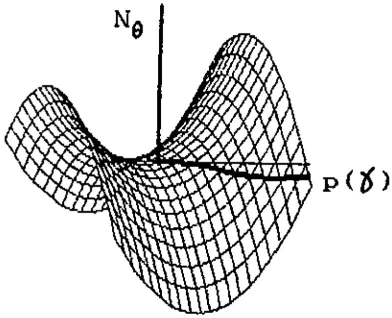
$$z = Re(e^{i\theta}.x^2) \ ,$$

z being the real coordinate on D_θ determined by the euclidean structure of $\mathbb{C}^2$ and the orientation cf D_{θ},.

Fig. Saddle and turned saddle



projection of the curve $y=x^2$ on T_mM+N_o



projection of the curve $y=x^2$ on T_mM+N_θ

Performing the change of variable $x' = e^{i\theta/2}.x$, we see that the projections C_θ are all isometric, more precisely that C_θ is deduced from C_0 (of equation $z = Re(x^2)$ by a rotation with a vertical axis and angle $-\theta/2$.

A section of C_0 by the vertical plane F_φ containing the real line Δ_φ of the x-complex line has the equation:

$$z = Re(a\rho^2.e^{2i\theta}) = \mid a \mid \ \rho^2 cos\ (2\varphi + \varphi_a)$$

where φ_a is the argument of a, therefore the maximal and minimal values of the curvature in $(0,0)$ of those curves are opposite and of absolute value $2. \mid a \mid$. This implies that at (0.0), C_0 has zero mean curvature and Gaussian curvature $4 \mid a \mid^2$.

Remark: The projections of the complex curve of equation $z = ax^n$ on the 3-spaces E_θ are obtained from the projection on E_0 by rotations of angle $-\theta/n$. As curvatures depend only on 2-jets at the point where they are computed, we have proved the following proposition:

Proposition 12.1.1 *Let C be a holomorphic curve of $\mathbb{C}^2$ then the orthogonal projections of C on the 3-spaces $E_\theta = T_m + D_\theta$, where T_m is the complex line tangent at m to C and where D_θ is a real line normal to C in m, have all the same Gaussian curvature at m and have all zero mean curvature at m.*

12.2 Complex Gauss map and critical points.

The normal space $N(m)$ of C at m is a complex line. This allows us to define a map $\gamma_{\mathbb{C}}$ of C to$\mathbb{C}P_1$ by $\gamma_{\mathbb{C}}(m) = N(m)$. At the point (1,0), the Fubini-Study metric of$\mathbb{C}P_1$ is the euclidean metric of the chart given by the map x/y.

Let $K(\Delta, m)$ be the gaussian curvature of the projection M_Δ of M on the space $E_\Delta = T_m M \oplus \Delta$.

The Lipschitz-Killing curvature at m, $K(m)$, of an even dimensional submanifold M of R^N is, up to a constant depending only on the dimensions involved, equal to the integral on the projective space on (real) lines of the normal space, of $K(\Delta, m)$:

$$K(m) = const \int_{\mathbb{P}N(m)} K(\Delta, m)$$

where const indicates a constant depending only on the dimensions involved.

Proposition 12.2.1 *The jacobian of the complex Gauss map satisfies :*

$$\mid det\ D\gamma_{\mathbb{C}}(m) \mid^2 = -K(\Delta, m) = const\ K(m)$$

where const is a universal constant.

Proof: It is enough to prove the proposition for the curve C of equation $y = ax^2$ at the origin as the numbers we shall compute depend only on 2-jets. Let $m(x)$ be the point (x, x^2). The complex normal line is generated by the vector $(-2a\bar{x}, 1)$, therefore, using the map x/y, the differential of the complex Gaussmap is $-2a.J$, where J is conjugation.

One gets $\mid det\ D\gamma_{\mathbb{C}}(0) \mid = 4 \mid a \mid^2$. □

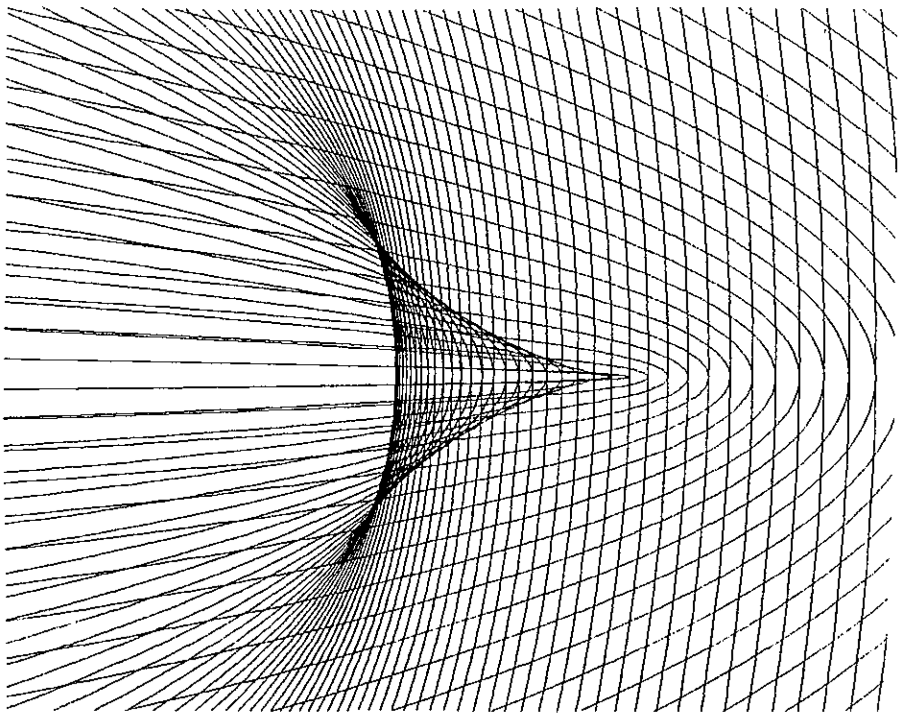
Let us now see what the counterpart of the existence of a complex Gauss map is when one looks at projections on complex lines. We will note $\pi_{L_{\mathbb{C}}}$ the orthogonal projection on the complex line $L_{\mathbb{C}}$. Let C be a holomorphic local parametrisation of the curve C. The differential $D(\pi_{L_{\mathbb{C}}}.C)$ is a linear complex map which implies that its real rank (as a real linear map) can be only 0 or 2. This implies that a point is a critical point of $\pi_{\mathbb{C}}.C$ if and only if it is a critical point of $\pi_D.C$, where D is a real line contained in $L_{\mathbb{C}}$.

Corollary 12.2.2 *Let* $|\mu|(C,D)$ *denote the number of critical points of the orthogonal projection of* C *on the real line* D *and* $|\mu|(C,L_{\mathbb{C}})$ *be the number of critical points of the projection of* C *on the complex line* $L_{\mathbb{C}}$*. For every real line* D *contained in a complex line* $L_{\mathbb{C}}$ *one has:*

$$|\mu|(C,D)=|\mu|(C,L_{\mathbb{C}}).$$

Remark: The critical values of the projection of a complex curve on a real 2-plane which is not a complex line may contain arcs. A nice study of this critical locus for a family of planes containing a complex line in the neighbourhood of non degenerate critical value of the projection on the complex line can be found in the book by Arnold, Gusein-Zade et Varchenko [A-G-V] p. 20-21, see fig. I.2.

Fig. projection of the complex curve $y=x^2$ on a real plane which is close to the complex y-axis



12.3 Polar curves.

We have already met polar varieties Γ_h and γ_h respectively the critical points and the critical locus of the orthogonal projection of a submanifold on the subspace h. They are equally important in the complex frame; (see for example Teissier [Tei2]. Slightly more generally, a polar variety is always (the closure of) the set of points where an incidence relation between the tangent subspaces to a certain object and a fixed subspace satisfy a given incidence relation. Let us now give the examples we shall use later.

Definition 12.3.1 *Let $\mathcal{F}$ be the foliation defined by an algebraic 1-form of $\mathbb{C}^2 : \omega = P.dx + Q.dy$. The tangent plane at a point (x, y) to the leaf of the foliation through (x, y) is the kernel of ω, when P and Q are not both zero. Let L be a complex line. The polar curve $\Gamma^L(\mathcal{F})$ is defined by:*

$$\Gamma^L(\mathcal{F}) = \overline{\{(x, y) \mid \omega(x, y)(L) = 0 \text{ and } \omega \neq 0\}}.$$

Observe the choice of upper indices; to be consistent with the previous chapters we need to define:

Definition 12.3.2

$$\Gamma_{(L^\perp)}(\mathcal{F}) = \overline{\{(x, y) \mid \omega \neq 0 \text{ and } p_{L^\perp}|L_{(x,y)} \text{ has a critical point at } (x, y)\}}.$$

Here $L_{(x,y)}$ is the leaf of $\mathcal{F}$ trough the regular point (x, y).

Of course, it is the curve $\Gamma^L(\mathcal{F})$

As in the real case,the name polar curve comes from the fact it is generically a curve except for a set of lines of measure zero. Again, we shall extensively use generic properties. In the algebraic context the measure zero bad set we should avoid is often a closed algebraic subset.Except for degenerate cases which we ignore, Γ_L is an algebraic curve whose equation is $P.a+Q.b = 0$, where (a, b) is a vector generating L.

A particular case is when $\mathcal{F}$ is the level foliation of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$. The intersection of the polar curve $\Gamma_L(\mathcal{F})$ with a nonsingular level $f = \lambda$ of the polynomial is the set of critical points $\Gamma_L(f = \lambda)$.

Theorem 12.3.3 ***Exchange theorem.*** *Let V be an open piece of a holomorphic curve, its total curvature satisfies:*

$$\int_V |K| = const. \int_{\mathbb{CP}(1)} |\mu| (V, L).$$

Proof: The theorem is a consequence of the exchange theorem proved before for codimension p submanifolds of $\mathbb{R}^N$, and of the corollary about numbers of critical points of the projection on real or complex lines proved above. □

A global consequence is the :

Proposition 12.3.4 *Let f be a polynomial of two complex variables of degree d. The total curvature of the algebraic curve C of equation $f = 0$ is less than of equal to $d(d-1)$.*

Proof: Let $\mathcal{F}$ be the foliation defined by the levels of the polynomial f. To each generic complex line L is associated a polar curve Γ_L which has degree less than or equal to $d-1$. By Bezout's theorem the intersection $\Gamma_L \cap C$ has at most $d.(d-1)$ points; these points are precisely the critical points of the projection of the curve on the complex line L. One deduces now the proposition from the exchange theorem. □

12.4 Isolated singularities

We shall show that when a sequence of smooth objects tends towards a singular one a distribution of curvature with support on the singular locus often arises naturally. The singularity will appear as a condensation at a point of the behaviour of compact submanifolds.

Let us first give a real algebraic example. The plane curve C of equation:

$$x^3 + y^2 = 0$$

is the limit of the family of curves C_λ of equations:

$$x^3 + y^2 = \lambda.$$

Let us consider the total curvature of the arc of C_λ contained in a small ball centered at the singular point.

Proposition 12.4.1 *The following limit :*

$$\lim_{\varepsilon->0} \lim_{\lambda->0} \int_{C_\lambda \cap B_\varepsilon} | k |$$

exists and is equal to π

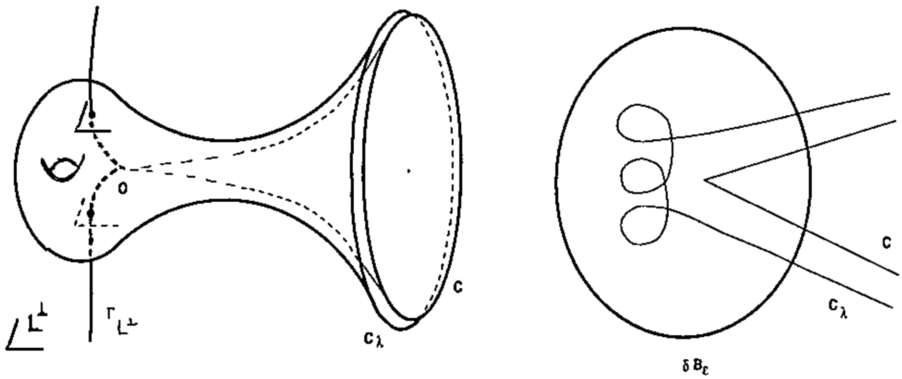


Figure 61: One dimension-faithfull picture and one codimension-faithfull one of C_λ

We shall show that such a phenomenon always occurs when one studies a sequence of levels of a complex polynomial having an isolated singularity or more generally of a polynomial map to $\mathbb{C}^p$ having an isolated singularity such that the zero level is a complete intersection. Let us first recall the topological and algebraic facts we will need. The study in the neighbourhood $B(0, \varepsilon)$ of an isolated singularity of the topology of the level $f = \lambda$ of a complex hypersurface has been done by Milnor [Mil3] .

Theorem 12.4.2 *[Mil3] . Let 0 be an isolated singularity of the complex polynomial* $f : \mathbb{C}^{n+1} \to \mathbb{C}$*. Then for* ε *small enough and* λ *(chosen after* ε*) small enough, the intersection* $B_\varepsilon \cap (f = \lambda)$ *of the level* $f = \lambda$ *with the ball of radius* ε *has the homotopy type of a wedge of* μ *spheres of real dimension* n*.*

Following Teissier we shall pose $\mu^{(n+1)} = \mu(f)$. The notation is justified by the following theorem :

Theorem 12.4.3 *[Tei1] or [Tei2]. There exists a measure zero analytic closed set of the Grassmann manifold $G_{n+1,1}$ such that, if $H \in G_{n+1,i} \backslash \gamma_i$, the Milnor number $\mu(f_{|H})$ takes the generic value $\mu^{(i)}$ independently of H.*

Let us first consider the case of a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ The levels of f form a foliation $\mathcal{F}$. At each regular point m of $\mathcal{F}$, $T_m\mathcal{F}$ is the tangent hyperplane to the level of f through m. Let us now fix a (vectorial) complex hyperplane h.

Definition 12.4.4 *The polar curve Γ^h is the closure of the set of regular points m such that $T_m\mathcal{F} = h$ (here we identify the affine space $T_m\mathcal{F}$ and the vector subspace which is parallel to it).*

Proposition 12.4.5 *[Le2] p. 263 and [Tei2] p. 269 (the polynomial f does not need to have in this proposition isolated singularities). The polar curve Γ^h is contained in an algebraic curve Γ'^h more precisely, if Σ is the singular locus of f, one has :*

$$\overline{\Gamma^h} = \overline{\Gamma'^h} \backslash \Sigma.$$

Proof: when the singularity is isolated It is enough to choose a base $e_1, \cdots, e_n$ of h The equations of Γ^h are in this case :

$$df(e_1) = df(e_2) = \cdots = df(e_n) = 0.$$

□

The following theorem about the total curvature is now a mere translation of the previous one, using the complex exchange theorem , [Lan1]:

Theorem 12.4.6 *[Lan1].Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial.*

$$\lim_{\epsilon -> 0} \lim_{\lambda -> 0} \int_{\mathbb{C}_\lambda \cap B_\epsilon} | K |= const\ (-1)^n (\mu^{n+1} + \mu^n)$$

where K is the Lipschitz-Killing curvature of the level C_λ and const a positive constant.

Remark: The first study of the curvature of levels $f = \lambda$ of the polynomial f near an isolated singular point is done in the thesis of L. Ness [Ne]. She shows in particular that the curvature of the levels is unbounded in the neighbourhood of the singularity.

Using more information about the polar curves than just the intersection number $\Gamma^h \cdot C_0$ we can give a more precise description of how the curvature of C_λ concentrates near the singular point. The geometric picture is that of concentrations of curvature near the vertices of regular polygons inscribed on circles whose radian are fractional powers of λ. The precise statement for non irreducible curves and the analysis of the phenomenon in terms of the contact of the branches of the generic polar curves and C_0 was done by Teissier [Tei3], after previous results in the irreducible case by the author [Lan4].

The seminal example is $f = x^3 - y^2$. Let us consider the polar curves $\Gamma_{a,b} = \{df(a,b) = 0\}$. Their equation is $3ax^2 - 2by = 0$. The intersection points of C_λ and $\Gamma_{a,b}$ satisfy :

$$\begin{cases} x^3 - y^2 = \lambda \\ 3ax^2 - 2by = 0. \end{cases}$$

Their absciss therefore satisfies : $x^3 - (3a/2b)^2 x^4 = \lambda$. The three intersection points of the polar curve and C_λ have abscissas close to $3\sqrt{\lambda}$ and ordinate of principal term $(3a/2b)x^2$. This is true, provided λ is small enough, for any point of $\mathbb{C}P_1$ different from $(0, b)$. Notice first that the cubic root of λ is much larger than the square root of λ, which is the order of the distance of the origin to the curve C_λ. In other words with a lens of strength $(\lambda)^{-1/2}$, one sees two parallel lines at finite distance from the origin :

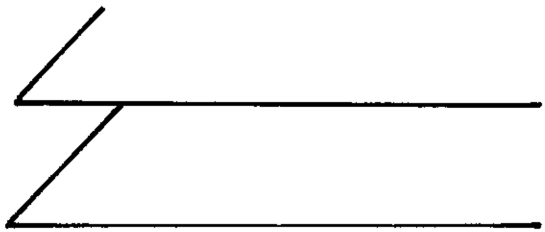


Figure. $\lim \lambda^{-\frac{1}{2}}(C_\lambda)$
as the lines $ax + by$ cut C_λ at points of ordinate of principal part $(\lambda)^{1/2}$

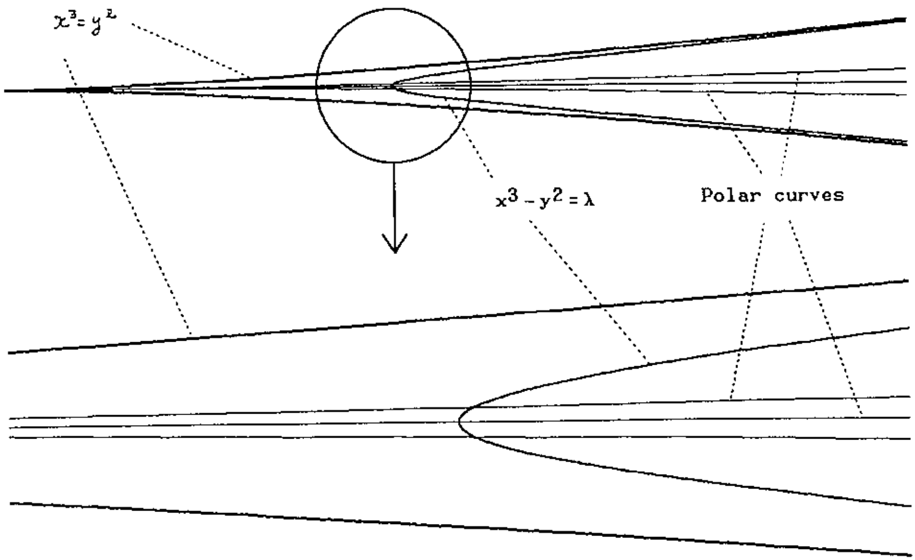
for all generic values of (a,b). With a weaker lens of strength $(\lambda)^{1/3}$, one sees three branch points :

$$\lim(\lambda)^{-1/3}(C_\lambda) = \left\{ \begin{matrix} & \cdot & \\ \cdot & & \cdot \end{matrix} \right\}$$

Each branch point of order two carries a distribution of gaussian curvature of total mass -2π. One can, applying the Gauss Bonnet theorem to the surface $C_\lambda \cap B((\lambda^{\frac{1}{3}}, O), \lambda^{\frac{1}{3}+\eta}$, for a suitably small positive η, check that its total curvature is very close to 2π. This property is true because this ball contains exactly one point of intersection with the polar curves $\Gamma_{a,b}$ for (a,b) not in a neighbourhood shrinking with λ, of the non generic direction $(0,b)$ of $\mathbb{C}P_1$.

Remark also that the Gauss Bonnet theorem applied to $C_\lambda \cap B((0,0),\varepsilon$ implies that the total curvature of this intersection, for a suitably small positive ε is very close to 6π.

The previous calculations prove that the picture of the real levels of $x^3 - y^2$ should look much more acute than usually drawn, as the turn should occur in a very small neighbourhood of the cubic root of λ. Rescaling we see a parabola. See figure below.



The general case needs more lenses, the strenth of which are determined using a theorem of Smith and Merle [Sm] et [Me]. See [Tei3].

Let us now give an intuitive justification of this multiscale phenomenon of concentration of curvature. For that consider a family of branches $\Gamma^q_{a,b}$ of the polar curves $\Gamma_{a,b} = \{m| < gradf(m) \subset \mathbb{R}(a,b) = 0$ $(a,b) \in A$ where A is the complement of small open discs centered on non generic-directions of $\mathbb{C}P_1$ with a given contact order with C_0 which is larger than one. Among those non-generic directions are the lines L such that the polar curve Γ_L has $L^\perp$ among its tangents at zero. See [Tei2].

Affirmation. Any complete complex curve, the complex Gauss image of which is contained in A should cross all the curves $\Gamma^q_{a,b}$, $(a,b) \in A$ provided it crosses one of them close enough to the origin.

Proof. The condition $(a,b) \in A$ implies that the angle of the curve and the polar branches is bounded away from 0, since in a small enough neighbourhood of the origin the tangent space to $\Gamma^q_{a,b}$, $(a,b) \in A$, is very close to the set of non generic directions. The curve C_λ through a point close enough from the origin has then to cross the family of branches, and this implies the Gauss image of the intersection of C_λ with the family of branches contains A.

The existence of a positive bound to the angle between the branches considered above of the polar curves and C_λ implies also that the size of the piece of intersection should be of the order of the "transverse size" of the family of branches (the transverse distance makes sense in the neighbourhood of a first intersection point of the curve with one of the branches of the polar curves considered above). See fig. $x^3 = y^2$ at two different scales.

Let us finally observe that, in the non-irreducible case, part of the curvature of C_λ may be spread over a ball of radius $C.(\lambda)^{1/m}$, for m large enough, and C a large enough constant. For example m is the multiplicity at 0 of C_0, if the polynomial f is homogeneous.

The study of P.Rouillé [Rou] of the geometry of a neighbourhood of an isolated complex singularity of a foliation by level curves of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ goes beyond integral geometry as he can even describe the shape of the renormalisation of $f = \lambda$ at a concentration of curvature.

Let us now consider a surjective polynomial map $f : \mathbb{C}^{n+p-} \to \mathbb{C}^p$. The levels of f form a singular foliation of $\mathbb{C}^{n+p}$ with singular locus Σ.

Definition 12.4.7 *The polar variety Γ^h is the closure of the set of regular points m such that $T_m\mathcal{F} \subset h$.*

Proposition 12.4.8 *There exists an algebraic variety Γ'^h such that :*

$$\overline{\Gamma}^h = \overline{\Gamma'^h} \backslash \Sigma.$$

Proof: when the singularity is isolated and the intersection is *complete* . Let u be a vector of $\mathbb{C}^p$. The equation $< f \mid u >=< \lambda \mid u >$ defines a hypersurface which contains the level $f = \lambda$. The level $f = \lambda$ is the intersection of the hypersurfaces $< f \mid u >=< \lambda \mid u >$ where u takes all values in $\mathbb{C}^p \backslash 0$. The set of hyperplanes tangent at m to the hypersurfaces containing $T_m\mathcal{F}$. Let us associate to each polynomial $< f \mid u >$ with value in $\mathbb{C}$ a polar curve $\Gamma(< f \mid u >, h)$.

The previous remark shows that the polar variety Γ^h is the closure of the intersection of the union of the polar curves $\Gamma(< f \mid u >, h)$ with the set of regular points of the foliation $\mathcal{F}$. Let us choose coordinates on $\mathbb{C}^{n+p}$ and $\mathbb{C}^p$, and let J be the jacobian matrix :

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial f_1} & \cdots & \dfrac{\partial f_1}{\partial z_{n+p}} \\ \dfrac{\partial f_p}{\partial z_1} & \cdots & \dfrac{\partial f_p}{\partial z_{n+p}} \end{pmatrix}$$

Let $e_1, \cdots, e_{n+p-1}$ be a basis of h. As the function $< f \mid u >$ can be written in the matrix form $\tilde{\tilde{u}}. \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$, the equations of $\Gamma(< f \mid u >, h)$ are :

$$0 =< \tilde{\tilde{u}}.J \mid e_1 >=< \tilde{\tilde{u}}.J \mid e_2 >= \cdots =< \tilde{\tilde{u}}.J \mid e_{n+p-1} > .$$

or :

$$(*) \qquad \tilde{\tilde{u}}.J.\bar{e}_1 = \tilde{\tilde{u}}.J.\bar{e}_2 = \cdots = \tilde{\tilde{u}}.J.\bar{e}_{n+p-1}.$$

The regular point m belongs to Γ^h if and only if there exists a vector u satisfying (*). This amounts to say that the system of vectors of $\mathbb{C}^p$:

$$g_1 = J.\bar{e}_1,\ g_2 = J.\bar{e}_2,\ \cdots,\ g_{n+p-1} = J.\bar{e}_{n+p-1}$$

is of rank smaller or equal to $(p-1)$. The equations of Γ^h are obtained by equating to zero the set of determinants which guarantees this rank condition.

The points of $\Gamma^h \cap [(f = \lambda)\backslash\Sigma]$ are exactly the critical points of the restriction to the smooth part of the leaf $f = \lambda$ of the orthogonal projection $p_{h^\perp}$ on the complex line $h^\perp$. □

Milnor's codimension 1 results were generalised by Hamm [Ha] and Giusti and Henry [G-H] for complete intersections.

Let now $f : \mathbb{C}^{n+1} \to \mathbb{C}^p$ be a surjective algebraic map such that the origin is an isolated singular point of f and such that the level $(f = 0)$ is a complete intersection. We will denote by C_λ the level variety $(f = \lambda)$. Let us state the algebraic results that we will need.

Theorem 12.4.9 *[Ha]. For ε small enough and $\lambda \neq 0$ (chosen after ε) small enough, the manifold with boundary $(C_\lambda \cap B_\varepsilon)$ has the homotopy type of a wedge of μ spheres of real dimension n.*

Theorem 12.4.10 *[G-H]. There exists a measure zero analytical closed set γ_i of the Grassmann manifold $G_{n+p,i+p-1}$ such that, if*
$H \in G_{n+p,i+p-1}\backslash\gamma_i$, the Milnor number $\mu(f_{|H})$ takes the generic value $\mu^{(i)}$, independently of H.

Generalising the codimension 1 case, see [Teil], Greuel [Gre] and Lê [Le2] independently proved :

Theorem 12.4.11 *The intersection multiplicity at 0 of the complete intersection and a generic polar variety Γ^h satisfies :*

$$\lim_{\epsilon \to 0} \lim_{\lambda \to 0} \sharp(C_\lambda \cap B_\varepsilon \cap \Gamma_h) = (C_0.\Gamma_h) = \mu^{n+1} + \mu^n.$$

The following theorem about the total curvature is now a mere translation, using an exchange theorem in codimension p, of the previous one, extending the codimension 1 result of [Lan1]:

Theorem 12.4.12 *[Lan5].Let $f : \mathbb{C}^{n+1} \to \mathbb{C}^p$ be a polynomial such that the level $f = 0$ is a complete intersection, then:*

$$\lim_{\epsilon \to 0} \lim_{\lambda \to 0} \int_{\mathbb{C}_\lambda \cap B_\epsilon} | K | = const\ (-1)^n (\mu^{n+1} + \mu^n)$$

where K is the Lipschitz-Killing curvature of the level C_λ and const a positive constant depending only on dimensions.

Remark: The study of other symmetric functions of curvature , in the codimension 1 case, was started by Griffiths [Gr], and continued by Kennedy [Ke] and Loeser [Lo].

Remark:(integral geometry in $\mathbb{CP}_n$) In this paragraph f will be a homogeneous polynomial map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ of degree greater or equal to two having only isolated singular points in $\mathbb{CP}_n$. Using a pencil of projective lines, one can define polar curves (see the chapter **spheres** and the chapter **foliation** for the construction of the curves of contact of a foliation with a pencil). Then adding the previous result: (there $f : \mathbb{C}^n \to \mathbb{C}$)

$$\lim_{\epsilon \to 0} \lim_{\lambda \to 0} \int_{C_\lambda \cap B_\epsilon} | K | = const\ (-1)^n (\mu^n + \mu^{n-1})$$

with Bezout's theorem one gets a geometric proof of Laumon's results [Lau] [Lan6]:

Proposition 12.4.13

$$degree(C^*) = [d(d-1)]^{n-1} - \sum_{m\ singular} (\mu^n + \mu^{n-1})(m)$$

13 The space of spheres

Let L be the Lorentz quadratic form defined by:

$$L(x_1, x_2, ..., x_n) = (x_1)^2 + (x_2)^2 + ... - (x_n)^2$$

We will call *light cone* the isotropic cone of L. We note also L the associated bilinear form, and call L-orthogonal vectors a, b such that $L(a, b) = 0$. One can prove that the set of oriented $(n-3)$-spheres of the sphere S^{n-2} admits a bijection on the set of points of the quadric Λ of equation L = 1.

Proposition 13.0.14 *Let c be a path in Λ. If at each point c(t) of the path, the tangent vector v(t) satisfies:*

$$L(v) > 0, \ (space - like\ curve),$$

the corresponding family of spheres admits an envelope;
if

$$L(v) < 0, \ (time - like\ curve),$$

at any point of the path, the spheres are nested.

Proof: As c(t) belongs to $\mathcal{S}$, that is satisfies $L(c(t)) = 1$, one has $L(c(t), v) = 0$. The condition for a 1-parameter family of spheres to admit an envelope is that the L-orthogonal space to the plane generated by c(t) and v(t) intersects the light cone. As c(t) and v(t) are L-orthogonal, it is equivalent to $L(v(t)) > 0$ □

Proposition 13.0.15 *If a speed one space-like curve satisfies:*

$$L(c'') - (L(c'', c))^2 > 0$$

their caracteristic circles admit an envelope.

more geometrically, if c is a speed one space-like curve, let

$$N_g = c'' - L(c'', c).c$$

the previous condition reads $L(N_g) > 0$. We will call *tendril* such a curve and *axis* the envelope of the caracteristic circles.

The length of the curve is then the total rotation of the family of spheres along its axis. Let us also observe that $L = -1$ endowed with the restriction to each tangent space to $L = -1$ is a model of H, the hyperbolic space. Each sphere σ of S^{n-2} is the "boundary at infinity" of a totally geodesic subspace h of H.

Let $\mathcal{G}$ be the group of linear isomorphisms of R^n leaving L invariant. Its restriction to H is the group of isometries of the hyperbolic space H. To chose a point z in H determines a metric on the sphere S^{n-2}. This metric is the projection on S^{n-2}, sphere at infinity of H, of the metric on $T_z(H)$ using the geodesic rays of origin z.

Different choices of the point z determine conformally equivalent metrics on the sphere S^{n-2}. The sphere does not even admit a measure invariant by the conformal group. Fortunately the sets of spheres of S^{n-2} do. In particular, Λ is endowed with a measure m invariant by $\mathcal{G}$. That measure can also be seen as the measure, invariant by the isometries of H, defined on the set of totally geodesic hyperplanes of H. Let us project the sphere S^{n-2} stereographically on an affine space $\mathbb{R}^{n-2}$. There, a sphere Σ is located with its center $x_1, x_2, ..., x_{n-2}$ and its radius r. the measure m is expressed by:

$$m = |[1/(r^4)]dx_1 \wedge dx_2 \wedge ... \wedge dx_{n-2} \wedge dr|$$

Remark: Let (v_1, v_2, v_3, v_4) be four vectors of $T_{v_0}\Lambda$. The volume of the parallelepiped constructed on these vectors is

$$|det(v_0, v_1, v_2, v_3, v_4)| = \sqrt{(- det(\mathcal{L}(v_i, v_j))}$$

13.1 Spheres of dimension 0

We will start with spheres of dimension 0 in S^1, and study their positions with respect to a "torus" T made of 4 distinct points. An oriented sphere σ disjoint from T bounds an interval I. We will say that σ is trivial if I contains two points of T. Informally we may say that the small enough spheres will all be trivial.

Proposition 13.1.1 *The torus T which minimises the measure of the set of non trivial spheres is the torus made of the four vertices of a square (or its image by the conformal group of the circle).*

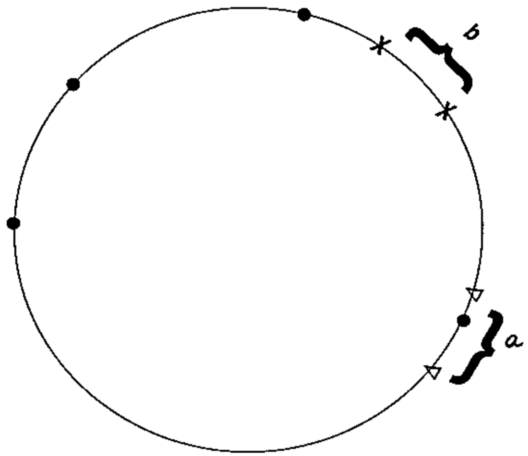


Figure 62: a nontrivial (a), and a trivial (b) 0-sphere

Proof: The domain Z of S formed by the non trivial spheres is bounded by segments of light rays formed by the spheres containing one of the four points of T.

That region is a chain of parallelograms the vertices of which are spheres formed of two points of T. Moreover the vertices common to two parallelograms are the spheres made of two non consecutive points of T. We can, performing if necessary a homography,suppose that those vertices common to two parallelograms are contained in the horizontal plane $x_3 = 0$. To move one of those vertices does not change the area of Z up to order one if the lengthes of the sides of the parallelograms which are consecutive on a light ray are equal. As the picture should be symmetrical with respect to the origin, that is possible only if the lengthes of the arcs of the circle $\Lambda \cap (x_3 = 0)$ bounded by the vertices common to the parallelograms are all equal. This shows that for our choice of the metric on the circle,T is the set of vertices of a square.

$\square$

As the only conformal invariant of a set of four points is their cross-ratio, The measure $m(Z)$ is a function of this cross-ratio.

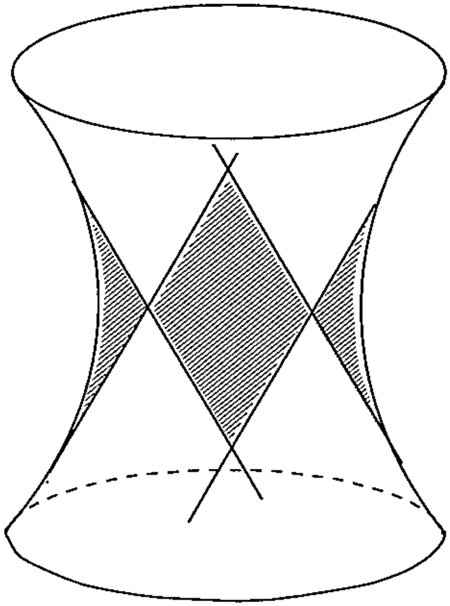


Figure 63: the set of non-trivial 0-spheres

13.2 The circles of S^2

The set of circles of S^2 is identified with the points of the 3-dimensional quadric $L = 1 \in \mathbb{R}^4$. The "torus" in that dimension is an embedding of $S^1 \times S^0$ that is two disjoint embedded circles. The two circles C_1 and C_2 bound two disjoint discs D_1 and D_2. A circle C is of *alternate double contact* if it is tangent to C_1 at x_1 ,to C_2 at x_2 and separates S^2 in two discs, one containing a neighbourhood of $x_1 \in D_1$ and the other a neighbourhood of $x_2 \in D_2$.

Remark: The circles of alternate double contact form a tendril.

Question Consider now for each alternate double contact circle C,the circle(s) Σ orthogonal to C in alternated tangency points $x_1 \in C_1$ and $x_2 \in C_2$ *doubly orthogonal circles.* Do they form a tendrill? Remark that, when C_1 and C_2 are circles, the doubly orthogonal circles form a pencil.

The circle two piece property

Definition 13.2.1 *A top circle $C \subset S^2$ for a closed subset $A \subset S^2$ is a circle intersecting A, bounding two discs D_1 and D_2 such that the closure of one , say $\overline{D}_1$ contains A, and the interior of D_2 is disjoint from A.*

We will call the intersection $C \cap A$ of a top circle with A a *topset* of A.

Definition 13.2.2 *A simple closed curve of S^2 has the circle two piece property , if it is divided by any circle in at most two connected components.*

Proposition 13.2.3 *A simple closed curve satisfying the circle two piece property is a (round) circle.*

The result is clear as, for any other simple closed curve, perturbing an osculating circle with minimal contact with the curve will give circles which contradict the circle two piece property. The circle two piece property is also meaningfull for 2-dimensional submanifolds with smooth boundary of S^2.

Proposition 13.2.4 *The only 2-dimensional manifolds W with smooth boundary of S^2 having the circle two piece property are obtained by removing from S^2 a finite number of disjoint closed discs D_i with boundary (round) circles.*

Lemma 13.2.5 *The top sets of a closed set A satisfying the circle two piece property are connected , that is they are either a point or an arc of the corresponding top circle. Conversely if all the topsets of a closed set are connected, then it satisfies the circle two piece property.*

Proof: Consider a sequence of increasing discs D_1^i converging to the disc D_1 of boundary C such that $A \subset \overline{D}_1$. If all the intersections $A \cap D_2^i$ of A whith the complement of D_1^i in Σ are void or connected the intersections $\partial D_1^i \cap A$ are also void or connected and would converge to at most one interval or a point of C, contradicting the hypothesis that the topset in C is not connected. If a circle cuts A in more than two pieces, a disc D_2 of boundary C will intersect the closed set A in at least two connected components. We can decrease D_2 , keeping two connected components in $\overline{D}_2^t$ till its boundary is a top circle (first reduce one component of $A \cap \overline{D}_2^t$ to a point p or an interval containing a point p, then proceed using circles tangent at p). Therefore ∂D_2^t provide the top circle intersecting a in two connected components. □

Proof: of the proposition If the boundary of W is not a union of circles, then , consider one component of ∂W which is not a circle. Performing a suitable inversion, this component can be seen as the outer boundary component of $w \subset \mathbb{R}^2$. Some circle bounding a disc in $\mathbb{R}^2$ containing W will be tangent to the outer component of ∂W defining a non-connected topset. The previous lemma provides a contradiction. The conditions of the proposition are sufficient because any circle C tranverse to ∂W intersects each circle ∂D_i in zero or two points. Then C and the discs D_i it crosses form a necklace the complement of which has two open connected components which are the components of $W \backslash C \cap W$. □

13.3 Spheres of dimension two

They form a 4-dimensional manifold. We can observe that time-like curves in $\mathcal{S}$ correspond to nested spheres, space-like curves to spheres enveloping a canal surface. A limit case is the family of osculating spheres to a surface M of R^3, along a line of curvature.The corresponding curve of $\mathcal{S}$ is everywhere tangent to the light cone.

Question Is a closed tendril of $\mathcal{S}$ of length bigger or equal to π ?

13.4 The spherical two-piece property

Definition 13.4.1 *A closed surface $M \subset S^3$ satisfies the sperical two piece property, S.T.P.P. if for any sphere Σ the difference $M \setminus (M \cap \Sigma)$ has at most two connected components. Such a surface is called taut*

In 1970 T.Banchoff proved the following theorem:

Theorem 13.4.2 *[Ban2] A surface embedded in S^3 satisfying the spherical two piece property is either a embedded round sphere or a Dupin cyclide, that is the conformal image of a torus of revolution of (complex) equation*

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Remark: The Dupin cyclides are in two different ways the envelopes of one dimensional families of spheres tangent to three spheres bounding three disjoint balls.

The proof of this theorem is analogous to the proof of Kuiper's result about tight immersions. One needs to consider *spherical topsets* and *top spheres*.

Definition 13.4.3 *A sphere Σ is a top sphere if it bounds two balls B_1 and B_2 such that:*
-the interior of say, B_2 does not meet M
-both $\overline{B}_2$ and $\overline{B}_1$ do meet M.

We can weaken that definition:

Definition 13.4.4 *A sphere Σ is said to be a local topsphere of M at $m \in M$ if m belongs to $\Sigma \cap M$ and if m has a neighbourhood $U \subset M$ which is contained in one, say B_1 of the balls B_1 and B_2 of boundary Σ. If the neighbourhood $U \subset M$ can be chosen to intersect Σ only in m then we say that the sphere Σ is a strict local topsphere.*

Proposition 13.4.5 *A surface $M \subset S^3$ has the spherical two piece property if and only if every local topsphere is a topsphere.*

Proof: Suppose it is not the case, then there exists a point $q \in int(B_2)$. For a sphere Σ' tangent to M at m, but bounding a closed ball B_1' which strictly contains B_1. It is a strict local topsphere of M at m, and the intersection $\overline{B}_2' \cap M$ has at least two connected components, one reduced to m, and one containing q. A third sphere Σ'' tangent in p to Σ', very close to Σ_1' and contained in B_1' contradicts the spherical two piece property. □

At a point m, we can consider the pencil of spheres tangent to M at m which, with the point m is a circle $\mathcal{P}(m)$. The support spheres of M form, if M is not a (round) sphere an interval of this pencil. Let us call $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ the boundary spheres of this interval. Applying this construction to nested neighbourhoods $U_i \subset M; i \in \mathbb{N}$ such that $\cap_{i\in\mathbb{N}} U_i = m$ we get spheres $\Sigma_i^+(M,m)$ and $\Sigma_i^-(M,m)$ which converge to the two osculating spheres of M at m: $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$. We can also define them using a stereographic projection of center different from m and the principal curvatures of $stereo(M)$

at *stereo*(m). This last observation implies that, when $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ are different, the intersection $\Sigma_1(M,m) \cap M$ is tangent to a line $L_1(m) \subset T_mM$ and the intersection $\Sigma_2(M,m) \cap M$ is tangent to a line $L_2(m) \subset T_mM$. We call these directions *principal directions.* A point where $\Sigma_1(M,m) = \Sigma_2(M,m)$ is called an *umbilic.*

Lemma 13.4.6 *If M is a taut smooth surface of S^3 then $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ coincide with $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$.*

Proof: The interval of $\mathcal{P}(m)$ containing the point sphere m and bounded by $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ is in that case equal to the set of topspheres. □

We are ready to prove the:

Theorem 13.4.7 *A smooth taut surface embedded in S^3 is either a (round) sphere or a smooth torus*

Proof:

First notice that a (round) circle of S^3 has the spherical two piece property.

If M has an umbilic m, then it "lies" between identical spheres $\Sigma_1(M,m) = \Sigma_2(M,m)$, and is therefore a sphere. If it does not have any umbilical point, then there exist two transverse line fields on M, $L_1(m)$ and $L_2(m)$. As M is embedded in S^3 it is orientable, and therefore is a torus. □

Proposition 13.4.8 *A Dupin cyclide is taut.*

Proof: The envelope of a time like curve in Λ is a canal surface, union of the caracteristic circles of the family. The directions tangent to this family of circles are principal directions. A Dupin cyclide is in two different ways a canal surface, and therefore admits two transverse foliations by circles (tangent to the principal directions). The components of $M \setminus \Sigma$ are the union of plaques of these two foliations. The circle two piece property applied to the leaves of the two foliations imply that they are cut in at most two intervals, and can match in at most two connected components. □

Then an essential lemma is:

Lemma 13.4.9 *A spherical top set of a taut embedded torus M satisfies the circle two piece property.*

Proof: As before B_1 is the ball of boundary a topsphere Σ which contains M in its closure and B_2 the other ball of boundary Σ. If the topset does not satisfy the two piece property,in the topsphere Σ we can find a circle C which is a topcircle of $\Sigma \cap M$ such that the intersection $C \cap \partial\Sigma \cap M = C \cap M$ is not connected. As before the intersection $M \cap \Sigma$ is contained in $\overline{D}_1$, a disc of boundary C, and the other disc D_2 of boundary C does not meet $M \cap \Sigma$. Choose a and c on different components of $C \cap M$ and b and d in different components of $C \backslash (C \cap M)$, so that these points are in cyclic order on C. Let γ be a geosdesic arc from b to d in D_2 and V a neighbourhood of γ in S^3 disjoint from M. Turning Σ around C we get a family Σ^t. We chose the rotation sign to leave γ out of the component, but chose the rotation small enough to garantee the existence of a continuous family of paths γ^t joining a to c in $\Sigma^t \cap V$ (B_1^t obtained by continuity from B_1). Then the points a and c will be in different components of $B_2^t \cap M$,as there is no path connecting a and c in $M \cap C = M \cap \Sigma \cap \Sigma^t$, and as any path in the union of the hemispheres Σ^t containing the arcs γ^t joining a and b should cross V. Therefore, for t small enough, (with the right sign), Σ^t cuts M in at least three connected components. (This last argument is quite analogous to Kuiper's for tight surfaces). □

Proposition 13.4.10 *If M is a taut torus in S^3 then for any topsphere Σ, $\Sigma \cap M$ is a point or a circle.*

Proof: We know by the previous proposition that the top set satisfies the circle two piece property. It cannot be Σ as M is a torus, nor contain interior points, which would be umbilical points of M, and imply again the equality $M = \Sigma$. The topset could apriori also be $\Sigma \setminus \{non\ finite\ family\ of\ round\ discs\}$. The boundary of those discs cannot bound a disc in M without contradicting tautness (consider a Poncelet pencil of spheres containing Σ), but then these boundary curves would be disjoint simple closed curves on M ; three disjoint simple curves on a torus always disconnect it into more than two pieces, so $M \cap int(B_1)$ would have at least two components. Moving Σ slightly into B_1 provides a sphere Σ' bounding a ball B_1' such that $M \cap \overline{B}_1'$ has at least two connected components. The only possibilities left are a point and a circle.
□

The interval of topspheres tangent at $m \in M$ to the taut torus M is bounded by the two osculating spheres at m, Σ_1 and Σ_2. Let us consider a sphere Σ tangent at m to M close to Σ_1 which is not a topsphere. It intersects M in a neighbourhood of m into two transverse arcs crossing at m the tangents of which are form a very acute angle and are close to the principal direction $L_1 \subset T_m M$. Suppose that the intersection $\Sigma_1 \cap M$ is the point m. Choose a neighbourhood $U \subset S^3$ of m such that the intersection $M \cap U$ is a small disc. For non topsphere Σ^t tangent to M at m close enough to Σ_1 the intersection $\Sigma^t \cap M$ is contained in U. As, at m there are four arcs of $\Sigma^t \cap M$ with distinct tangents, we can find two points p and q in $\Sigma^t \cap M$ such that any path from p to q in D_1^t passes through m. Choose in $U \cap \Sigma_1$ a very small circle σ centered at m, such that that the small disc $\delta_\sigma$it bounds does not contain any of the points p and q. In the pencil of spheres containing Σ_1 ,and following by continuity the ball B_1, some interval of spheres Σ^τ starting at $\Sigma_1$will be such that $\overline{B_1^\tau}$ contains p and q but does not contain m . For τ small enough and with the right sign, Σ^τ does not satisfy the two piece property. Then we can conclude that the osculating spheres intersect a taut torus M in circles. Those circles are necessarily lines of curvature, so M is a Dupin cyclide [Dar] . This ends the proof of the theorem giving the list of taut surfaces in S^3.

13.5 Intersection of surfaces and curves of the sphere S^3 with spheres

Let us now show that we can associate to a closed surface or a closed curve of S^3 a subset of $\mathcal{S}$ the measure of which is a conformal invariant of the surface or curve.

Let M be a compact surface embedded in S^3 . There exists a radius ϵ (depending on M) such that any sphere $\Sigma \subset S^3$ of radius smaller than ϵ either does not meet M or meets M in a point or a closed curve bounding a disc in M. Then the measure of the set of *nontrivial spheres*, that is the spheres which meet M in more than one curve, or in a curve which is not the boundary of a disc in M, is a conformal invariant. A smaller conformal invariant is the measure of the spheres which intersection with M contains a nontrivial component in the homology of M.

Definition 13.5.1

$$nt(M) = measure\{non\,trivial\,spheres\,for\,M\}$$

$$ntop(M) = measure\{\sigma \text{ intersecting } M \text{ non trivially in } H^1(M)\}$$

Let γ be a compact closed curve embedded in S^3. There exists a radius ϵ (depending on γ) such that any sphere $\Sigma \subset S^3$ of radius smaller than ϵ either does not meet γ or meets it in one or two points. Then the measure of the set of *nontrivial spheres for* γ , here spheres which meet γ in at least four points, is a conformal invariant of the curve γ We can define:

Definition 13.5.2

$$nt(\gamma) = measure\{non\,trivial\,spheres\,for\,\gamma\}$$

$$NT(\gamma) = \int_S (\sharp\gamma \cap \Sigma - 2)^+$$

where φ^+ *is the function equal to* φ *when* $\varphi \geq 0$ *and equal to 0 when* $\varphi \leq 0$

13.6 Questions

- **conjecture** there exists a positive constant α such that, when the closed embedded curve $\gamma \subset S^3$ is knotted,

$$nt(\gamma) \geq \alpha$$

- **conjecture** there exists a positive constant β such that, when the closed embedded surface $M \subset S^3$ is not a sphere,

$$nt(M) \geq \beta$$

- **The Willmore conjecture** the following 2-form on a surface M embedded or immersed in S^3 is invariant by the action of the conformal group on S^3:

$$dw = (k_1 - k_2)^2 \cdot dv$$

where k_1 and k_2 are the principal curvatures and dv the area form of M.

The integral on M of this form:

$$W(M) = \int_M dw$$

is then a conformal invariant of the immersed surface. Looking first at revolution tori of equation

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Conjecture [Wil1] [Wil2] When M is a torus:

$$W(M) \geq 2\pi^2$$

This conjecture has proved to be particulary rich in connection with other problems see [Wil1] [Wil2] [Li-Ya].

There may be an inequality linking $W(M)$ and the measure of the spheres with non trivail intersection with M.

- **Möbius energy** The author thanks D.Rolfsen for pointing out the reference [F-H-W] to him. Recently M.F.Freedman, Z-X.He and Z.Wang defined the Möbius energy of a rectifiable curve embedded in $\mathbb{R}^3$ by:

$$E(\gamma) = \int\int_{\gamma\times\gamma\setminus\Delta} \frac{1}{|\gamma(v) - \gamma(u)|^2} - \frac{1}{[dist_{\mathbb{R}^3}(\gamma(v), \gamma(u))]^2}$$

where Δ is the diagonal of the product $\gamma \times \gamma$. Separately the integral of the two fractions would diverge, but the sum converges. They prove that this function is invariant by the Möbius group.

Let $c([\gamma])$ be the crossing number of the knot type $[\gamma]$ (the infimum of the number of crossings of the projection of the knot $\gamma \in [\gamma]$ on a plane, when γ describes the isotopy class.

Theorem 13.6.1 *[F-H-W] The energy $E(\gamma)$ of a simple closed curve $\gamma \subset \mathbb{R}^3$ satisfies the inequality:*

$$E(\gamma) \geq 2\pi c([\gamma]) + 4$$

The equality $E(\gamma) = 4$ is achieved only when the curve is a (plane, round) circle.

Conjecture The Möbius energy and the measure of the set of spheres intersecting the curve γ in at least four points are linked by inequalities. One may have to take a multiplicity involving the number of intersection points into account.

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