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STOCHASTIC DYNAMICS OF DETERMINISTIC SYSTEMS

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Magna est veritas et .... Yes, when it gets a chance.
There is a law, no doubt – and likewise
a law regulates your luck in the throwing of dice.
It is not Justice, the servant of men, but accident,
hazard, Fortune – the ally of patient Time –
that holds an even and scrupulous balance.

Joseph Conrad, *Lord Jim.*

CONTENTS

1. Introduction

Let the time-evolution of some natural process be described by a transformation $f : M \to M$ on a manifold M. Physically observable quantities correspond to real, or complex, functions φ defined on the phase space M. Thus, experimental data on the system comes, usually, in the form of sequences of "measurements" $\varphi(f^j(z))$, where $z \in M$ and $j \geq 0$. From such data one tries to extract the main intrinsic properties of the underlying dynamical process.

Very often, these time-series $\varphi(f^j(z))$ behave in a rather complicated and "erratic" way as time j varies, even for simple evolution laws f. Moreover, time-series may depend very sensitively on the initial state of the system: arbitrarily small modifications of $z \in M$ typically lead to quite different values of $\varphi(f^j(z))$ for j large. These facts are illustrated by Figure 1.1, where the values of $\varphi(f^j(z))$ are plotted against time j, for the following case. The map f is the Hénon diffeomorphism of the plane $\mathbb{R}^2$, given by $f(x, y) = (1 - 1.4x^2 + y, 0.3x)$. The observable function φ is simply $\varphi(x, y) = x$. Time j increases in the horizontal axis from 0 to 100, and the two graphs correspond to two nearby choices of the initial state, respectively, $z = (0, 0)$ and $z' = (0.01, 0)$. Note that both sequences seem to behave quite randomly. Moreover, despite the fact that the values of $\varphi(f^j(z))$ and $\varphi(f^j(z'))$ are close to each other if j is small, just by continuity, the two sequences apparently become uncorrelated after only a few iterates.

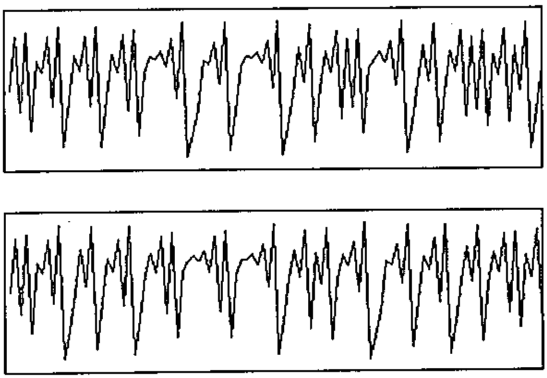

Figure 1.1: Time-series for two nearby initial states of the Hénon map

Such "chaotic" behaviour means that the dynamics may be hard to understand in deterministic terms, and that a stochastic analysis of the time-series may be a more fruitful approach. That is, one regards time-series as essentially random sequences, and focus on determining their statistical properties. Of particular interest are those properties which are intrinsic to the dynamical system, that is, independent of the choice of a (typical) initial state z, even more so if they are robust under small modifications of the system.

The present work is devoted to the study of the statistical properties of deterministic systems with chaotic dynamical behaviour. In the sequel we introduce some of the mathematical notions and problems involved in this program, as well as a few basic examples. We also give a brief outline of the historical background, see e.g. [PT93] for more information.

Rigorous statements and proofs are presented in Chapters 2 through 5, for various classes of systems covering most known cases. Along the way, we introduce several of the techniques devised for the study of such systems. Most of the time, we restrict ourselves to discrete-time systems, namely smooth or piecewise-smooth transformations on manifolds. However, a large share of what we do applies also to flows or semi-flows as we shall briefly comment. In Chapter 6 we discuss some of the recent developments and open problems in this area, including a program towards a global understanding of chaotic systems recently proposed by Palis [PT93], [Pa], largely inspired by a probabilistic viewpoint as we adopt here.

1.1. Physical measures.

A first, basic question concerns the existence of asymptotic time-averages

$$E_x(\varphi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

for "many" points $x \in M$. Clearly, $E_x(\varphi)$ exists whenever x is a periodic point of f, i.e. whenever $f^k(x) = x$ for some $k \geq 1$. More generally, Birkhoff's ergodic theorem asserts that asymptotic time-averages exist for almost every point, with respect to any f-invariant probability measure. This is most relevant if f is volume-preserving, that is, leaves invariant some smooth (Lebesgue) measure on the manifold M. However, arbitrary invariant measures may lack physical meaningfulness. In general, we take "many" above to mean "positive measure set" with respect to some Lebesgue measure.

Furthermore, one wants to understand if, and when, time-averages can be independent of the initial point. Suppose that, for every continuous function $\varphi : M \to \mathbb{R}$, the average $E_x(\varphi)$ exists and is independent of the point x taken in some positive measure set $B \subset M$. Then

$$\varphi \mapsto E(\varphi) = E_x(\varphi) \qquad (\text{any } x \in B)$$

defines a nonnegative linear operator on the space $C^0(M, \mathbb{R})$ of real continuous functions which, by the representation theorem, can be thought of as a Borel measure μ on M:

$$\int \varphi \, d\mu = E(\varphi) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \qquad (\text{any } x \in B).$$

Observe that such a measure μ can be "physically observed" by computing time-averages of continuous functions for randomly chosen points $x \in M$ (positive probability of getting $x \in B$).

This motivates the following definition. An f-invariant probability measure μ is a *physical*, or *SRB* (for *Sinai-Ruelle-Bowen*) measure for f if there exists a positive

Lebesgue measure set of points $x \in M$ such that

$$\int \varphi \, d\mu = E(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \qquad \text{for every } \varphi \in C^0(M, \mathbb{R}). \tag{1.1}$$

The set of points $x \in M$ satisfying this property is called the *(ergodic) basin* of μ, and is denoted $B(\mu)$. The previous considerations can then be summarized in

Problem 1. Given $U \subset M$ such that $f(U) \quad U$, investigate the existence of some SRB measure μ with support contained in U. Study the uniqueness and the ergodicity of μ. Describe its basin $B(\mu)$.

SRB measures are believed to exist in great generality (the assumption that such a measure exists is usually implicit in numerical studies of experimental systems), but actual constructions are known only for certain classes of systems, see Chapters 2 ff.. Also, the following simple counterexample, due to Bowen, shows that this is a matter of some subtlety.

Example 1.1. (see e.g. [Ta95]) This consists of a vector field in the plane with two saddle-points A, B exhibiting a double saddle-connection. The two saddle-connections bound an open region L containing another equilibrium point C, which is a source. Under appropriate assumptions on the eigenvalues at the saddle points A and B, the trajectory $X^t(z)$ of any point $z \in L \setminus \{C\}$ accumulates on the boundary of L as time $t \to +\infty$. See Figure 1.2. However, given any continuous φ with $\varphi(A) \neq \varphi(B)$, the time-average

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(X^t(z)) \, dt$$

does not exist (for any such z): both $\varphi(A)$ and $\varphi(B)$ are accumulation points.

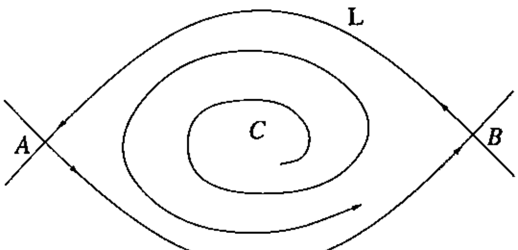

Figure 1.2: Bowen's counterexample

It is an important open question whether examples such as these can be made generic (Bowen's counterexample has codimension 2 in the space of flows). In the interval, [HK90] have given examples of quadratic maps without SRB measures, and of quadratic maps whose SRB measure is the Dirac measure supported on a *repelling* fixed point.

1.2. Correlation functions.

From now on we let $U \subset M$ be some open set with $f(U) \subset U$. We suppose that f admits a unique SRB measure μ with support contained in U, and we analyse the system $(f|U, \mu)$.

The next step is to try and understand whether, and how fast, memory of the past is lost by the system as time evolves. In more precise terms, one wants to know to what extent observations $\varphi(f^n(x))$ made at some instant $n \gg 1$ are affected by initial values $\psi(x)$ of some given observable ψ (possibly with $\psi = \varphi$). This is naturally expressed by means of the *correlation functions*

$$C_n(\varphi, \psi) = \int (\varphi \circ f^n)\psi \, d\mu - \int \varphi \, d\mu \cdot \int \psi \, d\mu.$$

Note that $C_n(\varphi, \psi) = 0$ corresponds, in probabilistic terms, to $\varphi \circ f^n$ and ψ being independent random variables. We say that (f, μ) is *mixing* if $C_n(\varphi, \psi) \to 0$ for every pair (φ, ψ): the value of $\varphi \circ f^n$ becomes less and less dependent of the value of ψ as time goes to infinity. We say that (f, μ) is *exponentially mixing* (or, *has exponential decay of correlations*) if this "loss of memory" occurs exponentially fast: there is $\tau < 1$ and for each (φ, ψ) there is $C > 0$ such that

$$|C_n(\varphi, \psi)| \le C\tau^n \qquad \text{for all } n \ge 1. \tag{1.2}$$

The following simple examples are meant to illustrate these ideas. First, a word of warning: one usually takes φ, ψ varying in some convenient Banach space of functions $\mathcal{F}$, and then the previous definitions are relative to that space (e.g. the existence and the value of τ may depend on $\mathcal{F}$). The particular choice of the Banach space varies with the context, $\mathcal{F}$ may not contain characteristic functions.

Example 1.2. Let $M = [0, 1]$, f be given by $f(x) = 1 - |2x - 1|$, and μ be Lebesgue measure. It is not difficult to find $\tau < 1$ such that given any pair of intervals $I, J \subset M$ there is $C > 0$ such that $\varphi = \chi_I$, $\psi = \chi_J$ satisfy (1.2). On the other hand, if $M = S^1$, f is a rigid rotation, and μ is Lebesgue measure, then $C_n(\chi_I, \chi_J)$ does not converge to zero, for any intervals $I, J \subset S^1$.

An important difference between these two examples concerns hyperbolicity: in the first case f is uniformly expanding, while in the second one f completely lacks hyperbolicity. In fact, an important theme here is that a small amount of hyperbolicity (together with topological mixing, say) suffices for exponential decay of correlations. The following example shows that this theme should be taken with some precaution.

Example 1.3. (see e.g. [CI96] and Section 3.5) Let $f : [0, 1] \to [0, 1]$ satisfying, for some $c \in (0, 1)$,

(i) f is increasing and C^2 on $[0, c]$ and on $(c, 1]$, with $f(0) = 0$ and $f(c^+) = 0$;

(ii) $f'(0) = 1$, but $|f'(x)| > 1$ for $x \neq 0$ (including $x = c^\pm$); moreover $f''(0) > 0$;

See Figure 1.3. Then f does not admit a *finite* invariant measure which is absolutely continuous with respect to Lebesgue measure, but it has an infinite absolutely continuous invariant measure μ. The system (f, μ) has polynomial decay of correlations, i.e., (1.2) holds if the righthand side is replaced by Cn^{-d} for some $d > 0$, but it is not exponentially mixing.

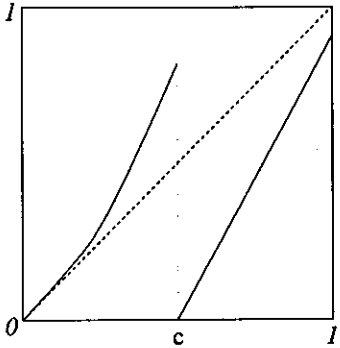

Figure 1.3: A map with polynomial decay of correlations

We shall see in Section 3.5 that μ is not an SRB measure for f. The map f does admit a unique SRB, but this is the Dirac measure supported on the neutral fixed point 0. So, despite the fact that the map looks prevalently expanding, its behaviour on typical orbits is not at all hyperbolic: the Lyapunov exponent $\lim n^{-1} \log |(f^n)'(x)|$ is equal to zero at Lebesgue almost every point x.

1.3. Randomness.

Other important characterization of almost independence (more precisely, weak correlation) of successive observations may be given through central limit and large deviations theorems. Both kinds of results describe the oscillations of finite-time averages

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x))$$

around their expected value $\int \varphi \, d\mu$. We begin by quoting classical statements for independent identically distributed random variables.

Theorem 1.1 (central limit theorem for i.i.d.r.v.). *Let $X_0, \ldots, X_n, \ldots$ be independent identically distributed random variables taking values in $\mathbb{R}$, with average $\bar{X} = E(X_n) < \infty$ and variance $\sigma^2 = E((X_n - \bar{X})^2) \in (0,1)$. Then, given any open interval $A \subset \mathbb{R}$, the probability of*

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(X_j - \bar{X}) \in A \quad \textit{converges to} \quad \frac{1}{\sqrt{2\pi}\sigma}\int_A e^{\frac{-t^2}{2\sigma^2}}\, dt$$

as $n \to +\infty$.

Theorem 1.2 (large deviations theorem for i.i.d.r.v.). *Let $X_0, \ldots, X_n, \ldots, \bar{X}$, and σ be as in Theorem 1.1. Assume, moreover, that $E(e^{tX_n}) < \infty$ for every $t \in \mathbb{R}$. Then, given any $\epsilon > 0$, the probability $\mathcal{P}(n, \epsilon)$ of*

$$|\frac{1}{n} \sum_{j=0}^{n-1} (X_j - \bar{X})| > \epsilon$$

converges to zero exponentially fast as $n \to +\infty$, in the sense that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mathcal{P}(n, \epsilon) < 0.$$

Each of these theorems is part of a whole family of related results which includes considerably more sophisticated statements. See Appendix B for proofs, additional information, and references to the literature.

Going back to our dynamical context, we say that an observable φ *satisfies the central limit theorem* for (f, μ) if there is $\sigma > 0$ such that, for every interval $A \subset \mathbb{R}$,

$$\mu(\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (\varphi(f^j(x)) - \int \varphi \, d\mu) \in A\}) \to \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{\frac{-t^2}{2\sigma^2}} \, dt,$$

as $n \to +\infty$. On the other hand, we say that φ *satisfies the large deviations theorem* for (f, μ) if, given any $\epsilon > 0$ there is $h(\epsilon) > 0$ such that

$$\mu(\{x \in M : |\frac{1}{n} \sum_{j=0}^{n-1} (\varphi(f^j(x)) - \int \varphi \, d\mu)| > \epsilon\}) \leq e^{-nh(\epsilon)}$$

for every large $n \geq 1$. As we shall see, properties such as these hold when φ has sufficiently fast decay of correlations. In some sense, this means that individual time-series behave in an essentially "random" way over large ranges of time.

Summarizing this discussion, we state

Problem 2. Determine whether (f, μ) satisfies the mixing properties, the central limit theorem, and or the large deviations theorem, for all the observables in some appropriate Banach space. Estimate the rate of decay of the correlation functions.

1.4. Stochastic stability.

Quite often, the mathematical formulation $f : M \to M$ of a given physical process involves simplifications, where a "main" part of the process is isolated (this is what f is meant to describe) and external influences are discarded as too complex to be taken in consideration and, hopefully, too small to be relevant. Clearly, this procedure requires a justification, specially if, as it frequently happens, the simplified system f turns out to be structurally unstable (meaning that arbitrarily close transformations g may have very different dynamical behaviour, see Section 1.5 for a more precise definition).

In many instances where such external influences are not completely understood, or are too complex to be effectively expressed in deterministic terms, one can think of them as a kind of random "noise". One then speaks of stochastic stability if the presence of small noise has only a small effect on the asymptotic behaviour of f. In more precise terms, for each small $\varepsilon > 0$ one considers iterates

$$x_j = f_j \circ \cdots \circ f_1(x), \qquad x \in U, \ \ j \geq 0,$$

where the f_i are chosen randomly and independently from each other in the ε-neighbourhood of f, according to some given distribution law. It is convenient to assume that $f(\bar{U}) \subset U$, to ensure that $f_i(U) \subset U$ for every i. Then, under general conditions, there exist probability measures μ_ε with $\operatorname{supp} \mu_\varepsilon \subset U$ and

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) = \int \varphi \, d\mu_\varepsilon$$

for all continuous $\varphi : M \to \mathbb{R}$ and "many" (positive probability) random trajectories $(x_j)_{j \geq 0}$ with $x_0 \in U$. We say that $(f|U, \mu)$ is *stochastically stable* if μ_ε converges to μ in the weak*-sense, that is,

$$\int \varphi \, d\mu_\varepsilon \to \int \varphi \, d\mu \quad \text{for all } \varphi \in C^0(M, \mathbb{R})$$

as $\varepsilon \to 0$ (if μ_ε is not unique then we require convergence to μ for all such stationary measures with support contained in U).

While structural stability requires very rigid constraints on the dynamical system, stochastic stability is likely to hold quite in general. Indeed, another informal theme is that systems with exponential decay of correlations tend to be stochastically stable: known counterexamples, such as the next one, are nongeneric in some way or another.

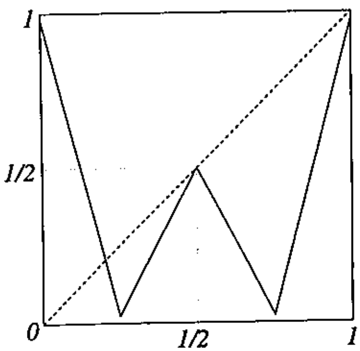

Figure 1.4: An expanding map that is not stochastically stable

Example 1.4. ([Ke82], see [BaY93]) The example is a continuous piecewise affine and expanding map $f : [0,1] \to [0,1]$: there are $c_1 = 0 < c_2 < c_3 = 1/2 < c_4 < c_5 = 1$ and $\sigma_i > 1$, $i = 1,2,3,4$, such that $f'(x) = (-1)^i \sigma_i$ for all $x \in (c_i, c_{i+1})$. See Figure 1.4. Moreover, $f(1/2) = 1/2$ and, due to the presence of this periodic turning point, the map f is not stochastically stable.

Problem 3. Find general conditions on the system ensuring stochastic stability.

Further understanding of the dynamical behaviour (resonances, distribution of periodic points, ...) can be obtained from other important invariants, such as the *correlation spectrum* or *dynamical zeta functions*. See [Ba95], [Ru94], and references therein. Although we do not treat these invariants explicitly here, their study is closely related to that of the problems stated above.

1.5. Brief history.

Daily life observations, e.g. of mechanical processes subject to friction, or of simple chemical reactions, may suggest that systems typically evolve to a steady (equilibrium) state. Indeed, it was believed for a while that most dynamical systems are gradient-like: a finite number of periodic motions, to which converges every trajectory of the system; in particular, there are only finitely many (periodic) attractors whose basins of attraction cover a full probability subset of the phase space.

On the other hand, more sophisticated natural phenomena, such as turbulence in the motion of viscous fluids, hinted at much richer forms of dynamical behaviour. Early attempts by Landau-Lifschitz to describe turbulence were based on the idea of high-dimensional invariant tori contained in the phase-space and carrying quasi-periodic flows: they suggested that presence of a large number of rationally independent frequences might explain the complicated patterns associated to turbulent motion. However, this was chalenged by Ruelle-Takens [RT71], who showed that the proposed mechanism for the formation of such tori does not actually lead to quasi-periodic motion. They suggested that, instead, turbulence is associated to the presence of some "strange" kind of attractor.

About a decade earlier, Smale had been astonished to find out that smooth flows and transformations may exhibit infinitely many periodic motions in a robust way: after any small perturbation of the system an infinite number of such periodic motions continues to exist. His efforts to understand this phenomenon lead him to discover the "horseshoe map", and thus to introduce in Dynamics the concept of hyperbolicity [Sm67]. His notion of hyperbolic (or Axiom A) systems, unified (generic) gradient-like systems, the horseshoe, and the class of systems introduced by Anosov [An67].

The theory of hyperbolic systems was developed to near completion throughout the sixties and the seventies, and provided a powerful framework for the understanding of complicated dynamics. In particular, (nonperiodic) hyperbolic attractors were the first "strange" attractors whose behaviour may be considered to be understood. The ergodic theory of these systems was most successfully built in the seventies, through the work of Sinai, Ruelle, Bowen, [Si72], [BR75], [Ru76]. Moreover, contributions by a large number of people, culminating with [Ma86], see also [Ha97], showed that hyperbolicity is the key condition for structural stability of a system. A system is structurally stable if there is a one-to-one correspondance between its orbits and the orbits of any nearby system, preserving the direction of time.

However, it was soon realized that nonhyperbolic (and thus, nonstable) systems also exist in a robust way. In particular, [Ne79] showed that many (Baire second category) diffeomorphims have infinitely many attracting periodic orbits, thus proving unfounded expectations that generic systems might have only finitely many attractors.

In fact, a number of simple models motivated by concrete problems in Nature were being found, which did not fit the framework of uniform hyperbolicity. This included the now famous Lorenz flows [Lo63], Hénon attractors [He76], and Feigenbaum-Coullet-Tresser cascades of period-doubling, [CT78], [Fe78]. Although structurally unstable, these phenomena are very robust: they persist for many (positive probability), or even all, small perturbations of the initial system. Lorenz observations pointed out that sensitive dependence on the initial state is a source of fundamental unpredictability in deterministic systems. All in all, these discoveries showed that the requirement of (uniform) hyperbolicity and (structural) stability is too restrictive, and that a broader and more flexible framework is necessary to encompass the relevant phenomena taking place in natural systems.

A large part of the recent work in Dynamics in recent years has, thus, been devoted to the study of the properties of chaotic systems having only some weak form of hyperbolicity. The program we described in the previous sections has been carried out for a large class of nonuniformly hyperbolic transformations of the interval, as we shall see in Section 5. A large deal of information is available also concerning Lorenz-like attractors of flows, which we shall mention in the Section 6.3. Moreover, this program has also been brought close to completion in the, more subtle, case of Hénon-like attractors, as we shall also see in Chapter 6.

Recently, Palis has been proposing a number of ideas and conjectures [PT93], [Pa], aiming at a global description of systems with complex dynamical behaviour. At the core of his program is the conjecture that every system can be approximated by another having only finitely many attractors which, in addition, have well-defined statistical properties: unique SRB measure, stochastic stability. A more precise formulation will be given in the last section.

Notes.

This text is organized as follows. In Chapter 2, we treat the class of smooth expanding maps, that is, maps whose derivative expands every tangent vector, with expansion rate uniformly bounded away from 1. Although comparatively simple, these maps have rather nontrivial features and serve as a good introduction to Chapter 4, where we give a unified presentation of ergodic properties of uniformly hyperbolic attractors of diffeomorphisms. The results are by now classical, but the approach we follow in this chapter, borrowing some key ideas from [Li95], is new.

Before that, in Chapter 3, we extend the results in Chapter 2 to piecewise smooth expanding maps in dimension 1, including Lorenz-like maps and maps with infinitely many smoothness domains. Besides their intrinsic interest, piecewise expanding maps are also instrumental in the study of many smooth nonuniformly hyperbolic systems, such as Lorenz-like attractors of flows or unimodal maps of the interval. Indeed, in Chapter 5 we decribe a tower construction associating a piecewise expanding map in a (noncompact) 1-dimensional space to each unimodal map in a large class of nonuniformly hyperbolic transformations of the interval. Combined with other tools (e.g. cocycles), this allows us to give a rather complete picture of the ergodic behaviour of such maps.

As we already said, Chapter 6 is devoted to a more informal discussion of recent developments, mostly about nonuniformly hyperbolic systems.

In a forthcoming version the discussion on Hénon-like attractors will be expanded into a whole new chapter, and more background material will be included. I also plan to add new sections to Chapter 3 (higher dimensions) and to Chapter 5 (stochastic stability). This will also be a chance to correct any injustices I may be commiting to people who have contributed to this field, despite the effort I have put into giving the correct references.

This work started as a set of handwritten notes for a course I lectured at IMPA in the (southern hemisphere) summer term of 1996. Remarks by participants in the course and other colleagues helped improving the initial text. It wish to thank J. F. Alves, V. Araújo, C. Bonatti, M. Benedicks, A. Castro, P. Collet, M. J. Costa, L. J. Díaz, M. A. Hertz, V. Horita, G. Keller, F. Ledrappier, C. Liverani, M. J. Pacifico, J. Palis, E. Pujals, among others, for helpful conversations on topics related to this work.

2. Expanding maps on manifolds

Let M be a compact connected manifold and $f: M \to M$ be a smooth map. We say that f is *expanding* if there exists $\sigma > 1$ such that

$$\|Df(x) \cdot v\| \geq \sigma \|v\| \quad \text{for all } x \in M \text{ and } v \in T_xM, \tag{2.1}$$

for some riemannian metric $\|\cdot\|$ (the particular choice of such a metric is not relevant here).

A standard class of examples is provided by the following construction. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $F(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Then there exists a unique map f on the n-dimensional torus $M = \mathbb{R}^n/\mathbb{Z}^n$ such that $f \circ \pi = \pi \circ F$, where $\pi : \mathbb{R}^n \to M$ is the canonical projection. Moreover, if all the eigenvalues of F have norm larger than 1 then this induced map f is expanding. Clearly, existence of a constant $\sigma > 1$ as in (2.1) is a C^1 open condition on the map. Thus, any smooth transformation of M close enough to f, in the C^1 sense, is also an expanding map.

In this chapter we prove

Theorem 2.1. *Let f be a $C^{1+\nu_0}$ expanding map, for some $\nu_0 \in (0,1]$. Then*

1. *f admits a unique invariant measure μ_0 which is absolutely continuous with respect to m; moreover, μ_0 is exact (thus ergodic) and $d\mu_0/dm$ is strictly positive; in particular, μ_0 is the unique SRB-measure of f;*
2. *(f, μ_0) is exponentially mixing and satisfies the central limit theorem, in the Banach space of ν-Hölder continuous functions, for any $\nu \in (0, \nu_0]$;*
3. *(f, μ_0) is stochastically stable under small random perturbations.*

The proof is based on the fact that ergodic properties of f may be derived from spectral properties of its *transfer* (or *Perron-Frobenius*) *operator*, acting in some convenient space of functions $\varphi: M \to \mathbb{R}$. The transfer operator $\mathcal{L}$ is defined by

$$(\mathcal{L}\varphi)(y) = \sum_{f(x)=y} \frac{\varphi(x)}{|\det Df(x)|} \tag{2.2}$$

(note that the sum is over a finite number of terms). The determinant is with respect to the riemannian metric we have chosen; observe that different choices lead to operators which are conjugate and, thus, have the same spectrum. The usefulness of $\mathcal{L}$ stems, in the first place, from the duality property

$$\int (\mathcal{L}\varphi)\psi \, dm = \int \varphi(\psi \circ f) dm \tag{2.3}$$

(whenever the integrals make sense), which is a direct consequence of the definition and the formula for the change of variables.

The relation (2.3) implies, in particular, that fixed points of $\mathcal{L}$ are directly related to absolutely continuous f-invariant measures. Indeed, if φ_0 is a nonnegative $L^1(m)$

function satisfying $\mathcal{L}\varphi_0 = \varphi_0$ then $\mu_0 = \varphi_0 m / \int \varphi_0 m$ is an f-invariant probability measure and, of course, $\mu_0 \ll m$. Conversely, if a finite f-invariant measure μ_0 is absolutely continuous with respect to m then $\varphi_0 = d\mu_0/dm$ satisfies $\mathcal{L}\varphi_0 = \varphi_0$.

In order to prove that such a fixed point does exist we use the notion of projective (or Hilbert) metric associated to a convex cone in a vector space, introduced by G. Birkhoff [Bi67]. This notion and its main properties are recalled in Section 2.1.

Then, in Section 2.2 we construct a cone C in the space of Hölder continuous functions, which is mapped strictly inside itself by the operator $\mathcal{L}$. It follows that $\mathcal{L}: C \to C$ is a contraction with respect to the projective metric θ associated to C and the first statements in the theorem are deduced from this fact, see Section 2.3.

The contraction property also enables us to further describe the spectral properties of $\mathcal{L}$ to show, in Section 2.4, that the system (f, μ_0) obtained in this way has exponential decay of correlations in the space of Hölder continuous functions. Moreover, it satisfies the central limit theorem in that same space. This last statement is derived from estimates obtained in the course of proving exponential mixing, together with an abstract central limit theorem for dynamical systems, Theorem 2.11, which we also prove in Section 2.5.

Finally, in Section 2.6 we develop a similar analysis for appropriate "perturbed" transfer operators, to prove stochastic stability under a very general random scheme. The method also yields a statement of *deterministic* stability: the absolutely continuous invariant measure depends continuously on the expanding map, in the $C^{1+\nu_0}$-topology.

2.1. Cones and Projective Metrics.

Let E be a vector space. By a *cone* in E we mean any subset $C \subset E \setminus \{0\}$ satisfying

$$v \in C \quad \text{and} \quad t > 0 \;\Rightarrow\; tv \in C.$$

The cone is *convex* if

$$v_1, v_2 \in C \quad \text{and} \quad t_1, t_2 > 0 \;\Rightarrow\; t_1 v_1 + t_2 v_2 \in C.$$

We define the *closure* $\overline{C}$ of C by

$$w \in \overline{C} \quad \Leftrightarrow \quad \begin{array}{l}\text{there are } v \in C \text{ and } (t_n)_n \searrow 0 \text{ such} \\ \text{that } (w + t_n v) \in C \text{ for all } n \geq 1.\end{array}$$

In all that follows we assume

$$\overline{C} \cap (-\overline{C}) = \{0\}. \tag{2.4}$$

Given $v_1, v_2 \in C$ we define

$$\alpha(v_1, v_2) = \sup\{t > 0 : v_2 - t v_1 \in C\} \quad \text{and} \quad \beta(v_1, v_2) = \inf\{s > 0 : s v_1 - v_2 \in C\}$$

(with the convention $\sup \emptyset = 0$, $\inf \emptyset = +\infty$, where $\emptyset$ denotes the empty set). Note that $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$ for all $v_1, v_2 \in C$. Indeed,

$$\begin{aligned} v_2 - tv_1 \in C, \quad sv_1 - v_2 \in C &\Rightarrow (s-t)v_1 \in C \text{ (by convexity)} \\ &\Rightarrow (s-t) \geq 0 \text{ (otherwise } -v_1 \in C\text{, contradicting (2.4)).} \end{aligned}$$

Moreover, $\alpha(v_1, v_2) < +\infty$ and $\beta(v_1, v_2) > 0$ for all $v_1, v_2 \in C$. Indeed,

$$\begin{aligned} \alpha(v_2, v_2) = +\infty \Rightarrow &\text{ there is } (t_n)_n \to +\infty \text{ so that } v_2 - t_n v_1 \in C \text{ for all } n \geq 1 \\ \Rightarrow &\text{ there is } (\hat{t}_n) \to 0 \text{ such that } \hat{t}_n v_2 - v_1 \in C \text{ for all } n \geq 1 \\ \Rightarrow &\ -v_1 \in \overline{C}, \text{ contradicting (2.4),} \end{aligned}$$

and $\beta(v_1, v_2) = 0 \Rightarrow -v_2 \in \overline{C}$ is proved in the same way. Now we let

$$\theta(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)} \quad \text{(with } \theta = +\infty \text{ if } \alpha = 0 \text{ or } \beta = +\infty\text{).}$$

In view of the previous remarks $\theta(v_1, v_2)$ is well-defined and takes values in $[0, +\infty]$.

Proposition 2.2. *$\theta(\cdot, \cdot)$ is a metric in the projective quotient of C, that is,*
a) $\theta(v_1, v_2) = \theta(v_2, v_1)$ for all $v_1, v_2 \in C$
b) $\theta(v_1, v_2) + \theta(v_2, v_3) \geq \theta(v_1, v_3)$ for all $v_1, v_2, v_3 \in C$
c) $\theta(v_1, v_2) = 0 \Leftrightarrow$ there exists $t > 0$ such that $v_1 = tv_2$.

Proof: We claim that $\alpha(v_2, v_1) = \beta(v_1, v_2)^{-1}$. Suppose first that $\alpha(v_2, v_1) > 0$. Then

$$\begin{aligned} \alpha(v_2, v_1) &= \sup\{t > 0 : v_1 - tv_2 \in C\} = \sup\{\frac{1}{s} > 0 : sv_1 - v_2 \in C\} \\ &= (\inf\{s > 0 : sv_1 - v_2 \in C\})^{-1} = \beta(v_1, v_2)^{-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha(v_2, v_1) = 0 \quad &\Leftrightarrow \quad v_1 - tv_2 \notin C, \quad \text{for all } t > 0 \\ &\Leftrightarrow \quad sv_1 - v_2 \notin C, \quad \text{for all } s > 0 \quad \Leftrightarrow \quad \beta(v_1, v_2) = +\infty, \end{aligned}$$

hence the claim holds also in this case. Analogously, $\beta(v_2, v_1) = \alpha(v_1, v_2)^{-1}$. Part a) of the proposition follows immediately.

To prove part b), we claim that $\alpha(v_1, v_2)\alpha(v_2, v_3) \leq \alpha(v_1, v_3)$ for all $v_1, v_2, v_3 \in C$. This is obvious if $\alpha(v_1, v_2) = 0$ or $\alpha(v_2, v_3) = 0$ and so we consider $\alpha(v_1, v_2) > 0$ and $\alpha(v_2, v_3) > 0$. Then there are $(r_n)_n \nearrow \alpha(v_1, v_2)$ and $(s_n)_n \nearrow \alpha(v_2, v_3)$ with

$$\begin{aligned} v_2 - r_n v_1 \in C \text{ and } v_3 - s_n v_2 \in C &\Rightarrow v_3 - s_n r_n v_1 \in C \text{ (by convexity)} \\ &\Rightarrow s_n r_n \leq \alpha(v_1, v_3) \text{ for all } n \geq 1, \end{aligned}$$

Passing to the limit as $n \to +\infty$ we prove the claim. Analogously, $\beta(v_1, v_2)\beta(v_2, v_3) \leq \beta(v_1, v_3)$. Part b) is now an easy consequence.

Finally,

$$\theta(v_1, v_2) = 0 \Leftrightarrow \alpha(v_1, v_2) = \beta(v_1, v_2) = \gamma \in (0, +\infty).$$

Then there are $(t_n)_n \nearrow \gamma$ and $(s_n)_n \searrow \gamma$ such that

$$\left.\begin{array}{lcl} v_2 - t_n v_1 \in C \text{ for all } n \geq 1 & \Rightarrow & v_2 - \gamma v_1 \in \overline{C} \\ s_n v_1 - v_2 \in C \text{ for all } n \geq 1 & \Rightarrow & \gamma v_1 - v_2 \in \overline{C} \end{array}\right\} \Rightarrow v_2 - \gamma v_1 = 0 \quad \text{(by (2.4))}.$$

This proves part c) of the proposition. □

We call $\theta(\cdot, \cdot)$ the *projective metric* associated to the convex cone C. Note that the projective metric depends in a monotone way on the cone. Indeed, let $C_1 \subset C_2$ be two convex cones in E and $\alpha_i(\cdot,\cdot)$, $\beta_i(\cdot,\cdot)$, $\theta_i(\cdot,\cdot)$ be the corresponding objects, $i = 1, 2$, as defined above. Clearly,

$$\alpha_1(v_1, v_2) \leq \alpha_2(v_1, v_2) \quad \text{and} \quad \beta_1(v_1, v_2) \geq \beta_2(v_1, v_2)$$

and so $\theta_1(v_1, v_2) \geq \theta_2(v_1, v_2)$ for all $v_1, v_2 \in C_1 \subset C_2$.

More generally, let E_1, E_2 be two vector spaces and $C_i \subset E_i$, $i = 1, 2$, be convex cones. Let $L: E_1 \to E_2$ be a linear operator and assume that $L(C_1) \subset C_2$. Then

$$\begin{aligned} \alpha_1(v_1, v_2) &= \sup\{t > 0 : v_2 - t\,v_1 \in C_1\} \\ &\leq \sup\{t > 0 : L(v_2 - t\,v_1) \in C_2\} \text{ (because } L(C_1) \subset C_2) \\ &= \sup\{t > 0 : L(v_2) - tL(v_1) \in C_1\} = \alpha_2(L(v_1), L(v_2)) \end{aligned}$$

and, analogously, $\beta_1(v_1, v_2) \geq \beta_2(L(v_1), L(v_2))$. Therefore,

$$\theta_1(v_1, v_2) \geq \theta_2(L(v_1), L(v_2)) \quad \text{for all } v_1, v_2 \in C_1.$$

In general, L need not be a strict contraction, with respect to θ_1 and θ_2, but the next proposition asserts that this is the case if $L(C_1)$ has finite θ_2-diameter.

Proposition 2.3. *Let $D = \sup\{\theta_2(L(v_1), L(v_2)) : v_1, v_2 \in C_1\}$. If $D < +\infty$ then*

$$\theta_2(L(v_1), L(v_2)) \leq (1 - e^{-D})\theta_1(v_1, v_2) \quad \textit{for all } v_1, v_2 \in C_1.$$

Proof: We may suppose $\alpha_1(v_1, v_2) > 0$ and $\beta_1(v_1, v_2) < +\infty$ for otherwise there is nothing to prove. Then there are $(t_n)_n \nearrow \alpha_1(v_1, v_2)$ and $(s_n)_n \searrow \beta_1(v_1, v_2)$ such that

$$\left.\begin{array}{l} v_2 - t_n v_1 \in C_1 \\ s_n v_1 - v_2 \in C_1 \end{array}\right\} \Rightarrow \theta_2(L(v_2 - t_n v_1), L(s_n v_1 - v_2)) \leq D \text{ for all } n \geq 1.$$

As a consequence, there are $(T_n)_n$ and $(S_n)_n$ such that $\lim\left(\log\frac{S_n}{T_n}\right) \le D$ and

$$\begin{aligned}
L(s_n v_1 - v_2) - T_n L(v_2 - t_n v_1) \in C_2 &\Leftrightarrow (s_n + t_n T_n) L(v_1) - (1+T_n) L(v_2) \in C_2 \\
&\Rightarrow \beta_2(L(v_1), L(v_2)) \le \frac{s_n + t_n T_n}{1+T_n} \\
S_n L(v_2 - t_n v_1) - L(s_n v_1 - v_2) \in C_2 &\Rightarrow \alpha_2(L(v_1), L(v_2)) \ge \frac{s_n + t_n S_n}{1+S_n}.
\end{aligned}$$

Then

$$\begin{aligned}
\theta_2(L(v_1), L(v_2)) &\le \log\left(\frac{s_n + t_n T_n}{1+T_n} \cdot \frac{1+S_n}{s_n + t_n S_n}\right) = \\
&= \log\left(\frac{s_n}{t_n} + T_n\right) - \log(1+T_n) - \log\left(\frac{s_n}{t_n} + S_n\right) + \log(1+S_n) \\
&= \int_0^{\log(s_n/t_n)} \left(\frac{e^x\,dx}{e^x + T_n} - \frac{e^x\,dx}{e^x + S_n}\right) \\
&\le \log\left(\frac{s_n}{t_n}\right) \cdot \sup_{x>0} \frac{e^x (S_n - T_n)}{(e^x + T_n)(e^x + S_n)} \\
&\le \log\left(\frac{s_n}{t_n}\right) \cdot \left(1 - \frac{T_n}{S_n}\right).
\end{aligned}$$

Passing to the limit as $n \to +\infty$ we conclude that

$$\theta_2(L(v_1), L(v_2)) \le \theta_1(v_1, v_2) \cdot \left(1 - e^{-D}\right).$$

The proof is complete. □

Example 2.1. Let $E = \mathbb{R}^2$ and $C = \{(x,y) : y > |x|\}$. The projective quotient of C can be identified in a natural way with $(-1,1) \times \{1\}$, thus, with the interval $(-1,1)$. Given $-1 < x_1 \le x_2 < 1$ we have

$$\begin{aligned}
\alpha(x_1, x_2) &= \sup\{t > 0 : (x_2, 1) - t(x_1, 1) \in C\} \\
&= \sup\{t > 0 : 1 - t > x_2 - t x_1\} = \frac{1 - x_2}{1 - x_1} \\
\beta(x_1, x_2) &= \frac{x_2 + 1}{x_1 + 1}
\end{aligned}$$

and so $\theta(x_1, x_2) = \log R(-1, x_1, x_2, 1)$, where R denotes the *cross-ratio* of four points $a < b \le c < d$ in the real line

$$R(a,b,c,d) = \frac{c-a}{b-a} \cdot \frac{d-b}{d-c}.$$

In this sense, projective metrics generalize the usual hyperbolic (Poincaré) metric in the interval (or the disk), see e.g. [MS93]. Moreover, Proposition 2.3 may be thought of as an extension of Schwartz lemma.

Example 2.2. Let X be a compact metric space and $E = C^0(X)$ be the space of continuous real-valued functions defined on X. Take

$$C = C_+ = \{\varphi \in E : \varphi(x) > 0 \text{ for all } x \in X\}.$$

Then, for any $\varphi_1, \varphi_2 \in C$

$$\begin{aligned}\alpha(\varphi_1,\varphi_2) &= \sup\{t > 0 : (\varphi_2 - t\,\varphi_1)(x) > 0 \text{ for all } x \in X\}\\ &= \inf\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in X\right\} \quad \text{and}\\ \beta(\varphi_1,\varphi_2) &= \sup\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in X\right\}.\end{aligned}$$

Therefore

$$\theta(\varphi_1,\varphi_2) = \log\frac{\sup(\varphi_2/\varphi_1)}{\inf(\varphi_2/\varphi_1)} = \log\sup\left\{\frac{\varphi_2(x)\varphi_1(y)}{\varphi_1(x)\varphi_2(y)} : x, y \in X\right\}.$$

Example 2.3. Let X and E be as before and take $C = C(a,\nu)$, the set of all $\varphi \in E$ such that $\varphi(x) > 0$ for all $x \in X$ and $\log\varphi$ is (a,ν)-Hölder continuous. This last condition means that

$$\exp(-ad(x,y)^\nu) \le \frac{\varphi(x)}{\varphi(y)} \le \exp(ad(x,y)^\nu) \quad \text{for all } x, y \in X.$$

Given $\varphi_1, \varphi_2 \in C$ and $t_1, t_2 > 0$ we have

$$\exp(-ad(x,y)^\nu) \le \frac{t_1\varphi_1(x) + t_2\varphi_2(x)}{t_1\varphi_1(y) + t_2\varphi_2(y)} \le \exp(ad(x,y)^\nu) \quad \text{for all } x, y \in X,$$

and so C is a convex cone. Now, $\alpha(\varphi_1,\varphi_2)$ is the supremum of all $t > 0$ satisfying, for all $x, y \in X$,

i) $(\varphi_2 - t\varphi_1)(x) > 0 \Leftrightarrow t < \dfrac{\varphi_2}{\varphi_1}(x)$

ii) $\dfrac{(\varphi_2 - t\varphi_1)(x)}{(\varphi_2 - t\varphi_1)(y)} \le \exp(ad(x,y)^\nu) \Leftrightarrow$

$$\Leftrightarrow t\,(\exp(ad(x,y)^\nu)\varphi_1(y) - \varphi_1(x)) \le \exp(ad(x,y)^\nu)\varphi_2(y) - \varphi_2(x)$$

$$\Leftrightarrow t \le \frac{\exp(ad(x,y)^\nu)\varphi_2(y) - \varphi_2(x)}{\exp(ad(x,y)^\nu)\varphi_1(y) - \varphi_1(x)}$$

$\dfrac{(\varphi_2 - t\varphi_1)(x)}{(\varphi_2 - t\varphi_1)(y)} \ge \exp(-ad(x,y)^\nu) \Leftrightarrow$

$$\Leftrightarrow t \le \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)}$$

In other words, $\alpha(\varphi_1, \varphi_2)$ equals

$$\inf\left\{\frac{\varphi_2(x)}{\varphi_1(x)}, \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} : x, y \in X, x \neq y\right\},$$

and $\beta(\varphi_1, \varphi_2)$ is given by a similar expression, just with supremum in the place of infimum.

The following simple example shows that, for general convex cones, the projectivization of $\{\varphi : \theta(\varphi, 1) < \infty\}$ need not be a complete metric space, cf. Proposition 2.6 in Section 2.3.

Example 2.4. Let X be a compact manifold and $E = C^1(X, \mathbb{R})$ be the Banach space of C^1 real functions on X endowed with the C^1-topology. Consider the convex cone $C = \{\varphi \in E : \varphi(x) > 0 \text{ for all } x \in X\}$. Let φ be a strictly positive function on X which is continuous but not differentiable, and let $(\varphi_n)_n$ be a sequence of strictly positive C^1 functions converging uniformly to φ. Clearly, the projective metric θ of the cone C is given by the same expression as in Example 2.2. Using this remark, it is easy to see that $(\varphi_n)_n$ is a θ-Cauchy sequence in the cone C. On the other hand, it can not be θ-convergent in C, since the θ-limit would also be a uniform limit and thus would coincide with φ.

2.2. Transfer operators and invariant cones.

Let $f\colon M \to M$ be a $C^{1+\nu_0}$ expanding map on a compact connected manifold M. Up to rescaling, we may suppose that the diameter of M is not larger than 1, and we do so once and for all. We also let m be the riemannian volume, normalized so that $m(M) = 1$.

The assumption of expansivity (2.1) implies, in particular, that the derivative Df is an isomorphism at every point, i.e., f is a local diffeomorphism. As a consequence, all the points $y \in M$ have a same number $k \geq 1$ of preimages (the *degree* of f). Moreover, given any preimage x of y, there exists a neighbourhood V of y and a map $g : V \to M$ such that $f \circ g = \text{identity}$ and $g(y) = x$. Since f is expanding, such a local inverse branch g must be contracting:

$$d(g(y'), g(y'')) \leq \sigma^{-1} d(y', y'')$$

for every y', y'' in V ($\sigma > 1$ is the expansion rate of f, recall (2.1)). Moreover, by compactness of M, the neighbourhood V may be chosen containing the ball of radius $\rho > 0$ around y, for some uniform constant $\rho > 0$. Summarising,

(i) there exists $k \geq 1$ such that every point $y \in M$ has exactly k pre-images under f;

(ii) there exist $\rho_0 > 0$ such that, given $y_1, y_2 \in M$ with $d(y_1, y_2) \leq \rho_0$, one may write $f^{-1}(y_j) = \{x_{j1}, \ldots, x_{jk}\}$, $j = 1, 2$, with

$$d(x_{1i}, x_{2i}) \leq \sigma^{-1} d(y_1, y_2) \quad \text{for each } i = 1, \ldots, k.$$

Now consider the linear operators $\mathcal{L}$ and U on $E = C^0(M, \mathbb{R})$, defined by

$$(\mathcal{L}\varphi)(y) = \sum_{i=1}^{k} \varphi(x_i)|\det Df(x_i)|^{-1} \qquad \text{and} \qquad (U\varphi)(x) = \varphi(f(x)).$$

Then (2.3) may be written

$$\int (\mathcal{L}\varphi)\psi \, dm = \int \varphi(U\psi)\, dm.$$

We fix $\rho_0 > 0$ as in (ii) and, for $a > 0$ and $0 < \nu \leq \nu_0$, we define $C(a,\nu)$ to be the convex cone of functions $x \in E$ satisfying

(1) $\varphi(x) > 0$ for all $x \in M$;

(2) $\log\varphi$ is (a,ν)-Hölder continuous on ρ_0-neighbourhoods, i.e.,

$$d(y_1,y_2) \leq \rho_0 \ \Rightarrow\ \varphi(y_1) \leq \exp(a\, d(y_1,y_2)^\nu)\varphi(y_2).$$

Proposition 2.4 (invariance). *There is $\lambda_1 < 1$ such that $\mathcal{L}(C(a,\nu)) \subset C(\lambda_1 a, \nu)$ for every sufficiently large $a > 0$.*

Proof: Clearly, $\varphi > 0 \ \Rightarrow\ L\varphi > 0$, because f is surjective, and so we only have to deal with condition (2). Let $y_1, y_2 \in M$ with $d(y_1,y_2) \leq \rho_0$ and write $f^{-1}(y_j) = \{x_{j1}, \dots, x_{jk}\}$, $j = 1,2$, as in (ii). Using the fact that $\log|\det Df(x)|$ is (a_0,ν_0)-Hölder for some $a_0 > 0$, we obtain, for every $\varphi \in C(a,\nu)$,

$$\begin{aligned}
(\mathcal{L}\varphi)(y_1) &= \sum_{i=1}^{k} \varphi(x_{1i})|\det Df(x_{1i})|^{-1} \\
&\leq \sum_{i=1}^{k} \varphi(x_{2i}) \exp(a\, d(x_{1i},x_{2i})^\nu) \cdot |\det Df(x_{2i})|^{-1} \exp(a_0\, d(x_{1i},x_{2i})^{\nu_0}) \\
&\leq \exp((a\sigma^{-\nu} + a_0) d(y_1,y_2)^\nu) \sum_{i=1}^{k} \varphi(x_{2i})|\det Df(x_{2i})|^{-1} \\
&\leq \exp(\lambda_1\, a\, d(y_1,y_2)^\nu) \cdot (\mathcal{L}\varphi)(y_2),
\end{aligned}$$

as long as $\lambda_1 \in (\sigma^{-1}, 1)$ and $a \geq a_0/(\lambda_1 - \sigma^{-\nu})$. We used the fact that $d(x_{1i},x_{2i})^{\nu_0} \leq (y_1,y_2)^{\nu_0} \leq d(y_1,y_2)^\nu$ (because $\nu \leq \nu_0$ and the diameter of M is less than 1). □

We denote $\theta = \theta_{a,\nu}$ the projective metric associated to the convex cone $C(a,\nu)$. Then, cf. Example 2.3, $\theta(\varphi_1,\varphi_2) = \log(\beta(\varphi_1,\varphi_2)/\alpha(\varphi_1,\varphi_2))$ where $\alpha(\varphi_1,\varphi_2)$ is given by

$$\text{(2.5)} \quad \inf\left\{ \frac{\varphi_2}{\varphi_1}(x), \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} : x,y \in M, x \neq y, \text{ and } d(x,y) \leq \rho_0 \right\}$$

and $\beta(\varphi_1, \varphi_2)$ has a similar expression, with inf replaced by sup. We also consider the cone

$$C_+ = \{\varphi \in E : \varphi(x) > 0 \text{ for all } x \in M\}.$$

The projective metric θ_+ associated to C_+ was calculated in Example 2.2, where we got $\theta_+(\varphi_1, \varphi_2) = \log(\beta_+(\varphi_1, \varphi_2)/\alpha_+(\varphi_1, \varphi_2))$ with

$$\alpha_+(\varphi_1, \varphi_2) = \inf\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in M\right\} \quad \text{and} \quad \beta_+(\varphi_1, \varphi_2) = \sup\left\{\frac{\varphi_2}{\varphi_1}(x) : x \in M\right\}.$$

Proposition 2.5 (finite diameter). *$D_1 = \sup\{\theta(\varphi_1, \varphi_2) : \varphi_1, \varphi_2 \in C(\lambda_1 a, \nu)\}$ is finite, for every $a > 0$, $\nu > 0$, $\lambda_1 < 1$.*

Proof: The proof has two steps: first we show that

$$\theta\text{-}\operatorname{diameter}(C(\lambda_1 a, \nu)) \leq \theta_+\text{-}\operatorname{diameter}(C(\lambda_1 a, \nu)) + K'(\lambda_1),$$

then we obtain

$$\theta_+\text{-}\operatorname{diameter}(C(a, \nu)) \leq K''(a),$$

where $K'(\cdot), K''(\cdot) < +\infty$. Given any $\varphi_1, \varphi_2 \in C(\lambda_1 a, \nu)$,

$$\begin{aligned}
&\frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} \\
&\qquad \geq \frac{\varphi_2}{\varphi_1}(x)\frac{\exp(ad(x,y)^\nu) - \exp(a\,\lambda_1\, d(x,y)^\nu)}{\exp(ad(x,y)^\nu) - \exp(-a\,\lambda_1\, d(x,y)^\nu)} \geq K_1 \frac{\varphi_2}{\varphi_1}(x),
\end{aligned}$$

where

$$K_1 = \inf\left\{\frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}} : z > 1\right\}.$$

Note that $K_1 \in (0,1)$, since

$$\lim_{z \to +\infty} \frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}} = 1 \quad \text{and} \quad \lim_{z \to 1+} \frac{z - z^{\lambda_1}}{z - z^{-\lambda_1}} = \frac{1 - \lambda_1}{1 + \lambda_1} < 1.$$

It follows that $\alpha(\varphi_1, \varphi_2) \geq K_1 \alpha_+(\varphi_1, \varphi_2)$. Analogously, $\beta(\varphi_1, \varphi_2) \leq K_2\, \beta_+(\varphi_1, \varphi_2)$, with

$$K_2 = \sup\left\{\frac{z - z^{-\lambda_1}}{z - z^{\lambda_1}} : z > 1\right\} \in (1, +\infty).$$

We obtain $\theta(\varphi_1, \varphi_2) \leq \theta_+(\varphi_1, \varphi_2) + \log K_2 - \log K_1$ and this concludes the first step of the proof.

Now, observe that

$$\theta_+(\varphi_1, \varphi_2) = \log \frac{\beta_+(\varphi_1, \varphi_2)}{\alpha_+(\varphi_1, \varphi_2)} = \log\sup\left\{\frac{\varphi_2(x)}{\varphi_1(x)}\frac{\varphi_1(y)}{\varphi_2(y)} : x, y \in M\right\}. \tag{2.6}$$

It is easy to see that if $\varphi \in C(a,\nu)$ then $\log\varphi$ is (b,ν)-Hölder continuous over the whole manifold M (not just on ρ_0-neighbourhoods), for some $b>0$. Indeed, by compactness and connectedness, there is $N \geq 1$ (depending only on M and ρ_0) such that given any $x,y \in M$ there are $z_0 = x$, $z_1, \ldots, z_N = y$ with $d(z_{i-1},z_i) \leq \rho_0$ for all $i = 1,\ldots,N$ and $\sum_{i=1}^N d(z_{i-1},z_i) \leq 2d(x,y)$. Then

$$\frac{\varphi(y)}{\varphi(x)} = \prod_{i=1}^N \frac{\varphi(z_i)}{\varphi(z_{i-1})} \leq \exp\left(\sum_{i=1}^N a\, d(z_{i-1},z_i)^\nu\right)$$
$$\leq \exp\left(N\,a\left(\sum_{i=1}^N d(z_{i-1},z_i)\right)^\nu\right) \leq \exp(N\,a\,2^\nu\, d(x,y)^\nu)$$

and one may take $b = 2Na$. It follows that, given any $\varphi_1,\varphi_2 \in C(a,\nu)$, we have

$$\log\frac{\varphi_2(x)}{\varphi_2(y)} \leq b\,d(x,y)^\nu \leq b \quad\text{and}\quad \log\frac{\varphi_1(y)}{\varphi_1(x)} \leq b$$

for every $x,y \in M$, hence $\theta_+(\varphi_1,\varphi_2) \leq 2\log b$. □

As a consequence of Propositions 2.2, 2.3, and 2.4, the operator $\mathcal{L} : C(a,\nu) \to C(a,\nu)$ is a Λ_1-contraction with respect to the metric $\theta = \theta_{a,\nu}$, with $\Lambda_1 = 1 - e^{-D_1}$.

2.3. Absolutely continuous invariant measure.

In order to prove that $\mathcal{L}$ has some fixed point φ_0 we take advantage of the fact that the cone C_+ is complete for the corresponding projective metric. More precisely, we have the following statement.

Proposition 2.6 (completeness). *Any θ_+-Cauchy sequence $(\varphi_n)_n$ in C_+ is θ_+-convergent in C_+. Moreover, if one normalizes $\int \varphi_n\,dm = 1$ for all $n \geq 1$, then $(\varphi_n)_n$ is also uniformly convergent.*

Proof: Let $(\varphi_n)_n$ be a θ_+-Cauchy sequence, normalized by $\int \varphi_n\,dm = 1$ for all $n \geq 1$. In particular, $(\varphi_n)_n$ is θ_+-bounded and so, recall (2.6), there exists $R_1 > 0$ such that

$$\frac{1}{R_1} \leq \frac{\varphi_n(x)\varphi_1(y)}{\varphi_n(y)\varphi_1(x)} \leq R_1$$

for all $x,y \in M$ and every $n \geq 1$. In particular,

$$\frac{1}{R_2} \leq \frac{\varphi_n(x)}{\varphi_n(y)} \leq R_2$$

for all $x,y \in M$ and every $n \geq 1$, where $R_2 = R_1 \sup\{\varphi_1(s)/\varphi_1(t) : s,t \in M\}$. On the other hand, the normalization $\int \varphi_n\,dm = 1$ implies $\inf\varphi_n \leq 1 \leq \sup\varphi_n$ and so

$$\frac{1}{R_2} \leq \varphi_n(x) \leq R_2 \quad\text{for every } x \in M \text{ and } n \geq 1.$$

Now, the Cauchy condition means that, given any $\varepsilon > 0$, there exists $N \geq 1$ such that for all $k, l \geq N$

$$\frac{\sup(\varphi_k/\varphi_l)}{\inf(\varphi_k/\varphi_l)} \leq e^\varepsilon \quad \Rightarrow \quad e^{-\varepsilon} \leq \inf \frac{\varphi_k}{\varphi_l} \leq 1 \leq \sup \frac{\varphi_k}{\varphi_l} \leq e^\varepsilon$$

(because $\int \varphi_k \, dm = \int \varphi_l \, dm$ implies $\inf(\varphi_k/\varphi_l) \leq 1 \leq \sup(\varphi_k/\varphi_l)$). It follows that

$$\sup |\varphi_k - \varphi_l| \leq \sup |\varphi_l| \cdot \sup \left| \frac{\varphi_k}{\varphi_l} - 1 \right| \leq R_2(e^\varepsilon - 1).$$

This means that $(\varphi_n)_n$ is a Cauchy sequence with respect to the uniform (supremum) metric, and so it is uniformly convergent. Let φ_0 be its uniform limit, observe that $\varphi_0 \geq R_2^{-1}$ and so $\varphi_0 \in C_+$. Passing to the limit as $l \to +\infty$ we get

$$e^{-\varepsilon} \leq \inf \frac{\varphi_k}{\varphi_0} \leq 1 \leq \sup \frac{\varphi_k}{\varphi_0} \leq e^\varepsilon.$$

for all $k \geq N$. This proves that both $\sup(\varphi_n/\varphi_0)$ and $\inf(\varphi_n/\varphi_0)$ converge to 1, and so $\theta_+(\varphi_n, \varphi_0)$ converges to zero, as $n \to \infty$. □

Remark 2.1. The form of the normalization condition in the previous proposition is somewhat arbitrary; it may be replaced, e.g., by $\sup \varphi_n = 1$, which makes sense in the more general context of Examples 2.2 and 2.3.

We apply the previous result to the sequence $\varphi_n = \mathcal{L}^n 1$. Since $\mathcal{L}$ is a θ-contraction $(\varphi_n)_n$ is θ-Cauchy and so it is also θ_+-Cauchy: recall that $\theta_+ \leq \theta$, since $C_+ \supset C(a, \nu)$. On the other hand,

$$\int \varphi_n \, dm = \int (\mathcal{L}^n 1) \cdot 1 \, dm = \int 1 \, dm = 1, \quad \text{for all } n \geq 1,$$

by (2.3). Hence, φ_n converges uniformly to some $\varphi_0 \in C_+$. In fact, $\varphi_0 \in C(\lambda_1 a, \nu)$, because the Hölder continuity condition in the definition of these cones is closed under uniform limits. Since $\mathcal{L}$ is a bounded operator with respect to the norm $||\cdot||_0$ of uniform convergence, (observe that

$$||\mathcal{L}\varphi - \mathcal{L}\psi||_0 \leq k \sup |\det Df|^{-1} ||\varphi - \psi||_0$$

where $k = \# f^{-1}(y)$ for any $y \in M$), it follows that φ_0 is a fixed point of $\mathcal{L}$. As a consequence, $\mu_0 = \varphi_0 m$ is an f-invariant measure of probability:

$$\int \varphi \, d\mu_0 = \int \varphi \varphi_0 \, dm = \int \varphi(\mathcal{L}\varphi_0) \, dm = \int (\varphi \circ f) \varphi_0 \, dm = \int (\varphi \circ f) \, d\mu_0$$

for every $\varphi \in C^0(M, \mathbb{R})$. Finally, $d\mu_0/dm = \varphi_0 \geq R_2^{-1} > 0$ and so μ_0 is equivalent to Lebesgue measure m.

2.4. Exponential mixing.

Let $\varphi \in C(\lambda_1 a, \nu)$. The previous arguments give

$$\theta_+(\mathcal{L}^n \varphi, \varphi_0) \le \theta(\mathcal{L}^n \varphi, \varphi_0) \le \theta(\varphi, \varphi_0)\Lambda_1^n \le D_1 \Lambda_1^n$$

and so

$$\sup |\mathcal{L}^n \varphi - \varphi_0| \le R_2(e^{D_1 \Lambda_1^n} - 1) \le R_3 \Lambda_1^n$$

for some constant $R_3 > 0$ and every $n \ge 1$. This may be seen as a statement of exponential loss of memory in the system: the iterates $(\mathcal{L}^n \varphi)m$ of an initial mass distribution φm converge exponentially fast to the equilibrium distribution $\varphi_0 m$. As we now show, a similar argument provides an exponential bound for correlation functions.

Proposition 2.7. *Given φ a ν-Hölder function and ψ an $L^1(m)$ function on M, there is $K_0 = K_0(\varphi, \psi) > 0$ such that*

$$\left| \int (\psi \circ f^n)\varphi \, dm - \int \psi \, d\mu_0 \int \varphi \, dm \right| \le K_0 \Lambda_1^n \text{ for all } n \ge 0.$$

Proof: Suppose first that $\varphi \in C(\lambda_1 a, \nu)$. It is no restriction to assume $\int \varphi \, dm = 1$. Then, denoting $||\psi||_1 = \int |\psi| \, d\mu_0$,

$$\left| \int (\psi \circ f^n)\varphi \, dm - \int \psi d\mu_0 \right| = \left| \int \psi \left(\frac{\mathcal{L}^n \varphi}{\varphi_0} - 1 \right) d\mu_0 \right| \le \left\| \frac{\mathcal{L}^n \varphi}{\varphi_0} - 1 \right\|_0 \cdot ||\psi||_1$$
$$\le (e^{D_1 \Lambda_1^n} - 1)||\psi||_1 \le R_3 ||\psi||_1 \Lambda_1^n.$$

Now let φ be a general ν-Hölder function and $A > 0$ be such that φ is (A, ν)-Hölder. For $B > 0$, we write

$$\varphi = \varphi_B^+ - \varphi_B^- \quad \text{where} \quad \varphi_B^\pm = \frac{1}{2}(|\varphi| \pm \varphi) + B.$$

Clearly, $\varphi_B^\pm$ are (A, ν)-Hölder continuous and $\varphi_B^\pm \ge B$. Taking $B = (A/\lambda_1 a)$ we get $\varphi_B^\pm \in C(\lambda_1 a, \nu)$ and so the proposition holds for $\varphi_B^\pm$. By linearity it holds for φ. □

Remark 2.2. Note, for future reference, that the constant $K_0 = K_0(\varphi, \psi)$ constructed in the proof of Proposition 2.7 has the form

$$K_0(\varphi, \psi) \le K_0' ||\psi||_1 (||\varphi||_1 + H_\nu(\varphi))$$

where $H_\nu(\varphi)$ denotes any number A such that φ is (A, ν)-Hölder, and $K_0' > 0$ is independent of φ and ψ. Indeed, the first part of the proof gives

$$K_0(\varphi_B^\pm, \psi) \le R_3 ||\psi||_1 \int \varphi_B^\pm \, dm \le R_3 ||\psi||_1 (\int |\varphi| \, dm + B) \le R_3 ||\psi||_1 (R_2 ||\varphi||_1 + \frac{A}{a\lambda_1}).$$

and our claim follows by noting that one may take $K_0(\varphi, \psi) \le K_0(\varphi_B^+, \psi) + K_0(\varphi_B^-, \psi)$.

Corollary 2.8 (exponential decay of correlations). *Given ν-Hölder continuous functions φ_1 and φ_2, there is $K = K(\varphi_1, \varphi_2) > 0$ such that*

$$\left|\int (\varphi_1 \circ f^n)\varphi_2 \, d\mu_0 - \int \varphi_1 \, d\mu_0 \int \varphi_2 \, d\mu_0\right| \le K\Lambda_1^n \text{ for all } n \ge 0.$$

Proof: Just take $\psi = \varphi_1$ and $\varphi = \varphi_2\varphi_0$. □

Let $\mathcal{F}_n = f^{-n}(\mathcal{F})$, for $n \ge 0$, where $\mathcal{F}$ is the Borel σ-algebra of M. A function $\xi: M \to \mathbb{R}$ is $\mathcal{F}_n$-measurable if and only if $\xi = \xi_n \circ f^n$ for some measurable ξ_n (if ξ is $\mathcal{F}_n$-measurable then every preimage $\xi^{-1}(y)$ is of the form $f^{-n}(A_y)$ for some $A_y \in \mathcal{F}$, just define $\xi_n | A_y \equiv y$). Moreover, $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_n \supset \cdots$ An f-invariant measure μ is called *exact* if the σ-algebra

$$\mathcal{F}_\infty = \bigcap_{n \ge 0} \mathcal{F}_n$$

is μ-trivial, in the sense that all $\mathcal{F}_\infty$-measurable functions are constant μ-almost everywhere. Note that exact measures are ergodic: if $A \subset M$ is f-invariant then $\chi_A \in \mathcal{F}_\infty$.

Corollary 2.9 (uniqueness and exactness). *The measure μ_0 is exact and it is the unique f-invariant measure which is absolutely continuous with respect to m.*

Proof: We have already shown that μ_0 and m are equivalent measures. Let $\psi \in L^1(\mu_0)$ be $\mathcal{F}_\infty$-measurable. Then for every $n \ge 0$ there is ψ_n a measurable function such that $\psi = \psi_n \circ f^n$. Note that $||\psi_n||_1 = ||\psi||_1 < \infty$. By Proposition 2.7 and Remark 2.2, given any ν-Hölder continuous function φ, there is $K_0''(\varphi) > 0$ such that

$$\left|\int (\psi - \int \psi \, d\mu_0)\varphi \, dm\right| = \left|\int (\psi_n \circ f^n)\varphi \, dm - \int \psi \, d\mu_0 \int \varphi \, dm\right|$$
$$\le K_0''(\varphi)||\psi_n||_1 \, \Lambda_1^n = K_0''(\varphi)||\psi||_1 \, \Lambda_1^n \to 0$$

and so

$$\int (\psi - \int \psi \, d\mu_0)\varphi \, dm = 0.$$

Hence $\psi = \int \psi \, d\mu_0$ almost everywhere, with respect to m and μ_0, and this proves that μ_0 is an exact measure. Finally, if μ is another invariant measure with $\mu \ll m$ then $\mu \ll \mu_0$ and so $\mu = \mu_0$ since μ_0 is ergodic. □

Let $L^2(\mathcal{F}_n) = \{\xi \in L^2(\mu_0) : \xi \text{ is } \mathcal{F}_n\text{-measurable}\}$, for each value of $n \ge 0$. Observe that $L^2(\mu_0) = L^2(\mathcal{F}_0) \supset L^2(\mathcal{F}_1) \supset \ldots \supset L^2(\mathcal{F}_n) \supset$ and

$$\mu_0 \text{ is exact} \quad \Leftrightarrow \quad \bigcap_{n \ge 0} L^2(\mathcal{F}_n) = \{\text{constants}\}.$$

Given $\varphi \in L^2(\mu_0)$ and $n \ge 0$, we denote $E(\varphi \mid \mathcal{F}_n)$ the orthogonal projection of φ to $L^2(\mathcal{F}_n)$.

Corollary 2.10. *For every ν-Hölder continuous function φ with $\int \varphi\, d\mu_0 = 0$ there is $R_0 = R_0(\varphi)$ such that $\|E(\varphi \mid \mathcal{F}_n)\|_2 \le R_0 \Lambda_1^n$ for all $n \ge 0$.*

Proof: It suffices to note that

$$\begin{aligned} \|E(\varphi \mid \mathcal{F}_n)\|_2 &= \sup\{\int \xi\varphi\, d\mu_0 : \xi \in L^2(\mathcal{F}_n) \text{ and } \|\xi\|_2 = 1\} \\ &= \sup\{\int (\psi \circ f^n)\varphi\,\varphi_0\, dm : \psi \in L^2(\mu_0) \text{ and } \|\psi\|_2 = 1\} \\ &\le K_0''(\varphi\,\varphi_0)\Lambda_1^n \end{aligned}$$

(since $\|\psi\|_1 \le \|\psi\|_2 = 1$ and we suppose $\int \varphi\, d\mu_0 = \int \varphi\,\varphi_0\, dm = 0$). □

Remark 2.3. The following description of the action of $\mathcal{L}$ in L^2 is contained in what we have done so far. Let φ_0 be as above and $H = \{\varphi \in L^2(m) : \int \varphi\, dm = 0\}$. We have $\mathcal{L}(\varphi_0) = \varphi_0$ and $\mathcal{L}(H) \subset H$, by (2.3). Consider the isometry $h : L^2(m) \to L_2(\mu_0)$, $h(\varphi) = \varphi/\varphi_0$, and introduce $\mathcal{P} : L^2(\mu_0) \to L^2(\mu_0)$, $\mathcal{P} = h \circ \mathcal{L} \circ h^{-1}$, and

$$N = h(H) = \{\psi \in L^2(m) : \int \psi\, d\mu_0 = 0\}.$$

It follows that $\mathcal{P}(1) = 1$ and $\mathcal{P}(N) \subset N$, and (2.3) asserts that $\mathcal{P}$ is the adjoint operator of $U : L^2(\mu_0) \to L^2(\mu_0)$, $U(\psi) = \psi \circ f$. Let us denote $L_0^2(\mathcal{F}_n) = N \cap L^2(\mathcal{F}_n)$. Then U and $\mathcal{P}$ are unitary operators with $U(L_0^2(\mathcal{F}_n)) = L_0^2(\mathcal{F}_{n+1})$ and $\mathcal{P}(L_0^2(\mathcal{F}_{n+1})) = L_0^2(\mathcal{F}_n)$. Exactness means that $L^2(\mu_0)$ splits as an orthogonal sum

$$L^2(\mu_0) = \{\text{constants}\} \oplus N = \{\text{constants}\} \oplus \left(\oplus_{n=0}^{\infty} \left[L_0^2(\mathcal{F}_n) \ominus L_0^2(\mathcal{F}_{n+1})\right]\right),$$

where $\ominus$ denotes orthogonal complement. The last corollary implies that the components of any $\psi \in L^2(\mu_0)$ in this splitting decrease exponentially fast as $n \to +\infty$.

2.5. Central limit theorem.

Based on the previous analysis, we now show that the oscillations of the Birkhoff sums of a ν-Hölder observable function around their expected value converge, in distribution, to a gaussian process. First, we state and prove an abstract central limit theorem for measurable (noninvertible) transformations.

Theorem 2.11. *Let $(M, \mathcal{F}, \mu)$ be a probability space, $f: M \to M$ be a measurable map such that μ is f-invariant and f-ergodic, and $\phi \in L^2(\mu)$ be such that $\int \phi\, d\mu = 0$. Let $\mathcal{F}_n$ denote the non-increasing sequence of σ-algebras $\mathcal{F}_n = f^{-n}(\mathcal{F})$, $n \ge 0$. Assume that*

$$\sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{F}_n)\|_2 < \infty.$$

Then $\sigma \ge 0$ given by

$$\sigma^2 = \int \phi^2\, d\mu + 2\sum_{n=1}^{\infty} \int \phi(\phi \circ f^n) d\mu \tag{2.7}$$

is finite and $\sigma = 0$ *if and only if* $\phi = u \circ f - u$ *for some* $u \in L^2(\mu)$. *On the other hand, if* $\sigma > 0$ *then, given any interval* $A \subset \mathbb{R}$,

$$\mu\left\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}}\, dt$$

as $n \to \infty$.

As an immediate consequence of this theorem and Corollary 2.10, we get the following central limit statement for expanding maps:

Proposition 2.12 (central limit theorem).
Let φ *be a* ν*-Hölder continuous function and*

$$\sigma^2 = \int \phi^2\, d\mu_0 + 2\sum_{j=1}^{\infty} \int \phi(\phi \circ f^j)\, d\mu_0, \quad \text{where } \phi = \varphi - \int \varphi\, d\mu_0.$$

Then σ *is well-defined and* $\sigma = 0$ *if and only if* $\varphi = u \circ f - u$ *for some* $\mu \in L^2(\mu_0)$. *If* $\sigma > 0$ *then for every interval* $A \subset \mathbb{R}$

$$\mu_0\left\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi\, d\mu_0\right) \in A\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}}\, dt,$$

as $n \to +\infty$.

In the remaining of this section we prove Theorem 2.11.

Proof: The general idea is to try and write $\phi = \eta + \zeta \circ f - \zeta$ for functions $\eta, \zeta \in L^2(\mu)$ such that

i) $\frac{1}{\sqrt{n}}(\zeta \circ f^n - \zeta)$ converges to zero in measure as $n \to +\infty$ (we prove a stronger fact: it converges to zero in $L^2(\mu)$);

ii) the (identically distributed) random variables $\eta \circ f^n$ satisfy a kind of independence condition, to be stated in (2.8).

The first property means that, given any $\varepsilon > 0$,

$$\lim \mu \left\{x \in M : \frac{1}{\sqrt{n}} |\zeta(f^n(x)) - \zeta(x)| > \varepsilon\right\} \to 0,$$

and implies that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (\phi \circ f^j) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (\eta \circ f^j) + \frac{1}{\sqrt{n}} (\zeta \circ f^n - \zeta)$$

has the same limit distribution as $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\eta\circ f^j$. It does not seem possible to take the random variables $\eta\circ f^n$ to be independent, in which case (most of) the theorem would follow directly from Theorem 1.1. Instead, we construct η in such a way that

$$E(\eta\circ f^n \mid \mathcal{F}_{n+1}) = 0 \quad \text{for every } n\geq 0, \tag{2.8}$$

then we invoke the fact that the conclusion of Theorem 1.1 remains true under this condition. In the language of probability theory, (2.8) means that $(\eta\circ f^n)_n$ is a *reversed martingale difference*, and the statement we have just made is usually called the martingale central limit theorem; see Appendix B.

Let us fill-in the details. We introduce the operator $U: L^2(\mu)\to L^2(\mu)$ given by $U\varphi=\varphi\circ f$, and we let $\mathcal{P}$ be the adjoint operator $\int(\mathcal{P}\psi)\varphi\,d\mu=\int\psi(U\varphi)d\mu$. Note that U is an isometry onto $L^2(\mathcal{F}_1)\subset L^2(\mathcal{F}_0)=L^2(\mu)$, and so $\mathcal{P}: L^2(\mathcal{F}_1)\to L^2(\mathcal{F}_0)$ is also an isometry. Moreover,

$$U(L^2(\mathcal{F}_n)) = L^2(\mathcal{F}_{n+1}) \quad\text{and}\quad \mathcal{P}(L^2(\mathcal{F}_{n+1})) = L^2(\mathcal{F}_n),$$

for each $n\geq 1$. Now we define

$$\zeta = -\sum_{j=1}^{\infty}\mathcal{P}^j(E(\phi\mid\mathcal{F}_j)) \quad\text{and}\quad \eta=\sum_{j=0}^{\infty}\mathcal{P}^j(E(\phi\mid\mathcal{F}_j)-E(\phi\mid\mathcal{F}_{j+1})).$$

Since $||\mathcal{P}^j(E(\phi\mid\mathcal{F}_j))||_2 = ||E(\phi\mid\mathcal{F}_j)||_2$, the hypothesis of the theorem ensures that the series defining ζ converges in $L^2(\mu)$. It also follows that

$$\left\|\frac{1}{\sqrt{n}}(\zeta\circ f^n-\zeta)\right\|_2 \leq \frac{1}{\sqrt{n}}(||\zeta\circ f^n||_2+||\zeta||_2) = \frac{2}{\sqrt{n}}||\zeta||_2\to 0 \text{ as } n\to+\infty.$$

As a consequence, $\frac{1}{\sqrt{n}}(\zeta\circ f^n-\zeta)$ converges to zero in measure. Next,

$$\begin{aligned}||\mathcal{P}^j(E(\phi\mid\mathcal{F}_j)-E(\phi\mid\mathcal{F}_{j+1}))||_2 &= ||E(\phi\mid\mathcal{F}_j)-E(\phi\mid\mathcal{F}_{j+1})||_2\\ &\leq ||E(\phi\mid\mathcal{F}_j)||_2\end{aligned}$$

(because $E(\phi\mid\mathcal{F}_j)-E(\phi\mid\mathcal{F}_{j+1})$ coincides with the orthogonal projection of $E(\phi\mid\mathcal{F}_j)$ to $L^2(\mathcal{F}_{j+1})^\perp$), and so the series defining η is also convergent in $L^2(\mu)$. Now we write

$$\begin{aligned}\eta &= E(\phi\mid\mathcal{F}_0)+\mathcal{P}(E(\phi\mid\mathcal{F}_1))+\mathcal{P}^2(E(\phi\mid\mathcal{F}_2))+\cdots\\ &\quad - E(\phi\mid\mathcal{F}_1)-\mathcal{P}(E(\phi\mid\mathcal{F}_2))-\mathcal{P}^2(E(\phi\mid\mathcal{F}_3))-\cdots\end{aligned}$$

Clearly, $E(\phi\mid\mathcal{F}_0)=\phi$. Moreover, for every $j\geq 1$,

$$\mathcal{P}^{j-1}(E(\phi\mid\mathcal{F}_j)) = U\mathcal{P}^j(E(\phi\mid\mathcal{F}_j)) = \mathcal{P}^j(E(\phi\mid\mathcal{F}_j))\circ f.$$

Indeed, $E(\phi \mid \mathcal{F}_j) \in L^2(\mathcal{F}_j)$ implies $\mathcal{P}^{j-1}(E(\phi \mid \mathcal{F}_j)) \in L^2(\mathcal{F}_1)$, and $U\mathcal{P} = \mathrm{id} \mid L^2(\mathcal{F}_1)$ because the adjoint operators U, $\mathcal{P}$ are isometries to their images. Therefore, we may rewrite

$$\begin{aligned}\eta &= \phi + \sum_{j=1}^{\infty} \mathcal{P}^j(E(\phi \mid \mathcal{F}_j)) - \sum_{j=1}^{\infty} \mathcal{P}^j(E(\phi \mid \mathcal{F}_j)) \circ f \\ &= \phi - \zeta + \zeta \circ f.\end{aligned}$$

Now, $E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1}) \in L^2(\mathcal{F}_{j+1})^\perp$ implies $\mathcal{P}^j(E(\phi \mid \mathcal{F}_j) - E(\phi \mid \mathcal{F}_{j+1})) \in L^2(\mathcal{F}_1)^\perp$, for all $j \geq 0$, and so

$$\eta \in L^2(\mathcal{F}_1)^\perp, \quad \text{that is,} \quad E(\eta \mid \mathcal{F}_1) = 0.$$

The martingale property (2.8) is an immediate consequence:

$$E(\eta \circ f^n \mid \mathcal{F}_{n+1}) = E(\eta \mid \mathcal{F}_1) \circ f^n = 0, \quad \text{for all } n \geq 0.$$

In particular, the variables $\eta \circ f^n$ are two by two orthogonal:

$$\int (\eta \circ f^k)(\eta \circ f^n)\, d\mu = 0 \quad \text{for every } k > n \geq 0,$$

because $\eta \circ f^k \in L^2(\mathcal{F}_{n+1})$. This gives

$$||\eta||_2^2 = \frac{1}{n} \sum_{j=0}^{n-1} ||\eta \circ f^j||_2^2 = \frac{1}{n} ||\sum_{j=0}^{n-1} \eta \circ f^j||_2^2 = ||\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j||_2^2.$$

Since

$$||\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j - \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \eta \circ f^j||_2 = ||\frac{1}{\sqrt{n}}(\zeta \circ f^n - \zeta)||_2 \to 0,$$

we conclude that

$$\begin{aligned}||\eta||_2^2 &= \lim ||\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi \circ f^j||_2^2 \\ &= \lim \frac{1}{n}\left(\sum_{k=0}^{n-1} \int (\phi \circ f^k)^2\, d\mu + 2 \sum_{0 \leq k < l \leq n-1} \int (\phi \circ f^k)(\phi \circ f^l) d\mu \right) \\ &= \lim \left(\int \phi^2\, d\mu + 2 \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \int \phi(\phi \circ f^j) d\mu \right) \\ &= \int \phi^2\, d\mu + 2 \sum_{j=1}^{\infty} \int \phi(\phi \circ f^j) d\mu.\end{aligned}$$

For the last equality note that

$$\begin{aligned}\int \phi(\phi\circ f^j)d\mu &= \int E(\phi \mid \mathcal{F}_j)(\phi\circ f^j)d\mu \quad (\text{because } \phi\circ f^j \in L^2(\mathcal{F}_j))\\ &\le \|E(\phi\mid\mathcal{F}_j)\|_2\,\|\phi\circ f^j\|_2 = \|E(\phi\mid\mathcal{F}_j)\|_2\,\|\phi\|_2\end{aligned}$$

and so

$$\begin{aligned}\|2\sum_{j=1}^{n}\frac{j}{n}\int \phi(\phi\circ f^j)d\mu + 2\sum_{j=n}^{\infty}\int\phi(\phi\circ f^j)d\mu\| \\ \le 2\|\phi\|_2\Big(\sum_{j=1}^{n-1}\frac{j}{n}\|E(\phi\mid\mathcal{F}_j)\|_2 + \sum_{j=n}^{\infty}\|E(\phi\mid\mathcal{F}_j)\|_2\Big)\\ \le 2\|\phi\|_2\Big(\varepsilon\sum_{j=1}^{\infty}\|E(\phi\mid\mathcal{F}_j)\|_2 + \sum_{j>n\varepsilon}\|E(\phi\mid\mathcal{F}_j)\|_2\Big)\end{aligned}$$

where the last expression can be made arbitrarily small by fixing $\varepsilon > 0$ close to zero and then taking $n \gg \varepsilon^{-1}$. In this way we have shown that

$$\|\eta\|_2^2 = \sigma^2,$$

recall that σ is defined in (2.7). In particular, we get that $\sigma < +\infty$. Moreover, $\sigma = 0$ implies $\eta = 0$ and so $\phi = \zeta\circ f - \zeta$. Conversely, if ϕ has the form $\phi = u\circ f - u$ then we may take $\zeta = u$ and $\eta = 0$ and then, by the previous arguments, $\sigma = \|\eta\|_2 = 0$. This proves the first part of the theorem. From now on we suppose $\sigma > 0$. Let $a < b$ be fixed. Given $\delta > 0$ we take $\varepsilon > 0$ such that

$$\Phi(a-\varepsilon, b+\varepsilon) \le \Phi(a,b) + \delta, \quad \text{where} \quad \Phi(r,s) = \frac{1}{\sigma\sqrt{2\pi}}\int_r^s e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

Then we take $n_1 \ge 1$ such that

$$\mu\Big\{x\in M : \Big|\frac{1}{\sqrt{n}}(\zeta\circ f^n - \zeta)(x)\Big| > \varepsilon\Big\} \le \delta \quad \text{for every } n \ge n_1.$$

By the martingale central limit theorem, see Appendix B, there exists $n_2 \ge n_1$ such that

$$\mu\Big\{x\in M : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\eta(f^j(x)) \in (a-\varepsilon, b+\varepsilon)\Big\} \le \Phi(a-\varepsilon, b+\varepsilon) + \delta.$$

Since $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi\circ f^j \in (a,b)$ implies

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\eta\circ f^j \in (a-\varepsilon, b+\varepsilon) \quad \text{or} \quad \Big|\frac{1}{\sqrt{n}}(\zeta\circ f^n - \zeta)\Big| > \varepsilon$$

we conclude that

$$\mu\left\{x \in M : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi(f^j(x)) \in (a,b)\right\} \le \Phi(a-\varepsilon, b+\varepsilon) + 2\delta \le \Phi(a,b) + 3\delta.$$

This shows that

$$\limsup \mu\left\{x \in M : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi(f^j(x)) \in (a,b)\right\} \le \Phi(a,b), \quad .$$

and a similar argument gives $\liminf \geq \Phi(a,b)$. □

2.6. Stochastic stability.

Next we prove that expanding maps $f\colon M \to M$ are stable under random perturbations. We consider parametrized families $f_t\colon M \to M$ of $C^{1+\nu_0}$-Hölder continuous maps, where t belongs in some metric space T. Suppose that there is $\tau \in T$ such that

$$f_\tau = f \quad \text{and} \quad T \ni t \mapsto f_t \in C^{1+\nu_0}(M, \mathbb{R}) \text{ is continuous at } \tau.$$

This means that if t is close to τ then f_t is uniformly close to f and Df_t is close to Df with respect to the ν_0-Hölder norm

$$|||G|||_{\nu_0} = \sup\{||G(x)|| : x \in M\} + \sup\left\{\frac{||G(x) - G(y)||}{d(x,y)^{\nu_0}} : x, y \in M, 0 < d(x,y) \le \rho_0\right\}.$$

An important particular case is T = some neighbourhood of f in $C^{1+\nu_0}$ and $f_t = t$. We also consider a family $(\theta_\varepsilon)_{\varepsilon>0}$ of regular probability measures in T such that $\operatorname{supp}\theta_\varepsilon \to \{\tau\}$ as $\varepsilon \to 0$. Then we are interested in comparing the asymptotics of random trajectories

$$x_j = f_{t_j} \circ \cdots \circ f_{t_1}(x)$$

where $t_1, \ldots, t_j, \ldots$ are independent random variables with distribution θ_ε, with the asymptotics of deterministic trajectories $f^j(x)$.

For that we introduce perturbed versions of the linear operators U and $\mathcal{L}$ we used before for the map f:

$$(U_t\,\varphi)(x) = \varphi(f_t(x)) \qquad (\mathcal{L}_t\varphi)(y) = \sum_{f_t(x)=y} \varphi(x)\,|Df_t(x)|^{-1}$$

and also

$$(\widehat{U}_\varepsilon\,\varphi)(x) = \int (U_t\,\varphi)(x)d\theta_\varepsilon(t) \qquad (\widehat{\mathcal{L}}_\varepsilon\varphi)(y) = \int (\mathcal{L}_t\varphi)(y)d\theta_\varepsilon(t),$$

acting on $E = C^0(M, \mathbb{R})$. By Fubini's theorem and (2.3)

$$\begin{aligned}\int (\widehat{\mathcal{L}}_\varepsilon\varphi)(y)\psi(y)dm(y) &= \int \left(\int (\mathcal{L}_t\varphi)(y)d\theta_\varepsilon(t)\right)\psi(y)dm(y)\\ &= \int \left(\int (\mathcal{L}_t\varphi)(y)\psi(y)dm(y)\right)d\theta_\varepsilon(t)\\ &= \int \left(\int \varphi(x)(U_t\psi)(x)dm(x)\right)d\theta_\varepsilon(t)\\ &= \int \varphi(x)\left(\int (U_t\psi)(x)d\theta_\varepsilon(t)\right)dm(x)\\ &= \int \varphi(x)(\widehat{U}_t\psi(x))dm(x).\end{aligned}$$

Hence, if φ_ε is a nonnegative $L^1(m)$ function with $\widehat{\mathcal{L}}_\varepsilon\varphi_\varepsilon = \varphi_\varepsilon$ and $\int \varphi_\varepsilon\, dm = 1$ then $\mu_\varepsilon = \varphi_\varepsilon\, m$ is a stationary probability measure:

$$\int (\widehat{U}_\varepsilon\psi)d\mu_\varepsilon = \int \psi\, d\mu_\varepsilon \quad \text{for all } \psi \text{ continuous.} \tag{2.9}$$

We proceed to show that there exists such a function φ_ε. Moreover, it is unique and it is close to φ_0 if ε is small.

Our assumptions imply that if $\varepsilon > 0$ is small enough then every f_t with $t \in \operatorname{supp}\theta_\varepsilon$ is a $C^{1+\nu_0}$ expanding map, with uniform bounds $\sigma > 1$ and $a_0 > 0$ for the rate of expansion of f_t and the Hölder constant of $\log|\det Df_t|$, respectively. This means that the estimates in the proof of Proposition 2.4 apply uniformly

$$\mathcal{L}_t(C(a,\nu)) \subset C(\lambda_1 a, \nu) \quad \text{for every } t \in \operatorname{supp}\theta_\varepsilon,$$

as long as $\sigma^{-1} < \lambda_1 < 1$ and $a \ge a_0/(\lambda_1 - \sigma^{-1})$. Therefore, by convexity and closedness of the cone $C(\lambda_1 a, \nu)$,

$$\widehat{\mathcal{L}}_\varepsilon(C(a,\nu)) \subset C(\lambda_1 a, \nu) \quad \text{for every small } \varepsilon > 0.$$

Arguing as we did before for $\varphi_n = \mathcal{L}^n 1$, we conclude that $\varphi_{\varepsilon,n} = \widehat{\mathcal{L}}_\varepsilon^n 1$ converges uniformly to some $\varphi_\varepsilon \in C(\lambda_1 a, \nu)$, which is a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. We take $\mu_\varepsilon = \varphi_\varepsilon\, m$. Note that this probability measure is equivalent to m.

Next, we show that μ_ε determines the asymptotics of almost all random trajectories $(x_j)_{j\ge 0}$, in the sense that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(x_j) \to \int \varphi\, d\mu_\varepsilon,$$

for m-almost all choices of the initial point x_0 and θ_ε-almost all choices of the perturbations $t_1, \ldots, t_j, \ldots$. First, we state a consequence of the previous arguments which is also interesting in itself.

Corollary 2.13 (exponential mixing for the random perturbations). *Given φ a ν-Hölder function and ψ an $L^1(m)$-function on M there is $K_0 = K_0(\varphi, \psi) > 0$ such that*

$$\left| \int (\widehat{U}_\varepsilon^n \psi) \varphi \, dm - \int \psi \, d\mu_\varepsilon \int \varphi \, dm \right| \leq K_0 \Lambda_1^n \quad \textit{for all } n \geq 0 \textit{ and } \varepsilon > 0 \textit{ small.}$$

Proof: Analogous to Proposition 2.7, just replace U, $\mathcal{L}$, μ_0, φ_0, by $\widehat{U}_\varepsilon$, $\widehat{\mathcal{L}}_\varepsilon$, μ_ε, φ_ε, respectively. □

Now consider the probability measure $\nu_\varepsilon = \mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}}$ defined on $M \times T^{\mathbb{N}}$ by

$$\nu_\varepsilon(A \times B_1 \times \cdots \times B_k) = \mu_\varepsilon(A) \times \theta_\varepsilon(B_1) \times \cdots \times \theta_\varepsilon(B_k)$$

for Borel sets $A \subset M$, $B_1, \dots, B_k \subset T$, and $k \geq 0$. We introduce the shift map

$$\sigma: M \times T^{\mathbb{N}} \to M \times T^{\mathbb{N}}, \quad \sigma(x, t_1, t_2, \dots) = (f_{t_1}(x), t_2, \dots).$$

It is easy to see that ν_ε is a σ-invariant measure, because μ_ε is stationary, recall (2.9). Hence, by the ergodic theorem,

$$\widetilde{\varphi}(x, t_1, t_2, \dots) = \lim \frac{1}{n} \sum_{j=0}^{n-1} (\varphi \circ \pi_0)(\sigma^j(x, t_1, t_2, \dots))$$

exists for ν_ε-almost every $(x, t_1, t_2, \dots) \in M \times T^{\mathbb{N}}$. Here π_0 is the canonical projection $\pi_0: M \times T^{\mathbb{N}} \to M$, $\pi_0(x, t_1, t_2, \dots) = x$. Note that $\pi_0(\sigma^j(x, t_1, t_2, \dots))$ is precisely what we have been denoting x_j. We are left to show that

$$\widetilde{\varphi}(x, t_1, t_2, \dots) = \int (\varphi \circ \pi_0) d\nu_\varepsilon = \int \varphi \, d\mu_\varepsilon \quad \nu_\varepsilon\text{-almost everywhere.}$$

For each $k \geq 0$ we define

$$\widetilde{\varphi}_k(x, t_1, \dots, t_k) = \int \widetilde{\varphi}(x, t_1, \dots, t_k, t_{k+1}, \dots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \dots .$$

Since $\widetilde{\varphi}$ is σ-invariant, i.e., $\widetilde{\varphi} \circ \sigma = \widetilde{\varphi}$,

$$\begin{aligned}
\widetilde{\varphi}_0(x) &= \int \widetilde{\varphi}(x, t_1, t_2, \dots) d\theta_\varepsilon(t_1) d\theta_\varepsilon(t_2) \dots \\
&= \int \widetilde{\varphi}(f_{t_1}(x), t_2, \dots) d\theta_\varepsilon(t_2) d\theta_\varepsilon(t_3) \dots d\theta_\varepsilon(t_1) \\
&= \int \widetilde{\varphi}_0(f_{t_1}(x)) d\theta_\varepsilon(t_1) = (\widehat{U}_\varepsilon \widetilde{\varphi}_0)(x)
\end{aligned}$$

for μ_ε-almost all $x \in M$. Then Corollary 2.13 implies

$$\int (\widetilde{\varphi}_0 - \int \widetilde{\varphi}_0 \, d\mu_\varepsilon)\varphi \, dm = \int (\widehat{U}_\varepsilon^n \, \widetilde{\varphi}_0)\varphi dm - \int \widetilde{\varphi}_0 \, d\mu_\varepsilon \int \varphi dm \to 0$$

for all ν-Hölder φ. Therefore,

$$\widetilde{\varphi}_0 = \int \widetilde{\varphi}_0 \, d\mu_\varepsilon = \int \widetilde{\varphi} \, d\nu_\varepsilon = \int (\varphi \circ \pi_0) d\nu_\varepsilon = \int \varphi \, d\mu_\varepsilon$$

m-almost everywhere, and so also μ_ε-almost everywhere. More generally, for $k \geq 1$,

$$\begin{aligned}\widetilde{\varphi}_k(x, t_1, \ldots, t_k) &= \int \widetilde{\varphi}(x, t_1, \ldots, t_k, t_{k+1}, \ldots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \ldots \\ &= \int \widetilde{\varphi}(f_{t_1}(x), t_2, \ldots, t_k, t_{k+1}, \ldots) d\theta_\varepsilon(t_{k+1}) d\theta_\varepsilon(t_{k+2}) \ldots \\ &= \widetilde{\varphi}_{k-1}(f_{t_1}(x), t_2, \ldots, t_k).\end{aligned}$$

It follows, by induction, that for every $k \geq 0$

$$\widetilde{\varphi}_k = \int \varphi \, d\mu_\varepsilon, \qquad (\mu_\varepsilon \times \theta_\varepsilon^k)\text{-almost everywhere.}$$

This gives $\widetilde{\varphi} = \int \varphi \, d\mu_\varepsilon$ at $(\mu_\varepsilon \times \theta_\varepsilon^{\mathbb{N}})$-almost every point, and so completes our argument.

Proposition 2.14 (stochastic stability). *φ_ε converges uniformly to φ_0 as $\varepsilon \to 0$. In particular, $\mu_\varepsilon \to \mu_0$ weakly* as $\varepsilon \to 0$.*

Proof: Proposition 2.7 gives, for some $R > 0$,

$$\left| \int \psi(\mathcal{L}^n 1) dm - \int \psi \, d\mu_0 \right| \leq R \|\psi\|_1 \Lambda_1^n$$

and a similar result for $\widehat{\mathcal{L}}_\varepsilon$ is deduced in precisely the same way, using Corollary 2.13,

$$\left| \int \psi(\widehat{\mathcal{L}}_\varepsilon^n 1) dm - \int \psi \, d\mu_\varepsilon \right| \leq R \|\psi\|_1 \Lambda_1^n .$$

Now, given $y \in M$ and $t \in \operatorname{supp} \theta_\varepsilon$ we write

$$f^{-1}(y) = \{x_1, \ldots, x_k\} \quad \text{and} \quad f_t^{-1}(y) = \{x_{1,t}, \ldots, x_{k,t}\},$$

with $\sup\{d(x_{i,t}, x_i) : 1 \leq i \leq k, \, t \in \operatorname{supp} \theta_\varepsilon, \, y \in M\} \to 0$ as $\varepsilon \to 0$. From

$$\mathcal{L}\varphi(y) = \sum_{i=1}^k \varphi(x_i) |\det Df(x_i)|^{-1} \quad \text{and} \quad \mathcal{L}_t\varphi(y) = \sum_{i=1}^k \varphi(x_{i,t}) |\det Df_t(x_{i,t})|^{-1}$$

one concludes easily that

$$1 - \xi(\varepsilon) \leq \frac{\mathcal{L}_t\varphi(y)}{\mathcal{L}\varphi(y)} \leq 1 + \xi(\varepsilon) \quad \text{for all } y \in M,\ t \in \operatorname{supp}\theta_\varepsilon,\ \varphi \in C(a,\nu),$$

where $\xi(\varepsilon)$ is independent of y, t, or φ, and converges to zero as $\varepsilon \to 0$. As a consequence, given any $\varphi \in C(a,\nu)$,

$$\left|\frac{\widehat{\mathcal{L}}_\varepsilon\varphi(y)}{\mathcal{L}\varphi(y)} - 1\right| \leq \xi(\varepsilon) \text{ for all } y \in M \text{ and so } \|\widehat{\mathcal{L}}_\varepsilon\varphi - \mathcal{L}\varphi\|_0 \leq \xi(\varepsilon)\|\mathcal{L}\varphi\|_0 .$$

Applying this to each $\varphi = \mathcal{L}^i 1$, $0 \leq i < n$, we get

$$\begin{aligned}
\left|\int (\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1)\psi\, dm\right| &= \left|\sum_{i=0}^{n-1} \int \widehat{\mathcal{L}}_\varepsilon^{n-i-1}(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)\cdot \psi\, dm\right| \\
&= \left|\sum_{i=0}^{n-1} \int (\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)(\widehat{U}_\varepsilon^{n-i-1}\psi) dm\right| \\
&\leq \sum_{i=0}^{n-1} \|(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)\|_0\ \|\widehat{U}_\varepsilon^{n-i-1}\psi\|_1\ \sup|\varphi_\varepsilon|^{-1} \\
&\leq \sum_{i=0}^{n-1} \xi(\varepsilon)\cdot \sup|\mathcal{L}^{i+1} 1| \cdot \|\psi\|_1 \cdot \sup|\varphi_\varepsilon|^{-1} \\
&\leq K n\, \xi(\varepsilon)\|\psi\|_1, \quad \text{where } K \text{ depends only on } f.
\end{aligned}$$

In the last inequality we use the fact that $\mathcal{L}^i 1 \in C(\lambda_1 a, \nu)$ and $\int \mathcal{L}^i 1\, dm = 1$, hence $\sup|\mathcal{L}^{i+1} 1|$ admits an upper bound independent of i, see the proof of Proposition 2.6. Altogether, the previous estimates imply

$$\left|\int (\varphi_\varepsilon - \varphi_0)\psi\, dm\right| = \left|\int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0\right| \leq (K n\, \xi(\varepsilon) + 2R\Lambda_1^n)\|\psi\|_1,$$

for every $n \geq 0$, $\varepsilon > 0$ small, and $\psi \in L^1(m)$. Therefore

$$\|\varphi_\varepsilon - \varphi_0\|_0 \leq (K n\, \xi(\varepsilon) + 2R\Lambda_1^n)$$

for every $n \geq 0$. We fix $n \geq 0$ such that $\lambda_1^n \geq \xi(\varepsilon) > \Lambda_1^{n+1}$ and then we get

$$\|\varphi_\varepsilon - \varphi_0\|_0 \leq K'\, \xi(\varepsilon) \log \xi(\varepsilon)$$

for some $K' > 0$ depending only on f. This proves the proposition. □

A special feature of these uniformly expanding systems is that all the previous arguments could be carried out with no assumption on the class of regular probability distributions θ_ε, (apart from $\operatorname{supp}\theta_\varepsilon \to \{\tau\}$). In particular, one may easily extract from Proposition 2.14 the following statement of stability of the absolutely continuous invariant measure under *deterministic* perturbations of the map.

Let f be an expanding map and $(g_n)_n$ be any sequence converging to f in $C^{1+\nu_0}(M)$. Define θ_ε to be the Dirac measure supported on g_n for all $\varepsilon \in (1/n+1, 1/n]$. Then, as a particular case of Proposition 2.14, the densities $\varphi_{1/n} = \varphi_{0,g_n}$ of the stationary measures $\mu_{1/n} = \mu_{0,g_n}$ converge uniformly to φ_0 as $n \to +\infty$. This proves that the absolutely continuous invariant measure varies continuously with the expanding map. In more precise terms,

Corollary 2.15 (deterministic stability). *Let f be as before and g be another expanding map, close to f in $C^{1+\nu_0}(M)$. Let $\varphi_{0,g}$ be the fixed point of the corresponding transfer operator and $\mu_{0,g} = \varphi_{0,g}m$. Then $\varphi_{0,g}$ is uniformly close to φ_0, in particular $\mu_{0,g}$ is close to μ_0 in the weak*-sense.*

Notes.

Most of this chapter is due to Ruelle, who developed the transfer (Perron-Frobenius) operator approach to smooth expanding maps, see [Ru89]. Projective metrics associated to cones were defined by G. Birkhoff [Bi67] and provide an elegant way to express spectral properties of the transfer operator, and were first used in this setting of Dynamics by [FS79], in their proof of the Perron-Frobenius-Ruelle theorem.

Central limit theorems for dynamical systems have been studied by a large number of people. The key idea in the proof of Theorem 2.11 (approximation by a martingale difference) is due to [Go69], and has now become standard, see e.g. [Du91]. Recent works include [DG86] [Ch95], [Li].

Stability of smooth expanding maps under (a somewhat different model of) random perturbations was first proved by [Ki86], [Ki86a]. Kifer's books [Ki86], [Ki88] are main references for this field. See also [Bl87].

3. Piecewise expanding maps

In the previous chapter we took the map f to be everywhere smooth. Here we weaken this assumption and study the ergodic properties of piecewise smooth expanding maps. In doing this we restrict ourselves to the one-dimensional setting, that is, maps of the interval $I = [0,1]$ or the circle $S^1 = [0,1]/(0 \sim 1)$, and we use M to represent both I or S^1. As mentioned before, piecewise expanding maps are interesting not only by themselves, but also as tools in the study of other class of systems. Such applications are given, e.g., in Chapter 5 and Section 3.5.

Some of the examples we have in mind are described in Figure 3.1. The *tent map* is among the simplest piecewise expanding maps. It has f' constant and strictly larger than 1 in absolute value, in each of the monotonicity intervals $[0,c)$ and $(c,1]$. In the second example, a *Lorenz-like map*, $|f'(x)| > \sigma > 1$ for all $x \neq 1/2$ and $|f'(1/2^{\pm})| = \infty$. This kind of maps appear naturally associated to the so-called geometric Lorenz attractors of flows in 3 dimensions, see [ABS77], [GW79]. Finally, the *Gauss map*, which plays a central role in the theory of continued fractions, is given by $f(x) = 1/x - [1/x]$ for $x \neq 0$ and $f(0) = 0$.

An important technical point is that, in general, the transfer operator $\mathcal{L}$ of these piecewise smooth maps does not preserve the space of (Hölder) continuous functions. On the other hand, under appropriate assumptions, $\mathcal{L}$ does preserve the space of observables with bounded variation. Moreover, the regularity of these observable functions, expressed in terms of variation, improves under the action of $\mathcal{L}$. This enables us to develop for such maps a version of the reasonings we used in Chapter 2.

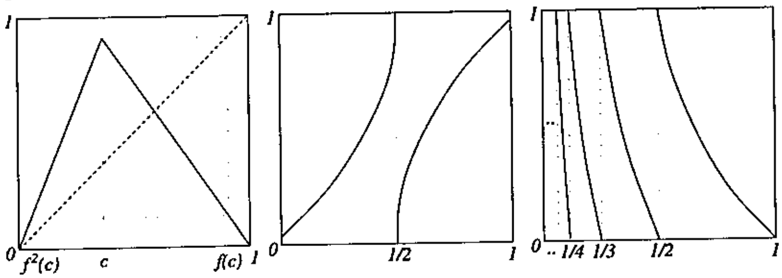

Figure 3.1: Tent map, Lorenz-like map, Gauss map

Recall that the *variation* $\operatorname{var}\varphi$ of a function $\varphi : M \to \mathbb{R}$ is defined by

$$\operatorname{var}\varphi = \sup \sum_{i=1}^{n} |\varphi(x_{i-1}) - \varphi(x_i)|$$

where the supremum is taken over all finite partitions $0 = x_0 < x_1 < \cdots < x_n = 1$, $n \geq 1$, of M (when $M = S^1 = [0,1]/(0 \sim 1)$, take $<$ to mean the orientation induced by the usual order in $[0,1]$). The variation $\operatorname{var}_\eta \varphi = \operatorname{var}(\varphi|\eta)$ of φ over an arbitrary interval $\eta \subset M$ is defined by a similar expression, with the supremum taken over all the

$x_0, x_1, \dots, x_n \in \eta$ with $\inf \eta \le x_0 < x_1 < \dots < x_n \le \sup \eta$. The following elementary properties will be used a number of times:

(v1) $\operatorname{var}_\eta(\varphi_1 + \varphi_2) \le \operatorname{var}_\eta \varphi_1 + \operatorname{var}_\eta \varphi_2$;

(v2) $\operatorname{var}_\eta(\varphi_1 \cdot \varphi_2) \le \operatorname{var}_\eta \varphi_1 \sup_\eta |\varphi_2| + \sup_\eta |\varphi_1| \operatorname{var}_\eta \varphi_2$;

(v3) $\operatorname{var}_\eta(\varphi \cdot \psi) \le \operatorname{var}_\eta \varphi \sup_\eta |\psi| + \sup_\eta |D\psi| \int_\eta |\varphi|\, dm$, if ψ is C^1;

(v4) $\operatorname{var}_\eta |\varphi| \le \operatorname{var}_\eta \varphi$;

(v5) $\operatorname{var}_\eta(\varphi \circ h) = \operatorname{var}_{h(\eta)} \varphi$, if $h : \eta \to h(\eta)$ is a homeomorphism;

(v6) $\operatorname{var}_\eta \int \varphi(t, \cdot)\, d\theta(t) \le \int \operatorname{var}_\eta \varphi(t, \cdot)\, d\theta(t)$ for every probability measure θ on a space T, and every $\varphi : T \times M \to \mathbb{R}$ with $\operatorname{var} \varphi(t, \cdot) < \infty$ for all $t \in T$.

One says that φ has *bounded variation* if $\operatorname{var} \varphi < \infty$. Then φ has at most countably many discontinuity points.

Let us explain what we mean by a *piecewise expanding map* $f: I \to I$ of the interval. The definition for circle maps is analogous. Most of the time we consider only maps with finitely many intervals of monotonicity, but we also discuss the infinite case in Sections 3.4 and 3.5. We always assume properties (E1) and (E2) below. Condition (E3) is introduced near the end of Section 3.1, and in Section 3.3 we use yet another condition (E4).

(E1) (regularity) There exist $0 = a_0 < a_1 < \dots < a_l = 1$ such that the restriction of f to each $\eta_i = (a_{i-1}, a_i)$ is of class C^1, with $|Df(x)| > 0$ for all $x \in \eta_i$ and $i = 1, \dots, l$. Moreover, the function $g_{\eta_i} = 1/|Df|\eta_i|$ has bounded variation for $i = 1, \dots, l$.

In particular, $(f|\eta_i)$ and g_{η_i} admit continuous extensions to $\bar\eta_i = [a_{i-1}, a_i]$, for each $i = 1, \dots, l$. Since modifying the values of a map over a finite set of points does not change its statistical properties, we may assume that f is either left-continuous or right-continuous (or both) at a_i, for each $i = 1, \dots, l$. Then let $\mathcal{P}^{(1)}$ be some partition of I into intervals η such that $\eta_i \subset \eta \subset \bar\eta_i$ for some i and $(f|\eta)$ is continuous. Moreover, for $n \ge 1$, let $\mathcal{P}^{(n)}$ be the partition of I such that $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y)$ if and only if $\mathcal{P}^{(1)}(f^j(x)) = \mathcal{P}^{(1)}(f^j(y))$ for all $0 \le j < n$. Given $\eta \in \mathcal{P}^{(n)}$, denote $g_\eta^{(n)} = 1/|Df^n|\eta|$. Then we assume, furthermore,

(E2) (expansivity) There exist $C_1 > 0$ and $\lambda_1 < 1$ such that $\sup g_\eta^{(n)} \le C_1 \lambda_1^n$ for all $\eta \in \mathcal{P}^{(n)}$ and all $n \ge 1$.

Example 3.1. Let $0 = a_0 < a_1 < \dots < a_l = 1$ and f be a function defined on $[0,1]$ such that each $f|(a_{i-1}, a_i)$, $i = 1, \dots, l$, admits a C^2 extension to $\bar\eta_i = [a_{i-1}, a_i]$ satisfying $|Df(x)| \ge \lambda_1 > 1$ for all $x \in \bar\eta_i$. Then, clearly, (E1) and (E2) hold for f.

In what follows we choose $C_1 > 0$ large enough so that $\operatorname{var} g_{\eta_i} \le C_1$ for all $i = 1, \dots, l$. The following consequence of (E2) will be useful later on. Given $\eta \in \mathcal{P}^{(n)}$ and $0 \le j < n$, let $\xi_j \in \mathcal{P}^{(j)}$, $\tilde\eta_j \in \mathcal{P}^{(1)}$, and $\zeta_j \in \mathcal{P}^{(n-j-1)}$ be defined by $\eta \subset \xi_j$, $f^j(\eta) \subset \tilde\eta_j$, and

$f^{j+1}(\eta) \subset \zeta_j$. Using property (v2) we conclude that

$$\begin{aligned} \operatorname{var} g_\eta^{(n)} &\le \sum_{j=0}^{n-1} \sup g_{\zeta_j}^{(n-j-1)} \cdot \operatorname{var} g_{\tilde\eta_j} \cdot \sup g_{\xi_j}^{(j)} \\ &\le \sum_{j=0}^{n-1} C_1 \lambda_1^{n-j-1} \cdot C_1 \cdot C_1 \lambda_1^j = (C_1^3/\lambda_1) n \lambda_1^n. \end{aligned}$$

We fix $\lambda_2 \in (\lambda_1, 1)$ and $C_2 = \sup\{(C_1^3/\lambda_1)\, n(\lambda_1/\lambda_2)^n : n \ge 1\}$ and conclude that

$$\operatorname{var} g_\eta^{(n)} \le C_2 \lambda_2^n \quad \text{for all } \eta \in \mathcal{P}^{(n)} \text{ and all } n \ge 1. \tag{3.1}$$

3.1. Absolutely continuous invariant measures.

Let $\mathcal{L}$ denote the Perron-Frobenius operator of f, defined by

$$\mathcal{L}\varphi = \sum_{\eta \in \mathcal{P}^{(1)}} \left[(g_\eta \varphi) \circ (f|\eta)^{-1}\right] \chi_{f(\eta)} = \sum_{\eta \in \mathcal{P}^{(1)}} \left[\frac{\varphi}{|Df|} \circ (f|\eta)^{-1}\right] \chi_{f(\eta)}.$$

Note that $(f|\eta)\colon \eta \to f(\eta)$ is strictly monotone. The product $\left[(g_\eta\varphi) \circ (f|\eta)^{-1}\right] \chi_{f(\eta)}$ is understood to be zero outside $f(\eta)$. The change of variables formula yields, once more,

$$\int (\mathcal{L}\varphi)\psi\, dm = \int \varphi(\psi \circ f)\, dm.$$

Proposition 3.1. *There are $C_0 > 0$ and $\lambda_0 < 1$ such that*

$$\operatorname{var}(\mathcal{L}^n \varphi) \le C_0 \lambda_0^n \operatorname{var}(\varphi) + C_0 \int |\varphi| dm$$

for every $n \ge 1$ and every function φ with bounded variation.

Proof: It follows from the definition that

$$\mathcal{L}^n \varphi = \sum_{\eta \in \mathcal{P}^{(n)}} (g_\eta^{(n)} \varphi) \circ (f^n|\eta)^{-1} \chi_{f^n(\eta)}, \text{ for each } n \ge 1.$$

Using properties (v1), (v2), (v5), we find

$$\operatorname{var}(\mathcal{L}^n\varphi) \le \sum_{\eta \in \mathcal{P}^{(n)}} \left[\left(\operatorname{var} g_\eta^{(n)} + 2 \sup g_\eta^{(n)}\right) \cdot \sup |(\varphi|\eta)| + \sup g_\eta^{(n)} \cdot \operatorname{var}(\varphi|\eta)\right],$$

note that $\operatorname{var} \chi_{f^n(\eta)} = 2$. Now use the following consequence of the mean value theorem and (v4) (see also Remark 3.1 below):

$$\sup |(\varphi|\eta)| \le \operatorname{var} |\varphi|\eta| + \frac{1}{m(\eta)} \int_\eta |\varphi| dm \le \operatorname{var}(\varphi|\eta) + \frac{1}{m(\eta)} \int |\varphi| dm. \tag{3.2}$$

Replacing above,

$$\begin{aligned}\operatorname{var}(\mathcal{L}^n\varphi) &\le \sum_{\eta\in\mathcal{P}^{(n)}}\left[(C_2\lambda_2^n+2C_1\lambda_1^n)\cdot\left(\operatorname{var}(\varphi|\eta)+\frac{1}{m(\eta)}\int|\varphi|dm\right)+C_1\lambda_1^n\operatorname{var}(\varphi|\eta)\right]\\ &\le 4C_2\lambda_2^n\sum_{\eta\in\mathcal{P}^{(n)}}\operatorname{var}(\varphi|\eta)+3C_2\lambda_2^2\sum_{\eta\in\mathcal{P}^{(n)}}\frac{1}{m(\eta)}\int|\varphi|dm\\ &\le C_3\lambda_3^n\operatorname{var}\varphi+K_3(n)\int|\varphi|dm,\end{aligned}$$

where $C_3 = 4C_2$, $\lambda_3 = \lambda_2$, and $K_3(n) = 3C_2\lambda_2^n(\#\mathcal{P}^{(n)})\sup\{m(\eta)^{-1} : \eta\in\mathcal{P}^{(n)}\}$. This is nearly what we want, but we must remove the dependence of K_3 on n. For that we fix $N\ge 1$ such that $C_3\lambda_3^N \le 1/2$ and denote $\widehat{K} = \max\{K_3(n) : 1\le n\le N\}$. Then, given any $n\ge 1$, we write $n = qN + r$ with $q\ge 0$ and $1\le r\le N$. Successive use of the previous bound gives

$$\begin{aligned}\operatorname{var}(\mathcal{L}^n\varphi) &\le \widehat{K}\int|\mathcal{L}^{n-N}\varphi|dm + C_3\lambda_3^N\operatorname{var}(\mathcal{L}^{n-N}\varphi)\\ &\le \widehat{K}\int|\varphi|dm+\frac{1}{2}\operatorname{var}(\mathcal{L}^{n-N}\varphi)\\ &\le \left(1+\cdots+\frac{1}{2^{q-1}}\right)\widehat{K}\int|\varphi|dm+\frac{1}{2^q}\operatorname{var}(\mathcal{L}^r\varphi)\\ &\le \left(1+\cdots+\frac{1}{2^q}\right)\widehat{K}\int|\varphi|dm+\frac{1}{2^q}C_3\lambda_3^r\operatorname{var}\varphi\end{aligned}$$

(we have used the fact that $|\mathcal{L}\psi|\le\mathcal{L}|\psi|$, hence $\int|\mathcal{L}\psi|dm\le\int|\psi|dm$ for all ψ). We conclude the proof by choosing $C_0\ge\max\{2\widehat{K}, C_3\}$ and $\max\{2^{-1/N},\lambda_3\}\le\lambda_0<1$. □

Remark 3.1. It may happen that some of the intervals $\eta\in\mathcal{P}^{(n)}$ consist of a single point, $\eta=[x,x]$, in which case (3.2) does not make sense, but this may be bypassed as follows. Let $\mathcal{Q}$ be the covering of $M = I, S^1$ obtained by replacing each $\eta=[x,x]\in\mathcal{P}^{(n)}$ by $\xi=[x-\varepsilon,x+\varepsilon]\cap M$, for some small $\varepsilon>0$, and leaving all the other $\eta\in\mathcal{P}^{(n)}$ unchanged. Then $m(\xi)>0$ for ever $\xi\in\mathcal{Q}$ and so the same arguments as before yield

$$\operatorname{var}(\mathcal{L}^n\varphi)\le 4C_2\lambda_2^n\sum_{\xi\in\mathcal{Q}}\operatorname{var}(\varphi|\xi)+K_3(n)\int|\varphi|\,dm,$$

as long as we replace $\eta\in\mathcal{P}^{(n)}$ by $\xi\in\mathcal{Q}$ in the definition of $K_3(n)$. Choosing $\varepsilon>0$ small enough we ensure that every point $x\in M$ belongs in at most two intervals $\xi\in\mathcal{Q}$. Then $\sum_{\xi\in\mathcal{Q}}\operatorname{var}(\varphi|\xi)\le 2\operatorname{var}\varphi$ and so the previous estimates remain valid, with $C_3 = 8C_2$.

Corollary 3.2. *Let $C(a)$ denote the cone of functions $\varphi\colon M\to\mathbb{R}$ such that $\varphi(x)\ge 0$ for all $x\in M$ and $\operatorname{var}\varphi\le a\int\varphi\,dm$. Then, if a is large enough, there is $N\ge 1$ such that*

$$\mathcal{L}^N(C(a))\subset C(a/2).$$

Proof: Take $\lambda = 1/2$ and $N \geq 1$ such that $C_0\lambda_0^N \leq 1/4$. Then, for $\varphi \in C(a)$,

$$\operatorname{var}(\mathcal{L}^n\varphi) \leq \frac{1}{4}\operatorname{var}\varphi + C_0\int\varphi\,dm \leq \left(\frac{a}{4}+C_0\right)\int\varphi\,dm \leq \frac{a}{2}\int\varphi\,dm,$$

as long as $a \geq 2C_0$. □

In the sequel we use the following compactness statement

Lemma 3.3 (Helly's theorem). *Let $\psi_n : M \to \mathbb{R}$, $n \geq 1$, be a sequence of functions on M and assume that there are constants $K_1 > 0$ and $K_2 > 0$ such that $\sup\psi_n \leq K_1$ and $\operatorname{var}\psi_n \leq K_2$ for every $n \geq 1$. Then there exists a subsequence $(\psi_{n_k})_k$ and a function $\psi_0 : M \to \mathbb{R}$ with $\sup\psi_0 \leq K_1$ and $\operatorname{var}\psi_0 \leq K_2$ such that $(\psi_{n_k})_k$ converges to ψ_0 as $k \to \infty$, m-almost everywhere and in $L^1(m)$.*

Proof: Clearly, it suffices to consider the case $M = [0,1]$. Write $\psi_n^+(x) = \operatorname{var}(\psi_n|[0,x])$ and $\psi_n^- = \psi_n^+ - \psi_n$. Then $(\psi_n^\pm)_n$ are uniformly bounded sequences of nondecreasing functions. We choose $(n_k)_k$ so that $\psi_{n_k}^\pm(q)$ converges to some real number $\psi_0^\pm(q)$ as $k \to \infty$, for every rational $q \in [0,1]$. Clearly, $\psi_0^\pm(q_1) \leq \psi_0^\pm(q_2)$ whenever $q_1 \leq q_2$. Then we extend $\psi_0^\pm$ to nondecreasing functions in the whole $[0,1]$ by setting

$$\psi_0^\pm(x) = \inf\{\psi_0^\pm(q) : q \in [x,1] \cap \mathbb{Q}\}.$$

We claim that $\psi_{n_k}^\pm(x)$ converges to $\psi_0^\pm(x)$ as $k \to \infty$, for every continuity point x of $\psi_0^\pm$ (a cocountable set). Indeed, given any such x and any $\delta > 0$, we may fix rational numbers $q_1 \leq x \leq q_2$ such that

$$\psi_0^\pm(x) - \delta \leq \psi_0^\pm(q_1) \leq \psi_0^\pm(x) \leq \psi_0^\pm(q_2) \leq \psi_0^\pm(x) + \delta.$$

Then, for every sufficiently large k,

$$\psi_0^\pm(x) - 2\delta \leq \psi_0^\pm(q_1) - \delta \leq \psi_{n_k}^\pm(q_1) \leq \psi_{n_k}^\pm(x) \leq \psi_{n_k}^\pm(q_2) \leq \psi_0^\pm(q_2) + \delta \leq \psi_0^\pm(x) + 2\delta$$

and this proves the claim. Next, let $\tilde{\psi}_0^\pm$ be right-continuous functions coinciding with $\psi_0^\pm$ at every point of continuity of $\psi_0^\pm$, and define $\psi_0 = \tilde{\psi}_0^+ - \tilde{\psi}_0^-$. It follows that ψ_{n_k} converges to ψ_0 except, possibly, on a countable set of points E. In particular, $\psi_{n_k} \to \psi_0$ m-almost everywhere and in $L^1(m)$. Finally,

$$|\psi_0(a)| = \lim_k |\psi_{n_k}(a)| \leq K_1 \qquad \text{and}$$

$$\sum_{j=1}^{s} |\psi_0(a_j) - \psi_0(b_j)| = \lim_k \sum_{j=1}^{s} |\psi_{n_k}(a_j) - \psi_{n_k}(b_j)| \leq \sup_k \operatorname{var}\psi_{n_k} \leq K_2,$$

for every a and $a_1 \leq b_1 \leq \cdots \leq a_s \leq b_s$ in $[0,1] \setminus E$. Since ψ_0 is right-continuous, this proves that $\sup\psi_0 \leq K_1$ and $\operatorname{var}\psi_0 \leq K_2$. □

Corollary 3.4. *The map f has some absolutely continuous invariant probability measure μ_0. Moreover, if μ is any such measure then $\mu = \varphi m$, where φ has bounded variation.*

Proof: Let $\varphi_n = \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j 1$. Proposition 3.1 implies that $(\varphi_n)_n$ has uniformly bounded variation:

$$\operatorname{var}\varphi_n \le \frac{1}{n}\sum_{j=0}^{n-1}\operatorname{var}(\mathcal{L}^j 1) \le \frac{1}{n}\sum_{j=0}^{n-1} C_0\int dm = C_0 .$$

In addition,

$$\int \varphi_n\,dm = \frac{1}{n}\sum_{j=0}^{n-1}\int \mathcal{L}^j 1\,dm = \frac{1}{n}\sum_{j=0}^{n-1}\int dm = 1 \text{ for all } n\ge 1$$

and so $(\varphi_n)_n$ is also uniformly bounded: cf. (3.2),

$$\sup\varphi_n \le \operatorname{var}\varphi_n + \int\varphi_n\,dm \le C_0+1.$$

Therefore, by Helly's theorem, there exists a subsequence $(\varphi_{n_k})_k$ converging in $L^1(m)$ to some function φ_0 with bounded variation. Write

$$\mathcal{L}\varphi_{n_k} = \varphi_{n_k} + \frac{1}{n_k}(\mathcal{L}^{n_k}1-1) \tag{3.3}$$

and note that $\|\mathcal{L}^n 1\|_1 = 1$ for all $n\ge 1$. Moreover, $\mathcal{L}$ is a bounded operator in $L^1(m)$:

$$\int|\mathcal{L}\varphi-\mathcal{L}\psi|dm \le \int\mathcal{L}(|\varphi-\psi|)dm = \int|\varphi-\psi|dm.$$

Hence, passing to the limit in (3.3) we get $\mathcal{L}\varphi_0=\varphi_0$, which means that $\mu_0=\varphi_0\,m$ is an invariant measure for f.

Next, let μ be an arbitrary f-invariant absolutely continuous probability measure. Then $\mu=\psi\,m$ with $\|\psi\|_1=1$ and $\mathcal{L}\psi=\psi$. We want to prove that ψ coincides Lebesgue almost everywhere with some bounded variation function φ. Let $(\psi_l)_l$ be a sequence of functions with bounded variation converging to ψ in $L^1(m)$. It is no restriction to suppose $\|\psi_l\|_1\le 2$ for all l and we do so. Then Proposition 3.1 yields

$$\operatorname{var}\left(\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\psi_l\right) \le \frac{1}{n}\sum_{j=0}^{n-1}C_0\left(\lambda_0^j\operatorname{var}\psi_l+\int|\psi_l|\,dm\right)\to C_0\int|\psi_l|\,dm\le 2C_0$$

and we also have $\|\frac{1}{n}\sum_{j=0}^{n}\mathcal{L}^j\psi_l\|_1\le\|\psi_l\|_1\le 2$. Hence, by the same arguments as before, for each fixed l the sequence $\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}^j\psi_l$ satisfies the assumptions of Helly's theorem. It follows that there exists a function $\bar{\psi}_l$ and a subsequence $(n_k)_k$ such that $\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}^j\psi_l$ converges to $\bar{\psi}_l$ in $L^1(m)$. Moreover, $\operatorname{var}\bar{\psi}_l \le 2C_0$ and $\|\bar{\psi}_l\|_1 \le 2$. These two last facts ensure that we may apply Helly's theorem also to $(\bar{\psi}_l)_l$, to conclude that some subsequence $(\bar{\psi}_{l_j})_j$ converges in $L^1(m)$ to a function φ with $\operatorname{var}\varphi \le 2C_0$. On the other hand,

$$\|\bar{\psi}_l - \psi\|_1 = \lim \|\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}^j(\psi_l - \psi)\|_1 \le \|\psi_l - \psi\|_1$$

implies that $\bar{\psi}_l$ converges in $L^1(m)$ to ψ. Thus, $\psi = \varphi$ Lebesgue almost everywhere, and so $\mu = \psi m = \varphi m$. □

Corollary 3.5. *f has only finitely many ergodic absolutely continuous invariant probability measures.*

Proof: Let μ be any such measure. Corollary 3.4 implies that there is some open interval J such that $\varphi = d\mu/dm > 0$ on J. Then the same must be true on $f^n(J)$, for every $n \ge 0$. In particular, m and μ are equivalent measures on each $f^n(J)$. As a consequence, Lebesgue almost every $x \in f^n(J)$ is generic for μ: the time-averages $n^{-1}\sum_{j=0}^{n-1}\varphi(f^j(x))$ converge to $\int \varphi\, d\mu$, for every continuous function φ. On the other hand, we must have $a_i \in f^{\bar{n}}(J)$ for some $\bar{n} \ge 0$ and some $i \in \{1, \cdots, l-1\}$ (respectively, $i \in \{1, \cdots, l\}$ in the case of circle maps). Otherwise, the length of $f^n(J)$ would be unbounded as $n \to \infty$ (because the map is expanding on each monotonicity interval) and this would, obviously, contradict the fact that $f^n(J) \subset I$. Now, the previous comments imply that intervals $f^{\bar{n}}(J)$ obtained in this way, for different ergodic measures, are necessarily disjoint. Thus, there are at most $l-1$ (respectively, l) such measures. □

Example 3.2. For tent maps and Lorenz-like maps, recall Figure 3.1, one may take $l = 2$. It follows that every absolutely continuous invariant measure μ_0 is ergodic, and so there is only one such measure. Indeed, suppose there is some measurable set A with $f^{-1}(A) = A$ and $0 < \mu_0(A) < 1$. Then

$$\mu_1(B) = \frac{\mu_0(B\cap A)}{\mu_0(A)} \quad \text{and} \quad \mu_2(B) = \frac{\mu_0(B\cap A^c)}{\mu_0(A^c)}$$

define two absolutely continuous invariant probability measures. The corresponding densities $\varphi_i = d\mu_i/dm$, $i = 1, 2$, are functions of bounded variation, and this implies that both A and A^c have nonempty interior. It follows that c_1 belongs in both A and A^c, a contradiction.

On the other hand, the following simple example shows that in general one can not expect μ_0 to be unique, nor $\mathcal{L}^N(C(a))$ to have finite diameter in $C(a)$, unless f satisfies some condition of dynamical indivisibility. Let $f : [0,1] \to [0,1]$ be given by $f(x) = (1/2) - |1/2 - 2x|$ for $x \in [0, 1/2]$ and $f(x) = 1 - |3/2 - 2x|$ for $x \in (1/2, 1]$. Then both $\chi_{[0,1/2]}\, m$ and $\chi_{[1/2,1]}\, m$ are invariant measures for f, absolutely continuous with respect to Lebesgue measure.

To ensure uniqueness of the absolutely continuous invariant measure, as well as the mixing properties to be studied in the forthcoming section, we introduce our next assumption on the expanding map f:

(E3) (topological mixing) There is an interval $I_* \subset I$ such that $f(I_*) = I_*$, every orbit $f^n(x)$, $x \in (0,1)$, eventually enters I_*, and $f|I_*$ is topologically mixing: for each interval $J \subset I_*$ there is $n \geq 1$ such that $f^n(J) = I_*$.

In the case of circle maps we simply require that for every subinterval J of $I_* = S^1$ there be some $n \geq 1$ such that $f^n(J) = S^1$.

Example 3.3. Let $f : I \to I$ be a tent map with $c = 1/2$ and $|Df(x)| = \sigma > \sqrt{2}$ for all $x \neq c$. Take $I_* = [f^2(c), f(c)]$. The first two conditions in (E3) are easy. We prove that $f|I_*$ is topologically mixing. Let $J \subset I_*$ be an interval. First, we claim that $f^n(J)$ must eventually contain the fixed point $p > c$ of $f|I_*$. Indeed, otherwise one would be able to construct a sequence of intervals

$$J = J_0 \supset J_1 \supset \cdots \supset J_n \supset \cdots$$

in the following way. If $f^{n-1}(J_{n-1})$ does not contain c, take $J_n = J_{n-1}$. Note that $m(f^n(J_n)) = \sigma\, m(f^{n-1}(J_{n-1}))$. If $f^{n-1}(J_{n-1})$ does contain c, then take $J_n \subset J_{n-1}$ such that $f^{n-1}(J_n)$ coincides with the largest of the two intervals $f^{n-1}(J_{n-1}) \cap [f^2(c), c]$ or $f^{n-1}(J_{n-1}) \cap [c, f(c)]$. Then $m(f^n(J_n)) \geq (\sigma/2) m(f^{n-1}(J_{n-1}))$. Moreover, $f^n(J_n)$ can not contain c, since we suppose that $p \notin f^{n+1}(J_n)$. Therefore, $J_{n+1} = J_n$ and so

$$m(f^{n+1}(J_{n+1})) \geq \sigma m(f^n(J_n)) \geq \frac{\sigma^2}{2} m(f^{n-1}(J_{n-1})).$$

As we suppose $\sigma^2 > 2$, it follows that the sequence $m(f^n(J_n))$ is unbounded, which is a contradiction. This proves that $p \in f^{n_1}(J_{n_1})$ for some $n_1 \geq 0$. Then, $p \in f^n(J)$ for every $n \geq n_1$ and, by expansivity, $[c,p] \subset f^{n_2}(J)$ for some $n_2 \geq n_1$. It follows that $[f^3(c), f(c)] \subset f^{n_2+3}(J)$. Now, it is easy to check that $f^3(c) < p$ for all $\sigma > \sqrt{2}$. If $f^3(c) \leq c$, we get $[f^2(c), f(c)] \subset f^{n_2+4}(J)$ and the claim is proved. Otherwise, there must be some odd number $k > 3$ such that $f^k(c) < c$ and $[f^k(c), f(c)] \subset f^{n_2+k}(J)$. Then $[f^2(c), f(c)] \subset f^{n_2+k+1}(J)$ and the claim is proved also in this case. We have proved that (E3) holds for tent maps as above.

In this context $\sigma > \sqrt{2}$ is also necessary for topological mixing. A similar argument shows that Lorenz-like maps with $|Df(x)| \geq \sigma > \sqrt{2}$ for all $x \neq 1/2$ satisfy (E3) for $I_* = I$. We also leave it to the reader to check that the Gauss map $f(x) = 1/x - [1/x]$ is topologically mixing on $I_* = I$.

Corollary 3.6. *If f is topologically mixing then it admits a unique absolutely continuous invariant probability measure μ_0. In addition, μ_0 is ergodic and its support coincides with I_*.*

Proof: Let φ_1, φ_2 be fixed points of the operator $\mathcal{L}$, with $\int \varphi_1\, dm = 1 = \int \varphi_2\, dm$. Let $X_1 = \{x : \varphi_1(x) \geq \varphi_2(x)\}$ and $X_2 = \{x : \varphi_1(x) < \varphi_2(x)\}$, and consider the pair of absolutely continuous measures $\mu_1 = (\varphi_1 - \varphi_2)\chi_{X_1} m$ and $\mu_2 = (\varphi_2 - \varphi_1)\chi_{X_2} m$. Since $\varphi_1 - \varphi_2$ is a fixed point of $\mathcal{L}$,

$$\int_{X_1} (\varphi_1 - \varphi_2)\, dm = \int_{X_1} \mathcal{L}(\varphi_1 - \varphi_2)\, dm = \int_{f^{-1}(X_1)} (\varphi_1 - \varphi_2)\, dm$$
$$\leq \int_{f^{-1}(X_1)\cap X_1} (\varphi_1 - \varphi_2)\, dm \leq \int_{X_1} (\varphi_1 - \varphi_2)\, dm.$$

This implies that $m(f^{-1}(X_1) \cap X_1) = m(X_1) = m(f^{-1}(X_1))$, and an analogous statement for X_2 can be proved in precisely the same way. Altogether, this ensures that both μ_1 and μ_2 are invariant under f.

Then, by Corollary 3.4, each density $d\mu_i/dm$, $i = 1, 2$, is a function with bounded variation. It follows that either $d\mu_i/dm$ is almost everywhere zero, or else its support contains some interval $J_i \subset I_*$. Furthermore, in this last case the support must contain the whole I_*, as a consequence of the assumption of topological mixing (E3). Since the supports of μ_1 and μ_2 are disjoint, we conclude that one of these two measures must be identically null, which means that either $\varphi_1 \geq \varphi_2$ Lebesgue almost everywhere or $\varphi_2 \geq \varphi_1$ Lebesgue almost everywhere. In any case, we have $\varphi_1 = \varphi_2$ Lebesgue almost everywhere, because these two functions have the same average value 1, and this proves the uniqueness statement.

Ergodicity is now an easy consequence. Let $A \subset I^*$ be any Borel subset such that $\mu_0(A) > 0$ and $f(A) = A$ (this is more general than $f^{-1}(A) = A$). Now let us define $\mu(B) = \mu_0(B \cap A)/\mu_0(A)$, for every Borel subset B. Then μ is an absolutely continuous f-invariant probability measure:

$$\mu(f^{-1}(B)) = \frac{\mu_0(f^{-1}(B) \cap A)}{\mu_0(A)} = \frac{\mu_0(f^{-1}(B \cap A))}{\mu_0(A)} = \frac{\mu_0(B \cap A)}{\mu_0(A)} = \mu(B).$$

For the second equality note that $f^{-1}(B \cap A) = f^{-1}(B) \cap f^{-1}(A)$, and the assumption on A implies $A \subset f^{-1}(A)$ and $\mu_0(f^{-1}(A) \setminus A) = 0$. Thus, $\mu = \mu_0$ and so $\mu_0(A) = 1$. □

3.2. Decay of correlations and central limit theorem.

Here we show that, under assumptions (E1)-(E3), the system (f, μ_0) has exponential decay of correlations and satisfies a central limit theorem in the space of functions with bounded variation. Moreover, the absolutely continuous invariant measure μ_0 is exact. Actually, we take two different approaches to the problem. For the first one we need the further assumption that $|Df|$ is bounded, in other words, there is $\delta > 0$ such that

$$g_\eta(x) \geq \delta \quad \text{for all } x \in \eta \text{ and } \eta \in \mathcal{P}^{(1)}. \tag{3.4}$$

This condition allows for substantial simplification of the arguments, but is not satisfied by some important examples, such as the Lorenz-like maps. That is why in the second part of this section we present a more general approach, requiring only our standing hypotheses (E1), (E2), (E3).

For the time being we suppose that (3.4) holds. On the other hand, in this case we need not assume either the existence or the uniqueness of μ_0, our arguments providing new proofs of these facts. Let I_* be as in (E3). We redefine $C(a)$ to be the cone of functions $\varphi: I_* \to \mathbb{R}$ such that $\varphi(x) \geq 0$ for all $x \in I_*$ and $\operatorname{var}_{I_*} \varphi \leq a \int_{I_*} \varphi \, dm$. This is a slight modification with respect to the notations in Corollary 3.2, but the statement and proof of the corollary remain valid. Let $\theta = \theta_a$ be the projective metric associated to $C(a)$. We also replace the partitions $\mathcal{P}^{(i)}$, $i \geq 1$, by their restrictions to I_*, which we denote in the same way, since no confusion can arise from doing so.

Proposition 3.7. *There is $k \geq 1$ such that $D = \sup\{\theta(\psi_1, \psi_2) : \psi_1, \psi_2 \in \mathcal{L}^{kN}(C(a))\}$ is finite.*

Proof: We begin by deriving a useful bound for $\theta(\psi_1, \psi_2)$, $\psi_1, \psi_2 \in C(a/2)$. It is no restriction to suppose $\int \psi_1 \, dm = \int \psi_2 \, dm = 1$ and we do so. Then we have

$$\operatorname{var}(\psi_2 - \frac{1}{3}\psi_1) \leq \operatorname{var}\psi_2 + \frac{1}{3}\operatorname{var}\psi_1 \leq \frac{a}{2}\int \psi_2 \, dm + \frac{a}{6}\int \psi_1 \, dm$$
$$\leq \frac{2a}{3} \leq a \int (\psi_2 - \frac{1}{3}\psi_1) \, dm.$$

Combined with $(\psi_2 - t\psi_1)(x) \geq 0 \Leftrightarrow t \leq (\psi_2/\psi_1)(x)$, this yields

$$\alpha(\psi_1, \psi_2) \geq \inf\left\{\frac{1}{3}, \frac{\psi_2}{\psi_1}(x) : x \in I_*\right\}.$$

Analogously,

$$\beta(\psi_1, \psi_2) \leq \sup\left\{3, \frac{\psi_2}{\psi_1}(x) : x \in I_*\right\}.$$

Now we claim that there are $k \geq 1$ and $\gamma > 0$ such that

$$\gamma \int \psi \, dm \leq \inf \psi \leq \sup \psi \leq \frac{1}{\gamma}\int \psi \, dm \tag{3.5}$$

for all $\psi \in \mathcal{L}^{kN}(C(a))$. Since $\mathcal{L}^{kN}(C(a)) \subset C(a/2)$, it follows that

$$\alpha(\psi_1, \psi_2) \geq \min\left\{\frac{1}{3}, \gamma^2\right\} \qquad \text{and} \qquad \beta(\psi_1, \psi_2) \leq \max\left\{3, \frac{1}{\gamma^2}\right\}$$

for all (normalized) $\psi_1, \psi_2 \in \mathcal{L}^{kN}(C(a))$. Choosing $\gamma \leq 1/\sqrt{3}$, we find $D \leq 4\log(1/\gamma)$.

Thus, we are left to prove (3.5). Observe that $\sup\{m(\eta) : \eta \in \mathcal{P}^{(n)}\}$ goes to zero as $n \to \infty$, because of (E2). We fix $q \geq 1$ such that $m(\eta) \leq 1/2a$ for all $\eta \in \mathcal{P}^{(qN)}$. Next, we choose $j \geq 0$ such that $f^{(j+q)N}(\eta) = I_*$ for every $\eta \in \mathcal{P}^{(qN)}$: existence of such a j is a consequence of (E3). Finally, we take $k = q + j$. Let $\psi = \mathcal{L}^{kN}\varphi$ with $\varphi \in C(a)$. The mean value theorem implies

$$\operatorname{var}(\varphi|\eta) \geq \frac{1}{m(\eta)}\int_\eta \varphi \, dm - \inf(\varphi|\eta) \geq 2a\left(\int_\eta \varphi \, dm - m(\eta)\inf(\varphi|\eta)\right)$$

for each $\eta \in \mathcal{P}^{(qN)}$. Adding over all η,

$$\operatorname{var}\varphi \geq 2a \int \varphi\, dm - 2a \sum_{\eta} m(\eta) \inf(\varphi|\eta) \geq 2a \int \varphi\, dm - 2a \max_{\eta} \inf(\varphi|\eta)$$

and then, using $\varphi \in C(a)$,

$$\max_{\eta} \inf(\varphi|\eta) \geq \frac{1}{2} \int \varphi\, dm.$$

In other words, $\inf(\varphi|\eta_0) \geq \frac{1}{2} \int \varphi\, dm$ for some $\eta_0 \in \mathcal{P}^{(qN)}$. Then, for any $y \in I_*$,

$$\begin{aligned} \psi(y) = (\mathcal{L}^{kN} \varphi)(y) &= \sum_{\zeta} (g_{\zeta}^{(kN)} \varphi) \circ (f^{kN}|\zeta)^{-1}(y) \\ &\geq \sum_{\zeta \subset \eta_0} (g_{\zeta}^{(kN)} \varphi) \circ (f^{kN}|\zeta)^{-1}(y), \end{aligned}$$

where the first sum is over all $\zeta \in \mathcal{P}^{(kN)}$ with $y \in f^{kN}(\zeta)$, and the second one is restricted to those of such ζ which are contained in η_0. Observe that, since $f^{kN}(\eta_0)$ covers I_*, this second sum involves at least one term. On the other hand, for any such term

$$(g_{\zeta}^{(kN)} \varphi)((f^{kN}|\zeta)^{-1}(y)) \geq \delta^{kN} \frac{1}{2} \int \varphi\, dm$$

by (3.4) and the choice of η_0. Altogether, we have shown that

$$\inf \psi \geq \frac{1}{2} \delta^{kN} \int \varphi\, dm = \frac{1}{2} \delta^{kN} \int \psi\, dm.$$

Moreover,

$$\sup \psi \leq \int \psi\, dm + \operatorname{var} \psi \leq (a+1) \int \psi\, dm,$$

and so we completed the proof of the claim. □

Let C_+ be the cone of nonnegative functions $\varphi\colon M \to \mathbb{R}$. The corresponding projective metric θ_+ is given by

$$\theta_+(\varphi_1, \varphi_2) = \log \frac{\sup(\varphi_2/\varphi_1)}{\inf(\varphi_2/\varphi_1)}$$

and satisfies $\theta_+(\varphi_1, \varphi_2) \leq \theta(\varphi_1, \varphi_2)$ for all $\varphi_1, \varphi_2 \in C(a)$. Proposition 3.7 implies that $\mathcal{L}^{kN}$ is a Λ-contraction with respect to the metric θ, with $\Lambda = 1 - e^{-D}$. As a consequence, the sequence $(\mathcal{L}^n 1)_n$ is Cauchy with respect to θ and so also with respect to θ_+. Moreover, recall (3.5), there is $\gamma > 0$ such that

$$\gamma \leq \inf \mathcal{L}^n 1 \leq \sup \mathcal{L}^n 1 \leq \frac{1}{\gamma} \quad \text{for all large enough } n \geq 1.$$

Hence, by the same argument as in Proposition 2.6, $(\mathcal{L}^n 1)_n$ converges uniformly to some function $\tilde{\varphi}_0 \in C(a)$. Of course, $\frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j 1$ must also converge to $\tilde{\varphi}_0$ and this proves that $\tilde{\varphi}_0$ coincides with the function φ_0 constructed in Corollary 3.4.

Proposition 3.8. *Given φ with bounded variation and ψ an $L^1(m)$ function on M, there is $C = C(\varphi,\psi) > 0$ such that $\left|\int(\psi\circ f^n)\varphi\,dm - \int\psi\,d\mu_0\int\varphi\,dm\right| \le C\Lambda^n$ for all $n \ge 0$.*

Proof: This is very similar to the proof of Proposition 2.7. Assume first that $\varphi \in C(a)$. It is no restriction to suppose $\int\varphi\,dm = 1$. Then

$$\left|\int(\psi\circ f^n)\varphi\,dm - \int\psi\,d\mu_0\right| = \left|\int\psi(\mathcal{L}^n\varphi - \varphi_0)dm\right| \le \|\psi\|_1 \sup|\mathcal{L}^n\varphi - \varphi_0|.$$

Moreover,

$$\begin{aligned}\sup|\mathcal{L}^n\varphi - \varphi_0| &\le \sup|\varphi_0|\sup\left|\frac{\mathcal{L}^n\varphi}{\varphi_0} - 1\right| \le \frac{1}{\gamma}\left(e^{\theta_+(\mathcal{L}^n\varphi,\varphi_0)} - 1\right)\\ &\le \frac{1}{\gamma}\left(e^{C'\Lambda^n} - 1\right) \le C''\Lambda^n\end{aligned}$$

where C', C'' are independent of φ, ψ, or n. In the third inequality we use

$$\begin{aligned}\theta_+(\mathcal{L}^n\varphi,\varphi_0) &\le \theta(\mathcal{L}^{n-kN}(\mathcal{L}^{kN}\varphi),\mathcal{L}^{n-kN}(\mathcal{L}^{kN}\varphi_0)) \le C'''\Lambda^{n-kN}\,\theta(\mathcal{L}^{kN}\varphi,\mathcal{L}^{kN}\varphi_0)\\ &\le C'''\,\Lambda^{n-kN}\,D \le C'\,\Lambda^n.\end{aligned}$$

This completes the proof when $\varphi \in C(a)$.

Now, given a general function φ with bounded variation, we write $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+(x) = \mathrm{var}(\varphi|[0,x]) + B$, where B is some large positive constant. Note that $\varphi^\pm$ are nondecreasing functions with $\mathrm{var}\,\varphi^\pm \le \mathrm{var}\,\varphi + |\varphi(1) - \varphi(0)| \le 2\,\mathrm{var}\,\varphi$. By choosing $B \ge \max\{2|\varphi(0)|, (4/a)\,\mathrm{var}\,\varphi\}$, we get $\inf\varphi_\pm = \varphi^\pm(0) \ge B/2$ and

$$\mathrm{var}\,\varphi^\pm \le 2\,\mathrm{var}\,\varphi \le a\frac{B}{2} \le a\int\varphi^\pm\,dm.$$

Then the proposition holds for $\varphi_B^\pm$ and so, by linearity, it holds for φ. □

Remark 3.2. Let us write $\|\varphi\|_{BV} = \int|\varphi|\,dm + \mathrm{var}\,\varphi$. The constant $C(\varphi,\psi)$ obtained in the proof above has the form

$$C(\varphi,\psi) = \text{ const } \|\psi\|_1(\sup|\varphi| + \mathrm{var}\,\varphi) \le \text{ const } \|\psi\|_1\,\|\varphi\|_{BV},$$

where the last equality is a consequence of (3.2):

$$\int|\varphi|\,dm + \mathrm{var}\,\varphi \le \sup|\varphi| + \mathrm{var}\,\varphi \le \int|\varphi|\,dm + 2\,\mathrm{var}\,\varphi \le 2\left(\int|\varphi|\,dm + \mathrm{var}\,\varphi\right).$$

Using the same arguments as in Chapter 2 (cf. Proposition 2.7 and Corollary 2.8), one concludes that (f,μ_0) has exponential decay of correlations

$$\left|\int(\varphi_1\circ f^n)\varphi_2\,d\mu_0 - \int\varphi_1\,d\mu_0\int\varphi_2\,d\mu_0\right| \le C(\varphi_1,\varphi_2)\Lambda^n \quad \text{for all } n \ge 0,$$

and a central limit property (see Corollary 2.10 and Section 2.5)

$$\mu_0\left\{x : \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\left(\varphi(f^j(x)) - \int \varphi\, d\mu_0\right) \in A\right\} \longrightarrow \int_A \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}\, dt \quad \text{as } n \to \infty,$$

for bounded variation functions φ_1, φ_2, φ. In this way one also proves that μ_0 is the unique f-invariant probability measure absolutely continuous with respect to Lebesgue measure, and that it is an exact measure. These conclusions will be recovered, in a more general setting, later on.

Now we explain how the previous conclusions can be derived without assuming the derivative of the map f to be bounded. The basic ingredient is to show that the spectrum of the transfer operator acting on the space of functions of bounded variation may be decomposed

(3.6) $\quad \operatorname{spec}(\mathcal{L}) = \{1\} \cup \Sigma_0, \quad$ with Σ_0 contained in a disc of radius $\Lambda < 1$.

As a direct consequence of this *quasi-compacity* property, we get that (f, μ_0) has exponential decay of correlations, with rate of decay not larger than Λ, and we also deduce a central limit theorem. A difference with respect to the previous approach is that we must prove exactness first, in order to obtain (3.6).

Let BV denote the space of functions with bounded variation, endowed with the complete norm

$$\|\varphi\|_{\mathrm{BV}} = \operatorname{var}\varphi + \int |\varphi|\, dm.$$

Proposition 3.1 ensures that $\mathcal{L}$ is a bounded operator on BV, actually, we get a stronger fact:

$$\begin{aligned}\|\mathcal{L}^n\varphi\|_{\mathrm{BV}} &= \operatorname{var}\mathcal{L}^n\varphi + \int |\mathcal{L}^n\varphi|\, dm \le C_0\lambda_0^n \operatorname{var}\varphi + (C_0+1)\int|\varphi|\, dm \\ &\le \max\{C_0\lambda_0^n, C_0+1\}\, \|\varphi\|_{\mathrm{BV}}, \quad \text{for every } n \ge 1.\end{aligned}$$

This implies that the norms $\|\mathcal{L}^n\|_{\mathrm{BV}}$ are uniformly bounded, and so the spectral radius of $\mathcal{L}$ is not larger than 1. Since we already constructed a fixed point φ_0 for $\mathcal{L}$, we conclude that the spectral radius is exactly equal to 1.

Next, we show that $\mathcal{L}$ is well approximated by compact (even finite rank) operators, in order to prove that its essential spectral radius is strictly smaller than 1. As before, let $\mathcal{P}^{(n)}$ be the partition of the ambient space into monotonicity intervals of f^n. We define linear operators $\pi_n : \mathrm{BV} \to \mathrm{BV}$, $n \ge 1$, by

$$\pi_n(\varphi)(x) = E(\varphi \mid \eta) = \frac{1}{m(\eta)}\int_\eta \varphi\, dm, \quad \text{for every } x \in \eta \text{ and } \eta \in \mathcal{P}^{(n)}.$$

Since $\mathcal{P}^{(n)}$ is a finite partition, the range of π_n is finite-dimensional, and so the same is true for the operator $\mathcal{L}^n \circ \pi_n$. On the other hand, cf. the proof of Proposition 3.1,

$$\begin{aligned}\operatorname{var}(\mathcal{L}^n\varphi - \mathcal{L}^n\pi_n\varphi) &= \operatorname{var} \sum_{\eta\in\mathcal{P}^{(n)}} \left(g_\eta^{(n)} \cdot (\varphi - E(\varphi|\eta))\right) \circ (f^n|\eta)^{-1}\chi_{f^n(\eta)} \\ &\leq \sum_{\eta\in\mathcal{P}^{(n)}} (\operatorname{var} g_\eta^{(n)} + 3\sup g_\eta^n)\operatorname{var}(\varphi|\eta) \\ &\leq (C_2\lambda_2^n + 3C_1\lambda_1^n)\operatorname{var}\varphi,\end{aligned}$$

note that $\sup_\eta |\varphi - E(\varphi|\eta)| \leq \operatorname{var}_\eta(\varphi - E(\varphi|\eta)) = \operatorname{var}_\eta \varphi$. Moreover,

$$\begin{aligned}\int |\mathcal{L}^n\varphi - \mathcal{L}^n\pi_n\varphi|\, dm \leq \sup|\mathcal{L}^n\varphi - \mathcal{L}^n\pi_n\varphi| \;&\leq\; \sum_{\eta\in\mathcal{P}^{(n)}} \sup g_\eta^{(n)} \cdot \sup_\eta |\varphi - E(\varphi|\eta)| \\ \leq C_1\lambda_1^n \sum_{\eta\in\mathcal{P}^{(n)}} \operatorname{var}(\varphi|\eta) \;&\leq\; C_1\lambda_1^n \operatorname{var}\varphi.\end{aligned}$$

It follows that $\|\mathcal{L}^n - \mathcal{L}^n\pi_n\|_{\mathrm{BV}} \leq (C_2\lambda_2^n + 4C_1\lambda_1^n) \leq C_3\lambda_3^n$, for $C_3 = C_2 + 4C_1$ and $\lambda_3 = \lambda_2 \geq \lambda_1$, and this implies that the essential spectral radius of $\mathcal{L}$ is at most $\lambda_3 < 1$.

As a consequence, we may write $\operatorname{spec}(\mathcal{L}) = \{\sigma_0 = 1, \sigma_1, \ldots, \sigma_k\} \cup \Sigma_0$, where each σ_i is an eigenvalue with finite multiplicity and norm 1, and Σ_0 is contained in some disc of radius $\Lambda < 1$. Here Λ may be either the essential spectral radius of $\mathcal{L}$, or the largest norm of an eigenvalue in the open unit disc. For the next step we need the following fact, whose proof we postpone:

(M) the measure μ_0 is mixing, and even exact, for the map f;

Recall that exactness means that every set in

$$\bigcap_{n=0}^{\infty} f^{-n}(\mathcal{B}), \qquad \mathcal{B} = \text{ Borel } \sigma\text{-algebra of } M,$$

has either or full measure. See [Ma87, Prop II.8.5] for a proof of the fact that every exact measure is mixing:

$$\mu\big(f^{-n}(A) \cap B\big) \to \mu(A)\mu(B) \quad \text{as } n \to +\infty \quad \text{for every } A, B \in \mathcal{B}.$$

Assuming (M) we can prove that $k = 0$, i.e., there are no eigenvalues with norm 1 other than $\sigma_0 = 1$. Indeed, suppose there is a nonzero function $\varphi_i \in \mathrm{BV}$ such that $\mathcal{L}\varphi_i = \sigma_i\varphi_i$ with $|\sigma_i| = 1$. Recall from Corollary 3.6 that the support of φ_0 is the whole I_*. Therefore, φ_i/φ_0 is defined almost everywhere on I_*, and $\varphi_i/\varphi_0 \in L^1(\mu_0)$

$$\int |\varphi_i/\varphi_0|\, d\mu_0 = \int |\varphi_i|\, dm \leq \sup|\varphi_i| < \infty.$$

Then the mixing property implies,

$$\int \varphi(\mathcal{L}^n\varphi_i)\,dm = \int (\varphi\circ f^n)\varphi_i\,dm = \int (\varphi\circ f^n)(\varphi_i/\varphi_0)\,d\mu_0 \to$$
$$\to \int \varphi\,d\mu_0 \int (\varphi_i/\varphi_0)\,d\mu_0 = \int \varphi\big(\varphi_0 \int \varphi_i\,dm\big)\,dm$$

for every $\varphi \in L^\infty(\mu_0)$. It follows that $\sigma_i^n\varphi_i = \mathcal{L}^n\varphi_i$ converges pointwise to $\varphi_0 \int \varphi_i\,dm$, but this can only happen if $\sigma_i = 1$ and $\varphi_i = \varphi_0 \int \varphi_i\,dm$. Thus, we have shown that 1 is indeed the only eigenvalue of $\mathcal{L}$ with norm 1, moreover, its eigenspace is 1-dimensional. Furthermore, its algebraic multiplicity must also be equal to 1: otherwise, there would be $\psi_0 \in \mathrm{BV}$ such that $\mathcal{L}^n\psi_0 = \varphi_0 + n\psi_0$ for every $n \geq 1$, contradicting the fact that the sequence $\|\mathcal{L}^n\|_{\mathrm{BV}}$, $n \geq 1$, is bounded.

Proposition 3.9 (quasi-compacity). *The spectrum of $\mathcal{L}$ acting on the space BV of functions with bounded variation may be written* $\mathrm{spec}(\mathcal{L}) = \{1\}\cup\Sigma_0$, *where 1 is a simple eigenvalue and Σ_0 is contained in a disc of radius $\Lambda < 1$. Moreover, the corresponding spectral splitting is given by $BV = \mathbb{R}\varphi_0 \oplus X_0$, with $X_0 = \{\varphi : \int \varphi\,dm = 0\}$.*

Proof: It only remains to check the last statement and this is easy. Just note that the splitting $\mathrm{BV} = \mathbb{R}\varphi_0 \oplus X_0$ is invariant under $\mathcal{L}$. Then $\mathrm{spec}(\mathcal{L}) = \{1\} \cup \mathrm{spec}(\mathcal{L}|X_0)$ and so we must have $\mathrm{spec}(\mathcal{L}|X_0) = \Sigma_0$. □

Corollary 3.10 (exponential decay of correlations). *Given any $\varphi_1, \varphi_2 \in BV$, there is $C = C(\varphi_1,\varphi_2) > 0$ such that*

$$\left|\int (\varphi_1\circ f^n)\varphi_2\,d\mu_0 - \int \varphi_1\,d\mu_0 \int \varphi_2\,d\mu_0\right| \leq C\Lambda^n \quad \textit{for all } n \geq 0.$$

Proof: Note that

$$\int (\varphi_1\circ f^n)\varphi_2\,d\mu_0 - \int \varphi_1\,d\mu_0 \int \varphi_2\,d\mu_0 = \int \varphi_1\Big(\mathcal{L}^n(\varphi_2\varphi_0) - \varphi_0 \int (\varphi_2\varphi_0)\,dm\Big)\,dm$$
$$= \int \varphi_1\mathcal{L}^n\big(\pi_0(\varphi_2\varphi_0)\big)\,dm,$$

where $\pi_0(\psi) = \psi - \varphi_0 \int \psi\,dm$ is the spectral projection onto $X_0 = \{\varphi : \int \varphi\,dm = 0\}$. Clearly, this is bounded from above by

$$\int |\varphi_1|\,dm \cdot \sup|\mathcal{L}^n(\pi_0(\varphi_2\varphi_0))| \leq \int |\varphi_1|\,dm \cdot \|\mathcal{L}^n(\pi_0(\varphi_2\varphi_0))\|_{\mathrm{BV}} \leq$$
$$\leq C'\Lambda^n \int |\varphi_1|\,dm \cdot \|\pi_0(\varphi_2\varphi_0)\|_{\mathrm{BV}} \leq C''\Lambda^n \int |\varphi_1|\,dm \cdot \|\varphi_2\varphi_0\|_{\mathrm{BV}}$$

for some $C', C' > 0$. Hence, we may take $C(\varphi_1,\varphi_2) = C'' \int |\varphi_1|\,dm\,\|\varphi_2\varphi_0\|_{\mathrm{BV}}$. □

The next result follows, in the same way as Corollary 2.10 and Proposition 2.12.

Corollary 3.11 (central limit theorem). *Let φ be a function with bounded variation and $\sigma^2 = \int \phi^2\, d\mu_0 + 2\sum_{j=1}^{\infty} \int \phi(\phi \circ f^j) d\mu_0$, where $\phi = \varphi - \int \varphi\, d\mu_0$. Then $\sigma < \infty$ and $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu_0)$. Moreover, if $\sigma > 0$ then for every interval $A \subset \mathbb{R}$*

$$\mu_0\left\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A\right\} \longrightarrow \int_A \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}\, dt \qquad \text{as } n \to +\infty.$$

In the remaining of the section we justify the claim (M) above. The next lemma asserts that, choosing N large, the iterates f^n have uniformly small distortion over a subset of monotonicity intervals of f^{n+N} covering most of the ambient M; moreover, those monotonicity intervals may be chosen with the additional property of being mapped by f^n onto (full) monotonicity intervals of f^N. Clearly, the relevance of such a statement of bounded distortion goes beyond the present application.

Lemma 3.12. *Given $\beta > 0$ there is $N \geq 1$ such that for every $n \geq 0$ there is a subset $\mathcal{Q}_{n+N} \subset \mathcal{P}^{(n+N)}$ of monotonicity intervals of f^{n+N} with $\mu_0\big(\bigcup_{\eta \in \mathcal{Q}_{n+N}} \eta\big) \geq 1 - \beta$ and*

(1) *$f^n(\eta) \in \mathcal{P}^{(N)}$ for every $\eta \in \mathcal{Q}_{n+N}$;*
(2)
$$e^{-\beta} \frac{\mu_0(\xi)}{\mu_0(\eta)} \leq \frac{\mu_0(f^n(\xi))}{\mu_0(f^n(\eta))} \leq e^{\beta} \frac{\mu_0(\xi)}{\mu_0(\eta)} \quad \textit{for every } \xi \subset \eta \in \mathcal{Q}_{n+N}.$$

Proof: The first step is to construct a subset $\mathcal{Q}^1_{n+N}$ of $\mathcal{P}^{(n+N)}$ such that (1) holds for every $\eta \in \mathcal{Q}^1_{n+N}$, and the remaining monotonicity intervals have small total measure if N is large. By definition, given any $k \geq 1$, an interval η belongs in $\mathcal{P}^{(k)}$ if and only if

(i) $f^i(x) \notin \{a_0, a_1, \ldots, a_l\}$ for every $x \in$ interior(η) and every $0 \leq i < k$;

(ii) for each $a \in \partial\eta$ there is some $0 \leq i < k$ for which $f^i(a) \in \{a_0, a_1, \ldots, a_l\}$.

(More acurately, an interval η satisfies (i) and (ii) if and only if there is some interval $\eta_0 \in \mathcal{P}^{(k)}$ such that interior$(\eta) \subset \eta_0 \subset$ closure(η); but such fine points are irrelevant in this context, and we disregard them from now on.) In particular, if $\eta \in \mathcal{P}^{(n+N)}$ satisfies

$$f^i(a) \notin \{a_0, a_1, \ldots, a_l\} \quad \text{for every } 0 \leq i < n \text{ and every } a \in \partial\eta, \tag{3.7}$$

then $f^n(\eta) \in \mathcal{P}^{(N)}$. We just take $\mathcal{Q}^1_{n+N}$ to be the set of all intervals $\eta \in \mathcal{P}^{(n+N)}$ for which (3.7) holds. Now, let $0 \leq i < n$ be fixed and suppose that η has

$$f^i(\partial\eta) \cap \{a_0, a_1, \ldots, a_l\} \neq \emptyset.$$

Then $f^i(\eta)$ is contained in some interval $\zeta \in \mathcal{P}^{(n+N-i)}$ whose boundary intersects the set $\{a_0, a_1, \ldots, a_l\}$. Of course, there are at most $2l$ such elements of $\mathcal{P}^{(n+N-i)}$, and $m(\zeta) \leq C_1 \lambda_1^{n+N-i}$ for each one of them. It follows that the μ_0-measure of their union is bounded by $2lC_1 \sup|\varphi_0| \lambda_1^{n+N-i}$ and then, by invariance, the same bound is valid for the μ_0-measure of all the $\eta \in \mathcal{P}^{(n+N)}$ not satisfying (3.7) for a given value of i. Hence, as we claimed, the total measure of the $\eta \notin Q^1_{n+N}$ is less than

$$\sum_{i=0}^{n-1} 2lC_1 \sup|\varphi_0|\lambda_1^{n+N-i} \leq 2lC_1 \sup|\varphi_0| \sum_{k=N+1}^{\infty} \lambda_1^k \leq \frac{\beta}{4},$$

as long as N is large enough.

Next, we want to show that (2) is true for every η in some (large) subset of Q^1_{n+N}. We need only to explain how to deduce the first inequality in (2), since the second one follows from a completely dual argument. We shall use the relation

$$\begin{aligned}\frac{\mu_0(f^n(\xi))}{\mu_0(f^n(\xi))} = \frac{\int_{f^n(\xi)} \varphi_0 \, dm}{\int_{f^n(\eta)} \varphi_0 \, dm} &= \frac{\int_\xi (\varphi_0 \circ f^n)|Df^n| \, dm}{\int_\eta (\varphi_0 \circ f^n)|Df^n| \, dm} \\ &\geq \frac{\inf_\eta \left((\varphi_0 \circ f^n)\,|Df^n|/\varphi_0\right)}{\sup_\eta \left((\varphi_0 \circ f^n)\,|Df^n|/\varphi_0\right)} \frac{\mu_0(\xi)}{\mu_0(\eta)}.\end{aligned}$$

Define $\psi(x) = 1/|Df(x)|$ for every $x \notin \{a_0, \ldots, a_l\}$. Note that our assumptions imply

$$\operatorname{var}\psi = \sum_{\eta \in \mathcal{P}^{(1)}} \operatorname{var} g_\eta \leq C_1\lambda_1 l < \infty$$

and

$$\int \frac{1}{\psi} \, d\mu_0 \leq \sup|\varphi_0| \sum_{\eta \in \mathcal{P}^{(1)}} \int_\eta |Df| \, dm \leq \sup|\varphi_0| \sum_{\eta \in \mathcal{P}^{(1)}} m(f(\eta)) \leq l \sup|\varphi_0| < \infty.$$

Let Q^2_{n+N} consist of all the intervals $\eta \in \mathcal{P}^{(n+N)}$ such that for every $0 \leq i < n$ the element ζ_i of $\mathcal{P}^{(n+N-i)}$ containing $f^i(\eta)$ satisfies

$$\frac{\inf(\psi|\zeta_i)}{\sup(\psi|\zeta_i)} \geq 1 - \lambda_1^{(n+N-i)/4}. \tag{3.8}$$

Then, for any $\eta \in Q^2_{n+N}$,

$$\frac{\inf_\eta |Df^n|}{\sup_\eta |Df^n|} \geq \prod_{i=0}^{n-1} \frac{\inf(\psi|\zeta_i)}{\sup(\psi|\zeta_i)} \geq \prod_{i=0}^{n-1} \left(1 - \lambda_1^{(n+N-i)/4}\right) \geq \prod_{k=N+1}^{\infty} \left(1 - \lambda_1^{k/4}\right) \geq e^{-\beta/2}$$

if N is large. Now let us estimate the total measure of the intervals $\zeta \in \mathcal{P}^{(n+N-i)}$ that do not satisfy (3.8). There are two cases to consider, depending on the value of the supremum. Clearly,

$$\int \frac{1}{\psi} \, d\mu_0 \geq \lambda_1^{-(n+N-i)/4} \sum_\zeta \mu_0(\zeta)$$

if the sum is taken over those ζ for which $\sup(\psi|\zeta) \leq \lambda_1^{(n+N-i)/4}$. On the other hand, $\sup(\psi|\zeta) \geq \lambda_1^{(n+N-i)/4}$ implies

$$\operatorname{var}(\psi|\zeta) \geq \sup(\psi|\zeta)\left(1 - \frac{\inf(\psi|\zeta)}{\sup(\psi|\zeta)}\right) \geq \lambda_1^{(n+N-i)/2}.$$

Therefore, the number of such intervals ζ does not exceed $\lambda_1^{-(n+N-i)/2} \operatorname{var}\psi$, and so their total μ_0-measure is bounded by

$$\lambda_1^{-(n+N-i)/2} \operatorname{var}\psi \cdot C_1 \lambda_1^{(n+N-i)} \sup|\varphi_0| = C_1 \lambda_1^{(n+N-i)/2} \operatorname{var}\psi \sup|\varphi_0|.$$

Putting these two estimates together we get that the μ_0-measure of the union of all the intervals $\zeta \in \mathcal{P}^{(n+N-i)}$ not satisfying (3.8) is less than

$$C_1' \lambda_1^{(n+N-i)/2}, \qquad \text{where} \quad C_1' = \int \frac{1}{\psi}\, d\mu_0 + C_1 \operatorname{var}\psi \sup|\varphi_0|.$$

Since μ_0 is invariant under every f^i, we conclude that the total μ_0-measure of the intervals $\eta \in \mathcal{P}^{(n+N)} \setminus \mathcal{Q}^2_{n+N}$ is bounded by

$$\sum_{i=0}^{n-1} C_1' \lambda_1^{(n+N-i)/2} \leq \sum_{k=N+1}^{\infty} C_1' \lambda_1^{k/2} \leq \frac{\beta}{4},$$

if N is large enough.

Now, note that $\operatorname{var}\varphi_0 < \infty$ and $\int (1/\varphi_0)\, d\mu_0 = \int dm = 1$. This means that all the previous reasonings apply also to $\psi = \varphi_0$. Thus, there exists a subset $\mathcal{Q}^3_{n+N}$ of $\mathcal{P}^{(n+N)}$ such that, as long as N is large enough,

$$\frac{\inf(\varphi_0|f^i(\eta))}{\sup(\varphi_0|f^i(\eta))} \geq 1 - \lambda_1^{(n+N-i)/4} \geq e^{-\beta/4}$$

for every $0 \leq i \leq n$ and $\eta \in \mathcal{Q}^3_{n+N}$, and the total μ_0-measure of $\mathcal{P}^{(n+N)} \setminus \mathcal{Q}^3_{n+N}$ is less than $\beta/4$ if is N large enough,

At this point we take $\mathcal{Q}_{n+N} = \mathcal{Q}^1_{n+N} \cap \mathcal{Q}^2_{n+N} \cap \mathcal{Q}^3_{n+N}$. By construction, the intervals in $\mathcal{P}^{(n+N)} \setminus \mathcal{Q}_{n+N}$ have total μ_0-measure less than β, and

$$\frac{\mu_0(f^n(\xi))}{\mu_0(f^n(\xi))} \geq e^{-\beta}\, \frac{\mu_0(\xi)}{\mu_0(\eta)},$$

for every $\xi \subset \eta \in \mathcal{Q}_{n+N}$. As we have already said, the other inequality in (2) is obtained in just the same way. $\square$

Proposition 3.13 (exactness). *If f is a piecewise expanding map f satisfying (E1), (E2), (E3) then the absolutely continuous invariant measure μ_0 is exact and so, in particular, it is mixing for f.*

Proof: Let $Z \subset I_*$ be such that for each $j \geq 1$ there is a Borel set Z_j with $Z = f^{-j}(Z_j)$. We show that if $\mu_0(Z) > 0$ then $\mu_0(Z) = 1$. Note that $Z_j = f^j(f^{-j}(Z_j)) = f^j(Z)$, since f is surjective (e.g., because it is topologically mixing). Fix $\beta \leq \mu_0(Z)/3$ and let $N = N(\beta)$ be as given by Lemma 3.12. By Lebesgue differentiation theorem, there exists $Z_0 \subset Z$ with $\mu_0(Z_0) = \mu_0(Z) \geq 3\beta$ and

$$\lim_{r \to 0} \inf_{|J|<r} \frac{\mu_0(Z \cap J)}{\mu_0(J)} = 1 \quad \text{for all } a \in Z_0,$$

where the infimum is taken over all closed intervals of length less than r that contain a. In particular, given any $\varepsilon > 0$ there exist $r > 0$ and a set $Z_\varepsilon \subset Z_0$ such that $\mu_0(Z_\varepsilon) \geq 2\beta$ and

$$\inf_{|J|<r} \frac{\mu_0(Z \cap J)}{\mu_0(J)} \geq 1 - \varepsilon \quad \text{for all } a \in Z_\varepsilon.$$

Take $n \geq 0$ large enough so that $|\eta| < r$ for every $\eta \in \mathcal{P}^{(n+N)}$, and let $\mathcal{Q}_{n+N}$ be given by the lemma. The fact that $\mu_0(Z_\varepsilon) \geq 2\beta$ implies that Z_ε intersects some $\eta_\varepsilon \in \mathcal{Q}_{n+N}$. Then

$$\frac{\mu_0(\eta_\varepsilon \setminus Z)}{\mu_0(\eta_\varepsilon)} \leq \varepsilon \qquad \text{and so} \qquad \frac{\mu_0(f^n(\eta_\varepsilon) \setminus Z_n)}{\mu_0(f^n(\eta_\varepsilon))} \leq e^\beta \varepsilon.$$

Recall also that $\zeta_\varepsilon = f^n(\eta_\varepsilon) \in \mathcal{P}^{(N)}$, which is a finite partition. Assumption (E3) implies that there is $q \geq 1$, depending only on $N = N(\beta)$, such that $f^q(\zeta_\varepsilon) = I_*$. Let $\kappa \geq 1$ be the number of elements of $\mathcal{P}^{(q)}$ that intersect ζ_ε, and let ζ_i, $1 \leq i \leq \kappa$, be those intersections of ζ with elements of $\mathcal{P}^{(q)}$. Then f^q is monotone on ζ_i, for every $0 \leq i \leq \kappa$, and the union of all the $f^q(\zeta_i)$ covers I_*. We claim that given $\delta > 0$ there exists $\varepsilon_1 > 0$ such that

$$\mu_0(A) \leq \varepsilon_1 \quad \Rightarrow \quad \mu_0(f^q(A)) \leq \delta$$

for every Borel set $A \subset \zeta_i$ and every $1 \leq i \leq \kappa$. Indeed, the similar statement with Lebesgue measure m in the place of μ_0 is a direct consequence of the smoothness of f^q, and so it suffices to recall that μ_0 is equivalent to m on I_*. Now, for any $\delta > 0$ let $\varepsilon_1 > 0$ be as before and choose $\varepsilon > 0$ above small enough so that $e^\beta \varepsilon \leq \varepsilon_1$. Then

$$\mu_0(\zeta_i \setminus Z_n) \leq \mu_0(\zeta_\varepsilon \setminus Z_n) \leq e^\beta \varepsilon \, \mu_0(\zeta_\varepsilon) \leq \varepsilon_1$$

gives

$$\mu_0(f^q(\zeta_i) \setminus Z_{n+q}) = \mu_0(f^q(\zeta_i \setminus Z_n)) \leq \delta.$$

Adding over $i = 1, \ldots, \kappa$, and recalling that μ_0 is an invariant measure

$$\mu_0(I_* \setminus Z) = \mu_0(I_* \setminus Z_{n+q}) \leq \sum_{i=1}^{\kappa} \mu_0(f^q(\zeta_i) \setminus Z_{n+q})) \leq \kappa \, \delta.$$

Since $\delta > 0$ is arbitrary and κ is bounded by the number of elements of $\mathcal{P}^{(q)}$, which does not depend on δ, we conclude that $\mu_0(I_* \setminus Z) = 0$, that is, $\mu_0(Z) = 1$. ☐

3.3 Stochastic stability.

Example 1.5 shows that *not all* piecewise expanding maps are stochastically stable. However, as we shall now show, stochastic stability does hold under a mild "generic" condition (E4) to be stated below. Observe that, unlike the smooth expanding maps treated in Chapter 2, piecewise expanding maps are, generally, not structurally stable.

The present statement of stochastic stability is analogous to the one we obtained before in the smooth case. We let T be any metric space and $T \ni t \longmapsto f_t$ be any parametrized family of piecewise expanding maps satisfying

$$f_t \to f_\tau = f \quad \text{as } t \to \tau$$

for some $\tau \in T$. (We shall explain below which topology we want to consider in the space of piecewise expanding maps). Moreover, let $(\theta_\varepsilon)_{\varepsilon>0}$ be an arbitrary family of probabilities on T, such that

$$\operatorname{supp}\theta_\varepsilon \to \{\tau\} \quad \text{as } \varepsilon \to 0.$$

We show that for each small $\varepsilon > 0$ there is a probability measure μ_ε on M such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(x_j) \to \int \varphi\, d\mu_\varepsilon \quad \text{as} \quad n \to +\infty,$$

for every continuous function $\varphi : M \to \mathbb{R}$ and $m \times \theta_\varepsilon^{\mathbb{N}}$-almost every random trajectory $x_j = f_{t_j} \circ \cdots \circ f_{t_1}(x)$. Furthemore, μ_ε converges to μ_0 as $\varepsilon \to 0$.

The precise content of the condition that $f_t \to f$ as $t \to \tau$ requires some words of explanation. Let us start by the circle case. We require that, for every t in a neighbourhood of τ, there exist

$$a_0(t) < a_1(t) < \cdots < a_{l-1}(t) < a_l(t) = a_0(t)$$

such that

(i) the restriction f_t to $(a_{i-1}(t), a_i(t))$ is a C^1 map, for each $i = 1, \ldots, l$;

(ii) $|Df_t| > 0$ on the complement of $\{a_1(t), \ldots, a_l(t)\}$.

For $i = 1, \ldots, l$, we denote $\eta_{i,t} = (a_{i-1}(t), a_i(t))$, and let $\phi_{i,t}\colon \eta_i \to \eta_{i,t}$ be the affine bijection mapping a_i to $a_i(t)$. Then we ask that there be some $\widetilde{C}_1 > 0$ close to C_1 do that, for every $i = 1, \ldots, l$,

(1) $a_i(t) \to a_i(\tau) = a_i$ as $t \to \tau$;

(2) $(1/|Df_t|\eta_{i,t}|) \circ \phi_{i,t}$ converges uniformly to $(1/|Df|\eta_i|)$ as $t \to \tau$;

(3) $\operatorname{var}(1/|Df_t|\eta_{i,t}|) \le \widetilde{C}_1$ if t is close to τ.

We shall denote $g_{\eta_i} = (1/|Df|\eta_i|)$, for each $i = 1, \ldots, l$.

Example 3.4. Let $a_0(t) < a_1(t) < \cdots < a_{l-1}(t) < a_l(t) = a_0(t)$ be continuous functions, defined for t in some metric space T and taking values in S^1. Let $(f_t)_{t\in T}$ be a family of transformations on M such that

(a) each $f_t|(a_{i-1}(t), a_i(t))$ has a C^2 extension to $[a_{i-1}(t), a_i(t)]$ with $|Df_t| \geq 1/\tilde{\lambda}_1 > 1$;

(b) each $(f_t \circ \phi_{i,t}) : [a_{i-1}, a_i] \to S^1$ varies continuously with t in the C^2-topology.

Then, clearly, $(f_t)_t$ is continuous at every $\tau \in T$, in the sense we have defined.

The definition of continuity for families of piecewise expanding maps of the interval is similar, but it is convenient to avoid the phenomenon of (random) trajectories escaping I_* and never returning. With that in mind, we always take the unperturbed map $f = f_\tau$ to extend to some larger interval $[\tilde{a}_0, \tilde{a}_l] \supset (\tilde{a}_0, \tilde{a}_l) \supset I_*$, in such a way that

(i) the restrictions of f to both $(\tilde{a}_0, a_1)$ and $(a_{l-1}, \tilde{a}_l)$ are of class C^1;

(ii) $|Df|\eta| > 0$ and $\text{var}(1/|Df|\eta|) \leq C_1$ for both $\eta = (\tilde{a}_0, a_1)$ and $\eta = (a_{l-1}, \tilde{a}_l)$;

(iii) $f([\tilde{a}_0, \tilde{a}_l]) \subset (\tilde{a}_0, \tilde{a}_l)$, and $f^q([\tilde{a}_0, \tilde{a}_l]) = I_*$ for some $q \geq 1$.

Then we require that, for each t close to τ, there exist

$$\tilde{a}_0 = \tilde{a}_0(t) < a_1(t) < \cdots < a_{l-1}(t) < \tilde{a}_l(t) = \tilde{a}_l$$

satisfying the analogs of conditions (1), (2), (3). In particular, $f_t([\tilde{a}_0, \tilde{a}_l]) \subset (\tilde{a}_0, \tilde{a}_l)$ for every t close enough to τ.

Example 3.5. Let $f: I \to I$ be a tent map and c be its turning point, recall Figure 3.1. As we have seen in Example 3.3, f is topologically mixing on $I_* = [f^2(c), f(c)]$ if $|Df| > \sqrt{2}$. In this case one may take $l = 2$, and it suffices to choose $0 < \tilde{a}_0 < f^2(c)$, $a_1 = c$, and $f(c) < \tilde{a}_2 < 1$, such that $f(\tilde{a}_0) = f(\tilde{a}_2) > \tilde{a}_0$.

Another important class of continuous families of piecewise expanding maps is provided by the following construction.

Example 3.6. Let $f_t : [0,1] \to [0,1]$, $t \in T$, be Lorenz-like maps given by

$$f_t(x) = \begin{cases} h_t^-(|x - 1/2|^{\alpha_t}) & \text{if } x < 1/2 \\ h_t^+(|x - 1/2|^{\alpha_t}) & \text{if } x > 1/2 \end{cases}$$

where $\alpha_t \in (0,1)$ varies continuously with t, and the $h_t^\pm$ are C^2 strictly monotone maps depending continuously on t in the C^2-topology. For the sake of definiteness, we let h_t^- be decreasing with $h_t^-(0) = 1$ and $h_t^-(1/2^{\alpha_t}) > 0$, and h_t^+ be increasing with $h_t^+(0) = 0$ and $h_t^+(1/2^{\alpha_t}) < 1$. We also take $h_t^\pm$ to be such that $|Df_t(x)| \geq 1/\tilde{\lambda}_1$ for some $\tilde{\lambda}_1 < 1$ and every $t \in T$ and $x \neq 1/2$, which ensures that the f_t are piecewise expanding maps. Then $(f_t)_t$ is continuous at every $\tau \in T$: it is clear that f may be C^2 extended to some $[\tilde{a}_0, \tilde{a}_2]$ as above, and C^2-closeness of $h_t^\pm$ to $h_\tau^\pm$ is easily seen to imply uniform closeness of $1/|Df_t|$ to $1/|Df|$, together with uniformly bounded variation for $1/|Df_t|$ on each of the intervals $(\tilde{a}_0, 1/2)$ and $(1/2, \tilde{a}_2)$.

Observe that if $(X_t)_{t \in T}$ is a continuous family of flows in three dimensions exhibiting geometric Lorenz attractors, cf. [GW79], then the corresponding family $(f_t)_{t \in T}$ of one-dimensional Lorenz-like maps as in this example. It follows that such a family is continuous in the sense defined above, which means that the topology we introduced above is natural also for these applications in the setting of flows.

We use $\underline{t} = (t_1, \dots, t_n, \dots)$ to denote the generic element of $T^{\mathbb{N}}$, with $\underline{\tau}$ representing $(\tau, \dots, \tau, \dots)$. Since modifying a map over a finite set of points does not affect its statistical properties, it is no restriction to suppose that every f_t is left-continuous at $a_i(t)$ or else every f_t is right-continuous at $a_i(t)$, for all $0 \le i \le l$ and t close to τ. Let $\mathcal{P}_t$ be any partition into subintervals η_t such that

(a) $\eta_{i,t} \subset \eta_t \subset \text{closure}(\eta_{t,i})$ for some $i \in \{1, \dots, l\}$;

(b) and $(f_t|\eta_t)$ is continuous, for every $\eta_t \in \mathcal{P}_t$.

For $n \ge 1$, we write $f_{\underline{t}}^n = f_{t_n} \circ \cdots f_{t_1}$ and let $\mathcal{P}_{\underline{t}}^{(n)}$ be the partition given by

$$\mathcal{P}_{\underline{t}}^{(n)}(x) = \mathcal{P}_{\underline{t}}^{(n)}(y) \quad \Leftrightarrow \quad \mathcal{P}_{t_{j+1}}(f_{\underline{t}}^j(x)) = \mathcal{P}_{t_{j+1}}(f_{\underline{t}}^j(y)) \text{ for every } 0 \le j < n.$$

It follows from (2) that we may choose $\tilde{\lambda}_1 < 1$ close to λ_1, so that

$$\text{(3.9)} \qquad \sup(1/|Df_{\underline{t}}^n|\eta_{\underline{t}}|) \le \widetilde{C}_1 \tilde{\lambda}_1^n \quad \text{for all } \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)} \text{ and } \underline{t} \text{ close to } \underline{\tau}\,.$$

Moreover, (2) and (3) imply, cf. (3.1),

$$\text{(3.10)} \qquad \text{var}(1/|Df_{\underline{t}}^n|\eta_{\underline{t}}|) \le \widetilde{C}_2 \tilde{\lambda}_2^n \quad \text{for all } \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)} \text{ and } \underline{t} \text{ close to } \underline{\tau}\,,$$

where $\widetilde{C}_2 > 0$ and $\tilde{\lambda}_2 < 1$ may be taken close to C_2 and λ_2, respectively.

To state our final hypothesis on the expanding map f, we introduce the notations

$$b_i^- = \lim_{x \to a_i^-} f(x) \quad \text{and} \quad b_i^+ = \lim_{x \to a_i^+} f(x),$$

for each $0 \le i \le l$ (except that, by convention, $b_0^- = b_0^+$ and $b_l^+ = b_l^-$ in the case $M = I$). Then we suppose

(E4) for every $n \ge 0$ and $0 \le i \le l$, we have $f^n(b_i^\pm) \notin \{a_1, \dots, a_{l-1}, a_l\}$, if $M = S^1$, respectively, $f^n(b_i^\pm) \notin \{\tilde{a}_0, a_1, \dots, a_{l-1}, \tilde{a}_l\}$, if $M = I$.

In fact a slightly weaker condition suffices for stochastic stability, namely, that no $b_i^\pm$ be a periodic point for f. Observe that, in either form, this is not an open condition, as it concerns the behaviour over an infinite time-scale.

Condition (E4) will used through the following direct consequences. The length $m(\eta)$ of each interval $\eta \in \mathcal{P}^{(n)}$ is positive, and so $\delta_n = \inf_{\eta \in \mathcal{P}^{(n)}} m(\eta) > 0$ for every $n \ge 1$. Then, for each fixed $n \ge 1$, there is a neighbourhood V_n of τ in T such that given any $t_1, \dots, t_n \in V_n$ there is a bijection

$$\mathcal{P}^{(n)} \ni \eta \to \eta_{\underline{t}} \in \mathcal{P}_{\underline{t}}^{(n)}, \qquad \text{with } \eta_{\underline{t}} \to \eta = \eta_{\underline{\tau}} \text{ as } \underline{t} \to \underline{\tau}.$$

In particular, if V_n is chosen small enough, $m(\eta_{\underline{t}}) \ge \tilde{\delta}_n$ for some $\tilde{\delta}_n > 0$ close to δ_n and every $\eta \in \mathcal{P}^{(n)}$. Moreover, up to further restricting V_n, we may suppose that the family of intervals $\{\eta \cup \eta_{\underline{t}} : \eta \in \mathcal{P}^{(n)}\}$ has overlap index 2: every point belongs in no more than 2 of its elements.

After these preliminaries, we may start proving that piecewise expanding maps satisfying (E1)-(E4) are stochastically stable. First, we introduce transfer operators corresponding to the perturbed maps f_t,

$$U_t\,\varphi(x) = \varphi(f_t(x)) \qquad\qquad \mathcal{L}_t\,\varphi(y) = \sum_\eta \frac{\varphi}{|Df_t|}\big((f_t|\eta_t)^{-1}(y)\big)$$

where the sum is over all the $\eta \in \mathcal{P}^{(1)}$ for which $y \in f_t(\eta_t)$. We also introduce their iterates

$$U_{\underline{t}}\,\varphi(x) = \varphi(f_{\underline{t}}^n(x)) \qquad\qquad \mathcal{L}_{\underline{t}}^n\,\varphi = \sum_\eta \frac{\varphi}{|Df_{\underline{t}}^n|}\big((f_{\underline{t}}^n|\eta_{\underline{t}})^{-1}(y)\big)$$

where the sum is over all $\eta \in \mathcal{P}^{(n)}$ such that $y \in f_{\underline{t}}^n(\eta_{\underline{t}})$. The same arguments as in the proof of Proposition 3.1 yield $\widetilde{C}_0 > 0$ and $\tilde{\lambda}_0 < 1$, close to C_0 and λ_0, respectively, such that

$$\operatorname{var}(\mathcal{L}_{\underline{t}}^n\,\varphi) \le \widetilde{C}_0\tilde{\lambda}_0^n \operatorname{var}\varphi + \widetilde{C}_0 \int |\varphi|\,dm \tag{3.11}$$

for every $t_1, \dots, t_n$ in V_n and φ with bounded variation.

Lemma 3.14. *There are $\widetilde{C} > 0$ and $\tilde{\lambda} \in (0,1)$, and for each $n \ge 1$ there is a neighbourhood Y_n of τ, such that*

$$||\mathcal{L}_{\underline{t}}^n\,\varphi - \mathcal{L}^n\,\varphi||_1 \le \widetilde{C}\tilde{\lambda}^n(\operatorname{var}\varphi + \int |\varphi| dm)$$

for every $t_1, \dots, t_n \in Y_n$ and every function φ with bounded variation.

Proof: For notational simplicity, we write f^{-n} and $f_{\underline{t}}^{-n}$ to mean $(f^n\eta)^{-1}$ and $(f_{\underline{t}}^n|\eta_{\underline{t}})^{-1}$, respectively. Then

$$\begin{aligned}
&\int |\mathcal{L}_{\underline{t}}^n\,\varphi - \mathcal{L}^n\,\varphi|\,dm \\
&\quad\le \sum_\eta \int_{f^n(\eta)\cap f_{\underline{t}}^n(\eta_{\underline{t}})} \big|(\varphi\circ f_{\underline{t}}^{-n})\,|Df_{\underline{t}}^{-n}| - (\varphi\circ f^{-n})\,|Df^{-n}|\big|\,dm \\
&\quad + \sum_\eta \Big(\int_{f^n(\eta)\setminus f_{\underline{t}}^n(\eta_{\underline{t}})} |\varphi\circ f^{-n}|\,|Df^{-n}|\,dm \\
&\qquad\qquad + \int_{f_{\underline{t}}^n(\eta_{\underline{t}})\setminus f^n(\eta)} |\varphi\circ f_{\underline{t}}^{-n}|\,|Df_{\underline{t}}^{-n}|dm\Big),
\end{aligned} \tag{3.12}$$

the sums being over all $\eta \in \mathcal{P}^{(n)}$. The first sum may be estimated as follows. If $t_1, \ldots, t_n$ are taken in some neighbourhood Y of τ then, at every point in $f^n(\eta) \cap f_{\underline{t}}^n(\eta_{\underline{t}})$,

$$\begin{aligned}
&\left|(\varphi \circ f_{\underline{t}}^{-n})\,|Df_{\underline{t}}^{-n}| - (\varphi \circ f^{-n})\,|Df^{-n}|\,\right| \\
&\quad \leq |(\varphi \circ f_{\underline{t}}^{-n}) - (\varphi \circ f^{-n})|\,|Df_{\underline{t}}^{-n}| + |\varphi \circ f^{-n}|\,|Df_{\underline{t}}^{-n} - Df^{-n}| \\
&\quad \leq \operatorname{var}(\varphi|\eta \cup \eta_{\underline{t}})\,\widetilde{C}_1\tilde{\lambda}_1^n + \sup|\varphi|\,\xi_n(Y)
\end{aligned}$$

where $\xi_n(Y) \to 0$ when $Y \to \{\tau\}$. Recalling that $\{\eta \cup \eta_{\underline{t}} : \eta \in \mathcal{P}^{(n)}\}$ has overlap index 2, and choosing Y small enough so that

$$\#\mathcal{P}^{(n)}\,\xi_n(Y) \leq \widetilde{C}_1\tilde{\lambda}_1^n,$$

we conclude that the first sum in (3.12) is bounded by

$$\begin{aligned}
\sum_\eta \left(\widetilde{C}_1\tilde{\lambda}_1^n \operatorname{var}(\varphi|\eta \cup \eta_{\underline{t}}) + \sup|\varphi|\,\xi_n(Y)\right) &\leq 2\widetilde{C}_1\tilde{\lambda}_1^n \operatorname{var}\varphi + \widetilde{C}_1\tilde{\lambda}_1^n \sup|\varphi| \\
&\leq 3\widetilde{C}_1\tilde{\lambda}_1^n \operatorname{var}\varphi + \widetilde{C}_1\tilde{\lambda}_1^n \int |\varphi| dm.
\end{aligned}$$

Moreover, using once again the continuity of f_t at $t = \tau$,

$$m(f^n(\eta) \setminus f_{\underline{t}}^n(\eta_{\underline{t}})) \leq \xi_n(Y) \quad \text{and} \quad m(f_{\underline{t}}^n(\eta_{\underline{t}}) \setminus f^n(\eta)) \leq \xi_n(Y),$$

where $\xi_n(Y)$ has the same meaning as before. It follows that the second sum in (3.12) is bounded by

$$\sum_\eta 2\sup|\varphi|\widetilde{C}_1\lambda_1^n\,\xi_n(Y) \leq 2\sup|\varphi|\widetilde{C}_1\tilde{\lambda}_1^n \leq 2\widetilde{C}_1\tilde{\lambda}_1^n \left(\operatorname{var}\varphi + \int |\varphi| dm\right),$$

as long as Y is small enough so that $\#\mathcal{P}^{(n)}\,\xi_n(Y) \leq 1$. To conclude the proof it suffices to take $\widetilde{C} = 5\widetilde{C}_1$ and $\tilde{\lambda} = \tilde{\lambda}_1$. □

Now we introduce the linear operators

$$\widehat{U}_\varepsilon\varphi(x) = \int U_t\varphi(x)\,d\theta_\varepsilon(t) \qquad \widehat{\mathcal{L}}_\varepsilon\varphi(y) = \int \mathcal{L}_t\varphi(y)\,d\theta_\varepsilon(t)$$

and note that

$$\int (\widehat{U}_\varepsilon\psi)\,\varphi\,dm = \int \psi\,(\widehat{\mathcal{L}}_\varepsilon\varphi)$$

for every $\varphi \in L^\infty(m)$ and every $\psi \in L^1(m)$. This is, simply, because the analogous duality property holds for U_t and $\mathcal{L}_t$ for every t (use Fubini's theorem in the same way as in Section 2.6). Moreover, (3.11) implies (recall property (v6) of the variation)

$$\operatorname{var}(\widehat{\mathcal{L}}_\varepsilon^n\varphi) \leq \widetilde{C}_0\tilde{\lambda}_0^n \operatorname{var}\varphi + \widetilde{C}_0 \int |\varphi|\,dm \tag{3.13}$$

for every $n \geq 1$ and every function φ with bounded variation. Then, cf. Corollary 3.4, there is a nonnegative function φ_ε which is fixed under the operator $\widehat{\mathcal{L}}_\varepsilon$, that is, $\widehat{\mathcal{L}}_\varepsilon \varphi_\varepsilon = \varphi_\varepsilon$. We normalize φ_ε by asking that $\int \varphi_\varepsilon \, dm = 1$, and denote $\mu_\varepsilon = \varphi_\varepsilon \, m$. Then μ_ε is a probability measure and it is a stationary measure:

$$\int (\widehat{U}_\varepsilon \psi) \, d\mu_\varepsilon = \int \psi \, (\widehat{\mathcal{L}}_\varepsilon \varphi_\varepsilon) \, dm = \int \psi \, \varphi_\varepsilon \, dm = \int \psi \, d\mu_\varepsilon$$

for every integrable function ψ.

Proposition 3.15 (exponential mixing for the random perturbations). *There are $\widetilde{\Lambda} < 1$ and $\widetilde{K} > 0$ such that, given φ with bounded variation and $\psi \in L^1(m)$,*

$$\left| \int (\widehat{U}_\varepsilon^n \, \psi) \varphi \, dm - \int \psi \, d\mu_\varepsilon \int \varphi \, dm \right| \leq \widetilde{K} \widetilde{\Lambda}^n \, \|\varphi\|_{\mathrm{BV}} \int |\psi| \, dm$$

for all $n \geq 0$ and every small enough $\varepsilon > 0$.

The approach we take for proving this proposition is to think of $\mathcal{L}_\varepsilon$ as a kind of perturbation of $\mathcal{L}$, to deduce that $\mathcal{L}_\varepsilon$ is quasi-compact from the fact that $\mathcal{L}$ is quasi-compact. The following perturbation lemma is what we need in this setting.

Lemma 3.16. *Let $C > 0$, $\lambda < 1$, $\Lambda < 1$, and $\mathcal{P}_\varepsilon : BV \to BV$, $\varepsilon \geq 0$, be a family of linear operators satisfying*

(1) $\int \mathcal{P}_\varepsilon \varphi \, dm = \int \varphi \, dm$ *and* $\varphi \geq 0 \Rightarrow \mathcal{P}\varphi \geq 0$;

(2) $\|\mathcal{P}_\varepsilon^n \varphi\|_{\mathrm{BV}} \leq C\lambda^n \|\varphi\|_{\mathrm{BV}} + C\|\varphi\|_1$;

for every $n \geq 1$, $\varepsilon \geq 0$, and $\varphi \in BV$. Suppose that

(a) given $n \geq 1$ there is $\varepsilon(n) > 0$ so that, for all $\varphi \in BV$ and $0 \leq \varepsilon \leq \varepsilon(n)$,

$$\|\mathcal{P}_\varepsilon^n \varphi - \mathcal{P}_0^n \varphi\|_1 \leq C\lambda^n \|\varphi\|_{\mathrm{BV}};$$

(b) $\operatorname{spec}(\mathcal{P}_0) = \{1\} \cup \Sigma_0$ *where 1 is a simple eigenvalue and* $\Sigma_0 \subset \{z \in \mathbb{C} : |z| \leq \Lambda\}$.

Fix $\sigma \in (\max\{\sqrt{\Lambda}, \sqrt{\lambda}\}, 1)$. Then, for any small enough $\varepsilon > 0$, $\operatorname{spec}(\mathcal{P}_\varepsilon) = \{1\} \cup \Sigma_\varepsilon$, *where 1 is a simple eigenvalue and $\Sigma_\varepsilon \subset \{z \in \mathbb{C} : |z| \leq \sigma\}$.*

Proof: As a first step, we claim that if n is sufficiently large and $0 < \varepsilon \leq \varepsilon(n)$ then $R(\mathcal{P}_\varepsilon^n, z^n) = (z^n I - \mathcal{P}_\varepsilon^n)^{-1}$ is a bounded operator on BV for any z with $\sigma \leq |z| < 1$. To prove this, we write

$$\begin{aligned} R(\mathcal{P}_\varepsilon^n, z^n) &= \left((z^n I - \mathcal{P}_0^n) + (\mathcal{P}_0^n - \mathcal{P}_\varepsilon^n)\right)^{-1} \\ &= \left((z^n I - \mathcal{P}_0^n) \cdot (I + R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_0^n - \mathcal{P}_\varepsilon^n))\right)^{-1} \\ &= \left(I + \sum_{k=1}^{\infty} \left(R(\mathcal{P}_0^n, z^n)(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\right)^k\right) R(\mathcal{P}_0^n, z^n). \end{aligned} \tag{3.14}$$

Of course, the operator $R(\mathcal{P}_0^n, z^n)$ is bounded, since z does not belong in the spectrum of $\mathcal{P}_0$. A main tool is the sequence $|||\cdot|||_n$ of norms on BV defined by

$$|||\varphi|||_n = \theta^n \|\varphi\|_{\mathrm{BV}} + \|\varphi\|_1,$$

where $\max\{\sqrt{\Lambda}, \sqrt{\lambda}\} < \theta < \sigma$. Note that $\theta^n \|\varphi\|_{\mathrm{BV}} \leq |||\varphi|||_n \leq 2\|\varphi\|_{\mathrm{BV}}$, and so each $|||\cdot|||_n$ is equivalent to $\|\cdot\|_{\mathrm{BV}}$. Then, in order to prove that the sum in (3.14) converges in the space of bounded operators on BV, it suffices to show that (for all large n)

$$(3.15) \qquad |||R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)|||_n < 1.$$

We start with a few simple observations. Assumption (1) implies that $X_0 = \{\varphi \in \mathrm{BV} : \int \varphi\, dm = 0\}$ is invariant under $\mathcal{P}_0$. Any accumulation point φ_0 of $\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{P}_0^j 1$ is a fixed point of $\mathcal{P}_0$ with $\int \varphi_0\, dm = 1$, and such accumulation points do exist, by Helly's theorem. In view of assumption (b), the accumulation point φ_0 is unique, and the spectral splitting corresponding to the decomposition $\mathrm{spec}(\mathcal{P}_0) = \{1\} \cup \Sigma_0$ must be $\mathrm{BV} = \mathbb{R}\varphi_0 \oplus X_0$. Condition (1) also implies $\int (\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\varphi\, dm = 0$ for every $n \geq 1$, $\varepsilon > 0$, $\varphi \in \mathrm{BV}$. In other words, the image of $\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n$ is contained in the eigenspace X_0 of $\mathcal{P}_0$. Now, for some $K > 0$ and any $\psi \in X_0$,

$$|||\mathcal{P}_0^n \psi|||_n \leq 2\|\mathcal{P}_0^n \psi\|_{\mathrm{BV}} \leq 2K\Lambda^n \|\psi\|_{\mathrm{BV}} \leq 2K\Lambda^n \theta^{-n} |||\psi|||_n \leq \frac{1}{2}|z|^n |||\psi|||_n$$

if n is large, recall that $|z| \geq \sigma > \theta > \sqrt{\Lambda}$. Then $|||(z^n I - \mathcal{P}_0^n)\psi|||_n \geq (1/2)|z|^n |||\psi|||_n$ for every $\psi \in X_0$, which gives

$$|||R(\mathcal{P}_0^n, z^n) | X_0|||_n \leq 2|z|^{-n} \leq 2\sigma^{-n}.$$

Next, from assumptions (2) and (a) we get

$$\begin{aligned} |||(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\varphi|||_n &\leq \theta^n \big(2C\lambda^n \|\varphi\|_{\mathrm{BV}} + 2C\|\varphi\|_1\big) + C\lambda^n \|\varphi\|_{\mathrm{BV}} \\ &\leq 2C\theta^n \big((\lambda^n + \lambda^n/2\theta^n)\|\varphi\|_{\mathrm{BV}} + \|\varphi\|_1\big) \\ &\leq 2C\theta^n |||\varphi|||_n, \end{aligned}$$

for all large n and $0 < \varepsilon \leq \varepsilon(n)$, recall that $\theta > \sqrt{\lambda}$. Combining these estimates we find

$$(3.16) \qquad |||R(\mathcal{P}_0^n, z^n)\,(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)|||_n \leq 4C\theta^n |z|^{-n} \leq 4C\theta^n \sigma^{-n}.$$

Since $\theta < \sigma$, this implies (3.15) and completes the proof of our claim.

As an immediate consequence, we get that the spectrum of $\mathcal{P}_\varepsilon$ contains no point z with $\sigma < |z| < 1$, whenever $0 \leq \varepsilon \leq \varepsilon(n)$ for some sufficiently large n. Let us denote $U_\varepsilon = \{z \in \mathrm{spec}(\mathcal{P}_\varepsilon) : |z| \geq 1\}$ and $\Sigma_\varepsilon = \{z \in \mathrm{spec}(\mathcal{P}_\varepsilon) : |z| \leq \sigma\}$. Note that U_ε is contained in the unit circle (and nonempty), as assumption (2) implies that the spectral radius of $\mathcal{P}_\varepsilon$ is equal to 1. For each $\varepsilon \geq 0$, let Y_ε be the eigenspace of $\mathcal{P}_\varepsilon$ associated to U_ε, and $\pi^\varepsilon : \mathrm{BV} \to Y_\varepsilon$ denote the corresponding spectral projection. We are left to prove that $U_\varepsilon = \{1\}$ and 1 is a simple eigenvalue or, in other words, that Y_ε has dimension 1. With that in mind, we show that $|||\pi^\varepsilon - \pi^0|||_n$ is small if $0 < \varepsilon \le \varepsilon(n)$ and n is large. Fix $\sigma < \sigma_1 < 1 < \sigma_2$ and let C_l^n denote the circle of radius σ_l^n around the origin, for $l = 1, 2$ and $n \ge 1$. Noting that π^ε is also the spectral projection for $\mathcal{P}_\varepsilon^n$ associated to $\operatorname{spec}(\mathcal{P}_\varepsilon^n) \cap \{w \in \mathbb{C} : \sigma_1^n < |w| < \sigma_2^n\}$,

$$\pi^\varepsilon = \frac{1}{2\pi i} \int_{C_1^n \cup C_2^n} R(\mathcal{P}_\varepsilon^n, w)\, dw.$$

for each $\varepsilon \ge 0$. Then, recall also (3.14), (3.16),

$$\begin{aligned}
|||\pi^\varepsilon - \pi^0|||_n &\le \frac{1}{2\pi} \sum_{l=1}^{2} \operatorname{length}(C_l^n) \cdot \sup_{w \in C_l^n} |||R(\mathcal{P}_\varepsilon^n, w) - R(\mathcal{P}_0^n, w)|||_n \\
&\le \sum_{l=1}^{2} \sigma_l^n \sup_{|z| = \sigma_l} \Big|\Big|\Big| \sum_{k=1}^{\infty} \big(R(\mathcal{P}_0^n, z^n)(\mathcal{P}_\varepsilon^n - \mathcal{P}_0^n)\big)^k \Big|\Big|\Big|_n \\
&\le \sum_{l=1}^{2} \sigma_l^n \sum_{k=1}^{\infty} (4C\theta^n \sigma_l^{-n})^k \le \sum_{l=1}^{2} \frac{4C\theta^n}{1 - (\theta/\sigma_l)^n} \le 16C\theta^n \le \frac{1}{2}
\end{aligned}$$

if n is large and $0 < \varepsilon \le \varepsilon(n)$. It follows that, given any $\varphi \in Y_\varepsilon \cap X_0$,

$$|||\varphi|||_n = |||\pi^\varepsilon \varphi||| = |||(\pi^\varepsilon - \pi^0)\varphi|||_n \le \frac{1}{2} |||\varphi|||_n$$

and so $\varphi = 0$. Since X_0 is a hyperplane, this proves that Y_ε is 1-dimensional. □

We apply this statement to $\mathcal{P}_0 = \mathcal{L}$ and $\mathcal{P}_\varepsilon = \widehat{\mathcal{L}}_\varepsilon$, any small $\varepsilon > 0$. Assumption (1) is clear from the definitions, (2) was proved in Proposition 3.1 and in (3.13), (a) is a direct consequence of Lemma 3.14 and Fubini's theorem, and (b) was obtained in Proposition 3.9. Then, Proposition 3.15, with $\widetilde{\Lambda} = \sigma$, follows from this lemma in just the same way as we derived Corollary 3.10 from Proposition 3.9.

Observe that, through the arguments following Corollary 2.13, the previous proposition implies

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) = \int \varphi \, d\mu_\varepsilon$$

for every continuous function φ and $m \times \theta_\varepsilon^{\mathbb{N}}$-almost all random trajectories $x_j = f_{\underline{t}}^j(x)$. At last, we are in a position to prove the main result in this section.

Proposition 3.17 (stochastic stability). *φ_ε converges to φ_0 in $L^1(m)$ as $\varepsilon \to 0$.*

Proof: Let $\widetilde{C} > 0$, $\tilde{\lambda} < 1$, and Y_n, $n \ge 1$, be as in Lemma 3.14, that is,

$$\|\mathcal{L}_{\underline{t}}^n 1 - \mathcal{L}^n 1\|_1 \le \widetilde{C} \tilde{\lambda}^n$$

for all $t_1, \dots, t_n \in Y_n$ and each $n \geq 1$. Given any $n \geq 1$, take $\varepsilon > 0$ to be small enough so that supp $\theta_\varepsilon \subset Y_n$. Then,

$$||\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1||_1 \leq \widetilde{C}\tilde{\lambda}^n .$$

On the other hand, Proposition 3.15 implies that

$$\left|\int (\widehat{\mathcal{L}}_\varepsilon^n 1 - \varphi_\varepsilon)\,\psi\,dm\right| = \left|\int (\widehat{U}_\varepsilon^n \psi)dm - \int \psi\,d\mu_\varepsilon\right| \leq \widetilde{K}\widetilde{\Lambda}^n \int |\psi|\,dm \leq \widetilde{K}\widetilde{\Lambda}^n \sup|\psi|$$

for every $n \geq 0$ and ε sufficiently small, and every bounded function ψ. Hence

$$||\widehat{\mathcal{L}}_\varepsilon^n 1 - \varphi_\varepsilon||_1 \leq \widetilde{K}\widetilde{\Lambda}^n \quad \text{for all } n \geq 0 \text{ and } \varepsilon > 0 \text{ small.}$$

In a similar way, using Corollary 3.10, $||\mathcal{L}^n 1 - \varphi_0||_1 \leq \widehat{K}\Lambda^n$ for all $n \geq 0$ and some $\widehat{K} > 0$. Altogether, this gives

$$||\varphi_\varepsilon - \varphi_0||_1 \leq \widehat{K}\Lambda^n + \widetilde{K}\widetilde{\Lambda}^n + \widetilde{C}\tilde{\lambda}^n$$

if ε is small enough, depending on n. The right hand side can be made arbitrarily small by choosing n large and so the proof is complete. □

Remark 3.3. Since every $\widehat{\mathcal{L}}_\varepsilon$ preserves $X_0 = \{\varphi \in \mathrm{BV} : \int \varphi\,dm = 0\}$, the spectral splitting corresponding to $\mathrm{spec}(\widehat{\mathcal{L}}_\varepsilon) = \{1\} \cup \Sigma_\varepsilon$ must be $\mathrm{BV} = \mathbb{R}\varphi_\varepsilon \oplus X_0$. It follows that the spectral projection π^ε is given by $\pi^\varepsilon\varphi = \varphi_\varepsilon \int \varphi\,dm$, and so Proposition 3.18 implies that $\|\pi_\varepsilon - \pi_0\|_1 = \|\varphi_\varepsilon - \varphi_0\|_1 \to 0$ as $\varepsilon \to 0$.

3.4 Infinitely many monotonicity intervals.

A large part of what we have been doing extends, under appropriate conditions, to piecewise expanding maps with countably many domains of smoothness. In this section we briefly describe some of the lines along which such an extension can be carried out, in particular, we prove a statement of existence of absolutely continuous invariant measure. The following example of a piecewise expanding map with no absolutely continuous invariant measure (either finite or infinite) shows that some set of assumptions is necessary for this.

Example 3.7. ([Ry83]) Let $f : [0,1] \to [0,1]$ be given by $f(0) = 0$ and

$$f(x) = 2x - 2^{-j+1}, \quad \text{for } x \in \eta_j = (2^{-j}, 2^{-j+1}] \text{ and each } j \geq 1.$$

Let μ be any f-invariant measure (possibly infinite). Suppose first that $\mu(\eta_1) > 0$. Then $\mu(f^{-n}(\eta_1)) = \mu(\eta_1) > 0$ for every $n \geq 1$. Since the preimages $f^{-n}(\eta_1)$ form a decreasing sequence of intervals whose intersection consists of the single point 1, we conclude that $\mu(\{1\}) = \mu(\eta_1) > 0$. Now suppose that $\mu(\eta_j) = 0$ for each $1 \leq j \leq k$, but $\mu(\eta_{k+1}) > 0$. The same arguments as before, applied to η_{k+1} and the restriction $f : [0, 2^{-k}] \to [0, 2^{-k}]$, shows that $\mu(\{2^{-k}\}) = \mu(\eta_{k+1}) > 0$. Finally, if $\mu(\eta_j) = 0$ for every $j \geq 1$ then $\mu(\{0\}) > 0$. Hence, in either case μ can not be absolutely continuous with respect to Lebesgue measure. In fact, the only invariant measures of f are Dirac measures supported on the fixed points $\{0, 2^{-k}, k \geq 0\}$. We leave to the reader to check that Lebesgue almost every orbit converges to 0, and so the Dirac measure at zero is the unique SRB measure of f.

Throughout, we suppose there is a countable partition of M into intervals η_j, $j \geq 1$, such that the restriction of f to each η_j is a C^1 diffeomorphism onto $f(\eta_j)$. We denote $g_{\eta_j} = 1/|Df|\eta_j|$ and $\lambda_j = \sup g_{\eta_j}$, and then we suppose that

(a) there is λ such that $\lambda_j \leq \lambda < 1$ for every $j \geq 1$.

For the time being, we assume that there are $K_1 > 0$ and $\delta > 0$ such that

(b1) for each $j \geq 1$, the restriction of f to η_j is of class C^2, with

$$\sup_{j\geq 1}\sup \frac{|D^2(f|\eta_j)|}{|D(f|\eta_j)|^2} = \sup_{j\geq 1}\sup |Dg_{\eta_j}| \leq K_1;$$

(c1) $m(f(\eta_j)) \geq \delta$ for every $j \geq 1$.

For the inequality in (b1) just note that

$$Dg_{\eta_j} = D\left(\pm \frac{1}{D(f|\eta_j)}\right) = \mp \frac{D^2(f|\eta_j)}{D(f|\eta_j)^2}.$$

Assumptions (b1), (c1) are unnecessarily strong, and we shall relax them later in this section. On the other hand, they already include some relevant examples, such as the Gauss map and small perturbations of it. Another interesting application is in the study of maps with neutral fixed points, as in Example 1.3 in the Introduction, which we discuss in the next section.

From the expression of the transfer operator

$$\mathcal{L}\varphi = \sum_{j\geq 1} (g_{\eta_j}\varphi) \circ (f|\eta_j)^{-1} \chi_{f(\eta_j)},$$

we get

$$\operatorname{var} \mathcal{L}\varphi \leq \sum_{j\geq 1} \Big(\operatorname*{var}_{f(\eta_j)} (g_{\eta_j}\varphi) \circ (f|\eta_j)^{-1}\Big) + 2 \sup_{f(\eta_j)} \left|(g_{\eta_j}\varphi) \circ (f|\eta_j)^{-1}\right|.$$

Using properties (v3), (v5) and assumptions (a) and (b1),

$$\begin{aligned}\operatorname*{var}_{f(\eta_j)} \left((g_{\eta_j}\varphi) \circ (f|\eta_j)^{-1}\right) = \operatorname*{var}_{\eta_j} \left(g_{\eta_j}\varphi\right) &\leq \sup_{\eta_j} g_{\eta_j} \cdot \operatorname*{var}_{\eta_j} \varphi + \sup_{\eta_j} |Dg_{\eta_j}| \int_{\eta_j} \varphi \, dm \\ &\leq \lambda \operatorname*{var}_{\eta_j} \varphi + K_1 \int_{\eta_j} \varphi \, dm.\end{aligned}$$

On the other hand, by (3.2),

$$\sup_{f(\eta_j)} \left|(g_{\eta_j}\varphi)\circ(f|\eta_j)^{-1}\right| \le \operatorname*{var}_{f(\eta_j)}\left((g_{\eta_j}\varphi)\circ(f|\eta_j)^{-1}\right)+ \\ + \frac{1}{m(f(\eta_j))}\int_{f(\eta_j)}|\varphi\circ(f|\eta_j)^{-1}||D(f|\eta_j)^{-1}|\,dm \\ \le \lambda \operatorname*{var}_{\eta_j}\varphi + (K_1+\frac{1}{\delta})\int_{\eta_j}|\varphi|\,dm.$$

Replacing above, we conclude that

$$\operatorname{var}\mathcal{L}\varphi \le 3\lambda \operatorname{var}\varphi + (3K_1+\frac{2}{\delta})\int|\varphi|\,dm \tag{3.17}$$

for every function φ with bounded variation.

Proposition 3.18. *Let f be a piecewise smooth map with countably many smoothness domains satisfying (a), (b1), (c1) above, and suppose that $3\lambda < 1$. Then f has some absolutely continuous invariant probability measure μ_0. Moreover, if μ is any such measure then $\mu = \varphi m$, where φ has bounded variation.*

Proof: Let us write $\widetilde{K}_1 = 3K_1 + 2/\delta$. Iterating (3.17) and using once more the fact that $\int|\mathcal{L}\psi|\,dm \le \int\mathcal{L}|\psi|\,dm = \int|\psi|\,dm$ for every integrable function ψ, we get

$$\operatorname{var}(\mathcal{L}^n\varphi) \le 3\lambda\operatorname{var}(\mathcal{L}^{n-1}\varphi) + \widetilde{K}_1\int|\varphi|\,dm \le (3\lambda)^n\operatorname{var}\varphi + \widetilde{K}_1\sum_{j=0}^{\infty}(3\lambda)^j\int|\varphi|\,dm,$$

for every $n \ge 1$ and every bounded variation function φ. This is analog to Proposition 3.1, and now the proposition can be proved in the same way as Corollary 3.4. □

Conditions (b1) and, specially, (c1) are too restrictive for certain applications, see Chapter 5 and Example 3.8 below. In the sequel we explain how the conclusion of Proposition 3.18 may be derived under more general hypotheses: we keep (a) but replace (b1) and (c1) by

(b2) $\operatorname{var}_{\eta_j} g_{\eta_j} \le K_2\lambda_j$ for each $j \ge 1$;

(c2) $\sum_{j\ge1}\lambda_j \le K_2$;

for some $K_2 > 0$. Clearly, (b2) is weaker than (b1), and it is easy to see that (c2) is a consequence of (b1) and (c1): for appropriate $x_j \in \eta_j$,

$$\frac{1}{\delta} \ge \sum_{j\ge1}\frac{m(\eta_j)}{m(f(\eta_j))} = \sum_{j\ge1} g_{\eta_j}(x_j) \ge \frac{1}{K_1}\sum_{j\ge1}\lambda_j.$$

The following example of a piecewise expanding transformation satisfying (a), (b2), (c2), but not (c1), is inspired by the study of unimodal maps of the interval, that we shall address in Chapter 5.

Example 3.8. Given $q \ge 1$ let $J_{-q} = [-1, -e^{-q})$ and $J_{+q} = (e^{-q}, 1]$. Moreover, for each $r \ge q$ denote

$$I_{-r} = [-e^{-r}, -e^{-r-1}) \qquad I_{+r} = (e^{-r-1}, e^{-r}].$$

Then let $\sigma > 1$, $K > 0$, and $f : [-1,1] \to [-1,1]$ be any map such that

(i) f is C^1 on each of the intervals J_{-q}, J_{+q}, and I_{-r}, I_{+r}, for $r \ge q$;

(ii) $|Df(x)| \ge \sigma$ for every $x \in J_{-q} \cup J_{+q}$, and $|Df(x)| \ge \sigma^r$ for every $x \in I_{-r} \cup I_{+r}$ and every $r \ge q$;

(iii) $\operatorname{var}\big(1/(Df|J_{\pm q})\big) \le K$, and $\operatorname{var}\big(1/(Df|I_{\pm r})\big) \le K\sigma^{-r}$ for every $r \ge q$.

Any such map f satisfies (a), (b2), (c2): condition (ii) implies (a), (iii) implies (b2), and (c2) is clear. It is also easy to see that maps satisfying (i), (ii), (iii) do exist. For instance, one may take f to be affine on each of the monotonicity intervals J_{-q}, J_{+q}, I_{-r}, I_{+r}, $r \ge q$, with $|Df| = \sigma$ on $J_{-q} \cup J_{+q}$ and $|Df| = \sigma^r$ on $I_{-r} \cup I_{+r}$ for each $r \ge q$. Note that in this case,

$$\operatorname{length}(f(I_{\pm r})) = \sigma^r(1 - e^{-1})e^{-r}$$

for all $r \ge q$. Therefore, if one chooses $\sigma < e$ then there is no lower bound for the length of the images of monotonicity intervals, which means that (c1) is not satisfied.

In the remaining of this section we show that the conclusion of Proposition 3.18 remains valid under these more general consitions (a), (b2), (c2). Let $n \ge 1$ and η be a monotonicity interval for f^n. For each $1 \le i \le n$, the $(i-1)$th iterate $f^{i-1}(\eta)$ is contained in some monotonicity interval $\eta_{j(i)}$ of f, and the "itinerary" $j(1), \dots, j(n)$ completely characterizes η. In more precise terms, given any monotonicity interval η of f^n we may write

$$\eta = \{x \in M : f^{i-1}(x) \in \eta_{j(i)} \text{ for every } 1 \le i \le n\}$$

for a unique choice of $j(1), j(2), \dots, j(n)$. Then $g_\eta^{(n)} = 1/|Df^n|\eta|$ has

$$\sup g_\eta^{(n)} \le \lambda_{j(1)} \cdots \lambda_{j(n)} \le \lambda^n \quad \text{and} \quad \operatorname{var} g_\eta^{(n)} \le K_2\, n\, \lambda_{j(1)} \cdots \lambda_{j(n)} \le K_2\, n\, \lambda^n$$

(using (b2), cf. also (3.1)). Then, as in the proof of Proposition 3.1,

$$\operatorname{var}(\mathcal{L}^n\varphi) \le \lambda^n \operatorname{var}\varphi + \sum_\eta (\operatorname{var} g_\eta^{(n)} + 2\sup g_\eta^{(n)}) \sup|(\varphi|\eta)|,$$

where the sum is over all the monotonicity intervals η of f^n.

The main difficulty in estimating this sum lies on the fact that the length of the monotonicity intervals is not bounded from below. To bypass this, we split the sum into two parts: first, we consider the sum over a convenient finite subset $G(n)$ of intervals η, and apply to it the same argument as in the finite case; then, we use assumption (c2) to bound the sum of the remaining terms. Let $k \gg n$ be fixed such that

$$K_2^{n-1} \sum_{j>k} \lambda_j \leq \lambda^n.$$

We take $G(n)$ to be the set of intervals η for which $\max\{j(1), \ldots, j(n)\} \leq k$, and denote $\delta_n = \inf\{m(\eta) : \eta \in G(n)\} > 0$ (recall Remark 3.1). Then,

$$\begin{aligned}
\sum_{\eta \in G(n)} &(\operatorname{var} g_\eta^{(n)} + 2 \sup g_\eta^{(n)}) \sup |(\varphi|\eta)| \\
&\leq \sum_{\eta \in G(n)} (K_2 n + 2)\lambda^n \left(\operatorname*{var}_\eta \varphi + \frac{1}{m(\eta)} \int_\eta |\varphi|\, dm \right) \\
&\leq (K_2 n + 2)\lambda^n \operatorname{var} \varphi + (K_2 n + 2)\lambda^n \frac{\#G(n)}{\delta_n} \int |\varphi|\, dm.
\end{aligned}$$

On the other hand,

$$\sum_{\eta \notin G(n)} (\operatorname{var} g_\eta^{(n)} + 2 \sup g_\eta^{(n)}) \sup |(\varphi|\eta)| \leq \sup |\varphi| \sum_{j>k} \sum_{l=1}^{n} \sum (K_2 n + 2)\lambda_{j(1)} \cdots \lambda_{j(n)},$$

where the last sum is over those $(j(1), \ldots, j(n))$ so that $j(l) = \max\{j(1), \ldots, j(n)\} = j$, and l is minimum with this property. It follows that

$$\begin{aligned}
\sum_{\eta \notin G(n)} (\operatorname{var} g_\eta^{(n)} + 2 \sup g_\eta^{(n)}) \sup |(\varphi|\eta)| &\leq \sup |\varphi| \sum_{j>k} \sum_{l=1}^{n} (K_2 n + 2)\lambda_j \, (\sum_{i=1}^{\infty} \lambda_i)^{n-1} \\
&\leq \sup |\varphi| \sum_{j>k} (K_2 n^2 + 2n)\lambda_j K_2^{n-1} \\
&\leq \sup |\varphi| \, (K_2 n^2 + 2n)\lambda^n
\end{aligned}$$

in view of our choice of k. Replacing these estimates above,

$$\operatorname{var}(\mathcal{L}^n \varphi) \leq (K_2 n + 3)\lambda^n \operatorname{var} \varphi + (K_2 n^2 + 2n)\lambda^n \sup |\varphi| + (K_2 n + 2)\lambda^n \frac{\#G(n)}{\delta_n} \int |\varphi|\, dm$$

Fix $\lambda_0 \in (\lambda, 1)$ and then choose $K_0 \gg K_2$, so that

$$(K_2 n + 3)\lambda^n \leq K_0 \lambda_0^n \quad \text{and} \quad (K_2 n^2 + 2n)\lambda^n \leq K_0 \lambda_0^n \quad \text{for every } n \geq 1.$$

Moreover, denote $L(n)$ the factor multiplying the integral. Then,

$$\operatorname{var}(\mathcal{L}^n \varphi) \leq K_0 \lambda_0^n (\operatorname{var} \varphi + \sup |\varphi|) + L(n) \int |\varphi|\, dm.$$

Now we only have to remove the dependence on n of the integral term, and this can be done using the same idea as in the last part of the proof of Proposition 3.1: fix $N \geq 1$ large enough so that $K_0\lambda_0^N \leq 1/2$ and then, given any $n \geq 1$, divide $n = QN + r$ with $q \geq 0$ and $1 \leq r \leq N$. In this way we find, increasing K_0 if necessary,

$$\text{var}(\mathcal{L}^n\varphi) \leq K_0\lambda_0^n\,(\text{var}\,\varphi + \sup|\varphi|) + K_0\int|\varphi|\,dm. \tag{3.18}$$

Using also $\int|\mathcal{L}\psi|\,dm \leq \int|\psi|\,dm$ and $\sup|\psi| \leq \text{var}\,\psi + \int|\psi|\,dm$, one concludes that the sequence $(\mathcal{L}^n 1)_n$ is uniformly bounded and has uniformly bounded variation. At this point the arguments in the proof of Corollary 3.4 carry on completely to the present context, proving the following generalization of Proposition 3.18.

Proposition 3.19. *If f is a piecewise smooth map with countably many smoothness domains satisfying (a), (b2), (c2) above, then the conclusions of Proposition 3.18 remain valid for f.*

The following simple criterium for finitude is now an immediate consequence, cf. the proof of Corollary 3.5. Let $C = \cup_{j=1}^{\infty}\partial\eta_j$, the set of points where f fails to be smooth. If $f^n(C)$ is finite for some $n \geq 1$ then f has only finitely many absolutely continuous invariant measures.

3.5 Maps with neutral fixed points.

In this section, we apply the previous ideas to the class of maps in Example 1.3. Recall that we let $f : [0,1] \to [0,1]$ satisfy, for some $c \in (0,1)$,

(i) f is increasing and C^2 on $[0,c]$ and on $(c,1]$, with $f(0) = 0$ and $f(c^+) = 0$;

(ii) $Df(0) = 1$, but $|Df(x)| > 1$ for $x \neq 0$ (including $x = c^{\pm}$); moreover $D^2f(0) > 0$;

We prove that f admits an absolutely continuous invariant measure μ which is σ-finite, but *not* finite. Moreover, μ is unique up to multiplication by a constant. On the other hand, given any continuous function φ

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \varphi(0) \tag{3.19}$$

for Lebesgue almost every point x, which means that the SRB-measure of f is the Dirac probability supported at the neutral fixed point 0 (and not μ). In particular, the Lyapunov exponent is zero

$$\lim_{n\to+\infty}\frac{1}{n}\log|Df^n(x)| = 0 \quad \text{at almost every point } x \in [0,1].$$

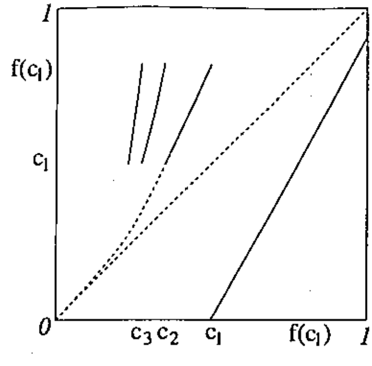

Figure 3.2: Inducing a piecewise expanding map

A first step is to replace f by a piecewise expanding map $\hat{f} : (0,1] \to (0,1]$ defined as follows. Denote $c_1 = c$, $\xi_0 = (0, c_1]$, $\xi_1 = (c_1, 1]$, $f_0^{-1} = (f|\xi_0)^{-1}$ and $f_1^{-1} = (f|\xi_1)^{-1}$. Let $\eta_0 = (c_1, f(c_1)] \subset \xi_1$ and, for each $j \geq 1$, let $c_{j+1} = f_0^{-1}(c_j)$ and $\eta_j = (c_{j+1}, c_j] \subset \xi_0$. Then define $\hat{f}$ by $\hat{f}|\xi_1 = f|\xi_1$ and $\hat{f}|\eta_j = f^j|\eta_j$ for $j \geq 1$; see Figure 3.2.

Note that $I_* = (0, f(c_1)]$ is invariant under $\hat{f}$, in the sense that $f(I_*) \subset I_*$. Moreover, $f^N(I) \subset I_*$, where $N \geq 1$ is the first integer such that $f^N(1) \leq f(c_1)$. For simplicity, we suppose $f^2(c_1) > c_1$ in all that follows. Then $\hat{f}$ is topologically mixing on I_*, as we shall see in the next paragraph: for any nonempty open subinterval J of I_* there exists $n \geq 1$ such that $f^n(J) = I_*$. Note that when $f^2(c_1) < c_1$ the map $\hat{f} : I_* \to I_*$ is not even surjective. However, one may still prove that the second iterate $\hat{f}^2$ is topologically mixing restricted to $(0, c_1)$. We leave the study of the case $f^2(c_1) \leq c_1$ as an exercise to the reader.

Let us prove our claim that f is topologically mixing on I_*. We start by noting that for any interval open $J \subset I_*$ there must be $m \geq 0$ such that $\hat{f}^m(J)$ contains some c_j, $j \geq 1$: otherwise, since $\hat{f}$ is uniformly expanding on each η_j, the length of the iterates of J would be unbounded. Then $\hat{f}^m(J) \supset (c_j, c_j + \varepsilon_1)$ for some $\varepsilon_1 > 0$. Moroever, replacing m by $m+1$ if necessary, we may suppose $j = 1$. This means that, up to replacing our interval by a convenient iterate, it is no restriction to suppose right from the start that J contains an interval $(c_1, c_1 + \varepsilon_1)$ with $\varepsilon_1 > 0$. Then $\hat{f}(J) \supset (0, \varepsilon_2)$ for some $\varepsilon_2 > 0$, and so $\hat{f}(J) \supset \eta_j$ for some $j \geq 1$. It follows that $\hat{f}^2(J) \supset (c_1, f(c_1)]$ and, as a consequence, $\hat{f}^3(J)$ contains $(0, f^2(c_1)] = (0, c_1] \cup (c_1, f^2(c_1)]$. On the one hand, this implies that $\hat{f}^4(J) \supset (c_1, f(c_1)]$, and so $\hat{f}^5(J) \supset (0, f^2(c_1)]$ On the other hand, as we suppose that $f^2(c_1) > c_1$, the interval $L = (c_1, f^2(c_1)]$ contains some interval $(c_1, c_1 + \varepsilon_3)$ with $\varepsilon_3 > 0$. Then, repeating a previous argument, $\hat{f}^5(J) \supset \hat{f}^2(L) \supset (c_1, f(c_1)]$. Altogether,

$$\hat{f}^5(J) \supset (0, f^2(c_1)] \cup (c_1, f(c_1)] = I_*,$$

which concludes the argument.

Now we show that $\hat{f}|I_*$ satisfies assumptions (a), (b1), (c1) of Section 3.4. Let $\lambda = \sup |Df|(\eta_1 \cup \eta_0)|^{-1}$. Then, by construction,

$$|D\hat{f}(x)| \geq \lambda > 1, \quad \text{at every } x \in I_*,$$

and this gives (a). Assumption (c1) follows immediately from the observation that $\hat{f}(\eta_j) = (c_1, f(c_1))$ for every $j \geq 1$, and so the family of intervals $\{f(\eta_j) : j \geq 0\}$ is finite. In order to check (b1), we begin by noting that our assumptions (i), (ii) give

$$|D(\log Df)| = \frac{|D^2 f|}{|Df|} \leq K_0 \quad \text{for some } K_0 > 0. \tag{3.20}$$

The following consequences are easy to deduce:

(1) There is $K_1 > 0$ such that, given any $j \geq 1$ and $x, y \in \eta_j$,

$$\frac{1}{K_1} \leq \frac{Df^j(x)}{Df^j(y)} \leq K_1.$$

Proof: Just use (3.20) together with the fact that $\sum_{i=0}^{j-1} |f^i(x) - f^i(y)| \leq m(I_*) \leq 1$.

(2) There is $K_2 > 0$ such that, given any $j \geq 1$ and $x_1 \in \eta_1, \ldots, x_j \in \eta_j$,

$$\sum_{l=1}^{j} \frac{1}{|Df^l(x_l)|} \leq K_2$$

Proof: Using (1) one gets $|Df^l(x_l)|^{-1} \approx m(\eta_l)/m(f^l(\eta_l)) \leq \text{const}\, m(\eta_l)$.

(3) There is $K_3 > 0$ such that $|D^2\hat{f}|/|D\hat{f}|^2 \leq K_3$.

Proof: It is clear that $|D^2\hat{f}|/|D\hat{f}|^2$ is bounded on η_0. For $x \in \eta_j$, $j \geq 1$, just use (3.20) and (2) in

$$\begin{aligned} \frac{D^2\hat{f}}{(D\hat{f})^2}(x) &= \frac{1}{D\hat{f}(x)} D(\log D\hat{f})(x) = \frac{1}{D\hat{f}(x)} \sum_{i=0}^{j-1} D\big(\log(Df \circ f^i)\big)(x) \\ &= \frac{1}{Df^j(x)} \sum_{i=0}^{j-1} \frac{D^2 f}{Df}(f^i(x)) Df^i(x) = \sum_{i=0}^{j-1} \frac{D^2 f}{Df}(f^i(x)) \frac{1}{Df^{j-i}(f^i(x))}. \end{aligned} \tag{3.21}$$

This proves property (b1) for $\hat{f}$.

Now we may conclude that $\hat{f}$ admits an absolutely continuous invariant probability measure $\hat{\mu} = \hat{\varphi} m$. If $\lambda < 1/3$ then this follows from a direct application of Proposition 3.18. To handle the general case $\lambda < 1$, it suffices to consider a convenient iterate of $\hat{f}$ as follows. Fix $N \geq 1$ such that $\lambda^N < 1/3$, and let $\hat{h} = \hat{f}^N$. Then $1/|D\hat{h}| \leq \lambda^N < 1/3$, wherever the derivative is defined. Moreover, $\hat{h}$ also satisfies conditions (b1) and (c1) (possibly for different values of K_1 and δ, but this is irrelevant). Indeed, from

$$\frac{D^2\hat{h}}{(D\hat{h})^2}(x) = \sum_{i=0}^{N-1} \frac{D^2\hat{f}}{D\hat{f}}(\hat{f}^i(x))\frac{1}{D\hat{f}^{N-i}(\hat{f}^i(x))}$$
$$= \sum_{i=0}^{N-1} \frac{D^2\hat{f}}{(D\hat{f})^2}(\hat{f}^i(x))\frac{1}{D\hat{f}^{N-i-1}(\hat{f}^{i+1}(x))},$$

cf. (3.21), we conclude that

$$\frac{|D^2\hat{h}|}{(D\hat{h})^2} \leq \sum_{i=0}^{N-1} K_1\lambda^{N-i-1} \leq K_1\sum_{l=0}^{\infty}\lambda^l,$$

which proves (b1) for $\hat{h}$. Next, observe that if η is a monotonicity interval for $\hat{h}$ then the boundary points of $\hat{h}(\eta)$ belong in

$$\bigcup_{i=1}^{N} \hat{f}^i(\{0,\ldots,c_2,c_1,f(c_1)\}) = \bigcup_{i=1}^{N} \hat{f}^{i-1}(\{0,c_1,f(c_1),f^2(c_1)\}).$$

Since this last set is finite, the family of all the images $\hat{h}(\eta)$ of monotonicity intervals of $\hat{h}$ is finite, and this implies property (c1) for $\hat{h}$. Then we may use Proposition 3.18 to conclude that $\hat{h}$ has some absolutely continuous invariant measure $\tilde{\mu}$. Finally, $\hat{\mu} = \sum_{i=0}^{N-1}(\hat{f}^i)_*\tilde{\mu}$ is an absolutely continuous invariant measure for $\hat{f}$: since $(\hat{f}^N)_*\tilde{\mu} = \tilde{\mu}$,

$$\hat{f}_*\hat{\mu} = \sum_{i=1}^{N}(\hat{f}^i)_*\tilde{\mu} = \sum_{i=0}^{N-1}(\hat{f}^i)_*\tilde{\mu} = \hat{\mu}.$$

Proposition 3.18 also states that the density of any $\hat{h}$-invariant absoutely continuous invariant measure is a function with bounded variation. In particular, if $\hat{\nu} = \hat{\psi}m$ is any $\hat{f}$-invariant measure then $\hat{\psi}$ has bounded variation: it suffices to note that $\hat{\nu}$ is also invariant under $\hat{h} = \hat{f}^N$. In particular, for any such $\hat{\nu}$ there exists an open interval $J \subset I_*$ such that $\inf(\hat{\psi}|J) > 0$ and then, by topological mixing, $\inf(\hat{\psi}|I_*) > 0$. This implies that all such measures are equivalent. Let $\hat{\mu}$ be the absolutely continuous invariant measure we have just exhibited. If A is a measurable set such that $\hat{f}^{-1}(A) = A$ and $\hat{\mu}(A) > 0$ then

$$\hat{\mu}_A(B) = \frac{\hat{\mu}(A\cap B)}{\hat{\mu}(A)}$$

defines an absolutely continuous $\hat{f}$-invariant measure with $\hat{\mu}_A(A^c) = 0$. Then, the two measures $\hat{\mu}$ and $\hat{\mu}_A$ being equivalent, $\hat{\mu}(A^c) = 0$. This proves that $\hat{\mu}$ is ergodic. As a consequence, it is the unique absolutely continuous invariant measure.

From $\hat{\mu}$ we may now construct an absolutely continuous measure μ invariant under the initial map f: let

$$\mu = (\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty}\sum_{i=0}^{j-1} f_*^i(\hat{\mu}|\eta_j). \tag{3.22}$$

Clearly, μ is σ-finite. It is also easy to see that it is f-invariant:

$$f_*\mu = f_*(\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty}\sum_{i=1}^{j} f_*^i(\hat{\mu}|\eta_j) = (\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty}\sum_{i=0}^{j-1} f_*^i(\hat{\mu}|\eta_j) = \mu$$

where the second equality results from

$$f_*(\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty} f_*^j(\hat{\mu}|\eta_j) = \hat{f}_*(\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty} \hat{f}_*(\hat{\mu}|\eta_j) = \hat{f}_*\hat{\mu} = \hat{\mu} = (\hat{\mu}|\eta_0) + \sum_{j=1}^{\infty}(\hat{\mu}|\eta_j).$$

Observe that each $f_*^i(\mu|\eta_j)$, $0 \le i < j$, is an absolutely continuous invariant measure supported on $f^i(\eta_j)$, with density $(\hat{\varphi}/|Df^i|) \circ (f_0^{-i})$. Here, f_0^{-i} is the inverse of the restriction of f^i to $[0, c_i]$, recall also that $\hat{\varphi}$ denotes the density of the $\hat{f}$-invariant measure $\hat{\mu}$. Then, for every y belonging in some η_j, $j \ge 1$,

$$\frac{d\mu}{dm}(y) = \sum_{i=0}^{\infty} \frac{\hat{\varphi}}{|Df^i|}(f_0^{-i}(y)).$$

Moreover, $(d\mu/dm)(y) = \hat{\varphi}(y)$ for $y \in \eta_0$. This proves that μ is absolutely continuous with respect to Lebesgue measure, with density given by

$$\varphi = \frac{d\mu}{dm} = \hat{\varphi}\chi_{\eta_0} + \left(\sum_{i=0}^{\infty} \frac{\hat{\varphi}}{|Df^i|} \circ f_0^{-i}\right)\chi_{\xi_0}. \tag{3.23}$$

Recall that $\xi_0 = (0, c_1] = \cup_{j=1}^{\infty}\eta_j$.

So far, we have not used the fact that 0 is a neutral fixed point, which is now needed to conclude that μ is an infinite measure. Let us begin by the following estimate.

(4) There is $\alpha > 0$ such that $c_j \ge \alpha/j$ for every $j \ge 1$.

Since we suppose $D^2f(0) > 0$, there are $a > 0$ and $k \ge 1$ such that $c_{j+1} \ge c_j - ac_j^2$ for every $j \ge k$. We take $\alpha = \min\{c_1, 2c_2, \dots, kc_k, 1/2a\}$, so that (4) is automatic for every $1 \le j \le k$. Now let $j \ge k$ and suppose that it satisfies $c_j \ge \alpha/j$. Then

$$c_{j+1} \ge c_j - ac_j^2 \ge \frac{\alpha}{j} - \frac{a\alpha^2}{j^2} = \frac{\alpha}{j+1}\left(1 + \frac{j - a\alpha(j+1)}{j^2}\right) \ge \frac{\alpha}{j+1}$$

(the last inequality uses $\alpha \le 1/2a$). This proves (4), by induction. It follows that

$$\mu(I) = \hat{\mu}(\eta_0) + \sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\hat{\mu}(\eta_j) = \hat{\mu}(\eta_0) + \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}\hat{\mu}(\eta_j)$$
$$= \hat{\mu}(\eta_0) + \sum_{i=0}^{\infty}\hat{\mu}([0,c_i]) \ge \sum_{i=1}^{\infty}\left(\inf\hat{\varphi}\right)\frac{\alpha}{i},$$

which diverges. Thus, the measure μ is indeed infinite, as stated.

Now we show that any absolutely continuous f-invariant measure $\nu = \psi m$ is a multiple of μ. In other words, there is some constant $C > 0$ such that

$$\psi = \hat{\psi}\chi_{\eta_0} + \left(\sum_{i=0}^{\infty}\frac{\hat{\psi}}{|Df^i|}\circ f_0^{-i}\right)\chi_{\xi_0} \quad \text{with } \hat{\psi} = C\hat{\varphi}, \tag{3.24}$$

compare (3.23). In particular, ν can not be finite. Given any function h, define

$$\mathcal{L}_0 h = \frac{h}{|Df|}\circ f_0^{-1} \quad \text{and} \quad \mathcal{L}_1 h = \frac{h}{|Df|}\circ f_1^{-1}$$

(understood as being identically zero outside $f(\xi_0)$ and $f(\xi_1)$, respectively). Then the transfer operators $\mathcal{L}$, of f, and $\widehat{\mathcal{L}}$, of $\hat{f}$, are given by

$$\mathcal{L}h = \frac{h}{|Df|}\circ f_1^{-1} + \frac{h}{|Df|}\circ f_0^{-1} = \mathcal{L}_1 h + \mathcal{L}_0 h$$

and

$$\widehat{\mathcal{L}}h = \frac{h}{|Df|}\circ f_1^{-1} + \sum_{j=1}^{\infty}\frac{h}{|Df|}\circ (f^j|\eta_j)^{-1} = \mathcal{L}_1 h + \sum_{j=1}^{\infty}\mathcal{L}_0^j\big(h\,\chi_{\eta_j}\big).$$

Let us introduce the function

$$\hat{\psi} = \psi - (\mathcal{L}_0\psi)\,\chi_{\xi_0} = \psi - \mathcal{L}_0\big(\psi\,\chi_{(0,c_2]}\big).$$

Then, by recurrence, we have

$$\psi = \hat{\psi} + (\mathcal{L}_0\psi)\,\chi_{\xi_0} = \hat{\psi} + (\mathcal{L}_0\hat{\psi})\,\chi_{\xi_0} + (\mathcal{L}_0^2\psi)\,\chi_{\xi_0} = \hat{\psi} + \sum_{i=1}^{n-1}(\mathcal{L}_0^i\hat{\psi})\,\chi_{\xi_0} + (\mathcal{L}_0^n\psi)\,\chi_{\xi_0}$$

for every $n \ge 1$. As part of proving (2) we got that $|Df^n|\circ f_0^{-n} \to \infty$ as $n \to \infty$. Then $\mathcal{L}_0^n\psi = (\psi/|Df^n|)\circ f_0^{-n}$ goes to zero as $n \to \infty$, and we conclude that

$$\psi = \hat{\psi} + \sum_{i=1}^{\infty}(\mathcal{L}_0^i\hat{\psi})\,\chi_{\xi_0} = \hat{\psi}\,\chi_{\eta_0} + \sum_{i=0}^{\infty}(\mathcal{L}_0^i\hat{\psi})\,\chi_{\xi_0}.$$

Now, to prove (3.24) we only have to show that $\hat{\psi} = C\hat{\varphi}$ for some constant C. Moreover, to do this it suffices to prove that $\hat{\psi}$ is a fixed point of the operator $\widehat{\mathcal{L}}$. Indeed, we already proved that $\hat{f}$ has a unique invariant probability measure $\hat{\mu}$, and this means that every positive function which is a fixed point of $\hat{\mathcal{L}}$ must be a multiple of $\hat{\varphi}$. The first step in the proof that $\widehat{\mathcal{L}}\hat{\psi} = \hat{\psi}$ is

$$\widehat{\mathcal{L}}\hat{\psi} = \mathcal{L}_1\psi - \mathcal{L}_1\big((\mathcal{L}_0\psi)\,\chi_{\xi_0}\big) + \sum_{j=1}^{\infty} \mathcal{L}_0^j\big(\psi\,\chi_{\eta_j}\big) - \sum_{j=1}^{\infty} \mathcal{L}_0^j\big((\mathcal{L}_0\psi)\,\chi_{\xi_0}\,\chi_{\eta_j}\big),$$

where we have just used the definitions of $\hat{\psi}$ and $\widehat{\mathcal{L}}$. Observe that

$$\mathcal{L}_1\big((\mathcal{L}_0\psi)\,\chi_{\xi_0}\big) = \frac{(\mathcal{L}_0\psi)\,\chi_{\xi_0}}{|Df|} \circ f_1^{-1} = 0$$

because $(\mathcal{L}_0\psi)\,\chi_{\xi_0}$ is zero on the image ξ_1 of f_1^{-1}. On the other hand,

$$\mathcal{L}_0^j\big((\mathcal{L}_0\psi)\,\chi_{\xi_0}\,\chi_{\eta_j}\big) = \mathcal{L}_0^j\big((\mathcal{L}_0\psi)\,\chi_{\eta_j}\big) = \mathcal{L}_0^{j+1}\big(\psi\,\chi_{\eta_{j+1}}\big),$$

for every $j \geq 1$. This means that the two sums above cancel each other out , except for the first term in the first sum, and so we get

$$\widehat{\mathcal{L}}\hat{\psi} = \mathcal{L}_1\psi + \mathcal{L}_0\big(\psi\,\chi_{\eta_1}\big) = \mathcal{L}_1\psi + (\mathcal{L}_0\psi)\,\chi_{\xi_1}\,.$$

As $\nu = \psi\, m$ is an f-invariant measure, ψ satisfies $\psi = \mathcal{L}\psi = \mathcal{L}_0\psi + \mathcal{L}_1\psi$. Using this in the previous equation,

$$\widehat{\mathcal{L}}\hat{\psi} = \psi - \mathcal{L}_0\psi + (\mathcal{L}_0\psi)\,\chi_{\xi_1} = \psi - (\mathcal{L}_0\psi)\,\chi_{\xi_0} = \hat{\psi},$$

as claimed. The proof of (3.24) is complete.

Finally, we show that typical trajectories spend most of the time close to the neutral fixed point 0. In precise terms,

$$\tau_n(k) = \frac{1}{n}\#\{0 \leq s < n : f^s(x) \in (c_{k+1}, f(c_1)]\} \to 0 \quad \text{as } n \to \infty, \tag{3.25}$$

for any $k \geq 1$ and Lebesgue almost any point $x \in I_*$. Observe that (3.19) is a direct consequence of this claim. First, we invoke the fact that $\hat{\mu}$ is an ergodic measure for $\hat{f}$, to conclude that

$$\theta_m(j) = \frac{1}{m}\#\{0 \leq i < m : \hat{f}^i(x) \in \eta_j\} \to \hat{\mu}(\eta_j) \quad \text{as } m \to \infty \tag{3.26}$$

for each $j \geq 0$ and $\hat{\mu}$-almost every point $x \in I_*$. Since $\hat{\mu}$ is equivalent to Lebesgue measure on I_*, it follows that Lebesgue almost every $x \in I_*$ satisfies (3.26). Now we want to express this information in terms of iterates of the map f. Let $k \geq 1$

be fixed, and x be any point satisfying (3.26). To deal with the two different time scales, corresponding to our two maps, we introduce the increasing sequence $l(i)$ given by $\hat{f}^i = f^{l(i)}$, for each $i \geq 0$. By the definition of $\hat{f}$,

$$l(i+1) = l(i) + 1 \text{ if } \hat{f}^i(x) \in \eta_0 \quad \text{and} \quad l(i+1) = l(i) + j \text{ if } \hat{f}^i(x) \in \eta_j\,,\ j \geq 1,$$

Given any $n \geq 1$, let m be the unique integer such that $l(m) \leq n < l(m+1)$. Up to time m the $\hat{f}$-trajectory of x hits each interval η_j exactly $m\theta_m(j)$ times. So,

$$n = (n - l(m)) + l(m) = (n - l(m)) + m\theta_m(0) + \sum_{j\geq 1} jm\theta_m(j).$$

Given $i \leq m$ let j be such that $\hat{f}^i(x) \in \eta_j$. If $0 \leq j \leq k$ then all the f-iterates of x in the time interval $[l(i), l(i+1))$ belong in $(c_{k+1}, f(c_1)]$. If $j > k$ then exactly k iterates of the f-orbit of x in the time interval $[l(i), l(i+1))$ belong in $(c_{k+1}, f(c_1)]$. In particular, in the time interval $[l(m), n)$ there can be at most k iterates in $(c_{k+1}, f(c_1)]$. Altogether, this means that the f-orbit of x spends not more than

$$m\theta_m(0) + \sum_{1\leq j\leq k} jm\theta_m(j) + \sum_{j>k} km\theta_m(j) + k$$

of its first n iterates in the interval $(c_{k+1}, f(c_1)]$. This gives, using $\sum_{j\geq 0} \theta_m(j) = 1$,

$$\tau_n(k) = \frac{\theta_m(0) + \sum_{1\leq j\leq k} j\theta_m(j) + \sum_{j>k} k\theta_m(j) + k/m}{(n - l(m))/m + \theta_m(0) + \sum_{j\geq 1} j\theta_m(j)} \leq \frac{2k}{\theta_m(0) + \sum_{j\geq 1} j\theta_m(j)}.$$

By (3.26), the denominator converges to $\hat{\mu}(\eta_0) + \sum_{j\geq 1} j\hat{\mu}(\eta_j) = \mu(I) = \infty$. It follows that $\tau_n(k) \to 0$ as $n \to \infty$, as claimed in (3.25).

Notes.

Existence of absolutely continuous invariant measures, Corollary 3.4, was proved by Lasota-Yorke [LY73] in the C^2 case (the map has a C^2 extension to the closure of each monotonicity interval). In the same setting, [LY78] obtained the finitude statement in Corollary 3.5. [Wo78] extended their conclusions to the bounded variation case. The Markov case was treated e.g. by [Bo79].

Propositions 3.7 and 3.8 are from [Li95a]. Lemma 3.12 was proved by [Bo77] in the C^2 case, and extended by [Ra78] to the bounded variation case. They used it to deduce a stronger version of Proposition 3.13.

A central limit theorem for one-dimensional piecewise expanding maps was proved by [Ke80].

Stochastic stability for these maps was first proved by [Ke82]. The approach we follow to get Propositions 3.15 and 3.17, including the perturbation Lemma 3.16, is due to [BaY93] (their arguments hold in the more general framework of perturbations we consider here).

Maps with infinitely many monotonicity intervals were studied by [Ry83], who extended conclusions of [LY73] and [HK82] to such maps. Transformations with neutral fixed points got the attention of several people, see e.g. [CI96] and references therein.

4. UNIFORMLY HYPERBOLIC ATTRACTORS

Now we carry on to the setting of uniformly hyperbolic attractors of diffeomorphisms the program developed in Chapter 2 for expanding maps. We consider $f : M \to M$ to be a diffeomorphism on the manifold M and $Q \subset M$ to be some positively invariant open set, in the sense that $f(\text{closure}(Q)) \subset Q$. We assume that the maximal invariant set

$$\Lambda = \bigcap_{n \geq 0} f^n(Q)$$

is transitive (i.e. contains dense orbits) and hyperbolic for f. This last property means that there is a splitting

$$T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$$

of the tangent bundle to M on Λ, and there are $\lambda_0 < 1$ and some riemannian metric $\|\cdot\|$ on M, such that

a) $Df(x) \cdot E^s_x = E^s_{f(x)}$ and $Df^{-1}(x) \cdot E^u_x = E^u_{f^{-1}(x)}$;

b) $\|Df(x)|E^s_x\| \leq \lambda_0$ and $\|Df^{-1}(x)|E^u_x\| \leq \lambda_0$ for every $x \in \Lambda$.

In Appendix A we recall some basic definitions and properties of hyperbolic sets which are used in the sequel. Our goal in the present chapter is to prove

Theorem 4.1. *Let Λ be a hyperbolic attractor for a C^2 diffeomorphism f as above. Then*

(1) *f admits a unique SRB-measure μ_0 supported on Λ and this measure is ergodic; moreover, the basin $B(\mu_0)$ contains a full Lebesgue measure subset of Q;*
(2) *(f, μ_0) is exponentially mixing and satisfies the central limit theorem in the Banach space of ν-Hölder continuous functions, for any ν in some interval $(0, \nu_1]$;*
(3) *(f, μ_0) is stochastically stable under small random perturbations.*

The idea to prove Theorem 4.1 is, once more, to derive the ergodic properties of f in the statement from spectral properties of a convenient transfer operator $\mathcal{L}$. However, there is one crucial difference if one compares this with the context treated in Chapter 2. In the expanding case the argument relies on local branches of f^{-1} contracting distances: then, taking advantage of the fact that $\mathcal{L}\varphi(y)$ is defined in terms of $\varphi|f^{-1}(y)$, one deduces that the transfer operator improves the regularity of functions, cf. Proposition 2.4. Clearly, this can no longer hold in the general hyperbolic case, where expansion and contraction coexist: on the contrary, one should expect the operator to worsen, rather than improve, regularity along the contracting (or stable) direction E^s. A related fact is that SRB-measures are, in general, singular with respect to Lebesgue measure (even when $\Lambda = M$).

There are two basic strategies to bypass this difficulty. A first one is to take the transfer operator to act on some larger Banach space containing objects more general than functions (e.g. the "density" of the SRB-measure). Alternatively, one may try and keep working with fairly unsophisticated Banach spaces by introducing some kind of quotient with respect to the contracting direction. The approach we adopt here is based on the idea of integrating along this stable direction: instead of dealing directly with observable functions we always consider their averages on local stable manifolds.

For the sake of clearness, most of the time we shall refer ourselves to a concrete model, namely solenoid embeddings of the (open) solid torus $Q = S^1 \times B^2 \hookrightarrow \mathbb{R}^3$, see Figure 4.1 and Appendix A. In more precise terms, we shall consider the map $f : Q \to Q$ to be in some small C^2-neighbourhood $\mathcal{N}$ of an embedding of the form

$$S^1 \times B^2 \ni (\theta, z) \mapsto (2\theta \bmod \mathbb{Z}, \phi(\theta) + Az) \in S^1 \times B^2 \tag{4.1}$$

where $\phi : S^1 \to B^2$ is of class C^2, and the linear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ has $\|A\| < 1$. We take advantage of some features of this model, such as globally defined coordinate systems, to rid the presentation of unnecessary technicalities and thus make the main ideas more transparent. But we make no essential use of specific properties of these maps, and all our arguments extend in fairly straightforward ways to the general context of uniformly hyperbolic attractors.

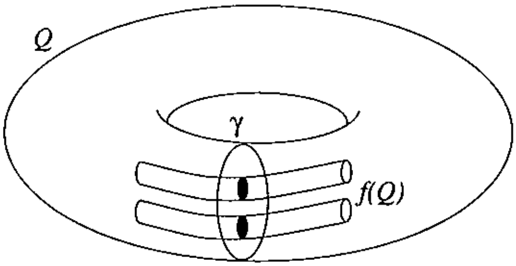

Figure 4.1: The solenoid on the solid torus Q

4.1 Transfer operators and invariant cones.

It is no restriction to suppose that the diameter of Q is at most 1, and we do so in all that follows. Let m be Lebesgue measure on Q, normalized by $m(Q) = 1$. We introduce linear operators

$$(U\varphi)(x) = \varphi(f(x))$$

and

$$(\mathcal{L}\varphi)(y) = \begin{cases} \varphi(f^{-1}(y))|\det Df(f^{-1}(y))|^{-1}, & \text{if } y \in f(Q); \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, changing variables $y = f(x)$,

$$\begin{aligned} \int_Q (\mathcal{L}\varphi)\psi\, dm &= \int_{f(Q)} \frac{\varphi(f^{-1}(y))}{|\det Df(f^{-1}(y))|} \psi(y)dm(y) \\ &= \int_Q \varphi(x)\psi(f(x))dm(x) = \int_Q \varphi(U\psi)dm. \end{aligned}$$

As mentioned before, we want to analyse the action of $\mathcal{L}$ on observable functions in terms of corresponding averages on local stable leaves. By a *local stable leaf* we mean any connected component γ of the intersection $W^s(\xi) \cap Q$ of the stable manifold of some point $\xi \in \Lambda$ with the open set Q; see Figure 4.1 and Appendix A. Of course, there is no canonical choice of probabilities supported on stable leaves. Instead, we average with respect to a whole class of measures, namely, the cone of Hölder continuous densities $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$, defined by

$$\mathcal{D}(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho(x) > 0 \text{ for all } x \in \gamma \text{ and } \log \rho \text{ is } (a, \mu) - \text{Hölder}\},$$

for each local stable leaf $\gamma \subset Q$. It is easy to check that $\mathcal{D}(\gamma)$ is a convex cone, let $\theta = \theta_\gamma$ denote the corresponding projective metric. We also need a similar cone

$$\mathcal{D}_1(\gamma) = \mathcal{D}(a_1, \mu_1, \gamma),$$

corresponding to better Hölder constants $a_1 > 0$ and $0 < \mu_1 < 1$: we shall take $a \gg a_1 \gg 1$ and $0 < \mu < \mu_1 < 1$, cf. Lemma 4.2, (4.7), (4.11) below. Finally, we also consider the projective metric $\theta_+ = \theta_{+,\gamma}$ associated to the cone of positive densities

$$\mathcal{D}_+(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho > 0\}.$$

Given $\varphi: M \to \mathbb{R}$ and $\rho \in \mathcal{D}(\gamma)$ we denote by $\int_\gamma \varphi\rho$ the integral of φ for the measure ρm_γ, where m_γ is the smooth measure induced on γ by the riemannian metric. We always suppose $\int_\gamma \rho = 1$, unless otherwise specified. Let $\varphi: M \to \mathbb{R}$ and $\mathcal{L}\varphi$ be as defined above. Given any stable leaf γ, let γ_1 and γ_2 be the stable leaves such that $f(\gamma_j) \subset \gamma$ for $j = 1, 2$. Then, for any $\rho \in \mathcal{D}(\gamma)$,

$$\begin{aligned}\int_\gamma (\mathcal{L}\varphi)\rho &= \sum_{j=1}^{2} \int_{f(\gamma_j)} \frac{\varphi(f^{-1}(y))}{|\det Df(f^{-1}(y))|}\rho(y) \\ &= \sum_{j=1}^{2} \int_{\gamma_j} \varphi(x) \frac{\det(Df \mid \gamma_j)(x)|}{|\det Df(x)|}\rho(f(x)) = \sum_{j=1}^{2} \int_{\gamma_j} \varphi\rho_j\end{aligned}$$

where $(Df \mid \gamma_j)$ denotes the restriction of Df to the tangent space of γ_j and

$$\rho_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho \circ f). \tag{4.2}$$

In the next lemma we use the fact that local stable leaves form a continuous family of C^2 embedded submanifolds of Q, in particular, they have have uniformly bounded curvature; see Appendix A.

Lemma 4.2. *There are $\lambda_1 < 1$ and $\Lambda_1 < 1$ such that, if a is large enough,*

a) $\rho \in \mathcal{D}(\gamma) \Rightarrow \rho_j \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ for $j = 1, 2$

b) $\rho', \rho'' \in \mathcal{D}(\gamma) \Rightarrow \theta_j(\rho'_j, \rho''_j) \leq \Lambda_1 \theta(\rho', \rho'')$ for $j = 1, 2$, *where θ and θ_j are the projective metrics associated to $\mathcal{D}(\gamma)$ and to $\mathcal{D}(\gamma_j)$, respectively.*

Proof: Clearly, $\rho > 0 \Rightarrow \rho_j > 0$. Let $K_1 > 0$, respectively $K_2 > 0$, be a Lipschitz constant for $\log|\det Df|$, respectively for $\log|\det(Df \mid \gamma_j)|$. Note that K_1 depends only on f, whereas K_2 depends also on some uniform bound for the curvature of stable leaves. Then

$$\begin{aligned}|\log\rho_j(x) - \log\rho_j(y)| &\leq |\log\rho(f\,x) - \log\rho(f\,y)| + |\log|\det Df(y)| - \log|\det Df(x)||\\ &\quad + |\log|\det(Df \mid \gamma_j)(x)| - \log|\det(Df \mid \gamma_j)(y)||\\ &\leq a\,d(f(x), f(y))^\mu + K_1\,d(x,y) + K_2\,d(x,y)\\ &\leq (a\,\lambda_s^\mu + K_1 + K_2)\,d(x,y)^\mu \quad \leq \quad a\,\lambda_1\,d(x,y)^\mu\end{aligned}$$

where $\lambda_s < 1$ is a uniform bound for the contraction of f along stable leaves. We suppose

$$a > \frac{K_1 + K_2}{1 - \lambda_s^\mu},$$

and then we fix $\lambda_1 \in (\lambda_s^\mu, 1)$ so that $a \geq (K_1 + K_2)/(\lambda_1 - \lambda_s^\mu)$. This proves a).

Now, in view of Proposition 2.3, to prove b) it suffices to show that $\mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ has finite diameter in $\mathcal{D}(\gamma_j)$. This is similar to the proof of Proposition 2.4. Let $\theta_{+,j} = \theta_{+,\gamma_j}$ denote the projective metric associated to the cone $\mathcal{D}_+(\gamma_j)$ of positive densities on γ_j. Observe that $\theta_{+,j}$ and θ_j are given by expressions analogous to Examples 2.3 and 2.4. Given $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$ and $x, y \in \gamma_j$, we have

$$\frac{\exp(a\,d(x,y)^\mu) - \rho''(y)/\rho''(x)}{\exp(a\,d(x,y)^\mu) - \rho'(y)/\rho'(x)} \geq \frac{\exp(a\,d(x,y)^\mu) - \exp(\lambda_1 a\,d(x,y)^\mu)}{\exp(a\,d(x,y)^\mu) - \exp(-\lambda_1 a\,d(x,y)^\mu)} \geq \tau_1$$

where $\tau_1 = \inf\{(z - z^{\lambda_1})/(z - z^{-\lambda_1}) : z > 1\} \in (0,1)$. Therefore,

$$\alpha_j(\rho', \rho'') \geq \tau_1\,\alpha_{+,j}(\rho', \rho'').$$

In just the same way, one finds $\tau_2 > 1$ such that $\beta_j(\rho', \rho'') \leq \tau_2\,\beta_{+,j}(\rho', \rho'')$. Thus,

$$\theta_j(\rho', \rho'') \leq \theta_{+,j}(\rho', \rho'') + \log(\tau_2/\tau_1)$$

for every $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j)$.

Next we find a uniform upper bound for $\theta_{+,j}(\rho', \rho'')$ with $\rho', \rho'' \in \mathcal{D}(\gamma_j)$. We normalize $\int_{\gamma_j} \rho' = 1 = \int_{\gamma_j} \rho''$ and then the mean value theorem gives

$$\frac{\rho''}{\rho'}(x) \geq \frac{\exp(-a\,(\operatorname{diam}\gamma_j)^\mu)}{\exp(a\,(\operatorname{diam}\gamma_j)^\mu)} \geq \exp(-2a).$$

Recall that we suppose diameter$(Q) \leq 1$. It follows that

$$\alpha_{+,j}(\rho', \rho'') \geq \exp(-2a) \quad \text{and} \quad \beta_{+,j}(\rho', \rho'') \leq \exp(2a)$$

and so $\theta_{+,j}(\rho',\rho'') \le 4a$, for all $\rho',\rho'' \in \mathcal{D}(\gamma_j)$. Altogether, we have shown that the θ_j-diameter of $\mathcal{D}(\lambda_1 a,\mu,\gamma_j)$ is bounded by $4a+\log(\tau_2/\tau_1)$. □

Let us recall a few facts concerning invariant foliations of hyperbolic attractors, see Appendix A for additional information. As already mentioned, local stable leaves form a continuous family $\mathcal{F}^s_{loc}$ of C^2 embedded submanifolds of Q. In particular, given any pair of nearby stable leaves γ and $\tilde\gamma$, there is a C^2 diffeomorphism

$$\pi = \pi(\tilde\gamma,\gamma) : \tilde\gamma \to \gamma,$$

C^2-close to the inclusion map of $\tilde\gamma$ in M. Of course, such a diffeomorphism is not unique, but for what follows we only need to know that it has been chosen satisfying (p1), (p2), (p3) below, which is always possible. For instance, in the case of the solenoid it suffices to take π to be the projection along the leaves of the horizontal foliation $\{S^1\times\{z\} : z \in B^2\}$ of $Q = S^1 \times B^2$. In other words, for each $z \in B^2$ the unique point $(\tilde\theta, z) \in \tilde\gamma$ is mapped by π to the unique point $(\theta, z) \in \gamma$.

Let γ_1, γ_2 be the connected components of $f^{-1}(\gamma)\cap Q$ and $\tilde\gamma_1$, $\tilde\gamma_2$ be the connected components of $f^{-1}(\tilde\gamma)\cap Q$ (numbered in such a way that $\tilde\gamma_1$ is closer to γ_1 than to γ_2), and let $\pi_j = \pi(\tilde\gamma_j,\gamma_j)$, for $j=1,2$. Let $d(y',y'')$ denote the distance between two points y', y'' belonging in a same horizontal leaf Γ, measured along Γ. Then there are constants $a_0>0$, $\nu_0>0$, $\lambda_u<1$, depending only on f, such that

(p1) π and $\log|\det D\pi|$ are a_0-Lipschitz maps;

(p2) $\log|\det D\pi(y)| \le a_0\, d(y,\pi(y))^{\nu_0}$ for every $y \in \tilde\gamma$;

(p3) $d(x,\pi_j(x)) \le \lambda_u\, d(f(x),\pi f(x))$ for every $x \in \tilde\gamma_j$ and $j=1,2$.

Indeed, (p1) follows directly from γ and $\tilde\gamma$ being the graphs of C^2 maps $B^2 \to S^1$ with uniformly bounded C^2 norm. Property (p2) also uses the fact that the tangent spaces to the leaves of $\mathcal{F}^s_{loc}$ form a Hölder continuous subbundle of the tangent bundle TQ, in particular,

$$\text{angle}(T_y\tilde\gamma, T_{\pi(y)}\gamma) \le A_0\, d(y,\pi(y))^{\nu_0}, \tag{4.3}$$

where $A_0>0$ and $\nu_0 \in (0,1]$ depend only on the map f.

To prove (p3), begin by noting that if ξ_0 is some curve joining x to $\pi_j(x)$ inside the horizontal leaf Γ_0 that contains x, then ξ_0 is expanded by iteration under f. Even more, the horizontal projection of $f(\xi_0)$ has length larger than $\sigma_u\,\text{length}(\xi_0)$, for some uniform $\sigma_u>1$. Let ξ_1 be a curve joining $f(x)$ to $\pi f(x)$ inside the corresponding horizontal leaf, with

$$d(f(x),\pi f(x)) = \text{length}(\xi_1).$$

Observe that the angle between $f(\Gamma_0)$ and the leaves of the horizontal foliation is bounded, at every point, by some constant $H>0$. On the other hand, the angle between γ and the leaves of the vertical foliation $\{\{\theta\}\times B^2 : \theta \in S^1\}$ is also uniformly bounded, by some constant $\delta>0$ which can be made arbitrarily small by taking f close enough to (4.1). It follows that we can take a curve ξ_2 joining $f(x)$ to $f\pi_j(x)$ inside $f(\Gamma_0)$, and such that the length of the horizontal projection of ξ_2 is smaller than

$$(1+H\delta)\,\text{length}(\xi_1) = (1+H\delta)\,d(f(x),\pi f(x)).$$

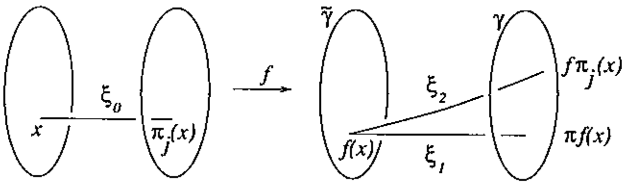

Figure 4.2: Expansion in the space of stable leaves

We suppose that $\delta > 0$ is so that $1 + H\delta < \sigma_u$, and then take $\sigma_u^{-1}(1+H\delta) \le \lambda_u < 1$. Then, denoting $\xi_0 = f^{-1}(\xi_2)$, we obtain (p2):

$$d(x,\pi_j(x)) \le \text{length}(\xi_0) \le \sigma_u^{-1}(1+H\delta)\,d(f(x),\pi f(x)) \le \lambda_u\,d(f(x),\pi f(x)).$$

For any γ, $\tilde\gamma$ as before, we define the *distance* between γ and $\tilde\gamma$ by

$$d(\gamma,\tilde\gamma) = \sup\{d(y,\pi(y)) : y \in \tilde\gamma\}. \tag{4.4}$$

As a direct consequence of (p3), the map induced by f in the space of local stable leaves is expanding for this distance:

$$d(\gamma,\tilde\gamma) \ge \lambda_u^{-1}\,d(\gamma_j,\tilde\gamma_j), \quad \text{for every } \gamma,\tilde\gamma, \text{ and } j=1,2. \tag{4.5}$$

Next, to every $\rho \in \mathcal{D}_1(\gamma)$ we associate the density $\tilde\rho\colon \tilde\gamma \to \mathbb{R}$ defined by

$$\tilde\rho(y) = \rho(\pi(y)) \cdot |\det D\pi(y)|. \tag{4.6}$$

Clearly, $\tilde\rho > 0$. Moreover, as a consequence of (p1), $\log\tilde\rho = \log\rho\circ\pi + \log|\det D\pi|$ is $(\tilde a_1,\mu_1)$-Hölder continuous with $\tilde a_1 = a_1 a_0^{\mu_1} + a_0$ Recall that diameter$(Q) \le 1$. This also implies that $\log\tilde\rho$ is $(\tilde a,\mu)$-Hölder continuous, for $\tilde a = a_1 a_0^{\mu_1} + a_0$. We suppose

$$\frac{a}{2} \ge \tilde a = a_1 a_0^{\mu_1} + a_0\,, \tag{4.7}$$

so that, in particular, $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \tilde\rho \in \mathcal{D}(\tilde\gamma)$. (This step is one of the reasons why we need the auxiliary cone $\mathcal{D}_1(\gamma)$, the other one is in the proof of Lemma 4.5.) Note also that $\int_\gamma \rho = \int_{\tilde\gamma}\tilde\rho$, by change of variables.

At last, we are in a position to introduce the invariant cone of observables we are interested in. Given $b>0$, $c>0$, and $\nu \in (0,1]$, we let $C(b,c,\nu)$ be the cone of bounded functions $\varphi : Q \to \mathbb{R}$ satisfying conditions (A), (B), (C) below:

(A) $\int_\gamma \varphi\rho > 0$ for every $\gamma \in \mathcal{F}^s_{loc}$ and every $\rho \in \mathcal{D}(\gamma)$;

(B) the map $\mathcal{D}(\gamma) \ni \rho \mapsto \log \int_\gamma \varphi\rho$ is b-Lipschitz, that is,

$$|\log \int_\gamma \varphi\rho' - \log \int_\gamma \varphi\rho''| \leq b\,\theta(\rho',\rho'')$$

for every $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$, and every $\gamma \in \mathcal{F}^s_{loc}$;

(C) the map $\mathcal{F}^s_{loc} \ni \gamma \mapsto \int_\gamma \varphi\rho$ is (c,ν)-Hölder, more precisely,

$$|\log \int_\gamma \varphi\rho - \log \int_{\tilde\gamma} \varphi\tilde\rho| \leq c\, d(\gamma,\tilde\gamma)^\nu$$

for every $\rho \in \mathcal{D}_1(\gamma)$ and every pair $\gamma, \tilde\gamma \in \mathcal{F}^s_{loc}$.

As a matter of fact, we want to think of the elements of $C(b,c,\nu)$ as equivalence classes of bounded functions for the equivalence relation

$$\varphi_1 \sim \varphi_2 \quad \Leftrightarrow \quad \varphi_1|\gamma = \varphi_2|\gamma \quad m_\gamma\text{-almost everywhere, for every } \gamma \in \mathcal{F}^s_{loc}.$$

However, replacing an equivalence class by any of its members never results in ambiguity, and so we ignore this formal distinction, in order not to overload the notations.

(A) and (C) are natural reformulations of the properties we required in the definition of the cone $C(a,\nu)$ in Section 2.2. Condition (B) is necessary to compensate for the fact that the functions $\varphi \in C(b,c,\nu)$ may take negative values. Observe, indeed, that (B) is automatically satisfied (with $b=1$) in the particular case when φ is nonnegative:

$$\frac{\int_\gamma \varphi\rho'}{\int_\gamma \varphi\rho''} \leq \frac{\sup\rho'}{\inf\rho''} \leq \frac{\sup\rho'/\inf\rho'}{\inf\rho''/\sup\rho''} = \exp(\theta_+(\rho',\rho'')) \leq \exp(\theta(\rho',\rho'')). \tag{4.8}$$

In the sequel we take b and c large, and ν close to zero, cf. Proposition 4.4 and (4.11).

Lemma 4.3. *$C(b,c,\nu)$ is a convex cone with $-\overline{C(b,c,\nu)} \cap \overline{C(b,c,\nu)} = \{0\}$.*

Proof: The convexity of the cone is a direct consequence of the convexity of the logarithm function. To prove the last statement we only have to show that

$$\int_\gamma \varphi\rho = 0 \text{ for all } \rho \in \mathcal{D}(\gamma) \text{ and all } \gamma \in \mathcal{F}^s_{loc} \quad \Rightarrow \quad \varphi = 0.$$

Let γ be fixed. Given any μ-Hölder continuous function $\psi : \gamma \to \mathbb{R}$ and any $B > 0$, we may write

$$\psi = (\psi^+ + B) - (\psi^- + B) \quad \text{where} \quad \psi^\pm(x) = \frac{1}{2}(|\psi(x)| \pm \psi(x)).$$

Now, it is easy to see that $(\psi^{\pm}+B)\in\mathcal{D}(\gamma)$ if B is large enough. Hence, by the linearity of the integral, $\int_\gamma \varphi\psi = 0$ for every μ-Hölder ψ. Since any bounded function can be L^1-approximated by μ-Hölder continuous functions, it follows that $\int_\gamma \varphi\psi = 0$ for every bounded function $\psi:\gamma\to\mathbb{R}$. Taking $\psi=\varphi|\gamma$ we conclude that $\varphi|\gamma=0$ at m_γ-almost every point, for arbitrary γ, and so $\varphi=0$. $\square$

Let us calculate the projective metric $\Theta=\Theta_{b,c,\nu}$ associated to $C(b,c,\nu)$. Given $\rho'\in\mathcal{D}(\gamma)$ with $\int_\gamma \rho'=1$,

$$\int_\gamma (\varphi_2 - t\varphi_1)\rho' > 0 \Leftrightarrow t < \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'}.$$

Next, for $\rho',\rho''\in\mathcal{D}(\gamma)$ with $\int_\gamma\rho'=1=\int_\gamma\rho''$,

$$\frac{\int_\gamma(\varphi_2-t\varphi_1)\rho''}{\int_\gamma(\varphi_2-t\varphi_1)\rho'} \le \exp(b\theta(\rho',\rho'')) \Leftrightarrow t \le \frac{\int_\gamma\varphi_2\rho'}{\int_\gamma\varphi_1\rho'}\frac{\exp(b\theta(\rho',\rho''))-(\int_\gamma\varphi_2\rho''/\int_\gamma\varphi_2\rho')}{\exp(b\theta(\rho',\rho''))-(\int_\gamma\varphi_1\rho''/\int_\gamma\varphi_1\rho')}.$$

We denote the expression in the last fraction by $\xi(\rho',\rho'',\varphi_1,\varphi_2)$. Then we also have

$$\frac{\int_\gamma(\varphi_2-t\varphi_1)\rho''}{\int_\gamma(\varphi_2-t\varphi_1)\rho'} \ge \exp(-b\theta(\rho',\rho'')) \Leftrightarrow t \le \frac{\int_\gamma\varphi_2\rho''}{\int_\gamma\varphi_1\rho''}\xi(\rho'',\rho',\varphi_1,\varphi_2).$$

Finally, given $\rho\in\mathcal{D}_1(\gamma)$ with $\int_\gamma\rho=1$, we have $\tilde\rho\in\mathcal{D}(\gamma)$ with $\int_{\tilde\gamma}\tilde\rho=1$, and

$$\frac{\int_{\tilde\gamma}(\varphi_2-t\varphi_1)\tilde\rho}{\int_\gamma(\varphi_2-t\varphi_1)\rho} \le \exp(c\,d(\gamma,\tilde\gamma)^\nu) \Leftrightarrow t \le \frac{\int_\gamma\varphi_2\rho\exp(c\,d(\gamma,\tilde\gamma)^\nu)-(\int_{\tilde\gamma}\varphi_2\tilde\rho/\int_\gamma\varphi_2\rho)}{\int_\gamma\varphi_1\rho\exp(c\,d(\gamma,\tilde\gamma)^\nu)-(\int_{\tilde\gamma}\varphi_1\tilde\rho/\int_\gamma\varphi_1\rho)}.$$

Let $\eta(\rho,\tilde\rho,\varphi_1,\varphi_2)$ denote the expression in the last fraction. Then, analogously,

$$\frac{\int_{\tilde\gamma}(\varphi-t\varphi_1)\tilde\rho}{\int_\gamma(\varphi_2-t\varphi_1)\rho} \ge \exp(-c\,d(\gamma,\tilde\gamma)^\nu) \Leftrightarrow t \le \frac{\int_{\tilde\gamma}\varphi_2\tilde\rho}{\int_{\tilde\gamma}\varphi_2\tilde\rho}\eta(\tilde\rho,\rho,\varphi_1,\varphi_2).$$

Therefore, $\alpha(\varphi_1,\varphi_2)$ is given by

$$\inf\left\{\frac{\int_\gamma\varphi_2\rho'}{\int_\gamma\varphi_1\rho'},\frac{\int_\gamma\varphi_2\rho'}{\int_\gamma\varphi_1\rho'}\xi(\rho',\rho'',\varphi_1,\varphi_2),\frac{\int_\gamma\varphi_2\rho}{\int_\gamma\varphi_1\rho}\eta(\rho,\tilde\rho,\varphi_1,\varphi_2),\frac{\int_{\tilde\gamma}\varphi_2\tilde\rho}{\int_{\tilde\gamma}\varphi_1\tilde\rho}\eta(\tilde\rho,\rho,\varphi_1,\varphi_2)\right\}$$

where the infimum runs over all $\rho'\in\mathcal{D}(\gamma)$, $\rho''\in\mathcal{D}(\gamma)$, $\rho\in\mathcal{D}_1(\gamma)$, and every pair of local stable leaves γ and $\tilde\gamma$. Moreover, $\beta(\varphi_1,\varphi_2)$ is given by a similar expression, with inf replaced by sup.

Proposition 4.4 (invariance). *There is $\lambda_2 < 1$ so that $\mathcal{L}(C(b,c,\nu)) \subset C(\lambda_2 b, \lambda_2 c, \nu)$ for every large enough b and c.*

Proof: Let $\rho \in \mathcal{D}(\gamma)$ and ρ_j be as defined above. Lemma 4.2(a) ensures that $\rho_j \in \mathcal{D}(\gamma_j)$ and so $\int_{\gamma_j} \varphi\rho_j > 0$ for each $j = 1,2$. As a consequence, $\int_\gamma (\mathcal{L}\varphi)\rho = \sum_j \int \varphi\rho_j > 0$, which proves the invariance of (A).

To prove the invariance of condition (B), let $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$. Denote, cf. (4.2),

$$\rho'_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho' \circ f) \quad \text{and} \quad \rho''_j = \frac{|\det(Df \mid \gamma_j)|}{|\det Df|}(\rho'' \circ f).$$

Moreover, let $\rho_j^- = \rho'_j / \int_{\gamma_j} \rho'_j$ and $\rho_j^= = \rho''_j / \int_{\gamma_j} \rho''_j$. Then, using condition (B) for φ, followed by Lemma 4.2(b),

$$\begin{aligned}\int_\gamma (\mathcal{L}\varphi)\rho'' &= \sum_j \int_{\gamma_j} \varphi\rho''_j = \sum_j \int_{\gamma_j} \rho''_j \int_{\gamma_j} \varphi\rho_j^= \le \sum_j \int_{\gamma_j} \rho''_j \cdot \exp(b\theta(\rho_j^-, \rho_j^=)) \int_{\gamma_j} \varphi\rho_j^- \\ &\le \sum_j \exp(b\theta(\rho'_j, \rho''_j)) \frac{\int_{\gamma_j} \rho''_j}{\int_{\gamma_j} \rho'_j} \int_{\gamma_j} \varphi\rho'_j \le \exp(b\Lambda_1\, \theta(\rho', \rho'')) \sum_j \frac{\int_{\gamma_j} \rho''_j}{\int_{\gamma_j} \rho'_j} \int_{\gamma_j} \varphi\rho'_j.\end{aligned}$$

By the same arguments as in (4.8),

$$\frac{\rho''_j}{\rho'_j}(x) = \frac{\rho''(f(x))}{\rho'(f(x))} \le \exp\left(\theta(\rho', \rho'')\right) \le \exp\left(\theta_+(\rho', \rho'')\right).$$

Thus, replacing above we conclude that

$$\begin{aligned}\int_\gamma (\mathcal{L}\varphi)\rho'' &\le \exp(b\Lambda_1\, \theta(\rho', \rho'') + \theta(\rho', \rho'')) \sum_j \int_{\gamma_j} \varphi\rho'_j \\ &\le \exp(b\lambda_2\, \theta(\rho', \rho'')) \int_\gamma (\mathcal{L}\varphi)\rho',\end{aligned}$$

as long as we fix $\lambda_2 \in (\Lambda_1, 1)$ and suppose $b \ge 1/(\lambda_2 - \Lambda_1)$.

Now we prove invariance of condition (C). Given two stable leaves γ and $\tilde\gamma$, let γ_j and $\tilde\gamma_j$, $j = 1,2$, be the connected components of $f^{-1}(\gamma) \cap Q$ and $f^{-1}(\tilde\gamma) \cap Q$, respectively. Let $\rho \in \mathcal{D}_1(\gamma) \subset \mathcal{D}(\gamma)$. As we have seen before,

$$\int_\gamma (\mathcal{L}\varphi)\rho = \sum_j \int_{\gamma_j} \varphi\rho_j \qquad \text{and} \qquad \int_{\tilde\gamma} (\mathcal{L}\varphi)\tilde\rho = \sum_j \int_{\tilde\gamma_j} \varphi(\tilde\rho)_j\,,$$

where $(\tilde\rho)_j : \tilde\gamma_j \to \mathbb{R}$ is defined by

$$(\tilde\rho)_j(x) = \tilde\rho(f(x)) \frac{|\det(Df \mid \tilde\gamma_j)(x)|}{|\det Df(x)|} = \rho(\pi f(x))\, |\det D\pi(f(x))|\, \frac{|\det(Df \mid \tilde\gamma_j)(x)|}{|\det Df(x)|}.$$

Since $\rho_j \in \mathcal{D}_1(\gamma_j)$, we may invoke property (C) for φ together with (4.5), to conclude that

$$|\log \int_{\gamma_j} \varphi\rho_j - \log \int_{\tilde{\gamma}_j} \varphi\widetilde{\rho_j}| \leq c\, d(\gamma_j, \tilde{\gamma}_j)^\nu \leq c\,\lambda_u^\nu\, d(\gamma, \tilde{\gamma})^\nu,$$

where

$$\widetilde{\rho_j}(x) = \rho_j(\pi_j(x))\,|\det D\pi_j(x)| = \rho(f\pi_j(x))\frac{|\det(Df \mid \gamma_j)(\pi_j(x))|}{|\det Df(\pi_j(x))|}\,|\det D\pi_j(x)|$$

with $\pi_j = \pi(\tilde{\gamma}_j, \gamma_j)$. Next, we use the following estimate, which is given by the auxiliary Lemma 4.5 below:

$$|\log \int_{\tilde{\gamma}_j} \varphi\widetilde{\rho_j} - \log \int_{\tilde{\gamma}_j} \varphi(\tilde{\rho})_j| \leq K_0\, d(\gamma, \tilde{\gamma})^\nu,$$

for some constant $K_0 > 0$ that does not depend on c. Altogether,

$$|\log \int_{\gamma_j} \varphi\rho_j - \log \int_{\tilde{\gamma}_j} \varphi(\tilde{\rho})_j| \leq (c\,\lambda_u^\nu + K_0)\, d(\gamma, \tilde{\gamma})^\nu,$$

for $j = 1, 2$, and so

$$|\log \int_{\gamma} (\mathcal{L}\varphi)\rho - \log \int_{\tilde{\gamma}} (\mathcal{L}\varphi)\tilde{\rho}| \leq (c\,\lambda_u^\nu + K_0)\, d(\gamma, \tilde{\gamma})^\nu \leq \lambda_2 c\, d(\gamma, \tilde{\gamma})^\nu,$$

as long as we take $\lambda_2 \in (\lambda_u^\nu, 1)$ and suppose $c \geq K_0/(\lambda_2 - \lambda_u^\nu)$. □

In this way we have reduced the proof of Proposition 4.4 to proving the following auxiliary statement.

Lemma 4.5. *There is $K_0 > 0$ depending only on f, a, a_1, b, such that*

$$|\log \int_{\tilde{\gamma}_j} \varphi\widetilde{\rho_j} - \log \int_{\tilde{\gamma}_j} \varphi(\tilde{\rho})_j| \leq K_0\, d(\gamma, \tilde{\gamma})^\nu$$

for every $\varphi \in C(b, c, \nu)$ and every $\gamma, \tilde{\gamma} \in \mathcal{F}^s_{loc}$, $\rho \in \mathcal{D}_1(\gamma)$, and $j = 1, 2$.

Proof: We use $K_1, \ldots, K_6$ to denote sufficiently large constants, depending only on f, a, a_1, b. The previous arguments prove that

(1) $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \tilde{\rho} \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}) \Rightarrow \rho' = (\tilde{\rho})_j \in \mathcal{D}(\lambda_1\tilde{a}_1, \mu_1, \tilde{\gamma}_j) \subset \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$;
(2) $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \rho_j \in \mathcal{D}(\lambda_1 a_1, \mu_1, \gamma_j) \subset \mathcal{D}(a_1, \mu_1, \gamma_j) \Rightarrow \rho'' = \widetilde{\rho_j} \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$.

where $\tilde{a}_1 = a_1 a_0^{\mu_1} + a_0$. In (1) we suppose that a_1 is large enough so that Lemma 4.2(a) holds with the cone $\mathcal{D}(\tilde{a}_1, \mu_1, \gamma)$ in the place of $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$.

It follows from (1) and (2) that both ρ' and ρ'' belong in $\mathcal{D}(\tilde{a}_1, \mu, \tilde{\gamma}_j)$ which, by (4.7), is contained in $\mathcal{D}(a/2, \mu, \tilde{\gamma}_j) \subset \mathcal{D}(\tilde{\gamma}_j)$. Hence, using condition (B) for the normalized densities $\rho'/\int_{\tilde{\gamma}_j}\rho'$ and $\rho''/\int_{\tilde{\gamma}_j}\rho''$,

$$|\log\int_{\tilde{\gamma}_j}\varphi\rho' - \log\int_{\tilde{\gamma}_j}\varphi\rho''| \le b\,\theta_j(\rho',\rho'') + |\log\int_{\tilde{\gamma}_j}\rho' - \log\int_{\tilde{\gamma}_j}\rho''|. \tag{4.9}$$

In order to bound the two terms on the right hand side, let us take a look at the relation

$$\frac{\rho'(x)}{\rho''(x)} = \frac{\rho(\pi f(x))}{\rho(f\pi_j(x))}\frac{|\det D\pi(f(x))|}{|\det D\pi_j(x)|}\frac{|\det(Df \mid \tilde{\gamma}_j)(x)|}{|\det(Df \mid \gamma_j)(\pi_j(x))|}\frac{|\det Df(\pi_j(x))|}{|\det Df(x)|}. \tag{4.10}$$

First, in view of (p3) and the fact that f is Lipschitz continuous, the distance between $\pi f(x)$ and $f\pi_j(x)$ is from above bounded by

$$\begin{aligned} d(\pi f(x), f(x)) + K_1\, d(x, \pi_j(x)) &\le (1 + \lambda_u K_1)\, d(\pi f(x), f(x)) \\ &\le (1 + \lambda_u K_1)\, d(\gamma, \tilde{\gamma}). \end{aligned}$$

Combining with the assumption $\rho \in \mathcal{D}_1(\gamma)$, we find

$$\big|\log\rho(\pi f(x)) - \log\rho(f\pi_j(x))\big| \le a_1(1+\lambda_u K_1)^{\mu_1}\, d(\gamma,\tilde{\gamma})^{\mu_1}.$$

On the other hand, by (p2),

$$\begin{aligned} \big|\log|\det D\pi(f(x))| - \log|\det D\pi_j(x)|\big| &\le a_0\, d(f(x), \pi f(x))^{\nu_0} + a_0\, d(x, \pi_j(x))^{\nu_0} \\ &\le a_0(1+\lambda_u^{\nu_0})\, d(\gamma,\tilde{\gamma})^{\nu_0}. \end{aligned}$$

Moreover, since $\log|\det Df|$ is Lipschitz continuous,

$$\big|\log|\det Df(\pi_j(x))| - \log|\det Df(x)|\big| \le K_2\, d(\pi_j(x), x) \le K_2\lambda_u\, d(\gamma,\tilde{\gamma}).$$

Next, using also the Hölder property (4.3) of the tangent bundle to $\mathcal{F}^s_{loc}$,

$$\begin{aligned} \big|\log|\det(Df \mid \tilde{\gamma}_j)(x)| - \log|\det(Df \mid \gamma_j)(\pi_j(x))|\big| &\le K_3\, d(x, \pi_j(x))^{\nu_0} \\ &\le K_3\lambda_u^{\nu_0}\, d(\gamma,\tilde{\gamma})^{\nu_0}. \end{aligned}$$

At this point we assume that

$$0 < \mu < \mu + \nu \le \mu_1 \le \nu_0. \tag{4.11}$$

Then, replacing the previous bounds in (4.10),

$$\big|\log\rho'(x) - \log\rho''(x)\big| \le K_4\, d(\gamma,\tilde{\gamma})^{\mu_1} \tag{4.12}$$

for some sufficiently large $K_4 > 0$, and every $x \in \tilde{\gamma}_j$ and $j = 1, 2$.

In particular,

$$|\log \int_{\tilde{\gamma}_j} \rho' - \log \int_{\tilde{\gamma}_j} \rho''| \le K_4\, d(\gamma, \tilde{\gamma})^{\mu_1} \tag{4.13}$$

and

$$\theta_{+,j}(\rho', \rho'') = \log \left(\frac{\sup_{\tilde{\gamma}_j}(\rho''/\rho')}{\inf_{\tilde{\gamma}_j}(\rho''/\rho')} \right) \le 2K_4\, d(\gamma, \tilde{\gamma})^{\mu_1}. \tag{4.14}$$

Inequality (4.13) provides the kind of bound we want for the last term in (4.9).

To bound the term $b\,\theta_j(\rho', \rho'')$, we combine (4.14) with the relation (recall e.g. the proof of Lemma 4.2)

$$\theta_j(\rho', \rho'') \le \theta_{+,j}(\rho', \rho'') + \log(\hat{\tau}_2/\hat{\tau}_1), \tag{4.15}$$

where

$$\hat{\tau}_1 = \inf \left\{ \frac{\exp(a\, d(x,y)^{\mu}) - \rho''(y)/\rho''(x)}{\exp(a\, d(x,y)^{\mu}) - \rho'(y)/\rho'(x)} : x, y \in \tilde{\gamma}_j \text{ with } x \neq y \right\}$$

and $\hat{\tau}_2$ is given by a similar expression, with inf replaced by sup. All that is left to do is to bound $|\log \hat{\tau}_1|$ and $|\log \hat{\tau}_2|$. Let us denote

$$B' = (\rho'(y)/\rho'(x)) \exp(-a\, d(x,y)^{\mu}) \quad \text{and} \quad B'' = (\rho''(y)/\rho''(x)) \exp(-a\, d(x,y)^{\mu}).$$

Clearly, $\rho' \in \mathcal{D}(a/2, \mu, \tilde{\gamma}_j)$ implies $\log B' \le -(a/2)\, d(x,y)^{\mu} < 0$, and analogously for ρ'' and B''. In particular,

$$|B' - B''| \le |\log B' - \log B''| = |\log \rho'(y) - \log \rho'(x) - \log \rho''(y) + \log \rho''(x)|.$$

On the one hand, (4.12) implies

$$|B' - B''| \le |\log \rho'(y) - \log \rho''(y)| + |\log \rho'(x) - \log \rho''(x)| \le 2K_4\, d(\gamma, \tilde{\gamma})^{\mu}. \tag{4.16}$$

On the other hand,

$$|B' - B''| \le |\log \rho'(y) - \log \rho'(x)| + |\log \rho''(y) - \log \rho''(x)| \le 2\tilde{a}_1\, d(x,y)^{\mu_1}, \tag{4.17}$$

because $\rho', \rho'' \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$. Since we are taking $\mu_1 \ge \mu + \nu$, it follows that

$$|B' - B''| \le K_5\, d(x,y)^{\mu}\, d(\gamma, \tilde{\gamma})^{\nu},$$

as long as $K_5 \ge \max\{2K_4, 2\tilde{a}_1\}$. Indeed, this last inequality is a direct consequence of (4.16) if $d(x,y) \ge d(\gamma, \tilde{\gamma})$, and of (4.17) in the case when $d(x,y) \le d(\gamma, \tilde{\gamma})$. Then,

$$\left| \log \frac{1 - B''}{1 - B'} \right| \le \frac{|B' - B''|}{1 - \max\{B', B''\}} \le \frac{K_5\, d(x,y)^{\mu}\, d(\gamma, \tilde{\gamma})^{\nu}}{1 - \exp(-(a/2)\, d(x,y)^{\mu})} \le K_6\, d(\gamma, \tilde{\gamma})^{\nu}.$$

Replacing in $\hat{\tau}_1$ and $\hat{\tau}_2$, we find

$$\log \hat{\tau}_1 \geq -K_6\, d(\gamma,\tilde{\gamma})^{\nu} \quad \text{and} \quad \log \hat{\tau}_2 \leq K_6\, d(\gamma,\tilde{\gamma})^{\nu},$$

and so, in view of (4.14) and (4.15),

$$\theta_j(\rho',\rho'') \leq 2K_4\, d(\gamma,\tilde{\gamma})^{\mu_1} + 2K_6\, d(\gamma,\tilde{\gamma})^{\nu}.$$

Finally, by (4.9) and (4.13),

$$|\log \int_{\tilde{g}_j} \varphi\rho' - \log \int_{\tilde{\gamma}_j} \varphi\rho''| \leq (2b+1)\, K_4\, d(\gamma,\tilde{\gamma})^{\mu_1} + 2bK_6\, d(\gamma,\tilde{\gamma})^{\nu} \leq K_0\, d(\gamma,\tilde{\gamma})^{\nu}$$

for some $K_0 > 0$, which concludes the proof of the lemma. □

In the proof of the next result we use the projective metric Θ_+ associated to the cone of bounded functions satisfying $\int_\gamma \varphi\rho > 0$ for every γ and every $\rho \in \mathcal{D}(\gamma)$. In the same way as we calculated Θ, one checks that $\Theta_+(\varphi_1,\varphi_2) = \log(\beta_+(\varphi_1,\varphi_2)/\alpha_+(\varphi_1,\varphi_2))$, with

$$\alpha_+(\varphi_1,\varphi_2) = \inf_{\rho,\gamma}\left\{\frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho}\right\} \quad \text{and} \quad \beta_+(\varphi_1,\varphi_2) = \sup_{\rho,\gamma}\left\{\frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho}\right\}$$

(taken over every $\rho \in \mathcal{D}(\gamma)$ and every stable leaf γ).

Proposition 4.6 (finite diameter). *For $b > 0$, $c > 0$, $\nu \in (0,1]$, the Θ-diameter $D_2 = \sup\{\Theta(\mathcal{L}\varphi_1,\mathcal{L}\varphi_2) : \varphi_1,\varphi_2 \in C(b,c,\nu)\}$ of $\mathcal{L}(C(b,c,\nu))$ is finite.*

Proof: As in the proof of Proposition 2.5, the argument has two parts. In a first step, we bound D_2 in terms of the Θ_+-diameter of $\mathcal{L}(C(b,c,\nu))$. Let $\varphi_1,\varphi_2 \in C(\lambda_2 b,\lambda_2 c,\nu)$. Given $\rho',\rho'' \in \mathcal{D}(\gamma)$ and $\rho \in \mathcal{D}_1(\gamma)$, we have

$$\begin{aligned}\xi(\rho',\rho'',\varphi_1,\varphi_2) &= \frac{\exp(b\theta(\rho',\rho'')) - \int_\gamma \varphi_2\rho''/\int_\gamma \varphi_2\rho'}{\exp(b\theta(\rho',\rho'')) - \int_\gamma \varphi_1\rho''/\int_\gamma \varphi_1\rho'}\\ &\geq \frac{\exp(b\theta(\rho',\rho'')) - \exp(b\lambda_2\,\theta(\rho',\rho''))}{\exp(b\theta(\rho',\rho'')) - \exp(-b\lambda_2\,\theta(\rho',\rho''))} \geq \tau_1\end{aligned}$$

where $\tau_1 = \inf\{(z - z^{\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (0,1)$. In just the same way,

$$\xi(\rho',\rho'',\varphi_1,\varphi_2) \leq \tau_2, \quad \eta(\rho',\rho'',\varphi_1,\varphi_2) \in [\tau_1,\tau_2], \quad \eta(\tilde{\rho},\rho,\varphi_1,\varphi_2) \in [\tau_1,\tau_2],$$

where $\tau_2 = \sup\{(z - z^{-\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (1,+\infty)$. As a direct consequence, $\alpha(\varphi_1,\varphi_2) \geq \tau_1\,\alpha_+(\varphi_1,\varphi_2)$ and $\beta(\varphi_1,\varphi_2) \leq \tau_2\,\beta_+(\varphi_1,\varphi_2)$, and so

$$\Theta(\varphi_1,\varphi_2) \leq \Theta_+(\varphi_1,\varphi_2) + \log(\tau_2/\tau_1)$$

for all $\varphi_1, \varphi_2 \in C(\lambda_2 b, \lambda_2 c, \nu) \supset \mathcal{L}(C(b,c,\nu))$.

Now we present the second and last step, where we show that the Θ_+-diameter of $\mathcal{L}(C(b,c,\nu))$ is finite, that is, there is a uniform upper bound for

$$\frac{\int_{\gamma''}(\mathcal{L}\varphi_2)\rho''}{\int_{\gamma''}(\mathcal{L}\varphi_1)\rho''}\Big/\frac{\int_{\gamma'}(\mathcal{L}\varphi_2)\rho'}{\int_{\gamma'}(\mathcal{L}\varphi_1)\rho'}, \quad \varphi_1,\varphi_2 \in C(b,c,\nu),\ \rho' \in \mathcal{D}(\gamma'),\ \rho'' \in \mathcal{D}(\gamma'').$$

In fact, we prove a bit more:

$$\frac{\int_{\gamma''}(\mathcal{L}\varphi)\rho''}{\int_{\gamma'}(\mathcal{L}\varphi)\rho'}, \quad \varphi \in C(b,c,\nu),\ \rho' \in \mathcal{D}(\gamma'),\ \rho'' \in \mathcal{D}(\gamma'') \text{ with } \int_{\gamma'}\rho' = \int_{\gamma''}\rho'' = 1,$$

is uniformly bounded over all γ', γ'' in $\mathcal{F}^s_{loc}$. Using the same notations as before,

$$\frac{\int_{\gamma''}(\mathcal{L}\varphi)\rho''}{\int_{\gamma'}(\mathcal{L}\varphi)\rho'} = \frac{\sum_{j=1}^2 \int_{\gamma_j''}\varphi\rho_j''}{\sum_{j=1}^2 \int_{\gamma_j'}\varphi\rho_j'}. \tag{4.18}$$

Now, observe that

$$\int_{\gamma_j'}\rho_j' = \int_{\gamma_j'}(\rho'\circ f)\,|\det(Df \mid \gamma_j')|\,|\det\, Df|^{-1} = \int_{f(\gamma_j')}\rho'|\det(Df^{-1})| \geq \Gamma_1 \inf\rho'$$

and, analogously,

$$\int_{\gamma_j''}\rho_j'' \leq \Gamma_2 \sup\rho'',$$

where $\Gamma_1 > 0$ and $\Gamma_2 > 0$, depend only on uniform bounds for the jacobian $|\det(Df^{-1})|$ and the riemannian volume of images of local stable leaves. Also, $\inf\rho' \geq \exp(-a)$ and $\sup\rho'' \leq \exp(a)$, as a consequence of the mean value theorem and $\int_{\gamma'}\rho' = 1 = \int_{\gamma''}\rho''$. Therefore,

$$\frac{\int_{\gamma_j''}\rho_j''}{\int_{\gamma_j'}\rho_j'} \leq \frac{\Gamma_2}{\Gamma_1}\exp(2a).$$

This means that normalizing ρ_j' and ρ_j'', that is, replacing them by, respectively,

$$\rho_j'/\int_{\gamma_j'}\rho_j' \quad \text{and} \quad \rho_j''/\int_{\gamma_j''}\rho_j'',$$

can affect the quotient in (4.18) only by some factor bounded by $(\Gamma_2/\Gamma_1)\exp(2a)$. Recall also that $\rho_j' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j')$ and $\rho_j'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j'')$, by Lemma 4.2. Hence our claim that the quotient in (4.18) is uniformly bounded will follow if we show that

$$\sup\frac{\int_{\gamma_2}\varphi\,\rho_2}{\int_{\gamma_1}\varphi\,\rho_1} < +\infty \tag{4.19}$$

where the supremum is taken over every $\varphi \in C(b, c, \nu)$, every $\rho_1 \in \mathcal{D}(\lambda_1 a, \mu, \gamma_1)$ and $\rho_2 \in \mathcal{D}(\lambda_1 a, \mu, \gamma_2)$ with $\int_{\gamma_1} \rho_1 = \int_{\gamma_2} \rho_2 = 1$, and every pair of local stable leaves γ_1, γ_2.

Let θ_1 and θ_2 be the projective metrics associated to $\mathcal{D}(\gamma_1)$, $\mathcal{D}(\gamma_2)$, respectively. First, we use condition (B) to get

$$\int_{\gamma_1} \varphi\rho_1 \geq \exp(-b\theta_1(\rho_1, \mathbf{1})) \int_{\gamma_1} \varphi\mathbf{1} \quad \text{and} \quad \int_{\gamma_2} \varphi\rho_2 \leq \exp(b\theta_2(\rho_2, \mathbf{1})) \int_{\gamma_2} \varphi\mathbf{1}$$

where a same symbol $\mathbf{1}$ denotes two slightly different objects, namely the constant function on γ_1 satisfying $\int_{\gamma_1} \mathbf{1} = 1$, and the constant function on γ_2 with $\int_{\gamma_2} \mathbf{1} = 1$. Let D_1 be some uniform upper bound for the θ-diameter of $\mathcal{D}(\lambda_1 a, \mu, \gamma) \subset \mathcal{D}(\gamma)$, $\gamma \in \mathcal{F}^s_{loc}$, cf. Lemma 4.2. Then

$$\exp(-D_1) \leq \exp(-b\theta_1(\rho_1, \mathbf{1})) \leq 1 \leq \exp(b\theta_2(\rho_2, \mathbf{1})) \leq \exp(D_1).$$

Finally, let $\tilde{\mathbf{1}}: \gamma_2 \to \mathbb{R}$ be given by $\tilde{\mathbf{1}}(x) = \mathbf{1}(\pi(x)) \,|\det D\pi(x)|$, where $\pi = \pi(\gamma_2, \gamma_1)$. By (p1), both $\mathbf{1}$ and $\tilde{\mathbf{1}}$ belong in $\mathcal{D}(a_0, 1, \gamma_2)$. On the other hand, recall (4.7),

$$\mathcal{D}(a_0, 1, \gamma_2) \subset \mathcal{D}(a_0, \mu, \gamma_2) \subset \mathcal{D}(a/2, \mu, \gamma_2).$$

Let D_0 be a uniform upper bound for the θ_2-diameter of $\mathcal{D}(a/2, \mu, \gamma_2) \subset \mathcal{D}(\gamma_2)$. Then, using conditions (B) and (C) for the function φ,

$$\frac{\int_{\gamma_2} \varphi\mathbf{1}}{\int_{\gamma_1} \varphi\mathbf{1}} \leq \frac{\int_{\gamma_2} \varphi\mathbf{1}}{\int_{\gamma_2} \varphi\tilde{\mathbf{1}}} \frac{\int_{\gamma_2} \varphi\tilde{\mathbf{1}}}{\int_{\gamma_1} \varphi\mathbf{1}} \leq \exp(b\theta_2(\mathbf{1}, \tilde{\mathbf{1}})) \exp(c\, d(\gamma_1, \gamma_2)^\nu) \leq \exp(b\, D_0 + c).$$

We conclude that

$$\frac{\int_{\gamma_2} \varphi\, \rho_2}{\int_{\gamma_1} \varphi\, \rho_1} \leq \exp(2b\, D_1 + b\, D_0 + c)$$

for all the φ, ρ_1, ρ_2, γ_1, γ_2 under consideration, which completes the proof of (4.19). Altogether, we have shown that

$$\frac{\int_{\gamma''} (\mathcal{L}\varphi)\rho''}{\int_{\gamma'} (\mathcal{L}\varphi)\rho'} \leq \Gamma_0, \qquad \Gamma_0 = \frac{\Gamma_2}{\Gamma_1} \exp(2a + 2b\, D_1 + b\, D_0 + c)$$

for every $\varphi \in C(b, c, \nu)$, leaves $\gamma', \gamma'' \in \mathcal{F}^s_{loc}$, and normalized densities $\rho' \in \mathcal{D}(\gamma')$ and $\rho'' \in \mathcal{D}(\gamma'')$. Then the Θ_+-diameter of $\mathcal{L}(C(b, c, \nu))$ is bounded by $\log \Gamma_0^2$. □

4.2 Sinai-Ruelle-Bowen measure.

In what follows we let $\Lambda_2 = (1 - e^{-D_2}) < 1$. Then, in view of Propositions 4.6 and 2.3, the operator $\mathcal{L}$ is a Λ_2-contraction for the projective metric Θ.

Similarly to what we did in the expanding case, we now use the sequence $(\varphi_n = \mathcal{L}^n 1)_n$ to construct an SRB-measure μ_0 for f on Λ. It follows, from $\mathcal{L}$ being a contraction, that

$$\Theta_+(\varphi_k, \varphi_l) \leq \Theta(\varphi_k, \varphi_l) \to 0 \quad \text{(exponentially fast) as } k, l \to \infty.$$

Note, however, that in the present case $(\varphi_n)_n$ can not be expected to converge to a limit function, since μ_0 and Lebesgue measure are mutually singular, in general. Instead, we shall use the statement of weak*-convergence given by the next proposition.

For the proof we need the important fact that the local stable foliation is absolutely continuous: projections along the leaves of $\mathcal{F}^s_{loc}$ are absolutely continuous maps with Hölder continuous jacobians. Before stating and proving the proposition, let us discuss this property a bit more, see also Appendix A. Denote $\tilde{m}$ the quotient measure induced by Lebesgue measure m in the space of local stable leaves, that is,

$$\tilde{m}(\tilde{A}) = m\Big(\bigcup_{\gamma\in\tilde{A}} \gamma\Big).$$

By a disintegration of m with respect to the local stable foliation, one means a family $(p_\gamma)_{\gamma\in\mathcal{F}^s_{loc}}$ such that each p_γ is a probability measure on γ and

$$\int_Q \psi\, dm = \int \Big(\int (\psi \mid \gamma)\, dp_\gamma\Big)\, d\tilde{m}(\gamma)$$

for every m-integrable function ψ. We use the following consequence of the absolute continuity of $\mathcal{F}^s_{loc}$: there are constants $a_0 > 0$ and $0 < \nu_0 \le 1$, and there exists a function $H : Q \to (0, +\infty)$ such that $\log H$ is (a_0, ν_0)-Hölder continuous, and $p_\gamma = (H|\gamma)m_\gamma$ defines a disintegration of m.

Proposition 4.7. *Given any Θ_+-Cauchy sequence $(\varphi_n)_n$ in $C(b, c, \nu)$, normalized by $\int_Q \varphi_n\, dm = 1$ for all $n \ge 1$, and given any continuous function $\psi : Q \to \mathbb{R}$, the sequence $\left(\int \varphi_n \psi\, dm\right)_n$ is Cauchy in $\mathbb{R}$.*

Proof: Consider first the case when $\psi > 0$ and $\log\psi$ is $(a/2, \mu)$-Hölder. In particular, $\log\psi$ is $(a/2, \mu)$-Hölder along each local stable leaf γ (with respect to the induced metric). We write

$$\int_Q \varphi_n\psi\, dm = \int \Big(\int_\gamma \varphi_n\psi\, dp_\gamma\Big)\, d\tilde{m}(\gamma) = \int \Big(\int_\gamma \varphi_n\, \psi\, H_\gamma\Big)\, d\tilde{m}(\gamma)$$

where $H_\gamma = H \mid \gamma$. Note that (ψH_γ) is strictly positive and $\log(\psi H_\gamma)$ is (a, μ)-Hölder, as long as we fix $a \ge 2a_0$ and $\mu \le \nu_0$, cf. (4.7) and (4.11). Moreover,

$$\int_Q \varphi_n\, dm = \int \Big(\int_\gamma \varphi_n\, H_\gamma\Big)\, d\tilde{m}(\gamma)$$

and $H_\gamma > 0$ with $\log H_\gamma$ an (a, μ)-Hölder function. Therefore, given $k, l \ge 1$,

$$\frac{\int_\gamma \varphi_k H_\gamma}{\int_\gamma \varphi_l H_\gamma} \ge \alpha_+(\varphi_k, \varphi_l) \quad \text{and} \quad \frac{\int_\gamma \varphi_k\psi H_\gamma}{\int_\gamma \varphi_l\psi H_\gamma} \le \beta_+(\varphi_k, \varphi_l) \qquad \text{for all } \gamma.$$

On the other hand, $\int_Q \varphi_k \, dm = 1 = \int_Q \varphi_l \, dm$ implies that $\int_{\hat\gamma} \varphi_k H_{\hat\gamma} \le \int_{\hat\gamma} \varphi_l H_{\hat\gamma}$ for some local leaf $\hat\gamma$. Thus,

$$\frac{\int_\gamma \varphi_k \psi H_\gamma}{\int_\gamma \varphi_l \psi H_\gamma} \le \frac{\beta_+(\varphi_k,\varphi_l)}{\alpha_+(\varphi_k,\varphi_l)} \frac{\int_{\hat\gamma} \varphi_k H_{\hat\gamma}}{\int_{\hat\gamma} \varphi_l H_{\hat\gamma}} \le e^{\Theta_+(\varphi_k,\varphi_l)} \quad \text{for all } \gamma,$$

implying that

$$\frac{\int_Q \varphi_k \psi \, dm}{\int_Q \varphi_l \psi \, dm} \le e^{\Theta_+(\varphi_k,\varphi_l)} \quad \text{for all } k,l \ge 1.$$

As a consequence,

$$\left|\int_Q \varphi_k \psi \, dm - \int_Q \varphi_l \psi \, dm\right| = \left|\int_Q \varphi_l \psi \, dm\right| \left|\frac{\int_Q \varphi_k \psi \, dm}{\int_Q \varphi_l \psi \, dm} - 1\right| \le \sup|\psi| \left(e^{\Theta_+(\varphi_k,\varphi_l)} - 1\right), \tag{4.20}$$

and the proposition is proved in this case.

Now, for an arbitrary μ-Hölder continuous function $\psi: Q \to \mathbb{R}$, we write

$$\psi = \psi_B^+ - \psi_B^-, \quad \text{where} \quad \psi_B^\pm = \frac{1}{2}(|\psi| \pm \psi) + B$$

and $B > 0$ is chosen large enough to ensure that $\log \psi_B^\pm$ is $(a/2, \mu)$-Hölder continuous. The previous argument applies to $\psi_B^\pm$ and so, by linearity, the proposition holds for ψ.

Finally, given any continuous function ψ and any $\varepsilon > 0$, we may take $\tilde\psi$ a μ-Hölder function such that $\sup|\psi - \tilde\psi| \le \varepsilon$. Then, for every $k, l \ge 1$,

$$\left|\int_Q \varphi_k \psi \, dm - \int_Q \varphi_l \psi \, dm\right| \le \left|\int_Q \varphi_k \tilde\psi \, dm - \int_Q \varphi_l \tilde\psi \, dm\right| + 2\varepsilon,$$

recall that we suppose $\int_Q \varphi_n \, dm = 1$ for all n. By the previous case, the right hand side is bounded by 3ε if k and l are large enough, and so we have proved that $\int_Q \varphi_n \psi \, dm$ is a Cauchy sequence also in this case. □

We are now in a position to introduce the SRB-measure μ_0 of the map f on Q. For that we consider $\varphi_n = \mathcal{L}^n 1$, for each $n \ge 1$. Then, by Proposition 4.7, $(\varphi_n)_n$ is a Θ-Cauchy sequence, and so it is also a Θ_+-Cauchy sequence. Moreover,

$$\int_Q \varphi_n \, dm = \int_Q (\mathcal{L}^n 1) 1 \, dm = \int_Q 1 \, (U^n 1) \, dm = \int_Q 1 \, dm = 1, \quad \text{for all } n \ge 1.$$

Then we define μ_0 to be the weak*-limit of $(\mathcal{L}^n 1) m = (f^n)_* m$:

$$\int \psi \, d\mu_0 = \lim \int_Q (\mathcal{L}^n 1) \psi \, dm = \lim \int_Q (\psi \circ f^n) \, dm,$$

for each continuous $\psi : Q \to \mathbb{R}$. Clearly, μ_0 is invariant under the map f: for any ψ,

$$\int (\psi \circ f)\, d\mu_0 = \lim \int_Q (\psi \circ f^{n+1})\, dm = \lim \int_Q (\psi \circ f^n)\, dm = \int \psi\, d\mu_0.$$

We shall see in the next section that

$$\frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \to \int \psi\, d\mu_0 \tag{4.21}$$

for every $\psi \in C^0(Q, \mathbb{R})$ and m-almost all $x \in Q$. As a consequence, μ_0 is indeed the (unique) SRB-measure for the attractor Λ of f in Q.

Remark 4.1. Given any $\varphi_0 \in C(b, c, \nu)$ with $\int_Q \varphi_0\, dm = 1$, consider the sequence $(\hat{\varphi}_n)_n$ defined by

$$\hat{\varphi}_{2k-1} = \mathcal{L}^k \varphi_0 \quad \text{and} \quad \hat{\varphi}_{2k} = \mathcal{L}^k 1, \quad \text{for } k \geq 1.$$

Then $(\hat{\varphi}_n)_n$ satisfies the hypotheses of Proposition 4.7: in particular,

$$\Theta_+(\hat{\varphi}_n, \hat{\varphi}_{n+1}) \leq \Theta(\hat{\varphi}_n, \hat{\varphi}_{n+1}) \leq \Lambda_2^{[n/2]} \max\left\{\Theta(1, \mathcal{L}\varphi_0), \Theta(\mathcal{L}\varphi_0, \mathcal{L}1)\right\} \leq \Lambda_2^{[n/2]} D_2,$$

and so the sequence is Θ_+-Cauchy. It follows that $(\hat{\varphi}_n)_n$ is weak*-Cauchy and so

$$\int \psi\, d\mu_0 = \lim \int_Q \hat{\varphi}_{2k} \psi\, dm = \lim \int_Q \hat{\varphi}_{2k-1} \psi\, dm = \lim \int_Q (\mathcal{L}^k \varphi_0)\, \psi\, dm,$$

for every continuous ψ. Thus, μ_0 is also the weak*-limit of $(\mathcal{L}^k \varphi_0)\, m = (f^k)_*(\varphi_0\, m)$ for any $\varphi_0 \in C(b, c, \nu)$.

For the proof of (4.21) we shall need the following lemma which, in rough terms, asserts that μ_0 behaves as an absolutely continuous measure (with respect to Lebesgue measure) *if one quotients out local stable leaves*. Let $\mathcal{F}_0$ be the σ-algebra of Borel sets which are union of local stable leaves: $B \in \mathcal{F}_0$ if and only if B is a Borel subset of M and, given any local stable leaf γ, either $\gamma \cap B = \emptyset$ or $\gamma \subset B$. Clearly, $\tilde{m}$ is just the restriction of m to $\mathcal{F}_0$.

Lemma 4.8. *There is $K > 0$ such that, for every $\psi \in L^1(\mathcal{F}_0)$,*

$$\frac{1}{K} \int_Q \psi\, dm \leq \int \psi\, d\mu_0 \leq K \int_Q \psi\, dm.$$

Proof: Let γ, $\tilde{\gamma}$ be local stable leaves and $H_\gamma = H|\gamma$ and $H_{\tilde{\gamma}} = H|\tilde{\gamma}$ be as in the proof of Proposition 4.7. In addition, let $\widetilde{H}_\gamma = (H_\gamma \circ \pi)\, |\det D\pi|$, with $\pi = \pi(\tilde{\gamma}, \gamma)$. Recall that $\log H$ is (a_0, ν_0)-Hölder, with (a_0, ν_0) depending only on f. Therefore, up to choosing a, a_1 larger than a_0, and μ, μ_1 smaller than ν_0, as in (4.7), (4.11), we have

$H_\gamma \in \mathcal{D}_1(\gamma)$ and $H_{\tilde\gamma}$, $\widetilde{H}_\gamma \in \mathcal{D}(a/2,\mu,\tilde\gamma)$. Then properties (B) and (C) give, for each $\varphi_k = \mathcal{L}^k 1$,

$$\frac{\int_\gamma \varphi_k H_\gamma}{\int_{\tilde\gamma} \varphi_k H_{\tilde\gamma}} = \frac{\int_{\tilde\gamma} \varphi_k \widetilde{H}_\gamma}{\int_{\tilde\gamma} \varphi_k H_{\tilde\gamma}} \frac{\int_\gamma \varphi_k H_\gamma}{\int_{\tilde\gamma} \varphi_k \widetilde{H}_\gamma} \le \exp(b\theta_+(\widetilde{H}_\gamma, H_{\tilde\gamma}) + c\,d(\gamma,\tilde\gamma)^\nu) \le \exp(bD_0 + c)$$

where D_0 is a uniform bound for the θ-diameter of $\mathcal{D}(a/2,\mu,\tilde\gamma) \subset \mathcal{D}(\tilde\gamma)$, see Lemma 4.2. For simplicity, we write $K = \exp(c + bD_0)$. Recalling that

$$\int \left(\int_{\tilde\gamma} \varphi_k H_{\tilde\gamma} \right) d\widetilde{m}(\tilde\gamma) = \int \varphi_k \, dm = 1$$

we conclude that $\int_\gamma \varphi_k H_\gamma \le K$ for every local stable leaf γ. Then

$$\int_Q \psi(\mathcal{L}^k 1)\, dm = \int \psi(\gamma) \Big(\int_\gamma (\mathcal{L}^k 1)\, H_\gamma \Big)\, d\widetilde{m}(\gamma) \le K \int \psi(\gamma)\, d\widetilde{m}(\gamma) = K \int_Q \psi \, dm,$$

note that functions $\psi \in L^1(\mathcal{F}_0)$ are constant on each stable leaf γ. Passing to the limit as $k \to \infty$, $\int \psi\, d\mu_0 \le K \int_Q \psi\, dm$. The dual inequality $\int \psi\, d\mu_0 \ge K^{-1} \int_Q \psi\, dm$ may be derived in just the same way, and so the argument is complete. □

4.3 Decay of correlations.

Our next goal is to prove the following statement of exponential decay of correlations with respect to Hölder continuous functions.

Proposition 4.9 (decay of correlations). *Given any ν-Hölder continuous function φ and any μ-Hölder continuous function ψ, there is $C_2 = C_2(\varphi,\psi) > 0$ such that, for all $n \ge 0$,*

a) $\left| \int_Q (\psi \circ f^n) \varphi \, dm - \int \psi \, d\mu_0 \int_Q \varphi \, dm \right| \le C_2 \Lambda_2^n$

b) $\left| \int (\psi \circ f^n) \varphi \, d\mu_0 - \int \psi \, d\mu_0 \right| \le C_2 \Lambda_2^n$

Proof: First we suppose that $\psi > 0$ and $\log\psi$ is $(a/2,\mu)$-Hölder. Then, as in (4.20),

$$\begin{aligned}
\left| \int_Q \psi(\mathcal{L}^n \varphi) dm - \int_Q \psi(\mathcal{L}^{n+k}\varphi) dm \right| &\le \sup\psi \int_Q (\mathcal{L}^{n+k}\varphi)\, dm \left(e^{\Theta_+(\mathcal{L}^n\varphi, \mathcal{L}^{n+k}\varphi)} - 1 \right) \\
&\le \sup\psi \int_Q \varphi\, dm \left(e^{\Lambda_2^{n-1}\Theta(\mathcal{L}\varphi, \mathcal{L}^{k+1}\varphi)} - 1 \right) \\
&\le \sup\psi \int_Q \varphi\, dm \left(e^{C_2' \Lambda_2^n} - 1 \right) \\
&\le C_2'' \Lambda_2^n \sup\psi \int_Q \varphi\, dm,
\end{aligned}$$

for every $\varphi \in C(b,c,\nu)$, where C_2', $C_2'' > 0$ are independent of ψ or φ. The third inequality uses Proposition 4.6. Since

$$\int_Q \psi(\mathcal{L}^n\varphi)dm = \int_Q (\psi\circ f^n)\varphi\,dm \quad\text{and}\quad \lim_{k\to\infty}\int_Q \psi(\mathcal{L}^{n+k}\varphi)dm = \int \psi\,d\mu_0 \int_Q \varphi\,dm$$

(by the definition of μ_0, together with Remark 4.1), we conclude that

$$\left|\int_Q (\psi\circ f^n)\varphi\,dm - \int \psi\,d\mu_0 \int_Q \varphi\,dm\right| \le C_2''\Lambda_2^n \sup\psi \int_Q \varphi\,dm \tag{4.22}$$

and this implies a) for the class of functions φ and ψ as above.

Now we prove b), under the further assumption that $\varphi > 0$ and $\log\psi$ is (c_1,ν)-Hölder continuous for some small $c_1 > 0$. We need the following statement, whose proof we postpone for a while (see Lemma 4.10 below):

$$\varphi\,(\mathcal{L}^l\,1) \in C(b,c,\nu) \quad \text{for every } l \ge 0, \tag{4.23}$$

as long as $c_1 > 0$ is small enough (depending only on f). Indeed, (4.23) allows us to replace φ by $\varphi(\mathcal{L}^l\,1)$ in (4.22), thus getting

$$\left|\int_Q (\psi\circ f^n)\varphi(\mathcal{L}^l\,1)dm - \int \psi\,d\mu_0 \int_Q \varphi(\mathcal{L}^l\,1)dm\right| \le C_2''\,\Lambda_2^n \sup\psi \int_Q \varphi(\mathcal{L}^l\,1)dm.$$

Passing to the limit as $l \to \infty$,

$$\left|\int (\psi\circ f^n)\varphi\,d\mu_0 - \int \psi\,d\mu_0 \int \varphi\,d\mu_0\right| \le C_2''\,\Lambda_2^n \sup\psi \int \varphi\,d\mu_0$$

for all $n \ge 0$.

So far we have proved a) and b) for strictly positive φ, ψ such that $\log\varphi$ is (c_1,ν)-Hölder and $\log\psi$ is $(a/2,\mu)$-Hölder. The general case is a straightforward consequence. Just write $\varphi = \varphi_B^+ - \varphi_B^-$ with

$$\varphi_B^\pm = \frac{1}{2}(|\varphi| \pm \varphi) + B$$

and $B > 0$, and decompose ψ in the same way. Take B large enough so that $\log\varphi_B$ be (c_1,ν)-Hölder and $\log\psi_B^\pm$ be $(a/2,\mu)$-Hölder. Then use the previous particular case, together with linearity of the integral, to complete the argument. □

That is, we reduced Proposition 4.9 to checking the claim (4.23). We restate this claim in the next lemma. The proof uses the same kind of ideas as that of Lemma 4.5.

Lemma 4.10. *There is $c_1 > 0$, depending only on the map f, such that given any function $\varphi > 0$ such that $\log \varphi$ is (c_1, ν)-Hölder continuous, then*

$$\varphi\,(\mathcal{L}^l\, 1) \in C(b, c, \nu) \quad \textit{for every } l \geq 0.$$

Proof: Indeed, $1 \in C(b, c, \nu)$ and so, by Proposition 4.4, $\mathcal{L}^l\, 1 \in C(b, c, \nu)$. It follows that

$$\int_\gamma \varphi\,(\mathcal{L}^l\, 1)\,\rho \geq \inf \varphi \int_\gamma (\mathcal{L}^l\, 1)\rho > 0,$$

for every $\rho \in \mathcal{D}(\gamma)$, which proves property (A). We have already observed that (B) is automatic for nonnegative functions, as long as $b \geq 1$.

To prove property (C), let γ, $\tilde{\gamma}$ be arbitrary local stable leaves, and let $\rho \in \mathcal{D}_1(\gamma)$. Fix $l \geq 0$ and let γ_J and $\tilde{\gamma}_J$, $J = 1, \ldots, 2^l$, the connected components of $f^{-l}(\gamma) \cap Q$ and $f^{-l}(\tilde{\gamma}) \cap Q$, respectively. Then,

$$\begin{aligned}\int_\gamma \varphi\,(\mathcal{L}^l\, 1)\,\rho &= \sum_{J=1}^{2^l} \int_{\gamma_J} (\varphi \circ f^l)\,(\rho \circ f^l)\,\frac{|\det(Df^l|\gamma_J)|}{|\det Df^l|}\\ &= \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi \circ f^l \circ \pi_J)\,(\rho \circ f^l \circ \pi_J)\,\frac{|\det(Df^l|\gamma_J)| \circ \pi_J}{|\det Df^l| \circ \pi_J}\,|\det D\pi_J|,\end{aligned}$$

where $\pi_J = \pi(\tilde{\gamma}_J, \gamma_J)$, and

$$\begin{aligned}\int_{\tilde{\gamma}} \varphi\,(\mathcal{L}^l\, 1)\,\tilde{\rho} &= \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi \circ f^l)\,(\tilde{\rho} \circ f^l)\,\frac{|\det(Df^l|\tilde{\gamma}_J)|}{|\det Df^l|}\\ &= \sum_{J=1}^{2^l} \int_{\tilde{\gamma}_J} (\varphi \circ f^l)\,(\rho \circ \pi \circ f^l)\,(|\det D\pi| \circ f^l)\,\frac{|\det(Df^l|\tilde{\gamma}_J)|}{|\det Df^l|}\,.\end{aligned}$$

Since all the functions involved here are positive, property (C) will follow if we show that

$$\log\left(\frac{\varphi(f^l\pi_J(x))}{\varphi(f^l(x))}\,\frac{\rho(f^l\pi_J(x))}{\rho(\pi f^l(x))}\,\frac{|\det(Df^l|\gamma_J)|(\pi_J(x))}{|\det(Df^l|\tilde{\gamma}_J)|(x)}\right.$$
$$\left.\frac{|\det Df^l|(x)}{|\det Df^l|(\pi_J(x))}\,\frac{|\det D\pi_J|(x)}{|\det D\pi|(f^l(x))}\right) \tag{4.24}$$

is bounded in norm by $c\,d(\gamma, \tilde{\gamma})^\nu$, at every $x \in \tilde{\gamma}$ and for every $J = 1, \ldots, 2^l$.

Let Γ_0 be the horizontal leaf containing x and $\xi_0 \subset \Gamma_0$ be a curve joining x to $\pi_J(x)$ such that $\text{length}(\xi_0) = (x, \pi_J(x))$. The key remark is that the angle of each iterate $f^i(\Gamma_0)$, $i \geq 0$, to the horizontal direction is bounded, at every point, by some constant $H > 0$ that depends only on f. It follows that

$$\text{dist}(f^l\pi_J(x), \pi f^l(x)) \leq H\, d(\gamma, \tilde{\gamma}), \tag{4.25}$$

and

$$\operatorname{dist}(f^l(x), f^l\pi_J(x)) \le \operatorname{length}(f^l(\xi_0)) \le (1+H)\,d(\gamma,\tilde\gamma). \tag{4.26}$$

More generally, the distance from $f^i(x)$ to $f^i\pi_J(x)$ is bounded by

$$\operatorname{length}(f^i(\xi_0)) \le (1+H)\,\lambda_u^{l-i}\,d(\gamma,\tilde\gamma) \tag{4.27}$$

for every $0 \le i \le l$. By the assumption on φ together with (4.26),

$$|\log\varphi(f^l\pi_J(x)) - \log\varphi(f^l(x))| \le c_1(1+H)^\nu\,d(\gamma,\tilde\gamma)^\nu \le c_1(1+H)\,d(\gamma,\tilde\gamma)^\nu.$$

Analogously, $\rho \in \mathcal{D}_1(\gamma)$ together with (4.25) give

$$|\log\rho(f^l\pi_J(x)) - \log\rho(\pi f^l(x))| \le a_1 H^{\mu_1}\,d(\gamma,\tilde\gamma)^{\mu_1}.$$

On the other hand, using (4.27) and the fact that $\log|\det Df|$ is Lipschitz continuous,

$$\big|\log|\det\, Df^l|(x) - \log|\det\, Df^l|(\pi_J(x))\big| \le K_7\,d(\gamma,\tilde\gamma)$$

for some large $K_7 > 0$. Recalling also (4.3),

$$\big|\log|\det(Df^l|\gamma_J)|(\pi_J(x)) - \log|\det(Df^l|\tilde\gamma_J)|(x)\big| \le K_8\,d(\gamma,\tilde\gamma)^{\nu_0},$$

for some $K_8 > 0$, and

$$\begin{aligned}\big|\log|\det\, D\pi_J|(x) - \log|\det\, D\pi|(f^l(x))\big| &\le A_0\,d(\gamma_J,\tilde\gamma_J)^{\nu_0} + A_0\,d(\gamma,\tilde\gamma)^{\nu_0}\\ &\le 2A_0\,d(\gamma,\tilde\gamma)^{\nu_0}.\end{aligned}$$

In view of our choice of μ, μ_1, ν in (4.11), we conclude that (4.24) is bounded by

$$c_1(1+H)\,d(\gamma,\tilde\gamma)^\nu + K_9\,d(\gamma,\tilde\gamma)^\nu$$

where the constant $K_9 > 0$ depends only on f and a_1. At this point, we assume that $c > 0$ has been taken large enough so that $K_9 \le c/2$, and then we choose any $c_1 \le c/(2(1+H))$. □

Remark 4.2. The form of the constant $C_2(\varphi,\psi)$ in Proposition 4.9 is relevant for the sequel. By the mean value theorem, $\sup|\varphi| \le \int_Q |\varphi|\,dm + \|\varphi\|_\nu(\text{diameter } Q)^\nu$ if φ is $(\|\varphi\|_\nu,\nu)$-Hölder continuous. Therefore, the argument in the proof of the proposition yields

$$C_2(\varphi,\psi) \approx \text{ const}\left(\int_Q |\varphi|\,dm + \|\varphi\|_\nu\right)\left(\int_Q |\psi|\,dm + \|\psi\|_\mu\right). \tag{4.28}$$

The Hölder term $\|\psi\|_\mu$ is essential, as shown by the following type of examples. Let $\psi: Q \to [0,1]$ be a C^1 function with

$$\int_Q \psi\, dm > 0 \quad \text{and} \quad \text{support}\,(\psi) \cap \Lambda = \emptyset.$$

Then $\psi_n = \psi \circ f^{-n}$ is a sequence of C^1 functions with $\int_Q (\psi_n \circ f^n)\, 1\, dm = \int_Q \psi\, dm > 0$, but $\int \psi_n\, d\mu_0 = 0$ for all $n \geq 0$. Note that $\|\psi_n\|_\mu$ is not bounded.

On the other hand, it is possible to improve the estimate of (4.28) in the following useful way. In the first step of the proof we took $\log \psi$ to be $(a/2, \mu)$-Hölder, to ensure that $\psi H_\gamma \in \mathcal{D}(\gamma)$ for every γ. Now, for this last conclusion it suffices that $\log \psi$ be $(a/2, \mu)$-Hölder *along stable leaves*. Thus, actually, the proof of Proposition 4.9 gives

$$C_2(\varphi, \psi) \approx \text{ const} \left(\int_Q |\varphi|\, dm + \|\varphi\|_\nu \right) \left(\int_Q |\psi|\, dm + \|\psi\|_\mu^s \right) \tag{4.29}$$

$\|\psi\|_\mu^s$ denoting any uniform Hölder constant for the restriction of ψ to each stable leaf.

Corollary 4.11. *Given any ν-Hölder continuous function φ, there is $C_3 = C_3(\varphi) > 0$ such that, for every $\psi \in L^1(\mathcal{F}_0)$ and all $n \geq 0$,*

a) $|\int_Q (\psi \circ f^n)\, \varphi\, dm - \int \psi\, d\mu_0 \int_Q \varphi\, dm| \leq C_3 \Lambda_2^n \int_Q |\psi|\, dm$

b) $|\int (\psi \circ f^n)\, \varphi\, d\mu_0 - \int \psi\, d\mu_0 \int \varphi\, d\mu_0| \leq C_3 \Lambda_2^n \int_Q |\psi|\, dm.$

Proof: This is a direct consequence of Proposition 4.9 and the last part of Remark 4.2. If ψ is $\mathcal{F}_0$-measurable then it is constant on each stable leaf, and so we may choose $\|\psi\|_\mu^s = 0$. Then take $C_3(\varphi) =$ const $(\int_Q |\varphi|\, dm + \|\varphi\|_\nu)$. □

Corollary 4.12 (ergodicity and SRB property). *The measure μ_0 is ergodic and satisfies*

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \to \int \varphi\, d\mu_0 \quad \text{as } n \to \infty,$$

for every continuous function $\varphi: Q \to \mathbb{R}$ and m-almost all $x \in Q$. In particular, μ_0 is the unique SRB-measure for f in Q.

Proof: Clearly, given any continuous φ and any pair of points x_1, x_2 belonging in a same stable leaf, then

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x_1)) \text{ converges} \quad \Leftrightarrow \quad \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x_2)) \text{ converges,}$$

and in that case the two limits are the same. Let A be the set of points $x \in Q$ such that $\lim n^{-1} \sum_{j=0}^{n-1} \varphi(f^j(x))$ does not exist, for some continuous φ. Then A is a union of stable leaves and the ergodic theorem gives $\mu_0(A) = 0$. Applying Lemma 4.8 to $\psi = \chi_A$ we get $m(A) = 0$.

Now, given any continuous φ, define $\tilde{\varphi}(x) = \lim n^{-1} \sum_{j=0}^{n-1} \varphi(f^j(x))$. We have just shown that $\tilde{\varphi}$ is defined almost everywhere, with respect to both measures μ_0 and m. Moreover, by the ergodic theorem, $\tilde{\varphi} \in L^1(\mathcal{F}_0)$ and $\tilde{\varphi} \circ f = \tilde{\varphi}$ at μ_0-almost every point. Lemma 4.8 for the characteristic function of $\{x \in Q : \tilde{\varphi}(f(x)) = \tilde{\varphi}(x)\}$ gives that $\tilde{\varphi} \circ f = \tilde{\varphi}$ at m-almost every point. Then part a) of Corollary 4.11 implies

$$\left| \int_Q (\tilde{\varphi} - \int \tilde{\varphi}\, d\mu_0) \phi\, dm \right| = \left| \int_Q (\tilde{\varphi} \circ f^n)\, \phi\, dm - \int \tilde{\varphi}\, d\mu_0 \int \phi\, dm \right| \le C_3 \Lambda_2^n \int_Q |\tilde{\varphi}|\, dm$$

for every $n \ge 0$ and every ν-Hölder function ϕ. Therefore,

$$\tilde{\varphi} = \int \tilde{\varphi}\, d\mu_0 = \int \varphi\, d\mu_0$$

m-almost everywhere and μ_0-almost everywhere. This proves ergodicity and the SRB-property, simultaneously. $\square$

4.4 Central limit theorem.

The main result in this section is the following abstract central limit theorem, which may be considered a version of Theorem 2.11 for invertible maps.

Theorem 4.13. *Let $(\Lambda, \mathcal{G}, \mu)$ be a probability space, $\phi \in L^2(\mu)$ be such that $\int \phi\, d\mu = 0$, and $f\colon \Lambda \to \Lambda$ be an invertible map such that both f and f^{-1} are measurable, and μ is f-invariant and f-ergodic. Let $\mathcal{G}_0 \subset \mathcal{G}$ be such that $\mathcal{G}_n = f^{-n}(\mathcal{G}_0)$, $n \in \mathbb{Z}$, is a non-increasing sequence of σ-algebras. Assume that*

$$\sum_{n=0}^{\infty} \|E(\phi \mid \mathcal{G}_n)\|_2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|\phi - E(\phi \mid \mathcal{G}_{-n})\|_2 < \infty, \tag{4.30}$$

and let

$$\sigma^2 = \int \phi^2\, d\mu + 2 \sum_{j=1}^{\infty} \int \phi(\phi \circ f^j)\, d\mu.$$

Then σ is finite, and $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu)$. Moreover, if $\sigma > 0$ then, for any interval $A \subset \mathbb{R}$,

$$\mu\left(\left\{ x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A \right\} \right) \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}}\, dt \qquad \text{as } n \to \infty.$$

Proof: The strategy is basically the same as in the noninvertible case, that is, we decompose $\phi = \eta + \zeta \circ f - \zeta$, where $\eta, \zeta \in L^2(\mu)$ are as in the proof of Theorem 2.11. In order to do that, we begin by writing $\phi = \phi^+ + \phi^-$, with $\phi^+ = E(\phi \mid \mathcal{G}_0)$ and

$\phi^- = \phi - E(\phi \mid \mathcal{G}_0)$, then we decompose each of the two terms ϕ^+ and ϕ^- separately. The argument for ϕ^+ is similar to the one in Theorem 2.11, based on the first part of (4.30), and for ϕ^- we use a dual version, relying on the second statement in (4.30).

For each j, we denote $\hat{E}(\phi \mid \mathcal{G}_j) = \phi - E(\phi \mid \mathcal{G}_j)$, the orthogonal projection of ϕ to $L^2(\mathcal{G}_j)^\perp$. We also let $\mathcal{P}$ be the adjoint operator of $U : L^2(\mu) \to L^2(\mu)$. Observe that U and $\mathcal{P}$ are unitary operators mapping, respectively, $L^2(\mathcal{G}_j)$ onto $L^2(G_{j+1})$ and $L^2(G_{j+1})$ onto $L^2(\mathcal{G}_j)$, for each j. Then, we introduce

$$\zeta^+ = -\sum_{j=1}^{\infty} \mathcal{P}^j(E(\phi \mid \mathcal{G}_j)) \quad \text{and} \quad \eta^+ = \sum_{j=0}^{\infty} \mathcal{P}^j(E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})),$$
$$\zeta^- = -\sum_{j=0}^{\infty} U^j(\hat{E}(\phi \mid \mathcal{G}_{-j})) \quad \text{and} \quad \eta^- = \sum_{j=1}^{\infty} U^j(\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j})).$$

Since $\mathcal{P}$ and U are isometries to their images, (4.30) ensures

$$\|\zeta^+\|_2 \le \sum_{j=1}^{\infty} \|E(\phi \mid \mathcal{G}_j)\|_2 < \infty \quad \text{and} \quad \|\zeta^-\|_2 \le \sum_{j=0}^{\infty} \|\phi - E(\phi \mid \mathcal{G}_{-j})\|_2 < \infty.$$

Moreover, $\mathcal{G}_{j+1} \subset \mathcal{G}_j$ implies that $E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})$ is orthogonal to $E(\phi \mid \mathcal{G}_{j+1})$, and so

$$\|\eta^+\|_2 \le \sum_{j=0}^{\infty} \|E(\phi \mid \mathcal{G}_j) - E(\phi \mid \mathcal{G}_{j+1})\|_2 \le \sum_{j=0}^{\infty} \|E(\phi \mid \mathcal{G}_j)\|_2 < \infty.$$

Also, $\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j}) = E(\phi \mid \mathcal{G}_{-j}) - E(\phi \mid \mathcal{G}_{-j+1}) \in L^2(\mathcal{G}_{-j})$ is orthogonal to $\hat{E}(\phi \mid \mathcal{G}_{-j}) \in L^2(\mathcal{G}_{-j})^\perp$, and so

$$\|\eta^-\|_2 \le \sum_{j=1}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-j+1}) - \hat{E}(\phi \mid \mathcal{G}_{-j})\|_2 \le \sum_{j=1}^{\infty} \|\hat{E}(\phi \mid \mathcal{G}_{-j+1})\|_2 < \infty.$$

We have shown that $\zeta^+, \eta^+, \zeta^-, \eta^-$ are in $L^2(\mu)$. On the other hand,

$$\begin{aligned} &\hat{E}(\phi \mid \mathcal{G}_{-j}) - \hat{E}(\phi \mid \mathcal{G}_{-j+1}) \in L^2(\mathcal{G}_{-j}) \ominus L^2(\mathcal{G}_{-j+1}) \text{ for all } j \ge 1 \\ &\quad \Rightarrow \quad U^j(\hat{E}(\phi \mid \mathcal{G}_j) - \hat{E}(\phi \mid \mathcal{G}_{j+1})) \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1) \text{ for all } j \ge 1 \\ &\quad \Rightarrow \quad \eta^- \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1). \end{aligned}$$

and, in a similar way, $\eta^+ \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1)$ (recall that $\ominus$ denotes orthogonal complement). As in the proof of Theorem 2.11, one obtains $\eta^+ = \phi^+ - \zeta^+ \circ f + \zeta^+$, moreover, a similar calculation yields

$$\begin{aligned}\eta^- &= U(\hat{E}(\phi \mid \mathcal{G}_0)) + U^2(\hat{E}(\phi \mid \mathcal{G}_{-1})) + U^3(\hat{E}(\phi \mid \mathcal{G}_{-2})) + \cdots \\ &\quad - U(\hat{E}(\phi \mid \mathcal{G}_{-1})) - U^2(\hat{E}(\phi \mid \mathcal{G}_{-2})) - U^3(\hat{E}(\phi \mid \mathcal{G}_{-3})) - \cdots \\ &= \hat{E}(\phi \mid \mathcal{G}_0) + U(\hat{E}(\phi \mid \mathcal{G}_0)) + U^2(\hat{E}(\phi \mid \mathcal{G}_{-1})) + U^3(\hat{E}(\phi \mid \mathcal{G}_{-2})) + \cdots \\ &\quad - \hat{E}(\phi \mid \mathcal{G}_0) - U(\hat{E}(\phi \mid \mathcal{G}_{-1})) - U^2(\hat{E}(\phi \mid \mathcal{G}_{-2})) - \cdots \\ &= \phi^- - U(\zeta^-) + \zeta^- = \phi^- - \zeta^- \circ f + \zeta^-.\end{aligned}$$

Finally, we set $\eta = \eta^+ + \eta^-$ and $\zeta = \zeta^+ + \zeta^-$. It follows that $\phi = \eta + \zeta \circ f - \zeta$ and

(i) $\frac{1}{\sqrt{n}}(\zeta \circ f^n - \zeta) \to 0$ in $L^2(\mu)$ and in measure, because $\zeta \in L^2(\mu)$;

(ii) $\eta \circ f^n \in L^2(\mathcal{G}_n) \ominus L^2(\mathcal{G}_{n+1})$ for every $n \geq 0$, and so $\eta \circ f^n$ is a reversed martingale difference for the sequence $(\mathcal{G}_n)_{n\geq 0}$.

At this point we have completely recovered the ingredients of the noninvertible case, and the proof proceeds in precisely the same way as that of Theorem 2.11. □

Returning to our setting of uniformly hyperbolic attractors of a diffeomorphism f, we obtain the following direct application of the previous theorem.

Proposition 4.14 (central limit theorem). *Let φ be a ν-Hölder continuous function and $\sigma \geq 0$ be given by*

$$\sigma^2 = \int \phi^2 \, d\mu_0 + 2 \sum_{j=1}^{\infty} \int \phi(\phi \circ f^j) \, d\mu_0,$$

where $\phi = \varphi - \int \varphi \, d\mu_0$. Then σ is finite and $\sigma = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^2(\mu_0)$. Moreover, if $\sigma > 0$ then, for any interval $A \subset \mathbb{R}$,

$$\mu_0\Big(\Big\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A\Big\}\Big) \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma^2}} \, dt \qquad \text{as } n \to \infty.$$

Proof: We consider $\Lambda = \cap_{n\geq 0} f^n(Q)$, $\mathcal{G}$ = Borel σ-algebra of Λ, and $\mu = \mu_0$. Moreover, we take $\mathcal{G}_0$ to be the restriction to the attractor Λ of the σ-algebra $\mathcal{F}_0$ (consisting of those Borel subsets which are union of local stable leaves). Clearly, $\mathcal{G}_n = f^{-n}(\mathcal{G}_0)$ is a decreasing sequence. We are left to check the hypothesis (4.30). The first statement is a direct consequence of Corollary 4.11(b):

$$\begin{aligned}\|E(\phi \mid \mathcal{G}_n)\|_2 &= \sup\Big\{\int \phi\xi \, d\mu_0 : \xi \in L^2(\mathcal{G}_n, \mu_0) \text{ with } \|\xi\|_2 = 1\Big\} \\ &\leq \sup\Big\{\int \phi(\psi \circ f^n) d\mu_0 : \psi \in L^2(\mathcal{G}_0, \mu_0) \text{ with } \|\psi\|_2 = 1\Big\} \\ &\leq C_3 \, \Lambda_2^n,\end{aligned}$$

because $\int \phi \, d\mu_0 = 0$ and $||\psi||_1 \le ||\psi||_2 = 1$. As for the second property, it follows directly from the Hölder continuity of φ. Indeed, $E(\phi \mid \mathcal{G}_{-n})$ is constant on each nth image $\eta = f^n(\gamma)$ of a stable leaf γ, and

$$\inf(\phi \mid \eta) \le E(\phi \mid \mathcal{G}_{-n})(\eta) \le \sup(\phi \mid \eta).$$

Since diameter $\eta \le C_s \lambda_s^n$, for uniform constants $C_s > 0$ and $\lambda_s \in (0,1)$, and the function ϕ is (A,ν)-Hölder, for some $A > 0$, we get that

$$||\phi - E(\phi \mid \mathcal{G}_{-n})||_2 \le ||\phi - E(\phi \mid \mathcal{G}_{-n})||_0 \le A\, C_s^\nu \, \lambda_s^{n\nu}.$$

This means that we are indeed in the conditions of Theorem 4.13, and so the proof is complete. □

4.5 Stochastic stability.

In this last section we prove that the maps $f\colon Q \to Q$ we have been considering are stable under small random perturbations. The setting is formally the same as in previous sections. We consider an arbitrary parametrized family $(f_t)_{t\in T}$ of C^2 maps from Q to Q, where T is any metric space. We suppose that, for some $\tau \in T$,

$$f_\tau = f \quad \text{and} \quad T \ni t \mapsto f_t \text{ is continuous at } \tau \quad \text{(with respect to the } C^2\text{-topology)}.$$

The basic example corresponds to T being some neighbourhood of f in the space of C^2 embeddings of Q into itself, with $f_t = t$ for each $t \in T$. We also consider an arbitrary family $(\theta_\varepsilon)_{\varepsilon>0}$ of regular probability measures on T such that

$$\operatorname{supp} \theta_\varepsilon \to \{\tau\} \quad \text{as} \quad \varepsilon \to 0.$$

Then we construct, for each small $\varepsilon > 0$, a probability measure μ_ε which is stationary under the random process associated to $((f_t)_{t\in T},\ \theta_\varepsilon)$:

$$\int \left(\int (\psi \circ f_t)\, d\mu_\varepsilon \right) d\theta_\varepsilon(t) = \int \psi \, d\mu_\varepsilon \tag{4.31}$$

for every continuous function $\psi : Q \to \mathbb{R}$. Moreover, this measure μ_ε determines the asymptotic time-averages of continuous functions over almost all random trajectories $x_j = f_{t_j} \circ \cdots f_{t_1}(x)$:

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(x_j) \to \int \psi \, d\mu_\varepsilon$$

for every continuous ψ and $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every $(x, t_1, \ldots, t_j, \ldots)$. Finally, we show that μ_ε converges weakly to the SRB-measure μ_0 of f when $\varepsilon \to 0$.

Our strategy for proving these statements is somewhat different from the one we used in the expanding case. In Section 2.6 we took advantage of the fact that the cone $C(a,\mu)$, that had been constructed for studying the unperturbed map f, is also invariant under transfer operators $\mathcal{L}_t$ associated to the perturbations f_t. As it turns out, the corresponding statement no longer holds in the present setting. The reason is that the definition of our cone $C(b,c,\nu)$ involves the stable foliation $\mathcal{F}^s_{loc}$, and the proof that $C(b,c,\nu)$ is invariant under the operator $\mathcal{L}$ makes use of the invariance of $\mathcal{F}^s_{loc}$ under the map f (more precisely, we needed the fact that the preimage of a local stable leaf is a union of local stable leaves).

One way to bypass this difficulty is to replace $C(b,c,\nu)$ by some other convex cone, invariant under the transfer operator of every f_t with t close to τ. Such a cone may be obtained, for instance, substituting in the definition of $C(b,c,\nu)$ the foliation $\mathcal{F}^s_{loc}$ by a larger class of submanifolds of Q with an invariance property with respect to every f_t.

An alternative approach, that we take here, relies on considering the skew-product map

$$(4.32)\qquad F: Q\times T^{\mathbb{N}} \to Q\times T^{\mathbb{N}}, \qquad F(x,t_1,t_2,t_3,\dots) = (f_{t_1}(x),t_2,t_3,\dots),$$

and developping for this F, and every small $\varepsilon > 0$, a theory similar to the one presented in the previous sections for f, with $m\times\theta_\varepsilon^{\mathbb{N}}$ in the role of Lebesgue measure m. The statements of stability made above are then easily deduced from this theory. This approach also provides information on the individual behaviour (e.g. correlation functions) of typical random iterates, although we do not pursue this aspect here.

A good part of this treatment of the "random" system $(F, m\times\theta_\varepsilon^{\mathbb{N}})$ consists in adapting arguments we used previously for f, profitting from their robustness under small perturbations: most of what we have done so far remains valid, with uniform estimates, when f^n is replaced by $f_{t_n}\circ\cdots\circ f_{t_1}$, for any $n\geq 1$ and any $f_{t_1},\dots,f_{t_n}$ in a sufficiently small C^2 neighbourhood $\mathcal{V}$ of f. We give the guidelines of each step but, as a rule, do not reproduce in detail those arguments which appeared already in the unperturbed case and which can be translated in straightforward ways to the present setting. The reader who has gone through the previous sections should find no difficulty in providing those details, and may find it a good exercise to do so.

The constants λ_1, λ_2, Λ_1, Λ_2, λ_u, λ_s, μ, $\nu\in(0,1)$, and a, b, $c>0$, have the same meaning as before, but may take slightly different values. More precisely, all these constants are *uniform*, meaning that they may be chosen depending only on f. We use $\underline{t}=(t_1,t_2,t_3,\dots)$ to represent a generic element of $T^{\mathbb{N}}$, and we also denote $\sigma(\underline{t})=(t_2,t_3,\dots)$. Up to replacing, right from the start, the metric space T by a sufficiently small neighbourhood of τ, we may suppose that every f_t, $t\in T$, belongs in a neighbourhood $\mathcal{V}$ of f as above (where the estimates of the unperturbed case remain valid). The precise conditions we need on $\mathcal{V}$ are stated at a few places along the way.

We start by introducing linear operators $\widehat{U}$ and $\mathcal{L}_{t_1}$, for $t_1\in T$, given by

$$(\widehat{U}\Phi)(x,\underline{t}) = (\Phi\circ F)(x,\underline{t}) = \Phi(f_{t_1}(x),\sigma(\underline{t}))$$

and

$$(\mathcal{L}_{t_1}\Phi)(y,\sigma(\underline{t})) = \begin{cases} \Phi(f_{t_1}^{-1}(y),\underline{t})\,|\det Df_{t_1}(f_{t_1}^{-1}(y))|^{-1}, & \text{if } y\in f_{t_1}(Q);\\ 0, & \text{otherwise.}\end{cases}$$

for every function $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$. We also define, for $\varepsilon > 0$ and Φ as before,

$$(\widehat{\mathcal{L}}_\varepsilon \Phi)(y, \sigma(\underline{t})) = \int (\mathcal{L}_{t_1} \Phi)(y, \sigma(\underline{t}))\, d\theta_\varepsilon(t_1).$$

Let $\pi_0 : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be the canonical projection $\pi_0(x, \underline{t}) = x$. We say that a function $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ *depends only on* x if it can be written $\Phi = \varphi \circ \pi_0$ for some $\varphi : Q \to \mathbb{R}$. Observe that if Φ depends only on x then so do $\mathcal{L}_{t_1}\Phi$ and $\widehat{\mathcal{L}}_\varepsilon \Phi$, for every $t_1 \in T$ and $\varepsilon > 0$. Thus, we may also think of these operators as acting on the space of functions defined on Q, and sometimes we do so.

A main property of $\widehat{U}$ and $\widehat{\mathcal{L}}_\varepsilon$ is the following duality relation, which follows directly from the definitions and Fubini's theorem, using the change of variables $y = f_{t_1}^{-1}(x)$:

$$\int \Phi(\widehat{U}\Psi)\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int (\widehat{\mathcal{L}}_\varepsilon \Phi)\Psi\, d(m \times \theta_\varepsilon^{\mathbb{N}}). \tag{4.33}$$

whenever the integrals make sense. This estalishes a close link between the operator $\widehat{\mathcal{L}}_\varepsilon$ and stationary measures of our random process, as illustrated by the following remarks.

Let $\hat{\mu}_\varepsilon$ be a probability measure on $Q \times T^{\mathbb{N}}$ given by $\hat{\mu} = \Phi\,(m \times \theta_\varepsilon^{\mathbb{N}})$, and suppose that $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ is a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. Then (4.33) gives, for any Ψ,

$$\int (\Psi \circ F)\, d\hat{\mu} = \int (\widehat{U}\Psi)\Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi\Phi\, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi\, d\hat{\mu}.$$

In other words, $\hat{\mu}$ is an invariant measure for F. The converse is proved in the same way: the density of an F-invariant measure absolutely continuous with respect to $m \times \theta_\varepsilon^{\mathbb{N}}$ is necessarily a fixed point of $\widehat{\mathcal{L}}_\varepsilon$. Now suppose that this fixed point Φ depends only on x, and write $\Phi = \varphi \circ \pi_0$. Then $\mu = \varphi m$ is a stationary probability measure on Q. To see this, let $\psi : Q \to \mathbb{R}$ be continuous, and define $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ by $\Psi(x, \underline{t}) = \psi(x)$. Then, since $\Phi(x, \sigma(\underline{t})) = \varphi(x) = \Phi(x, \underline{t})$, and $d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\underline{t}))\, d\theta_\varepsilon(t_1)$ represents just the same as $d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \underline{t})$,

$$\begin{aligned} \int \Big(\int \psi(f_{t_1}(x))\, d\mu_\varepsilon(x)\Big) d\theta_\varepsilon(t_1) &= \int \Psi(f_{t_1}(x), \sigma(\underline{t}))\Phi(x, \sigma(\underline{t}))\, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\underline{t})) d\theta_\varepsilon(t_1) \\ &= \int (\Psi \circ F)(x, \underline{t})\, \Phi(x, \underline{t})\, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \underline{t}) \\ &= \int (\Psi \circ F)\, d\hat{\mu}_\varepsilon = \int \Psi\, d\hat{\mu}_\varepsilon = \int \psi\, d\mu_\varepsilon. \end{aligned}$$

In general, $\widehat{\mathcal{L}}_\varepsilon$ need not have a fixed point and the stationary measure μ_ε need not be absolutely continous with respect to Lebesgue measure. However, the considerations we have just made motivate our approach for constructing the stationary measure. Similarly to what we did in the unperturbed case, cf. Proposition 4.7, we shall prove that successive push-forwards of $(m \times \theta_\varepsilon^{\mathbb{N}})$ form a sequence of probability measures

$$F_*^n(m \times \theta_\varepsilon^{\mathbb{N}}) = (\widehat{\mathcal{L}}_\varepsilon^n 1)\,(m \times \theta_\varepsilon^{\mathbb{N}})$$

that converges weakly to some measure $\hat{\mu}_\varepsilon$ on $Q \times T^{\mathbb{N}}$. Let μ_ε be the measure on Q defined by

$$\mu_\varepsilon = (\pi_0)_* \hat{\mu}_\varepsilon.$$

We shall seduce that μ_ε is the stationary measure we are looking for, from the fact that every $\widehat{\mathcal{L}}_\varepsilon^n 1$ depends only on x.

An important tool in this construction is the local stable foliation of the map F. The same kind of arguments as one uses for constructing the local stable foliation $\mathcal{F}^s_{loc}$ of the map f, see Appendix A, shows that there exists a (unique) foliation $\widehat{\mathcal{F}}^s_{loc}$ of $Q \times T^{\mathbb{N}}$ satisfying properties (1), (2), (3), (4) below. At this point we suppose that all the maps f_t are in a suffciently small neighbourhood $\mathcal{V}$ of f, cf. previous comments.

(1) Each leaf $\widehat{\mathcal{F}}^s_{loc}(x, \underline{t})$ through a point $(x, \underline{t})$ is a C^2 submanifold of $Q \times \{\underline{t}\}$, with uniformly bounded curvature.

(2) $F(\widehat{\mathcal{F}}^s_{loc}(x, \underline{t}))$ is contained in $\widehat{\mathcal{F}}^s_{loc}(F(x, \underline{t}))$, for every $(x, \underline{t}) \in Q \times T^{\mathbb{N}}$, and

$$F : \widehat{\mathcal{F}}^s_{loc}(x, \underline{t}) \to \widehat{\mathcal{F}}^s_{loc}(F(x, \underline{t}))$$

is a λ_s-contraction, for some uniform constant $\lambda_s \in (0, 1)$.

(3) Given $(y, \sigma(\underline{t})) \in Q \times T^{\mathbb{N}}$ and any $t_1 \in T$, the intersection

$$F^{-1}(\widehat{\mathcal{F}}^s_{loc}(y, \sigma(\underline{t}))) \cap (Q \times \{\underline{t}\})$$

has exactly two connected components, and they are also leaves of $\widehat{\mathcal{F}}^s_{loc}$.

(4) The foliation $\widehat{\mathcal{F}}^s_{loc}$ is absolutely continuous with respect to $(m \times \theta_\varepsilon^{\mathbb{N}})$.

Let us explain property (4) in more precise terms, before proceeding. Let $\widetilde{m}_\varepsilon$ be the measure induced by $(m \times \theta_\varepsilon^{\mathbb{N}})$ in the quotient space (the space of leaves) of $\widehat{\mathcal{F}}^s_{loc}$, that is,

$$\widetilde{m}_\varepsilon(\tilde{A}) = (m \times \theta_\varepsilon^{\mathbb{N}}) \left(\bigcup_{\gamma \subset \tilde{A}} \gamma \right)$$

for every measurable subset $\tilde{A}$ of the quotient space. Given any $\gamma \in \widehat{\mathcal{F}}^s_{loc}$, let m_γ be the smooth measure induced on γ by the riemannian metric of M. In (4) we mean that there exists a continuous function

$$H_\varepsilon : Q \times T^{\mathbb{N}} \to (0, +\infty),$$

bounded away from zero and infinity, such that $\{p_{\varepsilon,\gamma} = (H_\varepsilon \mid \gamma)\, m_\gamma : \gamma \in \widehat{\mathcal{F}}^s_{loc}\}$ defines a disintegration of $(m \times \theta_\varepsilon^{\mathbb{N}})$ along the leaves of the foliation: given any $\Psi \in L^1(m \times \theta_\varepsilon^{\mathbb{N}})$,

$$\int \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma (\Psi \mid \gamma) \, dp_{\varepsilon,\gamma} \right) d\widetilde{m}_\varepsilon(\gamma).$$

Moreover, H_ϵ may be taken such that $\log H_\epsilon$ is (a_0, ν_0)-Hölder continuous on every $Q \times \{\underline{t}\}$, with $a_0 > 0$ and $\nu_0 \in (0,1)$ depending only on the initial map f.

Given a leaf $\gamma \in \widehat{\mathcal{F}}^s_{loc}$ we define cones of Hölder continuous densities $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$ and $\mathcal{D}_1(\gamma) = \mathcal{D}(a_1, \mu_1, \gamma)$, in just the same way as before:

$$\mathcal{D}(\gamma) = \{\rho : \gamma \to \mathbb{R} \text{ such that } \rho(x) > 0 \text{ for all } x \in \gamma \text{ and } \log \rho \text{ is } (a, \mu) - \text{Hölder}\},$$

and similarly for $\mathcal{D}_1(\gamma)$. The constants a, μ, a_1, μ_1 are chosen as in (4.7), (4.11).

For any bounded function Φ defined on γ and any $\rho \in \mathcal{D}(\gamma)$ we let $\int_\gamma \Phi\rho$ denote the integral of Φ with respect to the measure ρm_γ supported on γ. The following simple consequence of Fubini's theorem will be useful later:

$$\int_\gamma (\widehat{\mathcal{L}}_\epsilon \Phi)\rho = \int \left(\int_\gamma (\mathcal{L}_{t_1}\Phi)\rho \right) d\theta_\epsilon(t_1). \tag{4.34}$$

For each $\underline{t} \in T^{\mathbb{N}}$ we denote $\widehat{\mathcal{F}}^s_{loc}(\underline{t})$ the restriction of the foliation $\widehat{\mathcal{F}}^s_{loc}$ to $Q \times \{\underline{t}\}$. We often identify $Q \times \{\underline{t}\}$ with Q, through the canonical bijection

$$\phi_{\underline{t}} : (Q \times \{\underline{t}\}) \ni (x, \underline{t}) \mapsto x \in Q,$$

thus thinking of each $\widehat{\mathcal{F}}^s_{loc}(\underline{t})$ also as a foliation of Q. Observe that after identification the action of F on the leaves of this foliation is described by the map f_{t_1}.

For $\gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{t}))$ and each $t_1 \in T$, let $\gamma_{1,t_1}, \gamma_{2,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\underline{t})$ be the connected components of

$$F^{-1}(\gamma) \cap (Q \times \{\underline{t}\}),$$

cf. property (3) above. Then, for any $\rho \in \mathcal{D}(\gamma)$ and any bounded function Φ,

$$\int_\gamma (\mathcal{L}_{t_1}\Phi)\rho = \sum_{j=1}^{2} \int_{\gamma_{j,t_1}} \Phi \rho_{j,t_1}, \qquad \rho_{j,t_1} = \frac{|\det(Df_{t_1} \mid \gamma_{j,t_1})|}{|\det Df_{t_1}|}(\rho \circ f_{t_1}), \tag{4.35}$$

compare (4.2). As in Lemma 4.2, there is some uniform constant $\lambda_1 < 1$ so that

$$\rho_{j,t_1} \in \mathcal{D}(\lambda_1 a, \mu, \gamma_{j,t_1}) \quad \text{for every } j = 1, 2.$$

Moreover, there is some other uniform constant $\Lambda_1 < 1$ so that

$$\theta_{j,t_1}(\rho'_{j,t_1}, \rho''_{j,t_1}) \le \Lambda_1 \theta(\rho', \rho''), \quad \text{for all } \rho', \rho'' \in \mathcal{D}(\gamma) \text{ and } j = 1, 2,$$

where θ and θ_{j,t_1} are the projective metrics associated to the cones $\mathcal{D}(\gamma)$ and to $\mathcal{D}(\gamma_{j,t_1})$, respectively.

The next step is to define a projection map $\pi = \pi(\tilde{\gamma}, \gamma)$ from a leaf $\tilde{\gamma}$ to another leaf γ of the local stable foliation of F, as well as a notion of distance $d(\gamma, \tilde{\gamma})$ between the two leaves. Let $\gamma \in \mathcal{F}^s_{loc}(\sigma(\underline{t}))$ and $\tilde{\gamma} \in \mathcal{F}^s_{loc}(\sigma(\underline{s}))$ for some $\sigma(\underline{t}) = (t_2, t_3, \ldots)$ and

$\sigma(\underline{s}) = (s_2, s_3, \dots)$ in $T^{\mathbb{N}}$. Identifying both $Q \times \{\sigma(\underline{t})\}$ and $Q \times \{\sigma(\underline{s})\}$ with Q, we may consider γ and $\tilde{\gamma}$ as submanifolds of Q, and then define π in just the same way as in Section 4.1. In more precise terms, we set

$$\pi = \pi(\tilde{\gamma}, \gamma) = \phi_{\sigma(\underline{t})}^{-1} \circ \pi(\phi_{\sigma(\underline{s})}(\tilde{\gamma}), \phi_{\sigma(\underline{t})}(\gamma)) \circ \phi_{\sigma(\underline{s})},$$

where $\pi(\phi_{\sigma(\underline{s})}(\tilde{\gamma}), \phi_{\sigma(\underline{t})}(\gamma))$ is as defined in Section 4.1. Analogously, given any $y \in \tilde{\gamma}$,

$$d(y, \pi(y)) = d(\phi_{\sigma(\underline{s})}(y), \phi_{\sigma(\underline{t})}(\pi(y))),$$

where the right hand side is meant as in Section 4.1.

For $t_1 \in T$, let $\underline{s} = (t_1, s_2, s_3, \dots)$, recall that $\underline{t} = (t_1, t_2, t_3, \dots)$. Let $\gamma_{j,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\underline{t})$ and $\tilde{\gamma}_{j,t_1} \in \widehat{\mathcal{F}}^s_{loc}(\underline{s})$, $j = 1, 2$, be the connected components of

$$F^{-1}(\gamma) \cap (Q \times \{\underline{t}\}) \quad \text{and} \quad F^{-1}(\tilde{\gamma}) \cap (Q \times \{\underline{s}\}),$$

respectively, and denote $\pi_{j,t_1} = \pi(\tilde{\gamma}_{j,t_1}, \gamma_{j,t_1})$. Then we have the following analogs of properties (p1) and (p3):

(q1) π and $\log|\det D\pi|$ are a_0-Lipschitz maps;

(q3) $d(x, \pi_{j,t_1}(x)) \le \lambda_0 \, d(f_{t_1}(x), \pi f_{t_1}(x))$ for all $x \in \tilde{\gamma}_{j,t_1}$, $j = 1, 2$, and $t_1 \in T$.

where $\lambda_u < 1$, $a_0 > 0$, and $\nu_0 \in (0, 1)$, depend only on the unperturbed map f. Indeed, (q1) is a consequence of the fact that stable leaves have uniformly bounded curvature, and (q3) is proved in the same way as (p3), with f_{t_1} in the place of f (and further restricting the neighbourhoof $\mathcal{V}$ of f, if necessary).

We also need an analog of property (p2), but this is more subtle. Indeed, (p2) relied on the Hölder property (4.3) of the tangent spaces to stable leaves, which has no straightforward analog in the present situation. To see this observe that, although the canonical identifications $\phi_{\sigma(\underline{t})}$ and $\phi_{\sigma(\underline{s})}$ allow us to think of both $\widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{t}))$ and $\widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{s}))$ as foliations of Q, in general these foliations do not coincide. For instance, a leaf γ of one foliation may intersect a leaf $\tilde{\gamma}$ of the other foliation transversely at some point y, in which case

$$d(y, \pi(y)) = 0 \quad \text{but} \quad \text{angle}(T_y\tilde{\gamma}, T_{\pi(y)}\gamma) > 0. \tag{4.36}$$

An exception occurs in the particular case $\sigma(\underline{t}) = \sigma(\underline{s})$, since a Hölder property similar to (4.3) can be proved for leaves γ_1, γ_2 within a same $Q \times \{\sigma(\underline{t})\}$, any $\sigma(\underline{t}) \in T^{\mathbb{N}}$: denoting $\pi = \pi(\gamma_1, \gamma_2)$,

$$\text{angle}(T_z\gamma_1, T_{\pi(z)}\gamma_2) \le a_0 \, d(z, \pi(z))^{\nu_0} \quad \text{for all } z \in \gamma_1 . \tag{4.37}$$

This property follows from the same methods as (4.3), and will be useful below. However, this particular case is not sufficient for our purposes (the statement of the analog of condition (C) in the definition of the new cone of observable functions in $Q \times T^{\mathbb{N}}$

must involve all the pairs of local stable leaves γ and $\tilde{\gamma}$ in $\widehat{\mathcal{F}}^s_{loc}$) and so we must deal with the difficulty expressed by (4.36).

The way we overcome this is by defining the distance $d(\gamma, \tilde{\gamma})$ between two general stable leaves in a careful way. Similarly to what we did before, we consider

$$d_1(\gamma, \tilde{\gamma}) = \sup\{d(y, \pi(y)) : y \in \tilde{\gamma}\}.$$

But a a key point is to take angles in consideration too, when determining how far apart two leaves are from each other. Given any point $z \in \gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{t}))$, let γ_z be the leaf of $\widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{s}))$ that contains z. We define

$$\partial(\gamma, \tilde{\gamma}) = \sup\{\text{angle}(T_z\gamma, T_z\gamma_z)^{1/\nu_0} : z \in \gamma\},$$

and define $\partial(\tilde{\gamma}, \gamma)$ in the same way, just reversing the roles of γ, $\tilde{\gamma}$ and $\sigma(\underline{t})$, $\sigma(\underline{s})$. Our definition also involves explicitly the distance between $\sigma(\underline{t})$ and $\sigma(\underline{s})$ (this will be useful in the proof of Lemma 4.16): we let

$$d_2(\gamma, \tilde{\gamma}) = \|\sigma(\underline{t}) - \sigma(\underline{s})\| = \sum_{i=1}^{\infty} 2^{-i} d(t_{i+1}, s_{i+1})$$

Finally, we define

$$d(\gamma, \tilde{\gamma}) = \max\{d_1(\gamma, \tilde{\gamma}), d_2(\gamma, \tilde{\gamma}), \partial(\gamma, \tilde{\gamma}), \partial(\tilde{\gamma}, \gamma)\}. \tag{4.38}$$

Of course, $\partial(\gamma, \tilde{\gamma}) = \partial(\tilde{\gamma}, \gamma) = d_2(\gamma, \tilde{\gamma}) = 0$ when $\sigma(\underline{t}) = \sigma(\underline{s})$, and so in that case $d(\gamma, \tilde{\gamma})$ coincides with the "usual" distance $d_1(\gamma, \tilde{\gamma})$.

Now, observe that $\pi = \pi(\tilde{\gamma}, \gamma) = \pi(\gamma_{\pi(y)}, \gamma) \circ \pi(\tilde{\gamma}, \gamma_{\pi(y)})$ and $\pi(\tilde{\gamma}, \gamma_{\pi(y)})(y) = \pi(y)$, for any $y \in \tilde{\gamma}$. So,

$$\log|\det D\pi|(y) = \log|\det D\pi(\gamma_{\pi(y)}, \gamma)|(\pi(y)) + \log|\det D\pi(\tilde{\gamma}, \gamma_{\pi(y)})|(y).$$

By definition, $\pi(y)$ belongs in $\gamma_{\pi(y)} \cap \gamma$, and so the first term on the right hand side is bounded by

$$\alpha_0 \,\text{angle}(T_{\pi(y)}\gamma, T_{\pi(y)}\gamma_{\pi(y)}) \leq \alpha_0 \partial(\gamma, \tilde{\gamma})^{\nu_0} \leq \alpha_0 d(\gamma, \tilde{\gamma})^{\nu_0}$$

for some universal constant $\alpha_0 > 0$. Also by definition, $\tilde{\gamma}$ and $\gamma_{\pi(y)}$ are both leaves of the foliation $\widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{s}))$. Hence, by (4.37), the second term on the right hand is bounded by

$$a_0 d(y, \pi(y))^{\nu_0} \leq a_0 d_1(\gamma, \tilde{\gamma})^{\nu_0} \leq a_0 d(\gamma, \tilde{\gamma})^{\nu_0}.$$

In this way we conclude the analog of (p2) we were looking for:

(q2) $\log|\det D\pi|(y) \leq (a_0 + \alpha_0) d(\gamma, \tilde{\gamma})^{\nu_0}$ for every $y \in \tilde{\gamma}$.

Now we also need to generalize the expansion property (4.5) to arbitrary pairs γ and $\tilde{\gamma}$ of leaves in the local stable foliation of F. This easily done, in the following way. Up to further restricting the metric space T, the leaves of the foliation $\widehat{\mathcal{F}}^s_{loc}$ (viewed as submanifolds of Q, via canonical identification) are uniformly close to the leaves of the stable foliation $\mathcal{F}^s_{loc}$ of f. Then, let $\gamma_1 \in \widehat{\mathcal{F}}^s_{loc}(\underline{t})$ and $\gamma_2 \in \widehat{\mathcal{F}}^s_{loc}(\underline{s})$ be any two leaves intersecting at some point z, with $\underline{t} = (t_1, t_2, t_3, \dots)$ and $\underline{s} = (t_1, s_2, s_3, \dots)$. Since the action of F on γ_1, γ_2, is described by a same map f_{t_1}, a small perturbation of f, we have

$$\text{angle}(T_{f_{t_1}(z)}F(\gamma_1), T_{f_{t_1}(z)}F(\gamma_2)) \geq \sigma\, \text{angle}(T_z\gamma_1, T_z\gamma_2) \tag{4.39}$$

for some constant $\sigma > 1$ depending only on f. Let $\gamma \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{t}))$ and $\tilde{\gamma} \in \widehat{\mathcal{F}}^s_{loc}(\sigma(\underline{s}))$ be arbitrary leaves, and let $j = 1, 2$ and $t_1 \in T$. Then,

$$\partial(\gamma, \tilde{\gamma}) \geq \sigma \partial(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1})$$

follows from applying the previous relation to $\gamma_1 = \gamma_{j,t_1}$, any $z \in \gamma_1$, and $\gamma_2 = \gamma_{j,t_1,z}$ (the leaf of $\widehat{\mathcal{F}}^s_{loc}(\underline{s})$ through z). Analogously, $\partial(\tilde{\gamma}, \gamma) \geq \sigma\partial(\tilde{\gamma}_{j,t_1}, \gamma_{j,t_1})$. On the other hand, (q3) gives

$$d_1(\gamma, \tilde{\gamma}) \geq \lambda_u^{-1} d_1(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1})$$

for every $j = 1, 2$ and $t_1 \in T$. Finally,

$$\begin{aligned} d_2(\gamma, \tilde{\gamma}) &= \|\sigma(\underline{t}) - \sigma(\underline{s})\| = \sum_{i=1}^{\infty} 2^{-i} d(t_{i+1}, s_{i+1}) \\ &= 2\sum_{i=2}^{\infty} 2^{-i} d(t_i, s_i) = 2\|\underline{t} - \underline{s}\| = 2d_2(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1}). \end{aligned}$$

It is no restriction to suppose $2 \geq \sigma \geq \lambda_u^{-1} > 1$ (decreasing σ in (4.39) and increasing λ_u in (q3), if necessary), and then these remarks give

$$d(\gamma, \tilde{\gamma}) \geq \lambda_u^{-1} d(\gamma_{j,t_1}, \tilde{\gamma}_{j,t_1}) \tag{4.40}$$

for every $j = 1, 2$ and $t_1 \in T$. This is the analog of (4.5) that we wanted.

Finally, given any leaves γ, $\tilde{\gamma}$ of $\widehat{\mathcal{F}}^s_{loc}$ and given any $\rho \in \mathcal{D}_1(\gamma)$, we let $\tilde{\rho}\colon \tilde{\gamma} \to \mathbb{R}$ be defined by

$$\tilde{\rho}(y) = \rho(\pi(y)) \cdot |\det D\pi(y)|.$$

In the same way as before, our choice of a, μ, a_1, μ_1 in (4.7) ensures that $\tilde{\rho}$ is in $\mathcal{D}(\tilde{\gamma})$.

Now we have all we need to give the definition of our cone $\widehat{C}(b, c, \nu)$ of observable functions in $Q \times T^{\mathbb{N}}$. At this point this is a direct translation of the definition of the cone $C(b, c, \nu)$ in Section 4.1. We let $\widehat{C}(b, c, \nu)$ consist of all the bounded functions $\Phi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ that satisfy

(AA) $\int_\gamma \Phi\rho > 0$ for every $\gamma \in \widehat{\mathcal{F}}^s_{loc}$ and every $\rho \in \mathcal{D}(\gamma)$;

(BB) $\log \int_\gamma \Phi\rho$ is b-Lipschitz as a function of $\rho \in \mathcal{D}(\gamma)$:

$$|\log \int_\gamma \Phi\rho' - \log \int_\gamma \Phi\rho''| \le b\,\theta(\rho', \rho'')$$

for every $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$, and every $\gamma \in \widehat{\mathcal{F}}^s_{loc}$;

(CC) $\int_\gamma \Phi\rho$ is (c, ν)-Hölder as a function of γ:

$$|\log \int_\gamma \Phi\rho - \log \int_{\tilde\gamma} \Phi\tilde\rho| \le c\, d(\gamma, \tilde\gamma)^\nu$$

for every $\rho \in \mathcal{D}_1(\gamma)$ and every pair $\gamma, \tilde\gamma \in \widehat{\mathcal{F}}^s_{loc}$.

The argument of Lemma 4.3 applies to $\widehat{C}(b, c, \nu)$, so that this is indeed a convex cone satisfying condition (2.4). We denote $\widehat\Theta$ the corresponding projective metric, and we also let $\widehat\Theta_+$ be the projective metric associated to the cone of strictly positive functions on $Q \times T^{\mathbb{N}}$. They can be calculated in the same way as Θ and Θ_+ in Section 4.1, in fact, one obtains similar expressions (up to replacing $\gamma \in \mathcal{F}^s_{loc}$ by $\gamma \in \widehat{\mathcal{F}}^s_{loc}$).

Let $\Phi \in \widehat{C}(b, c, \nu)$. The same argument as in the proof of Proposition 4.4 shows that there exists a uniform $\lambda_2 < 1$ such that, given any $t_1 \in T$ and any $\gamma, \tilde\gamma \in \widehat{\mathcal{F}}^s_{loc}$,

(1) $\int_\gamma (\mathcal{L}_{t_1}\Phi)\rho > 0$ for every $\rho \in \mathcal{D}(\gamma)$,

(2) for every $\rho', \rho'' \in \mathcal{D}(\gamma)$,

$$\exp(-b\lambda_2\,\theta(\rho', \rho'')) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho'' \le \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho' \le \exp(b\lambda_2\,\theta(\rho', \rho'')) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho''$$

(3) for every $\rho \in \mathcal{D}_1(\gamma)$,

$$\exp(-c\lambda_2\, d(\gamma, \tilde\gamma)) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho \le \int_{\tilde\gamma} (\mathcal{L}_{t_1}\Phi)\tilde\rho \le \exp(c\lambda_2\, d(\gamma, \tilde\gamma)) \int_\gamma (\mathcal{L}_{t_1}\Phi)\rho,$$

as long as $b > 0$ and $c > 0$ are fixed large enough. This means, in other words, that

$$\mathcal{L}_{t_1}(\widehat{C}(b, c, \nu)) \subset \widehat{C}(\lambda_2 b, \lambda_2 c, \nu)$$

for every $t_1 \in T$. Moreover, integrating with respect to $d\theta_\varepsilon(t_1)$, cf. (4.34), we conclude that the same relations (1), (2), (3) hold with $\widehat{\mathcal{L}}_\varepsilon$ in the place of $\mathcal{L}_{t_1}$. Therefore,

$$\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b, c, \nu)) \subset \widehat{C}(\lambda_2 b, \lambda_2 c, \nu).$$

A straightforward translation of the first part of the proof of Proposition 4.6 shows that there is $T > 0$, depending only on λ_2, so that

$$\widehat{\Theta}\text{-diameter}\,(\mathcal{C}) \leq \widehat{\Theta}_+\text{-diameter}\,(\mathcal{C}) + T$$

for every subset $\mathcal{C}$ of $C(\lambda_2 b, \lambda_2 c, \nu)$. In particular, we may take $\mathcal{C} = \mathcal{L}_{t_1}(C(b,c,\nu))$, any $t_1 \in T$, or $\mathcal{C} = \widehat{\mathcal{L}}_\varepsilon(C(b,c,\nu))$.

Besides, the same arguments as in the second part of the proof of Proposition 4.6 show that there is $\Gamma_0 > 0$ such that

$$\int_{\gamma''} (\mathcal{L}_{t_1}\Phi)\rho'' \leq \Gamma_0 \int_{\gamma'} (\mathcal{L}_{t_1}\Phi)\rho'$$

for all $t_1 \in T$, $\Phi \in \widehat{C}(b,c,\nu)$, $\gamma', \gamma'' \in \widehat{\mathcal{F}}^s_{loc}$, and normalized densities $\rho' \in \mathcal{D}(\gamma')$, $\rho'' \in \mathcal{D}(\gamma'')$. Integrating with respect to $d\theta_\varepsilon(t_1)$ we get

$$\int_{\gamma''} (\widehat{\mathcal{L}}_\varepsilon\Phi)\rho'' \leq \Gamma_0 \int_{\gamma'} (\widehat{\mathcal{L}}_\varepsilon\Phi)\rho'$$

for every Φ, γ', γ'', ρ', ρ'' as above. Consequently, the $\widehat{\Theta}_+$-diameter of $\mathcal{L}_\varepsilon(C(b,c,\nu))$ is bounded by $2\log\Gamma_0$. We conclude that, denoting $D_2 = 2\log\Gamma_0 + T$,

$$\widehat{\Theta}\text{-diameter}\,(\,\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b,c,\nu))\,) \leq D_2 < \infty,$$

and so $\widehat{\mathcal{L}}_\varepsilon$ is a Λ_2-contraction for the projective metric $\widehat{\Theta}$, with $\Lambda_2 = 1 - e^{-D_2}$.

This implies that $(\widehat{\mathcal{L}}^n_\varepsilon 1)$ is a Cauchy sequence for $\widehat{\Theta}$. Then it is also Cauchy for $\widehat{\Theta}_+$, since $\widehat{\Theta}_+ \leq \widehat{\Theta}$. It follows that the sequence of probability measures $(\widehat{\mathcal{L}}^n_\varepsilon 1)(m \times \theta^{\mathbb{N}}_\varepsilon)$ is weak*-Cauchy, meaning that for every continuous function $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$,

$$\int \Psi\,(\widehat{\mathcal{L}}^n_\varepsilon 1)\,d(m \times \theta^{\mathbb{N}}_\varepsilon), \quad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$. This last claim is proved in the same way as Proposition 4.7, using the absolute continuity of the foliation $\widehat{\mathcal{F}}^s_{loc}$ stated in property (4) above, including the fact that H may be taken uniformly Hölder continuous on leaves of the foliation.

On the other hand, it enables us to define a probability measure $\hat{\mu}_\varepsilon$ on $Q \times T^{\mathbb{N}}$ by letting

$$\int \Psi\,d\hat{\mu}_\varepsilon = \lim \int \Psi\,(\widehat{\mathcal{L}}^n_\varepsilon 1)\,d(m \times \theta^{\mathbb{N}}_\varepsilon)$$

for each continuous function $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$. It is easy to deduce from (4.33) that this $\hat{\mu}_\varepsilon$ is F-invariant: for every continuous Ψ,

$$\int (\Psi \circ F)\,d\hat{\mu}_\varepsilon = \lim \int (\Psi \circ F)\,(\widehat{\mathcal{L}}^n_\varepsilon 1)\,d(m \times \theta^{\mathbb{N}}_\varepsilon) = \lim \int \Psi\,(\widehat{\mathcal{L}}^{n+1}_\varepsilon 1)\,d(m \times \theta^{\mathbb{N}}_\varepsilon) = \int \Psi\,d\hat{\mu}_\varepsilon.$$

Moreover, the probability measure μ_ε on Q defined by

$$\int \psi \, d\mu_\varepsilon = \int (\psi \circ \pi_0) \, d\hat{\mu}_\varepsilon \quad \text{for each continuous } \psi : Q \to \mathbb{R},$$

is a stationary measure, recall (4.31). Indeed, let $\psi : Q \to \mathbb{R}$ be any continuous function, and $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be given by $\Psi(x, \underline{t}) = (\psi \circ \pi_0)(x, \underline{t}) = \psi(x)$. By definition and Fubini's theorem,

$$\begin{aligned}
&\int \left(\int \psi(f_{t_1}(x)) \, d\theta_\varepsilon(t_1) \right) d\mu_\varepsilon(x) \\
&= \int \left(\int \Psi(f_{t_1}(x), \sigma(\underline{t})) \, d\theta_\varepsilon(t_1) \right) d\hat{\mu}_\varepsilon(x, \sigma(\underline{t})) \\
&= \lim \int \left(\int \Psi(f_{t_1}(x), \sigma(\underline{t})) \, d\theta_\varepsilon(t_1) \right) (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \sigma(\underline{t})) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\underline{t})) \\
&= \lim \int (\Psi \circ F)(x, \underline{t}) \, (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \sigma(\underline{t})) \, d\theta_\varepsilon(t_1) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \sigma(\underline{t})).
\end{aligned}$$

Since $\widehat{\mathcal{L}}_\varepsilon^n 1$ depends only on x, we may write

$$\begin{aligned}
\int \left(\int \psi(f_{t_1}(x)) \, d\theta_\varepsilon(t_1) \right) d\mu_\varepsilon(x) &= \lim \int (\Psi \circ F)(x, \underline{t}) (\widehat{\mathcal{L}}_\varepsilon^n 1)(x, \underline{t}) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \underline{t}) \\
&= \lim \int \Psi(x, \underline{t}) (\widehat{\mathcal{L}}_\varepsilon^{n+1} 1)(x, \underline{t}) \, d(m \times \theta_\varepsilon^{\mathbb{N}})(x, \underline{t}) \\
&= \int \Psi(x, \underline{t}) \, d\hat{\mu}_\varepsilon(x, \underline{t}) = \int \psi(x) \, d\mu_\varepsilon(x),
\end{aligned}$$

which is precisely what stationarity means.

Next, we want to prove that μ_ε describes the asymptotic Birkhoff averages of every continuous function $\varphi : Q \to \mathbb{R}$ over $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every random trajectory. This is stated in a precise form in Proposition 4.17 below. For the proof we need the following lemma.

Lemma 4.16. *Let $\Psi : Q \times T^{\mathbb{N}} \to \mathbb{R}$ be any bounded function such that $\Psi > 0$ and $\log(\Psi|\gamma)$ is $(a/2, \mu)$-Hölder continuous along every leaf γ of the stable foliation $\widehat{\mathcal{F}}^s_{loc}$.*

(1) *For every $\Phi \in \widehat{C}(b, c, \nu)$ there exists $C(\Phi) > 0$ such that*

$$\left| \int \Psi \, (\widehat{\mathcal{L}}_\varepsilon^n \Phi) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Psi \, d\hat{\mu}_\varepsilon \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| \leq C(\phi) \Lambda_2^n \sup \Psi,$$

for every $n \geq 1$.

(2) *If Ψ satisfies $\Psi = \Psi \circ F$ at $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every point, then Ψ is almost everywhere constant:*

$$\Psi(x, \underline{t}) = \int \Psi \, d\hat{\mu}_\varepsilon \quad \textit{for } (m \times \theta_\varepsilon^{\mathbb{N}})\textit{-almost every } (x, \underline{t}) \in Q \times T^{\mathbb{N}}.$$

Proof: The proof of part (1) is based on the arguments leading to (4.20) and to the first part of Proposition 4.9. For $1 \le n < k$, denote $\Phi_n = \widehat{\mathcal{L}}_\varepsilon^n \Phi$ and $\Phi_k = \widehat{\mathcal{L}}_\varepsilon^k \Phi$. We write

$$\int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma \Phi_n \, (\Psi \, H_\varepsilon \mid \gamma) \right) d\tilde{m}_\varepsilon(\gamma),$$

and analogously for $\int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$. Moreover,

$$\int \Phi_n \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int_\gamma \Phi_n \, (H_\varepsilon \mid \gamma) \right) d\tilde{m}_\varepsilon(\gamma),$$

and analogously for $\int \Phi_k \, d(m \times \theta_\varepsilon^{\mathbb{N}})$. Recall that $\log H_\varepsilon$ is (a_0, ν_0)-Hölder continuous along leaves of the local stable foliation of F. In particular, $\log (H_\varepsilon | \gamma)$ is (a, μ)-Hölder along every stable leaf γ, recal (4.7), (4.11). Moreover, our assumptions on Ψ imply that $\log (\Psi H_\varepsilon | \gamma)$ is also (a, μ)-Hölder continuous along every leaf γ. Then, from the expression of $\widehat{\Theta}_+ = \log(\hat{\beta}_+ / \hat{\alpha}_+)$,

$$\frac{\int_\gamma \Phi_k \, (H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n \, (H_\varepsilon \mid \gamma)} \ge \hat{\alpha}_+(\Phi_k, \Phi_n) \quad \text{and} \quad \frac{\int_\gamma \Phi_k \, (\Psi H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n \, (\Psi H_\varepsilon \mid \gamma)} \le \hat{\beta}_+(\Phi_k, \Phi_n) \quad \text{for all } \gamma.$$

Moreover, using (4.33),

$$\int \Phi_n \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi(\widehat{U}^n 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \tag{4.41}$$

and, analogously, $\int \Phi_k \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$. As consequence, there must be some local stable leaf $\hat{\gamma}$ so that $\int_{\hat{\gamma}} \Phi_k \, (H_\varepsilon \mid \hat{\gamma}) \le \int_{\hat{\gamma}} \Phi_n \, (H_\varepsilon \mid \hat{\gamma})$, and then

$$\frac{\int_\gamma \Phi_k \, (\Psi H_\varepsilon \mid \gamma)}{\int_\gamma \Phi_n \, (\Psi H_\varepsilon \mid \gamma)} \le \frac{\beta_+(\Phi_k, \Phi_n)}{\alpha_+(\Phi_k, \Phi_n)} \frac{\int_{\hat{\gamma}} \Phi_k \, (H_\varepsilon \mid \hat{\gamma})}{\int_{\hat{\gamma}} \Phi_n \, (H_\varepsilon \mid \hat{\gamma})} \le \exp(\widehat{\Theta}_+(\Phi_k, \Phi_n))$$

for every γ. This implies

$$\frac{\int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})}{\int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})} \le \exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)),$$

and so

$$\begin{aligned} \left| \int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| &= \\ &= \left| \int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \right| \left| \frac{\int \Phi_k \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})}{\int \Phi_n \Psi \, d(m \times \theta_\varepsilon^{\mathbb{N}})} - 1 \right| \\ &\le \sup \Psi \int |\Phi_n| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \big(\exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)) - 1 \big). \qquad (4.42) \end{aligned}$$

It is easy to see from the definition of $\widehat{\mathcal{L}}_\varepsilon$ that $|\widehat{\mathcal{L}}_\varepsilon \Upsilon| \le \widehat{\mathcal{L}}_\varepsilon |\Upsilon|$ for every function Υ on $Q \times T^{\mathbb{N}}$. Then, using also the analog of (4.41) for $|\Phi|$,

$$\int |\Phi_n| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) \le \int \widehat{\mathcal{L}}_\varepsilon^n |\Phi| \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int |\Phi| \, d(m \times \theta_\varepsilon^{\mathbb{N}})$$

The fact that $\widehat{\mathcal{L}}_\varepsilon$ is a Λ_2-contraction for the projective metric $\widehat{\Theta}$ implies that

$$\widehat{\Theta}_+(\Phi_k, \Phi_n) \le \widehat{\Theta}(\Phi_k, \Phi_n) \le \Lambda_2^{n-1} \widehat{\Theta}(\Phi_{k-n+1}, \Phi_1) \le D_2 \Lambda_2^{n-1},$$

where $D_2 > 0$ is an upper bound for the $\widehat{\Theta}$-diameter of $\widehat{\mathcal{L}}_\varepsilon(\widehat{C}(b, c, \nu))$. It follows that

$$\exp(\widehat{\Theta}_+(\Phi_k, \Phi_n)) - 1 \le C_2 \Lambda_2^n$$

for some $C_2 > 0$ depending only on D_2 and Λ_2. Replacing in (4.42) and passing to the limit as $k \to \infty$ we obtain the conclusion of part (1) of the lemma, with $C(\Phi) = C_2 \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$.

In particular, if Ψ is such that $\Psi \circ = \Psi$ at $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost every point,

$$\int \Psi \, \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int (\Psi \circ F^n) \, \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \Psi \, (\widehat{\mathcal{L}}_\varepsilon^n \Phi) \, d(m \times \theta_\varepsilon^{\mathbb{N}})$$

for every $n \ge 1$. the last term converges to $\int \Psi \, d\hat{\mu}_\varepsilon \int \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}})$ as n goes to infinity, cf. Remark 4.1. Therefore,

$$\int \left(\Psi - \int \Psi \, d\hat{\mu}_\varepsilon \right) \Phi \, d(m \times \theta_\varepsilon^{\mathbb{N}}) = 0, \tag{4.43}$$

for every $\Phi \in \widehat{C}(b, c, \nu)$. We are left to explain why this implies

$$\Psi - \int \Psi \, d\hat{\mu}_\varepsilon = 0 \quad (m \times \theta_\varepsilon^{\mathbb{N}})\text{-almost everywhere.} \tag{4.44}$$

First, note that every bounded function on $Q \times T^{\mathbb{N}}$ can be approximated in $L^1(m \times \theta_\varepsilon^{\mathbb{N}})$ by a ν-Hölder function. Next, every ν-Hölder function may be written as the difference of two strictly positive functions whose logarithm is (c, ν)-Hölder continuous. Next, every function $\Upsilon > 0$ such that $\log \Upsilon$ is (c, ν)-Hölder belongs in the cone $\widehat{C}(b, c, \nu)$. Indeed, conditions (AA) and (BB) are automatic, since Υ is positive. To check (CC), observe that given $\gamma \in \mathcal{F}_{loc}^s(\underline{t})$ and $\tilde{\gamma} \in \mathcal{F}_{loc}^s(\underline{s})$, and given $\rho \in \mathcal{D}(\gamma)$,

$$\begin{aligned} \left| \log \int_\gamma \Upsilon \rho - \log \int_{\tilde{\gamma}} \Upsilon \tilde{\rho} \right| &= \left| \log \int_\gamma \Upsilon \rho - \log \int_{\tilde{\gamma}} \Upsilon (\rho \circ \pi) |\det D\pi| \right| \\ &= \left| \log \int_\gamma \Upsilon \rho - \log \int_\gamma (\Upsilon \circ \pi^{-1}) \rho \right| \\ &\le \sup\{ |\log \Upsilon(z, \underline{t}) - \log \Upsilon(\pi^{-1}(z), \underline{s})| : z \in \gamma \}. \end{aligned}$$

Hölder continuity of $\log \Upsilon$ implies that the last term is bounded by

$$c\left(\max\left\{d(z,\pi^{-1}(z)),\|\underline{t}-\underline{s}\|\right\}\right)^{\nu}\leq c\left(\max\left\{d_1(\gamma,\tilde{\gamma}),d_2(\gamma,\tilde{\gamma})\right\}\right)^{\nu}\leq cd(\gamma,\tilde{\gamma})^{\nu},$$

as we wanted to prove. In view of these remarks, (4.43) implies that

$$\int\left(\Psi-\int\Psi\,d\hat{\mu}_{\varepsilon}\right)\Phi\,d(m\times\theta_{\varepsilon}^{\mathbb{N}})=0$$

for every bounded function Φ. Taking $\Phi=\Psi-\int\Psi\,d\hat{\mu}_{\varepsilon}$ we obtain (4.44), thus completing the proof. □

Proposition 4.17. *Given any continuous function $\varphi: Q\to\mathbb{R}$, we have*

$$\lim\frac{1}{n}\sum_{j=0}^{n-1}\varphi(x_j)=\int\varphi\,d\mu_{\varepsilon}\tag{4.45}$$

for $(m\times\theta_{\varepsilon}^{\mathbb{N}})$-almost every $(x,\underline{t})\in Q\times T^{\mathbb{N}}$, where $x_j=f_{t_j}\circ\cdots\circ f_{t_1}(x)$.

Proof: Let $\Phi: Q\times T^{\mathbb{N}}\to\mathbb{R}$ be defined by $\Phi(x,\underline{t})=\varphi(x)$. Observe that $\varphi(x_j)$ is precisely the same as $\Phi(F^j(x,\underline{t}))$. Then, by the ergodic theorem, the set $B=B(\varphi)$ of points in $Q\times T^{\mathbb{N}}$ such that the limit in (4.45) exists has $\hat{\mu}_{\varepsilon}(B)=1$. It is easy to see that $(x,\underline{t})$ belongs in B if and only if $F(x,\underline{t})$ does. That is, the characteristic function χ_B satisfies $\chi_B=\chi_B\circ F$. Moreover, B is a union of entire local stable leaves of F, which means precisely that χ_B is constant on every stable leaf. Then the (positive) function χ_B+1 satisfies the assumptions of Lemma 4.16(2), and so it is constant

$$\chi_B+1=\int(\chi_B+1)\,d\hat{\mu}_{\varepsilon}=\hat{\mu}_{\varepsilon}(B)+1$$

$(m\times\theta_{\varepsilon}^{\mathbb{N}})$-almost everywhere. Then either $\chi_B=0$ almost everywhere or $\chi_B=1$ almost everywhere, with respect to the measure $(m\times\theta_{\varepsilon}^{\mathbb{N}})$. The first alternative would lead to

$$\hat{\mu}_{\varepsilon}(B)=\int\chi_B\,d\hat{\mu}_{\varepsilon}=\lim\int\chi_B(\hat{\mathcal{L}}_{\varepsilon}^{n}1)\,d(m\times\theta_{\varepsilon}^{\mathbb{N}})=0,$$

(the last integral is zero for every n), contradicting the ergodic theorem. Therefore, we must have $(m\times\theta_{\varepsilon}^{\mathbb{N}})(B)=1$. Now let $\beta: Q\times T^{\mathbb{N}}\to\mathbb{R}$ be the Birkhoff average of Φ. More precisely,

$$\beta(x,\underline{t})=\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\Phi(F^j(x,\underline{t}))=\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(x_j),\qquad x_j=f_{t_j}\circ\cdots\circ f_{t_1},$$

if $(x,\underline{t})\in B$, and $\beta(x,\underline{t})=0$ otherwise. Once again, it is easy to see that β is constant on stable leaves and satisfies $\beta\circ F=\beta$. Moreover, $\beta\geq\inf\varphi$ at every point in B. Then $\beta+|\inf\varphi|+1$ is a strictly positive satisfying the assumptions of Lemma 4.16(2), and so it is constant $(m\times\theta_{\varepsilon}^{\mathbb{N}})$-almost everywhere. Then β is also constant $(m\times\theta_{\varepsilon}^{\mathbb{N}})$-almost everywhere

$$\beta(x,\underline{t})=\int\beta\,d\hat{\mu}_{\varepsilon}=\int\Phi\,d\hat{\mu}_{\varepsilon}=\int\varphi\,d\mu_{\varepsilon}$$

for $(m\times\theta_{\varepsilon}^{\mathbb{N}})$-almost every point $(x,\underline{t})\in Q\times T^{\mathbb{N}}$ (the second equality is part of the ergodic theorem). □

Proposition 4.18 (stochastic stability). *As $\varepsilon \to 0$, the stationary measure μ_ε converges to the SRB measure μ_0 of f in the weak* topology:*

$$\int \psi \, d\mu_\varepsilon \to \int \psi \, d\mu_0, \quad \textit{for any continuous function } \psi : Q \to \mathbb{R}.$$

Proof: As every continuous function is uniformly approximated by μ-Hölder continuous functions, it suffices to consider the case when ψ is μ-Hölder continuous. Moreover, every μ-Hölder function ψ may be written $\psi = \psi^+ - \psi^-$ where $\psi^\pm > 0$ and $\log \psi^\pm$ are $(a/2, \mu)$-Hölder continuous. Therefore, we may assume right from the start that

$$\psi > 0 \quad \text{and} \quad \log \psi \text{ is } \left(\frac{a}{2}, \mu\right) - \text{Hölder}.$$

Then, as in (4.20), there is some uniform constant $C > 0$, so that

$$\begin{aligned}\left|\int_Q \psi(\mathcal{L}^n 1) dm - \int_Q \psi(\mathcal{L}^{n+j} 1) dm\right| &\le \sup|\psi| \left(\exp(\Theta_+(\mathcal{L}^n 1, \mathcal{L}^{n+j} 1)) - 1\right) \\ &\le \sup|\psi| \, C\Lambda_2^n\end{aligned}$$

for every $j \ge 1$ ($\mathcal{L}$ is the transfer operator of the unperturbed map f). Passing to the limit as $j \to \infty$,

$$\left|\int_Q \psi(\mathcal{L}^n 1) dm - \int \psi \, d\mu_0\right| \le C\Lambda_2^n \sup \psi.$$

Similarly, taking $\Phi = 1$ and $\Psi = \psi \circ \pi_0$ in Lemma 4.16(1),

$$\left|\int_Q \psi(\widehat{\mathcal{L}}_\varepsilon^n 1) \, dm - \int \psi \, d\mu_\varepsilon\right| = \left|\int_Q \Psi(\widehat{\mathcal{L}}_\varepsilon^n 1) \, d(m \times \theta_\varepsilon^{\mathbb{N}}) - \int \Psi \, d\hat{\mu}_\varepsilon\right| \le C \Lambda_2^n \sup \psi.$$

On the other hand,

$$\begin{aligned}\left|\int_Q \psi \, (\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1) \, dm\right| &= \left|\sum_{i=0}^{n-1} \int_Q \psi \, \widehat{\mathcal{L}}_\varepsilon^{n-i-1} (\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1) \, dm\right| \\ &\le \sum_{i=0}^{n-1} \left|\int_Q (\widehat{U}^{n-i-1} \psi)(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1) \, dm\right| \\ &\le \sum_{i=0}^{n-1} \sup \psi \int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)| \, dm.\end{aligned}$$

The first inequality uses (4.33), for functions that depend only on x. In the second one note that $\sup |\widehat{U}^{n-i} \psi| \le \sup \psi$. We claim that, for each fixed i,

$$\int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})(\mathcal{L}^i 1)| \, dm \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.46}$$

Let us assume this statement for a while, and explain why the proposition is now an easy consequence. Indeed, we find that

$$\left|\int_Q \psi(\widehat{\mathcal{L}}_\varepsilon^n 1 - \mathcal{L}^n 1)\, dm\right| \le \sum_{i=0}^{n-1} \sup \psi \cdot \xi_i(\varepsilon) \le \sup \psi \cdot \xi_n(\varepsilon)$$

where $\xi_j(\varepsilon)$ is a generic notation for a function of (j, ε) such that $\xi_j(\varepsilon) \to 0$ when $\varepsilon \to 0$, keeping j fixed, Together with the previous bounds this gives

$$\left|\int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0\right| \le (2C\Lambda_2^n + \xi_n(\varepsilon)) \sup \psi \text{ for every } n \ge 1.$$

Given any $\delta > 0$, fix n large enough so that $2C\Lambda_2^n \le \delta/2$. Then

$$\left|\int \psi\, d\mu_\varepsilon - \int \psi\, d\mu_0\right| \le \delta,$$

if $\varepsilon > 0$ is small enough so that $\xi_n(\varepsilon) \le \delta/2$. This completes the proof of the proposition, up to justifying claim (4.46).

Let $i \ge 0$ be fixed and denote $\varphi = \mathcal{L}^i 1$. We start by noting that

$$\begin{aligned}\int_Q |(\widehat{\mathcal{L}}_\varepsilon - \mathcal{L})\varphi|\, dm &= \int \left|\int_Q (\mathcal{L}_t - \mathcal{L})\varphi\, d\theta_\varepsilon(t)\right| dm \\ &\le \int \left(\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm\right) d\theta_\varepsilon(t).\end{aligned}$$

as a consequence of Fubini's theorem. Thus, in order to prove the claim, it suffices to show that

$$\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm \to 0 \quad \text{as } t \to \tau.$$

To do that, we disintegrate the integral with respect to the partition of Q into local stable manifolds $\gamma \in \mathcal{F}_{loc}^s$ of f:

$$\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm = \int \left(\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi|\, H_\gamma\right) d\tilde{m}(\gamma).$$

Moreover, for each $\gamma \in \mathcal{F}_{loc}^s$, we write

$$\begin{aligned}\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi|\, H_\gamma = &\int_{\gamma\cap f_t(Q)\cap f(Q)} |\tilde{\phi}_t - \tilde{\phi}|\, H_\gamma + \\ &+ \int_{\gamma\cap f_t(Q)\setminus f(Q)} |\tilde{\phi}_t|\, H_\gamma + \int_{\gamma\cap f(Q)\setminus f_t(Q)} |\tilde{\phi}|\, H_\gamma\end{aligned}$$

where $\tilde{\phi} = (\phi \circ f^{-1})|/ \det Df \circ f^{-1}|$ and $\tilde{\phi}_t = (\phi \circ f_t^{-1})/|\det Df_t \circ f_t^{-1}|$. Observe that $\tilde{\phi}_t$ converges uniformly to ϕ as $t \to \tau$, because we suppose that $f_t \to f$ in the C^2 topology (C^1 would be sufficient here). This ensures that the first term on the right hand side goes to zero as $t \to \tau$. Moreover, the riemannian volume of $\gamma \cap f_t(Q) \setminus f(Q)$ and of $\gamma \cap f(Q) \setminus f_t(Q)$ in γ converges uniformly to zero as $t \to \tau$. Thus, the other two terms also converge to zero. It follows that

$$\int_\gamma |(\mathcal{L}_t - \mathcal{L})\varphi| H_\gamma \to 0 \quad \text{as } t \to \tau,$$

uniformly in $\gamma \in \mathcal{F}^s_{loc}$, and so $\int_Q |(\mathcal{L}_t - \mathcal{L})\varphi|\, dm$ converges to zero as $t \to \tau$. As we already pointed out that this implies our claim (4.33). □

We conclude this section by deducing that the SRB measure μ_0 varies continuously with the difeeomorphim f. To explain this, let g be any map C^2 close to f. Then $g(Q) \subset Q$ and the maximal invariant set

$$\Lambda_g = \bigcap_{n \geq 0} g^n(Q)$$

is a hyperbolic attractor for g, see Appendix A. Everything we did here applies to g, if it is close enough to f, and so g has a unique SRB measure $\mu_{0,g}$ supported on Λ_g. Moreover,

Corollary 4.19 (deterministic stability). *The measure $\mu_{0,g}$ is close to $\mu_0 = \mu_{0,f}$ in the weak* topology, if g is C^2 close to f: given any continuous function $\psi : Q \to \mathbb{R}$,*

$$\int \psi \, d\mu_{0,g} \to \int \psi \, d\mu_0 \quad \text{as } g \to f.$$

Proof: This uses precisely the same argument as Corollary 2.15. Let $(g_n)_n$ be any sequence converging to f in $C^2(M)$. Define θ_ε to be the Dirac measure supported on g_n, for all $\varepsilon \in (1/n+1, 1/n]$. Then the stationary measure μ_ε, $\varepsilon \in (1/n+1, 1/n]$, coincides with the SRB measure μ_{0,g_n} of g_n and, as a particular case of Proposition 4.18, these stationary measures converge weakly to μ_0 as $n \to +\infty$. □

Notes.

Uniformly hyperbolic systems were introduced by Smale, see [Sm67], and have been studied by a large number of people both from the topological and the ergodic viewpoints. See e.g. [Sh87] and references therein.

SRB measures were first constructed by Sinai [Si72] for Anosov diffeomorphisms. This was extended by Ruelle [Ru76] for general hyperbolic (Axiom A) diffeomorphisms, and by Bowen-Ruelle [BR75] for Axiom A flows. See also [Bo75].

A key idea in our approach, "to integrate out local stable manifolds", is due to [Li95], and our presentation ows a good deal to his paper. He proved exponential decay of correlations for volume-preserving uniformly hyperbolic maps, taking some advantage of the a priori knowledge of the SRB measure (the riemannian volume). A main difference in the general hyperbolic attractors is that we have to start by finding the SRB measure (Proposition 4.7 and subsequent considerations).

We also push the arguments further to prove stochastic stability (a preprint version of [Li95] also contained a discussion of stochastic stability). Stability of uniformly hyperbolic attractors was proved by [Ki74], [Ki86a], for a different model of random perturbations. [Yo85] proved stochastic stability in the same sense as we consider here, under an assumption of absolute continuity on the probability distributions θ_ε.

The proof of the central limit theorem for invertible maps (Theorem 4.13) is inspired by [DG86].

5. Nonuniformly hyperbolic unimodal maps

In this chapter we study the statistical properties of a large class of nonuniformly hyperbolic maps of the interval. For simplicity, we state the results for the quadratic family

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = a - x^2, \qquad a \in \mathbb{R},$$

but all the arguments can be extended to general smooth unimodal maps with negative schwarzian derivative and nondegenerate critical point. The dynamics of these quadratic maps depends in a crucial way on the value of the parameter a. We begin by listing some main facts, referring the reader to [MS93] for definitions and more information.

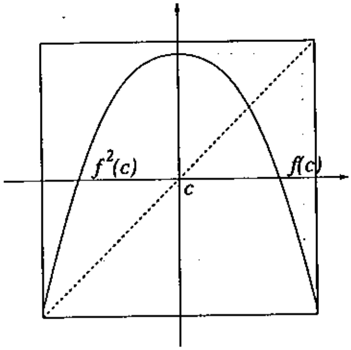

Figure 5.1: A quadratic map $f(x) = a - x^2$

It is easy to see that if $a < -1/4$ then every trajectory $f^n(x)$ goes to $-\infty$ as $n \to +\infty$. The same happens, for all typical trajectories, when $a > 2$: the exceptions form a Cantor set K with zero Lebesgue measure. Moreover, K is a uniformly hyperbolic set for f: there are $\sigma > 1$ and $N \geq 1$ such that $|(f^N)'(x)| \geq \sigma$ for every $x \in K$. The asymptotic dynamics of f is much richer if a is in between $-1/4$ and 2, which we always suppose from now on. Denoting $q = (-1 - \sqrt{1+4a})/2$, the fixed point with largest absolute value, the interval $I = [q, -q]$ is invariant under f, that is, $f(I) \subset I$. All the trajectories starting outside I go to $-\infty$ as $n \to +\infty$, but the behaviour of $f|I$ may take very different forms.

For an open and dense subset H of values of $a \in (-1/4, 2)$ the map admits a unique attracting periodic orbit $\mathcal{O}$, which attracts every typical trajectory in the invariant interval: the basin $B(\mathcal{O}) = \{x \in I : f^n(x) \to \mathcal{O}\}$ is an open, full Lebesgue measure subset of I. Moreover, $B(\mathcal{O})$ contains the critical point $c = 0$, and $K = I \setminus B(\mathcal{O})$ is a uniformly hyperbolic set for f. From a statistical point of view the situation is still very simple: the Dirac probability measure uniformly distributed along the periodic attractor is the SRB-measure of $f|I$.

Another interesting case corresponds to the orbit of the critical point being non-recurrent, that is,

$$\inf_{n \geq 1} |f^n(c) - c| > 0. \tag{5.1}$$

Then the map is expanding over its critical orbit, in the sense that there are $b > 0$ and $\lambda > 1$ such that

$$|(f^n)'(f(c))| \geq b\lambda^n \quad \text{for every } n \geq 1. \tag{5.2}$$

Moreover, a similar property holds for Lebesgue almost every point $x \in I$, if one allows b to depend on x. In this case, Lebesgue almost every orbit starting in I has a limit distribution, which is described by an absolutely continuous f-invariant probability measure μ_0. This measure μ_0 is unique and ergodic, and it is supported on a finite union of subintervals of I. Then, in particular, the trajectory of every typical point $x \in I$ is dense in those subintervals. Observe, however, that condition (5.1) holds only for an (uncountable) zero Lebesgue measure set of values of a.

Here we study another, much larger, set of parameter values $a \in (-1/4, 2) \setminus H$ for which the map exhibits complex asymptotic behaviour. As we shall see below, it suffices to assume (5.2) together with a much weaker form of (5.1), to ensure the existence of an SRB-measure μ_0 which is absolutely continuous with respect to Lebesgue measure and which has very rich statistical properties. A crucial point is that these weaker conditions are satisfied by a set of parameter values with positive Lebesgue measure. In the sequel we give a precise definition of the systems we shall be dealing with, as well as the statement of the main result.

Before that, let us mention that the two types of behaviour we have been discussing are typical for quadratic maps: a recent result of [Ly] asserts that for Lebesgue almost every value of the parameter $a \in (-1/4, 2)$ either f has an attracting periodic orbit or it admits an absolutely continuous invariant measure.

We make the following assumptions on the orbit of the critical point $c = 0$: there are constants $\lambda_c > 1 \gg \alpha > 0$ (the precise condition is at the beginning of Section 5.1) such that

(U1) $|(f^n)'(f(c))| \geq \lambda_c^n$ for every $n \geq 1$;

(U2) $|f^n(c) - c| \geq e^{-\alpha n}$ for every $n \geq 1$;

(U3) f is topologically mixing on the interval $I_* = [f^2(c), f(c)]$.

A few words of motivation are in order on these hypotheses. The Collet-Eckmann condition (U1) means that f is expanding on its critical orbit, and is our main hyperbolicity assumption on the map. Property (U2) should be compared to (5.2): we allow the critical orbit to be recurrent, but we impose a bound on the speed of the recurrence. Altogether, these two conditions ensure a certain amount of expanding behaviour for the map f, as we shall see. Both of them can be further weakened, for instance multiplying the right hand side by a positive constant, but we keep this formulation for the sake of simplicity. The topological mixing condition (U3) plays essentially the same role as in previous cases, and we just add that in the present setting it is equivalent to asking that f be non-renormalizable. Recall that f is called *renormalizable* if there exists a subinterval $J \subset I$ and an integer $k \geq 2$ such that

$$c \in \text{interior}\,(J), \quad c \notin f^i(J) \text{ for } 0 < i < k, \quad f^k(J) \subset J.$$

As already mentioned, (U1), (U2), (U3) hold, simultaneously, for a large (positive Lebesgue measure) set S of values of the parameter a. Observe that S is disjoint from H, since no $a \in H$ satisfies (U1). Indeed, if f has an attracting periodic orbit $\mathcal{O}$ then $c \in B(\mathcal{O})$ implies that $(f^n)'(f(c))$ converges to zero (exponentially fast).

Our goal is to prove

Theorem 5.1. *Under assumptions (U1), (U2), (U3),*

(1) *f admits a unique absolutely continuous invariant probability measure μ_0; moreover, μ_0 is ergodic and so it is an SRB-measure for f;*
(2) *(f, μ_0) has exponential decay of correlations and satisfies the central limit theorem in the space of functions with bounded variation;*

Moreover, (f, μ_0) is stochastically stable, in a strong sense, under certain random perturbations. The proof of this result uses a "perturbed" version of the arguments we shall develop in Sections 5.1 through 5.4 to prove Theorem 5.1, and we do not present it here. We just give the precise content of this stability statement, and refer the reader to [BaV96] for the proof and further information.

The class of random perturbations one considers in this setting is necessarily more restricted than in the situations we treated before. For instance, statements of deterministic stability analog to Corollary 2.15 or Corollary 4.19, are known to be false for general quadratic maps [HK90]. We consider perturbations within the quadratic family,

$$\mathbb{R} \ni t \mapsto f_t(x) = f(x) + t = (a + t) - x^2,$$

and we also impose certain conditions on the probability distributions $(\theta_\varepsilon)_{\theta>0}$. A main one is that θ_ε be absolutely continuous with respect to Lebesgue measure m, and supported on some subsinterval J_ε of $[-\varepsilon, \varepsilon]$. The two other conditions, of a somewhat more technical kind, are

$$\sup_{\varepsilon>0} \left(\varepsilon \sup \frac{d\theta_\varepsilon}{dm}\right) < \infty \qquad \text{and} \qquad \log \frac{d\theta_\varepsilon}{dm} \text{ concave on } J_\varepsilon.$$

Then we conclude that the random scheme admits a unique stationary measure μ_ε, that describes the asymptotic time averages of almost every random trajectory. Moreover, as the noise level ε goes to 0 the density $d\mu_\varepsilon/dm$ of the stationary measure converges in $L^1(m)$ to the density $d\mu_\varepsilon/dm$ of the SRB measure μ_0.

It is worth pointing out that the dynamics of these systems is very fragile under deterministic perturbations: the fact that H is dense in the parameter interval $(-1/4, 2)$ implies that the maps we are dealing with may be approximated by other quadratic maps having a periodic attractor and, thus, simple statistical features. This makes the stochastic stability statement all the more striking in the present setting.

In the sequel we sketch our approach to proving the ergodic properties in the statement of Theorem 5.1. It is, once more, based on studying the spectrum of convenient transfer operators. However, this time the "natural" operator

$$\mathcal{L}\varphi(y) = \sum_{f(x)=y} \frac{\varphi(x)}{|f'(x)|}$$

is not a right object to look at. To begin with, this is not well defined at $y = f(c)$. Moreover, the expansion properties of the dynamical system played a crucial role in all the situations we studied so far. This may lead one to suspect that for the present class of maps, which combine some amount of expansion with strong contraction (near c), the operator defined above may have poor spectral properties, and this is indeed so.

Instead, the basic strategy is to try and reduce this setting of nonuniformly hyperbolic dynamics to that of piecewise uniformly expanding maps treated in Chapter 3. More precisely, in Section 5.1 we describe a procedure associating to each quadratic map $f : I \to I$ satisfying (U1) and (U2), an expanding map $\hat{f} : \hat{I} \to \hat{I}$ defined on a countable union $\hat{I} = \cup_{k\geq 0}(B_k \times \{k\})$ of disjoint intervals. To make $\hat{f}$ expanding we have to consider an adapted riemannian metric on $\hat{I}$, of the form

$$\|\cdot\|_{(x,k)} = w_0(x,k)|\cdot|,$$

where $|\cdot|$ is the usual length (along each interval B_k) and w_0 is a convenient nonnegative function. Correspondingly, instead of the usual transfer operator, we consider its conjugate under multiplication by the *cocycle* w_0,

$$\mathcal{L}_0\varphi(\zeta) = \sum_{\hat{f}(\xi)=\zeta} \frac{1}{|\hat{f}'(\xi)|}\frac{w_0(\xi)}{w_0(\zeta)}\varphi(\xi), \tag{5.3}$$

where $\hat{f}'(\xi) = f'(x)$ for each $\xi = (x,k) \in \hat{I}$. This *tower extension* $\hat{f}$ is constructed in such a way that $\pi \circ \hat{f} = f \circ \pi$, where $\pi : \hat{I} \to I$ is the canonical projection given by $\pi(x,k) = x$. In other words, given any $(x,k) \in \hat{I}$ there is $l \geq 0$ so that $\hat{f}(x,k) = (f(x), l)$. Then

$$|\hat{f}'(\xi)|\frac{w_0(\hat{f}(\xi))}{w_0(\xi)}$$

is just the jacobian of $\hat{f}$ at ξ, with respect to the new metric. We introduce the measure $m_0 = w_0 m$ and then, by change of variables,

$$\int (\mathcal{L}_0\varphi)\psi\, dm_0 = \int \varphi(\psi \circ \hat{f})\, dm_0, \tag{5.4}$$

whenever the integrals make sense. We take $\mathcal{L}_0$ to act on the space $\mathrm{BV}(\hat{I})$ of functions with bounded variation on $\hat{I}$, whose precise definition will be shortly given. We prove that $\mathcal{L}_0$ is a quasi-compact operator, from which we deduce ergodic properties of $\hat{f}$. Their analogs for f follow, easily, using $\pi \circ \hat{f} = f \circ \pi$.

5.1. Towers, cocycles, transfer operators.

Here we give the precise definitions of the objects introduced in the paragraphs above. Throughout, we suppose that the constant α in (U2) has been taken small enough so that $e^{2\alpha} < \sqrt{\lambda_c}$. Then, starting the construction of the tower extension $\hat{f} : \hat{I} \to \hat{I}$, we fix $\beta \in (3\alpha/2, 2\alpha)$. We also fix $\rho > \lambda > 1$ such that $\rho > e^\alpha$ and $\lambda\rho e^\alpha < \sqrt{\lambda_c}$. We use two more constants $1 < \sigma \le \sigma_0$ and $0 < \delta \le \delta_0$, where $\sigma_0 \in (1, \lambda)$ and $0 < \delta_0 \ll \alpha$ are given by Lemma 5.2 below. We denote by C various large positive constants depending only on α and λ_c, and use $C(\cdots)$ for large positive constants depending also on other parameters involved in our constructions. For simplicity, we write $c_j = f^j(c)$ for each $j \ge 0$.

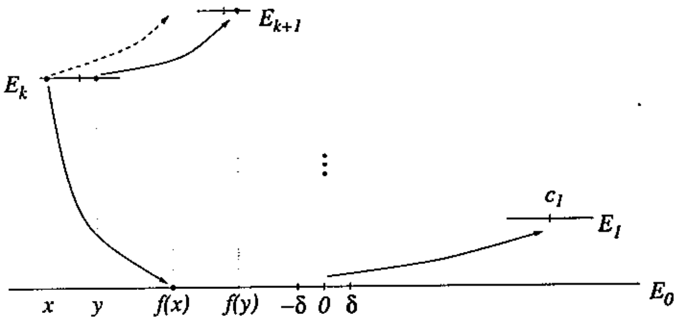

Figure 5.2: The tower extension $\hat{f} : \hat{I} :\to \hat{I}$

We define $B_k = [c_k - e^{-\beta k}, c_k + e^{\beta k}]$, for each $k \ge 1$, and $B_0 = I = [q, -q]$. Then we let $\hat{I} = \cup_{k\ge 0} E_k$, where $E_k = B_k \times \{k\}$. Observe that the critical point c is not contained in B_k, for any $k \ge 1$, since (U2) implies $|c_k| \ge e^{-\alpha k} > e^{-\beta k}$. This simple fact will be useful in a number of occasions. As we already said, we want $\hat{f}$ to be such that $\hat{f}(x, k) = (f(x), l)$ for some $l \ge 0$ depending on (x, k). The definition of $\hat{f}$ is given by the following pair of rules:

(1) whenever possible, $\hat{f}$ maps (x, k) one level higher in the tower, i.e., $l = k + 1$;
(2) if this is not allowed, that is, if $f(x) \notin B_{k+1}$, then $\hat{f}$ sends (x, k) directly to the ground level: $l = 0$.

Rule (1) admits an exception when $k = 0$: a point $(x, 0)$ goes up to level 1 only if x is close to zero, otherwise it remains in level 0. The precise expression is

$$\hat{f}(x, k) = \begin{cases} (f(x), 0) & \text{if either } f(x) \notin B_{k+1} \text{ or else } k = 0 \text{ and } |x| \ge \delta; \\ (f(x), k+1) & \text{otherwise.} \end{cases}$$

Typically, a point $(x, 0)$ moves around in the zeroth level E_0 for a while, until it hits $(-\delta, \delta) \times \{0\}$ at some time $m \ge 0$. Then it starts climbing the tower

$$\hat{f}^{m+j}(x, 0) = (f^{m+j}(x), j) \quad \text{for } 0 \le j \le h.$$

Unless $f^m(x)$ coincides with the critical point c, the integer h is finite and at the next iterate the orbit falls back to the ground level: $\hat{f}^{m+h+1}(x, 0) = (f^{m+h+1}(x), 0)$. Observe also that we must have $h \ge H(\delta)$, for some integer $H(\delta) \ge 1$ which can be made arbitrarily large by choosing δ small enough.

Next, we define our cocycle w_0. First, we set $w_0(x,0) = 1$ for every $x \in B_0$. Given any point $(x,k) \in E_k$, $k \geq 1$, there are two possibilities. If there is $z \in (0,\delta)$ such that $\hat{f}^k(z,0) = (x,k)$ then we define

$$w_0(x,k) = \lambda^k |(f^k)'(z)|^{-1}$$

(it is easy to see that z is unique, when it exists). If there is no such z then we set simply $w_0(x,k) = 0$. For each $k \geq 0$ we shall denote $W_k = \{x \in B_k : w_0(x,k) > 0\}$. Note that every W_k is an interval, whose closure contains c_k. We also write

$$W_* = \bigcup_{k \geq 0} (W_k \times \{k\}).$$

As mentioned before, we associate to w_0 the riemannian metric $\|\cdot\|_{(x,k)} = w_0(x,k)|\cdot|$ and the Borel measure $m_0 = w_0 m$.

Remark 5.1. By definition w_0 and m_0 are supported on W_*, reflecting the fact that points in $\hat{I} \setminus W_*$ are transient for $\hat{f}$, and so play no role as far as asymptotic behaviour is concerned. Let us note that certain points in the ground level E_0 are also transient: $\hat{f}(W_*)$ does not intersect $(f(\delta), f(c)] \times \{0\}$, and $\hat{f}^2(W_*) \subset \hat{f}(W_*)$ does not intersect $[f^2(c), f^2(\delta)) \times \{0\}$. To see this, suppose there exists $(x,k) \in W_*$ such that

$$\text{either} \quad \text{(i)} \quad \hat{f}(x,k) \in (f(\delta), f(c)] \times \{0\} \quad \text{or} \quad \text{(ii)} \quad \hat{f}^2(x,k) \in (f(\delta), f(c)] \times \{0\}.$$

Then either $f(x) \in (f(\delta), f(c)]$ or $f^2(x) \in [f^2(c), f^2(\delta))$, respectively. In both cases we must have $x \in (-\delta, \delta)$: this is immediate in (i), and easy to obtain in (ii), if one assumes that $\delta > 0$ is small enough so that $f^2(c) < f^2(\delta) < f^3(c)$. Now, in order that the ith iterate of (x,k) be in E_0, for $i = 1$ or $i = 2$, we must have $k + i > H(\delta)$. Assume that $0 < \delta < 1/64$ is small enough so that this implies length$(B_k) \leq 2e^{-\beta k} \leq 1/64$. Then, since $x \in B_k \cap (-\delta, \delta)$, the interval B_k must be contained in $(-1/32, 1/32)$. It follows that $|f'(y)| \leq 1/16$ for every $y \in B_k$, and so

$$\begin{aligned} |x - c_k| \leq e^{-\beta k} \;\Rightarrow\; & |f(x) - c_{k+1}| \leq \frac{1}{16} e^{-\beta k} \leq \frac{1}{8} e^{-\beta(k+1)} \\ \Rightarrow\; & |f^2(x) - c_{k+2}| \leq \frac{1}{2} e^{-\beta(k+1)} \leq e^{-\beta(k+2)} \end{aligned}$$

(because $|f'| \leq 4$ and $e^{\beta} < e^{2\alpha} < \sqrt{\lambda_c} \leq 2$). This means that $\hat{f}^i(x,k) \in E_{k+i}$ for $i = 1, 2$, contradicting the choice of (x,k).

For any point (y,l) such that $\hat{f}(y,l) \in W_*$, we denote

$$g(y,l) = \frac{1}{|f'(y)|} \frac{w_0(y,l)}{w_0(\hat{f}(y,l))}. \tag{5.5}$$

Clearly, $g(y,l) > 0$ if and only if (y,l) is in W_*. Moreover, as we already pointed out, in that case $1/g(y,l)$ is the jacobian of $\hat{f}$ at (y,l), with respect to the metric $\|\cdot\|$ (or, equivalently, with respect to the measure m_0).

Given a measurable function $\varphi : \hat{I} \to \mathbb{R}$ we define

$$\operatorname{var}\varphi = \sum_{k\geq 0}\operatorname{var}(\varphi|E_k) \qquad \sup\varphi = \sup_{k\geq 0}\sup(\varphi|E_k) \qquad \int \varphi\, dm_0 = \sum_{k\geq 0}\int_{E_k}\varphi w_0\, dm$$

where m denotes Lebesgue measure on $B_k \approx E_k$. Then we define the BV-norm of φ

$$\|\varphi\|_{\mathrm{BV}} = \operatorname{var}\varphi + \sup|\varphi| + \int |\varphi|\, dm_0,$$

and take $\mathrm{BV}(\hat{I})$ to be the (Banach) space of functions $\varphi : \hat{I} \to \mathbb{R}$ such that $\|\varphi\|_{\mathrm{BV}} < \infty$.

Finally, we describe the transfer operator $\mathcal{L}_0$ associated to $\hat{f}$. Given $\varphi \in \mathrm{BV}(\hat{I})$ and $(x,k) \in W_*$, we let

$$\mathcal{L}_0\varphi(x,k) = \sum_{\hat{f}(y,l)=(x,k)} \frac{\varphi(y,l)}{|f'(y)|}\frac{w_0(y,l)}{w_0(x,k)} = \sum_{\hat{f}(y,l)=(x,k)} (g\varphi)(y,l) \tag{5.6}$$

Observe that the sum involves exactly one term if $x \in W_k$, with $k \geq 2$, and exactly two terms if $x \in W_1$. For $k = 0$ there may be infinitely many terms: at most one for each value of $l \geq 1$, and at most two for $l = 0$. Then we extend $\mathcal{L}_0\varphi$ to $\hat{I}\setminus W_*$ by asking that it be constant on each connected component of $B_k \setminus W_k$, $k \geq 1$. More precisely, we let $a_k < b_k$ be the endpoints of the interval W_k, then we define

$$\mathcal{L}_0\varphi(x,k) = \begin{cases} \limsup_{y\to a_k^+} \mathcal{L}_0\varphi(y,k) & \text{if } x \leq a_k \\ \limsup_{y\to b_k^-} \mathcal{L}_0\varphi(y,k) & \text{if } x \geq b_k. \end{cases} \tag{5.7}$$

This seemingly unnatural definition is designed so that $\operatorname{var}(\mathcal{L}_0\varphi)$ and $\sup|\mathcal{L}_0\varphi|$ are left unchanged: the variation of $\mathcal{L}_0\varphi$ over B_k coincides with the variation of $\mathcal{L}_0\varphi$ over W_k, and a similar fact is true for the supremum. Of course, the same holds for $\int \mathcal{L}_0\varphi\, dm_0$, because m_0 is supported on W_*. In particular, the duality relation (5.4) is not affected by this convention.

Clearly, $\mathcal{L}_0$ is a nonnegative operator, in the sense that it maps nonnegative functions to nonnegative functions. Then, (5.4) also implies that $\mathcal{L}_0$ is nonincreasing with respect to the L^1-norm:

$$\int |\mathcal{L}_0\psi|\, dm_0 \leq \int \mathcal{L}_0|\psi|\, dm_0 = \int |\psi|\, dm_0 \quad \text{for every } \psi. \tag{5.8}$$

5.2 Uniform expansion for the tower map.

We shall see in the next sections that $\mathcal{L}_0$ is a bounded, and even quasi-compact, operator from $\mathrm{BV}(\hat{I})$ to itself. More precisely, 1 is a simple eigenvalue and the rest of the spectrum is contained in a disk of radius strictly smaller than 1. The proof of this relies on properties of uniform expansion of the map $\hat{f}: \hat{I} \to \hat{I}$ that we obtain in Lemma 5.4 below. First, we state and prove two key lemmas on the expanding behaviour of certain iterates of the map f.

Lemma 5.2. *There are $\sigma_0 > 1$, $b > 0$, and $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ there is $c(\delta) > 0$ such that, given any $x \in I$ and $n \ge 1$,*

(1) *if $x, f(x), \ldots, f^{n-1}(x) \notin (-\delta, \delta)$ then $|(f^n)'(x)| \ge c(\delta)\sigma_0^n$;*
(2) *(2) if, in addition, $f^n(x) \in (-\delta, \delta)$ then $|(f^n)'(x)| \ge b\sigma_0^n$.*

Proof: The arguments in the proof are now standard in one-dimensional dynamics. First, given $\delta_1 > 0$ there are $m \ge 1$ and $\sigma_1 > 1$ such that

$$|(f^m)'(y)| \ge \sigma_1^m \quad \text{whenever} \quad y, f(y), \ldots, f^{m-1}(y) \notin (-\delta_1, \delta_1). \tag{5.9}$$

Indeed, (U1) and [Si78] imply that all the periodic points of f are repelling, and then (5.9) is a consequence of [MS93, Section III.3]. In the sequel we fix $\delta_1 > 0$ small, depending only on α and λ_c, cf. (5.13)–(5.14). By (U1) and [No88], there are $\sigma_2 > 1$, $\delta_2 > 0$, $K_2 \ge 1$ such that, given any $1 \le l < m$,

$$|(f^l)'(y)| \ge \frac{1}{K_2}\sigma_2^l \quad \text{whenever} \quad f^l(y) \in (-\delta_2, \delta_2). \tag{5.10}$$

We take $\sigma_0 = \min\{\sigma_1, \sigma_2, \lambda\}$ and $\delta_0 = \min\{\delta_1, \delta_2\}$ and, for each $0 < \delta \le \delta_0$, we define $c(\delta) = \inf\{(|f'(x)|/\sigma_0)^m : x \in I \setminus (-\delta, \delta)\}$. The constant $b > 0$ will be defined below, with $b \le (1/K_2)$. Clearly, for all $0 \le l < m$,

$$|(f^l)'(y)| \ge c(\delta)\sigma_0^l \quad \text{if} \quad y, f(y), \ldots, f^{l-1}(y) \notin (-\delta, \delta). \tag{5.11}$$

Given n and x as in the statement, we denote $x_j = f^j(x)$, for $0 \le j \le n$. If $x_j \notin (-\delta_1, \delta_1)$ for all $0 \le j < n$ then the lemma follows immediately from (5.9), (5.10), (5.11). Indeed, writing $n = qm + l$, with $0 \le l < m$, we get

$$|(f^{qm+l})'(x)| = |(f^l)'(f^{qm}(x))|\,|(f^{qm})'(x)| \ge c(\delta)\sigma_0^l\,(\sigma_1^m)^q,$$

which gives (1). Moreover, if $f^n(x) \in (-\delta, \delta)$ then we may replace $c(\delta)\sigma_0^l$ by $K_2^{-1}\sigma_2^l$ in this estimate, thus proving (2).

From now on we suppose that the trajectory of x up to time n does intersect $(-\delta_1, \delta_1)$, and we define $0 \le \nu_1 < \cdots < \nu_s < n$ as follows. Let ν_1 be the smallest $j \ge 0$ with $x_j \in (-\delta_1, \delta_1)$. For each ν_i, $i \ge 1$, define

$$p_i = \max\{k \ge 1 : |x_{\nu_i + j} - c_j| < e^{-\beta j} \text{ for every } 1 \le j \le k\}.$$

Then let ν_{i+1} be the smallest $n > r > \nu_i + p_i$ for which $x_r \in (-\delta_1, \delta_1)$. For the time being we fix $1 \le i \le s$, and write $p = p_i$ and $\nu = \nu_i$. The previous definition and (U2) yield $|x_{\nu+j} - c_j| \le e^{-(\beta-\alpha)j}|c_j|$ and so

$$(1 - e^{-j(\beta-\alpha)})|f'(c_j)| \le |f'(x_{\nu+j})| \le (1 + e^{-j(\beta-\alpha)})|f'(c_j)|,$$

for all $1 \le j \le p$. Then

$$\frac{1}{C}|(f^p)'(c_1)| \le |(f^p)'(x_{\nu+1})| \le C|(f^p)'(c_1)|, \tag{5.12}$$

with $C^{-1} = \prod_{j \ge 1}(1 - e^{-j\alpha/2})$, recall that we take $\beta - \alpha > \alpha/2$. Moreover,

$$e^{-\beta(p+1)} \le |x_{\nu+p+1} - c_{p+1}| \le |x_{\nu+p} - c_p|(1 + e^{-p(\beta-\alpha)})|f'(c_p)|,$$

and so, by recurrence,

$$e^{-\beta(p+1)} \le |x_{\nu+1} - c_1| \prod_{j=1}^{p}(1 + e^{-j(\beta-\alpha)})|(f^p)'(c_1)| \le C|(f^p)'(c_1)||x_\nu|^2.$$

Combining this with (5.12) and (U1), we conclude that

$$|(f^{p+1})'(x_\nu)|^2 \ge \frac{1}{C}|(f^p)'(c_1)|^2|x_\nu|^2 \ge \frac{1}{C}(\lambda_c e^{-\beta})^{p+1}. \tag{5.13}$$

Up to taking δ_1 small enough with respect to α and λ_c, we may suppose the p_i (uniformly) sufficiently large so that (5.13) implies

$$|(f^{p_i+1})'(x_{\nu_i})| \ge \frac{1}{C}(\lambda_c e^{-\beta})^{(p_i+1)/2} \ge \frac{1}{C}(\lambda\rho)^{p_i+1} \ge K_2\lambda^{p_i+1}, \tag{5.14}$$

for each $1 \le i \le s$, recall that $e^{-\beta}\lambda_c > e^{-2\alpha}\lambda_c > \lambda^2\rho^2$. At this point we write $|(f^n)'(x)| = \prod_{j=0}^{n-1}|f'(x_j)|$ and partition the time range $[0, n)$ into subintervals $J \subset [0, n)$ as follows. Let $|J|$ denote the number of elements of each given J. First, we suppose $\nu_s + p_s < n$. For $J = [0, \nu_1)$ and for each $J = (\nu_i + p_i, \nu_{i+1})$, $1 \le i < s$, we have

$$\prod_{j \in J}|f'(x_j)| \ge K_2^{-1}\sigma_0^{|J|},$$

as a consequence of (5.9) and (5.10). The same holds for $J = (\nu_s + p_s, n)$ if $|x_n| < \delta$. In general, $J = (\nu_s + p_s, n)$ has

$$\prod_{j \in J}|f'(x_j)| \ge c(\delta)\sigma_0^{|J|},$$

by (5.9) and (5.11). Moreover,

$$\prod_{j\in J}|f'(x_j)| \geq K_2\sigma_0^{|J|}$$

for each $J = [\nu_i, \nu_i + p_i]$, $1 \leq i \leq s$, recall (5.14). Altogether, this proves both parts (1) and (2) of the lemma when $\nu_s + p_s < n$, since we take $b \leq (1/K_2)$. Now we treat the case $\nu_s + p_s \geq n$. We only have to consider $J = [\nu_s, n)$, as the previous estimates remain valid for all other subintervals involved. In general, (5.9) and (5.11) give

$$\prod_{j\in J}|f'(x_j)| \geq c(\delta)\sigma_0^{|J|}.$$

Part (1) follows, in the same way as before. In order to prove (2), we let $q = n - \nu_s - 1$. Then $0 \leq q < p_s$ and so, recall also (U2),

$$|x_{\nu_s}| \geq \delta > |x_n| \geq |c_{q+1}| - |x_n - c_{q+1}| \geq e^{-\alpha(q+1)} - e^{-\beta(q+1)} \geq \frac{1}{C}e^{-\alpha(q+1)}.$$

Moreover, (5.12) holds for $\nu = \nu_s$ and $p = q$. Hence,

$$|(f^{q+1})'(x_{\nu_s})| \geq \frac{1}{C}|(f^q)'(c_1)||x_{\nu_s}| \geq \frac{1}{C}(\lambda_c e^{-\alpha})^{q+1} \geq \frac{1}{C}\lambda^{q+1}.$$

We take $b = (CK_2)^{-1}$, for $C > 0$ as in the last term. □

As we already said at the beginning of this section, we take the constant δ in the definition of our tower satisfying $0 < \delta \leq \delta_0$, and we also fix $\sigma \in (1, \sigma_0]$.

Lemma 5.3. *There is $C > 0$ such that, given any $z \in (-\delta, \delta)$ and $k \geq 1$,*

(1) *if $|f^j(z) - c_j| \leq e^{-\beta j}$ for every $1 \leq j \leq k$, then*

$$\frac{1}{C} \leq \frac{|(f^k)'(f(z))|}{|(f^k)'(c_1)|} \leq C;$$

(2) *if, in addition, $|f^{k+1}(z) - c_{k+1}| \geq e^{-\beta(k+1)}$, then*

$$|(f^k)'(f(z))| \geq \frac{1}{C}\lambda_c^k \qquad \textit{and} \qquad |(f^{k+1})'(z)| \geq \frac{1}{C}\rho^k\lambda^k.$$

Proof: This follows from the same arguments as (5.12)–(5.14). Indeed, the assumption together with (U2) imply

$$(1 - e^{j(\alpha-\beta)})|f'(c_j)| \leq |f'(f^j(z))| \leq (1 + e^{j(\alpha-\beta)})|f'(c_j)|,$$

for every $1 \leq j \leq k$. Then, multiplying over $1 \leq j \leq k$ and taking C as in (5.12), we get the conclusion of (1):

$$\frac{1}{C}|(f^k)'(c_1)| \leq |(f^k)'(f(z))| \leq C|(f^k)'(c_1)|.$$

The first claim in (2) is a direct consequence of (1) and (U1), and the second one can be derived as follows. Given z and k as in the statement,

$$e^{-\beta(k+1)} \leq |f^{k+1}(z) - c_{k+1}| = |(f^k)'(y)||f(z) - c_1|$$

for some y in the interval bounded by c_1 and $f(z)$. Then $f^j(y)$ is in between $f^{j+1}(z)$ and c_{j+1} for each $0 \leq j < k$ and so (1) remains valid with y in the place of $f(z)$. In particular, $|(f^k)'(y)| \leq C|(f^k)'(c_1)|$ and so

$$e^{-\beta(k+1)} \leq C|(f^k)'(c_1)|\,|f(z) - c_1| \leq C|(f^k)'(c_1)|\,|z|^2. \tag{5.15}$$

This implies

$$|f'(z)|^2 \geq \frac{1}{C}|z|^2 \geq \frac{1}{C}e^{-\beta k}|(f^k)'(c_1)|^{-1}$$

and so, using the conclusion of (1) once again,

$$|(f^{k+1})'(z)| \geq \frac{1}{C}|(f^k)'(c_1)|\,|f'(z)| \geq \frac{1}{C}|(f^k)'(c_1)|^{1/2}e^{-\beta k/2} \geq \frac{1}{C}\lambda_c^{k/2}e^{-\beta k/2} \geq \frac{1}{C}\rho^k\lambda^k$$

(because $\beta < 2\alpha$ and $\lambda\rho e^\alpha < \sqrt{\lambda_c}$). □

We denote by $\mathcal{P}^{(n)}$ the partition of $\hat{I}$ into monotonicity intervals of $\hat{f}^n$, $n \geq 1$, characterized in the following way. For every $k \geq 1$, let

$$U_k = \{(x,k) \in E_k : \hat{f}(x,k) = (f(x), k+1)\}$$

and D_k^-, D_k^+ be the connected components of $E_k \setminus U_k$. That is, points in U_k are sent by $\hat{f}$ to an upper level of the tower, whereas points in $D_k^+ \cup D_k^-$ are mapped down to the ground level E_0. Note that U_k is a neighbourhood of c_k, but $D_k^\pm$ may be empty. For $k = 0$, we set

$$U_0^- = (-\delta, 0] \times \{0\}, \quad U_0^+ = (0,\delta) \times \{0\}, \quad D_0^- = [q, -\delta] \times \{0\}, \quad D_0^+ = [\delta, -q] \times \{0\}.$$

Then, we take

$$\mathcal{P}^{(1)} = \{U_k, D_k^-, D_k^+ : k \geq 1\} \cup \{U_0^-, U_0^+, D_0^-, D_0^+\}.$$

Then, for any $n > 1$, we take $\mathcal{P}^{(n)}$ to be the nth iterate of $\mathcal{P}^{(1)}$. That is, by definition, $\mathcal{P}^{(n)}(\xi_1) = \mathcal{P}^{(n)}(\xi_2)$ if and only if $\mathcal{P}^{(1)}(\hat{f}^i(\xi_1)) = \mathcal{P}^{(1)}(\hat{f}^i(\xi_2))$ for each $0 \leq i < n$.

In what follows we always assume that every $\eta \in \mathcal{P}^{(n)}$ has strictly positive length, moreover, the intersection of η with W_* is either empty or an interval with positive length. Note that in order to have this it suffices that the orbits of

$$(c, 0), \qquad (\pm\delta, 0), \quad \text{and} \quad (c_k \pm e^{-\beta k}) \quad k \geq 1,$$

be two-by-two disjoint injective sequences on $\hat{I}$, which can always be obtained by slightly modifying β and δ if necessary (so as to avoid a countable set of relations involving these two constants).

We shall also need the iterated versions $g^{(n)}$ of g, defined by

$$g^{(n)}(\xi) = g(\xi)\, g(\hat{f}(\xi)) \cdots g(\hat{f}^{n-1}(\xi)) = \frac{1}{|(f^n)'(x)|} \frac{w_0(\xi)}{w_0(\hat{f}^n(\xi))}$$

for every $\xi = (x, k)$ such that $\hat{f}^i(\xi) \in W_*$ for $1 \leq i \leq n$. For use in Lemma 5.5, let us observe that, cf. (5.5),

$$\mathcal{L}_0^n \varphi(\xi) = \sum_{\eta \in \mathcal{P}^{(n)}} \Big((g^{(n)}\varphi) \circ (\hat{f}^n|\eta)^{-1} \chi_{\hat{f}^n(\eta)} \Big)(\xi) \tag{5.16}$$

for every $n \geq 1$ and ξ with $\hat{f}^i(\xi) \in W_*$ for $1 \leq i \leq n$.

Remark 5.2. It follows from our definitions that if (x, k) belongs in $U_k \cap W_k$, $k \geq 1$, and $z \in (0, \delta)$ is such that $\hat{f}^k(z, 0) = (x, k)$, then

$$g(x, k) = \frac{1}{|f'(x)|} \frac{w_0(x, k)}{w_0(\hat{f}(x, k))} = \frac{1}{|f'(x)|} \frac{\lambda^k |(f^k)'(z)|^{-1}}{\lambda^{k+1} |(f^{k+1})'(z)|^{-1}} = \frac{1}{\lambda}.$$

The same remains true for $(x, 0) \in (-\delta, \delta) = U_0^+ \cup U_0^-$, On the other hand, if (x, k) is in $D_k \cap W_k$, $k \geq 1$, and z is as before,

$$g(x, k) = \frac{1}{|f'(x)|} \frac{w_0(x, k)}{w_0(\hat{f}(x, k))} = \frac{1}{|f'(x)|} \frac{\lambda^k |(f^k)'(z)|^{-1}}{1} = \frac{\lambda^k}{|(f^{k+1})'(z)|} \leq C\rho^{-k}$$

as a consequence of Lemma 5.3. Observe that $k \geq H(\delta)$, where $H(\delta)$ is the minimum height from which orbits starting in $(-\delta, \delta) \times \{0\}$ can fall down back to E_0, cf. Section 5.1. We suppose that $\delta > 0$ is small (and so $H(\delta)$ is large enough) so that this implies $C\rho^{-k} < 1/\lambda$. Moreover, Lemma 5.2 gives $g^{(n)}(x, 0) \leq 1/(c(\delta)\sigma_0^n)$ for every point $(x, 0)$ whose trajectory remains in E_0 up to time $n \geq 1$. These remarks express the uniformly expanding character of $\hat{f}$.

Lemma 5.4.

(1) *Let $\gamma \subset \eta \in \mathcal{P}^{(n)}$ be such that $\hat{f}^j(\gamma) \subset E_0$ for every $0 \leq j \leq n$. Then*

$$\sup_\gamma g^{(n)} \leq \begin{cases} C\sigma^{-n} & \text{if } \hat{f}^n(\gamma) \subset (-\delta, \delta) \times \{0\} \\ C(\delta)\sigma^{-n} & \text{in general.} \end{cases}$$

Moreover, $\operatorname{var}_\gamma g^{(n)} \le 2\sup_\gamma g^{(n)}$.

(2) *Let* $\gamma \subset \eta \cap W_*$ *for some* $\eta \in \mathcal{P}^{(n)}$ *and let* $0 \le l \le \min\{k, n-1\}$ *be such that* $\hat{f}^i(\gamma) \in E_{k-l+i}$ *for* $0 \le i \le l$ *and* $\hat{f}^i(\gamma) \in E_0$ *for* $l < i \le n$. *Then*

$$\sup_\gamma g^{(n)} \le \begin{cases} C\lambda^{-l}\rho^{-k}\sigma^{-n+l+1} & \text{if } \hat{f}^n(\gamma) \subset (-\delta,\delta)\times\{0\} \\ C(\delta)\lambda^{-l}\rho^{-k}\sigma^{-n+l+1} & \text{in general.} \end{cases}$$

Moreover, $\operatorname{var}_\gamma g^{(n)} \le 2\sup_\gamma g^{(n)}$.

(3) *Let* $\gamma \subset \eta \cap W_*$ *for some* $\eta \in \mathcal{P}^{(n)}$ *and let* $l \ge 0$ *be such that* $\hat{f}^i(\gamma) \in E_{l+i}$ *for* $0 \le i \le n$. *Then* $g^{(n)} = \lambda^{-n}$ *on* γ.

Proof: The first statement in (1) follows immediately from Lemma 5.2 and the observation that

$$g^{(n)}(x,k) = \frac{1}{|(f^n)'(x)|}$$

for every $(x,k) \in \gamma$. For the second statement we use the fact that f has negative schwarzian derivative:

$$Sf(x) = \frac{f'''}{f'}(x) - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2(x) = -\frac{3}{2x^2} < 0.$$

Indeed, since the class of maps with negative schwarzian derivative is closed under composition, we have $Sf^n < 0$. Then $(f^n)'''$ and $(f^n)'$ must have opposite signs at every local extremum of the first derivative, in other words, every local minimum of $(f^n)'$ is negative, and every local maximum of $(f^n)'$ is positive. Since the derivative is nonzero on the interior of each monotonicity interval, we conclude that $1/|(f^n)'(x)|$ has no local maximum, and so it has at most one local minimum on γ. As a consequence, $\operatorname{var}_\gamma g \le 2\sup_\gamma g$, as claimed.

For the first claim in (2), let $(x, k-l) \in \gamma$ and $z \in (0,\delta)$ with $\hat{f}^{k-l}(z,0) = (x,k-l)$. Then, by Lemma 5.3(2) and Lemma 5.2(1),

$$\begin{aligned} g^{(n)}(x,k-l) &= \frac{1}{|(f^n)'(x)|}\frac{\lambda^{k-l}\,|(f^{k-l})'(z)|^{-1}}{1} \\ &= \lambda^{k-l}\,|(f^{k+1})'(z)|^{-1}|(f^{n-l-1})'(f^{k+1}(z))|^{-1} \\ &\le \lambda^{k-l}\cdot C\rho^{-k}\lambda^{-k}\cdot C(\delta)\sigma^{-(n-l-1)}. \end{aligned}$$

Moreover, $C(\delta)$ may be replaced by C if

$$f^{n-l-1}(f^{k+1}(z)) = f^n(x) \in (-\delta,\delta).$$

The last statement in (2) follows from applying to $1/|(f^{n+k-l})'(z)|$ the same argument as we used in the previous case (1) for $1/|(f^n)'(x)|$.

Finally, (3) follows immediately from the first observation in Remark 5.2. □

5.3. Absolutely continuous invariant measures.

The main result in this section is Proposition 5.6, a version of Proposition 3.1 for the tower map $\hat{f}$. From it we deduce that $\hat{f}$ and f admit absolutely continuous invariant measures $\hat{\mu_0}$ and μ_0, respectively.

We begin by proving a partial statement, Lemma 5.5, which contains most of the technical difficulty of Proposition 5.6. The proof is fairly long, but the reader should find it useful to bear in mind that it is closely related to the calculation leading to Proposition 3.19. Indeed, the key fact which is implicit in the argument is that the first-return map of $\hat{f}$ to the ground floor E_0 is uniformly expanding, with properties akin to (a), (b2), (c2) of Section 3.4; see Remark 5.2.

Lemma 5.5. *There is $C > 0$ and, for each $n \geq 1$, there is $C(n) > 0$ such that for every $\varphi \in BV(\hat{I})$ and every interval $A \subset E_0$,*

$$\operatorname*{var}_A(\mathcal{L}_0^n\varphi) \leq \operatorname{var}(\chi_A\mathcal{L}_0^n\varphi) \leq C\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C(n)\int|\varphi|\,dm_0.$$

Proof: Fix $n \geq 1$ and any interval $A \subset E_0$. Whenever our estimates involve some constant $C(\delta)$, it is implicitly stated that the dependence on δ may be removed (that is, $C(\delta)$ may be replaced by C) if A is contained in $(-\delta, \delta) \times \{0\}$. For the sake of readability, we split the proof into several steps.

Step 1: We find a useful expression (5.18) for $\chi_A\mathcal{L}_0^n\varphi$, by decomposing backward orbits of points in A according to the instant when they have fallen down to the zeroth level E_0.

Let $\underline{\Gamma}(0)$ be the set of all nonempty intervals $\underline{\gamma}$ of the form $\underline{\gamma} = \eta \cap \hat{f}^{-1}(A)$, with $\eta \in \mathcal{P}^{(1)}$ and $\eta \subset E_0$, and let $\overline{\Gamma}(0)$ be defined in the same way, except that $\eta \subset E_k$ for some $k \geq 1$. Then, recalling (5.16) and the fact that $A \subset W_*$,

$$\chi_A\mathcal{L}_0^n\varphi = \sum_{\underline{\gamma}\in\underline{\Gamma}(0)}(g\mathcal{L}_0^{n-1}\varphi)\circ(\hat{f}|\eta)^{-1}\chi_{\hat{f}(\underline{\gamma})} + \sum_{\overline{\gamma}\in\overline{\Gamma}(0)}(g\mathcal{L}_0^{n-1}\varphi)\circ(\hat{f}|\eta)^{-1}\chi_{\hat{f}(\overline{\gamma})}\,. \tag{5.17}$$

Since g is zero outside W_*, we may just as well replace each $\overline{\gamma} \in \overline{\Gamma}(0)$ by $\gamma = \overline{\gamma} \cap W_*$. We call $\mathcal{G}(0)$ the set of all intervals γ obtained in this way. Now we repeat the same procedure for each $\underline{\gamma} \in \underline{\Gamma}(0)$ in the place of A. In this way we find sets of intervals $\underline{\Gamma}(1)$ and $\mathcal{G}(1)$ as follows: each $\underline{\gamma} \in \underline{\Gamma}(1)$ is given by

$$\underline{\gamma} = \eta \cap \hat{f}^{-2}(A)$$

for some $\eta \in \mathcal{P}^{(2)}$ such that $\eta \subset E_0$ and $\hat{f}(\eta) \subset E_0$, and each $\gamma \in \mathcal{G}(1)$ has the form

$$\gamma = \eta \cap \hat{f}^{-2}(A) \cap W_*$$

with $\eta \in \mathcal{P}^{(2)}$ contained in E_k for some $k \geq 1$ and $\hat{f}(\eta) \subset E_0$. Replacing in (5.17), we get that $\chi_A \mathcal{L}_0^n \varphi$ is given by

$$\sum_{\gamma \in \underline{\Gamma}(1)} \left(g^{(2)} \mathcal{L}_0^{n-2} \varphi\right) \circ (\hat{f}^2|\eta)^{-1} \chi_{\hat{f}^2(\underline{\gamma})} + \sum_{j=0}^{1} \sum_{\gamma \in \mathcal{G}(j)} \left(g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi\right) \circ (\hat{f}^{j+1}|\eta)^{-1} \chi_{\hat{f}^{j+1}(\gamma)} .$$

Repeating this operation n times, we obtain

$$\begin{aligned} \chi_A \mathcal{L}_0^n \varphi = &\sum_{\gamma \in \Gamma} (g^{(n)} \varphi) \circ (\hat{f}^n|\eta)^{-1} \chi_{\hat{f}^n(\gamma)} + \\ &+ \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \left(g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi\right) \circ (\hat{f}^{j+1}|\eta)^{-1} \chi_{\hat{f}^{j+1}(\gamma)} . \end{aligned} \tag{5.18}$$

Here $\Gamma = \underline{\Gamma}(n)$ is set of all intervals $\gamma = \eta \cap \left(\hat{f}^n\right)^{-1}(A)$ with $\eta \in \mathcal{P}^{(n)}$ and $\hat{f}^i(\eta) \subset E_0$ for $0 \leq i \leq n$, and $\mathcal{G}(j)$ is the set of all

$$\gamma = \eta \cap \left(\hat{f}^{j+1}\right)^{-1}(A) \cap W_* ,$$

where $\eta \in \mathcal{P}^{(j+1)}$ is contained in some E_k with $k \geq 1$, and $\hat{f}^i(\eta) \subset E_0$ for every $1 \leq i \leq j+1$. Then, using properties (v1), (v2), (v5), stated near the beginning of Chapter 3, we find

$$\begin{aligned} \operatorname{var}(\chi_A \mathcal{L}_0^n \varphi) &\leq \sum_{\gamma \in \Gamma} \operatorname{var}(\chi_\gamma g^{(n)} \varphi) + \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \operatorname{var}(\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi) \\ &=: S_1 + S_2 . \end{aligned} \tag{5.19}$$

Step 2: We bound the first term S_1 on the right hand side of (5.19), see (5.21). Suprema are bound in terms of variations and L^1-norms, through the mean value theorem. We use the fact that the lengths of the monotonicity intervals involved in S_1 are bounded from below and their number is bounded from above, by constants depending only on n.

Using property (v2) of Chapter 3,

$$S_1 \leq \sum_{\gamma \in \Gamma} \left(\operatorname*{var}_\gamma g^{(n)} + 2 \sup_\gamma g^{(n)} \right) \cdot \sup_\gamma |\varphi| + \sup_\gamma g^{(n)} \cdot \operatorname*{var}_\gamma \varphi. \tag{5.20}$$

For each $\gamma \in \Gamma$, we also denote $\hat{\gamma}$ the monotonicity interval η that contains γ (this is for notational coherence with the sequel of the calculations). Moreover, we let $\mathcal{G}(S_1)$ be the set of all these $\hat{\gamma} = \eta$. Since

$$\mathcal{G}(S_1) \subset \mathcal{P}^{(n,0)} = \{\eta \in \mathcal{P}^{(n)} \text{ such that } \hat{f}^i(\eta) \subset E_0 \text{ for all } 0 \leq i \leq n\},$$

and $\mathcal{P}^{(n,0)}$ is a finite set of nonempty intervals depending only on n, there is some large constant $C(n) > 0$ such that

$$\#\mathcal{G}(S_1) \le C(n) \qquad \text{and} \qquad \frac{1}{m_0(\hat\gamma)} = \frac{1}{|\hat\gamma|} \le C(n) \text{ for all } \hat\gamma \in \mathcal{G}(S_1).$$

Then, by the mean value theorem,

$$\sup_\gamma \varphi \le \sup_{\hat\gamma} \varphi \le \operatorname*{var}_{\hat\gamma} \varphi + \frac{1}{m_0(\hat\gamma)} \int_{\hat\gamma} |\varphi|\, dm_0 \le \operatorname*{var}_{\hat\gamma} \varphi + C(n) \int |\varphi|\, dm_0 .$$

Of course, we also have $\operatorname{var}_\gamma \varphi \le \operatorname{var}_{\hat\gamma} \varphi$. Replacing this and Lemma 5.4(1) in (5.19),

$$\text{(5.21)} \qquad \begin{aligned} S_1 &\le \sum_{\hat\gamma \in \mathcal{G}(S_1)} \Big(C(\delta)\sigma^{-n} \operatorname*{var}_{\hat\gamma} \varphi + C(\delta)C(n) \int |\varphi|\, dm_0 \Big) \\ &\le C(\delta)\sigma^{-n} \Big(\sum_{\hat\gamma \in \mathcal{G}(S_1)} \operatorname*{var}_{\hat\gamma} \varphi \Big) + C(\delta)C(n) \int |\varphi|\, dm_0 . \end{aligned}$$

Step 3: We decompose the last term S_2 in (5.19) into three parts, according to the height of the tower level containing the interval $\gamma \in \mathcal{G}(j)$.

For each $\gamma \in \mathcal{G}(j)$ we define $j(\gamma) = j$, and also $k(\gamma) = k$ if $\gamma \subset E_k$. Observe that $k(\gamma)$ is at least $H(\delta)$ since, by construction, all these intervals γ are contained in W_*. Then we split

$$\begin{aligned} S_2 &= \sum_{j=0}^{n-1} \sum_{\gamma \in \mathcal{G}(j)} \operatorname{var}(\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi) \\ &= \sum_{j=0}^{n-1} \Big(\sum_{k(\gamma) \ge N} + \sum_{n-j-1 \le k(\gamma) < N} + \sum_{k(\gamma) < n-j-1} \Big) =: s_1 + s_2 + s_3 , \end{aligned}$$

where $N \ge n$ is to be fixed below, as a function of n only.

Step 4: We bound the term s_1 in S_2, see (5.22). The main point is that use of Lemma 5.3 introduces in the estimates a factor ρ^{-N}, which can be made small by choosing N large.

For each γ with $k = k(\gamma) \ge N$ let $\tilde\gamma = (\hat f^{n-j-1})^{-1}(\gamma) \subset E_{k-(n-j-1)}$. Since $\hat f$ is monotone on each level E_k, $k \ge 1$, we have that $\tilde\gamma$ is an interval and $\hat f^n$ is monotone on $\tilde\gamma$. Let $\tilde\eta$ the atom of $\mathcal{P}^{(n)}$ containing $\tilde\gamma$. Then

$$\mathcal{L}_0^{n-j-1} \varphi = \big(g^{(n-j-1)} \varphi \big) \circ \big(\hat f^{n-j-1} | \tilde\eta \big)^{-1}$$

on γ, and so

$$\chi_\gamma g^{(j+1)} \mathcal{L}_0^{n-j-1} \varphi = (\chi_{\tilde\gamma} g^{(n)} \varphi) \circ (\hat f^{n-j-1} | \tilde\eta)^{-1}.$$

Then, using (v2), (v5), and applying Lemma 5.4(2) to $g^{(n)}$ (with $l = n - j - 1$),

$$s_1 \leq \sum_{j=0}^{n-1} \sum_{k(\gamma)\geq N} C(\delta)\lambda^{-(n-j-1)}\rho^{-k}\sigma^{-j}(\operatorname{var}_{\tilde{\gamma}} \varphi + \sup_{\tilde{\gamma}} |\varphi|)\,.$$

Since $\hat{f}^j|E_0$ is at most 2^j-to-1, and E_k contains at most two intervals of monotonicity mapped to E_0 by $\hat{f}$, the sum above ranges over at most 2^{j+1} intervals γ for each given value of $j = j(\gamma)$ and $k = k(\gamma)$. Therefore, the previous inequality gives

$$\begin{aligned} s_1 &\leq \sum_{j=0}^{n-1} \sum_{k\geq N} 2^{j+1} C(\delta)\lambda^{-n+j+1}\rho^{-k}\sigma^{-j}(\operatorname{var}\varphi + \sup|\varphi|) \\ &\leq C(\delta)2^n\sigma^{-n}\rho^{-N}(\operatorname{var}\varphi + \sup|\varphi|) \end{aligned} \tag{5.22}$$

(note that $\rho > 1$ and $\sigma < 2\lambda$).

Step 5: We bound the second term s_2 in S_2, see (5.23). The mean value theorem is invoked to bound suprema in terms of variations and L^1-norms. This uses the fact that the number of monotonicity intervals involved in s_2 is bounded from above, and the lengths of their intersections with W_* are bounded from below, by constants depending only on n and N.

Using Lemma 5.4(2) as in Step 4, we find

$$s_2 \leq \sum_{j=0}^{n-1} \sum_{n-j-1\leq k(\gamma)<N} C(\delta)\lambda^{-(n-j-1)}\rho^{-k}\sigma^{-j}(\operatorname{var}_{\tilde{\gamma}} \varphi + \sup_{\tilde{\gamma}} |\varphi|)$$

where $\tilde{\gamma} \subset E_{k(\gamma)-(n-j-1)}$ is defined as before. A slight difference with respect to the previous case is that $\tilde{\gamma}$ may not be an interval, if $k(\gamma) = n - j - 1$. However, this is not important, since in this exceptional case $\tilde{\gamma}$ is just the union of two (symmetric) subintervals of $U_0^- \times \{0\}$ and $U_0^- \times \{0\}$, respectively, and then it suffices to consider each of these subintervals separately. From this point on the argument is essentially the same as in Step 3. We let $\hat{\gamma} = \tilde{\eta} \cap W_*$, where $\tilde{\eta}$ is the element of $\mathcal{P}^{(n)}$ containing $\tilde{\gamma}$, and we call $\mathcal{G}(s_2)$ the set of all $\tilde{\eta}$ constructed in this way. By definition,

$$\mathcal{G}(s_2) \subset \mathcal{P}^{(n,N)} = \{\eta \cap W_* : \eta \in \mathcal{P}^{(n)} \text{ and } \hat{f}^i(\eta) \subset \cup_{l\leq N} E_l \text{ for all } 0 \leq i \leq n\}.$$

Since $\mathcal{P}^{(n,N)}$ is a finite set of nonempty intervals contained in W_* which depends only on n and N, we may find $C(n,N) > 0$ large enough so that

$$\#\mathcal{G}(s_2) \leq C(n,N) \qquad \text{and} \qquad \frac{1}{m_0(\hat{\gamma})} \leq C(n,N)$$

for all $\hat{\gamma} \in \mathcal{G}(s_2)$. Then, the same calculations as we used to deduce (5.21), give

$$s_2 \leq C(\delta)\sigma^{-n}\Big(\sum_{\hat{\gamma}\in\mathcal{G}(s_2)} \operatorname{var}_{\hat{\gamma}} \varphi\Big) + C(\delta)C(n,N)\int |\varphi|\, dm_0\,. \tag{5.23}$$

Step 6: We bound the last term s_3 in S_2, see (5.24). The reasonings are the same as in Step 5. Combined with the previous estimates, this gives the preliminary bounds for the variation of $\chi_A \mathcal{L}_0^n \varphi$ in (5.25), (5.26).

Using similar arguments and objects $\hat{\gamma} \in \mathcal{G}(s_3)$ for those intervals $\gamma \in \mathcal{G}(j)$ with $k(\gamma) < n - j - 1$, we find

$$s_3 \le \sum_{\hat{\gamma} \in \mathcal{G}(s_3)} C(\delta) \lambda^{-k} \rho^{-k} \sigma^{-l} \operatorname*{var}_{\hat{\gamma}}(\mathcal{L}_0^{n-l} \varphi) + C(\delta) C(n) \int |\varphi| dm_0 , \tag{5.24}$$

where $j = j(\gamma)$, $k = k(\gamma)$, and $l = j + k + 1 < n$. Note that we used once more the fact that $\mathcal{L}_0$ does not increase the L^1-norm, cf. (5.8).

Replacing (5.21) – (5.24) in (5.19) we conclude that

$$\begin{aligned} \operatorname*{var}_A(\mathcal{L}_0^n \varphi) \le C(\delta) \sigma^{-n} & \sum_{\hat{\gamma} \in \mathcal{G}(S_1) \cup \mathcal{G}(s_2)} \operatorname*{var}_{\hat{\gamma}} \varphi + \sum_{\hat{\gamma} \in \mathcal{G}(s_3)} C(\delta) \sigma^{-l} \operatorname*{var}_{\hat{\gamma}}(\mathcal{L}_0^{n-l} \varphi) \\ & + C(\delta) \left(2^n \rho^{-N}\right) \sigma^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C(\delta) C(n, N) \int |\varphi| \, dm_0 , \end{aligned} \tag{5.25}$$

for general $A \subset E_0$, and

$$\begin{aligned} \operatorname*{var}_A(\mathcal{L}_0^n \varphi) \le C \sigma^{-n} & \sum_{\hat{\gamma} \in \mathcal{G}(S_1) \cup \mathcal{G}(s_2)} \operatorname*{var}_{\hat{\gamma}} \varphi + \sum_{\hat{\gamma} \in \mathcal{G}(s_3)} 1 \cdot \sigma^{-l} \operatorname*{var}_{\hat{\gamma}}(\mathcal{L}_0^{n-l} \varphi) \\ & + C \left(2^n \rho^{-N}\right) \sigma^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C(n, N) \int |\varphi| \, dm_0 , \end{aligned} \tag{5.26}$$

if $A \subset (-\delta, \delta) \times \{0\}$. The factor in the second term of (5.26) is important for the last part of the argument. Note that the calculations we have just presented yield a factor $C \lambda^{-k} \rho^{-k}$, which one may replace by 1 as we did, since $k \ge H(\delta) \gg 1$.

Step 7: If the sum over $\hat{\gamma} \in \mathcal{G}(s_3)$ in (5.25) is not void, we proceed by recurrence: we apply the previous estimates to each such $\hat{\gamma}$ in the place of A.

A few notational arrangements are necessary at this point. We change the name of the $\mathcal{G}(\cdot)$, of the $\hat{\gamma}$, and of their indices k and j, to $\mathcal{G}_1$, $\hat{\gamma}_1$, k_1, and j_1, respectively. Corresponding objects appearing at the ith step (for each $\hat{\gamma}_{i-1}$) will be denoted $\mathcal{G}_i$, $\hat{\gamma}_i$, k_i, and j_i, and we also let $l_i = j_i + 1 + k_i$. By construction, every of these $\hat{\gamma}_i$ is a subset of $(-\delta, \delta) \times \{0\}$, and so we may apply (5.26) to it (rather than (5.25)). After one recurrence step,

$$\begin{aligned} \operatorname*{var}_A \mathcal{L}_0^n \varphi \le C(\delta) \sigma^{-n} \Big(\sum_{\hat{\gamma}_1} \operatorname*{var}_{\hat{\gamma}_1} \varphi + C \sum_{\hat{\gamma}_1, \hat{\gamma}_2} \operatorname*{var}_{\hat{\gamma}_2} \varphi\Big) + \sum_{\hat{\gamma}_1, \hat{\gamma}_2} C(\delta) \sigma^{-l_1 - l_2} \operatorname*{var}_{\hat{\gamma}_2}(\mathcal{L}_0^{n - l_1 - l_2} \varphi) \\ + C(\delta)(1 + \# \mathcal{G}_1(s_3)) \left[(2^n \rho^{-N}) \sigma^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C(n, N) \int |\varphi| \, dm_0 \right] , \end{aligned}$$

the sums running over $\hat{\gamma}_1 \in \mathcal{G}_1(S_1) \cup \mathcal{G}_1(s_2)$, over $\hat{\gamma}_1 \in \mathcal{G}_1(s_3)$, $\hat{\gamma}_2 \in \mathcal{G}_2(S_1) \cup \mathcal{G}_2(s_2)$, and over $\hat{\gamma}_1 \in \mathcal{G}_1(s_3)$, $\hat{\gamma}_2 \in \mathcal{G}_2(s_3)$, respectively. By construction, each $\hat{\gamma}_1 \in \mathcal{G}_1(s_3)$ is contained in some monotonicity interval $\eta_1 \in \mathcal{P}^{(l_1)}$ such that $\hat{f}^i(\eta_1) \subset \cup_{k \le k_1} E_k$ for all $0 \le i \le l_1$. Since the correspondence $\hat{\gamma}_1 \mapsto \eta_1$ is one-to-one, and $k_1 < l_1 \le n$, we conclude that $\#\mathcal{G}(s_3) \le C(l_1, k_1) \le C(n)$ for large enough $C(l_1, k_1)$ and $C(n)$. In fact, this same argument shows that $\#\mathcal{G}_j(s_3) \le C(n)$ for all $1 \le j \le n$. Hence, after at most n steps,

$$\begin{aligned} \operatorname*{var}_A \mathcal{L}_0^n \varphi \le C(\delta)\sigma^{-n} \sum_{i=1}^{n} \Big(C \sum_{\hat{\gamma}_1, \dots, \hat{\gamma}_{i-1}, \hat{\gamma}_i} \operatorname*{var}_{\hat{\gamma}_i} \varphi \Big) \\ + C(\delta)\, C(n) \Big((2^n \rho^{-N}) \sigma^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C(n, N) \int |\varphi| \, dm_0 \Big), \end{aligned} \tag{5.27}$$

the second sum being over $\hat{\gamma}_1 \in \mathcal{G}_1(s_3), \dots, \hat{\gamma}_{i-1} \in \mathcal{G}_{i-1}(s_3)$, $\hat{\gamma}_i \in \mathcal{G}_i(S_1) \cup \mathcal{G}_i(s_2)$. Observe that the intervals $\hat{\gamma}_i$ occurring in (5.27) are all contained in distinct atoms of the partition $\mathcal{P}^{(n)}$, and so they are two-by-two disjoint. Therefore, the variations of φ over such intervals add up, so that the first term on the right hand side of (5.27) is bounded by $C(\delta)\sigma^{-n} \operatorname{var} \varphi$. Now we fix $N \gg n$ in such a way that

$$C(n) \cdot 2^n \rho^{-N} \le 1,$$

and then the second term is also bounded by $C(\delta)\sigma^{-n}(\operatorname{var} \varphi + \sup |\varphi|)$. Moreover, once we have chosen N in this way, depending only on n, we may replace $C(n, N)$ by $C(n)$ in the last term. Finally, since δ is also fixed at this point, we may omit the reference to δ, replacing $C(\delta)$ by C in (5.27). □

Proposition 5.6. *Given any $\bar{\sigma} \in (1, \sigma)$ there is $C > 0$ such that*

(1) $\operatorname{var}(\mathcal{L}_0^n \varphi) \le C \bar{\sigma}^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C \int |\varphi| \, dm_0$;

(2) $\sup(\mathcal{L}_0^n \varphi) \le C \bar{\sigma}^{-n} (\operatorname{var} \varphi + \sup |\varphi|) + C \int |\varphi| \, dm_0$

for any function $\varphi \in BV(\hat{I})$ and any $n \ge 1$.

Proof: Let $1 < \bar{\sigma} < \tilde{\sigma} < \sigma$. First, we fix $n = n_0$ and decompose

$$\operatorname{var} \mathcal{L}_0^{n_0} \varphi = \sum_{k=0}^{\infty} \operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi.$$

For $k > n_0$ and $(x, k) \in W_k \times \{k\}$ there exists a unique $(y, k - n_0) \in E_{k-n_0}$ such that $\hat{f}^{n_0}(y, k - n_0) = (x, k)$. Then, by Lemma 5.4(3),

$$\mathcal{L}_0^{n_0} \varphi(x, k) = g^{(n_0)}(y, k - n_0) \varphi(y, k - n_0) = \lambda^{-n_0} \varphi(y, k - n_0),$$

and so

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0} \varphi = \operatorname*{var}_{W_k \times \{k\}} \mathcal{L}_0^{n_0} \varphi \le \lambda^{-n_0} \operatorname*{var}_{E_{k-n_0}} \varphi.$$

If $k \le n_0$ then the same argument gives

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0}\varphi \le 2\lambda^{-k} \operatorname*{var}_{E_0} \mathcal{L}_0^{n_0-k}\varphi$$

(the factor 2 accounts for the fact that f is 2-to-1 on E_0). Combining with Lemma 5.5, for $A = E_0$ and $n = n_0 - k < n_0$,

$$\operatorname*{var}_{E_k} \mathcal{L}_0^{n_0}\varphi \le 2\lambda^{-k}\big[C\sigma^{-(n_0-k)}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int |\varphi|\, dm_0\big] .$$

Recall that we are taking $\tilde{\sigma} < \sigma$ and that we have chosen $\sigma \le \lambda$. Thus, adding the previous estimates over all $k \ge 0$,

$$\begin{aligned}\operatorname{var} \mathcal{L}_0^{n_0}\varphi &\le \lambda^{-n_0} \operatorname{var}\varphi + n_0 C\sigma^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + n_0 C(n_0)\int |\varphi|\, dm_0 \\ &\le C\tilde{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int |\varphi| dm_0 ,\end{aligned} \tag{5.28}$$

as long as the constants C and $C(n_0)$ in the last term be fixed large enough with respect to the ones in the second term.

In order to prove part (1) of the proposition one must now remove the dependence on n_0 of the factor in the last term in (5.28). We start by proving a similar inequality for the supremum:

$$\sup|\mathcal{L}_0^{n_0}\varphi| \le C\tilde{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup|\varphi|) + C(n_0)\int|\varphi| dm_0 . \tag{5.29}$$

In doing this it is convenient to consider separately the suprema over the the upper and over the lower part of the tower. On the one hand, we note that

$$\sup_{\cup_{k>n_0} E_k} |\mathcal{L}_0^{n_0}\varphi| = \sup_{k>n_0} \big(\sup_{W_k\times\{k\}} |\mathcal{L}_0^{n_0}\varphi|\big) \le \sup_{k>n_0} \big(\lambda^{-n_0} \sup_{E_{k-n_0}} |\varphi|\big) \le \lambda^{-n_0} \sup|\varphi|,$$

as a direct consequence of Lemma 5.4(3) applied n_0 times. Since $\sigma \le \lambda$, this implies (5.29) when the supremum of $|\mathcal{L}_0^{n_0}\varphi|$ is attained over the union of the E_k with $k > n_0$. From now on we suppose otherwise, that is, we suppose that $\sup|\mathcal{L}_0^{n_0}\varphi|$ is attained on some level E_k with $k \le n_0$. Using the mean value theorem,

$$\sup|\mathcal{L}_0^{n_0}\varphi| = \sup_{E_k}|\mathcal{L}_0^{n_0}\varphi| \le \operatorname*{var}_{E_k} \mathcal{L}_0^{n_0}\varphi + \frac{1}{m_0(E_k)}\int_{E_k} |\mathcal{L}_0^{n_0}\varphi|\, dm_0.$$

Clearly, $m_0(E_l) > 0$ for every $l \ge 0$: it suffices to note that each W_l contains a neighbourhood of c_l. As a consequence, $\sup_{l\le n_0} 1/m_0(E_l)$ is less than some constant $C(n_0) > 0$ depending only on n_0. Combining this with (5.8), we obtain

$$\sup|\mathcal{L}_0^{n_0}\varphi| \le \operatorname{var} \mathcal{L}_0^{n_0}\varphi + C(n_0)\int |\varphi|\, dm_0 ,$$

so that now (5.29) follows from (5.28).

At this point we choose an integer q large enough so that $2C\bar{\sigma}^{-q} < \bar{\sigma}^{-q} < 1/2$, where C is the constant multiplying $\bar{\sigma}^{-n_0}(\operatorname{var}\varphi + \sup\varphi)$ in (5.28), (5.29). Then, given any $n \geq 1$, we write $n = pq + r$ with $0 \leq r < q$. Using the inequalities above p times with $n_0 = q$, and once more with $n_0 = r$,

$$\begin{aligned}\operatorname{var}(\mathcal{L}_0^n\varphi) &\leq \frac{1}{2}\bar{\sigma}^{-q}\big(\operatorname{var}\mathcal{L}_0^{n-q}\varphi + \sup|\mathcal{L}_0^{n-q}\varphi|\big) + C(q)\int|\varphi|\,dm_0\\ &\leq \frac{1}{2}\bar{\sigma}^{-pq}\big(\operatorname{var}\mathcal{L}_0^r\varphi + \sup|\mathcal{L}_0^r\varphi|\big) + C(q)\big(1+2^{-1}+\cdots+2^{-p+1}\big)\int|\varphi|\,dm_0\\ &\leq C\bar{\sigma}^{-pq-r}\big(\operatorname{var}\varphi + \sup|\varphi|\big) + \big(2C(q)+C(r)2^{-pq}\big)\int|\varphi|\,dm_0.\end{aligned}$$

Finally, as $r < q$ and q has already been fixed, we may bound $2C(q) + C(r)2^{-pq}$ by some constant C independent of n. □

For constructing absolutely continuous invariant probability measures for the maps f and $\hat{f}$ we also need

Lemma 5.7. *$m_0 = w_0 m$ is a finite measure on $\hat{I}$.*

Proof: Of course, $m_0(E_0) = m(E_0) = m(I)$ is finite. Moreover, for each $k \geq 1$,

$$m_0(E_k) = \int_{B_k} w_0(x,k)\,dm(x) = \int_{W_k}\frac{\lambda^k}{|(f^k)'(z)|}\,dm(x),$$

where $z \in (0,\delta)$ is uniquely defined by $\hat{f}^k(z,0) = (x,k)$. We change variables $x = f^k(z)$, and then we get

$$m_0(E_k) = \int_{Y_k}\lambda^k\,dm(z) = \lambda^k\,m(Y_k),$$

where $Y_k = \{z \in (0,\delta) : f^k(z) \in W_k\}$. Next, we observe that

$$2e^{-\beta k} \geq m(B_k) \geq m(W_k) \geq \frac{1}{C}|(f^{k-1})'(c_1)|\,m(f(Y_k)) \geq \frac{1}{C}\lambda_c^{k-1}\,m(Y_k)^2,$$

where the third inequality is a consequence of (5.16) and the mean value theorem. Replacing above, and recalling that we have chosen $\sqrt{\lambda_c} > \lambda\rho$ and $\beta > 0$,

$$m_0(E_k) \leq \sqrt{2C\lambda_c}\lambda^k\lambda_c^{-k/2}e^{-\beta k/2} \leq \sqrt{2C\lambda_c}\,\rho^{-k} \tag{5.30}$$

for every $k \geq 1$. Since $\rho > 1$, the claim follows immediately. □

Proposition 5.8.

(1) *$\hat{f}$ has some invariant probability measure $\hat{\mu}_0$ that is absolutely continuous with respect to m_0;*

(2) *f has a unique invariant probability measure μ_0 absolutely continuous with respect to Lebesgue measure m on I, and μ_0 is ergodic.*

Proof: Part (1) is quite similar to Corollary 3.4. Recall from Lemma 5.7 that $m_0(\hat{I})$ is finite. Proposition 5.6 implies that the sequence $\varphi_n = n^{-1}\sum_{j=0}^{n-1}\mathcal{L}_0^j(1/m_0(\hat{I}))$ is uniformly bounded and has uniformly bounded variation. By Helly's theorem (Lemma 3.3) there exists a subsequence φ_{n_k} converging in $L^1(m_0)$ to some function φ_0 in $\mathrm{BV}(\hat{I})$. The operator $\mathcal{L}_0$ being continuous with respect to the norm of $L^1(m_0)$, recall (5.8), it follows that φ_0 is a fixed point of $\mathcal{L}_0$. Then, using (5.4), the absolutely continuous measure $\hat{\mu}_0 = \varphi_0 m_0$ is $\hat{f}$-invariant, and

$$\int \varphi_n \, dm_0 = \frac{1}{n}\sum_{j=0}^{n-1}\int \mathcal{L}_0^j(1/m_0(\hat{I}))\, dm_0 = \frac{1}{n}\sum_{j=0}^{n-1}\int (1/m_0(\hat{I}))\, dm_0 = 1$$

for every $n \geq 1$. Therefore, $\int \varphi_0 \, dm_0 = 1$ and so $\hat{\mu}_0$ is a probability measure.

To prove (2), we take $\mu_0 = \pi_* \hat{\mu}_0$, where $\pi : \hat{I} \to I$ continues to denote the projection $\pi(x,k) = x$. In other words, for each Borel subset A of I,

$$\mu_0(A) = \hat{\mu}_0(\pi^{-1}(A)) = \sum_{k=0}^{\infty} \hat{\mu}_0\big((A \cap B_k) \times \{k\}\big).$$

Then μ_0 is a probability, and the relation $\pi \circ \hat{f} = f \circ \pi$ ensures that μ_0 is f-invariant:

$$\mu_0(f^{-1}(A)) = \hat{\mu}_0(\pi^{-1}(f^{-1}(A))) = \hat{\mu}_0(\hat{f}^{-1}(\pi^{-1}(A))) = \hat{\mu}_0(\pi^{-1}(A)) = \mu_0(A).$$

Moreover, μ_0 is easily seen to be absolutely continuous with respect to Lebesgue measure. In fact, if A has zero Lebesgue measure then the same is true for every $A \cap B_k$, for every $k \geq 0$. This implies $m_0((A \cap B_k) \times \{k\}) = 0$ for all $k \geq 0$, and so $\mu_0(A) = 0$.

A main ingredient to prove that μ_0 is unique and ergodic is the result of [BL89] asserting that *any unimodal map with negative schwarzian derivative, nondegenerate critical points, and no periodic attractors is ergodic with respect to Lebesgue measure: if $A \subset I$ satisfies $f^{-1}(A) = A$ then either $m(A) = 0$ or $m(A^c) = 0$.* Then such an A must have $\mu_0(A) = 0$ or $\mu_0(A^c) = 0$, which proves ergodicity of μ_0. Now, we claim that the measure μ_0 is equivalent to Lebesgue measure m on the interval $I_* = [f^2(c), f(c)]$. This can be seen as follows. Since φ_0 has bounded variation, and $\int \varphi_0 \, dm_0 = 1$, there is some interval $\gamma \subset W_*$ such that $\inf_\gamma \varphi_0 > 0$. Then the density of $\hat{\mu}_0$ with respect to the usual length is bounded away from zero on γ. As a consequence, $\inf_{\pi(\gamma)}(d\mu_0/dm) > 0$. On the other hand, the assumption of topological mixing (U3) ensures that $f^N(\pi(\gamma)) = I_*$ for some $N \geq 1$. It follows that

$$\inf_{I_*}\left(\frac{d\mu_0}{dm}\right) \geq \inf_{\pi(\gamma)}\left(\frac{d\mu_0}{dm}\right) \cdot \frac{1}{\sup |(f^N)'|} > 0, \tag{5.31}$$

which implies our claim. Finally, let ν be any f-invariant probability measure which is absolutely continuous with respect to Lebesgue measure. It is easy to see that the support of ν must be contained in I_*, and so ν is absolutely continuous with respect to μ_0. Since μ_0 is ergodic, it follows that $\nu = \mu_0$ (because ergodic measures are minimal for the relation of absolute continuity), proving uniqueness. □

Closing this section, we prove that the support of φ_0 contains

$$W_\delta = W_* \setminus \left([f^2(c), f^2(\delta)) \cup (f(\delta), f(c)]\right) \times \{0\}.$$

Then $\operatorname{supp}\varphi_0$ must coincide with W_δ: $\varphi_0 = \mathcal{L}_0^n\varphi_0$ implies that φ_0 is identically zero on $\hat{I} \setminus \hat{f}^n(\hat{I})$, for every $n \geq 1$, and we have seen in Remark 5.1 that $\cap_{n\geq 1}\hat{f}^n(\hat{I})$ is contained in W_δ.

Lemma 5.9.

(1) $\inf\left(\varphi_0|[f^2(\delta), f(\delta)] \times \{0\}\right) > 0$

(2) $\inf\left(\varphi_0|W_k\right) > 0$ *for every* $k \geq 1$.

Proof: Let $\gamma_1 \subset W_*$ be some open interval such that $\inf_{\gamma_1}\varphi_0 > 0$. By the topological mixing assumption (U3), there exists some $n_1 \geq 0$ such that

$$\pi(\hat{f}^{n_1}(\gamma_1)) = f^{n_1}(\pi(\gamma_1)) = I_*\,.$$

In particular, $\pi(\hat{f}^{n_1}(\gamma_1))$ contains the fixed point $p = (-1 + \sqrt{1+4a})/2 > 0$ of f. Moreover, up to slightly modifying β if necessary, we may suppose that no endpoint $(c_k \pm e^{-\beta k}) \times \{k\}$ of a level E_k, $k \geq 1$, projects down to p. Then there exists some open interval $\gamma_2 \subset \hat{f}^{n_1}(\gamma_1)$ such that $\pi(\gamma_2)$ contains p. Clearly, $\pi(\hat{f}^n(\gamma_2))$ must contain p for every $n \geq 1$. Now we suppose that $p \neq c_k$ for every $k \geq 1$ (if this happens to be false, we simply replace p by any other periodic orbit of f whose orbit does not intersect $(-\delta, \delta)$, and the argument proceeds along the same lines). Then, there exists some finite time $n_2 \geq 0$ at which the point $\xi \in \gamma_2$ satisfying $\pi(\xi) = p$ falls down to E_0: $\hat{f}^{n_2}(\xi) = (p, 0)$. Up to another arbitrarily small modification of β, we may suppose that the orbit of ξ does not pass through any of the boundary points of the monotonicity intervals in $\mathcal{P}^{(1)}$. Then $\hat{f}^{n_2}(\xi)$ contains some open neighbourhood γ_3 of $(p, 0)$ in E_0. Let $n_3 \geq 0$ be minimum such that $f^{n_3}(\pi(\gamma_3))$ intersects $(-\delta, \delta)$. Then $\hat{f}^{n_3}(\gamma_3) = f^{n_3}(\pi(\gamma_3)) \times \{0\}$ contains $[\delta, p] \times \{0\}$. Let us denote

$$\sigma_1 = \hat{f}([\delta, p] \times \{0\}) = [p, f(\delta)] \times \{0\} \quad \text{and} \quad \sigma_1 = \hat{f}^2([\delta, p] \times \{0\}) = [f^2(\delta), p] \times \{0\}.$$

Then $\sigma_1 \cup \sigma_2 = [f^2(\delta), f(\delta)] \times \{0\}$. We use the following property

$$\inf_{\gamma}\varphi_0 > 0 \Rightarrow \inf_{\hat{f}(\gamma)}\varphi_0 > 0,$$

which is a direct consequence of the fact that φ_0 is a fixed point for the transfer operator of $\hat{f}$. Since $\sigma_i \subset \hat{f}^{n+i}(\gamma_1)$, with $n = n_1 + n_2 + n_3$, we get that $\inf_{\sigma_i} \varphi_0 > 0$ for $i = 1, 2$. Part (1) of the lemma follows immediately.

Moreover, given $(y,k) \in W_k$, $k \geq 1$, and $z \in (0,\delta)$ such that $f^k(z) = y$,

$$(5.32)\quad \varphi_0(y,k) = (\mathcal{L}_0^k \varphi_0)(y,k) = \frac{\varphi_0(z,0)}{\lambda^k} + \frac{\varphi_0(-z,0)}{\lambda^k} \geq \frac{2}{\lambda^k} \inf \left(\varphi_0 | [f^2(\delta), f(\delta)] \times \{0\}\right).$$

This proves part (2). □

Remark 5.3. This last relation also yields another useful conclusion:

$$\varphi_0(y,k) \leq \frac{2}{\lambda^k} \sup \left(\varphi_0 | [f^2(\delta), f(\delta)] \times \{0\}\right) \leq \frac{2}{\lambda^k} \sup \varphi_0 \,,$$

and so

$$(5.33)\qquad \sum_{k=0}^{\infty} \sup(\varphi_0 | E_k) \leq \sum_{k=0}^{\infty} 2\lambda^{-k} \sup \varphi_0 < \infty.$$

5.4. Quasi-compacity and decay of correlations.

In this section we prove that the measures $\hat{\mu}_0$ and μ_0 we have just constructed are exact, and so also mixing, for the corresponding dynamical systems $\hat{f}$ and f (Proposition 5.13). As a consequence, the transfer operator $\mathcal{L}_0$ is quasi-compact and both systems $(\hat{f}, \hat{\mu}_0)$ and (f, μ_0) have exponential decay of correlations in corresponding spaces of functions with bounded variation. Proposition 5.13 also provides another proof of the ergodicity of μ_0 (besides implying that $\hat{\mu}_0$ is also ergodic). For the proof we need a few preparatory lemmas.

Lemma 5.10. *Given $\varepsilon > 0$ there exists $N \geq 0$, and for each $n \geq 1$ there exists a subset $\mathcal{Q}(n,N)$ of $\mathcal{P}^{(n+N)}$, such that*

(1) *$f^n(\eta) \in \mathcal{P}^{(N)}$ and $f^n(\eta) \subset \bigcup_{k=0}^{N} E_k$ for every $\eta \in \mathcal{Q}(n,N)$;*
(2) *the $\hat{\mu}_0$-measure of the union of the intervals $\eta \notin \mathcal{Q}(n,N)$ is at most ε.*

Proof: Let us denote ∂_k, $k \geq 0$, the set of boundary points of the elements of the partition $\mathcal{P}^{(1)}$ contained in E_k. That is,

$$\partial_0 = \{q, -\delta, 0, \delta, -q\} = \partial D_0^- \cup \partial U_0^- \cup \partial U_0^+ \cup \partial D_0^+$$

and, for each $k \geq 1$,

$$\partial_k = \partial D_k^- \cup \partial U_k \cup \partial D_k^+.$$

Observe that each ∂_k, $k \geq 1$, contains at most 4 points. For $n \geq 1$, $N \geq 1$, and $\eta \in \mathcal{P}^{(N+n)}$, let $(k(i))_i$, be the sequence given by

$$\hat{f}^i(\eta) \subset E_{k(i)}, \quad \text{for each } i \geq 0.$$

Let $\tau > 0$ be fixed in the following way: for what concerns the present lemma τ is arbitrary, but for the proof of Lemma 5.11 it is convenient to choose $\tau = \log(\lambda\rho)/\log 8$. Then define $\mathcal{Q}(n, N)$ to be the subset of intervals $\eta \in \mathcal{P}^{(N+n)}$ such that

(i) $k(i) \le N + (n-i)\tau$ for $0 \le i \le n$.

(ii) $\hat{f}^i(\partial\eta)$ is disjoint from $\partial_{k(i)}$ for every $0 \le i < n$;

Condition (ii) is an analog of (3.7): it implies that $\hat{f}^n(\eta)$ belongs in $\mathcal{P}^{(N)}$. The case $i = n$ in condition (i) means that $\hat{f}^n(\eta) \subset E_k$ for some $k \le N$. Thus, property (1) in the statement is satisfied by every element η of $\mathcal{Q}(n, N)$. Now we only have to show that the total measure of those intervals η for which either of (i) or (ii) fails can be made arbitrarily small by increasing N.

Let $0 \le i \le n$ be fixed. Then (i) fails for a given $\eta \in \mathcal{P}^{(N+n)}$ if and only if

$$\eta \subset \hat{f}^{-i}\Big(\bigcup_{k>N+(n-i)\tau} E_k\Big).$$

We have shown in Lemma 5.7 that the m_0-measure of the tower levels E_k decreases exponentially fast with k, recall (5.30). Then the same is true for the $\hat{\mu}_0$-measure, since $\hat{\mu}_0 = \varphi_0 m_0$ and φ_0 is a bounded function. So

$$\hat{\mu}_0\Big(\bigcup_{k>N+(n-i)\tau} E_k\Big) \le K_1 \rho^{-N-(n-i)\tau},$$

for some $K_1 > 0$. Since $\hat{\mu}_0$ is $\hat{f}$-invariant, it follows that the $\hat{\mu}_0$-measure of the union of all the $\eta \in \mathcal{P}^{(N+n)}$ such that (i) fails is also bounded by $K_1\rho^{-N-(n-i)\tau}$.

Keeping $0 \le i \le n$ fixed, let us also estimate the total measure of the elements of $\mathcal{P}^{(N+n)}$ that satisfy (i) but not (ii). Let $\zeta = \zeta_i$ be the element of $\mathcal{P}^{(N+n-i)}$ containing $\hat{f}^i(\eta)$. Since (ii) breaks down, some boundary point of ζ must be in $\partial_{k(i)}$. On the other hand, in view of (i), there are at most $8 + 6\big(N + (n-i)\tau\big)$ such intervals ζ: 8 inside E_0 and not more than 6 inside each E_k, $1 \le k \le N + (n-i)\tau$. Moreover, the m_0-measure of each one of them is bounded by $\lambda^{-N-(n-i)}$, because the jacobian of $\hat{f}^{N+(n-i)}$ with respect to m_0 is everywhere larger than $\lambda^{N+(n-i)}$, cf. Remark 5.2. Using once more the fact that φ_0 is bounded, we conclude that the $\hat{\mu}_0$-measure of the union of these intervals $\zeta \in \mathcal{P}^{(N+n-i)}$ is at most

$$K_2\big(8 + 6\big(N + (n-i)\tau\big)\big)\lambda^{-N-(n-i)} \le 14K_2(N+n-i)\lambda^{-N-(n-i)}$$

for some $K_2 > 0$. Then, because $\hat{\mu}_0$ is $\hat{f}$-invariant, the same bound applies to the $\hat{\mu}_0$ measure of the union of all the intervals $\eta \in \mathcal{P}^{(N+n)}$ for which (i) holds but (ii) fails.

We conclude that the total measure of the monotonicity intervals η of $\hat{f}^{N+n}$ for which either (i) or (ii) fails for some $0 \le i \le n$ is bounded by

$$\sum_{i=0}^{n} \left(K_1 \rho^{-N-(n-i)\tau} + 14K_2(N+n-i)\lambda^{-N-(n-i)}\right) \le$$
$$\le \rho^{-N}\Big(\sum_{j=0}^{n} K_1 \rho^{-j\tau}\Big) + \lambda^{-N/2}\Big(\sum_{j=0}^{n} 14K_2(N+j)\lambda^{-(N+j)/2}\Big)$$
$$\le K_3\rho^{-N} + K_3\lambda^{-N/2},$$

for some $K_3 > 0$. This can be made smaller than ε by choosing N sufficiently large, and so the proof is complete. □

Lemma 5.11. *Given $N \ge 1$ and $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ such that for any $n \ge 1$, any interval $\eta \in \mathcal{Q}(n, N)$, and any Borel subset $\xi \subset \eta$*

$$\frac{m(\xi)}{m(\eta)} \le \varepsilon_1 \quad \Rightarrow \quad m(\hat{f}^n(\xi)) \le \varepsilon_2 .$$

Proof: Most of the proof is based on the same ideas as Lemmas 5.2 and 5.3. The main new ingredient is to use condition (i) $k(i) \le N + (n-i)\tau$ in the definition of $\mathcal{Q}(n, N)$, cf. proof of Lemma 5.10, taking τ sufficiently small, e.g., $\tau = \log(\lambda\rho)/\log 8$.

Suppose first that $\eta \subset E_0$ and $\hat{f}^n(\eta) \subset E_0$. In this case we prove that $\hat{f}^n$ has uniformly bounded distortion on η (depending on N, but not on n nor on η). Let us consider the sequence of iterates $0 \le \nu_1 < \nu_1 + p_1 < \nu_2 < \cdots < \nu_s < \nu_s + p_s < n$ defined by

(a) $\hat{f}^j(\eta) \subset E_0$ for $0 \le j \le \nu_1$, for $\nu_i + p_i < j \le \nu_{i+1}$ and $1 \le i \le s-1$, and for $\nu_s + p_s < j \le n$

(b) $\hat{f}^j(\eta) \subset E_{\nu_i - j}$ for $\nu_i \le j \le \nu_i + p_i$ and $1 \le i \le s$;

Let $\gamma = \pi(\eta) \subset I$ and $x, y \in \gamma$. First we consider $0 \le j < \nu_1$. Using $f^j(\gamma) \cap (-\delta, \delta) = \emptyset$, and Lemma 5.2 together with the mean value theorem,

$$\sum_{j=0}^{\nu_1 - 1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \le \sum_{j=0}^{\nu_1 - 1} \frac{1}{\delta}|f^j(\gamma)| \le$$
$$\le \sum_{j=0}^{\nu_1 - 1} \operatorname{const} \sigma^{j-\nu_1} |f^{\nu_1}(\gamma)| \le \operatorname{const} |f^{\nu_1}(\gamma)|. \tag{5.34}$$

For the same reasons,

$$\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}-1} \big| \log|f'(f^j(x))| - \log|f'(f^j(y))| \big| \le \operatorname{const} |f^{\nu_{i+1}}(\gamma)| \tag{5.35}$$

for every $1 \le i \le s-1$, and

$$\sum_{j=\nu_s+p_s+1}^{n-1} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| \le \text{const}\,|f^n(\gamma)|. \tag{5.36}$$

Now let $j = \nu_i$, and denote $\Delta_i = d(f^{\nu_i}(\gamma), c)$. Then

$$\big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}$$

Next we consider $\nu_i < j \le \nu_i + p_i$. Lemma 5.3(1) implies

$$\frac{|f^{\nu_i+1}(\gamma)|}{\Delta_i^2} \le \text{const}\,\frac{|D_{p_i}^{\pm}|}{|U_{p_i}|} \le \text{const}\,\frac{e^{-\beta p_i}}{e^{-\beta(p_i+1)/4}} \le \text{const}\,.$$

As a consequence,

$$|f^{\nu_i}(\gamma)| \le \text{const}\,\Delta_i\,, \qquad \text{which implies} \qquad \frac{|f^{\nu_i+1}(\gamma)|}{\Delta_i^2} \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}\,.$$

Using Lemma 5.3(1) again

$$\frac{|f^j(\gamma)|}{e^{-\beta(j-\nu_i)}} \le \text{const}\,\frac{|f^{\nu_i+1}(\gamma)|}{\Delta_i^2} \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}$$

On the other hand, by definition the of B_k, E_k, and assumption (U2), $f^j(\gamma)$ does not intersect $(-\gamma e^{-\alpha(j-\nu_i)}, \gamma e^{-\alpha(j-\nu_i)})$, where $\gamma = 1 - e^{\alpha-\beta}$. It follows that

$$\begin{aligned}\sum_{j=\nu_i+1}^{\nu_i+p_i} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| &\le \sum_{j=\nu_i+1}^{\nu_i+p_i} \text{const}\,e^{\alpha(j-\nu_i)}|f^j(\gamma)| \\ &\le \sum_{j=\nu_i+1}^{\nu_i+p_i} \text{const}\,e^{(\alpha-\beta)(j-\nu_i)}\frac{|f^{\nu_i}(\gamma)|}{\Delta_i} \le \text{const}\,\frac{|f^{\nu_i}(\gamma)|}{\Delta_i}.\end{aligned} \tag{5.37}$$

Putting (5.34), (5.35), (5.36). (5.37) together, and noting that $\Delta_i \le 1$, we find

$$\sum_{j=0}^{n-1} \big|\log|f'(f^j(x))| - \log|f'(f^j(y))|\big| \le \text{const}\sum_{i=1}^{s}\frac{|f^{\nu_i}(\gamma)|}{\Delta_i} + \text{const}\,|f^n(\gamma)| \tag{5.38}$$

Of course, $|f^n(\gamma)| \le \text{const}$. Lemmas 2 and 5.3(2) imply

$$|f^{\nu_i}(\gamma)| \le \text{const}\,(\lambda\rho)^{\nu_i-n}|f^n(\gamma)| \le \text{const}\,(\lambda\rho)^{\nu_i-n}$$

for each $1 \le i \le s$, and (5.15) gives

$$\Delta_i^2 \ge \text{const}\, e^{-\beta(p_i+1)} |(f^{p_i})'(c_1)|^{-1} \ge \text{const}\, e^{-2\beta p_i} 4^{-p_i}.$$

Now, condition (i) in the definition of $\mathcal{Q}(n, N)$ implies

$$p_i = k(\nu_i + p_i) \le N + (n - \nu_i - p_i)\tau \le N + (n - \nu_i)\tau$$

and so, recall that $\tau = \log(\lambda\rho)/\log 8$ and $e^\beta \le \sqrt{\lambda_c} \le 2$

$$\sum_{i=1}^{s} \frac{|f^{\nu_i}(\gamma)|}{\Delta_i} \le \sum_{i=1}^{s} (\lambda\rho)^{\nu_i - n} (2e^\beta)^{p_i} \le \sum_{i=1}^{s} 4^N (\lambda\rho 4^{-\tau})^{\nu_i - n}$$
$$\le 4^N \sum_{i=1}^{s} 2^{(\nu_i - n)\tau} \le \text{const}\, 4^N.$$

Replacing in (5.38), we conclude that f^n has bounded distortion on γ

$$\sum_{j=0}^{n-1} \big| \log |f'(f^j(x))| - \log |f'(f^j(y))| \big| \le \text{const}\, 4^N. \tag{5.39}$$

In equivalent terms, $\hat{f}^n$ has bounded distortion on η, as we had claimed. In particular, in this case we may take $\varepsilon_1 = \big(\varepsilon_2/m(I)\big)\exp(-K_1)$, where $K_1 > 0$ denotes the right hand term in (5.39).

Now the remaining cases can be treated easily. If η is not contained in E_0 then we define $p_0 + 1 \ge 1$ to be the first iterate for which $\hat{f}^{p_0+1} \subset E_0$. Then we modify the first condition in (a) to $\hat{f}^j(\eta) \subset E_0$ for $p_0 + 1 \le j \le \nu_1$. The sum

$$\sum_{j=p_0+1}^{\nu_1 - 1} \big| \log |f'(f^j(x))| - \log |f'(f^j(y))| \big|$$

is estimated in just the same way as (5.34). For the sum over $0 \le j \le p_0$ we use a simpler version of (5.37): since $\hat{f}^j(\eta) \subset E_{k(0)+j}$,

$$\sum_{j=0}^{p_0} \big| \log |f'(f^j(x))| - \log |f'(f^j(y))| \big| \le \sum_{j=0}^{p_0} \text{const}\, e^{\alpha(j+k(0))} e^{-\beta(j+k(0))}$$
$$\le \sum_{k=1}^{\infty} \text{const}\, e^{(\alpha-\beta)k} \le \text{const}\,. \tag{5.40}$$

Thus, this last sum just adds a constant term to (5.38), and so does not affect the conclusion in (5.39): $\hat{f}^n$ has bounded distortion on η also in this case.

Finally, suppose that $\hat{f}^n(\eta)$ is not contained in E_0. Then we let $\nu = \nu_s$ be the last iterate for which $\hat{f}^\nu(\eta) \subset E_0$, and we do not define p_s. The previous case shows that $\hat{f}^\nu$ has bounded distortion on η, cf. (5.39).

$$\sum_{j=0}^{\nu-1} \big| \log |f'(f^j(x))| - \log |f'(f^j(y))| \big| \leq K_1 .$$

In general, we can not expect $\hat{f}$ to have bounded distortion on $\hat{f}^\nu(\eta)$: for instance, this last interval may contain the critical point c. But it is easy to see that, given $\varepsilon_3 > 0$ there is $\varepsilon_4 > 0$ such that for every $\zeta \subset \hat{f}^\nu(\eta)$

$$\frac{m(\zeta)}{m(\hat{f}^\nu(\eta))} \leq \varepsilon_3 \quad \Rightarrow \quad \frac{m(\hat{f}(\zeta))}{m(\hat{f}^{\nu+1}(\eta))} \leq \varepsilon_4 \tag{5.41}$$

Finally, $\hat{f}^{n-\nu-1}$ has bounded distortion on $\hat{f}^{\nu+1}(\eta)$: as in (5.40), we find

$$\sum_{j=\nu+1}^{n-1} \big| \log |f'(f^j(x))| - \log |f'(f^j(y))| \big| \leq K_2 ,$$

for some $K_2 > 0$ independent of ν, n, and η. Now the conclusion of the lemma can be deduced as follows. Given $\varepsilon_2 > 0$, take $\varepsilon_4 = \big(\varepsilon_2 / m(I)\big) \exp(-K_2)$. Next, take $\varepsilon_3 > 0$ as in (5.41). Finally, let $\varepsilon_1 = \varepsilon_3 \exp(-K_1)$. Then,

$$\frac{m(\xi)}{m(\eta)} \leq \varepsilon_1 \;\Rightarrow\; \frac{m(\hat{f}^\nu(\xi))}{m(\hat{f}^\nu(\eta))} \leq \varepsilon_3 \;\Rightarrow\; \frac{m(\hat{f}^{\nu+1}(\xi))}{m(\hat{f}^{\nu+1}(\eta))} \leq \varepsilon_4 \;\Rightarrow\; \frac{m(\hat{f}^n(\xi))}{m(\hat{f}^n(\eta))} \leq \frac{\varepsilon_2}{m(I)}$$

in particular, $m(\hat{f}^n(\xi)) \leq \varepsilon_2$. □

Let $\mathcal{B}$ be the Borel σ-algebra of I and $\widehat{\mathcal{B}}$ be the Borel σ-algebra of $\hat{I}$. By definition, the invariant measure μ_0 is *exact* for f if

$$B \in \mathcal{B}_\infty = \bigcap_{n=0}^{\infty} f^{-n}(\mathcal{B}) \quad \Rightarrow \quad \mu_0(B) = 0 \;\text{ or }\; \mu_0(I \setminus B) = 0.$$

Analogously, we say that $\hat{\mu}_0$ is exact for $\hat{f}$ if

$$\hat{B} \in \widehat{\mathcal{B}}_\infty = \bigcap_{n=0}^{\infty} f^{-n}(\widehat{\mathcal{B}}) \quad \Rightarrow \quad \hat{\mu}_0(\hat{B}) = 0 \;\text{ or }\; \hat{\mu}_0(\hat{I} \setminus \hat{B}) = 0.$$

We continue to denote $\pi : \hat{I} \to I$ the canonical projection.

Lemma 5.12.

(1) *If $A \subset I$ belongs in $\mathcal{B}_\infty$ then $\pi^{-1}(A) \subset \hat{I}$ belongs in $\widehat{\mathcal{B}}_\infty$.*

(2) *For any $\hat{A} \subset \hat{I}$ in $\widehat{\mathcal{B}}_\infty$ there are $A_- \subset A_+ \subset I$ so that $\pi^{-1}(A_-) \subset \hat{A} \subset \pi^{-1}(A_+)$ and $A_+ \setminus A_-$ is a countable set.*

Proof: The first part is easy: if $A = f^{-n}(A_n)$ for some Borel subset $A_n \subset I$, then

$$x \in \pi^{-1}(A) \Leftrightarrow \pi(x) \in A \Leftrightarrow \pi(\hat{f}^n(x)) = f^n(\pi(x)) \in A_n$$
$$\Leftrightarrow \hat{f}^n(x) \in \pi^{-1}(A_n) \Leftrightarrow x \in \hat{f}^{-n}(\pi^{-1}(A_n)),$$

that is, $\pi^{-1}(A) = \hat{f}^{-n}(\pi^{-1}(A_n))$.

To prove part (2), let $A_+ = \pi(\hat{A})$ and

$$A_- = A_+ \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{c_j : j \geq 0\}).$$

It is clear that $\hat{A} \subset \pi^{-1}(A_+)$, so let us prove that $\pi^{-1}(A_-) \subset \hat{A}$. Given any $z \in A_-$ there exists some $\xi \in \hat{A}$ such that $\pi(\xi) = z$. Then we only have to show that any other $\eta \in \hat{I}$ such $\pi(\eta) = z$ also belongs in $\hat{A}$. Now, the elements of $\widehat{\mathcal{B}}_\infty$ are characterized by the property

$$[\zeta_1 \in \hat{A} \text{ and } \hat{f}^n(\zeta_1) = \hat{f}^n(\zeta_2) \text{ for some } n \geq 1] \quad \Rightarrow \quad \zeta_2 \in \hat{A}.$$

Therefore, we are left to show that for any ξ and η as above there is $n \geq 1$ such that $\hat{f}^n(\xi) = \hat{f}^n(\eta)$. To this end, since $\pi(\hat{f}^n(\xi)) = \pi(\hat{f}^n(\eta))$ for every $n \geq 1$, it suffices to show that there exists $n \geq 1$ such that $\hat{f}^n(\xi)$ and $\hat{f}^n(\eta)$ belong both in E_0.

To prove this we introduce the following notion. Given $x \in (-\delta, \delta)$, we define the *falling time* $p(x)$ of x to be the smallest integer $j \geq 1$ such that $\hat{f}^{j+1}(x, 0) \in E_0$. The same kind of argument as in (5.15) gives, recall also (U1),

$$e^{-\beta p(x)} \geq |f^{p(x)} - c_{p(x)}| \geq \frac{1}{C}|(f^{p(x)-1})'(c_1)||c_1 - f(x)| \geq \frac{1}{C\lambda_c}\lambda_c^{p(x)}x^2. \tag{5.42}$$

Fix $\gamma = 1 - e^{\alpha-\beta} > 0$. Up to taking δ small, we may suppose that $p(x) \geq H(\delta)$ is large enough so that the previous relation implies

$$\lambda_c^{p(x)}x^2 \leq \gamma^2 \qquad (\text{in particular } x \neq 0 \ \Rightarrow \ p(x) < \infty). \tag{5.43}$$

Let us write $\xi = (z, k)$ and $\eta = (z, l)$. The definition of A_- ensures that the f-orbit of $z \in A_-$ is disjoint from the critical orbit, and so $p(f^n(z))$ is finite for every $n \geq 1$. Suppose there is no $n \geq 1$ such that both $\hat{f}^n(\xi)$ and $\hat{f}^n(\xi)$ are in E_0. Then each of their orbits must start climbing the tower again before the other one falls down to E_0. That is, there must be an infinite sequence of times $0 < \nu_1 < \nu_2 < \cdots$ such that $f^{\nu_i}(z) \in (-\delta, \delta)$ (one of the orbits moves from E_0 to E_1) and $\nu_{i+1} \le \nu_i + p(f^{\nu_i}(z))$ (while the other is still climbing up) for all $i \ge 1$. To check that this leads to a contradiction, we write $p_i = p(f^{\nu_i}(z))$ and note that

$$\begin{aligned} \nu_{i+1} - \nu_i \le p_i \quad &\Rightarrow \quad |f^{\nu_{i+1}}(z) - c_{(\nu_{i+1}-\nu_i)}| \le e^{-\beta_1(\nu_{i+1}-\nu_i)} \\ &\Rightarrow \quad |f^{\nu_{i+1}}(z)| \ge \gamma e^{-\alpha(\nu_{i+1}-\nu_i)} \ge \gamma e^{-\alpha p_i}. \end{aligned}$$

(in the last implication we use (U2)) Combining this with (5.43) and $e^{2\alpha} < \sqrt{\lambda_c}$,

$$\gamma^2 \ge \lambda_c^{p_{i+1}} |f^{\nu_{i+1}}(z)|^2 \ge \gamma^2 \lambda_c^{p_{i+1}} e^{-2\alpha p_i} \ge \gamma^2 \lambda_c^{p_{i+1}-(p_i/2)},$$

and so $p_{i+1} \le p_i/2$ for every $i \ge 1$. Since the p_i are positive integers, the sequence p_i can not be infinite. This gives us the contradiction we were looking for. □

Proposition 5.13 (exactness). *The measure $\hat{\mu}_0$ is exact for $\hat{f}$ and μ_0 is exact for f.*

Proof: First we prove that $\hat{\mu}_0$ is exact. Let $\hat{A} \in \widehat{\mathcal{B}}_\infty$, that is, for every $j \ge 1$ there exists a Borel set $\hat{A}_j$ such that $\hat{A} = \hat{f}^{-j}(\hat{A}_j)$. We want to show that if $\hat{\mu_0}(\hat{A}) > 0$ then $\hat{\mu_0}(\hat{I} \setminus \hat{A}) = 0$. Fix $\varepsilon > 0$ once and for all, small enough so that $\mu_0(\hat{A}) > 3\varepsilon$. Let θ be an arbitrary constant in $(0, 1)$. By Lebesgue differentiation theorem and the fact that $\hat{\mu}_0$ is absolutely continuous with respect to Lebesgue measure m on $\hat{I}$, there exist $\hat{B}_\theta \subset \hat{A}$ and $r > 0$ such that $\mu_0(\hat{B}_\theta) \ge 2\varepsilon$ and

$$\frac{m(J \cap \hat{A})}{m(J)} \ge 1 - \theta$$

for every closed interval J with length less than r containing some point $\xi \in \hat{B}_\theta$. Let $N = N(\varepsilon)$ be as in Lemma 5.10, and fix $n \ge 1$ large enough so that all the elements of $\mathcal{P}^{(n+N)}$ have length less than r. Since $\mu_0(\hat{B}_\theta) \ge 2\varepsilon$, some element η_θ of $\mathcal{Q}(n, N)$ must intersect $\hat{B}_\theta$, recall Lemma 5.10(2). Then

$$\frac{m(\eta_\theta \setminus \hat{A})}{m(\eta_\theta)} \le \theta.$$

Let $\varepsilon_2 > 0$ be any small number, and $\varepsilon_1 > 0$ be as given by Lemma 5.11. We choose $\theta \le \varepsilon_1$, so that the previous relation implies

$$m\big(\hat{f}^n(\eta_\theta) \setminus \hat{A}_n\big) = m\big(\hat{f}^n(\eta_\theta \setminus \hat{A})\big) \le \varepsilon_2$$

(given $\xi \in \eta_\theta$ then $\hat{f}^n(\xi)$ is in $\hat{A}_n$ if and only if ξ is in $\hat{A} = \hat{f}^{-n}(\hat{A}_n)$). By the construction of $\mathcal{Q}(n, N)$ in Lemma 5.10(1), $\zeta_\theta = \hat{f}(\eta_\theta)$ is an element of the partition $\mathcal{P}^{(n)}$ contained in some level E_l of the tower with $l \le N$. Observe that there are only finitely many such intervals ζ_θ. Hence, there is $q \geq 1$ depending only on N (and so completely determined by $\varepsilon > 0$) such that

$$\pi(\hat{f}^q(\zeta_\theta)) = f^q(\pi(\zeta_\theta)) = I_* \tag{5.44}$$

Let ζ_i, $1 \leq i \leq \kappa$, be the (nonempty) intersections of ζ_θ with elements of the partition $\mathcal{P}^{(q)}$. Since $\zeta_\theta \subset \cup_{l \leq N} E_l$, the number κ of such intersections is finite and depends only on N (since q is also determined by N). It is easy to see that given any $\varepsilon_3 > 0$ there exists $\varepsilon_2 > 0$ such that

$$m(\hat{B}) \leq \varepsilon_2 \quad \Rightarrow \quad m(\hat{f}^q(\hat{B})) \leq \varepsilon_3$$

for every subset $\hat{B}$ of the $\hat{I}$: just take $\varepsilon_2 = \varepsilon_3/4^q$ and use $|(f^q)'| \leq 4^q$. Then, from

$$m(\zeta_i \setminus \hat{A}_n) \leq m(\zeta_\theta \setminus \hat{A}_n) \leq \varepsilon_2$$

we find

$$m\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big) = m\big(\hat{f}^q(\zeta_i \setminus \hat{A}_n)\big) \leq \varepsilon_3 .$$

As a consequence,

$$m\Big(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big)\Big) \leq \kappa\varepsilon_3 \leq \varepsilon_4$$

where $\varepsilon_4 > 0$ can be made arbitrarily small by reducing ε_3 (which corresponds to reducing θ) without changing κ (that is, keeping ε and N fixed). Recall also that μ_0 is absolutely continuous with respect to Lebesgue measure m on I. It follows that, given any $\varepsilon_5 > 0$, one has

$$\mu_0\Big(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big)\Big) \leq \varepsilon_5 \tag{5.45}$$

if θ (and so also ε_4) are taken small enough.

At this point we are close to our goal, which is to prove that $\hat{I} \setminus \hat{A}$ has zero measure. Observe that

$$\hat{\mu}_0(\hat{I} \setminus \hat{A}) = \hat{\mu}_0(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}),$$

simply because $\hat{\mu_0}$ is $\hat{f}$-invariant and $\hat{A} = \hat{f}^{-(n+q)}(\hat{A}_{n+q})$. We claim that

$$\pi\big(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}\big) \subset \Big(\bigcup_{i=1}^{\kappa} \pi\big(\hat{f}^q(\zeta_i) \setminus \hat{A}_{n+q}\big)\Big) \cup \Big(\bigcup_{l \geq 0} f^{-l}(\{c_j : j \geq 0\}\Big). \tag{5.46}$$

This last set is only countable, and so the combination of (5.45) and (5.46) implies that

$$\hat{\mu}_0(\hat{I} \setminus \hat{A}) = \hat{\mu}_0(\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}) \leq \varepsilon_5.$$

Since ε_5 is arbitrary, this gives that $\hat{\mu}_0(\hat{I} \setminus \hat{A}) = 0$ as we wanted to show. Thus, all that remains to be done to conclude that $\hat{\mu}_0$ is exact is to prove the claim (5.46).

For this, let ξ_1 be any point in $\hat{f}^{n+q}(\hat{I}) \setminus \hat{A}_{n+q}$, and take $\tau_1 \in \hat{I}$ such that $\hat{f}^{n+q}(\tau_1) = \xi_1$. By (5.44) there exists $1 \le i \le \kappa$ and $\xi_2 \in \hat{f}^{n+q}(\zeta_i)$ such that $\pi(\xi_1) = \pi(\xi_2)$. Moreover, $\zeta_i \subset \hat{f}^{n+q}(\hat{I})$, by construction, and so we may take $\tau_2 \in \hat{I}$ such that $\hat{f}^{n+q}(\tau_2) = \xi_2$. We need to show that either

$$\pi(\xi_1) = \pi(\xi_2) \in \bigcup_{l \ge 0} f^{-l}(\{c_j : j \ge 0\}). \tag{5.47}$$

or else $\xi_2 \notin \hat{A}_{n+q}$. Suppose (5.47) does not hold. Then, as in Lemma 5.12, there exists $j \ge 0$ such that

$$\hat{f}^{n+q+j}(\tau_1) = \hat{f}^j(\xi_1) = \hat{f}^j(\xi_2) = \hat{f}^{n+q+j}(\tau_1)$$

As $\tau_1 \notin \hat{A}$, because $\xi_1 \notin \hat{A}_{n+q}$, it follows that $\tau_2 \notin \hat{A}$, and so

$$\xi_2 \in \hat{f}^{n+q}(\zeta_i) \setminus \hat{A}_{n+q}.$$

This establishes the claim (5.46). We have shown that $\hat{\mu}_0$ is an exact measure for $\hat{f}$.

The statement that μ_0 is exact is an immediate consequence: given any $A \in \mathcal{B}_\infty$ the preimage $\pi^{-1}(A)$ belongs in $\widehat{\mathcal{B}}_\infty$, by Lemma 5.12(1), and then $\mu_0(A) = \hat{\mu}_0(\pi^{-1}(A))$ is either 0 or 1. $\square$

Proposition 5.14 (quasi-compacity). *There exists $\tau < 1$ such that the spectrum of the operator $\mathcal{L}_0$ acting on $BV(\hat{I})$ may be written* $\operatorname{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$, *where* 1 *is a simple eigenvalue and Σ_0 is contained in the disk of radius τ. The corresponding invariant splitting is $BV(\hat{I}) = \mathbb{R}\varphi_0 \oplus X_0$, where $X_0 = \{\varphi \in BV(\hat{I}) : \int \varphi \, dm_0 = 0\}$. In particular, the spectral projection π_1 associated to the eigenvalue* 1 *is given by $\pi_1(\varphi) = \varphi_0 \int \varphi \, dm_0$.*

Proof: It follows from Proposition 5.6 that

$$\|\mathcal{L}_0^n \varphi\|_{\mathrm{BV}} \le C\bar{\sigma}^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int |\varphi| \, dm_0 \le C\|\varphi\|_{\mathrm{BV}}$$

for every $\varphi \in \mathrm{BV}(\hat{I})$, where $C > 0$ and $\bar{\sigma} \in (1, \sigma)$ do not depend on φ nor on n. Then $\|\mathcal{L}_0^n\|_{\mathrm{BV}} \le C$ for every $n \ge 1$, and so the spectral radius of $\mathcal{L}_0$ is at most 1. As we already constructed a fixed point φ_0 for $\mathcal{L}_0$, the spectral radius is exactly 1. Now the proof has two main steps. First we show that the essential spectral radius of $\mathcal{L}_0$ is bounded by $1/\bar{\sigma} < 1$. This implies that the spectrum is the union of a finite set of eigenvalues of finite multiplicity contained in the unit circle, with a compact subset of a disc of radius $\tau_0 < 1$: τ_0 is either the essential spectral radius of $\mathcal{L}_0$ or the norm of the second largest eigenvalue. Then we deduce from Proposition 5.13 that 1 is the only eigenvalue in the unit circle, and that it has multiplicity 1.

The basic idea to bound the spectral radius of $\mathcal{L}_0$ is to show that the iterates $\mathcal{L}_0^n$ are well approximated by certain operators with finite-dimensional range. We begin by noting that, as a consequence of Remark 5.2 and Lemma 5.4, there exists $C > 0$ such that

$$\sup g^{(n)} \le C\sigma^{-n} \quad \text{and} \quad \operatorname{var} g^{(n)} \le C\sigma^{-n} \quad \text{for all } n \ge 1.$$

Given any $n \ge 1$, fix $N \ge n$ such that $m_0(\cup_{k>N} E_k) < \bar\sigma^{-n}$, and then define

$$\alpha_n : BV(\hat I) \to BV(\hat I) \quad \text{and} \quad \alpha_{n,N} : BV(\hat I) \to BV(\hat I)$$

by choosing an arbitrary point x_η in each monotonicity interval $\eta \in \mathcal{P}^{(n)}$, and then setting

$$\alpha_n(\varphi) = \sum_{\eta \mathcal{P}^{(n)}} \varphi(x_\eta)\,\chi_\eta \qquad \text{and} \qquad \alpha_{n,N}(\varphi) = \alpha_n\big(\varphi \cdot \chi_{(\cup_{k\le N} E_k)}\big).$$

Observe that the range of $\alpha_{n,N}$ has finite dimension:

$$\dim \alpha_{n,N}\big(\mathrm{BV}(\hat I)\big) \le \#\{\eta \in \mathcal{P}^{(n)} : \eta \subset E_k \text{ for some } k \le N\} < \infty.$$

We claim that there is $C_0 > 0$ such that

$$\|\mathcal{L}_0^n - \mathcal{L}_0^n \alpha_{n,N}\|_{\mathrm{BV}} \le C_0 \bar\sigma^{-n} \tag{5.48}$$

for every $n \ge 1$. Since each $\mathcal{L}_0^n \alpha_{n,N}$ has finite-dimensional range, it follows that the essential spectral radius of $\mathcal{L}_0$ is not bigger than $1/\bar\sigma$, as we claimed.

To prove claim (5.48) we use the relation

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n \alpha_{n,N})\varphi\|_{\mathrm{BV}} \le \|\mathcal{L}_0^n(\varphi - \varphi_N)\|_{\mathrm{BV}} + \|(\mathcal{L}_0^n - \mathcal{L}_0^n \alpha_n)\varphi_N\|_{\mathrm{BV}}, \tag{5.49}$$

where $\varphi_N = \varphi \cdot \chi_{(\cup_{k\le N} E_k)}$. To bound $\|\mathcal{L}_0^n(\varphi - \varphi_N)\|_{\mathrm{BV}}$ we apply Proposition 5.6 to the function $\varphi - \varphi_N = \varphi \cdot \chi_{(\cup_{k>N} E_k)}$. Since $\operatorname{var}(\varphi - \varphi_N) \le \operatorname{var}\varphi$ and $\sup|\varphi - \varphi_N| \le \sup|\varphi|$, we find

$$\operatorname{var} \mathcal{L}_0^n(\varphi - \varphi_N) \le C\bar\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int |\varphi - \varphi_N|\, dm_0$$
$$\sup|\mathcal{L}_0^n(\varphi - \varphi_N)| \le C\bar\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) + C\int |\varphi - \varphi_N|\, dm_0.$$

Moreover, in view of our choice of N,

$$\int |\varphi - \varphi_N|\, dm_0 \le m_0(\cup_{k>N} E_k)\, \sup|\varphi| \le \bar\sigma^{-n} \sup|\varphi|.$$

It follows that

$$\|\mathcal{L}_0^n(\varphi - \varphi_N)\|_{\mathrm{BV}} \le 4C\bar\sigma^{-n}(\operatorname{var}\varphi + \sup|\varphi|) \le 4C\bar\sigma^{-n}\|\varphi\|_{\mathrm{BV}}. \tag{5.50}$$

Next, we estimate $\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\varphi_N\|_{\mathrm{BV}}$. Given a monotonicity interval η of $\hat{f}^n$ and given $y \in \hat{f}^n(\eta)$, we denote $y_\eta = (\hat{f}^n|\eta)^{-1}(y)$. Then, for every $\psi \in \mathrm{BV}(\hat{I})$,

$$\sup|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi| \le \sup_y \sum_{y\in\hat{f}^n(\eta)} g^{(n)}(y_\eta)|\psi(y_\eta) - \psi(x_\eta)|$$
$$\le \sum_\eta C\sigma^{-n} \operatorname*{var}_\eta \psi \quad \le C\sigma^{-n} \operatorname{var}\psi.$$

Let $\psi_\eta = \psi - \psi(x_\eta)$. Note that $\sup_\eta|\psi_\eta| \le \operatorname{var}_\eta\psi = \operatorname{var}_\eta\psi_\eta$. Then, recall (5.16),

$$\operatorname{var}(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi = \operatorname{var}\sum_\eta \left(g^{(n)}\,\psi_\eta\,\chi_\eta\right) \circ (\hat{f}^n|\eta)^{-1}$$
$$\le \sum_\eta \left(\operatorname*{var}_\eta g^{(n)} \sup_\eta|\psi_\eta| + \sup_\eta g^{(n)} \operatorname*{var}_\eta \psi_\eta + 2\sup_\eta g^{(n)} \sup_\eta|\psi_\eta|\right)$$
$$\le \sum_\eta \left(4C\sigma^{-n} \operatorname*{var}_\eta\psi\right) \quad \le 4C\sigma^{-n}\operatorname{var}\psi,$$

Finally,

$$\int |(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi|\,dm_0 \le m_0(\hat{I})\,\sup|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi| \le m_0(\hat{I})\,C\,\sigma^{-n}\operatorname{var}\psi.$$

Summarizing,

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\psi\|_{\mathrm{BV}} \le K\sigma^{-n}\operatorname{var}\psi \le K\bar{\sigma}^{-n}\|\psi\|_{\mathrm{BV}},$$

where $K = C(5 + m_0(\hat{I}))$. We use this relation for $\psi = \varphi_N$. Observing that $\|\varphi_N\|_{\mathrm{BV}} \le \|\varphi\|_{\mathrm{BV}}$, we get

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_n)\varphi_N\|_{\mathrm{BV}} \le K\bar{\sigma}^{-n}\|\varphi_N\|_{\mathrm{BV}} \le K\bar{\sigma}^{-n}\|\varphi\|_{\mathrm{BV}}. \tag{5.51}$$

Combining in (5.49), (5.50), (5.51), we obtain (5.48) with $C_0 = 4C + K$:

$$\|(\mathcal{L}_0^n - \mathcal{L}_0^n\alpha_{n,N})\varphi\|_{\mathrm{BV}} \le (4C + K)\bar{\sigma}^{-n}\|\varphi\|_{\mathrm{BV}}.$$

Now we proceed to the second step of the proof. Let $\lambda_1 \in S^1$ be an eigenvalue of $\mathcal{L}_0$ and $\varphi_1 \in BV(\hat{I})$ be a corresponding eigenfunction: $\mathcal{L}_0\varphi_1 = \lambda_1\varphi_1$. Then $\mathcal{L}_0^n\varphi_1 = \lambda_1^n\varphi_1$ and this implies that $\varphi_1 = 0$ at every point in the complement of $\hat{f}^n(\hat{I})$, for each $n \ge 1$. Thus, by Remark 5.1, φ_1 is identically zero in the complement of

$$W_\delta = W_* \setminus \left([f^2(c), f^2(\delta)) \cup (f(\delta), f(c)]\right) \times \{0\}.$$

On the other hand, we showed in Lemma 5.9 that the fixed point φ_0 of $\mathcal{L}_0$ is strictly positive on W_δ. Then we may write $\varphi_1 = \phi\varphi_0$ for some function ϕ. Observe that ϕ belongs in $L^1(\hat\mu_0)$:

$$\int |\phi|\, d\hat\mu_0 = \int |\varphi_1|\, dm_0 \le \|\varphi_1\|_{\mathrm{BV}} < \infty.$$

On the other hand, Proposition 5.13 implies that the measure $\hat\mu_0$ is mixing for the map $\hat f$. Then, in particular,

$$\int (\psi \circ \hat f^n)\phi\, d\hat\mu_0 \to \int \psi\, d\hat\mu_0 \int \phi\, d\hat\mu_0,$$

for every bounded function ψ. Now, the left hand side may be written

$$\int (\psi \circ \hat f^n)\phi\, d\hat\mu_0 = \int (\psi \circ \hat f^n)\varphi_1\, dm_0 = \int \psi(\mathcal{L}_0^n\varphi_1)\, dm_0 = \int \psi(\lambda_1^n\varphi_1)\, dm_0,$$

and the right hand side

$$\int \psi\, d\hat\mu_0 \int \phi\, d\hat\mu_0 = \int \psi(\varphi_0 \int \phi\, d\hat\mu_0) dm_0 = \int \psi(\varphi_0 \int \varphi_1\, dm_0) dm_0.$$

Thence, $\lambda_1^n\varphi_1$ converges to $\varphi_0 \int \varphi_1\, dm_0$ weakly in $L^1(m_0)$. Clearly, this implies that $\lambda_1 = 1$ and $\varphi_1 = \varphi_0 \int \hat\varphi_1\, dm_0$. This proves that 1 is the only eigenvalue in the unit circle, and that its eigenspace $\ker(\mathcal{L}_0 - \mathrm{id})$ has dimension 1.

In fact, one can say more: $\lambda_1 = 1$ has algebraic multiplicity 1, meaning that

$$\dim\ker\left((\mathcal{L}_0 - \mathrm{id})^n\right) = 1 \quad \text{for every } n \ge 1.$$

Otherwise, there would exist a nonzero function $\psi_0 \in \mathrm{BV}(\hat I)$ such that $\mathcal{L}_0\psi_0 = \psi_0 + \varphi_0$. Then, by recurrence,

$$\mathcal{L}_0^n\psi_0 = n\psi_0 + \varphi_0 \quad \text{for every } n \ge 1,$$

which would contradict the conclusion obtained previously that the norms $\|\mathcal{L}_0^n\|$ are uniformly bounded.

Thus far we have shown that $\mathrm{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$, where Σ_0 is contained in a disk of radius τ, for some $\tau < 1$, and 1 is a simple eigenvalue. Finally, the splitting

$$\mathrm{BV}(\hat I) = \mathbb{R}\varphi_0 \oplus X_0, \qquad X_0 = \{\varphi \in \mathrm{BV} : \int \varphi\, dm_0 = 0\}$$

is invariant under $\mathcal{L}_0$, with $\mathrm{spec}(\mathcal{L}_0|\mathbb{R}\varphi_0) = \{1\}$ and $1 \notin \mathrm{spec}(\mathcal{L}_0|X_0)$. Hence, it must be the spectral splitting associated to the decomposition $\mathrm{spec}(\mathcal{L}_0) = \{1\} \cup \Sigma_0$. Clearly, the projection π_1 onto the first factor is given by $\pi_1(\varphi) = \varphi_0 \int \varphi\, dm_0$. □

Proposition 5.15 (decay of correlations). *Let $\bar{\tau} \in (\tau, 1)$ where τ is as in Proposition 5.14. There exists $\bar{C} > 0$ such that*

(1) *given any $\hat{\varphi} \in BV(\hat{I})$, any $\hat{\psi} \in L^1(m_0)$, and any $n \geq 1$,*

$$\left| \int (\hat{\psi} \circ \hat{f}^n) \hat{\varphi}\, d\hat{\mu}_0 - \int \hat{\psi}\, d\mu_0 \int \hat{\varphi}\, d\hat{\mu}_0 \right| \leq \bar{C}\, \bar{\tau}^n\, \|\hat{\varphi}\|_{\mathrm{BV}}\, \|\hat{\psi}\|_1$$

for every $n \geq 1$;

(2) *given any $\varphi \in BV(I)$, any bounded function $\psi : I \to \mathbb{R}$, and any $n \geq 1$,*

$$\left| \int (\psi \circ f^n) \varphi\, d\mu_0 - \int \psi\, d\mu_0 \int \varphi\, d\mu_0 \right| \leq \bar{C}\, \bar{\tau}^n\, \|\varphi\|_{\mathrm{BV}}\, \sup |\psi|$$

Proof: To prove (1) we write

$$\begin{aligned}\int (\hat{\psi} \circ \hat{f}^n) \hat{\varphi}\, d\hat{\mu}_0 - \int \hat{\psi}\, d\mu_0 \int \hat{\varphi}\, d\hat{\mu}_0 &= \int \hat{\psi}\, \mathcal{L}_0^n(\hat{\varphi}\, \varphi_0)\, dm_0 - \int \hat{\psi} \varphi_0 \Big(\int \hat{\varphi}\, \varphi_0\, dm_0 \Big)\, dm_0 \\ &= \int \hat{\psi} \Big(\mathcal{L}_0^n(\hat{\varphi}\, \varphi_0) - \varphi_0 \Big(\int \hat{\varphi}\, \varphi_0\, dm_0 \Big) \Big) dm_0\end{aligned}$$

Observe that

$$\begin{aligned}\operatorname{var} \hat{\varphi}\varphi_0 &= \sum_{k \geq 0} \operatorname{var}(\hat{\varphi}\varphi_0 | E_k) \\ &\leq \sum_{k \geq 0} \operatorname{var}(\hat{\varphi} | E_k) \sup(\varphi_0 | E_k) + \sup |(\hat{\varphi} | E_k)| \operatorname{var}(\varphi_0 | E_k) \\ &\leq \big(\sup_{k \geq 0} \operatorname{var}(\hat{\varphi} | E_k) \big) \sum_{k \geq 0} \sup(\varphi_0 | E_k) + \big(\sup_{k \geq 0} \sup |(\hat{\varphi} | E_k)| \big) \sum_{k \geq 0} \operatorname{var}(\varphi_0 | E_k) \qquad (5.52) \\ &\leq \operatorname{var} \hat{\varphi} \sum_{k \geq 0} \sup |(\varphi_0 | E_k)| + \sup |\hat{\varphi}| \sum_{k \geq 0} \operatorname{var}(\varphi_0 | E_k)\end{aligned}$$

recall Remark 5.3. Moreover,

$$\sup |\hat{\varphi}\varphi_0| = \sup_{k \geq 0} \sup |(\hat{\varphi}\varphi_0 | E_k)| \leq \sup_{k \geq 0} \big(\sup |(\hat{\varphi} | E_k)| \sup(\varphi_0 | E_k) \big) \leq \sup |\hat{\varphi}| \sup \varphi_0$$

and

$$\int |\hat{\varphi}\varphi_0|\, dm_0 \leq \sup |\hat{\varphi}| \int \varphi_0\, dm_0 .$$

This proves that,

$$\|\hat{\varphi}\varphi_0\|_{\mathrm{BV}} \leq K_1 \|\hat{\varphi}\|_{\mathrm{BV}}, \qquad \text{and so} \qquad \hat{\varphi}\varphi_0 \in \mathrm{BV}(\hat{I}), \tag{5.53}$$

where $K_1 = \sum_{l\geq 0}\sup(\varphi_0|E_k) + \operatorname{var}\varphi_0 + \int \varphi_0\, dm_0$, recall Remark 5.3.

Now, φ_0 being a fixed point of $\mathcal{L}_0$,

$$\mathcal{L}_0^n(\hat\varphi\,\varphi_0) - \varphi_0\Big(\int \hat\varphi\,\varphi_0\, dm_0\Big) = \mathcal{L}_0^n\Big(\hat\varphi\,\varphi_0 - \varphi_0\int \hat\varphi\,\varphi_0\, dm_0\Big) = \mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big),$$

where $\pi_0(\phi) = \phi - \varphi_0\int \phi\, d\hat\mu_0$ is the projection onto the factor X_0 of the spectral splitting $\mathbb{R}\varphi_0 \oplus X_0$, recall Proposition 5.14. Since $\operatorname{spec}(\mathcal{L}_0|X_0) = \Sigma_0$, which is contained in the disk of radius τ, we get that

$$\sup\big|\mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big)\big| \leq \|\mathcal{L}_0^n\big(\pi_0(\hat\varphi\,\varphi_0)\big)\| \leq K_0\bar\tau^n\|\hat\varphi\,\varphi_0\|_{\mathrm{BV}} \leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\hat\varphi\|_{\mathrm{BV}}$$

for some $K_0 > 0$ and every $n \geq 1$. Replacing above,

$$\begin{aligned}\Big|\int (\hat\psi\circ \hat f^n)\hat\varphi\, d\hat\mu_0 - \int \hat\psi\, d\mu_0\int \hat\varphi\, d\hat\mu_0\Big| &\leq \int |\hat\psi|\, dm_0\cdot \sup\big|\mathcal{L}_0^n\big(\pi_0(\varphi\,\varphi_0)\big)\big| \\ &\leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\hat\varphi\|_{\mathrm{BV}}\,\|\psi\|_1\end{aligned}$$

and so it suffices to take $\bar C \geq K_0K_1\,\|\varphi_0\|_{\mathrm{BV}}$.

Now part (2) can be deduced easily. Given $\varphi,\psi\in \mathrm{BV}(I)$ define $\hat\varphi(x,k) = \varphi(x)$ and $\hat\psi(x,k) = \psi(x)$. Then

$$\int |\hat\psi|\, dm_0 \leq m_0(\hat I)\sup|\hat\psi| \leq m_0(\hat I)\sup|\psi|, \tag{5.54}$$

in particular $\hat\psi\in L^1(m_0)$. Moreover, the function $\hat\varphi$ is bounded and satisfies

$$\sup_{k\geq 0}\operatorname{var}(\hat\varphi|E_k) = \sup_{k\geq 0}\operatorname{var}(\varphi|B_k) \leq \operatorname{var}\varphi < \infty.$$

Then, as in (5.52), (5.53), $\|\hat\varphi\varphi_0\|_{\mathrm{BV}} \leq K_1\|\varphi\|_{\mathrm{BV}}$, which ensures that $\hat\varphi\varphi_0\in \mathrm{BV}(\hat I)$. So, in just the same way as in the previous situation,

$$\begin{aligned}\Big|\int (\psi\circ f^n)\varphi\, d\mu_0 - \int \psi\, d\mu_0\int \varphi\, d\mu_0\Big| &= \Big|\int (\hat\psi\circ \hat f^n)\hat\varphi\, d\hat\mu_0 - \int \hat\psi\, d\mu_0\int \hat\varphi\, d\hat\mu_0\Big| \\ &\leq (K_0K_1\,\|\varphi_0\|_{\mathrm{BV}})\,\bar\tau^n\,\|\varphi\|_{\mathrm{BV}}\, m_0(\hat I)\sup|\psi|\,.\end{aligned} \tag{5.55}$$

We just take $\bar C \geq K_0K_1\, m_0(\hat I)\,\|\varphi_0\|_{\mathrm{BV}}$. ☐

Corollary 5.16 (central limit theorem). *Let $\varphi\in BV(I)$ and*

$$\sigma^2 = \int \phi^2\, d\mu_0 + 2\sum_{j=1}^{\infty}\int \phi(\phi\circ f^j)d\mu_0, \quad \textit{where}\quad \phi = \varphi - \int \varphi\, d\mu_0.$$

Then $\sigma < \infty$ and $\sigma = 0$ if and only if $\phi = \mu_0 \circ f - u$ for some $u \in L^2(\mu_0)$. Moreover, if $\sigma > 0$ then for every interval $A \subset \mathbb{R}$

$$\mu_0\left\{x \in I : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \phi(f^j(x)) \in A\right\} \longrightarrow \int_A \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}\, dt \qquad \text{as } n \to +\infty.$$

Proof: This follows the steps of Corollary 2.10 and Proposition 2.12. It is no restriction to suppose $\int \varphi\, d\mu_0 = \int \varphi \varphi_0\, dm = 0$, and we do so. Let $\mathcal{F}_n = f^{-n}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of I. Then

$$\begin{aligned} \|E(\varphi \mid \mathcal{F}_n)\|_2 &= \sup\{\int \psi\varphi\, d\mu_0 : \psi \in L^2(\mathcal{F}_n) \text{ and } \|\psi\|_2 = 1\} \\ &= \sup\{\int (\psi \circ f^n)\varphi\, d\mu_0 : \psi \in L^2(\mu_0) \text{ and } \|\psi\|_2 = 1\}. \end{aligned}$$

Define $\hat\varphi(x,k) = \varphi(x)$ and $\hat\psi(x,k) = \psi(x)$, for each $\psi \in L^2(\mu_0)$. We claim that the function $\hat\psi \in L^1(m_0)$. Indeed, by Remark 5.1 and Cauchy-Schwarz inequality,

$$\begin{aligned} \int |\hat\psi|\, dm_0 = \int |\hat\psi| \varphi_0^{-1}\, d\hat\mu_0 &\le \left(\int \varphi_0^{-2}\, d\hat\mu_0\right)^{1/2} \left(\int |\hat\psi|^2\, d\hat\mu_0\right)^{1/2} \\ &= \left(\int (\varphi_0|W_*)^{-1}\, dm_0\right)^{1/2} \left(\int |\psi|^2\, d\mu_0\right)^{1/2}. \end{aligned}$$

Using (5.30) and (5.32), there exists some constant $K_3 > 0$ so that

$$\begin{aligned} \int (\varphi_0|W_*)^{-1}\, dm_0 &\le \sum_{k=0}^{\infty} m_0(E_k) \sup(\varphi_0|W_* \cap E_k)^{-1} \le \sum_{k=0}^{\infty} K_3 \lambda^{2k} \lambda_c^{-k/2} e^{-\beta k/2} \\ &\le \sum_{k=0}^{\infty} K_3 e^{-(\alpha+\beta)k} < \infty \end{aligned}$$

since we have chosen $\lambda < \rho$ and $\lambda\rho e^{\alpha} < \sqrt{\lambda_c}$. Thus, denoting K_4 the square root of the last term in the previous expression,

$$\int |\hat\psi|\, dm_0 \le K_4 \|\psi\|_2 < \infty,$$

proving our claim. Then, compare (5.55),

$$\begin{aligned} \|E(\varphi \mid \mathcal{F}_n)\|_2 &\le \sup\{\int (\hat\psi \circ \hat f^n)\hat\varphi\,\varphi_0\, dm_0 : \hat\psi \in L^2(m_0) \text{ and } \|\hat\psi\|_1 \le K_2\} \\ &\le (K_0 K_1 \|\varphi_0\|_{\mathrm{BV}}) \bar\tau^n \|\varphi\|_{\mathrm{BV}} K_2 \quad \le K_5 \bar\tau^n \|\varphi\|_{\mathrm{BV}}, \end{aligned}$$

where $K_5 = K_0 K_1 K_2 \|\varphi_0\|_{\mathrm{BV}}$. In particular $\sum_{n=0}^{\infty} \|E(\varphi \mid \mathcal{F}_n)\|_2^2$ is finite, and so the corollary follows directly from Theorem 2.11. □

Notes.

Abundance (positive Lebesgue measure set of parameter values) of quadratic maps with absolutely continuous invariant measure is a result of Jakobson [Ja81]. Several other proofs and extensions appeared since then, e.g. [CE80], [BC85], [BC91], [NS91].

Exponential decay of correlations was proved independently by [KN92] and [Yo92]. They used (other) tower extensions and cocycles, in much the same way we do here.

Stochastic stability was proved by [KK86] when the critical point is nonrecurrent, corresponding to an uncountable zero Lebesgue measure set of parameters, and by [BeY92] for a set of parameters with positive Lebesgue measure slightly different from the one we consider here. [KK86] considered a more general random perturbation scheme. Both papers dealt with stochastic stability in the weak sense: weak* convergence of the stationary measure to the absolutely continuous invariant measure. Strong stochastic stability (L^1 convergence) had been obtained in a unpublished paper of [Co84].

Our presentation is based on [BaV96], who proved the full statement of Theorem 5.1, together with a result of exponential decay of correlations for the random perturbations scheme. The approach is inspired by the treatment of uniformly expanding maps in [BaY93].

6. Recent developments and future perspectives

Here we discuss some of the recent results in the study of "chaotic" dynamical systems, in line with the ones we presented before. We make no attempt to "completeness", but we expect this overview to give the reader a fair idea of the progress attained in this field, and to indicate some likely trends of future research.

6.1 Hénon-like attractors.

The prototype for this class of systems is the Hénon model in the plane $M = \mathbb{R}^2$

$$f = f_{a,b} : M \to M, \qquad f(x,y) = (1 - ax^2 + y, bx),$$

introduced by [He76] as the simplest class of systems exhibiting complex dynamical behaviour, related to the presence of a "strange" attractor. Figure 6.1 describes the attractor for the parameter values $a = 1.4$, $b = 0.3$ initially considered by [He76].

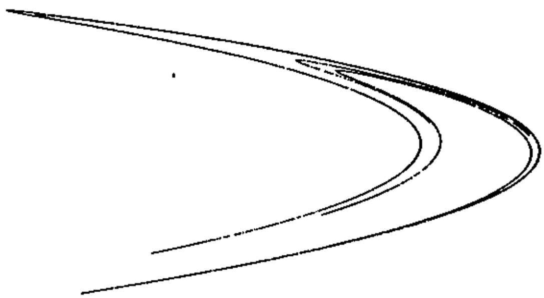

Figure 6.1: Hénon attractor

Some elementary features of Hénon maps are easy to derive. For instance, the jacobian of f is everywhere equal to $-b$, and f is a diffeomorphism for every $b \neq 0$, with inverse given by

$$f^{-1}(\xi,\eta) = (b^{-1}\eta, \xi - 1 + a(b^{-1}\eta)^2\ ).$$

It is also not difficult to see that for (a,b) in a fairly large portion of the parameter space the map f admits some invariant region U, in the sense that $f(\text{closure}(U)) \subset U$. This happens, for instance, for (a,b) close to $(1.4, 0.3)$ (another example is $1 < a < 2$ and b close to zero) and Hénon's experiments suggested that orbits in such a region U are attracted to an invariant set Λ with a complicated geometric structure.

The proof of Hénon's conjecture that this Λ is indeed a nontrivial attractor (not just an attracting periodic orbit with very high period) turned out to be a very hard problem. A proof that strange attractors do occur in the Hénon model came only a few years ago, through the work of Benedicks and Carleson.

Theorem 6.1 [BC91]. *There exists a subset $E \subset \mathbb{R}^2$ with positive Lebesgue measure such that for every $(a,b) \in E$ the map f admits a compact invariant set Λ such that*

(1) *the basin $B(\Lambda) = \{z \in M : f^n(z) \to \Lambda \text{ as } n \to +\infty\}$ has nonempty interior;*
(2) *there exists $z_1 \in \Lambda$ whose forward orbit $\{f^n(z) : n \geq 0\}$ is dense in Λ and there exist $c > 0$ and a tangent vector v to $M = \mathbb{R}^2$ at z_1, such that*

$$\|Df^n(z)v\| > e^{cn}\|v\| \quad \textit{for every } n \geq 1.$$

In particular, Λ can not contain periodic attractors. The parameter subset E they construct in the proof of this theorem is located near $(a,b) = (2,0)$. Existence of a global strange attractor close to $(a,b) = (1.4, 0.3)$ remains an important open problem.

A characteristic feature of these systems is that they combine two very different types of behaviour. At points (x,y) far from $\{x = 0\}$ the map f is essentially *hyperbolic*: it is not difficult to construct an unstable cone field (around the horizontal direction, i.e., the direction of $(1,0)$) and a stable cone field (around the direction of $(1, 2ax)$). However, these cone fields can not be extended to the whole phase-space, due to the *folding* taking place near $x = 0$: for $|x|$ small the derivative maps nearly horizontal tangent vectors at (x,y) to vectors inside the stable cone at $f(x,y)$.

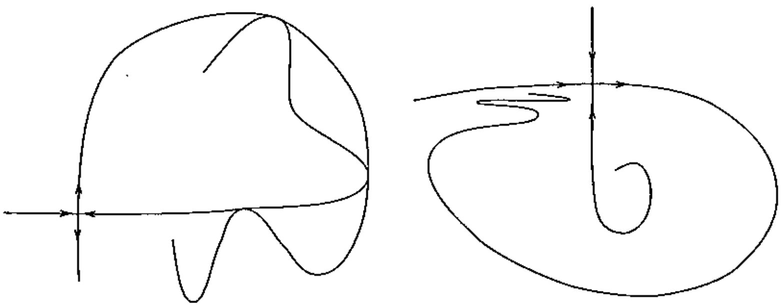

Fig 6.2: Homoclinic tangency and critical saddle-node cycle

Since then, attractors combining hyperbolic and folding behaviour have been shown to occur in very general contexts of bifurcations of dynamical systems. [MV93] proved that the generic unfolding of a homoclinic tangency by a parametrized family of surface diffeomorphisms is always accompanied by the formation of such attractors, for a positive Lebesgue measure set of parameters. A version of this result in manifolds of any dimension was proved in [Vi93].

A stronger conclusion was obtained by [DRV96] for a related bifurcation mechanism, critical saddle-node cycles: the bifurcation parameter is always a point of positive Lebesgue *density* for the set of parameters corresponding to nonhyperbolic attractors. Even more, in many cases the attractor has a *global* character: the basin contains a prescribed neighbourhood of the original cycle. Moreover, destruction of a hyperbolic set (a horseshoe) through collision with a periodic attractor also leads to formation of a global strange attractor [Co].

Here, *Hénon-like attractors* always refers to nonuniformly hyperbolic attractors such as those constructed in these papers. A precise definition of *strange attractor* is proposed in Section 6.4, inspired by the properties of these systems, but for the time being we avoid being too strict in the use of the term.

Theorem 6.2 [BeY93]. *Hénon-like strange attractors support a unique invariant SRB measure.*

More precisely, [BeY93] prove that there exists a unique invariant probability measure μ which is ergodic, has a positive Lyapunov exponent

$$\lambda_1 = \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n\| > 0 \quad \mu\text{-almost everywhere},$$

and *has absolutely continuous conditional measures along unstable manifolds* (absolute continuity is with respect to the riemannian measure on each unstable manifold). Then general arguments, e.g. from [PS89], show that μ is an SRB measure: given any continuous function φ

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \to \int \varphi\, d\mu \quad \text{as } t \to +\infty$$

for a positive (2-dimensional) Lebesgue measure subset of points x in the basin $B(\Lambda)$ of Λ.

An important problem in this setting, raised a number of years ago by Ruelle and by Sinai, concerns the exact relation between the topological basin and the ergodic basin of nonuniformly hyperbolic attractors. More precisely, given a nonuniformly hyperbolic attractor Λ supporting an SRB measure μ, is $B(\mu)$ always a full Lebesgue measure subset of $B(\Lambda)$?

The next theorem sets an affirmative answer to this question for Hénon-like attractors: *there are no "holes" in the topological basin* $B(\Lambda)$ (consisting of points whose asymptotics are not described by μ).

Theorem 6.3 [BeV1]. *Through Lebesgue almost every point in $B(\Lambda)$ there is a local stable manifold which intersects Λ. Moreover,*

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \to \int \varphi\, d\mu \quad \text{as } t \to +\infty,$$

for every continuous function φ and for Lebesgue almost every $x \in B(\Lambda)$.

As before, Lebesgue refers to the 2-dimensional Lebesgue measure. Also, by a stable manifold we mean a curve which is exponentially contracted under all positive iterates of f.

Proceeding with the study of the statistical properties of Hénon-like systems (f, μ), Benedicks and Young proved recently

Theorem 6.4 [BeY3]. *The system (f, μ) has exponential decay of correlations in the space of Hölder continuous functions.*

Theorem 6.4 [BeY3]. *The system (f, μ) satisfies a central limit theorem in the space of Hölder continuous functions.*

Their basic strategy is to construct a Markov extension of the initial map exhibiting many of the features of a uniformly hyperbolic set (a "horseshoe with infinitely many branches").

The next result brings this study close to conclusion, essentially completing the solution for Hénon-like attractors of Problems 1, 2, 3 stated in the Introduction.

Theorem 6.6 [BeV2]. *Hénon-like strange attractors are stochastically stable under certain random perturbations.*

The class of random perturbations we consider in this theorem includes

$$t \mapsto f_t(x, y) = f(x, y) + t,$$

where the random variable t takes values in an ε-neighbourhood of $0 \in \mathbb{R}^2$, according to an absolutely continuous probability distribution θ_ε such that $\varepsilon(d\theta_\varepsilon/dm)$ is uniformly bounded away from zero and infinity.

6.2. Multidimensional attractors.

In all the situations considered in the previous section the attractor is essentially a 1-dimensional object, both from a topological and a dynamical point of view, independently of the dimension of the ambient manifold. What we mean by this is that not only the topological dimension of all these attractors is equal to 1 but, moreover, the map exhibits a unique positive Lyapunov exponent (a single direction of expansion) on typical points. That is to say, sensitive dependence on initial conditions close to these points occurs only along one direction: in complementary directions nearby orbits tend to approach as time increases.

A class of high dimensional Hénon-like attractors was introduced in [Vi97]. Similarly to the systems we have been discussing, they combine hyperbolic behaviour and folding behaviour. Yet, these high dimensional attractors have substantially different properties, in particular, they are much more robust than their low dimensional counterparts. To explain this, let us consider the simplest model studied in [Vi97]. These are smooth maps f of the cylinder $M = S^1 \times \mathbb{R}$ given by

$$f(\theta, x) = (g(\theta), a(\theta) - x^2), \tag{6.1}$$

where $g(\cdot)$ is an expanding map of the circle, and $a(\cdot)$ is some Morse function on S^1. For instance, $g(\theta) = d\,\theta \mod \mathbb{Z}$ for some large $d \in \mathbb{N}$, and $a(\theta) = a_0 + \alpha \sin(2\pi\theta)$ where $a_0 \in (1, 2)$ is such that the critical point $c = 0$ is nonrecurrent for the quadratic map $h(x) = a_0 - x^2$, cf. (5.1). The parameter α is taken to be small, in particular, $a_0 + |\alpha| < 2$. Then its easy to find a compact interval $I \subset \mathbb{R}$ such that $f(S^1 \times I) \subset \text{interior}(S^1 \times I)$.

The next theorem states that *both Lyapunov exponents are positive* at almost every orbit in the invariant region, and the same is true *for any map in a neighbourhood of* f. In other words, these maps exhibit sensitive dependence on the initial state in all directions in phase-space.

Theorem 6.7 [Vi97]. *Let f be as above, with α sufficiently small. Then there exists a neighbourhood $\mathcal{N}$ of f is the space of all C^3 maps from $M = S^1 \times \mathbb{R}$ to itself, and there exists a constant $c_0 > 0$, such that*

$$\liminf_{n \to +\infty} \frac{1}{n} \log \|D\tilde{f}^n(z)v\| \geq c_0$$

for every $\tilde{f} \in \mathcal{N}$, every nonzero vector $v \in T_z M$, and Lebesgue almost every point $z \in S^1 \times I$.

A main point distinguishing these examples from the quadratic maps in Chapter 5 and the Hénon-like attractors in Section 6.1, is the nature of the *critical* region, i.e. the region in phase-space where the folding effect occurs.

For quadratic maps f of the interval there is a well-defined critical point $c = 0$. Then, to have the map display expanding behaviour (positive Lyapunov at most points) one must be certain that contraction that takes place every time an orbit passes close to the critical point does not accumulate too much. For instance, it is easy to check that if

$$|f^n(c) - c| < 4^{-n}$$

for some $n \geq 1$, then $f^n(\text{closure}(J)) \subset J$ and $|(f^n)'| < 1$, for $J = (-2\,4^{-n}, 2\,4^{-n})$. This implies that f has a periodic attractor, and then the Lyapunov exponent is negative at almost every bounded orbit. Thus, for the quadratic map to exhibit nonperiodic behaviour some condition on the parameter is needed, preventing the iterates of the critical point c from returning too fast to the neighbourhood of c. That is the meaning of (5.1) and of assumption (U2) in Chapter 5. As mentioned in there, this last condition is satisfied by a set of parameters with positive Lebesgue measure. A considerably more sophisticated form of this approach is used in the case of Hénon attractors, whose *critical set* [BC91] is, roughly speaking, of Cantor set with very small dimension.

For the systems in (6.1) the critical set $\mathcal{C}$ is easily identified: it is just the curve $S^1 \times \{0\}$, where the derivative of f fails to be an isomorphism. However, one can no longer expect to be able to prevent iterates of $\mathcal{C}$ from being close to $\mathcal{C}$ in the way we did for 1-dimensional quadratic maps in (U2). The reason is clear: both $\mathcal{C}$ and its iterates have topological dimension 1, in a 2-dimensional ambient manifold, and so having some $f^n(\mathcal{C})$ intersecting $\mathcal{C}$ is a robust phenomenon: it can not be avoided through any sort of small parameter modifications. Let us point out that this is not specific to these examples, rather it is a very general feature of multidimensional maps, as the critical set generically looks like a codimension 1 submanifold.

The method to prove the existence of two positive Lyapunov exponents for the maps in Theorem 6.7 must, therefore, be considerably different from those arguments one uses for systems where a control of the recurrence of the critical set is possible: one has to rely on a statistical analysis of the visits of points to the neighbourhood of the critical set. The way this is done can be sketched as follows. Given a point $z = (\theta, x) \in S^1 \times I$, the contracting effect of the derivative $Df(z)$ is estimated through the logarithm

$$\Delta(z) = \log|-2x|$$

of the derivative of f along the vertical direction (for nearby maps $\tilde{f}$ one uses the derivative along the leaves of a nearly vertical invariant foliation). Then the overall contraction accumulated on the orbit of z up to time $n \geq 1$ is given by

$$\sum_{\nu=0}^{n-1} \Delta_\nu(z), \qquad \text{where } \Delta_\nu(z) = \Delta(f^\nu(z)).$$

The following stochastic properties of these Δ_ν are crucial (see [Vi97] for more precise statements):

(1) the average value of Δ_ν, with respect to Lebesgue measure, is positive for each $\nu \geq 1$, and uniformly bounded away from zero;
(2) Δ_μ and Δ_μ are fairly independent if μ and ν are far apart, more precisely, the correlation between Δ_μ and Δ_ν decreases very fast as $|\mu - \nu|$ increases.

This permits to conclude, through a large deviations argument, that for almost every point z

$$\liminf \frac{1}{n} \sum_{\nu=0}^{n-1} \Delta_\nu(z) \geq c_0 > 0 \tag{6.2}$$

where $c_0 > 0$ is related to (smaller than) the lower bound in (1). This yields the conclusion of Theorem 6.7.

Substantial development of this approach allowed [Al] to construct an SRB measure for these maps.

Theorem 6.8 [Al]. *Let f and $\mathcal{N}$ be as before. Then every map $\tilde{f} \in \mathcal{N}$ has an absolutely continuous (with respect to 2-dimensional Lebesgue measure) finite invariant measure μ, which is ergodic and so is an SRB measure for $\tilde{f}$.*

In fact, these maps $\tilde{f}$ are ergodic with respect to Lebesgue measure, which implies that the absolutely continuous measure is unique; see the Appendix of [Al].

The following notion plays a crucial role in the proof of Theorem 6.8. Let $c_1 \in (0, c_0)$ be fixed. An integer $n \geq 1$ is a *hyperbolic time* for a point $z \in S^1 \times I$ if

$$\frac{1}{k} \sum_{\nu=n-k}^{n-1} \Delta_\nu(z) \geq c_1 \tag{6.3}$$

for every $1 \le k \le n$. A main point is that if n is a hyperbolic time for z then f^{-k} is contracting at $f^n(z)$, for every $1 \le k \le n$, uniformly on z and n and exponentially on k. Another important consequence is a bounded distortion property for the jacobian of f^{-n} close $f^n(z)$, as we shall see in a while.

One proves, using (6.2), that *almost every point has infinitely many hyperbolic times.* Another important step is to construct a partition $\{R_i : i \in \mathbb{N}\}$ of (a full Lebesgue measure subset of) M into rectangles, and a piecewise expanding map F satisfying

(a) F is smooth on each R_i, in fact $F_i|R_i = f^{n_i}|R_i$ where n_i is a hyperbolic time for some point in R_i;

(b) $F|R_i$ is expanding, with uniform expansion rate, and the distortion $\|D(\log J_i)\|$ of the jacobian $J_i = |\det(DF|R_i)^{-1}|$ is uniformly bounded;

(c) the images $F(R_i)$ are rectangles (domains bounded by the union of four smooth curves) with size and angles at the boundary vertices uniformly bounded from below.

In addition,

(d) the total area of the rectangles R_i for which n_i is larger than some given $n \ge 1$ decreases with n as $\exp(-\text{const}\sqrt{n})$.

Existence theorems for certain piecewise expanding maps with finitely many smoothness domains were proved, e.g. by [Ke79] and [GB89], using a high dimensional version of the notion of functions with bounded variation [Gi84]. An extension of these theorems for maps with infinitely many smooth domains, proved in [Al], applies to maps satisfying (a), (b), (c) above. It gives that F admits some absolutely continuous invariant measure μ_F. Then the measure

$$\mu = \sum_{n=0}^{\infty} f_*^n \left(\mu_F \mid \bigcup_{i:n_i>n} R_i\right)$$

is invariant under f and, using property (d) and the fact that the density of μ_F has bounded variation, one checks that μ is a finite measure.

Theorems 6.7 and 6.8 hold for variations of the previous construction where one takes g to be an expanding map on an arbitrary torus T^k. [Vi97] also treats another construction where the map f is a diffeomorphism, with a multidimensional strange attractor. SRB measures are expected to exist also for these attractors, but a construction has not yet been carried out.

In fact, these last examples are part of a large class of diffeomorphisms usually called partially hyperbolic. A C^1 diffeomorphism $f : M \to M$ on a compact manifold M is *partially hyperbolic* if there exists a continuous splitting $TM = E \oplus E^c$ of the tangent bundle to M, invariant under Df

(1) $Df(x)v \in E_{f(x)}$ for every $v \in E_x$, and $Df(x)v \in E^c_{f(x)}$ for every $v \in E^c_x$,

and there exist $\lambda < 1$ and a riemannian metric $\|\cdot\|$ on M, such that

(2s) either $\|Df|E_x\| \le \lambda$ and $\|Df|E_x\| \, \|(Df|E^c_x)^{-1}\| \le \lambda$ for every $x \in M$,

(2u) or $\|(Df|E_x)^{-1}\| \le \lambda$ and $\|(Df|E_x)^{-1}\| \, \|Df|E^c_x\| \le \lambda$ for every $x \in M$.

The first condition in (2s)-(2u) means that the derivative is uniformly hyperbolic (either contracting or expanding) on the subbundle E. The second condition allows for both contraction along the *central bundle* E^c, but it requires that the behaviour of Df on E *dominate* the behaviour on E^c: in case (2s) any contraction exhibited by $Df|E^c$ must be weaker than the contraction rate along the *strong-stable bundle* $E = E^{ss}$; analogously, in case (2s) any expansion exhibited by $Df|E^c$ must be weaker than the expansion rate along the *strong-unstable bundle* $E = E^{uu}$.

Topological and dynamical properties of partially hyperbolic systems have been studied for some time. [Sh71] used a skew-product construction to give the first examples of nonhyperbolic diffeomorphisms of T^4 which are C^1 *robustly transitive*: any C^1 close diffeomorphism has orbits dense in the ambient manifold. The maps he obtained in this way are partially hyperbolic.

Another construction, also yielding partially hyperbolic diffeomorphisms, enabled [Ma78] to reduce to three the minimum dimension of such examples. He starts with an Anosov diffeomorphism f_0 of $M = T^3$ having two expanding directions and one contracting direction. More precisely, Df_0 admits an invariant splitting into three subbundles $TM = E_0^{uu} \oplus E_0^u \oplus E_0^{ss}$ such that $Df_0|E_0^{ss}$ is uniformly contracting, and $Df_0|E_0^{uu}$ and $Df_0|E^u$ are uniformly expanding, with a larger expansion rate on E_0^{uu} say. Then he modifies the diffeomorphism in such a way as to render the *weak-unstable* direction E_0^u contracting in a neighbourhood of some periodic point. The new diffeomorphisms f also admit a Df-invariant splitting

$$TM = E^{uu} \oplus E^c \oplus E^{ss} \tag{6.4}$$

where $Df|E^{uu}$ is uniformly expanding, $Df|E^{ss}$ is uniformly contracting, and they both dominate $Df|E^c$. There are invariant foliations $\mathcal{F}^{uu}$, $\mathcal{F}^c$, $\mathcal{F}^{ss}$ tangent to each of these subbundles and, with the help of these foliations, one proves that these diffeomorphisms are transitive. The nature of $Df|E^c$ depends on the point, in particular, there are periodic points where it is contracting and other periodic points where it is expanding. Therefore, any such f can not be an Anosov diffeomorphism.

Recently, [BD96] provided new constructions of partially hyperbolic diffeomorphisms which are robustly transitive, including the first examples where the central direction has dimension larger than 1.

Even more recently, a result of [DPU] displays an intimate relation between partial hyperbolicity and C^1 persistence of transitiveness, at least for diffeomorphisms in three dimensions.

Theorem 6.9 [DPU]. *Let M be a 3-dimensional manifold, and f be a C^1 diffeomorphism of M which is C^1 robustly transitive. Then f is partially hyperbolic, i.e., it satisfies either (1)-(2s) or (1)-(2u).*

Examples by [Bo] show that C^1 robust transitiveness does not imply the stronger form of partial hyperbolicity in (6.4). Other such examples, of C^1 robustly transitive diffeomorphisms that do not admit an invariant splitting into three subbundles, can be obtained by a modification of [Ma78], see [BoV].

On the other hand, the ergodic properties of partially hyperbolic diffeomorphisms are still poorly understood. SRB measures were constructed by [Ca93] for diffeomorphisms derived from Anosov diffeomorphisms through (Hopf, saddle-node, period doubling) bifurcations of periodic points. In her context, SRB measures coincide with Gibbs u-states (invariant measures having absolutely continuous conditional measures along the leaves of the *strong-unstable* foliation $\mathcal{F}^{uu}$), constructed by [PS82] for general partially hyperbolic attractors. This approach to finding SRB measures works quite in general, when the derivative is "predominantly contracting" on the central direction E^c, see [BoV]. When the central direction E^c is, on the contrary, "predominantly expanding", Gibbs u-states are no longer SRB-measures. Situations of this kind, including examples of the type constructed by [Ma78], are dealt with in an ongoing work by Alves, Bonatti, and the present author.

6.3. Lorenz-like flows.

Attractors of flows present important new features with respect to the discrete time case, specially when they involve singularities (equilibrium points) interacting with regular orbits. Such singular attractors were first introduced by [ABS77], [GW79], as models for the observations made by Lorenz in [Lo63]. A key fact about these geometric Lorenz models is that they are robust: any flow close to the initial one has an attractor with similar properties.

There is now a vast literature on the geometric, dynamical, and ergodic properties of these attractors. See e.g. [Sp82], [Bu83], [CT88], [Pe92], [Sa92], and references therein. Curiously enough, the actual occurrence of a strange attractor in Lorenz original equations remains an unproven conjecture. On the other hand, similar types of robust attractors have been shown to appear in various types of bifurcations of flows [Ro89], [Ry89], [ACL95], [MP2], [MP1], [MPP1]. Recently, [MMP2] have been developing a theory of singular attractors of flows in three dimensions, characterizing robustness in terms of a hyperbolicity property. For a precise statement of their results we need a few definitions.

Let $(X_t)_{t\in\mathbb{R}}$ be a C^k flow on a manifold M, $k \geq 1$, and $\Lambda \subset M$ be compact and invariant under the flow. One calls Λ a *singular* (or *Lorenz-like*) *attractor* if

(a) the flow is transitive on Λ;

(b) there exists an open neighbourhood U of Λ such that $\Lambda = \cap_{t>0} X_t(U)$;

(c) Λ contains some singularity of the flow (and is not reduced to it).

Singular transitive sets are defined analogously, just taking the intersection over all $t \in \mathbb{R}$ in (b). A singular attractor is called C^1 *robust* if

$$\Lambda_Y = \bigcup_{t>0} Y_t(U)$$

is again a singular attractor, for any C^1 near flow $(Y_t)_{t\in\mathbb{R}}$. A similar notion applies to singular transitive sets, with the intersection taken over $t \in \mathbb{R}$.

We say that $\Lambda \subset M$ is a *singular hyperbolic set* for the flow if it is invariant and there exists a continuous splitting

$$T_\Lambda M = E^{ss} \oplus E^{cu}$$

of the tangent bundle over Λ, which is invariant under the flow and satisfies

(i) E^{ss} has dimension 1 and is uniformly contracting;

(ii) E^{cu} is volume expanding and is dominated by E^{ss}.

Denote $-X$ the flow obtained from $X = (X_t)_{t\in\mathbb{R}}$ by reversing the direction of time.

Theorem 6.10 [MMP2]. *Let Λ be a C^1 robust singular transitive set of a C^1 flow on a three dimensional manifold M. Then Λ is a singular hyperbolic set, and a singular attractor, for either X or $-X$.*

They also characterize the type of singularities that may be contained in a C^1 robust singular transitive set: the eigenvalues are necessarily real and satisfy

(6.5) $\qquad$ either $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$, or $\lambda_2 > \lambda_3 > 0 > -\lambda_3 > \lambda_1$;

moreover, all the singularities have the same number of contracting eigenvalues.

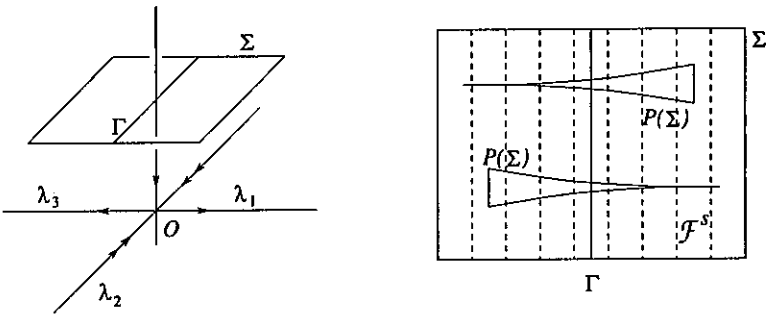

Fig 6.3: Poincaré map of a Lorenz-like flow

Statistical properties of Lorenz-like attractors of flows may be derived from the information on Lorenz-like maps we obtained in Chapter 3. We just mention a few main ideas in this direction; see e.g. [CT88] and references mentioned in it for further information and proofs. We begin by recalling some basic facts concerning the geometric Lorenz models of [ABS77], [GW79]. By construction, these systems have a two-dimensional submanifold Σ as a partial cross-section to the flow. More precisely, there is a curve $\Gamma \subset \Sigma$ and a well-defined Poincaré first-return map

$$P : \Sigma \setminus \Gamma \to \Sigma.$$

The curve Γ corresponds to the intersection of Σ with the stable manifold of the singularity O contained in the attractor, and future trajectories of points in Γ do not come back to Σ. The Poincaré map P is hyperbolic, in the following sense:

(1) P admits an invariant contracting smooth foliation $\mathcal{F}^s$ containing Γ as a leaf; that is, every leaf $\mathcal{F}^s_z$ is mapped by P completely inside some leaf $\mathcal{F}^s_{P(z)}$, and $P|\mathcal{F}^s_z$ is a uniform contraction;

(2) the space of leaves $\Sigma/\mathcal{F}^s$ of this foliation $\mathcal{F}^s$ is diffeomorphic to an interval, and the map f induced by P on $\Sigma/\mathcal{F}^s$ is a Lorenz-like map: it is uniformly expanding, with derivative tending to infinity as one approaches Γ.

Then, as we have seen in Chapter 3, f admits a unique invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure m. Moreover, μ is ergodic and so it is an SRB measure for f. Recall also that the density $d\mu/dm$ is a function with bounded variation, in particular it is bounded.

From μ we may now construct an SRB measure μ_P for the Poincaré return map P, through the following general procedure [Bo75]. Since μ is defined on the interval, identified to the space of leaves of the contracting foliation $\mathcal{F}^s$, we may also think of it as a measure on the σ-algebra of Borel subsets of Σ which are union of entire leaves of $\mathcal{F}^s$. Using the fact that P is uniformly contracting on leaves of $\mathcal{F}^s$, one concludes that the sequence

$$P_*^n \mu, \qquad n \geq 1,$$

of push-forwards of μ under P is weak*-Cauchy: given any continuous $\varphi : \Sigma \to \mathbb{R}$

$$\int \varphi \, d(P_*^n \mu) = \int (\varphi \circ P^n) \, d\mu, \qquad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$. Define, μ_P to be the weak*-limit of this sequence, that is,

$$\int \varphi \, d\mu_P = \lim \int \varphi \, d(P_*^n \mu)$$

for each continuous φ. Then μ_P is invariant under P, and it is an SRB measure for the Poincaré return map: this last statement follows from the fact that μ is an SRB measure for f, together with the remark that asymptotic time-averages of continuous functions $\varphi : \Sigma \to \mathbb{R}$ are constant on the leaves of $\mathcal{F}^s$.

Next, another general procedure yields an SRB measure μ_X for the flow. Denote $\tau : \Sigma \setminus \Gamma \to (0, +\infty)$ the return time to Σ, defined by,

$$P(z) = X_{\tau(z)}.$$

Then τ is bounded away from zero. Moroever,

$$P(z) \approx |\log d(z, \Gamma)|$$

for z close to Γ. Combining this with the definition of μ_P and the remark made above that $d\mu/dm$ is a bounded function, one may conclude that

$$\tau_0 = \int \tau \, d\mu_P < \infty. \tag{6.6}$$

Denote by $\sim$ the equivalence relation on $\Sigma \times \mathbb{R}$ generated by $(z, \tau(z)) \sim (P(z), 0)$. Let $N = (\Sigma \times \mathbb{R}) / \sim$ and $\nu = \pi_*(\mu_P \times dt)$, where $\pi : \Sigma \times \mathbb{R} \to N$ is the quotient map and dt

is Lebesgue measure in $\mathbb{R}$. Observe that (6.6) means that ν is a finite measure. Define $\phi : N \to M$ by $\phi(z,t) = X_t(z)$, and let $\mu_X = \phi_* \mu_P$. The measure μ_X is invariant under the flow, and one can check that it is an SRB for the flow X:

$$\frac{1}{T}\int_0^T \varphi(X_t(z))\,dt \to \int \varphi\,d\mu_X \quad \text{as } T \to +\infty$$

for every continuous function $\varphi : M \to \mathbb{R}$, and Lebesgue almost every point $z \in \phi(N)$.

Other types of singular attractors of flows have also been constructed in recent years, displaying more subtle forms of persistence under perturbations. [Ro93] considered a modification of the classical geometric Lorenz models where the eigenvalues at the singularity satisfy

$$\lambda_2 < \lambda_3 < 0 < \lambda_1 < -\lambda_3.$$

In other words, the expanding eigenvalue is dominated by both contracting eigenvalues, compare (6.5). The singular attractors he obtains in this way have rather different properties, if compared with the cases we have been discussing, in particular they are persistent only in a measure-theoretical sense: a singular attractor exists for a positive Lebesgue measure set of parameters, in generic parametrized families of flows through the initial one.

[LV1], [LV2], introduce an extended geometric model for Lorenz equations, which combines singular behaviour (presence of singularities) and critical behaviour (folding, in the sense of Section 6.1). In particular, these critical geometric Lorenz models do not admit an invariant contracting foliation, as do all the cases we have mentioned so far. Singular attractors occurring for these flows have the same kind of measure-theoretical persistence as those [Ro93]. The same kind of persistence holds also for the spiral attractors in [PRV1], [PRV2]. These are global singular attractors with very complicated geometric structure, related to the unfolding of homoclinic connections associated to saddle-focus singularities (two contracting complex eigenvalues and one real expanding eigenvalue, dominating the contracting ones). See Figure 6.4.

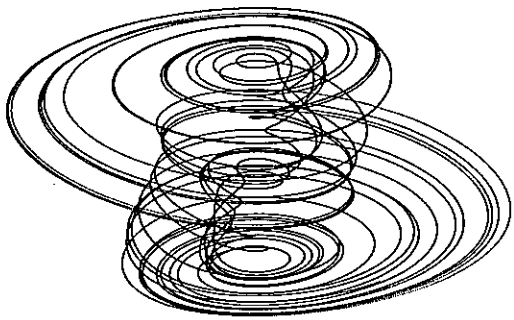

Figure 6.4: A spiral attractor

In a similar setting, [PR97] proved persistence of suspended Hénon-like attractors, and even gave the first examples of coexistence of infinitely many such attractors.

In all the cases we have mentioned so far, the singularity contained in the attractor has only one expanding eigenvalue (the second alternative in (6.5) corresponds to a repeller, i.e. an attractor for the reversed flow $-X$). The problem whether robust singular attractors may contain singularities whose unstable manifold has dimension larger than 1 was recently solved by [BPV]: given $k \geq 2$ and any $n \geq k+3$, there exists a manifold M with dimension n, and there exists a C^1-open set of vector fields exhibiting a transitive attractor that contains a hyperbolic singularity whose unstable manifold has dimension k. Under further assumptions of regularity, these *multidimensional singular attractors* support a unique SRB measure.

Closing this section, we mention that a statement of exponential decay of correlations was recently proved by [Yo] for yet another important class of systems: plane billiards with convex scatterers [Si70].

6.4. Finitude of attractors.

As we mentioned in the Introduction, it was believed for some time that most dynamical systems should be simple (e.g., should have only finitely many attractors) and should be stable (robust) under small perturbations. Attempts to make these ideas precise, mostly in topological terms, proved unsuccessful throughout the sixties and seventies. Here we want to close with a brief presentation of ideas recently advanced by Palis [Pa] which in a sense vindicates those early expectations of simplicity and stability in Dynamics, even if in new, more probabilistic, terms.

We restrict ourselves to the setting of diffeomorphisms on compact manifolds, although the scope of his program is more general (including noninvertible transformations, smooth flows and, possibly, evolution partial differential equations).

A main ingredient is the conjecture that the diffeomorphisms having finitely many attractors are dense: any diffeomorphism g may be approximated by another one f for which one may partition the ambient manifold M as $U_1 \cup \cdots \cup U_k \cup R$, so that

(1) each U_i, $1 \leq i \leq k$, is an open set invariant under f and R is a closed set with zero Lebesgue measure;
(2) every point $z \in U_i$, $1 \leq i \leq k$, converges under positive iteration to some compact set $\Lambda_i \subset U_i$ which is invariant and transitive (contains dense orbits) for f;
(3) each Λ_i, $1 \leq i \leq k$ is either an attracting periodic orbit or a strange attractor.

The definition of *strange attractor* is largely motivated by the examples we have been discussing, specially the Hénon-like attractors: in (3) one means that Λ_i

(1) contains some dense orbit along which the derivative of f grows exponentially fast, in norm, as $n \to +\infty$;
(2) contains a dense subset consisting of periodic saddles and coincides with the closure of the unstable manifold of some of those saddles;
(3) supports an ergodic SRB-measure μ_i having some positive Lyapunov exponent, and whose basin $B(\mu_i)$ has full Lebesgue measure in U_i.

(4) is *persistent* under perturbations of the system: generic parametrized families of diffeomorphisms f_s, $s \in \mathbb{R}^p$, with $f_0 = f$, exhibit a similar attractor for a positive Lebesgue measure set of parameters s close to zero.

Furthermore, conjecturedly, the dynamics of such systems with finitely many attractors is robust under small perturbations, in two different senses. A first one concerns metric stability of the basins of attraction: given a small perturbation $\tilde{f}$ of any f as above, for each $1 \leq i \leq k$ there are $n_i \geq 1$ and attractors $\tilde{\Lambda}_{i,1}, \ldots, \tilde{\Lambda}_{i,n_i}$, either periodic or strange, such that the union of the corresponding basins $\tilde{U}_{i,1}, \ldots, \tilde{U}_{i,n_i}$ coincides with $B(\Lambda_i)$ up to a small Lebesgue measure set. A second one has to do with persistence, under small random perturbations, of the statistical properties of such diffeomorphisms on the basin of each attractor Λ_i: the system $(f|U_i, \mu_i)$ is stochastically stable for all $1 \leq i \leq k$.

APPENDIX A: INVARIANT FOLIATIONS OF HYPERBOLIC SETS

In this appendix we review some basic properties of uniformly hyperbolic sets which are relevant for the previous chapters, specially Chapter 4. See [Sh87] and [Ma87] for a presentation of the theory of hyperbolic systems, including proofs of the results we quote here. The versions for random maps we mention at the end are explicitly stated and proved in [LQ95]

In all that follows f is a C^k diffeomorphism, $k \geq 1$, of a compact manifold M. We say that $p \in M$ is a *hyperbolic fixed point* for f if $f(p) = p$ and the spectrum of the derivative $Df(p) : T_pM \to T_pM$ is disjoint from the unit circle. This last condition implies, in particular, that $Df(p) - \mathrm{id}$ is an isomorphism of T_pM and so, by a standard application of the implicit function theorem, the point p has a *hyperbolic continuation* for diffeomorphisms close to f. What this means is that if $\tilde{f}$ is a diffeomorphism C^k close to f, then $\tilde{f}$ has a (unique) hyperbolic fixed point $\tilde{p}$ close to p.

We define the *stable manifold* $W^s(p)$ and the *unstable manifold* $W^u(p)$ of a fixed point p, by

$$W^s(p) = \{x \in M : \lim_{n\to+\infty} f^n(x) = p\} \quad \text{and} \quad W^s(p) = \{x \in M : \lim_{n\to-\infty} f^n(x) = p\}$$

Theorem A.1. *Let $p \in M$ be a hyperbolic fixed point of a C^k diffeomorphism f. Then $W^s(p)$ is an injectively immersed C^k submanifold of M. Moreover, if $\tilde{f}$ is C^k near f then the stable manifold $W^s(\tilde{p})$ of the hyperbolic continuation of p is C^k near $W^s(p)$ on compact parts.*

A dual result, for the unstable manifold, follows from this just by considering the inverse f^{-1} in the place of f. Moreover, these notions and conclusions extend to the case when p is a periodic point of f, that is, if $f^k(p) = p$ for some $k \geq 1$: it suffices to replace f by f^k.

Hyperbolic sets, cone fields.

Let Λ be a compact subset of M that is *invariant* under f, i.e., $f(\Lambda) = \Lambda$. We say that Λ is a *hyperbolic set* for f if there exists a splitting $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$ of the tangent bundle to M on Λ, and there are $\lambda_0 < 1$ and a riemannian metric $\|\cdot\|$ on M, so that

(1) $Df(x) \cdot E^s_x = E^s_{f(x)}$ and $Df^{-1}(x) \cdot E^u_x = E^u_{f^{-1}(x)}$;

(2) $\|Df(x)|E^s_x\| \leq \lambda_0$ and $\|Df^{-1}(x)|E^u_x\| \leq \lambda_0$ for every $x \in \Lambda$.

In practice, one seldom knows a priori the decomposition $T_\Lambda M = E^s \oplus E^u$. Fortunately, in order to prove that a compact invariant subset is hyperbolic it suffices to exhibit some reasonable approximation of the stable (E^s) and the unstable (E^u) subbundles. The precise formulation of this is in the next theorem, which involves the notion of *invariant (stable and unstable) cone fields.* Let $T_\Lambda M = F_1 \oplus F_2$ be an arbitrary continuous splitting of the tangent bundle. Given $a \leq 1$ and $b \leq 1$ (and a riemannian metric $\|\cdot\|$ on M) define cone fields $\Lambda \ni x \mapsto C^s_a(x)$, and $\Lambda \ni x \mapsto C^u_b(x)$ by

$$C^s_a(x) = \{v_1 + v_2 \in F_1 \oplus F_2 : \|v_2\| \leq a\|v_1\|\}$$

$$C^u_a(x) = \{v_1 + v_2 \in F_1 \oplus F_2 : \|v_1\| \leq b\|v_2\|\}.$$

Theorem A.2. *Let Λ be a compact invariant set for a C^1 diffeomorphism f. Suppose that there is a splitting $T_\Lambda M = F_1 \oplus F_2$ and there are constants $a \le 1$, $b \le 1$, and $\lambda < 1$ so that*

$$Df^{-1}(x)\cdot C^s_a(x) \subset C^s_{\lambda a}(f^{-1}(x)) \quad \text{and} \quad Df(x)\cdot C^u_b(x) \subset C^u_{\lambda b}(f(x))$$

and

$$\|Df^{-1}(x)w_s\| \ge \lambda^{-1}\|w_s\| \quad \text{and} \quad \|Df(x)w_u\| \ge \lambda^{-1}\|w_u\|$$

for every $x \in \Lambda$, $w_s \in C^s_a(x)$, $w_u \in C^u_b(x)$. Then Λ is a hyperbolic set for f.

The simplest example of a hyperbolic set is the set $\Lambda = \{p\}$ formed by a single hyperbolic fixed point. A hyperbolic splitting as in the definition is provided by the eigenspaces E^s_p, respectively E^u_p, of $Df(p)$ associated to the eigenvalues with norm smaller than 1, respectively larger than 1.

Example A.1 (The horseshoe) Let Q be a square in the plane and consider a decomposition $Q = R_1 \cup \cdots \cup R_5$ of Q into five horizontal strips. Let also D_1 and D_2 be two half disks at the top and bottom sides of Q. Consider an embedding f of $Q \cup D_1 \cup D_2$ into itself as described in the figure.

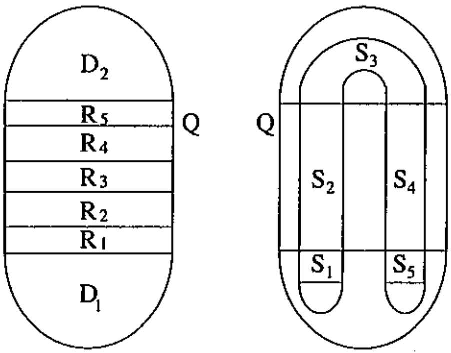

Fig A.1: A horseshoe

More precisely,

(a) f maps R_2, R_4 affinely into vertical strips S_2, S_4 crossing Q from top to bottom: it preserves the horizontal and the vertical directions, contracting by $\lambda < 1/2$ along the horizontal direction and expanding by $\rho > 2$ along the vertical direction;

(b) f sends D_1, R_1, D_2, R_2 inside D_1, and R_3 inside D_2.

Then f is easily extended to a diffeomorphism of the two-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, with a repelling fixed point outside $Q \cup D_1 \cup D_2$. Note that, by (b), the map f must also have a fixed point in D_1.

There are two possibilities for the orbit of a point $x \in Q$: either it eventually enters D_1, in which case it stays there forever, or it remains all the time in Q. The set

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(Q)$$

of points whose full orbit is inside Q is a hyperbolic set for f. To see this, note first that Λ is contained in $R_2 \cup R_4$. Then the hyperbolic splitting $T_\Lambda M = E^s \oplus E^u$ for f on Λ is given by

$$E^s_x = \mathbb{R} \times \{0\} \quad \text{and} \quad E^u_x = \{0\} \times \mathbb{R},$$

recall (a).

Actually, the requirements in (a) are unnecessarily strong: in particular, given any map g C^1 close to a map f as above,

$$\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(Q)$$

is a hyperbolic set for g. To see this, fix $a < 1$ consider the cone fields

$$C^s_a(x) = \{(\dot{x}_1, \dot{x}_2) \in T_x\mathbb{R}^2 : |\dot{x}_2| \leq a|\dot{x}_1|\}$$

$$C^u_a(x) = \{(\dot{x}_1, \dot{x}_2) \in T_x\mathbb{R}^2 : |\dot{x}_1| \leq a|\dot{x}_2|\}$$

defined on a neighbourhood U of $R_2 \cup R_4$. Clearly,

$$Df(x)\, C^*_a(x) \subset C^x_{(\lambda/\rho)a}(f(x)) \quad \text{for } * = s \text{ or } * = u,$$

and every $x \in U$ such that $f(x) \in U$. Moreover, vectors in $C^s_a(x)$ are uniformly expanded by $Df^{-1}(x)$ and vectors in $C^u_a(x)$ are uniformly expanded by $Df(x)$, at least if we fix a large enough. This means that these cone fields fulfill the hypotheses of Theorem A.2 for f. Most important, those hypotheses are robust under small C^1 perturbations: if they hold for some map then they also hold (possibly for a slightly different σ) for any C^1 close map. Thus these cone fields also satisfy the hypotheses of Theorem A.2 for the map g. This allows us to conclude that any g-invariant set contained in U is hyperbolic. In particular this applies to Λ_g.

Observe that the argument we have just used has little to do with the particular model we were considering. It shows that robustness under C^1 small perturbations is a general feature of hyperbolic sets (and a crucial one). We shall make these remarks more precise later.

Example A.2 (Anosov diffeomorphisms) : Consider a linear isomorphism A of $\mathbb{R}^2$ whose representation with respect to the canonical basis of $\mathbb{R}^2$ is a hyperbolic matrix with integer coefficients and determinant equal to 1. Then A preserves the lattice $\mathbb{Z}^2$, and so there exists a well-defined (unique) smooth map f from the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ to itself satisfying

$$\pi \circ A = f \circ \pi$$

where $\pi : \mathbb{R}^2 \to T^2$ is the canonical projection. Moreover, f is a diffeomorphism: its inverse may be obtained through the same construction, with A^{-1} in the place of A. Let λ and $\lambda' = 1/\lambda$ be the eigenvalues of A, with $|\lambda| < 1$. They are necessarily irrational numbers, and the corresponding eigenspaces $\hat{E}^s$ and $\hat{E}^u$ have irrational slope. Given any $w \in T^2$, choose $z \in \mathbb{R}^2$ such that $\pi(z) = w$, and then let

$$E^s_w = D\pi(z) \cdot E^s \quad \text{and} \quad E^u_w = D\pi(z) \cdot E^u.$$

Note that these objects do not depend on the choice of z. Then this defines subbundles $E^s = (E^s_w)_{w \in T^2}$ and $E^u = (E^u_w)_{w \in T^2}$ of the tangent space of T^2, and

$$TT^2 = E^s \oplus E^u$$

is a hyperbolic splitting for f: vectors in E^s_w are contracted by λ under $Df(w)$, and vectors in E^u_w are contracted by λ under $Df^{-1}(z)$. This proves that the whole ambient manifold $\Lambda = T^2$ is a hyperbolic set for f.

The same argument as in the previous example shows that T^2 is a hyperbolic set also for every map g which is C^1 close enough to f.

Example A.3 (The solenoid) Let $S^1 = \mathbb{R}/\mathbb{Z}$, D^2 be the closed unit disk in the complex plane, and let Q be the solid torus $Q = S^1 \times D^2$. Given $0 < \rho < 1/(2\pi)$ and $0 < \lambda < \rho$, let $f : Q \to Q$ be the map given by

$$f(\theta, z) = (2\theta \bmod \mathbb{Z}, \rho e^{2\pi i\theta} + \lambda z).$$

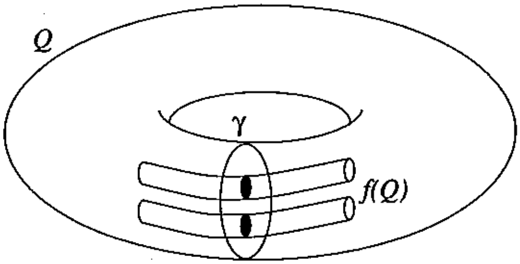

Figure A.2: The solenoid

Geometrically, f acts on the solid torus by stretching along the S^1 direction, contracting along the D^2 direction, and wrapping the image twice around the S^1 direction. The assumptions on ρ and λ ensure that f is an embedding of Q strictly into itself. Then the set

$$\Lambda = \bigcap_{n \geq 0} f^n(Q)$$

of those points whose orbit is defined for all times (both positive and negative) is a hyperbolic set for f. This may be seen by considering cone fields

$$C^s_a(p) = \{(\dot{\theta}, \dot{z}) \in T_p(S^1 \times D^2) : |\dot{\theta}| \leq a|\dot{z}|\}$$

$$C^u_b(p) = \{(\dot{\theta}, \dot{z}) \in T_p(S^1 \times D^2) : |\dot{z}| \leq b|\dot{\theta}|\}$$

with $b = 1$ and a sufficiently small, and using Theorem A.2.

Stability, invariant manifolds.

We noted before that a hyperbolic fixed point is a special case of a hyperbolic set. On the other hand, as we now explain, a hyperbolic set always corresponds to a hyperbolic fixed point of a convenient diffeomorphism on some Banach manifold. Let Λ be an invariant set for a diffeomorphism f, and $C(\Lambda, M)$ be the space of continuous maps from Λ to M. Let $\Gamma(\Lambda, M)$ be the space of continuous sections over Λ, i.e., the space of continuous maps

$$\sigma : \Lambda \longrightarrow T_\Lambda M \quad \text{such that} \quad \pi \circ \sigma = \text{id},$$

where π is the natural projection from $T_\Lambda M$ to Λ. Then $\Gamma(\Lambda, M)$ is a Banach space for the C^0 norm, and one may endow $C(\Lambda, M)$ with a natural structure of manifold over $\Gamma(\Lambda, M)$. The map $F_f : C(\Lambda, M) \longrightarrow C(\Lambda, M)$ given by $F_f(\varphi) = f\varphi f^{-1}$ is C^k if f is C^k, and it has the inclusion $i_\Lambda : \Lambda \longrightarrow M$ as a fixed point. Most important , this fixed point i_Λ is hyperbolic if and only if Λ is a hyperbolic set for f.

As a first consequence of this correspondence, we may derive the following result of persistence and stability of hyperbolic sets under small perturbations of the initial diffeomorphism.

Theorem A.3. *Let Λ be a compact invariant set for a C^1 diffeomorphism f on M. Then there is a a neighborhood $\mathcal{N}$ of f in the space of C^1 diffeomorphisms of M, and there is a continuous map $\phi : \mathcal{N} \longrightarrow C(\Lambda, M)$ with $\phi(f) = i_\Lambda$, such that for every $g \in \mathcal{N}$ the set $\Lambda_g = \phi(g)(\Lambda)$ is hyperbolic for g and we have $\phi(g) \circ f|\Lambda = g|\Lambda_g \circ \phi(g)$.*

Given any $x \in \Lambda$ and a map g C^1 close to f, we call the point $\tilde{x} = \phi(g)(x)$ given by this theorem the *hyperbolic continuation* of x for g.

Another important consequence of the correspondence made above is the existence, differentiability, and continuous variation with the dynamics, of stable and unstable manifolds associated to all points in a hyperbolic set Λ. To explain this we need some terminology. Fix some small $\varepsilon > 0$. We define the *local stable manifold* of size ε of a point $x \in \Lambda$

$$W^s_\varepsilon(x) = \{y \in M : \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0 \text{ and } d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}.$$

Analogously, the *local unstable manifold* of size ε of $x \in \Lambda$ is

$$W^u_\varepsilon(x) = \{y \in M : \lim_{n\to-\infty} d(f^n(x), f^n(y)) = 0 \text{ and } d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}.$$

The (global) *stable and unstable manifolds* of $x \in \Lambda$ are defined by

$$W^s(x) = \{y \in M : \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\}$$

and

$$W^u(x) = \{y \in M : \lim_{n\to-\infty} d(f^n(x), f^n(y)) = 0\},$$

respectively. It follows immediately from the definitions that

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_\varepsilon(f^n(x))) \quad \text{and} \quad W^u(x) = \bigcup_{n\geq 0} f^n(W^u_\varepsilon(f^{-n}(x)))$$

Theorem A.4. *Let $\Lambda \subset M$ be a hyperbolic set of a C^k diffeomorphism f on M, $k \geq 1$. Provided $\varepsilon > 0$ is small enough, every local stable manifold $W^s_\varepsilon(x)$ is a C^k disk embedded in M with $T_x W^s_\varepsilon(x) = E^s_x$. Moreover, it varies continuously with the point $x \in \Lambda$ in the C^k topology. Finally, if g is C^k-near f then the local stable manifold $W^s_\varepsilon(\bar{x})$ for g of the hyperbolic continuation $\bar{x} = \phi(g)(x)$ of x is uniformly close to $W^s_\varepsilon(x)$ in the C^k topology.*

Local product structure.

Let us restrict to an important subclass of hyperbolic sets, with the additional property of having local product structure. Take Λ to be a hyperbolic set of the diffeomorphism $f : M \longrightarrow M$, and let $\varepsilon > 0$ be fixed as before. Since the local stable and unstable manifolds of any given point in Λ are transverse, and these invariant manifolds depend continuously on the point, it follows that there exists $\delta > 0$ such that, for any pair of points $x, y \in \Lambda$ with $d(x, y) \leq \delta$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of a single point, which we denote by $[x, y]$. We say that Λ has a *local product structure* if $[x, y] \in \Lambda$ for all $x, y \in \Lambda$.

A crucial feature of hyperbolic sets with local product structure is the *shadowing property*, expressed in the next theorem. Given $\alpha > 0$, a sequence $\underline{x} = (x_k)_{k\in\mathbb{Z}}$ in M is called an *α-pseudo-orbit* for the diffeomorphism f if

$$d(f(x_k), x_{k+1}) \leq \alpha \quad \text{for every } k \in \mathbb{Z}.$$

If there is some $N > 0$ such that $x_{k+N} = x_k$ for every $k \in \mathbb{Z}$, we say that the α-pseudo-orbit $\underline{x}$ is periodic. Given $\beta > 0$, we say that $y \in M$ is a *β-shadow* for the α-pseudo-orbit if

$$d(f^k(y), x_k) \leq \beta \quad \text{for every } k \in \mathbb{Z}.$$

Theorem A.5. *Let f be a diffeomorphism on M and Λ be a hyperbolic set with local product structure. Given any $\beta > 0$ there is some $\alpha > 0$ such that every α-pseudo-orbit $\underline{x} = (x_k)_{k\in\mathbb{Z}}$ in Λ has a β-shadow $y \in \Lambda$.*

In the sequel we state some corollaries of this *shadowing lemma*. We say that an f-invariant set $\Lambda \subset M$ is *locally maximal* if there is a neighborhood U of Λ in M such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$.

Corollary A.6. *Let Λ be a hyperbolic set for the diffeomorphism f. Then Λ has a local product structure if and only if it is locally maximal.*

Let Λ be a hyperbolic set for the diffeomorphism f in M. We define the *stable set* $W^s(\Lambda)$ and the *unstable set* $W^u(\Lambda)$ of Λ by

$$W^s(\Lambda) = \{x \in M : \lim_{n\to+\infty} d(f^n(x), \Lambda) = 0\}$$

and

$$W^u(\Lambda) = \{x \in M : \lim_{n\to-\infty} d(f^n(x), \Lambda) = 0\}$$

Corollary A.7. *If Λ is a hyperbolic set with local product structure then*

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x) \qquad \textit{and} \qquad W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x).$$

Corollary A.8. *Let Λ be a hyperbolic set for the diffeomorphism f and suppose that the restriction of f to Λ is topologically mixing. Then*

(1) *$W^s(x)\cap\Lambda$ and $W^u(x)\cap\Lambda$ are dense in Λ for every $x\in\Lambda$.*
(2) *$W^s(x)\cap W^u(y)\cap\Lambda$ is dense in Λ for every $x, y\in\Lambda$.*
(3) *$W^*(x)$ is dense in $W^*(\Lambda)$ for every $x\in\Lambda$ and for both $*=s$ and $*=u$.*

We say that a compact set $\Lambda\subset M$ is an *attractor* for the diffeomorphism f on M, if $f(\Lambda)=\Lambda$ and there is some neighbourhood $Q\subset M$ of Λ for which

$$f(\text{closure}(Q))\subset Q \quad\text{and}\quad \Lambda=\bigcap_{n\geq 0} f^n(Q).$$

It follows from Corollary A.7 (or from Theorem A.9 below) that if an attractor Λ is a hyperbolic set for f then it has local product structure. In particular,

$$\bigcup_{x\in\Lambda} W^s(x) = W^s(\Lambda)$$

contains a whole neighbourhood Q of Λ.

Theorem A.9. *Let Λ be a hyperbolic set for the diffeomorphism f on M. Then Λ is an attractor for f if and only if for every $p\in\Lambda$ we have $W^u(p)\subset\Lambda$.*

We define the *local stable foliation* $\mathcal{F}^s_{loc}$ of a hyperbolic set $\Lambda\subset M$ to be the lamination defined by the local stable manifolds of points in Λ (given by Theorem A.4). If Λ is an attractor then $\mathcal{F}^s_{loc}$ is a proper foliation of a neighbourhood of Λ.

Absolute continuity.

In all that follows we assume that Λ is a hyperbolic attractor for a C^k diffeomorphism f on M, with $k\geq 2$. Let $\mathcal{F}^s_{loc}$ be the local stable foliation of Λ, and $W^s_{loc}\subset M$ be the union of the leaves of $\mathcal{F}^s_{loc}$. We use both $W^s_{loc}(x)$ and $\mathcal{F}^s_{loc}(x)$ to denote the leaf in $\mathcal{F}^s_{loc}$ through a point $x\in W^s_{loc}$. We say that a submanifold N of M is transverse to $\mathcal{F}^s_{loc}$ if

$$N\subset W^s_{loc} \quad\text{and}\quad T_xN\oplus T_xW^s_{loc}(x)=T_xM, \ \text{for every } x\in N\cap W^s_{loc}.$$

Given submanifolds N_1 and N_2 of M transverse to $\mathcal{F}^s_{loc}$, we say that $P: N_1\longrightarrow N_2$ is a *Poincaré map* for $\mathcal{F}^s_{loc}$ if it is injective and continuous, and

$$P(x)\in W^s_{loc}(x)\cap N_2 \quad \text{for every } x\in N_1$$

We say that a Poincaré map $P : N_1 \longrightarrow N_2$ is *absolutely continuous* if there is a continuous map $J : N_1 \longrightarrow \mathbb{R}$, the *jacobian* of P, such that

$$m_2(P(A)) = \int_A J dm_1 \quad \text{for every Borel subset } A \subset N_1,$$

where m_i denotes the smooth measure induced on N_i, $i = 1, 2$, by the riemannian metric.

Finally, we say that $\mathcal{F}^s_{loc}$ is absolutely (resp. Hölder) continuous if every Poincaré map for $\mathcal{F}^s_{loc}$ is absolutely (resp. Hölder) continuous.

Theorem A.10. *Let f be a C^2 diffeomorphism on M and $\mathcal{F}^s_{loc}$ be the local stable foliation associated to a hyperbolic attractor of f. Then, $\mathcal{F}^s_{loc}$ is absolutely continuous and Hölder continuous. Moreover, the jacobian of any Poincaré map for $\mathcal{F}^s_{loc}$ is also Hölder continuous.*

This theorem is used in Chapter 4 through the following corollary. Let us consider the disintegration [Ro66], [Ro67] $(p_\gamma)_{\gamma \in \mathcal{F}^s_{loc}}$ of the riemannian measure m with respect to the stable foliation. That is, each p_γ is a probability supported on the leaf γ, and

$$\int_Q \psi \, dm = \int \left(\int (\psi \mid \gamma) dp_\gamma \right) d\tilde{m}(\gamma)$$

for every integrable function ψ, where $\tilde{m}$ is the quotient measure on the space of leaves of $\mathcal{F}^s_{loc}$, defined by

$$\tilde{m}(\tilde{A}) = m\Big(\bigcup \{\gamma : \gamma \in \tilde{A}\} \Big).$$

We claim that one may take $p_\gamma = (H \mid \gamma) m_\gamma$, where m_γ is the riemannian metric on γ, and $H: Q \to \mathbb{R}$ is a strictly positive function having $\log H$ Hölder continuous.

Before we prove this result let us make a few comments on the stable foliation $\mathcal{F}^s_{loc}$. Let U be some small open set consisting of local stable leaves. Up to taking a local chart, we may pretend that U is an open subset of the euclidean space $\mathbb{R}^s \oplus \mathbb{R}^u$, where s and u are the dimensions of the invariant subbundles E^s and E^u, respectively. Then, with the help of Poincaré maps of $\mathcal{F}^s_{loc}$ we may define a homeomorphism

$$\Phi: D^u \times D^s \to U, \qquad \Phi(\xi, \eta) = (P(\xi, \eta), \eta)$$

where $D^s \subset \mathbb{R}^s$ and $D^u \subset \mathbb{R}^u$ are disks, satisfying

(1) $\Phi(D^u \times \{0\})$ is contained in some unstable manifold;

(2) $\{\Phi(\xi, \eta) : \eta \in D^s\}$ coincides with the stable leaf through $\Phi(\xi, 0)$, for all $\xi \in D^u$.

Moreover, in view of Theorems A.10 and A.4, Φ may be taken so that

(3) Φ and Φ^{-1} are Hölder continuous (in both variables) and Φ is C^2 on η;

(4) Φ is absolutely continuous, with Hölder continuous jacobian $J\Phi = J_\xi P$.

Now our claim can be deduced as follows. Let $\nu = \Phi_*^{-1} m$, that is, $d\nu = J\Phi \, d\xi \, d\eta$. Then

$$d\nu_\xi = J\Phi(\xi, \cdot) d\eta.$$

is a disintegration $(\nu_\xi)_\xi$ of ν with respect to the foliation $\{\{\xi\} \times D^s : \xi \in D^u\}$ of $D^u \times D^s$. Now, for each $\gamma = \Phi(\{\xi\} \times D^s)$ let

$$p_\gamma = (\Phi \mid \{\xi\} \times D^s)_* \nu_\xi \, .$$

That is, denoting $J_\eta \Phi$ the jacobian of the C^2 map $(\Phi \mid \{\xi\} \times D^s)$ with respect to the measures η and m_γ,

$$dp_\gamma = J\Phi(\xi, \cdot) \, (\Phi \mid \{\xi\} \times D^s)_* \, d\eta = \frac{J\Phi}{J_\eta \Phi}(\xi, \cdot) \, dm_\gamma \, .$$

The family of measures $(p_\gamma)_\gamma$ obtained in this way is a disintegration of m, and so we have shown that we may take

$$H(\Phi(\xi, \eta)) = \frac{J\Phi}{J_\eta \Phi}(\xi, \eta).$$

This proves our claim, since Φ^{-1}, $J\Phi$, $J_\eta \Phi$ are all Hölder continuous, and $J\Phi$, $J_\eta \Phi$ are bounded away from zero and infinity. Observe also that

$$\tilde{m}(\tilde{A}) = \nu\Big(\bigcup_{\tilde{A}} \{\xi\} \times D^s \Big) = \int_{\tilde{A}} J\Phi(\xi, \eta) \, d\xi \, d\eta.$$

where the union and the integral are over the $\xi \in D^u$ such that $\gamma = \phi^{-1}(\{\xi\} \times D^s)$ belongs in $\tilde{A}$ (and every $\eta \in D^s$).

Invariant foliations for random maps.

We close by giving precise statements of versions of the previous results for skew-product maps

$$F \colon Q \times T^{\mathbb{N}} \to Q \times T^{\mathbb{N}}, \quad F(x, t_1, t_2, \dots) = (f_{t_1}(x), t_2, \dots)$$

which are used in Section 4.5. Here T is just a metric space. We continue to suppose that $\Lambda = \cap_{n \geq 0} f^n(Q)$ is a hyperbolic attractor for f, and we assume that all the maps f_t, with $t \in T$, belong in a sufficiently small neighborhood of f.

We define the *tangent space* at a point $(x, \underline{t}) \in Q \times T^{\mathbb{N}}$ to be $T_x Q \times \{\underline{t}\}$. We say that $N \subset Q \times T^{\mathbb{N}}$ is a *submanifold* of $Q \times T^{\mathbb{N}}$ if for every $(x, \underline{t}) \in N$ the set $N_{\underline{t}} = N \cap (Q \times \{\underline{t}\})$ is a submanifold of $Q \times \{\underline{t}\} \approx Q$. Moreover, by definition, the tangent space to N at $(x, \underline{t}) \in N$ is $T_x N_{\underline{t}} \times \{\underline{t}\}$.

It is easy to check that

$$\hat{\Lambda} = \bigcap_{n \geq 0} F^n \left(Q \times T^{\mathbb{N}} \right)$$

is compact and F-invariant. We define the tangent bundle to $Q \times T^{\mathbb{N}}$ over $\hat{\Lambda}$ as

$$T_{\hat{\Lambda}}(Q \times T^{\mathbb{N}}) = \bigcup_{(x,\underline{t}) \in \hat{\Lambda}} (T_x Q \times \{\underline{t}\})$$

and endow it with the metric $\|\cdot\|$ given by the riemannian metric induced on each $T_x Q \times \{\underline{t}\}$. We also define the derivative of F over $\hat{\Lambda}$ by

$$DF: T_{\hat{\Lambda}}(Q \times T^{\mathbb{N}}) \to T_{\hat{\Lambda}}(Q \times T^{\mathbb{N}}), \qquad DF(x, \underline{t})(v, \underline{t}) = (Df_{t_1}(x)v, \sigma(\underline{t})).$$

Proposition A.11. *There is a continuous splitting $T_{\hat{\Lambda}}(Q \times T^{\mathbb{N}}) = \hat{E}^s \oplus \hat{E}^u$ invariant under DF, and there is a constant $0 < \lambda < 1$ for which*

$$\|DF|\hat{E}^s\| \leq \lambda \quad \textit{and} \quad \|DF|\hat{E}^u\| \geq \lambda^{-1}.$$

This proposition may be proved by using invariant cone fields for f, defined in a neighbourhood of Λ. Note that such cone fields area also invariant under every small perturbation of f.

Then, using this hyperbolic structure for $\hat{\Lambda}$ one can prove the existence of local stable manifolds for points in $\hat{\Lambda}$, in much the same way as one proves Theorem A.4.

Theorem A.12. *There is $\lambda_s < 1$ and for each $(x, \underline{t}) \in \hat{\Lambda}$ there is a C^k disk $W^s_{loc}(x, \underline{t})$ embedded in $Q \times \{\underline{t}\}$, such that*

(1) $T_{(x,\underline{t})} W^s_{loc}(x, \underline{t}) = \hat{E}^s_{(x,\underline{t})}$,

(2) $F(W^s_{loc}(x, \underline{t})) \subset W^s_{loc}(F(x, \underline{t}))$,

(3) $F: W^s_{loc}(x, \underline{t}) \to W^s_{loc}(F(x, \underline{t}))$ *is a λ_s-contraction.*

Moreover, the disk $W^s_{loc}(x, \underline{t})$ varies continuously with the point $(x, \underline{t}) \in \hat{\Lambda}$ in the C^k topology.

We call $W^s_{loc}(x, \underline{t})$ the *local stable manifold* of F at the point $(x, \underline{t}) \in \hat{\Lambda}$. Then we let $\hat{\mathcal{F}}^s_{loc}$ be the foliation of a neighbourhood $\widehat{W}^s_{loc}$ of $\hat{\Lambda}$ whose leaves are these local stable manifolds. We want to state that the foliation $\hat{\mathcal{F}}^s_{loc}$ is absolutely continuous, Theorem A.13 below, but this requires some words of explanation.

A submanifold N of $Q \times T^{\mathbb{N}}$ is said to be transverse to $\hat{\mathcal{F}}^s_{loc}$ if $N \subset \widehat{W}^s_{loc}$ and

$$T_{(x,\underline{t})} N \oplus T_{(x,\underline{t})} \widehat{W}^s_{loc} = T_x Q \times \{\underline{t}\}.$$

for every $(x, \underline{t}) \in N \cap \widehat{W}^s_{loc}$. Given submanifolds N_1 and N_2 of $Q \times T^{\mathbb{N}}$ transverse to $\hat{\mathcal{F}}^s_{loc}$, we say that $P: N_1 \to N_2$ is a *Poincaré map* for $\hat{\mathcal{F}}^s_{loc}$ if it is injective and continuous, and

$$P(x, \underline{t}) \in W^s_{loc}(x, \underline{t}) \cap N_2 \quad \text{for every } (x, \underline{t}) \in N_1.$$

A Poincaré map $P: N_1 \to N_2$ is *absolutely continuous* if there is a continuous map $J: N_1 \to \mathbb{R}$, the *jacobian* of P, such that for every $\underline{t} \in T^{\mathbb{N}}$ and every Borel subset $A \subset N_1 \cap (Q \times \{\underline{t}\})$ we have

$$m_2(P(A)) = \int_A J(\cdot, \underline{t})\, dm_1$$

where m_i is the riemannian measure on $N_i \cap (Q \times \{\underline{t}\})$ for $i = 1, 2$. We say that P is Hölder continuous if for fixed $\underline{t} \in T^{\mathbb{N}}$ the map $P(\cdot, \underline{t})$ is Hölder continuous, with uniform Hölder constants. Analogously, we define Hölder continuity of the jacobian J. Finally, we say that $\widehat{\mathcal{F}}^s_{loc}$ is absolutely (resp. Hölder) continuous if every Poincaré map for $\widehat{\mathcal{F}}^s_{loc}$ is absolutely (resp. Hölder) continuous.

Theorem A.13. *Let f and $F: Q \times T^{\mathbb{N}} \to Q \times T^{\mathbb{N}}$ be as before. The stable foliation $\widehat{\mathcal{F}}^s_{loc}$ associated to F is absolutely continuous and Hölder continuous, and the jacobian of any Poincaré map for $\widehat{\mathcal{F}}^s_{loc}$ is also Hölder continuous. Moreover, all the Hölder constants involved may be taken uniform (i.e., independent of the map) in a neighbourhood of f.*

Let θ_ε be a probability measure on T and $\widehat{m}_\varepsilon$ be the measure induced by $(m \times \theta_\varepsilon^{\mathbb{N}})$ on the quotient space (the space of leaves) of $\widehat{\mathcal{F}}^s_{loc}$:

$$\widehat{m}_\varepsilon(\tilde{A}) = (m \times \theta_\varepsilon^{\mathbb{N}})\Big(\bigcup \{\gamma : \gamma \in \tilde{A}\}\Big)$$

for every measurable subset $\tilde{A}$ of the quotient space. Recall that given any leaf $\gamma \in \widehat{\mathcal{F}}^s_{loc}$, there is $\underline{t} \in T^{\mathbb{N}}$ such that $\gamma \subset Q \times \{\underline{t}\}$. Let m_γ be the smooth measure induced on γ by the riemannian metric of $Q \approx Q \times \{\underline{t}\}$. Using the previous theorem and the same arguments as in the deterministic case, one proves that the measure $(m \times \theta_\varepsilon^{\mathbb{N}})$ admits a disintegration $(p_{\varepsilon,\gamma})_\gamma$ along the leaves γ of $\widehat{\mathcal{F}}^s_{loc}$ which have Hölder continuous densities with respect to the corresponding riemannian measure m_γ. More precisely, there are constants $a_0 > 0$ and $0 < \nu_0 < 1$, depending only on the map f, and there exists a continuous function

$$H_\varepsilon: Q \times T^{\mathbb{N}} \to (0, +\infty)$$

bounded away from zero and infinity, such that $\log H$ is (a_0, ν_0)-Hölder continuous on every $Q \times \{\underline{t}\}$, and $p_{\varepsilon,\gamma} = (H_\varepsilon \mid \gamma)\, m_\gamma$, $\gamma \in \widehat{\mathcal{F}}^s_{loc}$, defines a disintegration of $m \times \theta_\varepsilon^{\mathbb{N}}$:

$$\int \Psi d(m \times \theta_\varepsilon^{\mathbb{N}}) = \int \left(\int (\Psi \mid \gamma)\, dp_{\varepsilon,\gamma}\right) d\widehat{m}_\varepsilon(\gamma).$$

for any $\Psi \in L^1(m \times \theta_\varepsilon^{\mathbb{N}})$.

APPENDIX B: LARGE DEVIATIONS AND CENTRAL LIMIT THEOREMS

Here we prove the central limit theorem and (a special case of) the large deviations theorem for independent identically distributed random variables, stated as Theorems 1.1 and 1.2 in the Introduction. The arguments are standard and can be found in many probability texts, see e.g. [Br68], [Bi79], [El85], [Du91], but are included here for convenience of the reader. We also give the precise statement of the martingale central limit theorem, which was used in the proofs of Theorem 2.11 and Theorem 4.11. See [Ne65] for a proof.

For proving Theorem 1.1 it is no restriction to suppose $\bar{X} = 0$ and $\sigma = 1$ (it suffices to replace X_n by $(X_n - \bar{X})/\sigma$, if necessary), and we do so. Let $\mathcal{F}(x) \doteq \mathcal{P}(X_n \le x)$ and $\mathcal{G}_n(x) = \mathcal{P}(S_n \le x)$ be the distribution functions of X_n (any $n \ge 1$) and $S_n = (X_1 + \cdots + X_n)/\sqrt{n})$, respectively. Clearly, it suffices to prove that

$$\mathcal{G}_n(x) \to H(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad \text{as } n \to \infty,$$

for every $x \in \mathbb{R}$. We introduce the characteristic functions

$$f(t) = E\left(e^{itX_n}\right) = \int_{-\infty}^{+\infty} e^{itx}\, d\mathcal{F}(x) \quad \text{and} \quad g_n(t) = E\left(e^{itS_n}\right) = \int_{-\infty}^{+\infty} e^{itx}\, d\mathcal{G}_n(x)$$

of X_n and S_n, respectively. Since $X_1, \ldots, X_n$ are assumed to be independent

$$g_n(t) = E\left(e^{itS_n}\right) = \prod_{j=1}^{n} E\left(e^{itX_j/\sqrt{n}}\right) = f(t/\sqrt{n})^n.$$

Taylor expansion gives, for each $s \in \mathbb{R}$,

$$f(s) = \int_{-\infty}^{+\infty} \left(1 + isx - \frac{1}{2}s^2x^2 + o(s^2)\right) d\mathcal{F}(x) = 1 - \frac{1}{2}s^2 + o(s^2),$$

recall that $E(X_n) = \bar{X} = 0$ and $E(X_n^2) = \sigma^2 = 1$. As a consequence, $\log f(s) = -\frac{1}{2}s^2 + o(s^2)$ and so, replacing above,

$$g_n(t) = \exp\left(n\left(-\frac{1}{2}\frac{t^2}{n} + o\left(\frac{t^2}{n}\right)\right)\right) \to h(t) = e^{-t^2/2}, \quad \text{as } n \to \infty.$$

Now, observe that $h(t)$ is the characteristic function of the normalized gaussian law:

$$\int_{-\infty}^{+\infty} e^{itx}\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right) dx = e^{-t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-(x-it)^2/2}\, dx = e^{-t^2/2}.$$

It follows, using the inversion formula for the Fourier transform (and the fact that the gaussian distribution function H is continuous), that $\mathcal{G}_n$ must converge to H. This is what we wanted to prove. □

Now we prove Theorem 1.2 under an additional assumption, namely

$$E(X_n e^{tX_n}) < \infty \quad \text{and} \quad E(X_n^2 e^{tX_n}) < \infty \quad \text{for all } t \in \mathbb{R}.$$

This allows for substantial simplification of technical aspects of the proof, without changing its basic flavour. On the other hand, this condition is always satisfied in the bounded case, and so it represents no serious restriction in the context of Dynamics we are interested in. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the *free-energy function*

$$\phi(t) = \log E(e^{t X_n})$$

(any $n \geq 1$), note that $\phi(0) = 0$. Direct differentiation gives $\phi'(0) = \bar{X} = 0$, and

$$\begin{aligned}\phi''(t) &= \frac{E(e^{tX_n})\, E(X_n^2 e^{tX_n}) - E(X_n e^{tX_n})^2}{E(e^{tX_n})^2} \\ &= E\left(\left(X_n - E\left(X_n \frac{e^{tX_n}}{E(e^{tX_n})}\right)\right)^2 \frac{e^{tX_n}}{E(e^{tX_n})}\right) > 0.\end{aligned}$$

Note that the inequality is indeed strict: otherwise X_n would be constant, which would contradict $\sigma > 0$. Now we introduce the *entropy function* $h : \mathbb{R} \to \mathbb{R}$, defined to be the Legendre transform of ϕ, that is,

$$h(z) = \sup\{tz - \phi(t) : t \in \mathbb{R}\}.$$

The fact that ϕ is strictly convex implies

$$h(z) > 0 \quad \text{for every } z \neq 0.$$

It is also easy to see that the supremum in the definition of $h(z)$ may be restricted to $t > 0$ if $z > 0$, respectively $t < 0$ if $z < 0$. Now, given any $t > 0$,

$$\mathcal{P}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\Big)\, e^{tn\varepsilon} \leq E(e^{t\sum_{i=1}^{n} X_i}) = e^{n\phi(t)},$$

where the equality follows from the independence assumption. In other words,

$$\mathcal{P}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\Big) \leq e^{-n(t\varepsilon - \phi(t))}$$

for every $t > 0$, and so

$$\mathcal{P}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \varepsilon\Big) \leq e^{-nh(\varepsilon)}.$$

Using also $t < 0$ we conclude that, for every $n \geq 1$ and $\varepsilon > 0$,

$$\mathcal{P}\big(\big|\frac{1}{n}\sum_{i=1}^{n} X_i\big| \geq \varepsilon\big) \leq e^{-nh(\varepsilon)} + e^{-nh(-\varepsilon)} \leq 2e^{-n\hat{h}(\varepsilon)},$$

where $\hat{h}(\varepsilon) = \min\{h(\varepsilon), h(-\varepsilon)\} > 0$, from which the conclusion of the theorem follows immediately. □

Remark: More generally, see Theorem II.4.1 in [El85],

$$\limsup \frac{1}{n} \log \mathcal{P}\big(\frac{1}{n}\sum_{i=1}^{n} X_i \in K\big) \leq \inf\{h(z) : z \in K\}$$

for every compact set $K \subset \mathbb{R}$, and

$$\liminf \frac{1}{n} \log \mathcal{P}\big(\frac{1}{n}\sum_{i=1}^{n} X_i \in O\big) \geq \inf\{h(z) : z \in O\}$$

for every open set $O \subset \mathbb{R}$.

Before stating the martingale central limit theorem we recall a few notions from probability theory. Given a sequence X_n, $n \geq 0$, of real random variables on a probability space $(M, \mathcal{F}, \mu)$, consider the shift map

$$\tau : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}, \qquad (\omega_n))n \geq 0 \mapsto (\omega_{n+1})_{n\geq 0},$$

and the probability measure ν on $\mathbb{R}^{\mathbb{N}}$ given by

$$\nu\big(\{(\omega_n)_{n\geq 0} : \omega_k \in A_0, \ldots, \omega_{k+m} \in A_m\}\big) = \mu\big(X_k^{-1}(A_0), \ldots, X_{k+m}^{-1}(A_m)\big)$$

for each $k \geq 0$, $m \geq 0$, and intervals $A_0, \ldots A_m \subset \mathbb{R}$. Then, the sequence $(X_n)_{n\geq 0}$ is called *stationary* if the measure ν is τ-invariant, and *ergodic* if ν is τ-ergodic. A relevant particular case corresponds to having $X_n = X_0 \circ f^n$ for every $n \geq 0$, where X_0 is some measurable function and $f : M \to M$ is a measurable map: the sequence is stationary if μ is f-invariant, and it is ergodic if μ is f-ergodic.

Now let $\mathcal{F}_n$, $n \geq 0$, be a non-increasing sequence of σ-algebras on M. Then $(X_n)_{n\geq 0}$ is a *reversed martingale difference* for $(\mathcal{F}_n)_{n\geq 0}$ if

(a) X_n is $\mathcal{F}_n$-measurable for every $n \geq 0$;

(b+) $E(X_n \mid \mathcal{F}_{n+1}) = 0$ for every $n \geq 0$.

Condition (b+) just means that $\int_A X_n \, d\mu = 0$ for every $A \in \mathcal{F}_{n+1}$. *Direct martingale differences* are defined in a similar way, replacing "non-increasing" by "non-decreasing" and (b+) by

(b–) $E(X_n \mid \mathcal{F}_{n-1}) = 0$ for every $n \geq 1$, and $E(X_0) = 0$.

A special example corresponds to X_n being independent and identically distributed with zero expectation: then $(X_n)_n$ is a direct martingale difference for $\mathcal{F}_n = \sigma$-algebra generated by $\{X_j : j \leq n\}$, and a reversed martingale difference for $\mathcal{F}_n = \sigma$-algebra generated by $\{X_j : j \geq n\}$ (the σ-algebra generated by a set of functions is the smallest σ-algebra with respect to which all those functions are measurable).

Martingale central limit thorem. *Let $(X_n)_{n\geq 0}$ be a stationary, ergodic, direct or reversed, martingale difference, such that $\sigma^2 = E(X_0^2)$ is strictly positive and finite. Then, for every $z \in \mathbb{R}$,*

$$\mu\Big(\{x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} X_j(x) < z\}\Big) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2\sigma^2}\, dt, \quad \textit{as } n \to \infty.$$

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