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BIFURCATIONS OF PLANAR VECTOR FIELDS AND HILBERT'S 16TH PROBLEM

ROBERT ROUSSARIE

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ROBERT ROUSSARIE (Univ. Bourgogne, Dijon, França)

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INSTITUTO DE MATEMÁTICA PURA E APLICADA

Estrada Dona Castorina, 110 - Jardim Botânico

22460-320 - Rio de Janeiro, RJ, Brasil

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PREFACE

During the last decade, a new impulse was given to the study of planar vector fields, in particular by the introduction of ideas coming from Bifurcation Theory. In the study of planar vector fields the essencial difficulty is the control of the number of isolated periodic orbits (limit cycles) and of their bifurcations. This was already clearly formulated by Hilbert, for polynomial vector fields, in his Sixteenth Problem. In a given family of vector fields X_λ the possible compact subsets which are limits of sequences of limit cycles are called limit periodic sets. The study of their bifurcations is of primordial interest for the study of local bifurcations in the family X_λ. In particular, it is crucial to obtain an estimation for the number of limit cycles which can bifurcate from any limit periodic set Γ (the cyclicity of the unfolding (X_λ, Γ) : *Cycl* (X_λ, Γ)). A general conjecture, which would imply a positive answer to the Hilbert's 16^{th} Problem, is that the cyclicity of any analytic unfolding (X_λ, Γ) is finite.

In chapter 1, we recall some general properties of vector fields on surfaces of genus 0, such as the Poincaré-Bendixson property and make a first approach to their bifurcation theory.

Limit periodic sets, which are the main subject of the text, are introduced in chapter 2, where their general properties are established. We prove a partial result about their structure, in the spirit of the Poincaré-Bendixson theory, and define the cyclicity : *Cycl* (X_λ, Γ). We show that the Hilbert's problem reduces to a general conjecture on the finite cyclicity for analytic unfoldings. This reduction is described in detail for quadratic vector fields.

In chapter 3, we consider a single vector field (the "0-parameter case") and give a survey on the desingularization theory and on the solution of the Dulac Problem : To prove that any analytic vector field on S^2 has just a finite number of limit cycles. The reason for this survey is that some ideas introduced in chapter 3 will be extended to families in the next chapters. For the Dulac Problem, we restrict ourselves to the case of hyperbolic graphics which can be easily studied and generalized to families.

The last three chapters are devoted to the study of bifurcations of limit periodic sets and to partial proofs of the conjecture on finite cyclicity. In chapter 4, one considers regular limit periodic sets, i.e. elliptic singular points and periodic orbits. They can be studied by standard methods of differentiable and analytic geometry. Of particular interest is the notion of Bautin Ideal associated to analytic unfoldings of ∞-codimension vector fields of "centre type". This Ideal, first introduced by Bautin for the study of polynomial quadratic vector fields, is extended to general analytic unfoldings. Using it, one can obtain an estimation for the cyclicity. It is also related to Melnikov functions and some applications to quadratic vector fields are developed.

In chapter 5, one considers elementary graphics which are the limit periodic sets whose singular points are elementary. One proves a general structure result for the unfoldings of the transition map near a hyperbolic saddle. This result, together with the Bautin Ideal notion, allows a complete study of the saddle connection unfoldings. In the last part of the chapter, one presents a review on recent results obtained by Mourtada, El Morsalani, Il'Yashenko, Yakovenko and others, about general elementary graphics.

The last chapter is devoted to non elementary limit periodic sets and their desingularization. One treats in details the simplest case, a connection at a Bogdanov-Takens cusp point, and gives some indications about a general desingularization theory for vector field families.

The principal aim of this text was to put together results which are dispersed in previous articles, although some results appear here for the first time. For instance, theorems 4.2 and 5.13 give explicit bounds for the cyclicity in terms of the Bautin Ideal and are applied in theorem 5.14 to quadratic vector fields. The paragraph 6.2 is a presentation of results for generic bifurcations of the cuspidal loop, which will appear in a forthcoming publication.

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Chapter 1

Families of two dimensional vector fields.

1.1 Vector fields on surfaces of genus 0.

In this section we will consider individual vector fields, which may be considered as 0-parameter families. We will suppose that this vector field is at least of class C^1. This will be sufficient to assure existence and uniqueness for the flow $\varphi(t,x)$ (t is the time, $x \in S$, the phase space) and to have the qualitative properties we recall below.

1.1.1 The phase space.

We will suppose S to be a connected surface of genus 0, i.e a submanifold of the 2-sphere S^2. This surface may be compact or not. In the first case it may be S^2 or a surface with boundary, like the closed annulus $S^1 \times [0,1]$. In this case we will assume that the vector field is tangent or transversal to each component of the boundary ∂S. Later on, we will also consider boundaries with corners, such as the square $[0,1] \times [0,1]$, with some natural conditions for X along the boundary.

1.1.2 The Poincaré-Bendixson property.

The principal reason to restrict ourselves to phase space of genus 0 will be to avoid non-trivial recurrences.

For instance, if we consider the vector field $X = \frac{\partial}{\partial\theta_1} + \alpha\frac{\partial}{\partial\theta_2}$ on the 2-torus $T^2 = R^2/_{Z^2}$, with angular coordinates (θ_1, θ_2), each orbit of X is dense on T^2 when α is irrational. This implies that the ω-limit set of each orbit is the whole phase space. On the contrary, for a 0-genus phase space, one has the following :

Theorem 1 *(Poincaré-Bendixon)*

Let X a vector field on a compact surface S of genus 0.
Suppose that each singular point of X is isolated. Let γ be any orbit of X. Then if its ω-limit set ω_γ is non empty, it falls in one of the following 3 cases :

1) ω_γ is a singular point,

2) ω_γ is a periodic orbit,

3) ω_γ contains a non empty subset of singular points Σ and at least one regular orbit. The ω and α-limit set of each regular orbit in ω_γ is one point of Σ.

One can find a proof of the Poincaré-Bendixson theorem, for S compact, in [MP]. The proof for a non-compact S is very similar. I do not want to repeat it here. The basic idea of the proof is a very simple and natural topological argument, though some details are a bit delicate. This argument will be useful later for families, so I want to give it in the following lemma :

Lemma 1 *Let X a vector field on a surface of genus* 0. *(We don't suppose that the singular points are isolated). Let γ be any orbit and $\sigma \subset S$ any interval transversal to X (i.e, for any $x \in \sigma$, the vector $X(x)$ is transversal to the tangent space $T_x\sigma$). Then $\omega_\gamma \cap \sigma$ contains at most one point.*

Proof Suppose that $\omega_\gamma \cap \sigma$ contains at least two distinct points a, b. Let x some point on γ, $\varphi(t, x)$ the trajectory.
Because $a, b \in \omega_\gamma$, it is possible to find three times $t_1 < t_2 < t_3$ such that :

1) $a_1 = \varphi(t_1, x) \in]a, b[$ (interval on σ with end points a, b),

2) Let σ_a and σ_b the two open subintervals on σ with a_1 as end point and containing a and b respectively. Then $a_2 = \varphi(t_2, x)$ is the first return of $\varphi(t, a_1)$ on σ_a,

3) $a_3 \in \sigma_b$.

Let Γ the simple topological closed curve, union of the interval $[a_1, a_2]$ and the orbit arc $\varphi([t_1, t_2], x)$. Now, because S *is a surface of genus* 0, the curve Γ separates it into two connected components A, B, by Jordan theorem.

One can choose A to be the component which contains the points $\varphi(t, a_2)$ for all $t > 0$ and B, the one which contains points $\varphi(t, a_3)$ for all $t < 0$: but this is a contradiction, because we no longer can go from a_2 to a_3 by the flow of X . ∎

As a consequence of the lemma, any recurrent orbit cuts any transversal interval σ in at most one point. It is the case for periodic orbit. It implies that any recurrent orbit is a singular point or a periodic orbit : *on a surface of genus* 0*, a vector field has just trivial recurrence.*

If S is compact and if, as in the Theorem, all singular points of X are isolated, then these singular points are finite in number. Moreover, if X is analytic, the number of regular orbits which occur in item 3 of the Theorem, must be finite. We will prove this point in chapter 2, using the Desingularization Theorem. Then the third case in the theorem reduces to a graphic, according the following definition :

Definition 1 *Let X a vector field on a surface.*

A graphic Γ for X is a compact, non-empty, invariant subset which consist of a finite number of isolated singular points $\{p_1, \cdots, p_s\}$ and regular orbits $\{\gamma_1, \cdots, \gamma_\ell\}$ such that the ω and α-limit set of each of these regular orbits is one of the singular points. Moreover Γ is the direct image of a S^1-immersion, oriented by increasing time.

Finally, if Γ is the ω-limit set of some trajectory γ, one can choose a transversal segment $\sigma \simeq [0, 1[$, with $0 \in \Gamma$, on one side of γ, such that a return map P is defined on σ, with $P(0) = 0$. We will say that Γ is a *monodromic graphic.*

We can now formulate a more precise form for the Poincaré-Bendixson theorem in the analytic case :

Theorem 2 *(Poincaré-Bendixson theorem for C^ω vector fields).*
Let X an analytic vector field with isolated singular points on some compact surface of genus 0. Let γ any orbit and suppose that $\omega(\gamma) \neq \emptyset$. Then ω_γ is a singular point, a periodic orbit or a monodromic graphic.

1.1.3 Phase portrait.

A consequence of the Poincaré-Bendixson theory is that, in general, a vector field on a surface of genus 0, admits simple classification of its topological type. To make this precise, we need a notion of equivalence (of topological type) between vector fields.

Definition 2 *Let X, Y two vector fields on S. They are topologically equivalent if there exists a homeomorphism h from S onto S, which sends each orbit of X onto some orbit of Y, preserving the time-orientation.*

If X and Y are topologically equivalent, one says that they have the same phase portrait. (In fact one can identify the phase portrait of X with its equivalence class).

It seems hopeless to look for a classification of all possible phase portraits . To obtain interesting and useful results, one has to get rid off the too degenerate vector fields. So one looks either to generic smooth vector fields, or to polynomial or analytic ones.

1.1.3.1 Generic vector fields. See [MP] and appendix for details.

To simplify, we suppose now that S is compact. We call $\mathcal{X}^r(S)$, for $1 \leq r \leq \infty$, the space of all vector fields on S, with its usual C^r-topology for uniform convergence. Because we are interested in this section on individual vector fields, it is of no interest to look at vector field whose phase portrait may be destroyed by an arbitrary small perturbation. So the only vector fields of interest will be the stable ones, according to the following definition :

Definition 3 *Let $X \in \mathcal{X}^r(S)$. One says that X is C^s-structurally stable (for some $s : 1 \leq s \leq r$) if and only if there exists some neighborhood $\mathcal{U}$ of X in $\mathcal{X}^r(S)$, for the C^s-topology, such that any $Y \in \mathcal{U}$ is topologically equivalent to X.*

A simple class of vector fields is defined as follow :

Definition 4 *$X \in \mathcal{X}^r(S)$ is said Morse-Smale if and only if :*

1) all critical elements of X are hyperbolic (a critical element is a singular point or a periodic orbit),

2) there is no connection between saddle points of X.

The principal result for generic vector fields is :

Theorem 3 *(Andronov-Pontryaguin,Peixoto).*

Let any $r : 1 \leq r \leq \infty$. Then :

1) the set $MS^r(S)$ of all Morse-Smale vector fields in $\mathcal{X}^r(S)$, is open and dense in $\mathcal{X}^r(S)$,

2) a vector field $X \in \mathcal{X}^r(S)$ is C^r-structurally stable if and only if it is in $MS^r(S)$.

Each Morse-Smale vector field has just a finite number of critical elements. The singular points are hyperbolic sources, sinks or saddles. The periodic orbits may be hyperbolic attractive of repelling. It is possible, but rather tedious to give a complete classification of the possible phase portraits.

I recall that the theorem of Andronov-Pontryaguin and Peixoto was extended in 1962 by M. Peixoto to oriented compact surfaces of arbitrary genus M [P]. The proof is then much harder, due to the difficulties to get rid off non-trivial recurrences by small perturbations. This can be made by a C^r-perturbation for orientable M. But for non-orientable M, one needs a C^1-perturbation. The reason is that on this last case, it is not possible to construct directly a C^r-perturbation and one has to use the closing-lemma which has only been proved for the C^1-topology. See [MP] for a complete discussion and proofs about these difficult questions. On the other hand, the proof that any Morse-Smale vector field on M is C^r-structurally stable is very similar to the proof for surface of genus 0.

1.1.3.2 Analytic and polynomial vector fields.

Polynomial vector fields are also of particular interest, even if at first sight they seem to form a rather special class of vector fields. In fact, their popularity in many fields of application (electrotechnics, ecology, biology,...) comes from their apparent simplicity, and as consequence the belief that their properties would be also simple. We will see that this belief is somewhat misleading. Nevertheless, one clear advantage of polynomial vector fields is that they are easy to integrate numerically and so, they lead to simple modelisation.

Another important point is that polynomial vector fields of a given degree n, form an explicit family with finite parameters. We call it $\mathcal{P}_n$. Some questions which make no sense in the general context of generic vector fields, can be addressed now : Is it possible to find algorithms to locate singular points and periodic orbits ? Is it possible to search them by means of an algorithm which can be implanted on a computer ? We will come back to these questions in the context of families of vector fields.

Of course one can look as above to Morse-Smale vector fields, now inside the space $\mathcal{P}_n$, endowed with the topology of coefficients. The density question was solved by J.Sotomayor in [So2].But it appears that the question :*Is each structurally stable polynomial vector field a Morse-Smale one?*, remains unsolved in general. Moreover, it will be essential for the study of generic smooth unfoldings, to look at all polynomial vector fields and not just at the Morse-Smale ones, as it would be natural in the generic context, for individual vector fields. For instance, the unfoldings of Hamiltonian vector fields will be of interest.

Almost all the questions we will consider about polynomial vector fields will only make use of their analyticity.

We will take advantage of this for extending any polynomial vector field X on R^2, to an analytic vector field $\widetilde{X}$ on S^2. We refer at this extension, as the *Poincaré compactification* of X. The most straight way to obtain it, is explained as follows :

Identifying R^2 with C by taking $x + iy = z$, one can write the differential equation of $X \in \mathcal{P}_n$ as :

$$\dot{z} = P(z,\bar{z}) = \sum_{0\le i+j\le n} a_{ij}\ z^i\bar{z}^j \ with\ a_{ij} \in C \tag{1.1}$$

Let $Z = 1/z$ the chart at infinity in C (the two charts $z \in C$ and $Z \in C$ form an atlas of $S^2 = C \cup \{\infty\}$).

In this chart, the equation (1) writes :

$$\dot{Z} = -\frac{\dot{z}}{z^2} = -Z^2 \sum a_{ij}\ Z^{-i}\ \overline{Z}^{-j} \tag{1.2}$$

Of course the vector field goes to infinity when $Z \to 0$ or $z \to \infty$. But if we multiply X by the real analytic function $f_n(z) = \dfrac{1}{1+(z\bar{z})^n}$, it is easy to see that the vector field $\widetilde{X} = f_n X$ is analytic on S^2. This is clear in the chart z. On the chart Z, $\widetilde{X}$ writes :

$$\begin{aligned}\dot{Z} &= -\frac{Z^2}{1+(Z\overline{Z})^{-n}} \sum_{0\le i+j\le n} a_{ij}Z^{-i}\overline{Z}^{-j} = -\frac{Z^2(Z\overline{Z})^n}{1+(Z\overline{Z})^n}\sum a_{ij}Z^{-i}\overline{Z}^{-j} \\ \dot{Z} &= -\frac{Z^2}{1+(Z\overline{Z})^n} \sum_{0\le i+j\le n} a_{ij}\ Z^{n-i}\overline{Z}^{n-j}\end{aligned} \tag{1.3}$$

which is also analytic in Z.

In this compactification, one adds just one point to R^2. In the usual way to present the Poincaré compactification, one adds a circle of points to R^2 (the circle at infinity). One obtains a vector field $\widetilde{X_1}$ on D^2, and R^2 is identified to int (D^2). To obtain this vector field $\widetilde{X}_1$ from $\widetilde{X}$, it will suffice to blow-up the point ∞ of S^2. This blow-up procedure will be explained in chapter 3.

In chapter 3, we will also return in detail, to the study of polynomial and analytic vector fields.

1.2 A first approach to Bifurcation Theory.

In general, a vector field will depend on n parameters.

For a vector field modelling some natural phenomenon, the parameters will represent the exterior world : damping coefficients, frequencies,... in mechanics; rates of growth, rates of predations,... in ecology, and so on. So that, even when one just wants to consider an individual vector field, the parameters will be in fact present in some implicit way.

For instance, if we consider some given polynomial vector field X_0, of degree n, it will be natural to look at its coefficients as parameters. What happens when we modify the coefficients of X_0 is of first importance to decide on the interest of the modelisation of some phenomenon by the explicit vector field X_0.

The embedding of the vector field X_0 inside a whole family X_λ brings many interesting new bifurcation problems. Their study is precisely the subject of these notes. In this section, we will introduce some general definitions, which can be easily extended in a more general context, for dynamical systems in any dimension. Later on, in the next chapter, we will introduce some notions more specific to vector fields families on surfaces.

A vector field family (X_λ) on S is a map of a parameter space P in the space $\mathcal{X}^r(S) : \lambda \to X_\lambda$.

Here, $1 \leq r \leq \infty$ or $r = \omega$ for analytic families. We will also consider the already mentioned family $\mathcal{P}_n$ of vector fields of degree $\leq n$. For this family the parameter is the set of coefficients. It is the euclidian space $R^{N(n)}$, with $N(n) = (n+1)(n+2)$. In general, the parameter space P will be a manifold of finite dimension p, usually R^p. We will say that the family (X_λ) is C^k if it is given in a neighborhood of each $(x_0, \lambda_0) \in S \times P$ by expressions :

$$X_\lambda(x, \lambda) = \sum_{i=1}^{n} a_i(x, \lambda)\frac{\partial}{\partial x_i} \quad ,$$

where $a_i(x, \lambda)$ are C^k functions defined on $U_{x_0} \times W_{\lambda_0}$, where U_{x_0} and W_{λ_0} are respectively charts of S and P in a neighborhood of x_0 and λ_0. Here $k = 1, \cdots, \infty$ or ω.

Of course, one can see a C^k-family (X_λ) as a vector field defined on the total space $S \times P$, which is tangent to the fibers of the projection $\pi : S \times P \to P$, $(\pi(x, \lambda) = \lambda)$; i.e, (X_λ) is associated to the field $X(x, \lambda) = X_\lambda(x)$ on $S \times P$ which verifies $d\pi(x, \lambda)[X(x, \lambda)] \equiv 0$.

If we write $\chi^r(S;P)$ for the space of all C^r-families of vector fields with parameter in P, the previous remark means : $\chi^r(S;P) \subset \chi^r(S \times P)$. We put on $\chi^r(S;P)$ the topology induced by the C^r-topology on $\chi^r(S \times P)$.

Let (X_λ) some family. The set in the parameter space where X_λ is structurally stable is an open set $U(X_\lambda)$. On each connected component of $U(X_\lambda)$, the phase portrait of X_λ is constant. On the contrary, this phase portrait changes in general when we go across $\Sigma(X_\lambda) = P - U(X_\lambda)$. This is the reason why one calls this set : *bifurcation set* of the family (X_λ). This set is closed. It seems reasonable to think that this set has empty interior, in general. In fact, this is false as soon as dim $S \geq 3$, as it was proved by Smale [Sm]. (Smale considered the whole family $\chi^r(S)$, where the parameter space is the space $\chi^r(S)$ itself, and exhibited an open set of non structurally stable vector fields ; it follows that there exists generic finite parameter families, with any number of parameter, such that the bifurcation set has a non empty interior). When dim $S = 2$, this remains an open question, if the parameter dimension is larger than 2.

At least, one knows that the structural stability is a generic property, and that $\Sigma(X_\lambda)$ has generically an empty interior for 1-parameter families. It was proved by Sotomayor in [$So1$]. If Σ is the set of all non-structural stable vector fields in $\chi^r(X)$, $\Sigma((X_\lambda)) = \rho^{-1}(\Sigma)$ where ρ is the map $\lambda \to (X_\lambda)$. The structure of the sets $\Sigma((X_\lambda))$ depends on the structure of Σ. If one could apply the transversality theorem to Σ (i.e if Σ would be a stratified set, with finite codimension strata), then for each generic family X_λ, $\Sigma((X_\lambda))$ would have also a structure of stratified subset of S. In particular $\Sigma((X_\lambda))$ would be generically nowhere dense. But to establish this property for Σ, looks rather utopic. Up to know, one just knows the (codimension 1)-skeleton of Σ.[$So1$]. Describe the 2-skeleton seems rather complicated. Of course the study of the skeleton of Σ, up to codimension k is completely equivalent to the study of all generic families with less than k parameters. In these notes we want to prove some very partial results in these direction. In fact, we will emphasize some particular aspects of the bifurcation theory concerning more particularly the number of isolated periodic orbits, i.e the question of *finite cyclicity*. I introduce this concept of cyclicity in the next chapter. Now I want to give some basic and general definitions about families of vector fields.

1.2.1 General definitions.

Definition 5 *Let $(X_\lambda),(Y_\lambda)$ two C^r families of vector fields with the same parameter space P and the same phase space S ; $r = 1,\cdots,\infty$, or ω and let $s : 0 \le s \le r$. We say that (X_λ) and (Y_λ) are (C^0-fibre, C^s)-equivalent, if there exists a diffeomorphism φ of P, of class C^s, such that for each $\lambda \in P$, X_λ and $Y_{\varphi(\lambda)}$ are topologically equivalent. If the equivalence homeomorphism may be chosen so that to form a continuous family $h_\lambda(x)$, one says that (X_λ) and (Y_λ) are (C^0,C^s)-equivalent. If $\varphi = Id$, one says simply that (X_λ) and (Y_λ) are respectively C^0-fibre or C^0-equivalent.*

Remark 1
1) The (C^0,C^s)-equivalence may be strengthened in the notion of (C^ℓ,C^s) equivalence or even conjugacy (for $0 \le \ell, s \le r$). But, these equivalences are in general too strict to avoid problems of moduli between generic families which are (C^0-fibre, C^s)-equivalente. This occurs for the Hopf-Takens bifurcations with more than 4 parameters (see [R1]).

2) It will be important to have some smoothness for the change of parameter φ, since one is interested on the differentiable or even analytic structure of the bifurcation set : for applications it will be relevant to know if 2 lines of bifurcation have some flat contact for instance.

Definition 6 *Let (Y_λ), $\lambda \in P$ a C^r-family and $\varphi : Q \to P$, $\varphi(\mu) = \lambda$, a C^r-map. We say that the family $X_\mu, \mu \in Q$, given by $X_\mu = Y_{\varphi(\mu)}$ is induced from (Y_λ) by the map φ.*

Remark 2 *This operation of induction will be very important to select a "good" set of parameters, i.e for replacing a parameter μ by a larger one λ, where the properties are "unfolded". (See below the notion of versal unfolding).*

Definition 7 *A germ of family (X_λ, λ_0) at $\lambda_0 \in P$ is called unfolding of X_λ at λ_0. More generally, if $\Gamma \subset S$ is some compact non empty* *invariant subset for X_{λ_0}, we will also consider the germ of X_λ along $\Gamma \times \{\lambda_0\}$. We will write in short this unfolding (X_λ, Γ), or $(X_\lambda, \Gamma, \lambda_0)$ if we want to recall the parameter value.*

Unfoldings are represented by local families. For instance (X_λ, Γ) will be represented by a family $\widetilde{X}_\lambda$ on some neighborhood $U \times W$ of $\Gamma \times \{\lambda_0\}$ in $S \times P$. We can always assume that W is diffeomorphic to R^p, $(p = dim\ P)$, and U is diffeomorphic to R^2, if Γ is some point. So that it will be equivalent to speak of unfoldings or local families, i.e families defined on an arbitrary neighborhood of $\Gamma \times \{\lambda_0\}$.

Remark 3 *The two preceding definitions of equivalence and induction are easily translated for unfoldings, by taking representative local families.*

Definition 8 *Let Γ, some compact non-empty invariant subset for X_{λ_0}. We say that (X_λ, Γ) is a versal unfolding for the germ (X_{λ_0}, Γ), for the (C^0-fibre, C^s) equivalence, if :*

1) Any other unfolding (Y_μ, Γ) of (X_{λ_0}, Γ) (i.e, any unfolding (Y_μ, Γ), $\mu \in Q$ some parameter space, with Γ an invariant set for Y_{μ_0} where $(Y_{\mu_0}, \Gamma) \equiv (X_{\lambda_0}, \Gamma)$) is C^0-fibre equivalent to an unfolding induced from X_λ by a germ of C^s-map $(\varphi, \mu_0)\ :\ (Q, \mu_0) \to (P, \lambda_0)$.

2) dim (P) is minimal for the property 1.

1.2.2 Singularities of finite codimension. The example of saddle-node bifurcations.

I refer to [Ma],[GG] for details about transversality theory and notions related to it : genericity, codimension of a singularity, versality and structural stability of unfoldings. Here I just give a brief survey of the terminology.

One defines a *singularity Σ of codimension k* as a submanifold of codimension $n + k$, (if $n = dim\ S$), of some space of l–jets of vector fields on S. This submanifold is supposed to be invariant under the natural action of $(l + 1)$-jets of diffeomorphisms of S. A germ (X, x_0) is "a singularity of type Σ" if $j^l X(x_0) \in \Sigma$. Now, a consequence of the

Transversality Theorem of Thom is that generically any C^{l+1} family of vector fields has a l-jet extension $(x, \lambda) \in S \times P \to j^l X_\lambda(x)$, transversal to Σ. This means that the set $\Sigma(X_\lambda) = j^l X_\lambda^{-1}(\Sigma)$ is a submanifold of codimension $k + n$ in $S \times P$, and that in particular $\Sigma(X_\lambda) = \emptyset$, if $k > p = dim\ P$.

Any known versal family is constructed in the following way. The germ (X, x_0) is taken as some singularity Σ of codimension k and the versal unfolding $(X_\lambda, \{x_0\})$ of (X, x_0) is any k-parameter unfolding whose jet-extension is transversal to Σ. In all the known cases, the versal unfolding $(X_\lambda, \{x_0\})$ is also *structurally stable* in the following sense : any nearby k-parameter unfolding (Y_λ) has a jet-extension cutting Σ at some (x_1, λ_1) close to (x_0, λ_0) and the germ of $(Y_\lambda,\ \{x_1, \lambda_1\})$ is equivalent to the germ $(X_\lambda,\ \{x_0, \lambda_0\})$. This implies also that all the germs on Σ, near (X, x_0) are equivalent to (X, x_0). If Σ is defined in the space of K-jets, we can take in particular any germ with the same K-jet as (X, x_0) : this means that any germ of Σ near (X, x_0) is K-*determinated.*

In general one reverses the terminology : singularity of finite codimension k means singularity whose transversal unfoldings are versal and structurally stable. Then the singularity is finitely determinated and the codimension k is equal to the number of parameters of any versal unfolding.

A pragmatic way to construct a versal family is as follow : one selects a (finite determinated) singularity of codimension k and considers any transversal unfolding to it. This procedure is inspired by the unfolding theory of differentiable maps, as developed by Thom,Arnold,Mather and others. Unfortunately, there are no general results in this direction for unfoldings of vector fields (even on surfaces).

To illustrate the above remarks, let us consider *saddle-node bifurcation of codimension* k, $k \geq 1$.

We say that a germ $(X, 0)$ at $0 \in R^2$ is a saddle-node singularity of codimension k, $k \geq 1$ if :

1) $j^1X(0)$ has only one eigenvalue equal to 0,

2) Let W any central manifold through 0, of class larger than k, then

$$X \mid_W (x) = [\alpha x^{k+1} + O(x^{k+1})]\frac{\partial}{\partial x}$$

with $\alpha \neq 0$ where x is a parametrization of W, with $x = 0$ at the origin.

It is easy to verify that the set of all germs $(X, 0)$ with these properties defines a submanifold $SN(k)$ in the $(k+1)$-jet space of vector-fields at $0 \in R^2$. This submanifold is a codimension $k+2$. Using the Central Manifold Theory [CLW] and the Preparation Theorem [M], it is possible to prove that any C^∞ unfolding $(X_\lambda, 0)$ of $(X, 0)$, with $\lambda_0 = 0$, is, for any finite $l >> k$, C^l-equivalent (i.e, by a C^K family of diffeomorphisms and multiplicative functions) to :

$$\pm y \frac{\partial}{\partial y} \pm \left[x^{k+1} + \sum_{i=0}^{n-1} \alpha_i(\lambda) x^i\right] \frac{\partial}{\partial x}$$

where $\alpha_i(\lambda)$ are C^k germs of functions at $0 \in R^p$.
This means that the unfolding :

$$Y_\alpha = \pm y \frac{\partial}{\partial y} \pm \left[x^{k+1} + \sum_{i=0}^{n-1} \alpha_i \ x^i\right] \frac{\partial}{\partial x} \quad , \quad \alpha = (\alpha_0, \cdots, \alpha_{k-1})$$

is a versal unfolding for $(X_0, 0)$ for the (C^K, C^K) equivalence. Moreover this unfolding is structurally stable and it is easy to verify that :

- any germ in $SN(k)$ is $(k+1)$-determinate,
- the $(k+1)$-jet extension of Y_α is transversal to $SN(k)$.

One can find details for the proofs and the bifurcation diagrams in [D2].

1.2.3 Bifurcations of singular points versus bifurcations of periodic orbits. The Bogdanov-Takens bifurcation.

What makes easy the study of saddle node bifurcations is that no periodic orbit is contained in some neighborhood of the origin. If we consider for instance a polynomial family of vector fields (X_λ), $\lambda \in R^p$ some set of coefficients, all the properties concerning singular points are described by polynomial equations or inequations. Let $X_\lambda = A_\lambda \frac{\partial}{\partial x} + B_\lambda \frac{\partial}{\partial y}$ where A_λ, B_λ are polynomials of degree $\leq n$, in x, y depending linearly on the parameter λ. The singular set is given by the polynomial equations :

$$A_\lambda = B_\lambda = 0. \tag{1.4}$$

If we want to look to the set of degenerate singular points, we have to add the equation :

$$\frac{\partial(A,B)}{\partial(x,y)} = 0 \tag{1.5}$$

The set of parameters values where one has at least one degenerate singular point is obtained by the elimination of (x,y) between (1),(2) : it is a semi-algebraic subset in the parameter space. The set of parameters where X_λ has a saddle node point of codimension k is also a semi-algebraic set, defined as the projection on R^p of the semi-algebraic set $j^{k+1}\ X_\lambda^{-1}\ (SN(k))$, and so on.

On the other hand, if we want to study the periodic orbits, we have to integrate the vector field X_λ and the properties defined via the flow of X_λ are no longer given in general by algebraic algorithms, even if the family is polynomial. For instance, to study the periodic orbits cutting some line interval $\sigma \subset R^2$, proceed as follow. Suppose that for some value $\lambda_0 \in P$, we have a return on σ : there exists some line interval σ', $\sigma' \supset \overline{\sigma}$, transversal to X_{λ_0} and such that for each $u \in \sigma$, the trajectory through u has a return $P_{\lambda_0}(u)$ on σ'. It follows by continuity that there exists a neighborhood W of λ_0 in P, and a first return map $P_\lambda(u) : \sigma \times W \to \sigma'$. Now, the key remark is that we just know that $P_\lambda(u)$ is analytic, and in general we cannot deduce more information from the fact that X_λ is polynomial. So that, the equation $\{P_\lambda(u) - u = 0\}$ which gives the equation of pairs (u,λ) such that X_λ has a periodic orbit through u (i.e. the equation of periodic orbits cutting σ) is just analytic and one has not, in general, any algebraic algorithm to solve it.

To illustrate this point, I want to present now the Bogdanov-Takens bifurcation. This bifurcation is the most complex one of codimension 2. and we need to study it to have a complete list of all generic unfoldings with less than two parameters. For us it will be useful to illustrate the methods for the study of unfoldings along these notes.

A complete treatment of this bifurcation can be found in [Bo],[T1],[RW].

Bogdanov-Takens bifurcation.

Let $(X,0)$ be a germ such that $j^1X(0)$ is nilpotent, i.e. linearly conjugate to $y\,\frac{\partial}{\partial x}$. So we assume now that $j^1X(0) = y\,\frac{\partial}{\partial x}$. It is easy to show that, up to a quadratic diffeomorphism, one has :

$$j^2X(0) = y\,\frac{\partial}{\partial x} + ax^2\,\frac{\partial}{\partial y} + b\;xy\,\frac{\partial}{\partial y},\ \textit{with}\ a, b \in R.$$

Generically one can suppose that $a \neq 0$ and $b \neq 0$. Then by a linear change of coordinates one can reduce to $a = 1$ and $b = \pm 1$. So that one can suppose that :

$$j^2X(0) = y\,\frac{\partial}{\partial x} + x^2\,\frac{\partial}{\partial y} \pm xy\,\frac{\partial}{\partial y} \tag{1.6}$$

Germs $(X,0)$ with 2-jets equivalent to (1.4) form a singularity of codimension 2 in the space of 2-jets of vector fields with 2 connected components TB^+, TB^- (depending on the sign $\pm$ of $xy\,\frac{\partial}{\partial y}$). We will see in Chapter 3 that the germ $(X,0)$ has the same phase portrait as the Hamiltonian vector field $y\,\frac{\partial}{\partial x} + x^2\,\frac{\partial}{\partial y}$, with a cusp singular point at the level through the origin. So, one calls $(X,0)$: a *cusp singularity.*

Such a singular point is the simplest non-elementary singularity (see chapter 3).

Now let (X_λ) be any C^∞ unfolding of $(X,0)$, with $(X_0,0) = (X,0)$. (The unfolding is defined at $\lambda_0 = 0 \in R^\lambda$) with $(X_0,0) \in TB\pm$. Let :

$$\dot{x} = H_1(x,y,\lambda) = y + 0(\|m\|^2) + 0(|\lambda|)\ ,\ \dot{y} = H_2(x,y,\lambda) \tag{1.7}$$

the differential equation of (X_λ) ; $m = (x,y)$, $\|.\|$ a norm on R^2. One has $\frac{\partial H_1}{\partial y}(x,y,\lambda) \neq 0$. So that we can take as local coordinates : $Y = H_1(x,y,\lambda)$, $X = x$. Renaming again the coordinates by x, y, the equation (2) takes the form :

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= F(x,y,\lambda) = g(x,\lambda) + yf(x,\lambda) + y^2Q(x,y,\lambda) \end{aligned} \tag{1.8}$$

where g, f and Q are $\mathcal{C}^\infty$ functions.

By hypothesis : $g(x,0) = x^2 + 0(x^3)$, $f(x,0) = \pm x + 0(x^2)$.

The systems is now equivalent to a second order differential equation :

$$\ddot{x} = g(x,\lambda) + \dot{x}f(x,\lambda) + \dot{x}^2Q(x,\dot{x},\lambda). \tag{1.9}$$

It contains a Hamiltonian part : $\ddot{x} = g(x,\lambda)$ corresponding to the function $H(x,y,\lambda) = \frac{1}{2}\, y^2 + G(x,\lambda)$ where $G(x,\lambda) = -\int_0^x g(s,\lambda)ds$.

Now, because $G(x,0) = \dfrac{1}{3}\, x^3 + 0(x^3)$, the Hamiltonian $H(x,0,0)$ has a versal unfolding in the sense of Catastrophe Theory : there exists a $\mathcal{C}^\infty$ differentiable change of coordinates with parameter :

$$x = U(X,\lambda) = X + 0(X) + 0(\lambda) \tag{1.10}$$

such that :

$$-g(x,\lambda)dx = -(X^2 + \mu(\lambda))dX \tag{1.11}$$

for some $\mathcal{C}^\infty$ function $\mu(\lambda)$. (see [M]).

Using the $\mathcal{C}^\infty$ change of coordinates $x = U(x,\lambda)$, $y = y$, we obtain that (1.3) is differentiably equivalent (i.e, up a $\mathcal{C}^\infty$ diffeomorphism, and a multiplication by a $\mathcal{C}^\infty$ positive function), to :

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 + \mu(\lambda) + y\big(\nu(\lambda) \pm x + x^2h(x,\lambda)\big) + y^2Q(x,y,\lambda). \end{cases} \tag{1.12}$$

where $\mu(\lambda)$ and $\nu(\lambda)$ are $\mathcal{C}^\infty$ functions such that $\mu(0) = \nu(0) = 0$ and $h(x,\lambda)$, $Q(x,y,\lambda)$ are $\mathcal{C}^\infty$ functions.

At this point it seems interesting to choose μ, ν as new parameters. There are two ways to achieve this :

1) Supposing that X_λ is a generic 2-parameter unfolding in the sense that $\frac{\partial(\mu,\nu)}{\partial(\lambda_1,\lambda_2)}(0) \neq 0$. Then, by a $\mathcal{C}^\infty$ change of parameters, we can suppose that $\lambda = (\lambda_1, \lambda_2) = (\mu, \nu)$. Or,

2) Introducing the new family :

$$X^{\pm}_{\mu,\nu,\lambda} \begin{cases} \dot{x} &= y \\ \dot{y} &= x^2 + \mu + y\Big(\nu \pm x + x^2 h(x,\lambda)\Big) + y^2 Q(x,y,\lambda). \end{cases} \quad (1.13)$$

The initial family (1.10) is then induced by the $\mathcal{C}^\infty$ map $Q : \lambda \to (\mu(\lambda),\, \nu(\lambda),\, \lambda) \in R^{2+p}$.

If we prove that the parameter λ in (1.11) plays no role in the sense that there exists a $\mathcal{C}^\infty$ map of maximal rank $\pi(\mu,\nu,\lambda)$ on the space (μ,ν), with $\pi(0,0,0) = (0,0)$ such that for each (μ,ν,λ) near $(0,0,0)$, $X_{\mu,\nu,\lambda}$ is equivalent to $X^{N\pm}_{\pi(\mu,\nu,\lambda)}$ where :

$$X^{N\pm}_{\mu,\nu} \begin{cases} \dot{x} &= y \\ \dot{y} &= x^2 + \mu + y(\nu \pm x). \end{cases} \quad (1.14)$$

then, the initial family (1.10) will be ($\mathcal{C}^0$-fibre, $\mathcal{C}^\infty$) equivalent to the family induced by the map $\pi \circ \Phi$, and we will have proved the following :

Theorem 4 *(Bogdanov-Takens).*

The polynomial unfolding $X^{N\pm}_{\mu,\nu}$ is versal unfolding of the Cusp singularity (Bogdanov-Takens singularity defined by (1.1)) for the ($\mathcal{C}^0$-fibre-$\mathcal{C}^\infty$) equivalence.

Remark 4 *It is possible to obtain a $(\mathcal{C}^0,\mathcal{C}^\infty)$-equivalence ([DR1]).*

Of course if we just consider generic 2-parameter unfoldings, we obtain a weaker result : every generic two parameter unfolding of the cusp singularity is ($\mathcal{C}^0$-fibre, $\mathcal{C}^s$) equivalent to $X^{N\pm}_{\mu,\nu}$. But the result as given in the theorem is better in the sense that it applies to unfoldings with any dimension of parameter and, in particular, it is indispensable to study unfoldings of singularities of cod ≥ 3.

In order to begin the proof of Theorem 4, we return to the family unfolding (8). Note that if we change (x,y,μ,ν,λ) in $(x,-y,\mu,\nu,\lambda)$ the unfolding $X^+_{\mu,\nu,\lambda}$ changes into the unfolding $-X^-_{\mu,\nu,\lambda}$. So that it suffices to look to the case +. The equation for singular points is given by : $y = 0$, $x^2 + \mu = 0$. There exists no singular points for $\mu > 0$, and two singular points for $\mu < 0$: $e_\mu = (-\sqrt{-\mu}, 0)$ and $s_\mu = (\sqrt{-\mu}, 0)$. It is easy to verify that s_μ is a saddle point, and e_μ a node or a focus. Moreover, the line $\{\mu = 0\}$ for $\nu \neq 0$, is a line of (codimension 1)-saddle-node bifurcations : when we go across the axis 0ν, for $\nu \neq 0$, and in the direction of negative μ, it appears the pair of singular points $\{e_\mu, s_\mu\}$.

At any point $(x, 0)$, div $X_\Lambda(x, y, \Lambda) = \nu + x + x^2 h(x, \lambda)$ (we put $\Lambda = (\mu, \nu, \lambda)$).

In particular : div $X_\Lambda(e_\mu) = \nu\sqrt{-\mu} + x^2 h(x, \lambda)$.

So that the equation div $X_\Lambda(e_\mu) = 0$ defines (for all λ) a curve $H : \mu = \mu_h(\nu) = -\nu^2 + 0(\nu^2)$ for $\nu \geq 0$ in the case + and $\nu \leq 0$ in the case −. We only consider the case + from now on. We write $\nu = \nu_h(\mu)$, $\mu \leq 0$ for the inverse of $\mu_n(\nu)$. Along the line H, the singular point e_μ is elliptic (its eigenvalues are purely imaginary). In fact it is easy to prove that div $X_\Lambda(e_\mu)$ changes of sign regularly across H and that H is a generic line of Hopf bifurcations of codimension 1. (See chapter 4).

Crossing the line H with increasing ν, the focus e_μ becomes stable and it appears a small unstable periodic orbit γ_Λ around e_μ. We let it out for a moment to look at the left hand separatrices of the saddle point s_μ. If a small negative value $\mu_0 < 0$ is chosen, for a fixed value λ_0, it is easy to see that these two separatrices cross the axis $0x$ at points $a(\lambda)$ for the lower separatrix and $b(\nu)$ for the upper one. Now, observe that if ν increases, the vector $X_\Lambda(x, y)$ for $\Lambda = (\mu_0, \nu, \lambda_0)$ rotates in the positive sense, for each $(x, y) \in R^2$ with $y \neq 0$. As a consequence : $\frac{da(\nu)}{d\nu} > 0$ and $\frac{db}{d\nu}(\nu) < 0$.

For small values of ν, one has $a(\nu) < b(\nu)$ and for large values : $a(\nu) > b(\nu)$. Then there exists only one value $\nu_0 = \nu_c(\mu_0)$ where

$$a(\nu_0) = b(\nu_0).$$

For this value ν_0 one has a saddle connection. Moreover

$$\frac{d}{d\nu}(a - b)(\nu) \neq 0$$

and also

$$divX_{\Lambda_0}(s_{\mu_0}) \neq 0\Big(\Lambda_0(\mu_0, \nu_0, \lambda_0)\Big)$$

: this means that the saddle connection is a generic codimension 1 connection. (See chapter 5). When ν increases from values less than ν_0, a large periodic orbit exists which, for $\nu = \nu_0$ becomes the saddle connection and disappears for $\nu > \nu_0$. The curve $C = \{\nu_0 = \nu_c(\mu_0) \mid \mu_0 < 0$ small enough $\}$ is a generic line of codimension 1 saddle connections.

The two lines : $H = \{\nu = \nu_h(\mu)\}$ and $C = \{\nu = \nu_c(\mu)\}$ form the boundary of a conic region T in the parameter space. A small periodic orbit appears in this region near H and a large one disappears near C. It seems reasonable to think that it is the same orbit which appears on one side and disappears on the other side and also that for each $(\mu, \nu) \in T$ we have just one periodic orbit.

Moreover, there is no periodic orbit outside T, in a whole neighborhood of the origin in the parameter space. This will give the complete description of the bifurcation diagram and is the essential part of the proof of Theorem 4. But the proof of this point is unexpectedly delicate. It needs the use of a rescaling in phase space and parameter space (see chapter 6), asymptotic methods and a fine result on Abelian integrals (see chapters 4,5). The reason is that we have no simple algorithm to control the return map on $[e_\mu, s_\mu]$.

So that the above example illustrates the difficulties to study periodic orbits, even in the very simple family $X^{N\pm}_{\mu,\nu}$. We return to it several times along these notes, to illustrate different technical ideas and to achieve a complete proof of Theorem 4. In the next chapter, I want to focus on the principal subject of these notes : *how to study bifurcations of periodic orbits ?*

Chapter 2

Limit Periodic Sets.

As explained at the end of the previous chapter, the most difficult problem in the study of bifurcations in a family of vector fields on 0-genus surface, is the control of periodic orbits. In fact, in generic smooth families, the periodic orbits will be isolated for each value of parameter. For analytic families we have two different possibilities for each orbit : it may be isolated or belong to a whole annulus of periodic orbits. In this last case and for the value of parameter where one has infinitely many periodic orbits, the vector field has a local analytic first integral and the nearby vector fields in the family may be studied by the perturbation theory to be introduced in Chapter 4 : they have in general isolated periodic orbits. The interest for the study of isolated periodic orbits is also supported by tradition and applications.

Definition 9 *A limit cycle for a vector field X in dimension 2, is a (one side) isolated periodic orbit. (necessarily 2-sides isolated for analytic X).*

The most famous question about limit cycles was formulated by D. Hilbert in his inaugural talk at the first International Congress of Mathematics in Paris (1901). The 16^{th} problem of the list he submitted to the audience has a "part a" about the classification of ovals with polynomial equation $\{H(x,y) = 0\}$ and a "part b" about limit cycles of polynomial vector fields. Let us quote this part b of Hilbert's 16^{th} problem :

..In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (limit cycles) for a differential equation of the first order of the form

$$\frac{dy}{dx} = \frac{Y}{X}$$

where X and Y are rational integral functions of the n^{th} degree in x and y...

Formulated in modern terminology, the question of Hilbert is about the study of *bifurcations* of limit cycles in the family $\mathcal{P}_n$ of all polynomial vector fields of degree $\leq n$, parametrized by the space of coefficients. One has just retained the following weak question, known presently as Hilbert's sixteenth problem :

For any $n \geq 2$, there exists a number $H(n) < \infty$, such that any vector field of degree $\leq n$, has less than $H(n)$ limit cycles.

Of course one has discarded the case $n = 1$ as trivial: $(H(1) = 0)$. Recall that the problem remains opened even for $n = 2$.

As it was underlined is the Hilbert's formulation, one has to look at the bifurcations of limit cycles when the parameter varies. So to study this question we introduce now the central concept for these notes : the concept of *limit periodic set*, organizing center for limit cycle's bifurcations.

2.1 Organizing centers for bifurcations of limit cycles.

2.1.1 Definition of limit periodic sets.

Let X_λ any C^1 family of vector fields on a surface M not necessarily of genus 0.

Definition 10 *[R3] A limit periodic set for X_λ is a compact non empty subset Γ in M, such that there exists a sequence $(\lambda_n)_n \to \lambda_*$ in the* *parameter space P, and for each λ_n, X_{λ_n} has a limit cycle γ_{λ_n} with the following property :*

$\gamma_{\lambda_n} \to \Gamma$ *where* $n \to \infty$ *in the Hausdorff topology of the space* $C(M)$ *of all non empty compact subjects of* M.

Recall that if M is a metrisable space, the Hausdorff topology is defined in the set $C(M)$ of all compact non empty subsets of M, in the following way : let d be a distance on M, defining its topology. For $A, B \in C(M)$ let : $d_H(A,B) = \underset{x\in A, y\in B}{Sup} \left\{ \underset{z\in B}{Inf} d(x,z)\ ,\ \underset{z'\in A}{Inf} d(z',z) \right\}$.

It is not difficult to show that d_H is a distance on $C(M)$, and that this distance defines a topology on $C(M)$ independent of the choice of the distance d : the *Hausdorff topology on* $C(M)$. More delicate is to show that if (M,d) is a compact metric space, the same is true for $(C(M), d_H)$ (see [Ba] for instance).

Remark 5 *1) Once chosen the distance d on M, the convergence $\gamma_{\lambda_n} \to \Gamma$ is equivalent to : for any $\varepsilon > 0$, $\exists n(\varepsilon)$ such that if $n \geq n(\varepsilon)$ γ_{λ_n} enters the ε-neighborhoods of Γ and inversely, Γ enters the ε-neighborhoods of γ_{λ_n}.*

2) Definitions of limit periodic sets were proposed by many authors, in particular by Perko [Per], Françoise and Pugh [FP], and more recently by Ilyashenko, Yakovenko [IY2]. These definitions are more restrictive in the sense that Γ is supposed to be a limit of a 1-parameter family of limit cycles :
$\Gamma = \lim_{\varepsilon \to 0} \gamma_{\lambda(\varepsilon)}$ *for a continuous arc $\lambda(\varepsilon)$:$]0,1[\to P$. I have preferred to introduce a definition in terms of a discrete sequence $(\lambda_n)_n$, because it is better adapted to the proofs of topological properties of limit periodic sets. It is clear that the two definitions are not equivalent for C^∞ families. Take the 1-parameter family*

$$y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y} + \Big(\varphi(\varepsilon) - (x^2+y^2)\Big)\Big(x\,\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)$$

where $\varphi(\varepsilon) = sin\,(\frac{1}{\varepsilon})e^{-\frac{1}{\varepsilon^2}}$, $\varphi(0) = 0$. For this family, the origin of R^2 is a limit periodic set in the sense above, but it is not in the "continuous" definition. For analytic families, the equivalence of the two definitions is an open question. The answer would be yes if it was true that the *diagram of bifurcation of each limit periodic set in analytic families had a topological conic structure. But this is again an open question, surely a fundamental and among the most difficult ones of the whole subject*

First examples of limit periodic sets on a surface S of genus 0 :

- **Singular elliptic points :** They appear as limit periodic sets in Hopf-Takens bifurcations for instance.

- **Periodic orbits :** A multiple periodic orbit (for instance a double or semi-stable periodic orbit) may bifurcate in several hyperbolic ones.

These two first examples are called : regular limit periodic set. Their bifurcations can be studied, using the theory of bifurcations for smooth functions (catastrophe or singularity theory) or analytic geometry (for analytic vector fields, and center point or non isolated periodic orbits). We will treat them in chapter 4.

- **Saddle connection :** We have found a saddle connection in the Bogdanov-Takens bifurcation for instance. The study of their bifurcations brings new problems because the return map near such a connection is no longer differentiable. They are studied in chapter 5, together with more general elementary graphics. Later, in chapter 6 we look to more degenerate limit periodic sets. Let LC be the union, in the product space $S \times P$, of all limit cycles. This set is as smooth as the family, so is analytic if the family is analytic. But its closure $\overline{LC}$ is no longer an analytic subset of $S \times P$, except at the regular limit periodic sets. It is the reason why bifurcations of limit cycles can't be treated entirely by methods of analytic or differentiable geometry.

2.1.2 Structure of limit periodic sets.

Lemma 2 *Let Γ some limit periodic set in a C^1 family X_λ, defined on a 0-genus surface S, for some value $\lambda_* \in P$. Let $\sigma \subset S$, any interval transversal to X_{λ_*}. Then $\sigma \cap \Gamma$ contains at most one point.*

Proof Suppose that $\Gamma \cap \sigma$ contains at least two points a, b. For n large enough, the vector field X_{λ_n} is transversal to σ and γ_{λ_n} cuts σ at least in two points : a_n near a and b_n near b. Because $a \neq b$, one has also

$a_n \neq b_n$ for large n. But this is impossible by the same arguments as in lemma 1.1. ∎

So that a limit periodic set has the same basic property as ω-limit set of individual vector field : to be cut at most in one point by any transversal segment. As it was remarked in 1.1.1, this implies the conclusions of theorem 1.1. So we have proved :

Theorem 5 *(Poincaré-Bendixson for vector field families)*
Let any C^1 vector field family X_λ on a compact genus 0-surface. Let Γ a limit periodic set for this family at the parameter value $\lambda_* \in P$. Assume that all singular points of X_{λ_*} are isolated. Then Γ falls in one of the 3 following cases :

1) Γ is a singular point of X_{λ_*},

2) Γ is a periodic orbit,

3) Γ contains a subset Σ of singular points and at least one regular orbit. The ω and α limit set of each of these regular orbits in an element of Σ. Moreover if S is compact and X_λ analytic, Γ is a graphic.

The result is very similar with the Poincaré-Bendixson theorem for ω-limit sets. Nethertheless it is worth to notice the following differences :

- a periodic orbit which is ω-set must be isolated on one side. On the contrary a non isolated periodic orbit (for instance a level curve of a Hamiltonian vector field) may happen as limit periodic set.

- a graphic which appears as limit periodic set may be non monodromic. The simplest example of this phenomenon is the graphic Γ made by a central manifold connection at a saddle point of codimension 1. As it was recalled in chapter I, such a singularity unfolds in a 1 dimensional versal unfolding, which writes locally : $-y\,\dfrac{\partial}{\partial y} + (\lambda + x^2)\,\dfrac{\partial}{\partial x}$. We assume that for $\lambda = 0$, the separatrix which is locally $0x$ for $x > 0$ returns along the separatrix $0x$, $x < 0$ to make the graphic Γ. Then it appears one hyperbolic attracting limit cycle near this connection Γ, for $\lambda < 0$, near 0. So that Γ is a limit periodic set of the unfolding and it is not monodromic as it appears in figure 1.

We can now give a more accurate classification of possible limit periodic sets for families on compact surfaces of genus 0 :

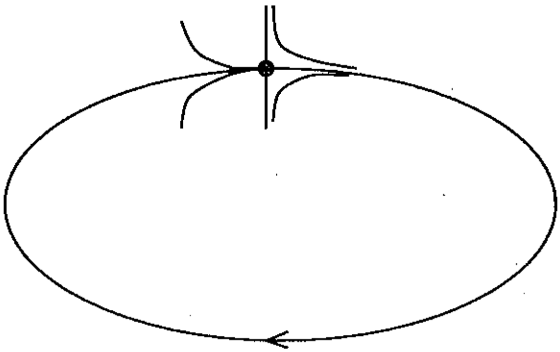

Figure 2.1: Non-monodromic limit periodic set

- **Regular limit periodic sets :** Elliptic singular points or periodic orbits. They may be of finite codimension or not. In this last case, we always assume that the family is analytic. Then, infinite codimension will mean that the return map for X_{λ_*} near Γ is equal to identity. (The elliptic point will be called a *centre*).

- **Elementary graphics :** Graphics whose all singular points are elementary. An elementary singular point is an algebraically isolated with at least one real non zero eigenvalue (they are the irreducible singular points in the sense of the desingularization theory developed in chapter 3). A elementary point may be hyperbolic, or semi-hyperbolic. In the two cases it must be of saddle type (or saddle-node type) to enter in a limit periodic set. In this case, sources or sinks are of course forbidden. If all the singular points are hyperbolic saddles, one says that the graphic is a *hyperbolic graphic.* Elementary graphics may be monodromic or not, isolated (among periodic orbits of X_{λ_*} or not). In chapter 5, we will study some of them.

- **Non-elementary graphics :**They are graphics with some non elementary singular points. We suppose again that all singular points in the graphic are isolated. The most simple example is obtained by connecting the two separatrices of a cuspidal singular point of Bogdanov-Takens type. We call it a cuspidal graphic and we return to it in details in chapter 6. In this chapter we will explain how, in some sense, to reduce non-elementary graphics to elementary ones. We can find many non-elementary graphics even in the family of quadratic vector fields as we explain at the end of this chapter. Elementary graphics may be monodromic or not, isolated or not from periodic orbits of X_{λ_*}.

- Limit periodic sets with non-isolated singular points.

We will see in chapter 3 that it is easy to get rid off non-isolated points of some individual analytic vector fields, so the restriction about the singular points is not too serious in Theorem 1.1. But it is not the case for families : one cannot easily replace the given family by a new one such that for each λ, all singular points of X_λ are isolated. Existence of non-isolated singular points is of course a non-generic phenomenon. But we will see that they appear in a systematic way when using desingularization methods in families (rescaling of variables for instance). Also, they are present in polynomial families. We look at these questions in chapter 6 .

A general theorem of structure for limit periodic sets (even for analytic families) is not known up to now. In chapter 6 again we will give some partial results in this direction. The *degenerate graphics* will be the simplest examples :

Definition 11 *A degenerate graphic Γ for a vector field X on a surface, is a compact, non empty invariant subset made by a finite number of isolated singular points $\{p_1, \cdots, p_s\}$, regular orbits $\{\gamma_1, \cdots, \gamma_\ell\}$ and arcs of non isolated singular points $\{r_1, \cdots, r_k\}$, such that the ω and α limit set of each regular orbits is a point in $\{p_1\} \cup \cdots \{p_s\} \cup r_1 \cup \cdots \cup r_k$. Moreover Γ is the direct image of a S^1-immersion, oriented by increasing time along the regular orbits.*

For instance, all limit periodic sets with non-isolated singular point in the study of quadratic vector fields are degenerate graphics (see [DRR1]).

2.2 The cyclicity property.

In this section, we will show that the problem to find a uniform bound for the number of limit cycles of a given family, for instance the Hilbert's

16^{th} problem, can be replaced by a local one about the number of limit cycles which bifurcate from each limit periodic set.

2.2.1 Definition of cyclicity for limit periodic sets.

A precise definition for the number of limit cycles which bifurcate from a limit periodic set is the following :

Definition 12 *Let Γ a limit periodic set of some C^1 family X_λ, defined at some value $\lambda_* \in P$. Denote by d some distances on S, P and d_H the induce a Hausdorff distance on $C(S)$. For each $\varepsilon, \delta > 0$ define :*

$$N(\delta,\varepsilon) = Inf\ \Big\{number\ of\ limit\ cycles\ \gamma\ of\ X_\lambda \mid d_H(\gamma,\Gamma) \leq \varepsilon$$

$$and\ d(\lambda,\lambda_\alpha) \leq \delta\Big\}.$$

Then the cyclicity of the germ (X_λ, Γ) is given by :

$$Cycl\ (X_\lambda,\Gamma) = \underset{\varepsilon,\delta}{Inf}\ N(\delta,\varepsilon).$$

As indicated in the definition, this bound $Cycl\ (X_\lambda, \Gamma)$ just depends on the germ X_λ along Γ, i.e. on the unfolding (X_λ, Γ). Of course $Cycl\ (X_\lambda, \Gamma)$ may be infinite. If it is finite, it represents in a precise way the local bound for the number of limit cycles which bifurcate from Γ in the given family X_λ.

A priori, if we change the unfolding (X_λ, Γ) of (X_{λ_*}, Γ), the cyclicity may change. It may happen that there exists a finite uniform bound for all the possible unfoldings of (X_{λ_*}, Γ). In this case, we call it : *absolute cyclicity* of (X_{λ_*}, Γ), or simply absolute cyclicity of Γ. It just depends on the germ of the unfolded vector field X_{λ_*} along Γ.

In the next paragraph, we will return to the general question of the relation between local bounds (finite cyclicity) and global bounds (as in the Hilbert's 16^{th} problem). Here, to conclude this paragraph, I want to emphasize that the computation of the cyclicity may be the crucial step in the obtaintion of bifurcation diagrams.

To illustrate this point, return to the Bogdanov-Takens bifurcation. Suppose known that any limit periodic set in this family *has cyclicity* *less than 1.* Then we can easily deduce from this the Theorem I-4. It works as follows : fix some value $\mu_0 < 0$, small enough. The effect of increasing the parameter ν is just to make a positive translation on the graph of the return map $P_\nu(x)$ on the interval $[e_{\mu_0}, s_{\mu_0}]$. This comes from the *rotating* property of the vector $X_\lambda(x,y)$. Then, the cyclicity hypothesis says that $\frac{\partial P_\nu}{\partial x} \geq 1$ at each $x \in [e_{\mu_x}, s_{\mu_0}]$. This implies that a periodic orbit has just one way to appear from a Hopf bifurcation and one way to disappear in a saddle connection bifurcation. It is not possible to create or to annihilate a pair (or more) of periodic orbits. As a consequence, there is no limit cycle for $\nu \notin]\nu_h(\mu_0),\ \nu_c(\mu_0)[$ and just one limit cycle inside this interval. This suffices to establish the bifurcation diagram and so to prove theorem 1.4.

2.2.2 The finite cyclicity conjecture. Local reduction of the Hilbert's 16^{th} problem.

It is easy to produce C^1 families where some limit periodic set have infinite cyclicity. But the author of these notes has the conviction that it cannot be the case for analytical families. Let us formulate this idea precisely :

- **Finite cyclicity conjecture** : Let Γ any compact invariant subset for an analytic vector field X, on some 0-genus surface. Then, for any analytic unfolding (X_λ, Γ) of (X, Γ), one has $Cycl\,(X_\lambda, \Gamma) < \infty$.

These notes are essentially devoted to a partial proof of this conjecture. In chapter 3, we will see that it is valid for 0-parameter families (Dulac's problem). In chapter 4, that it is valid for regular limit periodic sets. In chapter 5, that it is valid for unfoldings of generic elementary graphics . Of course, a complete proof of the conjecture remains an open question. We will show here, how it would imply a positive answer to Hilbert's 16^{th} problem.

First, let us remark that a direct consequence of the definition is that the cyclicity is an upper semi-continuous function on the set of all limit periodic sets :

Lemma 3 *Let $(\lambda_i) \to \lambda_*$ a converging sequence in P. Suppose that for each i, Γ_i is a limit periodic set for the value λ_i, such that $(\Gamma_i) \to \Gamma_*$*

as $i \to \infty$, in the Hausdorff sense. Then, Γ_* is a limit periodic set for the value λ_* and Cycl $(X_\lambda, \Gamma_*) \geq \limsup_{\lambda_i \to \lambda_*}$ Cycl (X_λ, Γ_i).

As a consequence one has the following :

Proposition 1 *[R3] Let X_λ a C^1 family of vector fields defined on some compact surface S of genus 0 with a compact set of parameters P. Then, there exists a uniform bound $H((X_\lambda)) < \infty$ for the number of limit cycles of each vector field X_λ, (each X_λ has less than $H((X_\lambda))$ limit cycles), if and only if each limit periodic set Γ of (X_λ) has a finite cyclicity in (X_λ).*

Proof Of course, if such a bound $H((X_\lambda))$ exists, it is trivial that each limit periodic set has a finite cyclicity in (X_λ). Suppose on the contrary that *Cycl* $(X_\lambda, \Gamma) < \infty$ for each limit periodic set in (X_λ) but a uniform bound $H((X_\lambda))$ does not exist. This means that one can find a sequence (λ_i) in P, and a limit periodic set Γ_i for each X_{λ_i} such that *Cycl* $(X_\lambda, \Gamma_i) \to \infty$ for $i \to \infty$.

Now, because P and $C(S)$ are compact spaces, one can find a subsequence λ_{i_j} such that $(\lambda_{i_j}) \to \lambda_*$ and such that $\Gamma_{i_j} \to \Gamma_*$ in the Hausdorff sense. One has that *Cycl*$(X_\lambda, \Gamma_{i_j}) \to \infty$ for $j \to \infty$. It follows from Lemma 3 that *Cycl* $(X_\lambda, \Gamma_*) = \infty$. This contradicts the hypothesis. ■

A family defined an a compact 0-genus surface with a compact set of parameters, will be called : *a compact family.* The preceding proposition implies that if the finite cyclicity conjecture is true, then any *analytic* compact family (X_λ), would have a uniform bound $H((X_\lambda)) < \infty$ for the number of the limit cycles of each X_λ. In particular, it would imply a positive answer to the Hilbert's 16 th problem.

Indeed, it is easy to replace the family $\mathcal{P}_n$ of polynomial vector fields of degree $\leq n$ by an analytic compact family on $S^2 \times S^{N-1}$ where S^{N-1} is the unit sphere in R^N with $N = 2n(n+1)$, the space of coefficients. The reason is as follows : for each $\mu \in R^+$, one has $X_{\mu\lambda} = \mu X_\lambda$ where $\lambda \in R^N$; so that $X_{\mu\lambda}$ and X_λ are equivalent and one can restrict λ to S^{N-1} ; next, we have seen in I-1.3.2, how to embed, up to some analytic positive function, any polynomial vector field X of degree n, into some analytic vector field $\widetilde{X}$ defined on S^2 (R^2 is identified with $S^2 - \infty$). The same formula embeds the whole family $(X_\lambda) = \mathcal{P}_n$ into an analytic family $(\widetilde{X}_\lambda)$, defined on $S^2 \times S^{N-1}$ (notice that the multiplicative function $(1+(z\overline{z})^n)^{-1}$ does not depend on λ). The vector field $\widetilde{X}_\lambda \mid int\ S^2$ is equivalent to X_λ, and so a bound $\widetilde{H}(n)$ for the family $(\widetilde{X}_\lambda)$ is also a bound for the polynomial family $\mathcal{P}_n(X_\lambda)$.

Remark 6 *The proof of the above proposition is just a compactness argument. It does not give an algorithm to compute $H((X_\lambda))$, even if we had an explicit bound for the cyclicity of every limit periodic set (Notice that we do not assume that there exists a uniform bound for the cyclicity of every limit periodic set ; this uniformity follows from the proof). So we have just a proof of the existence of the bound $H((X_\lambda))$. It is exactly the same as in the following simple example : suppose that π is the projection of some simple compact curve $\Gamma \subset R^2$ on some line δ, and that we know that any critical point of the projection is a generic fold point ; then there exists a bound $B < \infty$ such that for any $\lambda \in \delta$ the number of points in $\pi^{-1}(\lambda)$ is less than B. But, depending on the data Γ, π, δ, this bound B can take any finite value. Here, it is the same : the finite cyclicity conjecture would imply that for each n, the Hilbert's bound $H(n)$ exists, but it does not allow a computation of this bound. We refer to this problem : "prove that $H(n) < \infty$ exists" as the existential Hilbert's 16^{th} problem. It may be hoped that this problem is more tractable than the initial one which can be stated as : "prove that $H(2) = 4$".*

2.2.3 A program for solving the existential Hilbert's problem.

As it was said in the last paragraph, a general conjecture is that any analytic unfolding has a finite cyclicity. Because a direct attack to this conjecture seems somewhat utopic at this moment, a more reasonable way to address the question of the existence of a uniform bound $H((X_\lambda))$, for a given analytic family (X_λ) is to follow the program below:

- make a list of every limit periodic set which appears in the family

(X_λ),

- show that each such limit periodic set has a finite cyclicity.

In [DRR1], [DRR2] we have followed this program for a compact family equivalent to the family $\mathcal{P}_2$ of quadratic vector fields. Recall that the Hilbert's problem is not solved even in this case. In [DRR1], we achieved the first step of the program. In the second paper we collected all known results of finite cyclicity and added some new ones. I want to make a brief review of these two articles and indicate the new progress made from the time of their publication as well as to state the principal difficulties which remain open.

Before taking the first step, we have to chose a "good" compact family equivalent to $\mathcal{P}_2$. In paragraph 2.2 above, I showed how to obtain one such a family in general, for any $n \geq 2$. Here, for $n = 2$ it is easy to use the specific properties of quadratic vector fields to obtain a better compact family (with a minimum number of parameters). In fact, we are just interested in vector fields X which have at least one limit cycle γ. It is well known that this limit cycle bounds a disk D_γ in R^2, which contains just one singular point, necessarily a focus or a center [Ye]. So that, translating this singular point at the origin of R^2, and performing a linear change of coordinates, the vector field X has the following equation :

$$\left\{ \begin{array}{rcl} \dot{x} &=& \alpha x - \beta y + \varepsilon_1\, x^2 + \varepsilon_2\, xy + \varepsilon_3\, y^2 \\ \dot{y} &=& \beta x + \alpha y + \delta_1\, x^2 + \delta_2\, xy + \delta_3\, y^2 \end{array} \right. \tag{2.1}$$

with $\beta \neq 0$. Of course, one can suppose also that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3) \neq 0$. A time rescaling allows us to suppose that (α, β) belongs to S^1. Using the linear change of coordinates $(x, y) \to (x, -y)$, we can even suppose that $\beta \geq 0$, i.e. $(\alpha, \beta) \in P^1$. In this way we have included the non necessary value $(\alpha, \beta) = (1, 0)$, to have a compact domain for (α, β). Next, the transformation $(x, y) \to (\frac{x}{u}, \frac{y}{u})$ transforms the parameter $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ to the parameter $\frac{1}{u}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$. Hence, it is sufficient to study X_λ for $\lambda \in P^1 \times S^5$ $\Big((\alpha, \beta) \in P^1,\ (\varepsilon_1, \cdots, \delta_3) \in S^5\Big)$.

As it was explained in the paragraph 2.3, we can extend (X_λ) to a family $(\widetilde{X}_\lambda)$ in S^2, or better on D^2, blowing up the point at ∞ on S^2.

So that we have obtained an analytic compact family on $S = D^2$ with parameter in $P = P^1 \times S^5$.

Remark 7 *It is possible to reduce the quadratic part of (1.1) indexed by parameters in S^5, to a normal form indexed by parameters in S^4. Several such reductions are available : Kaypteyn's, Lienard's, Ye's normal forms. Each of them uses a rotation in the parameter space, to eliminate one coefficient of the quadratic part. But this rotation is not unique in general and, depending on the type of limit periodic set, it may be more useful to consider one or an other (See [DRR1]). So to describe the results it is better to remain in the 6 parameter family $(\widetilde{X}_\lambda)$, $\lambda \in P^1 \times S^5$.*

Because we have just to look at limit cycles surrounding the origin, we have not to take into consideration all limit periodic sets in the family, but we have just to prove that :

- Any limit periodic set of $\widetilde{X}_\lambda$ *surrounding the origin* has a finite cyclicity (such a limit periodic set may be equal to the origin itself, or bounds a disk containing the origin in its interior).

Comparing this question to the initial Hilbert's problem for $\mathcal{P}_2$, we have obtained a substantial reduction.

For instance, we do not need to study the following problems :

(1) quadratic perturbation of linear or constant vector fields (by the way, a question equivalent to the Hilbert's problem itself !),

(2) finite cyclicity of singular points of nilpotent linear part (for instance, of the cuspidal Takens-Bogdanov singular point),

(3) finite cyclicity of singular points with vanishing linear part,

(4) finite cyclicity of degenerate graphics with lines or curves of non-normally hyperbolic singular points,

(5) investigation of the number of zeros of Abelian integral on intervals of periodic solutions (because we look at the existential Hilbert's problem, the weak Hilbert's 16^{th} problem as defined by V. Arnold [I3] is not our aim).

This reduction looks a bit mysterious. For instance the quadratic Bogdanov-Takens family : $\dot{x} = y$, $\dot{y} = x^2 + \mu + y(\nu \pm x)$ is a subfamily of $\mathcal{P}_2$ and contains limit periodic sets. Of course all limit cycles of this family exist inside our family $\widetilde{X}_\lambda$. The fact is that, when $(\mu, \nu) \to (0,0)$, the corresponding parameter value $\lambda \to \lambda_0$ (after extracting a subsequence), due to the compacity of the parameter space. It happens that the corresponding limit cycle γ_λ converges toward a limit periodic set of $(\widetilde{X}_\lambda)$ which may be the origin, a periodic orbit or a saddle connection, or perhaps a limit periodic set containing a part of the circle at infinity (the origin in the phase space of the Bogdanov-Takens family, which is a limit periodic set of it, has been blown-up in our new family $(\widetilde{X}_\lambda)$, in the precise sense explained in chapter 6 below).

So let us look at limit periodic sets surrounding the origin. These possible limit periodic sets may be :

(a) limit periodic sets with isolated singular points uniquely,
(b) limit periodic sets with some non-isolated singular points.

For the first class, we can apply the Poincaré-Bendixson theorem 1.2. As possible limit periodic set we have the origin (and then $\alpha = 0$), periodic orbits or graphics. We will see in chapter 4 that regular limit periodic sets have finite cyclicity, so we have just to consider graphics. To obtain the list of such possible graphics, one uses the following information, available for quadratic systems :

- a quadratic vector field has at most 4 singular points in R^2, counted with multiplicity,
- a quadratic vector field has at most 6 singular points at infinity (counted with multiplicity), which appear in opposite symmetrical pairs,
- a line in R^2 has at most 2 contact points with a quadratic vector field, or is invariant,
- a polycycle (i.e. a monodromic graphic) with at least 2 singular vertices must contain the straight line segment joining any pair of vertices.

Using these properties, it is not difficult, but rather tedious to obtain a list of all possible graphics. To present them in a rational way we have introduced in [DRR1] some interesting subcategories of graphics. The figures 2.2, 2.3, 2.4 which present some of them come from [DRR1] :
- **finite graphics** (i.e. graphics contained in R^2). These graphics

has less than 3 vertices. They may be elementary (hyperbolic or not) or non-elementary. They may be monodromic or not. One has 10 such finite graphics. See figure 2.2.

- **infinite graphics** (i.e. containing a part of the circle at infinity). We have classified them by the number of their vertices at infinity , their total number of vertices, the nature of the vertices(the simplest ones, with a pair of opposite points at infinity are the 'hemispheres'. see fig. 2.3). These graphics form the large majority : 100 among a total of 121 .

Next we have to consider limit periodic sets with non-isolated singular points. Recall that the general structure of such limit periodic sets is unknown in any analytic family. But fortunately, they are not so frequent in our family $(\widetilde{X}_\lambda)$. In fact, if a vector field of $(\widetilde{X}_\lambda)$ has non-isolated singular points it is equivalent, up to some linear change of coordinates, to one of the following vector fields $\widetilde{X}$:

$$\text{(a)} \qquad \begin{cases} \dot{x} = (\lambda x - y)(x+1) \\ \dot{y} = (x + \lambda y)(x+1) \end{cases} \qquad \lambda \in R$$

$$\text{(b)} \qquad \begin{cases} \dot{x} = \lambda x - y + x^2 \\ \dot{y} = x + \lambda y + xy \end{cases} \qquad \lambda \in R$$

$$\text{(c)} \qquad \begin{cases} \dot{x} = x(x+1) \\ \dot{y} = y(x+1) \end{cases}$$

In the first case $\widetilde{X}$ has a line of singular points $\{x = -1\}$. In the second case, the singular set is the circle at infinity. Finally in the third case we have the union of $\{x = -1\}$ and the circle at infinity as set of singular points.

In each case, it is easy to see that the possible limit periodic sets are all degenerate graphics, according to the definition 2, made by the union of a regular orbit and one or two segments of singular points. Finally all the possibilities appear in five different pictures represented in figure 2.4. Notice that each picture may contain different degenerate graphics.

This achieves the first part of the program. A few of the 121 different limit periodic sets were known to have finite cyclicity at the time we wrote [DRR1] : the finite graphics $(F_1^1), (F_2^1), F(F_3^1), (F_1^2)$ in figure 2.2

and the hemispheres : $(H_1^1), (H_2^1)$ in figure 2.3. In [DRR2], we add 25 new cases to this list. All of them, as the already known ones, are elementary graphics, and have a cyclicity less than 2. (For some of them the result was only obtained under generic assumptions).

In chapters 4,5, I shall introduce methods to treat regular and elementary limit periodic sets. They are the methods used in [DRR2].

From the time of [DRR1] some new results were obtained : in [DER], Dumortier, El Morsalani and Rousseau proved the finite cyclicity of almost all elementary graphics of finite codimension; Mourtada, ElMorsalani and myself treated the case of some infinite codimension hyperbolic graphics with 2 vertices at infinity ([EMR]) ; this was next extended by the two first authors to finite graphics of the same type ([EM]) ; finally, Zoładek obtained the finite cyclicity for infinite codimension "triangle"([Z]).

At the present moment, none of the non-elementary or degenerate graphics have been studied. In chapter 6, I shall introduce a method of desingularization for vector field families. We will verify that using this method one can reduce the question in our family $\widetilde{X}_\lambda$ to a problem of finite cyclicity for singular elementary limit periodic set. These singular limit periodic sets are a little more general than those introduced in this chapter ; they will be defined in chapter 6.

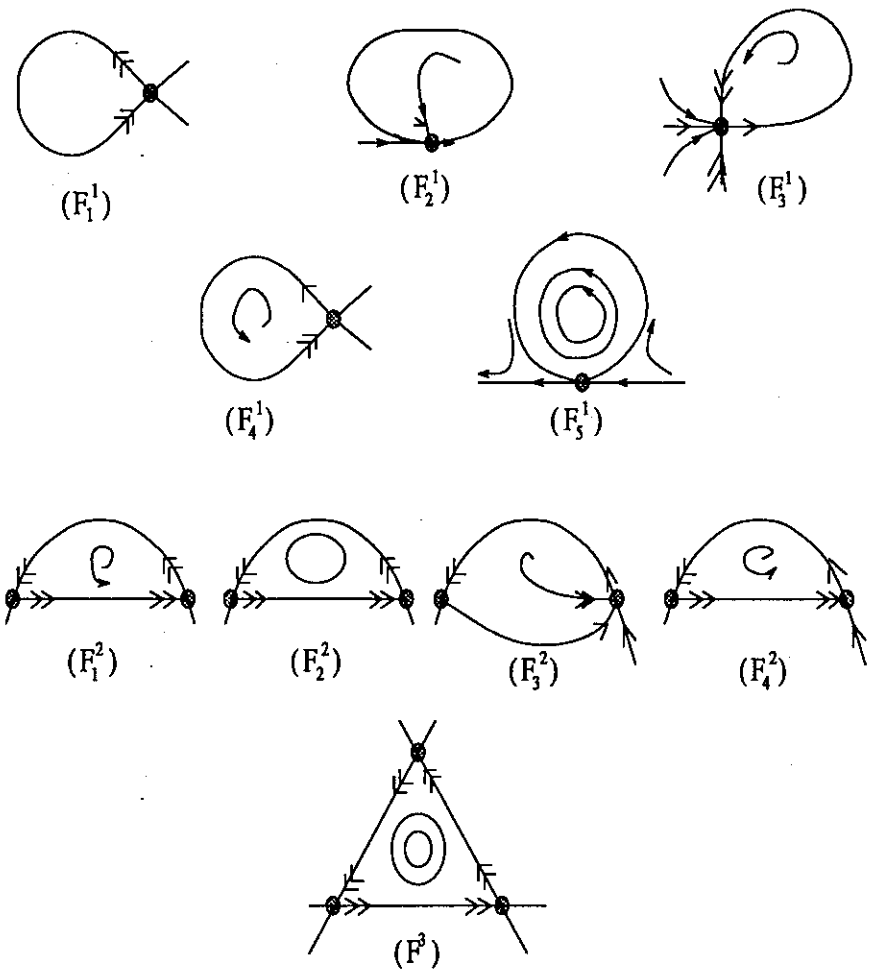

Figure 2.2:

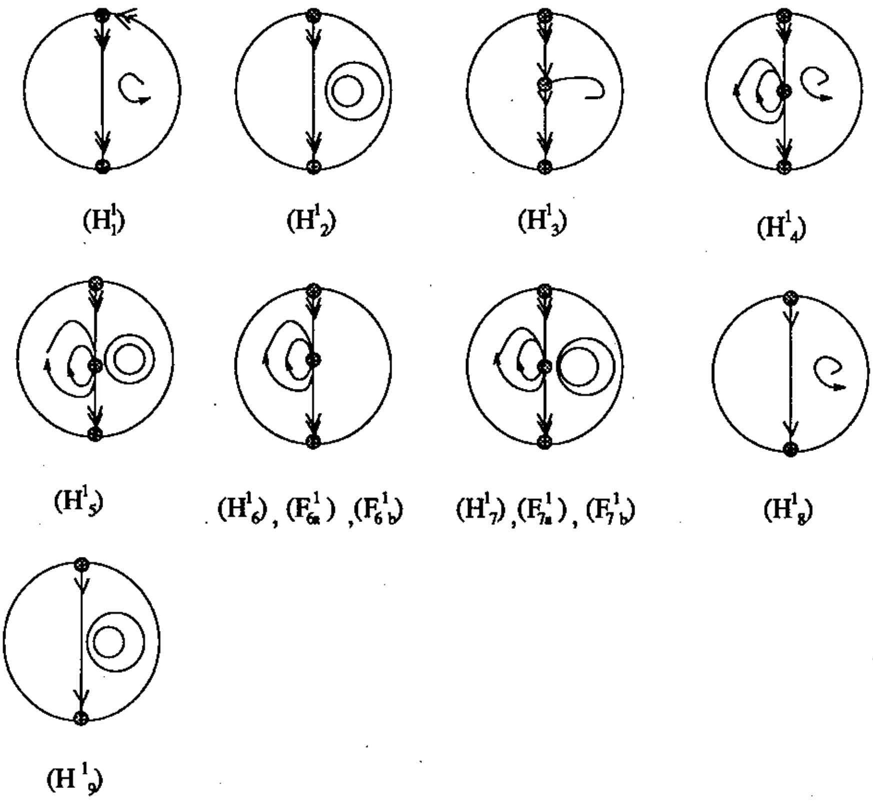

Figure 2.3:

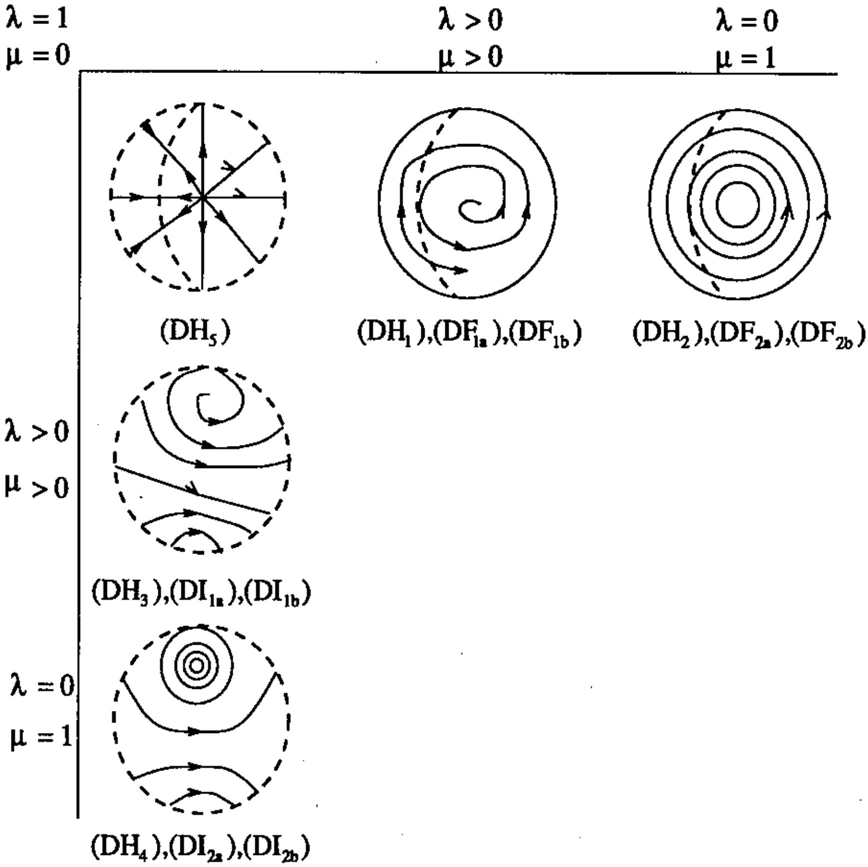
$\lambda = 1$
$\mu = 0$
$\lambda > 0$
$\mu > 0$
$\lambda = 0$
$\mu = 1$
(DH_5)
$(DH_1),(DF_{1a}),(DF_{1b})$
$(DH_2),(DF_{2a}),(DF_{2b})$
$\lambda > 0$
$\mu > 0$
$(DH_3),(DI_{1a}),(DI_{1b})$
$\lambda = 0$
$\mu = 1$
$(DH_4),(DI_{2a}),(DI_{2b})$

Figure 2.4:

Chapter 3

The 0-parameter case.

As an introduction to the theory of bifurcations I want to consider in this chapter individual vector field, i.e. family of vector field with a 0-dimensional parameter space. We will present in this chapter two fundamentals tools : the desingularization and the asymptotic expansion of the return map along a limit periodic set. In the particular case of an individual vector field these techniques have a complete achievement : the desingularization theorem says that any algebraically isolated singular point may be reduced to elementary singularities by a finite sequence of blowing ups. If X is some analytic vector field, on S^2 the return map of any elementary graphic has the property of isolated fixed point. As a consequence, in this special case one has no accumulation of limit cycles in the phase space. In other words each limit periodic set has a cyclicity equal to zero and an analytic vector field on the sphere has just a finite number of limit cycles.

In the following chapters, we will apply these techniques to families of vector fields. In this case, however, they have not the same degree of achievement and the main problems, which have a solution for individual vector fields, remain open for families.

The following text is just a survey about the subject and I include it in these notes to make them self-contained. I am going to follow closely the texts of F. Dumortier [D1], [D2] concerning the desingularization and also texts by Il'yashenko [I2], [I3] and Moussu [Mo] concerning the Dulac problem.

3.1 Blowing up singularities of vector fields.

In this paragraph, the vector field X is just studied locally in a neighborhood of a singular point. So, we can suppose that the phase space is R^2 and the singular point is the origin. We suppose that X is $\mathcal{C}^\infty$.

3.1.1 Polar and directional blow ups.

Let X a $\mathcal{C}^\infty$ vector field on R^2, such that $X(0) = 0$.

We consider the polar coordinate mapping $\overline{\Phi} : S^1 \times R \to R^2$ given by $\Phi(\theta, r) = (r \ \cos\ \theta, r\ \sin\ \theta)$. The pull-back $\widehat{X}$, with $\Phi_*(\widehat{X}) = X$ is a $\mathcal{C}^\infty$ vector field on $S^1 \times R$, called polar blow-up of X.

The smoothness of $\widehat{X}$ is clear from the direct computation of $\widehat{X}$. In fact, let be $X = X_1(x,y)\ \dfrac{\partial}{\partial x} + X_2(x,y)\ \dfrac{\partial}{\partial y}$ and look at $\widehat{X} = \eta_1(\theta,r)\ \dfrac{\partial}{\partial \theta} + \eta_2(\theta,r)\ \dfrac{\partial}{\partial r}$. Write $< u, v >= u_1v_1 + u_2v_2$ for $u = (u_1,u_2)$, $v = (v_1,v_2) \in R^2$. Since:

$$\Phi_*\Big(\frac{\partial}{\partial \theta}\Big) = x\ \frac{\partial}{\partial y} - y\ \frac{\partial}{\partial x}\ ,\ \Phi_1\Big(r\ \frac{\partial}{\partial r}\Big) = x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y} \tag{3.1}$$

one has :

$$\begin{aligned} < \widehat{X}, \frac{\partial}{\partial \theta} > &= r^2\ \eta_1 = < X,\ x\ \frac{\partial}{\partial y} - y\ \frac{\partial}{\partial x} > \\ < \widehat{X}, r\ \frac{\partial}{\partial r} > &= r^2\ \eta_2 = < X,\ x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y} > . \end{aligned} \tag{3.2}$$

This gives :

$$\begin{aligned} \eta_1(\theta,r) &= \frac{1}{r^2}\ (-r\sin\theta\ X_1(r\cos\theta,\ r\sin\theta) \\ &\quad +r\cos\theta\ X_2(r\cos\theta,\ r\sin\theta)) \\ \eta_2(\theta,r) &= \frac{1}{r^2}\ (+r\cos\theta X_1(r\cos\theta,\ r\sin\theta) \\ &\quad +r\sin\theta X_2(r\cos\theta,\ r\sin\theta)). \end{aligned} \tag{3.3}$$

Now, because $X_1(0,0) = X_2(0,0) = 0$, the term r^2 can be factorized in the two parentheses of (3.3) and η_1, η_2 turn out to be $\mathcal{C}^\infty$.

This follows from the Taylor formula with integral rest. The same idea proves that $\widehat{X}$ is C^{k-1} if X is C^k.

One verifies that $j^kX(0) = 0$ implies that $j^k\widehat{X}(u) = 0$ for all $u \in S^1 \times \{0\}$. This means that the degenerate singularity has been transformed into a whole circle of singularities.

In practice, one simplifies the calculations by looking at charts and performing the so-called "*directional*" blow-ups.

$$x - direction \ : \ (\bar{x}, \bar{y}) \to (\bar{x}, \bar{y}\ \bar{x}) \tag{3.4}$$

$$y - direction \ : \ (\bar{x}, \bar{y}) \to (\bar{x}\ y, \bar{y}). \tag{3.5}$$

On $\{x \neq 0\} = \left\{\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\right\}$, (3.4) is the same as polar blow-up, up to the analytic coordinate change $(\theta, r) \to (r \ \cos \ \theta, tg \ \theta)$.

Indeed, $(r\cos\theta, tg\theta . r\cos\theta) = (r\cos\theta, \ r\sin\theta)$.

The "pull-backs" of the directional blow-up maps are hence merely expressions of $\widehat{X}$ in well-chosen coordinate systems. Here the degenerate singularity is transformed into a line of singularities.

After blowing up the singularity to a circle, one can *desingularize* $\widehat{X}$ by considering $\overline{X} = \frac{1}{r^k}\ \widehat{X}$, k being the order of the largest zero jet of X at 0, i.e. : $j^\ell X(0) = 0$ if $\ell \leq k$ and $j^{k+1}\ X(0) \neq 0$.

For the directional blow-up we use $\frac{1}{\bar{x}^k}\ \widehat{X}$, resp. $\frac{1}{\bar{y}^k}\ \widehat{X}$. These last vector fields are no longer coordinate expressions of $\overline{X}$ but are equal to $\overline{X}$ up to an analytic coordinate change and *multiplication by a positive analytic function.* This positive factor does not constitute any problem since we are only concerned with the orbit structure (phase portrait) of X around the singularity.

Example 1.

Let $X = (x^2 - 2xy)\ \frac{\partial}{\partial x} + (y^2 - xy)\ \frac{\partial}{\partial y} + 0(\|(x,y)\|^2)$.

Putting

$$c = \cos\theta, \ s = \sin\theta,$$

one has :

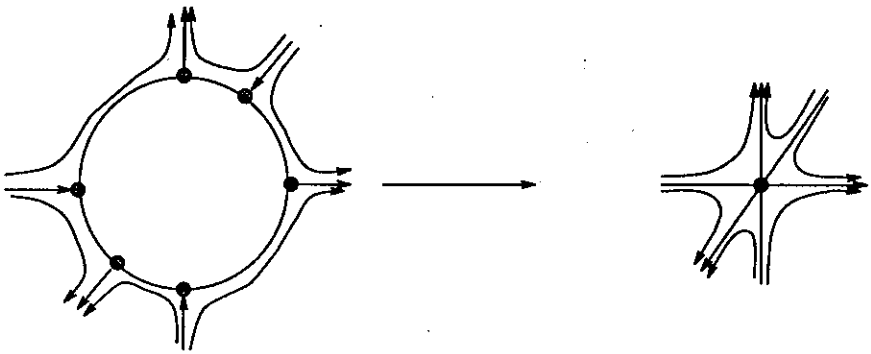

Figure 3.1: A singularity and its polar blow-up.

$$\overline{X} = \frac{1}{r}\,\widehat{X} = 2sc(s-c)\frac{\partial}{\partial\theta} + (c^3 + s^3 - cs(c+s))r\frac{\partial}{\partial r} \tag{3.6}$$

$\overline{X}$ has six singular points for $\theta = 0,\ \pm\frac{\pi}{2},\ \pi,\ \frac{\pi}{4},\ -\frac{3\pi}{4}$ which are all hyperbolic. The phase portraits for $\overline{X}$ near $S^1 \times \{0\}$ and X near the origin are shown in Figure 3.1.

3.1.2 Successive Blow-ups.

In example 1 above, one blowing-up was sufficient to determine the topological type of the germ. The reason is that after just one blowing-up, all the new singular points are hyperbolic and so have a well determined topological type. Then the different topological types glue up to determine the topological type of the germ X.

It is easy to think of examples of vector fields with singularities where one blow-up will not suffice to determine their topological type.

Example 2.

Let $Y_b = y\,\frac{\partial}{\partial x} + (x^2 + bxy)\,\frac{\partial}{\partial y} + 0(\|(x,y)\|^2)$. (A cusp singularity

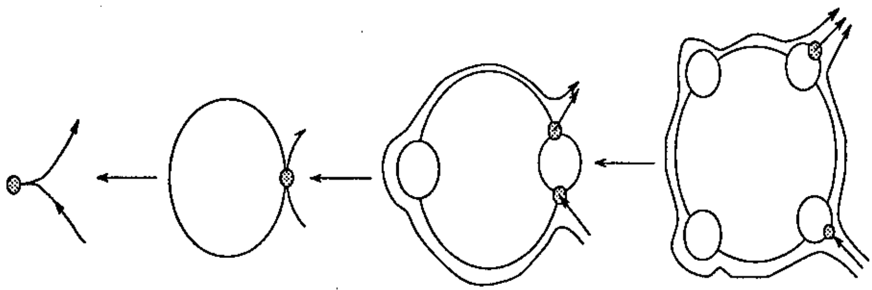

Figure 3.2: Successive blow-ups for the cusp singularity.

as defined in chapter 1). One requires three steps to desingularize it (see below for a precise definition of desingularization) and identify it topologically as a "cusp". See the details of computation in [*T*3]. For a picture of the three steps, (see Figure 3.2).

The *procedure of successive blow-ups* can be formulated as follows. We use the map :

$$\tilde{\Phi} : \left\{ z \mid \|z\| > \frac{1}{2} \right\} \subset R^2 \rightarrow R^2 \ : \ z \rightarrow z - \frac{z}{\|z\|} \tag{3.7}$$

and then divide out by a power of $(\|z\| - 1)$. To blow up a second time in a point z_0 on the unit circle, we translate it to the origin and apply again $\tilde{\Phi}$; the second blow-up mapping is therefore : $\Phi_2 = T_{z_0} \circ \tilde{\Phi}$ where $T_{z_0}(z) = z + z_0$. After a sequence of n blow-ups : $\Phi_1 \circ \cdots \circ \Phi_n$ (including the required divisions) we find a C^∞ vector field $\overline{X}^n$ defined on some open domain $U_n \subset R^2$.

Let $\Gamma_n = (\Phi_1 \circ \cdots \circ \Phi_n)^{-1}(0) \subset U_n$ and denote by A_n the connected component of $R^2 \backslash \Gamma_n$ with a non-compact closure. One verifies that $\partial A_n \subset \Gamma_n$; it is homeomorphic to S^1 and it consists of a finite number of regular C^∞-arcs meeting transversaly at end points. The effect of the divisions is seen as follows : there exists an analytic function $F_n > 0$ on A_n with $\widehat{X^n} = F_n \overline{X}^n$ and $\widehat{X^n} \mid A_n$ is analytically conjugate to $X \mid R^2 \backslash \{0\}$ by means of $(\Phi_1 \circ \cdots \circ \Phi_n)_{|A_n}$.

Definition 13 *A singular point is called elementary of one of the fol-*

lowing condition is fulfilled :

a) It is a hyperbolic singularity : the two eigenvalues have non-zero real part.

b) It is a non-degenerate semi-hyperbolic singularity : one eigenvalue is non-zero, the other is equal to zero but the infinite jet corresponding to any centre manifold is non-zero.

c) It is a germ of a line of normally hyperbolic singularities.

The topological type of an elementary singularity depends just on the sign of eigenvalues in the cases a), c), and also on principal part of jet on any centre manifold in the case b) (see chapter 1).

Moreover, elementary singularities cannot be simplified by blowing-up : if one blows up some of them one just produces new elementary singularities. So, it is natural to see them as the final state for the desingularization procedure.

A theorem of desingularization for general real vector field germ of R^2 was proved by Dumortier [D1]. To express the "generality" of the vector field we need the following definition :

Definition 14 *A vector field X on R^2, with $X(0) = 0$ satisfies a Lojasiewicz inequality if there exist $k \in N$, and $c > 0$ such that :*

$$\|X(x)\| \geq c\|x\|^k \textit{ for } \forall x \in U,$$

U is some neighborhood of 0.

This property is not exceptional for X. For instance a stronger property is :

Definition 15 *A C^∞ vector field has the origin as algebraic isolated singularity if the ideal generated by the components contains a power of the maximal ideal. Notice that this property is equivalent to the same for formal series. This property for analytic germs is equivalent to the following topological one : $0 \in C^2$ is isolated among the zeroes of the complexification $\widetilde{X}$ of X.*

I has been proved in [D1] that there exists a subset Σ_∞ in the space of ∞-jets of vector fields at 0 : $J^\infty V$, such that if $j^\infty X(0) \notin \Sigma_\infty$, then X

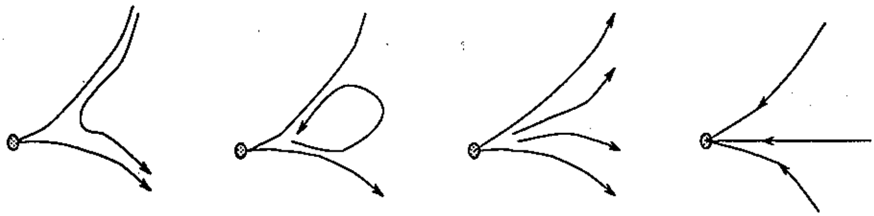

Figure 3.3:

has an algebraically isolated singularity at 0. As a consequence in any generic family with a finite number of parameters, all the singularities are algebraically isolated.

We can now state Dumortier's desingularization theorem.

Theorem 6 *If X is a C^∞ of vector field which satisfies a Lojasiewicz inequality, then there exists a finite sequence of blow-ups $\Phi_1 \circ \Phi_2 \cdots \circ \Phi_n$ leading to a vector field $\overline{X}^n$ along ∂A_n all whose singularities are elementary.*

Remark 8 *Because $0 \in R^2$ is an isolated singularity of X, all singularities of $\overline{X}^n$ in some neighborhoods V_n of ∂A_n in A_n are on ∂A_n. For instance the normally hyperbolic ones (case c) in the above definition) occure along smooth arcs of ∂A_n, or may be along all ∂A_n if ∂A_n is smooth (in this case $n = 1$ and $X \simeq (x^2+y^2)^k \left(x \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y}\right) + O\Big(\|(x,y)\|^{2k+1}\Big)$).*

A consequence of theorem 1 is that it is always possible to find a finite number of C^∞ invariant lines, each cutting ∂A_n in one point. In the case this number is not zero they divide small neighborhoods of ∂A_n into a finite number of sectors which, after *blowing down*, provide a decomposition of small neighborhoods of the singularity into hyperbolic (or saddle) sectors, elliptic sectors, and parabolic sectors of attracting or expanding type (see Figure 3.3).

The invariant C^∞ lines in the boundary of these sectors blow down to the so called *characteristic orbits (or lines)* : this means that the orbits tend to the singularity for $t \to +\infty$ or $-\infty$, with a well defined slope. Existence of a characteristic orbit may be read on the vector field $\overline{X}^n$ resulting from the blowing-up procedure. It is equivalent for $\overline{X}^n$ to have at least one singular point at a smooth point of ∂A_n. On the contrary, one has a well defined return map on a transversal segment to ∂A_n, which blows down to a return map for X, on a half segment through the origin. One says that X is of *monodromic type*. In this case, the determination of the topological type of X is more difficult because it is not determined in general by the finite jet of X which determines the desingularization, and vector fields with the same algebraically isolated ∞-jet may have different topological type : for instance X, with

$$j^\infty X(0) = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}.$$

3.1.3 Quasi-homogeneous blow-up and Newton diagram.

Although the method of successive blow-ups is sufficient to study singularities in general, in many cases we can significantly speed up the procedure using *quasi-homogeneous blow-ups.*

Definition 16 *A function $f : R^n \to R$ is quasi-homogeneous of type $(\alpha_1, \cdots, \alpha_n) \in N^n$ and degree k if and only if for any $r \in R$:*

$$f(r^{\alpha_1}\, x_1, \cdots, r^{\alpha_n}\, x_n) = r^k\, f(x_1, \cdots, x_n).$$

A vector field $X = X_1\,\frac{\partial}{\partial x} + X_2\,\frac{\partial}{\partial y}$ is called quasi-homogeneous of type (α_1, α_2) and degree $k+1$ if X_i is quasi-homogeneous of type α_i and degree $k + \alpha_i$, respectively.

Example : $(ax^2 - 2xy)\,\frac{\partial}{\partial x} + (y^2 - axy)\,\frac{\partial}{\partial y}$ is homogeneous of degree 2 : i.e. quasi-homogeneous of type (1,1), and degree 2.

Let us consider an example where the quasi-homogeneous part of lowest degree is determining. For general information on the method we refer to [BM], [Br].

We look again to the cusp singularity Y_b (which needed a 3 steps desingularization by homogeneous blow-up). For it, an appropriate quasi-homogeneous blow-up is :

$$\varphi : S^1 \times R \to R^2 \qquad (\theta, r) \to (r^2 \cos\theta,\ r^3 \sin\theta).$$

We perform this blow-up in the differential equation for Y_b :

$$Y_b \begin{cases} \dot{x} &= y + 0(\|(x,y)\|^2) \\ \dot{y} &= x^2 + byx + 0(\|(x,y)\|^2) \end{cases} \tag{3.8}$$

This gives : (writing $\cos\theta = c$, $\sin\theta = s$) :

$$\widehat{Y}_b \begin{cases} (3s^2+2c^2)\dot{\theta} &= r(2c^3+3c^2-3)+0(r^4) \\ (3s^2+2c^2)\dot{r} &= r^2 sc(1+c)+0(r^4) \end{cases} \tag{3.9}$$

and a desingularized vector field $\overline{Y}_b = \dfrac{1}{r(3s^2+2c^2)} \widehat{Y}_b$:

$$\overline{Y}_b \begin{cases} \dot{\theta} &= 2c^3+3c^2-3+0(r^3) \\ \dot{r} &= sc(1+c)r+0(r^3) \end{cases} \tag{3.10}$$

It is easy to verify that the polynomial $P(c) = 2c^3+3c^2-3$ has just one (simple) root c_0, $c_0 \in]0,1[$, and so $\overline{X}_b \mid S^1 \times \{0\}$ has two simple singular points $\theta_0 \in]0, \pi/2[$, $-\theta_0$, with $\cos\theta_0 = c_0$.

At these points, the radial eigenvalue $\pm s_0 c_0(1+c_0)$ ($s_0 = \sin\ \theta_0$) is not zero.

The phase portrait for $\overline{Y}_b$ and Y_b are given in Figure 3.4.

To detect determining quasi-homogeneous components there is the possibility to use *Newton's diagram.* The best way to define and also to memorize Newton's diagram is to work with the *dual 1-form* of the given vector field.

For a vector field $X_1 \dfrac{\partial}{\partial x} + X_2 \dfrac{\partial}{\partial y}$, its dual 1-form is $\omega = X_1\ dy - X_2\ dx$.

Take now :

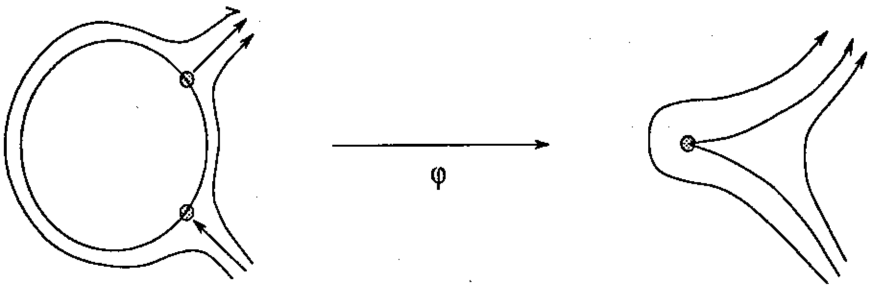

Figure 3.4:

$$j^\infty\,\omega(0) = \left(\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} a_{ij}\,x^i\,y^j\right)dx + \left(\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} b_{ij}\,x^i\,y^j\right)dy.$$

The *support* of ω (or X) is defined by :

$$S_\omega = \{(i+1,j) \mid a_{ij}\neq 0\} \;\cup\; \{(i,j+1) \mid b_{ij}\neq 0\}.$$

The *Newton polyhedron* of ω (or X) is the convex hull Γ_ω of the set :

$$P_\omega = \bigcup_{(r,s)\in S_\omega} \{(r,s)+R^2_+\}$$

while the Newton's diagram of ω (or X) is the union of the compact sides γ_k of Γ_ω. We obtain a quasi-homogeneous component by restricting $(i+1,j)$ and $(i,j+1)$ to some γ_k.

The Newton's diagram of the above vector field Y_b has one compact side, related to the quasi-homogeneous component $y\,\dfrac{\partial}{\partial x} + x^2\,\dfrac{\partial}{\partial y}$. We have obtained the weights (2,3) by taking the smallest entire vector orthogonal to the Newton's diagram side (see Figure 3.5).

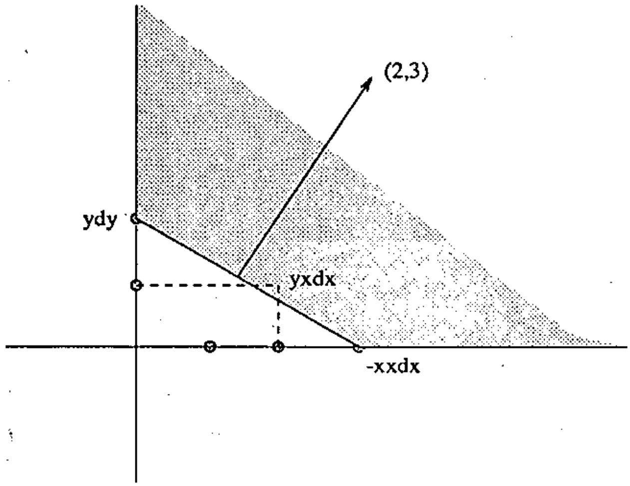

Figure 3.5:

3.2 The finiteness result for analytic vector fields on S^2.

A preliminary problem to the Hilbert's sixteenth one is to prove the following finiteness result :

- *Any polynomial vector field on R^2 has just a finite number of limit cycles.*

As we have said in chapter 1, any polynomial vector field can be extended to an analytic vector field on S^2 and we can ask the previous question for any analytic vector field on S^2.

This question was studied the first time by Dulac in 1923 [Du2] .He gave a proof which presents a gap as it was noticed a long time after by Il'yashenko [I1]. A correct proof was given for quadratic vector fields by Bamon [Bam]. Very recently, complete proofs of the finiteness result was obtained independently by Ecalle [E] and Il'yashenko [I2]. In this paragraph, I want to indicate how to reduce the question to the property of non-accumulation of limit cycles for a polynomial vector field : the so called *Dulac problem*, and in the next paragraph I want to give some indications of the proof of this last problem in the particular case of hyperbolic polycycles. This case is sufficient to obtain Bamon's result (see [Mo] for details).

So, let us consider any polynomial vector field $X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}$. Such a vector field can have non-isolated zeros, but in this case, the two components X_1 and X_2 have a non-trivial polynomial common factor. Let Q be the greater common factor of X_1, X_2. Then :

$$X = Q\left(\overline{X}_1 \frac{\partial}{\partial x} + \overline{X}_2 \frac{\partial}{\partial y}\right)$$

where $\overline{X}_1$, $\overline{X}_2$ are prime polynomials. As a consequence :

- *Any singular point of* $\overline{X} = \overline{X}_1 \frac{\partial}{\partial x} + \overline{X}_2 \frac{\partial}{\partial y}$ *is algebraically isolated.*

Next, if Γ is some periodic orbit for X, Γ doesn't contain singular points of X and so, it is also a periodic orbit of $\overline{X}$. So, X would have a finite number of limit cycles if the same result holds for $\overline{X}$. From now, we can suppose that our given vector field has just algebraically isolated singular points.

Suppose that such a polynomial vector field has infinitely many limit cycles. It is the same for the analytic vector field X obtained by extending it to S^2. Using the compacity of the space $\mathcal{C}(S^2)$ of all compact subsets of S^2, one can find a sequence of $(\gamma_n)_n$ of limit cycles converging toward some compact invariant subset Γ for X. In the terminology of chapter 2, Γ is a limit periodic set for the trivial family with 0-parameter, made by the single vector field X. So, to prove the finiteness result is equivalent to prove that X (as a family !) has *no limit periodic set.* Looking at theorem 2.5, we know that Γ is a singular point, a periodic orbit or contains at the same time singular points and regular orbits. The two first cases cannot occur, because the first return map along such a Γ is analytic and cannot have accumulation of fixed points (corresponding to the limit cycles γ_n). For the last case, one has claimed in theorem 2.5 that Γ must be *a graphic.*

We begin to prove this claim :

Lemma 4 *Let X_λ an analytic family and Γ a limit periodic set for some value λ_0 which contains singular points and regular orbits of X_{λ_0}. Suppose that each singular point in Γ is algebraically isolated. Then Γ is a graphic.*

Proof Because Γ is compact, it can just contain a finite number of singular points : $p_1, \cdots, p_k$. To prove that Γ is a graphic, it suffices to prove that Γ contains also just a finite number of regular orbits. Suppose on the contrary that Γ contains infinitely many regular orbits.

Then, for at least one of the singular points, say p_1, one has an infinite sequence of regular orbits $(\gamma_n)_{n\in N}$ such that $\alpha(\gamma_n) = p_1$.

By assumption, p_1 is algebraically isolated and we can apply the desingularization theorem 1 : the singular point p_1 has just a finite number of sectors. Clearly, the above orbits must belong to elliptic or expanding parabolic sectors. We can find a sub-sequence $(\gamma_{n_i})_{i\in N}$ in a same sector (which is elliptic or parabolic), and construct a transversal segment σ cutting each γ_{n_i} in one point at least. But this is in contradiction with lemma 2.2. ∎

Remark 9 *The same proof works for limit sets of analytic vector fields. On the contrary, it is possible to construct a smooth vector field which has some limit periodic set or limit set with infinitely many regular orbits (and non algebraically isolated singular points).*

We return now to our vector field X with an accumulation of limit cycles γ_n on some graphic Γ. This graphic must have a well defined returned map on some half interval σ transversal to Γ ($\sigma \simeq [0,1[$ and $\{0\} = \sigma \cap \Gamma$) on the side where one has the accumulation. Such a graphic is called a monodromic graphic or *polycycle.*

Now, at each singular point, we can apply the desingularization theorem. The desingularization mapping $\Phi_1 \circ \cdots \circ \Phi_n$ at each point is analytic, so that we can construct, gluing up local charts defined at each p_i, an analytic surface $\widetilde{U}$ and a proper map $\Phi : \widetilde{U} \to U$ on some neighborhood of Γ such that Φ is equal to the desingularization map above a neighborhood of each p_i and is an analytic diffeomorphism elsewhere.

The vector field X lifts up to a vector field $\widehat{X}$ on $\widetilde{U}$, which may be desingularized by division by functions defined locally. So, we obtain on $\widetilde{U}$ an analytic singular foliation defined by vector fields $\overline{X}_i$; each $\overline{X}_i$ is defined on an open set U_i, and $\overline{X}_i$, $\overline{X}_j$ differ by a positive analytic function on $U_i \cap U_j$ (we call such a foliation a "local vector field" in chapter 6). This foliation is oriented and has exactly the same qualitative properties as a vector field. The counter-image $\Phi^{-1}(\Gamma) = \tilde{\Gamma}$ is an elementary polycycle (each vertex in $\tilde{\Gamma}$ is elementary). Because Φ is an analytic diffeomorphism outside $\tilde{\Gamma}$, the infinite sequence of limit cycles in U which accumulates on Γ, lifts up in an infinite sequence in $\widetilde{U}$ accumulating $\tilde{\Gamma}$.

Finally, we are reduced to prove that such an accumulation is impossible. It is the so-called *Dulac problem* :

- An elementary polycycle of an analytic foliation cannot be accumulated by limit cycles.

3.3 The Dulac problem.

In this paragraph, I want to give some indications on the solution of the Dulac problem. As I have said above, this proof is quite recent and I am not sure to understand all its details. So, I am going to limit myself to the simpler case of *hyperbolic polycycles.*

A beautiful proof was given by Il'yashenko in 1985 in this case [I2]. This proof contains some of the ideas used for the general case and moreover we will see that it can be extended somewhat to general unfoldings of hyperbolic polycycles (see chapter 5). Here, I am going to follow partially the survey given by Moussu [Mo].

So, let be an analytic foliation F in a neighborhood U of some hyperbolic polycycle Γ. Let $p_1, \cdots, p_n$ the vertices labelled in cyclic order. Let σ' any half-segment transversal to Γ, such that return map P is defined from $\sigma \to \sigma'$ where $\sigma \subset \sigma'$ is some subsegment, neighborhoods of the base point $a \in \sigma' \simeq [a, b[$.

At each vertex p_i, one can choose local coordinates (x_i, y_i) such that $0x_i$, Oy_i are local unstable and stable manifolds and more precisely $0x_i^+$, $0y_i^+$ belong to Γ and the trajectories corresponding to the return map

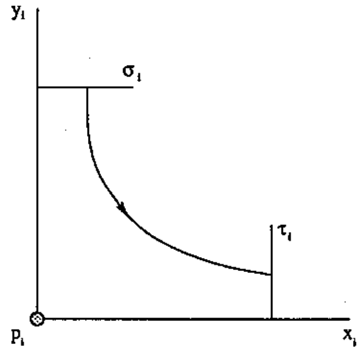

Figure 3.6:

near Γ are in the first quadrant.

Taking transversal segments σ_i, τ_i to $0y_i$, $0x_i$ respectively we can define a transition map $y_i = D_i(x_i)$ from σ_i^+ to τ_i (σ_i^+ corresponding to $x_i \geq 0$) near each saddle point and also a regular transition R_i along each side of Γ, from τ_{i-1} to σ_i.

Taking σ' in σ_1^+ for instance, we can write $P(x_1)$ as a composition :

$$P(x_1) = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1(x_1). \tag{3.11}$$

The maps R_i are analytic diffeomorphisms. We want to look more closely to the structure of each saddle transition. Let p any hyperbolic saddle and $D(x)$ its transition map. The structure of D depends strongly on whether the saddle is resonant (with a rational ratio of eigenvalues) or not.

In any case, we will prove in chapter 5 that there exists a formal series $\widehat{D}(x)$, the so-called *Dulac series of D* :

$$\widehat{D}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} \; P_i(Lnx)$$

where λ_i is a sequence of positive numbers : $\lambda_1 < \lambda_2 < \cdots < \lambda_1 < \cdots$ tending to infinity, with $\lambda_1 = r = \dfrac{-\mu_2}{\mu_1}$ (the ratio of hyperbolicity of p ;

μ_2, μ_1 being the eigenvalues), and a sequence of polynomials P_i, with $P_1 = A$ (a positive constant).

This series is asymptotic to $D(x)$ in the following way: for any s,

$$| D(x) - \sum_{i=1}^{s} x^{\lambda_i} P_i (Lnx) |= O(x^{\lambda_s}). \tag{3.12}$$

Definition 17 *A germ of map f at $0 \in R^+$ is said quasi-regular if :*

(i) f has a representative on $[0, X[$ which is C^∞ on $]0, X[$.

(ii) f is asymptotic to a Dulac series $\hat{f}$:

$$\hat{f}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(Lnx), \text{ with } 0 < \lambda_1 < \lambda_2 < \cdots \text{ a sequence}$$

of positive coefficients tending to ∞ and P_i a sequence of polynomials.

One says that f is a quasi regular homeomorphism if f is quasi-regular and if $P_1(x) \equiv A$ (a positive constant).

It is straightforward to verify that the set of all quasi-regular homeomorphisms is a group $\mathcal{D}$ (for the composition of maps) which contains the group $Diff_0$ of germs of diffeomorphisms fixing 0.

As a consequence :

Proposition 2 *The Poincaré map P is quasi-regular.*

Remark 10 *This result is also true for C^∞ vector fields or foliations.*

Now, suppose that $\widehat{P}(x) \not\equiv x$, we have that $P(x) - x$ has also a non-zero Dulac series and so : $P(x) - x$ is equivalent to some $\alpha x^\lambda\ Ln^k\ x$ for $\alpha \neq 0$, $\lambda > 0$ and $k \in N$.

But this implies that the equation $\{P(x) - x = 0\}$ has no roots in $]0, X]$, for some $X > 0$, contradicting the assumption of accumulation of limit cycles on Γ, and so, of roots of $\{P(x) - x = 0\}$ on $\{x = 0\}$. So, the Dulac series of P is identical to x. We want to prove that this implies that $P(x) \equiv x$. It is precisely this crucial step : $\widehat{P}(x) - x \equiv$

$0 \Rightarrow P(x) - x \equiv 0$ which seems to miss in Dulac's paper. This gap was filled up by Il'yashenko in [I2] in the hyperbolic case. The idea was to prove for P a more precise property of *quasi-analyticity* :

Definition 18 *Let $f : [0, X[\to R$ a function.*
One says that f is quasi-analytic if :

(i) f is quasi-regular.

(ii) The map : $X \to f \circ exp(-X)$ has a bounded holomorphic extension $F(Z)$ on some domain Ω_b of C, defined by : $\Omega_b = \{Z = (X+iY) \in C \mid X > b(1+Y^2)^{1/2}\}$ where b is a positive real number.

A consequence of the Phragmen-Lindelöf theorem is that for quasi-analytic function, the mapping $f \to \hat{f}$ is injective (see [Ch] :

Lemma 5 *If f is a quasi-analytic function, such that $\hat{f} \equiv 0$, then $f \equiv 0$.*

Proof For $b' > 0$ large enough, the image of $C^+ = \{Real\ (Z) \geq 0\}$ by $\varphi : Z \to \varphi(Z) = b'(1+Z)^{1/2} + Z$ with $\varphi(0) = b'$ is contained in Ω_b. Let $F(Z) = f \circ\ exp\ (-Z)$ as in the definition. The function $G = F \circ \varphi$ is a bounded holomorphic function on C^+, and, because $\hat{f} \equiv 0$, there exist real K, K_n for $\forall n \in N$ such that :

$$\begin{aligned} \mid G(Z) \mid &< K\ if\ Z \in C^+\ and \\ \mid G(X) \mid &< K_n\ exp\ (-nX)\ if\ X \in R^+. \end{aligned}$$

Now let be $G_n(Z) = G(Z).\ exp\ (nZ)$.

We apply two times the Phragmen-Lindelöf theorem . A first time to the sectors $\{Y \geq 0,\ X \geq 0\}$ and $\{Y \leq 0,\ X \geq 0\}$.

Because $\mid G_n(Z) \mid \leq K\ exp\ (n \mid Z \mid)$, one has for instance :

$$Sup(\mid G_n(X+iY) \mid;\ X \geq 0,\ \ Y \geq 0)$$
$$\leq Sup(\mid G_n(X+iY) \mid, X or Y = 0) \leq Sup\{K, K_n\}.$$

So that $\mid G_n(Z) \mid$ is bounded on C^+ and we can apply the Phragmen-Lindlöf theorem a second time :

$$Sup(\mid G_n(Z) \mid; Z \in C^+\} \leq Sup(\mid G_n(Z) \mid; Z \in \partial C^+\} = K.$$

Using this last inequality for $X \in R^+$:

$$|G_n(X)| \leq K \Longrightarrow |G(X)| \leq K exp(-\eta X), \ \forall n \in N, \ \forall X \in R^+.$$

Of course, this implies that $G(X) \equiv 0$. ∎

So, to prove that $P(x) \equiv x$ it suffices to prove that $P(x)$ is quasi-analytic. In the composition $P(x) = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1$ each R_i is real analytic at $x = 0$, and so it is a restriction of local holomorphic diffeomorphism at $\{z = 0\}$. Clearly, such a function is quasi-analytic.. The key point is to prove that :

Theorem 7 *The transition map D at a hyperbolic saddle singularity is quasi-analytic.*

We postpone a moment the proof of the theorem 2 to finish, using it, the proof of the Dulac problem for hyperbolic polycycles.

Let $\varphi(Z) = exp\,(-Z)$ and $\varphi^{-1}(z)$ the branch of $-Log\,(z)$ such that $\varphi^{-1}\,(1) = 0$.

For each mapping $g(x)$ in the composition $P(x) = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1$ the map : $G(z) = \varphi^{-1} \circ g \circ \varphi(Z)$ defines a holomorphic diffeomorphism of some domain Ω_b into another domain $\Omega_{b'}$ (because g is quasi-analytic).

So that we can lift up the composition P into a composition of holomorphic diffeomorphisms from domains Ω_{b_i} to $\Omega_{b_{i+1}}$, for $i = 1, \cdots, 2n$. Composing with φ, we obtain that P is quasi-analytic. This finishes the proof of Dulac problem.

Proof of theorem 2.

Write $g(x) = D(x)$, the transition map. We call $\widetilde{X}$ a complex extension of X in some neighborhoods $\widetilde{W}$ of $0 \in C^2$. Up to some multiplicative factor, the differential equation for $\widetilde{X}$ is :

$$\begin{cases} \dot{z} &= z \\ \dot{\omega} &= -r(1+a(z,\omega))\omega \end{cases} \tag{3.13}$$

$r \in R^+$ is the hyperbolic ratio, and $z = x + ix'$, $\omega = y + iy'$ are complex coordinates. We can suppose that $a(z,\omega)$ is holomorphic in a neighborhoods of a polydisk $D \times D$ or radius (1,1) and that $| a |< 1/2$; the trajectories of $\widetilde{X}$ define an holomorphic foliation F, transversal to the projection $\pi(z,\omega) = z$.

Any path $c : [0,1] \to D^+$, starting at $z \in D^+$ and ending at $1 \in D^+$ has a *partial lift* $\bar{c}$ for π, starting at $\bar{z} = (z,1)$, tangent to F ; this means that there exists $\eta > 0$ such that :

$$\bar{c} : [0,\eta] \to D^+ \times D,\ \bar{c}(0) = \bar{z},\ \pi \circ \bar{c} = c, \quad and\ \bar{c}(t)$$

is a trajectory of $\widetilde{X}$. If $\eta = 1$, we say that $\bar{c}$ is a *lift of* c. If $z = x \in R^+$ is small enough, the path $c_x = t \to (1-t)x + t$, $t \in [0,1]$ has a lift $\bar{c}_x$ and by definition of $g : \bar{c}_x(1) = (1, g(x))$.

Let $Z = X + iY$, and Γ_Z the composition of the two paths :

$$\Gamma_Z^1 : t \to (1-t)X + iY + t \ and\ \Gamma_Z^2 : t \to 1 + i(1-t)Y.$$

If $c_Z = \ exp \circ (-\Gamma_Z)$ has a lift $\bar{c}_Z$ one has $(1, G(Z)) = \bar{c}_Z(1)$.

It is clear that G is holomorphic and bounded in a neighborhood of $R^+ \subset C$. It remains to show that this neighborhood contains a domain Ω_b.

The path c_Z is the composition of the two paths $c_Z^1 = \ exp\ (-\Gamma_Z^1)$ and $c_Z^2 = \ exp\ (-\Gamma_Z^2)$. It is convenient to parametrize these two paths by the flow of the first line of (3.13) :

$$\begin{array}{lccl} & c_Z^1 & : & t \in [0, -Log \mid z \mid] \to ze^t \\ and\ : & & & \\ & c_Z^2 & : & t \in [0, Y] \to e^{-Y}e^{it} \end{array} \tag{3.14}$$

where $z = exp\ (-Z)$.

We have to find some inequality : $Y \leq \varphi(X)$ (for φ of smaller order than X^2) such that one can lift the path c_Z, i.e., lift the path c_Z^1 in a

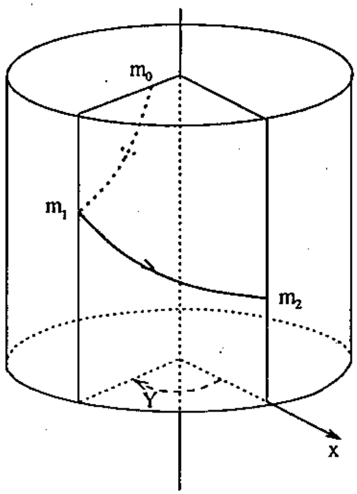

Figure 3.7:

path $\bar{c}_Z^1$ from $m_0 = (z, 1)$ to $m_1 = \left(\frac{z}{|z|}, \omega_1\right)$ and next c_Z^2 in a path $\bar{c}_Z^2$ from m_1 to $m_2 = (1, \omega_2)$. (See Figure 3.7).

To obtain $\bar{c}_Z^1$, we replace the complex time τ in (3.13) by $\tau = t \in [0, -\,Log\ |z|]$. Using that $|a| \leq 1/2$, we have that the solution of the second line of (3.13) verifies :

$$|\omega(0)|\, e^{\frac{1}{2}rt} \leq |\omega(t)| \leq |\omega(0)|\, e^{\frac{3}{2}rt}. \tag{3.15}$$

And in particular, c_Z^1, can be lifted and :

$$|z|^{1/2r} \leq |\omega_1| \leq |z|^{3/2r}. \tag{3.16}$$

To obtain $\bar{c}_Z^2$, we replace τ by $\tau = \theta$, $\theta \in [0, Y]$.
The solution $(z(\theta), \omega(\theta))$ verifies : $|z(\theta)| \equiv 1$ and :

$$\frac{d\omega}{d\theta} = ir(1+a)\omega(\theta). \tag{3.17}$$

If we write $\omega = \rho e^{i\varphi}$, $\rho \in R^+$, $\varphi \in R$, $\omega(\theta) = \rho(\theta)e^{i\varphi(\theta)}$, (3.17) is equivalent to :

$$\left(\frac{d\rho}{d\theta} + i\rho\,\frac{d\varphi}{d\theta}\right) = ir\,(1+a)\rho. \tag{3.18}$$

Or :

$$\begin{cases} \dfrac{d\varphi}{d\theta} &= r + r\,Re(a)\rho \\ \dfrac{d\rho}{d\theta} &= -Im\,(a)\rho \end{cases} \tag{3.19}$$

At this point, we need more information on a, because knowing that $\mid a \mid$ is bounded would not be sufficient. In fact, using the Dulac form (see chapter 5), it is possible to choose holomorphic coordinates (z,ω) such that $a(z,\omega) = 0(\mid z.\omega \mid)$. Here, $a(z,\omega) = a(z(\theta),\omega(\theta))$ with $\mid z(\theta) \mid \equiv 1$.

So that :

$$\mid a(z,\omega) \mid = 0(\rho). \tag{3.20}$$

And finally, the second line of (3.19) gives $\dfrac{d\rho}{d\theta} = 0(\rho^2)$ or, there exists $K > 0$ such that :

$$\left|\frac{d\rho}{d\theta}\right| \leq K\rho^2. \tag{3.21}$$

By integration, this differential inequality implies that :

$$|\rho(\theta)| \leq \frac{\rho(0)}{(1 - 3K \mid Y \mid (\rho(0))^3)^{1/3}} \tag{3.22}$$

Recall that $\rho(0) = \mid \omega_1 \mid$.

To have $\bar{c}_Z^2 \subset D \times D$, it will be sufficient to have :

$$1 - 3K|Y| \mid \omega_1 \mid^3 \geq \mid \omega_1 \mid^3. \tag{3.23}$$

Taking in account (3.15), it is sufficient to have :

$$\mid Y \mid \leq \frac{1}{3K}\,(e^{\frac{3}{2}rX} - 1). \tag{3.24}$$

Clearly, the domain defined by (3.24) contains domains Ω_b.

Remark 11 *If $r \notin R^+$ or if one just use the boundeness of $\mid a \mid$ in place of (3.20), we will obtain a linear inequality $\mid Y \mid \leq KX$ in place of (3.24). The domain so defined doesn't contain an Ω_b, and we could not apply the Phragmen-Lindelöf idea.*

Chapter 4

Bifurcations of regular limit periodic sets.

In this chapter, (X_λ) will be a smooth or in section 3 an analytic family of vector fields on a phase space S and with parameter $\lambda \in P$, as in chapter I. Periodic orbits and elliptic singular points which are limits of sequence of limit cycles are called *regular limit periodic sets.* A reason for this denomination is that for such a limit periodic set Γ one can define locally return maps on transversal segments, which are as smooth as the family itself. As a consequence, the limit cycles near Γ will be given by a smooth equation and the theory of bifurcations of limit cycles from Γ will reduce to the theory of unfoldings of differentiable functions. In fact, we just will need the Preparation Theorem and not the whole Catastrophe Theory to treat finite codimension unfoldings.

Section 3 will be devoted to ∞-codimension analytic unfoldings. In these cases the vector field X_{λ_0} we unfold is of center type ; i.e, it admits a whole annulus of closed orbits. If Γ is any of them, the finite cyclicity of (X_λ, Γ) could be deduced from a general theorem by Gabrielov. I prefer to deduce it in a simple way, using the notion of *Bautin Ideal*, which is interesting by itself and may be used also for singular limit periodic sets where methods from analytic geometry are not sufficient, as we will see in the next chapter. As we will see decompositions in the Bautin Ideal are in fact generalizations of the Melnikov asymptotic formula. Such centre-type vector fields appears in special analytic families, for instance the polynomial family $\mathcal{P}_n$; they may also result from use of rescaling formulas as we will see for instance in this chapter for the Bogdanov-Takens family.

4.1 The return map.

4.1.1 Return map for a periodic orbit.

Let Γ be some periodic orbit of X_{λ_0}. Let $x_0 \in \Gamma$ and σ' some smooth open interval imbedded in S, transversal to X_{λ_0} at any point and such that $x_0 \in \sigma'$. We can find a subinterval σ, neighborhoods of x_0 in σ' such that :

i) $\bar{\sigma} \subset \sigma'$,

ii) a (first) *return map* for the flow of X_{λ_0} is defined from σ into σ'. Let $h_{\lambda_0}(u) : \sigma \to \sigma'$ be this map.

As a consequence of the implicit function theorem, there exists a neighborhood W of λ_0 in P and a map :

$$h(u, \lambda) : \sigma \times W \to \sigma'$$

such that, for each $\lambda \in W$, $h_\lambda(u) = h(u, \lambda)$ is the first return map of X_λ, from σ to σ'. Here u is a smooth parametrization of σ', with $x_0 = \{u = 0\}$. This map is smooth and analytic if the family (X_λ) is analytic.

Let $\delta_\lambda(u) = \delta(u, \lambda) = h(u, \lambda) - u$ the *difference map* ; $\delta : \sigma \times W \to R$. The fixed points of h_λ, which are the roots of $\{\delta_\lambda = 0\}$, correspond to the intersections of periodic orbits of X_λ with σ. In this way, we obtain all closed orbits of X_λ cutting σ, for λ near enough λ_0. This is a consequence of Lemma I.1.2 which implies that each periodic orbit of X_λ cuts σ in at most one point and the fact that σ can be chosen as small as we need. Let us write explicitly this result :

Lemma 6 *For each $\varepsilon > 0$, one can find $\sigma(\varepsilon)$, a neighborhoods of x_0 in σ' such that : $u \in \sigma(\varepsilon)$ is a root of $\{\delta_\lambda = 0\}$ for $\lambda \in W$ if and only if the orbit γ of X_λ throw u is a periodic orbit with $d_H(\gamma, \Gamma) \leq \varepsilon$ (see definition of Hausdorff distance d_H in chapter II).*

If $N(\varepsilon,\lambda)$ is the number of isolated roots of $\{\delta_\lambda = 0\}$ in $\sigma(\varepsilon) = \{| u | \leq \varepsilon\}$, it follows from lemma 1 that :

$$Cycl\ (X_\lambda, \Gamma) = \underset{\substack{\lambda\to\lambda_0 \\ \varepsilon\to 0}}{Sup}\ \{N(\varepsilon,\lambda)\}$$

and that the study of $Cycl\ (X_\lambda,\Gamma)$ reduces to the study of the number of roots of the equation $\{\delta_\lambda = 0\}$ near $u = 0$, for λ near λ_0.

4.1.2 Return map near an elliptic point.

Recall that an elliptic point x_0 for X_{λ_0} is a singular point with complex eigenvalues. It may be a focus or centre type point (if surrounded by a whole disk of closed orbits). In any case, such a singular point is non degenerate and using implicit function in theorem, it is easy to see that the family is smoothly conjugate for (x,λ) near (x_0,λ_0) to a family with $X_\lambda(x_0) \equiv 0$, and with no other singular point other that x_0 in some neighborhoods of x_0. We will assume this from now on.

In some neighborhoods W_0 of λ_0, the eigenvalues of X_λ at x_0 are equal to $\beta(\lambda) \pm i\alpha(\lambda)$ with $\alpha(\lambda) \neq 0$.

Let be now a smooth interval $\sigma' \simeq [0, b'[$, imbedded in S with end point 0 at x_0, and transversal to X_{λ_0} at any point $u \neq 0$.

Lemma 7 *Let any $b \in]0, b'[$ and $\sigma = [0, b[$. Then there exists a neighborhood $W \subset W_0$ of λ_0 such that a return map $h_\lambda(u)$ is defined for X_λ from σ into σ' ; this map, extended by $h_\lambda(0) = 0$ is smooth as function of (u,λ) (analytic if (X_λ) is analytic).*

Proof Let Ω a coordinate chart containing $x_0 : \Omega \simeq R^2$ of coordinate (x,y) and $x_0 = (0,0)$. Let $\varphi(r,\theta) = (r\ cos\ \theta, r\ sin\ \theta)$ the polar coordinate map. As we have seen in chapter III, there exists a family $(\widehat{X_\lambda})$ in $(S^1 \times R^+) \times W$ such that $\varphi_*(\widehat{X_\lambda}) = X_\lambda \mid \Omega \times W$.

Recall that, if $X_\lambda(x,y) = X_1(x,y,\lambda)\frac{\partial}{\partial x} + X_2(x,y,\lambda)\frac{\partial}{\partial y}$ in $\Omega \times W$, the 'blown-up' family $(\widehat{X_\lambda})$ is equal to :

$$\widehat{X_\lambda} = \eta_1(r,\theta,\lambda)\ \frac{\partial}{\partial\theta}\ + \eta_2(r,\theta,\lambda)\ r\ \frac{\partial}{\partial r}$$

with :

$$\eta_1(r,\theta,\lambda) = \frac{1}{r^2}\Big(-rs\ X_1(rc,rs,\lambda) + rcX_2(rc,rs,\lambda)\Big)$$

and

$$\eta_2(r,\theta,\lambda) = \frac{1}{r^2}\ \Big(rcX_1(rc,rs,\lambda) + rsX_2(rc,rs,\lambda)\Big)$$

with :

$$c = cos\theta \quad and \quad s = sin\theta.$$

So that clearly $(\widehat{X}_\lambda)$ being smooth or analytic depending on (X_λ) being smooth or analytic.

We can suppose the coordinates are chosen such that :

$$j^1\ X_\lambda(0) = \alpha\Big(-y\ \frac{\partial}{\partial x} + x\ \frac{\partial}{\partial y}\Big) + \beta\Big(x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y}\Big).$$

In polar coordinates :

$$J^1\ \widehat{X}_\lambda(\theta,0) = \alpha\ \frac{\partial}{\partial\theta} + \beta r\ \frac{\partial}{\partial r} \qquad for\ \ \forall\theta\in S^1.$$

This implies that $\widehat{X}_\lambda$ is non singular for each $(\theta,0)$ and that the curve $S^1\times\{0\}$ is a periodic orbit. If $b'>0$ is chosen small enough the interval $\{0\}\times[0,b'[$ is transversal to $\widehat{X}_\lambda$ for all $\lambda\in W$ and one can choose $b\in]0,b'[$ such that the return map $\widehat{h}_\lambda$ of $\widehat{X}_\lambda$ is defined on $]0,b[\times W$.
Of course $\widehat{h}_\lambda(r)$ is a smooth (resp. analytic) function of (u,λ) if (X_λ) is smooth (resp. analytic). Under the mapping φ, intervals $\{0\}\times[0,b'[$, $\{0\}\times[0,b[$ are sent to the interval σ', σ resp. in the $0x$-axis and because $\varphi_*(\widehat{X}_\lambda)=X_\lambda$, the return map h_λ for X_λ, is defined from σ to σ' and is equal to $\widehat{h}_\lambda$. This concludes the proof. ∎

The vector field family $(\widehat{X}_\lambda)$ is in fact defined in a whole neighborhoods of $S^1\times\{0\}$ in $S^1\times R$ (It suffices to take $r\in R$ in the polar coordinate map), and the return map $\widehat{h}_\lambda$ extends in a whole neighborhoods Σ of $0\in R$. This return map is clearly equals to the return map for X_λ, defined on the $0x$-axis for negative values. This means that $h_\lambda(u)$ extends smoothly on a whole neighborhoods Σ of 0 (such that $\sigma=\Sigma\cap\{x\geq 0\}$). In the same way it is easy to see that the first return

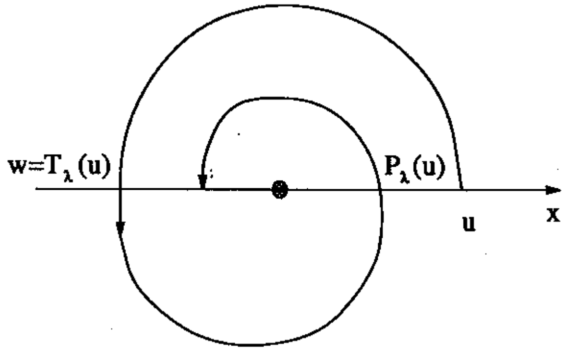

Figure 4.1:

of the flow by $u \in \Sigma$ on the whole axis $0x$ is a well defined smooth map $T_\lambda(u) : \Sigma \to R$ with $T_\lambda(u).u \leq 0$. Moreover one has locally in some neighborhoods of $0 \in \Sigma$:

$$h_\lambda \circ T_\lambda = T_\lambda \circ h_\lambda \ (\textit{see Figure } 4.1 \ :).$$

It follows from this, that P_λ on $\{u \leq 0\}$ is locally conjugate to h_λ on $\{u \geq 0\}$ and that if h_λ is a contraction (expansion) for $\{u \geq 0\}$ it is the same for $\{u \leq 0\}$. As a direct consequence ;

Lemma 8 *Suppose that $\delta_{\lambda_0}(u) = h_{\lambda_0}(u) - u$ is not flat at $u = 0$ (i.e : $\exists k$ such that $j^k\ \delta_{\lambda_0}(0) \neq 0$). Then $\delta_{\lambda_0}(u)$ has an odd order $2k+1$:*

$$\delta_{\lambda_0}(u) = (\beta(\lambda_0) - 1)u + 0(u) \ \textit{with} \ \beta(\lambda_0) \neq 1 \ \textit{or :}$$

$$\delta_{\lambda_0}(u) = +\alpha_{2k+1}\ (\lambda_0)\ u^{2k+1}\ + 0(u^{2k+1}) \ \textit{with} \ \beta(\lambda) = 1 \ \textit{and}$$

$$\alpha_{2k+1}\ (\lambda_0) \neq 0.$$

The case $\beta(\lambda_0) \neq 1$ corresponds to an hyperbolic focus. If $\beta(\lambda_0) = 1$ and $\delta_{\lambda_0}(u) = \alpha_{2k+1}\ (\lambda_0)^{u^{2k+1}} + 0(u^{2k+1})$ one says that x_0 is a *weak focus of order* k of X_{λ_0}.

4.2 Regular limit periodic sets of finite codimension.

4.2.1 Periodic orbit.

Let Γ some periodic orbit for X_{λ_0}, as in 4.1.2, with a transversal interval σ, $\delta_\lambda(u) = h_\lambda(u) - u$ the corresponding difference map, for $(u,\lambda) \in \sigma \times W(\{u=0\} = \Gamma \cap \sigma)$.

Definition 19 *Γ is said of codimension $k \geq 0$ if $\delta_{\lambda_0}(u)$ is of order $k+1$ at $u=0$, i.e :*

$$\delta_{\lambda_0}(u) = \alpha_{k+1}\, u^{k+1} + 0(u^{k+1}) \quad with \quad \alpha_{k+1} \neq 0.$$

Remark 12 *A finite codimension periodic orbit is necessarily a limit cycle. So, Γ is of order 0 if and only if Γ is a hyperbolic limit cycle. In this case, one can choose an annulus Ω around Γ and a neighborhood W' of λ_0 in W such that, for $\forall \lambda \in W'$, X_λ has an unique (hyperbolic) limit cycle Γ_λ, with $\Gamma = \Gamma_{\lambda_0}$. So that the Cycl $(X_\lambda, \Gamma) = 1$, in this case.*

This result of finiteness is easily generalized :

Lemma 9 *Let Γ a limit cycle of X_{λ_0} of codimension k. Then Cycl $(X_\lambda,\Gamma) \leq k+1$.*

Proof As we have seen in lemma 1, *Cycl* (X_λ,Γ) is equal to the number of local roots for the equation $\{\delta_\lambda(u) = 0\}$. But because $\frac{\partial^{k+1}\,\delta_{\lambda_0}}{\partial u^{k+1}}(0) \neq 0$, one can find $\sigma_1 : 0 \in \sigma_1 \subset \sigma$ and a neighborhood $\lambda_0 : W_1 \subset W$ such that $\frac{\partial^{k+1}\,\delta_\lambda}{\partial u^{k+1}}(u) \neq 0$ for $\forall (u,\lambda) \in \sigma_1 \times W_1$. It follows from Rolle's theorem that the function $u \to \delta_\lambda(u)$ has less than $k+1$ roots in σ_1 (for any $\lambda \in W_1$). ∎

Remark 13 *If the return map $h_\lambda(u) : \sigma \to \sigma'$ is defined for $\lambda \in W$, the set of parameter values $\lambda \in W$ for which at least one limit cycle of order k cuts σ is given by the equation :*

$$\{\delta_\lambda(u) = \cdots = \delta_\lambda^{(k)}(u) = 0 \ \ , \ \ \delta_\lambda^{(k+1)}(u) \neq 0\}.$$

The map which at each $\lambda \in W$ *associates* $h_\lambda(u) \in \mathcal{C}^\infty(\sigma, \sigma')$ *is a smooth map. More generally, if* $X_0 \in \chi^\infty(S)$ *has a return map* $h_{X_0}(u) : \sigma \to \sigma'$, *one can find a neighborhoods* W *of* X_0 *in* $\chi^\infty(S)$ *such that each* $X \in W$, *has a return map* $h_X : \sigma \to \sigma'$. *The map* $X \in \chi^\infty(S) :\to P_X \in \mathcal{C}^\infty(\sigma, \sigma')$ *is also smooth (in the sense of differentiable maps between Frechet spaces). It is also easy to prove that the above equations define a codimension k-submanifold of* LC_k $W \subset \chi^\infty(S)$. *(*$LC_k(\sigma)$ *limit cycles of codimension* k, *related to* σ*). We can call it a singularity as in chapter I : the singularity of vector fields with 1 limit cycle of cod.* k *(cutting the given interval* σ*). It is more general that the singularities defined in chapter I which were given by a submanifold in a jet space. Here,* $LC_k(\sigma)$ *is not defined in term of the jets of the vector fields but throw its return map on* σ.

A consequence is that it is difficult to find this set in a given family : for instance, the subset LC_k *of polynomial vector fields of degree* $\leq n$, *having at least 1 limit cycle of codimension* k *is an analytic subset of* $\mathcal{P}_n$, *but we know almost nothing about it. For instance, we do not know if* LC_k, *for* $k \geq 4$ *is empty or not in* $\mathcal{P}_2$.

It is easy to give a more precise description of unfoldings of codimension k limit cycle . If σ is a transversal segment to such a limit cycle Γ for the parameter value λ_0, one has : $\delta_{\lambda_0}(u) = \alpha_{k+1}\ u^{k+1} + 0(u^{k+1})$ ($\{u = 0\} = \Gamma \cap \sigma$). Then it follows from the *preparation theorem* that there exist functions $U(u, \lambda)$, with $U(0, \lambda_0) \neq 0$ and $\alpha_0(\lambda), \cdots, \alpha_{k-1}(\lambda)$ in a neighborhoods of $(0, \lambda_0)$ and λ_0 respectively such that :

$$\delta_\lambda(u) = U(u, \lambda)\Big(u^{k+1} + \sum_{j=0}^{k-1} \alpha_j(\lambda)u^j\Big) \tag{4.1}$$

If X_λ is analytic the functions U, α_j are also analytic [N]. If X_λ is $\mathcal{C}^\infty$, one can find U, α_j also $\mathcal{C}^\infty$. This is the "$\mathcal{C}^\infty$ preparation theorem" of Malgrange [M].

From (4.1) one has that the equation $\{\delta_\lambda(u) = 0\}$ is equivalent to the polynomial equation :

$$u^{k+1} + \sum_{j=1}^{k-1} \alpha_j(\lambda)u^j = 0 \tag{4.2}$$

This equation is factorized throw the universal unfolding of the monomial u^{k+1} :

$$P_{k+1}(u,\alpha) = u^{k+1} + \sum_{k=1}^{k-1} \alpha_j \, u^j = 0 \tag{4.3}$$

The diagram of bifurcation for the roots of P_{k+1} in terms of the parameter $\alpha = (\alpha_0, \cdots, \alpha_{k-1})$ is rather famous at least for $k \leq 4$, because it gives 4 on the 7"elementary catastrophes" (these reducing to phase space of dimension 1) : the fold for $k = 1$, the cusp for $k = 2$, the swallow tail for $k = 3$ and the butterfly for $k = 4$. We refer to the abundant literature on Catastrophe Theory for a description and also to [D2], for applications to vector fields.

The vector field X_λ is locally equivalent to a any vector field with difference function :

$$\delta_\lambda^{N\pm}(u) = \pm\Big(u^{k+1} + \sum_{j=0}^{k-1} \alpha_j(\lambda) \, u^j\Big) \tag{4.4}$$

So a first question is to construct such a vector field family near $\sigma \times \{\lambda_0\}$. It is a trivial exercise in the C^∞ case and I leave it to the reader :

Lemma 10 *(Lifting Lemma) : Let $h_\lambda(u) : \sigma' \times W \to \sigma$ a C^∞ family of diffeomorphisms of σ' into σ. Then one can find a C^∞ family of vector field on some annulus U (containing σ), with parameter in W, whose has $h_\lambda(u)$ as return map.*

Remark 14 *I do not know if such a result exists for analytic vector field families.*

Clearly, the initial family X_λ is induced throw the map $\alpha(\lambda) = \Big(\alpha_0(\lambda), \cdots, \alpha_{k-1}(\lambda)\Big)$, from the versal unfolding $X_\alpha^{k\pm}$ one can construct, using lemma 5, for the function $\delta_\alpha^{k\pm}(u) = \pm\Big(u^{k+1} + \sum_{k=0}^{k-1} \alpha_j \, u^j\Big)$. $X_\alpha^{k\pm}$ is a structurally stable unfolding of codimension k, and in any generic family with l parameters, any local unfolding of limit cycle is induced by some of them with $k \leq \ell$.

4.2.2 Elliptic focus.

Let us now consider some elliptic focus point x_0 for X_{λ_0}. As above one can suppose that x_0 is a non degenerate singular point for any $\lambda \in W$, some neighborhoods of λ_0, a transversal segment σ' by x_0 $\sigma' \sim [0, b'[$ $(x_0 = \{u = 0\})$ and $\sigma \subset \sigma'$, $\sigma = [0, b[$ and a return map $h_\lambda(u) : \sigma \times W \to \sigma'$, with $h_\lambda(0) \equiv 0$.

To simplify the study of X_λ and of its return map h_λ, the family is reduced to its normal form. We just recall this and refer to [D] for an existence proof :

Up to a $\mathcal{C}^\infty$ conjugacy (i.e a $\mathcal{C}^\infty$ coordinate change, depending on the parameter), X_λ is equivalent to :

$$X_\lambda^N = \Big(f(x^2+y^2,\lambda) + f_\infty\Big)\Big(-y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\Big)$$
$$+\Big(g(x^2+y^2,\lambda)+g_\infty\Big)\Big(x\,\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}\Big) \tag{4.5}$$

where $f(u,\lambda)$ and $g(u,\lambda)$ are $\mathcal{C}^\infty$, $f_\infty(x,y,\lambda)$, $g_\infty(x,\lambda,y)$ are $\mathcal{C}^\infty$ and are flat at the origin: $(j^\infty\, f_\infty(0,\lambda) = j^\infty\, g_\infty(0,\lambda) = 0)$.

We can write X_λ^N in polar coordinate :

$$X_\lambda^N = \Big(f(\rho^2,\lambda)+f_\infty(\rho,\theta,\lambda)\Big)\,\frac{\partial}{\partial\theta}+\Big(g(\rho^2,\lambda)+g_\infty(\rho,\theta,\lambda)\Big)\,\rho\,\frac{\partial}{\partial\rho} \tag{4.6}$$

with f_∞, g_∞ flat at $\rho = 0$. Of course, $f(0,\lambda) \neq 0$ for any $\lambda \in W$ and we can divide X_λ^N locally along $\{0\} \times W$ by the component in $\dfrac{\partial}{\partial\theta}$.

So, X_λ is $\mathcal{C}^\infty$ equivalent to a vector field family :

$$Y_\lambda = \frac{\partial}{\partial\theta} + (G(\rho^2,\lambda) + G_\infty(\rho,\theta,\lambda))\rho\,\frac{\partial}{\partial\rho} \tag{4.7}$$

To obtain the return map on σ (we choose it in $\{\theta = 0\}$), one has to integrate the differential equation of Y_λ :

$$\left\{\begin{array}{rcl}\dot\theta &=& 1\\ \dot\rho &=& (G+G_\infty)\rho\end{array}\right. \tag{4.8}$$

We can eliminate the time t, and look for the solution ρ in term of θ. It is solution of the equation :

$$\frac{d\rho}{d\theta} = \Big(G(\rho^2, \lambda) + G_\infty(\rho, \theta, \lambda)\Big)\rho \tag{4.9}$$

If $\rho(\theta, \lambda)$ is the solution with $\rho(0, \lambda) = u \in \sigma$, the return map is

$$h(u, \lambda) = h_\lambda(u) = \rho(2\pi, \lambda).$$

Now, because (4.9) in a C^∞ equation in ρ^2, up to a flat term, the return map has the following form :

$$h_\lambda(u) = \Big(\overline{h}_\lambda(u^2, \lambda) + h_\infty(u, \lambda)\Big) \tag{4.10}$$

where $\overline{h}_\lambda(u^2, \lambda)$ is C^∞, $h_\infty(u, \lambda)$ is flat at $u = 0$ and $\overline{h}_\lambda = e^{2\pi\beta(\lambda)} + 0(u^2)$ (where $\beta(\lambda) \pm i$ are the eigen-values of the 1-jet of (4.8)). But any flat function can be written as a C^∞ function of u^2 for $u \geq 0$, so that we can include the term h_∞ in $\overline{h}$:

$$h_\lambda(u) = u\ \overline{h}_\lambda(u^2, \lambda) \tag{4.11}$$

and :

$$\delta_\lambda(u) = u\ \overline{\delta}(u^2, \lambda) \tag{4.12}$$

with $\overline{\delta}_\lambda$ C^∞ in u^2 and λ, $\overline{\delta}(u^2, \lambda) = (e^{2\pi\beta(\lambda)} - 1) + 0(u^2)$. Let us suppose that X_{λ_0} has a weak focus of order k. This means that :

$$\overline{\delta}_{\lambda_0}(u^2) = \overline{\alpha}_k\ u^{2k} + 0(u^{2k}) \ \ with \ \ \overline{\alpha}_k \neq 0.$$

In this case, one has trivially :

Lemma 11 *If x_0 is a weak focus of order k, then Cycl $(X_\lambda, \{x_0\}) \leq k$.*

Proof Equation for limit cycles near the origin is $\{\overline{\delta}_\lambda(u^2) = 0\}$ and 0 is a zero of order k is u^2. So that applying k times the Rolle theorem to this function of u^2 gives the result : $\overline{\delta}_\lambda(u^2)$ has less than k zeros on $[0, U]$ for some $U > 0$ and for $\lambda \in W$. ■

Of course, as for periodic orbits, one can obtain a precise description for the bifurcation diagram. Applying the preparation theorem to the C^∞ function $\overline{\delta}$, one has :

$$\overline{\delta}_\lambda(u) = U(u,\lambda)\Big[u^{2k} + \sum_{j=0}^{k-1} \alpha_j(\lambda)\ u^{2j}\Big] \tag{4.13}$$

$U(0,\lambda_0) \neq 0$ and $\alpha_j(0) = 0$.

Notice that is not possible to kill the term in $u^{2(k-1)}$ by a translation in u because we have to preserve $\{u = 0\}$ which corresponds to the singular point of the vector field.

$\overline{\delta}_\lambda$ is then factorized, up to the unity U, throw the versal function :

$$\overline{\delta}_\alpha(u) = u^{2k} + \sum_{j=1}^{k-1} \alpha_j\ u^{2j}. \tag{4.14}$$

It is clear that the zeros of $\overline{\delta}_\alpha$ correspond to the limit cycles of the polynomial vector field family :

$$X_\alpha^{N\pm} = \frac{\partial}{\partial\theta} \pm \Big(\sum_{j=0}^{k-1} \alpha_j\ \rho^{2j} + \rho^{2k}\Big)\,\rho\,\frac{\partial}{\partial\rho} \tag{4.15}$$

($\pm$: sign of $\overline{\alpha_k}$).

Remark 15 *It is possible to construct a C^∞ conjugacy between the Poincaré map of $X_\alpha^{N\pm}$ and $u\overline{\delta}_\alpha(u^2)$, but in general, not possible "to kill" by a conjugacy the unity $U(u,\lambda)$. So that, if $X_{\alpha(\lambda)}^{N\pm}$ is equivalent to X_λ for each λ, it is not possible in general to construct a (C^0, C^0) equivalence for $k \geq 4$. One can construct a topological obstruction to this ! (see [R1]).*

One has obtained finally is that $X_\alpha^{N\pm}$ is the versal unfolding of the germ X_{λ_0}, for the (C^0-fibre, C^∞) equivalence relation. Recall that this means that one can find a C^∞ map $\alpha(\lambda)$ such that for each value of λ, X_λ is topologically equivalent to $X_{\alpha(\lambda)}^{N\pm}$. If X_λ is analytic, the map $\alpha(\lambda)$ is also analytic.

This result was proved for $k = 1$ by Hopf and extended for any $k \geq 2$ by F. Takens [T1]. In fact, Takens proved a somewhat stronger result : he obtained a smooth map $H(x, y, \lambda) : R^2 \times W \to R^2 \times W$ above the *map $\alpha(\lambda)$ which bring the limit cycles of X_λ into the limit cycles of the model $X_\alpha^{N\pm}$. It is why we will call the above unfolding : degenerate Hopf unfolding, or Hopf-Takens unfolding.*

4.3 Regular limit periodic set of infinite codimension.

We restrict now to an analytic family X_λ, and we suppose that for some value $\lambda_0 \in P$, there exists an interval σ' such that each orbit cutting it is periodic. We will say that X_{λ_0} is of *centre-type*. Of course, we may suppose that σ' is a maximal interval with this property and that the ends of σ' belong to singular limit periodic set. This limit periodic set may be reduced to a single point, a centre. (It is the reason why I say that X_{λ_0} is centre-type). The simpler case is when this centre is elliptic : the 1-jet of X_{λ_0} at this point is conjugate to a rotation, and this limit point is then a regular limit periodic set as the other orbits through the points of σ'. We will consider such a singular point in this paragraph. The end point of σ' may belong to a more complicate limit periodic set : a single non-elliptic point or a limit periodic set with a singular point and regular orbit. We will study this possibility in the next chapter.

If $e \in \partial\sigma'$ is an elliptic point, we will suppose as above that e is a non-degenerate singular point for any value of λ in some neighborhood W_0 of λ_0. In this case, we will call σ' the half-closed interval $\sigma' \cup \{e\}$.

In all cases (σ' half-closed or open), for any $\sigma \subset \sigma'$ ($e \in \sigma$ if σ' is half-closed), such that $\bar{\sigma} \subset \sigma'$, one can find some neighborhoods W of λ_0 in W_0 such that the return map $h_\lambda(u) : (u, \lambda) \in \sigma \times W \to \sigma'$ and the difference map $\delta_\lambda(u) = h_\lambda(u) - u : \sigma \times W \to R$ are analytic. That X_{λ_0} is centre type is equivalent to $\delta_{\lambda_0}(u) \equiv 0$. We want to study the cyclicity and bifurcation properties of the germ of X_λ along $\{\lambda_0\} \times \gamma_{u_0}$ when γ_{u_0} is the orbit of X_{λ_0} throw $u_0 \in \sigma$ (included the centre case : $u_0 = e$). To this, we introduce in the section 3.1 an ideal in the germs of analytic function of λ at λ_0 : the *Bautin Ideal* $\mathcal{I}$. We will see that the difference function δ_λ may be divided locally in this ideal, and we will give also some other properties for $\mathcal{I}$ and the division of δ_λ which allows to estimate the cyclicity (X_λ, γ_u) (section 3.3). Finally, we will see that the Melnikov asymptotic formula is special case of division in the ideal and that inversely Melnikov functions may be used to compute the division. We close this paragraph with applications of the Bautin Ideal to quadratic vector fields.

4.3.1 The Bautin Ideal.

For any $u_0 \in \sigma$, we can expand the analytic function $\delta(u,\lambda)$ in series in $u - u_0$:

$$\delta(u,\lambda) = \sum_{i=0}^{\infty} a_i(\lambda, u_0)(u-u_0)^i.$$

The functions $a_i(\lambda, u_0)$, are analytic in $W \times \sigma$. To simplify we will not indicate the dependence on u_0 and simply write $a_i(\lambda)$ for $a_i(\lambda, u_0)$. Let $\tilde{f}$ denote the germ at λ_0 of an analytic functions $f(\lambda)$ defined in a neighborhoods of λ_0. ($\tilde{f} \in \mathcal{O}$, ring of analytic germs at $\lambda_0 \in P$). We consider the ideal $\mathcal{I}^{u_0} \subset \mathcal{O}$, generated by the germs $\tilde{a}_i : \mathcal{I}^{u_0} = \mathcal{I}\{\tilde{a}_i\}_i$.

This ideal is Noetherian and so, is generated by a finite number of germs $\tilde{a}_i$:

$$\mathcal{I}^{u_0} = \mathcal{I}\{\tilde{a}_0, \cdots, \tilde{a}_N\}.$$

The functions a_i and the number N depend on u_0.

We have the following *division property* :

Proposition 3 *There exists a constant $R > 0$ such that for all $u_0 \in \sigma$, there exist a neighborhoods of λ_0 : $W_{u_0} \subset W$ and analytic functions $h_0(u,\lambda), \cdots, h_N(u,\lambda)$ defined on $([u_0 - R,\ u_0 + R] \cap \sigma) \times W_{u_0}$ and on this domain :*

$$\delta(u,\lambda) = \sum_{i=0}^{N} a_i(\lambda)\ h_i(u,\lambda). \tag{4.16}$$

Moreover : $h_i(u,\lambda) = (u-u_0)^i(1 + 0(u-u_0))$.

Remark 16 *Recall that $a_i(\lambda)$ and also N may depend on u_0. But the above constant R is independent of u_0.*

Proof We suppose that W is compact. Let K the union of all trajectories of X_λ between points in $\bar{\sigma}$ and the first return on σ', for $\forall\lambda \in W$; K is a compact subset of $S \times P$. One can extend the real analytic vector field family (X_λ), to a holomorphic vector field family $(\widehat{X}_{\hat{\lambda}})$ defined for $(\hat{x}, \hat{\lambda})$ in some neighborhood of K in the complexification $\widehat{S} \times \widehat{P}$ of $S \times P$. For this holomorphic family, one can choose sections $\hat{\sigma}$, $\hat{\sigma}' \subset \widehat{S}$, diffeomorphic to disks, such that $\overline{\hat{\sigma}}. \subset$ int $\hat{\sigma}'$, and $\hat{\sigma} \cap S = \sigma$, $\hat{\sigma}' \cap S = \sigma'$. One can also choose some compact extension $\widehat{W}$ of W in $\widehat{P}(W = \widehat{W} \cap P)$, such that $\widehat{X}_{\hat{\lambda}}$ has a holonomy map $\hat{h}(\hat{u}, \hat{\lambda})$: $\hat{\sigma} \times \widehat{W} \to \hat{\sigma}'$. Let $\hat{\delta}(\hat{u}, \hat{\lambda}) = \hat{h}(\hat{u}, \hat{\lambda}) - \hat{u}$. Then for all $\hat{u}_0 \in \hat{\sigma}$, one has series expansion :

$$\hat{\delta}(\hat{u}, \hat{\lambda}) = \sum_{i=0}^{\infty} \hat{a}_i(\hat{\lambda}, \hat{u}_0)(\hat{u} - \hat{u}_0)^i \tag{4.17}$$

The functions $\hat{a}_i(\hat{\lambda}, \hat{u}_0)$ are holomorphic on $\widehat{W} \times \overline{\hat{\sigma}}$ and extend the real function $a_i(\lambda, u_0)$. Because $\overline{\hat{\sigma}} \times \widehat{W}$ is compact, the expansion (1) has a convergence radius greater that some $2R > 0$, where R is independent from u_0 and λ. Also, there exists an independent constant $M > 0$ such that :

$$| \hat{a}_i(\hat{\lambda}, \hat{u}_0) | \leq M(2R)^{-i} \quad for \ \ \forall\, i \in N. \tag{4.18}$$

We want to find holomorphic functions $\hat{h}_i(\hat{u}, \hat{\lambda})$, $i = 0, \cdots, N$ defined on $D_R(\hat{u}_0) \times \widehat{W}_{\hat{u}_0}$ (where $D_R(\hat{u}_0) = \{u \in \hat{\sigma} \mid | \hat{u} - \hat{u}_0 | \leq R\}$ and $\widehat{W}_{\hat{u}_0}$ is some neighborhoods of $\hat{\lambda}_0 = \lambda_0$ in $\widehat{W}$), such that :

$$\hat{\delta}(\hat{u}, \hat{\lambda}) = \sum_{i=0}^{N} \hat{a}_i(\hat{\lambda})\ \hat{h}_i(\hat{u}, \hat{\lambda}) \tag{4.19}$$

The formula (4.16) follows, if one notices that for :

$$(x, \lambda) \in [u_0 - R, u_0 + R] \cap \sigma \times W_{u_0} \quad , \quad W_{u_0} = \widehat{W}_{u_0} \cap P$$

one has :

$$\delta(u, \lambda) = \sum_{i=0}^{N} a_i(\lambda) Re[\hat{h}_i(u, \lambda)].$$

So, it suffices to take $h_i(u,\lambda) = Re(\hat{h}_i(u,\lambda))$. To obtain formula (4.19), we have to use the following theorem in [H] (theorem 7, page 32) :

(D) Let $A_0, \cdots, A_N$ holomorphic functions on some domain V in C^Λ and let $\lambda_0 \in$ int V. Let $\mathcal{I} = \mathcal{I}(\tilde{A}_0, \cdots, \tilde{A}_N)$ the ideal generated by germs of the A_i at λ_0. Then there exists a polydisk $P \subset$ int V, with centre at λ_0, and a constant $K > 0$ such that : for any function φ holomorphic on P, such that $\tilde{\varphi} \in \mathcal{I}$, there exist functions $H_0, \cdots, H_N$ holomorphic on P such that :

$$\varphi = \sum_{i=0}^{N} A_i\ H_i \quad on \quad P$$

and $\mid H_i \mid_\rho \leq K \mid \varphi \mid_\rho$.
(Here $\mid . \mid_\rho$ is the sup norm for continuous functions on P).

We can apply this to $V = \widehat{W}$ and $\hat{a}_i = A_i$, $i = 0, \cdots, N$. Let $\widehat{W}_{u_0}$, the polydisk in (D). For each $j > N$ one can write :

$$\hat{a}_j(\hat{\lambda}) = \sum \hat{a}_i(\hat{\lambda}).\hat{h}_{ji}(\hat{\lambda}). \tag{4.20}$$

For holomorphic functions $\hat{h}_{ji}$ on $\widehat{W}_{u_0}$, such that :

$$\mid \hat{h}_{ji} \mid_{\widehat{W}_{u_0}} \leq K. \mid \hat{a}_j \mid_{\widehat{W}_{u_0}}. \tag{4.21}$$

One can extend formulas (4.20), (4.21) to any $j \geq 0$, taking $\hat{h}_{ji} = \delta_{ji}$ for $0 \leq i, j \leq \ell$, and replacing K by Sup $\{1, K\}$ in (4.21). Now, in the double sum :

$$\hat{\delta}(\hat{u},\hat{\lambda}) = \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{N} \hat{a}_i(\hat{\lambda})\hat{h}_{ji}(\hat{\lambda}) \Big) (\hat{u} - \hat{u}_0)^j, \tag{4.22}$$

we can commute the two summations. This is possible, because for all i,j :

$$\mid \hat{a}_i(\hat{\lambda}).\hat{h}_{ji}(\hat{\lambda})(\hat{u} - \hat{u}_0)^j \mid \leq MK(2R)^{-j} \ \mid \hat{u} - \hat{u}_0 \mid^j \tag{4.23}$$

■

Corollary 1 *The ideal $\mathcal{I}^{u_0}$ is independent of the choice of u_0 in σ.*

Proof Let any $u_0, u_1 \in \sigma$ such that $\mid u_0 - u_1 \mid < R$.

One can apply the formula (4.16) at u_1 and expand h_i in series of $u - u_1$.

It follows that, if :

$$\delta(u, \lambda) = \sum_{i=0}^{\infty} b_j(\lambda)(u - u_1)^j \tag{4.24}$$

one has : $\tilde{b}_j \in \mathcal{I}^{u_0}$. So that : $\mathcal{I}^{u_1} \subset \mathcal{I}^{u_0}$.

But this argument is symmetrical : we can write (4.16) at u_1 and expand it in series if $(u - u_0)$. We obtain $\mathcal{I}^{u_0} \subset \mathcal{I}^{u_1}$ and finally $\mathcal{I}^{u_0} = \mathcal{I}^{u_1}$ if $\mid u_0 - u_1 \mid < R$. The result follows from the connexity of σ. ∎

Definition 20 *We will call "Ideal of Bautin", the ideal $\mathcal{I}^{u_0}$ for any $u_0 \in \sigma$. This is an ideal of $\mathcal{O}_{\lambda_0}$, the ring of analytic germs at λ_0. It is associated to the germ of (X_λ) along $\sigma \times \{\lambda_0\}$.*

Remark 17 *$\mathcal{I} \neq \mathcal{O}_{\lambda_0}$ if and only if $\delta(u, \lambda_0) \equiv 0$. If $\mathcal{I} = \mathcal{O}_{\lambda_0}$, the function $\delta(u, \lambda_0)$ has a finite multiplicity at each $u_0 \in \sigma$. The set of zeros of $\mathcal{I}$: $Z(\mathcal{I})$ is the germ at λ_0, of values of the parameter for which X_λ has a centre type. Bautin computed this ideal for quadratic vector fields(see [B]). It is the reason to call it 'Bautin Ideal' in general. We will return to the result of Bautin in a forthcoming section.*

4.3.2 Properties of the Bautin Ideal.

If $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ is any set of generators for the Bautin Ideal $\mathcal{I}$, one can write :

$$a_i\ (\lambda) = \sum_{j=1}^{\ell} \varphi_j(\lambda) h_{ji}\ (\lambda) \quad for \quad i = 0, \cdots, N$$

on some neighborhood W_{u_0} of λ_0 and some analytic factors h_{ji} when a_i are the coefficients of (0) at u_0.

Putting it in (4.16), and factorizing, we see that we can write this formula with the functions $\varphi_1, \cdots, \varphi_\ell$.

Proposition 4 *Let $\varphi_1, \cdots, \varphi_\ell$ a set of analytic functions on W whose germs generate $\mathcal{I}$. Then, for any $u_0 \in \sigma$, there exists a neighborhood of λ_0, $W_{u_0} \subset W$ and analytic functions $h_1(u,\lambda), \cdots, h_\ell(u,\lambda)$ defined on $[u_0 - R, u_0 + R] \cap \sigma \times W_{x_0}$ such that, on this domain :*

$$\delta(u,\lambda) = \sum_{i=1}^{\ell} \varphi_i(\lambda)\ h_i(u,\lambda). \tag{4.25}$$

Remark 18 *Of course, we lost the control of order of h_i in $u - u_0$.*

Definition 21 *We say that $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ is a minimal set of generators for $\mathcal{I}$ if $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ is a basis of the vector space $\mathcal{I}/\mathcal{MI}$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{\lambda_0}$. We will call dimension of $\mathcal{I}$, $\ell(\mathcal{I}) = \dim_R \mathcal{I}/\mathcal{MI}$, the number of generators of any minimal system.*

It is possible to extract a minimal set of generators from any set of generators, for instance from the set $\{\tilde{a}_0, \cdots, \tilde{a}_N\}$ of the first coefficients at some point u_0.

Lemma 12 *Let $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ a minimal set of generators and $\tilde{f} \in \mathcal{I}$. Let $\tilde{f} = \sum_{i=1}^{\ell} \tilde{\varphi}_i\ \tilde{h}_i$ a decomposition of $\tilde{f}$ in this set. Then the vector $(h_i(0))_{i=1,\ldots,\ell}$ depends just of $\tilde{f}$ and $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ (Remark that the decomposition of $\tilde{f}$ is not necessarily unique).*

Proof $\tilde{f} = \sum_{i=1}^{\ell} h_i(0)\tilde{\varphi}_i \mod \mathcal{IM}$ so that $(h_1(0), \cdots, h_\ell(0))$ is the vector of $\tilde{f}$-components for $\tilde{f}$ in the basis $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ of $\mathcal{I}/\mathcal{IM}$, and are uniquely defined. ∎

Lemma 13 *Let $\{\tilde{\varphi}_i\}$ and $\{\tilde{\psi}_j\}_j$ $i,j = 1, \cdots, \ell(\mathcal{I})$ two minimal sets of generators. Then, there exists a matrix $\{\tilde{H}_{ij}\}$ with coefficients in $\mathcal{O}_{\lambda_0}$ such that $\tilde{\varphi}_i = \sum_{j=1}^{\ell} \widetilde{H}_{ij}\ \tilde{\psi}_j$ and the matrix $\{H_{ij}(\lambda_0)\}_{i,j}$ is invertible.*

Proof Let φ, ψ the vectors of germs : $\{\tilde{\varphi}_i\}$, $\{\tilde{\psi}_j\}$. Because these vectors are systems of generators of the same ideal $\mathcal{I}$, there exist matrices of germs H, L such that :

$$\varphi = H\psi \quad and \quad \psi = L\varphi$$

It follows that :

$$\varphi = HL\varphi \tag{4.26}$$

Then, as a consequence of lemma 8 : $H \circ L(\lambda_0) = H(\lambda_0) \circ L(\lambda_0) = \mathrm{Id}$ and the matrix $H(\lambda_0)$ is invertible. ∎

Proposition 5 *Let $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ a minimal system of generators for $\mathcal{I}$. Let $\delta(u,\lambda) = \sum_{i=1}^{\ell} \varphi_i(\lambda)\ h_i(u,\lambda)$ a division formula as in proposition 2, at some point $u_0 \in \sigma$.*

Then the functions $h_i(u) = h_i(u,\lambda_0)$ are independent of u_0 and so globally defined on σ. Moreover they are R-independent.

Proof The first part of the conclusion is consequence of lemma 8. It suffices to prove the independence of the germs h_i at *any point* $u_0 \in \sigma$, and it suffices to prove this for the factors associated to *any* minimal system of generators $\varphi = \{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$. In fact, if $\psi = \{\tilde{\psi}_1, \cdots, \tilde{\psi}_\ell\}$ is a second minimal systems of generators, it exists, from lemma 8, a matrix of germs H, such that $\varphi = M\psi$ and $H(0)$ is invertible. If $h = (\tilde{h}_1, \cdots, \tilde{h}_\ell)$ are factors for φ :

$$\delta = \sum_{i=1}^{\ell} h_i\ \varphi_i = < h, \varphi > .$$

But $< h, \varphi > = < h, H\psi > = < {}^tHh, \psi >$ for where ${}^tM$ is the transposed matrix. So that $h' = {}^tHh$ is a system of factor for ψ and because ${}^tH(0)$ is invertible, the component of h' are R-independent germs at u_0, if it is the case for h.

So, it suffice to prove the result for a minimal set of generators which is extracted from the system of generators of coefficient at u_0 : $\tilde{a}_0, \cdots, \tilde{a}_N$.

Proposition 1 gives a division :

$$\delta = \sum_{i=1}^{N} a_i\ h_i$$

and $H_i(u) = h_i(u, \lambda_0) \simeq (u - u_0)^i$. The last condition implies the gems h_i, $i = 0, \cdots, N$ are independent at u_0. Unfortunately, the system $\{\tilde{a}_0, \cdots, \tilde{a}_N\}$ is not minimal in general. We are going to extract a minimal system from this one by a finite number of steps such that, at each step we have a system of generators $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_k\}$, obtained from the last system by dropping one term and such that the associated factors $H_1, \cdots, H_k$ are R-independent . It suffices to prove the recurrence step because after $N - \ell$ steps we must arrive to a minimum set of generators.

Suppose that :

$$\delta = \sum_{i=1}^{k} \varphi_i \, h_i \qquad (k > \ell) \tag{4.27}$$

with $H_1(x), \cdots, H_k(x)$ R-independent, but such that $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_k\}$ is not minimal. This means that one of the $\tilde{\varphi}_i$, say $\tilde{\varphi}_1$, depends of the others mod $\mathcal{IM}$:

$$\tilde{\varphi}_1 = \sum_{j \geq 2} \tilde{s}_j \, \tilde{\varphi}_j \;\; mod \;\; \mathcal{MI}.$$

But this means that there exist $\tilde{m}_1, \cdots, \tilde{m}_l \in \mathcal{M}$ such that :

$$\tilde{\varphi}_1 = \sum_{j \geq 2} \tilde{s}_j \, \tilde{\varphi}_j + \sum_{i=1}^{\ell} \tilde{m}_i \, \tilde{\varphi}_i$$

$$(1 - \tilde{m}_1)\tilde{\varphi}_1 = \sum_{j \geq 2} (\tilde{s}_j + \tilde{m}_j) \, \tilde{\varphi}_j$$

$$\tilde{\varphi}_1 = \sum_{j \geq 2} \tilde{S}_j \, \tilde{\varphi}_j \;\; for\ some\ germs \;\; \tilde{S}_j.$$

Putting this in (4.27) :

$$\begin{aligned} \delta &= \Big(\sum_{j \geq 2} S_j \, \varphi_j\Big) k_1 + \sum_{j \geq 2} \varphi_j \, h_j \\ \delta &= \sum_{j \geq 2} k_j \, \varphi_j \;\; with \\ k_j &= h_j + S_j \, h_1. \end{aligned}$$

I say that the $K_i(u) = k_i(u, \lambda_0)$ are independent (germs).
Suppose, on the contrary, that there exists some non trivial relation :

$$\sum_{i=2}^{k} \alpha_i \ K_i(u) \equiv 0 \quad (\alpha_2, \cdots, \alpha_k) \in R^{k-1}.$$

This implies that :

$$\Big(\sum_{i=2}^{k} \alpha_i \ S_i(0)\Big) H_1(u) + \sum_{i=2}^{k} \alpha_i \ H_i \equiv 0.$$

But $\{H_1(u), \cdots, H_k(u)\}$ being an independent system, this implies that $\alpha_2 = \cdots = \alpha_k = 0$. This is impossible. ∎

The factor functions $H_1(u), \cdots, H_\ell(u)$ associated to any minimal system of generator $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell$ are analytic. Because they are R-independent, each $H_i \not\equiv 0$ and then, has some finite order at each $u_0 \in \sigma$. We are going to prove now that for any u_0, it is possible to choose minimal system to have strictly increasing order of the H_i.

Lemma 14 *Let any $u_0 \in \sigma$. Then it exists a minimal systems of generators $(\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell)$ such that :*

$$order \ H_1(u_0) < order \ H_2(u_0) < \cdots < order \ H_\ell(u_0) < \infty$$

$$(order \ f(u_0) = n \Longleftrightarrow f(u) = \alpha(u - u_0)^n + 0((u - u_0)^n), \ \alpha \neq 0).$$

Proof Let any minimal system of generators. Clearly, we can order it such that :

$$order \ H_1(u_0) \leq \cdots \leq order \ H_\ell(u_0).$$

We are going to construct a sequence of minimal set of generators, $\varphi^1, \cdots, \varphi^\ell$ such that φ^1 is the given one and such that for φ^s :

$$order \ H_1(u_0) < \cdots order \ H_{s-1}(u_0) \leq \cdots \leq order \ H_\ell(u_0)$$

for the associate system of factors $h_1, \cdots, h_\ell$.
We just give the recurrence step (how to pass from φ^s to φ^{s+1}, $s < \ell$) :

So, let be $H_1, \cdots, H_\ell$ the factors for φ^s.
If order $H_s(u_0) <$ order $H_{s+1}(u_0)$, we take $\varphi^{s+1} = \varphi^s$.
If order $H_s(u_0) =$ order $H_{s+1}(u_0) = \cdots =$ order$H_{s+\sigma}$ we take :

$$\begin{aligned} K_i(u) &= H_i(u) \; for \; s \leq i \; and \; i \geq s+\sigma \\ and \; K_i(u) &= H_i(u) - \frac{\alpha_i}{\alpha_s} h_s \; for \; s \leq i < s+\sigma. \end{aligned}$$

for $H_j(u) = \alpha_j(u-u_0)^{n_j} + \cdots$

These formulas defined an invertible matrix M such that : $K = (K_1, \cdots, K_\ell) = MH$. One has : $< \varphi^s, H >= \delta =< \varphi^s, M^{-1}K >=< {}^tM^{-1}\varphi^s, K >$.

So that, K is the factor vector for the minimal set ${}^tM^{-1}\varphi^s$. Moreover it is clear that up to a reordering of terms, one has :

$$order \; K_1(u_0) < \cdots < order \; K_{s+1}(u_0) \leq \cdots \leq order \; K_\ell(u_0).$$

■

Definition 22 *We say that a minimal system $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ such that order $H_1(u_0) < \cdots <$ order $H_\ell(u_0)$ is adapted to the point $u_0 \in \sigma$.*

4.3.3 Finite cyclicity of regular limit periodic sets.

Let $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ any minimal system of generators, and $H_1(u), \cdots, H_\ell(u)$ the associated factors. Because these functions are analytic and R-independent, at each $u_0 \in \sigma$, some finite jet of them are already R-independent.

Definition 23 *For any $u_0 \in \sigma$, we define the index $s_\delta(u_0)$ by :*

$$s_\delta(u_0) = Inf \; \{n \in N \mid \{j^n \; H_i(u_0)\}_i \; is \; a \; R-independent \; system\}.$$

As we remarked above, $s_\delta(u_0) < \infty$. Also, it follows from lemma 8 that this index is independent of the choice of the minimal system. Clearly enough, if $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ is adapted to u_0, $s_\delta(u_0) = \textit{order } H_\ell(u_0)$ (the maximal order at u_0, among the factors H_i). This gives a practical way to compute $s_\delta(u_0)$ by looking for an adapted minimal system of generators (see examples below). It follows also from this, that $s_\delta(u_0) \geq \ell - 1$ (Remark that order $H_1(u_0) = 0$ in general !).

When $e \in \partial\sigma$ is a centre singular point we need a slightly different definition. Of course, at a centre point we have the following :

Lemma 15 *Let* $\delta(u, \lambda) = \sum_{i=1}^{\infty} a_i(\lambda)u^i$ *the expansion at the centre point e (corresponding to* $u = 0$*). Then, for* $\forall p \geq 1 : \tilde{a}_{2p} \in \mathcal{I}(\tilde{a}_1, \cdots, \tilde{a}_{2p-1})$ *: the ideal generated by the coefficients of previous odd order.*

Proof This property may be obtained, looking at recurrence formulas for the coefficients a_i (see [B] for instance).

A more easy way is to notice that this property is independent of the choice of the transversal interval σ, the choice of parametrization and also a multiplication of X_λ by some analytic function $g(x, y, \lambda)$, $g(0, 0, \lambda) \neq 0$. So it suffices to prove the result when X_λ is written in normal form up to order $2N + 1 \gg 2p$. In this normal form, and in polar coordinate (r, θ) we have :

$$X_\lambda = \frac{\partial}{\partial\theta} + \Big(\sum_{i=0}^{N} \beta_i(\lambda) r^{2i} + O(r^{2N+2}) \Big) r \, \frac{\partial}{\partial r}$$

($O(r^{2N+2})$ being an analytic function in (r, θ, λ)).

A direct integration of the differential equation of X_λ gives :

$$\dot{\theta} = 1, \; \dot{r} = r\Big(\sum_{i=0}^{N} \beta_i \, r^{zi} + 0(r^{2N}) \Big)$$

This implies that:

$$\begin{aligned} \delta(r, \lambda) &= a\Big(\sum_{j=0}^{N} b_{2j+1}(\lambda) r^{2j} + O(r^{2N+2}) \Big) \\ b_1 &= (e^{2\pi\lambda_1} - 1)u, \cdots \end{aligned}$$

when $\{u = x = r\}$ is the parametrization of the $0x$-axis.

For these choices (of $\sigma, \cdots$) the result is trivial because $a_{2i} \equiv 0$ for $i \leq N$. ∎

In the division formula we can write each even a_{2p} as a combination of odd previous coefficients. It follows a formula :

$$\delta(u, \lambda) = u \sum_{j=1}^{\ell} a_{2j+1}(\lambda) h_{2j+1}(u, \lambda) \tag{4.28}$$

with :

$$h_{2j+1}(u, \lambda) = u^{2j}(1 + 0(u)) \tag{4.29}$$

Now, if we extract a minimal system of generators from the system

$$\{a_{2j+1}\}_{j=1,\cdots,\ell},$$

each of the corresponding factor has an odd order at $e = 0$ and it is also true for the adapted system of generators we can construct from it, as in lemma 9. As a consequence, for any minimal system of generators :

$$Inf \left\{n \mid \{j^n \ h_j(0)\}_j \ is \ R\text{-}independent\right\} is \ odd.$$

Definition 24 *If the above number is equal to* $2k+1$ *we define :* $s_\delta(e) = k$.

Finite cyclicity for regular limit periodic sets is a consequence of the following Gabrielov's theorem [G] :

Theorem 8 *(Gabrielov) : Let* C *a compact analytic real set and* $\pi : C \to \Sigma$ *a proper analytic map of* C *onto another real analytic set. Then, there exist a* $K < \infty$ *such that for any* $\lambda \in C$*, the number of connected component of* $\pi^{-1}(\lambda)$ *is bounded by* K.

Here, we take $C = \{(u, \lambda) \mid \delta(u, \lambda) = 0 \text{ and } (u, \lambda) \in \bar{\sigma} \times W\}$ and π the projection on the parameter space : $\pi(u, \lambda) = \lambda$.
The limit cycles for X_λ through points of σ correspond to the connected components of 0-dimension of $\pi^{-1}(\lambda)$.

The notion of Bautin ideal and the related index $s_\delta(u_0)$ allow us to obtain an explicit bound for the cyclicity.

Theorem 9 *Let an analytic family as above and $u_0 \in \sigma$ any point of a transversal for X_{λ_0} (u_0 may be a centre boundary point). Let γ_{u_0} the orbit through X_{λ_0} by u_0 ($\gamma_{u_0} = e$ if $u_0 = e$) :*

i) Cycl $(X_\lambda, \gamma_{u_0}) \leq s_\delta(u_0)$,

ii) if the Bautin Ideal $\mathcal{I}$ is regular (i.e : $\mathcal{I} = \{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ with $d\varphi_1(\lambda_0) \wedge \cdots \wedge d\varphi_\ell(\lambda_0) \neq 0$) then Cycl $(X_\lambda, \gamma_{\lambda_0}) \geq \ell - 1$.

Proof

Point i) : As it was proved in lemma 1, it suffices to obtain the bound $s_\delta(u_0)$ for $(u, \lambda) \in \sigma_1 \times W_1$ some compact neighborhood of (u_0, λ_0) in $\sigma \times W$. Consider first the case $u_0 \in$ int (σ) (u_0 is not a centre). By hypothesis, we know that :

$$\delta(u, \lambda) = \sum_{i=1}^{\ell} \varphi_i(\lambda) h_i(u, \lambda)$$

in a neighborhood $\sigma_1 \times W_1$ of (u_0, λ_0) with :

$$t_1 = order\ H_1(u_0) < t_2 = order\ H_2(u_0) < \cdots < t_\ell = order\ H_\ell(u_0) = s_\delta(u_0)$$

$$(h_i(u) = H_i(u, \lambda_0)).$$

Let $\Sigma = \{\lambda \in W_1 \mid \varphi_1(\lambda) = \cdots = \varphi_\ell(\lambda) = 0\}$ and $W^i = \{\lambda \in W_1 \mid \mid \varphi_i(\lambda) \mid \geq \mid \varphi_j(\lambda) \mid \text{ for } \forall j \neq i\}$, $i = 1, \cdots, \ell$.

Clearly, the zeros of $\delta(u, \lambda)$ are isolated if and only if $\lambda \in W_1 - \Sigma = W^1 \cup \cdots \cup W^\ell - \Sigma$.

We will show that for each $i = 1, \cdots, \ell$, there exists an interval σ^i, neighborhood of u_0 in σ_1 and a neighborhood W_1 such that for all $\lambda \in W^i - \Sigma$, the function $\delta(u, \lambda)$ has less than t_i isolated roots on σ^i. This interval will be obtained by a succession of restrictive conditions on σ_1 and on W_1, which will be stated by the claim "restricting u, λ" without more precisions.

So, fixing $i = 1, \cdots, \ell$, we are going to construct a sequence of functions $\delta^1 = \delta$, $\delta^2, \cdots, \delta^i$, recurrently defined as follow.

Restricting u, λ, one can suppose that $\partial^{t_1} h_1(u, \lambda) \neq 0$ for all (u, λ). (W_1 is supposed compact ; we note $\partial^s / \partial u^s = \partial^s$).

Then $\partial^{t_1} \delta / _{\partial^{t_1} h_1} = \varphi_1 + \varphi_2 \; h_2 / _{\partial^{t_1} h_1} + \cdots$.
Let $\delta^2 = \partial(\partial^{t_1} \delta / _{\partial^{t_1} h_1}) = \varphi_2 \; h_2^2 + \cdots + \varphi_\ell \; h_\ell^2$.
Clearly, the new functions h_j^2 so defined, verify :

$$h_j^2(u, \lambda_0) = \alpha_j^2 \; u^{t_j - t_1 - 1} \; (1 + 0(u)) \; with \; \alpha_j^2 \neq 0$$

and the function δ^2 is similar to δ^1, with one term less.

So, we can introduce the following recurrence hypothesis for

$$2 \leq j \leq i \quad :$$

$$\delta^j(u, \lambda) = \varphi_j(\lambda) h_j^j(u, \lambda) + \cdots + \varphi_\ell(\lambda) h_\ell^j(u, \lambda) \tag{4.30}$$

with :

$$h_s^j(u, \lambda_0) = \alpha_s^j \; u^{t_s - t_{j-1} - 1} \; (1 + 0(u)). \tag{4.31}$$

And a recurrence step defined by :

$$\delta^{j+1} = \partial\Big(\partial^{t_j - t_{j-1} - 1} \; \delta^j / \partial^{t_j - t_{j-1} - 1} \; h_j^j\Big) \tag{4.32}$$

for all $j \leq i - 1$.
After the last step we obtain the function :

$$\delta^i = \varphi_i \; h_i^i + \cdots + \varphi_\ell \; h_\ell^i$$

such that :

$$\partial^{t_i - t_{i-1} - 1} \; \delta^i = \varphi_i \; k_i + \cdots + \varphi_\ell \; k_\ell$$

where, restricting u, λ, one has : $k_i(u, \lambda) \neq 0$,

$$k_j(u, \lambda) = 0(u). \tag{4.33}$$

Then :

$$\mid \partial^{t_i - t_{i-1} - 1} \; \delta^i \mid \geq (\varphi_i)\Big(\mid k_i \mid - \sum_{j=i+1}^{\ell} \; \mid \frac{\varphi_\ell}{\varphi_i} \mid . \mid k_\ell \mid\Big). \tag{4.34}$$

Because $\mid \frac{\varphi_\ell}{\varphi_i} \mid \leq 1$ on W^i and (4.33), restricting u, λ, we can suppose that $\mid k_i \mid - \sum_{j=i+1}^{\ell} \mid \frac{\varphi_\ell}{\varphi_i} \mid \mid k_\ell \mid \geq b > 0$ for some constant b. But on $W^i - \Sigma$, $\mid \varphi_i \mid > 0$ and so $\partial^{t_i - t_{i-1} - 1}\ \delta^i\ (u, \lambda) \neq 0$ for $\forall (u, \lambda) \in (W^i - \Sigma) \times \sigma^i$.

This last function was obtained from δ by a sequence of $t_1 + 1 + (t_2 - t_1 - 1) + \cdots = t_i + 1$ derivations and some divisions by non-zero functions. So by successive applications of Rolle theorem we have that $\delta(u, \lambda)$ has less than t_i isolated roots on σ^i for all $\lambda \in W^i - \Sigma$.

This concludes the proof of point i) when u_0 is not a centre.

In the centre case, we use the formula (4.28) above : the roots of $\delta(u, \lambda)$, different from 0, are roots of :

$$\sum_{j=1}^{\ell} a_{2j+1}(\lambda) h_{2j+1}\ (u, \lambda).$$

Next, taking a normal form for X_λ of order N large, we can see easily that the factors $h_{2j+1}\ (u, \lambda) = k_j(U, \lambda)$ where $U = u^2$ and k_j is a C^k-function ; taking N large enough, this order k can be taken arbitrarily large. Now $k_j(U, \lambda_0) = U^j(1 + O(U))$ and we can repeat the above proof to obtain a bound of the cyclicity by $s_\delta(e)$.

Point ii) : If $\varphi_1, \cdots, \varphi_\ell$ are independent functions, we may suppose chosen local coordinates $\lambda_1, \cdots, \lambda_\ell, \cdots, \lambda_\Lambda$ in the parameter space, with $\varphi_i = \lambda_i$, $i = 1, \cdots, \ell$ and $\lambda_0 = (0, \cdots, 0)$.

Of course, we can always suppose that the minimal system $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_\ell\}$ is adapted to u_0 which is supposed translated to 0 :

$$\delta(u, \lambda) = \sum_{i=1}^{\ell} \lambda_i\ h_i(u, \lambda)$$

with $h_i(u, 0) = u^{n_i}(1 + 0(u))$, $n_1 < n_2 \cdots < n_\ell$.

The essential remark is that the sequence $\{h_1(u, 0), \cdots, h_\ell(u, 0)\}$ is a *Cheybicheff system* (see [J1]). This means that there exists some $U > 0$ such that on $[0, U]$ the function :

$$\tilde{\delta}(u,\tilde{\lambda}) = \sum_{i=1}^{\ell} \lambda_i \; h_i(u,0) \qquad \tilde{\lambda} = (\lambda_1,\cdots,\lambda_\ell)$$

has at most $\ell - 1$ roots counted with multiplicity. Moreover the bifurcation diagram for the roots of $\tilde{\delta}$ is the same as for the polynomial : $\sum_{i=1}^{\ell} \lambda_i \; u^i$ on $[0,U]$.

In particular, one has a simplex $\Delta \subset S^{\ell-1}$ in the unit sphere of R^ℓ such that for $\forall\tilde{\lambda} \in \mathring{\Delta}$, $\tilde{\delta}(u,\lambda)$ has $\ell - 1$ simple roots on $]0,U[$.

Now, let $\tilde{\lambda} = u\overline{\lambda}$, $\overline{\lambda} \in S^{\ell-1}$ and $u \in R^+$ and let $\lambda' = (\lambda_{\ell+1},\cdots,\lambda_\Lambda)$.

$$\delta(u,\lambda) = u\Big[\sum_{i=1}^{\ell} \bar{\lambda}_i \; h_i(u,0) + 0(\lambda)\Big]. \tag{4.35}$$

The equation $\delta(u,\lambda) = 0$ is equivalent to :

$$\sum_{i=1}^{\ell} \overline{\lambda}_i \; h_i(u,\lambda) + 0(\lambda) = 0. \tag{4.36}$$

If we take any open, non-empty compact subset of Δ, such that $\bar{Q} \subset \mathring{\Delta}$, we deduce that (4.36) has at least $\ell - 1$ simple roots in $]0,U[$ for :

$$\lambda = u\overline{\lambda},$$

$$(\overline{\lambda},u) \in S^{\ell-1} \times R^+$$

and u sufficiently small.

This proves that $\mathcal{C}ycl\ (X_\lambda,\gamma_{u_0}) \geq \ell - 1$ in this case (The proof works as well in the case that u_0 is a centre). ∎

Remark 19 *It is easy to generalize the result ii). For instance, we can replace the condition "$\mathcal{I}$ regular" by the following one :*

"The map $\lambda \to \varphi(\lambda)$ is locally surjective at $\lambda = \lambda_0''$.

It would be interesting to have algebraic characterizations for ideals $\mathcal{I}$ with this property. As an example of such ideal, we may consider $\mathcal{I} = \{\lambda_1^3,\ \lambda_2^5\}$.

4.3.4 Melnikov functions.

In this section we will consider 1-parameter analytic families X_ε with $\varepsilon \in R$ which unfold X_0 of centre type. In this case, it is possible to expand $\delta(u, \varepsilon)$ in terms of ε :

$$\delta(u, \varepsilon) = \sum_{i=1}^{\infty} M_i(u)\varepsilon^i. \tag{4.37}$$

The functions $M_i(u)$ are analytic on σ. We call M_i the i^{th} *Melnikov function.* The above series converges near each $u_0 \in \sigma$.

If the family is not identically trivial ($\delta(u, \varepsilon) \equiv 0$) there exists a k, such that $M_i(u) \equiv 0$ for $i \leq k-1$ and $M_k(u) \not\equiv 0$. In this case, the Bautin ideal of X_ε at $\varepsilon = 0$ is generated by $\varepsilon^k : \mathcal{I} = (\varepsilon^k)$.

Moreover the equation $\{\delta(u, \varepsilon) = 0\}$ is equivalent to :

$$M_k(u) + O(\varepsilon^2) = 0. \tag{4.38}$$

If $u_0 \in \sigma$, the cyclicity of γ_{u_0} is bounded by the order of M_k at u_0 (half the order minus one if u_0 is a centre). This is of course a trivial particular case of theorem 14.

We recall now how to compute the Melnikov functions. First, one can find an analytic function K defined in a neighborhood of any centre or periodic orbit of the centre type vector field X_0 such that KX_0 is an Hamiltonian vector field (div $(KX_0) \equiv 0$). This means that there exists a first integral H for X_0 which verifies : $-dH = +KX_0 \rfloor dx \wedge dy$. (It is trivial to find H, K along a closed orbit of X_0. At a centre point, one can write X_0 in polar coordinates and look for $H(r, \theta)$, such that $H(-r, \theta + \pi) = H(r, \theta)$).

Call ω_ε the dual form of $X_\varepsilon : \omega_\varepsilon = X_\varepsilon \rfloor dx \wedge dy$. Let σ a transversal interval, parametrized by the value $h = H(u)$. Then it is well known that :

$$M_1(h) = -\int_{\gamma_h} \omega_1 \quad if \quad \omega_1 = \left.\frac{\partial \omega}{\partial \varepsilon}\right|_{\varepsilon=0}$$

and γ_h is the periodic orbit of X_0 in $\{H = h\}$.

This formula was recently extended by J.P. Françoise [F] and S. Yakovenko [Y] at any Melnikov function M_i for linear perturbation

$$dH + \varepsilon\omega_1.$$

I give a more general version obtained by J.C. Poggiale [Po] for any 1-parameter family.

So, $\omega_\varepsilon = dH + \varepsilon\omega_1 + \cdots \varepsilon^{k+1}\omega_{k+1} + 0(\varepsilon^{k+1})$ (for any k). We choose a domain U, invariant by the flow of X_0 and containing the transversal σ.

Proposition 6 *Under the above assumptions, one supposes that $M_j(h) \equiv 0$ for $j \leq k$. Then :*

$$M_{k+1}(h) = \int_{\gamma_h} \Big(\sum_{i=1}^{k} g_i\ \omega_{k+1-i}\ - \omega_{k+1}\Big)$$

where the analytic functions g_i, $i = 1, \cdots, k$ are defined recursively on U by :

$$\omega_i - g_i\ dH = \sum_{j=1}^{i-1} g_j\ \omega_{k-j} + dR_i.$$

Proof We need the following result :

Lemma 16 *Let ω be an analytic 1-form defined in the neighborhoods U. Then $\int_{\gamma_n} \omega \equiv 0$ for $h \in \sigma$ if and only if there exist analytic functions g, R such that in this neighborhoods : $\omega = gdH + dR$.*

Proof of lemma 11 If γ_{h_0} is a closed orbit, one can adopt action-angle variables (H, θ) near γ_{h_0} (H near h_0 and $\theta \in S^1$). $\omega = gdH + bd\theta$, and $\int_{\gamma_M} \omega \equiv 0 \Longleftrightarrow \int_0^{2\pi} b(H,\theta)d\theta \equiv 0$. Then there exist a function $b(H,\theta)$ such that $b(H,\theta)d\theta = dR$ and $\omega = gdH + dR$. If γ_{n_0} is the centre, we use the same method : the function $R(H,\theta)$ is analytic in (x,y)-coordinate at the origin. ∎

Return now to the proof of proposition 5.

First, by integration of the formula $\omega_\varepsilon = dH + \varepsilon\omega_1 + 0(\varepsilon)$ we obtain that $M_1(h) = -\int_{\gamma_h} \omega_1$ (it is the usual Melnikov's formula at order 1).

We make now the following recurrence hypothesis : there exist functions $g_1, g_2, \cdots, g_{k-1}$, analytic on U, such that for all $j = 1, \cdots, k$:

$$M_j(h) = \int_{\gamma_h} \Big(\sum_{i=1}^{j-1} g_i \; \omega_{j-i} - \omega_j \Big) \equiv 0.$$

Using this relation for $j = k$ and applying lemma 11, one finds two analytic functions on U : g_k, R_k such that :

$$-\sum_{i=1}^{n-1} g_i \; \omega_{k-i} + \omega_k = g_k \; dH + dR_k$$

and then : $\omega_k - g_k \; dH = \sum_{i=1}^{k-1} g_i \; \omega_{k-i} + dR_k$.
This proves that the functions g_k, R_k may be constructed by recurrence. Next a direct expansion gives that :

$$\Big(1 - \sum_{i=1}^{k} g_i \, \varepsilon^i\Big)\omega_\varepsilon = d\Big(H - \sum_{i=1}^{k} \varepsilon^i \; R_i\Big) + \varepsilon^{k+1} \Big(\omega_{k+1} - \sum_{i=1}^{k} g_i \, \omega_{k+1-i}\Big) + 0(\varepsilon^{k+1}).$$

Because H is a Morse function we can find an analytic diffeormorphism in U, Id $+0(\varepsilon)$ which changes $H - \sum_{i=1}^{k} \varepsilon^i \; R_i$ in H and does not modify the coefficient in ε^{k+1}. The result for $M_{k+1}(h)$ follows by integration, like for $k = 1$. ∎

Remark 20

1) In the linear case $dH + \varepsilon\omega_1$ *the formula is simply* $M_{k+1}(h) = \int_{\gamma_h} g_k \; \omega_1$ *where the* g_i *are given recursively by :*

$$g_i \; dH + dR_i = -g_{i-1} \; \omega_1.$$

2) Proposition 5 is clearly true for smooth families.

If now X_λ is any unfolding of centre type vector field X_{λ_0} an important question is :

- can we deduce the cyclicity for X_λ by computing Melnikov functions for 1-parameter subfamilies $X_{\lambda(\varepsilon)}$ where $\lambda(\varepsilon)$ is any analytic arc in the parameter space with $\lambda(0) = \lambda_0$?

Of course, if γ is some periodic orbit or centre of X_{λ_0} $Cycl(X_\lambda, \gamma) \geq \underset{\lambda(\varepsilon)}{Sup}\, Cycl(X_{\lambda(\varepsilon)}, \gamma)$. The above question reduces to prove that we have the reverse inequality $Cycl(X_\lambda, \gamma) \leq \underset{\lambda(\varepsilon)}{Sup}\, \mathcal{O}_{\lambda(\varepsilon)}$ where $\mathcal{O}_{\lambda(\varepsilon)}$ is the order of the first non zero Melnikov function for $X_{\lambda(\varepsilon)}$. We can be more explicit. Suppose that $\{\widetilde{\varphi}_1, \cdots, \widetilde{\varphi}_\ell\}$ is a minimal set of generators for the Bautin Ideal. Then considering theorem 2, it would suffice to prove :

"There exists an analytic curve $\lambda(\varepsilon)$ throw λ_0 such that order $(\varphi_\ell \circ \lambda)(0) <$ order $(\varphi_i \circ \lambda)(0)$ for any $i \neq \ell$".

In fact, if the above claim was true $\mathcal{O}_{\lambda(\varepsilon)} \geq$ *order* $h_\ell(u_0) = s_\delta(u_0) \geq Cycl(X_\lambda, \gamma_{u_0})$ where $\gamma = \gamma_{u_0}$. This is trivially true when the Bautin Ideal is regular. I don't know if this is true in general.

4.3.5 Application to quadratic vector fields.

4.3.5.1 Bautin result.

If we are interested by quadratic vector fields with at least 1 limit cycle, it is sufficient to consider the Kaypten-Dulac family X_λ, with a focus or centre point at the origin. Using a rotation, we can eliminate one parameter and look at the following 6-parameter family :

$$X_\lambda \left\{ \begin{array}{lcl} \dot{x} & = & -y + \lambda_1 x - \lambda_3\, x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6\, y^2 \\ \dot{y} & = & x + \lambda_1 y + \lambda_2\, x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2\, y^2 \end{array} \right. \qquad (4.39)$$

For any $\lambda \in R^6$, the origin is a focus or a centre. One can consider the return map on the $0x$-axis, and the difference function $\delta(x, \lambda) = \sum_{i=1}^{\infty} a_i(\lambda) x^i$.

It is easy to compute $a_1(\lambda) = e^{2\pi\lambda_1} - 1$.

The other 7 first coefficients were computed in [Du1] for instance (see also [B], [Ye],...). We already know that it suffices to compute coefficients of odd order :

$a_3(\lambda) = c_3\ \lambda_5(\lambda_3 - \lambda_6)\ \ mod\ (a_1)$
$a_5(\lambda) = c_5\ \lambda_2\ \lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5(\lambda_3 - \lambda_6))\ \ mod\ (a_1, a_3)$
$a_7(\lambda) = c_7\ \lambda_2\ \lambda_4(\lambda_3 - \lambda_6)^2\ (\lambda_3\ \lambda - 2\lambda_6^2\ - \lambda_2^2)\ \ mod\ (a_1, a_3, a_5).$

For some constant c_3, c_5, $c_7 \neq 0$.
A difficult result of Bautin in [B] is that the ideal generated by the coefficient $\mathcal{I}(a_i)$ in the ring of analytic functions of R^6 is generated by a_1, a_3, a_5, a_7, and so the Bautin ideal at each λ_0 is generated by the germs of :

$$v_1 = \lambda_1,\ v_3 = \lambda_5(\lambda_3 - \lambda_6),\ v_5 = \lambda_2\ \lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5(\lambda_3 - \lambda_6))$$

$$and\quad v_7 = \lambda_2\ \lambda_4(\lambda_3 - \lambda_6)^2\ (\lambda_3\ \lambda_6 - 2\lambda_6^2 - \lambda_2^2).$$

Using proposition 2 and lemma 10, we can write locally near $x = 0$ and each $\lambda_0 \in R^6$:

$$\delta(x, \lambda) = v_1\ h_1(x, \lambda) + v_3\ h_3(x, \lambda) + v_5\ h_5(x, \lambda) + v_7\ h_7(x, \lambda)$$

$$with\quad h_i = x^i(1 + \psi_i(x, \lambda))\ and\ \varphi_i = 0(x).$$

This formula implies that at each $\lambda_0 \in R^6$, $s_\delta(0) \leq 3$. So that, by theorem 2, at most three limit cycles may bifurcate from the origin by perturbation of the parameter λ.

The set $Z = \{v_1 = v_3 = v_5 = v_7 = 0\}$ is the set of parameter values for which X_λ is centre type. In this case the origin is a centre surrounded by a "centre basin" B_λ of periodic orbits. This set was first described by Dulac [Du1].

It is an algebraic subset of $\mathbf{R}^6$ with four irreducible components :

$Q^H = \{\lambda_1 = \lambda_4 = \lambda_5 = 0\}$
$Q^R = \{\lambda_1 = \lambda_2 = \lambda_5 = 0\}$
$Q^D = \{\lambda_1 = \lambda_3 - \lambda_6 = 0\}$
$Q^M = \{\lambda_1 = \lambda_5 = \lambda_4 + 5(\lambda_3 - \lambda_6) = \lambda_3\ \lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0\}.$

Each of these components has a geometrical meaning : Q^H contains the Hamiltonian vector fields (equation : div $X_\lambda \equiv 0$) Q^R contains the reversible ones : each has a symmetrical phase portrait around one line through 0. The vector fields in Q^D contains 3 invariant lines (real or complex) and vector fields in Q^M contains an invariant parabola and an invariant cubic. Moreover for each centre one has explicit first integral and integrating factor (see [Du1] and [S1] for a more recent and complete study).

Outside the intersection of components, the set Z is a submanifold and the ideal of Bautin is regular.

Consider for instance $\lambda_0 \in Q_M \backslash Q^M \cup Q^R \cup Q^D$.

At such a point $\lambda_4 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 - \lambda_6 \neq 0$.

Then the ideal $\mathcal{I}^{\lambda_0}$ is generated by λ_1, λ_5, $\lambda_4 + 5(\lambda_3 - \lambda_6)$ and $\lambda_3\,\lambda_6 - 2\lambda_6^2 - \lambda_2^2$. So that $\mathcal{I}^{\lambda_0}$ is regular and $\ell(\mathcal{I}^{\lambda_0}) = 4$. Applying point ii) of theorem 2, we see that the cyclicity at such value λ_0 is greater that 3 and so equal to 3 : one can find values of λ near λ_0 with 3 "small" limit cycles.

In the same way one can prove that the cyclicity is larger than 2, 2 and 1 at regular points of Q^H, Q^R and Q^D respectively.

We have seen that Bautin obtained an estimate of the index s_δ at the origin for any $\lambda_0 \in Z$. For orbits of X_{λ_0} different from the centre point no general result is known except in some particular cases : for instance Horozov and Iliev [HI] proved that for the open subset of Q^H corresponding to generic Hamiltonian vector fields with three saddle points and one centre, then $s_\delta(u_0) = 2$ for any $u_0 \in \sigma$, transversal interval in the Centre basin. Because we know that the cyclicity is greater than 2, this implies that $Cycl(X_\lambda, \gamma_{u_0}) = 2$ in this case. For such a hamiltonian vector field, the basin B_λ is a disk bounded by a saddle-connection and these authors obtained in [HI] the whole bifurcation diagram. In particular, they proved that the total number of limit cycles for near by vector fields is less than 2. This includes the study of the saddle connection bifurcation we want to consider in the next chapter.

4.3.5.2 Bogdanov-Takens unfolding.

Some unfoldings can be reduced to perturbations of centre type vector fields. For instance, let us consider the Bogdanov-Takens unfolding introduced in chapter I, written in normal form :

$$X^+_{\mu,\nu,\lambda}\begin{cases} \dot{x} &= y \\ \dot{y} &= x^2+\mu+y(\nu+x)+yx^2\ h(x,\lambda)+y^2\ Q(x,y,\lambda)\end{cases}$$

where h, Q are smooth functions. To reduce this unfolding to a perturbation of a centre type vector field, one can use the following rescaling formulas :

$$x=\varepsilon^2\ \bar{x},\ y=\varepsilon^3\ \bar{y},\ \mu=-\varepsilon^4\quad \nu=\varepsilon^2\bar{\nu}. \tag{4.40}$$

Taking $(\bar{x},\bar{y})\in\overline{D}$, some compact domain in R^2 and $\bar{\nu}\in K$ some close interval to be defined below, $\varepsilon\in R^+$, these formulas transforms the family X^+ in a new one $\bar{X}_{\bar{\nu},\varepsilon,\lambda}=\frac{1}{\varepsilon}\widehat{X}$, where $\widehat{X}$ is the family X^+ written in coordinates $(\bar{x},\bar{y})$:

$$\overline{X}_{\bar{\nu},\varepsilon}\begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= \bar{x}^2-1+\varepsilon\bar{y}(\bar{\nu}+\bar{x})+0(\varepsilon^2)\end{cases} \tag{4.41}$$

This is an ε-perturbation of the hamiltonian vector field X_0 with Hamiltonian function $H_1=\frac{1}{2}\ \bar{y}^2+\bar{x}-\frac{\bar{x}^3}{3}$ (See Figure 4.2:).

To study limit cycles bifurcating for the hamiltonian cycles we take a disk $\overline{D}$ large enough to contain the disk bounded by the homoclinic loop through the saddle $s=(+1,0)$. For each $h\in\left[-\frac{2}{3},\frac{2}{3}\right]$, let γ_h be the cycle in $\{H=h\}$; γ_0 is the centre $(-1,0)$ and $\gamma_{2/3}$ the homoclinic loop. Let $\omega_{\nu,\varepsilon}$ the dual form of $\overline{X}_{\nu,\varepsilon}$:

$$\omega_{\nu,\varepsilon}=dH-\varepsilon(\nu-x)dy+O(\varepsilon^2). \tag{4.42}$$

From proposition 5, the difference map for the family $\overline{X}_{\nu,\varepsilon}$ has the following expansion in ε :

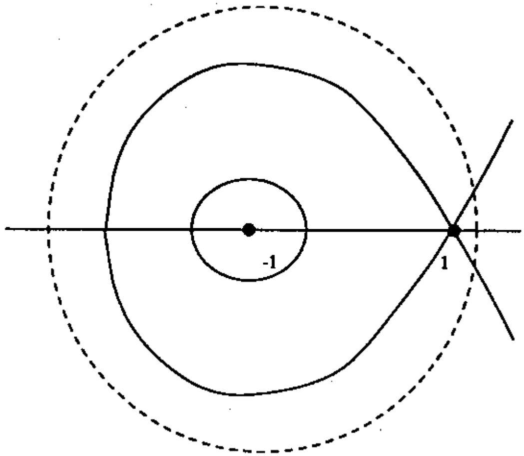

Figure 4.2:

$$\delta(h, \nu, \varepsilon) = \varepsilon M_1(h, \nu) + 0(\varepsilon)$$

where :

$$M_1(h, \nu) = \int_{\gamma_h} (\nu - x)dy = \nu I_0(h) - I_1(h) \tag{4.43}$$

We write :

$$I_i(h) = \int_{\gamma_h} x^i \, ydx. \tag{4.44}$$

$I_0(h)$ is equal to the area of the disk with boundary γ_h.
So, $I_0(h) > 0$ for $h \neq 0$ and $I_0(h) \sim h$.

Because $I_1(0) = 0$, the function $B_1(h) = \frac{I_1}{I_0}(h)$ is defined and analytic for $h \in [0, 1/_6[$. The equation for limit cycles $\{\delta = 0\}$ is equivalent to :

$$\frac{\delta}{\varepsilon I_0} = \nu - B_1(h) + 0(\varepsilon) = 0. \tag{4.45}$$

And the unicity of limit cycle in the Bogdanov Takens unfolding as claimed in chapter I, reduced to prove the following result :

Theorem 10 *(Bogdanov) : For all* $h \in \left[-\frac{2}{3}, \frac{2}{3}\right[$, $B_1'(h) < 0$ *and*

$$B_2'(h) \to \infty$$

as

$$h \to \frac{2}{3}.$$

We will prove this result in a next section about abelian integrals. The fact that $B_1'(0) < 0$ implies that the line H we have introduced in chapter I, is a line of generic Hopf bifurcations.

We will study the line of homoclinic loops in the next chapter. Here, the theorem, applied for $h \in [h_0, h_1]$ where h_0 is chosen near 0 and h_1 near $1/6$, $[h_0, h_1] \subset \left]-\frac{2}{3}, \frac{2}{3}\right[$, implies the unicity of the limit cycle in the interior of the tongue T between the two lines H, C.

4.3.5.3 An example of a non-regular ideal.

In the above example, the computation of cyclicity reduces to compute a Melnikov function. This is because the Bautin Ideal was regular (generated by ε in the Bogdanov-Takens unfolding). In this section, we will consider a case of non-regular ideal, for the quadratic family in the Kaypten-Dulac form (4.39). This means that we will choose λ_0 at intersection of centre-components. So let $\lambda_0 \in Q^H \cap Q^R - \{0\}$. This means that $\lambda_0 = (0, \cdots, 0, \lambda_6)$ with $\lambda_6 \neq 1$. Changing $(x, y) \to (\beta y, \beta x)$ with $\beta = \mid \lambda_6 \mid^{-1/2}$ and t in $sign(\lambda_6)t$, we can reduce to $\lambda_6 = -1$; X_{λ_0} is an hamiltonian vector field, with hamiltonian function $H_2(x, y) = \frac{1}{2}\, y^2 + \frac{1}{2}\, x^2 + \frac{1}{3}\, x^3$ as in section 3.5.2.

At the parameter value $\lambda_0 = (0, \cdots, 0, -1)$, the Bautin Ideal is generated by the germs of λ_1, λ_5, $\lambda_2\lambda_4$. So that, as consequence of proposition 2, we can divide $\delta(h, \lambda)$ near (h_0, λ_0), for $h_0 \in [0, 1/6[$, in the ideal :

$$\delta(h, \lambda) = \lambda_1\, h_1(h, \lambda) + \lambda_5\, h_2(h, \lambda) + \lambda_2\lambda_4\, h_3(h, \lambda). \tag{4.46}$$

Clearly, $\{\lambda_1,\ \lambda_5,\ \lambda_2\lambda_4\}$ is a minimal set of generators at λ_0. The functions $H_i(h) = h_i(h, \lambda_0)$ are analytic on $[0, 1/_6[$. To compute them, we can use some 1-parameter subfamilies (of (4.39), after the above substitution and replacing $\frac{\lambda_1}{|\lambda_\sigma|}$ and $\frac{\lambda_i}{|\lambda_6|^2}$, $i \geq 2$, by λ_1, λ_i respectively) :

- **Computation of H_1.**

One considers the subfamily :

$$\lambda(\varepsilon) = (\lambda_1 = \varepsilon,\ \lambda_2 = \cdots = \lambda_5 = 0, \lambda_6 = -1)$$

$$X_{\lambda(\varepsilon)} \begin{cases} \dot{x} &= y + \varepsilon y \\ \dot{y} &= -x - x^2 + \varepsilon y \end{cases} \tag{4.47}$$

Substituting $\lambda(\varepsilon)$ in (4.39), gives :

$$\delta(h, \lambda(\varepsilon)) = \varepsilon H_1(h) + O(\varepsilon).$$

This shows that $H_1(h) = M_1(h)$, the first Melnikov function for the family (4.47). Its dual 1 form is :

$$\omega_\varepsilon = dH + \varepsilon(-ydx + xdy).$$

From proposition 5, one obtains :

$$M_1(h) = \int_{\gamma_h} -ydx + xdy = 2I_0(h) \tag{4.48}$$

(with notation $I_i(h)$ introduced in (4.44)).

-**Computation of H_2.**

Take

$$\lambda(\varepsilon) = (\lambda_5 = \varepsilon, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_6 = -1)$$

$$X_{\lambda(\varepsilon)} = \begin{cases} \dot{x} &= y \\ \dot{y} &= -x - x^2 + \varepsilon xy \end{cases}$$

And then :

$$H_2(h) = M_1(h) = -I_1(h). \tag{4.49}$$

- **Computation of H_3.**

We consider the family :

$$\lambda(\varepsilon) = (\lambda_1 = \lambda_3 = \lambda_5 = 0, \lambda_2 = \lambda_4 = \varepsilon,\ \lambda_6 = -1).$$

$$X_{\lambda(\varepsilon)} \left\{ \begin{array}{rcl} \dot{x} & = & y + \varepsilon(y^2 + xy - x^2) \\ \dot{y} & = & -x - x^2 + 2\varepsilon xy \end{array} \right. \tag{4.50}$$

Now, $\delta(h, \lambda(\varepsilon)) = \varepsilon^2\ H_3(h) + 0(\varepsilon^2)$ and : $H_3(h) = M_2(h)$, the second Melnikov function for (4.50). Its dual 1-form is :

$$\omega_\varepsilon = dH + \varepsilon\bar{\omega}\ \ with\ \bar{\omega} = -2xydx + (y^2 + xy - x^2)dy. \tag{4.51}$$

We can verify that $M_1(h) \equiv 0$:

$$M_1(h) = \int_{\gamma_h} -2xydx + (y^2 + xy - x^2)dy = -\int_{\gamma_h} d\left(yx^2 + \frac{y^3}{3}\right) + \int_{\gamma_h} yxdy$$

and :

$$\int_{\gamma_h} xydy = \int_{\gamma_h} x\left(dH - (x + x^2)dx\right) \equiv 0.$$

As it was proved in 3.4, there exist functions g, R such that :

$$\bar{\omega} = gdH + dR \tag{4.52}$$

This equation for g is equivalent to :

$$d\bar{\omega} = dg \wedge dH\ \ ,\ i.e\ :$$

$$-ydx \wedge dy = dg \wedge [ydy + (x + x^2)dx] \tag{4.53}$$

Clearly $g(x, y) = x$ is a solution for (4.53), and by proposition 5, one has :

$$H_3(h) = M_2(h) = \int_{\gamma_h} x\bar{\omega} = \int_{\gamma_h} -2yx^2dx + x(y^2 + xy - x^2)dy \tag{4.54}$$

We want to compute the four integrals in (4.54). To simplify notations, we will write $\alpha \sim 0$ for $\int_{\gamma_h} \alpha \equiv 0$.

$$1) \qquad yx^2\, dx \sim y(dH - ydy - xdx) \sim -yxdx = -\omega_1 \qquad (4.55)$$

$$2) \qquad yx^2dy \sim x^2(dH - (x+x^2)dx) \sim 0 \qquad (4.56)$$

$$3) \qquad x^3dy \sim (3H - \frac{3}{2}\, y^2 - \frac{3}{2}\, x^2)dy \sim -\frac{3}{2}\, x^2\, dy$$

$$\text{and } x^2\, dy \sim -yd(x^2) = -2xydx = -2\omega_1.$$

$$\text{So that } : x^3dy \sim 3\omega_1. \qquad (4.57)$$

$$4) \quad xy^2\, dy \sim 2x(H - \frac{1}{2}\, x^2\, - \frac{1}{3}\, x^3)dy \sim -2h\omega_0 - 3\omega_1 - \frac{2}{3}\, x^4\, dy$$

$$x^4\, dy \sim -4yx^3\, dx \sim -4y(3H - \frac{3}{2}\, y^2\, - \frac{3}{2}\, x^2)dx$$

$$\sim -4h\omega_0 + 6y^3\, dx + 6yx^2\, dx \sim -4h\omega_0 - 6\omega_1 + 6y^3\, dx$$

$$\text{and } : y^3\, dx \sim -3y^2\, xdy.$$

Then, finally :

$$xy^2\, dy \sim -2h\omega_0 - 3\omega_1 - \frac{2}{3}\left(-4h\omega_0 - 6\omega_1 - 18xy^2\, dy\right)$$

$$-\, 11xy^2\, dy \sim \frac{2}{3}\, h\omega_0 + \omega_1. \qquad (4.58)$$

Collecting the different contributions (4.55)-(4.58), we obtains :

$$y\bar{\omega} \sim -\frac{2}{33}\, h\omega_0 + \frac{20}{11}\, \omega_1 \qquad (4.59)$$

and the following expression for $M_3(h)$:

$$H_3(h) = M_2(h) = -\frac{2}{33}\, hI_0(h) + \frac{20}{11}\, I_1(h). \qquad (4.60)$$

Taking the generators $\varphi_1 = -2\lambda_1$, $\varphi_2 = -\lambda_5$, $\varphi_3 = -\frac{2}{33}\, \lambda_2\, \lambda_4$ for the Bautin Ideal at λ_0, we have :

$$\delta(h,\lambda) = \varphi_1\, h_1(h,\lambda) + \varphi_2\, h_2(h,\lambda) + \varphi_3\, h_3(h,\lambda) \qquad (4.61)$$

for new factors h_i, such that $H_1(h) = I_0$, $H_2(h) = I_1$, and $H_3(h) = hI_0 - 3I_1$.

Proposition 7 *For each :*

$$h \in]0, 1/6[, \quad det \begin{pmatrix} H_1 & H_1' & H_1'' \\ H_2 & H_2' & H_2'' \\ H_3 & H_3' & H_3'' \end{pmatrix} \neq 0. \tag{4.62}$$

Proof To verify (4.62) we can replace the functions H_1, H_2, H_3 by some multiples : LH_1, LH_2, LH_3 where $L(h)$ is any analytic function on $]0, 1/6[$, everywhere non zero : $L(h) \neq 0$ pour $\forall h \in]0, 1/6[$. Because $I_0(h) > 0$ for $h \in]0, 1/6[$ we can take $L = I_0^{-1} = H_1^{-1}$. Let $B_2(h) = \frac{I_1}{I_0}(h)$ as above. Then (4.62) is equivalent to :

$$\begin{vmatrix} 1 & 0 & 0 \\ B_2 & B_2' & B_2'' \\ B_2 + h & B_2' + 1 & B_2'' \end{vmatrix} \neq 0, \ i.e \ : \ B_2''(1) \neq 0 \ for \ \forall h \in [0, 1/6.$$

We will prove this in the next section. ∎

A consequence of proposition 6 is that $s_\delta(h) = 2$ for $\forall h \in]0, 1/6[$. At the centre point $(0, 0)$, corresponding to $h = 0$, one has $h \sim x^2$. Then the fact that $B_2''(0) \neq 0$ (see next section) will clearly imply that $s_\delta(0) = 2$.

So that, as a consequence of theorem 2 one has :

$$Cycl(X_\lambda, \gamma_{h_0}) \geq 2 \ for \ \forall h \in [0, 1/6[.$$

In fact, $Cycl(X_\lambda, \gamma_h) = 2$. This can be proved using the remark after the proof of theorem 2 or deduced from the above result [HI] : using this paper one can find a sequence $(\lambda_i)_i \to \lambda_0$, $\lambda_i \in Q^H \backslash Q^R$ such that for each i, X_{λ_i} has some periodic orbit γ_i with cyclicity 2 and $\gamma_i \to \gamma_{h_0}$. Then, using the semi-continuity of the cyclicity, proved in lemma 2.3, one obtains that $Cycl \ (X_\lambda, \gamma_{n_0}) \geq 2$ and so that the cyclicity is 2. We have proved :

Theorem 11 *Let $\lambda_0 \in Q^R \cap Q^R - \{0\}$ and γ_{h_0}, $h_0 \in [0, 1/6[$ any regular limit periodic set for X_{λ_0}. Then in the Kaypten-Dulac family X_λ, $Cycl(X_\lambda, \gamma_{h_0}) = 2$.*

Remark 21 *From formula (4.46) and proposition 6, it is possible to deduce the bifurcation diagram for X_λ, near λ_0.*

4.3.6 Some properties of Abelian Integrals.

In the preceding sections we have seen that the properties of a family near a centre-type vector field are closely related to properties of abelian integrals, equal to integral of algebraic 1-forms of the cycles of an Hamiltonian function H : $\int_{\gamma_n} \bar{\omega}$. In our applications H and $\bar{\omega}$ are rather special. In particular H may be reduced to the form : $H(x,y) = y^2 - P(x)$ where $P(x) = \sum_{i=0}^{2g+1} p_i\, x^i$.

This special case, I want to present briefly to begin with. General results were proved by Petrov [Pe].

We suppose chosen a continuous family of closed curve γ_h, each of them in the level $\{H = h\}$; in general h must be chosen in the universal covering of $C - \Sigma$ where Σ is the set of critical values of H, but in our applications, we will take h in some interval image by H of an interval between a centre e and a next critical point s.
In this case, one may think $h \in [e, s] \subset R \subset C$.

For each meromorphic 1-form ω one can consider the abelian integral : $I_\omega(h) = \int_{\gamma_\lambda} \omega$. In particular, let :

$$\omega_i = x^i\, y dx \ , \ \alpha_i = \frac{x^i}{y}\, dx$$

and :

$$I_i(h) = \int_{\gamma_h} \omega_i \ , \ J_i = \int_{\gamma_h} \omega\alpha_i.$$

Now, the two more important results about abelians integrals for $H(x,y) = y^2 - P(x)$ are :

1) For each algebraic ω, there exist $2g$ polynomials in h : $Q_i(h)$, $i = 0, \cdots, 2g-1$, such that :

$$I_\omega(h) = \sum_{i=0}^{2g-1} Q_i(h) I_i(h).$$

2) There exist two $(2g-1)\times(2g-1)$ matrices C and M such that :

$$I = hJ + CJ - MI \quad and \quad \frac{d}{dh} I = -\frac{1}{2}\, J \tag{4.63}$$

where $I = (I_i)_{i=0,\cdots,2g-1}$, $J = (J_i)_{i=0,\cdots,2g-1}$.

One can eliminate J to obtain the linear differential system :

$$(Id + M)I + -\frac{1}{2}\,(h\ Id + C)\,\frac{dI}{dt} \tag{4.64}$$

This system, which is singular at critical values of H, is precisely the Gauss-Maning connexion of H.

We are going to give a short proof for point 2). This proof, communicated to me by S. Yakovenko, is based of a preprint of Givental.

First, we have : $d\omega_i = d(x^i\ ydx) = x^i\ dy \wedge dx$ so that : $d\omega_i = -\frac{1}{2}\ \frac{x^n}{y}\ dx \wedge dH = -\frac{1}{2}\ \alpha_i \wedge dH$, $\alpha_i = 0, \cdots, 2g-1$.

This is equivalent to : $\frac{dI}{dh} = -\frac{1}{2}\ J$.

Next, for any : $n = 0, \cdots, 2g-1$:

$$\omega_n = x^n\ ydx = \frac{x^n(P+h)}{y}\ dx = h\alpha_n + \frac{x^n\ P}{y}\ dx. \tag{4.65}$$

Dividing $x^n\ P$ by P' we have :

$$x^n\ P = \sum_{i=0}^{2g-1} C_{ni}\ x^i + Q(x)P'(x). \tag{4.66}$$

This formula defines the polynomial $Q(x)$, with degree : $deg\ Q = n + Lg + 1 - 2g = n + 1$. Let :

$$Q(x) = \sum_{j=0}^{n+1} q_{nj}\ x^j. \tag{4.67}$$

Substituting (4.66) in (4.65), one gets :

$$\omega_n = x^n\, ydx = h\alpha_n + \sum_{i=0}^{2g-1} C_{ni}\, \alpha_i + \frac{QP'}{y}\, dx. \qquad (4.68)$$

Now, writing $\Omega \sim 0$ for $\int_{\gamma_n} \Omega \equiv 0$, and using $P'dx \sim ydy$ in (4.68), one has :

$$\omega_n \sim h\alpha_n + \sum_{i=0}^{2g-1} C_{ni}\, \alpha_i + 2Qdy. \qquad (4.69)$$

But :

$$2Qdy \sim -2Q'ydx = -2\sum_{i=0}^{n} (i+1)q_{ni+1}\, \omega_i. \qquad (4.70)$$

So that :

$$\omega_n = h\alpha_n + \sum_{i=0}^{2g-1} C_{ni}\, \alpha_i - 2\sum_{i=0}^{n} (i+1)q_{ni+1}\, \omega_i. \qquad (4.71)$$

Putting :

$$C = (C_{ni})_{n,i} \; and \; M = 2((i+1)q_{ni+1})_{n,i} \qquad (4.72)$$

(4.71) is the system (4.63).

We want to apply these general considerations to obtain the results about the ratio $\frac{I_1}{I_0}$ we claimed in theorem 3 and in the proof of proposition 6 above.

First, notice that the hamiltonian $H_1(x_1, y_1) = \frac{1}{2}\, g_1^2 + x_1^2 - \frac{x_1^3}{3}$ in section 3.5.2 and the hamiltonian $H_2(x_2, y_2) = \frac{1}{2}\, y_2^2 + \frac{1}{2}\, x_2^2 + \frac{x_2^3}{3}$ are equivalent to the hamiltonian $H(x,y) = y^2 - x + x^3$. If we put :

$$\left\{ \begin{array}{lcl} x_2(x) & = & \frac{\sqrt{3}}{2}\, x - \frac{1}{2} \\ y_2(x) & = & \sqrt{2}y \end{array} \right. \qquad (4.73)$$

Then :

$$H_2(x_2, y_2) = \frac{\sqrt{3}}{8}\, H(x, y) + \frac{1}{12}. \tag{4.74}$$

And, with :

$$\begin{cases} x_1(x) &= -\sqrt{3}x \\ y_1(x) &= \sqrt{2}y \end{cases} \tag{4.75}$$

we have :

$$H_1(x_1, y_1) = \sqrt{3}\, H(x, y) \tag{4.76}$$

So that, the ratios $B_1(h_2)$, $B_2(h)$ of $\frac{I_1}{I_0}$ for the Hamiltonians H_1, H_2 are simply related to the ratio for H :

$$B_1(\sqrt{3}h) = -\sqrt{3}B(h) \; and \; B_2\Big(\frac{\sqrt{3}}{8}\, h + \frac{1}{12}\Big) = \frac{3}{4}\sqrt{2}B(h) - \frac{\sqrt{6}}{2}. \tag{4.77}$$

So, to prove the claims in theorem 3 and proposition 6, we just need to prove the following theorem, for the analytic function :

$$B(h) = \frac{I_1}{I_0}(h) \;\; : \Big[-\frac{2}{3\sqrt{3}} \, , \, \frac{2}{3\sqrt{3}}\Big[\to R. \tag{4.78}$$

Theorem 12 *$B'(h) < 0$ for $\forall h \in \Big[-\frac{2}{3\sqrt{3}} \, , \, \frac{2}{3\sqrt{3}}\Big[$ and $B'(h) \to -\infty$ for $h \to \frac{2}{3\sqrt{3}}$. Moreover $B''(h) < 0$ for $\forall h \in \Big[-\frac{2}{3\sqrt{3}} \, , \, \frac{2}{3\sqrt{3}}\Big[$.*

Proof First, we want to prove that the function $B(h)$ verify the Ricatti equation :

$$9\Big(\frac{4}{27} - h^2\Big)\, \frac{dB}{dh} = -7B^2 - 3hB + \frac{5}{3}. \tag{4.79}$$

This is an easy consequence of the general formula (4.63). Repeating the above proof we have here : $P(x) = x - x^3$, and then :

$$\omega_0 = ydx = \frac{h + P}{y}\, dx = h\alpha_0 + \frac{Pdx}{y}. \tag{4.80}$$

Writing : $P(x) = \frac{2}{3}\,x + \frac{1}{3}\,x(1-3x^2)$ and substituting in (4.80) we obtain :

$$\frac{5}{3}\,\omega_0 \sim h\alpha_0 + \frac{2}{3}\,\alpha_1. \tag{4.81}$$

In the similar way :

$$xP(x) = \frac{2}{9} + (-\frac{2}{9} + \frac{2}{3}\,x^2)(1-3x^2) \; and \; :$$

$$\omega_1 \sim h\alpha_1 + \frac{2}{9}\,\alpha_0 - \frac{4}{3}\,\omega_1 \tag{4.82}$$

and so :

$$\frac{7}{3}\,\omega_1 \sim \frac{2}{9}\,\alpha_0 + h\alpha_1. \tag{4.83}$$

Relations (4.81), (4.83) with $\frac{dI_i}{dh} = -\frac{1}{2}\,J_i$ gives :

$$\begin{cases} \frac{5}{3}\,I_0 &= 2hI_0' + \frac{4}{3}\,I_1' \\ \frac{7}{3}\,I_1 &= \frac{4}{9}\,I_0' + 2hI_1' \end{cases} \tag{4.84}$$

Which can be solved in I_0', I_1' :

$$\begin{cases} (\frac{4}{27} - h^2)I_0' &= -\frac{5}{6}\,hI_0 + \frac{7}{9}\,I_1 \\ (\frac{4}{27} - h^2)I_1' &= \frac{5}{27}\,I_0 - \frac{7}{6}\,hI_1 \end{cases} \tag{4.85}$$

Notice that the roots of $\frac{4}{27} - h^2$: $\pm\frac{2}{3\sqrt{3}}$ are the critical values of $H(y=0)$. Now (4.85) implies the Ricatti equation (4.79) for $B(h) = \frac{I_1}{I_0}(h)$.

This equation for $B(h)$ means that the graph of this function belongs to an orbit of the following vector field Z on the space R^2 of coordinates (h, B) :

$$Z = 9\left(\frac{4}{27} - h^2\right)\frac{\partial}{\partial h} + \left(-7B^2 - 3hB + \frac{5}{3}\right)\frac{\partial}{\partial B}. \tag{4.86}$$

This vector field has four critical points :

$$\alpha_0 = \left(-\frac{2}{3\sqrt{3}}\ ,\ \frac{1}{\sqrt{3}}\right)\ ,\ \alpha_1 = \left(\frac{2}{3\sqrt{3}}\ ,\ \frac{5}{7}\ ,\ \frac{1}{\sqrt{3}}\right),$$
$$\alpha_0' = \left(-\frac{2}{3\sqrt{3}}\ ,\ -\frac{5}{7}\ ,\ \frac{1}{\sqrt{3}}\right)\ ,\ \alpha_1' = \left(\frac{2}{3\sqrt{3}}\ ,\ -\frac{1}{\sqrt{3}}\right)$$

and admits the lines $\Delta_0 = \left\{h = -\frac{2}{3\sqrt{3}}\right\}$ and $\Delta_1 = \left\{h + \frac{2}{3\sqrt{3}}\right\}$ as invariant lines. Along these lines Z is normally hyperbolic and in restriction to Δ_0 and Δ_1 the critical points are also hyperbolic.

The four critical points are hence hyperbolic and it is easily checked that α_0 and α_1' are saddle points, while α_0' and α_1 are nodes, respectively unstable and stable. The phase portrait of Z in the vertical strip $U = \left\{B \geq 0\ ,\ -\frac{2}{3\sqrt{3}} \leq h \leq \frac{2}{3\sqrt{3}}\right\}$ can now easily be obtained taking into account the value of the vertical component of Z when $B = 0$ and when B is large (see Figure 4.3).

In particular, we notice the existence of a unique Z-orbit lying in the interior of U and having the saddle point $\alpha_0 = \left(-\frac{2}{3\sqrt{3}}, -\frac{1}{sqrt3}\right)$ as an α-limit point : it is the unstable separatrix Γ of α_0, which tends to α_1 for $t \to +\infty$.

If we notice that $B(h) \to \frac{1}{\sqrt{3}}$ for $h \to -\frac{2}{3\sqrt{3}}$, it follows that *the graph of $B(h)$ is equal ot Γ* (of course, this implies that $B(h) \to \frac{5}{7\sqrt{3}}$ for $h \to \frac{2}{3\sqrt{3}}$).

Let us show that $B'(h) < 0$ for all $h \in \left[-\frac{2}{3\sqrt{3}}\ ,\ \frac{2}{3\sqrt{3}}\right[$.

For $h = -\frac{2}{3\sqrt{3}}$, we have that $B'(-\frac{2}{3\sqrt{3}}) = -\frac{1}{8}$: this is simply obtained, computing the slope of the eigenspace at the saddle point α_0. For the other values of h, we make the following qualitative reasoning.

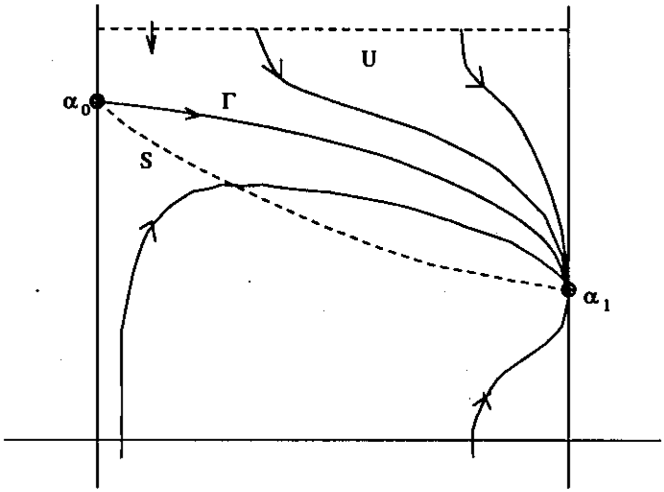

Figure 4.3:

We consider the equation $-7B^2 - 3hB + \frac{5}{3} = 0$, describing the point where Z is horizontal. This equation defines a hyperbola, whose two connected components are graphs of functions of h. The part of this hyperbola contained in the strip U is an arc S joining α_0 and α_1. Along S, we can solve h in term of B : $h = \frac{-7B^2 + 5/3}{3B}$ (since $B \neq 0$ on S). Hence Z is transverse to S along it and directed to the right. We now study the position of S with respect to Γ. At α_0, the tangent to S has a slope equal to $-\frac{1}{4}$, which is smaller than the slope $B'\left(-\frac{2}{3\sqrt{3}}\right) = -\frac{1}{8}$ of Γ at the same point. Then, in the neighborhoods of α_0, the separatrix Γ is above S. But as, along S, Z is transverse to S and directed to the right, the orbit Γ is not permitted to cut S again for $t \to +\infty$: the orbit Γ is hence entirely located in U, above S. But in this region, the vertical component of Z is negative. It follows that $B'(h) < 0$ for all $h \in \left[-\frac{2}{3\sqrt{3}}\ ,\ \frac{2}{3\sqrt{3}}\right[$. For $h \to \frac{2}{3\sqrt{3}}$, $B'(h) \to -\infty$ because an easy computation gives that the eigen-value at α_1 along Δ_1 is greater than the transversal eigen-value (the two eigenvalues are negative).

Let us now show that $B''(h) < 0$ for all $h \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right[$.

First of all, using a development up to order 2 of the equation (4.86) in $h = -\frac{2}{3\sqrt{3}}$, one obtains that $B''(-\frac{2}{3\sqrt{3}}) = -\frac{55}{2304}\sqrt{3} < 0$.

Let us for a moment suppose that $B''(h)$ would have a zero on $\left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right[$ and let $h_0 > -\frac{2}{3\sqrt{3}}$ be the minimum of such points : $B''(h_0) = 0$ and $B''(h) < 0$ for all $h \in \left[-\frac{2}{3\sqrt{3}}, h_0\right[$.

We show that this is impossible since Z is a *quadratic vector field.* Therefore consider D, the tangent of Γ in the point $m_0 = (h_0, B(h_0))$. As $B''(h_0) = 0$, the order of contact between D and Γ is at least 2. Let v be a vector orthogonal to D and $D(u)$ a linear parametrization of D. The function $\psi(u) =< Z(D(u)), v >$ ($< .,. >$ denoting the euclidean scalar product on R^2) has a zero of order at least 1 in u_0, with $D(u_0) = m_0$. As $B''(h) < 0$ for all $h \in \left[-\frac{2}{3\sqrt{3}}, h_0\right[$, the corresponding arc of Γ is situated below D. The line D hence cuts $\Delta_0 = \left\{h = -\frac{2}{3\sqrt{3}}\right\}$ at a point n_0 above α_0. At this point, Z is directed downwards. On the other hand, in the points of D with abscissa $< h_0$ but near h_0, Z is directed towards the half plane above D. From this it follows that the function $\psi(u)$ must have a zero at some $u_1 \neq u_0$ with $D(u_1) \in]n_0, m_0[$.

However, the vector field Z being quadratic, the function $\psi(u)$ is polynomial of second degree in u ; the existence of a double zero at u_0 and another zero u_1 implies then $\psi \equiv 0$ and hence that Γ *is a line segment.* This is of course not compatible with $B''\left(-\frac{2}{3\sqrt{3}}\right) < 0$, ending the proof of the theorem (see Figure 4.4:). ∎

To end this section, I want to give a very useful algorithm due to Petrov [P] to obtain a bound for the number of zeros of any algebraic integral $I = \int_{\gamma_h} \omega$ for the cubic hamiltonian $H(x,y) = y^2 - x + x^3$. From the point i) above, we know that there exist polynomials P, Q in h such that :

$$I = P(h)I_0(h) + Q(h)I_1(h).$$

One can find in [P] an algorithm to compute P, Q and also an estimate of the degrees of P, Q in term of the degree of ω. The number of zeros of $I(h)$ on $\left]-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right[$ is the same number as for the function $G = P + QB$.

Now let $K = G \subset D(P, Q)$ and write :

$$P = KP_0\ ,\ Q = KQ_0\ ,\ P_0,\ Q_0 \textit{ without common roots.}$$

The number of zeros for G is equal to the number of zeros of K plus the number for $G_0 = P_0 + Q_0B$.

But, because P_0, Q_0 have no common roots, no zero of P_0 is also zero of G_0. So that, the number of zeros of G_0 is the same as the number of zeros of $g = B + \frac{P_0}{Q_0}$.

Now, the function g is solution of a Ricatti equation :

$$9\left(\frac{4}{27} - h^2\right)g' = R_2\ g^2 + R_1\ g + R_0 \tag{4.87}$$

with :

$$R_0 = \frac{N}{Q_0^2} \textit{ with },$$

$$N = 9\left(\frac{4}{27} - h^2\right)(P_0'\ Q_0 - P_0\ Q_0') - 7Q_0^2 + 3hP_0\ Q_0 + \frac{5}{3}\ Q_0^2. \tag{4.88}$$

The crucial point is the following result of Petrov, based in Khovanskii's ideas :

Lemma 17 *Let* $\alpha < \beta$, α, $\beta \in \left]-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right[$, *two consecutive roots of* Q_0. *Then between two consecutive roots of* g *there exists at least one root of* N.

Proof (See Figure 4:4:)

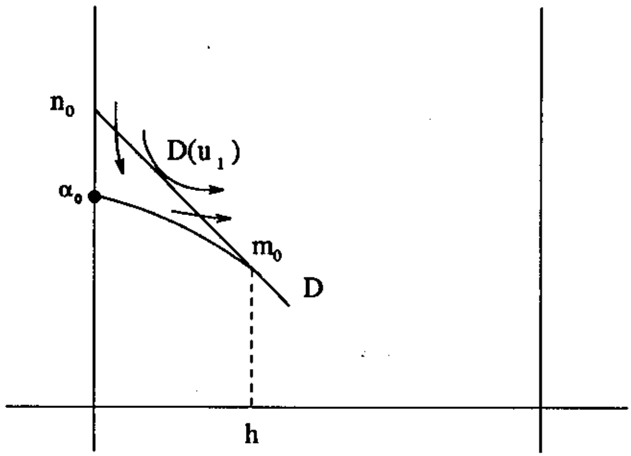

Figure 4.4:

Let two consecutive roots h_1, h_2 of g. If g' is also zero at any of these roots, it is also the case for R_0, following (4.87). Suppose that g' is not zero at h_1 and h_2.

Then $g'(h_1).g'(h_2) < 0$, and following (4.87), it is the same for R_0 and also for N. Then N has at least one root between h_1 and h_2. ∎

An easy consequence of the lemma 12 (extended to the case of multiple root of g), is that the number of roots of g between α and β, counted with multiplicity is less than the number of roots for N between the same points, plus one.

It follows that the total number of roots for g, with multiplicity, between $-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}$, is bounded by deg $(N) + deg(Q_0) + 2$. From (4.88) : deg $(N) \leq 2\,Sup\,(deg(P_0)$, deg $(Q_0)) + 1$ and so the number of roots of I is bounded by the above bound plus the number of roots of K. Finally, one obtains :

Corollary 2 *Suppose that* $I(h) = P(h)I_0(h) + Q(h) + Q(h)I_1(h)$, P, Q *polynomials in* h. *Then the number of roots of* I, *with multiplicity,* *between* $-\frac{2}{3\sqrt{3}}$ *and* $\frac{2}{3\sqrt{3}}$ *is bounded by :* $2\ Sup\ (deg\ (P),\ deg\ (Q))+1$.

Example.

Let us consider the family of section 3.5.3. As a consequence of result in this section, the cyclicity may be computed as the supremum of cyclicity for all subfamilies $\lambda(\varepsilon)=\Big\{f\{\lambda_1=\varepsilon\overline{\lambda}_1,\ \lambda_5=\varepsilon\overline{\lambda}_5,\ \lambda_2=\varepsilon\overline{\lambda}_2,\ \lambda_4=\varepsilon\overline{\lambda}_4 \mid (\overline{\lambda}_1,\ \overline{\lambda}_2,\ \overline{\lambda}_3,\ \overline{\lambda}_4,\ \overline{\lambda}_5)\in S^3\Big\}$.

For such a family :

$$\delta(h,\lambda(\varepsilon))=\varepsilon M_1(h,\overline{\lambda})+0(\varepsilon)$$

with :

$$M_1(h,\overline{\lambda})=(\overline{\varphi}_1+\overline{\varphi}_2\ \overline{\varphi}_4\ h)I_0(h)+(\overline{\varphi}_5-3\overline{\varphi}_2\ \overline{\varphi}_4)I_1(h). \qquad (4.89)$$

As a consequence of corollary 2, we have that the total number of zeros of M, and also the multiplicity at each $h_0\in\Big[-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}}\Big[$ is less than 3. We have proved above that this multiplicity is in fact less than 2.

Chapter 5

Bifurcations of elementary Graphics.

After the regular limit periodic sets, the simplest limit periodic sets are the *elementary graphics.*

As it was defined in chapter 2, an elementary graphic for X_{λ_0} is an invariant immersion of S^1, made by a finite number of regular orbits and elementary (i.e. hyperbolic or isolated semi-hype rbolic) singular points. Limit sets of each regular orbit are contained in the set of singular points and the immersion is oriented by the orbit orientation.

Such elementary graphic Γ may be monodromic, this means that it has a return map (defined on some interval $[a, b[$ with $a \in \Gamma$). In this case we often calls it a *polycycle.*

In this chapter we will deal with monodromic graphics with hyperbolic singular points and will refer to them as either as hyperbolic graphics or polycycles. The same methods apply also to the study of the non-monodromic case.

The simplest case corresponds to a graphic with just one hyperbolic saddle. We will call it a *saddle connection, or homoclinic loop.* The two first paragraphs are devoted to their study.

The most important fact is that the Poincaré map defined along Γ is not differentiable at points whose ω-limit is one of the singular point because the transition map near an elementary point is not differentiable.

In the first paragraph we will establish an expansion of the transition near an hyperbolic saddle using a natural unfolding of the logarithm. We will apply this expansion to study unfoldings of the saddle connexions of finite codimension in the second paragraph, and of analytic infinite codimension in the following one. In the last paragraph I will present some recent results concerning general elementary polycycles, due to Mourtada, El Morsalani, Ilyashenko and Yakovenko and also to point out some remaining open questions.

5.1 Transition map near a hyperbolic saddle point.

Consider X_λ a C^∞ unfolding at some hyperbolic saddle point s_{λ_0} of X_{λ_0} ; $\lambda \in P \simeq R^n$ the parameter space. Since we are just interested here in the germ of a family at the point $(s_{\lambda_0}, \lambda_0)$, without loss of generality we can suppose that the vector field family X_λ is defined in a neighborhood V of $s_{\lambda_0} = s = (0,0) \in R^2$, for parameter values λ in a neighborhood W of the origin in R^n, and has a hyperbolic saddle at s for $\forall\lambda \in W$. We can also suppose that the local unstable and stable manifolds are given by $W^u = 0x \cap V$ and $W^s = 0y \cap V$. Finally, we suppose that s is the unique singular point of X_λ in V.

5.1.1 Normal form of X_λ near the saddle point.

Eigenvalues at s are equal to $\lambda_2(\lambda)$, $\lambda_1(\lambda)$ with

$$\lambda_2(\lambda) < 0 < \lambda_1(\lambda)$$

for any $\lambda \in W$. Let $r(\lambda) = \dfrac{-\lambda_2(\lambda)}{\lambda_1(\lambda)}$. We call it : *ratio of hyperbolicity of* X_λ *at* s. Dividing X_λ by λ_1 (λ), we can suppose that the eigenvalues are 1, $-r(\lambda)$ and that the 1-jet of X_λ at s is equal to :

$$j^1 X_\lambda(0) = x\frac{\partial}{\partial x} - r(\lambda)y\frac{\partial}{\partial y}. \tag{5.1}$$

A first consequence of the hyperbolicity of s is the following result of finite determinacy :

Proposition 8 *[Bon]. There exists a function $K(k) : N \to N$ such that $K(k) \to \infty$ for $k \to \infty$ $K(k) > k$ such that if Y_λ is any germ of C^∞ vector field family along $\{s\} \times W$ with the property :*

$$j^{K(k)}\ (Y_\lambda - X_\lambda)(0) = 0. \tag{5.2}$$

Then, the two family germs X_λ and Y_λ are C^k-conjugate. (This means that there exist a C^k germ of family of diffeomorphisms : g_λ on a neighborhood of $\{s\} \times W_1$, such that $g_\lambda(Y_\lambda) = X_\lambda$ on this neighborhood).

Remark 22 *The result in [Bon] is proved for vector field families on R^p, and gives an explicit function $K(k)$ which just depends on $j^1\ X_\lambda(0)$. It is important to notice that this result does not depend on the possible resonances.*

Proposition 1 allows us to replace X_λ by a polynomial family, up to a C^k conjugacy. We want to prove now a variant of the Dulac-Poincaré normal form theorem for the vector field family X_λ :

Proposition 9 *Let X_λ a C^∞ family as above.*

1) Suppose that $r(\lambda_0) \notin Q$. Then there exist a sequence of neighborhoods W_i of λ_0 in W, $i \geq 1$: $\lambda_0 \in \cdots W_{i+1} \subset W_i \subset \cdots \subset W_1$, such that for any $N \in N$ and $\lambda \in W_{N+1}$:

$$j^{N+1}\ X_\lambda(s) \sim x\ \frac{\partial}{\partial x}\ + r(\lambda) y\ \frac{\partial}{\partial y}. \tag{5.3}$$

2) Suppose that $r(\lambda_0) = \dfrac{p}{q}$, p and q without common factors. Then there exist a sequence of neighborhoods as above and a sequence of smooth functions $\alpha_i(\lambda) : W_i \to R$, $\alpha_1(\lambda) = p - qr(\lambda)$ on W_1, such that for any
$N \in N$ and $\lambda \in W_{N+1}$:

$$j^{(p+q)N+1}\ X_\lambda(s) \sim x\ \frac{\partial}{\partial x} + \Big(-r(\lambda) + \frac{1}{q}\ \sum_{i=1}^{N}\ \alpha_{i+1}\ (\lambda)(x^p\ y^q)^i\Big) y\ \frac{\partial}{\partial y}$$

$$= x\ \frac{\partial}{\partial x} + \frac{1}{q}\ \Big(-p + \sum_{i=0}^{N}\ \alpha_{i+1}\ (\lambda)(x^p\ y^q)^i\Big) y\ \frac{\partial}{\partial y}. \tag{5.4}$$

Here the sign $\sim$ denotes equivalence of jets. Formulas (5.3), (5.4) are equivalent to the following : X_λ is C^N equivalent to $X_\lambda^N + P_\lambda^N$ where X_λ^N is the right hand polynomial vector field family and P_λ^N a C^∞ family on $V_1 \times W_{N+1}$ with a $N+1$ or $(p+q)N+1$ zero jet at s respectively, for any $\lambda \in W_{N+1}$.

Proof The proof given in [R4] for the resonant case $p = q = 1$, is easily extended. It is based on the following remark : for $\lambda = \lambda_0$, we have no resonance if $r(\lambda_0) \neq Q$ or all the resonance relations $\lambda_i - \sum_{j=1}^{2} n_j \lambda_j = 0$, $i = 1, 2$ are generated by the unique relation

$$p\lambda_1(\lambda_0) + q\lambda_2(\lambda_0) = 0$$

if $r(\lambda_0) = p/q$. By continuity, for each N one can find a neighborhood W_N of λ_0 in W_1 such that this remains valid for all $\lambda \in W_N$ (of course, the neighborhoods W_N form a decreasing sequence). One constructs the normal form up to order $(p+q)N+1$, using subspaces of homogeneous vector fields, which are independent of $\lambda \in W_N$. ∎

Combining the two above propositions, we see that, at each order of differentiability, one can replace the given family near the saddle point by a polynomial one :

Theorem 13 *Let a C^∞ family X_λ, as above defined near a saddle point $s_{\lambda_0} = s$ of X_{λ_0}. There exists a function $N(k) : N \to N$ such that in some neighborhood of s and for $\lambda \in W_{N(k)}$, the family X_λ is C^k-equivalent to the polynomial family :*

$$x \frac{\partial}{\partial x} + \left(r(\lambda) + \frac{1}{q} \sum_{i=1}^{N(k)} \alpha_{i+1}(\lambda)(x^p y^q)^i\right) y \frac{\partial}{\partial y} \tag{5.5}$$

if $r(\lambda_0) = p/q$; if $r(\lambda_0) \notin Q$, all the $\alpha_i(\lambda) \equiv 0$ for $i \geq 2$.

Proof It suffices to take $N(k)$ such that $(p+q)N(k)+1 > K(k)$ in case of resonance p/q and $N(k)+1 > K(k)$ when $r(\lambda_0) \notin Q$ and to apply the two above propositions. ∎

Remark 23 *The function $\alpha_1(\lambda)$ is uniquely defined by the relation $\alpha_1(\lambda) = p - qr(\lambda)$. It is not the same for the other resonant quantities $\alpha_i(\lambda)$, $i \geq 2$, in the resonant case $r(\lambda_0) = p/q$. Nevertheless, we have seen that one can choose smooth α_i if X_λ is C^∞, and clearly analytic α_i if X_λ is an analytic family.*

5.1.2 The structure of transition map for the normal family.

If $r(\lambda_0) \neq Q$ we have seen that X_λ if C^k equivalent to the linear family $X_\lambda^N = x\ \dfrac{\partial}{\partial x} - r(\lambda)\ y\ \dfrac{\partial}{\partial y}$, if one restricts λ in $W_{N(k)+1}$. The transition map for this linear family between $\sigma = [0,1[\ \times\{1\}$ and $\tau = \{1\}\times]-1,1[$ is : $x \to x^{r(\lambda)}$.

If now $r(\lambda_0) = p/q$, we have seen that X_λ factorizes up to C^k equivalence, through a polynomial family of special type.

More generally, we are going to consider the *analytic normal family* X_α^N :

$$X_\alpha^N = x\ \frac{\partial}{\partial x} + \frac{1}{q}\ \Big(-p + \sum_{i=0}^{\infty}\ \alpha_{i+1}\ (x^p\ y^q)^i\Big) y\ \frac{\partial}{\partial y} \tag{5.6}$$

where $P_\alpha(u) = \sum\limits_{i=0}^{\infty} \alpha_{i+1}\ u^{i+1}$ is an analytic entire function of $u \in R$, and $\alpha = (\alpha_1, \alpha_2, \cdots) \in A$, where A is the set :

$$A = \{\alpha = (\alpha_1, \alpha_2, \cdots) \mid \ \mid \alpha_1 \mid < \frac{1}{2}\ ,\ \mid \alpha_i \mid < M\ for\ i \geq 2\} \tag{5.7}$$

where $M > 0$ is some fixed constant.

X_α^N has the axes $0x$ and $0y$ as unstable and stable manifolds. If σ, τ are transversal segments as above, the flow of X_α^N defines a transition map $D_\alpha(x)$ from σ to τ, which extends continuously to $D_\alpha(0) = 0$. This map is analytic in (x, α) for $x \neq 0$. We want to study its properties at $\{x = 0\}$.

Making the singular change of variabl es : $u = x^p\ y^q$, $x = x$, the differential equation of X_α^N is brought in the following form :

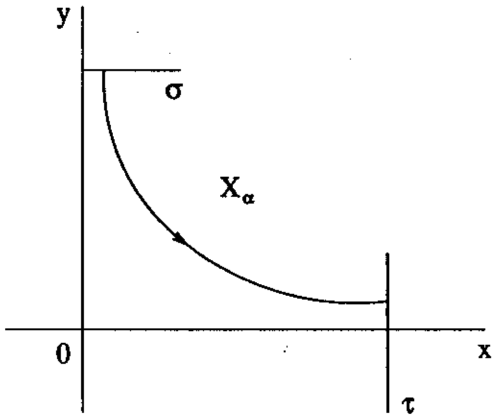

Figure 5.1:

$$\begin{cases} \dot{x} &= x \\ \dot{u} &= P_\alpha(u) = \sum_{i=1}^{\infty} \alpha_i \, u^i \end{cases} \tag{5.8}$$

We see that the variables are separated in (5.8). The first equation gives $x(t,x) = xe^t$. Let us look at the second one ; $P_\alpha(u)$ is analytic for $| u | \leq 1$ for $\alpha \in A$. Call $u(t,u)$ the solution such that $u(0,u) = u$ (see Figure 5.1).

We can expand it in series in u for each t :

$$u(t,u) = \sum_{i=1}^{\infty} g_i(t) u^i. \tag{5.9}$$

One has $g_1(t) = e^{\alpha_1 t}$ and $g_i(0) = 0$ for $i \geq 2$.

We want to study the form of the g_i and the convergence of the above series in function of t. For this, we are going to compare $u(t,u)$ to the solution of the hyperbolic equation :

$$\dot{U} = \frac{1}{2}\, U + M \sum_{i=1}^{\infty} U^{i+1}. \tag{5.10}$$

We have the following estimations :

Lemma 18 *Let $U(t,u) = \sum_{i=1}^{\infty} G_i(t) u^i$ the power series expansion of the trajectories of (5.10). Then, for each $i \geq 1$ and $t \geq 0$:*

$$| g_i(t) | \leq G_i(t) \quad for\ any \quad \alpha \in A.$$

Proof Substituting (5.9) in the equation : $\frac{\partial u}{\partial t}\,(t,u) = P_\alpha(u(t,u))$, we obtain recurrent equations for the $g_i(t)$, the system E_g :

$$\begin{cases} \dot{g}_1(t) &=& \alpha_1\, g_1 \\ \dot{g}_2(t) &=& \alpha_1\, g_2 + \alpha_2\, g_1^2 \\ \dot{g}_3(t) &=& \alpha_1\, g_3 + 2\alpha_2\, g_1\, g_2 + \alpha_3\, g_2^2 \end{cases}$$

And more generally :

$$\dot{g}_i = \alpha_1\, g_i + P_i(\alpha_2, \cdots, \alpha_i\ ;\ g_1, \cdots, g_{i-1})\ for\ i \geq 2$$

where P_i is a rational polynomial in $\alpha_2, \cdots, \alpha_i, g_1, \cdots, g_{i-1}$ with positive rational coefficients.

Now, let be $U(t,u)$ the trajectory of $\dot{U} = P_\alpha(U)$ with $\alpha = \Big(\frac{1}{2}, M, M, \cdots\Big)$. We have for $G_i(t)$ the system E_G :

$$\begin{aligned} \dot{G}_1 &= \frac{1}{2}\, G_1 \\ \dot{G}_2 &= \frac{1}{2}\, G_2 + M G_1^2 \end{aligned}$$

and more generally :

$$\dot{G}_i = \frac{1}{2}\, G_i + P_i(M, \cdots, M\ ;\ G_1, \cdots, G_{i-1})$$

(with the same polynomials P_i as above).

We can solve the system E_G by :

$$G_1(t) = e^{1/2t} \ , \ G_2(t) = \psi_2(t)e^{\frac{1}{2}t} \ with \ \psi_2(t) = \int_0^t e^{-\frac{1}{2}\tau} \ M_1 \ G_1^2(\tau)d\tau$$

and more generally :

$$G_i(t) = \psi_i(t)e^{\frac{1}{2}t} \ with$$

$$\psi_i(t) = \int_0^t e^{i-\frac{1}{2}\tau} \ P_i\Big(M, \cdots, M \ ; G_1(\tau), \cdots, G_{i-1}(tau)\Big)d\tau. \qquad (5.11)$$

It follows easily from these formulas, that $G_i(t) > 0$ for $t > 0$. Now we are going to show that

$$| \ g_i(t) \ |\leq G_i(t)$$

for each $t \geq 0$. First, this is trivially true for $i = 1$:

$$| \ g_1(t) \ |\leq e^{|\alpha_1|t} \ \leq e^{\frac{1}{2}t} \ = G_1(t). \qquad (5.12)$$

Suppose that we have shown : $| \ g_j(t) \ |\leq G_j(t)$ for each $j : 1 \leq j \leq i-1$, and any $t \geq 0$. We are going to compare the two equations :

$$\dot{g}_i(t) = \alpha_1 \ g_i + P_i \ (\alpha_2, \cdots, \alpha_i \ ; \ g_1, \cdots, g_{i-1})$$

and :

$$\dot{G}_i(t) = \frac{1}{2} \ G_i + P_i \ (M, \cdots, M \ ; \ G_1, \cdots, G_{i-1}).$$

Because the coefficients in P_i are positive, we have :

$$| \ P_i(\alpha_2, \cdots, \alpha_i \ ; \ g_1, \cdots, g_{i-1}) \ |\leq P_i(| \ \alpha_2 \ |, \cdots, | \ \alpha_i \ | \ ; \ | \ g_1 \ |, \cdots, | \ g_{i-1} \ |)$$

$$\leq \ P_i(M, \cdots, M \ ; \ G_1, \cdots, G_{i-1}). \qquad (5.13)$$

Now, for $t = 0$, we have $G_1(0) = 1$ and $G_i(0) = 0$ for $i \geq 2$. So, we have $\dot{G}_i(0) = P_i \ (M, \cdots, M \ ; \ G_1(0), \cdots, G_{i-1}(0)) = MG_1(0)^i = M$ and also $| \ \dot{g}_i(0) \ | \ \leq \ | \ \alpha_i \ | \ | \ g_1(0) \ |^i \ \leq \ | \ \alpha_i \ |< M$. So, for $t = 0$ we have :

$g_i(0) = G_i(0) = 0$ and $|\ \dot{g}_i(0)\ | \leq \dot{G}_i(0)$. By continuity, this gives, for t small enough : $|\ \dot{g}_i(t)\ | < \dot{G}_i(t)$.

We want to show now that this inequality holds for any $t \geq 0$ (and so, we will have : $|\ g_i(t)\ | \leq G_i(t)$ for any $t \geq 0$).

Suppose on the contrary that $t_0 > 0$ is the inferior bound of the values of t such that $|\ \dot{g}_i(t)\ | \geq \dot{G}_i(t)$. For all $t \in [0, t_0]$ we have : $|\ \dot{g}_i(t)\ | \leq \dot{G}_i(t)$. So, for all $t \in [0, t_0]$ we also have : $|\ g_i(t)\ | \leq G_i(t)$. Now, for $t = t_0$:

$$\begin{aligned} \dot{g}_i(t_0) &= \alpha_1\ g_i(t_0) + P_i(\alpha_2, \cdots, \alpha_i\ ;\ g_1(t_0), \cdots, g_{i-1}(t_0)) \\ \dot{G}_i(t_0) &= \frac{1}{2}\ G_i(t_0) + P_i(M, \cdots, M\ ;\ G_1(t_0), \cdots, G_{i-1}(t_0)). \end{aligned}$$

By induction on i, we know that $G_j(t_0) \geq g_j(t_0)$ for $2 \leq j \leq i-1$. By the choice of t_0, we already know that $G_i(t_0) \geq |\ g_i(t_0)\ |$. So, the inequality $|\ \alpha_1\ | < \frac{1}{2}$ implies that $|\ \dot{g}_i(t_0)\ | < \dot{G}_i(t_0)$. But, by continuity, this strict inequality is available for $t > t_0$, t near t_0 : this last point contradicts the definition of t_0. ∎

Next, we prove the following :

Lemma 19 *There exist constants C, $C_0 > 0$ such that :*

$$|\ g_i(t)\ | \leq C_0\ [Ce^{t/2}]^i \ \textit{for any}\ i \geq 1,\ t \geq 0\ \textit{and any}\ \alpha \in A.$$

Proof Using lemma 1, it is sufficient to show that $G_i(t) \leq C_0\ [Ce^{\frac{t}{2}}]^i$ for some constants C_0, C, and $i \geq 1$, $t \geq 0$, $\alpha \in A$. Recall that the function $U(t, u) = \sum_{i \geq 1} G_i(t) u^i$ is a trajectory of the hyperbolic 1-dimensional vector field $X = P(u)\ \frac{\partial}{\partial u}$ with $P(u) = \frac{1}{2}\ u + M \sum_{i=1}^{\infty} u^{i+1}$. From the analytic linearisation theorem of Poincaré, there exists an analytic diffeomorphism $g(u) = u + 0(u)$, converging for $|\ u\ | \leq K_1$ for some K_1, such that :

$$g_*\Big(P(u)\ \frac{\partial}{\partial u}\Big) = \frac{1}{2}\ u \frac{\partial}{\partial u}. \tag{5.14}$$

This diffeomorphism g sends the flow $U(t,u)$ of $P\,\frac{\partial}{\partial u}$ into $U_0(t) = ue^{\frac{1}{2}t}$, the flow of $\frac{1}{2}\,u\,\frac{\partial}{\partial u}$.

This means : $U_0(t, g(u)) = g \circ U(t,u)$ for $|\, u\,|, |\, U(t,u)\,| \leq K_1$.

Because g is invertible for $|\, u\,| \leq K_1$, there exist constants b, B, $0 < b < B$ such that :

$$b\,|\, u\,| \leq |\, g(u)\,| \leq B\,|\, u\,| \;\; for \;\; |\, u\,| \leq K_1. \tag{5.15}$$

Suppose that : $|\, u\,| \leq \frac{b}{B}\, K_1\, e^{-\frac{1}{2}t}$. Then :

$$|\, g(u)\,| \leq B\,|\, u\,| \leq b\, K_1\, e^{-\frac{1}{2}t} \tag{5.16}$$

and : $|\, U_0(t, g(u))\,| = |\, g(u)\,|\, e^{\frac{t}{2}} \leq bK_1$.

Now, one has : $U(t,u) = g^{-1} \circ U_0(t, g(u))$.

This implies that :

$$|\, U(t,u)\,| \leq \frac{1}{b}\,|\, U_0(t, g(u))\,| \leq K_1. \tag{5.17}$$

Using Cauchy inequalities for the coefficients $G_i(t)$, we find :

$$|\, G_i(t)\,| \leq \frac{Sup\,\{|\, U(t,u)\,|\,|\, u\,| = R(t)\}}{(R(t))^i} \leq \frac{K_1}{(R(t))^i} \tag{5.18}$$

where : $R(t) = \frac{b}{B}\, K_1\, e^{-\frac{t}{2}}$.

So, finally, we obtain :

$$|\, G_i(t)\,| \leq K_1\, \left(\frac{B}{b}\, K_1^{-1}\right)^i e^{i\frac{t}{2}},$$

which is the desired estimate, with $C_0 = K_1$ and $\frac{B}{b}\, K_1^{-1}$. ∎

We will show below that the g_i are analytic functions of $t > 0$. For the moment, we notice that the formula $\frac{\partial u}{\partial t}\,(t,u) = P_\alpha(u(t,u))$ shows that the series in u of $\frac{\partial u}{\partial t}$ has the same radius of convergence as $u(t,u)$. (Recall that $P_\alpha(u)$ is supposed to be an entire function). The same is true for any derivative $\frac{\partial^k u}{\partial t^k}(t,u)$, by induction on k. This remark gives an estimate for the coefficients $\frac{d^k g_i}{dt^k}(t)$ of this derivative.

Using Cauchy inequalities as above :

$$\left|\frac{d^k g_i(t)}{dt^k}\right| \leq \frac{Sup\ \{\mid \frac{\partial^k u}{\partial t^k}(t,u) \mid ;\ \mid u \mid = R(t)\}}{\mid R(t) \mid^i}$$

which gives : $\left|\frac{d^k g_i}{dt^k}(t)\right| \leq C_k\ (Ce^{\frac{t}{2}})^i$ for some $C_k > 0$.

So, we have proved :

Lemma 20 *For each $k \geq 1$, there exists a constant $C_k > 0$ such that :*

$$\left|\frac{d^k g_i}{dt^k}(t)\right| \leq C_k\ (C.e^{\frac{t}{2}})^i\ for\ any\ i \geq 1,\ t \geq 0,\ \alpha \in A \tag{5.19}$$

(C is the same as in lemma 2).

We will give now establish more precisely the form of the functions $g_i(t)$. For this, we introduce the function :

$$\begin{aligned} \Omega(\alpha_1, t) &= \frac{e^{\alpha_1 t} - 1}{\alpha_1}\ for\ \alpha_1 \neq 0\ and \\ \Omega(0, t) &= t. \end{aligned}$$

With this notation we have :

Proposition 10 *For each $k \geq 1$, $g_k(t) = e^{\alpha_1}\ Q_k(t)$ where Q_k is a polynomial of degree $\leq k-1$ in Ω. The coefficients of Q_k are polynomial in $\alpha_1, \cdots, \alpha_k$. More precisely :*

$$Q_k = \alpha_1 \Omega + \bar{Q}_k(\alpha_1, \cdots, \alpha_k, \Omega) \tag{5.20}$$

where $\bar{Q}_k$ is a polynomial of degree $\leq k-1$ in Ω, with coefficients in $\mathcal{I}(\alpha_1, \cdots, \alpha_{k-1}) \cap \mathcal{I}(\alpha_1, \cdots, \alpha_k)^2 \subset Z\ [\alpha_1, \cdots, \alpha_k]$. Here, $\mathcal{I}(u, v, \cdots)$ stands for the polynomial ideal generated by $u, v, \cdots$.

Proof Write again the system E_g for the g_i :

$$\begin{aligned} \dot{g}_1 &= \alpha_1\ g_1 \\ \dot{g}_2 &= \alpha_1\ g_2 + \alpha_2\ g_1^2 \\ &\vdots \\ \dot{g}_k &= \alpha_1\ g_k + P_k\ (\alpha_2, \cdots, \alpha_k\ ;\ g_1, \cdots, g_{k-1}). \end{aligned}$$

The polynomial P_k is obtained from the coefficient of u^k in the expansion $\sum_{j\geq 2} \alpha_j \Big[\sum_{i\geq 1} g_i\ u^i\Big]^j$. It follows easily from this, that P_k is homogeneous linear in $\alpha_2, \cdots, \alpha_k$.

Each polynomial :

$$g_1^{\ell_1} \cdots g_{k-1}^{\ell_{k-1}} \textit{ is such that } \sum_{j=1}^{k-1} \ell_j \geq 2 \textit{ and } \sum_{j=1}^{k-1} j.\ell_j = k. \tag{5.21}$$

First, we show $g_k(t) = e^{\alpha_1 t}\ Q_k(t)$ with Q_k a polynomial of degree $\leq k-1$ in Ω, and polynomial coefficients in $\alpha_1, \cdots, \alpha_k$ (in particular : $g_1(t) = e^{\alpha_1 t}$, $g_2(t) = \alpha_2\ e^{\alpha_1 t}\ \Omega, \cdots$).

Look at the equation for g_k :

$$\dot{g}_k = \alpha_1\ g_k + P_k\ (\alpha_2, \cdots, \alpha_k\ ;\ g_1, \cdots, g_{k-1}),$$

We use an induction on k. So let us suppose known that for each $j \leq k-1$: $g_j(t) = e^{\alpha_1 t}\ Q_j(t)$, as above. Notice that $e^{\alpha_1 t} = \alpha_1 \Omega + 1$. So, each g_j has degree $\leq j$ in Ω. Now, it follows from (5.23) that :

$$P_k(\alpha_2, \cdots, \alpha_k\ ;\ g_1, \cdots, g_{k-1}) = e^{2\alpha_1 t}\ X_k(\Omega),$$

where X_k is a polynomial of degree $\leq k-2$ in Ω (to see this point, replace in each monomial $g_1^{\ell_1} \cdots g_{k-1}^{\ell_{k-1}}$ of P_k, a product of two factors g_i g_j by $e^{2\alpha_1 t}\ Q_i\ Q_j$ and the other factors g_ℓ by $(\alpha_1 \Omega + 1) Q_\ell$).

Now, $g_k = e^{\alpha_1 t}\ Q_k$ with :

$$\begin{aligned} Q_k(t) &= \int_0^t e^{-\alpha_1 t}\ P_k\ (\alpha_2, \cdots, \alpha_k\ ;\ g_1, \cdots, g_{k-1}) d\tau \\ Q_k(t) &= \int_0^t e^{\alpha_1 \tau}\ X_k\ (\Omega) d\tau = \int_0^t X_k\ (\Omega)\ \dot{\Omega} d\tau \end{aligned} \tag{5.22}$$

(Because $\dot{\Omega} = e^{\alpha_1 t}$).

It follows from (5.22) that $Q_k(t)$ is a polynomial of degree $\leq k-1$ in Ω. From the induction, it follows easily that the coefficients are polynomials in $\alpha_1, \cdots, \alpha_k$. To obtain the precise form of the statement, notice that for $k \geq 2$:

$$P_k(\alpha_2, \cdots, \alpha_k \; ; \; g_1, \cdots, g_{k-1}) = \alpha_k \; g_1^k + \widetilde{P}_k$$

where $\widetilde{P}_k$ is linear in $\alpha_2, \cdots, \alpha_{k-1}$ and each monomial in $\widetilde{P}_k$ contains at least one of the g_i with $i \geq 2$. Moreover we know that the coefficients of such a g_i are in $\mathcal{I}(\alpha_1, \cdots, \alpha_i)$. So, the coefficients of $\widetilde{P}_k$ are in $\mathcal{I}(\alpha_1, \cdots, \alpha_{k-1}) \cap \mathcal{I}(\alpha_2, \cdots, \alpha_k)^2$.

Now :

$$Q_k = \alpha_k \int_0^t e^{(k-1)\alpha_1 \tau} \; d\tau + \int_0^t e^{-\alpha_1 \tau} \; \widetilde{P}_k(\tau) d\tau. \tag{5.23}$$

Look at the first term :

$$\int_0^t e^{(k-1)\alpha_1 \tau} \; d\tau = \frac{e^{(k-1)\alpha_1 t} - 1}{(k-1)\alpha_1}.$$

Using $e^{\alpha_1 t} = \alpha_1 \Omega(t) + 1$, we obtain :

$$e^{(k-1)\alpha_1 t} = 1 + (k-1)\alpha_1 \; \Omega + \alpha_1^2 \; S(\Omega) \tag{5.24}$$

where $S(\Omega)$ is a polynomial in Ω ; so that :

$$\alpha_k \int_0^t e^{(k-1)\alpha_1 \tau} \; d\tau = \alpha_k \; \Omega + \frac{\alpha_k \; \alpha_1}{k-1} \; S(\Omega). \tag{5.25}$$

The term $\int_0^t e^{-\alpha_1 \tau} \; \widetilde{P}_k \; d\tau$ gives a polynomial in Ω, with coefficients in $\mathcal{I}(\alpha_1, \cdots, \alpha_{k-1}) \cap \mathcal{I}(\alpha_2, \cdots, \alpha_k)^2$. So, we obtain finally $Q_k(t) = \alpha_k \; \Omega + \overline{Q}_k$ with $\overline{Q}_k$ as in the statement. ■

Now, we go back to the map $D_\alpha(x)$.

The time to go from σ to τ is equal to $t(x) = -Lnx$. Notice that $u_{|\sigma} = x^p$ and $u_{|\tau} = y^q$. So that we can compute $D_\alpha(x)$ using $u(t,x)$:

$$D_\alpha(x)^q = u(-Lnx, \; p) \; for \; x \geq 0 \; with D_\alpha(0) = 0 \tag{5.26}$$

(and $D_\alpha(0) = 0$).

There is no problem to see that D_α is well defined for $x \in [0, X]$ where X is some value > 0, and is analytic in (x, α) for $x \neq 0$, $\alpha \in A$. We want to study its behavior at $x = 0$. For this, we notice that the lemma 5 implies that for each $t > 0$, the convergence radius of the series $\Sigma\ g_i(t)u^i$ is greater that $\frac{1}{C}\ e^{-\frac{1}{2}t}$. So, for any x small enough, the series $\sum_i g_i(t)x^{p_i}$ converges for each $t < -2Lnx$ and in particular for $t = -Lnx$. So, we can use this series to compute $D_\alpha(x)$:

$$D_\alpha(x)^q = \sum_{i=1}^{\infty} g_i(-Lnx)x^{pi}. \tag{5.27}$$

The convergence is normal on an interval $[0, X]$ for some $X > 0$. We can now use estimates on g_i, $\frac{d^k g_i}{dt^k}$ in lemmas 1,2 to obtain the following :

Proposition 11 *Let any $k \in N$. There exists a $K(k)$ such that :*

$$D_\alpha(x)^q = \sum_{i=1}^{K(k)} g_i(-Lnx)x^{pi} + \psi_k \tag{5.28}$$

where $\psi_k(x, \alpha)$ is a C^{kp} function in (x, α), kp-flat at $x = 0$ (i.e : $\psi_k(0, \alpha) = \frac{\partial \psi_k}{\partial x}\ (0, \alpha) = \cdots = \frac{\partial^{kp} \psi_k}{\partial x^{kp}}\ (0, \alpha) = 0$).

Proof Given k, we want to find $K(k)$ such that :

$$(D_\alpha^q)^K\ (x) = \sum_{K+1}^{\infty} g_i(-Lnx)x^i$$

is a C^{Kp}, Kp-flat function.

We are going to see that the series $(D_\alpha^q)^K$ can be derived term by term. First, we have :

$$\frac{d}{dx}[g_j(-Lnx)x^{pj}] = -g_j^{(1)}(-Lnx)x^{pj-1} + pjq_j(-Lnx)x^{pj-1} \tag{5.29}$$

$\left(g_j^{(1)} = \frac{dg_j}{dx}\right)$.

Now, from the estimates of Lemma 2, we have :

$$\left|g_j^{(1)}(-Lnx)\right| \leq C_1 \mid Cx \mid^{-\partial/2}$$

and from lemma 1 :

$$\mid g_j(-Lnx) \mid \leq C_0 \mid C_1 \mid^{-j/2} .$$

So, for some constant $M_1 > 0$, we have :

$$\left|\frac{d}{dx} [g_j(-Lnx)x^{pj}]\right| \leq jM_1 \mid C.x \mid^{(p-\frac{1}{2})j-1} . \tag{5.30}$$

More generally, using lemma 2, we have, for each $s \leq j$:

$$\left|\frac{d^s}{dx^s} [g_j(-Lnx)x^{pj}]\right| \leq \frac{j!}{(j-s)!} M_s \mid C.x \mid^{(p-\frac{1}{2})j-s} \tag{5.31}$$

for some constant $M_s > 0$.

It follows from these estimates that if $(p-\dfrac{1}{2})K > k$ and if $0 \leq s \leq k$, the series :

$$\sum_{j \geq K+1} \frac{d^s}{dx^s} \left(g_j(-kn(x)x^{pj}\right)$$

converges and is equal to 0 at $x = 0$. So, we obtain that the function $(D_\alpha^q)^K = \sum_{j \geq K+1} \cdots$ is k-flat and C^k. ∎

We define the function $\omega(x, \alpha_1)$ by :

$$\begin{aligned} \omega(x, \alpha_1) &= \frac{x^{-\alpha_1} - 1}{\alpha_1} \quad if \ \alpha_1 \neq 0 \\ \omega(x, 0) &= -Lnx. \end{aligned} \tag{5.32}$$

This function is related with the above function $\Omega(\alpha_1, t)$ by :

$$\omega(x, \alpha_1) = \Omega(\alpha_1, -Lnx). \tag{5.33}$$

Note that for each $k > 0$, $x^k\omega \to -x^k \ Lnx$ as $\alpha_1 \to 0$ (uniformly for $x \in [0, X]$, for any fixed $X > 0$). We are going to consider finite combinations of functions $x^i\ \omega^j$ with $i, j \in N$ and $0 \leq j \leq i$. These functions $x^i\ \omega^j$ form a totally ordered set with the following order

$$x^i\omega^j \prec x^{i'}\omega^{j'}$$

$\Longleftrightarrow i' > i$ or $i = i'$ and $j > j'$. One has : $1 \prec x\omega \prec x \prec x^2\omega^2 \prec x^2\omega \prec x^2 \prec \cdots$.

The notation $x^i\omega^j + \cdots$ means that after the sign + one finds a finite combination of $x^{i'}\omega^{j'}$. Then, we have for the transition map $D_\alpha(x)$, the *following* (x, ω)*-expansion of order* k *:*

Theorem 14 *Let any* $k \in N$*. Then, the transition map* D_α *of* X_α^N *defined by* (5.28) *(relative to the segments* σ*,* τ *defined above) has the following* (x, ω)*-expansion of order* kp *:*

$$(D_\alpha(x))^q = x^p + \alpha_1[x^p\omega + \cdots] + \alpha_2[x^{2p}\omega + \cdots] + \cdots$$
$$+ \alpha_k[x^{Kp}\ \omega + \cdots] + \psi_K(x, \alpha) \qquad (5.34)$$

where $K(k)$ *is defined in proposition (5.28), each term between brackets is a finite combination of* $x^i\omega^j$ *(with the above convention) ; the coefficients of the non written* $x^i\omega^j$ *after the signs + are polynomial functions in the* α_s*, which are zero if* $\alpha = 0$*. The remaining term* ψ_k *is a* C^{kp} *function in* (x, α)*, which is kp-flat for* $x = 0$ *and any* α *:*

$$\left(\psi_k(0, \alpha) = \frac{\partial \psi_k}{\partial x}\ (0, \alpha) = \cdots = \frac{\partial^{kp}}{\partial x^{kp}}\ \psi_k(0, \alpha) = 0\right).$$

Proof The proposition 3 gives :

$$g_k(-Lnx) = e^{-\alpha_1 Lnx}\ Q_k(-Lnx)$$
$$= x^{-\alpha_1}\ \left(\alpha_k\omega + \bar{Q}_k(\alpha_1, \cdots, \alpha_k, \omega)\right) \qquad (5.35)$$

with $\overline{Q}_k$ of degree $\leq k - 1$ in ω and coefficients in $\mathcal{I}(\alpha_1, \cdots, \alpha_{n-1}) \cap \mathcal{I}(\alpha_2, \cdots, \alpha_k)^2$.

So, the general term $g_k(-Lnx)x^{pk}$ in $(D_\alpha(x))^q$ is equal to :

$$g_k(-Lnx)x^{pk} = x^{pk-\alpha_1}\ (\alpha_k\ \omega + \overline{Q}_k). \qquad (5.36)$$

Using $x^{-\alpha_1} = \alpha_1\omega + 1$ this term can be rewritten as :

$$g_k(-Lnx)x^{pk} = \alpha_k x^{pk}\omega + \alpha_1\alpha_k\ x^{pk}\ \omega^2 + (1+\alpha_1\omega)x^{pk}\ \overline{Q}_k \qquad (5.37)$$

$$for\ k \geq 2\ and\ x^p g_1(-Lnx) = x^{p-\alpha_1} = \alpha_1\ x^p\ \omega + x^p. \qquad (5.38)$$

So, we have :

$$(D_\alpha(x))^q = x^p + \alpha_1\ x^p\ \omega + \alpha_2\ x^{2p}\ \omega + \alpha_1\ \alpha_2\ x^{2p}\ \omega^2 + (1+\alpha_1\ omega)x^{2p}\ \overline{Q}$$

$$+\ \alpha_3\ x^{3p}\ \omega + \alpha_1\ \alpha_3\ x^{3p}\ \omega^2 + x^{3p}(1+\alpha_1\ \omega)\overline{Q}_3 + \cdots + \psi_K \qquad (5.39)$$

where $+\cdots$ is for the expansion of $x^{ps}g_s(-Lnx)$ for $4 \leq s \leq K(k)$. Now we rearrange the sum $\sum_{i=1}^{K(k)} g_i(-Lnx)x^{pi}$ in the following way : first, we take all the terms whose coefficients are divisible by α_1. Next, all the remaining terms (not divisible by α_1) but divisible by α_2 and so on, until α_K. We obtain the following expansion :

$$\begin{aligned} D_\alpha(x)^q = x^p\ &+\ \alpha_1[x^p\ \omega + \alpha_2\ x^{2p}\ \omega + x^{2p}\ \omega\overline{Q}_2 + \alpha_3\ x^{3p}\ \omega^2 + \cdots] \\ &+\ \alpha_2\ [x^{2p}\ \omega + terms\ in\ x^{3p}\ \overline{Q}_3, \cdots, x^{Kp}\ \overline{Q}_K \\ &\qquad\qquad divisible\ by\ \alpha_2,\ not\ by\ \alpha_1] \\ &\ \vdots \\ &+\ \alpha_K\ x^{Kp}\ \omega + \psi_k(x,\alpha). \end{aligned}$$

Looking at this expansion, it is clear that each term after $x^{sp}\,\omega$ in the bracket related to α_s is of order greater that $x^{sp}\,\omega$ and has a coefficient in $\mathcal{I}(\alpha_1,\cdots,\alpha_K)$ (because it comes from coefficients in $\mathcal{I}(\alpha_1,\cdots,\alpha_K)^2$ divided by α_s). The sum from 1 to K contains all monomial terms in x,ω coming from the expansion $\sum_{i=1}^{K} g_i(-Lnx)x^{ip}$ and we know that the remaining term ψ_k obtained in proposition 4 is C^k and k-flat at $x=0$. This ends the proof. ∎

5.1.3 The structure of the transition map of X_λ.

We return now to the initial C^∞ family X_λ. We suppose chosen a fixed system of coordinates (x, y) for which the saddle point s_λ is at the origin, the $0x$ and $0y$ axes are local unstable and stable manifolds of X_λ respectively, for each $\lambda \in W_1$ and the 1-jet of X_λ is given by (5.1).

Take now transversal segments σ, τ to $0y$ and $0x$ respectively : σ is parametrized by $x \in [0, X]$ and τ by $y \in [-Y, Y]$ for some $X, Y > 0$. For any $k \in N$, theorem 1 gives a C^k equivalence of X_λ with a polynomial normal form family $X^N_{\alpha(\lambda)}$; here $\alpha(\lambda) = (\alpha_1(\lambda), \cdots, \alpha_N(\lambda))$ and the family X^N_α are defined in section 1.1. This C^k equivalence defines C^k families of diffeomorphisms $\Phi_\lambda(x)$, $\psi_\lambda(y)$ on R, in a neighborhood s of $\Phi_\lambda(0) = \psi_\lambda(0) = 0$ such that if D_α is the transition map for X^N_α, as it is defined in section 1.2, one has :

$$D_\lambda(x) = \psi_\lambda \circ D_{\alpha(\lambda)} \circ \Phi_\lambda(x). \tag{5.40}$$

X^N_α is a linear vector field if $r(\lambda_0) \notin Q$ and the polynomial normal form family given in proposition 2 if $r(\lambda_0) = \frac{p}{q}$. In this last case, we apply to it the results of section 1.2 in particular in theorem 2. For any k, we have an (ω, x)-expansion at order k which depends only on $\alpha_1(\lambda), \cdots, \alpha_N(\lambda)$ because all the $\alpha_i(\lambda) \equiv 0$ for $i \geq N+1$:

$$\begin{aligned}(D_{\alpha(\lambda)}\ (x))^q &= x^p + \alpha_1(\lambda)[x^p\omega + \cdots] + \cdots \\ &+ \alpha_N(\lambda)[x^{Np}\ \omega + \cdots] + \psi_k\ (x, \lambda)\end{aligned} \tag{5.41}$$

where the conventions are the ones in theorem 2 and $\psi_k(x, \lambda)$ is C^k and k-flat at $x = 0$. Of course, if the resonant quantities may be chosen independent of k, it is not the case for the expansions in the brackets.

5.1.3.1 Dulac Series for D_{λ_0}.

In this subsection, I want to verify that the transition map near an hyperbolic saddle is quasi-regular (we have used this fact in chapter 3). To expand the transition $D(x)$ for the saddle point of a vector field X, we can use formulas (5.40) and (5.41) for a trivial family (X_λ constant and equal to X and $D_{\lambda_0} = D$). Then $\alpha_1 = 0$ and $\omega = -Ln\ x$.

If $r = r(0) \notin Q$ we have $D(x) = \psi \circ D_0 \circ \varphi(x)$ for ψ, φ C^k diffeomorphisms and $D_0(x) = x^r$.

If $r = \dfrac{p}{q}$:

$$D_0(x)^q = x^p + \alpha_2\, x^{2p}(-Ln\ x) + \cdots + \alpha_N\, x^{Np}\,(-Ln\ x) + \psi_k(x)$$

$\psi_k : C_1^k$ k-flat. Expanding $\psi \circ D_0 \circ \varphi$ and ordering the terms, we obtain that, for any k, there exists a sequence of coefficients λ_i : $\lambda_1 = r < \lambda_2 < \cdots < \lambda_{N(k)}$, $\lambda_{N(k)} \geq k$ and a sequence of polynomials $P_1 = A$ (a constant),..., $P_{N(k)}$ such that :

$$D(x) = \sum_{i=1}^{N(k)} x^{\lambda_i}\, P_i(Ln\ x) + \psi_k(x)$$

where ψ_k is a C^k and k-flat function.

The coefficients λ_i, and the polynomials P_i are well-defined, i.e. independent on k. This means that if we take $k' > k$, the sequence for k', truncated at order $N(k)$. This is similar to the unicity of Taylor series. Taking k arbitrarily large, we have a well defined infinite series $\widehat{D}(x) = \sum_{i=1}^{\infty} x^{\lambda_i}\, P_i(x)$ which is asymptotic to $D(x)$ is the following way :

- for any $k \in N - \{0\}$:

$$\mid D(x) - \sum_{i=1}^{s} x^{\lambda_i}\, P_i(Ln\ x) \mid = 0(x^{\lambda_s})$$

with $\lambda_1 = r < \lambda_2 < \cdots < \lambda_s < \cdots$ an infinite sequence of positive coefficients tending to ∞ and $P_1 = P_2, \cdots, P_s, \cdots$ an infinite sequence of polynomials.

The series $\widehat{D}$ is called : the *Dulac series* of the map D.

A C^∞ function on $]0, X[$, which admit at $x = 0$ a series as above is said to be *quasi-regular.*

Remark 24 *For the Dulac series of the transition, we have noticed that* $\lambda_1 = r$*, the hyperbolicity ratio, and* $P_1 = 1$ *so that* $D(x) = Ax^r + 0(x^r)$*. It is also easy to verify that :* $\lambda_i \in N + rN$ *for all i, and that* P_i *is constant for any i when* $r \notin Q$*. (Logarithmic terms just occur when one has a resonant saddle).*

5.1.3.2 $D_\lambda(x)$ when $p = q = 1$.

For the study in the next paragraph, we want to write now an expansion for $D_\lambda(x)$ in the case $p = q = 1$. So, we have to compute (5.40). To this end we need :

Lemma 21 *Let $\varphi_\lambda(x)$ a C^k parameter family of diffeomorphisms as above. Then, with the convention introduced in section 1.2 :*

$$\omega \circ \varphi_\lambda = c(\lambda)\omega + \cdots + \xi_k(x,\lambda) \tag{5.42}$$

where $c(\lambda) > 0$ for $\lambda \in W_1$ and ξ_k is C^k in (x,λ) and k-flat at $x = 0$ uniformly in $\lambda \in W_1$.

Proof Let :

$$\tilde{\omega} = \omega \circ \Phi_\lambda \text{ and } \Phi_\lambda(x) = u(\lambda)x(1 + \cdots + \tilde{\Phi}_k) \tag{5.43}$$

(again with the above convention) and $u(\lambda) > 0$.

$$\begin{aligned}\tilde{\omega} &= \frac{u^{-\alpha_1}\ x^{-\alpha_1}(1 + \cdots \tilde{q}_k)^{-\alpha_1} - 1}{\alpha_1} \\ &= u^{-\alpha_1}\ \frac{x^{-\alpha_1}\ (1 + \cdots + \tilde{\Phi}_k)^{-\lambda_1} - 1}{\alpha_1} + \frac{u^{-\alpha_1} - 1}{\alpha_1}\end{aligned}$$

$\varphi(\lambda) = \dfrac{u^{-\alpha_1} - 1}{\alpha_1}$ is a C^∞ function of λ and :

$$\frac{x^{-\alpha_1}(1 + \cdots + \tilde{\Phi}_k)^{-\alpha_1} - 1}{\alpha_1} = x^{-\alpha_1}\ \frac{(1 + \cdots + \tilde{\Phi}_k)^{-\alpha_1} - 1}{\alpha_1} + \omega$$

$\psi = \dfrac{(1 + \cdots \tilde{\Phi}_k)^{-\alpha_1} - 1}{\alpha_1}$ is a C^k function in (x,λ) and $x^{-\alpha_1} = \alpha_1\ \omega + 1$.
One obtains finally : $\tilde{\omega} = u^{-\lambda_1}(1 + \psi(x,\lambda))\omega + \cdots$ which has the desired form, once expanded. ∎

If we substitute $\Phi_\lambda(x)$ in $D_{\alpha(\lambda)}(x) = x + \alpha_1[x\omega + \cdots] + \cdots$ and use the above lemma, it is clear that we obtain a similar expansion as (5.34), but with new coefficients α_i which are now of class C^k.

Next, if :

$$\psi_\lambda(y) = \gamma_1(\lambda)y + \gamma_2(\lambda)y^2 + \cdots + \gamma_k(\lambda)y^k + 0(y^{k+1}) \qquad (5.44)$$

we obtain $\varphi_\lambda \circ D_{\alpha(\lambda)} \circ \Phi_\lambda$ substituting $D_{\alpha(\lambda)} \circ \varphi_\lambda$ in (5.44). It is clear that we can reorder the terms of this expansion to obtain again a similar expansion as (5.34). We have proved :

Proposition 12 *Let X_λ, σ, τ as above and $D_\lambda(x)$ the transition map of σ to τ. Let W_1 the neighborhood of λ_0 such that $D_\lambda(x)$ is defined from $\sigma \times W_1$ to τ. Then, there exists a sequence of neighborhoods of $\lambda_0 : W_1 \supset W_2 \supset \cdots \supset W_k \supset \cdots$ such that for all $k \in N$ there exist C^k functions $\alpha_1^k(\lambda), \cdots, \alpha_k^k(\lambda)$ and an expansion for $(x, \lambda) \in W_k$:*

$$D_\lambda(x) = x + \alpha \sum_{i=1}^{k} \alpha_i^k(\lambda)[x^i\ \omega + \cdots] + \psi_k(x, \lambda) \qquad (5.45)$$

with the conventions as in theorem 2. Here $\alpha_1^k \equiv \alpha_1 = r(\lambda) - 1$ for any k.

5.1.3.3 Mourtada's form for D_λ

The expression (5.45) will be used to study unfoldings of homoclinic saddle loop (see next paragraph). To study hyperbolic polycycles with more than 1 singular point, A. Mourtada has introduced a simpler expression, which is valid without assumption on $r(\lambda_0)$.

We consider transversal segments $\sigma = [0, X]$, $\tau = [-Y, Y]$ as above and let be $D_\lambda(x) : \sigma \times W_0 \to \tau$ the transition map.

Definition 25 *(1) Let $W_k \subset W_0$ some neighborhood of λ_0 and I_k be the set of functions $f : [0, X] \times W_k \to R$ with the following properties :*

(i) f is C^∞ on $]0, X] \times U_k$.

(ii) For each $j \le k$, $\varphi_j(x, \lambda) = x^j \dfrac{\partial^j f}{\partial x^j}(x, \lambda) \to 0$ for $x \to 0$, uniformly on λ (we will say that $\dfrac{\partial^s f}{\partial x^j} = o(x^{-j})$ uniformly at $\lambda = 0$).

(2) A function $f : [0, X] \times W_0 \to R$ is said to be of class I if f is C^∞ on $]0, X] \times W_0$ and if for all k there exists $W_k \subset W_0$, neighborhood of $\lambda_0 \in P$ such that f is of class I_k on W_k.

Theorem 15 *(Mourtada [M1]). Let X_λ, σ, τ, D_λ as above. Then, for $(x,\lambda) \in \sigma \times W_0$:*

$$D(x,\lambda) = x^{r(\lambda)} \left(A(\lambda) + \Phi(x,\lambda)\right) \tag{5.46}$$

with $\Phi \in I$ and $A(\lambda)$ a $\mathcal{C}^\infty$ positive function.

Proof I sketch briefly the ideas of the proof. The details can be found in [M1]. First, let us notice that,given σ and τ with their parametrizations, the functions A and Φ in formula (5.46) are *unique.*
This follows from the fact that $r(\lambda)$ is well defined and that :

$$A(\lambda) = \lim_{x\to 0} x^{-r(\lambda)}\ D(x,\lambda)$$

and also that : $\Phi(x,\lambda) = x^{-r(\lambda)}\ D(x,\lambda) - A(\lambda)$.

Next, let us notice that a function has an expression as in (5.46), with $A(\lambda)$ a $\mathcal{C}^k$ function and $\Phi \in I_k$ if and only if this is true after compositions on the right and on the left by $\mathcal{C}^k$ families of diffeomorphisms $\phi_\lambda(x)$ and $\psi_\lambda(x)$, with $\phi_\lambda(0) \equiv \psi_\lambda(0) = 0$ for all $\lambda \in W_0$.

So it suffices to prove that, in $\mathcal{C}^k$-normal form coordinates :

$$D_{\alpha(\lambda)}(x) = x^{r(\lambda)} \left(B(\lambda) + \psi(x,\lambda)\right) \tag{5.47}$$

with $B(\lambda)$ a $\mathcal{C}^k$ function and $\psi \in I_k$.

To prove this, we consider two cases :

(i) if $r(\lambda_0)$ is irrational, then $D_{\alpha(\lambda)}(x) = x^{r(\lambda)}$ and the result is trivial,

(ii) if $r(\lambda_0) = \dfrac{p}{q}$, we apply theorem 2, at some order $k' >> k$. In fact, we have to notice that the first bracket begins by a monomial $x^{sp}\ \omega^\ell$ with $s \geq 2$, $\ell \leq p$. This is also the case for the other brackets.

So that, we can write for any k' :

$$(D_{\alpha(\lambda)}(x))^q = x^p + \alpha_1(\lambda) x^{p\omega} + \sum_{\substack{\ell \leq s \leq K \\ s \geq 2}} \alpha_{s\ell}\ (\lambda) x^{sp}\ x^\ell + \psi_K(x,\lambda) \tag{5.48}$$

where the $\alpha_{s\ell}(\lambda)$ are $\mathcal{C}^{k'}$, $K(k') \in N$ and ψ_K is $\mathcal{C}^{k'}$ and k'-flat at $x = 0$.

Let us notice that $x^p + \alpha_1(\lambda)x^{p\omega} = x^{qr(\lambda)}$. We can rewrite (5.48) in the following form :

$$D_{\alpha(\lambda)}(x) = x^{r(\lambda)}\Big[1 + \sum_{\substack{\ell \le s \le K \\ s \ge 2}} \alpha_{s\ell}(\lambda)x^{sp-qr(\lambda)}\ \omega^\ell + x^{-qr(\lambda)}\ \psi_K(x,\lambda)\Big]^{\frac{1}{r(\lambda)}}. \tag{5.49}$$

Choosing k' large enough, $x^{-qr(\lambda)}\ \psi_K$ is C^k and k-flat at $x = 0$.
Now, $sp - qr(\lambda) \geq q\ \left(2\ \frac{p}{q} - r(\lambda)\right)$. For $\lambda = \lambda_0 : q\left(2\ \frac{p}{q} - r(\lambda_0)\right) = 2p$.
If we take $\lambda \in W_k$, some small neighborhood of λ_0, there exists a $c > 0$ such that $q\left(2\ \frac{p}{q} - r(\lambda)\right) > c$.
Expansion of the power $\frac{1}{q}$ of the bracket in (5.49) gives the desired form for $D_{\alpha(\lambda)}\ (x)$. ∎

Definition 26 *We call Mourtada's form, the expression (5.46) of the transition map and call $\mathcal{D}_k$ (respectively $\mathcal{D}$) the class of maps as in (5.46), when $\Phi \in I_k$ (resp.I).*

The importance of the classes I, and $\mathcal{D}$ comes from the following theorem, which is easily proved by direct computations :

Theorem 16 *(Mourtada [M1]).*

(i) I is an algebra : for f, g $\in I$, a,b $\in R$, we have $f.g \in I$ and $af + bg \in I$.

(ii) If $f \in I$ and $g \in \mathcal{D}$ then $f \circ g \in I$.

(iii) Maps of class $\mathcal{D}$ can be composed. More precisely if $D_{1,\lambda}(x) = D_1(x,\lambda) = x^{r_1(\lambda)}\ (A_1(\lambda) + \Phi_1(x,\lambda))$ and $D_{2,\lambda}\ (x) = D_2(x,\lambda) = x^{r_2(\lambda)}$ $(A_2(\lambda) + \Phi_2(x,\lambda))$, then : $D_{2,\lambda} \circ D_{1,\lambda}(x) = D_{3,\lambda}(x) = x^{r_3(\lambda)}(A_3(\lambda) + \Phi_3(x,\lambda))$ with $r_3(\lambda) = r_2(\lambda).r_1(\lambda)$, $A_3(\lambda) = A_2(\lambda).A_1(\lambda)^{r_2(\lambda)}$ and $\Phi_3 \in I$. Hence, $D_{3,\lambda} \in \mathcal{D}$.

(iv) If $D_\lambda \in \mathcal{D}$ then $D_\lambda^{-1} \in \mathcal{D}$.

(v) I is closed under the derivation $x\,\dfrac{d}{dx}$ *: if* $f \in I$ *then* $x\,\dfrac{df}{dx} \in I$.
As a consequence, if $D_\lambda = x^{r(\lambda)}(A(\lambda)+\Phi(x,\lambda))\mathcal{D}$
then :

$$\frac{\partial D_\lambda}{\partial x} = r(\lambda)x^{r(\lambda)-1}(A(\lambda)+\psi(x,\lambda)) \text{ with } \psi \in I. \tag{5.50}$$

(vi) Any smooth germ is in I; any smooth diffeomorphism germ g at 0, with $g(0)=0$ *is in* $\mathcal{D}$.

Remark 25 *The formula (5.50) is true even if* $r(\lambda_0) = 1$. *But, if* $r(\lambda_0) \neq 1$, *if we restrict* W_0 *such that* $r(\lambda) \neq 1$ *for all* $\lambda \in W_0$, *we will have that* $x\,\dfrac{\partial D_\lambda}{\partial x} \in \mathcal{D}$.

More generally $x^n\,\dfrac{\partial^n D_\lambda}{\partial x^n} \in \mathcal{D}$ *for* W_0 *small enough if* $r(\lambda_0) \neq n$.

It is for this reason that the Mourtada's form will be more useful under generic assumption : $r(\lambda_0) \neq n \leq N$ *for N large enough.*

5.2 Unfoldings of saddle connections in the finite codimension case.

Let X_λ be a $\mathcal{C}^\infty$ family of vector fields in S such that X_{λ_0} has some saddle connection Γ. We want to study the unfolding defined by X_λ along $\Gamma \times \{\lambda_0\}$ so that we can suppose that X_λ is restricted to some neighborhood $\mathcal{U}$ of Γ in S, diffeomorphic with an annulus (S is supposed to be orientable) and $\lambda \in W_0$ some neighborhoods of λ_0 in the parameter space. Let σ, τ some transversal sections near the saddle point s of X_{λ_0}, as above : we suppose chosen a local system of coordinates (x,y) near s, such that $s=(0,0)$ is the saddle point of X_λ for all $\lambda \in W_0$, $0x$, $0y$ are the unstable and stable local manifolds and σ, τ are parametrized respectively by $x \in [0,X[$ and $y \in]-Y,Y[$. We suppose chosen a section $\sigma' \supset \sigma$, parametrized by $]-X',X[$.

Let V this chart with coordinates (x,y) (see Figure 5.2)

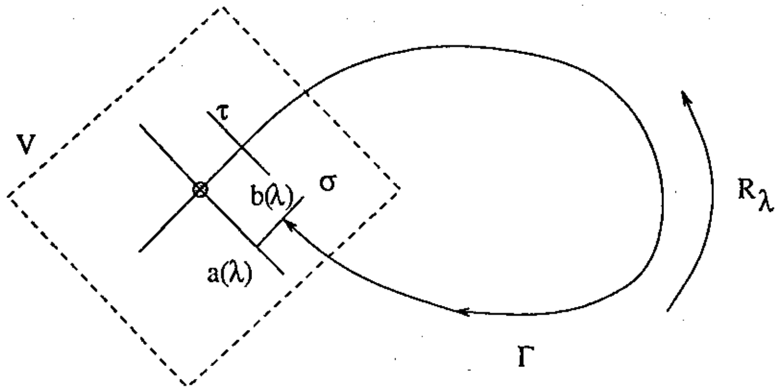

Figure 5.2:

The Poincaré map $P_\lambda(x) : \sigma \times W_0 \to \sigma' =]-X', X[$ may be obtained as the composition :

$$P_\lambda(x) = R_\lambda^{-1} \circ D_\lambda(x) \tag{5.51}$$

where R_λ is the $\mathcal{C}^\infty$ regular transition map from σ' to τ for $-X_\lambda$ (we suppose chosen W_0 and X small enough such that P_λ is defined on $\sigma \times W_0$).

5.2.1 The codimension 1 case.

As generic assumption for codimension 1 bifurcations we can suppose that $r(\lambda_0) \neq 1$. Using notations and results of section 1.3, $T_\lambda(\lambda) = R_\lambda^{-1} - \beta(\lambda)$, where $\beta(\lambda) = R_\lambda^{-1}(0)$, and D_λ are in $\mathcal{D}$. This gives :

$$P_\lambda(x) = x^{r(\lambda)}\ (A(\lambda) + \Phi(x,\lambda)) + \beta(\lambda),\ \textit{with}\ \Phi \in I \tag{5.52}$$

with $r(\lambda_0) \neq 1$ and $A(\lambda_0) > 0$.

It follows from theorem 3 that :

$$\frac{\partial P_\lambda}{\partial x} = r(\lambda) x^{r(\lambda)-1}\ (A(\lambda) + \Phi_1(x,\lambda)). \tag{5.53}$$

So that $\frac{\partial P_\lambda}{\partial x} \to 0$ or ∞ for $x \to 0, \infty$, uniformly in $\lambda \in W_0$ depending if $r(\lambda_0) > 1$ or $r(\lambda_0) < 1$, and from Rolle theorem, the equation $\{P_\lambda(x) - x = 0\}$ has at most one root in σ for any $\lambda \in W_0$, if σ is small enough.

Now, the roots x with $| x |$ sufficiently small corresponds to periodic orbits whose Hausdorff distance to Γ is sufficiently small. This is a quite obvious generalization of the lemma 1 in 4.1, that we formulate now without proof for general limit periodic sets :

Lemma 22 *Let Γ any limit periodic set for a family X_λ at the parameter value λ_0. Let σ, σ' transversal sections to Γ as above. Let $P_\lambda(x)$ the Poincaré map of X_λ from σ to σ' (we suppose that $\lambda \in W_0$ some neighborhoods of λ_0 in the parameter space).*

Let $\delta_\lambda(x) = P_\lambda(x) - x$. Then, for each $\varepsilon > 0$, one can find $\sigma(\varepsilon)$, a neighborhoods of $x_0 \in \Gamma \cap \sigma$ in σ such that : $x \in \sigma(\varepsilon)$ is a root of $\{\delta_\lambda(x) = 0\}$ for $\lambda \in W_0$ if and only if the orbit γ of X_λ through x is a periodic orbit with $d_H(\gamma, \Gamma) \leq \varepsilon$ (d_H : Hausdorff distance corresponding to a chosen distance on the phase space).

So, the computation of the cyclicity of Γ is equivalent to the computation of the number of roots of δ_λ, on $\sigma(\varepsilon)$ for ε and W_0 small enough.

Here, for the saddle connection, the periodic orbits cutting σ' *must cut* σ. The fact that $\delta_\lambda(x) = P_\lambda(x) - x$ has at most one root on $W_0 \times \sigma$ implies :

Proposition 13 *Let be Γ a saddle connection as above, with $r(\lambda_0) \neq 1$. Then $Cycl(X_\lambda, \Gamma) \leq 1$.*

In fact one can deduce from (5.52) a more precise result. It is always possible to construct a C^∞ 1-parameter family $\widetilde{X}_\beta$, near $\Gamma \times \{0\}$ with turn map :

$$\widetilde{P}_\beta(x) = x^r + \beta \tag{5.54}$$

$(r-1)(r(\lambda_0) - 1) > 0$. For instance, one can take a fixed linear vector field in the coordinate chart with transition map : $\widetilde{D}(x) = x^r$ and glue this chart with a second chart near a regular arc on $\Gamma - \{s\}$ such that the transition $\widetilde{R}^{-1}(y) = y + \beta$ (see [IY1]). Now, it is not difficult to

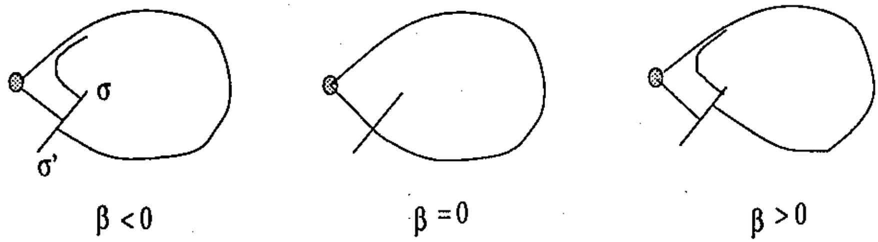

Figure 5.3:

prove that for all λ the two maps $P_\lambda(x)$ and $\tilde{P}_{p(\lambda)}(x)$ are $C°$-conjugate. Next, one can extend this conjugacy in an equivalence between $\widetilde{X}_{p(\lambda)}$ and X_λ for each $\lambda \in W_0$ in some neighborhoods of Γ . It is even possible but more difficult to obtain an equivalence depending continuously on λ [AAD] :

Theorem 17 *Let $\widetilde{X}_\beta$ an unfolding of saddle connection with return map $\tilde{P}_\beta(x) = x^r + \beta$ where $(r-1)(r(\lambda_0)-1) > 0$. Then, the unfolding (X_λ, Γ) is induced by the map $\beta(\lambda)$, up to $(C°, C°)$-equivalence ($\widetilde{X}_\beta$ is a versal unfolding of Γ).*

The bifurcation diagram of $\widetilde{X}_\beta$ is quite simple. One can suppose that $r > 1$ for instance (if not, change $\widetilde{X}_\beta$ in $-\widetilde{X}_\beta$) : a hyperbolic stable limit stable bifurcates from Γ for positive β (see Figure 5.3).

Of course, if X_λ is already a 1-parameter family such that $\beta'(\lambda_0) \neq 0$, one can replace λ by β, up a diffeomorphism in the parameter space, and the diagram of bifurcation of X_λ is the same as the one of $\widetilde{X}_\beta$:

Theorem 18 *Let (X_λ, Γ) a 1-parameter unfolding of saddle connection Γ, for the value λ_0, which verifies the generic assumptions : $r(\lambda_0) \neq 1$ and $\beta'(\lambda_0) \neq 0$.*

Then, the unfolding (X_λ, Γ) is $(C°, C°)$-equivalent to the "model" $(\tilde{X}_\beta, \Gamma)$. In particular, this unfolding is unique up to $(C°, C°)$ equivalence and change X_λ in $-X_\lambda$) and is structurally stable.

5.2.2 The k-codimension case, $k \geq 2$.

Up to now we suppose the $r(\lambda_0) = 1$. A saddle connection with this condition is of course of codimension greater than 2 (one condition is needed to express the connection and another one is $r(\lambda_0) = 1$). To make precise the notion of codimension, we will used the so-called Dulac expansion for the return map $P(x) = P_{\lambda_0}(x)$ for X_{λ_0} along Γ.

Using proposition 5 for D_λ and the Taylor expansion of $R_\lambda(x)$ we have for

$$\delta_\lambda(x) = P_\lambda(x) - x,$$

at any order $k \in N$:

$$\begin{aligned}\delta_\lambda(x) = \beta_0(\lambda) \; &+ \; \alpha_1(\lambda)[x\omega + \cdots] \\ &+ \; \beta_1(\lambda)x + \alpha_2(\lambda)[x^2\omega + \cdots] + \cdots \\ &+ \; \beta_{k-1}(\lambda)x^{k-1} + \alpha_k(\lambda)[x^k\omega + \cdots] + \cdots \psi_k(x, \lambda)\end{aligned} \tag{5.55}$$

$$\beta_0(\lambda) = P\lambda(0) = b(\lambda) - a(\lambda)$$

where $a(\lambda)$, $b(\lambda)$ are the just intersections of the unstable and the stable manifold of s with σ'.
$\alpha_1(\lambda) = 1 - r(\lambda)$, where $r(\lambda)$ is the hyperbolicity ratio. We have $\alpha_1(0) = p_0(0) = 0$.

The functions β_i come from the Taylor expansions of $R_\lambda(x)$ and are C^∞, but the $\alpha_i(\lambda)$ come from the formula for D_λ depend in general of k and are just C^k.

Taking $\lambda = \lambda_0$ in the formula (5.55), we obtain an expansion of $\delta_{\lambda_0}(x)$:

$$\delta_{\lambda_0}(x) = \beta_1 x + \alpha_2 x(-Lnx)^2 + \cdots$$

$$+ \beta_{k-1}\, x^{k-1} + \alpha_k\, x^k(-Lnx) + 0(x^k). \tag{5.56}$$

We notice that the coefficients β_i, α_j we obtain in this way are independent of k : if we write a similar expansion at order $k' \geq k$ for $\delta_{\lambda_0}(x)$, then the coefficients α_i, β_j for $i \leq k$, $j \leq k-1$ are the same. So, using expansion (5.56) at any order k, we obtain a well defined series :

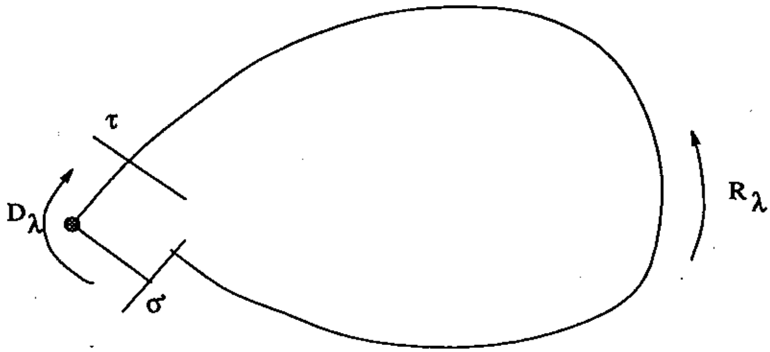

Figure 5.4:

$$\widehat{\delta}_{\lambda_0}(x) = \sum_{i=1}^{\infty} \Big(\beta_i \; x + \alpha_{i+1} \; x^{i+1} \; (-Lnx)\Big) \qquad (5.57)$$

which is asymptotic to $\delta_{\lambda_0}(x)$ in the following way for any $k \in N - \{0\}$:

$$\Big|\delta_{\lambda_0}(x) - \sum_{i=1}^{k} \Big(\beta_i x + \alpha_{i+1} \; x^{i+1}(-Lnx)\Big)\Big| = o(x^k). \qquad (5.58)$$

One calls this series, the Dulac series for $\delta_{\lambda_0}(x)$.

Definition 27 *We say that the saddle connection Γ of X_{λ_0} is of codimension $2k$ if $\beta_i = \alpha_j = 0$ for $1 \leq i \leq k-1$ and $2 \leq j \leq k$. We say that Γ is of codimension $2k+1$ if $\beta_i = \alpha_j = 0$ for $1 \leq i \leq k$ and $2 \leq j \leq k$.*

In other terms, codimension $2k$ means that $\delta_{\lambda_0}(x) \sim \beta_{2k} \; x^k$ with $\beta_k \neq 0$, and codimension $2k+1$ means that $\delta_{\lambda_0}(x) \sim \alpha_{k+1} \; x^{k+1} Lnx$ with $\alpha_{k+1} \neq 0$.

Theorem 19 *Let (X_λ, Γ) a C^∞ unfolding of codimension $\ell \geq 2$ (this means that $r(\lambda_0) = 1$ and that $P_{\lambda_0}(x) - x$ is equivalent to x^k if $\ell = 2k$, $x^{k+1} Lnx$ if $\ell = 2k+1$). Then : $Cycl(X_\lambda, \Gamma) \leq \ell$.*

To study the number of zeros of δ_λ, we have to extend somewhat the algebra generated by the $x^i\omega^j$. We introduce now the algebra of functions, continuous in (x,λ) which are finite combinations of the monomials $x^{\ell+n\alpha_1}\omega^n$, with ℓ, $n \in Z$, $m \in N$, $\alpha_1 = \alpha_1(\lambda)$. The coefficients are continuous functions of λ. We call it the *algebra of admissible functions.*

Of course, we consider also monomials as functions of (x,α_1), but when we consider combinations of monomials, α_1 is always replaced by the function $\alpha_1(\lambda)$.

Now, we introduce between the monomials, the following *partial* strict order :

$$x^{\ell'+n'\alpha_1}\,\omega^{m'} \prec x^{\ell+n\alpha_1}\,\omega^m \Longleftrightarrow \begin{cases} \ell' < \ell \text{ or} \\ \ell' = \ell,\ n'=n \text{ and } m'>m. \end{cases}$$

Notice that $x^{\ell+n'\lambda_1}\,\omega^{m'}$ and $x^{\ell+n\alpha_1}\,\omega^m$ with $n\neq n'$ are not ordered.

Later on, the notation : $f+\cdots$ where f is a monomial will mean that after the sign $+$ there is a (non written) finite combination of monomials g_i with $g_i \succ f$. (This definition extends the one uses in theorem 2 for instance).

We will also use the symbol $*$ to replace any continuous function of λ, non zero at $\lambda=\lambda_0$, and we write $\dot\Phi$ for the derivation in x : $\dot\Phi = \frac{\partial\Phi}{\partial x}$. With these conventions, we indicate now some easy properties of the algebra of admissible functions :

a) Let g, f two monomials with $g \succ f$, then $\frac{g}{f}(x,\alpha_1) \to 0$ for $(x,\alpha_1)\to(0,0)$. This follows from the two following observations : $\omega \geq Inf\ (\frac{1}{\mid\alpha_1\mid}\ ,\ -Lnx)$ and $x^{s(\alpha_1)}\ \omega^m \to 0$ (for any continuous function $s(\alpha_1)$, with $s(0)>0$), if $(x,\alpha_1)\to(0,0)$ and $m\in N$.

b) Let a monomial $f\succ 1$, then $f(x,\alpha_1)\to 0$ for $x\to 0$ (uniformly in α_1) : $f\succ 1$ means that $f = x^{\ell+n\alpha_1}\,\omega^m$ with $\ell\geq 1$, and we can use the same argument as in a).

c) $f_1 \succ f_2$ and any $g \Longrightarrow gf_1 \succ gf_2$.

d) Let $f = x^{\ell+n\alpha_1}\,\omega^m$. Then :

$$\dot{f} = (\ell + (n-m)\alpha_1)x^{\ell-1+n\alpha_1}\ \omega^m - mx^{\ell-1+n\alpha_1}\ \omega^{m-1}.$$

From this formula follows easily :

e) Let $f = x^{\ell+n\alpha_1}\ \omega^m$ with $\ell \neq 0$ and g any monomial such that $g \succ f$. Then $\dot{g}$ is a combination of two monomials g' and g'' and $\dot{f} = * f' + \cdots$ with $f' \succ g'$, $f' \succ g''$.

We shall also use rational functions of the algebra of the following type : $\frac{f+\cdots}{1+\cdots}$. We call them : *admissible rational functions.*

For them, we have :

f) $$\Big(\frac{x^{\ell+n\alpha_1}\ \omega^m}{1+\cdots\cdots\cdots}\Big)^{\bullet} = *\frac{x^{\ell+n\alpha_1}\ \omega^m}{1+\cdots\cdots\cdots}\ if\ \ell \neq 0.$$

We can give now a proof of theorem 7. We consider successively the two cases α_{k+1} and β_k.

Proof of theorem 7 in the case α_{k+1}.

Recall that one can write :

$$\delta_\lambda(x) = \beta_0 + \alpha_1[x\omega + \cdots] + \beta_1 x + \alpha_2[x^2\omega + \cdots] + \cdots$$

$$+ \alpha_k[x^k\ \omega + \cdots] + \beta_k\ x^k + \alpha_{k+1}\ x^{k+1}\ \omega + \cdots + \psi_K \tag{5.59}$$

where α_i, β_j are continuous functions, ψ_K is a C^K function of (x, α), K-flat in x, with $K > 2k+1$. We suppose that :

$$\beta_0(\lambda_0) = \cdots = \beta_k(\lambda_0) = 0,\ \alpha_1(\lambda_0) = \cdots = \alpha_k(\lambda_0) = 0\ and\ \alpha_{k+1}(\lambda_0) \neq 0.$$

From the property d) above it follows :

$$(x^j\ \omega)^{\bullet} = (j - \alpha_1)x^{j-1}\ \omega + \cdots if\ j \neq 0\ and\ \dot{\omega} = x^{-1-\alpha_1}.$$

So, differentiating δ_λ, we obtain, using also property e) :

$$\dot{\delta}_\lambda = \alpha_1[*\omega + \cdots] + \beta_1 + \alpha_2[*x\omega + \cdots] + \cdots + *\alpha_{k+1}\ x^k\ \omega + \cdots + \dot{\psi}_K. \tag{5.60}$$

If, we differentiate δ_λ, $k+1$ times, we find :

$$\delta_\lambda^{(k+1)} = \alpha_1[* \, x^{-k-\alpha_1} \; + \cdots] + \alpha_2[* \, x^{-(k-1)-\alpha_1} \; + \cdots] + \cdots$$

$$+ * \, \alpha_{k+1} \, \omega + \cdots + \psi_K^{k+1}. \tag{5.61}$$

All the monomials $\beta_j x^j$, for $j \leq k$ have disappeared. Multiplying by $x^{k+\alpha_1}$, we obtain (using property c)) :

$$x^{k+\alpha_1} \, \delta_\lambda^{(k+1)} = [*1 + \cdots] + \alpha_2[*x + \cdots] + \cdots$$

$$+ * \, \alpha_{k+1} \, x^{k+\alpha_1} \, \omega + \cdots + x^{k+\alpha_1} \, \psi_K^{(k+1)} \tag{5.62}$$

Above and in what follows each bracket designates an admissible function.

Locally (in some neighborhood of $(\lambda_0, 0)$), the zeros of $\delta_\lambda^{(k+1)}$ are zeros of the following function $\xi_1 = \dfrac{x^{k+\alpha_1} \, \Delta_\lambda^{(k+1)}}{[* \, 1 + \cdots\cdots]}$ where the denominator is first bracket in (5.62).

$$\xi_1 = \alpha_1 + \alpha_2 \frac{* \, x + \cdots}{* \, 1 + \cdots} + \cdots$$

$$+ \alpha_k \, \frac{* \, x^{k-1} + \cdots}{* \, 1 + \cdots} + \frac{* \, \alpha_{k+1} \, x^{k+\alpha_1} \, \omega + \cdots}{* \, 1 + \cdots\cdots} + \Phi_1. \tag{5.63}$$

Here $\Phi_1 = \dfrac{x^{k+\alpha_1} \, \psi_K^{(k+1)}}{* \, 1 + \cdots\cdots}$ is a $\mathcal{C}^{K-k-1}$ function, at least $K-k-1$ flat at $x = 0$. Using property f), we have :

$$\dot{\xi} = \alpha_2 \, \frac{* \, 1 + \cdots}{* \, 1 + \cdots} + \cdots$$

$$+ \alpha_k \, \frac{* \, x^{k-2} \; + \cdots}{* \, 1 + \cdots\cdots} + \frac{* \, \alpha_{k+1} \, x^{k-1+\alpha_1} \, \omega + \cdots}{* \, 1 + \cdots\cdots} + \Phi_2 \tag{5.64}$$

where $\Phi_2 = \dot{\Phi}_1$ is $\mathcal{C}^{K-k-2}$, $K-k-2$ flat in $x = 0$; $\dot{\xi}_1 = \alpha_2 \, u_1 + \cdots$ where u_1 is invertible as a rational admissible function. Let be $\xi_2 = u_1^{-1} \, \dot{\xi}_1$ and derive again ξ_2 :

$$\dot{\xi}_2 = \alpha_3 \, \frac{* \, 1 + \cdots}{* \, 1 + \cdots} + \cdots + \dot{\Phi}_2. \tag{5.65}$$

We have $\dot{\xi}_2 = \alpha_3 \ u_2 + \cdots$, where u_2 is invertible as admissible rational function. We define $\xi_3 = u_2^{-1} \ \dot{\xi}_2$ and so on.

In this way, we find a sequence of functions : $\xi_1, \cdots, \xi_k$ such that ξ_j is the product of $\dot{\xi}_{j-1}$ and some invertible admissible rational function. For the last one, ξ_k, we have :

$$\xi_k = \alpha_k + \frac{* \ \alpha_{k+1} \ x^{1+\alpha_1} \ \omega + \cdots}{* \ 1 + \cdots\cdots} + \Phi_k \tag{5.66}$$

where Φ_k is $\mathcal{C}^{K-2k}$, $(K-2k)$-flat.

Differentiating a last time, we obtain :

$$\dot{\xi}_k = \frac{* \ \alpha_{k+1} \ x^{\alpha_1} + \cdots}{* \ 1 + \cdots\cdots} + \dot{\Phi}_k. \tag{5.67}$$

Then, using the fact that $\dot{\Phi}_k$ is $(K-2k-1)$-flat, with $K-2k-1 > 0$, and the property a), we obtain :

$$x^{-\alpha_1} \ \omega^{-1} \dot{\xi}_k = +\alpha_{k+1} + o(1). \tag{5.68}$$

(Here $o(1)$ means a function $\psi(x, \lambda)$, such $\psi(x, \lambda) \to 0$ for $x \to 0$, uniformly in λ). The assumption $\alpha_{k+1} \ (\lambda_0) \neq 0$ implies that locally $x^{-\alpha_1} \omega^{-1} \dot{\xi}_k$ and also $\dot{\xi}_k$ are non zero for (λ, x), $x \geq 0$. So, the function ξ_k has at most one zero for (x, λ) near $(0, \lambda_0)$, ξ_{k-1} has at most two zeros, and so on : ξ_1 has at most k zeros, locally. Now, ξ_1 has at least the same number of zeros as $\delta_\lambda^{(k+1)}$, and finally the function $\delta_\lambda(x)$ has at most $2k+1$ zeros near 0, for λ near λ_0.

Proof in the case β_k.

We differentiate the map δ_λ only k times :

$$\begin{aligned} \delta_\lambda^{(k)}(x) &= \alpha_1 [* \ x^{-k+1-\alpha_1} \ + \cdots] + \cdots \\ &+ \alpha_k [* \ \omega + \cdots] + * \beta_k + \cdots + \psi_K^{(k)}. \end{aligned} \tag{5.69}$$

We introduce next :

$$\xi_1 = \frac{\delta_\lambda^{(k)} \ (x)}{[* \ x^{-k+1-\alpha_1} \ + \cdots]} = \alpha_1 + \alpha_2 \ \frac{* \ x + \cdots}{* \ 1 + \cdots} \ + \cdots$$

$$+\frac{*\,\alpha_k\,x^{k-1+\alpha_1}\,\omega + *\,\beta_k\,x^{k-1+\alpha_1}\ +\cdots}{*\,1+\cdots\cdots} + \Phi_1 \qquad (5.70)$$

where Φ_1 is $\mathcal{C}^{K-k}$, $(K-k)$-flat at $x=0$.

As in the previous case, we define a sequence of functions : $\xi_1, \cdots, \xi_{k-1}$ with ξ_j equal to $\dot{\xi}_{j-1}$ multiplied by an invertible admissible rational function. The last function ξ_{k-1} is equal to :

$$\xi_{k-1} = *\,\alpha_{k-1} + \frac{*\,\alpha_k\,x^{k-1+\alpha_1}\,\omega + *\,\beta_k\,x^{1+\alpha_1}\ +\cdots}{*\,1+\cdots\cdots} + \dot{\Phi}_{k-1} \quad (5.71)$$

and then :

$$\dot{\xi}_{k-1} = \frac{*\,\alpha_k\,x^{\alpha_1}\,\omega + *\,\beta_k\,x^{\alpha_1}\ +\cdots}{*\,1+\cdots\cdots} + \dot{\Phi}_{k-1} \qquad (5.72)$$

where $\dot{\Phi}_{k-1}$ is $\mathcal{C}^{K-2k+1}$, $(K-2k+1)$-flat.

We define now a function ξ_k :

$$\xi_k = x^{-\alpha_1}\,\omega^{-1}[*\,1+\cdots]\dot{\xi}_{k-1} = *\,\alpha_k + *\,\beta_k\,\frac{*\,1+\cdots}{*\,1+\cdots}\cdot\frac{1}{\omega} + \Phi_k \quad (5.73)$$

where the bracket is the denominator in (5.72). The function Φ_k is $\mathcal{C}^{K-2k}$, $(K-2k)$-flat.

If we derive ξ_k, we obtain :

$$\dot{\xi}_k = *\,\beta_k\,\frac{x^{-1-\alpha_1}\ +\cdots}{*\,1+\cdots\cdots}\cdot\frac{1}{\omega^2} + \dot{\Phi}_k. \qquad (5.74)$$

And :

$$\omega^2\,\frac{*\,1+\cdots}{*\,x^{-1-\lambda_1}\ +\cdots}\,\dot{\xi}_k = *\,\beta_k + \omega^2\,\frac{*\,1+\cdots}{*\,x^{-1-\lambda_1}\ +\cdots}\cdot\dot{\Phi}_k. \qquad (5.75)$$

The remaining term is $o(1)$. So, because $\beta_k(\lambda_0) \neq 0$, we have that $\dot{\xi}_k \neq 0$ for (λ, x) sufficiently near $(\lambda_0, 0)$. It follows, as in the previous case, that the map Δ_λ has at most $2k$ zeros for (λ, x) near $(\lambda_0, 0)$.

5.2.3 Bifurcation diagrams for generic unfoldings.

To cover more easily the two cases, we call now all the coefficients in the δ_λ-expansion : β_i. So we write :

$$\delta_\lambda(x) = \beta_0(\lambda) + \beta_1(\lambda)[x\omega + \cdots] + \cdots$$

$$+ \beta_{2m-1}(\lambda)[x^m\ \omega + \cdots] + \beta_{2m}\ (\lambda)x^m + \cdots + \psi_k. \qquad (5.76)$$

Now, even coefficients correspond to the monomials x^m and odd ones to monomials $[x^m\ \omega + \cdots]$.

Suppose that some saddle connection Γ for X_{λ_0} is of finite codimension n : $\beta_0(\lambda_0) = \cdots = \beta_{n-1}\ (\lambda_0) = 0$ and $\beta_n(\lambda_0) \neq 0$.

Then, the following result was proved by P. Joyal [J2] :

Theorem 20 *Let $n = 2m$ or $2m-1$. There exists a neighborhoods $[0,\varepsilon]\times W_0$ of $(0,\lambda_0)$ and a mapping $\alpha(\lambda) = (\alpha_0(\lambda), \cdots, \alpha_{n-1}\ (\lambda))$ defined on W_0 such that the zeros of the polynomial :*

$$P(x,\alpha) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}\ x^{n-1} + x^n$$

in $[0,\varepsilon]$ are those of (5.76).

Remark 26 *In fact, Joyal looked at $\Delta_\lambda(x) = D_\lambda(x) - R_\lambda(x)$ in place of $\delta_\lambda(x) = P_\lambda(x) - x$, but the two equations $\Delta_\lambda = 0$ and $\delta_\lambda = 0$ and equivalent.*

If X_λ is a generic family with ℓ parameters (dim $P = \ell$), we know that any saddle connection Γ at λ_0 has a codimension $n \leq \ell$ and also that the map $\beta(\lambda) = (\beta_0(\lambda) \cdots \beta_{n-1}\ (\lambda))$ is of maximal rank at λ_0. In the case that $\ell = n$, Joyal proved :

Theorem 21 *Let $n = 2m$ or $2m-1$. If $\beta(\lambda) = (\beta_0(\lambda), \cdots, \beta_{n-1}(\lambda))$ is a mapping of maximal rank (i.e : $\dfrac{D(\beta_0, \cdots, \beta_{n-1})}{D(\lambda_1, \cdots, \lambda_n)}(\lambda_0) \neq 0$), then there exists a neighborhoods $[0,\varepsilon]\times W_0$ of $(0,\lambda_0)$, and a homeomorphism $\alpha(\lambda)$ such that the zeros of (5.76) in $[0,\varepsilon]$ are the same as the zeros of the polynomial $P(x,\alpha)$ in $[0,\varepsilon]$.*

Remark 27

1) The proof of theorem 9 given by Joyal is rather difficult to understand. It would be important to find a more clear one.

2) It is easy to construct by gluing (as I explained in the codimension 1 case) a generic unfolding of codimension n. Theorem 8 says that such unfolding is versal for any other unfolding of codimension n saddle connection.

3) It is reasonable to think that theorem 9 extend to any generic unfolding : If $n \leq \ell$, it would be a topological submersion $\alpha(\lambda)$ near λ_0 of the parameter space onto the parameter space $(\alpha_0, \cdots, \alpha_{n-1})$, and the local bifurcation diagram should be equal to a topological trivial product of the diagram with n parameters.

The proof given by Joyal for these two theorems are based on the notion of Chebychev family.

Definition 28 *Let $f_0, \cdots, f_n$ real functions on some interval $[a, b] \subset R$. One says that $\{f_0, \cdots, f_n\}$ is a Chebychev system if any combination $\sum_{i=0}^{n} \alpha_i \, f_0$, $\alpha_i \in R$ has no more than n isolated zeros on $[a, b]$. It is called regular if any subsystems is also a Chebychev one.*

Remark 28 *The number of zeros is counted with multiplicity. One wants to consider also non differentiable functions ($x^k Ln^\ell x$ for instance) at least at the ends a, b of the interval. Joyal introduced some general definition of multiplicity for general function.*

For a function with a Dulac series expansion : $\sum_{\ell \leq k} a_{ij} x^k Ln^\ell x$ the multiplicity at 0, is defined using the well ordered sequence of the monomials $x^k Ln^\ell x$: one says that the multiplicity at zero is n if the first non zero coefficient in the Dulac expansion is of order n.

Now, the following well-known result for differentiable systems was extended by Joyal for general system [J1] :

Theorem 22 *Let $\{f_0, \cdots, f_n\}$ and $\{g_0, \cdots, g_n\}$ be regular systems in $[a, b]$. Let $\mu = (\mu_0, \cdots, \mu_{n-1})$ and $\nu = (\nu_0, \cdots, \nu_{n-1})$. If $\mu_n \neq 0$ and $\nu_n \neq 0$, then there exists a homeomorphism $\mu(\nu)$ such that $\mu_0 \, f_0(x) + \cdots + \mu_{n-1} \, f_{n-1}(x) + \mu_n \, f_n(x)$ and $\nu_0 \, g_0(x) + \cdots + \nu_{n-1} \, g_{n-1}(x) + \nu_n \, g_n(x)$ have the same zeros in $[a, b]$.*

In particular, $\{1, x, \cdots, x^n\}$ is a regular system in any interval $[a, b]$. So, the theorem implies that the bifurcation diagram of the zeros of $\sum_{i=0}^{n} \mu_i f_i$, $\mu_n \neq 0$, for any Chebychev system is homeomorphic to the one of the polynomial $x^n + \sum_{i=0}^{n-1} a_i x^i$.

Of course, one can apply these results for the germs at one end of the interval.

Any strictly increasing sequence of n monomials $x^i\ Ln^j\ x$, with $0 \leq j \leq i$, and ordered by increasing flatness, is a Chebychev system for any n. It may be proved by a "derivation-division" argument as in the proof of theorem 7.

To prove theorem 10, one can look at the extended unfolding :

$$\Delta_{\beta,\lambda,u} = \beta_0\ f_0 + \beta_1\ f_1 + \cdots + \beta_{n-1}\ f_{n-1} + f_n \tag{5.77}$$

where $f_i(x, u, \lambda)$ is the corresponding function factor of β_i where we put $\omega(x, u) = \dfrac{x^{-u} - 1}{u}$ and look f_i as independent function of $x, u, \lambda, \beta_0, \cdots, \beta_n$ are looked at as independent parameter, and $f_n(x, u, \lambda) = \beta_n[\cdots] + \psi_k$ includes the last term in the expansion of Δ and the remaining term ψ_k.

It is also a consequence of theorem 7 that the sequence $\{f_0, \cdots, f_n\}$ is a regular Chebychev system, and so the bifurcation diagram Σ of $\delta_{\beta,\lambda,u}$ is topologically transversal to the β-planes : $(u, \lambda) = \mathcal{C}$onstant.

The map δ_λ is equal to :

$$\delta_\lambda = \delta_{\beta(\lambda),u(\lambda)}. \tag{5.78}$$

So that, the bifurcation diagram of Δ_λ is just the counter-image $\varphi^{-1}(\Sigma)$, where $\varphi(\lambda) = (\beta(\lambda), \lambda), u(\lambda))$.

To prove theorems 9, one has to prove that the map φ is (topologically) transversal to Σ. This is not the case for *any* map φ. For instance, look at :

$$\delta_{u,\beta}(x) = \beta_0 + \beta_1\ x\omega + x.$$

It is easy to prove that for $u = 0$:

$$\delta_{0,\beta}(x) = \beta_0 + \beta_1 x(-Lnx) + x$$

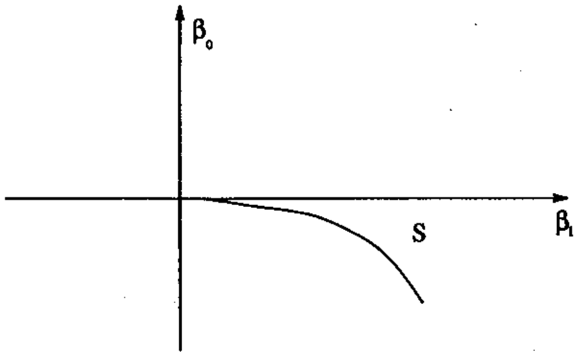

Figure 5.5:

has a bifurcation diagram on $[0, \infty]$ with a line d of double zeros and a sector S with 2 simple zeros (see Figure 5.5).

If we take $\lambda = (\beta_0, \beta_1)$ and define the map $\varphi(\lambda) = (\beta_0, \beta_1, u = \beta_1)$ one obtains :

$$\delta_\lambda(x) = \beta_0 + x^{1+\beta_1}$$

which has just 1 zero on $[0, \infty]$ for each $\lambda \in R^2$.

The proof of this point, i.e. the transversality of φ to the bifurcation diagram Σ is a very obscure point in the proofs of Joyal's theorem, and it is really difficult to check it. I just want to give a direct and independent proof for $n = 2$ which appeared in [DRS1].

So let a generic expansion of δ_λ at order 2 :

$$\delta_\lambda(x) = \beta_0(\lambda) + \beta_1(\lambda)x\omega + \beta_2(\lambda)x + (\lambda)x^2\omega^2 + j(\lambda)x^2\omega + \psi(x, \lambda) \quad (5.79)$$

with $\lambda \in R^2 \to (\beta_0(\lambda), \beta_1(\lambda))$ a local diffeomorphism at λ_0 and $0 < \beta_2(\lambda_0) < 1$. Locally, we can suppose $\lambda = (\beta_0, \beta_1)$ in a neighborhoods of 0.

$\omega = \dfrac{x^{-\beta_1} - 1}{\beta_1}$. Put $\xi_\lambda = \beta_1\omega + \cdots$ the equation $\delta_\lambda(x) = 0$ writes :

$$-\beta_0 = \xi_\lambda(x). \quad (5.80)$$

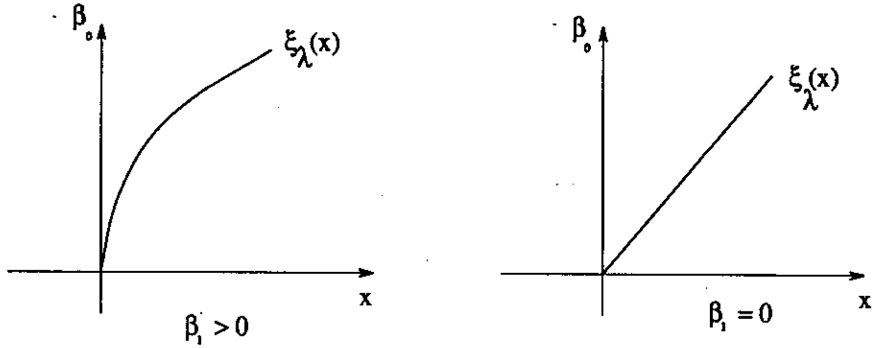

Figure 5.6:

One has :

$$\dot{\xi}_\lambda(x) = \beta_1[(1-\beta_1)\omega - 1] + \beta_2 + o(1). \tag{5.81}$$

To study the bifurcation diagram we look at different cases :

(i) Case $\beta_1 \geq 0$. Here $x^{-\beta_1} - 1 \geq 0$ and because our hypothesis $\beta_2(0) > 0$ and $\beta_1(0) = 0$ we see that :

$$\exists A > 0,\ \exists U > 0,\ \exists E > 0 \textit{ such that}$$

$$\forall \varepsilon \in]0, E[\textit{ and } \forall(\beta_1, x) \in [0, A] \times [0, U] : \dot{\xi}_\lambda > 0.$$

(Indeed for $\beta_1 > 0$: $\dot{\xi}_\lambda \to \infty$ when $x \to 0$; for $\beta_1 = 0$: $\dot{\xi}_\lambda \to \beta_2$ when $x \to 0$) (see Figure 5.6).

For $*\ \beta_0 \geq 0$, $x_0 > 0$ is solution of $\xi_\lambda(x) = -\beta_0$.

For $\beta_0 < 0$, $\exists x > 0$ solution of $\xi_\lambda(x) = -\beta_0$.

The bifurcation diagram for $\beta_1 \geq 0$ is given in figure 5.7.

For each $\beta_0 \geq 0$ fixed we have the creation of one stable limit cycle when β_0 goes from positive to negative values ; $\{\beta_0 = 0, \beta_1 \geq 0\}$ is a half line of saddle connections with non-zero divergence at the saddle point (codimension 1 bifurcation).

(ii) Case $\beta_1 < 0$.

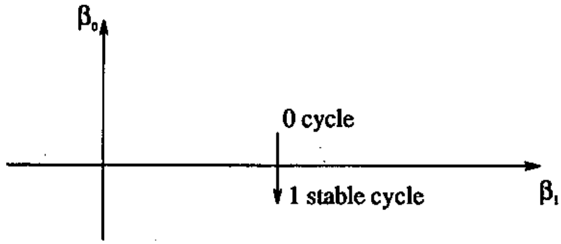

Figure 5.7:

$$
\begin{aligned}
(x^2\,\omega^2)^\bullet &= (2-2\beta_1)x\,\omega^2 - 2x\,\omega \\
(x^2\,\omega)^\bullet &= (2-\beta_1)x\,\omega - x
\end{aligned}
$$

Hence :

$$
\begin{aligned}
\dot{\xi}_\lambda = (1+\beta_1)(x^{-\beta_1}-1)+\beta_2-\beta_1 &+ \gamma[(2-2\beta_1)x\,\omega^2-2x\,\omega] \\
&+ \delta[(2-\beta_1)x\,\omega - x] + \dot{\psi}.
\end{aligned}
$$

As $(x\,\omega^2)^\bullet = (1-2\beta_1)\omega^2 - 2\omega$:

$$
\ddot{\xi}_\lambda = -\beta_1(1-\beta_1)x^{-\beta_1-1} + \gamma[(2-2\beta_1)[(1-2\beta_2)\omega^2-2\omega]-2(1-\beta_1)\omega+2]
$$

$$
\Big\lfloor +\delta[(2-\beta_1)(1-\beta_1)\omega-(2-\beta_1)-1]+\ddot{\psi}
$$

$$
\ddot{\xi}_\lambda = -\beta_1(1-\beta_1)x^{-\beta_1-1} + 2\gamma(1-\beta_1)(1-2\beta_1)\omega^2 + (1-\beta_1)[-6\gamma+(2-\beta_1)\delta]\omega
$$

$$
+[2\gamma+(\beta_1-3)\delta]+\ddot{\psi}.
$$

We can find a bounded function $O(1)$ so that :

$$
\ddot{\xi}_\lambda = -\beta_1(1-\beta_1)x^{-1-\beta_1} + \omega^2\,O(1).
$$

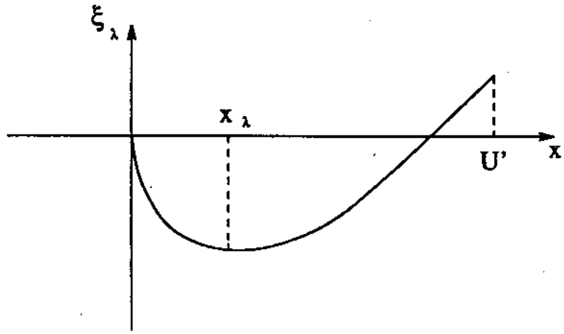

Figure 5.8:

From this, we will prove that : $\exists A' > 0$, $\exists U' > 0$, $\exists E' > 0$ such that for all $\varepsilon \in]0, E[$ and for all $\beta_1 \in] - A', 0[$, the graph of ξ_λ on $[0, U']$ looks like in figure 5.8.

We see that x_λ is a unique strict minimum : $\dot{\xi}_\lambda(x_\lambda) = 0$, $\ddot{\xi}_\lambda(x_\lambda) > 0$, $\dot{\xi} > 0$ on $]x_\lambda, U'[$ and $\dot{\xi}_\lambda < 0$ on $[0, x_\lambda[$ with $\dot{\xi}_\lambda \to -\infty$ for $x \to 0$.

Therefore, consider :

$$\frac{1}{\omega^2}\,\ddot{\xi}_\lambda = -\frac{\beta_1(1-\beta_1)}{\omega^2\; x^{1+\beta_1}} + O(1).$$

$\omega = \dfrac{x^{-\beta_1} - 1}{\beta_1} \leq \mid Lnx \mid$ for $U' < 1$ and E' sufficiently small.

Choosing some $\delta > 0$, taking $-\dfrac{\delta}{2} \leq \beta_1 < 0$, and U' sufficiently small :

$$\frac{1}{\omega^2}\,\ddot{\xi}_\lambda \geq \frac{\mid \beta_1 \mid}{(Ln\; x)^2\; x^{1-\delta/2}} + 0(1) \geq \frac{\mid \beta_1 \mid}{x^{1-\delta/1}} + 0(1).$$

Take $M > 0$ so that $0(1) \geq -M$ on $[0, U'] \times [-\delta, 0]$, then :

$$\frac{1}{\omega^2}\,\ddot{\xi}_\lambda \geq M \; if \; \frac{\mid \beta_1 \mid}{x^{1-\delta}} \geq 2M \iff x \leq \Big(\frac{\mid \beta_1 \mid}{2M}\Big)^{\frac{1}{1-\delta}}.$$

Let us write $x \leq C \mid \beta_1 \mid^\mu$ with $C = \frac{1}{(2M)^\mu}$ and $\mu = \frac{1}{1-\delta}$

For these values of U', δ, M let us consider $\dot{\xi}_\lambda$ on $[C \mid \beta_1 \mid^\mu, U']$:

$$\dot{\xi}_\lambda = \beta_1(1-\beta_1)\omega + \beta_2 - \beta_1 + o(1).$$

As $\beta_1 < 0$, the function $\beta_1\omega = x^{-\beta_1} - 1 = e^{-\beta_1 Ln\, x} - 1$ is strictly increasing and negative, so that for all $x \in [C \mid \beta_1 \mid^\mu, U']$:

$$\beta_1\omega(C \mid \beta_1 \mid^\mu) \leq \beta_1\omega(x) \leq 0.$$

And :

$$\beta_1\omega(C \mid \beta_1 \mid^\mu) = O(\mid \beta_1 Ln(C \mid \beta_1 \mid^\mu) \mid)$$

$$\Rightarrow \beta_1\omega(x) = 0(\mid \beta_1 Ln(C \mid \beta_1 \mid) Ln(C \mid \beta_1 \mid^\mu) \mid) \; for \; x \in [C \mid \beta_1 \mid^\mu, U'].$$

Hence :

$$\dot{\xi}_\lambda = (1-\beta_1)0(\mid \beta_1 \mid Ln(C \mid \beta_1 \mid^\mu)) + \beta_2 - \beta_1 + o(1).$$

For U' sufficiently small : $\mid o(1) \mid \leq B_{2/3}$; and for A' sufficiently small :

$$\mid -\beta_1 + (1-\beta_1)0(\mid \beta_1 Ln(C \mid \beta \mid^\mu) \mid) \mid \leq \beta_{2/3}$$

implying that : $\dot{\xi}_\lambda \geq \beta_{2/3} > 0$.

As a conclusion we see that for $-A' \leq \beta_1 < 0$ fixed, there is a bifurcation value (corresponding to a generic coalescence of limit cycles) for : $\xi_\lambda(x_\lambda) = -\beta_0(\lambda)$. This bifurcation occurs at $\beta_0 = \Gamma(\beta_1)$: Γ is a $\mathcal{C}^\infty$ function for $\beta_1 < 0$ because of the implicit function theorem applied to the Poincaré mapping of the $\mathcal{C}^\infty$ vector field X_λ in the neighborhoods of the semi-stable limit cycle (see Figure 5.9:).

Taking $\Gamma(0) = 0$, we will now say something about the behavior of $\Gamma(\beta_1)$ in the neighborhoods of $\beta_1 = 0$:

$$x_\lambda \; is \; given \; by \; \dot{\xi}_\lambda(x_\lambda) = 0$$

$$i.e. \; : \beta_1(1-\beta_1)\omega(x_\lambda) + \beta_2 - \beta_1 + o(1) = 0$$

or equivalently :

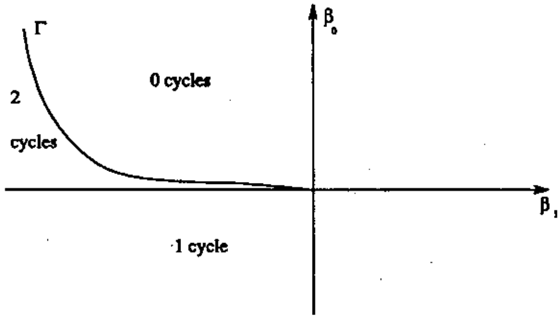

Figure 5.9:

$$|\beta_1|\,\omega(x_\lambda) = \frac{1}{1-|\beta_1|}\,(\beta_2 - |\beta|_1 + o(1)).$$

Now :

$$\begin{aligned}\beta_0 &= x_\lambda\Big[|\beta_1|\,\omega(x_\lambda) - (\beta_2 + 0(1))\Big]\\ &= x_\lambda\Big[\frac{\beta_1\,\beta_2}{1-\beta_1} - \frac{\beta_1}{1-\beta_1} + 0(1)\Big] \le x_\lambda.\end{aligned}$$

And, as $|\beta_1|\,|\,Ln\ x_\lambda| \ge |\beta_1|\,\omega(x_\lambda) \ge \frac{\beta_2}{2}$ for A' and U' sufficiently small we see that :

$$x_\lambda \le e^{-\frac{\beta_2}{2|\beta_1|}}$$

and hence :

$$\Gamma(\beta_1) \le e^{-\frac{\beta_2}{2|\beta_1|}}. \tag{5.82}$$

Meaning that Γ is ∞-flat for $\beta_1 = 0$.

Remark 29 *1) Above we just proved that $\Gamma(\beta_1)$ is ∞-flat at 0 in a C°-sense (formula (5.82)). It is reasonable to think that Γ is C^∞ at 0 and is ∞-flat in C^∞-sense (all derivatives tend to zero when $\beta_1 \to 0$).*

2) Unfoldings of codimension k saddle connections appear in cusp unfoldings of codimension k+1 (The Bogdanov-Takens bifurcation corresponds to k=1). In [R2] I proved that these unfoldings have a finite cyclicity. Next P.Joyal [J3] and P.Mardesic [Mar3] proved that generic unfoldings of finite codimension cusp singularities are versal and gave a complete description of their bifurcation diagrams. The saddle connection unfoldings appear inside the cusp unfoldings as perturbations of Hamiltonian vector fields. In this context an independent proof of similar theorems as theorems 8,9 above were given by P.Mardesic in [Mar1],[Mar3].

5.3 Unfoldings of saddle connections of infinite codimension.

In this paragraph, we restrict ourselves to analytic unfoldings (X_λ, Γ) where Γ is some saddle connection for the parameter value λ_0. As in the smooth case, we can suppose that the saddle point is a same point s for any $\lambda \in W_0$, some neighborhoods of λ_0 and that there exists an analytic chart (x, y) where $s = (0,0)$ and the axis $0x$, $0y$ are respectively unstable and stable local manifolds for X_λ, $\lambda \in W_0$. Let σ', τ analytic transversal segments to $0y$, $0x$: σ' parametrized by $x \in]-X', X[$, τ by $y \in]-Y, Y[$; let be $\sigma = [0, X[$.

Take X, Y small enough such that $D_\lambda : \sigma \times W_0 \to \tau$ and R_λ^{-1} : $\tau \times W_0 \to \sigma'$ are defined. Let $P_\lambda(x) = R_\lambda^{-1} \circ D_\lambda \; : \; \sigma \times W_0 \to \sigma'$ the Poincaré return map and $\delta_\lambda(x) = P_\lambda(x) - x : \sigma \times W_0 \to R$ the difference map.

Definition 29 *We will say that the unfolding (X_λ, Γ) is of infinite codimension if the Dulac series $\hat{\delta}_{\lambda_0}$ is equal to zero.*

For smooth family this condition means that δ_{λ_0} is C^∞ and flat at $x = 0$. For analytic family we have seen in chapter 3 that $\delta_{\lambda_0}(x)$ is quasi-analytic and the condition $\widetilde{\delta}_{\lambda_0} \equiv 0$ is equivalent to $\delta_{\lambda_0} \equiv 0$. This means that the nearby trajectories, on the side the return map is defined, are periodic orbits : the vector field X_{λ_0} is of centre type.

5.3.1 Finite cyclicity property of analytic unfoldings

I have studied analytic unfoldings of homoclinic connections in [$R5$]. Here I just want to explain the principal steps of this study and complete it by some new results (theorem 13 below). I refer the reader to [$R5$] for more details.

Firstly, we need a version of the asymptotic expansion at order k, given in formula (5.45) for analytic unfoldings :

Proposition 14 *Let analytic sections σ, σ' as above and X_λ any analytic unfolding with $r(\lambda_0) = 1$. (We don't suppose that $\delta_{\lambda_0}(x) \equiv 0$). Then, for any $k \in N - \{0\}$, there exist a neighborhoods of λ_0, $W_k \subset W_0$ and analytic maps $\beta_0^k, \cdots, \beta_k^k$, $\alpha_1^k, \cdots, \alpha_{k+1}^k$ from W_k to R, such that on $\sigma \times W_0$:*

$$\delta_\lambda(x) = \beta_0^k + \alpha_1^k[x\omega + \cdots] + \cdots + \beta_k^k\ x^k + \alpha_{k+1}^k\ x^{k+1}\ \omega + \psi_k(x,\lambda) \quad (5.83)$$

where ψ_k is C^k and k-flat at $x = 0$. Expressions in brackets are finite combinations of monomials $x^i\omega^j$, $0 \leq j \leq i \leq k$, with coefficients analytic in λ, zero at λ_0 ; any monomial in $+\cdots$ has an order strictly larger that the leading term's one.

We cannot use the proof given in the smooth case because the change of coordinates we use to reduce X_λ in normal form has a finite differentiability. As a consequence the coefficients β_i^k have also a finite differentiability.

So, to replace theorem 3 we used for smooth family, we use now the following result coming from Dulac :

Theorem 23 *(Dulac normal form). For any $N \in N$, up to an analytic equivalence, X_λ is equal to :*

$$X_\lambda = x\,\frac{\partial}{\partial x}\ y\Big(1 + \sum_{j=1}^{N}\ \alpha_i(\lambda)(xy)^i + (xy)^N\ G(x, y_i, \lambda)\Big)\ \frac{\partial}{\partial y} \qquad (5.84)$$

where α_i, G are analytic functions.

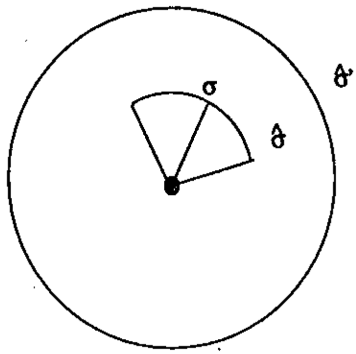

Figure 5.10:

Based on this Dulac normal form, I first established in [R5] the expansion for $D_\lambda(x)$ with analytic coefficients like in proposition 7, for the principal part of X_λ (up to the remainder term $(xy)^N G\frac{\partial}{\partial y}$) follows form theorem 9 ; next I proved that this remainder term just sum up a k-flat perturbation.

In fact, one has (and one needs !) a stronger result : the Dulac normal form works for holomorphic families and one can apply it to some complex extension of X_λ (in some neighborhoods $\widehat{V}$ of Γ in C^2) and for some neighborhoods $\widehat{W}_0$ of λ_0 in C^Λ (if the real parameter space is R^Λ). Using this complex extension of $\widehat{X_{\hat{\lambda}}}$ of X_λ we can prove that $\delta_\lambda(x)$ has also a complex extension : σ' may be extended in a disk $\hat{\sigma}'$ in C and σ in a sector $\hat{\sigma}$ at $0 \in C$, containing σ (see Figure 5.10).

Then, there exists a map $\widehat{\delta}_{\hat{\lambda}}(\hat{x}) : \hat{\sigma} \times \widehat{W}_0 \to C$ with a similar expansion as (5.83) where $\hat{x} \in \hat{\sigma}$, $\hat{\omega}$ is an holomorphic extension of ω on $\hat{\sigma} \times \widehat{W}_0$ and the coefficients $\hat{\beta}_i^k$, $\hat{\alpha}_j^k$ are holomorphic on $\widehat{W}_0$:

$$\widehat{\delta}_{\hat{\lambda}}(\hat{x}) = \hat{\beta}_\partial^k + \hat{\alpha}_1^k\,[\hat{x}\,\hat{\omega} + \cdots] + \cdots + \hat{\psi}_k\,(\hat{x}, \hat{\lambda}). \qquad (5.85)$$

The remaining term $\hat{\psi}_k : \hat{\sigma} \times \widehat{W}_0 \to C$ is just C^k-real and k-flat at $\hat{x} = 0$ (but of course holomorphic like all the other terms, on $\hat{\sigma} - \{0\}$).

From now on, we want to take in account that (X_λ, Γ) is an ∞-

codimension unfolding. In the last chapter we have introduced the Bautin Ideal $\mathcal{I}$: this ideal is generated by the coefficient germs $\widetilde{a}_i$ of the series expansion $\delta_\lambda(x) = \sum_i a_i(\lambda)(x - x_0)^i$ at any *regular* point $x_0 \in \sigma - \{0\}$.

The interest of the holomorphic extension comes from the fact that it permits we to pass to the limit $x_0 = 0$:

Lemma 23 *Let an expansion of $\delta_\lambda(x)$ at order k be like in proposition 7. Then for each coefficient germ $\widetilde{\beta}_i^k$, $\widetilde{a}_j^k$ belongs to $\mathcal{I}$.*

Proof First, deriving at order $k+1$ the expansion (5.85) we eliminate all the monomials x^i, $i \le k$:

$$\hat{\delta}_{\hat{\lambda}}^{(k+1)}\ (\hat{x}) = \hat{\alpha}_1^k\ [x^{-k-\lambda_1} + \cdots] + \cdots + +\hat{\alpha}_{k+1}^k\ \hat{x}^{k+\alpha_1}\ \hat{\omega} + \widehat{\psi}_k^{(k+1)}. \quad (5.86)$$

One divides this expression by the first bracket $x^{-k-\alpha_1}[*1 + \cdots]$:

$$\frac{x^{k+\alpha_1}\ \widehat{\delta}_{\hat{\lambda}}^{k-1}\ (\hat{x})}{[*\ 1 + \cdots\cdots]} = \hat{\alpha}_1^k + \hat{\alpha}_2^k\ \frac{[*\ x + \cdots]}{[*\ 1 + \cdots]}\ + \cdots = \hat{\alpha}_1^k + o(1). \quad (5.87)$$

The term $o(1) \to 0$ for $\hat{x} \to 0$ (uniformly in $\hat{\lambda}$).

For each $\hat{x} \in \hat{\sigma} - \{0\}$, the germ $\hat{\lambda} \to \delta_{\hat{\lambda}}\ (\hat{x})$ at λ_0 belongs to the ideal $\widehat{\mathcal{I}}$ obtained by the complexification of the Bautin Ideal $\mathcal{I}$. But, because this complex ideal $\widehat{\mathcal{I}}$ is *closed*, the germ of $(\delta_{\hat{\lambda}}^{(k+1)}\ (x), \lambda_0)$ belongs also to $\widehat{\mathcal{I}}$ and also the left member of (5.87) and its limit for $\hat{x} \to 0$. This means that $(\widehat{\alpha}_1^k, \lambda_0) \in \widehat{\mathcal{I}}$.

We can prove in the same way that any germ $(\hat{\alpha}_i^k, \lambda_0) \in \widehat{\mathcal{I}}$.

For instance, if $\widehat{\delta}_1$ is the left member of (5.87), one has that $(\widehat{\delta}_1 - \widehat{\alpha}_1^k, \lambda_0) \in \widehat{\mathcal{I}}$ and also $\hat{\alpha}_2^k(\lambda)$ because :

$$\frac{[*\ 1 + \cdots]}{[*\ x + \cdots]}\ (\widehat{\delta}_1 - \hat{\alpha}_1 - \hat{\alpha}_1^k) = \hat{\alpha}_2^k + o(1). \quad (5.88)$$

Next, we subtract the "α"-part of the expansion (5.86) :

$$\hat{\delta}_{\hat{\lambda}}(\hat{x}) - \sum_{i=1}^{k}\ \hat{\alpha}_i^k\ (\hat{\lambda})[*\ \hat{x}^{i+1}\ \hat{\omega} + \cdots] = \sum_{j=0}^{k}\ \hat{\beta}_j^k\ (\lambda)\hat{x}^j + o(x^k). \quad (5.89)$$

The first part of the proof implies that the left number of (5.89) has a germ at λ_0 which belongs to $\widehat{\mathcal{I}}$ for any $\hat{x} \neq 0$. Using that $\widehat{\mathcal{I}}$ is closed we prove as above that $(\widehat{\beta}_j^k, \lambda_0) \in \widehat{\mathcal{I}}$ for each $j \leq k$. ∎

The last lemma means that the principal part of (5.85) belongs to $\widehat{\mathcal{I}}$ for any $\hat{x} \neq 0$. Because it is the same for $\widehat{\delta}_{\widehat{\lambda}}$ itself, we have also the same property for the remainder term $\widehat{\psi}_k(\hat{x}, \hat{\lambda})$. This permits to divides the expansion (5.85) in the ideal $\widehat{\mathcal{I}}$:

If $\varphi_1, \cdots, \varphi_\ell$ is a system of generators for $\mathcal{I}$ and $\widehat{\varphi}_1, \cdots, \widehat{\varphi}_\ell$ their complex extension which form a system for $\widehat{\mathcal{I}}$ one can divide $\widehat{\delta}_{\widehat{\lambda}}$ (at order k) :

$$\widehat{\delta}_\lambda(\hat{x}) = \sum_{i=1}^{\ell} \widehat{\varphi}_i \, \hat{h}_i^k \tag{5.90}$$

with :

$$\hat{h}_i^k = \widehat{\beta}_{i0}^k + \hat{\alpha}_{i1}^k \, [\hat{x}\hat{\omega} + \cdots] + \cdots + \hat{\alpha}_{i,k+1}^k \, \hat{x}^{k+1} \, \hat{\omega} + \hat{\omega} + \widehat{\psi}_i^k \, (\hat{x}, \hat{\lambda}) \tag{5.91}$$

where the $\widehat{\beta}_{ij}^k$, $\hat{\alpha}_{ij}^k$ are holomorphic functions of $\widehat{\lambda}$ and $\widehat{\psi}_i^k \, (\hat{x}, \widehat{\lambda})$ is C^k, k-flat at $\hat{x} = 0$.

One establishes the formula (5.90) as for the regular case in chapter 4 using the theorem (D) (see [R5]), and finally restricting it to δ_λ real :

Theorem 24 *Let $\varphi_1, \cdots, \varphi_\ell$ analytic functions on W_0 whose germs $\widetilde{\varphi}_1, \cdots \widetilde{\varphi}_\ell$ generate the Bautin Ideal $\mathcal{I}$. Let $k \in N$, $k \geq 1$. There exists a neighborhood W_k of λ_0 (in W_0) and functions $h_i^k(x, \lambda)$, $1 \leq i \leq \ell$, with (ω, x)-expansion at order k :*

$$h_i^k(x, \lambda) = \beta_{i0}^k + \alpha_{i1}^k(\lambda)[x\omega + \cdots] + \cdots + \alpha_{i,k+1}^k \, x^{k+1} \, \omega + \psi_i^k(x, \lambda) \tag{5.92}$$

as in proposition 7. These functions are factors of the division of $\delta(x, \lambda)$ in $\varphi_1, \cdots, \varphi_\ell$. This means that on $[0, x_0] \times W_k$ for some $x_0 > 0$, one has :

$$\delta(x, \lambda) = \sum_{i=1}^{\ell} \varphi_i(\lambda) h_i^k(x, \lambda). \tag{5.93}$$

As in chapter 4, we now restrict ourselves to some *minimal set of generators* for $\mathcal{I}$. Recall that factors $h_i^k(x, \lambda_0)$ are independent of the choice of k and also of the base point x_0 where we take the expansion. So, we have functions $h_i(x)$, defined on σ by : $h_i(x) = h_i^k(x, \lambda_0)$ where $h_i^k(x, \lambda)$ are factors for any expansion defined at x. Taking the value at λ_0 for the expansions of $h_i^k(x, \lambda_0)$ for all k, we see that $h_i(x)$ has at $x = 0$ a Dulac series :

$$D^\infty h_i(x) = \sum_{j=0}^{\infty} (\beta_{ij}\ x^j + \alpha_{ij+1}\ x^{j+1}\ Lnx) \tag{5.94}$$

$D^\infty\ h_i(x)$ is the series of $h_i(x)$ is the following sense :

$$\mid h_i(x) - \Big(\sum_{j=0}^{N}\ \beta_{ij}\ x^i + \alpha_{ij+1}\ x^{j+1}\ Lnx\Big) \mid = o(x^N) \tag{5.95}$$

for $\forall N \in N$.

The functions h_i are analytic on $\sigma - \{0\}$ and are R-independent. This implies that each of them is non identically zero.

I conjecture that they are quasi-analytic. If this was the case : $h_i(x) \not\equiv 0 \Rightarrow D^\infty\ h_i(x) \not\equiv 0$. But I don't know if this is true. What was proved in [R5] was a weaker result which was sufficient to obtain the finite cyclicity. Here, I want to make precise this result by giving an explicit bound. To express it we consider quasi-regular functions : they are functions f, analytic on $\sigma - \{0\}$ which has at 0 a more general Dulac series than (5.94) :

$$D^\infty f = \sum_{0 \le \ell \le s} \alpha_{s\ell}\ x^s\ Ln^\ell\ x. \tag{5.96}$$

The monomials $x^s Ln^\ell x$ are totally ordered : $1 \prec xLnx \prec x \prec x^2Ln^2x \prec x^2Lnx \prec x^2 \prec \cdots$. We call the *order* of a monomial $x^s Ln^\ell x$ order its in this total order : order $(x^s Ln^\ell x) = \dfrac{s(s+1)}{2} - \ell$; for instance order $(1) = 0, \cdots,$ order $(x^{s+1} Lnx) = \dfrac{(s+1)(s+4)}{2}$.

We say that the order at 0 of a quasi regular function is k if the first non-zero coefficient in $D^\infty f$ is of order k. We write : $\text{order}_0(f) = k$. Of course, one says that $\text{order}_0(f) = \infty$ if $D^\infty f \equiv 0$ (This is equivalent to say that f is ∞-flat at 0). We write $D^N f$ for the Dulac series, truncated at order $N \in N$.

Using this order for quasi-regular functions, it is possible to propose a definition of an index $s_\delta(0)$ which extends the index $s_\delta(u_0)$ defined for regular points in chapter 4.

Let any minimal set of generators and $h_1(x), \cdots, h_\ell(x)$ the corresponding factors at $\lambda = \lambda_0$. Let s the dimension of the vector space generated by the Dulac series $\hat{h}_i : 0 \leq s \leq \ell$. Notice that the series $\hat{h}_i$ are R-dependent $\iff s < \ell \iff$ one of the $\hat{h}_i$ is zero.

There exists some N such that : dim $\{D^N f_i\}_i = dim\{D^\infty f_i\}_i$.

Definition 30

$$s_\delta(0) = Inf\ \{N \mid dim\ \{D^N f_i\}_i = s\}.$$

The dimension s and the index $s_\delta(0)$ is independent of the choice of the minimal set of generators.

If $s = 0$, this means that any factor is flat. In this case we assume $s_\delta(0) = 0$, but we will prove that this cannot occur.

As we have proved for a regular point in lemma 4.9, it is possible to choose a minimal set of generators for $\mathcal{I}$ such that the corresponding non-flat factors h_i are in strictly increasing order :

$$order_0(h_1) < \cdots < order_0(h_s) < \infty$$

and order $(h_j) = \infty$ if $j \geq s+1$.

We will say that this minimal set is *adapted to 0*. For such a minimal set, $s_\delta(0) = order_0(h_s)$. As a consequence, $s_\delta(0) \geq s - 1$.

The main result we want to prove in this section, which clarifies the finiteness result of [R5], is :

Theorem 25 *Let X_λ some analytic unfolding of a cuspidal connection Γ. For a section σ to Γ, $\sigma \simeq [0, X]$ with $\{0\} = \Gamma \cap \sigma$.*

Let be s, and $s_\delta(0)$ as above. Then :

(i) $Cycl(X_\lambda), \Gamma \leq s_\delta(0)$.

(ii) If $\mathcal{I}$ is regular, $Cycl(X_\lambda, \Gamma) \geq s - 1$.

As in [R5] we introduced a desingularization map : $\Phi : \widetilde{W}_0 \to W_0$ given by e Hironaka's theory [H]. This map is a proper analytic map with the property that at each $\widetilde{\lambda}_0 \in D = \Phi^{-1}(\lambda_0)$, the lifted functions $\widetilde{\varphi}_i(\widetilde{\lambda}) = \varphi_i \circ \Phi(\widetilde{\lambda})$ have a monomial form :

$$\widetilde{\varphi}_i(\widetilde{\lambda}) = u_i(\widetilde{\lambda}) \prod_{i=1}^{\Lambda} z_1^{p_1^i} \cdots z_\Lambda^{p_\Lambda^i} \qquad (5.97)$$

for local coordinates $(z_1, \cdots, z_n)$ at $\widetilde{\lambda}_0$; $\tilde{u}_i(\widetilde{\lambda}_0) \neq 0$.

The germs of the $\widetilde{\varphi}_i$ at $\widetilde{\lambda}_0$ does not form in general a minimal set of generators of the lifted ideal $\widetilde{\mathcal{I}}^{\widetilde{\lambda}_0} = (\mathcal{I} \circ \Phi)^{\widetilde{\lambda}_0}$ at $\widetilde{\lambda}_0$, but it is possible to find a subsequence $i_1 < \cdots < i_L$ such that $\widetilde{\varphi}_{i_1}, \cdots, \widetilde{\varphi}_{i_L}$, have this property ; moreover, if $H_1, \cdots, H_L$ are the corresponding factors, order$_0$ $H_j =$ *order*$_0$ h_{i_j}, $j = 1, \cdots, L$.

It follows from this that the corresponding index $s_{\widetilde{\delta}}^{\widetilde{\lambda}_0}$ at $0 \in \sigma$ for the lifted unfolding $X_{\Phi(\widetilde{\lambda})}$ at $\widetilde{\lambda}_0$ is less than $s_\delta(0)$. (We write $\widetilde{\delta}_{\widetilde{\lambda}}(x) = \delta_{\Phi(\widetilde{\lambda})}(x)$; this map is defined for $\widetilde{\lambda}$ in a neighborhood of $\widetilde{\lambda}_0$).

By the definition of cyclicity, there exist a neighborhood $W(\widetilde{\lambda}_0)$ of $\widetilde{\lambda}_0 \in D$ and a value $x_{\widetilde{\lambda}_0}$ such that :

$$Cycl\ (X_{\Phi(\widetilde{\lambda})},\ \Gamma\ ;\ \widetilde{\lambda}_0) = N(\widetilde{\lambda}_0)\ where$$

$N(\widetilde{\lambda}_0)$ is the maximum of the number of zeros for the equation $\{\widetilde{\delta}_{\widetilde{\lambda}}(x) = 0\}$ in $[0, x_{\widetilde{\lambda}_0}]$ for $\widetilde{\lambda} \in \widetilde{W}(\widetilde{\lambda}_0)$.

Extracting a finite subcovering from the covering $\{W(\widetilde{\lambda}_0)\}_{\widetilde{\lambda}_0}$ of the compact set D, one obtains that there exists some $\widetilde{\lambda}_0$ such that :

$$Cycl\ (X_\lambda\ ,\ \Gamma\ ;\ \lambda_0) = N(\widetilde{\lambda}_0) = Cycl(X_{\phi(\widetilde{\lambda})},\ \Gamma\ ;\ \widetilde{\lambda}_0) \qquad (5.98)$$

(Choose $\widetilde{\lambda}_0$ for which :

$$Cycl\ (X_{\Phi(\widetilde{\lambda})}, \Gamma\ ; \widetilde{\lambda}_0) = \underset{\widetilde{\lambda}_1 \in D}{Sup}\ Cycl\ (X_{\Phi(\widetilde{\lambda})},\ \Gamma\ ;\ \widetilde{\lambda}_1).$$

Now, as we have noticed above $s_{\widetilde{\delta}}^{\widetilde{\lambda}_0}(0) \leq s_\delta(0)$. To prove the part (i) of theorem 13 it is sufficient to prove it at any $\widetilde{\lambda}_0 \in D$:

Proof of part (i) of theorem 13.

Up to now, in this part of the proof, we forget the subscript. We just denote the family $X_{\Phi(\tilde{\lambda})}$ by X_λ, $\tilde{\lambda}_0$ by λ_0 and so on.

We suppose chosen a minimal system of generators $\varphi_1, \cdots, \varphi_\ell$ (ℓ in place of L) of the special form (5.97). A consequence of this special form, proved in [R5], is the following :

Proposition 15 *If a minimal set of generators $\varphi_1, \cdots, \varphi_\ell$ has the form (5.97) and is adapted at 0, then, there exists a r, $0 < r \leq 1$ such that if :*

$$V_i^r = \{\lambda \in W_0 \mid \mid \varphi_i(\lambda) \mid \geq r \mid \varphi_j(\lambda) \mid, \ \forall j = 1, \cdots, \ell\}.$$

Then :

$$\bigcup_{i=1}^{s} V_i^r = W_0. \tag{5.99}$$

Remark 30 *This proposition means that the sets V_i^r, which are related to the non-trivial series $\hat{h}_i (i \leq s)$ are sufficient to cover a neighborhoods of λ_0.*

Sketch of proof for proposition 8. (see details in [R5]).

Write $\varphi_i(\lambda) = u_i(\lambda)\ \psi_i(\lambda)$ where $u(\lambda_0) \neq 0$ and :

$$\psi_i(\lambda) = \prod_{i=1}^{\Lambda} z_1^{p_1^i} \cdots z_\Lambda^{p_\Lambda^i} \ (\lambda = (z_1, \cdots, z_\Lambda) \ and \ \lambda_0 = (0, \cdots, 0)).$$

Let $W_i = \{\lambda \in W_0 \mid \mid \psi_i(\lambda) \mid \geq \mid \psi_j(\lambda) \mid \text{ for } \forall j \neq i\}$. Let also $I = \{i \mid \lambda_0 \in \overline{int\ W_i}\}$.

It is easy to prove that :

(a) $U_{i \in I} W_i$ is a neighborhoods of λ_0 and also that $\overline{int\ W_i} = \overline{\mathring{W}_i}$ if $\mathring{W}_i = \{\lambda \in W_0 \mid \mid \varphi_i(\lambda) \mid > \mid \psi_j(\lambda) \mid \text{ for } \forall j \neq i\}$. Next, if $i \in I$, it is possible to find an analytic arc $\lambda(\varepsilon) = (\varepsilon^{n_1}, \cdots, \varepsilon^{n_\Lambda})$ for some $n_1, \cdots, n_\Lambda$ such that :

$$order\ (\psi_i \circ \lambda)_{|\varepsilon=0} < order\ (\psi_j \circ \lambda)_{|\varepsilon=0} \ for \ all \ j \neq i.$$

Taking the division of $\delta(x,\lambda)$ given in theorem 12, one has :

$$(b)\ \delta(x,\lambda)=\sum_{i=1}^{\ell}\ \varphi_i(\lambda)h_i\ (x,\lambda),$$

Substituting $\lambda(\varepsilon)$ we obtain that :

$$(c)\ \delta(x,\lambda(\varepsilon))=\alpha_i\ h_i(x)\varepsilon^{n_i}+0(\varepsilon^{n_i})$$

for some $\alpha_i\neq 0$.

At this point we use a generalization of Il'yashenko's theorem (theorem 3.4) saying that for any analytic unfolding, every partial derivative of $\delta(x,\lambda)$ in the parameter λ at $\lambda=\lambda_0$ is *also quasi-analytic* (see a proof in [MMR] an d also in [R] for the 1-parameter case).

The proof is quite similar to the one given in chapter 3. We can apply it to the 1-parameter family $X_{\lambda(\varepsilon)}$ with difference map $\Delta(x,\varepsilon)=\delta(x,\lambda(\varepsilon))$. We find that $h_i(x)=\dfrac{1}{\alpha_i}\dfrac{\partial^{n_i}}{\partial\varepsilon^{n_i}}\ \Delta(x,\varepsilon)_{\varepsilon=0}$ is quasi-analytic. As a consequence, we have that $I\subset\{1,\cdots,s\}$ (for each index in I, the corresponding function h_i is quasi-analytic and so $D^\infty h_i\not\equiv 0$). It follows from (a) that $\bigcup_{i=1}^{s} W_i$ is a neighborhood of λ_0 and that there exists a value r, $0<r\leq 1$ such that $\bigcup_{i=1}^{s} V_i^r$ is also a neighborhood of λ_0.

Let be $f_1,\cdots,f_n,\cdots$ the sequence of monomial $x^i\ \omega^j$, $0\leq j\leq i$, indexed by order : $f_1=1$, $f_2=x\omega$, and so on.

Let be a minimal set of generators, adapted to 0 , and $h_1,\cdots,h_\ell$ the corresponding functions on σ. Let be $n_i=\mathit{order}\ h_i(0)\ :\ n_1<n_2\cdots<n_s<\infty$ and $n_j=\infty$ for $j\geq s+1$.

We will use the decomposition of the difference map $\delta_\lambda(x)$ in the ideal the generators $\varphi_1,\cdots,\varphi_\ell$ at some high order of differentiability $k(k>>s_\delta(0)$, the index at 0) :

$$\delta_\lambda(x)=\sum_{i=1}^{\ell}\ \varphi_i(\lambda)h_i(x,\lambda) \tag{5.100}$$

(we don't write the subscript k).

We can write, with $N=s_\delta(0)$, for $i=1,\cdots,\ell$:

$$h_i(x,\lambda) = \sum_{j=1}^{N} \alpha_{ij}\ (\lambda) j_g\ (x,\lambda) + \cdots + \psi_i\ (x,\lambda) \tag{5.101}$$

where $+\cdots$ is as usual a finite combination of monomials $x^i\omega^j$ and $\psi_i(x,\lambda)$ a $\mathcal{C}^k$, k-flat function at $x=0$.

We know that $\alpha_{ij}(\lambda_0) = 0$ if $j < n_i$ for $\forall i = 1,\cdots,\ell$.

We first want to rearrange the combination (100) in a new combination :

$$\delta_\lambda(x) = \sum_{i=1}^{N} \Phi_i(\lambda)\ H_i(x,\lambda) \tag{5.102}$$

with the following properties :

(i) $H_i(x,\lambda) = f_i(x,\lambda) + \cdots + \tilde{\psi}_i(x,\lambda)\ \ i = 1,\cdots,N$ and $\tilde{\psi}_i(x,\lambda)$, a $\mathcal{C}^k$ and k-flat function at $x=0$.

(ii) If $W_i^r = \{\lambda \in W_0 \mid \mid \Phi_0(\lambda) \mid \geq r \mid \Phi_j(\lambda) \mid,\ \forall j = 1,\cdots,N\}$.
Then $\bigcup_{i=1}^{s} W_{n_i}^r = W_0$ in some new neighborhood $[0,x_0] \times W_0$.
We proceed in two steps :

(i) For any j we change φ_j in a combination :

$$\varphi_j(\lambda) + \sum_{n_\ell < j} \beta_{j\ell}(\lambda)\varphi_{n_\ell}\ (\lambda)\ where\ \beta_{j\ell}(\lambda_0) = 0,$$

such that for the new minimal system one has (101) with the extra condition : $\alpha_{jn}\ (\lambda) \equiv 0$ for all $1 \leq p \leq s$ and $j > n_p$.

Clearly, the new minimal system $\varphi_1,\cdots,\varphi_\ell$ is adapted and taking perhaps new W_0 and r, we have preserved the condition (99).

(ii) We define now the new set of generators $\Phi_1,\cdots,\Phi_N$ (no longer a minimal one), by :

$$\begin{aligned}\Phi_1 &= \sum_{i=1}^{\ell} \varphi_i\ \alpha_{i1},\cdots,\Phi_{n_1-1} = \sum_{i=1}^{\ell} \varphi_i\ \alpha_{in_1-1}\\ \Phi_{n_1} &= \varphi_1\end{aligned}$$

$$\Phi_j = \sum_{i=1}^{\ell} \varphi_i \, \alpha_{ij} \ for \ n_1 + 1 \leq j < n_2 \qquad (5.103)$$
$$\Phi_{n_2} = \varphi_2 \ and \ so \ on.$$

Using that all the coefficients α_{ij} which enter in the above formula are zero at λ_0, we see that, restricting W_0 enough, and choosing $0 < r' \leq 1$ small enough, one has, for $W_i^{r'}$ defined as above :

$$W_{n_i}^{r'} \supset V_i^r \ for \ i = 1, \cdots, s$$

which is the property (ii), writing r in place of r'.

The property (i) for the factors H_i follows form the construction : we can take $H_1 = f_1, \cdots, H_{n_1-1} = f_{n_1}$ $H_{n_1} = h_1$, $H_{n_1+1} = \sum_{j \geq 2} \alpha_{n_1+1j} \ (\lambda) f_j$ and so on.

The remaining of the proof mimics the proof of theorem 4.2 part (i) for regular periodic orbits : we will prove that for $\lambda \in W_{n_i}^r$, $i \leq s$ one has less than n_i roots for $\{\delta_\lambda(x) = 0\}$ (restricting λ, x).

For this, we construct by an algorithm of "*derivation-division*" (division by non-zero functions on $]0, x_0[$) a sequence of functions $\delta_0 = \delta_1, \cdots, \delta_{n_1}$ such that the last one is locally non-zero, and the bound for the number of roots will follow from a recurrent application of Rolle's theorem. For our present case, derivation will produce non-bounded functions like x^{-k}. It is the reason we have arrange the expansion of δ_λ as in (102) with factor $H_i(x, \lambda)$ equivalent to $f_i(x, \lambda)$ for *all* λ (and not only for $\lambda = \lambda_0$ as in the smooth case) because we do not want to "leave behind us" some non-bounded function (with small coefficient). A proof similar to the one we need was obtained by M. El Morsalani for general expansions : $\Sigma\alpha_{ij} \ (\lambda) x^i \ \omega^j + \psi_k$ of finite codimension [E2].

Here, I want to give a more direct proof which treats each term in the sum (103) in the same way. For this, one needs a more general algebra of admissible functions than the one introduced in section 2.2. It will be the algebra of finite combinations of monomials :

$$x^{\ell + n\alpha_1} \ \omega^m \ with \ \ell, \ n, \ m \in Z \ (m \in Z \ and \ not \ in \ N) \ \ \alpha_1 = \alpha_1(\lambda).$$

We order these monomials in a partial order as in section 2.2. We introduce rational admissible functions $f = \dfrac{* \, x^{\ell + n\alpha_1} \ \omega^m + \cdots}{* \, 1 + \cdots\cdots}$ as in section 22, with the same convention for $*$ (a continuous function of λ, which is non-zero at λ_0).

To make computations more easy we extend a little the convention : $+\cdots$. We mean now a finite combination of monomials of order strictly greater than the *dominant monomial* $x^{\ell+n\alpha_1}\,\omega^m$, plus some $\psi : C^k$ and k flat at $x = 0$ (for the k assumed in the beginning).

For these new functions, the properties $a) - f)$ written in section 2.2. remain valid. Moreover, we notice that $f)$ remains valid for ω^m, $m \neq 0$ as dominant term.

We now explain the algorithm. At each step we will produce a sequence of rational admissible functions. We just write their dominant terms.

We begin with the sequence of N dominant terms for the H_i :

$$1 \prec x\omega \prec x \prec x^2\omega^2 \prec x^2\omega \prec x^2 \prec x^3\omega^3 \prec x^3\omega^2 \prec x^3\omega \cdots$$

deriving once, we have the sequence of new dominations :

$$0\ ,\ \omega \prec 1 \prec x\omega^2 \prec\ x\omega \prec x \prec x^2\omega^3 \prec x^2\omega^2 \prec x^2\omega \cdots$$

next, dividing by the lowest one obtains :

$$1 \prec \omega^{-1} \prec x\omega \prec x \prec x\omega^{-1} \prec x^2\omega^2 \prec x^2\omega \prec x^2 \cdots$$

which contains one less term.

We now explain the induction step :

- one supposes a sequence of rational admissible functions with strictly increasing sequence of dominant terms :

$$x^{j+n\alpha_1}\,\omega^m \ \textit{with}\ j \geq 1,\ n,\ m \in Z \ \textit{or}\ j = 0 \ \textit{and}\ n = 0$$

(for instance ω^n for $m \in Z$). We have to look at two different possibilities for the sequence of dominant terms :

(i) First possibility :

$$1 \prec \omega^{-\ell_1} \prec \cdots \prec \omega^{-\ell_v} \prec \cdots \prec x^{j+n\alpha_1}\,\omega^m \tag{5.104}$$

with : $1 \leq \ell_1 < \ell_v$ $(v \geq 1)$ and $j \geq 1$, the same n for all terms ; the first function is 1.

Then a derivation followed by the division by the first remaining term gives :

$$1 \prec \cdots \prec \omega^{-(\ell_v-\ell_1)} \prec \cdots \prec x^{j+(h+1)\alpha_1}\ \omega^m. \tag{5.105}$$

(ii) Second possibility :

$$1 \prec x^{j_0+n\alpha_1}\ \omega^{m_0} \prec \cdots \prec \omega^{j+n\alpha_1}\ \omega^m. \tag{5.106}$$

Now we have no "pure" term in ω, $1 \leq j_0 < j$, n, $m \in Z$, the same n for each term and the first function is 1.

The operation of derivation-division gives :

$$1 \prec \cdots \prec x^{j-j_0+n\alpha_1}\ \omega^{m-m_0}. \tag{5.107}$$

In the two cases, one obtains a similar sequence as the initial one, with one less term.

After j steps, the first j^{th} functions become zero and the $(j+1)^{th}$ one is transformed into 1.

We apply the operation of *derivation-division* to the sequence $H_1, \cdots, H_N$ and by linear combination to $\delta_\lambda(x) = \sum_{i=1}^{N} \Phi_i\ H_i$. After n_p steps for $1 \leq p \leq s$ we obtain a final function :

$$\Delta_\lambda^p(x) = \Phi_{n_p}.H_{n_p}^p + \Phi_{n_p+1}\ H_{n_{p+1}}^p + \cdots + \Phi_N\ H_N^p \tag{5.108}$$

where $H_{n_p}^p \equiv 1$ and the dominant term of each H_j^p for $j > n_p$ is of order strictly larger than 0 (order of 1). This means that $H_j^p(x,\lambda) \to 0$ for $(x,\lambda) \to (0,\lambda_0)$.

Taking the size of W_0 and x_0 small enough, one can suppose that $\mid H_j^p(x,\lambda) \mid < \dfrac{1}{Nr}$. Now take $\lambda \in W_{n_p}^r$:

$$\mid \Delta_\lambda^p(x) \mid \geq \mid \Phi_{n_p}(\lambda) \mid - \sum_{j \geq n_p+1} \mid \Phi_j(\lambda) \mid \mid H_j^p(x,\lambda) \mid \geq \mid \Phi_{n_p}(\lambda) \mid\ (1-r) \tag{5.109}$$

for $(x,\lambda) \in [0,x_0] \times W_0$.

Let be $\Sigma = \{\Phi_1 = \cdots = \Phi_N = 0\}$, the set of zeroes of the Bautin Ideal in W_0. We have :

$$\Phi_{n_p}(\lambda) = 0, \ \lambda \in W^r_{n_p} \Rightarrow \lambda \in \Sigma \cap W^r_{n_p}.$$

This means that for $\lambda \in W^r_{n_p} - \Sigma$, the function $\Delta^p_\lambda(x)$ has no zero on $]0, x_0[$. But this function differs from δ_λ by n_p derivations and so, $\delta_\lambda(x)$ has no more than n_p isolated zeros on $]0, x_0[$ for $\lambda \in W^r_{n_p}$.

Applying this argument on each set $W^r_{n_p}$, $p = 1, \cdots, s$ we obtain the first part of theorem 13.

Proof of part (ii) of theorem 13.

The proof is very similar to the one given for the smooth case, in theorem 4.2 : if $\varphi_1, \cdots, \varphi_\ell$ are independent functions of λ, we can choose local coordinates $\lambda_1, \cdots, \lambda_\ell, \cdots, \lambda_\Lambda$ in the parameter space, with $\varphi_i = \lambda_i$, $i = 1, \cdots, \ell$ and $\lambda_0 = (0, \cdots, 0)$. Supposing that $\varphi_1, \cdots, \varphi_\ell$ was adapted at 0, one has :

$$\delta_\lambda(x) = \sum_{i=1}^{\ell} \lambda_i \ h_i(x, \lambda)$$

with order $h_i(0) = n_i$, $n_1 < n_2 < \cdots < n_s < \infty$, $n_j = \infty$ if $j \geq s+1$.

Restrict ourselves to the subfamily $\lambda_{s+1} = \cdots = \lambda_\ell = 0$; we go on to write λ for $(\lambda_1, \cdots, \lambda_s)$: $\delta_\lambda(x) = \sum_{i=1}^{s} \lambda_i \ h_i(x, \lambda)$ with the orders as above. As in the regular case, the key point is to observe that the sequence $h_1(x, 0), \cdots, h_s(x, 0)$ is a regular Chebychev family of germs at 0. This point was proved for these functions with dominant term $x^i(Lnx)^j$, $i \geq j \geq 0$ by P. Joyal [J2]. The end of the proof is exactly the same as in the smooth case, and we conclude that in this case, $Cycl(x_\lambda, \Gamma) \geq s - 1$.

5.3.2 An example in Quadratic vector fields

We finish this paragraph, giving an application of theorem 13 to quadratic vector fields. We return to the example introduced in 4.3.5.3. There we looked to the 6-parameter family of Kaypten-Dulac (4.39) for the value $\lambda_0(\lambda_6 = -1, \ \lambda_1 = \cdots = \lambda_5 = 0)$.

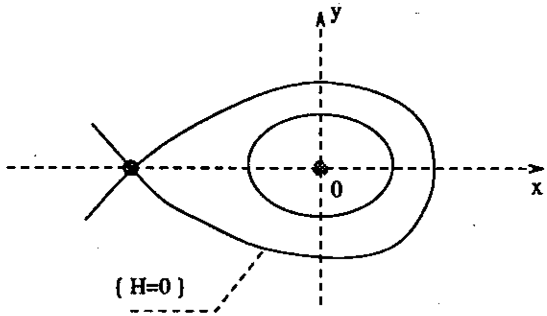

Figure 5.11:

For this parameter value, the vector field is hamiltonian with Hamilton function $H(x,y) = \frac{1}{2}\ y^2 + \frac{1}{2}\ x^2 + \frac{1}{3}\ x^3$. We have computed the cyclicity of the smooth cycles ofX_{λ_0}. Now, we want to look at the saddle connection for $X_{\lambda_0} : \Gamma \subset \{H = \frac{1}{6}\}$. (see Figure 5.11).

We have seen that the Bautin Ideal at λ_0 is generated by λ_1, λ_5, $\lambda_2\ \lambda_4$, so that :

$$\delta_\lambda(x) = \varphi_1(\lambda)h_1(h,\lambda) + \varphi_2(\lambda)h_2(h,\lambda) + \varphi_3(\lambda)h_3(h,\lambda)$$

with $\varphi_1 = -2\lambda_1$, $\varphi_2 = -\lambda_5$, $\varphi_3 = -\frac{2}{33}\ \lambda_2\lambda_4$ with :

$$h_1(h,\lambda_0) = I_0(h)\ ,\ h_2(h,\lambda_0) = I_1(h)\ and\ h_3(h,\lambda_0) = hI_0 - 3I_1.$$

Let $u = \frac{1}{6} - h$ a local parametrization of a transversal σ at Γ ($\Gamma \cap \sigma = \{u = 0\}$). It is easy to prove directly that any abelian integral $I(u) = \int_{\gamma_u} \omega$ for any algebraic 1-form ω has a Dulac series at $u = 0$:

$$I(u) = \gamma_0 + \gamma_1 uLnu + \gamma_2 u + o(u).$$

So that :

$$\begin{aligned} h_1 &= I_0(u) = \alpha_0 + a_1 uLnu + a_2 u + 0(u) \\ h_2 &= I_1(u) = b_0 + b_1 uLnu + b_2 u + 0(u) \\ h_3 &= (\frac{1}{6}a_0 - 3b_0) + (\frac{1}{6}a_1 - 3b_1)uLnu + (\frac{1}{6}a_2 - 3b_2 - a_0)u + o(u). \end{aligned}$$

It follows from these expansions that h_1, h_2, h_3 are independent at order 2 (up to the "$uLnu$"-order) if and only if the 3×3 determinant :

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ \frac{1}{6}a_0 - 3b_0 & \frac{1}{6}a_1 - 3b_1 & \frac{1}{6}a_2 - 3b_2\ a_0 \end{vmatrix} \neq 0.$$

This is equivalent to $a_0 \neq 0$ and :

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \neq 0$$

The first condition is trivially fulfilled : a_0 is equal to the area of the disk bounded by Γ. To verify the second condition, we recall that the ratio $B_2(h) = \frac{I_1}{I_0}(h)$ is equal to :

$$B_2(h) = \frac{3}{4}\ \sqrt{2}B(\frac{\sqrt{3}}{8}\ (h - \frac{1}{12})) - \frac{\sqrt{6}}{2}$$

where $B(h)$ is solution of a Ricatti equation :

$$9(\frac{4}{27} - h^2)\ \frac{dB}{dh}\ = -7B^2 - 3hB + \frac{5}{3}. \tag{5.110}$$

For $h = \frac{1}{6}$, $\frac{\sqrt{3}}{8}(h - \frac{1}{12}) = \frac{2}{3\sqrt{3}}$ and $B_2(u)$ has a Dulac series equivalent to the one of $B(\frac{2}{3\sqrt{3}} - u) = \overline{B}(u)$.

We have : $B_2(h) = \frac{b_1}{b_0} + \frac{a_1\ b_0 - a_1\ b_1}{b_0^2}\ uLnu + o(uLnu)$.

So that, to prove that $a_1\ b_0 - a_0\ b_1 \neq 0$ is equivalent to verify that :

$$\overline{B}(u) = \gamma_0 + \gamma_1 uLnu + o(uLnu),\ with\ \gamma_1 \neq 0. \tag{5.111}$$

To compute γ_1, we write the Ricatti equation in $\overline{B}$, deduced from (110) :

$$9u(\frac{4}{3\sqrt{3}}-u)\overline{B}'=7\overline{B}^2+3(\frac{2}{3\sqrt{3}}-u)\overline{B}-\frac{5}{3} \qquad (5.112)$$

and introducing expansion (111) in (110) allows to compute the coefficients γ_i by an induction on i and to verify that $\gamma_1 \neq 0$.

Now, the independence of h_1, h_2, h_3 at order 2 is equivalent to $s_6(0) = 2$. We deduce from this, that in the Kaypten-Dulac family :

$$Cycl(X_\lambda, \Gamma) \leq 2.$$

Now, as in chapter 4, we can used Horozov's result for "generic hamiltonian" : Horozov proved the cyclicity of the saddle connection for such a generic hamiltonian is 2 . The semi-continuity of cyclicity implies that $Cycl(X_\lambda, \Gamma) \geq 2$ and so, finally :

Theorem 26 *The cyclicity of the saddle connection for $\dot{x} = y$, $\dot{y} = x + x^2$ in the quadratic Kaypten-Dulac family is equal to 2.*

5.4 Unfoldings of elementary graphics.

I want to consider now the general elementary graphics, where one has more than one vertice and also the possibility to have semi-hyperbolic points among them.

5.4.1 Hyperbolic graphic with 2 vertices.

Suppose that X_λ is an unfolding in a neighborhood of a graphic Γ with two hyperbolic saddles p_1, p_2 as vertices for $\lambda = \lambda_0 = 0$. Let be $r_1(\lambda)$, $r_2(\lambda)$ the hyperbolicity ratios of p_1, p_2, which are supposed to be hyperbolic saddle for each $\lambda \in W_0$, some neighborhood of $0 \in P$, the parameter space. Let $r_1 = r_1(0)$, $r_2 = r_2(0)$. We have a first result, proved by Cherkas [C]. I follow here a proof given in [DRR2].

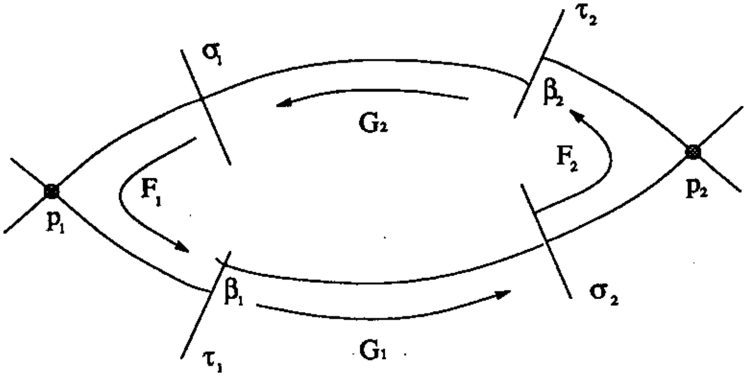

Figure 5.12:

Theorem 27 *Suppose fulfilled the following generic conditions*

$$r_1.r_2 \neq 1,$$

$r_1 \neq 1$, $r_2 \neq 1$. *Then Cycl* $(X_\lambda, \Gamma) \leq 2$.

Proof We can suppose that $r_1.r_2 < 1$. (This means that the Poincaré map is expanding). We can write the Poincaré map as a composition of four transitions : $P(x, \lambda_1) = G_2 \circ F_2 \circ G_1 \circ F_1(x_1, \lambda)$ (see Figure 5.12:).

Here x_1 is a coordinate on the transversal segment σ_1. The transitions G_1, G_2 are differentiable and the transitions F_1, F_2 have the Mourtada's form :

$$F_1(x_1) = x_1^{r_1(\lambda)}(1 + \Phi_1(x_1, \lambda)) \tag{5.113}$$
$$F_2(x_2) = x_2^{r_2(\lambda)}(1 + \Phi_2(x, \lambda)) \tag{5.114}$$

We have two different cases to consider :

1) $r_1 < 1$, $r_2 < 1$. In this case $\dfrac{\partial F_1}{\partial x_1} \sim x_1^{r_1(\lambda)-1}$ and $\dfrac{\partial F_2}{\partial x_2} \sim x_2^{r_2(\lambda)-1}$. So, the derivative $\dfrac{\partial P}{\partial x_1}(x_1, \lambda) \to \infty$ when $(x, \lambda) \to (0,0)$ and one has

at most one limit cycle near Γ, which is hyperbolic and expanding : $\mathcal{C}ycl\ (X_\lambda, \Gamma) \leq 1$.

2) **Up to the ordering of p_1, p_2 : $r_1 > 1$ and $r_2 < 1$.** In this case, instead of studying the solutions of $P(x_1, \lambda) = x_1$ we rather study the zeros of :

$$H(x_1, \lambda) = G_1 \circ F_1(x_1, \lambda) - F_2^{-1} \circ G_2^{-1}(x, \lambda) = 0. \tag{5.115}$$

We can write :

$$G_1(y_1) = y_1 + \beta_1(\lambda),\ G_2^{-1}(x_1) = x_1 + \beta_2(\lambda) \tag{5.116}$$

$$F_1(x_1) = x_1^{r_1(\lambda)}(A_1(\lambda) + \Phi_1(x_1, \lambda)) \text{ and } F_2^{-1}(x_2) = x_2^{s_2(\lambda)}(A_2(\lambda) + \Phi_2(x_1, \lambda)).$$

Here we put $s_2(\lambda) = \dfrac{1}{r_2(\lambda)}$. We have included the non-linear part of G_1, G_2^{-1} in F_1, F_2, using the fact that the smooth diffeomorphisms are in $\mathcal{D}$ (see theorem 4) ; Φ_1, $\Phi_2 \in I$ and $A_1(0)$, $A_2(0) > 0$. We obtain :

$$H(x, \lambda) = x^{r_1}(A_1 + \Phi_1(x, \lambda)) + \beta_1 - X^{s_2}(A_2 + \Phi_2(X, \lambda)). \tag{5.117}$$

Here $x = x_1$, $X = x_1 + \beta_2$ and $\beta_1(\lambda)$, $\beta_2(\lambda)$ are the shift functions on the transversal segment τ_1 and τ_2 (see Figure 5.12).

A first derivation of H gives :

$$H'(x_1) = r_1 x^{r_1 - 1}\ (A + \Phi_1(x, \lambda)) - s_2\ X^{s_2 - 1}\ (A_2 + \Phi_2(X, \lambda)) \tag{5.118}$$

with ψ_1, $\psi_2 \in I$.

Hence, we can observe that $H' \neq 0$ when $X < x$ and x sufficiently small, i.e. there is at most one limit cycle for that position of the separatrices.

Zeros of $H'(x)$ are the same as zeros of :

$$K(x) = x(B_1 + \xi_1(x, \lambda)) - \left(\frac{s_2}{r_1}\right)^{\frac{1}{r_1 - 1}}\ X^{\frac{r_2 - 1}{r_2 - 1}}\ (B_2 + \xi_2(X, \lambda)) \tag{5.119}$$

with ξ_1, $\xi_2 \in I$ and $B_1(0)$, $B_2(0) > 0$.

We show that $K'(x)$ does not vanish in a neighborhood of $(x, y) = (0,0)$ yielding a maximum of two zeros of $H(x)$. Indeed :

$$K'(x) = [B_1 + 0(1)]-$$

$$\left[\left(\frac{s_2}{r_1}\right)^{\frac{1}{r_1-1}} \frac{s_2 - 1}{r_1 - 1} X^{\frac{r_2-r_1}{r_1-1}} (B_2 + \xi_2(X, \lambda))\right]. \tag{5.120}$$

The first term stays larger than a constant $M > 0$ while the second one goes to zero as $(x, \lambda) \to (0,0)$. ∎

Generic two parameter families are defined by the condition : $\lambda \in R^2 \to (\beta_1(\lambda), \beta_2(\lambda))$ has rank 2 at $0 \in R^2$.

Such families are structurally stable and versal. One can find diagrams of bifurcation in [M3], [DRR2] for instance. The cyclicity of these versal families is 1 in the case $(r_1 - 1)(r_2 - 1) > 0$ and 2 in the case $(r_1 - 1)(r_2 - 1) < 0$.

A general study of the unfoldings for these 2-hyperbolic graphics was begun by Mourtada in [M3], [M4].

He classified the possibilities in the following cases :

$$\begin{aligned} (\mathcal{C}_1) &: r_1 r_2 \neq 1 \\ (\mathcal{C}_2) &: r_1 r_2 = 1 \textit{ and } r_1 \notin Q \\ (\mathcal{C}_3) &: r_1 r_2 = 1 \textit{ and } r_1 \in Q \end{aligned} \tag{5.121}$$

For the case $(\mathcal{C}_1)$ new problems arise when r_1 or $r_2 = 1$. One has to use the expansion in x, ω given in proposition 5. The result, rather unexpected, is that the cyclicity is ≤ 2. For 3-parameter generic families (if $r_1(0) = 1$ for instance, we suppose $r_1'(0) \neq 0$), a bifurcation diagram is given in [M3] and [DRR2].

In [M3] Mourtada looked at the case $(\mathcal{C}_2)$ under the assumption the Poincaré map was not flat to identity. He proved that (X_0, Γ) has an absolute finite cyclicity, i.e. with a bound depending only on (X_0, Γ) and not on the unfolding (X_λ, Γ). Moreover, he computed this cyclicity in term of the resonant quantities at the saddle and of the Dulac series of the return map along Γ. The more interesting fact is that this absolute cyclicity does not depend just on the return map. In [M4] Mourtada considered the case ($\mathcal{C}_2$) for an analytic unfolding (X_λ, Γ) such that X_0 has an identity return map. Using a division in the Bautin ideal (like in paragraph 3 for the homoclinic connection), he proved that (X_λ, Γ) has a finite cyclicity, of course not absolute but depending on the given unfolding (X_λ, Γ). As in paragraph 3 above he used a (x, ω) expansion of some shift function $\delta_\lambda(x)$, and divided it in the Bautin Ideal. Here the "compensator" $\omega(x, \lambda)$ is equal to $\omega(x, \lambda) = \dfrac{x^{-\alpha(\lambda)} - 1}{\alpha(\lambda)}$ where $\alpha(\lambda) = 1 - r_1(\lambda) r_2(\lambda)$.

The last case ($\mathcal{C}_3$) was studied in [EM]. One of the difficulties for this case is that one has to work with two independent "compensators" $\omega_1(x, \lambda)$ and $\omega_2(x, \lambda)$ associated to each vertice p_1, p_2. In [E.M], the authors used an idea, already introduced in El Morsalani's thesis [E1] : one eliminates one of the compensators by the "Khovanski-Moussu" procedure, I explain below. They obtained partial results for this case. For instance, if the shift function $\delta(x, 0)$ is equivalent to $x^n Log^m x$ for some n, $m \in N$ (they say that $\delta(x, 0)$ has a logarithmic order), then the cyclicity of any unfolding (X_λ, Γ) is bounded by $\dfrac{n(n+5)}{2}$.

So finally the proof that any analytic unfolding of hyperbolic 2-graphics is almost complete. It remains to look at the case $r_1(0) = r_2(\lambda)^{-1} \in Q$ and $\delta(x, 0) \sim x^n$ or equal to identity.

Similar results were obtained by A.Jebrane and H.Zoladek [JZ], and A.Jebrane and A.Mourtada [JM] for the "figure height"-graphic.

5.4.2 Generic unfoldings of hyperbolic k-graphics.

An hyperbolic k-graphic Γ will be an hyperbolic graphic with k vertices : $p_1, \cdots, p_k$. Let be X_λ an unfolding of such a graphic defined for X_0 (see Figure 5.13). We label the vertices in circular order ($p_{k+1} = p_1$). Taking σ_i a transversal segment between the vertices p_i, p_{i+1}, we have a well defined shift function $\beta_i(\lambda)$ on each σ_i, difference between the first intersection of the unstable separatrix of p_i and the first intersection of the stable separatrix p_{i+1}.

We look to generic k-parameter families. A first generic conditions is :

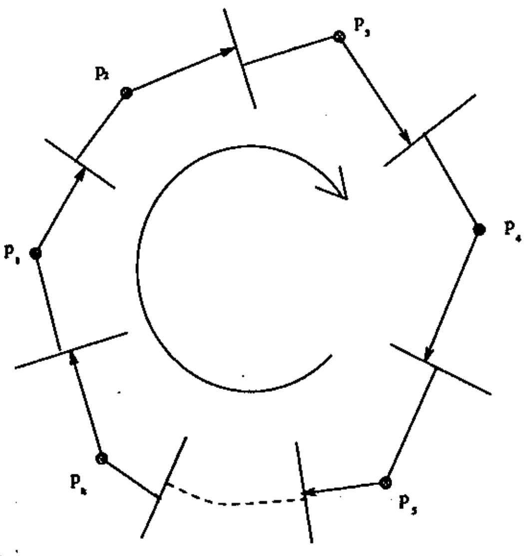

Figure 5.13:

(H1) : $\lambda \to (\beta_1(\lambda), \cdots, \beta_k(\lambda))$ has rank k at $0 \in R^k$. (Observe that $\beta_1(0) = \cdots = \beta_k(0) = 0$).

Next we suppose that the hyperbolic saddle points $p_1, \cdots, p_k$ remains for $\forall\lambda \in W_0$ (some neighborhoods of $0 \in R^k$). Let be $r_1(\lambda), \cdots, r_k(\lambda)$ the hyperbolic ratios.

Another type of generic conditions are put on (X_0, Γ) itself through the hyperbolic ratio $r_i = r_i(0)$. These conditions say that Γ and any subgraphics which may be created by unfolding have a "non-degenerate" return map :

(H2) : For each subset $J \subset \{1, \cdots, k\} : \prod_{j \in J} r_j \neq 1$.

The conditions (H1), (H2) are the generic conditions introduced above for the case $k = 2$. Mourtada in [M1], [M2] proved a general result of finite cyclicity for generic unfoldings of hyperbolic k-graphics :

Theorem 28 *Let (X_0, Γ) a generic smooth hyperbolic k-graphic. The generic conditions included the conditions (H2) among other explicit rational conditions on the ratios r_i. Let (X_λ) a generic unfolding of (X_0, Γ), i.e. satisfying (H1). Then $Cycl(X_\lambda, \Gamma)$ is finite.*

Moreover there exists a function $K(k) : N \to N$ such that

$$Cycl(X_\lambda, \Gamma) < K(k)$$

(the bound is uniform, independent on the unfolding X_λ). $K(2) = 2$, $K(3) = 3$, $K(4) = 5$ and $K(k)$ is given in a recursively way by a formula given in [M2].

Remark 31 *The cyclicity of (X_λ, Γ) is absolute in the sense that it depends only on (X_0, Γ) (in fact just on the number of vertices !). The generic conditions in theorem 16 are rational inequalities in $(r_1, \cdots, r_k)$. They define an open dense semi-algebraic subset U in R^k (space for $(r_1, \cdots, r_k)$) and for each connected component of U one has a given cyclicity. Mourtada made it explicit in his thesis for $k = 3, 4$. In the case $k = 2$, we have two connected components $U_1 = \{r_1 r_2 \neq 1,\ (r_1 - 1)\ (r_2 - 1) > 0\}$ with cyclicity 1 and $U_2 = \{r_1 r_2 \neq 1,\ (r_1 - 1)\ (r_2 - 1) < 0\}$ with cyclicity 2.*

A striking fact is that the cyclicity may be strictly bigger than the number of parameters, in contradiction with a previous conjecture of Sotomayor : for instance it may be equal to 5 for $k = 4$.

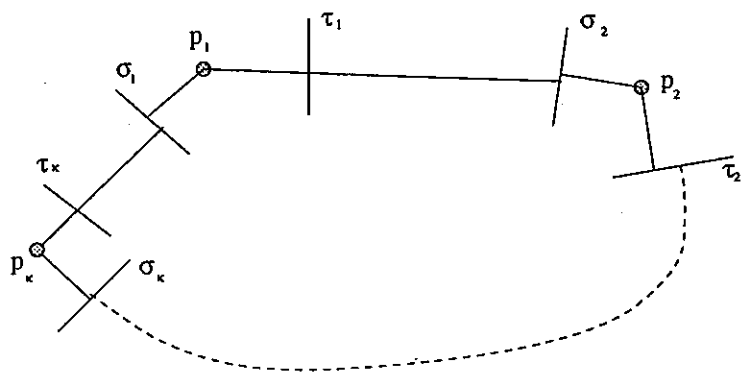

Figure 5.14:

It is not possible to give here even a partial proof of this result (this proof in [M1], [M2] cover more than 100 pages !). I just want to give a rough idea of some important steps.

We introduce $2k$ transversal segments σ_i, τ_i, $i = 1, \cdots, k$ (see Figure 5.14:) and want to look at the return map on σ_1.

To have a heuristic idea about how looks like the return map, suppose each transition near a saddle is : $x_i \to x_i^{r_i}$ (as if X_λ was linear near p_i) and that the regular map near τ_i and σ_{i+1} is a translation by β_i. (Of course one can suppose, under the assumed generic conditions, that $\lambda = (\beta_1, \cdots, \beta_k)$).

Then, the return map P is equal to :

$$P(x, \lambda) = \Big(\cdots((x_1^{r_1} + \beta_1)^{r_2} + \beta_2)\cdots\Big)^{r_k} + \beta_k. \tag{5.122}$$

In a first part of his proof, in [M1], Mourtada proved that this normal form (5.122) may be chosen, up to some non essential perturbation. To give this result more precisely, one has to generalize the property I^k introduced in paragraph 2. Suppose that $\eta(\lambda)$ is some continuous function with $\eta(0) = 0$. One says that a function $f(x, \lambda)$ defined in a neighborhoods of $(x, \lambda) = (0, 0)$ in the set $\{x \geq \eta(\lambda)\}$ is of "class I_λ^k for $(x - \eta(\lambda))$" if it is C^k, and verify the properties i), ii) in the definition of the class I^k, with x replaced by $x-\eta(\lambda)$ and the limit $(x-\eta)^j \frac{j^j f}{\partial x^j} \to 0$ supposed for $(x,\lambda) \to (0,0)$ (and not uniformly in λ !) on its domain of definition.

We can define two natural functions $\rho(\lambda)$, $\eta(\lambda)$ in the following way :

1) $\rho(\lambda) = Inf\ (x \in \sigma_1)\ P(x,\lambda) : \sigma_1 \to \sigma$ is defined }.

Of course this means that the trajectory of $\rho(\lambda) \in \sigma_1$ tends towards some p_i and $\rho(\lambda)$ is the largest value of x with this property ; $P(x,\lambda)$: $[\rho(\lambda), \varepsilon[$ for some fixed ε and $\rho(\lambda)$ is a continuous function with $\rho(0) = 0$.

2) We can define a function $\tilde{\eta}(\lambda)$ in a similar way as above for the trajectories of $-X_\lambda$, starting from τ_k. If $G_k(x,\lambda)$ is the transition map from τ_k to σ_1 one defines $\eta(\lambda) = G_k(\tilde{\eta}(\lambda),\lambda)$. It is a continuous function with $\eta(0) = 0$. Moreover $\rho(\lambda) \geq 0$, $\rho(\lambda) \geq \eta(\lambda)$ and we may have $\eta(\lambda) < 0$.

Let $F_1, \cdots, F_k$ the transition maps near the saddles, $F_i : \sigma_i \to \tau_i$ and $G_1, \cdots, G_k$ the regular transitions, $G_i : \tau_i \to \sigma_{i+1}$. For technical reasons, Mourtada prefer to replace the return $P(x,\lambda)$ by the shift :

$$F_k \circ \cdots \circ G_2 \circ F_2 \circ G_1 \circ F_1 - G_k^{-1} = \Delta(x_1,\lambda). \tag{5.123}$$

The roots of $\Delta(x_1,\lambda)$ on the domain $\mathcal{U} = \bigcup_{\lambda \in W_0}]\rho(\lambda), \varepsilon[\times \{\lambda\}$ correspond to limit cycles sufficiently near Γ.

It is now possible to give the first result of Mourtada :

Theorem 29 *(Reduction to the normal form [M1]).*
Let be $\rho(\lambda)$, $\eta(\lambda)$ and the domain $\mathcal{U}$ introduced above.
Then :

$$\Delta(x_1,\lambda) = \left(\cdots (x_1^{r_1} + b_1)^{r_2} + \cdots b_{k-1}\right)^{r_k} + b_k \varphi(x_1,\lambda). \tag{5.124}$$

Here $b_1, \cdots, b_k$ are continuous functions of λ ($b_i(\lambda) = 0 \iff \beta_i(\lambda) = 0$) ; $\varphi(x_1,\lambda) = x_1(\alpha(\lambda) + f(x_1,\lambda))$ where $\alpha(\lambda)$ is positive and continuous on W_0 and $f(x,\lambda)$, defined on $V = \bigcup_{\lambda \in W_0}]\eta(\lambda), \varepsilon[\times \{\lambda\}$, is of class I_λ^k for $(x - \eta(\lambda))$.

Proof A rough idea for the proof of theorem 17 is as follows : taking any coordinate y_1 on τ_1, the Mourtada's form for F_1 is $F_1(x,\lambda) = x^{r_1}\ (A_1(\lambda) + \Phi_1(x,\lambda))$, with $A_1(0) > 0$. Now, it is possible to change the coordinate y_1 by a change $\bar{y}_1 = y_1(1 + \psi_1(x,\lambda))$, with ψ_1 of class I^k such that F_1 writes : $x_1 \to \bar{y}_1 = x^{r_1}$. And so on, one can change the coordinate on each transversal segment to reduce each regular map to a translation and each saddle transition to $x_i \to x_i^{r_i}$. Of course the change of coordinate may have some singularity at other points than 0 : for this reason, one has to work with the larger class I_λ^k for some functions $x - u(\lambda)$. Also, the change of coordinates on transversal segments explains why the shift functions $\beta_i(\lambda)$ are replaced by the new functions $b_i(\lambda)$. At the end of the construction we have reduced the transition map $\sigma_1 \to \tau_k$ to its "normal form" and transition G_k^{-1} to $\varphi - b_k$. ∎

To obtain the computation of the number of roots of $\{\Delta(x_1,\lambda) = 0\}$ a rough idea is that in the computations, the term φ plays no role in comparison with the principal normal part. So that, one is reduced to study the roots of this principal part. This was done in [M2], which is devoted to the "finiteness algorithm" for the roots of $\{\Delta = 0\}$. I just indicate the first steps of this algorithm, in the case $k = 4$.

We define recurrently :

$$h_j(x,\lambda) = [h_{j-1}(x,\lambda)]^{r_j} + b_j$$

$$\textit{and}\ \Delta_\lambda(x) = \Delta(x,\lambda) = h_4 - x(1+f). \qquad (5.125)$$

1) **First step :** We eliminate b_4 by a derivation (in x) :

$$\Delta'_\lambda(x) = h_3^{r_4-1}\ h_2^{r_3-1}\ h_1^{r_2-1}\ h_0^{r_1-1}\ - (1 + f_{1,0}(x,\lambda)) \qquad (5.126)$$

$*$ is some constant and $f_{1,0}$ of I_λ^k.

2) **Second step :** (Linearization in h_3 and elimination:

$$\begin{aligned} \Delta'_\lambda(x) &= 0 \textit{ is equivalent to } : \\ \Delta_{1,\lambda}(x) &= *(1 + f_{1,1})h_2^{z_3}\ h_1^{z_2}\ h_0^{z_1} - h_3 = 0 \end{aligned} \qquad (5.127)$$

with $z_j = \dfrac{r_j - 1}{1 - r_4}$.

One can derive :

$$\Delta'_{1,\lambda}(x) = *(1+f_{2,0})h_2^{z_3-1}\, h_1^{z_2-1}\, h_0^{z_1-1} - r_1r_2r_3\, h_2^{r_3-1}\, h_1^{r_2-1}\, h_0^{r_1-1} \quad (5.128)$$

with :

$$\Sigma = r_1r_2z_3\, h_1^{r_2}\, h_0^{r_1} + r_1z_2h_2h_0^{r_1} + z_1(1+f_{2,1})h_2h_1.$$

3) **Third step.** (Simplify $\Delta'_{1,\lambda}$ and derive).

One divides $\Delta'_{1,\lambda}$ by $h_2^{r_3-1}\, h_1^{r_2-1}\, h_0^{r_1-1}$. Then $\Delta'_{1,\lambda}$ has the same roots as :

$$\Delta_{2,\lambda}(x) = *(1+f_{2,0})h_2^{y_3}h_1^{y_2}h_0^{y_1}\Sigma - r_1r_2r_3 \text{ } with \text{ } y_j = z_j - r_j. \quad (5.129)$$

The derivate of $\Delta_{2,\lambda}$ has the same roots as :

$$\Delta_{3,\lambda}(x) = Q^{2,2}(x,\lambda) = [h_2]^2\, Q^{2,1} + h_2\, S_2Q^{1,1} + (S_1)^2\, Q^{0,1} \quad (5.130)$$

$$with\ S_2 = h_1^{r_2}h_0^{r_1},\ S_1 = h_0^{r_1}. \quad (5.131)$$

$Q^{n,1}$ is a function which is homogeneous of degree n in (h_1, S_1), having as coefficients functions $Q^{n,0} = * + f_{i_m,j_m}$.

(Everywhere $*$: is a positive constant and $f_{i,j}$ a function of class I_λ^k).

I stop here this computation. I have just presented the text of Mourtada to show that the first two steps are of similar nature : one linearizes the expression in the parameter b_4 and next b_3, and one gets rid off it by a derivation. But unfortunately this process stops at the third step and the last expression is just quadratic in b_2. From now on, Mourtada uses a more sophisticated argument based on the fact that the expression is polynomial in the parameter with coefficients whose he can control in a recurrent way.

This change in the algorithm at the third step explains why the formula for $K(k)$ changes at $k = 3$:$K(2) = 2$, $K(3) = 3$, but $K(4) = 5$.

5.4.3 Generic elementary polycycles.

In [IY3], Il'yashenko and Yakovenko obtained a general result of finite cyclicity for elementary graphics.

Theorem 30 *For any $n \in N$, there exist a number $E(n) \in N$ (called "elementary bifurcation number") such that any generic unfolding o f elementary graphic with n parameters has a cyclicity bounded by $E(n)$.*

Remark 32 *This result seems to generalize the previous one of Mourtada which just looked at generic n-parameter unfoldings of hyperbolic n-graphics. In the result of Il'Yashenko-Yakovenko the genericity assumption implies that the number k of vertices is less than the number n of parameters. Mourtada obtained an explicit bound in function of k. On the contrary, although the function $E(n)$ is a "primitive recursive" function, i.e. computable by an algorithm, the practical computation of it remains an open question. Another interesting problem would be to obtain an explicit definition for the genericity one needs in the theorem 18. It will be important to know these generic conditions in order to apply the result in given bifurcation problems.*

I recall that in theorem 16 of Mourtada, the genericity conditions are explicit rational inequalities in the hyperbolicity ratios r_i and also the independence of the shift maps β_i.

The biggest interest of theorem 18 is to prove the existence of $E(n)$. We can also introduce more particular bounds like: $E(k,n)$ for the generic elementary graphics with k vertices and n parameters, $H(k,n)$, $H(n)$ for hyperbolic graphics. Of course $E(n) = Sup\ \{E(k,n) \mid k \leq n\}$, $H(n) = Sup\{H(k,n) \mid k \leq n\}$.

These numbers are known for small values of n. (For instance $H(2) = 2$). A review about these results and also a description of all generic elementary unfoldings for $n \leq 3$ was made by Kotova, Stanzo [K-S].

Sketch of proof.

The proof appeared in [IY3]. It is rather long and profound. A good introduction for it and related results was made in [IY2]. I just quote the sketch of proof from this article (quoted in italics). Personal comments are added between quotation marks :

(The proof) consists of four principal steps :

1) C^k-smooth normalization of the family near each elementary singularity. The main tool here is provided by the classification theorems from [YI1] ("see also paragraph 1, [Bon]"). The polynomial normal forms are integrable. We perform an explicit integration of normal forms in the class of Pfaffien functions introduced by Khovanskii [K] and show that the correspondence maps near each singular points in the normalization coordinates can be expressed through elementary transcendental functions which satisfy Pfaffian equations ("see remark below"). The degree and the total number of these equations can be estimated in terms of n.

2) Algebraization of the system of equations obtained at the previous step : the reduction procedure suggested in [K] allows to eliminate transcendental functions from the equations determining fixed points of the monodromy map. After this elimination, there appears a system of equations having the form of a chain map, a composition of a polynomial map and a jet of a generic smooth map.

3) Gabrielov-type finiteness conditions are established for a smooth map $F : R^k \to R^k$ to have a uniformly bounded number of regular preimages $\neq F^{-1}(y)$ when the point y varies over a compact subset of R^k. These conditions are automatically satisfied if a map F is real analytic. We introduce a topological complexity characteristics, the contiguity number, in terms of which an upper estimate for the number of preimage can be expressed.

4) Thom-Boardman-type construction allows us to prove that the above finiteness conditions can be expressed in terms of transversality of the jet extension of Γ to some semi-algebraic subsets of the jet space. Moreover, this construction can be generalized to cover chain maps of the form $P \circ (j^\ell F)$, where P in a polynomial, and $j^\ell F$ is the ℓ-jet extension of a generic smooth map. This is exactly the class of maps which appear after Khovanskii elimination procedure (step 2 above). The contiguity number of a chain map is expressed through the integer-valued data (degree of the polynomial P, order of the jet ℓ and dimension of the domain and target spaces).

Remark 33 *The Khovanskii elimination procedure was firstly used for analytic vector fields by Moussu and Roche in [MoR] to prove the non-accumulation of limit cycles on elementary graphics (see chapter 3) under the assumption : the normal form at each singular point in the graphic is convergent. For instance, we have seen above that at a singular hyperbolic saddle with resonance $p:q$ the normal form writes :*

$$\dot{x} = x \ , \ \dot{y} = \frac{1}{q}\Big(-p + \sum_{i=0}^{\infty} \alpha_{i+1}(x^p y^q)\Big)y.$$

Putting $u = x^p y^q$ this system is equivalent to :

$$\dot{x} = x, \quad \dot{u} = P(u) = \sum_{i=1}^{\infty} \alpha_i u^i. \tag{5.132}$$

Assumption is that P is convergent.One can obtain an analytic first integral for (5.132) : the dual 1-form of (5.132) $\omega = x du - P(u)dx = xP(u)dF$ with :

$$F(x,y) = Q(u) - Log x \ where Q(u) = \int_a^u \frac{ds}{P(s)}. \tag{5.133}$$

Now let $y = D(x)$ the transition map from $\{y = 1\}$ to $\{x = 1\}$. Using the first integral F, we have :

$$F(x,1) = F(1, D(x)). \tag{5.134}$$

So that $\{y = D(x)\}$ is graph of an integral curve of the following 1-form Ω :

$$\Omega = \frac{\partial F}{\partial x}(x,1)dx - \frac{\partial F}{\partial y}(1,y)dy.$$

Using the expression (5.132) for F, it is easy to compute that Ω is equal, up some analytic factor, to :

$$\Omega = (px^p - P(x^p))P(y^a)dx - yy^a xP(x^p)dy. \tag{5.135}$$

So, $\{y = D(x)\}$ is graph of integral curve of the analytic 1-form Ω. One says that the function $D(x)$ satisfies the Pfaffian equation $\Omega = 0$. The computation above is an example of the reduction procedure recalled in point 2 of the proof sketch.

Another example of the same procedure was used by El Morsalani in [E1] to eliminate one compensator $\omega = \dfrac{x^{-\alpha}-1}{\alpha}$ *from expressions. In this case, one remarks that* $\dfrac{\partial \omega}{\partial x} = -x^{-1-\alpha} = -x^{-1}(\alpha\omega+1)$*. So that* $x \to \omega(x,\alpha)$ *satisfies the Pfaffian equation :* $xd\omega + (\alpha\omega+1)dx = 0$.

To conclude this section I want to mention partial but interesting results about bifurcations studies and computation of cyclicity for elementary graphics :

- as recalled above Jebrane and Zoladek in [JZ] and others studied bifurcations of the "eight-figure" (at a saddle point) of finite codimension. The study of the symmetrical case was made by Rousseau and Zoladek [RZ].

- El Morsalani in [E3] applied the reduction methods to graphics with two semi-hyperbolic vertices. It was proved in [IY2] that a figure made by two "opposite" semi-hyperbolic points of codimension 1, (forming a "lip-figure") can have an arbitrary "global cyclicity" even under generic assumptions (the cyclicity of each graphic in this lip-figure is less than 2).

- In a recent preprint [DER], Dumortier, El Morsalani and Rousseau proved that any non-trivial elementary graphic in the family introduced in chapter 2, equivalent to the quadratic family, has a finite cyclicity (in general ≤ 2). Here, a trivial elementary graphic is a graphic with identical return map. In [Z], Zoladek announced that the cyclicity of the trivial hyperbolic triangles in the quadratic vector family is equal to 3.

Chapter 6

Desingularization theory and bifurcation of non-elementary limit periodic sets.

6.1 The use of rescaling formulas.

In the study of Bogdanov-Takens unfolding, we have introduce in 3.5.2 formulas of rescaling in the phase-space and in the parameter space :

$$x = r^2\ \bar{x},\ y = r^3\ \bar{y},\ \mu = -r^4,\ \nu = r^2\ \bar{\nu}.$$

The aim was the following : taking $(\bar{x}, \bar{y})$ is some compact disk $\overline{D}$ in R^2, and $\bar{\nu} \in K$, some closed interval, the Bogdanov-Takens family is transformed to a r-perturbation of a generic Hamiltonian vector field X_0. Generic means that the two singular points of X_0 are non-degenerate.

Of course, the whole operation reduces to take the counter-image of the family $X_{\mu,\nu}$, (we forget the extra-parameter λ) by the map $\Phi_\mu(\bar{x}, \bar{y}) = (r^2\ \bar{x},\ r^3\ \bar{y})$, making the change $Q^P(r, \bar{\nu}) = (\mu = r^4,$ $\nu = r^2\ \bar{\nu})$ in the parameter. The result of perturbation theories presented in the chapter 4 and 5 lead to the complete study of the new family $\overline{X}_{r,\bar{\nu}}\ (\bar{x}, \bar{y})$ on the disk $\overline{D}$ and for $(r, \bar{\nu}) \in [0, U] \times K$ for some

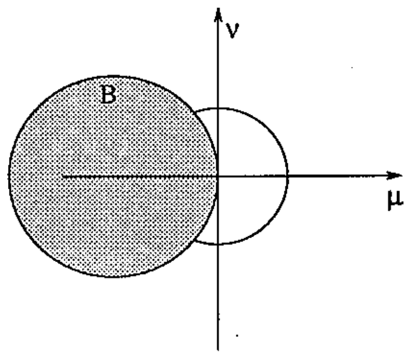

Figure 6.1:

small $U > 0$. (near regular cycles of the Hamiltonian vector field X_0 in chapter 4 and along the homoclinic loop in chapter 5).

The maps Φ_r^s and the parameter change Φ^P are *singular* at $\{r = 0\}$. This point has the following two consequences :

- The study of $\overline{X}$ just cover a conic sector in the parameter space :$B = \Phi^P[[0, U] \times K]$, bounded by a curve with quadratic contact with the 0ν-axis (see Figure 6.1). So, to complete the study we have to look at the family $X_{\mu,\nu}$ in a complement of B in some neighborhood of the origin in the parameter space (see [B], [RW] for details).

- For parameter $(\mu, \nu) \in B$, the study of $\overline{X}_{r,\bar{\nu}}$, on $\overline{D}$ gives the phase portrait of $X_{\mu,\nu}$ on the disk $D_r = \Phi_r^P(\overline{D})$ and the diameter of D_r goes to zero when $r \to 0$. But, we need to obtain the phase portrait on a fixed disk D, neighborhoods of the origin in the phase space.

In the case of the Bogdanov-Takens unfolding, it is easy to choose a disk D such that no singularities exists in $D \backslash D_r$. Then, as a consequence of Poincaré-Bendixon theory, one can prove that $X_{\mu,\nu} \mid D_r$ is topologically equivalent to $X_{\mu,\nu} \mid D$.

This problem happens as a rule when one uses rescaling formulas in unfoldings.

In general, rescaling formula for an unfolding (X_λ) at $0 \in R^p$ and at the origin in the parameter space R^k are formulas :

$$\begin{cases} (x,y) & = & \Phi_r^S(\bar{x}) = (r^{\alpha_i}\ \bar{x}, r^{\alpha_2}\ \bar{y}) \\ \mu & = & \Phi^P\ (\bar{\mu}, r) = (r^{\beta_i}\ \bar{\mu})_{i=1,\cdots,k} \end{cases} \tag{6.1}$$

where $\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_k)$.

Here we have $\lambda = (\mu, \Lambda)$. We have selected some important parameter $\mu = (\mu_1, \cdots, \mu_k)$ which enter in the formula (the parameter (μ, ν) is that of the Bogdanov-Takens unfolding).

The coefficients α_i, β_j are chosen by considerations of quasi-homogeneity.

For instance, we can choose the α_i, looking at the Newton diagram for the vector field X_0 $(\lambda = 0)$. Next, we can choose the coefficients β_j with the idea to have the maximal of monomials of the family at the lowest degree of quasi-homogeneity : it would be the same to take a face of the Newton diagram for the Taylor series of the family expanded in all variables x and λ. Let us consider :

$$\left(\Phi_r^S\right)_*^{-1}\ (X_{\Phi^P}) = r^\delta\ \overline{X}_{r,\bar{\mu},\lambda}\ (\bar{x}) \tag{6.2}$$

where δ is the biggest possible. Now, *rescaling the family* is to take $\bar{x} \in \overline{D}$, some compact domain, and $\bar{\mu} \in S^{k-1}$, the unit sphere and $r \in R^+$.

This choice covers a neighborhood in the parameter space (contrary to the restricted choice recalled above for the Bogdanov-Takens family), but it remains the problem that the disk $D_r = \Phi_r^S(\overline{D})$ shrinks when $r \to 0$. This fact has several inconveniences. Some non trivial bifurcation phenomenons could happen in the region $D \backslash D_r$ and we have to study them or to justify that $\overline{X}_{|D_r}$ is equivalent to $\overline{X}_{|D}$. But the problem is even more serious if the singular point where we use the rescaling formulas is a vertex of a limit periodic set we want to study.

Consider for instance a graphic Γ of X_0 with the point p as vertex. Then trajectories γ_1, with $\alpha(\gamma_1) = p$ and γ_2, with $\omega(\gamma_2) = p$ belong to Γ, and to study the unfolding (X_λ, Γ) we have to look at the transition map near p, $T_\lambda : \sigma_1 \to \sigma_2$ where σ_1, σ_2 are transversal segments to γ_1, γ_2, taken in ∂D where D is a fixed neighborhoods of $0 \in R^2$ (see Figure 6.2).

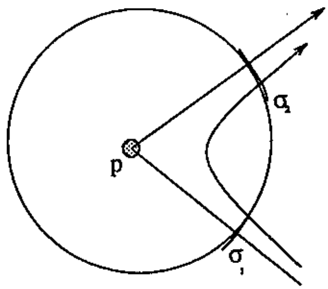

Figure 6.2:

But a study by rescaling just allows to study transition maps between transversal segments taken in the boundary of D_r, whose distance to the origin goes to zero when $r \to 0$. And we remain with an unsolved singular limit problem.

The idea to get rid off this difficulty was to consider the rescaling formulas as a partial chart in some *global generalized blowing up*. The global blowing-up is defined by the map $\Phi(\bar{x}, \bar{\mu}, r, \Lambda) = (x, \mu, \Lambda)$ given by the rescaling formulas (6.1) (i.e. $\Phi = (\Phi_r^S, \Phi^P, \Lambda)$) when one takes $(\bar{x}, \bar{\mu}) \in S^{k+1}$, $r \in R^*$, $\Lambda \in R^{p-k}$. We see that the domain of the rescaling $\overline{D} \times S^{k-1}$ is homeomorphic to a part of S^{k+1}. In fact, corresponding to the decomposition $R^{k+2} = R^2 \times R^k$, we have a related topological decomposition of S^{k+1} :

$$S^{k+1} = \overline{D} \times S^{k-1} \cup S^1 \times D^k. \tag{6.3}$$

In this topological decomposition, we replace the "round sphere" by $\partial(\overline{D} \times D^k)$ where $D^k \subset R^k$ is the disk with centre $0 \in R^k$).

We consider the family X_λ has a vector field X, defined in R^{2+p}.

We suppose that $X_\lambda(0) = 0$ for $\lambda = (0, \Lambda)$. So that there exists, in general, a smooth vector field $\widehat{X}$ such that $\Phi_*(\widehat{X}) = X$. (It is the case for homogeneous blow-up : $\alpha_i = \beta_i = 1$; if not $\widehat{X}$ is smooth after multiplication by r^{δ_0}, for some $\delta_0 \geq 0$). In any case there exists a bigger δ such that : $\frac{1}{r^\delta}\widehat{X} = \overline{X}$ is a smooth vector field. This vector field $\overline{X}$ will be called the *desingularized vector field* (by the blow-up Φ). Of course

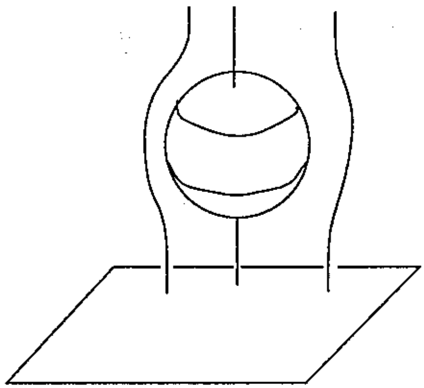

Figure 6.3:

$\overline{X}$ is not identically zero along the critical locus $S^{k+1} \times \{0\} \times R^{p-k}$ and we can expect that $\overline{X}$ has simpler singularities than X.

Taking $r \in [0, U]$, for some $U > 0$, $\Lambda \in W$, some neighborhoods of $0 \in R^{p-k}$, $(\bar{x}, \bar{\mu}) \in S^{k+1}$, we cover a whole neighborhood of $0 \in R^{p+k}$ in the (x, y, λ)-space. A new difficulty is that $\overline{X}$ is no longer a family : it remains a family in the old *rescaling domain* : with on $\overline{D}$, with parameters : $(\bar{\mu}, r, \Lambda) \in S^{k-1} \times [0, U] \times W$, but not in the other part : $S^1 \times D^k \times [0, U] \times W$ (see Figure 6.3:) we have called the *phase space domain.*

For practical computations, it is preferable in general to cover the two domains by an atlas of charts (we call them rescaling directional and phase-space directional charts). For instance, we can replace $(\bar{x}, \bar{y}) \in S^1$ by an atlas of directional charts : $\bar{x} \in K$ (some interval) $\bar{y} = \pm 1$, $\bar{x} = \pm 1$, $\bar{y} \in K$.

When we restrict the map Φ to these choices, we obtain on each chart C a vector field X_C which differs from the global one $\overline{X}$ (obtained by taking $(\bar{x}, \bar{\mu}) \in S^{k+1}$) by a positive smooth multiplicative function.

So that we replace $\overline{X}$ by the singular foliation it defines. We will call it below a *local vector field* to distinguish it from the dim 2−foliation produced by the blow-up of the fibration on the parameter space.

From the time I introduced this global blow-up in [R6] and systematized the method in a work in collaboration with Z. Denkowska [DeR], some applications have been developed concerning particular unfolding questions : nilpotent focus point of codimension 3 [DR2], and Van der Pol equation [DR3]. In these notes I want to return to the first example treated in [R6] : the cuspidal loop, because it is the simplest example of non-elementary graphic and also because some new progress was made recently about it. These new results will appear a paper in preparation [DRS3] and I want to present them here in the next paragraph. In the last paragraph, I want to explain how the global blow-up could enter into a theory of desingularization of families, along the lines presented in [DeR], also to compare it with the theory of Trifonov [Tr]. Finally I will point out that this subject has not yet arrived to a conclusion: a desingularization theory remains to be developed and I want to present some conjectures and some ideas to attack them.

6.2 Desingularization of unfoldings of cuspidal loops.

A cuspidal loop is a singular cycle consisting of a cusp point p and a connection Γ between the two branches of the cusp. We want to study generic unfolding of such a cuspidal loop. A first generic condition is that the cusp point is a codimension 2 singularity, i.e. :

$$j^2\, X_0(p) \sim y\,\frac{\partial}{\partial x} + \left(x^2 + \varepsilon xy\right)\frac{\partial}{\partial y}\ \textit{with}\ \varepsilon = \pm 1. \tag{6.4}$$

The connection adds an extra condition so that it is natural to look at generic 3-parameter unfoldings of (X_0, Γ).

We consider a segment Σ' transverse to the connection (see Figure 6.4:). Let be $P : \Sigma \to \Sigma'$ ($\Sigma \subset \Sigma'$, neighborhoods of $q = \Sigma' \cap \Gamma$), the Poincaré map. *It is a C^1-map* (see below) and we require that $\gamma = P'(q) \neq 1$. Changing X_0 in $-X_0$ we can suppose that $\gamma < 1$ (Γ is an attracting cycle).This is a generic condition (H1) for X_0.

Let us now consider X_λ to be a 3-parameter unfolding of X_0 near Γ for λ near $0 \in R^3$. As we have seen in chapter 1, one can choose

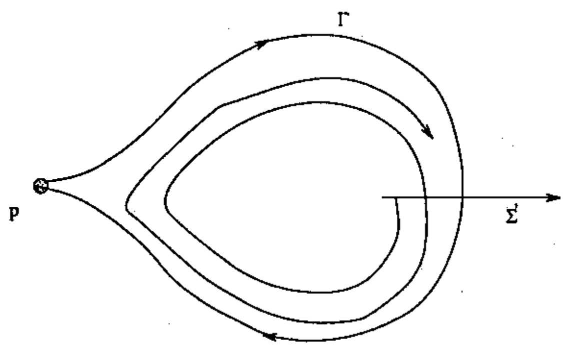

Figure 6.4:

coordinates (x,y) near p, with $p = (0,0)$ such that in a neighborhood of p, X_λ is C^∞ equivalent to :

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 + \mu(\lambda) + y(\nu(\lambda) + \varepsilon x + x^2 h(x,\lambda)) + y^2 Q(x,y,\lambda) \end{cases} \tag{6.5}$$

where h and Q are C^∞ functions.

A first generic condition for the family is (H2) : *the map : $\lambda \to (\mu(\lambda), \nu(\lambda))$ is of rank 2 at $\lambda = 0$.*.

As such, we maintain a cusp at 0 on a regular line L in the parameter space R^3, passing through the origin.

For $\lambda \in L$, it makes sense to define a *shift map* between the two separatrices of the cusp. In Σ' is oriented as in figure 6.4 and $u(\lambda)$, $s(\lambda)$ are the intersections of the unstable and the stable separatrix of p with Σ', respectively, one defines : $\sigma(\lambda) = u(\lambda) - s(\lambda)$.

Using the desingularization of p by the quasi-homogeneous blow-up $x = r^2\,\bar{x}$, $y = r^3\,\bar{y}$, it is possible to show that *σ is a C^∞ function of L.*

A second generic condition for X_λ is (H3) :*the map $\lambda \in L \to \sigma(\lambda)$ has a non zero derivative at $\lambda = 0$.* If we take *any* C^∞ extension of σ in a neighborhoods of $0 \in R^3$, the conditions (H2) and (H3) imply that the map $\lambda \to (\mu(\lambda),\ \nu(\lambda),\ \sigma(\lambda))$ is of rank 3 at $0 \in R^3$. So, up to a diffeomorphic change of parameter, we can suppose that $\lambda = (\mu, \nu, \sigma)$.

Definition 31 *In the following text, generic 3-parameter unfolding of cuspidal loop will mean an unfolding which verifies the three generic conditions (H1), (H2) and (H3).*

Remark 34 *Along the technical elaboration that follows I shall make a more precise choice of* σ*, depending on the construction that we will make at 0 (but I prefer to explain it later).*

Knowing the codimension 2 phenomena and using some heuristic arguments (like the famous "simplicity principle") it is not too hard *to predict* the possible bifurcation diagrams.

I say diagrams since it happens to be *two cases* depending on the sign $\pm$ in the Bogdanov-Takens bifurcation. I have pictured these diagrams in Figure 6.5 and Figure 6.6.

It is rather easy to prove the occurrence and genericity of the different saddle connections (lines L_r, L_ℓ, L_i, L_s). The hard part of the proof deals with the limit cycles. The fact that the small limit cycle that appears in the Bogdanov-Takens bifurcation is expanding in the case $\varepsilon = 1$ induces a slightly more complicated bifurcation diagram, exhibiting four limit cycles.

Remark 35 *1) This number 4 is not without importance. In codimension 1 and 2 in the plane, any generic limit periodic set generates a number of limit cycles bounded by the codimension. Here in the generic 3-parameter unfolding of the codimension 3 cuspidal loop, one may generates 4 limit cycles. It is a quite unexpected phenomenon, similar to the one observed by Mourtada for generic hyperbolic polycycles of codimension 4, which generates 5 limit cycles (see chapter 5 above).*

2) The figures 6.5 and 6.6 represent intersections of the cone-like 2-dimension bifurcations set with a 2-sphere S^2 *centered at* $0 \in R^3$*. One has removed a point on* S^2 *to make a planar picture.*

In fact, the most interesting part of the bifurcation set is situated in a small cylinder ($\sigma \simeq 0$). Indeed, for a fixed $\sigma(\sigma > 0$ or $\sigma < 0)$, it is clear that one can only expect to find the bifurcation diagram of the Bogdanov-Takens bifurcation (for $(\mu, \nu) \sim (0,0)$). So, it might be more natural during the study of the bifurcation set to intersect it with a "cylinder box" : $\{|\sigma| \leq S, x^2 + y^2 \leq r^2\}$.

The bifurcation diagrams in Figure 6.5, 6.6, which summarizes the results about bifurcations of the generic unfolding remains conjectural. What will be proved precisely in [DRS3] is that the study can be reduced to the properties of some function :

Theorem 31 *The diagrams in Figure 6.5, 6.6, are implied by the property of monotonicity (M) of a "transition time" function $t_\varphi(x)$. (We define $t_\varphi(x)$ and the property (M) below in subsection 6.2.3).*

Remark 36 *1) All phenomena of bifurcations concerning saddle connections are easy to obtain as we said before. Also the reduction I present below permit to study and check any codimension 2 phenomenon without the use of property (M), expect the occurrence of the triple limit cycle (TC). In fact, property (M) is just needed to justify the results about limit cycles : number, cyclicity, and existence of bifurcations for them : TC, DC^{out}, DC^{in} (see Figure 6.5, 6.6).*

2) C. Simó has obtained a good "numerical evidence" for the property (M.)

The next subsections are devoted to present the ingredients for the proof of theorem 1 whose details will appear in [DRS3]. These ingredients are needed to overcome the problem that separatrices of the cusp suddenly loose any geometrical meaning when $\mu > 0$, while on the region $\{\mu < 0\}$, they turn into separatrices of a saddle point, but in a non-differentiable way.

6.2.1 Global blow-up of the cusp unfolding.

At $(x, y, \mu, \nu) = (0, 0, 0, 0)$, we make the following global blow-up on :

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= x^2 + \mu + y(\nu + \varepsilon_1 x + x^2 h) + y^2 Q \end{cases} \tag{6.6}$$

where $\varepsilon_1 = \pm 1$, $h(x, \lambda)$, $Q(x, y, \lambda)$ are C^∞-functions and $\lambda = (\mu, \nu, \sigma)$.

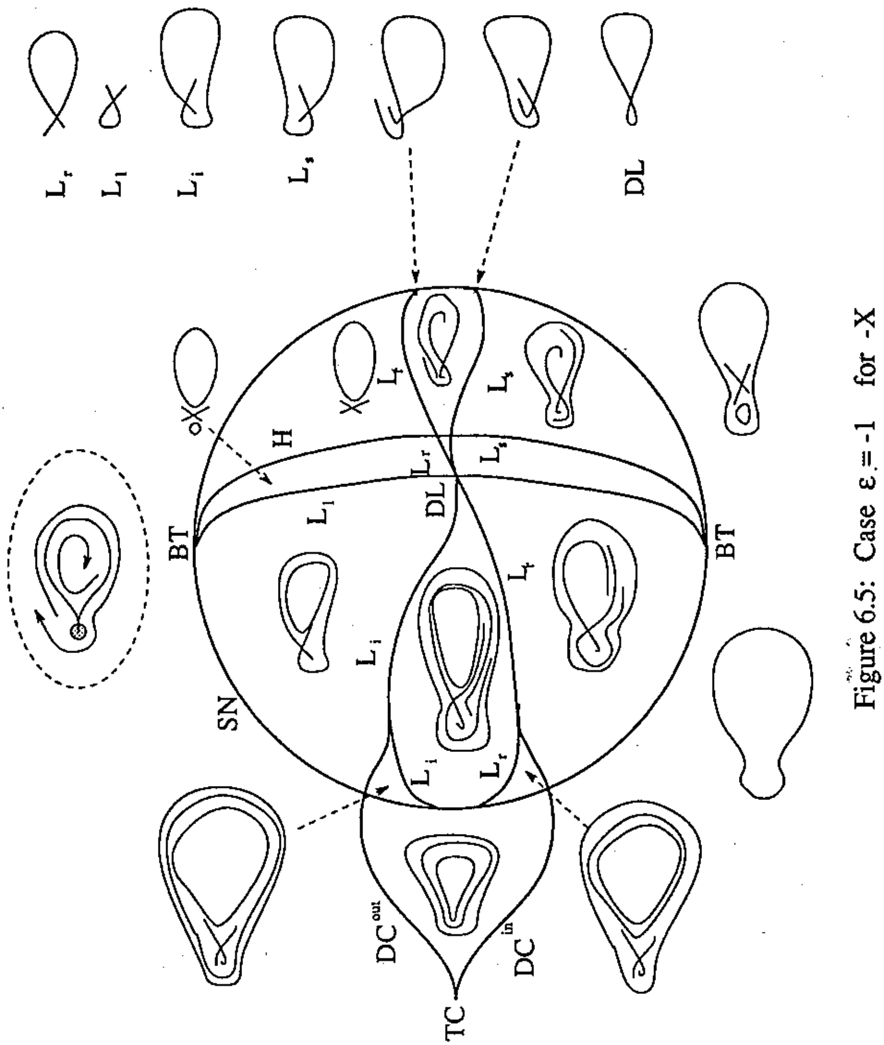

Figure 6.5:

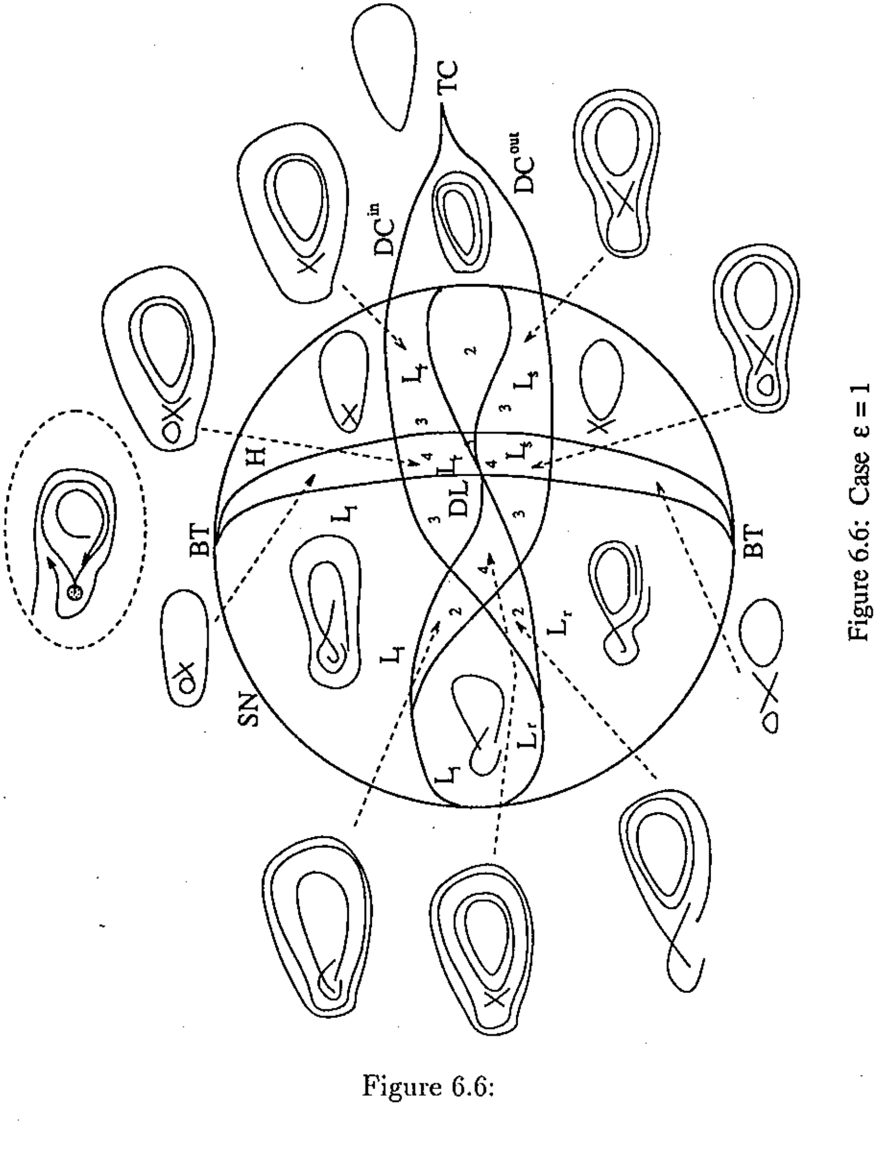

Figure 6.6:

Blow-up formulas :

$$\left\{\begin{array}{lcl} x & = & r^2\,\bar{x} \\ y & = & r^3\,\bar{y} \\ \mu & = & r^4\,v^4\cos\,\varphi \\ \nu & = & r\,v\sin\,\varphi \\ \sigma & = & \sigma \end{array}\right. \tag{6.7}$$

with $\bar{x}^2 + \bar{y}^2 + v^2 = 1$ (or $(\bar{x},\bar{y},v) \in \partial B$, where B is some "box" homeomorphic to D^3).

So, (6.7) defines a mapping $\Phi : S^2 \times R^+ \times S^1 \times R \to R^5$, $\Phi : ((\bar{x},\bar{y},v), r, \varphi, \sigma) \to (x,y,\mu,\nu,\sigma)$.

Remark 37 *Here, we compose the global blow-up with "small-parameter" r, as described in paragraph 1 with a polar type blow-up $\bar{\mu} = v^4.\cos\,\varphi$, $\bar{\nu} = v\,\sin\,\varphi$. So that the critical locus is not S^3 (for any fixed σ) but $S^2 \times S^1$ and one passes from one to the other by the branched covering map : $S^2 \times S^1 \to S^3$ which blows up one circle in S^3).*

We have now a C^∞ vector field : $\overline{X} = \dfrac{1}{r}\,\overline{X}$ with $\Phi_\lambda(\widehat{X}) = X$ and X is the family (X_λ), which we consider as a vector field in R^5.

For each constant value of (φ,σ), $\overline{X}$ induces a 3-dimensional vector field $\overline{X}_{(\varphi,\sigma)}$ defined on $S^2 \times R^+$ near $S^2 \times \{0\}$. So, the global blow up Φ changes our 3-parameter family into a (φ,σ)-family $\overline{X}_{(\varphi,\sigma)}$ we want to describe now. Several things can be observed making easier this description :

1) The set $\{v = 0\}$ is invariant and on it, the map Φ is the usual quasi-homogeneous blow of the cusp singularity ($\mu = \nu = 0$), studied in chapter 3 (see Figure 6.7:).

2) It suffices to study $\overline{X}_{(\varphi,\sigma)}$ on $\{v \geq 0\}$ to get a complete information in the (x,y,λ)-space near $0 \in R^5$.

3) For each (φ,σ), $\overline{X}_{(\varphi,\sigma)}$ leaves invariant the foliation given by $\{rv\} = C^{st}$. This is the foliation obtained by the blow up of the foliation of R^5 in the parameter space (μ,ν,σ). For $\{rv = u\}$, with $u > 0$, the

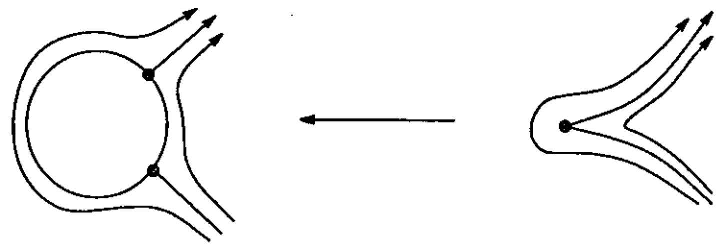

Figure 6.7:

leaf is a regular 2-manifold and for $\{rv = 0\}$, we get a stratified set as in figure 6.8, with 2 strata of dim 2 : $\widehat{F} \simeq S^1 \times R$ (the blow-up of the fiber $\mu, \nu = 0$) and the hemisphere $D_{\varphi,\sigma}$ in $\{\bar{x}^2 + \bar{y}^2 + \bar{v}^2 = 1,\ \bar{v} \geq 0\}$ in the critical locus.

We already know $\overline{X}_{(\varphi,\sigma)}$ on $\widehat{F} = \{v = 0\}$, and to have $\overline{X}_{(\varphi,\sigma)}$ on $D_{\varphi,\sigma} = \{r = 0\}$, we have to look at $\overline{X}_{(\varphi,\sigma)}$ on the half-sphere $D_{\varphi,\sigma}$.

Near $\{v = 0\}$ we use the : *phase directional rescaling.* This means that in (6.8) we take $\bar{x}^2 + \bar{y}^2 = 1$ and v near 0. (Or better, we use subcharts by taking $\bar{x} = \pm 1$, resp. $\bar{y} = \pm 1$).

Take just $\bar{x} = 1$: we obtain the phase portrait of the vector field $\overline{X}^x$ represented in figure 6.9, for each value (φ, σ) :

The singularities at p_1 and p_2 are hyperbolic saddles ; the eigenvalues at p_2 are : -6, -1, 1 (-6 along the $\bar{y}$-axis). At p_1 we have eigenvalues 6, 1, -1 (with 6 along the $\bar{y}$-axis). The relation between the eigenvalues at p_1 and p_2 is not a coincidence since in fact the two points are "the same". To see this, we use $\bar{y} = 1$ (instead of $\bar{x} = 1$), giving a vector field $\overline{X}^y$.

As in the global blow-up Φ, $y = r^3\,\bar{y}$, the chart $\bar{y} = 1$ includes $\bar{y} = -1$ by changing $(\bar{x}, r, v, \lambda, t) \to (\bar{x}, -r, -v, \lambda, -t)$.

We obtain a singular point p on the $0\bar{x}$ axis (see Figure 6.10:).

The study of $\overline{X}^y$ at p includes up to a C^∞ equivalence the situation of $\overline{X}^x$ near p_2 on $\{v \geq 0,\ r \geq 0\}$ and the situation near p_1, on $\{v \leq 0,\ r \leq 0\}$ (if we reverse time).

To complete the picture on the 2-sphere, we use the *"family rescaling"* (i.e. : the usual rescaling). In (6.7), we take $v = 1$, leading to a

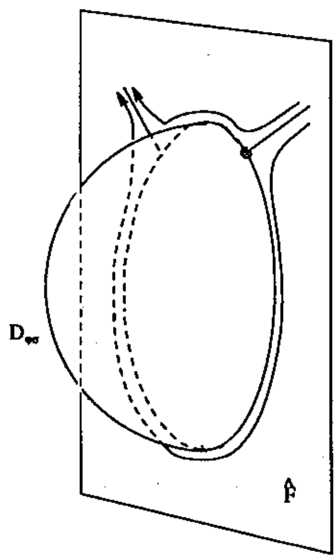

Figure 6.8:

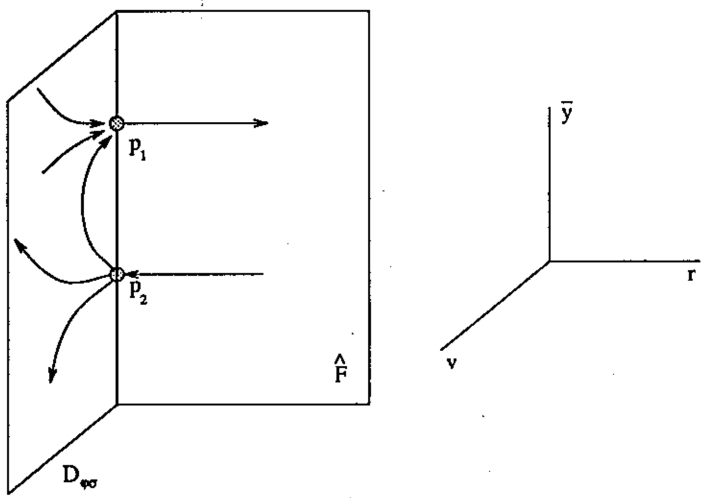

Figure 6.9:

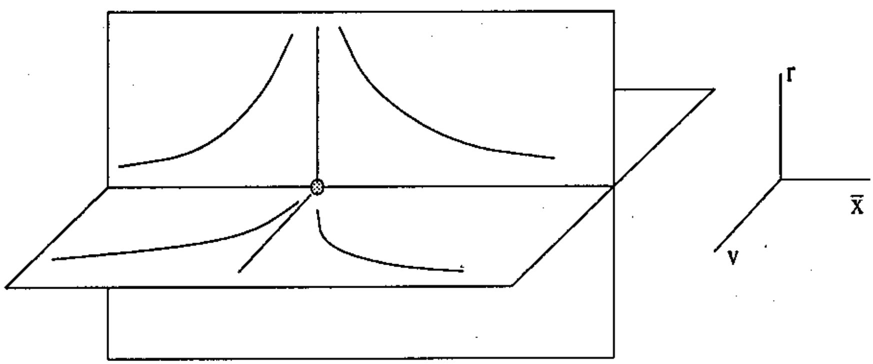

Figure 6.10:

(r, φ, σ) family of 2-dimension vector fields. On $\{u = 0\}$ this gives the following family of vector fields :

$$X_\varphi \begin{cases} \dot{\bar{x}} &= \bar{y} \\ \dot{\bar{y}} &= \bar{x}^2 + \cos\varphi + \bar{y}\sin\varphi \end{cases} \tag{6.8}$$

It is a very simple φ-parameter quadratic family of vector fields X_φ (it is independent of σ) ; we can draw the picture on the Poincaré disk. At ∞ (boundary of the Poincaré disk) we add the knowledge that we got by the phase directional rescaling. Notice that it corresponds to use the quasi-homogeneous compactification : $x = \dfrac{\cos\theta}{v^2}$, $y = \dfrac{\sin\theta}{v^3}$ ($\{v = 0\} = \infty$). (See Figure 6.11:).

For $\overline{X}$, we can define the limit periodic set as any limit in the sense of Hausdorff distance, of sequence of limit cycles. All of them are in $\{\sigma = 0\}$ inside $\widehat{F} \cup D_{\varphi,0}$ for some φ. They are closed curve made by the connection $\widehat{\Gamma}$ from p_1 to p_2 in $\widehat{F}$ (coming from Γ) and a connection between p_2 and p_1 in $D_{\varphi,0}$. They are all "elementary" in the sense that they just include the two singular hyperbolic points p_1, p_2 and perhaps a saddle point, as singular point. See in Figure 6.12, the different possible limit periodic sets of $\overline{X}$.

6.2.2 Asymptotic form for the shift map equation.

Let Σ_1, Σ_2 two transversal sections to orbits in $\{v = v_0\}$. These sections are rectangles parametrized by (θ, r). We consider the transition maps for the flow of $\pm\overline{X}_{\varphi,\sigma}$ from Σ_1 to Σ_2. First $G_{\varphi,\sigma} : \Sigma_1 \to \Sigma_2$ when we follow the flow of $\overline{X}_{\varphi,\sigma}$ ($G_{\varphi,\sigma}$ is the transition near the disk $D_{\varphi,\sigma}$) and next $R_{\varphi,\sigma}$ when we follow the flow of $-\overline{X}_{\varphi,\sigma}$ ($R_{\varphi,\sigma}$ is the transition near $\widehat{F}$).

Limit cycles cutting Σ_1 correspond to solution of the shift map equation :

$$\Delta_{\varphi,\sigma} = G_{\varphi,\sigma} - R_{\varphi,\sigma} = 0. \tag{6.9}$$

To obtain the whole information about the bifurcation diagram it will be sufficient to look at this equation $\{\Delta_{\varphi,\sigma} = 0\}$ when v_0 is chosen

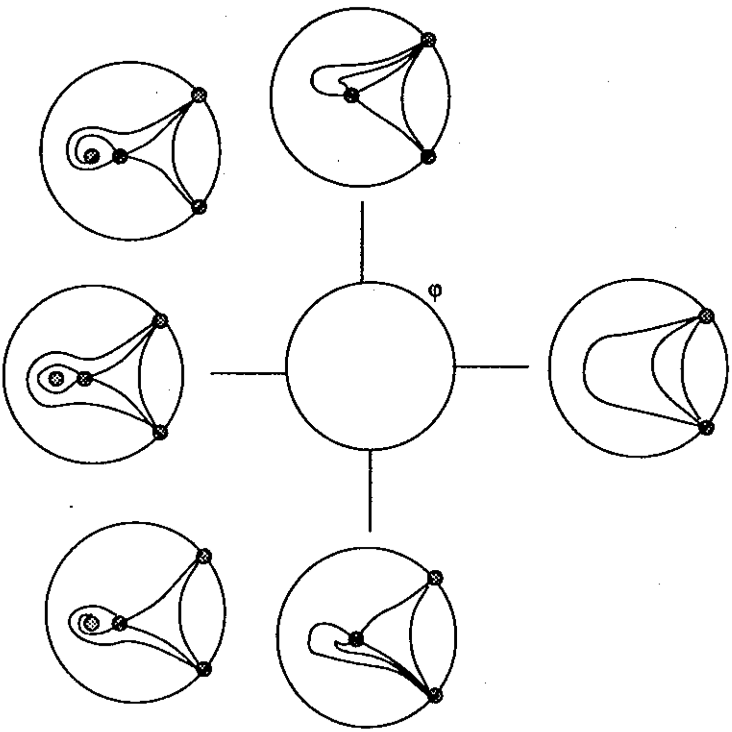

Figure 6.11:

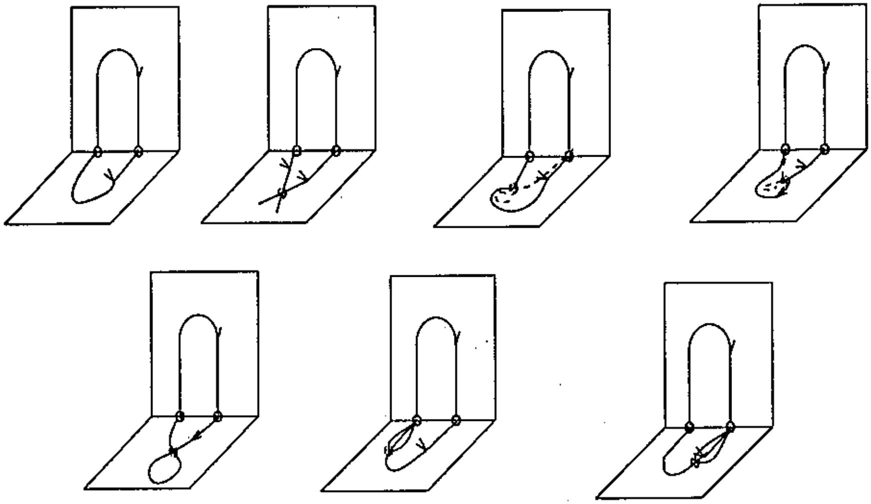

Figure 6.12: Limit periodic set of $\overline{X}$

small enough. Of course, the rest of the study will be completed by looking at similar shift map for sections Σ_1', Σ_2' chosen transversally to $\widehat{F}$, but it is easy to verify that any limit cycle cutting Σ_1' (small enough) must be a hyperbolic attracting cycle : one can prove that the integral of the divergence is always negative. So that no bifurcation may happen to such a limit cycle.

To obtain an interesting asymptotic form for $\Delta_{\varphi,\sigma}$, when the critical parameter $u = rv \to 0$, we introduce "good normal form" coordinate on Σ_1, Σ_2, coming from some normal form coordinates at the singular point p which gives birth simultaneously to p_1, p_2.

Recall that $\overline{X}_{\varphi,\sigma}$ has resonant eigenvalues : $-1, -6, 1$ at p and moreover the function $u = rv$ is a first integral. Moreover, up a multiplicative function, $\overline{X}_{\varphi,\sigma}$ is *linear in r and v.*

It is then possible to reduce $\overline{X}_{\varphi,\sigma}$ in a neighborhoods of p to a C^∞ normal formal by a C^∞ diffeomorphism ((φ,σ)-dependant) preserving in fact the two variables r, v (see details in [DRS3]). Let us formulate this special normal form theorem :

Theorem 32 *Let X_λ a parametrized C^∞ vector-field family in a neighborhood of $0 \in R^3$ ($\lambda \in R^p$, near $0 \in R^p$).*

Suppose that the differential equation for X_λ writes :

$$\dot{r} = -r,\ \dot{v} = v \text{ and } \dot{\Omega} = 6\Omega + \tilde{G}\ (\Omega,\ r,\ v,\ \lambda).$$

Then, for any $k \in N$, there exist a $N(k)$ and a C^k-family of diffeomorphisms $H_\lambda(r, v, \Omega) = (r, v, h_\lambda(r, v, \Omega))$ which brings X_λ to a $N(k)$-normal form :

$$\dot{r} = -r,\ \dot{v} = v,\ \dot{\Omega} = 6\Omega + \tilde{G}^N_\lambda(r, v, \Omega)$$

where $\tilde{G}$ is a resonant polynomial of degree $N = N(k)$:

$$\tilde{G}^N_\lambda(r,\ v,\ \Omega) = \sum_{\substack{i+j+\ell \le N \\ 6i-j+\ell=6}} \alpha_{ij\ell}\ (\lambda)\Omega^i\ r^j\ v^\ell.$$

Here one can show easily that $\tilde{G}^N = \alpha(\varphi)v^6 + O(r)$ where $\alpha(\varphi) = -\frac{12}{5}\sqrt{6}\ \sin^2\varphi\ (\frac{9}{625}\ \sin^4\varphi + \cos\varphi)$.

Remark 38 *This constant $\alpha(\varphi)$ is a resonant term at infinity for the quadratic vector field $X_\varphi = \overline{X}_{\varphi,\sigma}\ \mid r = 0$. It is clear that $\alpha(\varphi) = 0$ if $\varphi = 0, \pi$, when X_φ is a hamiltonian vector field. This value for α has been obtained directly by computation of the normal form with the help of Maple. A more clever way to obtain it would be to look at the complex extension of X_φ.*

From the normal form coordinates at p we deduce normal form coordinates for $\overline{X}_{\varphi,\sigma}$ in the "phase directional chart" ($\bar{x} = 1$), where we see the two points p_1, p_2. Let (r, v, Ω) be these coordinates near each point; $r = v = \Omega = 0$ correspond to p_1 and p_2 respectively.

We take as above two sections Σ_1, Σ_2 near p_1, p_2 respectively in $\{v = v_0\}$, v_0 small enough. We parametrize each one by (Ω, u) with $u = rv_0$ (the reason is that u is directly related to the initial parameters by : $\mu = u^4\ cos\varphi$, $\nu = u\ sin\varphi$). We can now give an asymptotic form for the transition map $R_{\varphi,\sigma}\ (\Omega, u)$:

Theorem 33 *Let be γ be the Poincaré exponent for the cuspidal loop and $\alpha(\varphi)$ the resonant term, defined above. Then :*

$$R_{\varphi,\sigma}\ (\Omega,u)=\gamma^{-1}\ \Omega+\sigma u^{-6}\ \alpha(\varphi)(\gamma^{-1}\ -1)Lnu+O(uLn^2u) \quad (6.10)$$

where the remainder term $O(u\ Ln^2\ u)$ is a function $\Phi(r,u,\sigma,\varphi)$ which is C^∞ in Ω and all whose partial derivatives $\frac{\partial^k\Phi}{\partial\Omega^k}=O(uLn^2u)$ (it is $O(uLn^2u)$ in a "$\mathcal{C}^\infty$-sense").

Proof I don't want to give a complete proof of this result, but just some rough idea of it. First, up to a C^1-change of coordinates we can linearize the vector field at p_1, p_2 (following a result of Belitskii; this is possible since we have no relation of the form $\lambda_i+\lambda_j=\lambda_k$ between the eigenvalues).

Taking transversal sections Σ_1', Σ_2' near p_1, p_2 in $\{r=r_0\}$ for some r_0, we can write R as a composition of three maps :

$$R=T_2\circ\overline{R}\circ T_1 \quad (6.11)$$

where T_1, T_2 are the transitions near p_1 and p_2 and $\overline{R}$ is a C^1 transition from Σ_1' to Σ_2'. We can write : $\overline{R}_{\varphi,\sigma}(\Omega,u)=\sigma+\gamma^{-1}\ \Omega+O(r^2)$ and :

$$T_1(\Omega)=(\frac{u}{u_0})^6\ \Omega\ ,\ T_2(\Omega)=(\frac{u}{u_0})^{-6}\ \Omega=T_1^{-1}\ (\Omega). \quad (6.12)$$

So that, R is conjugate to $\overline{R}$ by the linear map T_1, with coefficient $\left(\frac{u}{u_0}\right)^6\to 0$ when $u\to 0$. This conjugacy has a "*funnelling effect*" : $R_{\varphi,\sigma}(\Omega,u)$ has to tend toward an affine map when $u\to 0$. We find :

$$R_{\varphi,\sigma}\ (\Omega,u)=\sigma u^{-6}+\gamma^{-1}\ \Omega+O(u^6) \quad (6.13)$$

This argument which was used in [R6], gives only a "$\mathcal{C}^1$-control" on the remainder term $O(u^6)$. It would not be sufficient to study codimension 2 bifurcations and also bifurcation lines of double cycles. To obtain a more accurate estimate we have to use the normal form given in theorem 2. A first observation is that the integration of such normal form reduces to a 1-dimensional differential equation. Let $(r,\Omega)\in\Sigma\subset\{v=v_0\}$ and $\varphi(t)=(\Omega(t),r(t),v(t))$ the flow of $\overline{X}_{\varphi,\sigma}$ with these initial conditions. One has $r(t)=re^{-t}$, $v(t)=v_0e^t$ and for $\Omega(t)$ the equation : $\dot{\Omega}=6\Omega+\widetilde{G}^N(\Omega,r,v,\lambda)$ (here $\lambda=(\sigma,\varphi)$). We look for $\Omega(t)$ in the form :

$$\Omega(t) = e^{6t}\,\overline{\Omega}(t). \tag{6.14}$$

Substituting $r(t)$, $v(t)$ in (6.14), we obtain, with $\Omega(0) = \Omega$:

$$\dot{\Omega}(t) = +6\Omega(t) + e^{+6t}\,\dot{\overline{\Omega}}(t) = 6\Omega(t) + \sum_{i,j,\ell} \alpha_{ij\ell}\, e^{(6i-j+\ell)t}\, \Omega(t)^i r^j v_0^\ell \tag{6.15}$$

and so :

$$\dot{\overline{\Omega}}(t) = \sum_{i,j,\ell} \alpha_{ij\ell}\, e^{[-6+6i-j+\ell]t}\, \overline{\Omega}^i\, r^j\, v_0^\ell. \tag{6.16}$$

But, precisely, because $\widetilde{G}$ is a resonant polynomial, each coefficient of $-6+6i-j+\ell$ vanishes and $\overline{\Omega}$ is solution of an 1-dimension autonomous equation : $\dot{\overline{\Omega}}(t) = \widetilde{G}^N\,(\overline{\Omega}(t), r, v_0)$ (we no longer mention the dependence in $\lambda = (\varphi, \sigma)$).

Now, using classical estimates on the solution $\overline{\Omega}(t)$ we can prove that :

$$\overline{\Omega}(t) = \Omega + \alpha v_0^6\, t + O(rLn^2 r)$$

and then, $T_1(\Omega) = \left(\frac{u}{u_0}\right)^6 \overline{\Omega}\left(Ln\frac{u}{u_0}\right)$ with :

$$\overline{\Omega}\left(Ln\,\frac{u}{u_0}\right) = \Omega + \alpha\; Ln\; u + C + O(uLn^2 u). \tag{6.17}$$

Here, C is a constant term and $O(u^2 Lnu)$ must be understood in C^∞-sense. Up to this remainder term we see that the unique change with respect to the previous formula (6.12) is the introduction of a translation term $\alpha Lnu + C$. Using a similar result for $T_2(\Omega)$ we obtain for the composition $R = T_2 \circ \overline{R} \circ T_1$ the asymptotic expression in the theorem. ∎

Remark 39 *Let $\tilde{\sigma} = \sigma\; u^{-6}\; \alpha(\varphi)(\gamma^{-1}\; - 1)Lnu$ be the rescaled translation term which enter in the expression of $R_{\varphi,\sigma}$. This term has to remain bounded. So, assume chosen some $\tilde{\sigma}_0$ (depending on the choice of sections Σ_1, Σ_2) such that $\tilde{\sigma} \in [-\tilde{\sigma}_0,\; \tilde{\sigma}_0]$. We see that :*

$$\sigma \in \alpha(\varphi)(\gamma^{-1}\; - 1)u^6 Lnu + u^6[-\tilde{\sigma}_0,\; \tilde{\sigma}_0].$$

This specifies a conic region in the parameter space (μ, ν, σ) of size u^6 around the cone $\sigma = \alpha(\varphi)(1-\gamma^{-1})u^6 Lnu$ (recall that $\mu = u^4 cos\varphi$,

$\nu = u\ sin\varphi$), in which all the bifurcations occurring in a neighborhood of Γ are located. Of course, this remark doesn't concern the small cycle and the Bogdanov-Takens theory near the saddle point.

We look now at the function $G_{\varphi,\sigma}\ (r,u)$. If we put $g_\varphi(\Omega) = G_{\varphi,\sigma}\ (\Omega,0)$ (independent on σ), in equation (6.9) we obtain :

$$g_{\varphi,\sigma}\ (\Omega) = \gamma^{-1}\ \Omega + \tilde{\sigma}. \tag{6.18}$$

Because γ^{-1} is intrinsically defined, we also need to have an intrinsic definition of $\dfrac{\partial g}{\partial \Omega}$. It is indeed the case and follows from the fact that the variable Ω gives an intrinsic metric structure to the orbit space of X_φ. Look at this point more closely. The vector field $X_\varphi(\dot{x} = y$, $\dot{y} = x^2 + \cos\varphi + y \sin\varphi)$ is extended on the Poincaré disk D by the blow-up formula $x = \dfrac{\cos\theta}{v^2}$, $y = \dfrac{\sin\theta}{v^3}$ which follows from the global blow-up, and as we have noticed, the two singular points p_1, $p_2 \in \partial D^2$ have resonant linear part with eigenvalues $\pm(1,6)$. Now, we have to define the map $g_\varphi(\Omega)$ as the transition map from Σ_1 to Σ_2 which for $u = 0$, are intervals in $\{v = v_0\}$; Σ_1, Σ_2 are parametrized by the variable Ω, which together with v is a normal form coordinate at p_1 or p_2. This means that at each point p_1, p_2, the vector field X_φ has the following normal form expression :

$$X_\varphi \left\{ \begin{array}{rcl} \dot{v} & = & v \\ \dot{\Omega} & = & 6\Omega - \alpha v^6, \ (\textit{ at } p_1 \textit{ and the opposite at } p_2). \end{array} \right. \tag{6.19}$$

Suppose we consider at p_1 for instance two different reductions in normal form H_1, H_2. Then, $H_1 \circ H_2^{-1}$ is a diffeomorphism which leaves invariant the equation (6.19). It is easy to prove that such a diffeomorphism has to write :

$$H_1 \circ H_2^{-1}\ (\Omega, v) = (\Omega + \beta v^6, v) \tag{6.20}$$

for some $\beta \in R$.

As a consequence, the variable Ω is uniquely defined (i.e. independent of the normalizing diffeomorphism) up to a translation.

Another way to say this is as follows : let $\mathcal{O}_{p_1}$ the space of trajectories with α-limit in p_1, in the half space $\{v > 0\}$. Then, the map $\Omega \to \gamma_\Omega$

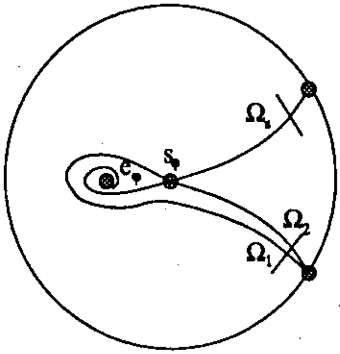

Figure 6.13:

(orbit in $\mathcal{O}_{p_1}$ through the point $(\Omega, 1)$ in the normal form charts) gives to the space $\mathcal{O}_{p_1}$ a well-defined metric (i.e. a parametrization defined up the choice of some origin). It is the same at the point p_2. Finally, we can see $g_\varphi(\Omega)$ has a map $g_\varphi(\Omega) : \mathcal{O}_{p_1} \to \mathcal{O}_{p_2}$ between the orbit-spaces at p_1 and p_2, defined by the transition. We call it the *"transition at ∞"* between p_1 and p_2. $\mathcal{O}_{p_1}, \mathcal{O}_{p_2}$ are isometric to R, with variable Ω. Strictly speaking the map $g_\varphi(\Omega)$ is defined outside the interval of orbits which tend toward the possible attracting focus e_φ (when $\varphi \in [-\pi, -\frac{\pi}{2}]$). So, let $D_\varphi \subset R$ the maximal domain of definition of $g_\varphi(\Omega)$:

$$D_\varphi = R \; is \; \varphi \in \left]-\frac{\pi}{2} \, , \, \frac{\pi}{2}\right[\, , \; D_\varphi = R - \Omega^1_\varphi \; if \; \varphi \in \left[\frac{\pi}{2} \, , \, \pi\right]$$

(where Ω^1_φ correspond the separatrix of s_φ) and $D_\varphi = R -]\Omega^1_\varphi, \Omega^2_\varphi[$ where Ω^1_φ, Ω^2_φ correspond to the two saddle separatrices of s_φ if $\varphi \in]-\pi, -\frac{\pi}{2}]$ (see Figure 6.13).

Finally, the equation (6.9) for limit cycles has the following asymptotic form :

$$g_\varphi(\Omega) - \gamma^{-1}\,\Omega - \tilde{\sigma} + O(uLn^2 u) = 0 \tag{6.21}$$

$$with\ \tilde{\sigma} = \sigma\ u^{-6}\ + \alpha(\varphi)(\gamma^{-1} - 1)Lnu$$

where the remainder term $0(uLn^2u)$ is $\mathcal{C}^\infty$ in Ω and is $0(u\ Ln^2\ u)$ in a $\mathcal{C}^\infty$ sense (any derivative in Ω is $O(u\ Ln^2\ u)$). So, the bifurcations in the family depends on the properties of $g_\varphi(\Omega)$.

6.2.3 The properties of $g_\varphi(\Omega)$.

First, we are going to obtain a more tractable expression for $g_\varphi(\Omega)$. Observe that the interval of definition of any trajectory γ with $\alpha(\gamma) = p_1$ and $\omega(\gamma) = p_2$ is bounded, i.e. for any $m \in \gamma$ one tends to p_1 for $t \to \tau_1$ and to p_2 for $t \to \tau_2$, where τ_1, τ_2 are finite. We can call $\tau_2 - \tau_1 = T(\gamma)$ the transition time from p_1 to p_2 along γ.

Each orbit γ as above belongs to $D_\varphi \subset \mathcal{O}_{p_1}$, with its natural parametrization by Ω. So that we will denote by $T_\varphi(\Omega)$ the transition time to go from p_1 to p_2 along the orbit of D_φ with parameter Ω . This function $T_\varphi(\Omega)$ is closely related to $g_\varphi(\Omega)$.

Proposition 16

$$\frac{\partial\ g_\varphi}{\partial\Omega}\ (\Omega) = \ exp(\sin\varphi T_\varphi(\Omega)). \tag{6.22}$$

Proof Let $\gamma_\varphi(\Omega)$ be the orbit of X_φ of D_φ, with parameter Ω. Let Σ_1, Σ_2 be two sections in normal form charts near p_1, p_2 in $\{v = v_0\}$. Let $G^{v_0}_\varphi(\Omega)$ denote the transition map between Σ_1 and Σ_2.

If we call $\|\ .\ \|$ the euclidian norm in the normal form charts, it follows from a well known variational formula that :

$$\frac{dG}{d\Omega}\ (\Omega) = \frac{\|X_\varphi(G^{v_0}_\varphi\ (\Omega))\|}{\|X_\varphi\ (\Omega)\|}\ exp\left(\int_0^{T^{v_0}}\ divX_\varphi dt\right) \tag{6.23}$$

where $\Omega \in \Sigma_1$, $G^{v_0}_\varphi(\Omega) \in \Sigma_2$ and T^{v_0} is the time to go from Ω to $G^{v_0}_\varphi(\Omega)$. Now, div $X_\varphi = \sin\ \varphi$ and $T^{v_0} = T_\varphi(\Omega) + 0(v_0)$.

Moreover it can be shown that :

$$\frac{\|X_\varphi(G^{v_0}\ (\Omega))\|}{\|X_\varphi\ (\Omega)\|}\ = 1 + O(v_0) \tag{6.24}$$

and also that :

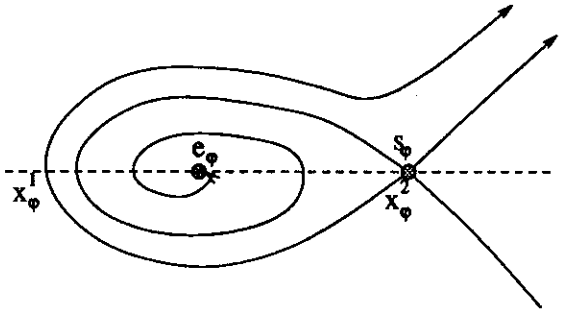

Figure 6.14:

$$\frac{\partial\, G_\varphi^{v_0}}{\partial\, \Omega}\,(\Omega) = \frac{\partial\, g_\varphi}{\partial\, \Omega}\,(\Omega) + O(v_0). \tag{6.25}$$

To prove for instance the last estimative (6.25), we observe that $G_\varphi^{v_0}$ is conjugate to g_φ.

Finally, we have that :

$$\frac{\partial\, g_\varphi}{\partial\, \Omega}(\Omega) = \; exp\;(\sin\varphi T_\varphi(\Omega)) + O(v_0) \tag{6.26}$$

and the result follows because $\dfrac{\partial\, g_\varphi}{\partial\, \Omega}\,(\Omega)$ is in fact independent on v_0. ■

Because the variable Ω is not explicitly defined, it is preferable to parametrize the transition time by the intersection point with the $0x$-axis. Let $M_\varphi \subset R$ ($0x$-axis) the domain of x such that the orbit through x has α and ω-limit in $\{p_1, p_2\}$. We have a transition diffeomorphism $\Omega_\varphi(x)$ from M_φ to D_φ and $M_\varphi = R$ if $\varphi \in]-\pi/2\;,\;\pi/2[$, $M_\varphi = R - x_\varphi^1$ (x_φ^1 : coordinate of the saddle s_φ) if $\varphi \in [\frac{\pi}{2}, \pi]$ and $M_\varphi = R -]x_\varphi^2,\, x_\varphi^1[$ if $\varphi \in]-\pi, -\pi/2]$ where x_φ^1 is the coordinate of intersection of the left hand instable separatrix with $0x$ and x_φ^1 is the coordinate of s_φ (see Figure 6.14:).

Definition 32 *We define $t_\varphi(x) = T_\varphi(\Omega_\varphi(x))$ the function time of transition for the orbit through $x \in M_\varphi$.*

We can now define the monotonicity property (M). Remark first that $t_\varphi(x) = t_{-\varphi}(g_{-\varphi})(x))$. This follows from the invariance of the differential equation of X_φ by : $\varphi \to -\varphi$, $y \to -y$, $t \to -t$. So that, it suffices to look at $\varphi \in [0, \pi]$:

Property (M) for $t_\varphi(x)$.

1) For each φ,the function :$x \to t_\varphi(x)$ has just one extremum (which is generic) at $x_\varphi \in R$ when $\varphi \in [0, \pi/_2[$; $t_\varphi(x)$ is monotonic when $\varphi \in [\pi/2, \pi]$.

2) $\dfrac{d}{d\varphi}\,[t_\varphi(x_\varphi)] > 0$ for $\varphi \in \left[0, \dfrac{\pi}{2}\right[$.

Remark 40 *It is easy to verify that $t_\varphi(x) \to 0$, when $x \to \pm\infty$. It follows from this and point 1) that x_φ is a maximum (in case $\varphi \in [0, \frac{\pi}{2}[$) and that $t_\varphi(x) \nearrow$ for $x \in]-\infty, x^1_\varphi[$ and $t_\varphi(x) \searrow$ for $x \in]x^1_\varphi, \infty[$ (in case $\varphi \in \left[\frac{\pi}{2}, \pi\right]$). Of course, $t_\varphi(x) \to \infty$ if $x \to x^1_\varphi, x^2_\varphi$).*

As we have said above, this property (M) has not been proved yet, but is supported by strong numerical evidence. Moreover it is possible to prove part 1) when $\varphi = 0$.

6.2.4 Monotonicity property for $t_0(x)$ and $t_\pi(x)$.

When $\varphi = 0$, the vector field is a hamiltonian vector field with hamiltonian function $H(x, y) = \dfrac{1}{2}\,y^2 - \dfrac{x^3}{3} - x$.

Up to an affine change in x, one can suppose that $H(x, y) = y^2 + x^3 + x$ and that the axis $0x$ is parametrized by the values h of the hamiltonian. Let be Γ_h the orbit by $h \in R$, i.e. : $\Gamma_h = \{H = h\} \subset R^2$. The transition time $t(h) = \displaystyle\int_{\Gamma_h} \frac{dx}{y}$.

Now, in the usual compactification of $C^2 \subset P_2(C)$ each Riemann surface $\{H = h\}$ has a regular point at infinity. So, on this surface, one can find a bounded cycle γ_n homotopic to Γ_h, and one can make this choice in a continuous way for $h \in R$.

Then : $\int_{\Gamma_h} \frac{dx}{y} = \int_{\gamma_h} \frac{dx}{y}$.

For such a continuous choice of $\gamma_h \subset \{H = h\}$, the abelian integral $J_0(h) = \int_{\gamma_h} \frac{dx}{y}$ has the properties which are described in 4.3.6. In particular, we have a Gauss-Manin system in the integrals I_0, I_1 or J_0, J_1 as a consequence of general formulas (4.63) in 4.3.6. For instance, for J_0, J_1 we obtain :

$$\begin{cases} \left(h^2 + \frac{4}{27}\right) J_0' = -\frac{1}{6}\, h\, J_0 - \frac{1}{9}\, J_1 \\ \left(h^2 + \frac{4}{27}\right) J_1' = -\frac{1}{27}\, J_0 + \frac{1}{6}\, h\, J_1. \end{cases} \tag{6.27}$$

We want to prove that $t'(h) = J_0'(h)$ has at most one simple root. To this end, we can use Petrov's lemma (lemma 12 in III.3.6.). First, from the first line of (6.27) we see that : $J_0' = 0$ if and only if $-\frac{1}{6}\, h\, J_0 - \frac{1}{9}\, J_1 = 0$ or :

$$g = \frac{J_1}{J_0} + \frac{3}{2}\, h = 0. \tag{6.28}$$

(Notice that J_0 has no root).

From the Ricatti equation for $\frac{J_1}{J_0}$:

$$\left(h^2 + \frac{4}{27}\right)\left(\frac{J_1}{J_0}\right)' = \frac{1}{9}\left(\frac{J_1}{J_0}\right)^2 + \frac{1}{3}\, h\left(\frac{J_1}{J_0}\right) - \frac{1}{27} \tag{6.29}$$

deduced from (6.27), we can write a Ricatti equation for g :

$$\left(h^2 + \frac{4}{27}\right) g' = \frac{3}{4}\, g^2 + \frac{3}{4}\, h^2 + \frac{5}{27}. \tag{6.30}$$

The polynomial $R_0(h) = \frac{3}{4}\, h^2 + \frac{5}{27}$ has no real root. So, it follows from lemma 12 in 4.3.6 that g has at most one simple root, and so that J_0' has also at most one simple root. Because $J_0 \to 0$ for $h \to \pm\infty$ it follows that J_0' has exactly one simple root where J_0 has a quadratic maximum.

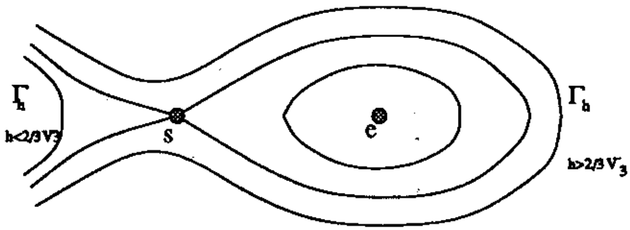

Figure 6.15:

In the case $\varphi = \pi$, we have to look to the hamiltonian $H(x,y) = y^2 - x + x^3$ already introduced in chapter 3. Now, the Ricatti equation for g has the form :

$$\left(\frac{4}{27} - h^2\right) g' = R_2(h)g + R_1(h) + R_0(h)$$
$$with\ R_0(h) = \frac{1}{4}\,h^2 - \frac{1}{27}. \tag{6.31}$$

The roots of R_0 are precisely the singular values of H : $\left\{\pm\frac{2}{3\sqrt{3}}\right\}$. The orbits Γ_h for $h \in \left[-\infty, +\frac{2}{3\sqrt{3}}\right[$ are on the left of the saddle point $s\left(x = -\frac{1}{3\sqrt{3}}\right)$, and surround the two critical points for $h > \frac{2}{3\sqrt{3}}$ (see Figure 6.15).

It is easy to verify that $t'(h) = J_0'(h) > 0$ for $h < \frac{2}{3\sqrt{3}}$. Now, because R_0 has no roots for $h > \frac{2}{3\sqrt{3}}$, J_0' has at most one simple root for $h \in \left]\frac{2}{3\sqrt{3}}, \infty\right]$. But, because $t(h) \to 0$ for $h \to \infty$ and $t(h) \to +\infty$ if $h \to \frac{2}{3\sqrt{3}}$ this implies that $t'(h)$ has also no root on $]\frac{2}{3\sqrt{3}}, \infty]$.

6.2.5 Indication for the proof of theorem 1.

We need the transition time $T_\varphi(\Omega)$ parametrized by Ω : $T_\varphi(\Omega) = t_\varphi(x_\varphi(\Omega))$ where $t_\varphi(x)$ is the transition time parametrized by $x \in M_\varphi$ and $x_\varphi(\Omega)$ the diffeomorphism $D_\varphi \to M_\varphi$. First, we remark that $T_\varphi(\Omega)$ has the same monotonicity property (M) as $t_\varphi(x)$:

1) $T_\varphi(\Omega)$ has just one extremum (a quadratic one) at $\omega_\varphi \in R$ when $\varphi \in [0, \frac{\pi}{2}[$; $T_\varphi(\Omega)$ is monotonic when $\varphi \in \left[\frac{\pi}{2}, \pi\right]$.

2) $\frac{d}{d\varphi}\left[T_\varphi(\Omega_\varphi)\right] > 0$ when $\varphi \in \left[0, \frac{\pi}{2}\right[$.

To prove it, we have just to notice that :

$$\frac{\partial T_\varphi}{\partial \Omega} = \frac{\partial t_\varphi}{\partial x} \cdot \frac{\partial x_\varphi}{\partial \Omega} \text{ and that } \frac{\partial x_\varphi}{\partial \Omega} > 0.$$

Next, $T_\varphi(\Omega_\varphi) \equiv t_\varphi(x_\varphi)$ and the point 2) is equivalent to the same property for t_φ.

From this property (M), it follows from $g_\varphi(\Omega)$, defined up to a constant by $\frac{\partial \Omega_\varphi}{\partial \Omega} = exp\,(\sin\ \varphi\ T_\varphi(\Omega))$, a behavior which is illustrated in figure 6.16, where we represent the graphs $g_\varphi : R \to R$ for different φ. The invariance of X_φ by : $\varphi \to -\varphi$, $y \to -y$, $t \to -t$ implies that $g_\varphi^{-1} = g_{-\varphi}$ so that it suffices to consider $\varphi \in [0, \pi]$. (In fact, for $\varphi \in]-\pi, 0[$ $\frac{\partial g_\varphi}{\partial \Omega} < 1$ and we have no bifurcation for these values of φ) (see Figure 6.16:).

Remark 41 *When φ goes from 0 to $\frac{\pi}{2}$ the value of the derivative at the unique inflexion point of g_φ increases monotonically from 1 to ∞. This follows from :*

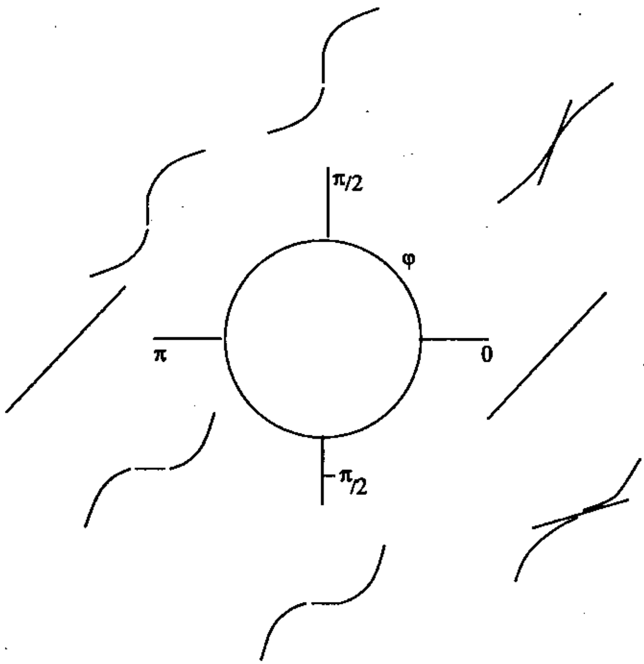

Figure 6.16:

$$\begin{aligned}\frac{\partial}{\partial\varphi}\left[\frac{\partial g_\varphi}{\partial\Omega}(\Omega_\varphi)\right] &= \frac{\partial}{\partial\varphi}\, exp\,(\sin\varphi t_\varphi(\Omega_\varphi)) \\ &= \left[\cos\ \varphi t_\varphi(\Omega_\varphi) + \sin\varphi\,\frac{\partial}{\partial\varphi}\,(t_\varphi(\Omega_\varphi))\right] \quad (6.32) \\ & \quad exp\,(\sin\varphi\ t_\varphi(\Omega_\varphi)).\end{aligned}$$

So that $\frac{\partial}{\partial\varphi}\left[\frac{\partial g_\varphi}{\partial\Omega}(\Omega_\varphi)\right] > 0.$

Recall that the equation of the fixed points for the return map is equivalent to the shift equation :

$$\Delta_{\varphi,\sigma}(\Omega,u) = g_\varphi(\Omega) - \gamma^{-1}\Omega - \tilde{\sigma} + O(uLn^2u) = 0.$$

Or, graphically, we have to look at the intersection of the graph $g_{\varphi,\sigma}(\Omega,u) = g_\varphi(\Omega) + 0(uLn^2u)$ with the line $:R = \gamma^{-1}\ \Omega + \tilde{\sigma}$, where $\tilde{\sigma}$ is related to σ by : $\tilde{\sigma} = u^{-6}\sigma + \alpha(\gamma^{-1}-1)Lnu$.

There exists a value $\varphi_1 \in \left]0, \frac{\pi}{2}\right[$ where the slope of the tangent at the inflexion point of g_{φ_1} is equal to γ^{-1}. For $\varphi < \varphi_1$ the slope of the tangent at g_φ, for any Ω is less than γ^{-1} and one has just 1 simple intersection point, and for $\varphi = \varphi_1$ a generic bifurcation of triple limit cycles.

For $u = 0$ this bifurcation is located at $(\varphi_1, \tilde{\sigma}_1)$ where $\tilde{\sigma}_1$ correspond to the intersection of the $\tilde{\sigma}$-axis with the tangent at the inflexion point of $g_{\varphi_1} : \tilde{\sigma}_1 = g_{\varphi_1}\ (\Omega_1) - \gamma^{-1}\ \Omega_{\varphi_1}$.

By the implicit function theorem, we have a line of bifurcation :

$$u \rightarrow (\tilde{\sigma}_1(u), \varphi_1(u))\ with\ \tilde{\sigma}_1(0) = \tilde{\sigma}_1,\ \varphi_1(0) = \varphi_1,$$

which defines a line TC of in the 3-parameter space :

$$u \rightarrow \left(\varphi_1(u),\ \sigma(u) = \alpha(\varphi_1(u))\ (\gamma^{-1}-1)u^6\ Ln\ u + u^6\ \tilde{\sigma}_1(u)\right)$$

or, explicitly :

$$(TC)\begin{cases} \mu(u) &= u^4\cos\,(\varphi_1(u)) \\ \nu(u) &= u\ \sin\,(\varphi_1(u)) \\ \sigma(u) &= \alpha(\varphi_1(u))(\gamma^{-1}-1)u^6 Ln\ u + u^6\tilde{\sigma}_1(u). \end{cases} \quad (6.33)$$

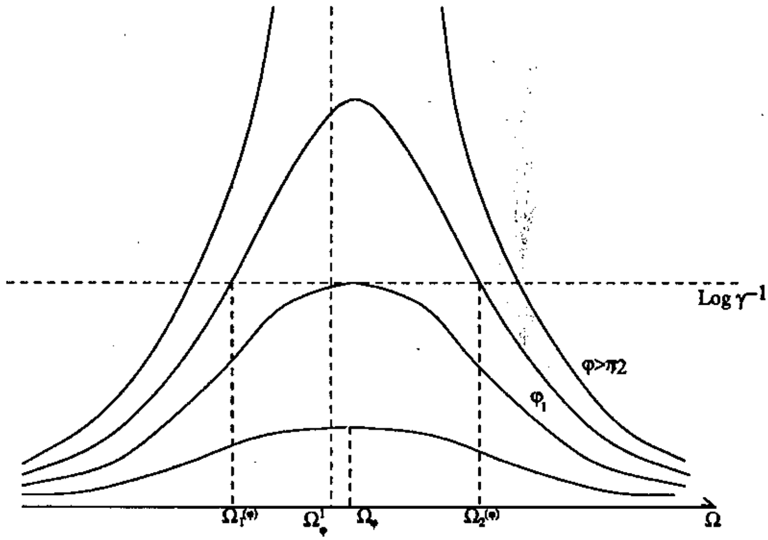

Figure 6.17:

From this line of triple cycles split two surfaces of double cycles. Their equation is given by :

$$\frac{\partial g_\varphi}{\partial \Omega} = \gamma^{-1} + 0(u\ Ln^2\ u).$$

Or :

$$\sin\varphi T_\varphi(\Omega) = Log\ \gamma^{-1} + 0(uLn^2u). \qquad (6.34)$$

The curve $\Omega \to \sin\varphi T_\varphi(\Omega)$ has an unique extremum whose value increases with φ and tends to ∞ when $\varphi \to \frac{\pi}{2}$ (see Figure 6.17).

For $\varphi > \varphi_1$, the graph of g_φ cuts the horizontal Log γ^{-1} in two points $\Omega_1(\varphi)$, $\Omega_2(\varphi)$ bifurcating from Ω_1, which define the two surfaces of double cycles which bifurcate from TC. These two surfaces may be extended for all values of $\varphi < \pi$.

When $\frac{\pi}{2} \leq \varphi < \pi$ the graph g_φ has a discontinuity at Ω_φ^1 and a vertical gap : $[g_\varphi^1, g_\varphi^2]$. The passage of the line $\gamma^{-1}\Omega + \tilde{\sigma}$ by the point

$(\Omega^1_\varphi, g^1_\varphi)$ corresponds to a saddle connection bifurcation L_s and for the passage by $(\Omega^1_\varphi, g^2_\varphi)$ to L_r.

And so it is easy to justify any bifurcations and the existence and number of limit cycles, outside a small neighborhoods of $u = 0$, $\varphi = \pi$.

The study in such neighborhoods is in fact entirely independent of the hypothesis (M) and just follows from the local properties of $g_\varphi(\Omega)$ for φ near π.

For $\varphi = \pi$ we have $g_\pi(\Omega) \equiv \Omega$, but the convergence : $g_\varphi(\Omega) \to \Omega$ is not regular when the value of Ω corresponds to a stable separatrix of the saddle s_π. To study the bifurcations at $\varphi = \pi$ we have to make a second directional blow-up in the parameter : $\varphi = \pi - u\xi$ with $\xi \in [-\xi_0, \xi_0]$, $u \in R^+$ small.

The vector field family $\overline{X}_{\varphi,u}$ is transformed in :

$$\overline{X}_{\xi,u} \begin{cases} \dot{x} &= y \\ \dot{y} &= x^2 - 1 + uy(\xi + \varepsilon x) + 0(u^2). \end{cases} \tag{6.35}$$

It is a u-perturbation of the hamiltonian vector field X_0 with hamiltonian function $H(x,y) = \frac{1}{2}\, y^2 + x - \frac{x^2}{3}$, which we have studied in chapter 4 for the Bogdanov-Takens theory.

The map $\widetilde{G}_{\xi,\sigma}(\Omega, u) = G_{\pi - u\xi,\sigma}(\Omega, u)$ is a u-perturbation of the identity. It may be studied by composition of regular maps with transitions near the saddle s on X_0. Before entering into the details of this computation, I draw the shape of the graph of $\widetilde{G}$ for $\widetilde{G}$, in figure 6.18 for some small $u > 0$.

For ξ small or large enough, we find back the shapes obtained for $g_\varphi(\Omega)$ for $\varphi > \pi$ and $\varphi < \pi$. These extremal shapes are independent on ε. But the transition of these two shapes are not the same in the two cases. For instance, in the case $\varepsilon = 1$, one has the horizontal level surrounded by large slopes in the central region: this will allow the possibility of a pair of double cycles and the possibility of four limit cycles.

Because $\widetilde{G} - \widetilde{G}_{|u=0}$ may be $O(u)$ (when Ω is a regular point) we need more precision on the remainder term $0(uLn^2u)$ in $R_{\varphi,\sigma}(\Omega, u)$. In fact, one can prove that this term writes :

$$\psi_0(u) + \psi_1(u)\Omega + \psi_2(u, \Omega, \sigma, \varphi)\Omega^2 \tag{6.36}$$

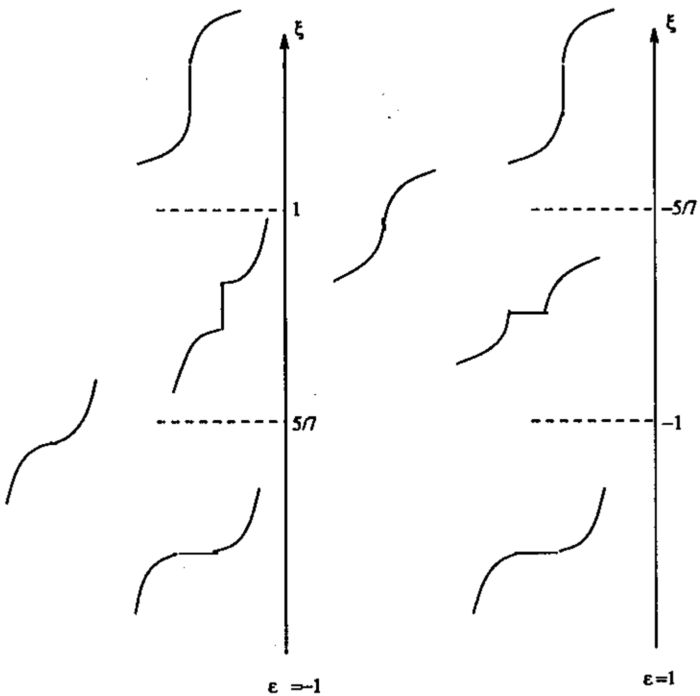

Figure 6.18:

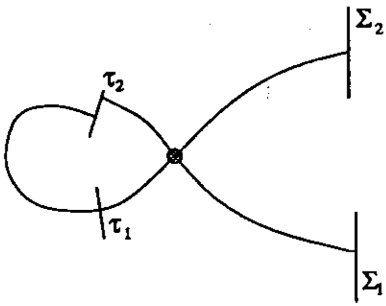

Figure 6.19:

$$with\ \psi_0(u) = O(uLn^2u),\ \psi_1(u) = O(uLn^2u)\ and\ \psi_2 = O(u^5) \quad (6.37)$$

(in a C^∞-sense).

So that we can write the transition R :

$$R_{\varphi,\sigma}(\Omega,u) = (\gamma^{-1}+\psi_1)\Omega+(\sigma u^{-6}+\alpha(\gamma^{-1}-1)Lnu+\psi_0)+O(u^5). \quad (6.38)$$

Here the remainder term is differentiable in Ω and can be incorporated in $\tilde{G}$; we will write $\tilde{R}_{\xi,\sigma}(\Omega,u)$ for R after the substitution $\varphi = \pi - u\xi$.

Bifurcations related to the double connection (DL) : (see Figure 6.19).

Because the separatrices of the saddle depend smoothly on the parameter for (φ, u) near $(\pi, 0)$, one can choose origins on Σ_1, Σ_2 such that $\Omega = 0$ corresponds to the separatrix intersection. Let Ω_1 be the coordinate on Σ_2. We introduce also sections τ_1, τ_2 as in figure 6.19, with coordinates z, z_1 respectively.

1) If $\Omega \geq 0$, $\tilde{G}$ is composition of three maps :

- From Σ_1 to τ_1 :

$$z = \Omega^{(1+u\bar{\alpha})}(1 + \Phi_0) \tag{6.39}$$

where $\bar{\alpha}(\xi, u, \cdots) = \xi + \varepsilon + 0(u)$ and Φ_0 is of class I (see chapter 5, section 5.1.3).

- From τ_1 to τ_2 :

$$z_1 = u\bar{\beta}_0 + (1 + u\bar{\beta}_1)z[1 + \Phi_1] \tag{6.40}$$

where $\bar{\beta}_0(\xi, u, \cdots) = a(\xi - \frac{5}{7}) + O(u)$.

$u\bar{\beta}_0$ is the shift between the 2 "small separatrices" on the left of the saddle. This quantity was computed in chapter 5 and $a = I_0(\frac{2}{3})$ is the area of the singular hamiltonian disk. $\Phi_1 = O(u)$ is smooth.

- From τ_2 to Σ_2 :

$$\Omega_1 = z_1^{1+u\bar{\alpha}}(1 + \Phi_2). \tag{6.41}$$

It is similar to the first transition.

By composition we obtain, for $\Omega \leq 0$:

$$\Omega_1 = \widetilde{G}_{\xi,\sigma}(\Omega, u) = (u\bar{\beta}_0 + (1 + u\bar{\beta}_1)\Omega^{1+u\bar{\alpha}}(1 + \widetilde{\Phi}_0))^{1+u\bar{\alpha}}(1 + \Phi_2). \tag{6.42}$$

The functions $\widetilde{\Phi}_0$, Φ_2 are of class I and are zero for $u = 0$.

2) If $\Omega \leq 0$, we have just the transition near the saddle :

$$\Omega_1 = \widetilde{G}_{\xi,\sigma}(\Omega, u) = \Omega^{(1+u\bar{\alpha})}(1 + \Phi_3). \tag{6.43}$$

Equation for the connection L_ℓ (small connection) is given by $\bar{\beta}_0 = 0$, i.e. :

$$\xi = \xi_0(u, \sigma) = \frac{5}{7} + O(u). \tag{6.44}$$

Equation for the right hand connection L_r is given by writing that $\widetilde{R}_{\xi,\sigma}(0, u) = 0$. This gives :

$$\sigma = \sigma_0(u, \xi) = \alpha(\pi - u\xi)u^6 Lnu(1 + O(u)). \tag{6.45}$$

The two equations (6.44), (6.45) gives two transversal surfaces. They cut along the line DL (Double loop).

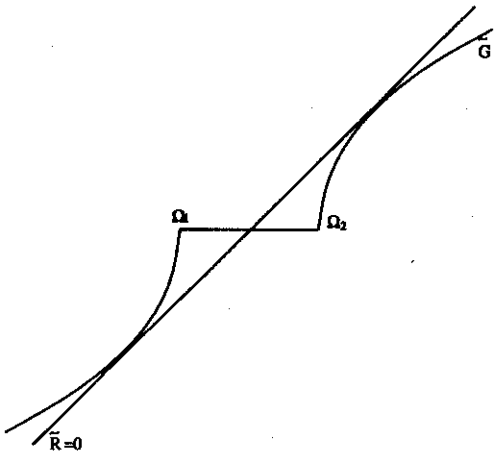

Figure 6.20:

To study the bifurcations near DL it is useful to introduce a local parameter $(\bar{\xi},\bar{\sigma})$ by :

$$\xi = \bar{\xi} + \xi_0, \ \sigma = \bar{\sigma} + \sigma_0. \tag{6.46}$$

In this local parametrization L_ℓ and L_r correspond to $\bar{\xi} = 0$ and $\bar{\sigma} = 0$ respectively. It is now easy to compute equations for the other surfaces of bifurcation. For instance, the "lower connection" L_i which corresponds to $\tilde{R} = \tilde{G} = 0$ has an equation :

$$(L_i) \ : \ \bar{\sigma} = \gamma^{-1} a u^{7-\bar{\alpha}_1 u} \ \bar{\xi}^{\frac{1}{1+\bar{\alpha}_1 u}} \ (1 + O(u)). \tag{6.47}$$

We have also : for L_s (the "upper connexion") given by $\Omega = 0$, $\tilde{G} = R$:

$$(L_s) \ : \ \bar{\sigma} = a u^{7+\bar{\alpha}_1 u} \ \bar{\xi}^{1+\bar{\alpha}_1 u} \ (1 + O(u)). \tag{6.48}$$

In the case $\varepsilon = 1$, we can look for pair of double limit cycles in the local chart $(\bar{\xi},\bar{\sigma})$. We present just an heuristic computation.

We want to write that the "line" $\tilde{R} = 0$ has double tangency with the $\tilde{G}$-graph (see Figure 6.20:).

The length of the horizontal level $[\Omega_1, \Omega_2]$ is of order $u \mid \beta_0 \mid^{\frac{1}{1+u\bar{\alpha}_1}}$. In first approximation the graph of $\widetilde{G}$ is symmetrical, and one has a double tangent when this tangent passes through the middle of $[\Omega_1, \Omega_2]$. Putting this in the equation of $\widetilde{G}$, we obtain :

$$\mid \bar{\xi} \mid = C_3(1+O(u))e^{\frac{-Log\gamma^{-1}}{u}}. \qquad (6.49)$$

for $C_3 = 2\gamma^{-1}a^{-1}$.

This is the $\bar{\xi}$-coordinate of the crossing point of $L_i \bigcap L_r$ (for some fixed value of u).

To obtain the $\bar{\sigma}$-coordinate of $L_i \bigcap L_r$, we have to compute the intersection of the double tangent with the $\bar{\sigma}$-axis.

One finds :

$$\bar{\sigma} = -\gamma^{-1}(1+O(u))u^7 e^{-\frac{Log\gamma^{-1}}{u}}. \qquad (6.50)$$

The more striking fact is that the two lines DL and $L_i \bigcap L_2$ have a *flat contact* of order $e^{\frac{Log\gamma^{-1}}{u}}$. So that the rectangular region in figure 6.5, when we have four large limit cycles is flat in u (the "radius" of the intersection sphere).

Saddle connection of codimension 2.

Look at the case $\varepsilon = 1$ for instance. The codimension 2 saddle connections correspond to Trace (saddle)=0. They correspond to surfaces which start from $u = 0$, $\xi = -1$ (we have that the hyperbolicity ratio is $u\bar{\alpha}_1$ with $\bar{\alpha}_1 = 1 + \xi + 0(u)$. Look at the *connection of type L_s* for instance : $\Omega = 0$ correspond to the entering separatrix. We can write :

$$\widetilde{G} = \Omega + u[\bar{\alpha}_0(\xi) + \bar{\alpha}_1(\xi)[\Omega\omega + \cdots] + \bar{\alpha}_2(\xi)\Omega + \cdots] + O(u^2). \qquad (6.51)$$

with the conventions introduced in chapter 5. We know that $\bar{\alpha}_0(1) > 0$.

Equation of limit cycles $\widetilde{G} = \widetilde{R}$ gives :

$$(h - 1 + \psi_1)\Omega + (\sigma u^{-6} - \alpha(\gamma^{-1} - 1)Lnu + \psi_0)$$

$$- u(\bar{\alpha}_0 + \bar{\alpha}_1[\Omega\omega + \cdots] + \bar{\alpha}_2\Omega) + O(u^2)O(\Omega) = 0. \qquad (6.52)$$

We can rearrange this equation in :

$$\tilde{\alpha}_0 + \tilde{\alpha}_1[\Omega\omega + \cdots] + \tilde{\alpha}_2\Omega + O(\Omega) = 0 \tag{6.53}$$

with :

$$\tilde{\alpha}_0 = \sigma u^{-6} - \alpha(\gamma^{-1} - 1)Lnu + O(uLn^2u) \tag{6.54}$$
$$\tilde{\alpha}_1 = -u\bar{\alpha}_1 + O(u^2) \tag{6.55}$$
$$\tilde{\alpha}_2 = \gamma^{-1} - 1 + O(u) \tag{6.56}$$

We can see that the bifurcation is a generic one for $u \neq 0$ ($\tilde{\alpha}_2 \neq 0$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_0$ are independent functions of σ and ξ). To verify this, it suffices to express the parameters σ, σ in function of the versal ones $\tilde{\alpha}_0$, $\tilde{\alpha}_1$:

$$\left\{ \begin{array}{lcl} \sigma & = & \alpha(\gamma^{-1} - 1)uLn^2u + u^6\tilde{\alpha}_0 + O(u^7Ln^2u) \\ \xi - 1 & = & -u^{-1}\tilde{\alpha}_1 + O(u). \end{array} \right. \tag{6.57}$$

A fixed rectangle in the $(\tilde{\alpha}_0, \tilde{\alpha}_1)$-plane corresponds to a rectangle in the (ξ, σ)-plane with a σ-dimension of order u^6 and the ξ one of order u^{-1} : this domain degenerates when $u \to 0$.

For the connection of type L_i, the corresponding formulas are :

$$\left\{ \begin{array}{lcl} \tilde{\alpha}_0 & = & u^{-6}\sigma - \alpha(\gamma^{-1} - 1)Lnu + O(uLn^2u) \\ \tilde{\alpha}_1 & = & -2u\bar{\alpha}_1 + O(u^2) \\ \tilde{\alpha}_2 & = & \gamma^{-1} - 1 + O(u) \end{array} \right. \tag{6.58}$$

Remark 42 *The formulas (6.58) for the L_i-connection are almost identical to the similar ones for the L_s-connection, up to the remainder term. They cannot permit to separate them. This corresponds to the fact that the coefficient $\bar{\alpha}_0(1)$ whose sign distinguishes the two cases is absorbed in the remainder term. To obtain a separation, we need a more precise computation which will be given in [DRS3].*

The regular values of (ξ, Ω).

If we take a value (ξ, Ω) where Ω is a regular value of $\tilde{G}_{\xi,\sigma}(\Omega, u)$ at $u = 0$, the function $\tilde{G}$ tends C^1 toward the identity for $u \to 0$. Because $\tilde{R}$ converges toward $\gamma^{-1}\Omega + \tilde{\sigma}$ with $\gamma^{-1} \neq 1$, we can just have simple roots in a compact domain which does not contain singular values of Ω : all the bifurcations at $\varphi = \pi$ arrive for $\xi = -\varepsilon, -\frac{5}{7}\varepsilon$ for the Ω-value of the entering separatrix. They are precisely the cases we have considered above.

This finishes the outline I wanted to give on the proof of theorem 1. A more complete study will be made in [DRS3].

6.3 A method of desingularization for analytic vector fields.

In this paragraph, we consider an analytic family (X_λ) on S^2. I want to explain how the generalized blow-up operation may be used to give a general method of desingularization for a family like (X_λ), as it was presented in [DeR]. The geometrical object we obtain by such desingularization is called *foliated local vector field*. Roughly speaking, it is given by a *local vector field* (another name for dimension 1 foliations with singularities). This local vector field is tangent to a singular 2-dimensional foliation, which comes from the blow-up of the fibration of R^{k+2} onto the parameter space R^k.

Next, we define precisely what are the *desingularization operations.* Apart from a generalized blow-up operation (which generalizes the above global blow-up used in the last paragraphs), we introduce the possibility *to divide* the local vector field by local functions (for instance the function r in the last paragraph) and also to replace a family by a new one *inducing* the first one. Finally, we want to propose general conjectures concerning desingularizations of analytic families, to look at relations with the Trifonov's theory and to formulate what are the hopes to succeed in proving the conjectures.

In what follows all ours objects such as families, maps and so on will be *real analytic.*

6.3.1 Foliated local vector fields.

Definition 33 *A local vector field is defined on a compact (maybe with boundary) analytic manifold E by a finite open covering $\{U_i\}$ of E*

with some analytic vector field X_i *on each* U_i, *verifying the following compatibility condition : for each pair of indices* i, j *such that*

$$U_i \cap U_j \neq \emptyset,$$

there exists an analytic function g_{ij} *defined and strictly positive in* $U_i \cap U_j$ *such that :*

$$X_i = g_{ij} X_j \text{ on } U_i \cap U_j.$$

Two collections $\{U_i, X_i\}$ and $\{V_j, Y_i\}$ as above are said to be equivalent if there exist positive analytic functions f_{ij} defined on $U_i \cap V_j$ such that $X_i = f_{ij} Y_j$. A *local vector field* on E is an equivalence class. Of course, each vector field defines a local vector field.

We denote by $Z(X)$ the union of sets of singular points of all X_i associated to a local vector field X. This set doesn't depend on the choice of the collection $\{U_i, X_i\}$.

Remark 43 *Such a local vector field is often called : oriented singular 1-dimensional foliation. We prefer the terminology "local vector field" because we reserve the term "foliation" for another purpose (see below).*

Definition 34 *We will call singular fibration a triple* (E, π, Λ) *consisting of :*

- a compact real analytic manifold E *of dimension* $k+2$,

- a compact real analytic manifold Λ *of dimension* k,

- an analytic surjective mapping $\pi : E \to \Lambda$ *such that for each* $x \in E$ *there are local coordinates* $x_1, x_2, \cdots, x_{k+2}$ *in a neighborhood of* $\pi(x)$, *sending* $\pi(x)$ *to 0, where* π *takes the form :*

$$\lambda_1 = \prod_{i=1}^{k+2} x_i^{p_1^i}, \cdots, \lambda_k = \prod_{i=1}^{k+2} x_i^{p_k^i} \text{ with } p_j^i \in N. \tag{6.59}$$

We suppose moreover, that π *is regular (rank* $(\pi) = k$*) on an open dense set* U_0. *We suppose also that each regular fiber of* π *in* U_0 *is diffeomorphic to a 2-dimensional compact submanifold of the 2-sphere (possibly with non-empty boundary), i.e. a surface of genus 0.*

Observe that for a foliation defined on an open dense set in E, any two extensions coincide on the intersection of their domains. Then there exists an unique maximal foliation extending the given one. We apply this remark here and we will denote by $\mathcal{F}$ the *maximal foliation* extending the foliation $\mathcal{F}_0$ defined by the connected components of the regular fibers of π on U_0. The domain of $\mathcal{F}$ will be denoted by U and the singular set of $\mathcal{F}$, $E\backslash U$ will be denoted by Σ.

Proposition 17 *[DeR]*
The maximal foliation $\mathcal{F}$, associated to a singular fibration (E, π, Λ) verifies the following properties :

1. The set Σ is an analytic subset of E and it is decomposed into two analytic manifolds Σ_1 and Σ_0, with cod $\Sigma_1 \geq 1$ and cod $\Sigma_0 \geq 2$, such that $\partial\Sigma_1 = \Sigma_0$.

2. If L is a leaf of $\mathcal{F}$, $\overline{L}$ is homeomorphic to a closed submanifold of S^2, with boundary and corners. For a given leaf L let $\partial_0 L$ stands for the set of corners and $\partial_1 L = \partial L \backslash \partial_0 L$.

3. We have $U_L \partial L = \Sigma$ and $\{\partial_1 L,\ L \in \mathcal{F}\}$ defines an analytic foliation of Σ_1. Also, $U\partial_0 L = \Sigma_0$, which can be seen foliated by points.

4. Let $x \in \Sigma$ and $\mathcal{L}_x$ be the collection of all the leaves L of $\mathcal{F}$ such that $\overline{L}$ is an analytic manifold with boundary in some neighborhoods of x. Suppose that $x \in \Sigma_1$ and let ℓ_x denotes the leaf of Σ_1 through x. We have :

$$\bigcap\{T_x L \mid L \in \mathcal{L}_x \text{ and } x \in \overline{L}\} = T_x \ell_x. \tag{6.60}$$

Similarly, for $x \in \Sigma_0$, we have :

$$\bigcap\{T_x L \mid L \in \mathcal{L}_x \text{ and } x \in \overline{L}\} = \{0\}. \tag{6.61}$$

Definition 35 *Given a singular fibration (E, π, Λ) and a point $x \in E$ a leaf through x is :*

- *the 2-dimension leaf of $\mathcal{F}$ containing x, if $x \in U$*
- *the 1-dimension leaf of Σ_1 containing x, if $x \in \Sigma_1$*
- *the point $\{x\}$, if $x \in \Sigma_0$.*

We will introduce the main object of this paragraph, generalizing the notion of family of vector fields :

Definition 36 *A foliated local vector field* $\mathcal{E} = (E, \pi, \Lambda, X)$ *is an object consisting of a singular fibration* (E, π, Λ) *and a local vector field defined of* E *such that* X *is tangent to the fibers, i.e.* $d\pi(x)[X(x)] = 0$ *for all* $x \in E$.

Remark 44 *An example of foliated local vector field is given by any analytic family* X_λ *on* S^2, *with* $\lambda \in \Lambda$, *some compact analytic manifold ;* $E = S^2 \times \Lambda$ *and* π *is the natural projection of* E *on* Λ *and* X *is the family seen as a vector field on* E.

Proposition 18 *A foliated local vector field* X *is tangent at each point* $x \in E$ *to the leaf through* x.

Proof The field X is tangent to the regular fibers by definition, so, by continuity and by the density of U, it is tangent to all the leaves of the maximal foliation $\mathcal{F}$.

Now, take a point $x \in \Sigma_1$. By continuity and using the point 4 of the proposition 5, we get that $X(x) \in T_x\ell_x$ where ℓ_x is the leaf through x. The same holds when $x \in \Sigma_0$. ■

Of course, the notion of foliated local vector field is motivated by the global blow-up as it was used in paragraph 6.2 above. For instance let $E = S^2 \times S^2$ and $R^2 \times R^2$ a local chart at some point $(x_0, \lambda_0) \in S^2 \times S^2$ (with $x_0 = 0$, $\lambda_0 = 0$). The usual blow-up at this point is a map $\Phi : \widehat{E} \xrightarrow{\Phi} E$. The new space $\widehat{E}$ has a singular projection $\widehat{\pi} = \pi \circ \Phi$ on $\Lambda \simeq S^2$. The critical locus of Φ is a 3-sphere D. For regular values of $\widehat{\pi}$, $\lambda \in S^2 - \{\lambda_0\}$ we have regular fibers $\widehat{\pi}^{-1}(\lambda) \simeq S^2$; on the contrary $\widehat{\pi}^{-1}(\lambda_0) = D \cup \widehat{F}$ (where $\widehat{F}$ is the blow-up of the fiber $F = \pi^{-1}(\lambda_0)$. The foliation $\mathcal{F}_0$, whose leaves are the $\widehat{\pi}^{-1}(\lambda)$ for $\lambda \neq \lambda_0$, extends in a maximal foliation $\mathcal{F}$. The set $\Sigma \simeq S^1$ is just the critical locus of fiber $\widehat{F} \simeq D^2$, with $\Sigma = \partial D^2$; $\Sigma \subset D$ an d $D \backslash \Sigma$ is made by 2-disks foliating $D \backslash \Sigma$ with boundary equal to Σ (see Figure 6.21).

6.3.2 Operations of desingularization.

A *limit cycle* of a foliated local vector field is a limit cycle of the restriction X_L of X to some leaf of $\mathcal{F}$.

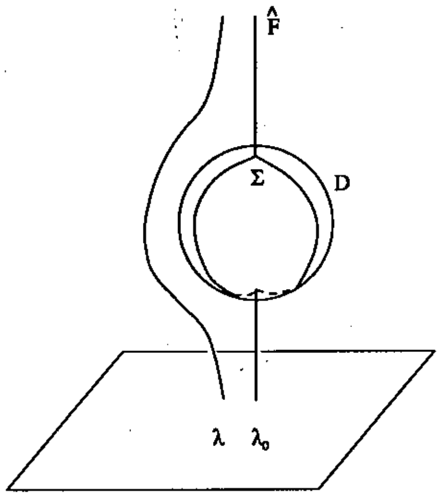

Figure 6.21:

In this paragraph, I want to define operations to pass from a foliated local vector field to another, keeping control on the number of limit cycles we have in each leaf.

Definition 37 *We say that K is a uniform bound on the number of limit cycles by leaves for a foliated local vector field $\mathcal{E}$ (in short a uniform bound for $\mathcal{E}$) if, for each L belonging to the maximum foliation $\mathcal{F}$, the number of limit cycles of the restricted field X_L is not greater than K. If such a bound exists for a given foliated local vector field $\mathcal{E}$, we say that $\mathcal{E}$ has the finiteness property.*

The most important property of the three operations described below is that *they preserve the finiteness property,* i.e. if a uniform bound exists for $\widetilde{\mathcal{E}}$, obtained from $\mathcal{E}$ by one of the desingularization operation, then $\mathcal{E}$ has also the finiteness property.

Induction.

Definition 38 *We say that $\mathcal{E} = (F\times\Lambda, \pi, \Lambda, X_L)$ is induced by another family $\widetilde{\mathcal{E}} = (F \times \widetilde{\Lambda}, \widetilde{\pi}, \widetilde{\Lambda}, \widetilde{X_{\widetilde{\lambda}}})$ if there exists an analytic map $\Phi : \Lambda \to \widetilde{\Lambda}$*

such that for any λ, (X_λ) *is topologically equivalent to* $(\widetilde{X}_{\Phi(\lambda)})$. *Observe that if* $\widetilde{K}$ *is a bound for* $(\widetilde{X}_{\widetilde{\lambda}})$, *then it is also a bound for* (X_λ).

Local division.

Definition 39 *Given two local vector fields* X, Y *on* E, *we say that* Y *is the result of local division of* X *if there is a finite open covering* $\{V_i\}$ *of* E, *for which both* X *and* Y *are defined by local vector fields* X_i, Y_i *and analytic functions* $f_i : V_i \to R$ *such that :* $X_i = f_i Y_i$ *on* V_i.

The functions f_i *may have zeros and* Y *may have more limit cycles than* X_i, *but any limit cycle of* X *is also a limit cycle for* Y. *So that if* Y *has the finiteness property, it is also the case for* X.

Generalized blow up.

First, we define the global blow-up at a point in the context of foliated local vector field. Let $\mathcal{E} = (\varepsilon, \pi, \Lambda, X)$ and $p \in E$. Suppose one chooses local coordinates $(x_1, \cdots, x_n)$ around p, $(\lambda_1, \cdots, \lambda_k)$ around $\pi(p)$ where π is monomial (as above). Suppose chosen a system of weights $(\alpha_1, \cdots, \alpha_n) \in N^n$. We can define a map in coordinates :

$$\varphi : S^{n-1} \times R^+ \to R^n \text{ , by } x = \tau^\alpha \, \bar{x} = (\tau^{\alpha_1} \, \bar{x}_1, \cdots, \tau^{\alpha_n} \, \bar{x}_n).$$

We define a blow-up space :

$$\widetilde{E} = (E \backslash \{p\}) \bigcup S^{n-1} \times R^+ / \sim$$

where $\sim$ is the identification $(\bar{x}, \tau) \in S^{n-1} \times R^+ \sim T \circ \varphi(\bar{x}, \tau) \in E$ and T is the coordinate map $T : R^n \to E$.

Now, the blow-up map $\overline{\Phi}$ is defined by the commutative diagram :

$$\begin{array}{ccc} S^{n-1} \times R^+ & \longrightarrow & R^n \\ i \big\downarrow & & \big\downarrow T \\ \widetilde{E} & \xrightarrow{\Phi} & R^n \end{array} \qquad (6.62)$$

We denote by D the set $i(S^{n-1} \times \{0\}) \subset \widetilde{E}$ which is the *critical locus* of Φ.

Now, we look at the effect of such a blowing-up on a vector field defined near p. So, let X a vector field in $W = T(R^n)$ with $X(p) = 0$. Then, an easy computation shows that there exists $s \in Z$ such that $r^s\ \Phi_*^{-1}(X) = \overline{X}$ is an analytic vector field on $S^{n-1} \times R^+$. We take the *minimal such s*. If $\mathcal{E} = (E, \pi, \Lambda, X)$ and $X(p) = 0$, the above operation gives a well-defined local vector field $\widetilde{X}$ on $\widetilde{E}$, such that $\widetilde{X}_{|\widetilde{E}\setminus D} \simeq X_{|E\setminus\{p\}}$. Finally, the *blown-up foliated local vector field of* $\mathcal{E}$ is equal to $\widetilde{\mathcal{E}} = (\widetilde{E}, \widetilde{\pi}, \widetilde{\Lambda}, \widetilde{X})$ where $\widetilde{\Lambda} = \Lambda$ and $\widetilde{\pi} = \pi \circ \Phi$.

One can generalize the blow-up operation by replacing p by some compact submanifold $C \subset E$. If cod $C = n$, this submanifold is subject to some restrictions : in order to blow-up with a given system of weights $(\alpha_1, \cdots, \alpha_n)$ we need that the embedding $C \subset E$ has the so-called *admissible trivialization* by an atlas of charts $W_i \simeq U_i \times R^n$, where U_i is a chart in C and moreover that the projection π on each W_i is monomial in the normal factor R^n. See definitions and details in [DeR].

If the above conditions are fulfilled and if $C \subset Z(X)$, the set of zeros of X), then we can blow-up E along C, using the weights $(\alpha_1, \cdots, \alpha_n)$. We produce a new foliated local vector field $\widetilde{\mathcal{E}} = (\widetilde{E}, \widetilde{\pi}, \Lambda, \widetilde{X})$ where :

- $\widetilde{E}$ is the blown-up space,

- $\widetilde{\pi} = \pi \circ \Phi$, where $\Phi : \widetilde{E} \to E$ is the blow-up map,

- $\widetilde{X}$ is a local vector field on $\widetilde{E}$ which is equal to $\Phi_*^{-1}(X)$ on $\widetilde{E}\setminus D$, where D is the critical locus of Φ (D is a fibered bundle over C, with fiber diffeomorphic to S^{n-1}).

Because $C \subset Z(X)$, we do not destroy any limit cycles of X. The blowing-up may disconnect some leaves, but each leaf of $\mathcal{E}$ is covered by an uniform finite number of leaves of $\widetilde{\mathcal{E}}$, so that, finiteness property for $\widetilde{\mathcal{E}}$ implies finiteness property for $\mathcal{E}$.

6.3.3 Conjectures.

Definition 40 *Let $\mathcal{E} = (E, \pi, \Lambda, X)$ be a foliated local vector field and $p \in E$ be a singular point of $X (p \in Z(X))$. We say that p is an elementary singular point, if for each $L \in \mathcal{L}_p$, the point p is an elementary singular point of X_L (i.e. the 1-jet $J^1X_L(p)$ has at least a non-zero real* *eigenvalue). Note that p may belong to the critical set* Σ *and in this case,* $p \in \overline{L}$ *means that* $p \in \partial L$. $\mathcal{L}_p$ *may contain many leaves L.*

We say that a compact invariant set Γ *is a limit periodic set of* $\mathcal{E}$ *if there exists a sequence of limit cycles of* $\mathcal{E}$ *which converges to* Γ *in the sense of the Hausdorff topology in* $C(E)$*, the topological space of compact, non-empty subsets in E. A limit periodic set is said elementary if each of its points is either regular* ($X(p) \neq 0$) *or is an elementary singular point.*

Remark 45 *It seems possible to show, using Poincaré-Bendixson theorem like in chapter 2, that each elementary limit set is formed by a finite number of arcs, each of them being either a regular trajectory or a normally hyperbolic arc of zeros, and a finite number of isolated singular points. Besides, each of these arcs would be contained in the closure of one of the leaves of* $\mathcal{F}$*, and the singular points in* Γ *would be in* Σ*.*

Such a curve is similar to the degenerate graphics of families introduced in chapter 2, with the difference now that Γ *can go through several leaves and the critical set* Σ*.*

Such elementary limit sets may be see in figure 6.12 above.

Definition 41 *A step of desingularization is a correspondence :*

$$\{\mathcal{E}_i\}_{i\in I} \to \{\mathcal{E}_{ij}\}_{(i,j)\in I,J}$$

between two collections of foliated local vector fields, satisfying the following conditions. Let be $\mathcal{E}_i = (E_i, \pi_i, \Lambda_i, X_i)$*. One supposes that there exists a collection* $\{\mathcal{E}^\circ_{ij}\}_{(i,j)\in I\times J}$, $\mathcal{E}^\circ_{ij} = (E_{ij}, \pi_{ij}, \Lambda_{ij}, X_{ij})$ *such that :*

(1) $E_{ij} \subset E_i$ *for all* $(i,j) \in I \times J$,

(2) for each $i \in I$*, every non elementary limit periodic set of* $\mathcal{E}_i$ *is contained in the interior of one of* E_{ij},

(3) for each $(i,j) \in I \times J$*, the maximal foliation of* $\mathcal{E}^\circ_{ij}$ *is the trace on* E_{ij} *of the maximal foliation of* $\mathcal{E}_i$*. Moreover, there is an analytic map* $\varphi_{ij} : \Lambda_{ij} \to \Lambda_i$ *such that* $\pi_{i|E_{ij}} = \varphi_{ij} \circ \pi_{ij}$,

(4) for each $(i,j) \in I \times J$, $\mathcal{E}_{ij}$ *is either equal to* $\mathcal{E}^\circ_{ij}$ *or is induced from* $\mathcal{E}^\circ_{ij}$ *by one of the three desingularization operations of the previous paragraph.*

Using the above conditions, we can now formulate two conjectures :

Desingularization Conjecture.

For any analytic family (X_λ), $\lambda \in \Lambda$ some compact analytic manifold, on S^2, a finite number of desingularization steps can be chosen such that in the resulting final collection $\{\mathcal{E}_i\}$ of foliated local vector fields any limit periodic set is elementary.

Reduced Local finite Cyclicity Conjecture.

Each elementary limit periodic set Γ of an analytic foliated local vector field $\mathcal{E} = (E, \pi, \Lambda, X)$ has a finite cyclicity property, i.e. there exists $\varepsilon > 0$ and $K \in N$ such that for each leaf $L \in \mathcal{F}$ the number of limit cycles of $\mathcal{E}$, at E-Hausdorff distance to Γ less than ε, is bounded by K (we suppose chosen some metric on E).

Now, as for the compact families in chapter 2, the finite cyclicity property for each limit periodic set implies the finiteness property for $\mathcal{E}$:

Lemma 24 *([DeR]). If each limit periodic set of a foliated local vector field $\mathcal{E}$ has a finite cyclicity, then $\mathcal{E}$ has the finiteness property.*

A finiteness conjecture for compact analytic families was formulated in chapter 2 (section 2.2). We can now replace it by a more elaborate one, in fact by the two above conjectures :

Proposition 19 *Desingularization Conjecture together with Reduced Local Finite Cyclicity Conjecture imply that each compact analytic family on $S^2 \times \Lambda$ as the finiteness property.*

Proof Suppose that after k desingularization steps we have obtained a final collection of foliated local vector fields whose limit periodic sets are elementary. It follows from the above lemma and the second conjecture that each foliated local vector field $\mathcal{E}$ in this collection has the finiteness property.

To finish the proof, we repeat this argument inductively. Suppose we have proved that in the s^{th} step of desingularization, $s \geq 1$, all foliated local vector fields have the finiteness property. Let $\mathcal{E}$ be one of the foliated local vector field of the $(s-1)^{th}$ step. Since the finiteness property is preserved by the three desingularization operations, each non elementary limit periodic set of $\mathcal{E}$ has the finite cyclicity property. Each elementary limit periodic set has it too, as it follows from the second conjecture. So, $\mathcal{E}$ has the finiteness property as a consequence of lemma 7. Therefore, by a finite induction, we obtain the required result. ∎

6.3.4 Final comments and perspective.

The desingularization method was useful to study several unfoldings and to obtain their diagrams of bifurcation. I have presented in the paragraph 2 a detailed review about the study of generic unfoldings of cuspidal loops, to appear in the forthcoming publication [DRS3]. Some other examples of the desingularization method were given in [DR2] for the unfolding of codimension 3 nilpotent focus and for the Van der Pol's singular perturbation equation in [DR3]. In each of these examples the singular points have a non-zero linear part : if degenerate, they must be nilpotent. Moreover the desingularization needed just a one-step global blow-up.

A first step in direction of the desingularization conjecture proof is to consider families where any singular point *has a non-zero linear part.* As said before all the already studied bifurcations verify this hypothesis. It includes singular point of finite codimension whose 1-jet is nilpotent (the codimension 3 was studied in [DRS2] and also the generic "turning points" of singular perturbation equations of any finite codimension. In [PR1] we proved that such families can be reduced to families with isolated nilpotent point (but perhaps with non-isolated elementary singular points) and we proved a Poincaré-Bendixson theorem in these last families : any limit periodic set must be a *graphic, degenerate or not* (as defined in chapter 2). In a forthcoming paper [PR2] we intend to prove the Desingularization Conjecture for families with isolated nilpotent points (the other points have to be elementary) : for instance, the local differential equations at a nilpotent point of finite codimension write :

$$\dot{x} = y,\ \dot{y} = (\alpha x^k + \cdots) + y(\beta x^\ell + \cdots) + y^2 Q,\ with\ \alpha, \beta \neq 0\ ; \quad (6.63)$$

by blowing-up one does not produce more complicated points than isolated nilpotent one (isolated among the other nilpotent ones) ; moreover it is possible to introduce at each singular point of the foliated local vector field we obtain by induction an index (k, ℓ, s) which decreases strictly (in the lexicographic order) at each step of blowing-up. (For nilpotent point of finite codimension (6.63), the index is $(k, \ell, 0)$).

A general theorem for the desingularization for holomorphic line fields (i.e. holomorphic singular foliation of dimension 1) was proved by Trifonov :

Theorem 34 *[Tr] Let $\pi : E \to \Lambda$ an holomorphic fibration with an holomorphic line fields ℓ tangent to the fibers. Then, there exist another fibration $\widetilde{\pi} : \widetilde{E} \to \widetilde{\Lambda}$ and holomorphic proper surjective maps $\Pi : \widetilde{E} \to E$, $\varphi : \widetilde{\Lambda} \to \Lambda$ such that the following diagram is commutative :*

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\Phi} & E \\ {\scriptstyle \widetilde{\pi}}\Big\uparrow & & \Big\uparrow{\scriptstyle \pi} \\ \widetilde{\Lambda} & \xrightarrow{\Phi} & \Lambda \end{array} \tag{6.64}$$

Moreover there exists a line field $\widetilde{\ell}$ on $\widetilde{E}$, whose image $\Phi(\widetilde{\ell})$ is ℓ (for each regular point $\widetilde{p}$ of $\widetilde{\ell}$: $d\Phi(\widetilde{p})(\widetilde{\ell}(\widetilde{p})) = \ell(\Phi(\widetilde{p}))$), and every singular point of $\widetilde{\ell}$ is elementary.

We can translate this theorem to real analytic line fields and also to analytic vector field families. But the Trifonov's theorem just looks at the foliation defined by the vector field : if f is an analytic non trivial function the vector fields X and fX define the same foliation.

So that the transcription of the theorem 9 for real analytic vector field families is :

Given an analytic vector field family (X_λ) defined on $E = S^2 \times \Lambda$ for instance, one has an analytic vector field family $(\widetilde{X_{\widetilde{\lambda}}})$ on $\widetilde{E} = \widetilde{S} \times \widetilde{\Lambda}$ and analytic proper surjective map $\overline{\Phi}$, Φ as above such that $\Phi_*(\widetilde{X_{\widetilde{\lambda}}}) = (X_{\Phi(\widetilde{\lambda})})$. Moreover for $(\widetilde{p}_0, \widetilde{\lambda}_0) \in \widetilde{E}$ there exists an analytic function $\widetilde{f}$ and a vector field $\widetilde{Y}$ defined in a neighborhood of $\widetilde{p}_0 \in \widetilde{S}$ such that $\widetilde{p}_0$ is a regular or an elementary singular point of $\widetilde{Y}$ and $\widetilde{X_{\widetilde{\lambda}_0}} = \widetilde{f}\widetilde{Y}$.

So, the zeros of $\widetilde{X}_{\tilde{\lambda}_0}$ may be non-isolated and moreover the factor $f(\tilde{p})$ cannot be extended in general in a neighborhood of $(\tilde{p}_0, \tilde{\lambda}_0)$ to divide the family $\widetilde{X}_\lambda$. As a consequence, the non-isolated zeros (in some fiber) are *unavoidable.* In fact, at such point, the family is equivalent at *singular perturbation equation* by a singular change of time (see [I]). As a simple example one can consider the family : $y \dfrac{\partial}{\partial y} + \varepsilon \dfrac{\partial}{\partial x}$ which has a normally hyperbolic line of zeros at $\varepsilon = 0$.

At a nilpotent point p, a vector field X writes $X = fY$ which $df(p) \neq 0$ and $Y(p) \neq 0$. So that, nilpotent points are elementary points in the sense of the Trifonov result. The aim in [PR2] is precisely to complete the desingularization of such unfoldings, and in particular to get rid off non normally hyperbolic lines of zeros.

In the Trifonov theory one just looks at X up to a multiplicative function f, because one is just interested in the foliation. It seems rather clear that the methods used in [Tr] may be used to desingularize simultaneously Y and f i.e. to obtain a desingularized family where at each point $(\tilde{p}_0, \tilde{\lambda}_0)$ $X_{\tilde{\lambda}_0} = f.Y$ with p_0 a regular or an elementary singular point, and f non zero or such that $\{f = 0\}$ has normal crossing at $\tilde{p}_0$, transversal or tangent to the field Y. If this result was true, we will have a finite simple list of possibilities to study. In each case it is possible to write a simple analytic normal form for the germ of X at $p = 0 \in R^2$:

(a) $X = y^k \dfrac{\partial}{\partial x}$ $k \geq 1$ (for $k = 1$: line of nilpotent points).

(b) $X = y^k \dfrac{\partial}{\partial y}$ $k \geq 1$ (for $k = 1$: line of normally hyperbolic points).

(c) $X = x^k y^\ell \Big(x \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y}\Big)$, $k \geq 1$, $\ell \geq 1$ (f has a normal crossing and $Y \neq 0$ is transversal to $\{f = 0\}$).

(d) $X = x^k y^\ell \dfrac{\partial}{\partial x}$, $k \geq 1$, $\ell \geq 1$ (f has a normal crossing and $Y \neq 0$ in tangent to $\{f = 0\}$).

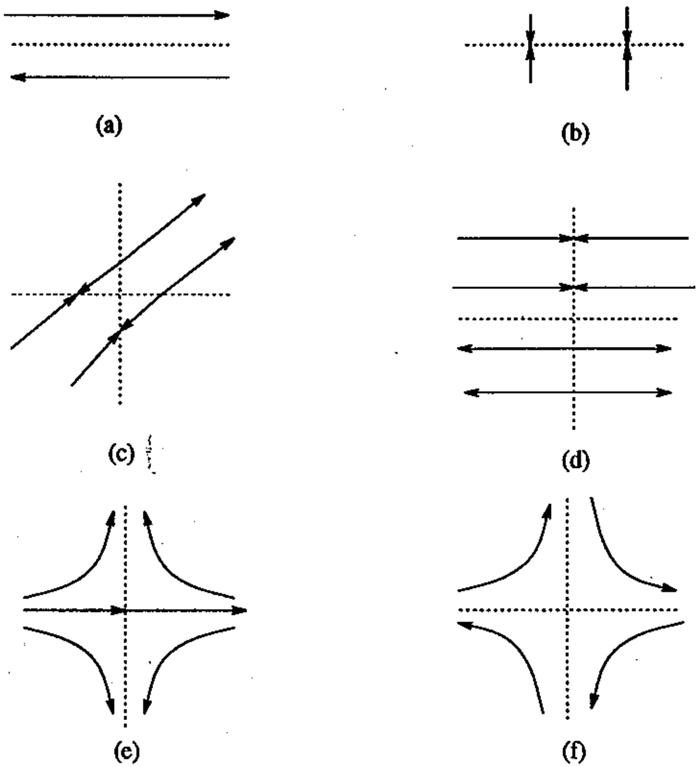

Figure 6.22:

(e) $X = x^k\, Y,\ k \geq 1$ (Y is finite codimension semi-hyperbolic point and is tangent to $\{x = 0\}$).

(f) $X = x^k\, y^\ell Y$, $k \geq 1$, $\ell \geq 1$ (Y is as in (e) and tangent to $\{x = 0\}$ and $\{y = 0\}$) (see Figure 6.22:).

The point (b) for $k = 1$ is already elementary in the sense of this chapter and (a), for $k = 1$, was treated in [PR1]. The first step, to treat the other cases would be to obtain a *good normal form theory for the unfoldings*, and next to apply the desingularization method explained in the last paragraph.

Once the desingularization conjecture is proved it will remain the second conjecture : prove the finite cyclicity conjecture for elementary limit periodic sets which appear in a foliated local vector field $\mathcal{E}$.

In the chapter 5, we have looked at unfolding of elementary graphics. They correspond to elementary limit periodic sets Γ which belong to the interior of some leaf of $\mathcal{E}$, i.e., such that $\Gamma \bigcap \Sigma = \emptyset$: in this case $\mathcal{E}$ is equivalent in a neighborhood of Γ to an usual unfolding. The study of the cuspidal loop in paragraph 2 of this chapter gives some ideas on difficulties and ideas we can use in the general case. A new problem, as we have seen, is to take into account transitions near the elementary points in Γ which are located in the singular set Σ of $\mathcal{E}$.

BIBLIOGRAPHY

General references :

[ALGM] - A. Andronov, E. Leontonich, I. Gordon, A. Maier, *Theory of Bifurcations of Dynamical Systems on a Plane*, Israel Program for Scientific Translations, Jerusalem (1971).

[AA] - D. Anosov, V.I. Arnold, *Dynamical systems I*, Encyclopaedia of Mathematical Sciences, Vol.1, Springer-Verlag (1988).

[A] - V.I. Arnold, *Supplementary Chapters to the Theory of Ordinary Differential Equations*, Nauka, Moscow (1978).

[CH] - S.-N. Chow, J. Hale, *Methods of bifurcation theory*, Springer-Verlag (1982).

[CLW] - S.-N. Chow, Chengzhi Li, Duo Wang, *Normal Forms and Bifurcations of Planar vector fields*, Cambridge University Press (1994).

[D] - F. Dumortier, *Singularities of vector fields*, Monografias de Matemática 32, I.M.P.A., Rio de Janeiro (1978).

[GG] - M. Golubitsky, V. Guillemin, *Stable mappings and their singularities*, GTM 14, Springer-Verlag (1973).

[GH] - J. Guckenheimer, P. Holmes, *Non-linear oscillations, dynamical systems, and bifurcations of vector fields*, Appl. Math. Sc. 42, Springer-Verlag (1983).

[MP] - W. de Melo, J. Palis, *Geometric theory of dynamical systems*, Springer-Verlag (1982).

[S] - D. Schlomiuk, *Bifurcations and Periodic Orbits of Vector Fields*, Nato ASI Series C : Mathematical and Physical Sciences, Vol. 408.

[So] - J. Sotomayor, *Lições de equações diferenciais ordinárias*, Projeto Euclides, CNP_q (1979).

References :

[AAD] - H.Annabi, M.L.Annabi, F.Dumortier, *Continuous dependence on parameters in the Bogdanov-Takens bifurcation*, Pitman Research Notes in Mathematics Series 222, (1992), 1-21.

[Bam2] - R.Bamon, *Quadratic vector fields in the plane have a finite number of limit cycles*, Publ. IHES 64 (1986), 111-142.

[Ba] - M. Barnsley, *Fractals Everywhere*, Academic Press, San Diego (1988).

[B] N. Bautin, *On the number of limit cycles which appear by variation of coefficients from an equilibrium position of focus or center type*, Trans. Amer. Math. Soc. 100, p. 181-196 (1954).

[Bo] - R.I. Bogdanov, *Versal deformation of a singularity of a vector field on the plane in the case of zero eigenvalues*, Seminar Petrovski (1976) (Russian), Selecta Math. Soviet. 1 (1981), 389-421 (English).

[Bon] - P. Bonckaert, *Partially hyperbolic fixed points with constraints*, preprint.

[B.M.] - M. Brunella, M. Miari, *Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra*, J. Differential Equation 85 (1990), 338-366.

[Br] - A.D. Brjuno, *Local Methods in Non-linear Differential Equations*, Springer Ser. Soviet Math., Springer-Verlag, Berlin - Heidelberg - New York (1989).

[C] L.A. Cherkas, *On the stability of singular cycles*, Diff. Equ. 4 (1968), 1012-17 (Russian).

[Ch] -B. Chabat, *Introduction à l'analyse complexe,tome 1*, Mir (ed. française: 1990).

[DeR] - Z. Denkowska, R. Roussarie, *A method of desingularization for analytic two-dimensional vector field families*, Bol. Soc. Bras. Mat.

Vol. 22, n°1(1991), p. 93-126.

[Du1] - H. Dulac, *Recherche sur les points singuliers des équations différentielles*, J. Ecole Polytechnique (2), cahier 9 (1904), 1-25.

[Du2] H. Dulac, *Sur les cycles limites*, Bulletin Soc. Math. France 51 (1923), 45-188.

[D1] - F. Dumortier, *Singularities of vector fields on the plane*, J. Diff. Equ., 23, n°1 (1977), 53-106.

[D2] - F. Dumortier, *Local study of planar vector fields : singularities and their unfoldings*, in : Structures in Dynamics, Finite Dimensional Deterministic Studies (H.W. Broer et al., eds), Stud. Math. Phys. 2, North-Holland, Amsterdam (1991), 161-241.

[D3] - F. Dumortier, *Techniques in the Theory of Local Bifurcations : Blow-Up, Normal Forms, Nilpotent Bifurcations, Singular Perturbations*, Notes written by B. Smits, in [S].

[DER] - F. Dumortier, M. El Morsalani, C. Rousseau, *Hilbert's 16^{th} Problem for quadratic vector fields and cyclicity of elementary graphics*, Preprint, Université de Valenciennes (1995).

[DR1] - F. Dumortier, R. Roussarie, *On the saddle loop bifurcation*, in : Lectures Notes in Math. n° 1455, "Bifurcations of planar vector fields", J.P. Françoise, R. Roussarie, eds. (1990), 44-73.

[DR2] - F. Dumortier, R. Roussarie, *Tracking limit cycles escaping from rescaling domains*, in : Proceedings of International Conference on Dynamical Systems and Related Topics, september 1990 - Nagoya Japon - Ed. K. Shiraiva. Adv. series in dyn. syst., Vol.9, World Scientific, p. 80-99, (1992).

[DR3] - F. Dumortier, R. Roussarie, *Canard cycles and centre manifolds*, Preprint n°34, Laboratoire de Topologie, Université de Bourgogne (1994). To appear in Memoirs of A.M.S..

[DRR1] - F. Dumortier, R. Roussarie, C. Rousseau, *Hilbert's 16^{th} prob-*

lem for quadratic vector fields, Journal Differential Equations, Vol. 110, n°1, p. 86-133, (1994).

[DRR2] - F. Dumortier, R. Roussarie, C. Rousseau, *Elementary Graphics of cyclicity 1 and 2*, Nonlinearity 7, p. 1001-1043, (1994).

[DRS1] - F. Dumortier, R. Roussarie, J. Sotomayor, *Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case*, Ergodic Theory Dynamical Systems 7 (1987), 375-413.

[DRS2] - F. Dumortier, R. Roussarie, J. Sotomayor, *Generic 3-parameter families of planar vector fields, unfoldings of saddle, focus and elliptic singularities with nilpotent linear parts*, in : Bifurcation of Planar Vector Fields : Nilpotent Singularities and Abelian Integrals (F. Dumortier et al. eds.), Lecture Notes in Math. 1480, Springer-Verlag, Berlin - Heidelberg - New York (1991), 1-164.

[DRS3] - F. Dumortier, R. Roussarie, J. Sotomayor, *Generic unfoldings of cuspidal loops*, (in preparation).

[E] - J. Ecalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Hermann, Paris (1992).

[E1] - M. El Morsalani, *Sur la cyclicité de polycycles dégénérées*, Thèse, Université de Bourgogne (1993).

[E2] - M. El Morsalani, *Bifurcations de polycycles infinis de champs de vecteurs polynomiaux*, to appear in Ann. Toulouse.

[E3] - M. El Morsalani, *Perturbations of graphics with semi-hyperbolic singularities*, Preprint, Laboratoire de Topologie, Université de Bourgogne (1995).

[EM] - M. El Morsalani, A. Mourtada, *Degenerate and non-trivial hyperbolic 2-polycycles : appearance of 2 independent Ecalle-Roussarie compensators and Khovanskii's theory*, Non-Linearity 7 (1994), p. 1593-1604.

[EMR] - M. El Morsalani, A. Mourtada, R. Roussarie, *Quasi-regularity* *property for unfoldings of hyperbolic polycycles*, Preprint, Laboratoire de Topologie, Université de Bourgogne n°13 (1992). Astérisque : "Complex analytic methods in dynamical system". C. Camacho and al ed., n° 222, p. 303-326, (1994).

[F] - J.P. Françoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields*, Preprint, Paris 6 (1993).

[FP] - J.P. Françoise, C.C. Pugh, *Keeping track of limit cycles*, J. Differential Equations 65 (1986), 139-157.

[G] - A.M. Gabrielov, *Projections of semi-analytic sets*, Functional Anal. Appl. 2 (1968), 282-291.

[H] - M. Hervé, *Several Complex Variables*, Oxford University Press (1963).

[HI] - E. Horozov, I.D. Ilev, *On saddle-loop bifurcations of limit cycles in perturbations of quadratic Hamiltonian systems*, J.Diff.Eq., vol.113, no 1, (1994) 84-105 .

[I1] - Yu. Il'Yashenko, *An example of equation* $dw/dz = P_n(z,w)/Q_n(z,w)$ *having...* Math. Sbor., 80, 3, (1969), 365-378.

[I2] - Yu. Il'Yasenko, *Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane*, Funk. Anal. Ego. Pri., 18, 3,(1984), 32-34. (Func. Ana. and Appl., 18,3 (1985), 199-209).

[I3] - Yu. Il'Yashenko, *Finiteness theorems for limit cycles*, Amer. Math. Soc., Providence, RI (1991).

[I4] - Yu. Il'Yashenko, *Local Dynamics and Nonlocal Bifurcations*, notes by A.M.Arkhipov and A.I.Shilov in [S].

[IY1] - Yu. Il'Yashenko, S. Yakovenko, *Finitely-smooth normal forms for local families of diffeomorphisms and vector fields*, Russian Math. Surveys 46 (1991), n° 1, 1-43.

[IY2] - Yu. Il'Yashenko, S. Yakovenko, *Around Hilbert sixteenth Prob-*

lem, to appear in Advances in Sov. Math. (1994).

[IY3] - Y. Il'Yashenko, S. Yakovenko, *Finite cyclicity of elementary polycycles in generic families*, to appear in Advances in Sov. Math. (1994).

[JM] - A. Jebrane, A. Mourtada, *Cyclicité finie des lacets doubles non triviaux*, Non-Linearity 7, n° 22 (1994), 1349-1365.

[JZ] - A. Jebrane, H. Zoladek, *A non symetric perturbation of symetric hamiltonian vector field*, Advan. and Appl. Math. n° 15, 1-12, (1994).

[J1] - P. Joyal, *Un théorème de préparation pour les fonctions a développement tchébychévien*, Erg. Theory Dynamical Sys. (to appear).

[J2] - P. Joyal, *The Generalized Homoclinic Bifurcation*, J.D.E. 107, (1994), 1-45.

[J3] - P. Joyal, *The Cusp of order N*, J. Diff. Eq. 88 (1990), 1-14.

[K] A. Khovanskii, *Fewnomials*, Amer. Math. Soc., Providence, RI (1991).

[KS] A. Kotova, V. Stanzo, *On few-parametric generic families of vector fields on the two-dimensional sphere*, to appear in Advances in Soviet Mathematics.

[M] - B. Malgrange, *Ideals of differentiable functions*, Oxford University Press (1966).

[Mar1] P. Mardesić, *The number of limit cycles of polynomial deformations of a Hamiltonian vector field*, Ergodic Theory Dynamical Systems 10 (1990), 523-529.

[Mar2] P. Mardesić, *An explicit bound for the multiplicity of zeros of generic Abelian integrals*, Nonlinearity 4 (1991), 845-852.

[Mar3] - P. Mardesić, *Le déploiement versel du cusp d'ordre n*, Thèse, Université de Bourgogne (1992).

[Ma] J. Martinet, *Singularités des Fonctions et applications différentia-*

bles, P.U.C. (1977).

[M1] - A. Mourtada, *Cyclicité finie des polycycles hyperboliques des champs de vecteurs du plan : mise sous forme normale*, in : Bifurcations of Planar Vector Fields (J.P. Françoise and R. Roussarie, eds.), Lecture Notes in Math. 1455, Springer-Verlag, Berlin - Heidelberg - New York (1990), 272-314.

[M2] - A. Mourtada, *Cyclicité finie des polycycles hyperboliques de champs de vecteurs du plan : algorithme de finitude*, Ann. Inst. Fourier (Grenoble) 41 (1991), 719-753.

[M3] - A. Mourtada, *Degenerate and non-trivial hyperbolic polycycles with two vertices*, Preprint, Laboratoire de Topologie, Université de Bourgogne (1991).

[M4] - A. Mourtada, *Analytic Unfolding of Irrational and Trivial 2-polycycle*, Preprint, Laboratoire de Topologie n° 16, (1992).

[Mo] - R. Moussu, *Le problème de la finitude du nombre de cycles limites*,Astérisque 145-146 volume 1985/1986, expoés 651-668, (1987), 89-99.

[MoR] - R. Moussu, C. Roche, *Théorème de Khovanskii et problème de Dulac*, Invent. Math., 105 (1991), 431-441.

[N] - R.Narashiman, *Introduction to the Theory of Analytic Spaces*, Lect.Notes in Math. 25 (1966).

[PR1] - D. Panazzolo, R. Roussarie, *Desingularization of analytic families of planar vector fields without singular points of vanishing linear part*, To appear in Bol. Soc. Br. Mat.

[PR2] - D. Panazzolo, R. Roussarie, *Desingularization of Analytic families of planar vector fields without singular point of vanishing linear part*, (in preparation).

[P] - M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology 1 (1962).

[Per] - L.Perko, Differential Equations and Dynamical Systems.

[Pe] - G.S. Petrov, *Elliptic integrals and their non-oscillation*, Functional Anal. Appl. 20 (1985), 37-40.

[Po] - J.C. Poggiale, *Applications des variétés invariantes à la modélisa+tion de l'hétérogénéité en dynamique des populations*, Thèse, Université de Bourgogne.

[RE] -R. Roussarie, *Techniques in Theory of Local Bifurcations: Cyclicity and Desingularization*, 347-382 in [S]

[R1] - R. Roussarie, *Weak and continuous equivalences for families of line diffeomorphisms in Dynamical Systems and Bifurcation theory*, Camacho, Pacifico ed., Longman, Scientific and Technical, Pitman Research Notes in Math. Series 160, p. 377-385, (1987).

[R2] - R. Roussarie, *Déformations génériques des cusps*, Astérisque n° 150/151 : Singularités d'Equations Différentielles, Dijon 1985, p. 151-184, (1987).

[R3] - R. Roussarie, *A note on finite cyclicity property and Hilbert's 16^{th} problem*, Lecture Notes in : Math. n° 1331, "Dynamical Systems-Valparaiso 1986", R. Bamon, R. Labarca, J. Palis, eds. (1988).

[R4] - R. Roussarie, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*, Bol. Soc. Bras. Mat., 17, (1986), no. 2, 67-101.

[R5] - R. Roussarie, *Cyclicité finie des lacets et des points cuspidaux*, Nonlinearity, fasc. 2, (1989), 73-117.

[R6] - R. Roussarie, *Desingularization of unfoldings of cuspidal loops*, in : "Geometry and analysis in nonlinear dynamics" (with H.W. Broer, F. Takens) Eds. Pitman Research Notes in Math. Series, n° 222, Longman Scientific and Technical (1992), p. 41-55.

[RW] - R. Roussarie , F. Wagener, *A study of the Takens-Bogdanov bifurcation*, Preprint n°43 (1994), Laboratoire de Topologie, Université de Bourgogne, to appear in : Resenhas (São Paulo).

[Ro] - C. Rousseau, *Bifurcation methods in polynomial systems*, 383-428 in [S].

[RoZ] - C. Rousseau, H. Zoladek, *Zeroes of complete elliptic integrals for 1:2 resonance*, J.Diff.Eq 94 (1991), 41-54.

[S1] - D. Schlomiuk, J. Guckenheimer, R. Rand, *Integrability of plane quadratic vector fields*, Exposition. Math. 8 (1990), 3-25.

[Sm] - S. Smale, *Structurally stable systems are non dense*, Amer. Jour. Math., 88, (1966).

[So1] - J. Sotomayor, *Generic one-parameter families of vector fields on two-dimensional manifolds*, Publ. Math. I.H.E.S. 43 (1974), 5-46.

[So2] - J. Sotomayor, *On Stable Planar Polynomial Vector Fields*, Rev. Mat. Iberoamericana, 2 (1985), 15-23.

[T1] - F. Takens, *Unfoldings of certain singularities of vector fields. Generalized Hopf bifurcations*, J. Diff. Equa. 14 (1973), 476-493.

[T2] - F. Takens, *Forced oscillations and bifurcations*, in : Applications of Global Analysis I, Comm. Math. Inst. University of Utrecht 3 (1974), 1-59.

[T3] - F. Takens, *Singularities of vector fields*, Publ. IHES 43 (1974), 48-100 .

[Tr] S. Trifonov, *Desigularization in Families of analytic Differential Equations*, Ann. Math. Soc. Trans. (2), Vol. 165 (1995), 97-129.

[Y] - S. Yakovenko, *A geometric proof of Bautin theorem*, to appear in Advances in Soviet Mathematics (1995).

[Ye] - Y. Ye, *Theory of Limit Cycles*, Amer. Math. Soc., Providence, RI (1986). Monographs, Vol. 66, Trans. Math.

[Z] - H. Zoladek, *The cyclicity of triangles and segments in quadratic systems*, to appear in J.D.E..

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